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# Tilting Sheaves on Brauer-Severi Schemes and Arithmetic Toric Varieties

Youlong Yan

*The University of Western Ontario*

Supervisor

Dhillon, Ajneet

*The University of Western Ontario*

Graduate Program in Mathematics

A thesis submitted in partial fulfillment of the requirements for the degree in Doctor of Philosophy

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# Tilting Sheaves on Brauer-Severi Schemes and Arithmetic Toric Varieties

(Thesis format: Monograph)

by

**Youlong Yan**

Department of Mathematics

A thesis submitted in partial fulfillment  
of the requirements for the degree of  
Doctor of Philosophy

The School of Graduate and Postdoctoral Studies  
The University of Western Ontario  
London, Ontario, Canada

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# Abstract

The derived category of coherent sheaves on a smooth projective variety is an important object of study in algebraic geometry. Over the past decades, a lot of techniques have been invented to study the structure of derived category of coherent sheaves. One important device relevant for this study is the notion of tilting sheaf, which was first introduced by Baer [Ba].

This thesis is concerned with the existence of tilting sheaves on some smooth projective varieties. The main techniques we use in this thesis are Galois descent theory and Proposition 4.1.8: a bundle on a smooth projective variety is a tilting bundle if it is a tilting bundle after a finite Galois extension. First we construct a tilting bundle on a general Brauer-Severi variety. Our major result shows the existence of tilting bundles on some Brauer-Severi schemes. As an application, we prove that there are tilting bundles on an arithmetic toric variety whose split toric variety has a splitting fan. Our findings extend the works of Costa and Miró-Roig [CM1] and Blunk [Bl].

**Keywords:** Derived category of coherent sheaves, tilting sheaf, Brauer group, Brauer-Severi schemes, arithmetic toric varieties, descent.

*To my parents*

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# Introduction

The study of the derived categories of coherent sheaves on homogeneous varieties dates back to the late 1970s when Beilinson [Be] first described the derived categories of projective spaces. Since then, it has become an increasingly popular subject in algebraic geometry, and a lot of mathematicians have been working on this field. Their efforts have put it in the forefront of modern algebraic geometry. In this thesis, we always assume that  $X$  is a smooth projective variety defined over a field  $K$  and denote by  $D^b(X)$  the bounded derived category of coherent sheaves on  $X$ , which is naturally a triangulated category. The bounded derived category  $D^b(X)$  has come to be understood as a homological replacement for the variety  $X$  and it is one of the most important algebraic invariants of a smooth projective variety  $X$ . For example, Bondal and Orlov showed that smooth projective varieties with ample canonical or anticanonical bundles are uniquely determined by their derived categories [BO2]. At the beginning of the 1980s, it was discovered that the derived category of coherent sheaves has connections with a variety of fields of mathematics, like the Homological Mirror Symmetry Conjecture of Kontsevich [Ko], which states that there is an equivalence of categories between the derived category of coherent sheaves on a Calabi-Yau variety and the derived Fukai category of its mirror. Thus it is vital for us to understand the structure of  $D^b(X)$ .

Over the past decades, a lot of techniques have been invented to study the



structure of the derived category of coherent sheaves. One important device relevant for this study is the notion of an exceptional collection. Exceptional collections provide a way to break up  $D^b(X)$  into simple components. Let  $X$  be a smooth projective variety over field  $K$ . An object  $E$  in  $D^b(X)$  is said to be *exceptional* if

$$\mathrm{Hom}(E, E) = K \quad \text{and} \quad \mathrm{Hom}(E, E[k]) = 0 \quad \forall k \neq 0.$$

An *exceptional collection* in  $D^b(X)$  is an ordered collection  $(E_0, E_1, \dots, E_n)$  of exceptional objects, satisfying

$$\mathrm{Hom}(E_j, E_i[k]) = 0 \quad \text{for all } k \text{ when } 0 \leq i < j \leq n.$$

Finally, an exceptional collection is *full* if the smallest triangulated subcategory of  $D^b(X)$  containing all the objects of the collection is  $D^b(X)$  itself.

The first example of a full exceptional collection is the collection

$$\{\mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}(1), \dots, \mathcal{O}_{\mathbb{P}^n}(n)\} \subset D^b(\mathbb{P}^n)$$

in the bounded derived category of coherent sheaves on  $\mathbb{P}^n$ . It was constructed by Beilinson in his pioneering work [Be]. After that, in his series of papers [K1], [K2], [K3] and [K4], Kapranov constructed a vast number of full exceptional collections on some projective homogeneous varieties (Grassmannians, flag and quadric varieties). These results naturally suggested the following conjecture.

**Conjecture 1.** Let  $X$  be a projective homogeneous variety of a split semisimple linear algebraic group over an algebraically closed field of characteristic zero. Then there exists a full exceptional collection of vector bundles in  $D^b(X)$ .

Up to now, the conjecture remains largely unsolved. Partial results in this direction can be found in [K4, Sa1, Ku1, Ku2, PS, Ku3, M, FM] and [AAGZ]. In [Ku3, §1.1], Kuznetsov and Polishchuk listed the known results up to that point according to types of simple algebraic groups classified by Dynkin diagrams.

Recently, Kawamata [Ka] proved the existence of a complete exceptional collection of sheaves on an arbitrary projective toric variety by means of minimal model theory.

A generalization of the notion of full exceptional collection is the notion of semi-orthogonal decomposition, which was introduced by Bondal in [Bon]. Let  $\mathcal{B}$  be a full triangulated subcategory of a triangulated category  $\mathcal{D}$ . The *right orthogonal* to  $\mathcal{B}$  is the full triangulated subcategory  $\mathcal{B}^\perp \subset \mathcal{D}$  consisting of the objects  $C$  such that  $\text{Hom}(B, C) = 0$  for all  $B \in \mathcal{B}$ . A sequence of triangulated subcategories  $(\mathcal{B}_0, \dots, \mathcal{B}_n)$  in a triangulated category  $\mathcal{D}$  is said to be *semi-orthogonal* if  $\mathcal{B}_j \subset \mathcal{B}_i^\perp$  whenever  $0 \leq j < i \leq n$ . If a semi-orthogonal sequence generates  $\mathcal{D}$  as a triangulated category, then we say it is a *semi-orthogonal decomposition* of  $\mathcal{D}$ . Using the tools developed in [BK], Orlov gave semi-orthogonal decompositions for projective bundles, Grassmann bundles and flag bundles. Later, Böhning [Boh] gave a semi-orthogonal decomposition for quadric bundles. All of these results generalize the full exceptional collections on the corresponding varieties by Beilinson [Be] and Kapranov [K4]. Further developments in this direction are the generalization to flat proper morphisms as in [Sa2] and the extensions to the twisted case, like a semi-orthogonal decomposition for Brauer-Severi Schemes in [Ber] and semi-orthogonal decompositions for twisted Grassmann bundles and flag bundles in [B].

Another important approach to determine the structure of  $D^b(X)$  is to construct tilting sheaves. This notion was first introduced by Baer [Ba]. A coherent sheaf  $\mathcal{T}$  of  $\mathcal{O}_X$ -modules on a smooth projective variety  $X$  is called a *tilting sheaf* (or, a *tilting bundle* if it is locally free) if

- (i) it has no higher self-extension, i.e.  $\text{Ext}_{\mathcal{O}_X}^i(\mathcal{T}, \mathcal{T}) = 0$  for all  $i > 0$ ,
- (ii) the endomorphism algebra of  $\mathcal{T}$ ,  $\text{Hom}_{\mathcal{O}_X}(\mathcal{T}, \mathcal{T})$ , has finite global homological dimension,

(iii) the direct summands of  $\mathcal{T}$  generate the bounded derived category  $D^b(X)$ .

The importance of tilting sheaves relies on the fact [Ba, Theorem 3.1.2, 3.1.3] that they can be characterized as those sheaves  $\mathcal{T}$  on  $X$  such that the functors  $Rhom_X(\mathcal{T}, -) : D^b(X) \rightarrow D^b(A)$  and  $- \otimes_X^L \mathcal{T} : D^b(A) \rightarrow D^b(X)$  define mutually inverse equivalences of the bounded derived categories of coherent sheaves on  $X$  and of finitely generated right  $A$ -modules, where  $A := Hom_{\mathcal{O}_X}(\mathcal{T}, \mathcal{T})$ . The existence of a tilting sheaf also plays an important role in the problem of characterizing the smooth projective varieties  $X$  [Cr, Corollary 2.6]: if  $X$  has a tilting sheaf, then its Grothendieck group  $K_0(X)$  is a free finitely generated abelian group.

There is a strong connection between tilting sheaves and full exceptional collections. Recall that we say a full exceptional collection  $(\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_n)$  is *strong* if it satisfies the conditions  $Hom^k(\mathcal{F}_i, \mathcal{F}_j) = 0$  for all  $i$  and  $j$ , with  $k \neq 0$ . In [Bon], Bondal first proved that if  $(\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_n)$  is a full strong exceptional collection of the bounded derived category  $D^b(X)$  of coherent sheaves on a smooth manifold  $X$ , then  $D^b(X)$  is equivalent to the bounded derived category  $D^b(mod - A)$  of right finite-dimensional modules over the algebra  $A := Hom(\mathcal{F}, \mathcal{F})$ , with  $\mathcal{F} := \bigoplus_{i=0}^n \mathcal{F}_i$ . By Baer's theorems, we know that  $\mathcal{F}$  is actually a tilting sheaf. And we can construct tilting sheaves from full strong exceptional collections. The first few examples of varieties that have tilting sheaves are projective spaces and Grassmannian manifolds, as these full exceptional collections given in [Be] and [K4] are actually full strong exceptional collections.

With full generality, the problem of constructing tilting sheaves seems out of reach and only some particular cases have been addressed (most of them are decomposable as line bundles): projective spaces [Be], Grassmannians, flag and quadric varieties [K4, FM], some fibrations [CRM], some toric varieties [CM1, CM2, Cr, CM3, CM4, CRM, BT, DLM, U, LM], some Fano varieties [BO3, GK, Ku4], some

rational surfaces [Ki, HP], etc. Further development in this direction is the existence of tilting bundles on some twisted varieties. Recently, Blunk, Sierra and Smith [BSS] proved that there is a tilting sheaf on a degree 6 del Pezzo surface over an arbitrary field, and Blunk [Bl] showed that there exist tilting bundles on some twisted homogeneous varieties.

The aim of this thesis, in the context of the works mentioned above, is to show the existence of tilting bundles on some twisted smooth projective varieties. Using Galois descent theory, we construct tilting bundles on twisted projective spaces (Theorem 4.1.12), which are different from those constructed by Blunk in [Bl]. We also show that there is a tilting bundle on the twisted projective bundles under some mild conditions (Theorem 4.1.16). As an application, we show the existence of tilting bundle on an arithmetic variety whose split toric variety corresponds to a splitting fan (Theorem 4.2.1), which generalizes the result obtained by Costa and Miró-Roig in [CM1].

## Thesis Organization

**Chapter 1:** This chapter is basically a survey of the well-known fact that the isomorphism classes of Azumaya algebras are in a one-to-one correspondence with isomorphism classes of Brauer-Severi schemes. It starts with a brief introduction of non-abelian cohomology, and then lists some facts about descent theory and twisted forms. Next, we give general information about Brauer groups and Brauer-Severi schemes. The first important result is the relation between the first Galois cohomology group and twisted forms (Propositions 1.1.16, 1.1.20). Using this result, we give a brief proof of the well-established fact that the isomorphism classes of

Azumaya algebras are in one-to-one correspondence with isomorphism classes of Brauer-Severi schemes (Theorems 1.2.19, 1.2.31). Another important fact is that the Brauer group of a rational smooth projective variety is equal to the Brauer group of its base field (Theorem 1.2.28).

**Chapter 2:** In this chapter, we devote ourself to a quick overview of toric varieties from the viewpoint of combinatorics. We begin with the construction of a split toric variety from a fan and establish the important fact that the equivariant morphisms of two toric varieties are in a one-to-one correspondence with the maps of their fans (Theorem 2.1.9). Next, with special interest, we study toric fibrations. After describing how to construct a fan of projective bundles from the fan of the base toric variety (Corollary 2.1.16), a characterization of toric variety with splitting fan is given (Corollary 2.1.24). In the final part of this chapter, we briefly introduce the arithmetic toric varieties, which are twisted forms of split toric varieties.

**Chapter 3:** This chapter begins with a quick introduction to the construction of the derived category of an abelian category and derived functors. Then we focus our discussion on the study of the bounded derived category of coherent sheaves. We first introduce the important devices for this study, namely, full (strong) exceptional collections, semi-orthogonal decomposition, tilting sheaf, etc. After this, we give a sketch of the proof of Beilinson’s outstanding result on projective space (Theorem 3.1.10) and recall its generalization—Orlov’s theorem about a semi-orthogonal decomposition of projective bundles (Theorem 3.1.21). Then we give Costa and Miró-Roig’s improvement of this result—a full strong exceptional collection for projective bundles (Theorem 3.1.27), which is the special case of toric variety with splitting fan (Theorem 3.1.28). Finally, we introduce Blunk’s construction of a tilting sheaf on Brauer-Severi variety (Theorem 3.1.38).

**Chapter 4:** This is the most important chapter of this thesis. We first prove an

important fact which states that a locally free sheaf is a tilting bundle on a smooth projective variety if it is a tilting bundle after a finite Galois extension (Proposition 4.1.8). Based on this result and using Galois descent or pushforward of sheaf (Remark 4.1.13), we give a tilting sheaf on Brauer-Severi variety (Theorem 4.1.12), which is different from the one given by Blunk (Theorem 3.1.38). An advantage of this method is that we may use it to construct a tilting bundle on twisted projective bundles under some mild conditions and obtain our main theorem (Theorem 4.1.16). After this, two special cases are given (Corollary 4.1.18, 4.1.19). Then, as an application, we show the existence of tilting sheaf on an arithmetic toric variety whose split toric variety has splitting fan (Theorem 4.2.1), which generalize Costa and Miró-Roig's result (Theorem 3.1.28). At the end of this chapter, we give the conclusion of this thesis and some further interesting questions.

# Chapter 1

## Brauer Groups and Brauer-Severi schemes

It is well-known that isomorphism classes of central simple algebras over a field are in one-to-one correspondence with isomorphism classes of Brauer-Severi varieties over it. More generally, the isomorphism classes of Azumaya algebras are in a one-to-one correspondence with isomorphism classes of Brauer-Severi schemes. This chapter is essentially a brief survey of these facts. We also introduce the Brauer group in this chapter.

### 1.1 Non-abelian Cohomology and Twisted Forms

#### 1.1.1 Non-abelian Cohomology

We first recall some basic facts about non-abelian group cohomology (cohomology with non-abelian coefficients of discrete groups) in this subsection. The main reference is [Se2] and all the results presented below can be found in detail there.

**Definitions 1.1.1.** Let  $G$  be a finite group. A  $G$ -group is a group  $A$  equipped with a

left  $G$ -action. If we write  ${}^g a$  for the image of  $a \in A$  under  $g \in G$ , then  ${}^g(a \cdot b) = {}^g a \cdot {}^g b$  for every  $g \in G$  and  $a, b \in A$ . We call  $A$  a  $G$ -module if it is abelian. The *morphism* of  $G$ -groups, a  $G$ -morphism for short, is a map  $f : A \rightarrow B$  of  $G$ -groups such that the diagram

$$\begin{array}{ccc} G \times A & \longrightarrow & A \\ id \times f \downarrow & & \downarrow f \\ G \times B & \longrightarrow & B \end{array}$$

commutes.

Now let us define the non-abelian cohomology  $H^0$  and  $H^1$ .

**Definitions 1.1.2.** Let  $G$  be a finite group and  $A$  a  $G$ -group. We define the *zeroth cohomology group of  $G$  with coefficients in  $A$*  as  $H^0(G, A) := A^G$ , the elements of  $A$  invariant under the action of  $G$ . A *cocycle*  $\mathbf{c}$  of  $G$  in  $A$  is a map  $G \rightarrow A, g \mapsto c_g$  such that  $c_{gh} = c_g \cdot {}^g c_h$  for each  $g, h \in G$ . Two cocycles  $\mathbf{c}$  and  $\mathbf{c}'$  are said to be *cohomologous* if there exists  $b \in A$  such that  $c'_g = b^{-1} \cdot c_g \cdot {}^g b$  for all  $g \in G$ . This is an equivalence relation on cocycles and the set of equivalence class, *the first cohomology set of  $G$  with coefficients in  $A$* , and is denoted by  $H^1(G, A)$ .

Unlike the abelian case,  $H^1(G, A)$  is not a group, but a pointed set, with a distinguished element the class of the unit cocycle  $\mathbf{1}$ , where  $\mathbf{1}_g = 1$ , for all  $g \in G$ .

When  $G$  acts trivially on  $A$ , then a cocycle is simply a group homomorphism and  $H^1(G, A)$  is the set of conjugacy classes of homomorphisms.

The cohomology sets  $H^0(G, A)$  and  $H^1(G, A)$  are functorial in both  $G$  and  $A$ , i.e. contravariant in  $G$  and covariant in  $A$ , and they fit into a natural exact sequence of pointed sets as follows. The *kernel* of a map of pointed sets  $f : (X, x) \rightarrow (Y, y)$ , where  $x, y$  are distinguished elements of  $X$  and  $Y$  respectively, is  $f^{-1}(y)$ . Let  $A$  and  $B$  be two  $G$ -groups and  $A \subset B$  be a normal  $G$ -subgroup, then there is a natural



exact sequence of pointed sets

$$1 \rightarrow H^0(G, A) \rightarrow H^0(G, B) \rightarrow H^0(G, B/A) \xrightarrow{\delta} H^1(G, A) \rightarrow H^1(G, B) \rightarrow H^1(G, B/A).$$

The connecting homomorphism  $\delta$  is defined as follows: let  $c \in H^0(G, B/A) = (B/A)^G$ , then we choose  $b \in B$  such that  $c = bA$ . Since  $c \in (B/A)^G$ , if  $g \in G$ , we have  $c_g := b^{-1} \cdot {}^g b \in A$ , and this defines a cocycle of  $G$  in  $A$ .

Further, if  $G$ -group  $A$  is abelian, we can continue the above exact sequence and define

$$H^1(G, B/A) \xrightarrow{\delta} H^2(G, A),$$

where the abelian group  $H^2(G, A)$  is regarded as a pointed set.

**Remark 1.1.3.** If the  $G$ -group  $A$  is abelian, then the definitions above coincide with usual group cohomology. As in [Gir], one could also define  $H^2(G, A), H^3(G, A)$ , etc. in for nonabelian  $A$ .

Let  $G, G'$  be two finite groups and  $h : G' \rightarrow G$  be a group homomorphism. Then for any  $G$ -group  $A$ , we have natural pullback maps  $h^* : H^i(G, A) \rightarrow H^i(G', A), i = 0, 1$ , which are morphisms of pointed sets. If  $h$  is the canonical projection on a quotient group, then  $\text{inf}_G^{G'} := h^*$  is said to be the *inflation* map. The composition of  $\text{inf}_G^{G'}$  with some extension of the  $G'$ -group is also called inflation.

Using inflation maps, non-abelian group cohomology can easily be extended to the case where  $G$  is a profinite group and  $A$  is a discrete  $G$ -group on which  $G$  acts continuously. Indeed, we can define

$$H^i(G, A) := \varinjlim_{G'} H^i(G/G', A^{G'}), \quad i = 0, 1,$$

where the direct limit is taken over the inflation maps and  $G'$  runs through all the normal open subgroups  $G'$  of  $G$  such that the quotient group  $G/G'$  is finite; moreover, the maps  $H^1(G/G', A^{G'}) \rightarrow H^1(G, A)$  are injective.

Now we will discuss non-abelian Čech cohomology briefly. The main reference is the first three chapters of J. S. Milne’s book [Mi1].

We will first recall some concepts. We say a homomorphism of rings  $h : A \rightarrow B$  is *flat* if the functor  $- \otimes_A B$  from  $A$ -modules to  $B$ -modules is exact. A morphism  $f : X \rightarrow Y$  of schemes is *flat* if for all points  $x \in X$ , the induced ring homomorphism  $\mathcal{O}_{Y,f(x)} \rightarrow \mathcal{O}_{X,x}$  is flat. We say  $f : X \rightarrow Y$  is *faithfully flat* if  $f$  is flat and surjective and  $f : X \rightarrow Y$  is *étale* if  $f$  is flat and unramified.

Let  $E$  be a class of morphisms of schemes that satisfies the conditions: all isomorphisms are in  $E$ , the composition of two morphisms in  $E$  is in  $E$  and any base change of a morphism in  $E$  is in  $E$ . Denoted by  $E/X$  the full subcategory of  $\mathbf{Sch}/X$  of  $X$ -schemes whose structure morphisms are in  $E$ . Now fix a scheme  $X$ , a class of morphism  $E$  and a full subcategory  $C/X$  of  $\mathbf{Sch}/X$  that is closed under fiber products and is such that, for any morphism  $Y \rightarrow X$  and any morphism  $U \rightarrow Y$  in  $E$ , the composition  $U \rightarrow X$  is in  $C/X$ . An  $E$ -covering of an object  $Y$  of  $C/X$  is a set  $\{U \xrightarrow{p} Y\}$  of morphisms in  $E$  which are jointly surjective in the sense that  $Y$  equals to the union of set-theoretic images, i.e.  $Y = \bigcup p(U)$ . The class of all such coverings of all objects in  $C/X$  is the  $E$ -topology on  $C/X$ . The  $E$ -site over  $X$  is the category  $C/X$  together with the  $E$ -topology and we denote it by  $(C/X)_E$ , or simply  $X_E$ . The *small  $E$ -site on  $X$*  is  $(E/X)_E$  and, in the case in which all morphism in  $E$  are locally of finite type, the *big  $E$ -site on  $X$*  is  $(\mathbf{LFT}/X)_E$ , where  $\mathbf{LFT}/X$  is the full subcategory of  $\mathbf{Sch}/X$  of  $X$ -schemes whose structure morphisms are locally of finite type.

The three particularly important examples are as follows:

- $E$  is  $(Zar)$  of all open immersions, we get the *Zariski site*  $X_{Zar}$  and the small  $(Zar)$ -site  $((Zar)/X)_{Zar}$ ;
- $E$  is  $(ét)$  of all étale morphisms of finite type, we get the *étale site*  $X_{ét}$  and the

small  $(\acute{e}t)$ -site  $((\acute{e}t)/X)_{\acute{e}t}$ ;

- $E$  is (fl) of all flat morphisms that are locally of finite type, we get the *flat site*  $X_{\text{fl}}$  and the big (fl)-site  $(\mathbf{LFT}/X)_{\text{fl}}$ .

Once we have the  $E$ -topology, then as for ordinary topological space, we can define presheaves and sheaves on the site and a Čech cohomology theory. Notice that here the morphisms in  $E$  play the role of the open subsets in the  $E$ -topology.

A *presheaf*  $\mathcal{P}$  of groups (not necessary commutative) on a site  $(C/X)_E$  is a contravariant functor  $(C/X)_E \rightarrow \text{Grp}$ , where  $\text{Grp}$  denotes the category of groups. A presheaf  $\mathcal{P}$  on  $(C/X)_E$  is a *sheaf* if it satisfies the following two conditions:

(S<sub>1</sub>) if  $s \in \mathcal{P}(U)$  and there is a covering  $(U_i \rightarrow U)_{i \in I}$  of  $U$  such that the restriction morphism  $\text{res}_{U_i, U}(s) = 0$  for all  $i \in I$ , then  $s = 0$ ;

(S<sub>2</sub>) if  $(U_i \rightarrow U)_{i \in I}$  is a covering of  $U$  and the family  $(s_i)_{i \in I}, s_i \in \mathcal{P}(U_i)$  satisfies

$$\text{res}_{U_i \times_U U_j, U_i}(s_i) = \text{res}_{U_i \times_U U_j, U_j}(s_j)$$

for all  $i, j \in I$ , then there exists an  $s \in \mathcal{P}(U)$  such that  $\text{res}_{U_i, U}(s) = s_i$  for all  $i \in I$ .

Let  $\mathcal{G}$  be a sheaf of groups on  $X_E$  and  $\mathcal{U} = (U_i \xrightarrow{\phi_i} X)_{i \in I}$  be a covering of  $X$ . A *cocycle* for  $\mathcal{U}$  with values in  $\mathcal{G}$  is a family  $(g_{ij})_{i \times j}, g_{ij} \in \mathcal{G}(U_{ij})$ , such that

$$(g_{ij}|_{U_{ijk}})(g_{jk}|_{U_{ijk}}) = (g_{ik}|_{U_{ijk}}).$$

Where  $U_{ij} = U_i \times_X U_j$  and  $U_{ijk} = U_i \times_X U_j \times_X U_k$ .

Two cocycles  $g$  and  $g'$  are said to be *cohomologous* if there is a family  $(h_i)_{i \in I}, h_i \in \mathcal{G}(U_i)$ , such that

$$g'_{ij} = (h_i|_{U_{ij}})g_{ij}(h_j|_{U_{ij}})^{-1}.$$

This is an equivalence relation on cocycles and the set of equivalence classes, and we call it the *first cohomology set* of  $\mathcal{U}$  with values in  $\mathcal{G}$  and denote it by  $\check{H}^1(\mathcal{U}/X, \mathcal{G})$ .

It is a pointed set with distinguished element  $(g_{ij})$  where  $g_{ij} = 1$  for all  $i, j \in I$ .

Another covering  $\mathcal{V} = (V_j \xrightarrow{\varphi_j} X)_{j \in J}$  of  $X$  is called a *refinement* of  $\mathcal{U} = (U_i \xrightarrow{\phi_i} X)_{i \in I}$  if there is a map  $\tau : J \rightarrow I$  such that for each  $j \in J$ ,  $\varphi_j$  factors through  $\phi_{\tau(j)}$ , that is,  $\varphi_j = \phi_{\tau(j)}\eta_j$  for some  $\eta_j : V_j \rightarrow U_{\tau(j)}$ . Denote by  $C^p(\mathcal{U}, \mathcal{G}) = \prod_{I^{p+1}} \mathcal{G}(U_{i_0 \dots i_p})$ ,  $p \geq 0$ . The map  $\tau$ , together with the family  $(\eta_j)_{j \in J}$ , induces a map  $\tau^p : C^p(\mathcal{U}, \mathcal{G}) \rightarrow C^p(\mathcal{V}, \mathcal{G})$  as follows: if  $g = (g_{i_0 \dots i_p}) \in C^p(\mathcal{U}, \mathcal{G})$ , then

$$(\tau^p g)_{j_0 \dots j_p} = \text{res}_{\eta_{j_0} \times \dots \times \eta_{j_p}}(g_{\tau(j_0) \dots \tau(j_p)}).$$

Clearly, the map  $\tau^1$  maps cohomologous cocycles to cohomologous cocycles, hence it induces maps on the first cohomology set,

$$\mu(\mathcal{V}, \mathcal{U}, \tau) : \check{H}^1(\mathcal{U}/X, \mathcal{G}) \rightarrow \check{H}^1(\mathcal{V}/X, \mathcal{G}).$$

**Lemma 1.1.4.** *The map  $\mu(\mathcal{V}, \mathcal{U}, \tau)$  does not depend on the choices of  $\tau$  or the family  $(\eta_j)$ .*

*Proof.* Suppose that  $\tau' : J \rightarrow I$ , together with family  $(\eta'_j)$ , is another map such that for each  $j \in J$ ,  $\varphi_j$  factor through  $\phi_{\tau'(j)}$ . Define  $\kappa : C^1(\mathcal{U}, \mathcal{G}) \rightarrow C^0(\mathcal{V}, \mathcal{G})$ . For  $g \in C^1(\mathcal{U}, \mathcal{G})$ ,  $(\kappa g)_j = \text{res}_{\eta'_j \times \eta_j}(g_{\tau'(j)\tau(j)})$ . Hence if  $g$  is a cocycle for  $\mathcal{U}$ , then

$$\begin{aligned} & (\tau'^1 g)_{j_0 j_1} (\kappa g)_{j_1} \\ &= \text{res}_{\eta'_{j_0} \times \eta'_{j_1}}(g_{\tau'(j_0)\tau'(j_1)}) \text{res}_{\eta'_{j_1} \times \eta_{j_1}}(g_{\tau'(j_1)\tau(j_1)}) \\ &= \text{res}_{\eta'_{j_0} \times \eta'_{j_1} \times \eta_{j_1}}(g_{\tau'(j_0)\tau(j_1)}|_{U_{\tau'(j_0) \times \tau'(j_1) \times \tau(j_1)}}) \\ &= \text{res}_{\eta'_{j_0} \times \eta'_{j_1} \times \eta_{j_0} \times \eta_{j_1}}(g_{\tau'(j_0)\tau(j_1)}|_{U_{\tau'(j_0) \times \tau'(j_1) \times \tau(j_0) \times \tau(j_1)}}) \\ &= \text{res}_{\eta'_{j_0} \times \eta_{j_0} \times \eta_{j_1}}(g_{\tau'(j_0)\tau(j_1)}|_{U_{\tau'(j_0) \times \tau(j_0) \times \tau(j_1)}}) \\ &= \text{res}_{\eta'_{j_0} \times \eta_{j_0}}(g_{\tau'(j_0)\tau(j_0)}) \text{res}_{\eta_{j_0} \times \eta_{j_1}}(g_{\tau(j_0)\tau(j_1)}) \\ &= (\kappa g)_{j_0} (\tau^1 g)_{j_0 j_1} \end{aligned}$$

Thus we have

$$(\tau'^1 g)_{j_0 j_1} = (\kappa g)_{j_0} (\tau^1 g)_{j_0 j_1} (\kappa g)_{j_1}^{-1}.$$

Therefore  $\mu(\mathcal{V}, \mathcal{U}, \tau)$  and  $\mu(\mathcal{V}, \mathcal{U}, \tau')$  are the same map.  $\square$

Hence, if  $\mathcal{V}$  is a refinement of  $\mathcal{U}$ , we get a map  $\mu(\mathcal{V}, \mathcal{U}) : \check{H}^1(\mathcal{U}/X, \mathcal{G}) \rightarrow \check{H}^1(\mathcal{V}/X, \mathcal{G})$ , which depends only on  $\mathcal{U}$  and  $\mathcal{V}$ . It follows that if, moreover,  $\mathcal{W}$  is a covering of  $X$  such that  $\mathcal{W}$  is a refinement of  $\mathcal{U}$  and  $\mathcal{V}$ , then  $\mu(\mathcal{W}, \mathcal{U}) = \mu(\mathcal{W}, \mathcal{V})\mu(\mathcal{V}, \mathcal{U})$ . Thus we can define the *first Čech cohomology set of  $\mathcal{G}$  over  $X$*  to be  $\check{H}^1(X, \mathcal{G}) = \varinjlim_{\mathcal{U}} \check{H}^1(\mathcal{U}/X, \mathcal{G})$ , where the direct limit is taken over all coverings  $\mathcal{U}$  of  $X$ .

A sequence  $1 \rightarrow \mathcal{G}' \rightarrow \mathcal{G} \rightarrow \mathcal{G}'' \rightarrow 1$  of sheaves of groups is said to be *exact* if for every  $U \in C/X$ ,  $\mathcal{G}'(U)$  is the kernel of the homomorphism  $\mathcal{G}(U) \rightarrow \mathcal{G}''(U)$  and every  $s \in \mathcal{G}''(U)$  can be locally lifted to a section of  $\mathcal{G}$ . As for non-abelian group cohomology, there is an exact sequence of pointed sets associated to the above exact sequence:

$$1 \rightarrow \mathcal{G}'(X) \rightarrow \mathcal{G}(X) \rightarrow \mathcal{G}''(X) \xrightarrow{\delta} \check{H}^1(X, \mathcal{G}') \rightarrow \check{H}^1(X, \mathcal{G}) \rightarrow \check{H}^1(X, \mathcal{G}'').$$

The connecting homomorphism  $\delta$  is defined as follows: let  $g'' \in \mathcal{G}''(X)$ , and let  $(U_i \rightarrow X)$  be a covering of  $X$  such that for each  $g_i \in \mathcal{G}(U_i)$ ,  $g_i$  maps to  $g''|_{U_i}$  under the map  $\mathcal{G}(U_i) \rightarrow \mathcal{G}''(U_i)$ . Then we may define  $(\delta g'')_{ij} = (g_i|_{U_{ij}})^{-1}(g_j|_{U_{ij}})$ .

### 1.1.2 Descent

In this subsection, we sketch parts of descent theory and list some facts needed in this dissertation without proof. We first list some facts about faithfully flat descent, for which the interested reader can refer to [Gr2, VIII] for more details. As a special case of faithfully flat descent, we also list some facts of Galois descent. In [J, §2], Jörg Jahnke gives a very good summary of Galois descent theory.

**Definition 1.1.5.** Let  $f : Y \rightarrow X$  be faithfully flat and quasi-compact, and let  $F$  be a sheaf of modules (schemes ...) over  $Y$ . A *descent datum* on  $F$  for  $f$  is an isomorphism  $\phi : p_1^*F \rightarrow p_2^*F$  satisfying condition

$$p_{31}^*(\phi) = p_{32}^*(\phi)p_{21}^*(\phi),$$

where  $p_i$  are the various projections  $Y \times_X Y \rightarrow Y$  and  $p_{ij}$  are the various projection  $Y \times_X Y \times_X Y \rightarrow Y \times_X Y$ . We call this condition the *cocycle condition*.

**Proposition 1.1.6. (Descent for modules)** *Let  $f : Y \rightarrow X$  be faithfully flat and quasi-compact, and let  $\mathcal{F}$  be a quasi-coherent sheaf over  $Y$ . Then every descent datum on  $\mathcal{F}$  for  $f$  arises from a quasi-coherent sheaf on  $X$ , that is, there exists a quasi-coherent sheaf  $\mathcal{F}'$  on  $X$  such that  $\mathcal{F} \simeq f^*\mathcal{F}'$  and the descent datum on  $\mathcal{F}$  for  $f$  is induced from the following commutative diagram*

$$\begin{array}{ccc} Y \times_X Y & \longrightarrow & Y \\ \downarrow & & \downarrow f \\ Y & \xrightarrow{f} & X. \end{array}$$

**Proposition 1.1.7. (Descent for schemes)** *Let  $f : Y \rightarrow X$  be faithfully flat and quasi-compact, and let  $Y'$  be a scheme over  $Y$  and  $\mathcal{L}'$  be a very ample invertible bundle over  $Y'$  relative to  $Y$ . Suppose  $\phi$  is a descent datum on  $Y'$  for  $f$ . If  $\phi$  also induces an isomorphism of the two base changes of  $\mathcal{L}'$  satisfying the same cocycle condition, then  $\phi$  arises from a scheme on  $X$ , that is, there exists a scheme  $X'$  over  $X$  such that  $Y' \simeq X' \times_X Y$  and the descent datum  $\phi$  on  $Y'$  for  $f$  is induced from the following commutative diagram*

$$\begin{array}{ccc} Y \times_X Y & \longrightarrow & Y \\ \downarrow & & \downarrow f \\ Y & \xrightarrow{f} & X. \end{array}$$

**Proposition 1.1.8. (Galois descent for algebras)** *Let  $L$  be a field and  $K \subset L$  be a subfield such  $L/K$  is a finite Galois extension. Suppose  $A$  is a (central simple) algebra over  $L$  equipped with a left  $\text{Gal}(L/K)$ -action,  $T : \text{Gal}(L/K) \times A \rightarrow A$ , such that the action of  $g$  is a  $g$ -linear map  $T_g : A \rightarrow A$  for every  $g \in \text{Gal}(L/K)$ . Then there exists a (central simple) algebra  $A'$  over  $K$  such that there is an isomorphism  $f : A' \otimes_K L \xrightarrow{\sim} A$ , where  $A' \otimes_K L$  is naturally equipped with the left twisted  $\text{Gal}(L/K)$ -action induced by the canonical one on  $L$ , and the isomorphism  $f$  is compatible with the action of  $\text{Gal}(L/K)$  on them.*

Please see Definition 1.2.1 for the definition of central simple algebra.

