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Leibniz's Puzzle And The Smooth Continuum: A Study In The Philosophy Of Mathematics

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LEIBNIZ'S PUZZLE AND THE SMOOTH CONTINUUM
a study in the philosophy of mathematics

by

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ABSTRACT

Kant distinguished between sensible and intellectual representation. The intellect represents mathematical objects as composed of their parts and so the continuum must be represented intellectually as a collection of punctual parts. However, an influential line of argument, advanced by Aristotle, Kant, and others contends that the continuum cannot be composed of parts, and so not determined by the intellect. Thus an intuition of space and time must be used in addition to intellection to determine mathematical objects.

The semantic tradition, in contrast, holds that intuition is not needed in order to determine objects. The closely related approach of transfinite set theory and the development of measure theory, topology, and mathematical logic has precluded the need for intuition of space and time by constructing continuua out of sets (of well distinguished objects). However this refutation of Kant is not decisive if Leibniz's infinitesimal calculus is taken seriously. For, underlying his calculus is the idea that each curve is locally straight and contains infinitesimal elements indistinguishable from zero. Hence, since there are no such objects in the universe of sets, such a curve cannot be a set. Thus a "Leibnizian puzzle" can be formulated with the consequence that intuition is needed in order to determine such a continuum.

However, this puzzle can be resolved by noting that it is possible, using the concepts of category theory, to widen the notion of set to that of variable set varying smoothly over a space. In such a model each curve is locally straight and the infinitesimal calculus can be developed. Thus the semantic philosophy can be extended to solve Leibniz's puzzle.

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INTRODUCTION

"I prized only that part of mathematics which was at the same time philosophy."

Bernard Bolzano.

"... from the seventeenth to the nineteenth century, the history of the philosophy of mathematics is largely identical with the history of the foundations of the calculus."

Abraham Robinson

The foundation of Leibniz's calculus rests upon an intriguing conception of the continuum (let us call it "the smooth continuum"): every curve can be regarded as an infinite-angled polygon. A curve is to be regarded, then, as possessing an infinity of points (the vertices of the polygon) joined together by infinitesimally small straight lines (the edges of the polygon). Leibniz also believed that mathematical knowledge was of clear and distinct objects of thought. In clear and distinct knowledge it is possible to enumerate the characteristics which distinguish an object from another object. Thus, he held a thesis which will be of central concern, the *decidability thesis*: $\forall x \forall y[(x = y) \vee (x \neq y)]$; i.e., the identity of any pair of actual mathematical objects is decidable (such objects will be called "decidable objects").

Leibniz did not believe that the parts of an object were themselves decidable and this fact is closely related to his solution to the problem of the composition of the continuum. The problem is whether the continuum should be regarded as composed of its parts, the parts being given as objects prior to the continuum, or whether the continuum is given as a whole which is prior to its parts. Taking the continuum to be composed of an infinite number of objects results in a number of paradoxes which were noted by Zeno, Aristotle, Leibniz and others. Leibniz's central problem appeared to be how we can grasp an infinitely divisible quantity as a whole, given the mind's finiteness.

Leibniz's solution consists in distinguishing between actual wholes, which are simple and indivisible substances (monads), and ideal wholes which arise as the *appearance* of an aggregate of monads. Ideal (as opposed to real) wholes contain their parts potentially whereas actual (real) wholes have no parts. We have a perception of the former, but our perception of it does not contain in itself the perception of these monads. Such phenomena constitute an ideal whole rather than a real whole and the parts of the continuum are indeterminate, arbitrary and ideal. Thus the continuum of Leibniz is non punctual inasmuch as it is not composed of actual objects.

The modern reaction to a non punctual model of the continuum is well known. Rescher exhibits this attitude when he says of Leibniz's view that

> One can, today, afford to be hard on Leibniz's treatment of the continuum problem. Subsequent developments in mathematics - the theory of transfinite numbers, point set topology, measure theory - have shown that Leibniz's method of attack was poor. (Rescher, 1967, p. 111).

Rescher is correct in pointing out that modern mathematical methods have cast doubt on a non-punctual model of the continuum. Modern mathematics has been quite successful at silencing the arguments of Aristotle, Leibniz, Newton and Kant. But it is Leibniz's conception of curves as infinitangular polygons which is particularly resistant to being treated as a composition of objects. The reason is fairly straightforward but was not explicitly proven until recently: infinitesimals are not decidable objects. This fact is not surprising, since in the seventeenth and eighteenth centuries the apparently contradictory nature of the infinitesimal (the fact that it seemed to be treated as both 0 and not 0 in calculations) made it difficult to accept that the identity of infinitesimals was decidable, and hence that any such analysis of curves into (decidable) objects was possible.

Arising from Leibniz's thought, then, is something which is at once both a puzzle and a challenge to promoters of punctual continua. *Leibniz's puzzle*: mathematical objects are decidable, but infinitesimal parts of a continuum are not decidable. Therefore some parts of a continuum are not mathematical objects. So, a continuum is not a collection of objects, and cannot be represented using the concepts of set theory. Of course this is not a paradox for Leibniz, since he considered the continuum as a confused perception of a multiplicity rather than a perception of a real whole object. Although the modern set theoretical approach has adopted the view that every mathematical object is decidable, rather than simply admitting that set theory is inherently limited by its inability to represent the smooth continuum, non-punctual continua are deemed incoherent.

Such an attitude towards infinitesimals is not completely unreasonable given the master's own attitude. Although the existence of the infinitesimal was accepted by a number of leading mathematicians including L'Hospital, Bernoulli, Fontenelle and Nieuwentijt, Leibniz himself was reluctant to treat infinitesimal and infinitangular polygons as actual objects, and at times, denied their existence. Instead he treated them as devices to shorten traditional mathematical reasoning, either by invoking a principle of continuity or the method of exhaustion.

Traditional mathematical reasoning relied heavily on the use of intuition. Newton's calculus, in particular, was founded upon the intuition of flowing non-punctual quantities. It was during the nineteenth century, beginning with Bolzano, that the use of intuition in mathematical reasoning was rejected. The banishment of the use of intuition as a method of mathematical proof, and its gradual replacement with a model theoretic approach, brought with it the concomitant rejection of the intuitive non-punctual continuum in favour of a punctual continuum. The newer model of the continuum is primarily due to the work of Cauchy, Weierstrass, Dedekind and Cantor who, in the nineteenth century, constructed the real numbers out of abstract sets. Refinements and developments, which continued into the twentieth century, allowed for the development of the conception of a "smooth" manifold, and the overcoming of traditional objections to a punctual continuum.

In contrast to the idea of a non-punctual continuum a *"smooth" manifold* is a collection of points which looks locally like an n-dimensional collection of real numbers endowed with enough derivatives, (see MacLane, 1986, ch. 6). Sometimes a continuum is thought of as being a compact, connected metric space, without requiring that it be differentiable. At any rate, on both these accounts a continuum is regarded as a set of well-distinguished points with some additional structure. These points are not regarded as potential cuts or possibilities of division, rather the continuum, on this view, is like a collection of building blocks. The "smoothness" of the manifold is due to a complex construction out of sets in which certain kinds of selections of blocks are ruled out so that the smoothness of the intuitive continuum is effectively mimicked.

As I mentioned, in the nineteenth century the belief in the autonomy of mathematics from other disciplines led the majority of mathematicians (Klein is a notable exception), to abandon the use of intuition of space and time in proofs, and this led to the belief that the continuum, as well as every mathematical object, is a collection of its elements. In 1925, in a paper given in honour of Weierstrass, Hilbert summed up the result of this process as ridding mathematics of the confused notion of the infinitesimal (Hilbert, 1989, p. 183).

> As a result of his penetrating critique, Weierstrass has provided a solid foundation for mathematical analysis. By elucidating many notions, in particular those of minimum, function, and differential quotient, he removed the defects which were still found in the infinitesimal calculus, rid it of all confused notions about the infinitesimal, and thereby completely resolved the difficulties which stem from that concept.

Historians of mathematics have often read the theme of this drama, which occurred over a period of about three centuries, into the history of mathematics writ large. For instance Boyer writes (1959, p. 4 - 5):

> Some twenty-five hundred years of effort to explain a vague instinctive feeling for continuity culminated thus in precise concepts which are logically defined but which represent extrapolations beyond the world of sensory experience. Intuition, or putatively immediate cognition of an element of experience which ostensibly fails of adequate expression, in the end gave way, as a result of reflective investigation, to those well-defined abstract mental constructs which science and mathematics have found so valuable as aids to the economy of thought.

There is a widespread tendency among mathematicians, philosophers, and especially historians, to think that the historical process which "gave way" to the set theoretic model of the continuum justifies this view of the continuum. ("If it was good enough for Dedekind, Cantor, Hilbert ... it's good enough for me.") But the result of a historical process is not necessarily the conclusion of a sound argument, especially when the result is itself subject to extensive dispute.

And the view that the continuum is composed of points has remained in dispute. As illustrious and varied a group as Aristotle, Zeno, Brouwer, Leibniz, Kant, Brentano, Poincare, Veronese, Weyl, Bergson, Godel, Lawvere, Galileo, Wittgenstein, Peirce, and Thom have objected to the punctual model of the continuum. A sampling of remarks give some of the flavour of the objections:

> ...no continuum can be made up out of indivisibles, as for instance a line out of points, granting that the line is continuous and the point indivisible. (Aristotle, trans. 1984, *Physics* 6.1)

Commenting on Cantor's theory of the continuum Poincare remarked that

> The continuum thus conceived is nothing but a collection of individuals arranged in a certain order. This is not the ordinary conception in which there is supposed to be, between the elements of the continuum, a sort of intimate bond which makes a whole of them, in which the point is not prior to the line, but the line to the point. Of the famous formula, the continuum is unity in multiplicity, the multiplicity alone subsists, the unity has disappeared. (quoted in Russell 1903, § 326)

> ... a true continuum has no points. (Thom, quoted in Bell, 1995)

> Space, like time, is a certain order - which embraces not only actuals, but possibles also. Hence it is something indefinite, like every continuum whose parts are not actual, but can be taken arbitrarily - space is something continuous but ideal , mass is discrete, namely an actual multitude, or being by aggregation, but composed of an infinity of units. In actuals, simples are prior to aggregation, in ideals the whole is prior to the part. The neglect of this consideration has brought forth the labyrinth of the continuum. (Leibniz, quoted in McGuire, 1992, p. 38)

> According to this intuitive concept, summing up all the points, we do not get the line; rather the points form some kind of scaffold on the line. (Godel, quoted by Wang, 1974, p. 86)

> Space and time are quanta continua ... points and instants mere positions ... and out of mere positions viewed as constituents capable of being given prior to space and time neither space nor time can be constructed. (Kant, 1784/1965, A170/B212)

Can *they all* be wrong? Perhaps, but it is unreasonable, even ridiculous, to believe that the notion of the continuum alluded to in the above remarks is incoherent. This suggests that the history of the rigorization of the calculus is not that of overcoming incoherent notions and establishing a conceptually clear view of the continuum but is rather a case of one conceptually clear view being traded for another. Restricting to Leibniz's view of the continuum, one may say that before such a shift the continuum consisted of points "intimately bonded" by infinitesimals and on the latter view it consisted solely of points.

The possibility of describing the points of a curve was suggested by Descartes's geometry which correlated points on a curve with numbers. As Weyl picturesquely put it, the continuum is like a gooey fluid, but in doing analysis only the points of the continuum are considered and the "gooeyness" (the infinitesimal) is left behind. The mathematician

> ... selects from the flowing goo .. a heap of individual points. The continuum is smashed into isolated elements and the interconnectedness of all its parts replaced by certain relations between the isolated elements. When doing Euclidean geometry it suffices to use the system of points whose coordinates are Euclidean numbers. The continuous 'space-sauce' that flows between them does not appear. (quoted in Feyerabend, 1983, p. 85)

So, in the sixteenth and seventeenth century it became possible to describe the points of a curve and the development of analysis allowed for proofs about the points of curves and many properties of the curve could be described in terms of its points. By the nineteenth century, however, it was believed that a curve *must* be described in terms of its points and that these points were not given in sensible intuition but rather, in Cantor's case, determined by concepts alone. It may suffice for certain purposes to use the system of points of the continuum, but it is little consolation for the Leibnizian calculus which uses both the points and infinitesimal line segments.

For many, the difficulties just cited would no doubt be thought of as ancient quibbles held as a result of a false philosophy of mathematics which regards mathematics as an intuition of forms in nature and consequently regards these objections based on intuition of the "true continuum" as irrelevant to mathematics. These objections are simply part of the old-fashioned view according to which a model must be given in intuition prior to its description by axioms. But on the modern model-theoretic view, the description of the continuum by axioms is independent of the selection of a model (which then may be guided by intuition). There is no "true" continuum, but only axiomatic descriptions of continua which may be true in one structure but not in another.

Continuing this line of argument one would say that requiring the mathematical descriptions to be limited to forms given in intuition, whether empirical or *a priori*, shackles mathematics to concepts and facts which are, in fact, irrelevant to mathematics. The outstanding example, of course, is how spatial intuitions limited the study of geometries to Euclidean geometries. But the examples may be multiplied. Limiting functions to those which represent the continuous motion of bodies could be thought of as hampering the development of analysis. Boyer (1959, p. 4 and 309) again expresses the standard historical view:

> It was nature which thrust upon mathematicians the problems of the continuum.... Nevertheless, in the rigorous formulation and elaboration of such concepts as have been introduced, mathematics must necessarily be unprejudiced by any irrelevant elements in the experiences from which they have arisen. Any attempt to restrict the freedom of choice of its postulates and definitions is predicated on the assumption that any preconceived notion of the nature of the relationships involved is necessarily valid.

In spite of this criticism, the necessity of intuition in mathematics has a venerable history and is not easily overcome. Aristotle and Kant both held that intuition was a necessary component of mathematical reasoning. Moreover, it is not only prominent philosophers who have regarded intuition as a necessary component of mathematical reasoning but a distinguished set of mathematicians from Euclid, to Barrow, Newton, Norris, Wallis, McLaurin, Colson, Poincare, Weyl, Brouwer and many others.

The words of Colson, an eighteenth century follower of Newton, are a good example of someone who thought that geometry and mechanics provided an ontological foundation for variable quantities (quoted in Guicciardini, 1989, p. 57):

> The foregoing principles of the doctrine of fluxions being chiefly abstracted and analytical, I shall here endeavour, after a general manner, to shew something analogous to them in geometry and mechanicks; by which they may become not only objects of the understanding, and of the imagination, (which will only prove their possible existence) but even of sense too, by making them actually exist in a visible and sensible form.

There is a standard philosophical explanation for why intuition was thought to be necessary for mathematical knowledge in the writings of Kantian interpreters such as Russell (1903), Allison (1983), Beck (1955), Hintikka (1992), Brittan (1992), and Friedman (1992) among others. It is that intuition offered a kind of reasoning that was unavailable in thought alone. The implication of this line of thought is that now, with a more powerful logic, intuition is no longer necessary.

Russell introduced this idea in the *Principles of Mathematics*, and it is worth quoting fairly fully since Russell links the observation (as I wish to do) to the problem of the composition of the continuum (or the problem of infinity and continuity as he calls it) (1903, § 24). Russell is not sympathetic with the intuitive conception of the continuum at all. For Russell modern logic does not only show that intuition is unnecessary but, like Hilbert, that the notion of infinitesimal is "unnecessary, erroneous, and self contradictory" (1903, §324). And for Leibniz's calculus he had these words: "And by his emphasis on the infinitesimal, he gave a wrong direction to speculation as to the calculus, which misled all mathematicians until Weierstrass.... and all philosophers down to the present day" (1903, §303). (The role of logic was not only a central theme in Russell's philosophy but also in the interpretation of other philosophers: the philosophies of Leibniz and Kant were thought to follow largely from their logical viewpoints.)

In describing the situation first he mentions the transformation of the problem in the hands of Weierstrass and Cantor:

> We come now to what has been generally considered the fundamental problem of mathematical philosophy - I mean, the problem of infinity and continuity. This problem has undergone, through the labours of Weierstrass and Cantor, a complete transformation. Since the time of Newton and Leibniz, the nature of infinity and continuity has been sought in discussions of the so-called infinitesimal calculus. But it has been shown that the calculus is not, as a matter of fact, in any way

> concerned with the infinitesimal, and that a large and important branch of mathematics is logically prior to it. (1903, §24)

Then, in a somewhat castigating tone, he construes this transformation as one which overcomes Kant's doctrine that mathematical objects are given in intuition:

> It was formerly supposed - and herein lay the real strength of Kant's mathematical philosophy - that continuity had an essential reference to space and time, and that calculus (as the word fluxion suggests) in some way presupposes motion or at least change.... All that has been changed by modern mathematics. What is called the arithmetization of mathematics has shown that all the problems presented, in this respect by space and time, are already present in pure arithmetic .... we shall find it possible to give a general definition of continuity, in which no appeal is made to a mass of unanalyzed prejudice which Kantians call "intuition" ....(1903, §24)

It has become increasingly clear, in retrospect, that it was more than a mere "mass of unanalyzed prejudice" which led Kant to adopt his position. As Russell well knew, Kant had a fundamental argument for the necessity of intuition in mathematics. Following Britain (1992) I label this argument Kant's *master argument* and briefly outline it. First of all, Kant says that:

> To know a thing completely, we must know every possible [property], and must determine it thereby, either affirmatively or negatively. (1784/1965, A573/B610)

Thus by determining an object, we may say (by the identity of indiscernibles) whether for any pair of objects that their identity is decidable. The master argument can be briefly stated as this: mathematical objects are determined. These objects are determined either by intuitions or by concepts. But concepts cannot determine intuitions completely; therefore intuition is needed to complete the determination of objects. Kant's strategy in the *Critique of Pure Reason* is to examine what is given in our cognition of mathematical objects and then to show that what is given in our cognition cannot arise wholly from the intellect, but requires intuition in addition (Falkenstein, 1991).

Not only is the conclusion of the master argument fundamental to Kant's mathematical philosophy but it bears a striking resemblance to

Aristotle's view regarding geometrical objects. This is important because Aristotle's understanding of geometrical objects was deeply ingrained in the mathematics of the seventeenth century (Jesseph, 1994; Gray, 1992). According to Aristotle, mathematical objects are the abstracted forms of physical bodies and of images produced in the imagination. The forms which are impressed upon the sensory organs of the body are then separated in thought. This proposal is made in opposition to Plato's view that mathematical objects are objects of reason alone. Thus it appears that we may say that for Aristotle too thought alone is not sufficient to determine mathematical objects (Lear, 1982).

Modern mathematics has offered new theories which are especially relevant to the understanding of the continuum. The unity of the whole continuum and the multiplicity of its parts is accounted for by considering collections of objects to have a unity which a mere multiplicity of parts cannot have. In addition the theory of transfinite numbers has allowed infinite quantities to be considered to be completed wholes. Real analysis, topology and measure theory have developed techniques which make it possible to show that certain properties (infinite divisibility, connectedness, and possessing a length or measure) *emerge* out of suitable collections of elements. These and other new methods of determining objects have not gone unnoticed. Russell's view has been given further ammunition by recent Kantian interpreters, such as Friedman (1992) who shows in some detail how the lack of mixed quantification and polyadic relations in logic forced the use of an intuitively conceived flowing continuum.

Of course Bolzano, Dedekind and Cantor thought intuition was irrelevant to mathematics. In adopting an alternative to intuition, they held that mathematical concepts are a *free creation of the mind*. In direct defiance of Kant, Cantor and Dedekind believed that intuition was not needed to determine objects. Objects need not be given in intuition but are determined purely logically when the identity of any pair of individuals is decidable. Dedekind made this point in 1872 by requiring that objects be "things" and Cantor made it in 1882 by requiring that the elements of sets be well-distinguished. Frege later made the same point in 1874 regarding numbers and required, in addition, that numbers have a criterion of identity. Hilbert made a similar point in his axiom of the "existence of an intelligence" in 1905. Finally, decidability is a provable consequence of Zermelo's axiomatization of set theory of 1908, and of Lawvere's axiomatization of the category of sets.

In spite of their differences, the method of Cantor as well as Pasch, Frege, Hilbert, Russell and others, may be broadly conceived as part of a movement to establish a semantic conception of mathematics (Coffa, 1991; Hallett, 1990, 1994; Demopoulos, 1994; Mayberry, 1994; Hylton, 1990) provide valuable recent writings on this topic). This philosophy of mathematics may be viewed as an alternative to the philosophical position of Kant on the necessity of intuition for a proper interpretation of the calculus. Intuition is not needed in order to supply content to mathematical assertions because there exist objective meanings which are independent of the (subjective) representations supplied by our faculty of intellect. Moreover, once the bond linking axioms to an intuitive interpretation is broken one is free to propose alternative models of axioms, thus making models independent of axiomatic description. Freedom of choice reigns in a way that no one interpretation is singled out as the "correct" interpretation. There are several kinds of continua just as there are several kinds of geometry. This approach may be called the "model theoretic" view.

It is worthwhile to explore this transition to the model-theoretic viewpoint because it is unclear to what extent its lessons have been fully assimilated. One leading mathematician, for instance, appears to commit himself to an abstractionist point of view rather than to a model-theoretic one. In MacLane's view mathematics studies an abstraction "in itself" where the phenomena from which it is abstracted is extraneous:

> ...Mathematics deals with a heaping pile of successive abstractions, each based upon parts of the ones before, referring ultimately (but at many removes) to human activities or to questions about real phenomena. The advance of mathematical understanding depends on the contemplation of each abstraction in itself. (1986, p. 448)

Thus it is appears that for MacLane axioms are not independent of their interpretation.

The independence of axioms from any particular interpretation in the model-theoretic outlook leads to the idea that the question as to which is the "correct" or "true" continuum is misconceived. Goldblatt suggests the problem in our context:

> Particularly striking is the fine-tuning that has been given to the modern logical/set-theoretical articulation of the conceived continuum (which to Euclid was not a set of points at all, let alone an object in a topos). Indeed it seems that the deeper the probing goes the less will be the currency given to the definite article in reference to "the continuum." (1984, p. xii-xiii)

On this reading the problem of the composition of the continuum is something of a pseudo-problem. Whether "the" continuum is the one described by Leibniz or by Cantor has meaning only relative to the model one is working within. This is indeed correct; but it only fuels the claim I wish to make. If mathematics is to be free, as Cantor says, then surely the guiding intuitions of Leibniz, Newton, L'Hospital, Bernoulli, Varignon and others are not to be dismissed as incoherent. And here is where the observation of Russell and the Kantian interpreters cuts both ways. For just as the intuitive non-punctual conception of the continuum arose because of the inadequacy of logic and mathematics to determine mathematical objects in Kant's sense, the punctual conception of the continuum of Dedekind and Cantor arose precisely because of the limitations of set theory to represent the smooth continuum. Thus, insofar as such a limitation is viewed as a *criticism* of the use of intuition and the resultant non punctual conception of the continuum, it is equally a *criticism* of set theory and its associated punctual conception of the continuum. This latter limitation is crystallized in the fact (noted in Leibniz's puzzle) that it is impossible that any curve is an infinitangular polygon in the category of sets. This fact makes it possible for a Kantian to argue that intuition must be called upon to represent the smooth continuum. How can one maintain the model-theoretic viewpoint and also represent the smooth continuum purely conceptually?

Notwithstanding the deep commitment to set theory in modern mathematics, the possibility of a smooth conception of the continuum has been rekindled recently as a result of the discovery of toposes in the 1960s by Grothendieck. As a first approximation a topos can be thought of as a kind of mathematical framework which shares fundamental structural similarities to the framework of set theory. Grothendieck discovered that the world of continuous spaces and maps and the world of discrete spaces and maps share fundamental similarities: they both can be modelled in toposes. So a topos is a mathematical framework sufficiently broad to encompass both the smooth objects (and smooth maps) and discrete mathematical objects (and discrete maps). An important consequence of this discovery is that it has made possible the construction of smooth spaces and maps out of discrete spaces and maps.

The idea that topos theory is able to unite the frameworks of smooth and the discrete is put into epic terms by Grothendieck.

> The theme of toposes ... is the 'bed' or 'deep river' in which are wedded geometry and algebra, topology and arithmetic, mathematical logic and category theory, the world of continuous and that of 'discontinuous' or 'discrete' structures. ... it is the vastest thing I have conceived, to capture an 'essence' common to situations the most removed from each other coming from one region or another in the vast universe of mathematical objects. (quoted in McLarty, 1992, p. 39)

There is, thus, an alternative to the punctual model of the continuum, and so the punctual model of the continuum has lost its historical inevitability. Moreover, as I mentioned above, and as Hallett has pointed out, it seems that the very *methods* used by those who established the set theoretical model of the continuum has resulted in a continuum which is truly smooth.

> ... mathematicians just do very often proceed by the method of Dedekind, Cantor and Hilbert. A nice example is provided by recent work on infinitesimals. The infinitesimals were originally treated as ideal elements, and, at least by Leibniz as useful but dispensable "fictions". But... *with the work on synthetic differential geometry, infinitesimals are admitted* as, in Robinson's words, "neither more nor less real than, for example, the standard irrational numbers" (1990, p. 249, my emphasis)

Just as the smooth continuum was the foundation of Leibniz's infinitesimal calculus, the new smooth continuum also allows for the development of the calculus (and differential geometry). What category theory does is extend the model - theoretic viewpoint to a wider possibility of structures. The universe of sets is no longer the unique or "absolute" model for mathematical axioms; categories which are very unlike the universe of sets are admitted as structures. It was to be expected that this development would not go unchallenged. Mayberry (1994) and Feferman (1973) and Bell (1981), in particular, have been strong critics of the role of categories as a substitute for the universe of sets while Bell (1986) defends the substitution. In particular Mayberry takes the position that the universe of sets is a *fixed* background ontology and *not a structure* which can satisfy some variable in the language of category theory. "The universe of sets is not a structure: it is the world that all mathematical structures inhabit, the sea in which they all swim (1994, p. 35)."

This criticism may be misleading because, in fact, there is something of a rift in the foundations of category theory. Excepting those who don't believe in the foundational role of categories at all, there are two profoundly different attitudes. Bell (1986) has argued that the absolute universe of sets, a kind of *mathematica magna* for mathematics, needs to be replaced - relativized - to a plurality of categories, namely toposes, with their corresponding local set theory. These toposes act as worlds of discourse or frameworks for mathematics. The language of topos theory may be defined in terms of some standard set theory. On the other hand, Lawvere (1966) and McLarty (1992) prefer an "absolute" category of categories which does not presuppose a universe of sets. Here I am presuming the former approach.

Nevertheless, the recent construction of smooth frameworks out of sets and functions has led to a significant reevaluation of the rigorization of the calculus movement. In hindsight it appears that the framework of smooth spaces and maps is the natural framework of the calculus of Leibniz. Bell claims that this framework allows "the virtually complete incorporation of the methods of the early calculus" (1988, p. 315).

The hope of revitalizing the Leibnizian calculus, in particular, is not unique to topos theory. It is well known that Abraham Robinson thought that his theory developed Leibniz's theory of the calculus, and, in fact, fully vindicated that theory, and that in light of Robinson's non-standard analysis that the history of the calculus must be rewritten (Robinson, 1966; 1969). Lakatos, as well, has expressed agreement that non-standard analysis revolutionizes the picture of the history of calculus. But this point of view has not gone undisputed.

The foremost historian of Leibnizian mathematics suggests that Leibniz's theory can be "rehabilitated" without adopting a new Robinsonian interpretation.

> If the Leibnizian calculus needs a rehabilitation because of too severe treatment by historians [not to mention mathematicians!] in the past half century, as Robinson suggests (1966, p. 260), I feel that the legitimate grounds for such a rehabilitation are to be found in the Leibnizian theory itself. (Bos, 1974, p. 82)

Edwards goes further. Non-standard analysis does not successfully elucidate the historical calculus of Leibniz but "merely" finds ways to reinterpret what he was saying in a new framework and therefore cannot justify *Leibniz's* calculus. Edwards says that

> It is true, as the above discussion suggests, that non-standard analysis can be employed to convert most of the intuitive infinitesimal arguments of the seventeenth and eighteenth century into logically precise arguments. But this is an *a posteriori* interpretation in terms of twentieth century mathematical thought rather than a vindication of the seventeenth and eighteenth centuries on their own terms. (1979. p. 346)

Edward's complaint here is that the interpretation of the calculus is quite different in *substance* than the mathematics of Leibniz. But this objection ignores the most salient aspect of the model theoretic viewpoint since, according to the model theoretic view, mathematical statements are not to be regarded as having fixed interpretations. Thus one should be able to "vindicate" the Leibnizian calculus by finding any interpretation under which it is valid.

There are serious problems, however, with the claim that non-standard analysis provides an interpretation of Leibniz's theory of the calculus. It is remarkable that no one, as far as I know, has pointed out that non-standard analysis does not implement Leibniz's fundamental idea of a curve being an infinitangular polygon; nor can it. This fact in itself makes Robinson's claims highly doubtful. Briefly put, non standard analysis is in contradiction with the main principle of Leibniz's calculus, namely, that every curve be considered an infinitangular polygon since Robinson's "hyperreal continuum" contains a property (the field property) which Leibniz's continuum cannot have.

I hope to show, in passing, that this latter criticism is not applicable to the interpretation of Leibniz's calculus within the framework of smooth spaces since the basic idea of the framework of smooth spaces, unlike that of non-standard analysis, *is* that the continuum is an infinitangular polygon. In addition, Leibniz did not adopt any particular interpretation (although he discussed a couple of possibilities) of this fundamental principle. His reticence to adopt an interpretation does not seem to arise out of sheer ignorance, but seems rather to be an attitude similar to the modern model-theoretic position. Mathematical reasoning has no need of an intuitive interpretation. Therefore it may not be an objection to a reformulation of Leibniz's calculus that it is given a modern interpretation that he could not have given it.

As a consequence of the development of a framework in which the smooth continuum exists one may see the rigorization movement, not as a flight from the incoherence of the smooth continuum which contains non - punctual infinitesimal parts to a coherent set theoretical continuum which contains only punctual parts; but as a concentration on the punctual parts of smooth spaces (which are sets) and punctual maps between such spaces (which are functions). McLarty has emphasized this connection.

> In the nineteenth century sets were first conceived as sets of points of spaces [punctual parts of smooth spaces], and then various assumptions and discoveries were made relating spaces to their sets of points until eventually spaces could be defined as sets of points with some structure. Then set theoretic thinking displaced geometric intuition in the foundations of mathematics. (1988, p. 75)

I will modify this proposal in three essential aspects. First, the rigorization of the calculus which displaced intuition for set theoretic thinking forced one to consider the punctual parts of smooth spaces to be sets (and not the other way around as McLarty seems to indicate). Otherwise there could be no continuity from intuitive thinking to set theoretic thinking. Secondly, it follows from this last observation that the shift from smooth spaces occurred gradually from the seventeenth to the nineteenth century. It did not occur as suddenly as one might presume from reading McLarty. Thirdly, McLarty believes the change of mathematical frameworks is pretty much a simple change in (internal) logic from intuitionistic to classical logic by adding the principle of excluded middle. This is correct, in a sense, but the situation is more subtle than this indicates because, as I will show, a central tenet of Leibniz, Kant, Bolzano, Dedekind and Cantor was that the identity of all mathematical objects is decidable (and this implies excluded middle). More precisely certain entities (points for instance) were not considered to be actual objects (as opposed to potentially existing objects or ideals) and so were not subject to the principle of excluded middle.

At any rate, if this line of thought is correct, then it shows that, although the rigorization movement may have released mathematics from the bonds of physical concepts of space, time and motion, it has now chained itself to models of the discrete mathematical framework of sets. Of course most thinkers simply assume that the models satisfying an axioms system are sets. Mayberry claims that the universe of sets "is the world that all mathematical structures inhabit" (1994, p. 35). The essential problem with the model-theoretic tradition (and an internal source of tension), then, is that the requirement that objects be well-distinguished, limits the kinds of models that can be constructed. In particular there can be no models of an intuitively smooth continuum, since the intuitive continuum contains objects which are not well-distinguished. More technically, there are no models of the topos of smooth spaces in the topos of sets.

Now we can return to the epistemological puzzle which was originally put forward. It seems that the Leibnizian continuum cannot be a set since the identity of every set is decidable by examining its members, but the identity of infinitesimals is undecidable. Is the Leibnizian continuum punctual? Yes and No. No because in the framework of sets and functions there are no infinitesimals. Yes because we may shift to a framework of smooth spaces and maps in which every curve contains infinitesimals. Moreover it is possible to construct a framework of smooth spaces and maps from a *base* framework of discrete spaces and maps. Thus, while the objects of the smooth framework are not necessarily decidable in that framework, they are composed of or emerge from distinct objects of the base framework. Thus, the continuum is non punctual but when *fully analysed* into components of the base framework, the continuum can be seen to be composed of distinct objects, and distinct maps. To put it in Leibnizian terms, the smooth continuum arises as the confused perception of an aggregate of monads (sets in the base topos). But if someone's (say God's) representation (of the smooth topos) contained within it the representation of every concept of which it was composed, then that representation would be of discrete objects.

I have given the gist of the argument of the thesis. In the first chapter I describe the traditional problem of the composition of the continuum and the non-punctual continuum as it arises in Aristotle and show how set theory attempts to answers these problems. Then, in the second chapter I ask whether the Leibnizian smooth continuum can be given a set-theoretic account and whether it is punctual as well. Leibniz believed that all mathematical objects were decidable but the puzzle that arises is that the smooth continuum seems to contain undecidable objects, and so it cannot be represented as a collection of decidable objects. In chapter three I present Kant's master argument for the necessity of an intuitive continuum. The important point that Kant raises is that the infinite divisibility of the continuum cannot be represented by the intellect alone because as finite creatures we cannot have a representation of each of its infinite parts. The rigorization of analysis is described as an attempt to defeat Kant's view that intuition is necessary for mathematical reasoning by developing a semantic philosophy which holds that the intellect may, by set theoretical reasoning, objectively represent the continuum as infinitely divisible without our thereby having a subjective representation of each its parts. The resulting rigorization is a shift, discussed by McLarty, from the investigation of the smooth continuum by Leibniz and others, to an investigation of the punctual parts of the smooth continuum by Cantor and others. This shift makes it impossible to have a smooth continuum as a model, since the smooth continuum contains non punctual infinitesimals and set theoretical reasoning cannot represent a continuum which contains non punctual elements. Thus Kant's argument can be extended: it is impossible to represent the smooth continuum by the concepts of set theory, and so intuition must be used. So, in the final chapter I briefly describe how the smooth continuum may be regained within the semantic tradition by constructing models in the *category* of smooth spaces; and thus shifting back from a discrete framework to a smooth framework.

ARISTOTLE'S OBJECTIONS AND SET THEORY'S SOLUTIONS

...no continuum can be made up out of indivisible, as for instance a line out of points..."

Aristotle

The traditional problem of the composition of the continuum is whether continuous quantities are composed of points. There is a tradition which holds that the continuum cannot be composed of points and Aristotle is one of its key figures. In what follows I briefly review three Aristotelian arguments against a punctual continuum in order to see how these objections can be handled in set theory. I report some solutions which draw on developments in the areas of transfinite set theory, topology and measure theory which attempt to show that the continuum can be composed of points after all. My intention is not to go into any great detail but merely to suggest the kind of solutions that set theory can offer and to give some background for later discussions.

The modern approach is quite removed from Aristotle's thinking. In ancient mathematics the objects of mathematics were considered to be abstracted from the forms of ordinary objects in nature, as a surface is abstracted from the surface of a table. These objects exist in nature as completed wholes and possess certain attributes which are given in perception. Thus the fact that a line segment is infinitely divisible, has a positive finite measure, and is cohesive were simply *given* as properties of the perceived object. Moreover, in the case of the properties mentioned above, every part of the continuum inherits those properties of the continuum.

Set theory implicitly offers a fundamentally different approach. According to one variant, the iterative view of sets, mathematics starts from given well-distinguished objects and collects them into a whole. Thus instead of extracting the parts from the intuitively given whole, set theory starts from given objects and compounds them into a whole, apparently united in virtue of the concept they fall under. So, the whole may possess properties that the parts do not have and likewise the parts may not inherit the properties of the whole. On this approach it is not assumed that every part of a line, for instance, has length, continuity, or cohesion, rather these properties *emerge* from suitable collections of elements which themselves do not have these properties.

If one is prepared to accept that parts of continua do not inherit all the properties of the whole then this approach can be seen as somewhat successful in answering the three *Aristotelian* objections. This success leads to an important question: is the Leibnizian continuum amenable to a set-theoretic analysis? *Is the Leibnizian continuum a punctual continuum?* Does the same strategy work? One might think that the rigorization of analysis settled this question by showing that Leibniz's (and Newton's) calculus and hence his view of the continuum, is ultimately based upon set theory. But this is a superficial reading of the history and philosophy of the movement, for instead of reinterpreting the view of the continuum and the use of infinitesimals, the infinitesimals were simply shunted aside by set theory and driven underground to be used only as "heuristic devices."

The apparent success of set theory in answering the Aristotelian objections inspires confidence that it can do the same for the Leibnizian continuum. But as I report in the next chapter, this confidence is quickly shattered when it is realized that the presence of undecidable elements guarantees that the Leibnizian continuum cannot be a collection of well-distinguished objects. But the concepts of set theory are usually taken to exhaust mathematical concepts. Thus it would seem that since we do, presumably, have the concept of a Leibnizian continuum, and it cannot be given by the concepts of set theory, we must rely on intuitive reasoning as Aristotle asserted.

Aristotle on continuity

Aristotle thinks of a continuous quantity as a unified whole consisting of an interior and its boundary such a line and its endpoints. The boundary of a continuum, such as a point or surface, is either potential or actual. If actual then it is the limit or extremity of the body, since an actual point has no interior. If the boundary is potential it indicates a position where a division may occur and an actual boundary would result. Such a boundary is a shared boundary of the continuum that it divides. A boundary serves two functions, then, since it allows for divisions and it bounds the interior of a region. "...the point, too, both makes the length continuous and bounds it" (trans. 1987, *Physics*, 4.11, 220a10-11).

The most important property of a continuum, is infinite divisibility. "The infinite is the first thing that presents itself to view in the continuous" (Trans. 1987, *Physics* 3.1, 16-21). This is important, for it indicates that continuity is a property which is simply given in intuition from the outset and so does not need to be proved to exist. Aristotle attributes infinite divisibility to continuous quantities in the following description of magnitudes and multitudes in Metaphysics 5.13:

> We call a quantity that which is divisible into constituent parts of which each is by nature a one and a "this". A quantity is a multitude if it is numerable, a magnitude if it is measurable. We call a multitude that which is divisible potentially into non-continuous parts, a magnitude that which is divisible into continuous parts; a magnitude, that which is continuous in one dimension is length, in two breadth, in three depth. Of these limited multitude is number, limited length is a line, breadth a surface, and depth a solid (Aristotle, 1020 7 - 14, quoted in Stein 1995, p. 336).

In short, the parts of a continuum are divisible into continuous parts. Thus its parts are divisible *ad infinitum*. For if not, then it would be divisible into a non-continuous part. This contrasts, notably, with a multitude, not every part of which is divisible into further parts. The Aristotelian stricture against a continuum being a multitude is, at first glance, true by definition. A magnitude is something contrasted with multitude. But this contrast evaporates if it is possible to have collections containing infinitely many members. In this case the collection will be infinitely divisible, but also actually divided into an infinite number of parts. So it will be a magnitude and a multitude.

Aristotle's view that parts of continuous quantities are potentially infinite rather than actually infinite represents a compromise between the apparent existence of the infinite and its paradoxical consequences. The apparent existence of an infinite number of atomic parts had seemed to follow from Democritus's argument in *Generation and Corruption* 1.2, 316a 13ff (Furley, 1969). This argument tries to prove the existence of atomic parts by proving its contradictory, that magnitude is divisible everywhere, false. For suppose that magnitude is divisible everywhere. If it is divisible everywhere what will be the products of such a division? They cannot be magnitudes, because such parts are further divisible. Nor can they be parts without magnitude, since such parts cannot be added together to make up a magnitude. So magnitude is not divisible everywhere, and so there must be indivisible atomic magnitudes.

But the existence of an infinite number of objects was difficult to accept because of the numerous contradictions in which the infinite was shrouded. One of the more amusing and persistent examples of a contradiction is what was called "Aristotle's wheel".

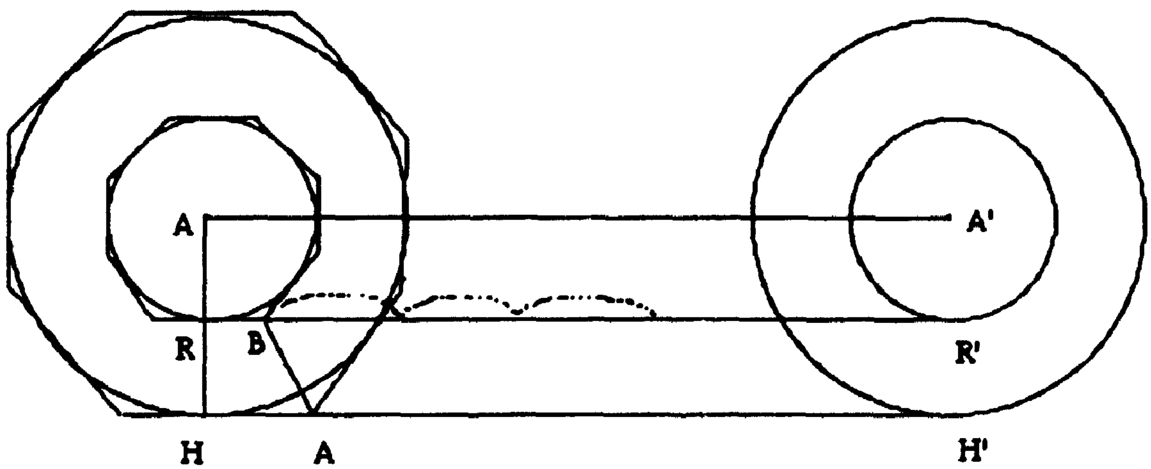


Figure 1

Consider the two concentric circles to be composed of their points. The circumference of a circle with radius two is twice as long as that with radius one, so if the lines are composed of points, the outer circle should (we intuitively think) contain a larger infinity of points than the latter. But by drawing radii as indicated one can see that each point R on the small circle corresponds one to one with a point H on the larger circle. Thus, we have the paradoxical situation that the two infinities are simultaneously of distinct size, and yet equal because their points are paired one to one. This argument against a punctual continuum was given by Duns Scotus in the thirteenth century (Moore, 1990, p. 50).