**Proposition 1.1.9. (Galois descent for quasi-projective schemes)** *Let  $L$  be a field and  $K \subset L$  be a subfield such  $L/K$  is a finite Galois extension. Suppose  $X$  is a quasi-projective scheme over  $L$  equipped with a left  $\text{Gal}(L/K)$ -action by twisted morphisms, i.e. the following diagrams*

$$\begin{array}{ccc} X & \xrightarrow{\rho_g} & X \\ \downarrow & & \downarrow \\ \text{Spec } L & \xrightarrow{t_g} & \text{Spec } L \end{array}$$

*commute, where  $t_g : \text{Spec } L \rightarrow \text{Spec } L$  is the morphism of affine schemes induced by  $g^{-1} : L \rightarrow L$ . Then there exists a quasiprojective scheme  $Y$  over  $K$  such that there is an isomorphism of  $L$ -schemes  $f : Y \times_{\text{Spec } K} \text{Spec } L \xrightarrow{\sim} X$ , where  $Y \times_{\text{Spec } K} \text{Spec } L$  is naturally equipped with the left twisted  $\text{Gal}(L/K)$ -action induced by the one on  $\text{Spec } L$ , and the isomorphism  $f$  is compatible with the action of  $\text{Gal}(L/K)$  on them.*

In this case, we say the scheme  $X$  over  $L$  descends to the scheme  $Y$  over  $K$ .

**Proposition 1.1.10. (Galois descent for quasi-coherent sheaves)** *Let  $K$  be a field and  $L/K$  a finite Galois extension. Let  $X$  be a scheme over  $K$ ,  $\pi : X \times_{\text{Spec } K} \text{Spec } L \rightarrow X$  the canonical morphism and  $\rho_g : X \times_{\text{Spec } K} \text{Spec } L \rightarrow X \times_{\text{Spec } K} \text{Spec } L$*

the naturally twisted morphism induced by  $t_g : \text{Spec } L \rightarrow \text{Spec } L$ . Suppose  $\mathcal{F}$  is a quasi-coherent sheaf over  $X \times_{\text{Spec } K} \text{Spec } L$  with a system  $(\iota_g)_{g \in \text{Gal}(L/K)}$  of isomorphisms  $\iota_g : \rho_g^* \mathcal{F} \rightarrow \mathcal{F}$  satisfying the relation  $\iota_h \circ \rho_h^*(\iota_g) = \iota_{gh}$  for all  $g, h \in \text{Gal}(L/K)$ . Then there exists a quasi-coherent sheaf  $\mathcal{G}$  over  $X$  such that there is an isomorphism of sheaves  $f : \pi^* \mathcal{G} \xrightarrow{\sim} \mathcal{F}$  over  $X \times_{\text{Spec } K} \text{Spec } L$  and for each  $g \in \text{Gal}(L/K)$ , the following diagram

$$\begin{array}{ccc} \rho_g^* \pi^* \mathcal{G} & \xrightarrow{\rho_g^* f} & \rho_g^* \mathcal{F} \\ \iota_g \downarrow & & \downarrow \iota_g \\ \pi^* \mathcal{G} & \xrightarrow{f} & \mathcal{F} \end{array}$$

commutes, where  $\iota_g : \rho_g^* \pi^* \mathcal{G} \cong (\pi \rho_g)^* \mathcal{G} = \pi^* \mathcal{G} \xrightarrow{\text{id}} \pi^* \mathcal{G}$ .

In this case, we say the sheaf  $\mathcal{F}$  over  $X \times_{\text{Spec } K} \text{Spec } L$  descends to the sheaf  $\mathcal{G}$  over  $X$ .

**Proposition 1.1.11. (Galois descent for homomorphisms of algebras)** *Let  $L$  be a field and  $K \subset L$  be a subfield such  $L/K$  is a finite Galois extension. Then to give a homomorphism of (central simple) algebras  $f : A \rightarrow B$  over  $K$  is equivalent to giving a homomorphism of (central simple) algebras  $f_L : A \times_K L \rightarrow B \times_K L$  over  $L$  that is compatible with the  $\text{Gal}(L/K)$ -actions, that is, for each  $g \in \text{Gal}(L/K)$ , the following diagram*

$$\begin{array}{ccc} A \otimes_K L & \xrightarrow{f_L} & B \otimes_K L \\ g \downarrow & & \downarrow g \\ B \otimes_K L & \xrightarrow{f_L} & B \otimes_K L \end{array}$$

commutes.

**Proposition 1.1.12. (Galois descent for morphisms of schemes)** *Let  $L$  be a field and  $K \subset L$  be a subfield such  $L/K$  is a finite Galois extension. Then to give a morphism of  $K$ -schemes  $f : X \rightarrow Y$  is equivalent to giving a morphism of*



$L$ -schemes  $f_L : X \times_{\text{Spec } K} \text{Spec } L \rightarrow Y \times_{\text{Spec } K} \text{Spec } L$  that is compatible with the  $\text{Gal}(L/K)$ -actions, that is, for each  $g \in \text{Gal}(L/K)$ , the following diagram

$$\begin{array}{ccc} X \times_K \text{Spec } L & \xrightarrow{f_L} & Y \times_K \text{Spec } L \\ g \downarrow & & \downarrow g \\ X \times_K \text{Spec } L & \xrightarrow{f_L} & Y \times_K \text{Spec } L \end{array}$$

commutes.

### 1.1.3 Twisted Forms and the First Cohomology Set

We will first establish a relationship between the  $L/K$ -forms of an object (algebra, scheme, ...) over a field  $K$ , where  $L/K$  is a finite Galois extension, and a Galois cohomology set, and then sketch its generalization, that is, a relationship between twisted forms of an object (scheme, sheaf of modules, algebras ...) over a scheme  $X$  and a first Čech cohomology set over  $X$ .

For the first part, the main references are Serre's two books [Se1] and [Se2]. We begin with a new description of the first cohomology set  $H^1(G, A)$ .

**Definition 1.1.13.** Let  $G$  be a finite group. Let  $E$  be a left  $G$ -set and  $A$  be a  $G$ -group. We say  $E$  is a *principal homogeneous space* (or *torsor*) over  $A$  if there is a right  $A$ -action on  $E$

$$\begin{aligned} E \times A &\longrightarrow E \\ (x, a) &\longmapsto x \cdot a \end{aligned}$$

such that

- (i) for each pair  $x, y \in E$ , there exists a unique  $a \in A$  such that  $y = x \cdot a$ ;
- (ii) the action is  $G$ -equivariant, i.e.  ${}^g(x \cdot a) = {}^g x \cdot {}^g a$  for every  $g \in G, x \in E$  and  $a \in A$ .

Denote by  $\mathbf{PHS}(A)$  the set of isomorphism classes of principal homogeneous spaces over  $A$ . Let  $E \in \mathbf{PHS}(A)$  and  $e \in E$ , then for each  $g \in G$ , there exists  $c_g \in A$  such that  ${}^g e = e \cdot c_g$ . Thus  $e$  determines a map  $\mathbf{c} : G \rightarrow A$ . Moreover, for  $g, h \in G$ , we have

$$e \cdot c_{gh} = {}^{gh} e = {}^g({}^h e) = {}^g(e \cdot c_h) = {}^g e \cdot {}^g c_h = e \cdot c_g \cdot {}^g c_h.$$

Thus  $c_{gh} = c_g \cdot {}^g c_h$ , so that  $\mathbf{c}$  is a cocycle. If we choose a different  $e' \in E$ , then we have  $e' = e \cdot \beta$  for some  $\beta \in A$ . Let  $\mathbf{c}' : G \rightarrow A$  be the cocycle corresponding to  $e'$ . Then for every  $g \in G$ , we have

$$e' \cdot c'_g = {}^g e' = {}^g(e \cdot \beta) = {}^g e \cdot {}^g \beta = e \cdot c_g \cdot {}^g \beta = e' \cdot \beta^{-1} \cdot c_g \cdot {}^g \beta,$$

thus  $c'_g = \beta^{-1} \cdot c_g \cdot {}^g \beta$ , that is,  $\mathbf{c}'$  and  $\mathbf{c}$  are cohomologous cocycles. Hence the association  $E \mapsto \mathbf{c}$  defines a map

$$\theta : \mathbf{PHS}(A) \longrightarrow H^1(G, A).$$

On the other hand, let  $\mathbf{c} : G \rightarrow A$  be a cocycle, denote by  $P_{\mathbf{c}}$  the set of group  $A$  on which  $G$  acts by the following formula:  ${}^g a := c_g \cdot {}^g a$ . Let  $A$  act on the right on  $P_{\mathbf{c}}$  by translations, then  $P_{\mathbf{c}}$  is a principal homogeneous space over  $A$ . If  $\mathbf{c}'$  is a cocycle cohomologous to  $\mathbf{c}$  such that  $c'_g = b^{-1} \cdot c_g \cdot {}^g b$  for some  $b \in A$ , then the map  $f : P_{\mathbf{c}'} \rightarrow P_{\mathbf{c}}, a \mapsto b \cdot a$ , is an isomorphism. Thus the association  $\mathbf{c} \mapsto P_{\mathbf{c}}$  defines a map

$$\eta : H^1(G, A) \longrightarrow \mathbf{PHS}(A).$$

One checks easily that  $\theta \circ \eta = 1$  and  $\eta \circ \theta = 1$ , and we get

**Proposition 1.1.14.** *[Se2, Chapter I §5 Proposition 33] The map  $\theta$  is a bijection.*

Now let  $K$  be a field, and  $L/K$  be a finite Galois extension with Galois group  $\text{Gal}(L/K)$ . For a  $\text{Gal}(L/K)$ -group  $A$ , write  $H^1(L/K, A)$  for the cohomology set  $H^1(\text{Gal}(L/K), A)$ , the *Galois cohomology set* of  $L/K$  with coefficients in  $A$ .

**Definition 1.1.15.** Let  $K$  be a field, and  $L/K$  be a finite Galois extension. Let  $X$  be a scheme (algebra, central algebra...) over  $K$ . We say another object  $Y$  over  $K$  of the same type is a  $L/K$ -form of  $X$  if after a field extension  $L/K$  they become isomorphic, i.e.  $X_L \simeq Y_L$ . Denote by  $E(L/K, X)$  the set of classes of  $L/K$ -forms of  $X$  under the equivalence relation defined by the  $K$ -isomorphisms.

Denote by  $Aut_L(X_L)$  the group of  $L$ -automorphisms of  $X_L$ . Let us define a natural action of  $Gal(L/K)$  on  $Aut_L(X_L)$  first. If  $g \in Gal(L/K)$  and  $\alpha \in Aut_L(X_L)$ , define  ${}^g\alpha := g \cdot \alpha \cdot g^{-1}$ , where  $g$  acts in the usual way on  $X_L$  and the product is simply composition of maps. We thus have a  $Gal(L/K)$ -group structure on  $Aut_L(X_L)$ , and the cohomology set  $H^1(L/K, Aut_L(X_L))$  is well defined.

Let  $Y$  be a  $L/K$ -form of  $X$  and denote by  $Iso_L(X_L, Y_L)$  the set of  $L$ -isomorphisms  $\varphi : X_L \xrightarrow{\sim} Y_L$ . Then  $Gal(L/K)$  naturally acts on  $Iso_L(X_L, Y_L)$ . If  $g \in Gal(L/K)$  and  $\varphi \in Iso_L(X_L, Y_L)$ , then define  ${}^g\varphi := g \cdot \varphi \cdot g^{-1}$ . Thus  $Iso_L(X_L, Y_L)$  is a  $Gal(L/K)$ -set. For  $\alpha \in Aut_L(X_L)$  and  $\varphi \in Iso_L(X_L, Y_L)$ , the composition  $\varphi \cdot \alpha$  gives a transitive and faithful right action of  $Aut_L(X_L)$  on  $Iso_L(X_L, Y_L)$ , that is,  $Iso_L(X_L, Y_L)$  is a principal homogeneous space over  $Gal(L/K)$ -group  $Aut_L(X_L)$ . Further, two isomorphic  $L/K$ -forms of  $X$  give two  $L$ -isomorphic principal homogeneous spaces over  $Gal(L/K)$ -group  $Aut_L(X_L)$ . Hence we have a map

$$\theta_{L/K} : E(L/K, X) \rightarrow H^1(L/K, Aut_L(X_L)).$$

**Proposition 1.1.16.** *The map  $\theta_{L/K}$  is injective. Moreover, when  $X$  is a (central simple) algebra, or  $X$  is a quasiprojective scheme,  $\theta_{L/K}$  is also surjective.*

*Proof.* For the injection, let  $Y, Y' \in E(L/K, X)$ , and suppose that  $\varphi \in Iso_L(X_L, Y_L)$  and  $\varphi' \in Iso_L(X_L, Y'_L)$  determine the same cocycle. Then we have  $\varphi^{-1}g\varphi = \varphi'^{-1}g\varphi'$  for every  $g \in Gal(L/K)$ . Consider the isomorphism  $\varphi\varphi'^{-1} : Y'_L \rightarrow Y_L$ , we have  ${}^g(\varphi\varphi'^{-1}) = \varphi\varphi'^{-1}$ . Then by Proposition 1.1.12 for morphism of schemes, or by

Proposition 1.1.11 for homomorphism of (central simple) algebras,  $\varphi\varphi'^{-1}$  is actually an isomorphism  $Y' \rightarrow Y$ .

For the surjection, let  $\mathbf{c}$  be a cocycle of  $Gal(L/K)$  in  $Aut_L(X_L)$ , and for each  $g \in Gal(L/K)$ , let  $g$  act on  $X_L$  by  $c_g \cdot g$  (here  $g : Spec L \rightarrow Spec L$  is given by  $g^{-1} : L \rightarrow L$  when  $X$  is a scheme), then for any  $g, h \in Gal(L/K)$ , we have

$$(c_g \cdot g) \cdot (c_h \cdot h) = c_g \cdot g \cdot c_h \cdot g^{-1} \cdot g \cdot h = c_g \cdot {}^g c_h \cdot gh = c_{gh} \cdot gh,$$

which shows this defines a left twisted action of  $Gal(L/K)$  on  $X_L$ . Then by Proposition 1.1.8 for  $X$  a (central simple) algebra over  $K$  and by Proposition 1.1.9 for  $X$  a quasiprojective scheme over  $K$ , there exists an object  $Y$  over  $K$  of the same type and an isomorphism  $f : X_L \rightarrow Y_L$  such that these twisted actions are induced by  $f$ . And the isomorphism  $f \in Iso_L(X_L, Y_L)$  determines the cocycle  $\mathbf{c}$ . Hence  $\theta_{L/K}$  is surjective when  $X$  is a (central simple) algebra, or  $X$  is a quasiprojective scheme. □

**Definition 1.1.17.** Let  $K$  be a field and  $X$  be a scheme (algebra, central algebra...) over  $K$ . We say another object  $Y$  over  $K$  of the same type is a  $K$ -form of  $X$  if after some finite Galois field extension  $L/K$  they become isomorphic, i.e.  $X_L \simeq Y_L$ . Denote by  $E(K, X)$  the set of classes of  $K$ -forms of  $X$  under the equivalence relation defined by the  $K$ -isomorphisms.

We have  $E(K, X) = \bigcup_{L/K} E(L/K, X)$  with  $L/K$  all finite Galois field extension. Moreover, if  $L'/L$  is a finite field extension such that  $L'/K$  is also a Galois extension, then the natural inclusion  $E(L/K, X) \subset E(L'/K, X)$  is compatible with the inflation map  $inf_{Gal(L/K)}^{Gal(L'/K)} : H^1(Gal(L/K), Aut_L(X_L)) \rightarrow H^1(Gal(L'/K), Aut_{L'}(X_{L'}))$ , that

is, the following diagram

$$\begin{array}{ccc}
 E(L/K, X) & \xrightarrow{\theta_{L/K}} & H^1(\text{Gal}(L/K), \text{Aut}_L(X_L)) \\
 \text{nat.incl.} \downarrow & & \downarrow \text{inj}_{\text{Gal}(L/K)}^{\text{Gal}(L'/K)} \\
 E(L'/X, X) & \xrightarrow{\theta_{L'/K}} & H^1(\text{Gal}(L'/K), \text{Aut}_{L'}(X_{L'}))
 \end{array}$$

commutes. Thus there is a unique natural map

$$\theta_K : E(K, X) \rightarrow H^1(\text{Gal}(K^{\text{sep}}/K), \text{Aut}_{K^{\text{sep}}}(X_{K^{\text{sep}}})) .$$

By Proposition 1.1.16, we can easily get

**Corollary 1.1.18.** *The map  $\theta_K$  is injective. Moreover, when  $X$  is a (central simple) algebra, or  $X$  is a quasiprojective scheme,  $\theta_K$  is also surjective.*

For the second part, the main reference is [Mi1, Chapter III, §4].

**Definitions 1.1.19.** Let  $Y$  be a scheme (sheaf of module, algebra ...) over a scheme  $X$ . We say another object  $Y'$  of the same type over  $X$  is a *twisted form* of  $Y$  for the étale topology on  $X$  if there exists a covering  $\mathcal{U} = (U_i \rightarrow X)$  for the étale topology such that  $Y \times_X U_i \simeq Y' \times_X U_i$  for all  $i$ . In this case, we say the covering  $\mathcal{U}$  *splits* the twisted form  $Y'$  of  $Y$ .

Denote by  $E(\mathcal{U}/X, Y)$  the set of all isomorphism classes of twisted forms of  $Y$  over  $X$  which are split by the étale cover  $\mathcal{U}$  of  $X$ .

Let  $\mathcal{A}ut(Y)$  be the sheaf of groups associated with the presheaf  $U \mapsto \text{Aut}_U(Y \times_X U)$ . Now suppose  $Y' \in E(\mathcal{U}/X, Y)$ , then there exists a étale covering  $(U_i \rightarrow X)$  of  $X$  such that we have a family of isomorphisms  $(Y \times_X U_i \xrightarrow{\phi_i} Y' \times_X U_i)$ . Setting  $\alpha_{ij} = \phi_i^{-1} \phi_j$  (omitting the restriction signs), then  $(\alpha_{ij})$  is a cocycle for  $\mathcal{U}$  with values in  $\mathcal{A}ut(Y)$ . Thus  $Y'$  determines an element, say  $\theta_{\mathcal{U}}(Y')$ , in  $\check{H}^1(\mathcal{U}/X, \mathcal{A}ut(Y))$ . It is easy to see that if  $Y''$  is another twisted form of  $Y$  which is isomorphic to  $Y'$  over

$X$  and split by the covering  $\mathcal{U}$  of  $X$ , then  $\theta_{\mathcal{U}}(Y'') = \theta_{\mathcal{U}}(Y')$ . Hence the association  $Y' \mapsto \theta(Y')$  defines a map

$$\theta_{\mathcal{U}} : E(\mathcal{U}/X, Y) \rightarrow \check{H}^1(\mathcal{U}/X, \mathcal{A}ut(Y)),$$

where the isomorphism class of  $Y$  maps to the distinguished element.

Denote  $U = \bigsqcup U_i$ , then every element, say  $(\phi_{ij})$ , in  $\check{H}^1(\mathcal{U}/X, \mathcal{A}ut(Y))$  defines a descent datum on  $Y \times U$  for the morphism  $U = \bigsqcup U_i \rightarrow X$ . Thus we have the following:

**Proposition 1.1.20.** *The map  $\theta_{\mathcal{U}}$  defined above is injective, and it is surjective if every descent datum on  $Y \times U$  arises from a twisted form.*

Similarly, we denote by  $E(X_{\acute{e}t}, Y)$  the set of all the isomorphic classes of twisted forms of  $Y$  over  $X$  under the étale topology, then we have a map

$$\theta : E(X_{\acute{e}t}, Y) \rightarrow \check{H}^1(X_{\acute{e}t}, \mathcal{A}ut(Y)).$$

**Corollary 1.1.21.** *The map  $\theta$  defined above is injective. Moreover, it is surjective if every descent datum on some étale covering of  $X_{\acute{e}t}$  arises from a twisted form.*

## 1.2 Brauer Groups and Brauer-Severi Schemes

### 1.2.1 Brauer Groups of a Field

Throughout this dissertation, an algebra over a field  $K$  is always assumed to have an unit together with a copy of  $K$  in the center of  $A$ .

**Definition 1.2.1.** Let  $K$  be a field and  $A$  be a finite dimensional  $K$ -algebra. We call  $A$  *simple* if it has no nontrivial two sided ideals. Moreover, we call it a *central simple algebra over  $K$*  if its center equals  $K$ .

**Example 1.2.2.** (1) The matrix algebra  $M_n(K)$  is obviously a central simple algebra over  $K$ .

(2) A finite dimensional division algebra  $D$  over  $K$  is a central simple algebra. Indeed, denote by  $Z(D)$  the centre of  $D$ . For any  $x \in D, y \in Z(D)$ , inverting the relation  $xy = yx$ , we get  $y^{-1}x^{-1} = x^{-1}y^{-1}$ . Hence  $Z(D)$  is a field and  $D$  is a central simple algebra over  $Z(D)$ .

We will state the main theorem on simple algebras over a field which says that every finite dimensional central algebra over  $K$  is a matrix algebra over a division algebra over  $K$ . But before that, let us recall some facts from module theory.

Let  $R$  be a ring. Recall that an  $R$ -module  $M$  is *Artinian* if all descending  $R$ -submodule chains in  $M$  stabilize. If  $R$  is a *simple ring*, then it has no nontrivial two sided ideals.

**Lemma 1.2.3.** [Dr] Let  $R$  be a simple ring and  $I$  be a minimal left ideal of  $R$ . Assume that  $M$  is a left Artinian  $R$ -module, then we have an isomorphism  $M \simeq \bigoplus_{i=1}^n I$  of left  $R$ -modules for some integer  $n$ .

*Proof.* Consider the two sided ideal of  $R$  generated by  $I : I' = \sum_{r \in R} Ir$ . Since  $R$  is simple, we have  $I' = R$ . Thus we have

$$M = \sum_{m \in M} Rm = \sum_{m \in M} I'm = \sum_{m \in M} \sum_{r \in R} Irm = \sum_{m \in M} Im.$$

Since  $M$  is an Artinian left  $R$ -module, there exist some minimal integer  $n$  and  $m_i \in M, 1 \leq i \leq n$  such that

$$M = \sum_{m \in M} Im = \sum_{i=1}^n Im_i.$$

So we can use this to construct a surjective left  $R$ -module homomorphism

$$h : \bigoplus_{i=1}^n I \rightarrow M, \quad (r_1, r_2, \dots, r_n) \mapsto \sum_{i=1}^n r_i m_i.$$

We claim  $h$  is also injective. Assume  $\sum_{i=1}^n r_i m_i = 0$  and  $r_1 \neq 0$ , then we have  $Rr_1 = I$  since  $I$  is minimal. Therefore we have  $Im_1 = Rr_1 m_1 \subseteq \sum_{i=2}^n Rr_i m_i \subseteq \sum_{i=2}^n Im_i$ , contradicting the minimal choice of  $n$ . Hence  $h$  is an isomorphism and  $M \simeq \bigoplus_{i=1}^n I$ . □

By Lemma 1.2.3, we can readily get the following two corollaries.

**Corollary 1.2.4.** *[Dr] All minimal left ideals of a simple ring are isomorphic.*

**Corollary 1.2.5.** *[Dr] Every left ideal of a simple ring is a direct sum of minimal left ideals.*

**Example 1.2.6.** Let us describe the simple left modules over the full matrix ring  $M_n(D)$ , where  $D$  is a division algebra. Denote by  $E_{ij}, 1 \leq i, j \leq n$ , the matrix with  $(i, j)$ -entry 1 and zero elsewhere. Then each element of  $M_n(D)$  is a  $D$ -linear combination of the  $E_{ij}$ . We claim that  $M_n(D)$  is simple. Indeed, let  $I$  be a two sided ideal of  $M_n(D)$ . It suffices to show that  $E_{ij} \in I$  for all  $1 \leq i, j \leq n$ . Notice we



have relation  $E_{ki}E_{ij}E_{jl} = E_{kl}$ , so it is enough to show that  $E_{ij} \in I$  for some  $i, j$ . Let  $0 \neq M = [m_{ij}] \in I$ , and suppose  $m_{ij} \neq 0$ , then we have  $m_{ij}^{-1}E_{ii}ME_{jj} = E_{ij}$ . Thus  $M_n(D)$  is a simple ring. Now for  $1 \leq l \leq n$ , consider the subring  $I_l \subset M_n(D)$ , which consists all the matrices  $M = [m_{ij}]$  with  $m_{ij} = 0$  for  $j \neq l$ . Obviously, these are left ideals of  $M_n(D)$ . Using a similar argument as above with matrices  $E_{ij}$ , we can show that they are also minimal left ideals. Note also that all of them are isomorphic to  $D^n$  as left  $M_n(D)$ -ideals.

Let  $R$  be a ring and  $M$  be a left  $R$ -module. An *endomorphism* of  $M$  is an  $R$ -homomorphism  $M \rightarrow M$ . It is easy to show that the set of all the endomorphisms of  $M$ ,  $End_R(M)$ , forms a ring, with multiplication given by composition of homomorphisms and addition by the rule  $(\phi + \psi)(m) = \phi(m) + \psi(m)$ , where  $\phi, \psi \in End_R(M)$  and  $m \in M$ . With this ring structure, the module  $M$  can be regarded as a left module over  $End_R(M)$ , with multiplication given by the rule  $\phi \cdot m = \phi(m)$  for  $m \in M, \phi \in End_R(M)$ . In particular, when  $A$  is an algebra over a field  $K$ ,  $End_R(M)$  is a  $K$ -algebra, too. Multiplication by an element of  $K$  defines an element in the centre of  $End_R(M)$ .

Recall that a non-zero  $R$ -module is *simple* if it has no nontrivial  $R$ -submodules.

**Lemma 1.2.7.** [GS](Schur) *Let  $M$  be a simple module over a  $K$ -algebra  $A$ . Then the set of endomorphisms  $End_A(M)$  is a division algebra.*

*Proof.* As mentioned above,  $End_A(M)$  is an algebra. To prove it is a division algebra, it suffices to show that every element of  $End_A(M)$  has an inverse. Let  $\phi : M \rightarrow M$  be a non-zero endomorphism. Consider the kernel  $ker(\phi)$ , which is an  $A$ -submodule. Since  $ker(\phi) \neq M$ , we have  $ker(\phi) = 0$ . Similarly, its image must equal to  $M$ . Thus  $\phi$  is an isomorphism, which means it has an inverse in  $End_A(M)$ .

□

Now let  $M$  be a left  $R$ -module and denote by  $R'$  the endomorphism ring. As mentioned above,  $M$  is naturally a left  $R'$ -module, too. So we can consider a new endomorphism ring  $End_{R'}(M)$ . We define a ring homomorphism:  $\lambda_M : R \rightarrow End_{R'}(M)$  by sending  $r \in R$  to the endomorphism in  $End_{R'}(M)$  which sends  $m \in M$  to  $rm$ . We claim this map is a ring homomorphism. Indeed, let  $\phi : M \rightarrow M$  be a homomorphism in  $R'$ , we have  $\phi \cdot \lambda_M(r)(m) = \phi(rm) = r\phi(m) = \lambda_M(r) \cdot \phi(m)$  for all  $m \in M$ . Thus  $\phi \cdot \lambda_M(r) = \lambda_M(r) \cdot \phi$  and  $\lambda_M(r)$  is indeed a  $R'$ -homomorphism.

**Lemma 1.2.8.** *[GS](Rieffel) Let  $R$  be a simple algebra and  $L$  be a non-zero left ideal of  $R$ . Denote by  $R' = End_R(L)$ . Then the ring homomorphism  $\lambda_L : R \rightarrow End_{R'}(L)$  defined above is an isomorphism.*

*Proof.* Obviously,  $\lambda_L$  is injective, since it is a non-zero map and its kernel is a two sided ideal of  $R$ , while  $R$  is a simple algebra.

To show that it is surjective, we prove first that  $\lambda_L(L)$  is a left ideal of  $End_{R'}(L)$ . Let  $\phi \in End_{R'}(L)$  and  $l \in L$ , then the composition  $\phi \cdot \lambda_L(l)$  is the homomorphism which maps  $x \in L$  to  $\phi(lx)$ . On the other hand, for any  $x \in L$ , observe the map  $\iota_x : L \rightarrow L, l \mapsto lx$ . It is easy to see that  $\iota_x$  is an  $R$ -endomorphism of  $L$ , that is,  $\iota_x \in R'$ . As  $\phi$  is a  $R'$ -endomorphism of  $L$ , we have  $\phi(lx) = \phi(\iota_x(l)) = \iota_x \cdot \phi(l) = \phi(l)x$ . So we have  $\phi \cdot \lambda_L(l) = \lambda_L(\phi(l))$ , which is an element in the image  $\lambda_L(L)$ . Thus  $\lambda_L(L)$  is a left ideal of  $End_{R'}(L)$ . Now consider the two sided ideal  $LR$  of  $R$ . Obviously  $LR \neq 0$ , thus  $LR = R$ . In particular, we have  $1 = \sum l_i r_i$  for some  $l_i \in L$  and  $r_i \in R$ . So for  $\phi \in End_{R'}(L)$ , we have  $\phi = \phi \cdot 1 = \phi \cdot \lambda_L(1) = \phi \cdot \sum \lambda_L(l_i r_i) = \sum \phi \cdot \lambda_L(l_i) \cdot \lambda_L(r_i) = \sum \lambda_L(\phi(l_i)) \cdot \lambda_L(r_i)$ . Hence  $\phi \in \lambda_L(R)$ . □

Now we state our main theorem:

**Theorem 1.2.9.** *[GS](Wedderburn's Theorem) Let  $A$  be a central simple algebra*

over a field  $K$  of finite rank, then

$$A \simeq M_n(D)$$

for some integer  $n \geq 1$  and some central division  $K$ -algebra  $D$ . Moreover, the division algebra  $D$  is uniquely determined up to isomorphism.

*Proof.* Since  $A$  is of finite dimension over  $K$ , every descending chain of left ideals of  $A$  must stabilize eventually. So we can assume  $L$  is a minimal left ideal of  $A$ , then it is a simple  $A$ -module. Denote  $D = \text{End}_A(L)$ . By Lemma 1.2.7,  $D$  is a division algebra. And by Lemma 1.2.8, we have an isomorphism  $A \simeq \text{End}_D(L)$ . As  $L$  is also a left  $D$ -module and  $D$  is a division algebra, we can regard  $L$  as a left vector space over  $D$ . Let  $n$  be the dimension of  $L$  over  $D$ . After choosing a basis of  $L$ , then every endomorphism of  $L$  over  $D$  can be represented as a matrix in  $M_n(D)$ ; conversely, every matrix in  $M_n(D)$  can be realized as an endomorphism of  $L$  over  $D$ . So we have an isomorphism  $\text{End}_D(L) \simeq M_n(D)$ .

Next we show that  $D$  is unique up to isomorphism. Assume that  $D'$  is another division algebra such that  $A \simeq M_m(D')$  for some integer  $m$ . Since  $L$  is a minimal left ideal of  $A$ , as shown in Example 1.2.6, we have isomorphisms  $L \simeq D^n$  and  $L \simeq D'^m$ . Thus we have isomorphisms  $D \simeq \text{End}_A(D^n) \simeq \text{End}_A(L) \simeq \text{End}_A(D'^m) \simeq D'$ .  $\square$

**Remark 1.2.10.** The converse is also true. If  $D$  is a central division algebra over a field  $K$ , then the matrix algebra  $M_n(D)$  is a central simple algebra over  $K$ . Since the matrix algebra  $M_n(D)$  is simple and its centre is same as the centre of  $D$ .

Let  $A$  and  $B$  be two central simple algebras over a field  $K$  with  $A \simeq M_n(D)$  and  $B \simeq M_m(D')$ , where  $D$  and  $D'$  are two division algebras over  $K$ . We say  $A$  is equivalent to  $B$  if the corresponding division algebras are isomorphic to each other, i.e.  $D \simeq D'$ . This is an equivalence relation and we denote by  $Br(K)$  the set of equivalence classes of central simple  $K$ -algebras.

Let  $A, B$  be two algebras over a field  $K$ , denote by  $A \otimes_K B$  the algebra over  $K$  with multiplication  $(a \otimes b) \cdot (a' \otimes b') = aa' \otimes bb'$  for all  $a, a' \in A$  and  $b, b' \in B$ . We know that every element in  $A \otimes_K B$  has the form  $a_1 \otimes b_1 + a_2 \otimes b_2 + \cdots + a_n \otimes b_n$  for some integer  $n$  with all  $a_i \in A, b_i \in B, 1 \leq i \leq n$ .

**Lemma 1.2.11.** *[Te] Let  $A, B$  be two finite dimensional algebras over a field  $K$ , then we have  $Z(A \otimes_K B) = Z(A) \otimes_K Z(B)$ . Moreover, if both  $A$  and  $B$  are simple and  $Z(A) = K$ , then  $A \otimes_K B$  is also simple with centre  $Z(B)$ .*

*Proof.* Clearly we have  $Z(A) \otimes_K Z(B) \subseteq Z(A \otimes_K B)$ . Conversely, since  $B$  is finite dimensional over  $K$ , suppose its dimension is  $n$  and let  $w_1, w_2, \dots, w_n$  be a basis of  $B$  over  $K$ . Then we have

$$A \otimes_K B = A \otimes_K (\oplus_{i=1}^n K w_i) = \oplus_{i=1}^n A w_i$$

as  $K$ -vector spaces, and any element  $c \in A \otimes_K B$  can be uniquely represented as  $c = a_1 \otimes w_1 + a_2 \otimes w_2 + \cdots + a_n \otimes w_n$ , where  $a_i \in A$  for  $1 \leq i \leq n$ . In particular, let  $c \in Z(A \otimes_K B)$ , we have

$$(a \otimes 1) \cdot c = c \cdot (a \otimes 1) \iff$$

$$aa_1 \otimes w_1 + aa_2 \otimes w_2 + \cdots + aa_n \otimes w_n = a_1 a \otimes w_1 + a_2 a \otimes w_2 + \cdots + a_n a \otimes w_n$$

for any  $a \in A$ , thus by the uniqueness of the above representation, we have  $aa_i = a_i a$  for  $1 \leq i \leq n$ , that is,  $a_i \in Z(A)$  for all  $i$ . So we have  $c \in Z(A) \otimes_K B \subseteq A \otimes_K B$ . Similarly, by switching the roles of the first and second entries, we can get  $c \in Z(A) \otimes_K Z(B) \subseteq Z(A) \otimes_K B$ . Therefore, we have  $Z(A \otimes_K B) = Z(A) \otimes_K Z(B)$ .

Now assume that both  $A$  and  $B$  are simple and  $Z(A) = K$ . Let  $I$  be a non-zero two sided ideal of  $A \otimes_K B$  and let  $0 \neq c = a_1 \otimes b_1 + a_2 \otimes b_2 + \cdots + a_n \otimes b_n \in I$  be an element with smallest  $n$ . If  $n = 1$ ,  $c = a \otimes b, a \neq 0, b \neq 0$ . Since  $A$  and  $B$  are simple, the two sided ideals generated by  $a$  in  $A$  and  $b$  in  $B$  equal to  $A$  and  $B$ , respectively. That is, there exist  $a_i, a'_i \in A$  for  $1 \leq i \leq m$  and  $b_j, b'_j \in B$  for  $1 \leq j \leq l$  such that

$\sum_{i=1}^m a_i a a'_i = 1$  and  $\sum_{j=1}^l b_j b b'_j = 1$ . So we have

$$\sum_{j=1}^l (1 \otimes b_j) \cdot \left( \sum_{i=1}^m (a_i \otimes 1) \cdot (a \otimes b) \cdot (a'_i \otimes 1) \right) \cdot (1 \otimes b'_j) = \left( \sum_{i=1}^m a_i a a'_i \right) \otimes \left( \sum_{j=1}^l b_j b b'_j \right) = 1 \otimes 1.$$

Thus  $1 \otimes 1 \in I$ , and hence  $I = A \otimes_K B$ . If  $n > 1$ , without loss of generality, we can assume the  $a_i$  (and  $b_j$ ) are linearly independent over  $K$ . Indeed, if, say,  $a_n = \sum_{i=1}^{n-1} \lambda_i a_i$ , then  $c = \sum_{i=1}^n a_i \otimes b_i = \sum_{i=1}^{n-1} a_i \otimes (b_i + \lambda_i b_n)$ , contradicting the minimality of  $n$ . With the same argument as above, we can assume  $a_1 = 1$ , then  $a_2 \notin K$ , otherwise  $a_1$  and  $a_2$  would be linearly dependent over  $K$ . Since  $Z(A) = K$ , there exist  $a \in A$  such that  $aa_2 \neq a_2a$ . Now consider the element

$$(a \otimes 1) \cdot c - c \cdot (a \otimes 1) = (aa_2 - a_2a) \otimes b_2 + \cdots + (aa_n - a_n a) \otimes b_n \in I.$$

Since  $aa_2 - a_2a \neq 0$  and the  $b_i$  are linearly independent over  $K$ , we have  $(a \otimes 1) \cdot c - c \cdot (a \otimes 1) \neq 0$ , which contradicts the minimality of  $n$ . Therefore we must have  $n = 1$ . So  $A \otimes_K B$  is a simple algebra with centre  $Z(A) \otimes_K Z(B) = Z(B)$ .  $\square$

From the above lemma, we can easily get the following corollary.

**Corollary 1.2.12.** *Let  $A$  and  $B$  be two central simple algebras over a field  $K$ , then  $A \otimes_K B$  is also a central simple algebra over  $K$ .*

The opposite algebra of an algebra  $A$  over a field  $K$ , which we denote by  $A^\circ$ , is the  $K$ -algebra which has the same underlying set and addition as  $A$ , but multiplication is defined as  $a \cdot b = ba$  for all  $a, b \in A^\circ$ . It is easy to see that  $(A^\circ)^\circ = A$  and  $A^\circ = A$  if and only if  $A$  is a commutative algebra.

**Lemma 1.2.13.** *Let  $A$  be a central simple algebra over a field  $K$ , let  $A^\circ$  be the opposite algebra of  $A$ , then  $A^\circ$  is also a central simple algebra over  $K$ , and  $A \otimes_K A^\circ \simeq M_n(K)$  for some  $n$ .*

*Proof.* It is obvious that the centre of  $A^\circ$  is  $K$ . Let  $I$  be a nontrivial two-sided ideal of  $A^\circ$ . Assume  $0 \neq a \in I$ , we have  $A \cdot a \cdot A = AaA$  as sets, but  $AaA = A$  since  $A$  is simple. Thus  $I = A^\circ$ ,  $A^\circ$  is simple.

Now considering a  $K$ -linear map  $A \otimes_K A^\circ \rightarrow \text{End}_{K\text{-mod}}(A)$  by sending  $a \otimes b$  to the map  $\phi \in \text{End}_{K\text{-mod}}(A)$  with  $\phi(x) = axb$  for all  $x \in A$ . This map is clearly nonzero, and thus injective, because  $A^\circ$  is simple by the above argument, and  $A \otimes_K A^\circ$  is also simple by Corollary 1.2.12. But both  $A \otimes_K A^\circ$  and  $\text{End}_{K\text{-mod}}(A)$  have the same dimension as  $K$ -vector spaces, hence this is an isomorphism. On the other hand, we have  $\text{End}_{K\text{-mod}}(A) \simeq M_n(K)$ , where  $n$  is the dimension of  $A$  over  $K$ . Therefore we have  $A \otimes_K A^\circ \simeq M_n(K)$ .  $\square$

Now, we can define an abelian group structure on the set  $Br(K)$ . Define

$$[A] \cdot [A'] = [A \otimes_K A'].$$

By Corollary 1.2.12 and Lemma 1.2.13, this is well-defined. We call  $Br(K)$  the *Brauer group of  $K$* .

**Corollary 1.2.14.** *Let  $K$  be an algebraically closed field, then  $Br(K) = 0$ .*

*Proof.* By Wedderburn's theorem, it suffices to show that there is no finite dimensional division algebra over  $K$  other than  $K$  itself.

Indeed, if  $D$  is a division algebra other than  $K$ , choose  $d \in D \setminus K$  and consider the set  $S = \{1, d, d^2, \dots\}$ . Since  $D$  is of finite dimension over  $K$ ,  $S$  is linearly dependent, so there exists a polynomial  $f(x)$  in  $K[x]$  such that  $f(d) = 0$ . As  $D$  is a division algebra, it has no zero divisors and we may assume  $f$  is irreducible. Thus we get a  $K$ -algebra homomorphism  $K[x]/f(x) \rightarrow D$  whose image contains  $d$ . But since  $K$  is algebraically closed, we have  $K[x]/(f(x)) \cong K$ , which contradicts  $d \in D \setminus K$ .  $\square$

Next, we will discuss central simple algebras under field extension.

**Theorem 1.2.15.** *[GS] Let  $A$  be an algebra of finite rank over a field  $K$ , then  $A$  is a central simple algebra if and only if there exists a finite Galois field extension  $L/K$  such that  $A \otimes_K L$  is isomorphic to the matrix ring  $M_n(L)$  for some integer  $n$ .*

*Proof.* For sufficiency, if  $I$  is a nontrivial two-sided ideal of  $A$ , then  $I \otimes_K L$  is also a nontrivial two-sided ideal of  $A \otimes_K L \simeq M_n(L)$ , which is a contradiction, since  $A \otimes_K L$  is simple by Lemma 1.2.11. Again, by Lemma 1.2.11, we have  $L = Z(A \otimes_K L) = Z(A) \otimes_K L$ , thus  $Z(A) = K$ . Hence  $A$  is a central simple algebra over  $K$ .

Conversely, let  $A$  be a central simple algebra over  $K$ , we show first that there exists a finite field extension  $K'/K$  such that  $A \otimes_K K' \simeq M_n(K')$  for some  $n \geq 1$ . Indeed, denote by  $\bar{K}$  the algebraic closure of  $K$ , then by Corollary 1.2.14, we have  $A \otimes_K (\bar{K}) \simeq M_n(\bar{K})$  for some  $n \geq 1$ . Now observe that every finite field extension  $K'$  of  $K$  is contained in  $\bar{K}$ , and the inclusion  $K' \subset \bar{K}$  induces an injective map  $A \otimes_K K' \hookrightarrow A \otimes_K \bar{K}$  and  $A \otimes_K \bar{K}$  is the union of the  $A \otimes_K K'$  in this way. Let  $e_1, e_2, \dots, e_{n^2} \in A \otimes_K \bar{K}$  be the elements which correspond to the standard basis element in  $M_n(\bar{K})$  via the isomorphism  $A \otimes_K \bar{K} \simeq M_n(\bar{K})$ . Assume that  $e_i e_j = \sum_{k=1}^{n^2} a_{ijk} e_k$  for  $1 \leq i, j \leq n^2$ . Since the set  $\{e_i, a_{ijk} : 1 \leq i, j, k \leq n^2\}$  is finite, there exists a sufficient large finite field extension  $K'/K$  such that  $A \otimes_K K'$  contains  $\{e_i : 1 \leq i \leq n^2\}$  and  $K'$  contains  $\{a_{ijk} : 1 \leq i, j, k \leq n^2\}$ . Mapping the  $e_i$  to the standard basis elements of  $M_n(K')$  we have an isomorphism  $A \otimes_K K' \simeq M_n(K')$ .

Next we show that we can choose a finite separable extension  $K'/K$  such that  $A \otimes_K K' \simeq M_n(K')$ . Otherwise, by the same argument as above, we have  $A \otimes_K K^{sep} \neq 0$  as an element in  $Br(K^{sep})$ , where  $K^{sep}$  is the separable closure of  $K$ . Thus, by Theorem 1.2.9,  $A \otimes_K K^{sep} \simeq M_n(D)$ , where  $D$  is a central division algebra over  $K^{sep}$  different from  $K^{sep}$  and  $n$  is some integer. Then by Corollary 1.2.14 we have  $D \otimes_{K^{sep}} \bar{K} \simeq M_d(\bar{K})$  for some  $d > 1$ . Now regarding the elements of

$M_d(\bar{K})$  as  $\bar{K}$ -points of the affine space  $\mathbb{A}_{\bar{K}}^{d^2}$ , then the elements of  $D$  correspond to the points of  $\mathbb{A}_{\bar{K}}^{d^2}$  defined over  $K^{sep}$ . Considering the map sending an element of  $M_d(\bar{K})$  viewed as an element of  $\mathbb{A}_{\bar{K}}^{d^2}$  to its determinant, it is given by a polynomial  $\wp$  in the variables  $x_1, x_2, \dots, x_{d^2}$  with all its coefficients 1 or  $-1$ . So we have  $\wp \in K^{sep}[x_1, x_2, \dots, x_{d^2}]$ . Since  $D$  is a division algebra, its non-zero elements give rise to invertible matrices in  $M_d(\bar{K})$ , that is, they have non-zero determinant. Thus the hypersurface determined by the polynomial  $\wp$  contains no points over  $K^{sep}$  except the origin, which contradicts fact the  $K^{sep}$ -rational points of the above hypersurface is dense [Sp, Theorem 11.2.7]. Therefore, there exists a finite separable extension  $K'/K$  such that  $A \otimes_K K' \simeq M_n(K')$ .

Finally, as every finite separable extension can be embedded into a finite Galois extension, there exists a finite Galois extension  $L/K$  such that  $K' \subset L$ . Hence we have  $A \otimes_K L \simeq M_n(K') \otimes_{K'} L \simeq M_n(L)$ .  $\square$

Thus an algebra over  $K$  is a central simple algebra if and only if it is a  $K$ -form of  $M_n(K)$  for some integer  $n$ .

**Definition 1.2.16.** Let  $A$  be a central simple algebra over a field  $K$ . A field extension  $L/K$  such that  $A \otimes_K L \simeq M_n(L)$  for some integer  $n$  is called a *splitting field* for  $A$ . We also say  $A$  *splits* over  $L$  or  $L$  *splits*  $A$ . We call the integer  $n = \sqrt{\dim_k A}$  the *degree* of  $A$  over  $K$ .

From the above theorem, we can easily get the following corollary.

**Corollary 1.2.17.** *Let  $K$  be a field and denote by  $K^{sep}$  its separable closure, then  $K^{sep}$  splits every central simple algebra over  $K$ , and we have  $Br(K^{sep}) = 0$ .*



### 1.2.2 Central Simple Algebras and Brauer-Severi Varieties

A variety we always mean an integral, separated scheme of finite type over a field throughout this thesis.

**Definitions 1.2.18.** Let  $X$  be a scheme over a field  $K$ . We call  $X$  a *Brauer-Severi variety* if there exists a finite Galois extension  $L/K$  such that  $X \otimes_K L \simeq \mathbb{P}_L^n$  for some  $n \in \mathbb{N}$ , that is,  $X$  is a  $L/K$ -form of  $\mathbb{P}^n$ . In this case,  $L$  is said to be a *splitting field* for  $X$ . We also say  $X$  *splits* over  $L$  or  $L$  *splits*  $X$ .

From the above definition, we know that a scheme over  $K$  is a Brauer-Severi variety if and only if it is a  $K$ -form of  $\mathbb{P}_K^n$  for some integer  $n$ .

Now we will give the close relation between Brauer-Severi varieties and central simple algebras through Galois cohomology theory.

**Theorem 1.2.19.** *There is one-to-one correspondence between the set of isomorphic classes of Brauer-Severi varieties of dimension  $n - 1$  over a field  $K$  and the set of isomorphism classes of central simple algebras of degree  $n$  over  $K$ .*

*Sketch of proof.* For any field  $L$ , the Skolem-Noether Theorem [GS, Theorem 2.7.2] asserts that  $\text{Aut}_L(M_n(L)) \simeq \text{PGL}_n(L) = \text{GL}_n(L)/L^*$ , and  $\text{Aut}_L(\mathbb{P}_L^{n-1}) \simeq \text{PGL}_n(L)$ , as shown in [H2, Chapter II, Example 7.1.1]. By Corollary 1.1.18, both sets in the theorem equal to  $H^1(\text{Gal}(K^{\text{sep}}/K), \text{PGL}_n(K^{\text{sep}}))$ . □

For a detailed proof of this theorem and more, the readers may refer to [J, §3, 4, 5].

Given a central simple algebra over a field  $K$ , we give the construction of the corresponding Brauer-Severi variety as follows [Ar, Se1]:

Let  $A$  be a central simple algebra of rank  $n^2$  over a field  $K$ , then the associated Brauer-Severi variety is the set of all left ideals  $L$  of  $A$  of rank  $n$ . If we fix a basis

for  $A$  over  $K$ , then it is embedded in  $Grass(n, n^2)$  as a closed subvariety, defined by the relations stating that each  $L$  is a left ideal of  $A$ .

**Example 1.2.20.** [Ar] Let  $A = M_n(K)$ . Denote by  $E_{ij}$  the  $n \times n$  matrix with 1 in the  $(i, j)$ -th position and 0 elsewhere, and set  $E_i = E_{ii}$ . Then for every left ideal  $L$  of rank  $n$ , we have a decomposition  $L = E_1L \oplus \cdots \oplus E_nL$ , with  $\dim_K E_iL = 1$  for each  $1 \leq i \leq n$  and  $E_{ji}E_iL = E_jL$ . Choose  $x \neq 0$  in  $E_1L$ , then  $x$  may be written as  $x = \sum a_j E_{1j}$  for some  $(a_1, \dots, a_n) \in K^n \setminus \{0\}$ . For another choice of  $x$ , we have  $x' = \lambda x$ . It follows that each left ideal  $L$  of  $A$  of rank  $n$  corresponds to a point  $(a_1, \dots, a_n)$  in  $\mathbb{P}_K^{n-1}$ . On the other hand, for every point  $(a_1, \dots, a_n) \in \mathbb{P}_K^{n-1}$ , let  $l = \sum a_j E_{1j}$ , then  $L = Kl \oplus KE_{21}l \oplus \cdots \oplus KE_{n1}l$  is the corresponding left ideal of rank  $n$  in  $A$ . Thus the Brauer-Severi variety associated to  $M_n(K)$  is projective space  $\mathbb{P}_K^{n-1}$ .

### 1.2.3 The Brauer Group of a Scheme

We discussed Brauer group over a field in subsection 1.2.1. In this subsection, we will sketch its generalization to relative case. The main references are Milne's book [Mi1] and Grothendieck's series of papers [Gr1].

Let us begin with the Azumaya algebras of a local ring first.

**Definition 1.2.21.** [Mi1] Let  $R$  be a commutative local ring and  $A$  a  $R$ -algebra with the map  $R \rightarrow A, r \mapsto r1$ , identifying  $R$  with a subring of the center of  $A$ . We say  $A$  is an *Azumaya algebra* over  $R$  if it is free of finite rank as an  $R$ -module and there is an isomorphism  $A \otimes_R A^o \rightarrow \text{End}_{R\text{-mod}}(A)$  that sends  $a \otimes a'$  to  $(x \mapsto axa')$ .

**Remark 1.2.22.** If  $R$  is a field, by Lemma 1.2.11 and Lemma 1.2.13, it is easy to see that  $A$  is an Azumaya algebra if and only if it is a central simple algebra. Moreover, similar to central simple algebras, using the tensor product, we can also define the Brauer group of  $R$ ,  $Br(R)$ , as in [Mi1, Chapter IV, §1].

Let  $A$  be an Azumaya algebra over a commutative local ring  $R$ . Denote by  $\tilde{A}$  the sheafification of  $A$  as the notation in [H2], the sheaf of algebras  $\mathcal{A} = \tilde{A}$  is called a sheaf of *Azumaya algebra* over  $\text{Spec } R$ .

**Definition 1.2.23.** [Mil] Let  $X$  be a scheme over  $K$ , and  $\mathcal{A}$  be a coherent sheaf on  $X$ . We call  $\mathcal{A}$  an *Azumaya algebra* over  $X$  if  $\mathcal{A}$  is an  $\mathcal{O}_X$ -algebra and for all closed points  $x$  of  $X$ ,  $\mathcal{A}_x$  is an Azumaya algebra over the local ring  $\mathcal{O}_{X,x}$ .

**Remark 1.2.24.** It follows that  $\mathcal{A}$  is locally free of finite rank as  $\mathcal{O}_X$ -module as shown in [Mi1, Chapter I, Theorem 2.9], and  $\mathcal{A}_x$  is an Azumaya algebra over  $\mathcal{O}_{X,x}$ , for any point  $x$  of  $X$  as in [Mi1, Chapter IV, Proposition 1.2].

Now we can define the Brauer group of a scheme  $X$ . Two Azumaya algebras  $\mathcal{A}$  and  $\mathcal{B}$  over  $X$  are said to be *similar* if there exist locally free  $\mathcal{O}_X$ -modules  $\mathcal{E}$  and  $\mathcal{E}'$ , of finite rank over  $\mathcal{O}_X$ , such that

$$\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}) \simeq \mathcal{B} \otimes_{\mathcal{O}_X} \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}').$$

As for any two locally free  $\mathcal{O}_X$ -modules  $\mathcal{E}$  and  $\mathcal{F}$ , of finite rank over  $\mathcal{O}_X$ , we have  $\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}) \otimes_{\mathcal{O}_X} \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{F}) \simeq \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F})$ , the similarity relation is an equivalence relation.

Let  $\mathcal{A}$  and  $\mathcal{B}$  be two Azumaya algebras over a scheme  $X$ . Then by definition, for all closed points  $x$  of  $X$ ,  $\mathcal{A}_x$  and  $\mathcal{B}_x$  are Azumaya algebras over the local ring  $\mathcal{O}_{X,x}$ . Since  $(\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{B})_x \simeq \mathcal{A}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{B}_x$ , and by Remark 1.2.22,  $\mathcal{A}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{B}_x$  is an Azumaya algebra over  $\mathcal{O}_{X,x}$ , thus  $\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{B}$  is an Azumaya algebra over  $X$  by definition. We can easily see that the tensor product operation is compatible with the similarity relation defined above.

Finally, let  $\mathcal{A}$  be an Azumaya algebra over a scheme  $X$ , by Remark 1.2.24,  $\mathcal{A}$  is locally free as an  $\mathcal{O}_X$ -module and for any point  $x$  of  $X$ ,  $\mathcal{A}_x$  is an Azumaya algebra over  $\mathcal{O}_{X,x}$ . We have  $(\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{A}^o)_x = \mathcal{A}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{A}_x^o \simeq \mathcal{E}nd_{\mathcal{O}_{X,x}}(\mathcal{A}_x)$  and

$\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{A})_x = \text{End}_{\mathcal{O}_{X,x}}(\mathcal{A}_x)$  for every  $x$  of  $X$ . Thus the canonical homomorphism  $\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{A}^\circ \rightarrow \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{A})$  is an isomorphism, by [H2, Proposition II 1.1].

Hence, the set of similarity classes of Azumaya algebra on a scheme  $X$  forms a group under the tensor product operation  $[\mathcal{A}][\mathcal{B}] = [\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{B}]$ :  $[\mathcal{O}_X]$  is the identity element and  $[\mathcal{A}]^{-1} = [\mathcal{A}^\circ]$ . We call it the *Brauer group* of  $X$  and denote it by  $Br(X)$ . And obviously  $Br(-)$  is a functor from schemes to abelian groups.

Next we will show that for a rational smooth projective variety, its Brauer group is the Brauer group of its ground field. We know that for a variety  $X$  over a field,  $Br(X)$  is torsion [Mi1, Chapter IV, proposition 2.7]. Using the Kummer sequence and Hochschild-Serre spectral sequence, one can prove this if the characteristic of the ground field is 0. It can also be shown for prime to  $p$  components if the characteristic of the ground field is a prime  $p$ , but it is hard to deal with the  $p$ -torsion part. To overcome this difficulty, we need to introduce the unramified Brauer group. The main reference is Saltman’s notes [S2, Chapter 10] and paper [S1]. Here the author is very thankful for the kindness of Professor Daniel Krashen to point out the proper reference.

Recall that a discrete valuation ring is a principal ideal domain with exactly one non-zero maximal ideal. Auslander-Goldman proved that for a regular domain  $R$  with fraction field  $K$ , the natural map  $Br(R) \rightarrow Br(K)$  is injective [S2, Theorem 9.6]. Thus we can give the following definition:

**Definition 1.2.25.** [S2] Let  $K \subset L$  be two fields and denote by  $R_{L/K}$  the set of all discrete valuation rings containing  $K$  with field of fractions  $L$ . We define the *unramified Brauer group*  $Br_u(L/K)$  to be the intersection of  $Br(R)$  for all  $R \in R_{L/K}$ .

**Proposition 1.2.26.** [S2] Let  $X$  be a regular projective variety over a field  $K$  with function field  $L$ , then  $Br(X) = Br_u(L/K)$ .

*Sketch of proof.* Note that  $U \mapsto Br(U)$  defines a sheaf on the Zariski topology of

$X$  [Mi1, page 147], we have  $Br(X) = \bigcap Br(\mathcal{O}_{X,P})$  for all  $P \subset X$  irreducible of codimension one. Thus we have  $Br_u(L/K) \subseteq Br(X)$ . On the other hand, let  $R \in R_{L/K}$ . By the Valuative Criterion of Properness [H2, Chapter II, Theorem 4.7],  $R$  is over some curve  $C \subseteq X$  and so  $\mathcal{O}_{X,C} \subseteq R$ . Thus  $Br(X) \subseteq Br(\mathcal{O}_{X,C}) \subseteq Br(R)$ . □

As we know that birationally equivalent varieties have isomorphic function fields [H2, Corollary I 4.5], thus  $Br(X)$  is a birational invariant.

**Proposition 1.2.27.** *[S1] Assume  $L/K$  be a purely transcendental extension of field, then  $Br_u(L/K) = Br(K)$ .*

*Proof.* See the proof of Proposition 1.7 in [S1]. □

Notice that the function field of a projective space is a purely transcendental extension of its base field, thus, combine the above two propositions, we have

**Theorem 1.2.28.** *Let  $X$  be a rational smooth projective variety over a field  $K$ , then we have  $Br(X) = Br(K)$ .*

### 1.2.4 Azumaya Algebras and Brauer-Severi Schemes

In §1.2.2, we discussed Brauer-Severi varieties and established that the one-to-one correspondence between the isomorphism classes of Brauer-Severi varieties of dimension  $n - 1$  over  $K$  and the isomorphism classes of central simple algebras of degree  $n$  over  $K$ . In this subsection, we will see that this definition and result can be generalized to a ground scheme. The main reference is [Gr1, I, §5, 7, 8]. The reader may also read the end of §4 in [Mi1, Chapter III] and the proof of the first step of Theorem 2.5 in [Mi1, Chapter IV, §2].

Recall that a morphism of schemes  $f : X \rightarrow Y$  is étale if  $f$  is flat and unramified [Mi1]. And an étale cover of a scheme  $X$  is a set  $\{U \xrightarrow{p} X\}$  of étale morphisms of

finite type which are jointly surjective in the sense that  $X$  equals to the union of set-theoretic images, i.e.  $X = \bigcup p(U)$ .

The local structure of an Azumaya algebra over a scheme is given by the following theorem.

**Theorem 1.2.29.** *Let  $X$  be a scheme,  $\mathcal{A}$  be a sheaf of  $\mathcal{O}_X$ -algebra which is locally free of finite rank as an  $\mathcal{O}_X$ -module. Then  $\mathcal{A}$  is a sheaf of Azumaya algebra over  $X$  if and only if there is an étale covering  $\{U \rightarrow X\}$  such that for each map  $U \rightarrow X$ , we have  $\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{O}_U \simeq M_n(\mathcal{O}_U)$  for some  $n$ .*

*Proof.* See [Mil, Chapter IV, Proposition 2.1]. □

From the theorem above, we get that  $\mathcal{A}$  is a sheaf of Azumaya algebras over  $X$  if and only if it is a twisted form of  $M_n(\mathcal{O}_X)$  for the étale topology for some integer  $n$ .

Now we are going to talk about Brauer-Severi schemes, which generalizes the notion of Brauer-Severi varieties discussed in §1.2.2, and they are closely related to Azumaya algebras through étale cohomology theory.

**Definition 1.2.30.** Let  $f : P \rightarrow X$  be a morphism of schemes. We say  $P$  is *Brauer-Severi scheme* over  $X$  if it is locally isomorphic to a projective space  $\mathbb{P}_X^n$  over  $X$  in the étale topology of  $X$  for some integer  $n$ , that is,  $P$  is a twisted form of  $\mathbb{P}_X^n$  for the étale topology.

**Theorem 1.2.31.** *There is a one-to-one correspondence between the set of isomorphism classes of Azumaya algebra over  $X$  of rank  $n^2$  and the set of isomorphism classes of Brauer-Severi schemes over  $X$  of relative dimension  $n - 1$ .*

*Sketch of proof.* Every Azumaya algebra over  $X$  of rank  $n^2$  is a twisted form of  $M_n(\mathcal{O}_X)$  for the étale topology, and Auslander-Goldman theorem [Gr1, §5, Theorem

5.10] asserts that  $\mathcal{A}ut(M_n(\mathcal{O}_X)) \simeq PGL_n$ . As every étale covering is also a (fl)-covering, by Corollary 1.1.21 and Proposition 1.1.6, the set of isomorphism classes of Azumaya algebra over  $X$  of rank  $n^2$  is equal to  $\check{H}^1(X_{\acute{e}t}, PGL_n)$ .

Similarly, a Brauer-Severi schemes over  $X$  of relative dimension  $n - 1$  is a twisted form of  $\mathbb{P}_X^{n-1}$ , and  $\mathcal{A}ut(\mathbb{P}_X^{n-1}) \simeq PGL_n$  as showed in [Mu, Chapter 0, §5]. Notice that  $\mathcal{O}_{\mathbb{P}_X^{n-1}}(1)$  is a very ample invertible bundle over  $\mathbb{P}_X^{n-1}$  relative to  $X$ , then by Corollary 1.1.21 and Proposition 1.1.7, the set of isomorphism classes of Brauer-Severi schemes over  $X$  of relative dimension  $n - 1$  is equal to  $\check{H}^1(X_{\acute{e}t}, PGL_n)$ .  $\square$

# Chapter 2

## Toric Varieties

### 2.1 Split Toric Varieties

We recall some basic facts about toric varieties (we will define a variety to be an integral separated scheme of finite type over a field) that are needed in this thesis, which can be found in many standard texts, such as [D], [F], [O2] and [CLS].

Denote by  $\mathbb{G}_m = \text{Spec } K[t, t^{-1}]$  the affine algebraic group endowed with co-multiplication  $t \mapsto t \otimes t$  on the coordinate ring. An *algebraic torus*  $T$  is an algebraic group isomorphic to  $\mathbb{G}_m^n$  where  $n$  is an integer  $\geq 1$ . A *toric variety* is a normal variety  $X$  that contains a torus  $T$  as a dense Zariski open subset, together with an action  $T \times X \rightarrow X$  of  $T$  on  $X$  that extends the natural action of  $T$  on itself.

In [D], Michel Demazure first constructed toric varieties as schemes over  $\text{Spec } \mathbb{Z}$  from the data of a unimodular fan. Later documents, such as [O2, F], start with fans in lattices and construct varieties over algebraically closed field. These latter constructions in fact give schemes over  $\text{Spec } \mathbb{Z}$  and can be applied to any field. In the following, we will follow this treatment.

Let  $N$  be a finitely generated free abelian group of rank  $n$ , that is,  $N \simeq \mathbb{Z}^n$ , and



$M = \text{Hom}(N, \mathbb{Z})$  denotes the dual of  $N$ . We have a canonical  $\mathbb{Z}$ -linear pairing

$$\langle \cdot, \cdot \rangle : M \times N \rightarrow \mathbb{Z}.$$

By scalar extension to the field  $\mathbb{R}$  of real numbers, we have  $\mathbb{R}$ -vector spaces  $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$  and  $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$  with a canonical  $\mathbb{R}$ -linear pairing  $\langle \cdot, \cdot \rangle : M_{\mathbb{R}} \times N_{\mathbb{R}} \rightarrow \mathbb{R}$ .

In the following, we limit our discussions to strong convex rational polyhedral cones. For more general convex polyhedral cones, the reader may refer to [F, §1.2].

**Definitions 2.1.1.** A convex subset  $\sigma \subset N_{\mathbb{R}}$  is called a *strong convex rational polyhedral cone* if it has an apex at the origin and there exists a  $\mathbb{R}$  linear independent subset  $\{n_1, \dots, n_d\} \subset N$  such that

$$\sigma = \{a_1 n_1 + \dots + a_d n_d : a_i \in \mathbb{R}, a_i \geq 0\}.$$

The dimension of  $\sigma$  is the dimension of the smallest subspace of  $N_{\mathbb{R}}$  that contains  $\sigma$ , that is,  $\dim(\sigma) := \dim(\sigma + (-\sigma))$ .

We will often refer to a strong convex rational polyhedral cone as a cone, when there is no possibility for confusion. Here rational means that  $\sigma$  is *generated* by  $\{n_1, \dots, n_d\}$ , a subset of elements in  $N$ , and we say they are *generators* of  $\sigma$ .

If  $\sigma \subset N_{\mathbb{R}}$  is cone, we define the *dual* of  $\sigma$ :

$$\sigma^{\vee} = \{u \in M_{\mathbb{R}} : \langle u, v \rangle \geq 0, \forall v \in \sigma\}.$$

By a fundamental fact from the theory of convex sets [F, page 9], we have  $(\sigma^{\vee})^{\vee} = \sigma$ , and  $\sigma^{\vee}$  is also rational [F, page 12], i.e. its generators can be taken from  $M$ .

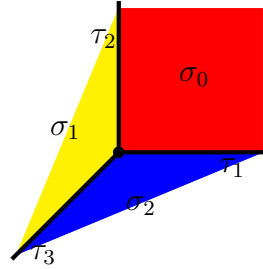
A *face*  $\tau$  of  $\sigma$  is a subset of  $\sigma$  of the form

$$\tau = \sigma \cap u^{\perp} = \{v \in \sigma : \langle u, v \rangle = 0\}$$

for some  $u \in \sigma^{\vee}$ . Actually, such  $u$  can be chosen in  $M \cap \sigma^{\vee}$  [O2, Chapter 1, Proposition 1.3], thus a face  $\tau$  is also a cone generated by those  $n_i$  in a generating set for

$\sigma$  such that  $\langle u, n_i \rangle = 0$ , and it is denoted by  $\tau \preceq \sigma$ . Note that any intersection of faces is also a face.

**Example 2.1.2.** Let  $N = \mathbb{Z}^2$  with a fixed basis  $\{n_1 = (1, 0), n_2 = (0, 1)\}$ , and  $n = -(n_1 + n_2)$ . Look at the following figure:



- (a)  $\sigma_0$  is a cone generated by  $\{n_1, n_2\}$  with faces  $\{\tau_1, \tau_2, \{0\}\}$ , and  $\{0\}$  is the intersection of  $\tau_1$  and  $\tau_2$ ;
- (b)  $\sigma_1$  is a cone generated by  $\{n, n_2\}$  with faces  $\{\tau_2, \tau_3, \{0\}\}$ , and  $\{0\}$  is the intersection of  $\tau_1$  and  $\tau_2$ ;
- (c)  $\sigma_2$  is a cone generated by  $\{n, n_1\}$  with faces  $\{\tau_1, \tau_3, \{0\}\}$ , and  $\{0\}$  is the intersection of  $\tau_1$  and  $\tau_2$ .

If  $\sigma$  is a cone in  $N_{\mathbb{R}}$ , it determines a commutative semigroup

$$S_{\sigma} = \sigma^{\vee} \cap M = \{u \in M : \langle u, v \rangle \geq 0, \forall v \in \sigma\}.$$

This semigroup is finitely generated [F, §1.2 Gordon's Lemma]. Let  $K[S_{\sigma}]$  be the commutative  $K$ -algebra generated by the set  $S_{\sigma}$ . It consists of linear combination of forms  $\chi^u$  for  $u \in S_{\sigma}$ , with multiplication given by  $\chi^u \cdot \chi^{u'} = \chi^{u+u'}$ . Thus  $K[S_{\sigma}]$  is a finitely generated  $K$ -algebra and it corresponds to an affine variety:

$$U_{\sigma} = \text{Spec } K[S_{\sigma}].$$

**Example 2.1.2 (continued).** Let  $M$  be the dual of  $N$ , with basis  $\{n_1^*, n_2^*\}$ . Consider  $\sigma_0$ , a computation shows that the semigroup  $S_{\sigma_0} = \sigma_0^\vee \cap M = \mathbb{Z}_{\geq 0}\{n_1^*, n_2^*\}$ . If we write  $X = \chi^{n_1^*}$  and  $Y = \chi^{n_2^*}$ , then the corresponding group algebra  $K[S_{\sigma_0}]$  is  $K[X, Y]$ .

Similarly, we can get the following table:

cone $\sigma$	generators of $S_\sigma$	$K[S_\sigma]$
$\sigma_0$	$n_1^*, n_2^*$	$K[X, Y]$
$\sigma_1$	$-n_1^*, -n_1^* + n_2^*$	$K[X^{-1}, X^{-1}Y]$
$\sigma_2$	$n_1^* - n_2^*, -n_2^*$	$K[XY^{-1}, Y^{-1}]$
$\tau_1$	$n_1^*, \pm n_2^*$	$K[X, Y, Y^{-1}]$
$\tau_2$	$\pm n_1^*, n_2^*$	$K[X, X^{-1}, Y]$
$\tau_3$	$n_1^* - n_2^*, -n_1^* + n_2^*, -n_1^* - n_2^*$	$K[XY^{-1}, X^{-1}Y, X^{-1}Y^{-1}]$
$\{0\}$	$\pm n_1^*, \pm n_2^*$	$K[X, X^{-1}, Y, Y^{-1}]$

Now let's see how the torus acts on an affine toric variety. If  $\sigma$  is a cone in  $N_{\mathbb{R}}$ , the torus  $T_N = U_{\{0\}}$  acts on  $U_\sigma$ ,

$$T_N \times U_\sigma \rightarrow U_\sigma,$$

as follows: A point  $t \in T_N(K)$  can be identified with a map  $M \rightarrow K^*$  of groups, and a point  $x \in U_\sigma(K)$  with a map  $S_\sigma \rightarrow K$  of semigroups; the product  $t \cdot x$  is the map of semigroups  $S_\sigma \rightarrow K$  given by

$$u \rightarrow t(u)x(u).$$

This gives the dual map on algebras:  $K[S_\sigma] \rightarrow K[S_\sigma] \otimes K[M]$ ,  $\chi^u \mapsto \chi^u \otimes \chi^u$  for all  $u \in S_\sigma$ . We can see that this is just the ordinary product of the algebraic group  $T_N$  when  $\sigma = \{0\}$ . Thus this action extends the action of  $T_N$  on itself.

If  $\tau$  is a face of  $\sigma$ , then  $S_\sigma$  is contained in  $S_\tau$ , so  $K[S_\sigma]$  is a subalgebra of  $K[S_\tau]$ , which gives a morphism of varieties  $U_\tau \rightarrow U_\sigma$ . In fact,  $U_\tau$  is a principal open subset of  $U_\sigma$ : if we choose  $u \in S_\sigma$  such that  $\tau = \sigma \cap u^\perp$ , then  $S_\tau = S_\sigma + \mathbb{Z}_{\geq 0} \cdot (-u)$  [F, §1.2, Proposition 2] and  $U_\tau \simeq \{x \in U_\sigma : u(x) \neq 0\} = (U_\sigma)_{x^u}$ . Thus we have a natural order-preserving correspondence from cones to affine varieties. And from the above argument we know that the actions of  $T_N$  on  $U_\sigma$  and  $U_\tau$  are compatible with the open inclusion  $U_\tau \rightarrow U_\sigma$ .

**Definition 2.1.3.** A *fan*  $\Sigma$  in  $N_{\mathbb{R}}$  is a finite collection of cones in  $N_{\mathbb{R}}$  such that

- (a) Any face of a cone in  $\Sigma$  is also a cone in  $\Sigma$ .
- (b) The intersection of any two cones in  $\Sigma$  is a common face of each (hence also in  $\Sigma$ ).

If  $\Sigma$  is a fan in  $N_{\mathbb{R}}$ , the *support* of  $\Sigma$  is  $|\Sigma| = \bigcup_{\sigma \in \Sigma} \sigma \subset N_{\mathbb{R}}$ , and we say it is *complete* if  $|\Sigma| = N_{\mathbb{R}}$ . Denote by  $\Sigma(r)$  the set of  $r$ -dimensional cones of  $\Sigma$ .