We can think of the larger circle as the rim and the smaller the hub of a wheel. The paradox is most vivid if we let the wheel rotate for one revolution. In this case the rim traces out a line *RR'* and the hub a line *HH'*. Since the wheel is rigid, and the three points are collinear at the beginning of the rotation, they will still be collinear at the finish. As the wheel rotates there will be no slipping: no point on the hub of th: wheel will slide through several points. Instead each point on the wheel will touch exactly one point on *HH'*. Similarly, there will be no skipping: no point on the line *HH'* will fail to be touched by the hub of the wheel as it rolls along.

The manner in which the hub rotates is exactly like the manner in which the rim rotates along *RR'*. If the circles are composed of points, how can the inner wheel make this revolution without skipping? For the inner concentric circle with a smaller circumference than the outer circle must match the movement of the outer circle over a path as it proceeds point by point, and finally cover a distance equal to the perimeter of the outer circle. The paradox suggests that the inner wheel must skip in order to cover the same distance as the larger.

Galileo offered his own resolution to this problem which involved giving the smaller length the apperance of a longer length by adding an infinite number of infinitesimally small line segments to serve as gaps between each point. The line is stretched out by the infinitesimal gaps, but the gaps are so small as to be unnoticeable. Galileo conceived of the circles as being "polygons of infinitely many sides", for after all "how ... without skipping can the smaller circle run through a line so much longer than its circumference" (Quoted in Smith, 1976, p. 575)?

The idea can be understood by considering both circles to be regular polygons with few sides, say octagons. As the outer octagon turns and pivots on a vertex *A*, the vertex *B* of the inner octagon must rise above the line *RR'* for a short distance *d*, a distance which is equal to the length of the side of the inner polygon, since *A B* is greater than distance from *A* to the line *RR'*. Thus, as the length of the sides becomes infinitesimally small, and the circle becomes an infintangular polygon, the gaps on *RR'* become infinitely small and give *RR'* the appearance of being equal to *HH'*.

Galileo also applied this idea to surface and to solid bodies:

> What is said of simple lines is to be understood also of surfaces and of solid bodies, considering those as composed of infinitely many unquantifiable atoms .... In this way there would be no contradiction in expanding, for instance, a little globe of gold into a very great space

> without introducing quantifiable void spaces - provided, however, that gold is assumed to be composed of infinitely many indivisibles (quoted in Smith, 1976, p. 575).

Aristotle's own answer to the Greek atomists did not involve the use of indivisibles. Instead he distinguished between the potential and actual infinite. On this reading, the paradox of Aristotle's wheel results from considering the set of points of the concentric circles to be actually infinite totalities. There is no sense in which we can compare the size of the points on the two lines because this would involve the simultaneous existence of an infinite number of points.

Yet, although the concept of the actual infinite leads to paradox, it is clear to Aristotle that the infinite must, in some sense, be said to exist because magnitudes are characterized by their infinite divisibility. What sense is it? Aristotle says that a continuous quantity can contain an infinite number of points "in addition" (by adding points successively) or "by division" (by making successive cuts in the quantity).

> But if there is, unqualifiedly, no infinite, it is clear that many impossible things will result. For there will be a beginning and end of time, and magnitudes will not be divisible into magnitudes, and number will not be infinite. Now when the alternatives have been distinguished thus and it seems that neither is possible, an arbitrator is needed and it is clear that in a sense the infinite is and in a sense it is not. "To be", then may mean "to be potentially" or "to be actually" ; and the infinite is either in addition or in division. It has been stated that magnitude is not in actual operation infinite; but it is infinite in division -it is not hard to refute indivisible lines - so that it remains for the infinite to be potentially. (Aristotle, trans. 1987, *Physics* 3.6 206a 10-15)

One can see from the above passage that "existence" is ambiguous for Aristotle, "existence" can mean potential existence or actual existence. Thus it is not that there are two kinds of infinity, the potential and the actual, one of which exists and the other does not. Potential and actual are kinds of existence; not kinds of infinity. The infinite, then, exists (potentially). This is the sense of existence in which the Olympic Games exist or in which a day exists. These are not actualized simultaneously in their entirely but are actualized successively, one by one. As Aristotle says: "The infinite exists when one thing can be taken after another endlessly, each thing taken being finite"(trans. 1984, *Physics* 3.6 206a25-30). Infinity is, therefore, not applied to individual things but to sequence of events or things (trans. 1984, *Physics* 3.4, 206a32-3).

Therefore Aristotle's conception of the continuity of a line is that of a potentially (unending) division of the line into further lines. To conceive of the division to have ended, is to conceive of the line as a completed infinity of individuals, and this result involves unsolvable paradoxes. This is a tactful evasion of the problems of the actually existing infinite, and leads to the use of construction in intuition as a necessary part of mathematics. For it is by construction of figures in thought, that they become actually existent. In Metaphysics IX where he describes the standard Euclidean proof that the sum of the angles of a triangle is equal to 180 degrees he relates: "It is by an activity also that geometrical constructions are discovered; for we find them by dividing. If they had already been divided, the constructions would have been obvious; but as it is they are present only potentially."

Almost all mathematicians up until Cantor followed Aristotle's approach to the infinite, that is to evade the problems by the use of his actual/ potential distinction. Gauss is often cited as an authoritative source of opposition to the actual infinite.

> But concerning your proof, I protest above all against the use of an infinite quantity as a completed one, which in mathematics is never allowed. The infinite is only a *facon de parler*, in which one properly speaks of limits. (Quoted in Dauben, 1979, p. 121)

Set theory attempts to overcome these problems by maintaining that there is an actual infinite number of points in the continuum. Traditionally, collections were viewed as pluralities, or multiplicities of units. Cantor replaced this idea by considering numbers to be true unities, because in them "a multiplicity and manifold of ones is joined together" (quoted in Hallet, 1984, p. 124) thereby turning the set into an object of study in its own right. The ability to conceive of collections as complete and unified where formerly they were mere multiplicities, leads to astounding possibilities. Cantor creates new transfinite numbers by generating them. First, we may add a new unity to an exiting unity. This generates all the natural numbers. Then we create a new transfinite number ω, the least ordinal number, as the limit of this sequence by considering the natural numbers as a whole. Thereafter we may add unities and take limits to generate new numbers. Cantor attempted to justify the existence of such transfinite numbers by showing that they each express a well ordering on their underlying sets. Thus, since finite sets are well ordered, to accept finite numbers is to implicitly accept the existence of transfinite numbers as well.

Accordingly Cantor replaces the notion of absolute (unending) infinity with actual infinity. He then divides the actual infinite into the increasable infinite, which he calls the "transfinite" and the "unincreasable" or "absolute" infinite. Or perhaps, it is better to say that, he extends the notion of finite to that of transfinite. It is the absolute infinite which is beyond mathematical determination. In Cantor's 1891 paper he presented his diagonal proof which showed that the size of the set of subsets of a set V is always greater than that of the set V. It followed that although the algebraic and hence rational numbers could be put into one-one correspondence with the natural numbers the same does not hold for the real numbers. Cantor drew from this result the conclusion that there is a "clear difference between a so called continuum and a set of the nature of the entire algebraic [and hence also natural] numbers." Later he went on to further characterize the continuum in ordinal terms.

The well known philosophical point which is drawn from this proof is that it allows for a distinction between two criteria for the size of collections, the part criterion and the correlation criterion. The parts criterion has it that if a collection is a part of a (more extensive) whole, then the part is of smaller size: the whole is greater than the part. According to correlation criterion, collections are of the same size if they can be put into a one-one correspondence. This condition is natural because it implies that there are the same number of people in a concert hall as there are seats if and only if every seat is occupied by one person. But it also has the "paradoxical" consequence that there are the same number of even numbers, odd numbers, algebraic numbers, and rational numbers, as there are natural numbers. So a collection which is a part of a collection can be of the same size as the more extensive whole. Therefore what the paradoxes show is that collections of different extensiveness are, nevertheless, the same size.

On Cantor's view, then, the paradoxes result from attempting to apply both criteria for the size of collections to transfinite sets, whereas they only apply together to finite sets.

> All so-called proofs against the possibility of actually infinite numbers are faulty, as can be determined in every particular case, and as can be concluded on general grounds as well..... from the outset they expect or impose all the properties of finite numbers upon the numbers in question, while on the other hand the infinite numbers ... must (in contrast to finite numbers) constitute an entirely new kind of number, whose nature is entirely dependent upon the nature of things and is an object of research, but not of our arbitrariness or prejudices. (Cantor, quoted in Dauben, 1979, p. 125)

The Cantorian solution to the paradoxes is well-entrenched, but it leaves some room for an Aristotelian to mount a challenge to the conception of a line as composed of points. For Cantor divided the infinite into the increasable and the unincreasable but Aristotle did not. For Aristotle the points on the line were, in essence, unincreasable. Thus it is not obvious how to apply the Cantorian solution to Aristotle. A Cantorian solution is usually forced upon Aristotle by saying that infinite divisibility for Aristotle is merely density: for any two points there is an intermediate point (of division). Since Cantor showed that there are actual dense collections of numbers, Aristotle was simply wrong to think that the continuum could not be composed of points.

But this interpretation of Aristotle is misleading. For Aristotle a multitude is divisible into parts which are indivisible, but a magnitude is divisible into *continuous* parts so that they are again divisible *ad infinitum.* In Aristotle's words, the continuum is "divisible everywhere". What would Aristotle say today if confronted with the set of real numbers? If we consider the real numbers, we find that they are dense, but contain indivisible real numbers as parts. It is tempting to believe that he would say that the collection of real numbers cannot be a magnitude since a magnitude is such that *every* part is divisible into parts *ad infinitum.*

Another look at Democritus's argument in *Generation and Corruption* for indivisibles seems to confirm this interpretation. In this argument the assumption is that an actual division, of a quantity which is divisible everywhere must leave nothing.

> Suppose then that it is divided; now what will be left? Magnitude? No that cannot be, since there will then be something left which is not divided, whereas it was everywhere divisible. But if there is to be no body or magnitude [left] and yet [this] division is to take place, then either the whole will be made of points, and then [parts] of which it is composed will have no size, or [that which is left] will be nothing at all. (A2, 316a 24-30, quoted in Furley, 1967)

A similar objection to Aristotle's rejection of punctual continua was given by Peirce. Peirce thought that because one cannot show that there is a *set* of "all the points there are" the continuum cannot be considered to be a collection of points. This is closely related to the fact established by both Cantor and Peirce that there can be no set of all sets. Since if V is the set of all sets it must contain all of its subsets. But Cantor's diagonal theorem show that the size of the subsets of a set V is always greater than that of the set V. Cantor used the diagonal method to show that the set of real numbers (i.e. the power set of natural numbers) is not in one - to - one correspondence with the natural numbers.

Unlike Cantor, Peirce drew from this result a conclusion that threatens to undermine the belief that the set of real numbers is a continuum. He observes that if a line is truly continuous (and so contains no gaps) it must contain "all the points there are", an unincreasable or absolute infinity of points, and so cannot be a set. By Cantor's (and Peirce's) theorem there can be no set of all sets, and so it follows that points cannot be constituent parts of a line:

> For if they were so, they would form a collection; and there would be a multitude greater than that of the points determinable on a line. We must, therefore, conceive that there are only so many points on the line as have been marked, or otherwise determined upon it. Those do form a collection, but even a greater collection remains determinable upon the line. All the determinable points cannot form a collection, since, by postulate, if they did the multitude of that collection would not be less than another multitude. The explanation of their not forming a collection is that all the determinable points are not individuals, distinct, each from all the rest.... (1960, p. 363)

If this line of thought is correct that the "totality" of all points is not a set, what kind of thing is it? Peirce's answer seems to be simply that it is, of course, a line which is grasped intuitively as a whole. Its continuity consists in the fact that it has an unincreasable infinity of points on the line. No matter how many points are determined on the line (by bisection, for instance), like Hilbert's hotel there is always room for more.

> It may be asked, "if the totality of the points determinable on a line does not constitute a collection, what shall we call it?" The answer is plain: the possibility of determining more than any given multitude of points, or in other words, the fact that there is room for any multitude at every part of the line, makes it *continuous*. Every point actually marked upon it breaks its continuity, in one sense. (Peirce, 1960, p. 363)

But so far this can hardly be considered an objection to a punctual model of the continuum. For it accepts that the real number continuum is punctual, it is just not punctual enough! Although there is always room for more points, the continuum becomes "overcrowded" in the sense that identity of the points is no longer decidable. It is this latter feature which transforms "all the points" into a line. In his words:

> A supermultitudinous [absolute or increasable infinite] collection sticks together by logical necessity. Its constituent individuals are no longer distinct and independent subjects. They have no existence - no hypothetical existence - except in their relations to one another. They are not subjects, but phrases expressive of the properties of the continuum (1976, Vol. 3, p. 95).

So Peirce is not thinking of a punctual continuum after all. The "constituent individuals" are not well-distinguished objects. At the root of Peirce's view, then, is an agreement with Leibniz's puzzle. The line cannot be a collection of points because it is not composed of well distinguished objects. Consider a finite line segment. No points are actually determined on a line until some construction, such as a bisection occurs. After a bisection occurs a new shared point, the boundary of the two distinguished parts exists. Peirce's idea that there must be an inexhaustible supply of points, or an absolute infinity of points, is simply a consequence of Aristotle's idea that every part must be divisible into an divisible part. One could argue that this highlights the point common to Peirce, Brouwer, Bergson and Kant, that intuitive cognition is able to conceive of absolute infinities as completed wholes, (such as the cognition of a given line) whereas intellectual (discursive) cognition cannot.

How can set theory deal with this problem? A defender of a punctual model of the continuum might well admit that there can be no "absolutely continuous" set. But rather than admit that a line, with its absolute infinity of points, is given in perception as a particular whole they could deny that we can have any such notion. Such a notion of an unending infinity synthesized into a whole is itself inconsistent. As Cantor says:

> A collection can be so constituted that the assumption of a "unification" of all its elements into a whole leads to a contradiction, so that it is impossible to conceive of the collection as a unity, as a "complete object." Such collections I call absolute infinite or inconsistent collections. (quoted in Dauben, 1979, p. 245)

Accordingly, what Cantor's argument against a set of all sets shows is that we cannot consider a given line segment in either intuition or the intellect as containing a completed absolute infinite, since either contains a contradiction. But we can coherently conceive of a transfinite number of points on a line, and for this we do not have to resort to intuition.

The continuum and cohesiveness

A further property of the continuum is its cohesion: the elements of the continuum cohere or stick together as a continuous, unified, whole. This is closely related to Aristotle's view that continuous quantities are given in sense perception and that sense perception is always of the particular. We sense a line as a continuous whole, and its parts as cohering by virtue of the fact that when we make a cut in the continuum the parts share and are thereby welded together by this common boundary. This property has been of paramount importance in Aristotle, Leibniz, Kant, Brouwer, Peirce and others. Aristotle uses the property of cohesion to argue that a continuum cannot be punctual. In *Physics* 6.1 he says, as I quoted earlier, that " it is impossible that something that is continuous be constituted from indivisibles, e.g. a line from points if the line is continuous but the point indivisible" (231a24-26, quoted in White, 1988, p. 2). In *Physics* 5.3 Aristotle defines continuity in terms of succession. "A thing succeeds something when it is distinguished as coming after the first with respect to place or kind or some other respect and there is nothing of the same sort between it and what it succeeds" (226b34-227a1, quoted in White, 1988, p. 4). Later he adds:

> The contiguous is that which, being in succession, touches [what it succeeds]. The continuous, then, is a species of the contiguous. I call something 'continuous' whenever the limit of both things at which they touch become one and the same, and as the word implies, they are "stuck together". But this is not possible if the extremities are two. It is clear from this definition that continuity pertains to those things from which there naturally results a sort of unity from their being joined together. (227 16, 227a10-15, quoted in White, 1988, p.4)

So on Aristotle's conception two curves are continuous when their extremities or boundaries touch and become one. As White (1988) points out, the fundamental idea here is that what is continuous is a natural unity; consequently any parts or divisions which we may want to distinguish within it will share a common boundary. Thus, there can be no "natural joints, seams or articulations" in the continuous with which to make a clean division into well-distinguished parts. As Aristotle remarked elsewhere: " Of things called one in their own right some are so called from being continuous" (trans. 1987, *Metaphysics* 5.6 1015b36 -1016a1).

The natural unity of the continuum, perhaps, has been the most emphasized aspect of the intuitive continuum. Thus it has often been observed that points cannot be considered in isolation from the continuum but are always part of a whole. In this sense the whole is prior to the points and the points arise as mere limits of the whole. Let me note a few instances of this belief to show how pervasive it is.

The student of Aristotle, Franz Brentano, put it this way:

> The spatial point cannot exist or be conceived of in isolation. It is just as necessary for it to belong to a spatial continuous whole as it is for the moment of time to belong to a temporal continuous whole. (1874/1973, p. 356)

Kant contrasted our intuitive cognition of particulars with intellectual (conceptual representation). He emphasized that in the representation of continuous magnitudes, such as time, the parts are represented through limitation of the whole intuition and therefore must be given in intuition:

> The infinitude of time signifies nothing more than that every determinate magnitude of time is possible only through limitations of one single time that underlies it. (1781/1965, A32/B48)

According to Brouwer, the continuum is given in our experience of moments as boundaries of past and future in which the past and present are joined together. I am calling this property "cohesion." Brouwer calls it the intuition of "two-oneness" and like Aristotle he sees this as the foundation of continuity. In Brouwer's Inaugural address at the University of Amsterdam he says this:

> This neointuitionism considers the falling apart of moments of life into qualitatively different parts, to be reunited only while remaining separated by time, as the fundamental phenomenon of the human intellect .... This basal intuition [of two -oneness], in which the connected and the separate, the continuous and the discrete are united, gives rise immediately to the intuition of the linear continuum, i.e. of the "betweenness", which is not exhaustible by the interposition of new units and which therefore can never be thought of as a mere collection of units. (1983, p. 80)

This experience of a boundary is not exhaustible by pure units because the continuity is not "in the units" but in their "betweenness" or the fact that they were joined together. Of course any set is exhaustible, or has a complete and definite number of members. Therefore Kant was right, Brouwer thinks, to consider arithmetic and geometry to be synthetic *a priori*.

On this view, then, the continuum is a unity and may be thought of as an idealized "perfect fluid", a homogeneous whole which has extension and duration, but has *no natural joints or gaps to individuate all of its parts.* One cannot arbitrarily "reach in" to a continuum and "pull out" a well-distinguished object. Like a fluid, it will simply flow through one's hands. Consequently, a continuum is non punctual, because any point would serve as natural joints to distinguish its parts.

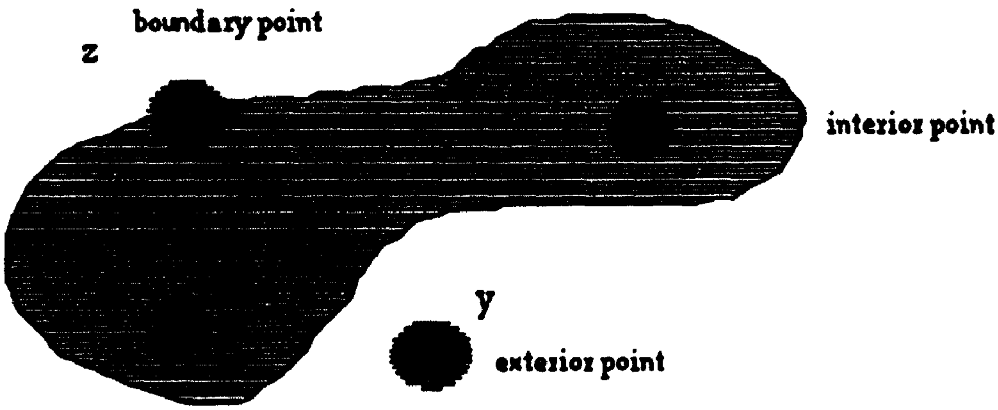
This notion of a natural unity is remarkably similar to that of a *connected* topological space. Thus it is not surprising that it has been argued by White (1988) that Aristotle's requirement that a continuum be a natural whole is met by the property of the *connectedness* of a topological space. The fundamental idea is to build into the continuum a kind of cohesiveness by constructing the continuum out of (open) sets such that, while the points may not be individually continuous, collections of them will be continuous. This perspective accords with Aristotle's that the points of a continuum will not be individually continuous, but is not consonant with Aristotle's assumption that cohesiveness cannot emerge from a larger collection of points. In particular, open sets of points, on an Aristotelian view, are not properly parts or "chunks" of the continuum. For chunks of the continuum must have (non-empty) interiors and definite boundaries. But if this topological interpretation is correct, we can extend the notion of a thing to include open sets of things, and then set theory can account for the natural unity of continua, and therefore Aristotle's objection can be overcome. In order to assess this claim we need some basic notions of topology.

Suppose we are given an n-dimensional Euclidean space $R^n = \{x = (x_1\ x_2, \ldots, x_n) : x_i \in R\}$. For instance R is the real line, and R^2 is the Euclidean plane. Intuitively, a *neighbourhood* of a point x is the set of points which are sufficiently close to x. These neighbourhoods define any "natural joints" which the space may have. Let A be a subset of points in R^n. A point x is an *interior* point of A if there is a neighbourhood N of x such that $x \in N \subseteq A$, i.e., the disc is totally enclosed in A. A point is *exterior* to A if there is a neighbourhood N such that $x \in N \subseteq R^n - A$, i.e. the disc is totally outside of A. In other words $N \cap A = \varnothing$. A point x of R^n is a limit point of A if every neighbourhood N of x contains at least one point of A *distinct from* x, i.e. $N \cap A \neq \varnothing$.

The importance that each point of the continuum be a limit point was emphasized by Bolzano:

> ...a continuum is present when we have an aggregate of simple entities (instances or points or substances) so arranged that each individual member of the aggregate has, at each individual and sufficiently small distance from itself, at least one other member of the aggregate for a neighbour. When this does not obtain, when so much as a single point is not so thickly surrounded by neighbours as to have at least one at each individual and sufficiently small distance from it, then we call such a point isolated, and say for that reason that our aggregate does not form a continuum. (Quoted in Wilder, 1978, p. 721)

Note that, according to this definition, every point of A is a limit point of A. Also, if x itself is in A and is the only point in A which lies in a neighbourhood of N (i.e.. $N \cap A = \{x\}$), then x is called an *isolated* point of A. If x is a limit point of A such that every neighbourhood of x contains a point $y \notin A$, then x is a boundary point of A. So every boundary point is also a limit point. Since if a point x is a limit point but not a boundary point, then it is not the case that every neighbourhood of x contains a point $y \notin A$; so there is a neighbourhood of x which contains no points not in A and hence there is a neighbourhood of x which contains only points of A. So x is an interior point of A. We can picture this situation something like a golf course, complete with putting greens and holes.



{limit points} = {boundary points}∪{interior points}

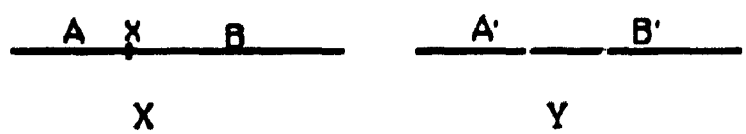
Figure 2

A set A is said to be *open* if every $x \in A$ is an interior point. A set A is said to be *closed* if every $x \notin A$ is an exterior point. A *topological space* is an ordered pair (X, O) of which the first element is a set of points and the second element O, the topology of X, is a family of open sets meeting the following conditions:

(1) $\varnothing$ and X are in O.
(2) the union of the elements of any subset of O are in O.
(3) the intersection of the elements of any finite subset of O are in O.

As White (1988) maintains, the open sets of X represent the possibility of a natural articulation or joint in the topological space. In a topological space the family of open sets constitutes the parts or regions of space that are subject to being "pulled out" or "excised" while leaving the remainder of the space intact. In general we can think of a joint as a "natural" or a "clean division" if, whenever we pull out a set A out of O, leaving the remainder in behind, we "take with it" only the elements of A. On the other hand, a messy division results if, whenever we pull a set A out of O, we must "take along with" A elements (limit points of A) which are not in A (the boundary points of A). Thus the remainder of the space is not left intact. However, it may not be the case that any sets can be "pulled out" of the topology while leaving the remaining space intact, and in this sense a topology will have no "natural seams or joints". In order to formulate the condition that a continuum has no natural joints, then, there must be a condition whereby whenever we "pull" an open set A from O we have a messy division. This condition is expressed by the *connectedness* of a topological space.

S is a *connected* set if whenever S is separated into two non-empty disjoint sets A and B such that their union is S, then either A contains a limit point of B or B contains a limit point of A. Therefore any way of slicing a connected set results in a messy division, one which does not respect the natural joints established by the open sets. S must be considered to be a natural unity or to consist of only one part.



X is connected but Y is not

Figure 3

In the above figure X and Y have been sliced into two parts, where B looks exactly like B' and A looks exactly like A' *with the point* x *adjoined.* A and B are now stuck together. The point x "glues" A and B together to make X connected and, being absent from Y constitutes a gap which allows Y to break into two parts. Since x is in the set A and is a limit point of A and B, so that x is infinitesimally close to B; in this sense, x, and also A cannot be separated from B. We note that x is a limit point of both A and B. Another way of stating the property of connectedness is to say that a continuum S cannot be the union of two non-empty disjoint subsets which are open relative to S. S admits no separation.

The intuition that the continuum must be cohesive in this sense was thought by Dedekind to be the "essence of continuity". The Dedekind cut property requires that:

> If all points of the straight line fall into two classes such that every point of the first class lies to the left of every point of the second class, then there exists one and only one point which produces this division of all points into two classes.... (1963, p. 11)

Call this point b. The infinite divisibility of the line entails that it is divisible at a point b to obtain two parts X and Y. According to the Dedekind cut property, b is going to have to be either in the left part (X) or the right part (Y) but not both. The property of cohesiveness guarantees that X and Y share a common boundary point, the point where they are together. More explicitly, it can be proved that any set which has the Dedekind cut property is connected.

So far I have attempted to ascertain whether modern topology is able to escape Aristotle's objections against a punctual continuum by interpreting "punctual" in a more liberal manner. A continuum could be said to be punctual simply if it was a set. But Aristotle had an argument against a punctual continuum in *Physics* 6.1 which was designed to show that "it is impossible that something that is continuous be constituted from indivisibles, [like], a line from points if the line is continuous but the points indivisible" (quoted in White, 1988, p. 4). It is interesting to assess what remains of this argument from the topological perspective.

In order to do this it is helpful to reformulate the notion of a topology of a set in terms of a basis set of elements for a topology rather than by describing the entire collection of open sets. To give a basis for the topology we need two conditions:

(1) for each $x \in A$, there is at least one basis element B containing x

(2) if $x \in (B_1 \cap B_2)$, then there is a basis element B_3 containing x such that $B_3 \subset (B_1 \cap B_2)$.

Now it is possible to *generate* a topology from the basis set by defining a set U of A to be *open* in A if, for each $x \in U$, there is a basis element B such that $x \in B$ and $B \subset U$. In other words every open set is the union of members of the basis. (For proof consult any topology text such as Munkres, 1975.) For example, the set of all circular regions of the plane satisfies the conditions for being the basis of a topology.

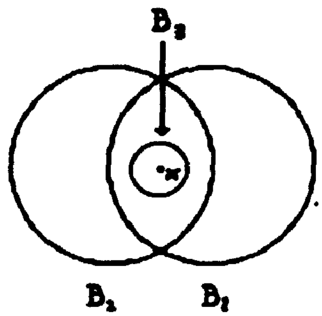


Figure 4

Aristotle's assertion that a continuum in Physics 6.1 cannot be composed of points can be reinterpreted by adherents of the punctual continuum as saying that a basis for the topology on the continuum cannot be the singleton sets of points (White, 1988). On this point the adherents of the punctual continuum agree. For by taking the singleton set of points as a basis for a topology we have, in fact, produced a topological space which is everywhere jointed. Such a basis produces the discrete topology, the set of all subsets of A. Here it is possible to separate any set into two disjoint open sets, since each open set (trivially) contains all its limit points (so it is closed), and so the topology is not connected. For if we consider any subset A of R^n, and x of R^n, there is an open set, namely $\{x\}$, which contains x, but does not contain any other point of A distinct from x. Thus x is not a limit point.

Now consider how this applies to Aristotle's argument in Physics 6.1. One rendering of Aristotle's argument against the view that the continuum is composed of indivisibles, that given by White (1988), goes as follows. Things are continuous only if their extremities are one; and things are touching only if their extremities are together. But since indivisibles do not have parts, they cannot be divided into parts with extremities and parts without. Now there are two possibilities: either (a) it is not the case that the extremity of one indivisible is one with or is touching the extremity of another indivisible; or (b) an indivisible touches another as "whole to whole." If (a) is the case, then pairs of indivisibles are neither continuous nor touching. But Aristotle believes that it is necessary that if the continuous were composed of points, then each indivisible would be "touching one another" or continuous (trans. 1984, 231a29-b1). In other words, points must be individually continuous. So if (a), then the continuous is not composed of indivisibles. If (b), then if the continuous were composed of indivisibles, none of these indivisibles would be distinguishable from any other constituent indivisible since they would all be whole to whole. Aristotle requires that whatever is continuous has distinct parts and may be divided in terms of these, which are therefore different and separate in place (trans. 1984, 231b4-6). Therefore, if (b), the continuous is not composed of indivisibles. So, by constructive dilemma, the continuous is not composed of the discrete, as a line from its points.

The premise which is questionable from the topological perspective contained in (a) is that pairs of points must touch and "become one." Points must be regarded as *individually continuous* or in contact with other points. As I have discussed this conception has also been endorsed by Peirce. One could also attribute it to Leibniz on the basis of his law of continuity, that "nature makes no jumps":

> Matter, according to my hypothesis, would be divisible everywhere and more or less easily with a variation which would be insensible in passing from one place to another neighbouring place; whereas according to the atoms, we make a leap from one extreme to the other, and *form a perfect incohesion,* which is in the place of contact, we pass to an infinite hardness in all other places. And these leaps are without example in nature. (quoted in Mates, 1986, p. 165, my emphasis)

From a topological perspective it may be that continuity fails at an individual level but emerges at a collective level when a sufficiently large number of points are collected. Aristotle's anti - emergent view follows quite evidently from his statement of the infinite divisibility of quantities in *Metaphysics* V.

For if continuity is an emergent property there is a level of parts at which a line is not continuous. But, on Aristotle's view, a magnitude is that which is divisible into continuous parts. Thus continuity must go "all the way down." There can be no parts which are not continuous.

Thus the basic objection is much the same as the one directed at Aristotle's definition of infinite magnitude. Because of this collective conception, topology apparently parts company with Aristotle, and with intuition. For while a sensed spatial region must consist of thing with a (non-empty) interior and a boundary, no such stricture applies to topology. Topology is free to conceive of things in a wider sense because it is not tied to intuition. Then cohesion emerges at a higher level.

Meaurement and the continuum

The paradox of measure was not actually raised by Aristotle although reports of it are contained in his work. In *Generation and Corruption* 316a14 -34 Aristotle discusses a well known argument against the continuum being composed of points. Nevertheless the argument concludes that the continuum must be composed of indivisible magnitudes (which are not points). This Aristotle vehemently denied. According to Vlastos (1968, p. 371) it is highly probable that this argument is originally due to Zeno and forms part of his paradox of measure or plurality. This paradox rests upon the assumption that a magnitude should be *additive*. That is to say, the measure of the whole should be the sum of the measures of the parts since the whole contains all of its parts. Nowadays it is useful to classify Zeno's argument as an example of a "paradoxical decomposition" (Wagon, 1985) in the same genre as the (Banach - Tarski) paradox that one can decompose a ball into two balls each of which is equal in volume to the original ball.

Of course the paradoxical decomposition of a ball would be impossible in ancient mathematics because every part of a magnitude was itself a magnitude. Put the other way around, the whole had to be the sum of its parts because parts are the lengths that remain after a line has been divided. If it was not the sum of its parts, then some part was simply unaccounted for.

In a nutshell the paradox of measure is this. Consider a line of unit length. Suppose, in addition, that the line is a collection of an infinite number of points. It is assumed that the length of any of these indivisibles is decidable; in particular, that all the atoms will either have positive or zero magnitude. So, if the line is composed of an infinite number of atoms then it will either have zero or infinite magnitude. The sum of the atoms is therefore either infinite or zero; but it is definitely not one. So it cannot be composed of atoms. This is a modern reconstruction of the paradox of measure that Zeno gives. For precise historical niceties see Vlastos (1968). (An analogous paradox can be given for dimension (see Grunbaum, 1952).)

Zeno's argument presents the challenge for mathematics to come up with a measure that works in the same way that our intuitive conception of measure works. One could argue, in a way which is becoming familiar, that if we cannot come up with a measure of sets which gives a unit line measure one, and each of its infinite number of punctual parts zero measure, then our reasoning regarding length must rely on intuition. But, of course, modern measure theory *does* accomplish this.

More celebrated than Zeno's paradox of measure are his arguments against motion. These four paradoxes of motion (the racecourse, Achilles and the tortoise, the arrow and the stadium) are often treated, following Aristotle's treatment of them, as independent of the paradox of measure, and as proving the separate Parmenidian doctrine that motion is impossible. It is more plausible, though, to treat the arguments about motion not as paradoxes, but as part of a grand argument designed to support the contention that a continuum cannot be composed of atomic parts. Instead of questioning the existence of motion on the basis of a punctual model of the continuum, Zeno may be taken as granting the existence of motion and arguing that this demands that time and space are not composed of an infinity of points. A number of authors have backed this interpretation, including Tait (unpublished), Skyrms (1983), and Owen (1957 - 8). (Note that Tait confusingly calls "the racecourse" the "dichotomy" and the "stadium" the "racecourse". See Vlastos (1968) for the correct terminology.)

Let me briefly consider the new interpretation. The racecourse asserts that a runner starting at a point S *cannot* reach the goal G except by traversing successive halves of the distance, subintervals of SG, each of length $SG/2^n$ for $n=1,2,3...$ Thus given M as the midpoint of SG, the runner must first traverse SM; if N is the midpoint of SM the runner must next traverse SN, and so on. It is then argued that the runner cannot traverse all the subintervals because this would involve successively completing an infinite number of tasks (Vlastos, 1968).

But on Tait's interpretation the point of the argument is this. Suppose, for simplicity, that we have the runner occupying exactly one point of space. Let $S = position(s)$ refer to the position of the runner at time s. Now suppose that the runner *moves* linearly during a time interval sg from $S = position(s)$ to $G = position(g)$. Then the racecourse asserts that each point P of the path SG is given by $P = position(t)$, for a unique t , $s < t < g$. (On this interpretation the fact that the runner must traverse the midpoint before traversing the whole course is purely arbitrary: the argument is meant to suggest that this fact holds for any point in the path SG.)

The arrow establishes the converse result, but the traditional reading is this. The arrow cannot move in a place where it is not. But neither can it move in the place where it is. For at a given instant the arrow occupies an area equal to itself, and a thing is at rest when it occupies an area equal to itself. Thus an arrow is always at rest (Vlastos, 1968). At this point commentators usually point out that the arrow is in motion in the sense that the instantaneous velocity at x is the limit of the average velocity of each interval of a sequence of progressively smaller intervals around x.

Here is an alternative interpretation. Suppose that the arrow is moving along a path SG and suppose that there is an instant t between s and g. Then there must be a point P between S and G on the path with $P = position(t)$. For if $P = S$, then there was no motion between s and t; and, if $P = G$, then there was no motion between t and g. But we are assuming the arrow is in motion. In short, the racecourse and the arrow establish that there is a bijective correspondence between positions in space and instants in time.

Now consider the paradox of Achilles and the tortoise. As Achilles starts from position S toward A, the tortoise, already at A, moves ahead. Suppose that her speed is r times that of Achilles (where r is some small fraction, say, 1/100). Then in the time, t, that it takes Achilles to traverse SA (of length s), she will traverse AB (of length sr). For the same reason, in the time, tr, it takes to travel AB she will traverse BC (of length sr^2). In this manner we obtain an unending sequence of intervals.

Intervals	run 1	run 2	run 3 ...
Achilles	$SA\ (=s)$	$AB\ (=sr)$	$BC\ (=sr^2)$ ...
tortoise	$AB\ (=sr)$	$BC\ (=sr^2)$	$CD\ (=sr^3)$ ...
temporal sequence	t	tr	tr^2

So Achilles will catch up to the tortoise if and only if an interval traversed by Achilles, and an interval traversed by the tortoise reach the same point at the same instant. But since, by assumption, the n^{th} interval traversed by the tortoise is identical to the $(n+1)$th traversed by Achilles, the tortoise will always be one interval ahead. So Achilles will never catch up (Vlastos, 1968).

But let's suppose that Achilles does, after all, catch the tortoise at Z. Let A be a point strictly between S and Z. Suppose that Achilles traverses SZ at a constant velocity in time interval sz, and the tortoise traverses the interval AZ at a constant velocity in time interval sz. So, under these assumptions, Achilles catches the tortoise at time z. Now it is possible to show that both SZ and sz contain an infinite number of atoms. Let $position(t)$ be the point in AZ at which the tortoise is located at time t in sz, and $time(p)$ the time at which Achilles is at P in SZ. This is justified by the isomorphism established by the racecourse and the arrow. Now the existence of an infinite sequence

$$P_0 < P_1 < P_2 \ldots \qquad\qquad t_0 < t_1 < t_2 \ldots$$

of points in SZ and instants in sz, respectively, may be defined as follows:

$$P_0 = A \qquad t_n = time(P_n) \qquad P_{n+1} = position(t_n)$$

Thus, from the assumption that Achilles starts his run at S and the tortoise at A, and Achilles catches the tortoise at Z, it is possible to show that SZ contains an infinite number of atoms in SZ and in sz.

So Zeno's "paradoxes of motion" appear to prove that the line *contains* an infinite number of points. More significantly the line must be considered to contain a completed infinity of points because the motion which generates them successively is complete. Achilles does catch the tortoise. There is no paradox of measure yet because it has only been proved that a line contains and infinite number of atoms. The paradox of measure arises when one considers the line to be *composed* of its unit sets of atoms, or punctual parts.

The Cantorian theory of the real number continuum allowed Zeno's problem of measure to be raised anew. Beginning in 1895, a number of French philosophers became interested in Cantor's work. In particular Hannequin, a young Kantian philosopher, had written a book criticizing the use of atoms in mathematics and physics. Although Hannequin conceded the existence of a least infinite ordinal ω, he rejected Cantor's theory of the continuum because of its inability to deal with metric considerations (Moore, 1983). He perceptively remarked: "Thus Cantor's researches have served only to render more obvious the ancient conflict between the continuum on the one hand and the notion of [real] number on the other (quoted in Moore, 1983, p. 131)."

On Cantor's and Dedekind's view of the continuum it consists of a set of real numbers, one for each point of the geometric continuum. It may have other properties, such as being a complete ordered field, but this need not concern us. Zeno argued that if a line segment is divisible *ad infinitum* it can be partitioned into an infinite number of punctual parts. Let me cite Democritus's argument again in *Generation and Corruption.*

> Suppose then that it is divided; now what will be left? Magnitude? No that cannot be, since there will then be something left which is not divided, whereas it was everywhere divisible. But if there is to be no body or magnitude [left] and yet [this] division is to take place, then either the whole will be made of points, and then [parts] of which it is composed will have no size, or [that which is left] will be nothing at all (A2, 316a 24-30, quoted in Furley, 1967, p. 84).

One way of construing this construction is as follows. (Here I basically follow Skyrms's (1984) construction.)

1: Partition the line into two segments obtained by bisecting it.
2: Refine the partition obtained at stage $(n-1)$ by bisecting each member of it.
ω: Take the common refinement of all partitions obtained at finite stages of the process.

In the last step Zeno's construction makes a bold leap from a potential infinity of parts to an actual infinity of parts. According to the interpretation adopted here this interpretation is justified because motion generates an actual infinity of points. Let us apply this construction to the set of real numbers. It is obvious how to construct each finite partition n. Consider the unit interval $[0,1]$. Bisecting this line into two parts gives: $[0,1/2);[1/2,1]$, with the midpoint arbitrarily placed in the right half. Step ω is not obvious. In order to complete step ω construct a sequence of sets: select the first member from the first partition ... the n^{th} member from the n^{th} partition, such that for each n, the $(n+1)^{th}$ set is a subset of the n^{th} set; such a sequence will be referred to as a chain. We can do this if we assume the axiom of choice. Zeno's bold leap amounts to this: each intersection of such a chain is a part of the line at level ω. The collection of such "ω parts" is considered to be an infinite partition of the unit line.

The parts of the line which we obtain will be the empty set together with the unit set of each point of the real line. To see this notice that each point on the line is a member of some ω part. First, for any point consider the set containing for each finite n, the member of the n^{th} level partition of which that point is a member. This set is a chain, and the point in question is in its intersection. Second, each ω part contains no more than one point. For between any two points on the real continuum there is some finite distance. Thus, there is some finite stage of the construction at which the points fall into different elements of the partition. So they fall into separate ω parts since if they were to fall in the same ω part they would have to both be members of the chain of which that ω part is the intersection. However some ω parts are empty, for consider the intersection of the chain $[0,1]$, $[0,1/2],[0,1/2]$ $[1/4,1/2)$, $[3/8,1/2)$, $[7/16,1/2)$ and so on is empty. The point $1/2$ is not a member of the chain and any point distinct from the midpoint is eventually dropped. Putting these features together shows that the collection of ω parts form a partition of R.

Perhaps it is wishful thinking to think that Zeno could have thought of such a construction but it does give a more precise sense to the idea of producing a collection of unit sets from the set of real numbers by "dividing through and through". Now let's look at how this construction leads to a paradox. Zeno argues that either all parts have zero magnitude or all parts have equal positive magnitude. Then he poses the dilemma:

(A) If the parts had a non zero finite magnitude, the whole would have an infinite magnitude.

(B) If the parts had zero magnitude, then the whole would have zero magnitude.

The conclusion can be questioned in a number of ways. One might follow Aristotle in denying that points are parts of lines; hence cannot be measured. For Aristotle points cannot be parts of lines by his very conception of lines as divisible into divisibles, and so each part thereby possesses a length. Moreover, Aristotle argued that a line consisting solely of points would lack cohesion. So Aristotle does not deny that parts are measurable, he denies that such a construction can be given. For, if one starts with a continuous quantity one must end up with a continuous quantity after successive divisions.