Given a fan  $\Sigma$  in  $N_{\mathbb{R}}$ , let us construct the corresponding toric variety  $X_\Sigma$ . First we take the disjoint union of the affine toric varieties  $\coprod_{\sigma \in \Sigma} U_\sigma$ , then glue them as follows: for cones  $\sigma$  and  $\sigma'$ , the intersection  $\sigma \cap \sigma'$  is a face of both of them, so  $U_{\sigma \cap \sigma'}$  is identified as a principal open subvariety of  $U_\sigma$  and of  $U_{\sigma'}$ ; glue  $U_\sigma$  and  $U_{\sigma'}$  by this identification on these subvarieties. The compatibility conditions [H2, Ex II 2.12] comes from the order-preserving nature of the correspondence from cones to affine varieties. Denote by  $X_\Sigma$  the gluing variety. The separability of  $X_\Sigma$  comes from the fact that  $S_{\sigma \cap \sigma'} = S_\sigma + S_{\sigma'}$  [F, §1.2 Proposition 3]. The actions of  $T_N$  on each  $U_\sigma$  patch together to give an action of  $T_N$  on  $X_\Sigma$ . Thus we have a toric variety which corresponds to the fan  $\Sigma$  in  $N_{\mathbb{R}}$ .

**Remark 2.1.4.** Let  $\Sigma$  be a fan in  $N_{\mathbb{R}}$  and  $\sigma, \sigma' \in \Sigma$ , and let  $\tau = \sigma \cap \sigma'$ . The Separation lemma [F, §1.2, (12)] asserts that there exists a  $u$  in  $\sigma^\vee \cap (-\sigma')^\vee$  such

that  $\tau = \sigma \cap u^\perp = \sigma' \cap u^\perp$ . So we have an isomorphism  $(U_\sigma)_{X^u} \simeq (U_{\sigma'})_{X^{-u}}$  which is the identity on  $U_\tau$ .

**Example 2.1.2 (continued).** Let  $\Sigma$  be the fan consisting cones  $\sigma_0, \sigma_1, \sigma_2$  and their faces. Then the toric variety  $X_\Sigma$  is covered by the affine opens

$$U_{\sigma_0} = \text{Spec}(K[X, Y]),$$

$$U_{\sigma_1} = \text{Spec}(K[X^{-1}, X^{-1}Y]),$$

$$U_{\sigma_2} = \text{Spec}(K[XY^{-1}, Y^{-1}]).$$

Moreover, the gluing data on the coordinate rings is given by

$$K[X, Y]_X \simeq K[X^{-1}, X^{-1}Y]_{X^{-1}},$$

$$K[X, Y]_Y \simeq K[XY^{-1}, Y^{-1}]_{Y^{-1}},$$

$$K[X^{-1}, X^{-1}Y]_{X^{-1}Y} \simeq K[XY^{-1}, Y^{-1}]_{XY^{-1}}.$$

If we use the usual homogeneous coordinates  $(x_0, x_1, x_2)$  on  $\mathbb{P}^2$ , then  $X \mapsto \frac{x_1}{x_0}$  and  $Y \mapsto \frac{x_2}{x_0}$  identify the standard affine open  $U_i \subset \mathbb{P}^2$  with  $U_{\sigma_i} \subset X_\Sigma$ . Hence  $X_\Sigma$  is the projective space  $\mathbb{P}^2$ .

We state the following theorem without proof. The reader may refer to [O2] for its proof.

**Theorem 2.1.5.** [O2, Theorem 1.10] *The toric variety  $X$  associated to a fan  $\Sigma$  in  $N_{\mathbb{R}}$  is smooth if and only if for each  $\sigma \in \Sigma$ ,  $\sigma$  is generated by a subset of a basis of  $N$ .*

In this case, we say the fan is *smooth*.

For each  $\sigma \in \Sigma$ , we have a closed point of  $U_\sigma$  defined by

$$m \in S_\sigma \mapsto \begin{cases} 1, & m \in S_\sigma \cap \sigma^\perp = \sigma^\perp \cap M \\ 0, & \text{otherwise.} \end{cases}$$

We denote this point by  $\gamma_\sigma$  and call it the *distinguished point* corresponding to  $\sigma$ .

The fan  $\Sigma$  is closely related to the structure of the toric variety  $X_\Sigma$ . Now let us consider the one-to-one correspondence between the cones in  $\Sigma$  and the orbits for the action of  $T_N(K)$  on the toric variety  $X_\Sigma(K)$ . For each  $\sigma \in \Sigma$ , denote by  $O(\sigma)$  the orbit containing the distinguished point  $\gamma_\sigma$ .

Let  $\sigma \in \Sigma$ , we define  $N_\sigma$  to be the sublattice of  $N$  generated (as a group) by  $\sigma \cap N$ , i.e.  $N_\sigma = (\sigma + (-\sigma)) \cap N$ , and  $N(\sigma) = N/N_\sigma$ ,  $M(\sigma) = \sigma^\perp \cap M$ . It is easy to see that dual pairing  $\langle \cdot, \cdot \rangle : M \times N \rightarrow \mathbb{Z}$  induces a perfect pairing

$$\langle \cdot, \cdot \rangle : M(\sigma) \times N(\sigma) \rightarrow \mathbb{Z}.$$

From which we have  $T_{N(\sigma)}(K) = \text{Spec } K[M(\sigma)](K) = \text{Hom}_{\mathbb{Z}}(M(\sigma), K^*) = N(\sigma) \otimes_{\mathbb{Z}} K^*$ . And  $T_N(K)$  acts on  $T_{N(\sigma)}(K)$  transitively via the projection  $T_N(K) \rightarrow T_{N(\sigma)}(K)$ , which induced by  $N \rightarrow N(\sigma)$ . On the other hand,

$$T_N(K) \cdot \gamma_\sigma = \{\gamma : S_\sigma \rightarrow K \mid \gamma(m) \neq 0 \Leftrightarrow m \in \sigma^\perp \cap M\} \simeq \text{Hom}_{\mathbb{Z}}(M(\sigma), K^*),$$

and it is compatible with the action of  $T_N(K)$ . Thus we have  $O(\sigma) = T_N(K) \cdot \gamma_\sigma \simeq \text{Hom}_{\mathbb{Z}}(M(\sigma), K^*)$ .

**Theorem 2.1.6.** *Let  $X_\Sigma$  be the toric variety of the fan  $\Sigma$  in  $N_{\mathbb{R}}$ . Then we have*

(i) *There is a one-to-one correspondence*

$$\begin{aligned} \{\text{cones in } \Sigma\} &\longleftrightarrow \{T_N(K)\text{-orbits in } X_\Sigma\} \\ \sigma &\longleftrightarrow O(\sigma) \simeq \text{Hom}_{\mathbb{Z}}(\sigma^\perp \cap M, K^*). \end{aligned}$$

(ii) *For each  $\sigma \in \Sigma$ , the corresponding affine open subset  $U_\sigma$  is the union of orbits*

$$U_\sigma(K) = \bigcup_{\tau \preceq \sigma} O(\tau).$$

(iii) For  $\sigma, \tau \in \Sigma$ ,  $\tau$  is a face of  $\sigma$  if and only if  $O(\sigma)$  is contained in the closure of  $O(\tau)$ , i.e.  $O(\sigma) \subseteq \overline{O(\tau)}$ , and  $\overline{O(\tau)} = \bigcup_{\tau \preceq \sigma} O(\sigma)$ .

*Proof.* For (i), we only need to show that for every  $T_N(K)$ -orbit  $O$ ,  $O = O(\sigma)$  for some cone  $\sigma \in \Sigma$ . Indeed, as  $X_\Sigma(K)$  is covered by the  $T_N(K)$ -invariant affine open subsets  $U_\sigma(K) \subset X_\Sigma(K)$  and  $U_\sigma(K) \cap U_\tau(K) = U_{\sigma \cap \tau}(K)$ , there exist a unique minimal cone  $\sigma \in \Sigma$  such that  $O \subseteq U_\sigma$ . We claim  $O = O(\sigma)$ .

Notice that  $U_\sigma(K) = \text{Hom}_{\text{semi.group}}(S_\sigma, K)$ , let  $\gamma : S_\sigma \rightarrow K$  be a point in  $O$ . Then  $\gamma^{-1}(K^*) = \sigma^\vee \cap \tau^\perp \cap M$  for some face  $\tau$  of  $\sigma$ . Thus  $\gamma \in U_\tau(K)$  and  $O = T_N(K) \cdot \gamma \subset U_\tau(K)$ . By the minimality of  $\sigma$ , we have  $\tau = \sigma$ . Thus  $\gamma^{-1} = \sigma^\perp \cap S_\sigma = M(\sigma)$ , and  $O = T_N(K) \cdot \gamma = O(\sigma)$ .

For (ii), we know that  $U_\sigma(K)$  is a union of orbits. If  $\tau$  is a face of  $\sigma$ , then  $O(\tau) \subseteq U_\tau(K) \subseteq U_\sigma(K)$ , so we have  $O(\tau) \subseteq U_\sigma(K)$ . On the other hand, from the proof of part (i), we see that any orbit contained in  $U_\sigma(K)$  must equal to  $O(\tau)$  for some face  $\tau \preceq \sigma$ . Hence we have  $U_\sigma(K) = \bigcup_{\tau \preceq \sigma} O(\tau)$ .

For (iii), if  $O(\sigma)$  is contained in the closure of  $O(\tau)$ , then the open neighborhood  $U_\sigma(K)$  intersects  $O(\tau)$ , and hence contains  $O(\tau)$ . Thus  $\tau$  is a face of  $\sigma$  by part (ii). Hence  $\bigcup_{\tau \preceq \sigma} O(\sigma) \subseteq \overline{O(\tau)}$ . We claim that  $\bigcup_{\tau \preceq \sigma} O(\sigma)$  is a closed subset. Indeed, as  $X_\Sigma(K)$  is covered by the  $T_N(K)$ -invariant affine open subsets  $U_\sigma(K) \subset X_\Sigma(K)$ , and once again, by part (ii), we have  $\bigcup_{\tau \preceq \sigma} O(\sigma) = X_\Sigma(K) \setminus (\bigcup_{\sigma \in \Sigma, \tau \not\preceq \sigma} U_\sigma(K))$ .  $\square$

**Definition 2.1.7.** Let  $X_{\Sigma_1}, X_{\Sigma_2}$  be two toric varieties, with  $\Sigma_1$  a fan in  $(N_1)_\mathbb{R}$  and  $\Sigma_2$  a fan in  $(N_2)_\mathbb{R}$ . We say a morphism  $\varphi : X_{\Sigma_1} \rightarrow X_{\Sigma_2}$  is *equivariant* if the restriction  $\varphi|_{T_{N_1}}$  is a group homomorphism of the torus  $T_{N_1} \rightarrow T_{N_2}$  and the following diagram

$$\begin{array}{ccc} T_{N_1} \times X_{\Sigma_1} & \longrightarrow & X_{\Sigma_1} \\ \varphi|_{T_{N_1}} \times \varphi \downarrow & & \downarrow \varphi \\ T_{N_2} \times X_{\Sigma_2} & \longrightarrow & X_{\Sigma_2} \end{array}$$

commutes.

**Definition 2.1.8.** A *map of fans*  $\phi : (\Sigma_1, N_1) \rightarrow (\Sigma_2, N_2)$  is a  $\mathbb{Z}$ -linear homomorphism  $\phi : N_1 \rightarrow N_2$  with scalar extension  $\phi : (N_1)_{\mathbb{R}} \rightarrow (N_2)_{\mathbb{R}}$  satisfying the following condition: for each  $\sigma \in \Sigma_1$ , there exists  $\sigma' \in \Sigma_2$  such that  $\phi(\sigma) \subseteq \sigma'$ .

**Theorem 2.1.9.** [CLS, Theorem 3.3.4] *Let  $X_i, i = 1, 2$ , be toric varieties, corresponding to fans  $(\Sigma_i, N_i), i = 1, 2$ . Then there is a one-to-one correspondence between the set of equivariant morphisms  $\varphi : X_1 \rightarrow X_2$  and the set of maps of fans  $\phi : (\Sigma_1, N_1) \rightarrow (\Sigma_2, N_2)$ .*

*Proof.* A  $\mathbb{Z}$ -linear map  $\phi : N_1 \rightarrow N_2$  gives rise to its dual  $\phi' : M_2 \rightarrow M_1$ , which induces a homomorphism of rings  $K[M_2] \rightarrow K[M_1]$ . Thus we have a homomorphism of algebraic tori  $\varphi_{\{0\}} : T_{N_1} \rightarrow T_{N_2}$ . Now let  $\sigma_1 \in \Sigma_1$ , there is a cone  $\sigma_2 \in \Sigma_2$  such that  $\phi(\sigma_1) \subset \sigma_2$ . Then we have  $\phi'(\sigma_2^\vee) \subset \sigma_1^\vee$  and  $\phi'(S_{\sigma_2}) \subset S_{\sigma_1}$ . Thus we have an equivariant morphism  $\varphi_{\sigma_1} : U_{\sigma_1} \rightarrow U_{\sigma_2}$ . By gluing affine pieces together, we obtain an equivariant morphism  $\varphi : X_{\Sigma_1} \rightarrow X_{\Sigma_2}$ .

Conversely, let  $\varphi : X_{\Sigma_1} \rightarrow X_{\Sigma_2}$  be an equivariant morphism. Then by composition with the homomorphism of algebraic tori  $\varphi|_{T_{N_1}} : T_{N_1} \rightarrow T_{N_2}$ , we have a homomorphism of the character group  $M_2 \rightarrow M_1$  and its dual  $\mathbb{Z}$ -homomorphism  $\phi : N_1 \rightarrow N_2$ .

It remains to show that  $\phi$  is a map of fans. Since  $\varphi$  is an equivariant morphism, the image under  $\varphi$  of each  $T_{N_1}(K)$ -orbit in  $X_{\Sigma_1}(K)$  is contained in a  $T_{N_2}(K)$ -orbit in  $X_{\Sigma_2}(K)$ . Let  $\tau_1 \preceq \sigma_1$  be cones in  $\Sigma_1$ . Consider the  $T_{N_1}(K)$ -orbits  $O(\sigma_1)$  and  $O(\tau_1)$ . By part (i) of Theorem 2.1.6, there exist cones  $\sigma_2, \tau_2 \in \Sigma_2$  such that  $\varphi(O(\sigma_1)) \subset O(\sigma_2)$  and  $\varphi(O(\tau_1)) \subset O(\tau_2)$ .

We claim that  $\tau_2$  is a face of  $\sigma_2$ . Indeed, since  $\varphi$  is continuous and by part (iii) of Theorem 2.1.6,  $O(\sigma_1) \subset \overline{O(\tau_1)}$ , we have  $\varphi(\overline{O(\tau_1)}) \subset \overline{O(\tau_2)}$ . Thus  $O(\sigma_2) \subset \overline{O(\tau_2)}$ . But the only orbits contained in the closure of  $O(\tau_2)$  are the orbits corresponding to cones that have  $\tau_2$  as a face. Hence  $\tau_2$  is a face of  $\sigma_2$ .



Consequently, we have  $\varphi(U_{\sigma_1}) \subset U_{\sigma_2}$ . Look at the corresponding homomorphism  $\phi'$  of rings, we have  $\phi'(S_{\sigma_2}) \subset S_{\sigma_1}$  and  $\phi'(\sigma_2^\vee) \subset \sigma_1^\vee$ , equivalently,  $\phi(\sigma_1) \subset \sigma_2$ . Hence we obtain a map of fans  $\phi : (\Sigma_1, N_1) \rightarrow (\Sigma_2, N_2)$ , which obviously induces the equivariant morphism  $\varphi$ .  $\square$

**Definition 2.1.10.** Let  $X, Y, Z$  be toric varieties and the associated fan of  $X$  is  $\Sigma$ . An *equivariant fiber bundle* with typical fiber  $Y$  is an equivariant morphism  $\varphi : Z \rightarrow X$  such that for each  $\sigma \in \Sigma$ , we have  $\varphi^{-1}(U_\sigma) \simeq Y \times U_\sigma$ .

There are close relations between the associated fans. The simplest case is the product of toric varieties, as stated below:

**Proposition 2.1.11.** [CLS, Proposition 3.1.14] Let  $\Sigma_1$  in  $(N_1)_\mathbb{R}$  and  $\Sigma_2$  in  $(N_2)_\mathbb{R}$  be fans, then

$$\Sigma_1 \times \Sigma_2 = \{\sigma_1 \times \sigma_2 \mid \sigma_1 \in \Sigma_1, \sigma_2 \in \Sigma_2\}$$

is a fan in  $(N_1)_\mathbb{R} \times (N_2)_\mathbb{R} = (N_1 \times N_2)_\mathbb{R}$  and

$$X_{\Sigma_1 \times \Sigma_2} \simeq X_{\Sigma_1} \times X_{\Sigma_2}.$$

More generally, we have the following description of equivariant fiber bundles.

**Proposition 2.1.12.** [O2, Proposition 1.33] Consider an equivariant morphism  $\varphi : X \rightarrow X'$ , corresponding to a map of fans  $\phi : (\Sigma, N) \rightarrow (\Sigma', N')$ . Let  $N''$  be the kernel of the  $\mathbb{Z}$ -linear mapping  $\phi : N \rightarrow N'$  and  $\Sigma''$  be a fan in  $N''_\mathbb{R}$ . Then  $\varphi : X \rightarrow X'$  is an equivariant fiber bundle with  $X_{\Sigma''}$  as typical fiber if and only if the following is satisfied:  $\phi : N \rightarrow N'$  is surjective and there exists a subfan  $\tilde{\Sigma}' \subset \Sigma$  such that  $\phi$  maps each cone  $\tilde{\sigma}' \in \tilde{\Sigma}'$  bijectively to a cone  $\sigma' \in \Sigma'$  such that  $\phi(\tilde{\sigma}' \cap N) = \sigma' \cap N'$  and it induces a homeomorphism  $|\tilde{\Sigma}'| \rightarrow |\Sigma'|$ , and furthermore,  $\Sigma = \{\tilde{\sigma}' + \sigma'' : \tilde{\sigma}' \in \tilde{\Sigma}', \sigma'' \in \Sigma''\}$ . In this case, the open set  $X_{\tilde{\Sigma}'} \subset X$  is a principal  $T_{N''}$ -bundle over  $X'$ .

*Proof.* For each  $\sigma' \in \Sigma'$ , let  $\Sigma(\sigma') = \{\sigma \in \Sigma \mid \phi(\sigma) \subseteq \sigma'\}$ , then we have  $\varphi^{-1}(U_{\sigma'}) = X_{\Sigma(\sigma')}$ .

For sufficiency, we only need to show that  $X_{\Sigma(\sigma')} \simeq X_{\Sigma''} \times U_{\sigma'}$ . By assumption, there exist a cone  $\tilde{\sigma}' \in \tilde{\Sigma}'$  such that  $\phi(\tilde{\sigma}') = \sigma'$  and  $\phi(\tilde{\sigma}' \cap N) = \sigma' \cap N'$ , thus there exist a  $\mathbb{Z}$ -linear map  $\phi' : N' \rightarrow N$  that splits  $\phi : N \rightarrow N'$ . It induces an isomorphism  $N'' \times N' \simeq N$ , and the scalar extension  $N''_{\mathbb{R}} \times N'_{\mathbb{R}} \simeq N_{\mathbb{R}}$  carries the product fan  $(\Sigma'', N'') \times (\Sigma', N')$  to the fan  $(\Sigma(\sigma'), N)$ . Thus by Proposition 2.1.11 we have

$$X_{\Sigma(\sigma')} \simeq X_{\Sigma''} \times U_{\sigma'}.$$

For necessity, we note for each  $\sigma' \in \Sigma'$ , we have  $X_{\Sigma(\sigma')} \simeq X_{\Sigma''} \times U_{\sigma'}$ . Denote by  $\underline{\sigma}'$  the fan in  $N'_{\mathbb{R}}$  containing all the faces of  $\sigma'$  and let  $\phi_{\sigma'} : \Sigma'' \times \underline{\sigma}' \rightarrow \Sigma(\sigma')$  be its corresponding map of fan, then  $\tilde{\Sigma}' = \{\phi_{\sigma'}(\{0\} \times \sigma') \mid \sigma' \in \Sigma'\}$  satisfies all the conditions we required.  $\square$

Now we can give description of a special equivariant fiber bundle: toric  $\mathbb{P}^{r-1}$ -bundle  $\mathbb{P}(\mathbb{E}) \rightarrow X$ , where  $X$  is a toric variety and  $E$  is a rank  $r$  equivariant vector bundle on  $X$ .

**Lemma 2.1.13.** *[DS, Lemma 1.1] Let  $\mathbb{E}$  be a vector bundle over a normal toric variety  $X$ . Assume that  $\mathbb{P}(\mathbb{E})$  is toric over  $X$ , then  $\mathbb{E} = \bigoplus \mathbb{L}_i$  with  $\mathbb{L}_i$  equivariant line bundles on  $X$ .*

*Sketch of Proof.* Consider the bundle  $\mathbb{P}(\mathbb{E}) \rightarrow X$  with fiber  $F = \mathbb{P}^{r-1}$  where  $r = \text{rank } \mathbb{E}$ . Every fiber has  $r$  fixed points which defines an unramified  $r$  to one cover of  $X$ ,  $p : Y \rightarrow X$ .  $X$  is simply connected as  $X$  is normal, thus we have  $Y = \bigcup X_i$  and  $\mathbb{E} = \bigoplus \mathbb{L}_i$ .  $\square$

As in [O2, §2.1], in what follows we assume all the fans are smooth and finite. Let  $X$  be a smooth toric variety associated with the fan  $(\Sigma, N)$ . As usual, denote by

$Div(X)$  the commutative group of Weil divisors on  $X$ , i.e., the free abelian group generated by all closed integral subschemes in  $X$  of codimension 1,  $CDiv(X)$  the group of Cartier divisors, i.e., the locally principal Weil divisors,  $\mathcal{L}(X)$  the group of invertible sheaves on  $X$  and  $LB(X)$  the group of line bundles over  $X$ . Since  $X$  is smooth, it is well known that  $Div(X) = CDiv(X)$  [H2, Proposition II 6.11], and by [H2, Proposition II 6.13], there is an isomorphism between  $\mathcal{L}(X)$  and  $CDiv(X)$ . Moreover, if we associate each line bundle with the dual sheaf of the sheaf of its sections, we get an isomorphism  $LB(X) \rightarrow \mathcal{L}(X)$  [H2, Ex II 5.18]. Thus we have an isomorphism  $Div(X) \simeq LB(X)$ . Furthermore, if we denote by  $TDiv(X)$  the group of equivariant Weil divisors and  $ELB(X)$  the group of equivariant line bundles over  $X$ , then we have an isomorphism  $TDiv(X) \simeq ELB(X)$ . For each  $\sigma \in \Sigma(1)$ , denote by  $V(\sigma) = \overline{O(\sigma)}$ , then by Proposition 2.1.6, we have  $TDiv(X) = \bigoplus_{\sigma \in \Sigma(1)} \mathbb{Z}V(\sigma)$ . Hence we have an isomorphism

$$\bigoplus_{\sigma \in \Sigma(1)} \mathbb{Z}V(\sigma) \xrightarrow{\simeq} ELB(X)$$

given by

$$D = \sum_{\varrho \in \Sigma(1)} a_{\varrho} V(\varrho) = \{(U_i, f_i)\} \mapsto \{g_{ij} = f_j/f_i\},$$

where  $\mathcal{U} = \{U_i\}$  is an open cover of  $X$  and  $div(f_i) = D|_{U_i}$ .

**Definition 2.1.14.** [O2, §2.1] Let  $\Sigma$  be a fan in  $N_{\mathbb{R}}$ , a real valued function  $h : |\Sigma| \rightarrow \mathbb{R}$  is said to be a  $\Sigma$ -linear support function if it is linear in each cone of  $\Sigma$  and integral with respect to the lattice  $N$ ,  $h(|\Sigma| \cap N) \subseteq \mathbb{Z}$ .

The set of all such support functions is denoted by  $SF(N, \Sigma)$ , and this is an additive group. For each  $h \in SF(N, \Sigma)$ , by definition, it is easy to see that it is determined by the set  $\{h(n(\sigma)) \mid \sigma \in \Sigma(1) \text{ and } n(\sigma) \text{ is the minimal generator of } \sigma \text{ in } \sigma \cap N\}$ , and thus we obtain an injective homomorphism

$$SF(N, \Sigma) \rightarrow \mathbb{Z}^{\Sigma(1)}.$$

On the other hand, let  $h \in SF(N, \Sigma)$ , there exists  $m_\sigma \in M$  for each  $\sigma \in \Sigma$  such that  $h(n) = \langle m_\sigma, n \rangle$ , for  $n \in \sigma \cap N$  and that  $\langle m_\sigma, n \rangle = \langle m_\tau, n \rangle$  holds whenever  $n \in \tau \preceq \sigma$ . Note that  $m_\sigma \in M$  is a solution in  $M$  of the system of equations  $\{\langle m_\sigma, n(\varrho) \rangle = h(n(\varrho)) : \varrho \in \Sigma(1), \varrho \preceq \sigma\}$ . Since  $\Sigma$  is smooth,  $\{n(\varrho) \mid \varrho \in \Sigma(1), \varrho \preceq \sigma\}$  is a part of a  $\mathbb{Z}$ -basis of  $N$ . Thus the above system of equations always has a solution. Hence we have an isomorphism

$$SF(N, \Sigma) \xrightarrow{\simeq} \mathbb{Z}^{\Sigma(1)}.$$

**Remark 2.1.15.** When  $h \in SF(N, \Sigma)$  is given, the above  $\{m_\sigma : \sigma \in \Sigma\} \subset M$  may not be uniquely determined, since  $\{m'_\sigma : \sigma \in \Sigma\}$  satisfying  $m_\sigma - m'_\sigma \in M \cap \sigma^\perp$  for each  $\sigma \in \Sigma$  gives rise to the same  $h$ .

Therefore, there exists an isomorphism

$$SF(N, \Sigma) \xrightarrow{\simeq} \bigoplus_{\varrho \in \Sigma(1)} \mathbb{Z}V(\varrho)$$

given by  $h \mapsto -\sum_{\varrho \in \Sigma(1)} h(n(\varrho))V(\varrho)$ .

Combining the above two isomorphisms, we obtain an isomorphism

$$SF(N, \Sigma) \xrightarrow{\simeq} ELB(X)$$

given by

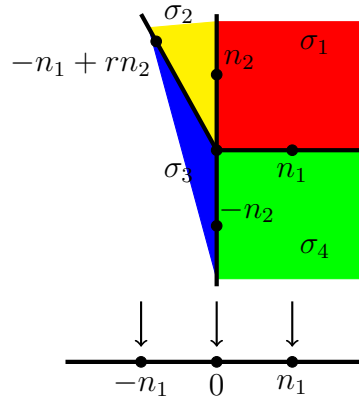
$$h = \{m_\sigma \mid \sigma \in \Sigma(1)\} \mapsto \{g_{\sigma\tau} = \chi^{m_\sigma - m_\tau}\}.$$

Now let  $D = \sum_{\varrho \in \Sigma(1)} a_\varrho V(\varrho)$  be a equivariant divisor corresponding to  $\Sigma$ -linear support function  $h = \{m_\sigma \mid \sigma \in \Sigma\}$ , i.e.  $\langle m_\sigma, n(\varrho) \rangle = -a_\varrho$  for all  $\varrho \in \underline{\sigma}(1)$  and  $\sigma \in \Sigma$ . We construct a new fan in  $N_{\mathbb{R}} \times \mathbb{R}$  as follows: for each  $\sigma \in \Sigma$ , let  $\tilde{\sigma} = \{(x, -h(x)) \mid x \in \sigma\}$ . Define  $\Sigma' = \{\tilde{\sigma} + (\{0\} \times \mathbb{R}_{\geq 0}) \mid \sigma \in \Sigma\}$ , then by Proposition 2.1.12, the toric variety  $X_{\Sigma'}$  associated to the fan  $\Sigma'$  is an equivariant line bundle  $X_{\Sigma'} \rightarrow X_\Sigma$ .

Now, we can give a detailed description of the associated fan of an equivariant projective bundle over a toric variety. We state it below:

**Corollary 2.1.16.** [O2, page 59] *Let  $X = \mathbb{P}(\mathbb{L}_1 + \cdots + \mathbb{L}_l)$  be a projective bundle over  $X'$ , where  $\mathbb{L}_1, \dots, \mathbb{L}_l$  are equivariant line bundles over toric variety  $X_{\Sigma'}$ . Suppose  $\Sigma'$ -linear support functions  $h_1, \dots, h_l$  give rise to the equivariant line bundles  $\mathbb{L}_1, \dots, \mathbb{L}_l$  on  $X'$ , respectively. Let  $N''$  be a free  $\mathbb{Z}$ -module with a basis  $\{n''_2, \dots, n''_l\}$  and let  $N := N' + N''$  and  $n''_1 := -(n''_2 + \cdots + n''_l)$ . Denote by  $\tilde{\sigma}'$  the image of each  $\sigma' \in \Sigma'$  under the map  $N'_{\mathbb{R}} \rightarrow N_{\mathbb{R}}$  which sends  $y' \in N'_{\mathbb{R}}$  to  $(y', -\sum_{1 \leq j \leq l} h_j(y')n''_j)$ . We then let  $\tilde{\Sigma}' := \{\tilde{\sigma}' : \sigma' \in \Sigma'\}$ . On the other hand, let  $\sigma''_i := \mathbb{R}_{\geq 0}n''_1 + \cdots + \mathbb{R}_{\geq 0}n''_{i-1} + \mathbb{R}_{\geq 0}n''_{i+1} + \cdots + \mathbb{R}_{\geq 0}n''_l$  for each  $1 \leq i \leq l$  and let  $\Sigma''$  be the fan in  $N''$  consisting of the faces of  $\sigma''_1, \dots, \sigma''_l$ . Then we have  $\Sigma := \{\tilde{\sigma}' + \sigma'' : \tilde{\sigma}' \in \tilde{\Sigma}', \sigma'' \in \Sigma''\}$  and  $X \simeq X_{\Sigma}$ .*

**Example 2.1.17.** Let  $X' = \mathbb{P}^1$  and  $l = 2$ . For a positive integer  $r$ , consider the rational ruled surface  $X = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(r))$ , which is called a Hirzebruch surface. The corresponding map of fans looks like this:



We always assume in the following that all the fans are complete and smooth.

**Definition 2.1.18.** [Bat] We call a nonempty subset  $\mathcal{P} = \{x_1, \dots, x_k\} \subseteq \Sigma(1)$  a *primitive collection* if for each element  $x_i \in \mathcal{P}$ , the remaining elements  $\mathcal{P} \setminus \{x_i\}$

generate a  $(k - 1)$ -dimensional cone in  $\Sigma$ , while  $\mathcal{P}$  itself does not generate any  $k$ -dimensional cone in  $\Sigma$ .

Let  $\mathcal{P} = \{x_1, \dots, x_k\} \subseteq \Sigma(1)$  be a primitive collection in  $\Sigma(1)$ . The *focus*  $\sigma(\mathcal{P})$  of  $\mathcal{P}$  is the smallest cone in  $\Sigma$  that contains  $x_1 + \dots + x_k$  (such cone exists as  $\Sigma$  is complete).

**Proposition 2.1.19.** *[Bat] Let  $\mathcal{P} = \{x_1, \dots, x_k\} \subseteq \Sigma(1)$  be a primitive collection in  $\Sigma(1)$ . Then  $\sigma(\mathcal{P}) \cap \mathcal{P} = \emptyset$ .*

*Proof.* Let  $\{y_1, \dots, y_m\}$  be the set of generators of  $\sigma(\mathcal{P})$ . It suffices to show that  $\mathcal{P} \cap \{y_1, \dots, y_m\} = \emptyset$ .

If not, assume, without loss of generality, that  $x_1 = y_1$ , consider the element  $x = x_2 + \dots + x_k$ , then  $x$  is in the interior of the cone  $\sigma$  generated by  $\{x_2, \dots, x_k\}$ . On the other hand, let  $x_1 + \dots + x_k = n_1 y_1 + \dots + n_m y_m$ , where  $n_1, \dots, n_m$  are positive integers, then  $x = (n_1 - 1)y_1 + n_2 y_2 + \dots + n_m y_m$  and  $x$  is in the interior of the cone  $\sigma'$  generated by  $\{y_1, \dots, y_m\}$  if  $n_1 > 1$ , or  $\{y_2, \dots, y_m\}$  if  $n_1 = 1$ . In both cases, we have  $\sigma = \sigma'$ . If  $\sigma'$  is generated by  $\{y_1, \dots, y_m\}$ , then  $y_1$  must be an element of  $\{x_2, \dots, x_k\}$ , this contradicts the assumption that  $x_1, \dots, x_k$  are different generators of  $\Sigma$ ; If  $\sigma'$  is generated by  $\{y_2, \dots, y_m\}$ , then  $\mathcal{P} = \{y_1, \dots, y_m\}$ , and this contradicts the fact that  $\mathcal{P}$  does not generate a cone in  $\Sigma$ .  $\square$

Let  $\Sigma \subseteq \mathbb{R}^n$  be a complete smooth fan. As we showed above, every generator  $x \in \Sigma(1)$  determines a  $T$ -invariant divisor, which is also a toric variety, and its corresponding fan  $\Sigma_x$  consists of images of all cones in  $\Sigma$  containing  $x$  via the natural projection  $\mathbb{R}^n \rightarrow \mathbb{R}^n / \mathbb{R}\langle x \rangle$ .

**Proposition 2.1.20.** *[Bat] (i) The set  $\Sigma_x(1)$  of all generators for  $\Sigma_x$  consists of the images  $\bar{x}' \in \mathbb{R}^n / \mathbb{R}\langle x \rangle$  of all generators  $x'$  such that  $\{x, x'\}$  generate a 2-dimensional cone in  $\Sigma$ .*

(ii) If  $\{\bar{x}_1, \dots, \bar{x}_k\}$  is a primitive collection in  $\Sigma_x(1)$ , then

$$\text{either } \{x, x_1, \dots, x_k\}, \text{ or } \{x_1, \dots, x_k\}$$

is a primitive collection in  $\Sigma(1)$ .

*Proof.* (i) is obvious from the construction of  $\Sigma_x$ . For (ii), let  $\{\bar{x}_1, \dots, \bar{x}_k\}$  be a primitive collection in  $\Sigma_x$ , then  $\{x, x_1, \dots, x_k\}$  does not generate a cone in  $\Sigma$ . Thus, there exists a primitive collection in  $\mathcal{P} \subseteq \{x, x_1, \dots, x_k\}$ . Since  $\{x, x_1, \dots, x_k\} \setminus \{x_i\}$  generates a cone in  $\Sigma$  for  $1 \leq i \leq k$ , we have  $\{x_1, \dots, x_k\} \subseteq \mathcal{P}$ . Thus,  $\mathcal{P} = \{x, x_1, \dots, x_k\}$ , or  $\mathcal{P} = \{x_1, \dots, x_k\}$ .  $\square$

Using the new terminologies, we can transform Corollary 2.1.16 as follows.

**Corollary 2.1.21.** [Bat, Proposition 4.1] *A smooth complete  $n$ -dimensional fan  $\Sigma$  corresponds to a toric variety  $X_\Sigma$  which is a toric  $\mathbb{P}^k$ -bundle over a smooth  $(n-k)$ -dimensional toric variety if and only if there exists a primitive collection  $\mathcal{P} = \{x_1, x_2, \dots, x_{k+1}\} \subseteq \Sigma(1)$  such that the corresponding primitive relation is  $x_1 + x_2 + \dots + x_{k+1} = 0$  and any other primitive collection in  $\Sigma(1)$  does not intersect with  $\mathcal{P}$ .*

**Definition 2.1.22.** [Bat] A smooth complete  $d$ -dimensional fan  $\Sigma$  is called a *splitting fan* if any two different primitive collections in  $\Sigma(1)$  are disjoint.

**Theorem 2.1.23.** [Bat, Theorem 4.3] *Let  $X$  be a toric variety associated with a splitting fan  $\Sigma$ , then  $X$  is a projectivization of a decomposable bundle over a toric variety which is associated with a splitting fan of a smaller dimension.*

*Proof.* By Corollary 2.1.21, we need only to show that there exists a primitive collection in  $\Sigma(1)$  with zero focus. We may prove this by induction on the number of element in  $\Sigma(1)$ .

Let  $\#\Sigma(1) = m, m \geq 2$ . When  $m = 2$ , assume  $\Sigma(1) = \{x_1, x_2\}$ , then we have  $x_1 = -x_2$  and  $\Sigma(1)$  itself is a primitive collection with zero focus. Thus the statement is true for  $m = 2$ .