But this objection simply begs the question at issue. Since we are questioning whether a collection of points can be a continuum it is not appropriate to rule out the possibility that continuity can emerge from that collection. In other words it cannot be assumed that successive divisions of a continuous quantity cannot result in a discontinuous quantity.

A second objection which Aristotle can give is that the construction can take place up to any level n, but cannot pass to level ω. He says: "A thing is infinite only potentially, i.e., the dividing of it can continue indefinitely ..." (*On Generation and Corruption* 318a, 21, quoted in Skyrms, 1983). But as I have pointed out, Zeno has available a devastating counter to someone, like Aristotle, who thinks that the construction can proceed for each finite n, but not to level ω on the grounds that the latter presupposes an actually completed infinite collection of n stages. How can one move from A to B? How does Achilles actually catch the tortoise? He cannot catch him at any finite stage n. But he does catch him, so he must have traversed an infinite number of points.

One might deny the conclusion by questioning the exhaustiveness of the dichotomy between (A) and (B); in particular by supposing that some of the parts of a line have different magnitudes. Suppose that an infinite number of the parts have zero magnitude and a finite number have a positive magnitude. This would defeat the paradox. Or suppose that a line is a succession of segments converging at an endpoint. Certain series of this type, for instance geometric series, will converge at the endpoint and so a sum can be established. So this assumption (B) may seem to be a simple blunder of thinking that an infinite collection of magnitudes must have an infinite magnitude. But it is not, and in fact Zeno's own paradoxes of Achilles and the Dichotomy provide a counterexample. The point is that there is such a partition of a line, and other possible partitions are ruled out by the assumption that the partition is invariant, that is, that each part has equal magnitudes. The idea is not that every partition is invariant, just that there is an invariant partition of the reals so that Zeno's construction is possible.

This raises a more subtle issue. We are assuming that on the basis of the decidability of magnitudes, either (A) or (B) must hold. Of course, according to the usual (non-constructivist) view, decidability does hold for real numbers; in particular, every real number is either 0 or distinct from 0. It might seem unquestionable to assume decidability at this stage, but, as I pointed out in the introduction (and as will be discussed later), Leibniz's infinitesimals have the feature that "they can't decide whether they are zero or not." In other words, it is not the case that any infinitesimal is equal to zero, nevertheless one cannot thereby conclude that some infinitesimal is unequal to zero. Undecidability of magnitudes could defeat the paradox at this point because, even if we agree that we are dealing with quantities which have a measure, we cannot make the inference from the fact that no part has zero magnitude, to the conclusion that every part has a positive magnitude. The parts in question may be undecidable magnitudes. Since we are attempting to give a paradox concerning Cantor's real numbers, however, it is appropriate to impose the constraint of decidability. In other words undecidability of magnitudes is not a way out of the paradox as far as the real number continuum is concerned.

The argument relies on the intuitive idea that the whole is equal to the sum of its parts, and the assumption that this idea continues to hold when the parts are infinite in number. This requires that there be some sort of coherent notion of the sum of an *infinite* number of magnitudes. We can use the following principle of ultra-additivity (Skyrms, 1983). Let S be *any* infinite set of magnitudes, and let S^* be the set of finite partial sums of magnitudes in S. A real number is an upper bound for S^* if and only if it is greater than or equal to every member of S^*. The principle of Ultra-additivity requires that the sum of S is the least upper bound of S^* if a least upper bound exists, and is ∞ otherwise.

The argument makes the additional assumption that there are no positive non-Archimedean magnitudes, since if there were their sum would be finite. The infinitesimal mathematics of Cavalieri, Leibniz, and others considered the length of a curve or the area under a curve to be the sum of an infinite number of non-Archimedean infinitesimal quantities. But, of course, this is ruled out by Cantor since real numbers are Archimedean. For if they were not R would not be complete: it would not be such that every non-empty set of real numbers with an upper bound has a least upper bound. For completeness implies the Archimedean condition (see MacLane, 1986, p. 103 for a proof). It follows that S^* will have no upper bounds and (B) will be false.

The inability of Cantor's theory of real numbers to deal with metric considerations in the late nineteenth century allowed measure theory to become a central area of investigation in French mathematics. The development of measure theory has allowed a certain kind of answer to Zeno's measure paradox that is similar, in spirit, to the answer to the problem of cohesiveness. This kind of answer is implicit in the mathematics. It was made explicit in a well known paper by Grunbaum who claims that "Zeno's mathematical paradoxes are avoided in the formal part of a geometry built on Cantorean foundations" (1952, p. 301). The philosophical point at the basis of this judgement is that singleton sets have measure zero, a collection of singleton's will have a non-zero measure which *emerges* from the collection of singleton sets. Grunbaum contrasts points with singleton sets (of points) and puts the point about emergence as follows:

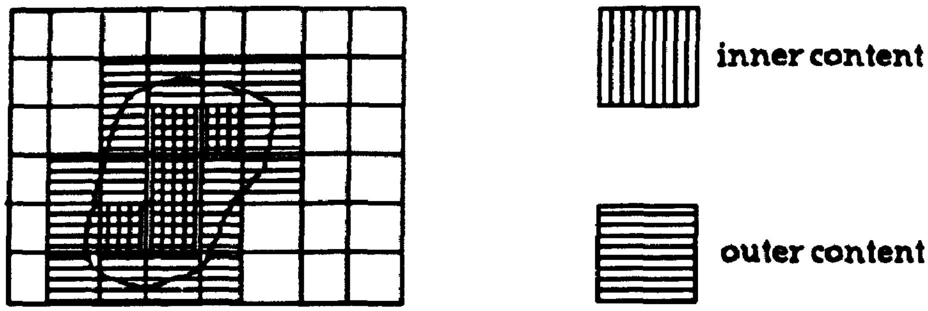
> While it is both logically correct and even of central importance to our problem that we treat a line interval of geometry as a set of point-elements, the definition of "length" renders it strictly incorrect to refer to such an interval as an "aggregate of unextended points." For the property of being unextended characterizes unit point - sets but is not possessed by their respective individual point elements, just as temperature is a property only of aggregates of molecules and not of individual molecules. (1952, p. 301)

How does measure theory allow one to escape from Zeno's paradox of measure? According to a Grunbaum-Skyrms view, modern measure theory questions the assumption that ultraadditivity is the correct sense in which the measure of a whole is the sum of a measure of the parts. For ultraadditivity fails in general for measurable sets. Consider the Peano - Jordan definition of a measurable set. A measure is not a property of an object given in our perception; it is an assignment of a unique number, called "the area" to a set.

The intuition is that a figure is measurable when its area can be compressed between sequences of polygons. In 1883 Peano noted that (I) the class of polygons contained in a given area (such as a circle) should be less than or equal to (II) the class of polygons containing the circle and conversely that (II) should be greater than or equal to (I) (see Hawkins, 1970). When these conditions determine a unique number, then it is the area of the region, and if not "then the concept of area would not apply in this case" (quoted in Hawkins, 1970, p. 87).

Thus, on the real line, an interval $[a,b]$ is assigned a length $b-a$ as a measure. Degenerate intervals $[a,a]$ are assigned 0 measure. These measures are the fundamental starting point for assigning measures, and the concept of measure is extended to include other sets. This occurs as follows. Consider finite sets of intervals which cover the set of points in an interval in the sense that the points are contained in their union. We associate with each such cover the sum of the lengths of the intervals contained in it. The greatest lower bound of these lengths is the *outer content* of the set. We can also consider the finite sets of non-overlapping (pairwise disjoint) intervals whose union is contained in the set at issue. We can associate with each such set the sum of the lengths of its members. The least upper bound of these lengths is the *inner content* of the set. A set is *measurable* in the sense of Peano and Jordan when the inner and outer content of the s are equal, and that content is its measure.

This definition of measure is easier to visualize with areas. Lay a grid over a plane containing the figure to be measured. If by halving one proceeds to a finer grid, the new inner approximation contains the old one and is usually larger because of the addition of new smaller squares to the old region, while the new outer region results by deleting new smaller squares from the old region. Thus the difference between inner and outer content becomes smaller. If continuing refinement of the figure brings the inner and outer content arbitrarily close to a number then this number is the content of the figure; if not, the figure is non measurable.



Approximation of Peano - Jordan content
Figure 5

Jordan was able to prove that such a measure is finitely additive. That is to say, if each of a finite collection of mutually disjoint sets is measurable, then their union is also measurable and is the sum of their measures. But the stronger sense of countable additivity fails for Peano-Jordan measure. For example, the set of rational points in [0,1] is not Jordan-Peano measurable since its outer content is 1 while its inner content is 0. But it is the union of a denumerable collection of unit sets.

Again this can be visualised better with areas. Consider a square *ABCD* on whose upper side *CD* we construct a perpendicular at every point whose distance from *C* is a rational number. In this case the outer content is twice as large as the inner content, and the two do not tend to a common limit as the grid is refined, because area *CC′D′D* always belongs to the outer approximation and only to that one. Every grid square lying in *CC′D′D*, no matter how small, contains both points that do, and others that do not belong to the figure.

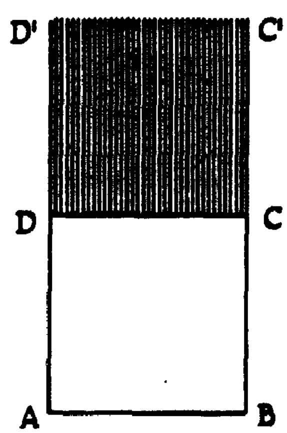


Figure 6

This approach to measure was generalised by Lebesgue who formulated the measure problem for plane regions as follows:

> The Measure Problem for plane regions bounded by simple closed curves can be posed in the following manner .... : To associate with each such region a number, to be called an area, such that congruent regions have equal areas, and such that the region formed from the union of finitely many regions, which have part of their boundary in common and which do not overlap, has as its area the sum of the areas of the component regions.... (quoted in Moore, 1983, p. 136)

In answer to this problem Lebesgue generalized the concepts of inner and outer content in such a way that the countable additivity of measure could be established. It is not necessary to go into the details of this concept of measure. The important point is that by applying Lebesgue measure to Zeno's partition of the real line does not lead to a paradox because there *is an additive measure* which assigns zero to each punctual part and assigns one to the unit line.

But, as Skyrms has pointed out, the spirit of Zeno is still capable of mischief. The spirit of Zeno would insist that such an approach to solving Zeno's paradox would call for a positive answer to Lebesgue's measure problem: every possible partition of the line must be measurable. However, a negative solution to Lebesgue's measure problem was given by Vitali in 1905. This, Vitali did, and in the process, he produced an example of a set which is non (Lebesgue) measurable.

But if the spirit of Zeno is malicious, then any paradoxical decomposition shows that the continuum cannot be regarded as being composed of points. Just as considering a unit line segment to be composed of points generates Zeno's paradox of measure, considering a unit cube to be composed of points leads to Hausdorff's paradox and its generalization due to Banach and Tarski (Wagon, 1985). Hausdorff showed that one cannot have a finitely additive measure which assigns the unit cube measure 1, assigns congruent sets equal measure and assigns a measure to all the subsets of the unit cube. Banach and Tarski were able to show (using the axiom of choice) that one can cut a ball into a finite number of pieces that can be rearranged so that one obtains two cubes of the same size as the original ball. The pieces are simply non measurable sets (Wagon, 1985; Jech, 1977).

But these argument can be questioned. As Moore (1983, p. 141) recounts, Lebesgue did not accept that Vitali had established the existence of a non measurable set because his proof relied upon the use of the axiom of choice. Lebesgue was not alone in this attitude. Hausdorff's 1914 paradox was taken by Borel to vindicate his (Borel's) opposition to the axiom of choice:

> The contradiction has its origin in Zermelo's axiom of choice. The set A is homogeneous on the sphere; but it is at the same time a half and a third of it .... The paradox results from the fact that A is not defined, in the logical and precise sense of the words defined. If one scorns precision and logic, one arrives at contradictions. (quoted in Moore, 1983, p. 142)

On the basis of the view of the French Intuitionists, e.g., Borel, Baire and Lebesgue, Moore has argued that the history of measure theory contains a contradiction between the expressed philosophy of a group of mathematicians and the type of mathematics they developed. On the one hand the aforementioned mathematicians stood opposed to the axiom of choice in mathematics and, on the other hand, they developed theories of measure which depended upon such an axiom. As Moore puts it, Lebesgue's "work reveals how a mathematician of the first rank may subtly fail to see that he is fundamentally violating his own philosophical scruples in his own work" (1983, p.149).

Perhaps their attitude toward the axiom of choice was justified because of the subsequent discovery of the Banach - Tarski paradox and Solovay's proof that the existence of a non measurable set depends in an essential way upon the axiom of choice. One can show, moreover, that in Solovay's model of set theory all the standard theorems of Lebesgue measure theory hold (Jech, 1977). This discovery resolves the "inherent contradiction" in the thinking of the French Intuitionists because the theory of measure can be developed without the axiom of choice - although not without the *countable* axiom of choice which *does* hold in Solovay's model. I think the moral to be drawn, then, is that in order to represent our intuitive conception of measure the axiom of choice has to be set to one side.

LEIBNIZ'S PUZZLE AND THE SMOOTH CONTINUUM

A curvilinear figure must be considered to be the same as a polygon with infinitely many sides.

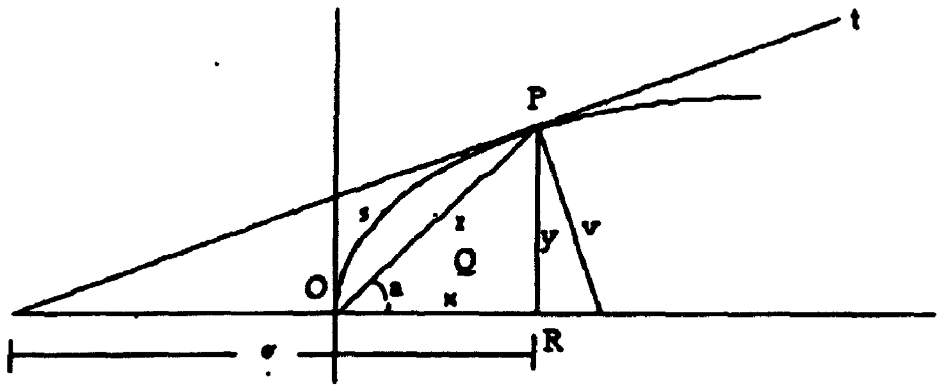
G. Leibniz

The previous chapter was devoted to a brief survey of how modern set theory attempts to solve the challenges raised by Aristotle to conceiving the continuum as composed of points. Aristotle doubted that collections of units could be conceived as having the properties that continua have. Modern set theory, by conceiving sets to be unities in their own right, allows for complex collections of points. At a sufficient degree of complexity the properties possessed by continua emerge from the discrete.

Is it possible to give a set theoretic account of the continuum? Like Aristotle, Leibniz believed that the continuum was perceived as a whole. But he differed in that he also believed that our conception of mathematical objects was clear and distinct. Hence, the continuum was nothing more than a confused perception of the underlying discrete monads. But Leibniz's view of the continuum had an additional structural feature which was central in his infinitesimal calculus. This feature is that every curve is to be considered an assemblage of straight lines. It follows from this viewpoint, that the positions along these straight lines have a peculiar feature, namely that the identity of any pair of positions is undecidable. The difficulty in adopting a set theoretic approach will be that the identity of every set is decidable; so the infinitesimal positions will not be representable by sets. It will become clear later that in order to represent these infinitesimals the notion of set will have to be generalized to that of variable set.

The principle of infinitesimal linearity and undecidable objects

Seventeenth century analysis was a corpus of analytical tools for the study of geometrical objects, in particular, curves. A curve embodies relations between several variable geometrical quantities such as ordinate, abscissa, arc length, polar arc, radius, subtangent, normal, areas between curve and axes, tangent, circumscribed rectangle. The main problems of mathematics involved determining the relations, such as tangent and quadrature, between variable quantities. These relations are illustrated in the following figure.



x: abscissa, y: ordinate, s: arc length, r: radius, a: polar arc, σ: subtangent, t: tangent, v: normal, Q = OPR: area between curve and x axis, xy: circumscribed rectangle.

Figure 7

Cartesian analysis had introduced the idea that relations between variable quantities be expressed by equations whenever possible. These relations were not functional relations since there was no sense of one variable being dependent upon another variable. Hence, a relation between x and y was a single relation rather than two separate mappings $x \longrightarrow y$ and $y \longrightarrow x$. The curve was *not* considered a graph of a function but rather a geometrical figure which embodies relationships between variable quantities.

It was technically impossible prior to Descartes to consider the variables of geometric analysis to be real numbers. For geometric quantities, as conceived by mathematicians up to the seventeenth century, lacked a multiplicative structure and a unit element. This is due to the fact that quantities were conceived of as having dimension, which could be that of a line, an area or a solid. Higher powers were uninterpretable because they were thought not to be interpretable in actual space, so quantities were not closed under multiplication. Descartes, in his *Geometrie*, found a way to multiply quantities without assuming that the quantities are of different dimension (Mancosu, 1992). Later Riemann introduced the idea of an n - fold extended space so that it became possible to interpret quantities as having a dimension greater than that of actual space (Nowak, 1989).

The most notable event in seventeenth century mathematics was a rebirth of infinitesimal techniques in the hands of Kepler, Cavalieri, Norris, Wallis, Barrow, Newton and Leibniz among others (Jeseph, 1989; Baron, 1969; Edwards, 1979). According to this conception surfaces were considered to be composed of indivisible surfaces, and lines of indivisible lines. But many were reluctant to take such a realist approach to indivisible elements. Cavalieri, one of the premiere practitioners of infinitesimal mathematics wrote to Galileo in 1639 that "I have not dared to say that the continuum was composed of these [infinitesimals]..."(quoted in Anderson, 1975, p. 307). Earlier in 1634 he had declared even more emphatically to Galileo that "I absolutely do not declare to compose the continuum by indivisibles" (quoted in Anderson, 1975, p. 307). In spite of the emphasis Cavalieri placed on not affirming the continuum to be composed of indivisibles it seems that he never took a definite view on whether the continuum was, in fact, composed of indivisibles.

This ambivalence is found in other prominent mathematicians as well. Huygen's opinion of the infinitesimal techniques was characteristic of many. He believed that infinitesimal techniques were methods of discovery but not methods of proof. These techniques could not be regarded as rigorous, but they could, nevertheless, make for good labour saving devices and be heuristically satisfying:

> As to the Cavalierian methods: one deceives oneself if one accepts their use as a demonstration but they are useful as a means of discovery preceding a demonstration.... Nevertheless that which comes first and which matters most is the way in which the discovery has been made. It is this knowledge which gives most satisfaction and which seems, therefore, preferable to supply the idea through which the results first came to light and through which it will be most readily understood. We will thereby save ourselves much labour and writing and the others the reading; it is necessary to bear in mind that mathematicians will never have enough time to read all the discoveries in geometry ... if they continue to be presented in a rigorous form according to the manner of the ancients. (quoted in Baron, 1969, p. 223)

Leibniz's attitude appeared to express a decidedly realist view of infinitesimals. By 1684 the Leibnizian calculus had reached a somewhat definite form and Leibniz made it clear that the fundamental idea of his calculus was that *every curve is an infinitangular polygon* (quoted in Bos, 1974, p. 14):

> I feel that this method and others in use up till now can all be deduced from a general principle which I use in measuring curvilinear figures, *that a curvilinear figure must be considered to be the same as a polygon with infinitely many sides.*

This principle was not completely novel. It had been alluded to by Protagoras of Abdera, Bryson, Antiphon, Descartes, and Galileo (Baron, 1969; Mentzeniotis, 1986). What was new was that it was the central notion of a comprehensive mathematical theory - the infinitesimal calculus. For instance Antiphon's ancient approach to computing the quadrature of a circle was to inscribe a square in the circle, and from this square construct to an octagon, and a 16 - gon and so on, until the circle and polygon coincide.

> And then in the same way of cutting the sides of the 16-gon and joining the lines and doubling the inscribed polygon, and doing this always, so that at some time, the area being exhausted, a certain polygon would be inscribed in the circle whose sides on account of smallness would coincide with the arc of the circle. (quoted in Mentzeniotis, 1986, p. 15)

Even Descartes pointed out that if a circle rolls on a straight line, the circle can be considered as a polygon made up of "cent mil millions" of sides and the tangent at each point will be perpendicular to the line joining that point to the generating line with the base line (Baron 1969, p. 164).

The intuition behind this principle was straightforward. For the Greeks rectilinear and curvilinear figures were different kinds of quantities, just as areas and lengths are different kinds of quantities. Thus it was difficult to understand how comparisons of curvilinear and rectilinear quantities could be made. It was obvious how to calculate the area of a square, say with sides of unit length, since the area is *defined* as the square of unit length. But, suppose we are given a circle of unit length and asked to compute the area of the circle in rectilinear units. Since a circle is a kind of curvilinear figure it is not immediately obvious how to calculate the area in rectilinear units. How is this gap between curvilinear and rectilinear to be bridged? Antiphon's simple solution begins by denying that there is a gap to be bridged between curvilinear and rectilinear figures. If the curve were actually a kind of rectilinear figure, then computing the area would, in principle, be the same as calculating the area of a rectilinear figure.

As simple as this solution seemed, it was rejected unequivocally by Aristotle because curvilinear and rectilinear figures were different kinds of quantity. Circles and infinitangular polygons can be distinguished in two simple ways. First, no matter how many sides a polygon had, it could never be a circle, since a straight line and a circle are tangent only at a point, and hence cannot coincide with an edge of a polygon. Building on this point, and the infinite divisibility of magnitude, the difference between the area of a circle and that of any member of a sequence of inscribed polygons can never become zero.

A similar attitude to the reintroduction of the infinitangular polygon in the seventeenth century was expressed by Berkeley who, in a manner reminiscent of Aristotle, considered the idea of a circle being an infinitangular polygon to be a perversion of language.

> What do the Mathematicians mean by Considering Curves as Polygons? either they are polygons or they are not. If they are why do we give them the name of curves? Why do not they constantly call them Polygons & treat them as such. If they are not polygons I think it is absurd to use polygons in their stead. What is this but to pervert language to adapt an idea to a name that belongs not to it but to a different idea. (quoted in Jesseph, 1993, p. 159)

It is apparent that the view that a curve can be considered to be an infinitangular polygon ("polygon infintanguli") has had a checkered intellectual history and, until recently, was fated to being viewed as an incoherent notion. Even today, respectable philosophers continue to speak of the "contradictory being of Leibniz's infinite sided polygons, at once continuous and discrete, geometric and combinatorial, infinitary and finite" (Grosholz, 1992, p. 133). This attitude could have been permanently changed because of the great results which the infinitesimal calculus brought about but it was not, largely because of the inability of Leibniz's calculus to survive the "rigorization" of the calculus. The inability of Leibniz's conception to survive is due to the fact, as we shall see, that infinitangular polygons cannot exist in the framework of a universe of sets.

A straightforward interpretation of Leibniz's writings is to take his principle at face value; and many who had read and comprehended Leibniz's papers, such as Bernoulli, Varignon, Nieuwentijt, and L'Hospital interpreted and applied the principle literally. In order to attempt to explain the concept of an infinitangular polygon, it is helpful to consider the case in which a curve is approximated by a finite polygon *OABCDE*.

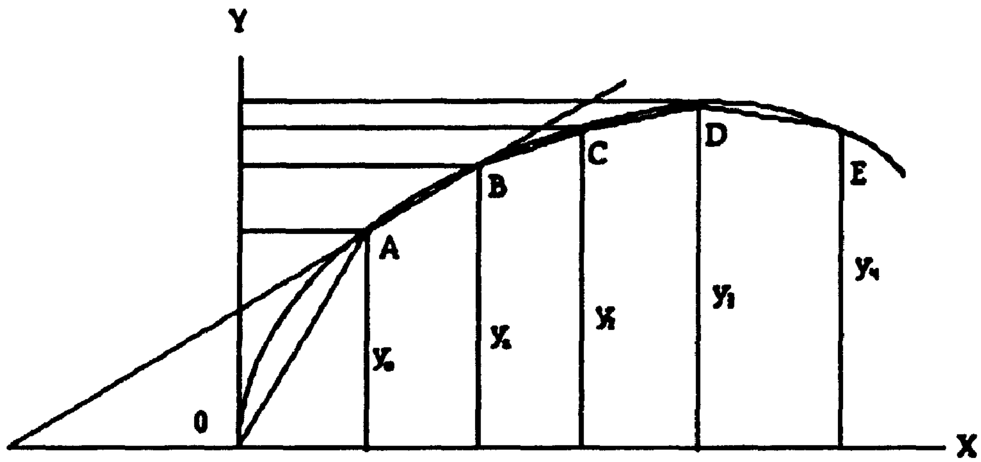


Figure 8

We suppose that, in approximating the curve, the vertices of the polygon are on the curve. The curve is, therefore, considered to be approximated by a sequence of points (the vertices of the polygon) joined together by straight lines. This occasions considering the quantities as varying with respect to the points of the curve. The approximating polygon gives rise to a *sequence* of variable quantities. Leibniz recognized that if the polygon is chosen so that the difference of the successive x values is equal to 1, then the sum of the ordinates provides an approximation of the quadrature and the differences of successive ordinates are approximations to the slope of the tangents for the corresponding points on the curve (Bos, 1974, p. 13).

Leibniz had observed in his early studies on number sequences that the operations of summing sequences and taking the difference of sequences were reciprocal operations: if one takes the successive differences of the sum sequence one obtains the original sequence, and similarly, if the one takes the successive sums of the difference sequence, one obtains the terms of the original sequence (diminished by a_1).

---

sequence

$a_1, a_2, a_3, a_4 \ldots$

sum sequence:	difference sequence:
$s_1, s_2, s_3, s_4, \ldots$ with	$d_1, d_2, d_3, d_4, \ldots$ with
$s_1 = a_1$	$d_1 = a_2 - a_1$
$s_2 = a_1 + a_2$	$d_2 = a_3 - a_2$
$s_3 = a_1 + a_2 + a_3$	$d_3 = a_4 - a_3$
$s_4 = a_1 + a_2 + a_3 + a_4$	$d_4 = a_5 - a_4$
etc.	etc.

He wrote:

> *Foundations of the calculus*: Differences and sums are reciprocal to each other, that is, the sum of the differences of a sequence is the term of the sequence, and the difference of the sums of a sequence is also the term of the sequence. The former I denote thus: $\int dx = x$; the latter thus $d\int x = x$ (quoted in Bos, 1986, p. 86).

A finite polygon will approximate a curve more nearly if the differences of the terms are made smaller and smaller. The Leibnizian calculus results from the daring hypothesis that when a finite polygon which approximates a curve becomes an infinitangular polygon, it *coincides* with the curve when the differences are taken to be infinitesimally near. A curve, then, is an infinite set of points "joined" by linear segments - geometric infinitesimals. The geometric infinitesimals linking vertices of the polygon are not punctual infinitesimals nor indivisibles but smoothly varying infinitesimal quantities, or "continua in the small".

One should notice immediately that there is an ambiguity in the notion of an infinitangular polygon since it is unclear whether Leibniz thinks of every curve as being an actually existing infinitangular polygon or a potentially existing infinitangular polygon. It may seem that Leibniz is saying that every curve is an *actually* completed infinitangular polygon but it is possible to make Leibniz's point in a way which does not state outright that every curve is an (actual) infinitangular polygon. This is the principle of *local straightness of curves*: each point on a curve lies on a straight line (compare Bell, 1988). Here a point may be interpreted as a *potential* locus of division such that when such a division is made, it will lie on a straight line segment.

On the other hand, how can we envisage an actual infinitangular polygon? The extrapolation from a finite polygon to an infinitangular polygon is not only daring but difficult to understand. One suggestion is to try to visualize this extrapolation as the result of a limit process of increasing the number of sides of the polygon. But visualizing this extrapolation as a limit process seems not to be of much help. For, suppose that we attempt to find the area under a curve by summing up the inscribed rectangles under the curve. This approach to quadrature was given by Leibniz in a 1677 manuscript. (It should be noted that one of the main advances of Leibniz's approach over Cavalieri's use of indivisible lines and surfaces was his use of differential *triangles*.)

> I represent the area of a figure by the sum of all the rectangles contained by the ordinates and the differences of the abscissa [the sum of the rectangles of the figure].... For the narrow triangles [the differential triangles]... since they are infinitely small compared with the said rectangles, may be omitted without risk; and thus I represent in my calculus the area of the figure by $\int y dx$, or the rectangle contained by each y [height] and the dx [base] that corresponds to it. (quoted in Mentzeniotis, 1986, p. 71)

This conception of finding the area under a curve is intimately related to the principle that every curve is a polygon. Since in order for the area to be equal to the sum of the areas of the rectangles the differential triangles must be equal to zero. One way of attempting to visualize this is to decrease the area of the triangles by making the areas smaller and smaller by making the ordinates closer and closer so that they approach zero. Leibniz observed that the problem with this traditional approach to limits is that we never get to infinitesimals as the limit of such a process. But what if the difference in ordinates *does* becomes zero? If we view this as the result of a limit process, where the differences between the terms of the sequence become zero, the ordinates fill the entire area between the curve and the axis, the rectangular areas of the polygon disappear and the polygon "collapses".

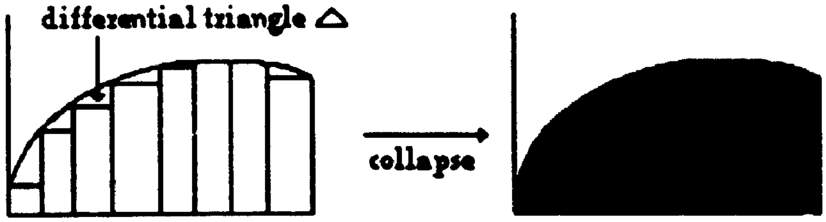


Figure 9

Thus in order for the area to be the sum of the rectangles, the ordinates are required to be distinct, so the polygon does not collapse, and yet the area of the differential or characteristic triangle must be equal to zero. In other words the base of the triangle must be non-zero, while the area Δ is zero. Leibniz dealt with this problem of collapse in a two different ways, each of which amounted to refusing to let the characteristic triangle equal zero. As we shall see it is a simple consequence of the principle of the local straightness of curves that the area of the differential triangle must be zero while the base is non zero.

This failure to understand an infinitangular polygon by means of a limit process is symptomatic of the general problem of attempting to understand geometrical concepts through the consideration of limit processes, and vice-versa. There are many other examples of apparent failures and an amusing example will help illustrate the problem (Giaquinto, 1994). Consider the following sequence of curves: first we draw a semicircle on a line segment; secondly we divide the segment into halves and form two semicircles, one on the upper side of the left half, and the other on the lower side of the right half; we then repeat step two *ad infinitum*. The curve gets closer and closer to the original line segment, so one might reason that the limit of the curves is the original line segment. So our visualization of this infinite process is the length of the curve becoming smaller and smaller, and gradually "winding itself" around the original line segment. Finally, in the limit, the curve is wound so "tightly" that it becomes the original line segment.

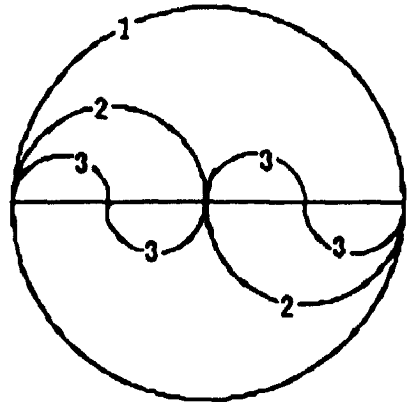


figure 10

Here our intuitive visualization is misleading because, in fact, the length of every curve is the same. If the original segment, the diameter of the first semicircle, has length d, that semicircle has length $\pi d/2$; the next curve, two semicircles on diameters half the length, has length $2(\pi d/4) = \pi d/2$. The next curve will have length $4\pi d/8 = \pi d/2$. Thus the length of the curve does not decrease contrary to expectations based upon visual intuition.

Because of the difficulty of visualizing infinitangular polygons Leibniz preferred to concentrate on operations on the sequence of points of the polygon. Thus Leibniz was led to consider infinite sums and sequences induced by the polygon. These difference are the arithmetic representations of the infinitesimal quantities, and they are regarded as zero compared to elements of the original sequence. One can proceed further and calculate a second order difference sequence from the difference sequence itself. Each n - order difference is regarded as incomparably small (zero, for all intents) in comparison with quantities at level $n-1$.

As we shall see the symbolic nature of reasoning is characteristic of Leibniz's approach to mathematical reasoning. In order to understand a mathematical concept we do not need not have a representation of each part of the concept, and for Leibniz it sufficed to study the arithmetical sequence which the polygon induced. But this approach can give rise to misunderstanding even among the most sensitive Leibniz scholars. In a finite polygon a finite number of values are associated with the variable. That is, the variable is considered to range over the values connected with the vertices of the polygon. When we extrapolate from a finite polygon to an infinitangular polygon the variable is conceived to range over an infinite sequence of values associated with vertices of the infinitangular polygon. Since the curve is supposed to be an infinitangular polygon Bos suggests that the "sequence and the variable now coincide; the variable is the sequence along which it ranges" (1986, p. 88) . Thus Bos contrasts Leibniz's approach to curves as ranging over an infinite sequence of (static) points rather than flowing along a continuum of values as Newton's continuum does.

But this is premature and departs from Leibniz's central tenet. What happened to the infinitesimal segments joining the vertices when the extrapolation took place? Did they disappear? A different answer which, of course, is difficult to visualize, is that when the number of points becomes infinite, the infinitesimal line segments joining the points become "absorbed" within the points. Thus the sides of the polygon become shorter and shorter and finally become "absorbed" within the points. The coincidence of the variable quantity and its points is due to the fact that the infinitesimal D is of non-zero length, and so may have very strange order properties. It turns out that, in modern models of Leibniz's continuum, we have $D \subseteq [0,0]$. As well, if $x \in [a,b]$, then $(x+d) \in [a,b]$. Therefore, since for each point a, $a \in [a,a]$, it follows that the infinitesimal distance around a, is entirely enclosed within a, $(a+d) \in [a,a]$. Thus the infinitesimally small "linelets"don't quite disappear; but are hidden within the points. The strangeness of this situation is evident from the fact that, according to standard measure theory, the measure of the segment $[a,a]$ is zero.

This suggestion is only the beginning of the radical consequences of accepting the Leibnizian conception of the continuum. Here I will draw attention to two such consequences. First, the principle that every curve is an infinitangular polygon implies that there exist non-degenerate square zero infinitesimals. Leibniz did not draw the conclusion that there were square zero infinitesimals; this was left to Nieuwentijt in his treatise *Analysis Infinitorium* which appeared *before* that of L'Hospital in 1695. Nieuwentijt said "anything that, if multiplied by an infinite quantity, does not produce a given [finite quantity], however small, cannot be reckoned among the beings, and must in geometry be counted as zero" (quoted in Vermij, 1989, p. 71). Nieuwentijt also attempted to demonstrate the calculus from the principle that considered curves as polygons (Vermij, 1989, p. 70). The publication of this work sparked an important dispute over the existence of higher- order differentials with Nieuwentijt claiming that there could not be such entities and Leibniz claiming there must be. This dispute reveals the confusion that existed over the basic points of how to apply the principles of the infinitesimal calculus. Nieuwentijt eventually conceded Leibniz's point, but he did not do so lightly:

> Every one who has had the trial of it, knows how mortifying it is to give up an hypothesis which he has believed for so many years to be true, upon which he has pored and meditated so many nights, with which he has blotted so much paper, and for the sake of it ran thro' so many books; and lastly, by the help of which, he fancies to himself, that he has arrived to the top of all wisdom, or at least that he shall soon reach it. (quoted in Vermij, 1989, p. 85)

The point can be illustrated with a simple problem. If $y = x^2$ and x takes the value $x + dx$ for some infinitesimal value dx, then y becomes $x^2 + 2xdx + dx^2$. If the change in x is dx, the change in y is $2xdx + dx^2$,then the ratio of the change is $2xdx + dx^2 / dx = 2x + dx$. But the differential quotient of x^2 is said to be $2x$ and not $2x + dx$. What justifies subtracting the dx? Nieuwentijt claimed that it was justified by the fact that $dx^2 = 0$. Hence the square of infinitesimals must equal zero.

Leibniz disagreed. He said:

> ... I accept not only infinitely small lines such as dx, dy, ... as true quantities in their own sort, but also their squares and rectangles, such as $dx\ dx$, $dy\ dy$, $dx\ dy$. And I accept cubes and other higher powers and products as well, primarily because I have found them useful for reasoning and invention. (quoted in Bos, 1974, p. 64)

This response must have been somewhat disconcerting, because Leibniz simply responded to a theoretical explanation with a practical answer.

In Leibniz's early papers, as well as in l'Hospital's and Johann Bernoulli's, neglecting the dx was justified because it was infinitesimally small compared to finite quantities: $2x + dx$ and $2x$ are equal. In this way Leibniz is prepared to simply extend the notion of equality to that of infinitesimal nearness. Later, Berkeley observed that errors in mathematics can never be so small as to be ignored. But on Leibniz's view there are no errors since If $2x + dx$ and $2x$ are not strictly equal we simply change the definition of "equal".

> Of course I hold with Euclid (Book V, definition 5) that all homogeneous quantities are comparable which can be made to exceed one another through multiplication by a finite number. (quoted in Jeseph, 1989, p. 241)

But also:

> I think that those things are equal not only whose difference is absolutely nothing, but also whose difference is incomparably small; and although this difference need not be called absolutely nothing, neither is it a quantity comparable with those whose difference it is. Just as when you add a point of one line to another line or a line to a surface you do not increase the magnitude.... (quoted in Jeseph, 1989, p. 240)

Thus, when applied to the problem of the collapse of the infinitangular polygon, the differential triangles do not become absolutely zero, but only incomparably small with respect to finite quantities. Leibniz would not commit himself to the idea that any order of infinitesimal was *absolutely* zero; this would imply the collapse of the polygon. This answer was never fully accepted by Nieuwentijt. Nieuwentijt thought that, in the end, all controversy regarding infinity comes down to what he regarded as an insoluble problem:

> Whether something that is infinitely small, or better: smaller than someone can determine, remains something in then end; or whether it should be kept to a mere nothing? (quoted in Vermij, 1989, p. 84)

It is just this problem, which the new framework of smooth spaces and maps solves by showing that there are nilpotent quantities of any power, i.e. $x^{n-1} \neq 0$ but $x^n = 0$ for any given n.

Nieuwentijt did not prove that the square of an infinitesimal is zero, but such a proof is available based upon the consideration that every curve is a polygon. Suppose we wish to find the tangent to a curve which is described by the equation $y = x^2$. Given the principle of local straightness of curves it is clear that:

.... to find a tangent is to draw a straight line joining two points of the curve which have an infinitely small distance to one another; or the produced side of the infinitangular polygon which for us is equivalent to the curve. This infinitely small distance, however can always be represented by a given differential, such as dv, or by a relation to it, that is, by a given tangent. (quoted in Bos, 1974, p. 63)

An intuitive proof of the existence of square zero infinitesimals is given by considering the graph of $y = x^2$.

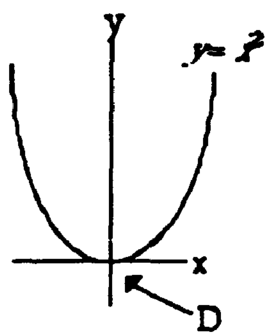


Figure 11

We consider the elements along the abscissa x to be numbers, the point at unit length is 1 , and the origin point is 0. D is the infinitesimal line segment which is the intersection of the parabola and x axis. Thus D is the collection of all points x such that $x^2 = 0$, or the domain of an "equalizer" map of the zero map and the square map, i.e. D can be regarded as the domain on which the maps agree. Now consider any element $d \in D$ on the x - axis at a given distance from the origin. Clearly, it is not the case that, for any d, $d = 0$, i.e., D cannot consist of 0 alone. For if it did, consider the part of the parabola $g = D \rightarrow R$, given by $g(d) = d^2$. Then $g(d) = g(0) + db$ for *any* b which violates the uniqueness of b.

Let's formulate the principle of local straightness of curves more precisely. Consider an arbitrary curve $y = f(x)$ (below).

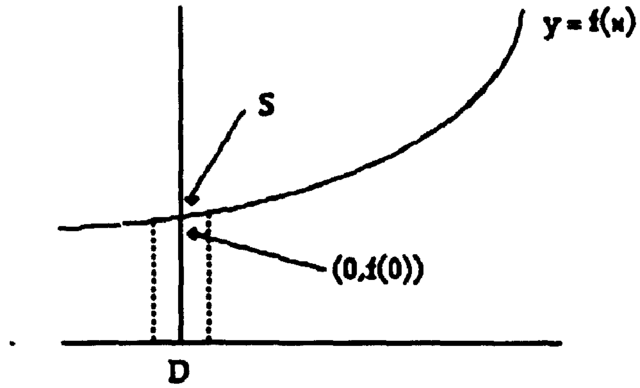


Figure 12

We have imagined the infinitesimals to be squeezed between points so that D lies around the point 0. Since the curve is locally straight, there is a *linear infinitesimal portion* S of the curve $y = f(x)$ around the point $(0, f(0))$ which coincides with the tangent line at that point. Suppose that we are in a mathematical framework in which every curve is represented by a polynomial. If the curve f were a polynomial function, then S may be taken to be the image $f[D]$of D under f. (Given a polynomial $f(x) = a_0 + a_1x + a_2x^2 + ... a_nx^n$ with $x^2 = 0$, we have $f(x) = a_0 + a_1x$) Thus if we consider the restriction g of f to D, then this postulate is equivalent to the assertion that the graph of g is linear on D. In short, a principle of *infinitesimal linearity* applies to D: for any map $g: D \longrightarrow R$ there is a unique slope $b \in R$ such that for all $d \in D$ $g(d) = g(0) + db$.

The consequence which will be most important for this thesis is that *the identity of elements of* D *is not decidable.* It is this fact, in conjunction with Leibniz's commitment to the decidability of objects which leads to what I have called "Leibniz's puzzle". For if a continuum is to be considered a collection of objects and every object is decidable, then infinitesimals cannot be contained in the continuum. Thus no curve can be considered infinitesimally linear, and so cannot be represented by the intellect. I am not claiming that Leibniz ever formulated this puzzle as it pertains to square-zero infinitesimals, nor that he was aware of the undecidability of infinitesimals. But such a puzzle can be culled from his work and fits nicely with his idea that infinitesimals are merely "ideal".

The undecidability of infinitesimals is a fairly immediate consequence of the principle of infinitesimal linearity. Suppose that D is decidable, that is

$\forall d_1 \forall d_2[(d_1 = d_2) \vee (d_1 \neq d_2)]$, i.e. any pair of infinitesimals is decidable. In particular each d is distinguishable from 0, $\forall d(d = 0) \vee (d \neq 0)$. Thus a function f may be defined by $f(0) = 0$ if $d = 0$ and if $d \neq 0$ then $f(d) = 1$. Since it is not the case that, for any d, $d = 0$, consider a $d \neq 0$. Then by the definition of the function f, $f(d) = 1 = db$. But then $1 = (db)^2 = 0$, a contradiction. Thus we conclude that $\neg(d \neq 0)$. Therefore by the initial assumption of decidability we obtain $d = 0$. This holds for all d, and so contradicts the former conclusion that $d \neq 0$. The conclusion follows. Note that this immediately entails that the classical law of excluded middle cannot hold, since that principle states that for any formula ω, we have $\omega \vee \neg\omega$,

Leibniz, I think, would not have been shocked at this last result. One may argue that this violates Leibniz's fundamental law of non-contradiction. That is, Leibniz presupposed the principle of excluded middle in order to apply his supreme principle of non-contradiction. As Leibniz put it:

> First of all, I assume that every judgement (that is, affirmation of negation) is either true or false and that if the affirmation is true the negation is false, and if the negation is true the affirmation is false .... All these are usually included in one designation, the principle of contradiction. (quoted in Mates, 1986, p. 153)

The reason that he would not have been shocked is that, as I will discuss, the parts of a continuum are not real but ideal. But the law of excluded middle should only applies to actual objects and not possible objects. This is certainly the line that Pierce took:

> Now if we are to accept the common idea of continuity ... we must say that a continuous line contains no points or we must say that the principle of excluded middle does not hold of these points. The principle of excluded middle applies only to an individual ... but places being mere possibilities without actual existence are not individuals (1976, p. xvi).

At any rate it has been revealed that Leibniz's fundamental idea leads quite directly to the fact that the infinitesimal parts of the continuum are undecidable in the sense that the identity of infinitesimals cannot be distinguished. This is an important structural difference between Aristotle's continuum and Leibniz's and it leads to a puzzle for anyone who wants to regard the continuum as composed of points. As we shall see in the next section, Leibniz believed that every mathematical object was decidable. But the infinitesimal parts of Leibniz's continuum are not decidable. Thus, on Leibniz's view, the parts of the smooth continuum cannot be objects. So, a continuum is not a collection of objects.

The "sublime geometry" and the problem of its interpretation

Let's approach Leibniz's conception of the continuum from a different direction. Namely, what the continuum must be like in order to solve the problems associated with the composition of the continuum. The basis of the solution to the problem of the composition of the continuum in the writings of Leibniz, Kant, Bolzano and Cantor is intimately related to their understanding of how mathematical objects are represented and thereby understood by us. It was a standard view in the seventeenth century that our knowledge of mathematical objects was mediated by representations. In Leibniz's words human souls "perceive what passes without them by what passes within them" (Alexander, 1956, p. 83). According to the usual story (which is highly influenced by Kant) in the Cartesian tradition representations are of a single faculty - intellection (as opposed to sensibility). Thus Descartes, Leibniz and Wolff assimilated all representation to a single faculty. As Kant would put it, Leibniz "intellectualized" appearances.

Descartes had taken it as a rule that "the things we conceive clearly and distinctly [are] true" (Cottingham, 1984. Vol. I, p. 193-4). Intuition on this view, is "the conception of a clear and attentive mind, which is so easy and distinct that there can be no room for doubt about what we are understanding ... [it] proceeds solely from the light of reason" (Cottingham, I, p. 14). For Leibniz it was analysis that rendered concepts distinct and the model of cognition was often blind and symbolic rather than intuitive (Leibniz, 1969b, p. 292). Thus knowledge could be classified as clear or obscure, distinct or confused, and symbolic or intuitive (Leibniz, 1969b, p. 291-292). Knowledge is *clear* when it is possible to recognize the object represented as one that has been seen before even if one cannot enumerate the properties which distinguish it from another object. Thus we may recognize two trees as distinct but be unable to enumerate the leaves on the tree which distinguish the two. Knowledge is *distinct* when it is possible to enumerate the characteristics which allow the object represented to be distinguished from another object. Thus an object which has been represented clearly and distinctly is what I have called a "decidable object." If every ingredient which enters into a distinct concept is itself known distinctly then knowledge is *intuitive* (Leibniz, 1969b, p. 291-292.)

Kant criticized Leibniz in taking sensory representations to differ from intellectual representations only in their greater degree of confusion, which he took to be the logical form of representations and not their content (1784/1965, B62). For Kant, our intuitive and conceptual representations have different part - whole structures and so originate in different faculties. Leibniz, in fact, holds that there are three levels of concepts, according to whether they arise from particular sense, common sense or the intellect.

> Thus there are three levels of concepts: those which are sensible only, which are produced by each sense in particular; those which are at once sensible and intelligible, which appertain to the common sense; and those which are intelligible only, which belong to the understanding. The first and second are imaginable, but the third lie beyond the imagination. The second and third are intelligible and distinct, but the first are confused, although they may be clear and recognizable. (Leibniz, quoted in McRae, 1994, p. 181)

The third kind of concept such as the I who perceives or acts is neither sensible nor imaginable but purely intellectual. Besides the representation of sensible qualities which are of particular external sense there are some which come from multiple senses in which these particular representations are found united. This sense is the imagination; it comprises both the concepts of particular senses which are clear yet confused and the concepts of the common sense which are clear and distinct. "And these clear and distinct ideas which are subject to the imagination are the objects of the *mathematical sciences* ...." (Leibniz, quoted in McRae, 1994, p. 180). Leibniz repeats this idea in other places as well. "Mathematics is the science of imaginable things. Universal mathematics should treat of the method of *determining* exactly what falls under the imagination or that which I call the logic of the imagination" (Leibniz quoted in McRae, 1994, p.182). On the basis of the fact that *clear and distinct* objects are the subject matter of mathematics I am attributing to Leibniz the view that every mathematical object is decidable.

A strong motive, perhaps the main motive, of Leibniz's for distinguishing between confused and distinct perceptions was precisely the need to solve the problem of the composition of the continuum (McRae, 1994, p. 178ff). Leibniz's chief concern appears to be how we can grasp the concept of the continuum as a single whole given that the continuum is infinitely divisible. To spell this out, our concept of the continuum must contain the concept of each of its parts, but such a *complete* concept which gives us intuitive knowledge can only be had by a divine intellect.

His solution begins by distinguishing between perception and apperception, the former may be confused or distinct but the latter is always distinct. The distinction between the two is a matter of degree. A perception may begin as clear yet confused but it is as sufficiently heightened and distinct "as when rays of light are concentrated by means of the shape of the humours of the eye and act with greater force that perceptions are noticed and thereby become objects of our consciousness or apperception" (Leibniz, quoted in McRae, 1994, p.180). Thus Leibniz makes the distinction between perception, which is "the internal state of the monad representing external things" and apperception which is "consciousness or the reflexive knowledge of this internal state itself" (Leibniz quoted in McRae, 1994, p. 179).

Secondly he distinguishes between appearances of substances and substances themselves. Appearances are infinitely divisible and comprise such continua as space, time and extended bodies, while substances must be indivisible unities. For Leibniz "that which is not truly one being is not truly a being" (quoted in McGuire, 1992, p. 32). So the only real or actual things are individual substances; appearances are merely ideal. An aggregation or collection of substances into extended bodies, for instance, appears to have unity. But this is an illusion, its unity is only mental.

> This unity of the idea of collection is only a congruity or relation, whose foundation is in what is found in each simple substance by itself. And so these beings by aggregation have no other complete unity but what is mental or phenomenal, like that of a rainbow. (quoted in McGuire, 1992, p. 40)

In holding this belief he differs drastically from those, such as Frege and Godel, who believe that the unity of an aggregation given by the concept under which it falls provides a *real* unity.

For ideal wholes, those whose unity is entirely in the mind and not in reality, the whole is prior to the parts. By dividing an appearance we *create* parts, and so the parts of the appearance exist potentially rather than actually. Because of the priority of whole to part we have no need of apperception to represent appearances because there is no need to individuate their parts in order to individuate the whole appearance. The appearance arises as a confused perception of an aggregate of actual simple substances.

But in order to recognize an aggregation as a plurality of substances, simple substances must be distinguished in representations by means of their internal qualities and relations. Otherwise by the identity of indiscernibles they would collapse into one substance. Thus there must be a plurality of affections and relations within the indivisible unity which is the simple substance or monad. So in order to distinguish between simple substances by means of our representations, the representations of those substances must contain *within themselves* the representation of the affections and relations.

However no human mind can grasp such a complete concept. "It is impossible for us to know individuals or to find any way of precisely determining the individuality of anything" (Leibniz, quoted in McRae, 1994, p. 190). The reason is that to individuate a given thing is to ascribe to it an infinity of attributes, and so only someone who is capable of grasping the infinite could know the principle of individuation of a given thing. Thus the problems of an actual infinite that arise from apperception of things is left for the Divine Intellect. There can be no problem arising from our grasping the existence of an actual completed infinite because we are unable to do so.

Ideal wholes are truly continuous but they are not real; actual wholes are simple and indivisible. Thus the question of whether the continuum is composed of actual indivisibles is due to confusing actual wholes, the monads, with ideal wholes, confused perceptions of monads. Leibniz puts the solution this way in a letter to Des Bosses:

> A continuous quantity is something ideal which pertains to possibles and actuals insofar as they are possible. A continuum that is, involves indeterminate parts, but on the other hand, there is nothing indefinite in actuals, in which every division that can be made, is made. Actuals are composed as a number is composed of unities, ideals as a number is composed of fractions; the parts are actual in the real whole but not in the ideal whole. But we confuse ideals with real substances when we seek for actual parts in the order of possibles, and indeterminate parts

> in the aggregate of actuals, and so entangle ourselves in the Labyrinth of the continuum and in inexplicable contradictions. (quoted in McGuire, 1992, p. 38 - 39)

The indeterminacy or undecidability of the parts of a continuum was considered by Leibniz to be due to the fact that its parts were not actual, but were confused perceptions of actual monads. Thus the solution to the problem of the composition of the continuum is that *neither the continuum nor its parts are actually real objects*. We do not need divine powers to account for our representation of a single real actually infinite collection of parts because our perception is only of a confused perception of an actually infinite multiplicity.

What Leibniz did *not* realize was that *the indeterminacy of parts was also implied by the conception of a continuum as infinitesimally linear*. This fact would not have particularly troubled Leibniz because he did not think of the parts of the continuum as actual objects. But it will be a problem for the semantic tradition because it *does* consider the parts of the continuum to be actually distinct objects. The conception of a curve as infinitesimally linear implies that even if the parts of the continuum are conceived to be actual they are still undecidable. Thus, there are two routes to the undecidability of elements of the continuum in Leibniz's thought: his conception of perception *and* the structure of the continuum itself which is used in the infinitesimal calculus. It is the implications of this latter aspect that I wish to consider.

Leibniz's view of the continuum implies that the infinitesimal parts of the continuum are merely ideal. However such a notion of the infinitesimal would come as a great shock to Leibniz's adherents in the Paris Academy of Sciences. In delivering L'Hospital's eulogy, Fontenelle (The secretary of the Paris Academy of Sciences) described the differential calculus as the "sublime geometry" and L'Hospital as possessing a map to "Le Pays de l'Infini". Given this tribute it is difficult to imagine that there was a bitter dispute in the French Academy over the proper interpretation of the calculus. As I noted, Leibniz managed to evade invoking an interpretation of an infinitangular polygon by relying on results concerning the infinite sequence it induced. However, Nieuwentijt, L'Hospital, as well as Varignon and the Bernoullis clearly believed in the existence of infinitesimal quantities and were under the impression that they were in agreement with Leibniz's interpretation of the calculus. This sparked a controversy between Leibniz and French mathematicians surrounding the proper interpretation of infinitesimals. The outcome of this dispute reveals that, for Leibniz, not only were infinitesimal parts of the continuum ideal, but the language of mathematics was held to be independent of its interpretation. Thus Leibniz anticipates the model theoretic viewpoint in mathematics which emphasizes the independence of the language of mathematics and its interpretations.

The centrality of the use of infinitesimals, and, in particular, the principle of infinitesimal linearity, was recognized by Bernoulli and made explicit in the notes dictated by Bernoulli to L'Hospital *L'Analyse des Infiniment Petit pour L'Intelligence des Lignes Courbes* (quoted in Jesseph, 1994, p. 139ff). The work opens with the following definitions and principles and provides the paradigm of a realist approach to the Leibnizian calculus.

> Definition I: Variable quantities are those which increase or diminish continually; and constant quantities are those which remain the same while others change. Thus in a parabola the ordinate and abscissa are variable quantities, while the parameter is a constant quantity.
>
> Definition II: The infinitely small portion by which a variable small quantity continually increases or diminishes is called the difference.
>
> Postulate or supposition: It is postulated that one can take indifferently for one another two quantities which differ from one another by an infinitely small quantity: or (which is the same thing) that a quantity which is augmented or diminished by another quantity infinitely less less than it, can be considered as if it remained the same.
>
> Postulate or supposition: It is postulated that a curved line can be considered as an infinite collection of right lines, each infinitely small: or (which is the same thing) as a polygon of an infinite number of sides, each infinitely small, which determine the curvature of the line by the angles they make with one another.

Some of Leibniz's comments made it appear that he did not believe in infinitesimals. For example, in a letter to John Bernoulli he says:

> As concerns infinitesimal terms, it seems not only that we never get to such terms, but that there are none in nature, that is, that they are not possible. Otherwise, as I have already said, I admit that if I could

> concede their possibility I could concede their being (Leibniz, 1969a, p. 511).

This comment is closely related, of course, to the problem of visualizing a complete infinitangular polygon. Elsewhere he writes: "I consider infinitesimals to be useful fictions" (quoted in Rescher, 1967, p. 106). Again: "To tell the truth, I am not so persuaded myself that is necessary for us to consider infinity and infinitesimals as anything other than ideal things or well founded fictions" (quoted in Earman, 1975, p. 238 my translation). An ideal thing for Leibniz is not a real thing but merely the appearance of a real thing which is caused by our confused perception of aggregations of real things. But it is well founded because the confused perception arises from an actual plurality of monads and is not a mere dream or hallucination. Thus, for instance, we may take a thousand sided polygon to be an infinitangular polygon, and its sides to be infinitesimal rather than finite, because of our inability to have a distinct perception of the polygon.

These statements reflect an ambivalence on Leibniz's part regarding the existence of infinitesimal quantities rather than an outright denial. In his letter to Bernoulli his denial that there are any infinitesimals in nature does not contradict this ambivalence, since Leibniz simply thought that there being "none in nature" was a consequence of his attitude. For, as Leibniz reasons, admitting the possible non existence of infinitesimals is tantamount to asserting that they don't exist at all. In this case, perhaps ambivalence is better served by being completely non committal.

Considering the straightforward interpretation of the principle of local straightness of curves that existed in French mathematical circles it is not surprising that the idea that infinitesimals were fictions caused some consternation and confusion. In a letter to his friend Pinson, Leibniz does not deny the existence of infinitesimals but says that they are unnecessary for the calculus. The differential may be supposed to stand to the variable in the proportion of a grain of sand to the earth.

> In our calculations there is no need to conceive the infinite in a rigorous way. For instead of the infinite or the infinitely small, one takes quantities as large or as small, as necessary in order that the error be smaller than the given error, so that one differs from Archimedes'

> style only in the expressions, which are more direct in our method and conform more to the art of invention. (quoted in Horvath, 1986, p. 66)

Opponents of the calculus used the letter to Pinson to attack Varignon by reciting Leibniz's own apparent admission that differentials were very small *fixed* finite quantities as Archimedes' method of exhaustion required rather than infinitesimal quantities. Varignon requested clarification on Leibniz's interpretation of the calculus and Leibniz responded that infinitesimals have the effect of being infinitely small because they *become* arbitrarily small:

> These incomparable quantities are not at all fixed or determined but can be taken as small as we wish in our geometrical reasoning and so have the effect of the infinitely small in the rigorous sense. If any opponent tries to contradict this proposition, it follows from our calculus, that the error will be less than any possible assignable error, since it is in our power to take that incomparably small quantity small enough that for that purpose, inasmuch as we can always take a quantity as small as we wish. May be that is what you Sir, mean by the notion when you speak about that of inexhaustion, moreover, there is no doubt, that this idea establishes the rigorous demonstration of the infinitesimal calculus. (quoted in Horvath, 1986, p. 66)

Leibniz's response emphasizes the fact that the differentials are not to be regarded as fixed, but as being variable, so that as they approach zero they have the *effect* of being infinitely small. Leibniz tells us that incomparable quantities are what we now refer to as non - Archimedean quantities: quantities which are such that for any a and b with $a < b$, there is a c such that $ac > b$ (Horvath, 1986, p. 63). In other words, for any b, there is a c such that $a + ... + a$ summed c times exceeds b. The effect that the quantities are as small as we wish entails that they cannot be Archimedean since a multiple of an arbitrarily small quantity will remain arbitrarily small.

Thus Leibniz's use of the method of exhaustion is quite different from the traditional one, since traditionally the method of exhaustion was wedded to the use of Archimedean quantities. Thus it is inaccurate to ascribe to Leibniz, as Bos (1974, p. 55) does, the view that one of his interpretations of the calculus was that of the traditional method of exhaustion. For the method of exhaustion uses only finite quantities. Bos has it the wrong way around:

Leibniz does not invoke the traditional method of exhaustion as an interpretation of the calculus; instead he uses the method of exhaustion to prove the existence of infinitesimals.

Let me explain. The method of exhaustion solved the problem of finding areas enclosed by curvilinear figures within the Euclidean theory of ratios and proportions of Euclid's Elements. The essence of the method is to take a given magnitude and construct another magnitude which bears the desired relation to the original. Fuller explanations are in Jeseph (1994) and references contained therein; Stein (1995) and in Edwards (1979). Consider the following definitions:

> 3. A *ratio* is a sort of relation in respect of size between two magnitudes of the same kind.
>
> 4. Magnitudes are said to *have a ratio* to one another when they are capable, when multiplied, of exceeding one another.
>
> 5. Magnitudes are said to *be in the same ratio,* the first to the second and the third to the fourth, when, if any equimultiples whatever be taken of the first and third, and any equimultiples whatever of the second and fourth, the former equimultiples alike exceed, are alike equal to, or alike fall short of, the latter equimultiples, respectively, taken in corresponding order
>
> 6. Let magnitudes which have the same ratio be called *proportional.*

These definitions allow for the comparison of magnitudes within each species of magnitude by forming ratios and constructing proportions. The finite nature of the theory of ratio and proportion is apparent from the fact that the multiplications in Definitions 4 and 5 are finite multiplications. Non-Archimedean (what Leibniz calls non comparable) quantities are explicitly barred, since Definition 5 would not hold if there were non-Archimedean quantities.

The theory of ratio and proportion gives rise to the fundamental method of proof in classical geometry: proof by exhaustion. The main idea in a proof by exhaustion consists in showing that the unknown ratio between two magnitudes can be determined by considering sequences of inscribed and circumscribed quantities which approximate the unknown quantity to within any degree of magnitude. When it can be shown that the unknown quantity is compressed between inscribed and circumscribed quantities, the proof is completed by a double *reductio ad absurdum* which shows that the unknown can be neither greater than nor less than a given amount. It is clear that no infinitesimal quantities are required in an exhaustion proof, since they are barred by Definition 5.

However, it was common in the seventeenth century to consider the method of exhaustion as equivalent to infinitesimal techniques and the existence of infinitesimals as proved by the method of exhaustion. We have already read Leibniz's words to that effect. Consider also Wallis's remark:

> The method of Exhaustions, (by inscribing and Circumscribing Figures, till their difference becomes less than any assignable) is a little disguised, in (what hath been called) *Geometria Indivisibilium* ... which is not, as to the substance of it, really different from the Method of Exhaustions, (used both by Ancients and Moderns,) but grounded on it, ***and demonstrable by it***: But is only a shorter way of expressing the same notion in other terms. (quoted in Jeseph, 1989, p. 234, my emphasis)

We can see, then, that Leibniz is not merely using the calculus as an abbreviated form of proof by exhaustion, instead he is interpreting the method of exhaustion as proving that there are incomparable magnitudes.

So Leibniz's first line of response is that we have no need of "rigorously" infinitely small quantities, since we use incomparably small quantities. These quantities are not intrinsically or rigorously small since "we never get to them" but insofar as we treat them as actual infinitesimal quantities, they are fictions. These incomparably small quantities are variably finite quantities, and are used in proofs by exhaustion.

Leibniz did not need to derive the existence of infinitesimals from the method of exhaustion since it can be derived from his principle that each curve is an infinitangular polygon. It has already been shown that the principle of infinitesimal linearity implies that there are square zero infinitesimals. One can then prove from the existence of square zero infinitesimals that the continuum is non Archimedean. Let $d \in D$ where D contains the infinitesimal parts of a curve R, i.e. $D = \{x \in R : x^2 = 0\}$; and b is a non infinitesimal part of a curve, i.e. $b \in R$ but $b \notin D$. Clearly, multiplication by an infinitesimal is a closed operation, that is, $(\forall d, b) db \in D$. (Since $(db)^2 = 0$ and so is in D as well.) So if the elements of R were Archimedean, then for any d, e with $d < e$, there is $b \in R$ such that $db > e$. But this would contradict the fact that $db \in D$.

Leibniz distinguished between two distinct questions: (1) whether infinitesimal quantities actually exist; and (2) whether analysis by means of differentials and the rules of the calculus leads to the correct solutions to the problems to which it is applied. This is confirmed in a letter to Varignon, where Leibniz says that "it is unnecessary to make mathematical analysis depend upon metaphysical controversies" (Leibniz, 1969b, 542 - 3). Bos (1974) agrees that Leibniz did not commit himself on the question of the existence of infinitesimals. Therefore, Bos reasons, if Leibniz thought that the infinitesimal calculus was justified he could not invoke infinitesimals to justify the calculus; so Leibniz *must* have treated infinitesimal as fictions.

> ... he could not invoke the existence of infinitesimals in answer to the objections to the validity of the calculus. Instead he had to treat infinitesimals as fictions which need not correspond to actually existing quantities, but which nevertheless can be used in the analysis of problems. (Bos 1974, p. 54)

Edwards (1979, p. 264) draws a similar conclusion. But this is going too far. Leibniz is not saying that we *must* treat infinitesimals as fictions. In fact, the interpretation of infinitesimals as fictions *was* one of the metaphysical views in question, and thus one of the views which was held to be independent of the justification of the calculus. The ambivalence regarding the existence of infinitesimals does not force him to adopt an interpretation which does not need them. Leibniz simply says that we may use infinitesimals as *ideal concepts* if we refuse to take them as real.

> .... Moreover even if someone refuses to admit infinite and infinitely small lines in a rigorous metaphysical sense and as real things, he can still use them with confidence as ideal concepts which shorten the reasoning. (Leibniz, 1969a, p. 543)

The confusion of other commentators, such as Bos, Earman and Ishiguro, is due, I think, to failing to come to grips with the fact that Leibniz believed his calculus was just that - a calculus; and as such the interpretation of the calculus was independent of its deployment as a calculus. It would be perfectly natural for a mathematician who worked in the seventeenth century (or a historian or philosopher studying this period) to assume that the reference of a mathematical expression is a form abstracted from physical objects. But this isn't Leibniz's view. Mancosu (1989) makes a related point in his assessment of Leibniz's interpretation of "infinitesimal". After discussing the interpretation of Leibniz offered by his French colleagues Mancosu says:

> Leibniz was in fact attempting to define a more subtle position by considering his infinitesimals as well founded fictions.... In effect, Leibniz was proposing a sophisticated "formalistic" foundation for his algorithm. However, by considering the infinitesimal as a well founded fiction, he was introducing a gap between the formal apparatus and [its interpretation]. (Mancosu, 1989, p. 238)

Mancosu's position is closer to the truth than that of Earman or Ishiguro. By denying that infinitesimals must be taken as real, he makes a decisive break with the accepted view that mathematical objects are abstracted from, and are the forms of, real objects. But Leibniz goes further than considering infinitesimals to be well founded fictions. He is not introducing a "gap between the formal apparatus and the [interpretation]" by introducing fictions. The gap is introduced because Leibniz believed that there was simply no need for any particular interpretation to justify the calculus. It was justified by its problem solving ability. Jeseph has recently come to a similar conclusion.

> Thus Leibniz's concern with matters of rigor leads him to propound a very strong thesis indeed, namely no matter how the symbols "*dx*" and " *dy*" are interpreted, the basic procedures of the calculus can be vindicated. Such vindication could take the form of a new science of infinity, or it could be carried out along classical lines, but in either case the new methods will be found completely secure. (1989, p. 243)

Thus Jesseph places Leibniz within the model theoretic tradition. On the modern model theoretic approach, no *particular* model is needed to justify a proof, but only that it must be true that whenever the premises are true in a model the conclusion is true as well. Fortunately recent developments in category theory have produced models in which Leibniz's reasoning is, in fact, correct.

Leibniz's point of view becomes clearer if we understand how he thought that mathematical reasoning works in general. The separation of language and models in Leibniz s philosophy is connected to his views about thought and meaning. Much of mathematical reasoning does not require that we have a particular interpretation or meaning in mind when we use a word, although we do know the meaning of the words involved.

> Thus when I think of a chiliagon, or of a polygon with a thousand equal sides, I do not always consider the nature of a side and of equality and of a thousand (or the cube of ten), but I use words which I have for them, because I remember that I know the meaning of the words but their interpretation is not necessary for the present judgment. Such thinking I usually call *blind* or *symbolic* .... (Leibniz, 1969b, p. 292)

The reason that mathematical reasoning can be blind and still be meaningful as well as increase our knowledge is a consequence of Leibniz's views on meaning in general. For Leibniz meanings are not private images located in one's mind, but objective entities. In *Alice in Wonderland* Humpty Dumpty thought that words meant whatever he wanted them to mean. This "humpty dumpty" approach to meaning contends that communication depends on our successfully transferring our subjective representations to one another. Or in the case of mathematical reasoning, we reason from step to step by transforming mental images. But it is unlikely that there would be much communication or such reasoning if we had to rely solely on the transference of private meanings between speakers. As Putnam (1975) has observed, there is a division of labour in language which allows experts to know the deeper meaning of specialized terms such as "gold", "temperature" or "atom", while, at the same time, allowing ordinary individuals to understand what is meant by such terms. In other words, the subjective representation of the meaning of such natural kind terms may differ among users; identity in objective meaning allows communication to take place.

An objective representation is required for communication because communication depends on attaching a common meaning to our expressions. In the case of mathematical reasoning, we often simply do not have a complete understanding of the object under investigation. Yet in order to analyze it we must suppose that the object of the imagination which we are analyzing does stay the same. Thus Leibniz thinks that we each have a confused, and therefore imperfect, understanding of the contents of our thoughts; but these contents themselves are objective and unchanging (Compare with Mates, 1986, p. 102). By adopting this point of view Leibniz is a forerunner of the semantic approach to meaning which requires that we distinguish between the subjective representation and the objective meaning of things.

Leibniz offered a second interpretation of the calculus through his law of continuity. Many of the same points emerge from studying this method. This law is formulated as:

> If any continuous transition is proposed terminating in a certain limit, then it is possible to form a general reasoning, which covers also the final limit. (Quoted in Bos, 1974, p. 56)

Elsewhere he compresses the law into the maxim that "Nature makes no jumps", or "No transition happens by a leap" (quoted in Mates, 1986, p. 163). Leibniz's favourite examples of the principle of continuity came from geometry and mechanics. The example which occurs most frequently in his writings concerns the fact that the various conic sections can be continuously transformed into one another by gradually tilting the intersecting pla... (Mates, 1986, p. 163 - 4).

However the example which concerns the infinitesimal calculus directly is that of conceiving of a circle as an infinitangular regular polygon by continually transforming a polygon into a circle. Because "nature makes no leaps" we are able to proceed continuously from polygons to circles.

> Although it is not at all rigorously true that rest is a kind of motion or that equality is a kind of inequality, any more than it is true that a circle is a kind of regular polygon, it can be said, nevertheless, that rest, equality and the circle terminate the motions, the inequalities, and the regular polygons which arrive at them by a continuous change and vanish in them. And although these terminations are excluded, that is, are not included in a rigorous sense in the variable they limit, they nevertheless have the same properties as if they were included in the series, in accordance with the language of infinities and infinitesimal, which takes the circle, for example, as a regular polygon with an infinite number of sides. Otherwise the law of continuity would be violated, namely, that since we can move from polygons to a circle by a continuous change and without making a leap, it is also necessary to

> make a leap in passing fı ɔm the properties of polygons to those of a circle (Leibniz, 1969, p. 546).

Here Leibniz argues that the principle of continuity can be invoked to prove that a circle is an infinitangular polygon.

Leibniz and his French supporters also argued, on the basis of his principle of continuity, that infinitesimals were necessary in order to implement limits. Leibniz gives the following argument for this in his Letter to Varignon (Loemker, 1969a, p. 545-6).

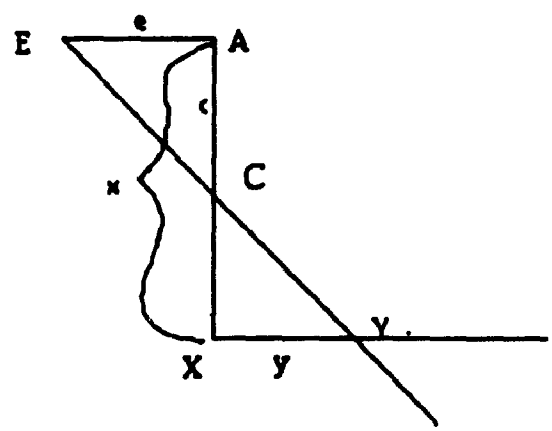


Figure 13

Assume that the angle ECA is not equal to 45 degrees. Thus the ratio $c/e \neq 1$. Since ECA is similar to YCX, we obtain $(x-c)/y = c/e$. Now suppose that the line EY is transported parallel to itself towards A. The angle ECA and, thus, the ratio c/e remain constant. But as EY passes over A, c and e will vanish. But according to Leibniz neither c nor e is zero. Rather they should be understood as evanescent or vanishingly small quantities. This is due to the fact that if the quantities did become zero, then the ultimate ratio would reduce to $0/0$ but $0/0 = 1$, which contradicts the assumption that the angle ECA is not 45 degrees.

> Hence c and e are not taken for zeros in this algebraic calculus, except comparatively in relation to x and y; but c and e still have an algebraic relationship to each other. And so they are treated as infinitesimal, exactly as are the elements of our differential calculus recognizes in the ordinates o: curves for momentary increments and decrements. (Lcemker, 1969, p. 545)

Here we find an application of Leibniz's law of continuity that in any continuous transition, ending in any terminus, it is permissible to *include* the final terminus.

An alternative approach: Newton and the intuition of variable quantities

I have placed Leibniz's approach to the calculus firmly within the semantic tradition which eschewed the need for any interpretation of the calculus. However, not everyone was convinced by Leibniz's view, and it was believed by many, especially those in England, that an interpretation of the calculus was to be found in spatial and temporal intuition. In particular, the Newtonian view of the calculus was that the foundation of the calculus was the intuition of motion. The fact that mathematical arguments were often couched in spatial and temporal terms no doubt influenced Kant's position that intuition was necessary for mathematical proof. In order to appreciate the Kantian argument for the necessity of intuition, and the negative reaction to the use of intuition by subsequent mathematicians, then, it is necessary to be acquainted the kind of temporal approach to the calculus that was adhered to by many others in England.

Newton foreshadows an important distinction drawn by Kant between intuitive representation of quantities and conceptual representation of quantities. Representation by the concepts has a compositional structure in which the objects are considered to be composed of their parts, whereas in intuitive representation a variable quantity is not composed of parts but, instead, is generated by a continual motion. Newton describes his own position as follows:

> I don't consider Mathematical quantities as composed of Parts extremely small but as generated by a continual motion. Lines are described and by describing are generated, not by any apposition of Parts, but by a continual motion of Points. Surfaces are generated by the motion of Lines, Solids by the motion of Surfaces, Angles by the rotation of their legs, Time by a continual flux, and so in the rest. These Geneses are founded upon Nature, and are every Day seen in the motion of Bodies (quoted in Friedman, 1992, p. 74).

Newton's calculus, as the name suggests, was based upon the conception of a curve as flowing along a continuum of values. The focus on flowing quantities represents one approach to geometric problem solving. This idea was in keeping with the Cartesian revolution in geometry which solved geometric problems by means of tracing a curve and intersecting it with a straight line, circle, or other curve rather than relying simply on ruler and compass constructions. Newton regarded the curve $f(x,y)=0$ (the *fluent* f as determined by moving lines x and y) as the locus of the intersection of two moving lines , one vertical and one horizontal. The x and y coordinates of the moving point are functions of time, specifying the location of the vertical and horizontal lines, respectively.

The motion is therefore the composition of the horizontal and vertical velocity vectors, the tangent velocity vector being given by the parallelogram sum of the vectors. If one takes a line of length k, there are products $k\dot{x}$ and $k\dot{y}$ which will stand in the same proportion as $\dot{x}$ and $\dot{y}$. So one may take the sum of those vectors in order to give the tangent vector to the fluent. So Newton considers a geometrical model of two or more points A and B travelling distance x and y along different straight lines in equal periods of time, so that $f(x,y)=0$ at each time, with speeds given by $\dot{x}$ and $\dot{y}$ respectively.

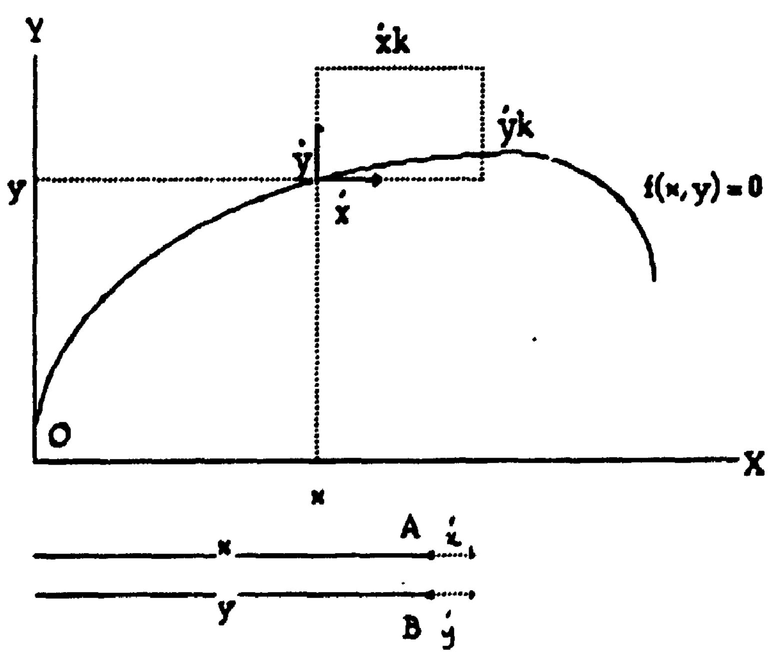


Figure 14

The fundamental problem of the calculus was this: given the fluents and their relations, to find the relations between their fluxions, and conversely. Newton's method of prime and ultimate ratios is the temporal analog of the modern use of limits. The prime ratios of nascent quantities are those which hold between quantities as the quantities are just beginning to be generated, while the ultimate ratios of evanescent quantities are those which hold between quantities as they diminish to zero and vanish. Newton describes the method in his 1693 "Treatise on Quadrature":

> Fluxions are very nearly as the augment of Fluents, generated in equal, but infinitely small parts of Time; and to speak exactly, are in the Prime Ratio of the nascent Augments: but they may be expounded by any Lines that are proportional to 'em (quoted in Kitcher, 1984, p. 238).

Consider the following exercise given in the Treatise. Suppose that a quantity (a fluent) x flows uniformly, i.e., its fluxion is 1. What is the fluxion of $y = x^n$? In the same time that the quantity x by flowing becomes $x+o$, the quantity x^n will become $(x+o)^n$. This last expression may be expanded by the binomial theorem to obtain:

$$(*) \quad (x+o)^n = x^n + nx^{n-1}o + \frac{n(n-1)x^{n-2}}{2!}o^2 + \ldots$$

and subtracting $y = x^n$ Newton obtained the change in x^n which corresponds to the change o in x. Now one may form the ratio of the change in x to the change in x^n. The augments o and $nx^{n-1}o + \frac{n(n-1)x^{n-2}}{2!}o^2 + \ldots$ are to one another as 1 is to $nx^{n-1} + \frac{n(n-1)x^{n-2}}{2!}o + \ldots$ Now let the augments vanish by setting $o^2 = 0$, and their ultimate ratio will be 1 to nx^{n-1}. So when the fluxion of $x = 1$, the fluxion of $y = x^n$ is nx^{n-1}.

Here we arrive at the same problem that provoked the dispute between Leibniz and Nieuwendijt. What allows one to eliminate all terms apart from $nx^{n-1}o$ (and so setting $o^2 = 0$ in (*))? It is not construed as a square zero infinitesimal, as Nieuwentijt believed it should. Rather it is justified by the temporal intuition that a process which ceases, ceases *with* a particular velocity. Thus if we take the ratios of the component fluxions at the end we obtain the ultimate ratio.

Newton realized that his method of prime and ultimate ratios may be met with some skepticism. The problem involves the interpretation of beginning and ceasing. One may argue that there are no such evanescent quantities because the proportion, before the quantities vanish, is not ultimate, and when they have vanished there is no proportion. Similarly, one may argue that a body arriving at a certain place and stopping has no ultimate velocity since before it has stopped it has not attained its "ultimate" velocity and when it has stopped there is no velocity. This criticism is premised upon the assumption that a temporal continuum is composed of instants, such that for any instant, there is an earlier instant. Newton digs in his heels with an appeal to the *temporal intuition* of a limit as that velocity *with which* a quantity stops .

> But the answer is easy; for by the ultimate velocity is meant that with which the body is moved, neither before it arrives at its last place and the motion ceases, nor after, but at the very instant it arrives; that is that velocity with which the body arrives at its last place, and with which the motion ceases. And in like manner, by the ultimate ratio of evanescent quantities (i.e. ones that are approaching zero) is to be understood the ratio of the quantities not before they vanish, nor afterwards, but with which they vanish. In like manner the first ratio of nascent quantities is that with which they begin to be. And the first or last sum is that which they begin and cease to be (or to be augmented or diminished). There is a limit which the velocity at the end of the motion may attain, but not exceed. This is the ultimate velocity. And there is the like limit in all quantities and proportions that begin and cease to be (1729/1955,*Principia* I, 1, Scholium).

Thus the problem of finding a tangent is understood as finding an ultimate ratio of varying quantities. Newton seems to recognize the inherent problems in understanding the notion of an ultimate ratio in an intuitively given continuum for, after giving the above - quoted answer, he then clarifies (or, at least, modifies) the notion of ultimate ratio in terms of *convergence* to a limit:

> Those ultimate ratios with which the quantities vanish are not truly the ratios of ultimate quantities but limits towards which the ratios of quantities decreasing without limit do always converge; and to which

> they approach nearer than by any give difference, but never go beyond, nor in effect attain to, till the quantities are diminished *ad infinitum.*

But how do we know that quantities converge to a limit? Here again temporal intuition seems necessary to allow the mathematical argument to proceed. If one traces out a curve by a finite continuous motion, and the motion gradually diminishes, then the line is completed and there is a point at which it ends. Any finite motion must be enclosed within limits, and so terminate at a point. Conversely, if the tracing of a line ends at a point, then the motion of the tracing has diminished to that limit. Thus every sequence which converges, converges to a limit point and, conversely, every point is the limit of some converging sequence. The necessity for these temporal intuitions, Kant argues, is due to the inability of the intellect alone to represent an infinite collection. It is to this argument that we now turn.

FROM SMOOTH SPACES TO SETS

"In actuals, simples are prior to aggregates, in ideals the whole is prior to the part. The neglect of this consideration has brought forth the labyrinth of the continuum."

G. Leibniz

In a previous chapter I examined the responses of modern mathematics to Aristotle's argument that the continuum is not composed of points. This led to the question of whether Leibniz's smooth continuum admits of a similar analysis. Such a set-theoretic analysis of the continuum runs counter to a powerful tradition in mathematical and philosophical thought. According to this tradition the primary reason for thinking that no continuum can be punctual is a consequence of Kant's "master argument" which concludes that only construction *in intuition* can provide *determinate* objects of knowledge for mathematics. Since intuition is a faculty which does not admit of an infinite analysis of its contents into distinguishable parts the continuum cannot be regarded as composed of points.

It was left to Bolzano and subsequent rigorizers to show that Kant's position regarding the necessity of intuition in mathematical knowledge was wrong by giving purely analytic proofs of significant propositions of analysis which hitherto required intuition. Bolzano is part of the movement to rigorize the calculus and is at the beginning of that tradition, the "semantic tradition", as it has been called by Coffa, which emphasizes the distinction between representations which are independent of our mind and representations which are mind dependent. This tradition attempted to rid the calculus of the need for intuition by invoking the use of mind independent representations. Thus the concepts of derivative and integral were given definitions independent of our intuitions, and underlying this "solution" is the assumption that the continuum is punctual. But a side-effect of this tradition was that in the final "rigorized" form of the "infinitesimal" calculus the infinitesimals were nowhere to be found.