Assume the statement is true for  $m < n$ . Now let  $\#\Sigma(1) = n$ . For every generator  $x_0 \in \Sigma(1)$ , consider the corresponding fan  $\Sigma_{x_0}$  of  $T$ -invariant divisor  $D_{x_0}$ .  $\Sigma_{x_0}$  is also a splitting fan and  $\#\Sigma_{x_0}(1) < n$ , thus there exists at least one primitive collection in  $\Sigma_{x_0}$  with zero focus. For every primitive collection  $\{\bar{x}_1, \dots, \bar{x}_k\}$  in  $\Sigma_{x_0}$  having zero focus, we have  $x_1 + \dots + x_k = ax_0$  for some integer  $a$ . By Proposition 2.1.20 (ii), we only need to consider two cases.

Case 1.  $\mathcal{P} = \{x_0, x_1, \dots, x_k\}$  is a primitive collection in  $\Sigma(1)$  for some  $x_0 \in \Sigma(1)$  and some primitive collection  $\{\bar{x}_1, \dots, \bar{x}_k\}$  in  $\Sigma_{x_0}$  with zero focus. Then we have  $S(\mathcal{P}) = x_0 + x_1 + \dots + x_k = (a+1)x_0$ . By Proposition 2.1.19,  $S(\mathcal{P})$  cannot be a positive multiple of  $x_0$ , thus  $a+1 \leq 0$ . If  $a+1 < 0$ , then  $a < -1$  and  $S(\mathcal{P})$  is in the interior of the cone  $\sigma \in \Sigma$  generated by  $\{x_1, \dots, x_k\}$ . By Proposition 2.1.19 again, this is impossible. Thus  $a+1 = 0$  and  $\mathcal{P} = \{x_0, x_1, \dots, x_k\}$  is a primitive collection in  $\Sigma(1)$  having zero focus.

Case 2. For any  $x_0 \in \Sigma(1)$  and any primitive collection  $\{\bar{x}_1, \dots, \bar{x}_k\}$  in  $\Sigma_{x_0}$  with zero focus,  $\mathcal{P} = \{x_1, \dots, x_k\}$  is a primitive collection in  $\Sigma(1)$ . Since every primitive collection contains at least two generators, the number of primitive collections in  $\Sigma(1)$  is at most a half of the number of generators in  $\Sigma(1)$ . Thus, there exist two different generators  $x, y \in \Sigma(1)$  and a primitive collection  $\mathcal{P} = \{x_1, \dots, x_k\}$  such that  $S(\mathcal{P}) = x_1 + \dots + x_k$  is an integral multiple of both  $x$  and  $y$ . This is possible only if  $x = -y$ . Thus  $\{x, y\}$  is a primitive collection in  $\Sigma(1)$  with zero focus.

Hence the statement is also true for  $\#\Sigma(1) = n$ . The induction is done.  $\square$

**Corollary 2.1.24.** *[Bat, Corollary 4.4] A smooth complete toric variety is produced from a projective space by a sequence of projectivizations of decomposable bundles if*



*and only if its corresponding fan is a splitting fan.*

## 2.2 Arithmetic Toric Varieties

In this section, we briefly introduce arithmetic toric varieties, the main reference is [ELST].

Let  $X_\Sigma$  be a split toric variety associated with a fan  $\Sigma \subseteq N_{\mathbb{R}}$  with torus  $T_N$ .

**Definition 2.2.1.** A *toric automorphism* of  $X_\Sigma$  is a pair of  $(\alpha, \phi)$ , where  $\alpha$  is an automorphism of the variety  $X_\Sigma$  and  $\phi$  is a group automorphism of the torus  $T$ , such that we have the following commutative diagram

$$\begin{array}{ccc} T_N \times X_\Sigma & \longrightarrow & X_\Sigma \\ \phi \times \alpha \downarrow & & \downarrow \alpha \\ T_N \times X_\Sigma & \longrightarrow & X_\Sigma. \end{array}$$

In particular, if  $t \in T_N(K)$  and  $x \in X_\Sigma(K)$  then

$$\alpha(tx) = \phi t \alpha(x),$$

where  $\phi t$  is the image of  $t$  under  $\phi$ . Since  $N = \text{Hom}(\mathbb{G}_m, T_N)$ , any automorphism of  $T_N$  is naturally induced by an automorphism  $\phi$  on  $N$ , and we use the same notation for both.

Similar to the proof of Theorem 2.1.9, if  $(\alpha, \phi)$  is a toric automorphism of  $X_\Sigma$ , we have  $\sigma \simeq \phi(\sigma)$  for every  $\sigma \in \Sigma$ , thus  $\phi$  is in the group  $\text{Aut}_\Sigma^N$  of automorphisms of  $N$  that preserve the fan  $\Sigma$ .

Since  $O(\{0\})$  is the unique dense orbit of  $T_N(K)$  on  $X_\Sigma(K)$  and  $T_N(K)$  acts freely on  $O(\{0\})$ , given a toric automorphism  $(\alpha, \phi)$  of  $X_\Sigma$ , there exist a unique  $t_\alpha \in T_N(K)$  such that

$$\alpha(\gamma_{\{0\}}) = t_\alpha \gamma_{\{0\}}.$$

where  $\gamma_{\{0\}}$  is the distinguished point of  $X_\Sigma$  corresponding to  $\{0\} \in \Sigma$ . If  $(\beta, \psi)$  is another toric automorphism, then

$$\beta \alpha(\gamma_{\{0\}}) = \beta(t_\alpha \gamma_{\{0\}}) = \psi t_\alpha t_\beta \gamma_{\{0\}},$$

and so  $t_{\beta\alpha} = t_\beta^\psi t_\alpha$ . Thus the map  $(\alpha, \phi) \mapsto (t_\alpha, \phi)$  is a homomorphism from the group of toric automorphisms of  $X_\Sigma$  to the semidirect product  $T_N(K) \rtimes \text{Aut}_\Sigma^N$ .

**Lemma 2.2.2.** [ELST, Lemma 2.1] *The map  $(\alpha, \phi) \mapsto (t_\alpha, \phi)$  is a group isomorphism from the group of toric automorphisms of  $X_\Sigma$  to  $T_N(K) \rtimes \text{Aut}_\Sigma^N$ .*

*Proof.* It suffices to show that the map  $(\alpha, \phi) \mapsto (t_\alpha, \phi)$  is invertible.

Given  $(t, \phi) \in T_N(K) \rtimes \text{Aut}_\Sigma^N$ , with  $t : M \rightarrow K^*$  a point in  $T_N(K)$ . For every  $\sigma \in \Sigma$ ,  $(t, \phi)$  defines a  $K$ -algebra homomorphism  $\phi_\sigma : K[\phi(S_\sigma)^\vee] \rightarrow K[S_\sigma]$  by sending  $u \in K[\phi(S_\sigma)^\vee]$  to  $t(u)\phi^{-1}(u)$ . Thus we have a morphism  $\alpha_\sigma : U_\sigma \rightarrow U_{\phi(\sigma)}$ . By gluing these affine pieces together, we have an automorphism  $\alpha$  of  $X_\Sigma$ . Furthermore, the action  $T_N \times X_\Sigma \rightarrow X_\Sigma$  induced by the homomorphisms  $K[S_\sigma] \rightarrow K[S_\sigma] \otimes K[M], u \mapsto u \otimes u$  satisfies  $\alpha(tx) = {}^\phi t \alpha(x)$ . Thus the assignment  $(t, \phi) \mapsto (\alpha, \phi)$  is the desired inverse.  $\square$

**Definition 2.2.3.** [ELST] Let  $K$  be a field. An *arithmetic torus over  $K$  of rank  $n$*  is an algebraic group  $\mathcal{T}$  over  $K$  such that  $\mathcal{T}_L \simeq T_{N,L}$  for some finite Galois extension  $L/K$  and lattice  $N$  of rank  $n$ . That is,  $\mathcal{T}$  is a  $L/K$ -form of the split torus  $T_{N,L}$ .

As  $T_N$  is affine and  $\text{Aut}(T_N) \simeq \text{Aut}(N)$ , the set of such  $L/K$ -forms is in natural bijection with the Galois cohomology set  $H^1(L/K, \text{Aut}(N))$ , by Proposition 1.1.16.

Since the Galois group  $\text{Gal}(L/K)$  acts on  $\text{Aut}(N)$  trivially, we have the following classification.

**Proposition 2.2.4.** [ELST, Proposition 2.5] *The  $L/K$ -forms of the torus  $T_N$  are given by conjugacy classes of homomorphisms  $\phi : \text{Gal}(L/K) \rightarrow \text{Aut}(N)$ .*

Let  $\phi : \text{Gal}(L/K) \rightarrow \text{Aut}(N)$ , the corresponding torus  $\mathcal{T}_\phi$  satisfies  $\mathcal{T}_\phi(L) = T_N(L)$ . Now let us describe the twisted action of the Galois group  $\text{Gal}(L/K)$  explicitly. For  $a \in \text{Aut}(N)$  we will also write  $a$  for its adjoint in  $\text{Aut}(M)$ . Given

$t \in T_N(L) = \text{Hom}(M, L^*)$ , for any  $g \in \text{Gal}(L/K)$ ,  ${}^g\phi t : M \rightarrow L^*$  is defined by the following composition

$${}^g\phi t : M \xrightarrow{\phi_g} M \xrightarrow{t} L^* \xrightarrow{g} L^*.$$

**Definition 2.2.5.** [ELST] An *arithmetic toric variety* over a field  $K$  is a pair  $(Y, \mathcal{T})$ , where  $\mathcal{T}$  is an arithmetic torus over  $K$  and  $Y$  is a normal variety over  $K$  equipped with a faithful action of  $\mathcal{T}$  which has a dense orbit.

Let  $L/K$  be a finite Galois extension over which the arithmetic torus  $\mathcal{T}$  splits, so that  $\mathcal{T}_L \simeq T_{N,L}$ , where  $N$  is the lattice of one-parameter subgroups of  $\mathcal{T}$ . By Proposition 2.2.4, there is a conjugacy class of group homomorphisms

$$\phi : \text{Gal}(L/K) \longrightarrow \text{Aut}(N)$$

such that  $\mathcal{T} = \mathcal{T}_\phi$ . Then  $Y_L$  is a normal variety over  $L$  and the split torus  $\mathcal{T}_L$  acts faithfully on  $Y_L$  and it has a dense orbit. As we mentioned at the beginning of the previous section, every split toric variety can be realized from some fan, thus  $Y_L$  is isomorphic to a split toric variety  $X_\Sigma$ , for some fan  $\Sigma \subseteq N_{\mathbb{R}}$ .

Thus we have an isomorphism of pairs

$$\psi : (Y_L, \mathcal{T}_L) \xrightarrow{\simeq} (X_{\Sigma,L}, T_{N,L}).$$

And we can use this isomorphism to obtain a  $\text{Gal}(L/K)$ -action on  $(X_{\Sigma,L}, T_{N,L})$  through the  $\text{Gal}(L/K)$ -action on  $(Y_L, \mathcal{T}_L)$ , and thus get a  $\text{Gal}(L/K)$ -action on the fan  $\Sigma \subseteq N_{\mathbb{R}}$ , and thus the homomorphism  $\phi : \text{Gal}(L/K) \rightarrow \text{Aut}(N)$  for which  $\mathcal{T} = \mathcal{T}_\phi$  may be chosen so that  $\phi(\text{Gal}(L/K)) \subset \text{Aut}_\Sigma^N$ . For  $g \in \text{Gal}(L/K)$ ,  $t_g \in T_N(L)$  is defined by

$${}^g\gamma_{\{0\}} = t_g \gamma_{\{0\}},$$

where  $\gamma_{\{0\}}$  is the distinguished point of  $X_\Sigma$  corresponding to  $\{0\} \in \Sigma$ .

Just as we showed in §1.1.3, there is a relationship between the  $L/K$ -forms of the pairs  $(X_{\Sigma,L}, T_{N,L})$  and the first cohomology group, the interested reader may refer to [ELST, §3] for more information.

To finish this section, we give two examples of arithmetic toric varieties corresponding to the algebra of quaternions over  $\mathbb{R}$  and cyclic algebra.

**Example 2.2.6.** Let  $\mathbb{H}$  be the algebra of quaternions, a 4-dimensional algebra with basis  $1, i, j, k$  over the real numbers field  $\mathbb{R}$ , the multiplication being determined by the rules

$$i^2 = -1, j^2 = -1, ij = -ji = k.$$

This is in fact a division algebra over  $\mathbb{R}$ , hence a central simple algebra over  $\mathbb{R}$ . We have an isomorphism

$$\phi : \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow M_2(\mathbb{C})$$

with

$$\phi(i \otimes 1) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \phi(j \otimes 1) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \phi(k \otimes 1) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

where  $i \in \mathbb{C}$  satisfying  $i^2 = -1$ .

Let  $Gal(\mathbb{C}/\mathbb{R}) = \{e, g\}$ , then for any  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{C})$ , we have

$$g \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \begin{pmatrix} \bar{d} & -\bar{c} \\ -\bar{b} & \bar{a} \end{pmatrix}.$$

By Example 1.2.20, the projective curve  $\mathbb{P}_{\mathbb{C}}^1$  is the variety associated to  $M_2(\mathbb{C})$ , and for  $t \in \mathbb{P}_{\mathbb{C}}^1$ , we have  $g(t) = -\bar{t}^{-1}$ .

By Lemma 2.2.2, the toric automorphism group of  $\mathbb{P}_{\mathbb{C}}^1$  is  $\mathbb{C}^* \rtimes \{\pm I\}$ , where  $\{\pm I\}$  acts on  $\mathbb{C}^*$  by  $-I$  sending  $t \in \mathbb{C}^*$  to  $t^{-1}$ . Thus the arithmetic toric variety associated

to  $\mathbb{H}$  is determined by the following homomorphism

$$\begin{aligned} \mathbf{c} : Gal(\mathbb{C}/\mathbb{R}) &\longrightarrow \mathbb{C}^* \rtimes \{\pm I\} \\ g &\longmapsto (-1, -I). \end{aligned}$$

In the following example, for the description of cyclic algebra, the reader may refer to [Dr] and [GS].

**Example 2.2.7.** Let  $L/K$  is a cyclic extension,  $n := |L:K|$  ( $(char(K), n) = 1$ ),  $G := Gal(L/K) = \langle \sigma \rangle$  and  $a \in K^*$ , then we have a cyclic algebra

$$(a, L/K, \sigma) := \bigoplus_{i=0}^{n-1} Le^i$$

with multiplication  $e^n = a$ ,  $le = e\sigma(l)$  for all  $l \in L$ .

$(a, L/K, \sigma)$  is a central simple algebra over  $K$  with  $L$  as its splitting field, i.e.  $(a, L/K, \sigma) \otimes_K L \cong M_n(L)$ .

Let

$$\tilde{F}(a) = \begin{pmatrix} 0 & 0 & \cdots & 0 & a \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \in GL_n(L).$$

A computation shows that  $\tilde{F}(a)^n = aI_n$ . If we denote by  $F(a)$  its image in the group  $PGL_n(L)$ , we have  $F(a)^n = 1$ . And the 1-cocycle associated to  $(a, L/K, \sigma)$  is

$$\mathbf{c}(a) : G \longrightarrow PGL_n(L)$$

$$\sigma \longmapsto F(a).$$

Thus the twisted action of  $\sigma$  on  $M_n(L)$  sends  $M$  to  $\tilde{F}(a) \cdot \sigma(M) \cdot \tilde{F}(a)^{-1}$  for any  $M \in M_n(L)$ .

By Example 1.2.20, the projective space  $\mathbb{P}_L^{n-1}$  is the variety associated to  $M_n(L)$ , and for  $[l_1, \dots, l_n] \in \mathbb{P}_L^{n-1}$ , we have  $\sigma([l_1, \dots, l_n]) = [a^{-1}\sigma(l_n), \sigma(l_1), \dots, \sigma(l_{n-1})]$ . Thus the arithmetic toric variety associated to  $(a, L/K, \sigma)$  is determined by the following homomorphism

$$\mathbf{c}(a) : G \longrightarrow (L^*)^n \rtimes \text{Aut}_\Sigma^N$$

$$\sigma \longmapsto ((a, \dots, a), A),$$

where  $(\Sigma, N)$  is the standard fan associated to  $\mathbb{P}^n$  and  $A = \begin{pmatrix} 0 & 0 & \cdots & 0 & -1 \\ 1 & 0 & \cdots & 0 & -1 \\ 0 & 1 & \cdots & 0 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -1 \end{pmatrix}$ .

# Chapter 3

## Derived Categories of Coherent sheaves

### 3.1 Derived Categories of Coherent sheaves

The derived category is a rather complicated object and it was introduced by Grothendieck at the beginning of the 1960's. Later its internal structure was axiomatized by Verdier through the notion of triangulated category in his 1967 thesis [V1] and [V2]. Roughly speaking, given an abelian category  $\mathcal{A}$ , its derived category gives a transparent and compact way to handle the totality of cohomological data attached to  $\mathcal{A}$  and equates a given object of  $\mathcal{A}$  to all of its resolutions. For a quick skimming of the derived categories of sheaves, the reader may refer to [C]; for more thorough introduction to derived categories, the reader may refer to [GM] and [Hu].

Let us first briefly introduce how we construct the derived category  $D(\mathcal{A})$  from an abelian category  $\mathcal{A}$ . Let  $\mathcal{A}$  be an abelian category, then the *category of complexes* of  $\mathcal{A}$ ,  $Kom(\mathcal{A})$ , is the category whose objects are complexes of objects of  $\mathcal{A}$ , and morphisms between complexes are chain maps. Any morphism  $f : A^\bullet \rightarrow B^\bullet$  induces



natural homomorphisms  $H^i(f) : H^i(A^\bullet) \rightarrow H^i(B^\bullet), i \in \mathbb{Z}$ . A short exact sequence  $0 \rightarrow A^\bullet \rightarrow B^\bullet \rightarrow C^\bullet \rightarrow 0$  in  $Kom(\mathcal{A})$  induces a long exact sequence

$$\dots \rightarrow H^i(A^\bullet) \rightarrow H^i(B^\bullet) \rightarrow H^i(C^\bullet) \rightarrow H^{i+1}(A^\bullet) \rightarrow \dots$$

We say two morphisms  $f, g : A^\bullet \rightarrow B^\bullet$  in  $Kom(\mathcal{A})$  are *homotopy equivalent*,  $f \sim g$ , if there exists a collection of homomorphisms  $h^i : A^i \rightarrow B^{i-1}, i \in \mathbb{Z}$ , such that

$$f^i - g^i = h^{i+1} \circ d_A^i + d_B^{i-1} \circ h^i.$$

This is an equivalence relation, and if  $f \sim g$ , we have  $H^i(f) = H^i(g)$  for all  $i \in \mathbb{Z}$ . Modulo the homotopy equivalent relation, then we get a new category, the *homotopy category*  $K(\mathcal{A})$  of  $\mathcal{A}$  with objects  $Ob(Kom(\mathcal{A}))$  and morphisms  $Hom_{K(\mathcal{A})}(A^\bullet, B^\bullet) = Hom_{Kom(\mathcal{A})}(A^\bullet, B^\bullet) / \sim$ .

We say a morphism of complexes  $f : A^\bullet \rightarrow B^\bullet$  is a *quasi-isomorphism* if it induces isomorphisms of cohomological groups  $H^i(f) : H^i(A^\bullet) \xrightarrow{\cong} H^i(B^\bullet)$  for all  $i \in \mathbb{Z}$ . By a process called *localization* with respect to the class of quasi-isomorphisms in  $K(\mathcal{A})$  (treat the quasi-isomorphisms in  $K(\mathcal{A})$  as isomorphisms), we may obtain a new category – the *derived category*,  $D(\mathcal{A})$ , of the abelian category  $\mathcal{A}$ . For details, the reader may refer to [GM], we summarize it as follows [Hu, Theorem 2.10]:

There exists a category  $D(\mathcal{A})$ , the *derived category* of  $\mathcal{A}$ , and a functor

$$Q : Kom(\mathcal{A}) \rightarrow D(\mathcal{A})$$

such that:

- i) If  $f : A^\bullet \rightarrow B^\bullet$  is a quasi-isomorphism, then  $Q(f)$  is an isomorphism in  $D(\mathcal{A})$ .
- ii) Any functor  $F : Kom(\mathcal{A}) \rightarrow D$  satisfying property i) factorizes uniquely over  $Q : Kom(\mathcal{A}) \rightarrow D(\mathcal{A})$ , i.e. there exists a unique functor (up to isomorphism)

$G : D(\mathcal{A}) \rightarrow D$  with  $F \simeq G \circ Q$ :

$$\begin{array}{ccc} \text{Kom}(\mathcal{A}) & \xrightarrow{Q} & D(\mathcal{A}) \\ & \searrow F & \swarrow \text{dotted } G \\ & & D \end{array}$$

Moreover, we have the following facts [Hu, Corollary 2.11]:

- i) The objects of the two categories  $\text{Kom}(\mathcal{A})$  and  $D(\mathcal{A})$  are identical.
- ii) The cohomology objects  $H^i(A^\bullet)$  of an object  $A^\bullet \in D(\mathcal{A})$  are well-defined.
- iii) Viewing any object in  $\mathcal{A}$  as a complex concentrated in degree zero yields an equivalence between  $\mathcal{A}$  and the full subcategory of  $D(\mathcal{A})$  that consists of all complexes  $A^\bullet$  with  $H^i(A^\bullet) = 0$  for  $i \neq 0$ .

The derived category  $D(\mathcal{A})$  (and  $K(\mathcal{A})$ ) has two fundamental operations built in: shifting:  $A^\bullet \mapsto A^\bullet[k]$ ,  $k \in \mathbb{Z}$  such that  $(A^\bullet[k])^i = A^{k+i}$  and cones: for any morphism of complexes  $f : A^\bullet \rightarrow B^\bullet$ , its mapping cone is the complex  $C(f)^\bullet$  with

$$C(f)^i = A^{i+1} \oplus B^i \quad \text{and} \quad d_{C(f)}^i := \begin{pmatrix} d_A^{i+1} & 0 \\ f^{i+1} & d_B^i \end{pmatrix}.$$

Then we may define the maps  $g : B^\bullet \rightarrow C(f)^\bullet$  and  $h : C(f)^\bullet \rightarrow A^\bullet[1]$  in the obvious way and get a *distinguished triangle*

$$A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C(f)^\bullet \xrightarrow{h} A^\bullet[1].$$

The categories  $K(\mathcal{A})$  and  $D(\mathcal{A})$  are no longer abelian categories, but are triangulated categories [GM]. Instead of short exact sequences, we deal with *exact triangles*, triangles that are isomorphic to distinguished triangles.

Analogously, given an abelian category  $\mathcal{A}$ , we can construct  $\text{Kom}^*(\mathcal{A})$ ,  $K^*(\mathcal{A})$  and  $D^*(\mathcal{A})$ , with  $*$  = +, −, or  $b$ , be the categories with objects of complexes  $A^\bullet$  such that  $A^i = 0$  for  $i \ll 0$ ,  $i \gg 0$ , respectively  $|i| \gg 0$ .

Now let us focus on the derived category of coherent sheaves of a scheme  $X$ , we denote by  $D^b(X)$  the bounded derived category of the abelian category  $\mathbf{Coh}(X)$ , i.e.  $D^b(X) = D^b(\mathbf{Coh}(X))$ .

When dealing with sheaves, we are always interested in functors like  $f_*$ ,  $f^*$ ,  $\text{Hom}$ ,  $\otimes$ ,  $\Gamma(X, -)$ , etc. Now we are going to give a very brief description of the derived functors between the derived categories (a functor  $RF$  in the case of a left exact functor  $F$ ; or  $LF$  in the case a right exact functor  $F$ ).

As an example, we give the brief construction of  $Rf_*$ , the right derived functor of  $f_*$ , which also works for other left exact functors. For simplicity, we assume  $f : X \rightarrow Y$  is a projective (or proper) morphism of noetherian schemes. Then  $f_* : \mathbf{Qcoh}(X) \rightarrow \mathbf{Qcoh}(Y)$  and  $f_* : \mathbf{Coh}(X) \rightarrow \mathbf{Coh}(Y)$  are both left exact, where the latter one follows from [H2, Theorem III 8.8].

**Proposition 3.1.1.** [Hu] *Let  $\mathcal{A}$  be an abelian category with enough injectives, then the natural functor*

$$\iota : K^+(\mathcal{I}_{\mathcal{A}}) \longrightarrow D^+(\mathcal{A})$$

*is an equivalence, where  $\mathcal{I}_{\mathcal{A}}$  are the injectives of  $\mathcal{A}$ .*

*Proof.* See [Hu]; Proposition 2.40. □

It is well known that the category  $\mathbf{Qcoh}(X)$  has enough injectives [H1, II, 7.18]. For  $f_* : \mathbf{Qcoh}(X) \rightarrow \mathbf{Qcoh}(Y)$ , by Proposition 3.1.1 we may define its right derived functor

$$Rf_* : D^+(\mathbf{Qcoh}(X)) \longrightarrow D^+(\mathbf{Qcoh}(Y))$$

through the composition

$$\begin{aligned} D^+(\mathbf{Qcoh}(X)) &\xrightarrow{\iota^{-1}} K^+(\mathcal{I}_{\mathbf{Qcoh}(X)}) \xrightarrow{\hookrightarrow} K^+(\mathbf{Qcoh}(X)) \\ &\xrightarrow{K(f_*)} K^+(\mathbf{Qcoh}(Y)) \xrightarrow{Q_{\mathbf{Qcoh}(Y)}} D^+(\mathbf{Qcoh}(Y)), \end{aligned}$$

where  $Q_{\mathbf{Qcoh}(Y)}$  is the localization functor.

Note that for any sheaf  $\mathcal{F} \in \mathbf{Qcoh}(X)$ , regarded as a complex in  $D^+(\mathbf{Qcoh}(X))$  with just  $\mathcal{F}$  in position 0 and zero elsewhere, then  $\iota^{-1}(\mathcal{F})$  is just the classical injective resolution of  $\mathcal{F}$ . What we gained here is that  $\iota^{-1}$  maps every complex to a complex with injective objects that is quasi-isomorphic to itself.

Now, let us state a relation between  $D^b(X)$  and  $D^b(\mathbf{Qcoh}(X))$ .

**Proposition 3.1.2.** *[Hu] Let  $X$  be a noetherian scheme. Then the natural inclusion functor*

$$D^b(X) \longrightarrow D^b(\mathbf{Qcoh}(X))$$

*defines an equivalence between the bounded derived category  $D^b(X)$  and the full triangulated subcategory  $D_{Coh}^b(\mathbf{Qcoh}(X))$  of bounded complexes of quasi-coherent sheaves with coherent cohomology.*

*Proof.* See [Hu]; Proposition 3.5. □

To construct the right derived functor  $Rf_* : D^b(X) \rightarrow D^b(Y)$ , we need two more results of higher direct images of sheaves. We state them below.

**Theorem 3.1.3.** *[Hu, Theorme 3.22, Corollary 2.68] Let  $f : X \rightarrow Y$  be a morphism of noetherian schemes and  $\mathcal{F}$  a quasi-coherent sheaf on  $X$ . Then the higher direct images  $R^i f_* \mathcal{F}$  are trivial for  $i > \dim(X)$ , and  $Rf_*(\mathcal{F}^\bullet) \in D^b(\mathbf{Qcoh}(Y))$  for any  $\mathcal{F}^\bullet \in D^b(\mathbf{Qcoh}(X))$ .*

**Theorem 3.1.4.** *[Hu, Theorem 3.23] Let  $f : X \rightarrow Y$  be a projective (or proper) morphism of noetherian schemes, then the higher direct images  $R^i f_* \mathcal{F}$  of a coherent sheaf  $\mathcal{F}$  on  $X$  are again coherent sheaves on  $Y$ .*

Finally, we can give the right derived functor  $Rf_* : D^b(X) \rightarrow D^b(Y)$ , which is

summarized in the following diagram:

$$\begin{array}{ccc}
 D^+(\mathbf{Qcoh}(X)) & \xrightarrow{Rf_*} & D^+(\mathbf{Qcoh}(Y)) \\
 \uparrow \wr & & \uparrow \wr \\
 D^b(\mathbf{Qcoh}(X)) & \xrightarrow{\text{Theorem 3.1.3}} & D^b(\mathbf{Qcoh}(Y)) \\
 \uparrow \wr & & \uparrow \wr \\
 D^b(X) & \xrightarrow{\text{Theorem 3.1.4}} & D^b(Y)
 \end{array}$$

In sheaf theory, while we have enough injectives and we may deal with the right derived functors of left exact functors like above, there are almost never have enough projectives. To get around this problem, we may replace injective objects by acyclic objects. When the class of acyclic objects is rich enough to be able to arrange every complex in  $D^-(\mathbf{Qcoh}(X))$  is quasi-isomorphic to a complex of acyclic objects, then we may use a similar construction for left exact functors as above to define  $LF$ , the left derived functor of a right exact functor  $F$ .

In the following we list some great technical advantages of using the derived category [Hu]. We will make use of them later.

i) Projection formula: let  $f : X \rightarrow Y$  be a proper morphism of projective schemes.

For any  $\mathcal{F}^\bullet \in D^b(X)$ ,  $\mathcal{E}^\bullet \in D^b(Y)$ , there exists a natural isomorphism

$$Rf_* \mathcal{F}^\bullet \otimes^L \mathcal{E}^\bullet \xrightarrow{\simeq} Rf_*(\mathcal{F}^\bullet \otimes^L Lf^* \mathcal{E}^\bullet).$$

ii) Let  $\mathcal{F}^\bullet \in D^-(X)$ , we have

$$R\Gamma \circ R\mathcal{H}om(\mathcal{F}^\bullet, -) = R\mathcal{H}om(\mathcal{F}^\bullet, -).$$

iii) Flat base change: if we are given a cartesian diagram

$$\begin{array}{ccc}
 X \times_Z Y & \xrightarrow{v} & Y \\
 g \downarrow & & f \downarrow \\
 X & \xrightarrow{u} & Z
 \end{array}$$

with  $u : X \rightarrow Z$  flat and  $f : Y \rightarrow Z$  proper. Then there exists a functorial isomorphism:

$$u^* Rf_* \mathcal{F}^\bullet \xrightarrow{\cong} Rg_* v^* \mathcal{F}^\bullet$$

for any  $\mathcal{F}^\bullet \in D(\mathbf{Qcoh}(Y))$ .

As  $u$  and, therefore,  $v$  are flat, both functors  $u^*$  and  $v^*$  are exact and need not be derived.

**Definition 3.1.5.** Let  $\mathcal{D}$  be a triangulated category and  $\mathcal{S}$  be a set of objects in  $\mathcal{D}$ , we denote by  $\langle \mathcal{S} \rangle$  the minimal full triangulated subcategory of  $\mathcal{D}$  containing all the objects in  $\mathcal{S}$ , i.e. it contains all the objects in  $\mathcal{S}$  and is closed under shifting and taking cones. We say  $\mathcal{S}$  *generates*  $\mathcal{D}$  if  $\langle \mathcal{S} \rangle = \mathcal{D}$ , that is,  $\langle \mathcal{S} \rangle$  is equivalent to  $\mathcal{D}$ .

Actually,  $\langle \mathcal{S} \rangle$  is the intersection of all full triangulated subcategories of  $\mathcal{D}$  containing all the objects in  $\mathcal{S}$ .  $\langle \mathcal{S} \rangle$  exists since  $\mathcal{D}$  contains  $\mathcal{S}$ .

**Definitions 3.1.6.** Let  $\mathcal{D}$  be a  $K$ -linear triangulated category. An object  $E$  is said to be *exceptional* if

$$\mathrm{Hom}(E, E) = K \quad \text{and} \quad \mathrm{Hom}(E, E[l]) = 0 \quad \forall l \neq 0.$$

An *exceptional collection* in  $\mathcal{D}$  is an ordered collection  $(E_0, E_1, \dots, E_n)$  of exceptional objects, satisfying

$$\mathrm{Hom}(E_j, E_i[l]) = 0 \quad \text{for all } l \text{ when } 0 \leq i < j \leq n.$$

If in addition

$$\mathrm{Hom}(E_j, E_i[l]) = 0 \quad \text{for } 0 \leq j \leq i \leq n, l \neq 0,$$

we call  $(E_0, E_1, \dots, E_n)$  a *strong exceptional collection*. The collection is *full* (or *complete*) if it generates  $\mathcal{D}$ .

When dealing with  $D^b(X)$ , we will always limit our discussion of exceptional objects within the set of coherent sheaves on  $X$ . For coherent sheaves  $\mathcal{E}$  and  $\mathcal{F}$ , we have [Hu, Proposition 2.56]

$$\mathrm{Hom}_{D^b(X)}(\mathcal{E}, \mathcal{F}[k]) = \mathrm{Ext}_{\mathcal{O}_X}^k(\mathcal{E}, \mathcal{F}).$$

**Definition 3.1.7.** [Ba] A coherent sheaf  $\mathcal{T}$  of  $\mathcal{O}_X$ -module on a smooth projective variety  $X$  is called a *tilting sheaf* if

- (i) it has no higher self-extensions, i.e.  $\mathrm{Ext}_{\mathcal{O}_X}^i(\mathcal{T}, \mathcal{T}) = 0$  for all  $i > 0$ ,
- (ii) the endomorphism algebra of  $\mathcal{T}$ ,  $\mathrm{End}_{\mathcal{O}_X}(\mathcal{T})$ , has finite global homological dimension,
- (iii) the direct summands of  $\mathcal{T}$  generate the bounded derived category  $D^b(X)$ .

If  $\mathcal{T}$  is locally free, then it is called a *tilting bundle*.

The reason why the notion of tilting sheaf is so important is the following result.

**Theorem 3.1.8.** [Ba, Theorem 3.1.2] *Let  $X$  be a smooth projective variety and  $\mathcal{T}$  be a tilting sheaf over  $X$ , with associated algebra  $A := \mathrm{End}_{\mathcal{O}_X}(\mathcal{T})$ . Then the functors*

$$F := \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{T}, -) : \mathbf{Coh}(X) \rightarrow \mathrm{mod}\text{-}A$$

*(here  $\mathrm{mod}\text{-}A$  is the category of finitely generated right  $A$ -modules) and*

$$G := - \otimes_A \mathcal{T} : \mathrm{mod}\text{-}A \rightarrow \mathbf{Coh}(X)$$

*induce equivalences of triangulated categories*

$$RF : D^b(X) \rightarrow D^b(\mathrm{mod}\text{-}A)$$

$$LG : D^b(\mathrm{mod}\text{-}A) \rightarrow D^b(X)$$

*that are quasi-inverse to each other.*

And the reason why full strong exceptional collections are interesting is the following lemma.

**Lemma 3.1.9.** *Let  $(\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_n)$  be a full strong exceptional collection of coherent sheaves on  $X$  over a field  $K$ , then  $\mathcal{T} = \bigoplus_{i=0}^n \mathcal{F}_i^{\oplus l_i}$ ,  $l_i \geq 1$  is an integer for  $0 \leq i \leq n$ , is a tilting sheaf on  $X$ .*

*Proof.* As *Ext* functor commutes with finite direct sum, we only need to show that the endomorphism algebra of  $\mathcal{T}$ ,  $End(\mathcal{T})$ , has finite global homological dimension. We can get this by induction on  $n$  using Proposition 3.2.6.

Notice that

$$End(\mathcal{T}) = \begin{pmatrix} M_{l_0}(K) & 0 & \cdots & 0 \\ Hom(\mathcal{F}_0, \mathcal{F}_1^{\oplus l_1}) & M_{l_1}(K) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ Hom(\mathcal{F}_0, \mathcal{F}_n^{\oplus l_n}) & Hom(\mathcal{F}_1^{\oplus l_1}, \mathcal{F}_n^{\oplus l_n}) & \cdots & M_{l_n}(K) \end{pmatrix}.$$

When  $n = 0$ ,  $End(\mathcal{T}) = M_{l_0}(K)$ . By Morita theory, the category of  $M_{l_0}(K)$ -modules is equivalent to the category of  $K$ -vector space. As  $K$  has finite global dimension, so does  $M_{l_0}(K)$ .