I want to emphasize that in overcoming Kant's argument, the model theoretic tradition has, in fact, embraced a central idea: mathematical objects must be decidable in the sense that $\forall x \forall y[(x = y) \vee (x \neq y)]$. Whereas formerly the parts of the continuum were not regarded as mathematical objects and so were represented as objects of intuition or as merely confused objects of the intellect, in the hands of Bolzano, Cauchy and Weierstrass, the domain of objects was widened to include the parts of the continuum as (decidable) objects. However, since infinitesimals are not decidable they were eliminated as possible objects of mathematical knowledge and only the punctual parts of smooth spaces were retained. So there was a change in the conception of a continuum from an object which contains undecidable parts to one which is an object composed of decidable parts. Leibniz's puzzle is thus ultimately resolved by the semantic tradition by declaring infinitesimals to be non entities, and by giving the (purportedly) purely conceptual construction of the continuum by Cantor and Dedekind.

Kant's master argument

The problem of the composition of the continuum in Kant's thought arises in a similar way as it does in both Aristotle and Leibniz's work. Aristotle considered the idea of completing an infinite division of a continuous quantity, and held that this was impossible. Both Leibniz and the early Kant held a kind of theocentric view of knowledge (Allison, 1984). According to this viewpoint our knowledge is mediated by or filtered through an infinitely deep hierarchy of concepts, with each concept only fully understood by God. If we consider Aristotle's question in the theocentric context of Leibniz the natural problem that arises is how one is able to consider a magnitude to be a completed whole if we, as finite human intellects cannot analyze the concepts into their infinite constituent concepts.

Kant expresses the main problem as follows. First he distinguishes between synthesis and analysis or regression. The former arrives at the concept of a substantial compound by successively adding or compounding parts to parts, whereas the latter arrives at simples by taking away the concept of composition. Thus the concept of a whole arises through synthesis, that of a simple through analysis.

The problem is this:

> ... in the case of a continuous magnitude, the regression from the whole to the parts, which are able to be given and in the case of an infinite magnitude, the progression from the parts to the whole, have in each case no limit. Hence it follows that in one case, complete

> analysis, and in the other case, complete synthesis will be impossible. (1770/1992, §1)

It follows that in the case of analysis we cannot have the idea of a completed object while in the case of synthesis we cannot have the idea of a totality of objects.

Kant notes that the representation of a continuous magnitude as a whole is frequently rejected as impossible on the basis of the above argument (1992, §1). Indeed Leibniz believed that the appearance of a continuum as a whole was simply a well-founded fiction. It was not an accurate representation of real phenomena but a confused perception of discrete monads. But, unlike Leibniz, Kant did not reject the real existence of a unified continuum existing on the basis that our perception of continua is confused. Kant says of people who argue that there cannot be infinite wholes that they are "perverse" and are "guilty of the gravest errors" (1992, ID §1). The defect of such an argument, for Kant, is that it depends upon the assumption that all thinking is intellectual.

At the base of this reaction was the fact that Kant rejected the idea that the distinction between intellect and sensation is the same as the distinction between distinct and confused perceptions. Kant held, in contrast to the Cartesian - Leibnizian - Wolff tradition, that representations are of two types - concepts and intuitions - and that we have two kinds of faculties, sensibility and intellect. The distinction between intuition and intellect, is based upon their different sources rather than their degree of confusion. The distinction between these two faculties is summed up by Kant in terms of their peculiar functions. In the first sentence of the Introduction to the Transcendental Logic he writes:

> Our cognition springs from two fundamental sources of the mind. The first of these is that by which representations are sensed ... the second the capacity to cognize an object by means of representations ... through the first an object is given to us; through the second this object is thought. (1781/1965, A50/B74)

Of course the distinction between intellect and sensation is central to the *Critique of Pure Reason*. But one looks in vain in the *Critique* for the reason for such a distinction between faculties (Falkenstein, 1991). However if one looks at the *Inaugural Dissertion* (ID) one finds that the very motive for the distinction between the faculties of sensibility and intellect was to solve the problem of the continuum just outlined (Kant, 1770/1994, §1 and §2; Falkenstein, 1991). Leibniz had made a "grave error", Kant thought, because the need to consider the perception of the continuum as confused is due solely to the false assumption that all thinking is intellectual and so must allow for analysis and synthesis.

Thus Kant's solution in the ID is quite different from Leibniz's. Whereas Leibniz considered the continuum as merely ideal, Kant's idea is that the problem of representing a continuous magnitude arises only when the composites in space and time arise by analysis or synthesis. Intellectual representation is compositional in nature whereas intuitive representation is not. Intuitive and intellectual representation are not distinguished by their degree of clarity but by a fundamental difference in the relation of part and whole. Put more explicitly a concept such as "animal" has related concepts "invertebrate" and "vertebrate" which are formed by *distinguishing* between various *kinds* of animals. Concepts contain other concepts in the form of a hierarchy of kinds in a species - genus relationship. We do not have the representation of a vertebrate in our representation of an animal in the way that we have a representation of our finger in our representation of our hand. In the former case the representation has a hierarchical or compositional structure of whole to part whereby the part is a kind of the whole, while in the latter case the part is conceived *in* the whole.

Because of the compositional structure of intellectual representation, the representation of a continuous quantity must contain, in itself, the representation of each of its infinite parts. Moreover, the representation "is composed of component concepts in the same way in which the entire represented thing is composed of component parts" (Kant, quoted in Coffa, 1992, p 31). But no complete analysis or synthesis can be given because, as finite creatures (unlike a divine, infinite creature) we cannot arrive at the simples of which a composite concept is composed. We could only do this if we could complete an infinite analysis of a whole to the fundamental simples of which it composed. How does Kant solve this problem?

For Kant, it is only the intellect and not sense which demands that something be subject to a thorough-going analysis and hence be represented as containing an infinity of concepts. It is not demanded of sensibility that what is observed should admit of analysis into distinguishable parts. Thus if it were the case that the objects given in the faculty of sense and the faculty of intellect were cognized in different ways, the problem would not arise. The continuity of space and time do not give rise to paradoxes because there is no requirement that they admit of an analysis into (an infinite number of) distinguishable parts.

Kant's solution, thus, depends upon the assumption that the cognition of time and space depends only upon one faculty - sensitive intuition. Thus he formulates his solution as follows:

> Let him who is to extricate himself from this thorny question note that neither the successive nor the simultaneous coordination of several things (since both co-ordinations depend on concepts of time) belong to a concept of a whole which derives not from the understanding but only to the conditions of sensitive intuition. (1770/1994, ID, §1)

Thus Kant only needs to prove that space and time are cognized by sensation, and so cannot be legitimately subject to an infinite analysis. The focal point of the ID is the argument for the assumption that the cognition of space and time depends only upon sensitive intuition. Kant's strategy is to show that there is something about the nature of space and time that renders it incapable of intellectual representation. The main point that he makes is that space and time are singular or are particular. Time is singular because different parts of time are united and ordered in a *single* time. A specific time cannot be distinguished from another time by means of a specific feature which may be abstracted (by the intellect) from it, but only in terms of its relations to other times. Finally, if time were a universal which could be abstracted by the intellect, it would have some common feature which would define a kind under which distinct time intervals formed differing subkinds. But time intervals are not kinds of time, as a Jaguar is a kind of animal. Time intervals are in time, and make up time, rather than being subkinds of a kind. The same points hold for space.

As to why sensation is always of particulars Kant tells us:

> There is (for man) no intuition of what belongs to the understanding [intellectualium], but only a symbolic cognition; and thinking is only possible for us through universal concepts in the abstract, not by means of a singular concept in the concrete. For all our intuition is bound to a

> certain principle of form under which form alone that anything can be apprehended by the mind immediately as singular, and not merely conceived discursively by means of general concepts. But this formal principle of our intuition (namely space and time) is the condition under which something can be the object of our senses. Accordingly this formal principle, as the condition of sensitive cognition, is not a means to intellectual intuition..... Divine intuition, however, which is the principle of objects, and not something governed by a principle, is an archetype and for that reason perfectly intellectual. (1770/1992, ID, §10)

Kant's purported answer to why we cannot have intellectual intuitions of particulars is that the cognition of objects is bound to the conditions of space and time - namely singularity and immediacy. But the whole point of the exercise was to prove that space and time can *only* be cognized by sense and not intellect. Thus unless there is some independent reason to adopt the two-faculty theory of cognition Kant appears to be arguing in a circle (Falkenstein, 1991). Moreover this passage is doubly embarrassing because Kant claims that we have a "symbolic cognition" through the intellect. But if this were so, then as Leibniz held, there is no reason to require that such a symbolic cognition admits of an infinite analysis into its parts as a concept would. Indeed, later he rejects the intuitive/symbolic distinction which Leibniz had originated. "It is a contrary and incorrect use of the word symbolic to contrast symbolic with intuitive modes of representation (as the new logicians have done) for the symbolic is properly a species of the intuitive" (Kant, quoted in Falkenstein, 1991, p. 178.)

But Falkenstein fails to appreciate Kant's remark regarding the distinction between human and divine intellect. For divine intellect can represent particulars by intellectual cognition, whereas human intellection can't. Thus the reason for this inability is the reason why *our* intellect cannot represent particulars. The distinguishing feature between human and divine intellect is that human intellect is finite, it cannot engage in an infinite act of analysis. But the fact that an infinite being could engage in an infinite act entails that a divine being could represent particulars intellectually. Although the nature of intellectual representation is that it is inherently general, the more features thought in a concept the narrower the extension of the concept. Thus if a concept of a thing contains so many features that its extension is only one object, then a particular has been represented intellectually. But

Kant maintains that it is impossible for a human to have a representation of a concrete particular because this would require the ability to complete a synthesis and have a complete concept in Leibniz's sense. The finitude of human cognitive capacity prevents this.

The *Critique* offers a very different solution to the problem of the paradoxes of the continuum. Whereas, in the ID each faculty separately cognizes different domains, in the *Critique* "neither concepts without intuitions in some way corresponding to them, nor intuitions without concepts can provide a cognition" (1781/1965, A50/B74). The strategy in ID was to isolate the distinguishing characteristic of a faculty and then to show that space and time could only be cognized by that faculty. But in the *Critique* our representations are irrevocably tied together, and now our knowledge of a continuum, such as a rainbow, is not the confused perception of the individual droplets; instead we have no perception of individual droplets as things in themselves at all (1781/1965, A44-46/B62-63).

Human cognition arises only as a result of such a synthesis of intuitions and concepts. The necessity of such a synthesis is the conclusion of a complex transcendental deduction in the *Critique* which starts from the premise that all of our representations are unified in a single subject. Moreover, such a union is necessary not only for mathematical knowledge but for meaning as well. Concepts provide the logical form of the objects which they represent, whereas the content of the form of the object is provided by sensibility. If there were no union of intuition and concept, then our concepts would be empty of content.

> We demand in every concept, first the logical form of a concept (of thought) in general, and second, the possibility of giving it an object to which it may be related. In the absence of such object, it has no sense and is completely empty in respect to content, though it may contain the logical function which is required for making a concept out of any possible data. Now the object cannot be given to a concept otherwise than in intuition. (A239/B298, quoted in Friedman, 1992, p. 96)

This presents a problem for Kant. The *Critique* cannot offer the same solution as the ID to the paradoxes of the continuum since there can be no representation of a particular without the representation being a concept and an intuition. Because the defining feature of intuitive cognition is no longer that of sensing particulars, it is no longer possible to argue that space and time must be intuitions because they are particulars.

The relevance of all this to Leibniz's view of the continuum can be shown rather quickly. Recall that Leibniz believed that the cognition of mathematical objects does *not* require a synthesis of intuition and concept. Instead, our mathematical reasoning proceeds without any particular interpretation at all. Leibniz believed that much of mathematical reasoning was purely symbolic, and therefore depended only upon the logical form of the concept rather than upon an object to which it might be related in perception. We do not, in this way, intuit the nature of the object, but rely in reasoning only on the sign for the thing rather than its nature. Leibniz contrasts symbolic with intuitive reasoning in the following remark.

> Yet for the most part, especially in a longer analysis, we do not intuit the entire nature of the subject matter at once but make use of signs instead of things, though we usually actually omit the explanation of these signs in any actually present thought for the sake of brevity, knowing or believing that we have the power to do so. Thus when I think of a chilliagon, or of a polygon with a thousand equal sides, I do not always consider the nature of a side and of equality and of a thousand (or the cube of ten), but I use words which I have of them, because I remember that I know the meaning of the words but their interpretation is not necessary for the present judgment. Such thinking I usually call *blind* or *symbolic*; we use it in algebra and arithmetic, and indeed almost everywhere. When a concept is very complex we certainly cannot think simultaneously of all the concepts which compose it. But when this is possible, or at least insofar as it is possible, I call this knowledge intuitive (Leibniz, 1969b, p. 292).

It is not much of a stretch to see the relevance of this comment to the infinitesimal calculus. The fundamental idea of the infinitesimal calculus was that each curve could be regarded as an infinitangular polygon. But "one cannot go to infinity in his proofs" and so we cannot have an intuitive representation of an infinitangular polygon (quoted in McRae, 1994, p. 192). So reasoning about curves in mathematics took exactly the form of symbolic reasoning. One need not have in one's mind the idea of an infinitangular polygon, one need only blindly (and blithely Berkeley might say) apply certain techniques.

Kant, of course, is opposed to any such conception of reasoning. Such a method which does not involve an object given in sensible intuition Kant sees as specious because it is empty of content.

> There is however, something so tempting in the possession of an art so specious, through which we give to all our knowledge, however poor and empty we yet may be with regard to its content, the form of understanding, that general logic, which is merely a canon of judgment, has been employed as if it were an organon for the actual production of at least the semblance of objective assertions, and thus in fact has been misapplied. (1781/1965, A60 - 61/B85, quoted in Friedman, 1992, p. 97)

It is not that difficult to see what Kant is getting at. Kant is concerned that a form of reasoning which does not use intuition lacks objective content or meaning. As Friedman (1992, p. 98 - 99) points out, much of the recent work of interpreting Kant's philosophy of mathematics can be seen to relate directly to this "formalist" conception of mathematics suggested in the above remark of Leibniz. If all there were to mathematical reasoning were the blind manipulation of symbols, then mathematics would be without content. The notion of synthesis itself implies that we must go beyond purely conceptual representation in order to attain knowledge.

> For the notion of synthesis clearly indicates that something outside of the given concept must be added as a substrate which makes it possible to go beyond the concept with my predicate. Thus the investigation is directed to the possibility of a synthesis of representations with regard to knowledge in general, which must soon lead to the recognition of intuition as an indispensable condition for knowledge, and pure intuition for a priori knowledge. (quoted in Allison, 1973, p. 155)

Leibniz was undaunted by such concerns. "No one should fear that the contemplation of signs themselves will lead us away from the things in themselves; on the contrary, it leads us into the interior of things" (quoted in Ishiguro, 1990, p. 44). The reason that we have confused perceptions is because the signs we are using are badly arranged, but with proper arrangement we have a "mechanical thread of meditation" with which any representation can be resolved into its component representations (Ishiguro, 1990, p. 44). Indeed an early project was to arrive at a Universal Characteristic or calculus with which all philosophical problems could be resolved by calculation. Leibniz could not contain his missionary zeal in the 1670s:

> Where this language can once be introduced by missionaries, the true religion, which is in complete agreement with reason, will be established, and apostasy will no more be feared in the future than the apostasy of men from the arithmetic or geometry which they have once learned. (quoted in Brown, 1984, p. 57)

Although one may be sympathetic to Kant's reservations regarding Leibniz's program, it presents a problem for his analysis of the continuum, and for a proper response to Leibniz. For how can we make out that space and time are given in intuition and then acted upon by the intellect if every representation comes *already* synthesized? The basic strategy must be this: one must show that our concepts of space and time could not have arisen from intellect alone, but require intuition.

Thus a basic theme of Kant's mathematical philosophy in the *Critique* (as a number of commentators have pointed out) is that a *merely* conceptual determination of an object of experience is never adequate; intuitions are required. The fact that objects must be determined is supposed to follow from the argument of the transcendental deduction. For instance, the concept of the self as subject of experience necessitates, among other things, that the objects of experience are spatial, endure through time, and are causally connected. This idea is most problematic in the case of mathematics, since it is here where conceptual determination seems unavailable.

Following Brittan (1992) this theme can be crystallized into Kant's *master argument* which argues for the view that intuitions are required in order to complete the determination of objects.

Determination thesis: Mathematical objects are completely determinable.

Conceptual underdetermination thesis: Concepts do not completely determine mathematical objects.

Framework thesis: Either concepts or intuition determine objects.

Necessity of intuition: Therefore, intuitions are required to complete the determination of mathematical objects.

This conclusion is often referred to as Kant's principle of synthetic judgments. Kant attributes determination to both propositions and to objects claiming that every well formed mathematical proposition is determinable. Mathematics, says Kant, may "...demand and expect none but assured answers to all questions within its domain, although up to the present they have perhaps not been found" (1781/1965, A480/B508). As far as objects go, Kant says that "... to know a thing completely, we must know every possible [property], and must determine it thereby, either affirmatively or negatively" (1781/1965, A573/B610).

Thus, to determine an object a with respect to a property P is to verify or falsify the proposition "a is P". Kant's point is directly relevant to the mathematics of the seventeenth century since the main problem of the infinitesimal calculus of this period was to determine the relations between variable quantities such as the area under a curve, the length of a curve, and the tangent to a curve. It is not surprising, then, that how we determine mathematical objects was of central concern to Kant. Indeed mathematical objects provided Kant with his paradigm illustrations of why intuition was needed in order to determine objects.

Given Kant's notion of determination we can say something more about objects. To determine an object is to decide, for every property, whether the object has the property or not. If every object is determined, and we are given the law of the identity of indiscernibles $\forall x \forall y[\forall F(Fx \Leftrightarrow Fy) \Rightarrow (x = y)]$, then we can say whether any pair of objects a and b whether $a = b$ or not. Thus, the thesis that every mathematical object is determined entails the decidability thesis: $\forall x \forall y[(x = y) \vee (x \neq y)]$, i.e. the identity of any pair of individuals is decidable. Conversely, by the law of indiscernibility of identicals $\forall x \forall y[(x = y) \Rightarrow \forall F(Fx \Leftrightarrow Fy)]$, we can say that two objects are determined by the same properties.

For Kant, the situation in the empirical sciences is quite different than that in mathematics. "In the explanation of natural appearances much must remain uncertain, and many questions [remain] insoluble, because what we know of nature is by no means sufficient, in all cases, account for what has to be explained" (1781/1965, A477/B505). Apparently for Kant whereas the spontaneity of pure intuitions guarantees the determinateness of mathematical objects, the receptivity of empirical intuition fails to complete the determination of empirical objects (see Allison, 1984, chapter 13). Unlike mathematical objects, no empirical object can ever be completely determined since (in light of the definition of determination above) "The complete determination is thus a concept, which in its totality, can never be exhibited in *concreto* "(1781/1965, A573/B610).

Again, this differs quite significantly from Leibniz. Since Leibniz believes that the content of a subject *always* includes that of the predicate, and that mathematical statements are of this subject - predicate form, he holds that mathematical objects can be determined by concepts alone. The concept of the subject must always include the concept of the predicate in such a manner that if one has a perfect understanding of the subject, then it will be known which properties apply to it. Therefore concepts alone are sufficient to determine an object.

> ... thus the subject term must always include that of the predicate, so that whoever understood the notion of the subject perfectly would also judge that the predicate belongs to it. That being so, we can say that the nature of an individual substance or complete being is such a complete notion as to include and entail all the predicates of the subject that notion is attributed to. (Leibniz, 1988, §8, 13)

Although the concept of the subject contains the concept of the predicate, that fact does not entail that we have a perfect understanding of the subject, since our representation may be purely symbolic. Only God has a full understanding of a (complete) concept.

How, then, are intuitions able to complete the determination of mathematical objects? Some sort of explanation is needed of how this determination is to take place. It is useful to suppose, roughly following Brittan (1992), that all mathematical propositions can be determined as the conclusion of a demonstration. I leave open what form the demonstration takes. A demonstration may take the form of a syllogism, as in Aristotle's demonstrations in the *Posterior Analytics*, or it may take the form of an argument in first order logic, or an argument in set theory, or even an argument in the internal language of a topos.

Given this understanding of determination, mathematical objects may fail to be determined in one of two ways. Either, there are not enough premises in the argument for it to be a cogent demonstration, or the form of the demonstration may not be powerful enough to determine an object, whatever premises are used. It is this failure to determine an argument because of a lack of premises within a theory of demonstration which forces the use of intuition to determine objects. This is the well known view of Beck (1955) which Friedman (1992) calls the anti-Russellian view. This anti-Russellian interpretation has been heavily criticized by Friedman (1992) and I will not discuss it further.

A different interpretation of determination which stems from Russell is that objects cannot be determined by concepts alone only because of some inadequacy of logic. A prominent passage which supports this reading is the following:

> Suppose a philosopher be given the concept of a triangle and he be left to find out, in his own way, what relation the sum of its angles bear to a right angle. He has nothing but the concept of a figure enclosed by three straight lines, along with the concept of just as many angles. However long he meditates on these concepts, he will never, produce anything new. He can analyze and clarify the concept of a straight line or of an angle or of the number three, but he can never arrive at any properties not already contained in these concepts. Now let the geometer take up this question. He at once begins by constructing a triangle.... [Kant here goes on to describe the standard construction.] In this fashion, *through a chain of inferences guided throughout by intuition*, he arrives at a solution of the problem that is simultaneously full, evident, and general (1781/1965 A715-717/B743-745, quoted in Friedman, 1992, p. 57).

The general point is that philosophy reasons using concepts alone, whereas mathematics exhibits the concepts in *concreto* (1781/1965, A715-717/B743-745). Philosophy is confined to general concepts; whereas mathematics is able to prove nothing by using concepts alone. Instead mathematics considers the concept which it constructs *in concreto,* although not empirically. Then it is reasoned that whatever follows from the general conditions of the construction must hold for the object of the concept thus constructed.

What logical limitations lead mathematics to resort to intuition? In particular, what is there in the concept of the continuum that cannot be given solely by the intellect? In connection with the problem of the composition of the continuum, the part-whole structure is fundamentally different in the faculty of intellect from that of sense. In sense the parts are contained in the whole, as a part of a picture is contained in it, and the continuum is thus unified and unbounded. But in our intellectual conception the parts precede the whole. The striking point is that we do perceive the continuum as a whole, but the whole is prior to its constituent parts, which are mere limitations of the single whole. Thus the continuum must be given in intuition. Kant puts the argument like this:

> Space and time are quanta continua, because no part of them can be given without being enclosed between limits (points and instants), and therefore only in such fashion that this part is itself again a space or a time. Space consists only of instants, time consists only of times. Points and instants are only limits, that is, mere places of their limitation. But places always presuppose the intuitions which they limit or determine; and out of mere places, viewed as constituents capable of being given prior to space or time, neither space nor time can be composed (A169-170/B211-212, quoted in Friedman, 1992, p. 74).

This passage is quite clearly saying that the concept of an unbounded continuum must rely on intuition in order to complete its determination. Again he says:

> The infinitude of time signifies nothing more than that every determinate magnitude of time is possible only through limitations of one single time that underlies it. The original representation, time, must therefore be given as unlimited. But when an object is so given that its parts, and every quantity of it, can be determinately represented only through limitation, the whole representation cannot be given through concepts, since they contain only partial representations; on the contrary such concepts must themselves rest upon immediate intuition (1781/1965, A32/B48).

This can be explained in greater detail. I have already mentioned that conceptual representation is compositional whereas intuitive representation is not. This leads to the fact that concepts and intuitions differ in their whole-part relations. The difference is established by the way that the *content* of the concept is divided. To begin with, concepts are general because by means of them one can represent many objects. To be exact, concepts are not representations of complete objects, but only of properties which may be common to many objects. The generality of a concept is manifest in the way that a part-whole relationship is derived from it. General concepts are organized in a hierarchical relation su·h that they bear a type-subtype relation. For instance, the concepts "physical object", "animal" and "left-handed" can be arranged in such fashion. "Physical object" is the highest, or most general concept, and contains the other concepts under it. Conversely, the concept "left-handed" presupposes the concepts "physical object" and "animal". The content of the concept "physical object" is contained in the concept "animal".

On Kant's view the extension of the higher concept is divided on the basis of characteristics which differentiate the objects. This allows for further lower order classifications of objects by dividing the objects into those objects which posses a further characteristic, such as left-handedness and those that do not. Thus the concept "animal" has a smaller extension than the concept "physical object" but a greater intension. Lower order concepts are obtained by adding content. So for concepts, the parts are greater (in content) than the whole.

This whole-part relation differs for intuitions. First of all, the whole part relation is not hierarchical or compositional. Instead, the part is contained within the whole, not as a subtype. My perception of a house contains the perception of the windows in it, rather than falling under it. The perception of the house is not a type of which the window is a subtype. In this case the content of the whole is greater than the content of any of its parts. Thus the contrast between intellectual representations and intuitions is in terms of their part whole relations.

The feature that Kant thought apparently clinches the difference between the whole - part structure is the fact that representations of infinitely divisible quantities cannot be given in the intellect. On this matter Friedman (1992) has advanced a compelling interpretation of the following passage of Kant's *Critique* according to which Kant is expressing the impossibility that a conceptual representation could contain an infinity of objects in its very idea. As Kant put it in the *Critique* :

> Space is represented as an infinite given quantity. Now one must certainly think every concept as a representation which is contained in

> an infinite aggregate of different possible representations (as their common characteristic), and it therefore contains these under itself. But no concept, as such, can be so thought as if it were to contain an infinite aggregate of representations in itself. Space is thought in precisely this way, however (for all parts in space *in infinitum* exist simultaneously). Therefore the original representation of space is an *a priori* intuition, not a concept (B40, quoted in Friedman, 1992, p. 64).

Friedman's gloss on this passage is that since the "laws of thought" of the (monadic) logic of Kant and Aristotle was not able to force the representation to contain an infinity of concepts within itself it could only be done intuitively by an iterative process of construction. The implication of this interpretation is that Leibniz was wrong to think that we could have a symbolic representation of an infinitangular polygon, or of any infinitely divisible quantity, since no concept which conforms to monadic logic can force its models to be infinite. Thus, since our concepts cannot force its models to be infinite, our representation of space cannot be conceptual.

This point can be made more explicit by considering Euclid's proof of proposition I of Book I: an equilateral triangle can be constructed with any given line segment as base. In order to prove this proposition within Euclid's method of demonstration we start with three basic operations: (i, drawing a line segment connecting any two given points; (ii) extending a line segment by any given line segment; (iii) drawing a circle with any given point as a centre and any given line segment as radius. Given AB, by postulate three we construct the circles C1 and C2 with AB as their radius. let C be a point of intersection of C1 and C2, and by postulate 1 draw lines AC and BC. Thus, by the definition of a circle (Definition 15) AC = AB = BC. So ABC is equilateral.

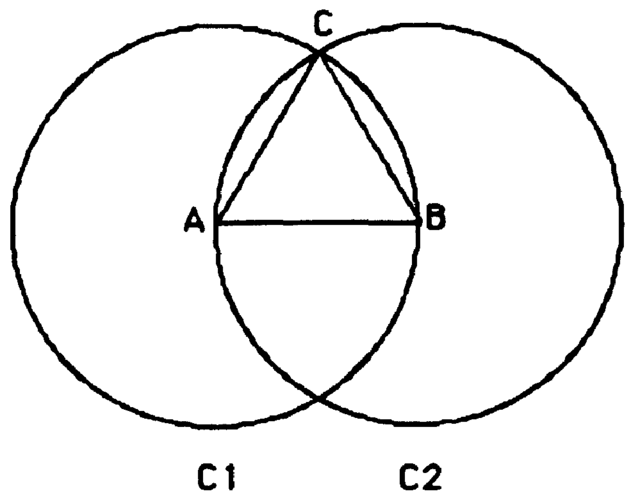


Figure 15

There is a difficulty with this proof and Friedman (1992) has drawn attention to its implications for Kant's views on intuition. The difficulty is that it contains a gap, since the existence of point C has not been proved. We have "let" C be a point of intersection of C1 and C2. But there is no guarantee from the axioms alone that there is a point of intersection. Of course, given the intuitive conception of the continuum, there cannot be a gap between which C1 and C2 "slip through." For one thing, C1 and C2 are both infinitely divisible. But consider the following case. We cover the Euclidean plane with Cartesian coordinates such that A has coordinates (-1/2, 0), and B has coordinates (1/2, 0). Thus the midpoint of AB has coordinates (0, 0) and point C has coordinates $(0,\sqrt{3/2})$. Now throw away all the points in the Euclidean plane with irrational coordinates. This model satisfies Euclid's axioms, but it does not even contain the point $(0,\sqrt{3/2})$. So Euclid's axioms do not guarantee that there is an intersection of C1 and C2.

The point can be put more generally. Monadic logic does not have the expressive power to force its models to be infinite. In particular, no set of monadic sentences with k primitive predicates, and with n variables can determine a model with more than $2^k \cdot n$ objects. The basic idea of the proof (for one variable) consists in showing that all models are generated by a binary choice tree. Suppose we are given k primitive predicates $P_1, P_2, P_3, \ldots P_k$. The class of subsets of any model can be partitioned into 2^k subclasses $P_1^*, P_2^*, P_3^*, \ldots P_k^*$ with $1 \leq i \leq k$, such that for each P_i^*, is either $P_i \vee \neg P_i$. So a tree is generated where each branch is of the following type: $P_1^* \wedge P_2^* \wedge P_3^* ... \wedge P_k^*$; since there are 2^k such subclassess we require 2^k elements for them to be distinct. Thus the conceptual resources of monadic logic are insufficient to force the concept of a potential or actual infinite number of objects.

The post Fregean approach is quite different. In Hilbert's axiomatization of geometry or the standard axiomatization of the real numbers (MacLane (1986), chapter III or Friedman (1992)) certain axioms exhibit a quantifier dependence, the logical form, $\forall x \exists y$, allows us to capture an iterative process of production formally: for any x, there exists a y such that .... This allows for the production of some y which can then be fed back into the formula to produce a new x. For instance, in the context of a theory of order, an axiom which states that, for each a there is b such that $a < b$, can only have infinite models.

This discussion bears comparison with Aristotle's own interpretation of geometrical axioms in *Metaphysics* IX where he describes the standard Euclidean proof that the sum of the angles of a triangle equal 180 degrees. Aristotle emphasizes the need to construct figures in the imagination. Philosophical reasoning does not *produce* anything new, whereas mathematical reasoning is able to obtain new knowledge because it can produce new objects in intuition. The similarity to Kant is obvious. Objects which can't be determined by thought alone need intuition to determine by producing an object.

> It is by an activity also that geometrical constructions are discovered; for we find them by dividing. If they had already been divided, the constructions would have been obvious; but as it is they are present only potentially. Why are the angles of a triangle equal to two right angles? Because the angles about one point are equal to two right angles. If, then, the line parallel to the side had been drawn upwards, it would have been evident why [the triangle had such angles] to anyone as soon as he saw the figure. Obviously, therefore, the potentially existing things are discovered by being brought to actuality; the explanation is that thinking is an actuality. (trans. 1987)

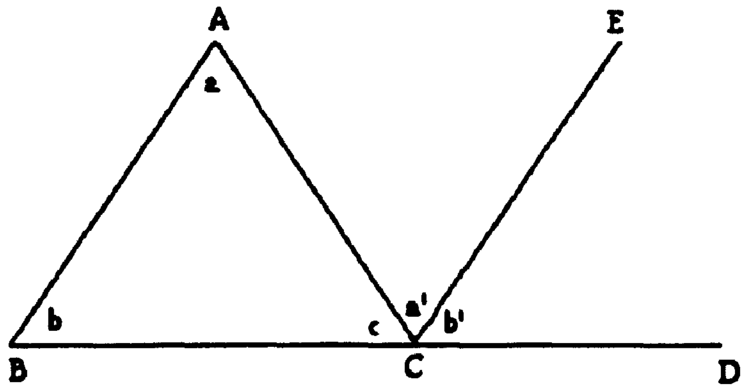


figure 16

In order to prove the theorem one simply notes that $a = a'$, $b = b'$ and $a + b + c = a' + b' + c$. It is from this observation that Aristotle was able to use the two faculty theory to solve the Platonic problem of the origin of our mathematical concepts. Objects are not given by our perception of some ideal realm, rather they are brought into existence by constructing them in thought.

The rigorization of the calculus

One of the central figures in the rigorization of the calculus was the Bohemian Monk Bernard Bolzano. Bolzano's work was near the very beginning of a remarkable change in the character of mathematics. In 1781, three years after the publication of the first *Critique,* the Berlin Academy offered a prize for anyone who could successfully prove propositions of the calculus without reverting to the use of infinitely small and infinitely large quantities. Mathematicians were not shy to take up this challenge and it led to what has been called a revolution in mathematical thought. As Gray (1992, p. 245) explains, "there was a revolution in mathematics in the nineteenth century because, although the objects of study remained superficially the same, the way they were defined, analyzed theoretically, and thought about intuitively was entirely transformed." Nevertheless, this revolution was deceptive since

> ...the very success of the new method [of arithmetizing analysis] disguised from its practitioners the revolutionary nature of the change it had effected. It was often seen as a vindication, rather than a

> repudiation of the traditional methods which it replaced. ... The theorems of Newton and his 18th century successors were retained - though, to be sure, their traditional proofs were expanded and made rigorous - and their applications remained the same, but their ultimate meanings were completely altered. (Mayberry, 1988, p. 331)

Bos offers a similar assessment of the Lebnizian calculus: "... the [modern] calculus, one may say, replaced the techniques, reinterpreted the notation, and lost the concepts" (1986, p. 91).

These assessments are more sympathetic to the mathematics of the time than is usually the case. Instead of seeing the rigorization of analysis, as suggested by the Berlin Academy question, as giving new proofs of the same mathematical results without using (incoherent) infinitesimal and infinitely large quantities, it is better to understand the rigorization as shifting from one mathematical framework to another. In this chapter I hope to further flesh out this shift as one from a framework of smooth spaces and maps to a framework of discrete spaces and maps.

The rigorization of the calculus was not a mere mathematical exercise but had a philosophical motivation. Bolzano's work, and the subsequent work of Cauchy and Weirstrass, showed how mathematical knowledge could be obtained by reasoning entirely based upon an analysis of concepts : "All mathematical truths can and must be proven by concepts" (Bolzano, quoted in Coffa, 1991, p. 22). This was carried out in opposition to Kant's claim that mathematical objects needed intuition in order to complete their determination. Newton and his followers, who regarded the calculus as founded upon an intuition of motion were also part of the target of the rigorization movement. As I have noted, spatial and temporal intuition was the foundation of Newton's calculus, unlike the calculus of Leibniz.

This emphasis on spatial and temporal intuition was the central point of contention for rigorizers. For instance, Colin McLaurin's attempted defense of Newton's use of motion in the calculus was criticized by d'Alembert in 1789 when he said that motion "is a foreign idea and one which is not necessary in the demonstration" (Grabiner, 1982, p.83). Bolzano in 1817 also wrote: "The concepts of time and motion are just as foreign to general mathematics as the concept of space (1980, p.161). This tradition continued unabated so that over half a century later Dedekind in elaborating his theory of the continuum relates that instead of using geometric concepts " it will be necessary to bring out clearly the corresponding purely arithmetic properties *in order to avoid even the appearance [that] arithmetic [is] in need of ideas foreign to it*" (1963, p.5).

The deepest critic of the use of intuition in mathematics was undoubtedly Bolzano who, in a sketch of an autobiography, challenged Kant's viewpoint that intuition is needed to prove mathematical truths:

> From very early on he dared to contradict him [Kant] directly on the theory of time and space, for he did not comprehend or grant that our synthetic *a priori* judgments must be mediated by intuition and, in particular, he did not believe that the intuition of time lies at the ground of the synthetic judgments of arithmetic, or that in the theorems of geometry it is allowable to rest so much on the mere claim of the visual appearance, as in the Euclidean fashion. He was all the more reluctant to grant this, since very early on he found a way to derive from concepts many geometric truths that were known before only on the basis of mere visual appearance. (quoted in Coffa, 1991, p. 28)

Bolzano's response consisted not only in criticizing Kant but in giving an analytic proof of the intermediate value theorem (sometimes referred to as Bolzano's theorem) (Russ, 1980). In giving such a proof Bolzano gives an example of establishing a theorem without the aid of intuition.

Prior to Bolzano's purely analytic proof of the intermediate value theorem an intuitive proof might go like this: If f is continuous on $[a,b]$ and $f(a)$ and $f(b)$ have opposite signs, then $f(c)=0$ for some c between a and b. We are given an uninterrupted line, the x axis. Any continuous function on $[a,b]$ on the x axis, with $f(a)<0<f(b)$ has an uninterrupted curve from a point below the x axis to a point above. The x axis is infinite in extension in both directions, so it cannot go around the x axis. Because of the continuity of the line, the curve cannot slip through, nor can the curve ascend to another dimension and tunnel through: the curve stays in the plane. Any uninterrupted curve from a point below the x axis to a point above must cross the x axis. As Lagrange himself put it: the curve "will little by little, approach the axis before cutting it and approach it, consequently by a quantity less than any given quantity" (quoted in Bottazini, 1986, p. 97). The y coordinate at the x axis is zero, so any function whose curve meets the x axis has a zero value. Here we seem to have a proof of Bolzano's theorem based

upon spatio-temporal intuition of continuity (A good discussion is in Giaquinto, 1994). What is wrong with it?

First of all, intuitions were considered to be irrelevant to mathematics because mathematics dealt with all possible forms, and not those which happened to be instantiated in the actual world. Even if one cannot raise any objection to a geometrical proof, Bolzano said, "... it is just as clear that it is an insufferable offense against right method to want to derive truths of pure (or general) mathematics (that is, arithmetic , algebra or analysis) from considerations that belong to a purely applied (or special) part of it, namely to geometry" (quoted in Bottazini, 1986, p. 98).

An example of an apparently faulty judgment is the supposition that any continuous function is also differentiable. Temporal intuition, in fact, guarantees that this is true for curves described by continuous motion, for, if a curve is generated by a continuous motion of an object (say a curve drawn by a pencil), then it automatically has a *direction* of motion or tangent at every point, namely the direction of the moving point (see Whiteside, 1962, p. 349). Moreover, one may interpret Leibniz's principle of continuity which asserts that "nature has no gaps" as entailing that each curve is straight for an infinitesimally small period over its domain of variation. There can be no sudden jumps, gaps, peaks or valleys in the production of a quantity. Putting these two ideas together we may say that the direction which a curve has at a point *is* the infinitesimal tangent vector which is coincident with the point.

However, in order to represent physical phenomena, such as the motion of a vibrating string, the idea of a function has to be enlarged to one which is not described by continuous motion. "... the first vibration [of a string] depends on our pleasure, since we can, before letting the string go, give it any shape whatsoever. This means that the vibratory nature of the string can vary infinitely, according to whether we give the string such and such a shape at the beginning of the movement" (Euler, quoted in Bottazini, 1986, chap. 1).

Moreover as a result of the limitation of a mathematics which is bound to intuition one may end up with incorrect results. For suppose we allow a smooth curve to be "pieced together" so that it has "cusps" or "sharp corners". Complicated examples were given by Weierstrass and Riemann. An elegant example of a continuous but nowhere differentiable function of this type was given by Koch in 1903. Take a horizontal line segment AB and divide it into three equal parts. Erect an equilateral triangle CED on the middle segment CD and erase the open segment CD. Repeat the same procedure on each of the remaining segments AC, CE, ED, DB. Finally, continue to iterate this process indefinitely on each remaining segment. This curve is continuous, in the modern sense, but there is no tangent at any given point. Thus it is not continuous in Leibniz's sense. At a particular finite stage of construction it looks something like this:

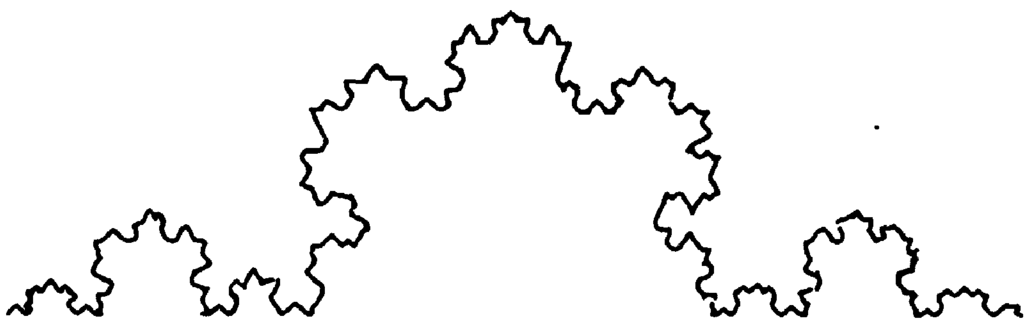


Figure 17

The construction described is a bit misleading since it starts from a curve and gradually builds peaks and valleys, which makes it seem that if one "zooms in" sufficiently, a smooth curve will be found. But the point is that no matter how much one zooms in there is no part of the curve which is smooth. It has sharp turns at every point. Such a function cannot be imagined as curve at all. (Notice that this latter judgment is similar to the usual response to the infinitangular polygon.) And so the intuitive proof that continuity implies differentiability, and its reliance on the correspondence between a function and a curve, does not work.

Bolzano's proof of the intermediate value theorem relies on a new definition of continuity in terms of points or *fixed* values. This conception originated in opposition to Newton's temporally influenced conception of a limit as the evanescent quantity *with which* a variable quantity ceases, but neither before nor after it ceases. This early notion of a limit, given by Newton, was not accepted by Bolzano, Cauchy or the other rigorizers of analysis because of its appeal to temporal notions. The main objection, as pointed out by Newton himself, was the problem of understanding what this evanescent state "with which" a quantity ceases is. For, if the quantity has either ceased or not, the value is either not the ultimate limit of the quantity, or it has a value of zero. Notice how this argument presupposes the decidability of magnitudes, and concludes that there can be no such evanescent state.

Both Bolzano and Cauchy took up the idea of defining "continuity", "derivative" and "integral" in terms of limits. Cauchy defined the concept of limit as follows: "When the successively attributed values of the same variable indefinitely approach a fixed value, so that finally they differ from it as little as desired, the last is called the limit of all others" (in Grabiner, 1981, p. 81). The modern definition, due to Weierstrass, goes further and eschews temporal intuition completely by replacing the idea of a variable quantity successively acquiring values. A sequence of points S_n has the number b as a limit if and only if to each point $\varepsilon > 0$ there exists a number k with the property: if $n > k$, then $|S_n - b| < \varepsilon$.

Cauchy's definition of the continuity of a function (and Bolzano's) is close to ours: "In other words, the function $f(x)$ is continuous with respect to x between the given limits if, an infinitely small increment in the variable produces an infinitely small increment in the function itself "(Grabiner, 1982, p. 87). In other words the continuity of a function at a point is one which preserves infinitesimal nearness of points. But an infinitesimal was for Cauchy, as it was for Leibniz, a subsisting entity rather than an existing entity because it is not an entity but stands for the property of being infinitely small (Cleave, 1978). The concept of an "infinitely small quantity" or "infinitesimal" is defined this way: "one says that a variable becomes infinitely small when its numerical value decreases indefinitely in such a way as to converge towards the value zero" (Edwards, 1979, p.310).

Thus Cauchy reverses the conceptual origin of continuity by defining continuity in terms of convergence to a limit, rather than the reverse as with Newton. Continuity can now be defined in terms of the preservation of convergence. Notice that at this stage that the notion of "infinitely small increment" and "variable quantity" still ties Cauchy and his contemporaries to the use of temporal concepts (Cleave, 1978). The modern notion of continuity can be obtained by simply replacing Cauchy's notion of convergence with that of the modern Weierstrassian notion given above. Thus a function f is continuous at a point b if f maps every sequence S_n converging to b into a sequence $f(S_n)$ converging to $f(b)$.

So far the continuity and convergence of a variable quantity have been analyzed in terms of the points of that quantity. The problem that arises immediately is that intuitively every finite motion which gradually diminishes, terminates at a point; so every series which converges (internally, so to speak) converges to a point. Can this be established without recourse to temporal intuition?

Cauchy tried. To define the sum of the sequence let $S_n = u_0 + u_1 + u_2 + \ldots + u_{n-1}$. To say that such a series converges is to say that "if, for increasing values of n, the sum S_n indefinitely approaches a certain limit S, the series is called convergent, and the limit in question is called the sum of the series" (in Grabiner, 1982, p.102). Cauchy then called attention to the differences between the first and the successive partial sums defined by

$$S_{n+1} - S_n = u_n$$
$$S_{n+2} - S_n = u_n + u_{n+1}$$
$$S_{n+3} - S_n = u_n + u_{n+1} + u_{n+2}$$

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Cauchy stated the following criterion of convergence: for any given n, the different sums have finished by constantly having an absolute value less than any assignable limit. In modern parlance such a sequence is said to have the *Cauchy property* and so is a *Cauchy sequence*: if to every point $\varepsilon > 0$ there is a number k such that for all $n > k$ and $m > k$ imply $|S_m - S_n| < \varepsilon$. Thus Cauchy attempted to show that every Cauchy sequence converges. Cauchy proved the necessity of this condition for convergence but as for sufficiency, Cauchy apparently thought that this was guaranteed by the temporal intuition that a quantity which gradually diminishes, ceases at some instant.

The problem is that the proof of the sufficiency of Cauchy's criterion requires very strong assumptions about the existence of points, for example that every real number is represented in the continuum. Here again temporal intuition was needed because of the inability of logic to guarantee the completeness of the real numbers. The necessity of the production of points in intuition in mathematical proofs was recognized by mathematicians and philosophers. Two important features of these quanta continua are that they are always conceived within limits and that they possess a flowing quality. These qualities were pointed out by Kant:

> Space and time are quanta continua, because no part of them can be given without being enclosed between limits. .... Such quantities may also be called flowing, since the synthesis (of the productive imagination) in their generation is a progression in time, whose continuity is most properly designated by the expression of flowing (flowing away) (1781/1965, A169-170/B211-212).

Moreover, as Kant put it, an intuition can only be obtained by drawing the continuum in thought:

> I cannot represent to myself a line, however, without drawing it in thought, that is gradually generating all its parts from a point. Only in this way can the intuition be obtained (1781/1965, A162-163/B203-204).

Both features of quantities are evident in Newton's fundamental lemma in the *Principia*:

> Quantities and the ratio of quantities which in any finite time converge continually to equality, and before the end of that time approach nearer to each other by any given difference, become ultimately equal (Book I, §1, Lemma 1).

The temporal intuition suggested above seems to guarantee completeness. Friedman (1992) provides an example of how this can be done. One can construct a line of length $\sqrt{2}$ by a continuous process that take one unit of time. At $t = 1/2$ a line of length 1.4 has been produced; at $t = 2/3$ a length of 1.41 has been produced and in general at time $t = n/(n+1)$ a line of length s_n is produced, where s_n is the decimal expansion of $\sqrt{2}$ carried out to n places. At $t = 1$ a line of length $\sqrt{2}$ has be constructed, and a point which can be identified as $\sqrt{2}$ exists, because, on the basis of intuition, a finite motion, occurring in a part of time, must be enclosed between temporal limits. One can repeat this procedure for any real number.

The proof of the intermediate value theorem given by Bolzano and Cauchy was only partially successful since it depended upon the proof of the sufficiency of the Cauchy condition. But the point that is highlighted by Bolzano's and Cauchy's proofs is that the logic of the intellect is not as limited as Kant would have one believe. Cauchy was able to go on and define the concepts of derivative and integral in a way which, when the concept of limit was clarified, relied upon no temporal intuition. The derivative of a function at a point x, is the limit of a ratio $\Delta y / \Delta x$, as Δx approaches zero, written, dy / dx. The integral is the limit as Δx approaches zero, of a sum $\sum_{k=1}^{n} f(C_k) \cdot \Delta x$, intuitively the product of the base and height of inscribed rectangles equal to the area under a curve f from a to b, written, $\int_a^b f(x)dx$.

But at this point Kant has a powerful response. Bolzano's and Cauchy's mathematics are empty of content, meaningless. On this view Bolzano and Cauchy have missed Kant's point. The determination of mathematical objects by intuition is what gives mathematics its content - no amount of thinking can produce such content. Bolzano was the first to understand what was wrong with Kant's conceptual underdetermination thesis. Kant had evidently conflated synthetic judgments with judgments which extend our knowledge. Here I largely follow Coffa's (1991) interpretation of Bolzano's response to Kant. In essence, Bolzano's solution is a development of Leibniz's point that we do not need to have an intuitive thought of an object in order to think of that object. We may only have clear and indistinct knowledge of some object and then use analysis to resolve the components of our representation of it into its distinct parts. On the basis of this analysis we have an intuitive concept and we have increased our understanding. Bolzano calls intuitive knowledge "thinking of a certain representation in itself" and puts his point this way:

> We think a certain representation in itself, i.e. we have a corresponding mental representation, only if we think of all the parts of which it consists, i.e. if we also have a mental representation of all these parts. But it is not necessarily the case that we are always clearly conscious of, and able to disclose, what we think. Thus it may occur that we think a complex representation in itself and are conscious that we think it, without being conscious of the thinking of its individual parts or able to indicate them (quoted in Coffa, 1991, p. 69).

Bolzano develops the point by drawing out the implication that there are two kinds of representation. His point can be illustrated with Leibniz's own example. We may think of a 1000 sided polygon without thereby thinking of all its parts. Nevertheless there is a sense in which we still have a representation of a thousand-sided polygon. This is the objective representation - the existence of the objective representation, the *content* of our thought, is independent of any mental state we may have. These are the "meanings" or "sense" of later philosophy. This objective representation is distinct from the object of our representation, an actual 1000 sided polygon. Finally, we have a subjective representation, a state of mind of thinking of an object such as a polygon.

Kant's confusion between the subjective and the objective representation is the basis for another confusion between ampliative and synthetic judgments. For he would say that in order to have the idea of an infinitangular polygon we must go beyond our intellectual representation of it. For "it is evident that from mere concepts only analytic knowledge is obtained" (1781/1965, A47/B64-5). Therefore "I must advance beyond the given [subject] concept, viewing as related to it something entirely different from what was thought in it" (1781/1965, A154/B193). But this follows *only* if our representations are taken to be subjective representations. For the parts of the infinitangular polygon may be thought in it without our thereby having a subjective representation of it. Thus it is possible to increase understanding by increasing the distinctness of our subjective concept through analysis. We only have need of intuition if there are no such objective representations.

But the existence of objective representations is virtually asserted by Kant himself. For Kant analysis is the process whereby indistinct representations become distinct. "... to analyze a concept [is] to become conscious to myself of the manifold which I always think in that concept" (1965, A7/B11). We may have an indistinct representation of a house in the distance, but may not be consciously aware of its doors and windows. But according to Kant, "we must necessarily have a representation of the different parts of the house.... For if we did not see the parts, we would not see the house either. But we are not conscious of this presentation of the manifold of its parts" (quoted in Coffa, 1991, p. 10).

Therefore, on Kant's understanding of analysis, our *understanding* of the concept (say tree) which is being analyzed changes whereas the concept does not. For in order to render a concept distinct by analysis we must admit that the same concept can be considered as indistinct, before analysis, and distinct after analysis. Kant remarks:

> When I make a concept distinct, then my cognition does not in the least increase in its content by this mere analysis ... [through analysis] I learn to distinguish better or with greater clarity of consciousness what already was lying in the given concept. Just as by the mere illumination of a map nothing is added to it, so by the mere elucidation of a given concept by means of analysis of its marks no augmentation is made to this concept itself in the least (quoted in Coffa, 1991, p. 11).

As Coffa points out, Kant's views on analysis called for a distinction between the act of representing to oneself an object and the content of that representation. Kant, nonetheless, disregards that distinction when he makes his argument for the necessity of intuition.

Kant's view is that, in the case of the representation of the continuum we must have a representation of each of its infinite parts. Otherwise our concept of the continuum would have no content. This representation cannot be thought by the (human, finite) intellect alone and so requires intuition. But Bolzano's and Leibniz's point would surely be that the content of the representation is always there - independent of our thinking of it, and Kant has simply confused the subjective representation with its content, the objective representation. The upshot of Kant's view is that the continuum is non - punctual in character. For in spatial and temporal representation the whole precedes the parts. On the other hand if Bolzano is correct, then the continuum must be compositional in character. For in conceptual representation the representation of the parts (such as the concept of point in the definition of continuity) precedes the definition of the whole (the continuum defined by such concepts).

Dedekind and Cantor on Autonomy and Real numbers

Cantor's and Dedekind's views, like those of Bolzano, are developed in explicit reaction to the use of intuition in geometry. Their ideas are important because each of them developed a view of the linear continuum as a set of real numbers and which is now regarded as standard. Yet there are significant difference both in philosophy and in their mathematical construction of real numbers. Cantor and Dedekind can both be understood as opposing Kant's view that sensible intuition is needed to determine objects. Cantor thinks that intellect is sufficient to determine objects, and aligns himself with the philosophy of Plato who has it that the concepts are already implanted in our mind. Dedekind is more extreme. He believes that intellect is able to create objects as if by divine intervention, and thus intellectual intuition determines objects. In this way Dedekind carries on the theocentric view of knowledge that Leibniz held, according to which only an infinite being could grasp the infinite, but contends that humans are gods in this respect. In spite of these difference they both agree that every mathematical object need to be determinate.

Let's begin with Dedekind since his view of the continuum is most accessible. Dedekind claimed Eudoxus's theory of quantity and ratios as the source of his theory or real numbers rather than that of Cantor or Heine (Dedekind, 1888/1963, p. 39-40). Just as the definition of proportionality of ratios induced a cut in the rationals, Dedekind would *create* an irrational number by this method. Although Dedekind rejected appeal to "geometric evidence" he, in fact, begins with the idea that the linear continuum is infinitely richer in points than the domain of rational numbers. His idea is to construct the irrational numbers from the rationals in such a way that the reals have the same completeness of numbers as the line has completeness of points (1888/1963, p.9).

According to Dedekind the "essence of the continuity of the straight line" consists in the following principle:

> If all points of the straight line fall into two classes such that every point of the first class lies to the left of every point of the second class, then there exists one and only one point which *produces* this division of all points into two classes... (1888/1963, p. 11).

The first thing to point out regarding this statement is that it may appear as if Dedekind is simply abstracting the continuity of the line from the line itself so that he is investigating the continuity of a class of objects through the use of space and time. But Dedekind indicates that the converse is the case:

> Only by means of the purely logical construction of the science of numbers, and the continuous number-domain achieved with it, are we

> in a position to investigate precisely our conceptions of space and time, by tying these to just that number domain created in our intellect (1888/1963, p. 32).

Here Dedekind is claiming that the structure of space and time are investigated by arithmetic and that arithmetic is part of logic. Dedekind says that "in speaking of arithmetic (algebra, analysis) as a part of logic I mean to imply that I consider the number concept entirely independent of the notions of space and time, that I consider it an immediate result from the laws of thought" (1963, p.31). The striking thing about both of these passages is that they both speak of producing or creating numbers and the latter adds that this is done *in the intellect*. A natural reading of Dedekind's position in relation to Kant's master argument is that he accepts that intuition is needed in order to determine objects, but insists that there is *intellectual* intuition.

This reading is confirmed by the following passage where Dedekind claims that a thing is *determined by thought alone* .

> A thing is completely determined by all that can be affirmed or thought about it. A thing a is the same as b (identical with b), and b the same as a, when all that can be thought concerning a can also be thought concerning b, and when all that is true of b can also be thought of a .... If the above coincidence of the thing denoted by a with the thing denoted by b does not exist, then are the things a, b said to be different, a is another thing than b, b another thing than a: there is some property belonging to the one that does not belong to the other (1888/1963, p. 44-5).

Here Dedekind states that a thing is determined (apparently in the sense of Kant) by thought; he then states the identity of indiscernibles. Thus he relates the determination thesis directly to the decidability of things.

Of course, in the semantic tradition, the idea that one *creates* mathematical objects by thought is a joke. Mayberry's reaction is indicative of this view. Concerning Dedekind's creation of a number by a cut he makes an amusing remark: "This, of course, is utter nonsense. However could one undertake to "create" even one irrational number, let alone uncountably many. Whatever could be the source of this vast "creative" power? You may call spirits from the vasty deep; but will they come when you do call for them" (1994, p. 24)? But, to be fair, this position moves away from Kant's use of sensible intuition; the emphasis on freely creating new objects in the intellect suggests that there is no requirement that new theories refer to any specific previously existing physical objects or abstractions of such objects. Thus, in addition to the attitude against the use of intuition there is also the idea, central to the model theoretic viewpoint, that mathematics has no particular objects of study.

Consequently, the modern use of Dedekind cuts drops the philosophical overtones inherent in the idea that objects are a product of the intellect and substitutes a definition of a modified Dedekind cut property: for each partition of the rational numbers into two non-empty classes A_1 and A_2 such that every element of A_1 is less than every element of A_2, there exists one and only one element of R less than or equal to every element of A_2 and larger than or equal to every element of A_1. For instance $\sqrt{2}$ is constructed, by forming the set of *rational* numbers $\{A,B\}$ where $A=\{x:x^2<2\}$ and $B=\{y:y^2\geq 2\}$.

However Dedekind would not have allowed such an emendation to his viewpoint. Thus one might suggest, as his friend Weber did, that no new numbers are actually created by Dedekind's cut in the rationals, rather the irrational is the cut itself as the above example would have it. Dedekind's response shows that he was deeply committed to idea that numbers are a creation of the intellect.

> If one wishes to pursue your way - and I would strongly recommend that this be carried out in detail - I should still ad that by number ... there be understood not the class (the system of mutually finite systems), but rather something new (corresponding to this class) which the mind creates. We are of divine species and without doubt possess creative power not merely in material things (railroads, telegraphs), but quite specially in intellectual things. This is the same question of which you speak at the end of your letter concerning my theory of irrationals, where you say that the irrational number is nothing else than the cut itself, whereas I prefer to create something new (different from the cut), which corresponds to the cut .... We have the right to claim such a creative power, and besides it is much more suitable, for the sake of the homogeneity of all numbers, to proceed in this manner (quoted in Stein, 1995, p. 248).

Of course the (epistemological) right to claim such a divine power is exactly what Kant argued against. Kant wrote in a letter to his friend Marcus

Hertz that there was a kind of divine intelligence, *intellectus archetypus* "on whose intuitions the things in themselves are grounded" (quoted in Walsh, 1968). The concepts of this type are able to determine objects because in this sort of case thinking of objects brings the objects into existence as "the ideas in the divine mind are the archetype of things" (quoted in Walsh, 1968). Kant, however, did not think that humans had any such divine creative reasoning ability. Dedekind disagreed.

Whereas Dedekind draws on the idea that the mind has special (and seemingly divine) powers of creation and could thereby create mathematical objects, the modified procedure has become a standard device of mathematics. But this creative power has led to strange results. Here I am referring to Dedekind's notorious proof of the existence of a simply infinite system, or the infinity of the natural numbers. Dedekind believed that by thinking alone one could have the idea of an infinity of natural numbers. Again, for those trained in the model-theoretic tradition, the reaction is predictable. Boolos describes this argument as "one of the strangest pieces of argumentation in the history of logic"(1995, p. 234). Dedekind argues that "the totality of things which can be objects of my thought" is infinite; for given such an object s, we can let S(s) to be the thought that s can be the object of my thought, and this will be a new object of my thought. S is therefore a one-one mapping of the potential objects of my thought into themselves and therefore this totality is infinite.

This seems closely related to Dedekind's discussion of the power of the mind to refer to objects. In one passage Dedekind gives a transcendental justification of the ability to refer to things:

> If we scrutinize closely what is done in counting an aggregate or number of things, we are led to the ability of the mind to relate things to things, to let a thing correspond to a thing, or to represent a thing by a thing, an ability without which no thinking is possible. Upon this unique and absolutely indispensable foundation ... must, in my judgment, the whole science of numbers be established (1888/1963, P. 32)

Nor is Dedekind alone among mathematicians in thinking that this capability is fundamental, or that the things referred to must be determinate in the sense that we can recognize them again (Leibniz's notion of a clear thought) and so distinguish them from other things; and also that these things occur in determinate laws. Hilbert calls this capability an axiom of thought or of existence of an intelligence:

> ... an Axiom of Thought or , as one might say, an Axiom of the existence of an Intelligence, which can be formulated approximately as follows: I have the capability to think things and to denote them through simple signs (a, n: ..., X, Y, ...; ...) in such a fully characteristic way that I can always unequivocally recognize them again. My thinking operated with these things in the designation in a certain way according to determinate laws, and I am capable of learning these laws through self-observation, and of describing them completely (quoted in Hallett, 1984, p. 179).

However, neither thinker gives the slightest explanation of these amazing powers.

Cantor's position shares similarities with Dedekind's. He shares the view that in order to describe the continuum we can not simply observe continuous quantities and abstract from them their mathematical form. He differs from Dedekind, however, in thinking that intellect creates objects. Instead the intellect awakens preexisting concepts with which we refer to objects. After agreeing with the position of Spinoza that the order of things and the order of ideas were the same he takes his cue from Leibniz:

> The same epistemological principle is hinted at even in Leibniz's philosophy. Only since the new empiricism, sensualism and skepticism, and the Kantian criticism that emerged from it, has it been believed that the source of knowledge and certainty is located in the senses or in the so-called form of pure intuition of the world of ideas and must be restricted to these. According to my conviction, however, these elements do not at all furnish certain knowledge. This can only be obtained through concepts and ideas, which are at best only stimulated by outer experience, but which are principally formed through inner induction and deduction, like something which, so to speak, already within us and is only awakened and brought to consciousness. (quoted in Hallett, 1984, p. 15).

Leibniz had a nice analogy for this viewpoint. The mind is not like a blank slate (as Locke thought) but like a block of marble - not a uniform block, but one that has veins, so that the blows of the sculptor reveal the underlying shape. Similarly the "blows" of sensory stimulus help us uncover the fundamental truths within us. Thus our knowledge of space and time is not grounded in intuition but is found within the mind itself.

Our concept of time is not from experience:

> ... I have to declare that in my opinion reliance on the concept of time or the intuition of time in the much more basic and more general concept of the continuum is quite wrong. In my opinion, time is an idea whose clear explication presupposes the independent concept of continuity, and which even with the help of this latter can be conceived neither objectively as a substance, nor subjectively as a necessary *a priori* form of intuition. Rather, it is nothing other than a relational concept, by whose aid the relation between various motions we perceive in nature is determined (Hallet, 1984, p.15).

Cantor also makes the same point with respect to space. Cantor is quite specific that the concept of continuity is not a necessary *a priori* form of intuition in Kant's sense nor is it abstracted from an intuition of space or time. Instead it is imposed upon nature by intellect alone. Thus Cantor and Dedekind reject the idea that intuition is needed for knowledge or to supply content to mathematics. However the sense of "free creation" in Dedekind and "awakening" concepts is troublesome. This aspect of their thought has been pursued by a few Platonists, such as Godel but has never been adequately understood. The ideas of free creation and awakening of existing concepts is related in spirit to the model theoretic tradition. The emphasis on *freely creating* new concepts suggests that there is no requirement that new theories refer to any specific previously existing objects. This orientation is in keeping with the idea that mathematics is not concerned with any objects in particular.

Cantor makes a similar claim as Dedekind regarding the creation or "introduction" of mathematical objects. In order to give content to mathematical statements one lays down a concept and its predicates. This concept must be definite, and hence the objects be *distinguished* by means of those concepts alone.

> In particular one is only obliged with the introduction of new numbers to give definitions of them through which they achieve such a

> definiteness and possibly such a relation to the older numbers that in given cases they can be distinguished from one another. As soon as a number fulfills all these conditions, it can and must be considered in mathematics as existent and real (Hallett, 1984, p. 17).

A fundamental element of Cantor's view of sets is that, in virtue of the principle of excluded middle, the identity of any elements of a set is decidable, and the membership relation is decidable.

> I call an aggregate (a collection, a set) of elements, which belong to any domain of concepts, *well-defined*, if it must be regarded as *internally determined* on the basis of its definition and in consequence of the logical principle of the excluded middle. It must also be *internally determined*, whether any object belonging to the same domain of concepts belongs to the aggregate in question as an element or not, and whether two objects belonging to the set, despite formal differences, are equal to one another or not (Dauben, 1979, p. 83.)

Here one might wonder under what circumstances a concept is *internally* determined as opposed to externally determined. Cantor distinguishes between internally determined and externally determined. By "internally determined" he meant that a mathematical statement may not be actually or externally determined because of lack of mathematical resources. But a mathematical statement is internally determined where it may be determined with a perfection of resources. For instance in Cantor's time the set of all numbers satisfying $x^n + y^n = z^n$ was not known, but apparently the problem has now been solved. So while the problem was not externally determined in Cantor's time it was internally determined because an actual determination can be made with a "perfection of resources" (Dauben, 1979, p.83).

In building up a set of real numbers Cantor had to start with already given numbers, and defined new numbers in relation to the old, natural numbers. But, if Cantor wanted to show that intuition is not needed in order to determine mathematical objects he had to show that it is not needed to determine the natural numbers. Thus the foundations of calculus ultimately depend, as Frege skillfully pointed out, on the foundations of arithmetic. At this point Cantor applies a two-fold act of abstraction. One considers a set *M* and the intellect abstracts from the nature and then the order of the elements to arrive at an ordinal number (Hallett, 1984, p. 128). However, Cantor's method of determining new objects was rightly subject to strong criticism by Frege. Frege raises an amusing problem. If one considers a white cat and a black cat and disregards the properties that serve to distinguish them, then one obtains the concept of "cat". If one then proceeds to abstract all distinguishing characteristics in order to obtain "pure units", the cats themselves, whatever we think of them, still remain the same. Moreover, we will no longer have the idea of a number, or plurality of units but of one unit, because all distinguishing characteristics have been abstracted away (1984, p. 45). Dummett gives the standard view of Frege's argument when he comments that Frege's argument refutes the abstractionist view of number "brilliantly, decisively, and definitively" (1992, p. 82).

Cantor never solved this problem and it was left for others, such as Frege and Zermelo, to attempt to solve. But supposing that we are given the concept of a natural number independent of intuition, then the problem is to show that we can generate a continuum from these concepts. Cantor's idea was to construct a set of objects - real numbers - which can be placed into one-one correspondence with the points of a line. This isomorphism to the points of the intuitively given continuum would then show that the constructed continuum was a real continuum. But Cantor was unable to prove that the real numbers were isomorphic to the points of the intuitive line, and so he took this correspondence as an axiom.

The assumption that space is continuous amounts to the supposition that the set of real numbers is isomorphic to the linear continuum. It was clear to Cantor that every point on the line corresponded to a real number b because each point could be represented as a distance b from the origin. But it was not clear that each real number corresponded to a point on the linear continuum. Thus Cantor took it as an axiom that "also, conversely, to every number there corresponds a definite point of the line, whose coordinate is equal to that number" (quoted in Dauben, 1979, p. 40).

Given this correspondence between the points of the smooth continuum and the constructed continuum Cantor was able to sustain the idea that continuity is an autonomous notion. Referring to the assumption that physical space is continuous Cantor says:

> According to the simultaneous but completely independent investigations of Dedekind and the author, this assumption consists in nothing other than the assumption that every point whose coordinates x, y, z are given by absolutely any definite real, rational or irrational numbers, in a rectangular coordinate system, is thought of as actually belong to space. There is no inner compulsion to this, but rather it must be seen as a free act of our activity of mental construction. The hypothesis of the continuity of space is hence nothing other than the assumption, arbitrary in itself, of the complete one to one reciprocal correspondence between the three dimensional purely arithmetical continuum (x, y, z) and the space which lies at the basis of the world of appearance. (quoted in Hallet, 1990)

Given this construction Cantor was faced with the problem of distinguishing between the continuous and the discrete, i.e. between the rational numbers and the real numbers. In other words, Cantor believed he had constructed a continuum and he wanted to characterize the "essence" of the continuum. Cantor went on to give a characterization of the continuum in terms of its "perfection", the fact that a set is equal to its limits points, and its order. No kind of characterization of the real numbers is nowadays recognized as essential (MacLane, 1986, ch. 4). It can be described as a complete, archimedean-ordered, field, or a complete, unbounded, ordered, set with a denumerably dense subset. These differing axiomatizations reflect different aspects of the continuum, its algebraic, continuity and order properties in the former case and its order, and continuity properties in the latter case. The essential point is that these axiomatizations of the real numbers allow for the description, in a purely formal manner, of concepts, such as continuous, infinite divisibilty, derivative, integral, and the proofs of theorems concerning these concepts without recourse to intuition.

Although, Cantor and Dedekind both found methods of completing the rational with irrationals Cantor differed with Dedekind on the construction of the real numbers. Whereas Dedekind "created" irrational numbers from Dedekind cuts, Cantor defined them as "fundamental sequences" or "Cauchy sequences" of rational numbers. Cantor's construction relies on the principle (usually called the "Cauchy condition") that every fundamental sequence has a limit b. By this Cantor meant that each sequence was associated with a symbol "b" which could be considered a real number. For instance the sequence $\{1, 1.4, 1.41, \ldots\}$ is identified with $\sqrt{2}$. Since, for any real number *b*, there many sequences which converge to *b*, one identifies the sequences which converge to *b*, and defines the collection of all such equivalence classes of fundamental sequences to be the set of real numbers *B*.

This different methods of constructing the reals is usually taken to be of no consequence mathematically because Dedekind reals and Cauchy reals can be shown to be isomorphic in set theory (MacLane, 1986. § 4.5). An interesting point which emerges, however, is that the two constructions are not, in general, isomorphic (Johnstone, 1977, § 6.6). So while the isomorphism of the two constructions led to confidence that concepts alone could describe the actual continuum, this result only holds generally if one assumes that the only framework is that of set theory.

Leibniz's puzzle revisited: the master argument extended

It is time to revisit Leibniz's puzzle. The puzzle arose as a result of two of Leibniz's cherished views: the identity of mathematical objects is decidable, and every curve is an infinitangular polygon. Of course, the problem is that the infinitesimal parts of an infinitangular polygon are not decidable. Therefore the infinitesimal parts of a continuum are not mathematical objects. So, a continuum cannot be a set, if it is to contain infinitesimals. For Leibniz the solution would be straightforward. Intuition was simply a confused perception of well-distinguished wholes. Thus Leibniz's solution was to consider the continuum to be a well founded fiction rather than a real object. It is a "fiction" because it is not a single whole, but a confused representation of a multiplicity which is taken to be a whole, and "well-founded" because it is not simply a hallucination or a dream. The continuum and its parts are not real objects at all; so the puzzle is solved. Every *real* object is decidable.

For Kant intuition was a completely different faculty; one which has a completely different function mandated by the unity of our apperception. The faculty of intuition grasps the continuum as a whole prior to the perception of its parts. Leibniz had erred, according to Kant, in denying that the continuum can be grasped as a whole because of his misunderstanding of the function of the faculty of sense. Even if this were true in Kant's time, Cantor's discoveries alter the situation. Prior to Cantor, who solved the paradoxes of the infinite, the intellect was regarded as unable to grasp a completed infinite because of the alleged inconsistencies in the concept of a completed infinite. But the construction of a punctual continuum of real numbers within set theory seemed to undercut the need for intuition to provide a representation of it. Although, Kant was, perhaps, correct that intuition *was* needed to grasp an infinite whole because of the inconsistencies in our concept of the infinite, Cantor's theory has swept away this barrier to intellectual representation of the infinite. Moreover, as Russell concluded, while the inadequacy of Kant's logic to force the representations of infinite magnitudes *was* a barrier to intellectual representation of such magnitudes, modern logic has overcome this barrier.

But Kant's master argument can be altered in a way which counters Russell's argument. According to the set theoretic approach every object is decidable. Thus, insofar as intellectual representation is mediated by the concepts of set theory, we are limited to representing decidable objects. Kant's argument against the ability of the intellect to determine mathematical objects was heavily based upon Newton's account of the calculus and its dependence upon intuition. But ironically, it seems that Leibniz's approach would have served him better in response the objections of the rigorizers. For if one takes seriously the perception that the continuum is smooth, i.e. that each curve is locally straight, then parts cannot be decidable, and no conceptual representation can be had. In particular there is no model of a smooth continuum in the universe of sets.

Thus Leibniz's puzzle is not a puzzle for a Kantian. It is simply another illustration that mathematics must use intuition. Friedman, following Russell, argued that Kant's logic was too weak to determine objects, but one can now argue that the implicit (classical) logic of set theory is too strong because it does not allow the existence of (undecidable) infinitesimals. Thus the infinitesimal techniques of Leibniz cannot work in a conceptual framework where every object is decidable. But every framework which is thought through the intellect is just such a framework. Thus, the faculty of intuition must be used.

This observation leads to a related problem. McLarty has claimed that Dedekind and Cantor created sets by abstracting the punctual parts of smooth spaces to form sets (1988). Of course Cantor was unable to prove that the set of real numbers was isomorphic to the continuum and took the correspondence as an axiom. But if we take the smooth continuum seriously, it is possible to understand that there was no way that Cantor could have proven, *in set theory,* that the punctual parts of spaces are sets, and that the continuum of real numbers are isomorphic to the smooth continuum. Such a proof requires that we be able to represent the smooth continuum by set theory - which we can't.

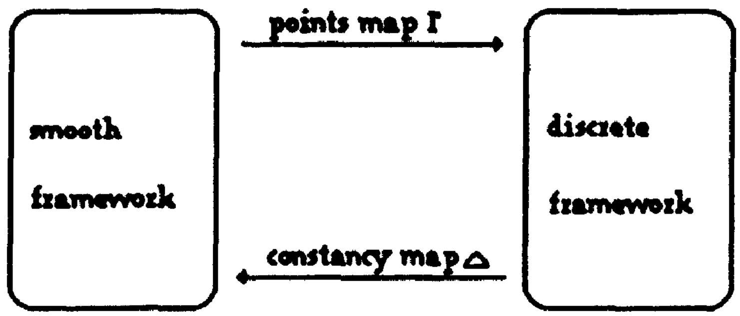


Figure 18

Understanding the transition from the Leibnizian conception of the continuum to that of Cantor and Dedekind requires a wider kind of framework pictured above. Given such a picture the rigorization of the calculus is understood as extracting only the points of smooth objects and maps in order to give a discrete framework. But, conversely, it is now possible to build a model of the smooth framework by constructing it out of constant sets and constant maps. When we view the shift from this wider framework it will become apparent that there is a solution to Leibniz's puzzle. I will argue that this shift to smooth spaces is an extension of the semantic approach to mathematics, and that it is required in order to answer Leibniz's puzzle.

FROM SETS TO SMOOTH SPACES

"Proofs of the simplest truth lie hidden very deeply and can at first only be brought to light in a way very different from how one originally sought them."

Gauss

In the last chapter I showed how Bolzano, Cantor and Dedekind overcame Kant's argument that the continuum *had* to be given in intuition by actually constructing a punctual continuum in thought. This might be thought of as simply supplying an alternative continuum to the intuitive one, but, from Russell's point of view, the freedom that Cantor introduced only serves to show that the earlier conceptions of the continuum were simply due to the limitation of Kant's logic. I have argued that this criticism applies equally well to the set theoretical analysis of the continuum of Cantor and Dedekind since, from the point of view of category theory, the idea that an object is a set is highly restrictive itself. In particular it does not allow variable quantities, undecidable objects, including infinitesimals. Thus Leibniz's puzzle has not really been answered by the semantic tradition as presently understood. In this chapter I discuss how category theory allows one to build variable sets from (constant) sets which offer a less restrictive framework in which the Leibnizian continuum can be regained. I argue that this pushes the semantic tradition further and allows Leibniz's puzzle to be solved.

This extension of the semantic tradition is accomplished by generalizing the notion of set to that of a set smoothly varying over a space or a *smooth* space. I attempt to motivate this idea by considering McTaggart's view of temporal flow. In passing I consider two objections to such a framework, (i) McTaggart's argument that there can be no variation; (ii) the more general objections to the use of categories rather than sets as mathematical models. In the framework of smooth spaces every curve is an infinitangular polygon and the infinitesimal calculus may be developed in such a framework. Given the development of such a smooth framework the rigorization of analysis can be understood as shifting from the framework of smooth space to the framework of discrete spaces and maps by taking the punctual parts of smooth spaces and maps. The extension which I am proposing is to shift from the framework of discrete spaces and maps back to the framework of smooth spaces and smooth maps.

Variable Sets

Leibniz, Newton, and their followers - even Cauchy- conceived of the calculus as a calculus of variable quantities. Kant, Newton and Leibniz all considered the continuum to be generated by the motion of points. It is this variation or temporal flow which distinguishes the continuum from a collection of discrete points. The set-theoretical model of the continuum elaborated by Cantor and Dedekind did not completely silence dispute on this issue. Indeed at least one of their critics, Du Bois Raymond, took issue with the static nature of their continuum. He relates that points themselves have no length and need to be moving in order to generate a length, thereby raising a point common to Kant and Newton.

> The conception of space as static and unchanging can never generate the notion of a sharply defined, uniform line from a series of points however dense, for after all, points are devoid of size, and hence no matter how dense a series of points may be, it can never become an interval, which always must be regarded as a sum of intervals between points. (Quoted in Ehrlich, 1994, p. x)

The majority opinion, however, was not with dissenters such as Du Bois Raymond, and by the twentieth century the mistrust of temporal intuition was so firmly entrenched that Russell could, for instance, ridicule the conception in the following remark:

> Originally, no doubt, the variable was conceived dynamically, as something which changed with the lapse of time, or as is said, as something which successively assumed all values of a certain class. This view cannot be too soon dismissed. If a theorem is proved concerning n, it must not be supposed that n is a kind of arithmetical Proteus, which is 1 on Sundays and 2 on Mondays and so on. (1903, § 87)

Thus, the rigorization of the calculus produced a change in the conception of variable quantities. A variable quantity was no longer regarded as something which is in a state of transition, such as a point flowing from point to point; rather it is captured as a collection of constant points. For Russell "Motion consists in the fact that, by the occupation of different places at different times, a correlation is established between places and times.... ", and when such correlation exists, there is motion. Continuous motion consists in the fact that the function defined on these domains of points and times is continuous (1903, §446). From the point of view of the "active" conception of flowing quantities and flowing objects such a correlation is merely the result of "stopping" or evaluating the flowing quantity at certain moments.

The modern antagonism toward variable quantities, which was a part of the rigorization of the calculus, gained support from unexpected quarters: the British idealist philosopher J. M. E. McTaggart. McTaggart's position, argued in his brilliant (1908) paper, can be summed up starkly: there is no coherent sense in which events vary with respect to time. Rather, there are only timeless quantities which we order by means of direct experience, anticipation and recollection. Those events which are directly experienced are present; those which are fading further and further in our memory are further and further in the past; and those events which are more and more faintly anticipated are further into the future. Set-theorists might well find themselves applauding this idea, for it suggests that they were right all along to abandon variable quantities for constant sets.

Recently, however, this apparent elimination of variable quantities in favour of static quantities by mathematicians such as Dedekind and Cantor and philosophers such as McTaggart has been the subject of severe criticism. The American mathematician F. W. Lawvere has vigorously argued that the notion of variable quantity is

> a notion which was taken quite seriously by the founders of analysis and which has not been eliminated by set theory any more than continuity has been eliminated by the 'arithmetization of analysis' (which is just that and not analysis itself). (1975, p. 135)

According to this view the motion of a variable quantity is needed in order to express the continuity of space and time, and even to understand constancy or lack of motion. Lawvere continues:

> ... the concept of motion as the presence of one body at one place at one time, in another place at a later time, describes only the result of motion. Every notion of constancy is relative, being derived perceptually or conceptually as a limiting case of variation. (1975, p. 136)

This was certainly the conception of motion that was derided by Russell in the previously quoted remark, but is there any reason for thinking that variable quantities are coherent after all? In answer to this Lawvere further proposes that the notion of a "variable set" is able to recapture the variability that was ignored in the arithmetization of the calculus. Lawvere's motivation is admittedly vague, but worth pursuing insofar as it promises to give a new account of the variable quantities of the seventeenth century.

Lawvere's stated dissatisfaction with the Russellian view of motion as recording the result of motion rather than characterising motion is similar to McTaggart's reasons for introducing an A series of temporal properties. Namely, a B series of static points does not exhibit any motion. It will be helpful, then, to briefly examine McTaggart's conception of temporal flow in order to understand the motivation for variable sets .

The central intuition about time flow for McTaggart is simply this: events which are present, were once future and will become past. Thus, he gives an analysis of the flow of time as that of an A series of temporal properties varying with respect to a B series of temporal instants. Much as the motion of an object in space is a succession of positions with respect to time, the flow of time is the successive acquisition of A series properties of pastness, presentness and futurity by B series events; properties which every event must have. In order for a quantity to be varying it must possess both A series and B series relations and properties.

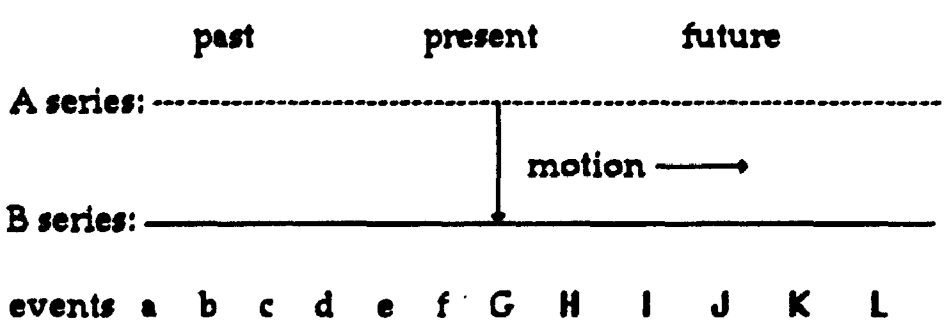


Figure 19

Put more abstractly the point is that, as the diagram portrays, variation requires that the A series and B series supply an alternative systems of coordinates for events. The coordinates in the A series are the properties of being present, and various degrees of being future and being past. In the B series events are coordinatized with respect to an earlier-than relation. In effect, the B series is a static record of events, and the A series properties moves across the B series endowing events with temporal properties. McTaggart writes that " ... this characteristic of presentness should pass along the series in such a way that all positions on the one side of the Present have been present, and all the positions on the other side of it will be present" (1908, p. 118).

McTaggart considered an A series varying with respect to a B series to be necessary for the flow of time. He argued as follows: Events are located in a B series only if time exists; time exists only if there is genuine change; but there is genuine change only if events are located in a real A series. Therefore, events are in a B series only if they are (also) located in an A series. One might be suspicious of the idea of "genuine" change. Indeed one might argue that the supposition that "genuine" change requires an A series is gratuitous. Why can't there be genuine change in a B series? The change of an event F into an event G is simply the fact that F is earlier in time than G.

What is problematic in any purely B series account of change is that a B series cannot exhibit genuine change, since B series facts express relations which are fixed and eternal. If F is before G, it was so a million years ago, it is now and will be a million years hence (McTaggart, 1908, p. 459). In order for there to be genuine change there must be a genuine difference in properties of events at different times, rather than merely relations which hold for all time. One must be able to say which event *is* (now) present, and therefore which events are past and future in an absolute way rather than simply expressing fixed relations between events.

Let's try to mathematically model an A space varying with respect to a B space. A first idea is to think of time as a B - indexed family of sets A, but this would just seem to be reproducing the B series constancy in the A series, since a family of sets exhibits no genuine variation. The A series would not "move" along the B series. Instead, drawing on Lawvere's idea of variable set, let's build a new model out of sets *and* functions, where a function from a set

X to a set Y introduces the idea of X *becoming* Y, so that it is the movement of the A series across the B series which is being described and not simply the result of such motion.

In order to explain this new framework it is necessary to begin with the notion of a category. A category is just a collection of objects and arrows, for instance a collection of sets and functions. Each arrow goes from one object (domain) to another object (codomain). To say that f goes from A to B we write $f: A \longrightarrow B$. In some cases "goes" will be taken quite literally as involving motion. Two arrows f and g with domain of f = codomain of g are *composable*. If f and g are composable arrows, then they have a composite arrow $f \circ g$. There are three axioms:

Domain and codomain: For every composable pair f and g, the composite $f \circ g$ goes from the domain of g to the codomain of f.

Identity: For each object A the identity arrow 1_A goes from A to A.

Associativity: composition is associative.

These are quite general axioms. From the point of view of category theory it is natural to regard these axioms as providing a mathematical framework which has as much freedom as possible, and any additional axioms as restricting mathematics to a particular framework. These axioms are evidently true for sets and functions (providing that we regard functions as going from a domain to a prescribed codomain and not just to the range of the function).

These axioms are also satisfied by other structures such as the category of pairs of sets and pairs of maps, or the category of partially ordered sets and order preserving maps. They are also true for more complicated categories such as the category of topological spaces and continuous maps. The feature which is most relevant for our purposes is the fact that new categories can be built from (old) categories. One can use this feature to construct more complex categories from sets and functions which will share many features of the category of sets but which, unlike the category of sets, will vary.