If we write

$$End(\mathcal{T}) = \begin{pmatrix} A & 0 \\ M & M_{l_n}(K) \end{pmatrix},$$

where

$$A = \begin{pmatrix} M_{l_0}(K) & 0 & \cdots & 0 \\ Hom(\mathcal{F}_0, \mathcal{F}_1^{\oplus l_1}) & M_{l_1}(K) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ Hom(\mathcal{F}_0, \mathcal{F}_{n-1}^{\oplus l_{n-1}}) & Hom(\mathcal{F}_1^{\oplus l_1}, \mathcal{F}_{n-1}^{\oplus l_{n-1}}) & \cdots & M_{l_{n-1}}(K) \end{pmatrix},$$

$$M = \left( Hom(\mathcal{F}_0, \mathcal{F}_n^{\oplus l_n}) \quad Hom(\mathcal{F}_1^{\oplus l_1}, \mathcal{F}_n^{\oplus l_n}) \quad \cdots \quad Hom(\mathcal{F}_{n-1}^{\oplus l_{n-1}}, \mathcal{F}_n^{\oplus l_n}) \right).$$



By induction, the algebra  $A$  has finite global dimension. And  $M_n(K)$  has finite global dimension. So by Proposition 3.2.6, we conclude that  $\text{End}(\mathcal{T})$  has finite global dimension.  $\square$

There is a partial converse for this lemma. If the tilting bundle  $\mathcal{T}$  is a direct sum of line bundles, then its summands give rise to a full strong exceptional collection [CM1, Lemma 4.5], also see [Cr, Proposition 2.7].

**Theorem 3.1.10.** [Be] *The derived category  $D^b(\mathbb{P}^n)$  is generated by the strong exceptional collection*

$$\{\mathcal{O}(-n), \mathcal{O}(-n+1), \dots, \mathcal{O}(-1), \mathcal{O}\}.$$

*Sketch of proof.* That this is a strong exceptional collection follows from [H2, III Proposition 6.3, 6.7] and [H2, Theorem 5.1].

To prove this collection is full, we need the Beilinson's resolution of the diagonal, i.e. the Koszul resolution of the diagonal  $\Delta$  on  $\mathbb{P}^n \times \mathbb{P}^n$ :

$$0 \rightarrow p_1^* \Omega^n(n) \otimes p_2^* \mathcal{O}(-n) \rightarrow \dots \rightarrow p_1^* \Omega^1(1) \otimes p_2^* \mathcal{O}(-1) \rightarrow \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^n} \rightarrow \mathcal{O}_\Delta \rightarrow 0$$

where  $p_i : \mathbb{P}^n \times \mathbb{P}^n \rightarrow \mathbb{P}^n$ ,  $i = 1, 2$ , are the projections.

Split off this resolution into short exact sequences

$$0 \rightarrow p_1^* \Omega^n(n) \otimes p_2^* \mathcal{O}(-n) \rightarrow p_1^* \Omega^{n-1}(n-1) \otimes p_2^* \mathcal{O}(-n+1) \rightarrow M_{n-1} \rightarrow 0$$

$$0 \rightarrow M_{n-1} \rightarrow p_1^* \Omega^{n-2}(n-2) \otimes p_2^* \mathcal{O}(-n+2) \rightarrow M_{n-2} \rightarrow 0$$

$\vdots$

$$0 \rightarrow M_1 \rightarrow \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^n} \rightarrow \mathcal{O}_\Delta \rightarrow 0.$$

Each of these short exact sequences can be regarded as a distinguished triangles in  $D^b(\mathbb{P}^n \times \mathbb{P}^n)$ . For an object  $F \in D^b(\mathbb{P}^n)$  [Hu, Ex 2.27], tensor product with  $Lp_2^* F$

and direct image under the first projection  $p_1$  yields distinguished triangles on the second factors (i.e.  $\Phi_M(-) = Rp_{2*}(M \otimes^L Lp_1^*(-))$ )

$$\Phi_{M_{i+1}}(F) \rightarrow \Phi_{p_1^*\Omega^i(i) \otimes p_2^*\mathcal{O}(-i)}(F) \rightarrow \Phi_{M_i}(F) \rightarrow \Phi_{M_{i+1}}(F)[1].$$

By the projection formula, we have  $\Phi_{\mathcal{O}_\Delta}(F) \simeq F$  and  $\Phi_{p_1^*\Omega^i(i) \otimes p_2^*\mathcal{O}(-i)}(F) \simeq R\Gamma(\mathbb{P}^n, F \otimes \Omega^i(i)) \otimes_K \mathcal{O}(-i)$ , a complex generated by  $\mathcal{O}(-i)$  (actually, it has all differentials zero and has  $\dim R^k\Gamma(\mathbb{P}^n, F \otimes \Omega^i(i))$  copies of  $\mathcal{O}(-i)$  in position  $k$ ).

Therefore, for any object  $F \in D^b(\mathbb{P}^n)$ ,  $F$  is generated by the set  $\{\mathcal{O}(-n), \mathcal{O}(-n+1), \dots, \mathcal{O}(-1), \mathcal{O}\}$ .  $\square$

**Remark 3.1.11.** For a more a detailed proof, the reader may refer to §3 in [C] and §8.3 in [Hu].

Later, in a series of papers [K1], [K2], [K2], [K4], Kapranov gave full strong exceptional collections for Grassmann, flag and quadric varieties.

Usually, it is hard to find a full strong exceptional collection in a derived category. It is occasionally useful to split a derived category into more manageable building blocks before starting to look for complete exceptional sequences. This is the motivation for giving the following definitions from [Bon, BK].

**Definitions 3.1.12.** Let  $\mathcal{B}$  be a full triangulated subcategory of triangulated category  $\mathcal{D}$ . The *right orthogonal* to  $\mathcal{B}$  is the full triangulated subcategory  $\mathcal{B}^\perp$  consisting of the objects  $C$  such that  $\text{Hom}(B, C) = 0$  for all  $B \in \mathcal{B}$ . The *left orthogonal*  ${}^\perp\mathcal{B}$  is defined analogously.

Let  $\mathcal{B}$  be a strictly full triangulated subcategory of triangulated category  $\mathcal{D}$ . We say that  $\mathcal{B}$  is *right admissible* (resp. *left admissible*) if for each  $D \in \mathcal{D}$ , there is a distinguished triangle  $B \rightarrow D \rightarrow C$ , where  $B \in \mathcal{B}$  and  $C \in \mathcal{B}^\perp$  (resp.  $C' \rightarrow D \rightarrow B$ , where  $C' \in {}^\perp\mathcal{B}$  and  $B \in \mathcal{B}$ ). A subcategory is said be *admissible* if it is left and right admissible.

**Lemma 3.1.13.** *Let  $\mathcal{A}, \mathcal{B}$  be two full triangulated subcategories of a triangulated category  $\mathcal{D}$ . Suppose  $\mathcal{A} \subseteq \mathcal{B}^\perp$ . Let  $\mathcal{C}$  be the full subcategory whose objects are those  $X$  that fit into triangles  $B \rightarrow X \rightarrow A$  with  $B \in \mathcal{B}$ ,  $A \in \mathcal{A}$ . Then  $\mathcal{C}$  is closed under shifts and taking cones. Hence  $\mathcal{C} = \langle \mathcal{A}, \mathcal{B} \rangle$ .*

*Proof.* It is clear that  $\mathcal{C}$  is closed under shifts. Let  $f : X \rightarrow X'$  be a morphism of objects in  $\mathcal{C}$ . Then, there exist unique morphisms  $\phi : B \rightarrow B'$  and  $\varphi : A \rightarrow A'$  up to isomorphisms such that the following diagram

$$\begin{array}{ccccc} B & \xrightarrow{\phi} & B' & \longrightarrow & C_\phi \\ \downarrow g & & \downarrow & & \\ X & \xrightarrow{f} & X' & \longrightarrow & C_f \\ \downarrow & & \downarrow & & \\ A & \xrightarrow{\varphi} & A' & \longrightarrow & C_\varphi \end{array}$$

commutes, where  $C_h$  denotes the cone of a morphism  $h$ ;  $B, B' \in \mathcal{B}$  and  $A, A' \in \mathcal{A}$ . Indeed, applying the functor  $\text{Hom}(B, -)$  to the second column triangle, we have  $\text{Hom}(B, B') = \text{Hom}(B, X')$ . In that case  $\phi$  is the preimage of  $f \circ g$  under this isomorphism. And thus there exists a morphism  $\varphi$  such that the above diagram commutes. By the generalized octahedron axiom [BBD, Proposition 1.1.11], the above diagram can be closed using a distinguished triangle in the last column. This triangle is the desired triangle.  $\square$

**Proposition 3.1.14.** [Bon, Lemma 3.1] *Let  $\mathcal{B}$  be a strictly full triangulated subcategory of a triangulated category  $\mathcal{D}$ . Then the following are equivalent:*

- (i)  $\mathcal{B}$  is right (resp. left) admissible.
- (ii)  $\mathcal{D}$  is generated by  $\mathcal{B}$  and  $\mathcal{B}^\perp$  (resp. by  ${}^\perp\mathcal{B}$  and  $\mathcal{B}$ ) as a triangulated category.

*Proof.* It is obvious that (i) implies (ii), and the converse follows from Lemma 3.1.13.  $\square$

**Corollary 3.1.15.** *Let  $\mathcal{A}, \mathcal{B}$  be two full triangulated subcategories of a triangulated category  $\mathcal{D}$  such that  $\mathcal{D}$  is generated by  $\mathcal{A}$  and  $\mathcal{B}$ . Suppose that  $\mathcal{A} \subseteq \mathcal{B}^\perp$ , then  $\mathcal{B}$  is right admissible and  $\mathcal{A} = \mathcal{B}^\perp$ .*

*Proof.* That  $\mathcal{B}$  is right admissible follows from Lemma 3.1.13 and Proposition 3.1.14, since  $\mathcal{A}$  and  $\mathcal{B}$ , and hence  $\mathcal{B}$  and  $\mathcal{B}^\perp$  generate  $\mathcal{D}$ . Let  $B^\perp$  be an object in  $\mathcal{B}^\perp$ . Then by Lemma 3.1.13, we have a distinguished triangle  $B \xrightarrow{0} B^\perp \xrightarrow{\varphi} A$ , where  $B \in \mathcal{B}$  and  $A \in \mathcal{A}$ . Thus we have the following commutative diagram

$$\begin{array}{ccccc}
 0 & \longrightarrow & B^\perp & \xlongequal{\quad} & B^\perp \\
 \downarrow & & \parallel & & \downarrow \varphi \\
 B & \xrightarrow{0} & B^\perp & \xrightarrow{\varphi} & A \\
 \downarrow & & \parallel & & \downarrow \phi \\
 0 & \longrightarrow & B^\perp & \xlongequal{\quad} & B^\perp
 \end{array}$$

Now consider the morphism from the first triangle to the third triangle, by 2-out-of-3 property, the morphism  $\phi \circ \varphi$  is an isomorphism. Thus  $\varphi$  is an isomorphism, and  $\mathcal{A} = \mathcal{B}^\perp$ . □

We list some more results in the following. The interested reader may refer the corresponding paper for details.

**Proposition 3.1.16.** *[BK, Proposition 1.12] Let  $\mathcal{B}$  be a strictly full triangulated subcategory of a triangulated category  $\mathcal{D}$ . Let  $\mathcal{B}_1$  be a right admissible subcategory of  $\mathcal{B}$ , and  $\mathcal{B}_2 = (\mathcal{B}_1^\perp)_{\mathcal{B}}$ . If  $\mathcal{B}_1$  and  $\mathcal{B}_2$  are both left (resp. right) admissible in  $\mathcal{D}$ , then  $\mathcal{B}$  is also left (resp. right) admissible in  $\mathcal{D}$ .*

**Definition 3.1.17.** [BK] Let  $\mathcal{D}$  be a triangulated category of finite type (i.e., for any  $A, B \in \mathcal{D}$ , each  $Ext^i(A, B)$  is finite-dimensional and almost all  $Ext^i = 0$ ). We say that  $\mathcal{D}$  is *right (resp. left) saturated* if every contravariant (resp. covariant) cohomology functor  $\mathcal{D} \rightarrow Vect$  is representable.

**Proposition 3.1.18.** [BK, Proposition 2.6] *Let  $\mathcal{B}$  be right (resp. left) saturated. Suppose  $\mathcal{B}$  is imbedded in a triangulated category  $\mathcal{D}$  as a full triangulated subcategory. Then  $\mathcal{B}$  is right (resp. left) admissible.*

**Theorem 3.1.19.** [BK, Theorem 2.14] *Let  $X$  be a smooth projective variety. Then  $D^b(X)$  is right and left saturated.*

The concept of an exceptional collection is a very important special case of the concept of a semiorthogonal set of subcategories:

**Definition 3.1.20.** A set of admissible subcategories  $(\mathcal{B}_0, \dots, \mathcal{B}_n)$  of a derived category  $\mathcal{D}$  is said to be *semiorthogonal* if the condition  $\mathcal{B}_j \subset \mathcal{B}_i^\perp$  holds when  $0 \leq j < i \leq n$  and  $\mathcal{B}_j \subset {}^\perp\mathcal{B}_i$  for  $j > i$ . In addition, a semiorthogonal set is said to be *complete* if it generates the category  $\mathcal{D}$ , and in this case, we say  $(\mathcal{B}_0, \dots, \mathcal{B}_n)$  is a *semiorthogonal decomposition* of  $\mathcal{D}$ .

Let  $\mathcal{E}$  be a vector bundle of rank  $r$  over a smooth projective variety  $X$ . Then there exists a projective bundle  $\mathbb{P}(\mathcal{E})$  with projection  $p : \mathbb{P}(\mathcal{E}) \rightarrow X$ . Using a resolution similar to Beilinson's resolution, Orlov gave a semiorthogonal decomposition for projective bundles.

**Theorem 3.1.21.** [Or, Theorem 2.6] *Let  $D(X)_k$  be the full and faithful subcategory of  $D^b(\mathbb{P}(\mathcal{E}))$  whose objects are all objects of the form  $p^*A \otimes \mathcal{O}_X(k)$  for an object  $A$  in  $D^b(X)$ . Then the set of admissible subcategories*

$$(D^b(X)_{-r+1}, \dots, D^b(X)_0)$$

*is a semiorthogonal decomposition of the bounded derived category  $D^b(\mathbb{P}(\mathcal{E}))$ .*

**Corollary 3.1.22.** [Or] *If there exists a complete exceptional set in the derived category  $D^b(X)$ , then the derived category  $D^b(\mathbb{P}(\mathcal{E}))$  also possesses a complete exceptional set. More explicitly, if  $(F_0, F_1, \dots, F_n)$  is a full exceptional collection of*

coherent sheaves on  $X$ , then

$$(p^*F_0 \otimes \mathcal{O}_{\mathcal{E}}(-r+1), p^*F_1 \otimes \mathcal{O}_{\mathcal{E}}(-r+1), \dots, p^*F_n \otimes \mathcal{O}_{\mathcal{E}}(-r+1), \dots, p^*F_0, \dots, p^*F_n)$$

is a full exceptional collection of coherent sheaves on  $\mathbb{P}(\mathcal{E})$ .

*Proof.* See [Or]; Corollary 2.7. □

In §3 of the same paper [Or], Orlov gives a semiorthogonal decomposition for the bounded derived category of Grassmann bundles and flag bundles, which generalizes Kapranov's results.

Later, Bernardara extended Theorem 3.1.21 to twisted projective bundles.

**Theorem 3.1.23.** [Ber] *Let  $f : X \rightarrow S$  be a Brauer-Severi scheme of relative dimension  $r$  over a locally Noetherian scheme  $S$ . Let  $\alpha$  be the corresponding class in  $H^2(S, \mathbb{G}_m)$ . Let  $D(S, \alpha)$  be the bounded derived category of the abelian category of  $\alpha$ -twisted coherent sheaves on  $S$ . Then there exist admissible full subcategories  $D(S, X)_k$  of  $D^b(X)$ , such that  $D(S, X)_k$  is equivalent to the category  $D(S, \alpha^{-k})$  for all  $k \in \mathbb{Z}$ . Moreover, the set of admissible subcategories*

$$(D(S, X)_0, \dots, D(S, X)_r)$$

*is a semiorthogonal decomposition for the category  $D^b(X)$  of bounded derived category of  $X$ .*

*Proof.* See [Ber]; Theorem 4.1. □

In a similar way, Baek generalizes the semiorthogonal decomposition of Grassmann bundles to twisted Grassmann bundles in [B], as stated below.

**Theorem 3.1.24.** [B] *Let  $p : Gr(k, \mathcal{A}) \rightarrow X$  be a twisted Grassmann bundle, where  $\mathcal{A}$  is a sheaf of Azumaya algebra has rank  $n^2$  over  $X$  and  $1 \leq k < n$ . Then there exists a semi-orthogonal decomposition of  $D^b(Gr(k, \mathcal{A}))$ .*

For toric varieties, by means of minimal model theory, Kawamata in [Ka] gives a description of their full exceptional collections. We state it below:

**Theorem 3.1.25.** *[Ka] Any smooth projective toric variety has a full exceptional collection.*

Now let us return to the topic of how to find a tilting sheaf. We know in general it is hard to find a tilting sheaf on a smooth projective variety, not even to mention to give a full strong exceptional collection. But in some special cases, we may do this, just as L. Costa and R. M. Miró-Roig did in their paper[CM1], which generalizes Corollary 3.1.22 in Orlov's paper [Or], where the collection does not satisfy the strong condition.

**Lemma 3.1.26.** *[CM1] Let  $(\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_n)$  be a full strong exceptional collection of locally free sheaves on a smooth projective variety  $X$  and let  $\mathcal{E}$  be a rank  $r$  vector bundle on  $X$ . Denote by  $S^a \mathcal{E}$  the  $a$ -th symmetric power of  $\mathcal{E}$  and assume that for any integer  $a$ ,  $0 \leq a \leq r - 1$ , and any  $l, m$ ,  $0 \leq l \leq m \leq n$ ,*

$$H^i(X, S^a \mathcal{E} \otimes \mathcal{F}_m \otimes \mathcal{F}_l^\vee) = 0, \quad i > 0.$$

*Then,*

$$(p^* \mathcal{F}_0 \otimes \mathcal{O}_{\mathcal{E}}(-r+1), p^* \mathcal{F}_1 \otimes \mathcal{O}_{\mathcal{E}}(-r+1), \dots, p^* \mathcal{F}_n \otimes \mathcal{O}_{\mathcal{E}}(-r+1), \dots, p^* \mathcal{F}_0, \dots, p^* \mathcal{F}_n)$$

*is a full strong exceptional collection of locally free sheaves on  $\mathbb{P}(\mathcal{E})$ .*

*Sketch of proof.* By Corollary 3.1.22, it is sufficient to show that for any  $k, j, l, m$  with  $0 \leq k < j \leq r - 1$  and  $l \leq m$  or  $0 \leq k = j \leq r - 1$  and  $l < m$ , we have

$$\text{Ext}^i(p^* \mathcal{F}_l \otimes \mathcal{O}_{\mathcal{E}}(k - r + 1), p^* \mathcal{F}_m \otimes \mathcal{O}_{\mathcal{E}}(j - r + 1)) = 0, \quad i > 0,$$

or equivalently

$$H^i(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathcal{E}}(j - k) \otimes p^*(\mathcal{F}_m \otimes \mathcal{F}_l^\vee)) = 0, \quad i > 0.$$

By the projection formula [H2, Ex III 8.3], for any locally free sheaf  $\mathcal{F}$  on  $X$ , we have

$$R^i p_*(\mathcal{O}_{\mathcal{E}}(a) \otimes p^* \mathcal{F}) \simeq R^i p_* \mathcal{O}_{\mathcal{E}}(a) \otimes \mathcal{F}.$$

On the other hand, by [H2, III Ex 8.4], we have  $R^i p_* \mathcal{O}_{\mathcal{E}}(a) = 0$  for  $0 < i < r - 1$  and all  $a \in \mathbb{Z}$ , and  $R^{r-1} p_* \mathcal{O}_{\mathcal{E}}(a) = 0$  for all  $a > -r$ . Therefore, for  $i \geq 0$  and  $a > -r$ , we have

$$H^i(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathcal{E}}(a) \otimes p^* \mathcal{F}) \simeq H^i(X, p_* \mathcal{O}_{\mathcal{E}}(a) \otimes \mathcal{F}).$$

Substitute the assumption, we obtain what we want.  $\square$

**Proposition 3.1.27.** [CM2] *Let  $\mathcal{E}$  be a rank  $r$  vector bundle on a smooth projective variety  $X$ . Assume that  $X$  has a full strong exceptional collection of locally free sheaves. Then,  $\mathbb{P}(\mathcal{E})$  has a full strong exceptional collection of locally free sheaves.*

*Proof.* Assume that  $(\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_n)$  is a full strong exceptional collection of locally free sheaves on  $X$ , then by Serre's theorem [H2, III Theorem 5.2], there exists a line bundle  $\mathcal{L} = \mathcal{O}_X(k)$ ,  $k \gg 0$ , on  $X$  such that for any integer  $a$ ,  $0 \leq a \leq r - 1$ , and any pair of integers  $l, m$ ,  $0 \leq l \leq m \leq n$ , we have

$$H^i(X, S^a(\mathcal{E} \otimes \mathcal{L}) \otimes \mathcal{F}_m \otimes \mathcal{F}_l^\vee) = 0, \quad i > 0.$$

Hence, it follows from Lemma 3.1.26 that

$$(p^* \mathcal{F}_0 \otimes \mathcal{O}_{\mathcal{E} \otimes \mathcal{L}}(-r+1), p^* \mathcal{F}_1 \otimes \mathcal{O}_{\mathcal{E} \otimes \mathcal{L}}(-r+1), \dots, p^* \mathcal{F}_n \otimes \mathcal{O}_{\mathcal{E} \otimes \mathcal{L}}(-r+1), \dots, p^* \mathcal{F}_0, \dots, p^* \mathcal{F}_n)$$

is a full strong exceptional collection of locally free sheaves on  $\mathbb{P}(\mathcal{E} \otimes \mathcal{L})$ . Since  $\mathbb{P}(\mathcal{E} \otimes \mathcal{L}) \simeq \mathbb{P}(\mathcal{E})$  [H2, Lemma II 7.9], we conclude that  $\mathbb{P}(\mathcal{E})$  also has a full strong exceptional collection of locally free sheaves.  $\square$

Combine Theorem 3.1.10 and Corollary 2.1.24, we obtain



**Theorem 3.1.28.** [CM1] *Any smooth, complete toric variety  $V$  with a splitting fan has a tilting bundle whose summands are line bundles.*

**Definitions 3.1.29.** Let  $\mathcal{D}$  be a triangulated category, a subcategory of  $\mathcal{D}$  is *thick* (*épaisse*) if it is closed under isomorphisms, shifts, taking cones of morphisms, and taking direct summands of objects. If  $\mathcal{S}$  is a set of objects in  $\mathcal{D}$ , we denote by  $\langle \mathcal{S} \rangle^{\mathcal{K}}$  the smallest thick full triangulated subcategory of  $\mathcal{D}$  containing  $\mathcal{S}$ . We say  $\mathcal{S}$  *classically generates*  $\mathcal{D}$  if  $\langle \mathcal{S} \rangle^{\mathcal{K}} = \mathcal{D}$ , that is,  $\langle \mathcal{S} \rangle^{\mathcal{K}}$  is equivalent to  $\mathcal{D}$

Let  $\mathcal{S}$  be a set of objects in a triangulated category  $\mathcal{D}$ , by the *right orthogonal*  $\mathcal{S}^\perp$  in  $\mathcal{D}$  we denote the full subcategory of  $\mathcal{D}$  whose objects  $A$  have the property that  $\text{Hom}_{\mathcal{D}}(E, A[i]) = 0$  for all  $E \in \mathcal{S}$  and  $i \in \mathbb{Z}$ . We may define the *left orthogonal*  ${}^\perp\mathcal{S}$  analogously. We say  $\mathcal{S}$  is a *right spanning class* of  $\mathcal{D}$  (or *right spans*  $\mathcal{D}$ ) if  $\mathcal{S}^\perp = 0$ . The *left spanning class* of  $\mathcal{D}$  is defined analogously. And we say  $\mathcal{S}$  is a *spanning class* of  $\mathcal{D}$  if it is both right and left spanning class of  $\mathcal{D}$ .

The following proposition gives one of the most common spanning class in  $D^b(X)$ .

**Proposition 3.1.30.** [Hu] *Let  $X$  be a smooth projective variety, then the set  $\{\mathcal{O}_x : x \in X \text{ closed point}\}$  is a spanning class of the bounded derived category  $D^b(X)$ .*

*Proof.* See [Hu]; Proposition 3.17. □

**Definition 3.1.31.** Let  $\mathcal{D}$  be a triangulated category, we say an object  $C \in \mathcal{D}$  is *compact* if the functor  $\text{Hom}_{\mathcal{D}}(C, -)$  commutes with direct sums. We denote by  $\mathcal{D}^c$  the full subcategory of  $\mathcal{D}$  consisting of the compact objects. This is a thick subcategory. If  $\mathcal{D}^c$  right spans  $\mathcal{D}$ , we say  $\mathcal{D}$  is *compactly spanned*.

**Remark 3.1.32.** In papers such as [BV, BSS, Bl], they say  $\mathcal{S}$  generates  $\mathcal{D}$  if  $\mathcal{S}^\perp = 0$ . To avoid confusions with Definition 3.1.5, we say  $\mathcal{S}$  right spans  $\mathcal{D}$  if  $\mathcal{S}^\perp = 0$ . For the same reason, we say  $\mathcal{D}$  is compactly spanned instead of compactly generated if  $\mathcal{D}^c$  right spans  $\mathcal{D}$ .

Clearly, if  $\mathcal{D}$  is compactly spanned and  $\mathcal{E} \subseteq \mathcal{D}^c$ , if  $\langle \mathcal{E} \rangle^{\mathcal{K}} = \mathcal{D}^c$ , then  $\mathcal{E}$  right spans  $\mathcal{D}$ . The following theorem tells us that the converse is also true.

**Theorem 3.1.33.** *[N1, Ravenel and Neeman] Let  $\mathcal{D}$  be a compactly spanned triangulated category. Then a set of objects  $\mathcal{E} \subset \mathcal{D}^c$  right spans  $\mathcal{D}$  if and only if  $\langle \mathcal{E} \rangle^{\mathcal{K}} = \mathcal{D}^c$ .*

*Proof.* See [BV]; Theorem 2.1.2. □

Denote by  $D(\mathbf{Qcoh}(X))$  the derived category of quasi-coherent sheaves over  $X$ .

**Proposition 3.1.34.** *[N2] Let  $X$  be a quasi-compact, separated scheme. Then the category  $D(\mathbf{Qcoh}(X))$  is compactly spanned.*

*Proof.* See [N2]; proposition 2.5. □

An object of  $D(\mathbf{Qcoh}(X))$  is *perfect* if it is locally quasi-isomorphic to a bounded complex of free sheaves of finite rank. We denote by  $X\text{-perf}$  the full subcategory of  $D(\mathbf{Qcoh}(X))$  of perfect complexes. This is a thick subcategory of  $D^b(X)$ . If  $X$  is quasi-projective, then a complex is perfect if and only if it is quasi-isomorphic to a bounded complex of vector bundles. The variety  $X$  is regular if and only if  $D^b(X) = X\text{-perf}$ . [R, §3.2.3]

**Lemma 3.1.35.** *[R] Let  $C \in D(\mathbf{Qcoh}(X))$ . Then  $C$  is perfect if and only if it is compact.*

*Proof.* See [R]; Lemma 3.5. □

Let  $X$  be a smooth projective variety, by the above argument and Lemma 3.1.35, we have  $D(\mathbf{Qcoh}(X))^c = X\text{-perf} = D^b(X)$ .

**Corollary 3.1.36.** *Let  $X$  be a smooth projective variety. Then  $D^+(\mathbf{Qcoh}(X))$  is compactly spanned and  $(D^+(\mathbf{Qcoh}(X)))^c = D^b(X)$ .*

*Proof.* Since  $D(\mathbf{Qcoh}(X))^c = D^b(X) \subseteq D^+(\mathbf{Qcoh}(X))$  and  $D(\mathbf{Qcoh}(X))$  is compactly spanned (Proposition 3.1.34), we have  $D^+(\mathbf{Qcoh}(X))$  is compactly spanned.

Now let  $\mathcal{S} \subset D^b(X)$  such that  $\mathcal{S}$  right spans  $D^+(\mathbf{Qcoh}(X))$ . By Theorem 3.1.33, we have  $\langle \mathcal{S} \rangle^{\mathcal{K}} = (D^+(\mathbf{Qcoh}(X)))^c \supseteq D(\mathbf{Qcoh}(X))^c = D^b(X)$ . But  $\langle \mathcal{S} \rangle^{\mathcal{K}} \subseteq D^b(X)$  as  $\mathcal{S} \subset D^b(X)$ , so we have  $\langle \mathcal{S} \rangle^{\mathcal{K}} = (D^+(\mathbf{Qcoh}(X)))^c = D^b(X)$ .  $\square$

Thus, by Theorem 3.1.33, we have

**Corollary 3.1.37.** *Let  $X$  be a smooth projective variety and  $\mathcal{S} \subset D^b(X)$ . Then  $\langle \mathcal{S} \rangle^{\mathcal{K}} = D^b(X)$  if and only if  $\mathcal{S}$  right spans  $D^+(\mathbf{Qcoh}(X))$ .*

Recently, Blunk generalized Beilinson and Kapranov's results to the twisted case.

**Theorem 3.1.38.** *[Bl] Let  $X := SB(A)$  be the Severi-Brauer variety of the algebra  $A$  of rank  $n^2$  over field  $K$ . Let  $\mathcal{I}$  be the 'tautological' sheaf on  $X$  and  $\mathcal{T} := \mathcal{O}_X \oplus \mathcal{I}^1 \oplus \mathcal{I}^2 \oplus \dots \oplus \mathcal{I}^{n-1}$ . Then  $\mathcal{T}$  is a tilting sheaf on  $X$ .*

*Sketch of proof.* Notice that for some Galois extension  $L/K$ , we have canonical morphism  $p : X_L \simeq \mathbb{P}_L^{n-1} \rightarrow X$  and  $p^*\mathcal{I} \simeq \bigoplus^n \mathcal{O}_{\mathbb{P}^{n-1}}(-1)$ . Then the results follows from Proposition 4.1.8, Theorem 3.1.10 and Lemma 3.1.9.  $\square$

In the same paper, using a similar method, Blunk also constructs a tilting sheaf for twisted Grassmanns and flag varieties.

## 3.2 Global Dimension

**Definitions 3.2.1.** [W] Let  $R$  be a ring and  $A$  be a left  $R$ -module. The *projective dimension*  $pd_R(A)$  is the minimum integer  $n$  (if it exists) such that there is a resolution of  $A$  by left projective  $R$ -modules

$$0 \rightarrow P_n \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow A \rightarrow 0.$$

The *global (homological) dimension* of  $R$ ,  $gldim(R)$ , is the supremum of  $pd_R(A)$  over all left  $R$ -modules  $A$ .

**Remark 3.2.2.** The above definition of global dimension is actually left global dimension. We can also define the right global dimension similarly. But they are same when  $R$  is left and right Noetherian [W], and in our case this is always true.

We give a useful lemma below:

**Lemma 3.2.3.** [W, Lemma 4.16] *Let  $A$  be a left  $R$ -module, then  $pd_R(A) \leq d$  if and only if  $Ext_R^{d+1}(A, B) = 0$  for any left  $R$ -module  $B$ .*

*Proof.*  $\Rightarrow$ ) Since  $Ext^*(A, B)$  can be computed using a projective resolution of  $A$ , it is clear that  $Ext_R^{d+1}(A, B) = 0$  for any left  $R$ -module  $B$ .

$\Leftarrow$ ) Suppose we have an exact sequence  $0 \rightarrow M \rightarrow P_{d-1} \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow A \rightarrow 0$  with the  $P$ 's projective, then we have  $Ext^{d+1}(A, B) \simeq Ext^1(M, B)$  by dimension shifting, thus  $Ext^1(M, B) = 0$  for all  $B$  and  $M$  is projective. Thus we have  $pd_R(A) \leq d$ .  $\square$

**Lemma 3.2.4.** *Let  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  be a short exact sequence of  $R$ -modules. If any two of  $pd_R(A)$ ,  $pd_R(B)$ ,  $pd_R(C)$  are finite, then the third one is finite. More specifically,  $pd_R(A) \leq \max\{pd_R(B), pd_R(C) - 1\}$ ,  $pd_R(B) \leq \max\{pd_R(A), pd_R(C)\}$  and  $pd_R(C) \leq \max\{pd_R(A) + 1, pd_R(B)\}$ .*

*Sketch of proof.* We can get this easily from Lemma 3.2.3 and the induced long exact sequence of the functor  $Ext^*(-, M)$  of the above short exact sequence for any left  $R$ -module  $M$ .  $\square$

In the following, all the rings we consider will have unit  $1 \neq 0$  and all the modules will be unital modules. Let  $T$  and  $U$  be two rings and  $M$  be a nonzero  $U$ - $T$ -bimodule, that is,  $M$  is a left  $U$ -module and right  $T$ -module such that  $(um)t = u(mt)$  for all  $u \in U, t \in T$  and  $m \in M$ . Consider the formal triangular matrix ring  $\Lambda = \begin{pmatrix} T & 0 \\ M & U \end{pmatrix}$ . Let  $\Omega$  be the category whose objects are the triples  $(A, B, f)$  with  $A$  a left  $T$ -module,  $B$  a left  $U$ -module and  $f : M \otimes_T A \rightarrow B$  a  $U$ -morphism. The morphisms between two objects  $(A, B, f)$  and  $(A', B', f')$  are pairs of morphisms  $(\alpha, \beta)$  where  $\alpha : A \rightarrow A'$  is a  $T$ -morphism and  $\beta : B \rightarrow B'$  is a  $U$ -morphism, such that the diagram

$$\begin{array}{ccc} M \otimes_T A & \xrightarrow{M \otimes \alpha} & M \otimes_T A' \\ f \downarrow & & f' \downarrow \\ B & \xrightarrow{\beta} & B' \end{array}$$

commutes. It is well-known [FGR, G] that the category  $\Omega$  is equivalent to the category of left  $\Lambda$ -modules. The  $\Lambda$ -module corresponding to the object  $(A, B, f) \in \Omega$  is the additive group  $(A \oplus B)_f$  with the left  $\Lambda$ -action given by

$$\begin{pmatrix} t & 0 \\ m & u \end{pmatrix} (a, b) = (ta, f(m \otimes a) + ub).$$

If  $(\alpha, \beta) : (A, B, f) \rightarrow (A', B', f')$  is a map in  $\Omega$  the associated map  $\phi : (A \oplus B)_f \rightarrow (A' \oplus B')_{f'}$  is given by  $\phi(a, b) = (\alpha(a), \beta(b))$  for any  $a \in A, b \in B$ . It is clear  $\phi$  is injective (resp. surjective) if and only  $\alpha, \beta$  are injective (resp. surjective).

Now we can give a characterization of projective modules over  $\Lambda$ .

**Theorem 3.2.5.** [HV, Theorem 3.1]  $(A \oplus B)_f$  is projective if and only if  $A$  is a

projective  $T$ -module,  $f : M \otimes_T A \rightarrow B$  is monic and  $B = f(M \otimes_T A) \oplus P$  with  $P$  projective  $U$ -module.

**Proposition 3.2.6.** *Let  $\Lambda = \begin{pmatrix} A & 0 \\ M & B \end{pmatrix}$  be a formal triangular matrix ring. If  $\text{gldim}(T) = m$  and  $\text{gldim}(U) = n$ , then*

$$\text{gldim} \begin{pmatrix} T & 0 \\ M & U \end{pmatrix} \leq m + n + 1$$

.

*Proof.* It suffices to show that for any object  $(A, B, f) \in \Omega$ , we have  $\text{pd}_\Lambda((A \oplus B)_f) \leq m + n + 1$ .

First consider the short exact sequence

$$0 \rightarrow (0, \ker(f), 0) \hookrightarrow (A, M \otimes_T A, 1_{M \otimes_T A}) \xrightarrow{(1_A, f)} (A, \text{Im}(f), f) \rightarrow 0.$$

By Theorem 3.2.5, we have  $\text{pd}_\Lambda((0 \oplus \ker(f))_0) = \text{pd}_U(\ker(f)) \leq n$  and  $\text{pd}_\Lambda((A \oplus M \otimes_T A)_{1_{M \otimes_T A}}) \leq \text{pd}_T(A) + \text{gldim}(U) + 1 \leq m + n + 1$ , thus by Lemma 3.2.4, we have  $\text{pd}_\Lambda((A \oplus \text{Im}(f))_f) \leq m + n + 1$ .