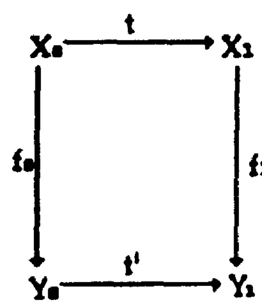
The category of sets and functions is our first example of a *topos*. For the moment think of a topos as a category which is sufficiently like the category of sets. In other words, a topos is a category with the following features. Any two objects A and B have a product, $A \times B$, comparable to the Cartesian product in a model of set theory. There is a terminal object 1, such that for any object X there is a unique map $X \longrightarrow 1$. In the universe of sets these are the singleton sets. For any pair of objects A and B, there is an exponential object B^A of all mappings $A \longrightarrow B$. This corresponds to the set of all functions from A to B in the category of sets. Finally there is an object of truth values Ω such that there is a natural correspondence between subobjects of X and arrows $X \longrightarrow \Omega$. In the universe of sets Ω is $\{0,1\}$ and maps $X \longrightarrow \Omega$ are characteristic functions or properties.

The difficulty with developing Leibniz's calculus with the concepts of set theory is that the classical logic which is presumed is sufficiently strong to prevent the existence of infinitesimals. However, the generalization of the concept of set to that of topos will allow one to define an "internal language" which is weaker than first order logic, and which allows infinitesimals to exist. Russell's defence of set theory against Kant has rebounded upon him. Just as he criticized Kant's logic as being too weak to allow proofs in the calculus without the aid of intuition, it turns out that the set theory which he defended was too strong to develop the Leibnizian calculus.

A rough idea of how such an internal language is defined is as follows. In the category of sets arrows are well pointed in the sense that for every parallel pair of arrows $f,g\colon A \longrightarrow B$ either $f = g$ or there is a map $x\colon 1 \longrightarrow A$ such that $fx \neq gx$. Hence an arrow from $1 \longrightarrow A$ may be considered an element of type A. However there are not "enough" arrows in an arbitrary topos to make it well pointed. Nevertheless, in an arbitrary topos arrows are determined by their "generalized elements" $u\colon U \longrightarrow A$ in the sense that for a parallel pair of arrows $f,g\colon A \longrightarrow B$, either $f = g$ or there is a map $u\colon U \longrightarrow A$ for some U such that $fu \neq gu$. Arrows $1 \longrightarrow PA$, for arbitrary objects A are set - like in that they satisfy a kind of bounded Zermelo - Frankel set theory. This is the so - called *local set theory* of a topos. Arrows from $1 \longrightarrow N$ are natural numbers for they satisfy Peano's postulates for natural numbers. Arrows to Ω are formulas. Moreover arrows $\Omega \longrightarrow \Omega$ define natural logical operations of conjunction, disjunction, negation, implication and even quantification.

The subobjects of any object classified by Ω form an algebra which is weaker than the usual algebra of first order logic and is called a *Heyting algebra*. For instance, the algebra of open sets of a topological space have this algebra. The most notable aspect of this algebra is that the the logical operations it defines are intuitionistic rather than classical, and it does not follow that every object is decidable (see Bell, 1988 and Borceaux, 1994 for details). A Heyting algebra in which the law of excluded middle is valid is called a *Boolean algebra*. Thus, in generalizing from set to topos one effects a generalization of the truth value object from a Boolean algebra to a Heyting algebra, and a corresponding weakening of the logical operations.

For the simplest case of a variable set consider a set varying over two moments of time, 0 (o'clock if you will) and 1 (MacLane, 1973, 1986; Bell, 1993). This is our second example of a topos (Set^C for any category C is still a topos, Bell (1988, p. 60)). Let 2 be the category with objects {0,1} and let arrows be from $A \longrightarrow B$ when $A < B$. The objects of the category are the *functions* $t:X_0 \longrightarrow X_1$ and $t':Y_0 \longrightarrow Y_1$, (functions always specify their domain and codomain), and the arrows of the category are $f:(t:X_0 \longrightarrow X_1) \longrightarrow (t':Y_0 \longrightarrow Y_1)$ the pair of functions such that the diagram



commutes. (That is $t' f_0 = f_1 t$.)

One may view this category as a set of events varying over two times, 0 and 1. X_0 is earlier than X_1, so X_0s time index is smaller than X_1s time index. But a variable set is not simply a static sequence X_0, X_1, for, in addition, it is *changing* from X_0 to X_1. The object t (an arrow) is X_0 changing into to X_1, so we may think of X as a single variable object standing for t together with its domain and codomain. The fact that the object is an arrow (which always has a domain and codomain) highlights the fact that it is the motion of the set which is being described and not the resulting domains. As Bell has emphasized, "In category theory the morphisms (arrows) between structures (objects) play an autonomous role which is in no way subordinate to that played by the structures themselves. So category theory is a language in which the verbs are on equal footing with the nouns" (1988, p. 236). Or, one might add, in category theory variable quantities are on an equal footing with constant quantities.

A new and clearer perspective is given if we consider an *object* $t{:}X_0 \longrightarrow X_1$ to be pictured as a space X over 2. According to this viewpoint the category we have been describing is a category of maps (called functors) between categories. Thus is the category of all maps from 2 to Set.

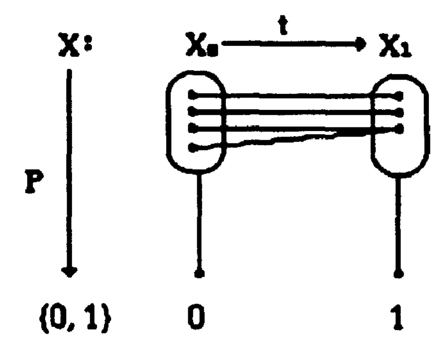


Figure 20

In this picture there is a map P mapping X to 2 and for each $i \in 2$, the inverse image $p^{-1}(i)$ is called the fibre of X over i. This emphasises that a set is embedded into a variable set as a (constant) set varying over a one point space. Here is where the A series enters the picture. X may be regarded as the A series and 2 as the B series. Thus t is a map which describes the events in X_i changing into X_j: events which were future are becoming present. We regard P as assigning X_i to 0 and X_j to 1 when X_i is in the past of X_j. So the transition map t: $X_i \longrightarrow X_j$ holds when X_i is in the past of X_j. The B series is generated, then, by the way that P partitions events into times. X_i is earlier than X_j is just the relation that X_i and X_j have with respect to the base space of times.

A *cross-section* of a space $p{:}X \longrightarrow 2$ over 2 is a map $s{:}2 \longrightarrow X$ such that ps is the identity map. The map s can be viewed as picking out one element from each fibre. So the set of all cross sections is thus the set of all arrows from $1 \longrightarrow X$, or the *points* of X. Therefore this points map Γ is a functor from $Set^2 \longrightarrow Set$ which effectively takes the points $(1 \longrightarrow X)$ of Set^2 into Set. Conversely, the construction of Set^2 takes constant sets and embeds them in Set^2.

Essentially this reveals a relationship between two distinct mathematical frameworks: variable sets and constant sets which are related as follows:

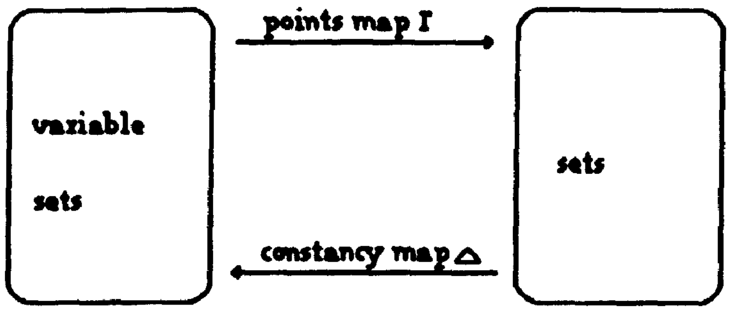


Figure 21

The mathematical essence of the rigorization of analysis, I will argue, is just this shift from variable sets to sets. But because of the inability of set theoretical concepts to represent the smooth continuum of Leibniz, this rigorization must be extended further by shifting back to variable sets.

A digression: on the paradox of temporal variation

In spite of the fact that I have begun to give a mathematical development of temporal variation in terms of Lawvere's conception of variable sets we now come to a serious problem. For, notoriously, McTaggart denied that this conception of change was possible because it harbours inherent contradictions. So, at this point a defender of McTaggart would likely object to the construction of variable sets by claiming that there is no variation in X. X is simply a static sequence of elements which possesses different properties at different times. According to this objection X (i. e., the function $t: X_0 \longrightarrow X_1$) is only a static sequence of X_i s at each t_i , for each i is fixed, eternal and constant. X_i is not becoming X_{i+1}. Let me digress for a moment to discuss this issue, since I think it is necessary to consider McTaggart's well known objections in order to vindicate the idea that X is varying and not just a more elaborate system of constant quantities.

McTaggart argues as follows (1908, p. 468): If events are located in a real A series then *each* event acquires the *absolute* properties of past, present and future, since, after all, what is present, was once future and will become past.

Past, present and future cannot be relations, because such relations would hold for all time, and so each event in such a series would be static and constitute a B series. This is because if X is past relative to Y it will always be past relative to Y; hence static. On the other hand there is a contradiction in supposing that any event has (timelessly) any two of these *absolute* properties. Therefore, a real A series cannot exist. But since the existence of the earlier than relation (the B series) depends upon the existence of a real A series, there is no B series. That is, time is unreal; or, in other words there is no variation.

Of course one should immediately object, as McTaggart himself points out, that the A series properties are contradictory only when attributed *simultaneously*. It is never true that an event *is* past, present and future - but *is* present, *was* future and *will become* past. They are not contradictory if they are attributed *successively*. But the fact that such events are successive cannot come from the B series, because this series is static. And so the successive attribution of A series properties is done relative to the A series itself. To state this again: an event *is* present, *was* future and *will become* past.

McTaggart is not slow to point out the danger of this response and what he says is potentially devastating:

> What we have done is this - in order to meet the difficulty that my writing of this article has the characteristics of past, present, and future, we may say that it is present, has been future, and will be past. But "has been" is only distinguished from "is" by being existence in the future. Thus our statement amounts to this - that the event in question is present in the present, future in the past, past in the future. And it is clear that there is a vicious circle here if we endeavour to assign the characteristics of present, past and future by the criterion of the characteristics of the present past and future. (1908, p. 468)

So McTaggart's response is that the price of attributing A series temporal properties to events is a new higher-level contradiction, and in order to attempt to vitiate this contradiction one must engage in a vicious regress.

Nonetheless, one may argue in response that when one says, as in McTaggart's preceding example, that an event "has been" when it is future in the past, we are not using future and past *at the same level*. Therefore there is no vicious circle of A series attributes. Instead, there is a hierarchy of levels of temporal properties. Each time an event is said to possess an A series property it must be qualified by means of an A series property *at a higher level.*

But in that case, says McTaggart, the fallacy will exhibit itself as a vicious regress rather than a vicious circle since each attribution of an A series property will have to be qualified at a higher level. Thus, the contradiction of being present, past and future at level one is resolved by ascending to level two so that, for instance, an event is not present *simpliciter*, but is present in the present, future in the past and past in the future. Of course, if this is so, a similar contradiction arises at the second level as it did on the first level, since an event must have all the second level temporal properties unless it is specified that it has them in succession at a third level. In each case the new A series will face the same incompatibility of attributes and a new A series will need to be constructed. So "You can never get rid of the contradiction, for by the act of removing it from what is to be explained, you produce it over again in the explanation" (1908, p. 469).

There is a strong inclination, as Dummett (1960) has pointed out, to regard McTaggart's denial of flowing time (and thus variable quantities) as a deeply confused sophism based upon a misunderstanding of the indexical nature of temporal predication. For consider other instances of indexical predication. Every place can be designated as here and there, near and far, and each person can be designated I and you. A Zebra may be designated as black and white. These indexical predicates may be *applied* to things in various circumstances. One might go on to argue, therefore, that there can be no space, personality, nor colour since here and there, near and far, I and you, black and white are *incompatible predicates* . But the sense of incompatibility is that there is no circumstance in which they *both* apply to the same entity. So incompatible predicates do apply to the same entity (in different circumstances), but this incompatibility does not, thereby, result in a contradiction. The confusion that is displayed is to neglect that indexical predicates are implicitly relativized to circumstances.

Yet I agree with Dummett's observation that this "solution" "rests on a grave misunderstanding" (p. 1960, p. 501). The point is underscored by the fact that McTaggart hasn't the slightest inclination to make these arguments regarding space, personality, or the colour of zebras. McTaggart's viewpoint hinges on the apparent fact that, unlike the cases of personality colour and space, a complete description of events taking place in time is impossible unless indexical expressions enter into it. We can coordinatize the whole of space and completely describe the position of objects in space in terms of those coordinates. We can coordinatize time and describe exactly when each event occurs in time. But such a description would be incomplete and *static*, since it would not describe which event is occurring *now*. Nevertheless when we attempt to give a complete description of time, then the use of A series indexical expressions will then introduce contradictions.

If this is correct so far, then we have a choice between giving up the reality of temporal flow or giving up the idea that a complete description of reality is possible, but still maintaining that temporal flow is, indeed, real. McTaggart gives up the reality of temporal flow and Dummett gives up the ability to offer a complete description. But I think that variable sets present us with a third option by allowing us to give an incomplete description of time flow which is nevertheless not a static description. Where McTaggart goes wrong, I think, is to believe that we must think of individual objects as constant objects, and so temporal attributions are made to such objects. When these attributions are made to a constant object a contradiction results. Thus, in order to avoid a contradiction, A series predications must be made only to *varying objects*.

But how can we treat a variable set X as truly varying, rather than simply a sequence of objects $X_1, X_2, \ldots X_n$? In set theory the maps $r: X_0 \longrightarrow X_1$ and $f: Y_0 \longrightarrow Y_1$, are treated as reducible to sets, i.e., to the objects of the category. Thus any map together with its domain and codomain is merely a fanciful description of a constant set. Category theory, on the other hand, permits arrows to be irreducible to the objects of the category. There is the possibility that a map represents motion. Moreover since maps are not forcibly reduced to sets, and so we may consider the map $f: Y_0 \longrightarrow Y_1$ together with its objects as itself an object in its own right, and one which is not a set. To think of a map as a variable set is to treat it as a unity, or an object. In other words, it extends the idea in set theory of taking a collection of objects to be a unity, to taking objects *and arrows* together to be a unity. Since the arrows are not taken to be automatically reducible to objects, new objects are formed which do not exist in the category of sets. What justification exists for taking such a step? The justification is that, since the objects of a topos share deep structural similarities with sets, then one has just as much reason to treat an arrow $r: X_0 \longrightarrow X_1$ as an object, as one has to treat a collection of objects as an object.

Category theory and the continuum: a semantic approach

The main purpose of this section is to suggest that category theory is part of the semantic tradition in mathematics - more precisely, of the model theoretic approach to mathematics. The semantic tradition can be seen as an attempt to show that intuition is not needed in order to determine mathematical objects. Recall that Kant believed that mathematical statements could not be thought without an *a priori* intuition to provide their meaning, as well as to ground or justify them. In response to this idea the semantic tradition beginning with Bolzano began to develop a punctual continuum in the form of a set of real numbers which could be thought without relying on intuition. Category theory generalizes this trend by showing how a non punctual continuum can be thought without the aid of intuition.

I have concentrated on the views of Cantor and Dedekind because their view of the continuum has now become standard and they can both be understood as motivated by the semantic approach to mathematics. However, the history of the semantic approach is long and complex and will not be retold at any length here. It is sufficient to note that Cantor and Dedekind's construction of the linear continuum, Russell and Frege's logicism, and Hilbert's axiomatic approach all owe much to the semantic tradition. I have already extensively quoted from Russell who intended his works *Principles of Mathematics* and *Principia Mathematica* to prove that Kant's view of mathematics was faulty (Hylton, 1990). Frege repeatedly insists that it is not necessary to refer to "foreign elements" such as space and time in order to justify arithmetic (Demopoulos, 1994; Dummett, 1992; Hallett, 1994). For instance Frege states, in the *Begriffsschrift* that by using his "concept script" in mathematical proof:

> ... we can see how pure thought, irrespective of any content given by the senses or even by an intuition a priori, can, solely from the content that results form its own constitution, bring forth judgements that at first sight appear to be possible only on the basis of some intuition. (quoted in Demopoulos, 1994, p. 229)

Hilbert can also be considered to be in this tradition because of his stress on the self - sufficiency of the intellect in mathematics.

> ... in the further development of a mathematical discipline, the human intellect becomes conscious of its self - sufficiency, encouraged by the achievement of solutions in the past. Out of itself, and often without recognizable stimulation from without, intellect creates new and fruitful problems, through purely logical combinations, through generalization, through specialization, through separating and collecting concepts in the cleverest ways, and thus steps forward itself as the real questioner. (quoted in Hallett, 1994, p. 161)

Within the semantic tradition there is a more specialised subapproach to meaning, the model theoretic approach. This approach accepts the distinction between subjective and objective representations of mathematical statements but pushes the distinction one step further by considering objective content to be relative rather than fixed. In contrast with the model theoretic viewpoint Russell and Frege took logic to be an umbrella theory which deals with the general operations of any concepts and which governs all of our thinking. In this sense logic is supposed to be a universal language, which supplies the "laws of thought" for every domain. The universe of discourse must be "everything that exists." Indeed Russell argued that logic must be a universal language because to think otherwise would lead to absurdities.

His basic argument was repeated on several occasions. Russell's argument is that if we are to have a restricted universe of discourse, something other than simply *the* universe, then *we* must establish such a universe of discourse by means of a statement which restricts the variables to a specific domain. But in *that* statement we have no reason to assume that we are using a restricted universe of discourse. We can make such an assumption only if we make a prior statement in which the restriction of the variables are made explicit. But then the same point will apply to this statement. Thus in order to have all variables restricted we will require an infinite regress of restricting statements. Since we can't have such a regress, the use of restricted variables presupposes unrestricted variables which range over everything that there is (Hylton, 1990, p. 145). Moreover, since logical propositions must be thought of as completely general, and unconditionally true only the unrestricted propositions are to be counted as part of logic.

The model theoretic viewpoint differs with this point of view by taking the objective content of mathematical statements to be relative rather than fixed. The distinction is most succinctly introduced by very briefly considering the basis of the dispute between Frege and Hilbert on the proper use of axiom systems. For Hilbert the non logical constants of a language, such as "point", "shoe", "beer mug", have a variable reference. In Hilbert's most memorable statement he makes reference to this variability by claiming that "point", "line" and "plane" might refer to table, chair and beermug. The point is that once an interpretation is given, the meaning of the non logical terms is fixed, but not prior to a choice of interpretation (Demopoulos, 1994; Hallett, 1994; Hodges, 1986). Thus the reference of the non logical terms is allowed to vary.

The process at work here in the introduction of the model theoretic approach is that of replacing the non logical constant by a variable, and its associated relativisation of the notion of truth from truth (in *the* universe) to truth *in a structure*. Hodges (1986) take the discovery by Tarski and Vaught of the notion of structure to be the discovery of a new indexical term like "here" or "now". Tarski argued, notably, that the notion of truth must be relativised in this way in order to solve semantic paradoxes such as that of the liar (Etchemendy, 1988). A notable consequence of this approach is that it allowed for a clear definition of *logical consequence* which did not depend upon the Kantian idea of content. No longer is a proposition B a logical consequence of A in virtue of its containing the content of A. Rather B is a logical consequence of A because in every structure in which A is true, B is also true.

While Tarski succeeded in isolating the notion of truth in a structure, it has been widely assumed that such structures must be drawn from a *fixed* domain of sets - the so called "absolute" universe of sets. Now, category theory allows one to proceed further in the replacement of constants by variables. Instead of seeing the structures of mathematics being drawn from the domain of an absolute universe of sets, this universe is itself seen as one among many categories which play the role of the structures of category theory. In other words category theory allows the category to vary rather than being a fixed category of sets.

Of course, this view of category theory as an extension of the model theoretic outlook did not originate with Eilenberg or MacLane, whose intentions were purely mathematical. They were considering certain functors which assign certain groups to topological spaces. For this reason they introduced the idea of a category as the domain and codomain of functors. In their initial presentation they viewed the notion of category as merely auxiliary, and not as introduced as a substitute for the universe of sets.

> It should be observed that the whole concept of a category is essentially an auxiliary one. Our basic concepts are essentially those of a functor and of a natural transformation .... The idea of a category is required only by the precept that every functor should have a definite class as domain and a definite class as range, for the categories are provided as the domain and ranges of functors. (quoted in Bell, 1981, p. 351)

Thus the ontological status of a category was uncertain at its inception because such an inclusive notion of domain and codomain was not needed. As they noted maps need not be defined for "all topological spaces" but only for given pairs of topological spaces. But the notion of a category caught on, and MacLane characterises the rise of category theory as the "idea of looking not at one group or one homomorphism, but at all the groups and all the homomorphisms - that is, the category of all groups ..." (1982, p. 24 - 25).

Yet many may be skeptical of the coherence of the idea of a category. Perhaps the idea that there can be categories such as variable sets is misconceived, since after all, the view that sets are the fundamental objects and that every object is a set (and so composed of discrete elements) is the fundamental foundational belief of the twentieth century. The work of Bourbaki has convinced many that all mathematical theories must be regarded as extensions of the theory of sets. Indeed this idea is so deeply ingrained that it allows for an even more recalcitrant stance. It has been expressed by Mayberry who puts it "Set theory is not really, or not just, a foundation for mathematics. It simply *is* modern mathematics "(1988, p. 353).

One might be tempted to think that Mayberry's predilection for sets may be a result of a lack of clear alternatives. The intuitive conception of varying quantity may be all well and good one might say, but it is not a mathematical conception, only an outdated metaphysical conception entirely derived from intuition. But this is not so. As I have shown, it is possible to generalise constant sets to variable sets. This leads to two related ways of looking at this development. On the first approach, associated with Reyes (1980), the notion of set is enlarged to that of continuously variable set which allows for the development of the calculus. As Moerdijk and Reyes make clear in their treatise on smooth infinitesimal analysis their purpose is meant to generalise set theory to topos theory in a way which is *compatible* with intuitive proofs in analysis, and later differential geometry (1991, p.v and p.166).

The second approach is to consider the notion of set as inherently flawed by its underdetermined character, which is evidenced by the fact that there are truths which are independent of the concept of set. Bell (1986, 1988) is a good representative of this approach. According to the view spelled out by Bell, the usual model theoretic viewpoint holds that there is one "absolute universe of sets" which provides the materials for every model, but the independence proofs of Cohen led to the idea that the concept of set was underdetermined. Mostowski noted in 1965 in response to Cohen's discoveries that we will probably have different notions of sets just as we have different notions of space and our discussion of set theory will be relative to the kinds of sets which we wish to study. Bell concludes from this state of affairs that

> In this event it becomes natural, even mandatory, to seek for the set concept a formulation that takes account of its underdetermined character, that is, one which does not bind it so tightly to the absolute universe of sets with its rigid hierarchical structure. (1988, p. 238)

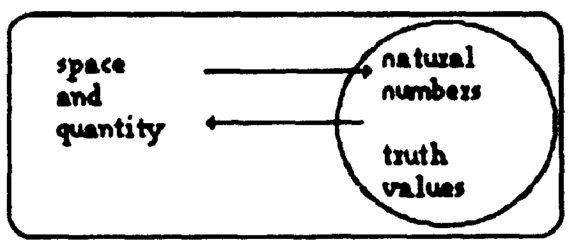
Bell goes on to say that category theory furnishes the required formulation of a determinate set concept through the concept of a topos and its associated internal language - *local set theory* (1988, p. 238). Thus Bell draws on the fact that the power objects of a topos are set-like. Thus it would seem that Bell's approach is to widen models to include toposes other than that of the topos of sets. One would think that the approach of Reyes and that of Bell should be naturally interpreted as an extension of the semantic tradition. But after noting the flexibility of a plurality of these local frameworks Bell concludes that:

> So the local interpretation of mathematics implicit in category theory accords closely with the unspoken belief of many mathematicians that their science is ultimately concerned, not with abstract sets, but with the structure of the real world. (1986, p. 425)

So rather than drawing the model theoretic conclusion, Bell takes the development of topos theory to confirm the view that mathematics is about "the structure of the real world."

Lawvere states his concern openly: the content of mathematics is the space and quantitative relationships in the world. However, the central purpose of the semantic outlook is to provide an approach to *a priori* proof in mathematics that *does not* rely upon intuition. It is, therefore, quite peculiar that Lawvere does not apparently take his own approach to the foundations of the calculus as a triumph of the semantic approach. Instead he takes it as an instance of how *intuition of space and time* allows for the development of the calculus (and synthetic differential geometry). His thinking appears to be the result of a Marxist materialist ideology that only a non-Russian mathematician could espouse. Let me cite a passage from Lawvere.

> As many have pointed out, the essential object of study in mathematics is space and quantitative relationships. Thus an essential part of the scientific world-picture, we have the mathematical world-picture



> whose links with the remainder of the scientific world-picture should never be forgotten. Consideration of this picture shows clearly, by the way, just how wrong was the banker Kronecker and his followers who claim that the continuum is only a mental construction from N and Ω (the subjective idealizations of iteration and truth respectively), rather than primarily a concept derived from our historical-scientific experience with the world of matter in motion. (1980, p. 378)

I find this a highly strange description of the significance of the development of variable sets which contain smooth objects. For Lawvere could have argued along a Kantian line that intuition is necessary because of our inability to cognize the smooth continuum through the intellect alone. He does not and, in fact, his work shows precisely the *opposite*: it *is* possible to represent the smooth continuum by the intellect alone if one is prepared to admit variable sets as mathematical structures. That is to say, the natural interpretation of Lawvere's axiomatization of elementary toposes is that it allows for an extension of the model theoretic tradition in which toposes other than the category of sets are models.

Mayberry raises a deep objection to the natural interpretation of category theory as that of simply extending the model theoretic approach to include models which are not sets. Mayberry sees it otherwise and regards this idea as confused.

> The *fons et origio* of all confusion here is the view that set theory is just another axiomatic theory and the universe of sets another mathematical structure. And it makes no difference whether that structure is taken to be the relational structure (V, $\in$) consisting of the cumulative hierarchy of sets V, equipped with the membership relation, $\in$, or to be the category Set whose object comprise *all* sets and whose arrows comprise *all* mappings. There are no such structures. The universe of sets is not a structure: it is the world that all mathematical structure inhabit, the sea in which they swim. (1994, p. 35)

The point that Mayberry is getting at can be made as follows. When an axiom is satisfied by a model that model is itself a set. But the universe of "all" sets is not itself a model of any axioms. No axioms could pick out the "universe of sets" as an object because postulating its existence is inconsistent. Thus the universe of sets is not a *model* but rather the "sea in which they swim". Yet, as Mayberry points out, there is a natural tendency for category theory to form inclusive categories of objects and maps.

The problem of forming large complete categories thus remains unsolved since we appear to be faced with the problems of grasping the unincreasable infinite by intellect alone. This reasoning applies to Lawvere's proposal to understand categories as given in a category of categories instead of as collections of objects and arrows. On such an approach a category is not defined as a collection of objects and arrows but as an object within a category of categories CAT and with maps as functions between categories. The intuition for such an object is readily introduced. A function between categories A and B is a structure preserving map from A to B. Every category A has an identity function $1_A : A \longrightarrow A$ which leaves the arrow and object unchanged, and given functions $F: A \longrightarrow B$, and $G: B \longrightarrow C$, there is a composite function $GF: A \longrightarrow C$.

> So it is natural to speak of a category of all categories, which we call CAT, the object of which are all the categories and the arrows of which are all the functors. This raises genuine problems. Is CAT a category in itself? Our answer here is to treat CAT as a regulative idea; that is, an inevitable way of thinking about categories, but *not strictly a legitimate entity*. (McLarty, 1992, p. 5, my emphasis)

Thus, when we try to form large categories (categories in which the objects and arrows do not form a set) we must face dealing with categories of *all* topological spaces and continuous maps, or *all* sets and functions, and ultimately, *all* categories. Unlike set theory, in which sets are limited in size, CAT, the category of categories, must be unlimited in size. But as McLarty *admits*, our concept of CAT determines no object. Rather such a concept is an idea of reason, and like an object seen in a mirror, is a "mere idea, a *focus imaginarius*, from which, since it lies outside the bounds of possible experience, the concepts of the understanding do not in reality proceed" (1871/1965, A645/B673). Thus McLarty's "solution" is a facile answer to the epistemological problem. Instead of answering the question "is our knowledge of mathematical objects justified by thought alone?", he just denies that we have any mathematical knowledge of objects at all!

But I don't think there is any need for either Mayberry's skeptical position or McLarty's view that categories are merely ideal. There is a third option which is very much like the earlier response to Peirce's objection. Recall that Peirce thought that in order to capture the infinite divisibility of a line we must be able to conceive of the line as consisting of "all the points there are". But this is impossible because it would require completing the uncompletable (i.e. the absolute infinite). However, if Cantor is correct, it is possible to conceive of a (completed) transfinite number of elements of the real number continuum which is infinitely divisible. Likewise we may consider a transfinite collection of "all sets", "all rings", and in general "all objects and arrows". In short the concept of category is defined in terms of the concept of set.

The model theoretic conception is closely related to another conception, the structuralist view in mathematics. Since many commentators have taken category theory to be the culmination of structuralism rather than the semantic approach to mathematics it is important to see that they both agree on the fundamental point that category theory relativizes the notion of structure to that of category. Thus, it takes Tarski's discovery one point further: "category" is taken to be indexical term.

A succinct statement of the structuralist doctrine is given by Resnik where again we find the basic point that mathematical objects are determined by means of their relations to other objects rather than their internal composition (1981, p. 530).

> In mathematics, I claim, we do not have objects with an internal composition arranged in structures, we have only structures. The objects of mathematics, that is, the entities which our mathematical constants and quantifiers denote, are structureless points or positions in structures. As positions in structures, they have no identity or features outside of a structure.

The structuralist motive is not unique to smooth spaces but pervades all of category theory. It is expressed at its most extreme by Lawvere who defends category theory as a foundation for mathematics in terms of a similar structuralist ambition:

> In the mathematical development of recent decades one sees clearly the rise of the conviction that the relevant properties of mathematical objects are those which can be stated in terms of their abstract structure rather than those in terms of which the objects were thought to be made of [sic]. The question naturally arises whether one can give a foundation for mathematics which expresses wholeheartedly this conviction concerning what mathematics is about and in particular in which classes and membership in classes do not play any role. (quoted in Feferman, 1977, p. 149 - 50)

The interpretation of category theory as the culmination of a structuralist programme in modern mathematics is not uncommon. It is endorsed by Dieudonne (1979), MacLane (1982) and Bell (1986) for example. In spite of the endorsement by Dieudonne, it is certainly not the culmination of Bourbaki's programme, as Corry (1992) has shown, since the notion of category was eschewed by Bourbaki. But here again we can see the same dialectical process of replacing constants by variables that was evident in the development of the conceptual structure of model theory.

According to Bell, category theory bears the same relation to abstract algebra as the latter does to elementary algebra. Elementary algebra results from the replacement of constant quantities by variable quantities while keeping the operations on these quantities fixed. Abstract algebra carries forward the same idea by allowing operations to vary while the structures (group, ring and so on) are fixed. Finally, category theory, in its turn, allows even the structures to vary, which gives rise to a general theory of structures themselves. Therefore, structuralism is seen as part of an agenda to replace constants by variables. Dieudonne likewise sees the theory of categories and functions as not only the culmination of the "difficult birth of mathematical structures" but as a mathematical framework which replaces the framework of sets. "[Category theory] also gives, nowadays, a framework and a guide which are as useful in modern mathematics as the set-theoretic approach was for the theories of the century" (1979, p. 22).

It may seem that category theory can solve the problem of the composition of the continuum simply by adopting the structuralist view, according to which an object is described by means of its relation to other objects rather than to what is "inside" it. This would allow the continuum to be considered a relational structure rather than an object with an internal constitution of objects much as Leibniz considered a continuum to involve indeterminate parts rather than actual parts. This approach is expressed clearly by Kock. By a "smooth object" Kock is referring to the smooth continuum and objects constructed from it.

> We want to have an axiomatic mathematical theory of smooth objects, by giving a theory of the totality of such. So we would not start by saying "a smooth object is a set of elements (atoms) with some additional structure"; but rather, smoothness is a property of how these objects relate to each other, or map to each other (1981, p. 52).

Of course we should not *start* describing smoothness in terms of atomic parts such as sets because there are no smooth spaces in the universe of sets. But simply giving a structural description of objects will not result in a "totality of smooth spaces" either since one can describe the category of sets in this manner and there are no smooth spaces in it. Instead there must a sufficiently rich framework to contain infinitesimals.

The idea that there can be linear infinitesimals in a framework of objects which are generalised (variable) sets has been adopted by Lawvere and his followers Kock, Reyes, Lavendhomme, McLarty, Bell and others. The solution to the problem, as was outlined in the preceding sections, is to regard points as, in a trivial sense, varying . According to Bell the inherent contradiction of the infinitesimal in the seventeenth century resulted from trying to represent infinitesimals in an unnatural way in discrete spaces. Thus in order to describe infinitesimals properly, we need to treat them as varying quantities:

> The infinitesimal methods commonly used in the 17th and 18th century to solve analytical problems had a great deal of elegance and intuitive appeal. But the notion of infinitesimal itself was flawed by contradictions. These arose as a result of attempting to represent change in terms of static conceptions. Now, one may regard infinitesimals as the residual traces of change after the process of change has terminated. The difficulty was that these residual traces could not logically coexist with the static quantities traditionally employed by mathematics. The solution to this difficulty, as it turns out, is to regard these quantities as also being subject to change, for then they will have the same nature as the infinitesimal representing the residual traces of change, and will become, *ipso facto*, compatible with these latter. (1988b, p. 314)

As I have argued the essence of the contradiction is to be found in the attribution of temporal properties to constant objects, and this contradiction is overcome by attributing temporal properties to variable objects. Thus in order to avoid the contradictions inherent in the seventeenth century mathematics it is necessary to provide a framework of variable objects; but this is not sufficient, for a more complicated variable set than that of a set varying over a discrete space is needed in order for infinitesimals to exist. Logically speaking, the contradiction resulted from the supposition that a variable quantity has either zero value or not zero value, but certainly has no intermediary state. By throwing away the presupposition that quantities must be decidable, it is possible to develop a framework in which variable quantities exist. Such a framework is a topos of sheaves of sets, and as MacLane recounts this framework holds it self out as a foundation of mathematics in competition with the universe of sets:

> Now the idea has appeared that set theory can be replaced by sheaf theory: The fundamental object of mathematics is not a set composed of elements but a sheaf of functions on some nonspecified space or locale. This proposes a foundation of mathematics with a geometric flavour replacing the usual analytical one. (1980, p.191)

A sheaf model of Spaces

The simple example of Set^2 described in a previous section does not account for more than two times. Consider the natural numbers as a category; that is, there is an arrow from m to n whenever $m < n$, and objects are the numbers. A slight generalisation is the category Set^N of sets varying over discrete time. Here the objects are all countable sequences of maps $X_0 \longrightarrow X_1 \longrightarrow X_2,\ldots$ and the arrows are all strings of maps $f_0, f_1, f_2,\ldots$ such that the diagram

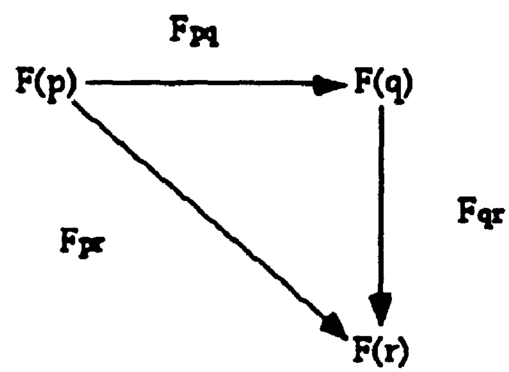
$$\begin{array}{lccccccccc}
t: & X_0 & \longrightarrow & X_1 & \longrightarrow & X_2 & \longrightarrow & X_3 & \longrightarrow & \cdots \\
 & \downarrow f_0 & & \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 & & \\
t': & Y_0 & \longrightarrow & Y_1 & \longrightarrow & Y_2 & \longrightarrow & Y_3 & \longrightarrow & \cdots
\end{array}$$

commutes.

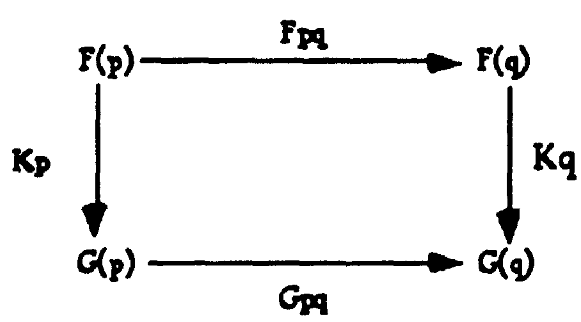
This framework is still too simple to represent variable quantities, for the calculus of the seventeenth century was concerned with smooth variation which is given by the principle of local straightness of curves. This section sketches a generalisation of Set^N to that of a smoothly varying sheaf of sets, a model which satisfies the principle of local straightness of curves. For proof that such sheaves are toposes consult MacLane and Moerdijk, chapter 2.8.

As in the case of Set^N, the objects of the model are constructed out of sets and functions, but are variable sets rather than sets. As a result the properties of the objects of the new category may be quite unlike those of the old category. In particular, the objects of set are decidable, whereas not every object of Set^N is decidable. Thus, since undecidable objects may be constructed from decidable objects, the clarity and distinctness which is at the cornerstone of Cantorian set theory has allowed fuzziness to be reintroduced into thought in a clear and distinct way.

A partially ordered set is a set together with a transitive, reflexive, and antisymmetric relation. A slight generalisation of Set^N replaces N with a partially ordered set P in order to obtain Set^P. This category has as *objects* functors $P \longrightarrow Set$, i.e. maps F which assign to each p in P a set $F(p)$ and to each $p, q,$ in P such that $p \leq q$ a map $F_{pq}:F(p) \longrightarrow F(q)$ such that $p \leq q \leq r$ implies

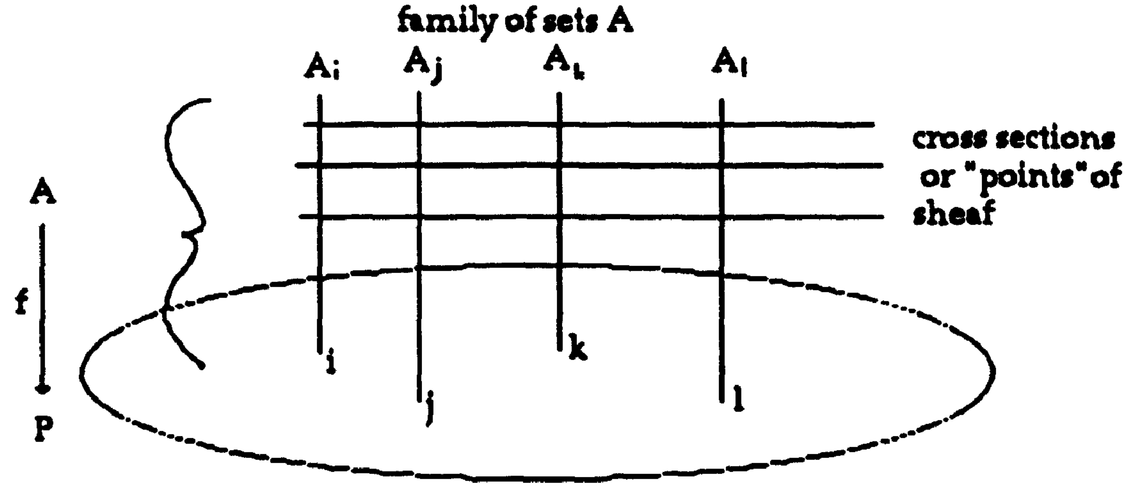


commutes, and F_{pp} is the identity map on $F(p)$. An *arrow* $K:F \longrightarrow G$ in Set^P is an assignment $p \longrightarrow Kp$ of a map $K_p:F(p) \longrightarrow G(p)$ to each $p \in P$ such that $p \leq q$ implies the diagram



commutes.

In order to get an intuitive picture of this situation it is helpful to look at the underlying set theoretic structure, to be called a *bundle* A over P (where A is in Set).

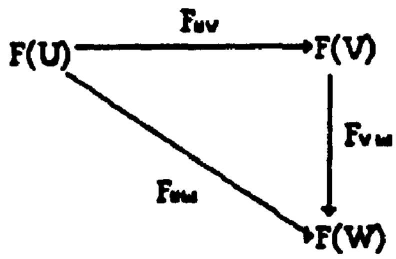


Partially ordered set P

Figure 22

Here is a collection $A = \{A_i : i \in P\}\}$ of pairwise disjoint $A_i s$. The set A_i is a *fibre* or stalk over i. The members of A_i are the *germs* at i. The whole structure of A_is "sprouting up" from P is the *bundle* of sets over the *base* space P. The aptness of the terminology is evident from the picture or by consulting your nearest gardener. A bundle of sets A over P is virtually just a function with codomain P. For let $A = \{x : x \in A_i\}$, for some $i \in P$. For $x \in A$, there is exactly one A_i such that $x \in A_i$, by the pairwise disjointness condition. Set $f(x) = i$. Thus all the members of A_i are mapped to i for each $i \in P$. So each fibre A_i can be recaptured as the inverse image of $\{i\}$ under f, and all A_j and so on in the same way. So given an arbitrary function $f : A \longrightarrow P$, we can define A_i to be $f^{-1}(\{i\})$ for each i, and likewise define $A = \{f^{-1}(\{i\}) : i \in P\} = \{A_i : i \in P\}$. Thus it is convenient to think of a bundle as a P indexed family of fibres $f^{-1}(\{i\})$, for each $i \in P$.

Consider P as a category. If we reverse the arrows in the category P to obtain P^{op} then we obtain $Set^{P^{op}}$. When P is the partially ordered set $O(X)$ of open sets of a topological space X, then objects in $Set^{O(X)^{op}}$ are called *presheaves on* X. So a presheaf F on X is an assignment to each pair of open sets V, U, such that $V \subset U$ a map $F_{UV} : F(U) \longrightarrow F(V)$ such that $W \subset V \subset U$ implies that



commutes. If F is a presheaf on X, and U, V open sets of X such that $V \subseteq U$ and $s \in F(U)$, a section of F on U, then write $s|V$ for $F_{UV}(s)$ (the restriction of s to V). A presheaf on a topological space B is a *sheaf* when it meets a patching or collating condition: for every covering $\{U_i : i \in I\}$ of an open set U in X and any family $\{S_i : i \in I\}$ such that $s_i \in F(U_i)$ for all $i \in I$ and if for any $i, j \in I$, we have $s_i|U_i \cap U_j = s_j|U_i \cap U_j$, there is a unique $s \in F(U)$ such that $s_i = s|U_i$ for all $i \in I$. In other words, the sections are uniquely collatable.

This explanation is quite abstract, but it can be made more concrete by considering the following example. The standard example of a sheaf is the set of real valued continuous functions $C(U)$ on a topological space. Such sheaves of continuous functions naturally arise in manifolds when these manifolds are defined "intrinsically". The idea of an intrinsic description goes back to Gauss who believed that a description of the geometry of a surface should be made without reference to an ambient space in which it is embedded. Consider the simple example of a sphere S^2. It can be described as the manifold of all solutions (x, y, z) of the equation $x^2 + y^2 + z^2 = 1$ in R^3. So it is a collection of points with a certain structure. But it can also be given an intrinsic description, that is, a description without reference to the ambient space R^3. If one omits the north pole of the sphere the stereographic projection is a homeomorphism $p: S^2 - \{n\} \longrightarrow R^2$, and similarly, $q: S^2 - \{s\} \longrightarrow R^2$, for the south pole. All of S^2 may be obtained by taking these two homeomorphic copies and pasting them together along the common part $S^2 - \{s, n\}$.

Any manifold may be described in a similar way by this pasting technique whereby local pieces of R^n are pasted together in order to obtain an intrinsic description of a manifold. In general, an n-dimensional manifold M is a topological space such that each point q in M has an open neighbourhood V homeomorphic to an open set $W \subset R^n$. Such a homeomorphism $\sigma: V \longrightarrow W \subset R^n$ is referred to as a *chart* for M. [Note that a function ϕ is continuous on $V \subset M$ when its inverse image is continuous on $W \subset R^n$; and in this way the chart defines a sheaf of continuous functions on M.] An *atlas* for M is an indexed set $\{\sigma_i : V_i \longrightarrow W_i\}$ of charts such that the domains V_i cover M. Any such atlas defines M as a topological space. Two charts σ_i and σ_j of an atlas may overlap on the set $V_i \cap V_j$, as in the diagram

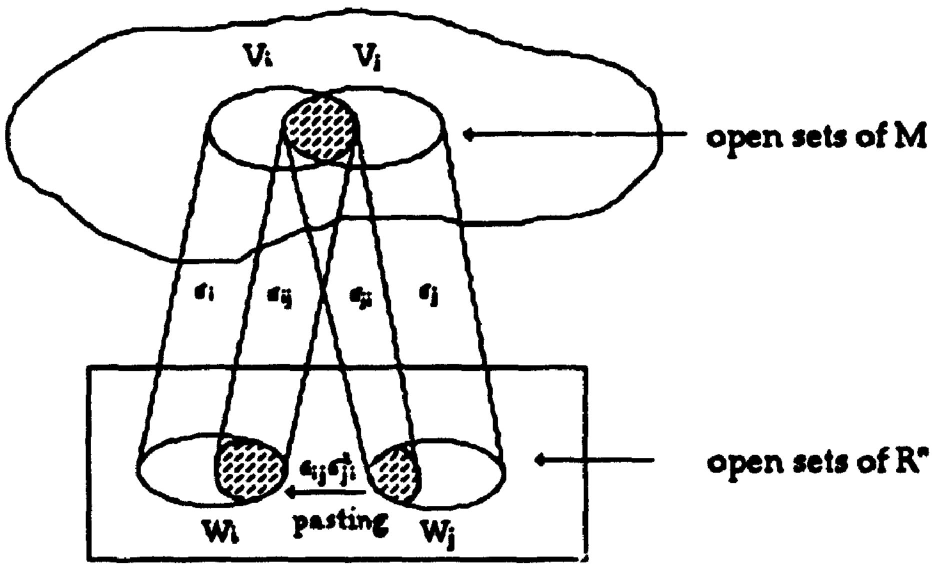


Figure 23

The chart σ_i gives by composition with inclusion $V_i \cap V_j \subset V_i \longrightarrow W_i$ a homeomorphism $\sigma_{ij} : V_i \cap V_j \cong W_{ij}$ from the overlap to some open set $W_{ij} \subset W_i \subset R^n$, and σ_j gives a (distinct) homeomorphism $\sigma_{ji} : V_i \cap V_j \cong W_{ji}$ to a distinct open set $W_{ji} \subset W_j \subset R^n$. Thus for each ordered pair of indices (i, j) there is a composite "transition" function $\sigma_{ij}\sigma_{ji}^{-1} : W_{ji} \longrightarrow W_{ij}$, such that $i, j \in I$ which homeomorphically maps one open set of R^n into another open set. In other words, M is obtained by taking all the open sets W_i of R^n and pasting the open sets $W_{ji} \subset W_j$ to $W_{ij} \subset W_i$ together by the "pasting" or "transition" maps. The "pasting" function codes the way in which two charts are to be pasted together on the manifold.

Smooth manifolds are built up in a similar way. Intuitively smooth means there is a set C^k of functions with k derivative, possibly with $k = \infty$. So a smooth manifold can be described as an atlas of charts with *smooth overlaps*. [Again the chart determines the sheaf C of continuous functions on M; and in the case of a smooth manifold a subsheaf of C, of smooth functions on M

Intuitively, this object is the linear continuum in a smooth topos.] To say that overlaps are smooth requires that the composite of smooth functions is smooth; that a smooth map remain smooth when its domain is restricted; and that a function put together from smooth pieces is itself smooth.

The set of smooth functions can be added and multiplied in such a way that they form a commutative ring. A smooth map is given by a pair $<a,b>$ where a is the base point, and b the slope, or as the constants and linear coefficients of a Taylor series. These coefficients should add and multiply as the Taylor coefficients do; that is: $\langle a_1 + b_1 \rangle + \langle a_2 + b_2 \rangle = \langle a_1 + a_2, b_1 + b_2 \rangle$ and $\langle a_1, b_1 \rangle . \langle a_2, b_2 \rangle = \langle a_1 . a_2, (a_1 . b_2 + b_1 . a_2) \rangle$. This is an example of the so-called C^∞ rings that are at the basis of synthetic differential geometry *and which we identity with the linear continuum R.*

The concept of a sheaf has been described but this is really only the oeginning of a complicated construction. Briefly, the next step is to specify a certain category theoretic notion called a "Grothendieck topology" and then a topos G of smooth spaces is defined as the category of sheaves for this topology. Roughly speaking the notion of a sheaf is extended to the category of C^∞ rings and the smooth topos G is then defined to be the resulting category of sheaves. (See Mac Lane and Moerdijk, 1992 §III.2; McLarty, 1992 §24.1 or Moerdijk and Reyes, 1992 chapter three.)

A curve on a manifold is a map $c{:}U \longrightarrow M$ with U an open interval in R. In order for the curve to have a direction at an arbitrary point it is necessary specify a tangent vector at each point. Intuitively a tangent vector to a curve c is a short segment around p pointing along c at p. If we think of each point in the Euclidean place $R \times R$ as specifying a location and a direction, i.e., a possible tangent vector to a curve, and D as the space of square zero infinitesimals, the existence of an isomorphism between $R \times R$ and the space R^D of all maps $D \longrightarrow R$ indicates that D may be regarded as a *generic tangent vector.* In other words, the only possible effect of a map $D \longrightarrow R$ is to translate and rotate it so that it is coincident with a tangent vector at a particular point. The fundamental result of synthetic differential geometry is that in smooth spaces there is such an isomorphism $R \times R \xrightarrow{\alpha} R^D$ defined by $\alpha(a,b) = [d \longrightarrow a + db]$ (Moerdijk and Reyes, 1992). It follows that for any map $g{:}D \longrightarrow R$ there is a unique slope $b \in R$ such that for all $(\forall d \in D) g(d) = g(0) + db$ (Moerdijk and Reyes, 1992, or McLarty 1994).

Unlike the usual manifold, such a "synthetic" manifold will contain infinitesimal elements, the loci of quotient rings of polynomials such as $R[x]/x^2$ (see Moerdijk and Reyes, 1992 for discussion). This idea of collapsing all polynomials that agree on their linear coefficients can be explained by analogy with the way 12:00 collapses into 0:00 on a twelve hour clock. When we add times on such a clock we treat multiples of 12 as congruent. For instance, 6 hours past 8:00 A.M. is 2:00 P.M. and so the two are congruent. By this method we may form classes of congruent times. In other words, given a polynomial function in a ring of smooth polynomial functions, we identify all those polynomials whose linear coefficients agree. For these polynomials, $x \neq 0$ but $x^2 = 0.$, so the infinitesimals are both linear and nilpotent. By the same method we can define higher order infinitesimals $x^{n-1} \neq 0$ but $x^n = 0$ as $R[x]/x^n$.

The topos of smooth spaces

I have suggested that an acceptable interpretation of the calculus, although obviously not the one that Leibniz actually gave, is given by smoothly varying sets and have given a rough sketch of such a model. The framework consisting of smooth mathematical objects and smooth maps is formalised in a topos Spaces. In this section I give axioms which fill out Leibniz's calculus and which define the idea of a smooth mathematical framework. I will not prove that the axioms hold in a model but if the reader wished to verify this he or she is urged to consult Moerdijk and Reyes (1991). This exercise reveals that it is possible to give proofs in the style of the Leibnizian calculus purely conceptually, thereby "rigorizing" the Leibnizian calculus. By generalizing the framework of concepts in such a manner one thereby solves "Leibniz's puzzle" by showing how it is possible to represent the smooth continuum soley by the intellect, and so an appeal to intuition is needed in proofs.

No system of axioms has become standard. I draw on the axioms in McLarty (1992) in which he gives a short presentation of the smooth calculus, from Bell (1988b) and Lavendhomme (1987). The latter work contains an elegant and readable discussion of the infinitesimal calculus and differential geometry in the context of smooth spaces. In addition, Bell (1988b) contains a very intuitive discussion of the smooth calculus which gives applications and some philosophical discussion.

Here a particular topos Spaces will be described. The intended model is the ring described in the previous section. The linear continuum is a ring of smooth functions, N is a subset of the linear continuum, products are products of this ring, the terminal object is the one point space 1 of the ring, the truth value object is some Heyting algebra. We refer to the objects of Spaces as smooth spaces, to its arrows as smooth maps, and to the global elements $x{:}1 \longrightarrow A$ as points of the smooth space A. We assume the

Existence of a linear continuum: Spaces has a (smooth) space R with selected points $0{:}1 \longrightarrow R$ and $1{:}R \longrightarrow R$ and maps $-{:}R \longrightarrow R$, $+;R \times R \longrightarrow R$, and $\times{:}R \times R \longrightarrow R$ that makes R a (non-trivial) ring.

Intuitively R is a line and the selected point are numbers. We need not assume that R is a collection of points but merely assume that once $R \times R$ is given, the operations described above can be defined by means of a Euclidean construction. It is assumed that R is non-trivial, that is, $0 \neq 1$. It has an arithmetic structure similar to the usual real numbers but also includes infinitesimals. Note that an element x does not contain a multiplicative inverse when $\neg\neg(x = 0)$, or equivalently, when $x \in D$. Otherwise, consider $d \in D$, a subspace of R. Then $1/d$ exists and $d \cdot (1/d) = 1$. Thus $0 = 0 \cdot (1/d) = d^2 \cdot (1/d) = d \cdot (d/d) = d \cdot 1 = d$. But, since this holds for any element d, this is not compatable with the assumption that $D \neq \{0\}$ which was shown earlier to be implied by the principle of infinitesimal linearity.

The following axiom says that if we except the nilpotent elements (i.e., when $\neg\neg\neg(x = 0)$, R is a field.

R is a field: $\neg(x = 0) \Rightarrow (\exists y \in R) xy = 1$

There is a constant map $x{:}R \longrightarrow R$, and a squaring map $x^2{:}R \longrightarrow R$. The equalizer of these two maps, i.e. the domain where they agree on inputs, is a space $D = \{x \in R{:}x^2 = 0\}$. This is the space of square zero infinitesimals. The central axiom describes D in terms of how maps behave on D.

Principle of infinitesimal linearity: For any map $g:D \longrightarrow R$ there is a unique $b \in R$, its slope, such that for all $(\forall d \in D) g(d) = g(0) + db$.

This says that every map f from D to R is an affine transformation, i.e., it preserves colinearity around a infinitesimal neighbourhood of 0. D is a linear segment of R which unlike a curve in the Euclidean plane, cannot be bent or broken under any transformation. In the Dedekind-Cantor continuum the points behave like rigid bodies, but in the smooth space R it is the space of infinitesimals D that behaves like an infinitesimally rigid body.

As was demonstrated in chapter three, D is not $\{0\}$ as it is in the continuum of real numbers, that is, it not the case that for all $d \in D$ $(d = 0)$. But it does not follow that any $d \in D$ is definitely unequal to zero because D is not decidable. Stronger still, "d cant' decide whether it is zero or not" in the sense that it is not the case that $\forall d \in D[(d = 0) \vee (d \neq 0)]$. In Bell's (1995) words, D is a "pure synthesis of location and direction". Since D is a subspace of R, R is not, in general, decidable either.

Let us say say that a and b are distinguishable when $\neg(a = b)$, and indistinguishable when $\neg\neg(a = b)$. We can say, moreover, that every element of D is indistinguishable from 0. For assume $\neg(d = 0)$. Since R is a field (in the sense specified above) and D a subspace of R we conclude for all d, $\neg(d = 0) \Rightarrow \exists y \in R(dy = 1)$. It follows that $\neg(d = 0) \Rightarrow (\exists y \in R) 0 = d^2y^2 = 1^2 = 1$. This contradicts the non -triviality of R, so we have $\neg\neg(d = 0)$. Notice that this immediately contradicts the decidability of infinitesimals, since we could conclude, assuming decidability, that for all $d \in D$ $(d = 0)$. Thus the linear continuum of Spaces is very much unlike that of the continuum of real numbers.

Not only is every map from D to R affine, it follows that every map from R to R is locally affine at every point. Intuitively since the image of any map from D to R is linear around $f(0)$ we translate D along R so that the origin lies on the x coordinate of the point in question. More precisely we may define the derivative of an arbitrary function $f:R \rightarrow R$ at an arbitrary point $x \in R$ by considering the interval $\{x + d : d \in D\}$. For any $f:R \rightarrow R$ and any $x \in R$, there is a function $g:D \longrightarrow R$ given by $g(d) = f(x + d)$ and therefore (by the principle of infinitesimal linearity) $f(x + d) = g(d) = g(0) + db = f(x) + db$ for some unique b in R and all $d \in D$. Since each x has a unique b, there is a unique map $f':R \longrightarrow R$ such that $f(x + d) = f(x) + df'(x)$ for all $x \in R$ and

$d \in D$. This equation may be considered the fundamental equation of the differential calculus. (Notice that it may be rewritten as $f(x+d)-f(x)/d=f'(x)$ and that the quantity $df'(x)$is exactly the change in the value of y $(=f(x))$ as x changes from x to $x+d$.)

The principle of infinitesimal linearity immediately implies a principle of infinitesimal cancellation: for any a,b in R, if $da=db$ for all $d \in D$, then $a=b$. Just consider the function $g:D \longrightarrow R$ defined by $g(d)=da=db$. The slope b must be unique and so $a=b$. From the definition of a derivative and the principle of infinitesimal cancellation it is possible to derive the usual rules of the calculus (See Bell, 1988; Lavendhomme, 1987; or McLarty 1992).

Given the concept of linear infinitesimal one can define continuity quite intuitively. Let us say that a is (infinitesimally) close to b whenever a and b differ by an infinitesimal, that is $a-b=d \in D$ and write $a \approx b$. This allows continuity to be defined directly as a infinitesimal preserving transformation preserving transformation in the space. We say that f is continuous just in case $f(a) \approx f(b)$ whenever $a \approx b$. It follows from the principle of infinitesimal linearity that every function is continuous. For given $x \in R$ define $g:D \longrightarrow R$ $g(d)=f(x+d)$ for all $d \in D$. Then by the principle of infinitesimal linearity $f(x+d)=g(d)=g(0)+db$ for some b in R and all d in D. Since $db \in D$, the result follows.

I have already quoted Leibniz as having said that "a curvilinear figure must be considered to be the same as a polygon with infinitely many sides." Leibniz, I have argued, preferred to reason symbolically without need of an interpretation. Initially one might reject this axiom out of hand: as Berkeley said, considering a curve to be a polygon is simply an abuse of language. No matter how many sides a polygon has it can never equal a circle, for a circle has a different tangent for each distinct point; whereas a given polygon has the same tangent at distinct points. Again the circle can only be approximated by a sequence of polygons. This description of the circle approximated by a *static* sequence of polygons neglects the fact that the tangent line to the curve is *varying* over the curve.

For consider the example in Bell (1988b, p. 306). Given a curve $y=f(x)$, consider a point P with x coordinate x_0 and let P move to an infinitesimally near point Q with x coordinate x_0+d, with $d \in D$. Moving the origin of coordinates to P transforms the variable (x,y) to (u,v) given by $u=x-x_0$, $v=y-f(x_0)$. Writing $f(x_0)=a$, $f'(x_0)=b$, and considering the fundamental equation of the differential calculus, the equation of the tangent to the curve at P (in terms or the new coordinates) is (I) $v = au$; and that for Q is (II) $v = (a + bd)u$. Note that both of these lines pass through P and Q and yet the lines are distinct, since (assuming that $b \neq 0$), I has slope a and II has slope $a + bd \neq a$.

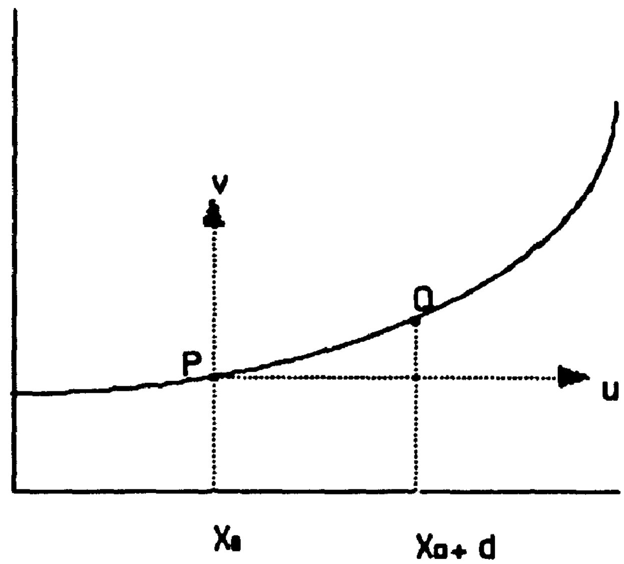


Figure 24

As one passes from P to Q the locally straight portion of the curve is subject to an increase in slope db. This reveals how a curve may be an infinitangular polygon, for the curvature is *manifested* in the infinitesimal rotation of the locally straight segment of the curve as one "moves" along the curve.

R is a preorder. There is a transitive, reflexive relation $\leq$ on R which is compatible with the following conditions:

a) $\forall x, y, z \in R(x \leq y) \Rightarrow (x + z \leq y + z)$

b) $\forall x, y \in R[(0 \leq x) \wedge (0 \leq y)] \Rightarrow (0 \leq xy)$

c) $(0 \leq 1) \wedge \neg(1 \leq 0)$

d) $\forall d \in D(0 \leq d) \wedge (d \leq 0)$

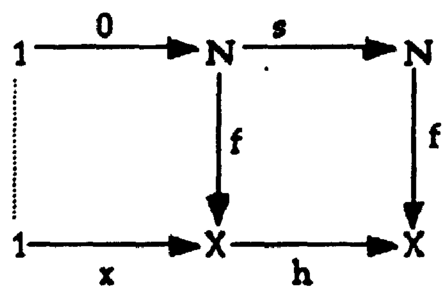
It can be proved from this axiom that a closed interval $[a,b]$ is stable under addition of infinitesimals: i.e. if $x \in [a,b]$ then, $(x + d) \in [a,b]$,) The existence of the required function f depends upon this fact.

This last condition d helps us to understand how to avoid the phenomenon of collapse of an infinitangular polygon. Now we define $[a,b] = \{x \in R : a \leq x \leq b\}$. By condition d we have $D \subseteq [0,0]$. Thus as the number of edges of a finite polygon increases the edges become smaller. In passing from a finite polygon to an infinite polygon, the edge D does not become identical to 0, since this would contradict the principle of infinitesimal linearity; nevertheless D is entirely contained within 0. In actuality, though, the infinitesimal portion D of the edge of the polygon was always within 0 even in a finite polygon.

For the sake of completeness, though irrelevant to our concern at this point, I will mention that Spaces must possess a natural number object. Intuitively there is a subobject of R that behaves like the natural numbers.

Existence of natural numbers: There is a natural number object N.

A natural number object is an object N and arrows $0{:}1 \longrightarrow N$, (0 is number) $s{:}N \longrightarrow N$ (the successor of a number is a number) such that to every diagram $1 \longrightarrow X \longrightarrow X$ there is a unique arrow which makes the following diagram commute.



This axiom is essentially the categorical version of the Dedekind-Peano postulates for natural numbers. It follows straightforwardly that the natural number object N is decidable. The diagram is just that defining the function $f(0) = x$, $fs = hf$. So the axiom states that natural numbers may be defined by recursion. See MacLane (1986, §2.2) for details.

These principles may be applied to solve traditional problems by using the infinitesimal reasoning of the seventeenth century. By using the concept of the smooth continuum we are able to determine areas, lengths, curvature and other features of quantities The following examples are drawn from Bell (1988b).

Example 1. The fundamental theorem of the calculus.

Leibniz considered finding a tangent to be drawing a straight line coincident with an infinitesimal line segment joining the vertices of an infinitangular polygon and finding the area under a curve as summing the inscribed rectangles under the curve. The principle of infinitesimal cancellation allows for a direct proof of the fundamental theorem of the calculus in a way which reveals its original meaning, namely that summing (or integration as Bernoulli later called it) and finding a tangent are mutually inverse operations. A fundamental insight of Leibniz was to apply this idea to geometric figures. "The consideration of differences and sums in number sequence had given me my first insight, when I realised that differences correspond to tangents and sums to quadratures" (quoted in Bos, 1974, p. 106).

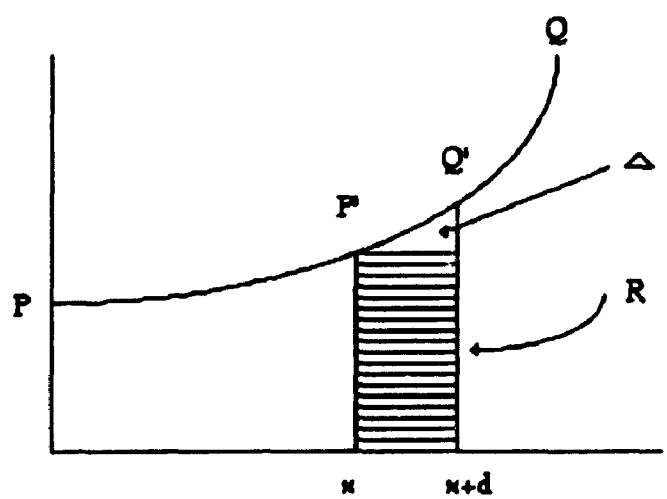


Figure 25

One could assume on the basis of spatial intuition that there is an area under the curve. But for rigorous purposes it is necessary to give a principle of Integration:

Integration Principle: For any $f:[0,1] \longrightarrow R$, there is a definite function $g:[0,1] \longrightarrow R$ such that $g' = f$ and $g(0) = 0$.

This principle formalizes the intuition that for any function $f:[0,1] \longrightarrow R$, there is definite (area function) $g:[0,1] \longrightarrow R$, such that for any x in the interval $[0,1]$, $g(x)$ is the area under the curve $y = f(x)$ from 0 to x. The area under f is symbolized as $\int_0^x f(t)dt$ and referred to as the *definite* integral of f over $[0,x]$. It is also possible to generalize this principle to arbitrary intervals to allow for integration over arbitrary intervals.

We can assume by the principle of integration that there is a well defined area $A(x)$ under the curve defined by the function $y = f(x)$ bounded by the x and y axes and by the abscissa x parallel to the y axis. In order to find how this area is related to the function $y = f(x)$ we begin by considering a slightly greater area under a curve bounded instead by $x + d$, with $d \in D$. Then $A(x+d) - A(x) = dA'(x)$, and writing R for the shaded rectangle, $A(x+d) - A(x) = R + \Delta$. But the infinitesimal portion PQ of the curve is straight, so Δ is a triangle of base d and height $df'(x)$. Hence the area of Δ is $1/2 \cdot d^2 = 0$, since $d^2 = 0$. Thus $dA'(x) = A(x+d) - A(x) = df(x)$. Since this holds for arbitrary d, we may cancel the d on each side to obtain $A'(x) = f(x)$. This is the fundamental theorem of the calculus and expresses the inverse nature of summing and taking derivatives.

In order to apply the fundamental theorem of the calculus it is necessary to assume a constancy principle. It follows from the order axiom that there is such a

Constancy principle: If $f:R \longrightarrow R$ and $f' = 0$, then f is constant (and conversely). It follows immediately from this principle that for $f,g:R \longrightarrow R$, if $f' = g'$, then f and g differ at most by a constant.

Example 2. Exhausting a circle.

The standard objection against a circle being an infinitangular polygon is connected with the method of exhaustion. According to Euclid's account there should be a difference between the area of any given inscribed polygon and a circle. It follows that if we inscribe a polygon and then partition the polygon into triangles, the sum of these triangles will not exhaust the circle. In short the area $1/2 radius \cdot circumferance$ cannot be derived from considering a circle to be an infinitangular polygon.

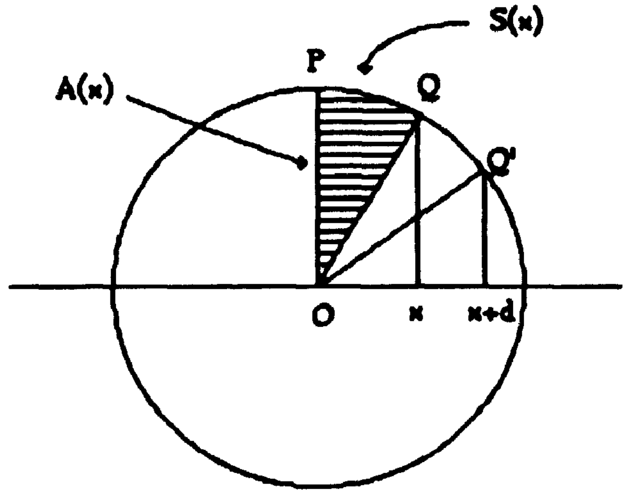


Figure 26

But consider the following reasoning. Let $A(x)$ be the area of the sector OPQ of the circle, where Q has abscissa x. Let $s(x)$ be the length of the arc PQ. If we permit x to change to $x+d$, with $d \in D$ (and with it Q to Q'), we obtain $dA'(x) = A(x+d) - A(x) = areaOQQ'$. But QQ' is a straight line of length $s(x+d) - s(x) = ds'(x)$. The triangle OQQ' then has an area $1/2r\overline{QQ'} = 1/2rs'(x)$ (where r is the radius of the circle). Therefore, $dA'(x) = 1/2 \cdot drs'(x)$. By cancelling ds we obtain $A'(x) = 1/2rs'(x)$. By the constancy principle we obtain $A(x) = 1/2rs(x)$, since $A(0) = s(0) = 0$. Thus taking $x = r$, the area of the quadrant OPR is $1/2 \cdot r \cdot \overline{PR}$. Since there are four quadrants, we multiply both sides by four and arrive at the conclusion that the area of a circle is $1/2rc$. Thus we obtain the right answer by treating the circle as an infinitangular polygon.

Example 3. Arc length.

We can use infinitesimal techniques rather than the method of limits to derive the arc length of a curve. Let $S(x)$ be length of the curve $y = f(x)$ measured from a given point on the curve.

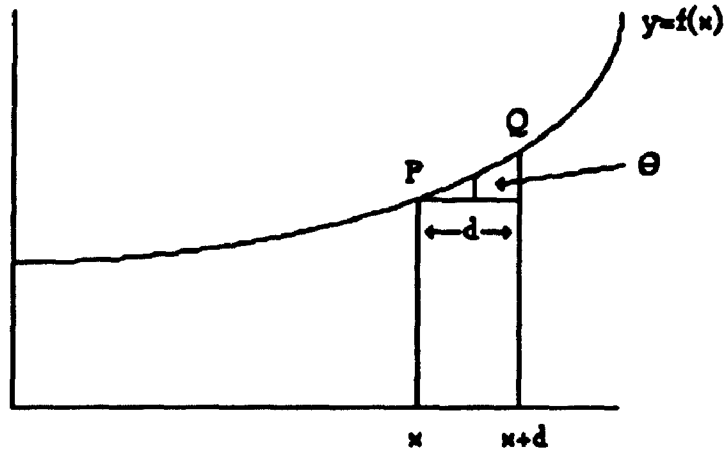


Figure 27

Given a point P on the curve with abscissa x, consider a nearby point Q on the curve with abscissa $x+d$, with $d \in D$. Then PQ is a small straight line segment of length $s(x+d)-s(x)=ds'(x)$. If we write $f'(x)=\tan\theta$, we have length of $PQ = d\sec\theta = d\sqrt{1+\tan^2\theta} = d\sqrt{1+f'(x)^2}$. Hence, since the length of $PQ = ds'(x)$, we have arrived at $ds'(x)=s(x+d)-s(x)=d\sqrt{1+f'(x)^2}$. Applying the principle of infinitesimal cancellation gives $s'(x)=\sqrt{1+f'(x)^2}$. From this formula it is also possible to derive the curvature of a curve at a given point (see Bell (1988b)).

Defining sets as punctual parts of spaces

Since we have sketched the construction of a smooth framework from a discrete framework of sets it is only natural that we can shift back to the discrete framework by disassembling spaces into sets. This fact makes it possible to describe the nineteenth century rigorization of the calculus as a shift from a smooth framework to a discrete one by taking the punctual parts and punctual maps of Spaces to form a category of sets. In this way it is possible to describe the "meta - mathematics" of the rigorization of analysis. The suggestion that there is a shift from smooth spaces to sets in the nineteenth century is originally due to McLarty (1987; 1988) but this suggestion needs to be qualified in several ways.

According to McLarty the transformation took place in the nineteenth century and was followed by the displacement of geometric thinking (1988, p. 75). This suggests that the displacement of geometric thinking arose as a result of the shift from spaces to sets effected by the development of set theory. Of course the decisive change came with the mathematics of Weierstrass, Cantor and Dedekind, but they could not have accomplished this feat without the prior philosophical viewpoint and mathematical efforts of rigorizers such as Bolzano and Cauchy, and even Leibniz. Thus it is more reasonable to regard the shift from smooth spaces to sets as occurring over a period of centuries and occurring as a *result* of the efforts to banish intuition from the calculus.

McLarty argues that the change in conception was a change in logic from intuitionistic logic to classical logic, and cites Cantor's and Dedekind's commitment to the decidability of objects and to excluded middle in support of that contention. This is a peculiar thing to say because Leibniz was also committed to the principle of the excluded middle and to the decidability of magnitudes; in fact both were among his main epistemological principles. So this gives rise to an obvious objection to regarding the rigorization of analysis as a shift from smooth spaces to discrete spaces. It is this: Leibniz, Cantor, Dedekind were committed to the decidability of objects and the principle of excluded middle. Thus one cannot describe the rigorization of analysis as a shift from a framework in which the principle of excluded middle applied to objects to one in which it doesn't.

The answer to this objection is that *in such a reconstruction we are trying to represent conceptually what was only represented intuitively or confusedly*. In the intuitive representation of the continuum, the points were not regarded as actual objects but were given as potential limits of the continuum as a whole. Aristotle regarded parts as potential, and they were thought of by Kant and Newton as limits of the whole which presuppose a representation of the whole and by Leibniz as ideal. So the parts of the continuum were not even thought of as a domain of objects which to which logic would apply. Thus it is only when we construe these parts as *objects*, in our conceptual reconstruction, that excluded middle cannot be held to apply to them, and this reflects their different status in the mathematics of that time.

This section gives an *axiomatic* description of the shift from Spaces to Sets. An interpretation of this shift in terms of sheaves is further described in MacLane and Moerdijk (1992, chapter VII) as an example of a "geometric morphism" from one topos to another. But instead of assuming that smooth spaces have an interpretation in sheaves, it is possible to describe a subcategory of discrete spaces within the category of Spaces so that sets can be defined as punctual parts of space and functions as punctual maps. In other words, by adding appropriate axioms to the axioms for smooth spaces already given it is possible to axiomatize a category of sets as a subcategory of Spaces. This map Γ will be called "points" because the discrete framework is obtained, essentially, by taking the points of the smooth framework. One can visualize the continuum of Spaces as viewed through a pair of "polarized" glasses whereby only the decidable points of spaces are noticed and any temporal variation is arrested at those points. The effect of such a shift on a Leibnizian curve is to "smooth out" its locally straight segments.

Thus, the result of banishing intuition from mathematics was to effect a "geometric morphism" $\Gamma{:}Spaces \longrightarrow Sets$ which takes the punctual parts of spaces; and whereby the continuum came to be understood as a (punctual) object in the category of sets. Conversely the constructions of smooth spaces is a geometric morphism $\Delta{:}Sets \longrightarrow Spaces$. Δ embeds the constant sets and constant maps into the variable category Spaces. So the rigorization movement, as described here, not only involves the elimination of intuition. It is a kind of dialectical process of shifting between frameworks.

The first step in the shift from smooth spaces to sets is to obtain a subcategory of spaces by taking the punctual parts of spaces.

Punctual parts exist: Every space M has a unique punctual part ΓM (i.e one such that every point of M is in ΓM). There are also maps between such spaces (i.e. for each map $f{:}M \longrightarrow N$ there is a map $\Gamma f{:}\Gamma M \longrightarrow \Gamma N$.

The next axiom must ensure that this subcategory behaves like a category of sets. Briefly this can be done as follows. Intuitively we want the object to be well distinguished. In the category of sets, sets are distinguished by means of their elements. In other words, for sets A and B, $A \subseteq B$ iff every member of A is a member of B. In category theory one says that a topos is extensional iff for every object o, the following condition holds for subobjects of o:

Punctual parts are distinguishable: $f \subseteq g$ iff for all $x{:}1 \longrightarrow o$, $x \in f$ implies $x \in g$.

(An equivalent formulation is that maps are well-pointed in the sense that for all $f, g{:}A \longrightarrow B$, either $f = g$ or there is an $x{:}1 \longrightarrow A$ such that $fx \neq gx$.) A fundamental result of topos theory is that an extensional (or well-pointed) topos satisfies the axioms of a set theory which is like ZF, except that quantifiers are restricted or bounded.

Here we may think of a shift from Spaces to Sets along the lines of a coordinate transformation in physics. Consider the concept "smooth function in sheaf X of smooth functions" interpreted within a framework of discrete sets. Any such function may be regarded as real number "varying" smoothly over X. In this framework everything is constant. Now consider the framework Spaces. Here everything *is* varying. Shifting from Spaces to Sets amounts to placing oneself in a coordinate system which is "comoving" with the varying smooth real numbers over X. So the variation of real numbers in Spaces is not "noticed"; and so the varying reals are regarded as satisfying the conditions for being a real number.

McLarty has described the development of the real continuum by Cantor and Dedekind as such a shift:

> Dedekind and Cantor each defined a set of real numbers, $\Re$, with arithmetic structure and an order relation; and both postulated that this represented the set of points on the geometric line [R] with their arithmetic and order relation. In our terms they defined $\Re$ within Set and added an axiom $\Gamma R = \Re$ plus others for arithmetic and order. (1988, p. 85)

After this had been accomplished it became unnecessary to deal with the smooth continuum R, and so the problem of how to represent the smooth continuum was thereby avoided. It is only since the development of topos theory that the relation between the smooth continuum and the real number continuum could be understood and that a coherent solution to Leibniz's puzzle be found.

CONCLUSION: LEIBNIZ'S PUZZLE RESOLVED

"...there is no such thing as philosophy - free science; there is only science whose philosophical baggage is taken on board without examination."

D. Dennett

A venerable view of the continuum is that it is not composed of points in which the parts are prior to the whole but is such that the whole is prior to its parts. Aristotle and others would have agreed. A multitude of points cannot be continuous, cohesive, or possess a measure. Kant's explanation for these features of the continuum was based upon the distinction between intuitive and intellectual representation. The form of pure intellection is such that concepts are composed of their constituent concepts in a species - genus relationship and that the objects they represent mirror the part-whole structure of conceptual representation. In purely intuitive representation the parts are contained in their whole by forming boundaries or limitations within the whole.

Kant argued, in addition, that it was necessary to use intuition in order to determine mathematical objects since our intuition of continuous magnitudes has a whole-part structure which could only be supplied in conjunction with intuitive representation. His most important idea was that the concept of the continuum was of an infinite quantity, which cannot be given purely intellectually because we lack the power to conceive of a concept solely by the intellect which contains within itself an infinite number of distinguished constituents. Thus intuition must be called in to have a representation of the continuous. Kant argued, furthermore, that even if one *could* find a way to represent such quantities by the intellect, such representations would be purely formal and, he thought, thereby empty of content.

The development of the concept of continuity and the construction of the real number continuum and its eventual axiomatization allowed for a continuum which could be grasped purely conceptually. Later, the development of the concepts of connectedness, and of Lebesgue measurable set, further increased the ability to represent the continuum solely by the intellect. So, It would seem, on balance, that the arguments of Kant and

Aristotle on this matter were extremely astute for their time, but are no longer sound.

The development of a semantic outlook took on Kant's objection that mathematics without *a priori* intuition would be empty of content and mathematical arguments uninformative. The semantic approach has gradually pried apart the notion of subjective content which is in the mind and may be of a purely symbolic character, and objective content which is independent of the mind and given by mathematical structures drawn from the universe of sets. The model theoretic approach shows that our ability to extend our knowledge comes, not from adding intuition to concepts in order to extend our subjective representation, but consists in proving that a statement, B, is a logical consequence of another, A, because B is true in a structure, whenever A is true in it.

But the success of this programme to refute Kant is faced with a puzzle which is generated by considering Leibniz's notion of the continuum. The rigorization of analysis was primarily devoted to ridding the calculus of the need for the intuition of space and time. But this emphasis on the use of spatial and temporal intuition was fundamentally a Newtonian conception of the calculus. The fundamental idea of Leibniz's calculus, is that each curve is an infinitangular polygon, and that the interpretation of the notion of infinitangular polygon is irrelevant to mathematics. The elimination of temporal and spatial concepts, and the use of temporal and spatial intuition in proofs finally resulted, in the present century, in conceptualizing math in terms of the concept of set. But how does this eventual "solution" to the problem of rigorization bear on the Leibnizian idea that each curve is an infinitangular polygon?

A committed Kantian could argue at this point that, since we consider a set to be a collection of *well distinguished* objects of thought, we cannot regard a smooth continuum as a collection of objects, that is, of points. The smooth continuum contains undecidable objects, and so there is no way that the infinitesimal parts of the continuum could be represented as a set. Thus there are no infinitesimally linear curves in the universe of sets. But, as Leibniz argued, we do have a such a notion of a continuum, and its validity is shown by the results of successful application of the calculus rather than by any intuitive representation we may have of such a notion. But if our concept of a continuum can only come from intuition or intellect, then our concept of a smooth continuum must require the use of intuition. Thus, in order to represent the smooth continuum we must go beyond what can be represented solely throught the intellect.

At this point the adherent of the semantic view has a choice. Either admit that Kant, Aristotle and others were right, after all, that intuition must be used in order to represent mathematical objects, or we can pursue the idea that our notion of conceptual representation or model must be widened from that of set to something which can encompass smooth curves. The second option is more favourable. It turns out that, if we generalize the notion of model or structure to that of category, then it is possible to represent the smooth continuum in a topos of smooth spaces and to prove the results of the calculus in the style of Leibniz. This option, then, generalizes the model theoretic tradition by allowing models to be categories which may be very unlike the category of sets.

There is some irony in this development when considered in light of Russell's criticism of Kant's use of intuition. Russell's view was that Kant's logic was simply too weak and, as a result, limited his representation of mathematical objects. However this criticism cuts both ways, for set theory *itself* limits the kind of objects that it can represent. Thus, it is ironic that the very criticism that drove the rigorization of the calculus to eliminate intuition in mathematical reasoning, and drove the infinitesimal out of the domain of mathematics, inevitably leads to the construction of smooth spaces in which the existence of the smooth continuum is regained.

The main difficulty of this approach is that if models are extended to include categories, how are we to represent these categories? If we represent them as objects of a further category, then we have a category of all categories CAT. But this notion is bound to fall prey to the paradoxes involved in trying to have completed totalities of objects. If this were the case we would have to invoke intuition of space and time (as Kant said) in order to grasp a complete infinite totality, or regard CAT as an idea of pure reason, a *focus imaginarius.* But there is a third option. First we consider a base category of discrete sets and well - pointed maps. Then it is possible to build variable sets by concatenating discrete sets and maps to form a category of smooth spaces which are sets, smoothly varying over a category of rings. In this framework every map is infinitesimally linear.

In this case we obtain a solution to the problem of the composition of the continuum reminiscent of Leibniz. The paradoxes of the continuum were evaded by saying that our representation of the continuum was a confused perception of underlying monads, just as a perception of a rainbow is a confused perception of its droplets. The objects of the variable sets may be regarded as the confused perception of the underlying monads (sets and well-pointed maps) of the base topos of sets. So the undecidable objects emerge from the decidable objects. But if, as is suggested by topos theory, we regard the variable sets as objects in their own right, then unlike Leibniz, we consider the objects of the smooth topos as *real* rather than *ideal*. In this way Leibniz's puzzle is resolved and a purely intellectual model of the smooth continuum vindicated.

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