Now consider the short exact sequence

$$0 \rightarrow (A, \text{Im}(f), f) \hookrightarrow (A, B, f) \rightarrow (0, B/\text{Im}(f), 0) \rightarrow 0.$$

Again, we have  $\text{pd}_\Lambda((0 \oplus B/\text{Im}(f))_0) = \text{pd}_U(B/\text{Im}(f)) \leq n$ . Thus by Lemma 3.2.4, we have  $\text{pd}_\Lambda((A \oplus B)_f) \leq m + n + 1$ .  $\square$

# Chapter 4

## Main results

### 4.1 Main Results

Throughout this chapter  $K$  will be a field and  $R$  will be a (not necessarily commutative)  $K$ -algebra with a unit. We remind the reader what this means. To be a  $K$ -algebra means that  $R$  is a  $K$ -vector space and a ring such that the vector space structure and multiplication on  $R$  are compatible in the following sense

$$x(ab) = (xa)b = a(xb) \quad \text{for all } x \in K, a, b \in R.$$

Notice that this implies that  $K$  is contained in the center of  $R$ . For any field extension  $L/K$  we write  $R_L$  for the  $L$ -algebra  $L \otimes_K R$ , once again  $L$  will be in the center of  $R_L$ . This fact will be used repeatedly below. Finally for a left  $A$ -module, we write  $A_L$  for the  $R_L$  module  $L \otimes_K A$ .

**Lemma 4.1.1.** *Let  $A, B$  be left  $R$ -modules. If  $L/K$  is a field extension, then there is an isomorphism of  $L$ -vector spaces*

$$\Phi : \text{Hom}_R(A, B) \longrightarrow \text{Hom}_{R_L}(A_L, B_L).$$

*Proof.* Observe that if  $f \in \text{Hom}_R(A, B_L)$  then the map

$$\begin{aligned} L \times A &\longrightarrow B_L \\ (x, a) &\longmapsto xf(a) \end{aligned}$$

is  $K$ -bilinear so there is an associated map  $\Phi(f) : A_L \rightarrow B_L$  which sends  $r \otimes a$  to  $rf(a)$  for  $r \in R_L$  and  $a \in A$ . It is easy to see that  $\Phi(f)(x(r_1 \otimes a_1) + r_2 \otimes a_2) = xr_1f(a_1) + r_2f(a_2) = x\Phi(f)(r_1 \otimes a_1) + \Phi(f)(r_2 \otimes a_2)$  for all  $x, r_1, r_2 \in R_L$  and  $a_1, a_2 \in A$ . Thus  $\Phi(f)$  is a homomorphism of left  $R_L$ -modules, and  $\Phi$  is well-defined.

As  $L$  is central in  $R_L$ ,  $\text{Hom}_R(A, B_L)$  is naturally regarded as a  $L$ -vector space in the following way: for  $l \in L$  and  $f \in \text{Hom}_R(A, B_L)$ ,  $l \cdot f$  is the homomorphism of left  $R$ -modules which sends  $a \in A$  to  $lf(a)$ . It is easy to see that  $\Phi$  is a homomorphism of  $L$ -vector spaces.

Observe that  $\Phi$  is injective as  $A \hookrightarrow A_L$  and  $\Phi(f)|_{1 \otimes A} = f$ . Given  $g : A_L \rightarrow B_L$  then  $g|_{1 \otimes A}$  is a homomorphism of left  $R$ -modules and  $\Phi(g|_{1 \otimes A}) = g$ . Hence  $\Phi$  is an isomorphism of  $L$ -vector spaces.  $\square$

**Lemma 4.1.2.** *Suppose that  $L/K$  is a field extension. If  $A$  and  $B$  are left  $R$ -modules, then there is a canonical homomorphism of  $L$ -vector spaces*

$$\Lambda : L \otimes_K \text{Hom}_R(A, B) \longrightarrow \text{Hom}_R(A, B_L).$$

*Moreover, if  $L/K$  is a finite extension then the above map is in fact an isomorphism.*

*Proof.* Consider the pairing

$$L \times \text{Hom}_R(A, B) \longrightarrow \text{Hom}_R(A, B_L)$$

sending  $(x, f)$  to the map  $a \mapsto xf(a)$ . This map is  $K$ -bilinear and hence the map  $\Lambda$  exists. As in the proof of previous lemma, we can check that  $\Lambda$  is well-defined and it is a homomorphism of  $L$ -vector spaces.



When  $L/K$  is a finite extension of degree  $d$ , with basis  $e_1, e_2, \dots, e_d$  we have  $B_L \cong \bigoplus_1^d e_i \otimes B$ . As the  $\text{Hom}$  functor commutes with finite direct sums, every homomorphism  $g : A \rightarrow B_L$  decomposes uniquely as  $g = (e_1 \cdot g_1, \dots, e_d \cdot g_d)$  for some homomorphisms  $g_i : A \rightarrow B$ ,  $1 \leq i \leq d$ . It follows that  $\Lambda(\sum_{i=1}^d e_i \otimes g_i) = g$  and thus  $\Lambda$  is surjective. On the other hand, every  $f \in L \otimes_K \text{Hom}_R(A, B)$  decomposes uniquely as  $\sum_{i=1}^d e_i \otimes g_i$  we see that  $\Lambda$  is also injective. Hence  $\Lambda$  is an isomorphism of  $L$ -vector spaces.  $\square$

**Corollary 4.1.3.** *Let  $R$  be a  $K$ -algebra and  $A, B$  be left  $R$ -modules. Then for any finite field extension  $L/K$ , we have a canonical isomorphism of  $L$ -vector spaces*

$$\text{Hom}_{R_L}(A_L, B_L) \rightarrow L \otimes_K \text{Hom}_R(A, B).$$

*Proof.* Combine the isomorphisms in Lemma 4.1.1 and Lemma 4.1.2.  $\square$

Consider the two contravariant families of  $\delta$ -functors on  $R$ -modules

$$\text{Ext}_{R_L}^i((-)_L, B_L) \quad \text{and} \quad L \otimes_K \text{Ext}_R^i(-, B).$$

The above argument shows that they agree for  $i = 0$ . Further, they are both coexact as they vanish on free modules for  $i > 0$ . Hence

**Proposition 4.1.4.** *Let  $R$  be a  $K$ -algebra and  $A, B$  be two left  $R$ -modules, then for any finite field extension  $L/K$ , we have natural isomorphisms*

$$\text{Ext}_{R_L}^i(A_L, B_L) \cong L \otimes_K \text{Ext}_R^i(A, B).$$

*Proof.* See [H2]; Theorem III 1.3A.  $\square$

**Lemma 4.1.5.** *Let  $R$  be a  $K$ -algebra and  $L/K$  be a finite field extension. If  $R_L$  has finite global dimension, then  $R$  has finite global dimension.*

*Proof.* Assume that  $\text{gldim}(R_L) \leq d$ , then by Lemma 3.2.3, we have  $\text{Ext}_{R_L}^{d+1}(M, N) = 0$  for any left  $R_L$ -modules  $M$  and  $N$ . Let  $A, B$  be left  $R$ -modules, by Proposition 4.1.4, we have an isomorphism

$$\text{Ext}_{R_L}^i(A_L, B_L) \simeq L \otimes_K \text{Ext}_R^i(A, B)$$

for any  $i \geq 0$ . But by assumption,  $\text{Ext}_{R_L}^{d+1}(A_L, B_L) = 0$ , hence  $\text{Ext}_R^{d+1}(A, B) = 0$ , and by Lemma 3.2.3 again, we have  $\text{gldim}(R) \leq d$ .  $\square$

**Lemma 4.1.6.** *Let  $X_K$  be a smooth projective variety and  $\mathcal{F}$  be a locally free coherent sheaf on  $X_K$ . If  $L/K$  is a separable field extension and  $v : X_L \rightarrow X_K$  is the canonical morphism, then  $\mathcal{F}$  has no higher self-extensions if and only if  $v^* \mathcal{F}$  has no higher self-extensions.*

*Proof.* Consider the following cartesian square:

$$\begin{array}{ccc} X_L & \xrightarrow{v} & X_K \\ q \downarrow & & p \downarrow \\ \text{Spec}(L) & \xrightarrow{u} & \text{Spec}(K) \end{array}$$

By flat base change, the natural map

$$u^* R^i p_* \mathcal{F} \rightarrow R^i q_* v^* \mathcal{F}$$

is an isomorphism of functors of quasi-coherent sheaves  $\mathcal{F}$  on  $X_K$  [H2, Proposition III 9.3].

Thus we have

$$u^* R^i p_*(\mathcal{F}^\vee \otimes \mathcal{F}) = R^i q_*(v^* \mathcal{F}^\vee \otimes v^* \mathcal{F}).$$

Since

$$\text{Ext}_{X_K}^i(\mathcal{F}, \mathcal{F}) = H^i(X_K, \mathcal{F}^\vee \otimes \mathcal{F}) = R^i p_*(\mathcal{F}^\vee \otimes \mathcal{F})$$

and

$$\mathrm{Ext}_{X_L}^i(v^*\mathcal{F}, v^*\mathcal{F}) = H^i(X_L, v^*\mathcal{F}^\vee \otimes v^*\mathcal{F}) = R^i q_* (v^*\mathcal{F}^\vee \otimes v^*\mathcal{F}),$$

we have

$$\mathrm{Ext}_{X_L}^i(v^*\mathcal{F}, v^*\mathcal{F}) = 0 \quad \text{if and only if} \quad u^*\mathrm{Ext}_{X_K}^i(\mathcal{F}, \mathcal{F}) = 0.$$

As  $u$  is faithfully flat, we get

$$u^*\mathrm{Ext}_{X_K}^i(\mathcal{F}, \mathcal{F}) = 0 \quad \text{if and only if} \quad \mathrm{Ext}_{X_K}^i(\mathcal{F}, \mathcal{F}) = 0.$$

Thus  $\mathcal{F}$  has no higher self-extensions if and only if  $v^*\mathcal{F}$  has no higher self-extensions.  $\square$

**Lemma 4.1.7.** *Let  $X_K$  be a smooth projective variety and  $L/K$  be a finite Galois extension. Let  $v : X_L \rightarrow X_K$  be the canonical morphism. Let  $\mathcal{F}$  be a locally free coherent sheaf on  $X_K$ . Then  $\langle v^*\mathcal{F} \rangle^{\mathcal{K}} = D^b(X_L)$  if and only if  $\langle \mathcal{F} \rangle^{\mathcal{K}} = D^b(X_K)$ .*

*Proof.* Assume  $\langle v^*\mathcal{F} \rangle^{\mathcal{K}} = D^b(X_L)$ . By Corollary 3.1.37, we have that  $\{v^*\mathcal{F}\}$  right spans  $D^+(\mathbf{Qcoh}(X_L))$ , that is, for any  $\mathcal{M} \in D^+(\mathbf{Qcoh}(X_L))$ ,  $\mathrm{Hom}(v^*\mathcal{F}, \mathcal{M}[i]) = 0$ ,  $i \in \mathbb{Z}$ , implies  $\mathcal{M} = 0$ . For the same reason, to show that  $\langle \mathcal{F} \rangle^{\mathcal{K}} = D^b(X_K)$ , it is equivalent to showing that  $\{\mathcal{F}\}$  is a right spanning class of  $D^+(\mathbf{Qcoh}(X_K))$ , i.e. for any  $\mathcal{M} \in D^+(\mathbf{Qcoh}(X_K))$  such that  $\mathrm{Hom}(\mathcal{F}, \mathcal{M}[i]) = 0$ ,  $i \in \mathbb{Z}$ , we have  $\mathcal{M} = 0$ .

Since we have  $\mathrm{Hom}(\mathcal{F}, \mathcal{M}[i]) = \mathrm{Ext}^i(\mathcal{F}, \mathcal{M}) = H^i(\mathrm{RHom}(\mathcal{F}, \mathcal{M}))$  for all  $i \in \mathbb{Z}$  [Hu, Remark 2.57], the condition  $\mathrm{Hom}(\mathcal{F}, \mathcal{M}[i]) = 0$ ,  $i \in \mathbb{Z}$  is equivalent to  $\mathrm{RHom}(\mathcal{F}, \mathcal{M}) = 0$ . Notice that  $\mathcal{F}$  (resp.  $v^*\mathcal{F}$ ) is locally free,  $\mathcal{H}om_{X_K}(\mathcal{F}, -)$  and  $\mathcal{F}^\vee \otimes_{X_K} -$  (resp.  $\mathcal{H}om_{X_L}(v^*\mathcal{F}, -)$  and  $v^*\mathcal{F}^\vee \otimes_{X_L} -$ ) are exact functors on  $\mathbf{Qcoh}(X_K)$  (resp.  $\mathbf{Qcoh}(X_L)$ ). Thus, for example,  $\mathrm{RHom}(\mathcal{F}, \mathcal{M})$  can be computed on  $D^+(\mathbf{Qcoh}(X_K))$  by applying  $\mathcal{H}om_{X_K}(\mathcal{F}, -)$  to each individual term in  $\mathcal{M}$ .

Now consider the following cartesian square:

$$\begin{array}{ccc} X_L & \xrightarrow{v} & X_K \\ q \downarrow & & \downarrow p \\ \text{Spec}(L) & \xrightarrow{u} & \text{Spec}(K) \end{array}$$

By flat base change (see page 68, iii)), the natural map

$$u^* R p_* \rightarrow R q_* v^*$$

is an isomorphism of functors [Hu, (3.18)].

Let  $RH\text{om}_{X_K}(\mathcal{F}, \mathcal{M}) = 0$ , we have

$$\begin{aligned} 0 &= u^* RH\text{om}(\mathcal{F}, \mathcal{M}) \\ &= u^* R p_* RH\text{om}(\mathcal{F}, \mathcal{M}) \quad (\text{see page 68, ii}) \\ &= R q_* v^* RH\text{om}(\mathcal{F}, \mathcal{M}) \\ &= R q_* v^* (\mathcal{F}^\vee \otimes^L \mathcal{M}) \\ &= R q_* (v^* \mathcal{F}^\vee \otimes^L L v^* \mathcal{M}) \\ &= R q_* RH\text{om}(v^* \mathcal{F}, L v^* \mathcal{M}) \\ &= RH\text{om}(v^* \mathcal{F}, L v^* \mathcal{M}). \end{aligned}$$

Thus we have  $v^* \mathcal{M} = 0$ . As  $v$  is faithfully flat, we get  $\mathcal{M} = 0$ .

Conversely, assume  $\langle \mathcal{F} \rangle^{\mathcal{K}} = D^b(X_K)$ . To complete the proof, it suffices to show any coherent sheaf on  $X_L$  is a summand of some coherent sheaf on  $X_L$  contained in  $\langle v^* \mathcal{F} \rangle^{\mathcal{K}}$ . Let  $\mathcal{F}$  be a coherent sheaf on  $X_L$  and  $G$  be the Galois group of the finite Galois extension  $L/K$ . For  $g \in G$ , let

$$\rho_g : X_L = X_K \otimes_{\text{Spec} K} \text{Spec} L \rightarrow X_L$$

be the morphism of schemes induced by  $\rho_g : \text{Spec} L \rightarrow \text{Spec} L$ , i.e. by  $g^{-1} : L \rightarrow L$ .

Denote

$$\mathcal{F}' = \bigoplus_{g \in G} \rho_g^* \mathcal{F}.$$

For each  $g \in G$  there are canonical identifications

$$\iota_g : \rho_g^* \mathcal{F}' = \bigoplus_{g' \in G} \rho_g^* \rho_{g'}^* \mathcal{F} = \bigoplus_{g' \in G} \rho_{g'g}^* \mathcal{F} \xrightarrow{id} \bigoplus_{g' \in G} \rho_{g'}^* \mathcal{F} = \mathcal{F}'$$

and it is easy to see that they satisfy the relations  $\iota_g \circ \rho_g^*(\iota_{g'}) = \iota_{g'g}$  for any  $g, g' \in G$ . By Galois descent theory, there is a coherent sheaf  $\mathcal{E}$  on  $X_K$  such that  $\mathcal{F}' \cong v^* \mathcal{E}$ . So we have  $\mathcal{F}' \in v^* \langle \mathcal{T} \rangle^{\mathcal{K}} \subseteq \langle v^* \mathcal{T} \rangle^{\mathcal{K}}$ , further  $\mathcal{F}$  is a summand of  $\mathcal{F}'$ ,  $\mathcal{F} \in \langle v^* \mathcal{T} \rangle^{\mathcal{K}}$ . So we have  $\langle v^* \mathcal{T} \rangle^{\mathcal{K}} = D^b(X_L)$ .  $\square$

Combining Lemmas 4.1.5, 4.1.6 and 4.1.7, we can show

**Proposition 4.1.8.** *Let  $X_K$  be a smooth projective variety and  $\mathcal{T}$  a locally free coherent sheaf on  $X_K$ . Let  $L/K$  be a finite Galois extension and  $v : X_L \rightarrow X_K$  the canonical morphism. If  $v^* \mathcal{T}$  is a tilting sheaf on  $X_L$ , then  $\mathcal{T}$  is a tilting sheaf on  $X_K$ .*

*Proof.* We need only to show that

$$End_{\mathcal{O}_{X_L}}(v^* \mathcal{T}) \cong End_{\mathcal{O}_{X_K}}(\mathcal{T}) \otimes_K L.$$

Notice that  $End_{\mathcal{O}_{X_L}}(v^* \mathcal{T}) \cong \Gamma(\mathcal{H}om_{\mathcal{O}_{X_L}}(v^* \mathcal{T}, v^* \mathcal{T}))$  and  $End_{\mathcal{O}_{X_K}}(\mathcal{T}) \otimes_K L \cong \Gamma(v^* \mathcal{H}om_{\mathcal{O}_{X_K}}(\mathcal{T}, \mathcal{T}))$ , thus it suffices to show that

$$\mathcal{H}om_{\mathcal{O}_{X_L}}(v^* \mathcal{T}, v^* \mathcal{T}) \cong v^* \mathcal{H}om_{\mathcal{O}_{X_K}}(\mathcal{T}, \mathcal{T}).$$

By [H2, Ex II 5.1], we have

$$\mathcal{H}om_{\mathcal{O}_{X_L}}(v^* \mathcal{T}, v^* \mathcal{T}) \cong v^* \mathcal{T}^\vee \otimes_{\mathcal{O}_{X_L}} v^* \mathcal{T}$$

and

$$v^* \mathcal{H}om_{\mathcal{O}_{X_K}}(\mathcal{T}, \mathcal{T}) \cong v^*(\mathcal{T}^\vee \otimes_{\mathcal{O}_{X_K}} \mathcal{T}) \cong v^* \mathcal{T}^\vee \otimes_{\mathcal{O}_{X_L}} v^* \mathcal{T}.$$

This completes the proof.  $\square$

In general, the converse may be not true.

**Definition 4.1.9.** Recall that  $R$  was a  $K$ -algebra with a unit. Suppose  $R$  has finite global dimension, say  $R$  is *stable of finite global dimension* if  $R_L$  has finite global dimension for every finite separable extension  $L/K$ .

As it is well known that every finite separable extension can be embedded into a finite Galois extension. Then by Lemma 4.1.5, we can get

**Lemma 4.1.10.** *Suppose  $R$  has finite global dimension over  $K$ , then  $R$  is stable of finite global dimension if and only if  $R_L$  has finite global dimension for any finite Galois extension  $L/K$ .*

The above lemma, together with Lemma 4.1.6 and 4.1.7, induce

**Corollary 4.1.11.** *Let  $X_K$  be a smooth projective variety and  $\mathcal{T}$  be a tilting bundle on  $X_K$  whose endomorphism algebra,  $\text{End}(\mathcal{T})$ , is stable of finite global dimension. Then for any finite Galois extension  $L/K$ ,  $v^*\mathcal{T}$  is a tilting sheaf on  $X_L$ , where  $v : X_L \rightarrow X_K$  is the canonical morphism.*

**Theorem 4.1.12.** *Let  $X$  be a Brauer-Severi variety over field  $K$ , then there exists a tilting bundle on  $X$ .*

*Proof.* Let  $L/K$  be a finite Galois extension such that  $X \otimes_K L \simeq \mathbb{P}_L^n$  and let  $\pi : \mathbb{P}_L^n \rightarrow X$  be the canonical morphism. By Proposition 4.1.8, it suffices to prove there exists a tilting bundle on  $\mathbb{P}_L^n$  which descends to a bundle on  $X$ .

Let  $G = \text{Gal}(L/K)$ . For  $g \in G$ , let  $\rho_g : \mathbb{P}_L^n \rightarrow \mathbb{P}_L^n$  be the morphism of schemes induced by  $\rho_g : \text{Spec } L \rightarrow \text{Spec } L$ , i.e. by  $g^{-1} : L \rightarrow L$ .

Denote

$$\mathcal{I} = \bigoplus_{g \in G} \rho_g^* \mathcal{O}_{\mathbb{P}_L^n}(-1).$$

For each  $g \in G$  there are canonical identifications

$$\iota_g : \rho_g^* \mathcal{I} = \bigoplus_{g' \in G} \rho_g^* \rho_{g'}^* \mathcal{O}_{\mathbb{P}_L^n}(-1) = \bigoplus_{g' \in G} \rho_{g'g}^* \mathcal{O}_{\mathbb{P}_L^n}(-1) \xrightarrow{id} \bigoplus_{g' \in G} \rho_{g'}^* \mathcal{O}_{\mathbb{P}_L^n}(-1) = \mathcal{I}$$

and it is easy to see that there is the relation  $\iota_g \circ \rho_g^*(\iota_{g'}) = \iota_{g'g}$ . By Galois descent for locally free sheaves,  $\mathcal{I}$  descends to a locally free sheaf on  $X$ .

Let

$$\mathcal{T} = \bigoplus_{i=0}^n \mathcal{I}^{\otimes i},$$

then  $\mathcal{T}$  descends to a locally free sheaf on  $X$ . We claim  $\mathcal{T}$  is a tilting bundle on  $\mathbb{P}_L^n$ .

Indeed, since for each  $g \in G$ ,  $\rho_g$  is an isomorphism of  $\mathbb{P}_L^n$  over  $K$ , it maps hyperplane sections to hyperplane sections. Thus  $G$  acts on  $Pic(\mathbb{P}_L^n)$  trivially and we have  $\mathcal{I} \simeq \bigoplus_{g \in G} \mathcal{O}_{\mathbb{P}_L^n}(-1)$ . Hence we have

$$\mathcal{T} = \bigoplus_{i=0}^n \mathcal{I}^{\otimes i} \cong \mathcal{O}_{\mathbb{P}_L^n} \oplus_{i=1}^n \mathcal{O}_{\mathbb{P}_L^n}(-i)^{\oplus |G|^i}.$$

Beilinson's Theorem 3.1.10 states that the collection

$$\{\mathcal{O}_{\mathbb{P}_L^n}(-n), \mathcal{O}_{\mathbb{P}_L^n}(-n+1), \dots, \mathcal{O}_{\mathbb{P}_L^n}(-1), \mathcal{O}_{\mathbb{P}_L^n}\}$$

is a full strong exceptional collection, thus by Lemma 3.1.9,  $\mathcal{T}$  is a tilting bundle on  $\mathbb{P}_L^n$ . □

**Remark 4.1.13.** The bundle  $\mathcal{I}$  constructed above actually descends to the pushforward of  $\mathcal{O}_{\mathbb{P}_L^n}(-1)$ , that is,  $\pi_* \mathcal{O}_{\mathbb{P}_L^n}(-1)$ . Thus the pushforward of tilting bundle  $\bigoplus_{i=0}^n \mathcal{O}_{\mathbb{P}_L^n}(-i)$  on  $\mathbb{P}_L^n$  is a tilting bundle on  $X_K$ .

**Remark 4.1.14.** (1) The endomorphism algebra of the tilting bundle on a Brauer-Severi variety constructed above is stable of finite global dimension, as any finite separable extension of  $K$  can be embedded into a finite Galois extension that contains the field  $L$  in the above proof.

(2) In [Bl], Blunk constructs a tilting bundle on Brauer-Severi variety (Theorem 3.1.38), which is different from the one we construct above, as the rank of  $\mathcal{L}$  in Blunk's construction depends on the dimension of the variety, while ours also depends on the order of the Galois group. His method is to use the 'tautological' sheaf, but this method may not generalize to Brauer-Severi schemes, as Example 4.1.21 shows. However, it may give a semi-orthogonal decomposition for twisted Grassmann bundles, similar to the semi-orthogonal decompositions given in [Ber] and [B].

Before discussing the relative case, we prove a lemma first. This lemma and the idea its proof are provided by Professor Patrick Brosnan.

**Lemma 4.1.15.** *Suppose  $X$  is a rational smooth projective variety over a field  $K$ . Then the Picard group of  $X$ ,  $Pic(X)$  is torsion free.*

*Proof.* Let  $n$  be a positive integer. Set

$$C(X) := \left\{ \alpha \in \frac{K(X)^\times}{K(X)^\times K^\times} : \operatorname{div} \alpha \equiv 0 \pmod{n} \right\}.$$

Define a group homomorphism

$$\begin{aligned} \pi : C(X) &\longrightarrow Pic(X)[n] \\ \alpha &\longmapsto \mathcal{O}\left(\frac{\operatorname{div} \alpha}{n}\right). \end{aligned}$$

It is easy to see that  $\pi$  is well-defined. We claim that  $\pi$  is isomorphism. Indeed, if  $\mathcal{L}$  is an  $n$ -torsion line bundle in  $Pic(X)$  and  $D$  is a Cartier divisor associated to it, then  $nD$  is rationally equivalent to 0. So there is an  $\alpha \in K(X)^\times$  such that  $\operatorname{div} \alpha = nD$ . Hence  $\pi(\alpha) = D$  and  $\pi$  is surjective. On the other hand, notice that  $\pi(\alpha) = 0$  if and only if  $\operatorname{div} \alpha = \operatorname{div} \beta^n$  for some  $\beta \in K(X)^\times$ . This implies that  $\operatorname{div}(\alpha/\beta^n) = 0$ . Since  $X$  is projective, we have  $\alpha/\beta^n \in K^\times$ , which implies that  $\alpha$  is 0 in  $C(X)$ , and  $\pi$  is injective.



Let  $(f, U) : X \dashrightarrow \mathbb{P}^m$  be a birational map, where  $U$  is a non-empty Zariski open subset of  $X$  with complement of codimension at least 2, then there is a group homomorphism  $\bar{f} : C(\mathbb{P}^m) \rightarrow C(X)$  sending  $\alpha$  to  $\alpha \circ f$ . Since  $\mathbb{P}^m = \mathbb{Z}$ , to show  $\text{Pic}(X)$  is torsion free, it is sufficient to show that  $\bar{f}$  is an isomorphism.

Since  $X$  and  $\mathbb{P}^m$  are birational, we have  $K(X) = K(\mathbb{P}^m)$ . Note that, if  $V$  is a non-empty Zariski open subset of  $X$  with complement of codimension at least 2, then  $C(X) \rightarrow C(V)$  induced by inclusion  $V \hookrightarrow X$  is clearly equal. If we write  $C(K(X))$  for  $K(X)^\times / (K(X)^{\times n} K^\times)$ , then  $C(K(X))$  is just the union of  $C(W)$  taken over all non-empty Zariski opens in  $X$ . Thus,  $C(X)$  and  $C(\mathbb{P}^m)$  both inject into  $C(K(X))$ , and to show  $\bar{f}$  is an isomorphism, it suffices to show that they have the same images there. Suppose  $\alpha \in C(\mathbb{P}^m)$ . Then, via the morphism  $f : U \rightarrow \mathbb{P}^m$ , we see that it is also in  $C(U)$  which equals to  $C(X)$ . So  $C(\mathbb{P}^m)$  is contained in  $C(X)$ . Similarly,  $C(X)$  is contained in  $C(\mathbb{P}^m)$ .  $\square$

In certain situations tilting bundles can be obtained via descent theory. Let  $X_K$  be a smooth projective variety that becomes a rational variety after some finite separable extension. Considering  $p : Y_K \rightarrow X_K$ , a Brauer-Severi scheme of relative dimension  $r$ , corresponding to a Azumaya algebra  $\mathcal{A}$  (see Theorem 1.2.31). By theorems 1.2.28 and Corollary 1.2.17, there exists a finite Galois extension  $L/K$  such that we have the following cartesian square:

$$\begin{array}{ccc} \mathbb{P}(\mathcal{E}) & \xrightarrow{v} & Y_K \\ q \downarrow & & p \downarrow \\ X_L & \xrightarrow{u} & X_K \end{array}$$

where  $\mathcal{E}$  is a vector bundle on  $X_L$  such that  $u^* \mathcal{A} \cong \mathcal{E} \text{nd}(\mathcal{E})$ . Denote  $G = \text{Gal}(L/K)$ . For  $g \in G$ , let  $\rho_g : X_L \rightarrow X_L$  and  $\rho_g : \mathbb{P}(\mathcal{E}) \rightarrow \mathbb{P}(\mathcal{E})$  be the morphisms of schemes induced by  $\rho_g : \text{Spec } L \rightarrow \text{Spec } L$ , i.e. by  $g^{-1} : L \rightarrow L$ .

**Theorem 4.1.16.** *Let  $p : Y_K \rightarrow X_K$  be a morphism as above, and suppose the Galois group  $G$  acts on  $\text{Pic}(X_L)$  trivially. If there exists a tilting bundle on  $X_K$  whose endomorphism algebra is stable of finite global dimension, then there exists a tilting bundle on  $Y_K$ .*

*Proof.* To show there is a tilting bundle on  $Y_K$ , it suffices to prove that there exists a tilting bundle on  $\mathbb{P}(\mathcal{E})$  which descends to a bundle on  $Y_K$  by Proposition 4.1.8.

Let

$$\mathcal{I}_i = \bigoplus_{g \in G} \rho_g^*(\mathcal{O}_{\mathcal{E}}(i) \otimes q^* \mathcal{L}^{\otimes i}), \quad i \neq 0,$$

where  $\mathcal{L}$  is some line bundle on  $X_L$  specified below. For each  $g \in G$  there are canonical identifications

$$\begin{aligned} \iota_g : \rho_g^* \mathcal{I}_i &= \bigoplus_{g' \in G} \rho_g^* \rho_{g'}^*(\mathcal{O}_{\mathcal{E}}(i) \otimes q^* \mathcal{L}^{\otimes i}) = \bigoplus_{g' \in G} \rho_{g'g}^*(\mathcal{O}_{\mathcal{E}}(i) \otimes q^* \mathcal{L}^{\otimes i}) \\ &\xrightarrow{\text{id}} \bigoplus_{g' \in G} \rho_{g'}^*(\mathcal{O}_{\mathcal{E}}(i) \otimes q^* \mathcal{L}^{\otimes i}) = \mathcal{I}_i \end{aligned}$$

and it is easy to see that the relation  $\iota_g \circ \rho_g^*(\iota_{g'}) = \iota_{g'g}$ . By Galois descent for locally free sheaves,  $\mathcal{I}_i$  descends to a locally free sheaf on  $Y_K$ .

Let  $\mathcal{T}$  be the tilting bundle on  $X_K$  whose endomorphism algebra is stable of finite global dimension. Write  $\mathcal{I}_0 = \mathcal{O}_{\mathcal{E}}$ , let

$$\mathcal{S} = \bigoplus_{i=0}^r q^* u^* \mathcal{T} \otimes \mathcal{I}_i,$$

then  $\mathcal{S}$  descends to a locally free sheaf on  $Y_K$ . We claim  $\mathcal{S}$  is a tilting bundle on  $\mathbb{P}(\mathcal{E})$  for some line bundle  $\mathcal{L}$  on  $X_L$ .

We have  $\text{Pic}(\mathbb{P}(\mathcal{E})) = q^* \text{Pic}(X_L) \times \mathbb{Z}\xi$ ,  $\xi = \mathcal{O}_{\mathcal{E}}(1)$  [H2, Ex II 7.9]. We will show  $G$  acts on  $\text{Pic}(\mathbb{P}(\mathcal{E}))$  trivially. Since  $G$  acts on  $\text{Pic}(X_L)$  trivially, it suffices to show  $\rho_g^* \mathcal{O}_{\mathcal{E}}(1) \simeq \mathcal{O}_{\mathcal{E}}(1)$  for each  $g \in G$ . Suppose  $\rho_g^* \mathcal{O}_{\mathcal{E}}(1) \simeq \mathcal{O}_{\mathcal{E}}(k) \otimes q^* \mathcal{L}'$  for some line bundle  $\mathcal{L}'$  on  $X_L$ . Since on the fiber,  $\rho_g$  is an isomorphism of  $\mathbb{P}^r$ , we have  $k = 1$ . Let the order of  $g$  is  $l$ , then we have  $\mathcal{O}_{\mathcal{E}}(1) = \rho_{g^l}^* \mathcal{O}_{\mathcal{E}}(1) \simeq \mathcal{O}_{\mathcal{E}}(1) \otimes q^* \mathcal{L}'^{\otimes l}$ . But by

Lemma 4.1.15,  $Pic(X_L)$  is torsion free, so we must have  $\mathcal{L}' \simeq \mathcal{O}_{X_L}$ . Thus we have  $\rho_g^* \mathcal{O}_{\mathcal{E}}(1) \simeq \mathcal{O}_{\mathcal{E}}(1)$ , and  $\mathcal{I}_i \simeq \bigoplus_{g \in G} \mathcal{O}_{\mathcal{E}}(i) \otimes q^* \mathcal{L}^{\otimes i}$  for  $i \neq 0$ .

To complete the proof, we will show that  $\mathcal{S}$  satisfies the conditions in Definition 3.1.7.

For no higher self-extension:

We need to show that

$$Ext_{\mathbb{P}(\mathcal{E})}^k(\mathcal{S}, \mathcal{S}) = 0, \text{ for all } k > 0.$$

Notice that

$$\begin{aligned} \mathcal{S} &= \bigoplus_{i=0}^r q^* u^* \mathcal{T} \otimes \mathcal{I}_i \\ &\simeq \bigoplus_{i=0}^r \bigoplus_{g \in G} q^* u^* \mathcal{T} \otimes \mathcal{O}_{\mathcal{E}}(i) \otimes q^* \mathcal{L}^i, \end{aligned}$$

We have

$$\begin{aligned} &Ext_{\mathbb{P}(\mathcal{E})}^k(\mathcal{S}, \mathcal{S}) \\ &= \bigoplus_{0 \leq i, j \leq r} \bigoplus_{g \in G} Ext_{\mathbb{P}(\mathcal{E})}^k(q^* u^* \mathcal{T} \otimes \mathcal{O}_{\mathcal{E}}(j) \otimes q^* \mathcal{L}^{\otimes j}, q^* u^* \mathcal{T} \otimes \mathcal{O}_{\mathcal{E}}(i) \otimes q^* \mathcal{L}^{\otimes i}) \\ &\simeq \bigoplus_{0 \leq i, j \leq r} \bigoplus_{g \in G} Ext_{\mathbb{P}(\mathcal{E})}^k(\mathcal{O}_{\mathbb{P}(\mathcal{E})}, q^*(u^* \mathcal{T} \otimes u^* \mathcal{T}^\vee) \otimes \mathcal{O}_{\mathcal{E}}(i-j) \otimes q^* \mathcal{L}^{\otimes(i-j)}) \\ &\simeq \bigoplus_{0 \leq i, j \leq r} \bigoplus_{g \in G} H^k(\mathbb{P}(\mathcal{E}), q^*(u^* \mathcal{T} \otimes u^* \mathcal{T}^\vee) \otimes \mathcal{O}_{\mathcal{E}}(i-j) \otimes q^* \mathcal{L}^{\otimes(i-j)}) \\ &\simeq \bigoplus_{0 \leq i, j \leq r} \bigoplus_{g \in G} H^k(X_L, u^* \mathcal{T} \otimes u^* \mathcal{T}^\vee \otimes \mathcal{L}^{\otimes(i-j)} \otimes q_* \mathcal{O}_{\mathcal{E}}(i-j)). \end{aligned}$$

Thus, to show  $Ext_{\mathbb{P}(\mathcal{E})}^k(\mathcal{S}, \mathcal{S}) = 0$ ,  $k > 0$ , is equivalent to show that

$$H^k(X_L, u^* \mathcal{T} \otimes u^* \mathcal{T}^\vee \otimes \mathcal{L}^{\otimes(i-j)} \otimes q_* \mathcal{O}_{\mathcal{E}}(i-j)) = 0$$

for  $k > 0$  and  $0 \leq i, j \leq r$ .

Since  $q_* \mathcal{O}_{\mathcal{E}}(l) \simeq S^l(\mathcal{E})$  for  $l \geq 0$  and  $q_* \mathcal{O}_{\mathcal{E}}(l) = 0$  for  $l < 0$  [H2, Ex III 8.4], it is sufficient to show that

$$H^k(X_L, u^* \mathcal{T} \otimes u^* \mathcal{T}^\vee \otimes \mathcal{L}^l \otimes S^l(\mathcal{E})) = 0$$

for  $0 \leq l \leq r$ .

For the case  $l = 0$ , this is true because  $u^* \mathcal{T}$  is a tilting bundle on  $X_L$  by Corollary 4.1.11. For the case  $0 < l \leq r$ , we can choose an ample line bundle  $\mathcal{L}$  such that the above equations are satisfied [H2, Proposition III 5.3].

For finite global dimension:

Denote

$$\mathcal{S}_i = q^* u^* \mathcal{T} \otimes \mathcal{L}^{\otimes i}, \quad 0 \leq i \leq r,$$

Write  $A = \text{End}(u^* \mathcal{T})$ , then  $A$  has finite global dimension since  $u^* \mathcal{T}$  is a tilting sheaf on  $X_L$  by Corollary 4.1.11.

For  $0 \leq i < j \leq r$ , we have

$$\begin{aligned} & \text{Hom}_{\mathcal{O}_{\mathbb{P}(\mathcal{E})}}(\mathcal{S}_j, \mathcal{S}_i) \\ &= \oplus \text{Hom}_{\mathcal{O}_{\mathbb{P}(\mathcal{E})}}(q^*(u^* \mathcal{T} \otimes \mathcal{L}^{\otimes j}) \otimes \mathcal{O}_{\mathcal{E}}(j), q^*(u^* \mathcal{T} \otimes \mathcal{L}^{\otimes i}) \otimes \mathcal{O}_{\mathcal{E}}(i)) \\ &= \oplus \Gamma(\mathbb{P}(\mathcal{E}), q^*(u^* \mathcal{T} \otimes u^* \mathcal{T}^\vee \otimes \mathcal{L}^{\otimes(i-j)}) \otimes \mathcal{O}_{\mathcal{E}}(i-j)) \\ &= \oplus \Gamma(X_L, u^* \mathcal{T} \otimes u^* \mathcal{T}^\vee \otimes \mathcal{L}^{\otimes(i-j)} \otimes q_* \mathcal{O}_{\mathcal{E}}(i-j)) \\ &= 0. \end{aligned}$$

Similarly,

$$\text{Hom}_{\mathcal{O}_{\mathbb{P}(\mathcal{E})}}(\mathcal{S}_i, \mathcal{S}_i) \cong M_{|G|}(A) \text{ for all } 1 \leq i \leq r.$$

Thus, we have

$$\text{End}_{\mathcal{O}_{\mathbb{P}(\mathcal{E})}}(\mathcal{S}) = \begin{pmatrix} A & 0 & \cdots & 0 \\ \text{Hom}_{\mathcal{O}_{\mathbb{P}(\mathcal{E})}}(\mathcal{S}_0, \mathcal{S}_1) & M_{|G|}(A) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \text{Hom}_{\mathcal{O}_{\mathbb{P}(\mathcal{E})}}(\mathcal{S}_0, \mathcal{S}_r) & \text{Hom}_{\mathcal{O}_{\mathbb{P}(\mathcal{E})}}(\mathcal{S}_1, \mathcal{S}_r) & \cdots & M_{|G|}(A) \end{pmatrix}.$$

We show that  $\text{End}_{\mathcal{O}_{\mathbb{P}(\mathcal{E})}}(\mathcal{S})$  has finite global dimension by induction in  $r$ .

When  $r = 0$ , this is true as  $\text{End}_{\mathcal{O}_{\mathbb{P}(\mathcal{E})}}(\mathcal{S}) = A$ .

If we write

$$\text{End}_{\mathcal{O}_{\mathbb{P}(\mathcal{E})}}(\mathcal{S}) = \begin{pmatrix} R & 0 \\ M & M_{|G|}(A), \end{pmatrix}$$

where

$$R = \begin{pmatrix} A & 0 & \cdots & 0 \\ \text{Hom}_{\mathcal{O}_{\mathbb{P}(\mathcal{E})}}(\mathcal{S}_0, \mathcal{S}_1) & M_{|G|}(A) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \text{Hom}_{\mathcal{O}_{\mathbb{P}(\mathcal{E})}}(\mathcal{S}_0, \mathcal{S}_{r-1}) & \text{Hom}_{\mathcal{O}_{\mathbb{P}(\mathcal{E})}}(\mathcal{S}_1, \mathcal{S}_{r-1}) & \cdots & M_{|G|}(A) \end{pmatrix}$$

$$M = \left( \text{Hom}_{\mathcal{O}_{\mathbb{P}(\mathcal{E})}}(\mathcal{S}_0, \mathcal{S}_r) \quad \text{Hom}_{\mathcal{O}_{\mathbb{P}(\mathcal{E})}}(\mathcal{S}_1, \mathcal{S}_r) \quad \cdots \quad \text{Hom}_{\mathcal{O}_{\mathbb{P}(\mathcal{E})}}(\mathcal{S}_{r-1}, \mathcal{S}_r) \right)$$

By induction, the algebra  $R$  has finite global dimension. By Morita theory, the category of  $M_{|G|}(A)$ -modules is equivalent to the category of  $A$ -modules. As  $A$  has finite global dimension, so does  $M_{|G|}(A)$ . Thus we conclude that  $\text{End}_{\mathcal{O}_{\mathbb{P}(\mathcal{E})}}(\mathcal{S})$  has finite global dimension by Proposition 3.2.6.

Finally, to prove completeness, we need to show that  $\langle \mathcal{S} \rangle^{\mathcal{K}} = D^b(\mathbb{P}(\mathcal{E}))$ . We know that  $\mathcal{S} = \bigoplus_{i=0}^r q^* u^* \mathcal{T} \otimes \mathcal{S}_i$ ,  $\mathcal{S}_0 = \mathcal{O}_{\mathcal{E}}$  and  $\mathcal{S}_i \simeq \bigoplus_{g \in G} \mathcal{O}_{\mathcal{E}}(i) \otimes q^* \mathcal{L}^{\otimes i}$  for  $i \neq 0$ , we have  $\langle \mathcal{S} \rangle^{\mathcal{K}} \supseteq \mathcal{D} := \langle q^* D^b(X_L) \otimes \mathcal{O}_{\mathcal{E}}(-r), \dots, q^* D^b(X_L) \otimes \mathcal{O}_{\mathcal{E}}(-1), q^* D^b(X_L) \rangle$ . Thus it suffices to show that  $\mathcal{D} = D^b(\mathbb{P}(\mathcal{E}))$ .

First we claim that  $\mathcal{D}$  is admissible in  $D^b(\mathbb{P}(\mathcal{E}))$ . Indeed, since all  $q^* D^b(X_L) \otimes \mathcal{O}_{\mathcal{E}}(-i)$  are equivalent to  $q^* D^b(X_L)$  and are full faithful subcategory of  $D^b(\mathbb{P}(\mathcal{E}))$ , and  $X_L$  is a smooth projective variety, by Theorem 3.1.19 and Proposition 3.1.18,  $q^* D^b(X_L) \otimes \mathcal{O}_{\mathcal{E}}(-i)$  is admissible in  $D^b(\mathbb{P}(\mathcal{E}))$  for all  $0 \leq i \leq r$ . And the set of the admissible subcategories  $(q^* D^b(X_L) \otimes \mathcal{O}_{\mathcal{E}}(-r), \dots, q^* D^b(X_L) \otimes \mathcal{O}_{\mathcal{E}}(-1), q^* D^b(X_L))$  is semiorthogonal by Theorem 3.1.21. Let  $\mathcal{D}(-1) = \langle q^* D^b(X_L) \otimes \mathcal{O}_{\mathcal{E}}(-1), q^* D^b(X_L) \rangle$ . Then by Corollary 3.1.15, we have  $q^* D^b(X_L) \otimes \mathcal{O}_{\mathcal{E}}(-1) = (q^* D^b(X_L)^\perp)_{\mathcal{D}(-1)}$ . Thus by Proposition 3.1.16, we obtain that  $\langle q^* D^b(X_L) \otimes \mathcal{O}_{\mathcal{E}}(-1), q^* D^b(X_L) \rangle$  is admissi-

ble in  $D^b(\mathbb{P}(\mathcal{E}))$ . Iteratively, we can show that  $\langle q^*D^b(X_L) \otimes \mathcal{O}_{\mathcal{E}}(-2), q^*D^b(X_L) \otimes \mathcal{O}_{\mathcal{E}}(-1), q^*D^b(X_L) \rangle, \dots, \mathcal{D}$  are all admissible in  $D^b(\mathbb{P}(\mathcal{E}))$ .

Thus, by Proposition 3.1.14, that to show that  $\mathcal{D} = D^b(\mathbb{P}(\mathcal{E}))$  is equivalent to show that the right (or left) orthogonal of  $\mathcal{D}$  in  $D^b(\mathbb{P}(\mathcal{E}))$  is zero. Hence it is sufficient to show that  $\mathcal{D}$  contains all the objects  $\mathcal{O}_x$ ,  $x \in \mathbb{P}(\mathcal{E})$  a closed point, as the set  $\{\mathcal{O}_x : x \in \mathbb{P}(\mathcal{E}) \text{ closed point}\}$  is a spanning class for  $D^b(\mathbb{P}(\mathcal{E}))$  by Proposition 3.1.30. Every closed point  $x \in \mathbb{P}(\mathcal{E})$  lies in the fibre  $F_s = v^{-1}(s) \simeq \mathbb{P}^r$  for some closed point  $s \in X_L$ . Since  $\mathcal{O}_{\mathcal{E}}(i)|_{F_s} \simeq \mathcal{O}_{\mathbb{P}^r}(i)$ , by Theorem 3.1.10, we have  $D^b(F_s) = \langle \mathcal{O}_{\mathcal{E}}(i)|_{F_s} : -r \leq i \leq 0 \rangle$ . And hence  $\mathcal{O}_x \in \langle \mathcal{O}_{\mathcal{E}}(i)|_{F_s} : -r \leq i \leq 0 \rangle$ . Notice that  $\mathcal{O}_{\mathcal{E}}(i)|_{F_s} \simeq \mathcal{O}_{\mathcal{E}}(i) \otimes \mathcal{O}_{F_s}$  and  $\mathcal{O}_{F_s} = q^*\mathcal{O}_s \in q^*D^b(X_L)$ , so we have  $\mathcal{O}_x \in \mathcal{D}$ .

This completes the proof.  $\square$

**Remark 4.1.17.** In [Or, Theorem 2.6], using the Koszul resolution of the diagonal, Orlov showed that the semiorthogonal set  $(q^*D^b(X_L) \otimes \mathcal{O}_{\mathcal{E}}(-r), \dots, q^*D^b(X_L) \otimes \mathcal{O}_{\mathcal{E}}(-1), q^*D^b(X_L))$  is complete in  $D^b(\mathbb{P}(\mathcal{E}))$ . Here we give a different proof.

A special case of the above theorem is that the base scheme is a Brauer-Severi variety, on which, as we already showed (Theorem 4.1.12 and Remark 4.1.14), there is a tilting bundle whose endomorphism algebra is stable of finite global dimension.

**Corollary 4.1.18.** *Let  $p : Y_K \rightarrow X_K$  be a Brauer-Severi scheme of relative dimension  $r$  on a Brauer-Severi variety  $X_K$ , corresponding to a Azumaya algebra  $\mathcal{A}$ . Then there exists a tilting bundle on  $Y_K$ .*

*Proof.* There exists a finite Galois extension  $L/K$  such that the following is a cartesian square:

$$\begin{array}{ccc} \mathbb{P}(\mathcal{E}) & \xrightarrow{v} & Y_K \\ q \downarrow & & \downarrow p \\ \mathbb{P}_L^r & \xrightarrow{u} & X_K \end{array}$$

where  $\mathcal{E}$  is a vector bundle on  $\mathbb{P}_L^n$  such that  $u^*\mathcal{A} \cong \mathcal{E}nd(\mathcal{E})$ .

Denote  $G = Gal(L/K)$ . Let

$$\mathcal{I} = \bigoplus_{g \in G} \rho_g^*(\mathcal{O}_{\mathcal{E}}(-1) \otimes q^*\mathcal{L}),$$

where  $\mathcal{L}$  is some line bundle on  $\mathbb{P}_L^n$ . Since the Galois group  $G$  acts on  $Pic(\mathbb{P}_L^n) = \mathbb{Z}$  trivially, it acts on  $Pic(\mathbb{P}(\mathcal{E})) = q^*Pic(\mathbb{P}_L^n) \times \mathbb{Z}\xi$ ,  $\xi = \mathcal{O}_{\mathcal{E}}(1)$ , trivially, too. Thus we have  $\mathcal{I} \simeq \bigoplus_{g \in G} \mathcal{O}_{\mathcal{E}}(-1) \otimes q^*\mathcal{L}$ .

Let

$$\mathcal{S} = v^*p^*\mathcal{I} \otimes (\bigoplus_{i=0}^r \mathcal{I}^{\otimes i}),$$

where  $\mathcal{I}$  is a tilting bundle on  $X_K$  as constructed in Theorem 4.1.12, then we can show that  $\mathcal{S}$  is a tilting bundle on  $\mathbb{P}(\mathcal{E})$  which descends to a locally free sheaf on  $Y_K$  for some proper line bundle  $\mathcal{L}$  on  $\mathbb{P}_L^n$  as in the proof in the above theorem.  $\square$

Moreover, when the base scheme is also an arithmetic toric variety, we may give a concrete description of the line bundle  $\mathcal{L}$ :

**Corollary 4.1.19.** *Let  $p : Y_K \rightarrow X_K$  be a Brauer-Severi scheme, which is also a toric morphism, of relative dimension  $r$  on a arithmetic toric variety  $(X_K, \mathcal{T})$ , whose split toric variety is a projective space, corresponding to a Azumaya algebra  $\mathcal{A}$ . If there exists a tilting bundle  $\mathcal{I}$  on  $X_K$ , then there exists a tilting bundle on  $Y_K$ .*

*Proof.* There exists a finite Galois extension  $L/K$  such that we have the following cartesian square:

$$\begin{array}{ccc} \mathbb{P}(\mathcal{E}) & \xrightarrow{v} & Y_K \\ q \downarrow & & \downarrow p \\ \mathbb{P}_L^n & \xrightarrow{u} & X_K \end{array}$$

where  $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}_L^n}(k_0) \oplus \mathcal{O}_{\mathbb{P}_L^n}(k_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}_L^n}(k_r)$  with  $k_0 \leq k_1 \leq \cdots \leq k_r$  such that  $u^*\mathcal{A} \cong \mathcal{E}nd(\mathcal{E})$ .

Then a line bundle  $\mathcal{L} \simeq \mathcal{O}_{\mathbb{P}^n_L}(k)$  with  $k \geq -k_0$  works for the constructions as given in the above theorem.  $\square$

**Remark 4.1.20.** It is easy to see that if we choose  $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^n_L}(k_0) \oplus \mathcal{O}_{\mathbb{P}^n_L}(k_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^n_L}(k_r)$  with  $0 \leq k_0 \leq k_1 \leq \cdots \leq k_r$ , then  $\mathcal{S} = \bigoplus_{g \in G} \rho_g^* \mathcal{O}_{\mathcal{E}}(-1)$  works.

For projective space  $\mathbb{P}^n$ , we always choose  $\{\mathcal{O}_{\mathbb{P}^n}(1)\}$  as a basis of  $Pic(\mathbb{P}^n) = \mathbb{Z}$ ; for projective bundle  $\pi : \mathbb{P}(\mathcal{E}) \rightarrow \mathbb{P}^n$ , we always choose  $\{\pi^* \mathcal{O}_{\mathbb{P}^n}(1), \mathcal{O}_{\mathcal{E}}(1)\}$  as a basis of  $Pic(\mathbb{P}(\mathcal{E})) = \mathbb{Z} \oplus \mathbb{Z}$ ; and so on.

In the above corollary, if we assume the line bundle summands of  $\mathcal{E}$  are in  $\mathbb{Z}_{\geq 0}$  and let  $\mathcal{O}(i, j) := \pi^* \mathcal{O}_{\mathbb{P}^n}(i) \otimes \mathcal{O}_{\mathcal{E}}(j)$ , then the collection

$$\{\mathcal{O}(-n, -r), \mathcal{O}(-n-1, -r), \dots, \mathcal{O}(0, -r), \dots, \mathcal{O}(-n, 0), \dots, \mathcal{O}(0, 0)\}$$

is a full strong exceptional collection on  $\mathbb{P}(\mathcal{E})$ . Let

$$\mathcal{T} := \mathcal{O}_{X_L} \oplus (\bigoplus \rho_g^* \mathcal{O}(i, j)),$$

where the second summand is the sum over the set  $\{(i, j) : -n \leq i \leq 0, -r \leq j \leq 0, i + j \leq -1\}$  then  $\mathcal{T}$  is a tilting sheaf on  $\mathbb{P}(\mathcal{E})$  by Lemma 3.1.9 and  $\mathcal{T}$  descends to  $Y_K$ .

**Example 4.1.21.** Let  $p : \mathbb{P}(\mathcal{E}) \rightarrow \mathbb{P}^n$  be a projective bundle with  $\mathcal{E} = \mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(k_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^n}(k_r)$ ,  $0 \leq k_1 \leq \cdots \leq k_r$  and at least one  $k_i > 0$ . Then

$$\mathcal{S} = \mathcal{O}_{\mathcal{E}}(-1) \oplus (\mathcal{O}_{\mathcal{E}}(-1) \otimes p^* \mathcal{O}_{\mathbb{P}^n}(-k_1)) \oplus \cdots \oplus (\mathcal{O}_{\mathcal{E}}(-1) \otimes p^* \mathcal{O}_{\mathbb{P}^n}(-k_r))$$

is the 'tautological' sheaf on  $\mathbb{P}(\mathcal{E})$ . Let  $\mathcal{T} = \mathcal{O}_{\mathbb{P}^n}(-n) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^n}$  be the tilting bundle on  $\mathbb{P}^n$ . We will show that

$$\mathcal{S} = \bigoplus_{i=0}^r q^* u^* \mathcal{T} \otimes \mathcal{I}_i \otimes p^* \mathcal{L}^{\otimes i}$$

is not a tilting bundle on  $\mathbb{P}(\mathcal{E})$ .



Though  $\mathcal{S}$  satisfies conditions (ii) and (iii) in Definition 3.1.7, it fails condition (i) and  $\mathcal{S}$  has higher self-extensions. Indeed, with a similar argument as in the proof of Theorem 4.1.16, for all  $k > 0$ , the equation  $\text{Ext}_{\mathbb{P}(\mathcal{E})}^k(\mathcal{S}, \mathcal{S}) = 0$  is equivalent to the following equations

$$H^k(\mathbb{P}^n, \mathcal{I} \otimes \mathcal{I}^\vee \otimes \mathcal{L}^{\otimes(i-j)} \otimes \mathcal{O}_{\mathcal{P}^n}(\sum_{l=1}^r (s_l - t_l)k_l) \otimes S^{i-j}(\mathcal{E})) = 0,$$

where  $s_l, t_l \geq 0$  for all  $0 \leq l \leq r$  and  $\sum_{l=1}^r s_l \leq i$ ,  $\sum_{l=1}^r t_l \leq j$  for all  $0 \leq j \leq i \leq r$ .

Notice that  $\mathcal{O}_{\mathcal{P}^n}(-n)$  is a summand of  $\mathcal{I} \otimes \mathcal{I}^\vee$ . Consider the case  $t_r = i = j > 0$  and all the other  $t_l, s_l$  are zero. Then by Theorem III 5.1 [H2], we know that  $H^n(\mathbb{P}^n, \mathcal{O}_{\mathcal{P}^n}(-n - jk_r)) \neq 0$ .

Hence  $\mathcal{S}$  is not a tilting bundle on  $\mathbb{P}(\mathcal{E})$ .

## 4.2 An Application

As an application, we generalize Corollary 4.1.19 to a special class of arithmetic toric varieties. Consider an arithmetic toric variety  $(X_K, \mathcal{T})$ , whose split toric variety  $X_L$  corresponds to a splitting fan  $\Sigma$  in a lattice  $N$ , where  $L/K$  is a Galois extension with Galois group  $G$ . Then by Theorem 2.1.23, we have a projectivization  $X_L = \mathbb{P}(\mathcal{E}) \rightarrow X'_L$ , which corresponds to a primitive collection  $\mathcal{P} = \{x_1, x_2, \dots, x_{k+1}\} \subseteq \Sigma(1)$  with primitive relation  $x_1 + x_2 + \dots + x_{k+1} = 0$  by Corollary 2.1.21. Let  $\varphi : G \rightarrow \text{Aut}(N)$  be a conjugacy class of group homomorphisms such that  $\mathcal{T} = \mathcal{T}_\varphi$  and  $\varphi(G) \subseteq \text{Aut}_\Sigma$ . As  $\Sigma(1)$  generates  $\Sigma$ , the action of  $G$  on  $\Sigma$  is determined by the action of  $G$  on  $\Sigma(1)$ . For any  $g \in G$ , the action of  $g$  on  $\Sigma(1)$  preserves the primitive relationship. Since  $\mathcal{P}$  has no intersection with any other primitive collection in  $\Sigma(1)$ , we must have either  $g(\mathcal{P}) = \mathcal{P}$  or  $g(\mathcal{P}) \cap \mathcal{P} = \emptyset$ . Let the distinguished primitive collections  $\mathcal{P}_1, \dots, \mathcal{P}_m$  be the images of  $\mathcal{P}$  under the action of  $G$ .

Again, by Corollary 2.1.21, each of these primitive collections  $\mathcal{P}_1, \dots, \mathcal{P}_m$  induces a projective bundle, so we have a series of projective bundles  $\mathbb{P}(\mathcal{E}_1^{id}) \rightarrow \mathbb{P}(\mathcal{E}_2^{id}) \rightarrow \dots \rightarrow \mathbb{P}(\mathcal{E}_m^{id}) \rightarrow Y_L$ , where  $Y_L$  is a toric variety with splitting fan by Theorem 2.1.23. By Corollary 2.1.16, we may assume that the fan  $\Sigma$  is built up from the fan  $\Sigma_{Y_L}$ . The Galois  $G$ -action on  $X_L$  induces an Galois  $G$ -action on  $Y_L$ . Let  $Y_L$  descends to  $(Y_K, \mathcal{T}')$ , then we have a compatible commutative diagram:

$$\begin{array}{ccc} X_L & \longrightarrow & (X_K, \mathcal{T}) \\ \downarrow & & \downarrow \\ Y_L & \longrightarrow & (Y_K, \mathcal{T}'). \end{array}$$

Actually, for every  $\tau \in S_m$ , the permutation group of the set  $\{1, 2, \dots, m\}$ , we have a series of projective bundles  $\mathbb{P}(\mathcal{E}_{\tau_1}^\tau) \rightarrow \mathbb{P}(\mathcal{E}_{\tau_2}^\tau) \rightarrow \dots \rightarrow \mathbb{P}(\mathcal{E}_{\tau_m}^\tau) \rightarrow Y_L$ . Thus each of these primitive collections  $\mathcal{P}_1, \dots, \mathcal{P}_m$  induces a projective bundle  $\mathbb{P}(\mathcal{E}_i) \rightarrow Y_L, i = 1, \dots, m$ . The  $G$ -action on  $X_L$  induces commutative diagrams

$$\begin{array}{ccc} \mathbb{P}(\mathcal{E}_i) & \xrightarrow{\rho_{g_{i,j}}} & \mathbb{P}(\mathcal{E}_j) \\ \downarrow & & \downarrow \\ Y_L & \xrightarrow{\rho_{g_{i,j}}} & Y_L \end{array}$$

for  $1 \leq i, j \leq m$ . So we may assume that  $\{\rho_g^*(\mathcal{E}_1) : g \in G\} = \{\mathcal{E}_1, \dots, \mathcal{E}_m\}$ .

We claim that  $X_L \simeq \mathbb{P}(\mathcal{E}_1) \times_{Y_L} \dots \times_{Y_L} \mathbb{P}(\mathcal{E}_m)$ .

Indeed, if we denote by  $X'_L = \mathbb{P}(\mathcal{E}_1) \times_{Y_L} \dots \times_{Y_L} \mathbb{P}(\mathcal{E}_m)$ , by Theorem 2.1.9, it suffices to prove  $\Sigma_{X_L} \simeq \Sigma_{X'_L}$ .

First, assume  $\Sigma_{Y_L}$ -linear support functions  $h_{i,1}, \dots, h_{i,k+1}$  give rise to the equivariant sheaves  $\mathcal{E}_i$  for  $1 \leq i \leq m$ . Let  $N_i$  be a free  $\mathbb{Z}$ -module with a basis  $\{n_{i,2}, \dots, n_{i,k+1}\}$  and  $n_{i,1} = -(n_{i,2} + \dots + n_{i,k+1})$  for  $1 \leq i \leq m$ . By Corollary 2.1.16, the fan  $\Sigma_{\mathbb{P}(\mathcal{E}_i)}$  is determined by  $\mathbb{R}$ -linear map  $N_{Y_L \mathbb{R}} \rightarrow N_{\mathbb{P}(\mathcal{E}_i) \mathbb{R}} := N_{Y_L \mathbb{R}} + N_i \mathbb{R}$  which sends  $y \in N_{Y_L \mathbb{R}}$  to  $(y, -\sum_{1 \leq j \leq k+1} h_{i,j}(y)n_{i,j})$ .

Notice that we have

$$X'_L = \mathbb{P}((p_{m-1} \circ \cdots \circ p_1)^* \mathcal{E}_m) \xrightarrow{p_m} \cdots \xrightarrow{p_3} \mathbb{P}(p_1^* \mathcal{E}_2) \xrightarrow{p_2} \mathbb{P}(\mathcal{E}_1) \xrightarrow{p_1} Y_L,$$

thus the fan  $\Sigma_{X'_L}$  is determined by the  $\mathbb{R}$ -linear map

$$N_{Y_L \mathbb{R}} \rightarrow N_{X'_L \mathbb{R}} := N_{Y_L \mathbb{R}} + N_{1\mathbb{R}} + \cdots + N_{m\mathbb{R}}$$

which sends  $y \in N_{Y_L \mathbb{R}}$  to  $(y, -\sum_{1 \leq j \leq k+1} h_{1,j}(y)n_{1,j}, \cdots, -\sum_{1 \leq j \leq k+1} h_{m,j}(y)n_{m,j})$ .

On the other hand, as we showed above, there is a series of projective bundles  $\mathbb{P}(\mathcal{E}'_m) \rightarrow \mathbb{P}(\mathcal{E}'_{m-1}) \rightarrow \cdots \rightarrow \mathbb{P}(\mathcal{E}'_1) \rightarrow Y_L$ . Let  $N'_i$  be a free  $\mathbb{Z}$ -module with a basis  $\{n'_{i,2}, \cdots, n'_{i,k+1}\}$  and  $n'_{i,1} = -(n'_{i,2} + \cdots + n'_{i,k+1})$  for  $1 \leq i \leq m$ . By Corollary 2.1.16, we may construct a fan  $\Sigma_{X_L}$  of  $X_L$  through the following maps

$$N_{Y_L \mathbb{R}} \rightarrow N_{Y_L \mathbb{R}} + N'_{1\mathbb{R}} \rightarrow \cdots \rightarrow N_{Y_L \mathbb{R}} + N'_{1\mathbb{R}} + \cdots + N'_{m\mathbb{R}}.$$

Without causing confusion, we use the same symbol  $n'_{i,j}$  to denote its image in  $N_{Y_L \mathbb{R}} + N'_{1\mathbb{R}} + \cdots + N'_{m\mathbb{R}}$  for all  $1 \leq i \leq m$  and  $1 \leq j \leq k+1$ . Choose  $\{n'_{i,j} | 1 \leq i \leq m, 2 \leq j \leq k+1\}$  as a part a basis of  $N_{Y_L \mathbb{R}} + N'_{1\mathbb{R}} + \cdots + N'_{m\mathbb{R}}$ , then the composition of above maps  $N_{Y_L \mathbb{R}} \rightarrow N_{Y_L \mathbb{R}} + N'_{1\mathbb{R}} + \cdots + N'_{m\mathbb{R}}$  sends  $y \in N_{Y_L \mathbb{R}}$  to  $(y, -\sum_{1 \leq j \leq k+1} h'_{1,j}(y)n'_{1,j}, \cdots, -\sum_{1 \leq j \leq k+1} h'_{m,j}(y)n'_{m,j})$ , where  $h'_{i,j}$  is a  $\Sigma_{Y_L}$ -support function for all  $1 \leq i \leq m$  and  $1 \leq j \leq k+1$ . As we showed above, for each  $i \in \{1, \cdots, m\}$ , the induced map  $N_{Y_L \mathbb{R}} \rightarrow N_{Y_L \mathbb{R}} + N'_{i\mathbb{R}}$  sends  $y \in N_{Y_L \mathbb{R}}$  to  $(y, -\sum_{1 \leq j \leq k+1} h'_{i,j}(y)n'_{i,j})$  determines a fan in  $N_{Y_L \mathbb{R}} + N'_{i\mathbb{R}}$  whose corresponding toric variety is isomorphic to  $\mathbb{P}(\mathcal{E}_l)$  for some  $l \in \{1, \cdots, m\}$ . Thus we may replace  $h'_{i,j}$  by  $h_{l,j}$  for all  $1 \leq j \leq k+1$ . Rearranging the order, we obtain that the  $\Sigma_{X_L}$  is isomorphic to a fan determined by map  $N_{Y_L \mathbb{R}} \rightarrow N_{Y_L \mathbb{R}} + N'_{1\mathbb{R}} + \cdots + N'_{m\mathbb{R}}$  which sends  $y \in N_{Y_L \mathbb{R}}$  to  $(y, -\sum_{1 \leq j \leq k+1} h_{1,j}(y)n'_{1,j}, \cdots, -\sum_{1 \leq j \leq k+1} h_{m,j}(y)n'_{m,j})$ .

Therefore, we have  $\Sigma_{X_L} \simeq \Sigma_{X'_L}$ , and hence  $X_L = \mathbb{P}(\mathcal{E}_1) \times_{Y_L} \cdots \times_{Y_L} \mathbb{P}(\mathcal{E}_m)$ .

Thus we get the following compatible commutative diagram

$$\begin{array}{ccc} X_L = \mathbb{P}(\mathcal{E}_1) \times_{Y_L} \cdots \times_{Y_L} \mathbb{P}(\mathcal{E}_m) & \longrightarrow & (X_K, \mathcal{T}) \\ \downarrow & & \downarrow \\ Y_L & \longrightarrow & (Y_K, \mathcal{T}'). \end{array}$$

Iteratively, we get the following diagram:

$$\begin{array}{ccc} X_{l,L} = \mathbb{P}(\mathcal{E}_{l,1}) \times_{X_{l-1,L}} \cdots \times_{X_{l-1,L}} \mathbb{P}(\mathcal{E}_{l,m_l}) & \longrightarrow & (X_{l,K}, \mathcal{T}_l) = (X_K, \mathcal{T}) \\ \downarrow & & \downarrow \\ \vdots & & \vdots \\ \downarrow & & \downarrow \\ X_{2,L} = \mathbb{P}(\mathcal{E}_{2,1}) \times_{X_{1,L}} \cdots \times_{X_{1,L}} \mathbb{P}(\mathcal{E}_{2,m_2}) & \longrightarrow & (X_{2,K}, \mathcal{T}_2) \\ \downarrow & & \downarrow \\ X_{1,L} = \times_L^{m_1} \mathbb{P}(\mathcal{E}_1) & \longrightarrow & (X_{1,K}, \mathcal{T}_1) \\ \downarrow & & \downarrow \\ X_{0,L} = \text{Spec } L & \longrightarrow & \text{Spec } K \end{array}$$

where  $\mathcal{E}_1$  is a decomposable vector bundle of rank  $r_1 + 1$  over  $X_{0,L}$  and  $\mathcal{E}_{i,j_i}$  is a decomposable vector bundle of rank  $r_i + 1$  over  $X_{i-1,L}$  and  $\{\rho_g^*(\mathcal{E}_{i,1}) : g \in G\} = \{\mathcal{E}_{i,1}, \dots, \mathcal{E}_{i,m_i}\}$  for  $2 \leq i \leq l$  and  $1 \leq j_i \leq m_i$ .

As we know that  $\text{Pic}(X_{i,L}) \simeq \mathbb{Z}^{\oplus(m_1+\dots+m_i)}$ , we may assume all the line bundle summands of  $\mathcal{E}_i$  are in  $(\mathbb{Z}_{\geq 0})^{\oplus(m_1+\dots+m_{i-1})}$  for all  $2 \leq i \leq l$ .

Without causing confusion, we use the same notation  $\mathcal{O}_{\mathbb{P}(\mathcal{E}_{i,j})}(k)$  ( $1 \leq i \leq l$ ,  $1 \leq j \leq m_i$ ) to denote the corresponding component in  $\text{Pic}(X_{h,L})$  for all  $i \leq h \leq l$ .

Denote by

$$\begin{aligned} & \mathcal{O}(j_{1,1}, \dots, j_{1,m_1}, \dots, j_{l,1}, \dots, j_{l,m_l}) \\ = & (\mathcal{O}_{\mathbb{P}(\mathcal{E}_1)}(j_{1,1}), \dots, \mathcal{O}_{\mathbb{P}(\mathcal{E}_1)}(j_{1,m_1}), \dots, \mathcal{O}_{\mathbb{P}(\mathcal{E}_{l,1})}(j_{l,1}), \dots, \mathcal{O}_{\mathbb{P}(\mathcal{E}_{l,m_l})}(j_{l,m_l})), \end{aligned}$$

where  $-r_i \leq j_{i,k_i} \leq 0$  and  $1 \leq k_i \leq m_i$  for  $1 \leq i \leq l$ .

Then the set

$$\{\mathcal{O}(j_{1,1}, \dots, j_{1,m_1}, \dots, j_{l,1}, \dots, j_{l,m_l}) : -r_i \leq j_{i,k_i} \leq 0, 1 \leq k_i \leq m_i\}$$

is a full strong exceptional collection of  $D^b(X_L)$  by the lexicographical order on  $(j_{1,1}, \dots, j_{1,m_1}, \dots, j_{l,1}, \dots, j_{l,m_l})$ . For any  $g \in G$ , we have

$$\begin{aligned} & \rho_g^* \mathcal{O}(j_{1,1}, \dots, j_{1,m_1}, \dots, j_{l,1}, \dots, j_{l,m_l}) \\ &= \mathcal{O}(j_{1,\tau_{1,g}(1)}, \dots, j_{1,\tau_{1,g}(m_1)}, \dots, j_{l,\tau_{l,g}(1)}, \dots, j_{l,\tau_{l,g}(m_l)}), \end{aligned}$$

where  $\tau_{i,g}$ ,  $1 \leq i \leq l$ , are permutations of the corresponding sets  $\{1, \dots, m_i\}$ . So it is also in the same set.

Let

$$\mathcal{T} = \bigoplus \rho_g^* \mathcal{O}(j_{1,1}, \dots, j_{1,m_1}, \dots, j_{l,1}, \dots, j_{l,m_l}),$$

where the summand is the sum over the set  $\{(j_{1,1}, \dots, j_{1,m_1}, \dots, j_{l,1}, \dots, j_{l,m_l}) : -r_i \leq j_{i,k_i} \leq 0\}$ , then  $\mathcal{T}$  is a tilting sheaf on  $X_{l,L}$  by Lemma 3.1.9 and  $\mathcal{T}$  descends to  $X_K$ . Thus by Proposition 4.1.8, we have proved:

**Theorem 4.2.1.** *Let  $(X, \mathcal{T})$  be an arithmetic toric variety, whose split toric variety corresponding to a splitting fan, then there exists a tilting bundle on  $X$ .*

### 4.3 Conclusion and Further Questions

Overall, using Galois descent theory, we give constructions of tilting bundles on Brauer-Severi varieties (Theorem 4.1.12), which are different from the one constructed by Blunk [Bl]. We also show that for certain families  $Y \rightarrow X$  of Brauer-Severi Schemes over special rational smooth projective varieties  $X$ , the existence of a tilting bundle on  $Y$  depends on the existence of one on  $X$  (Theorem 4.1.16). This result can be viewed as a relative version of the result on Brauer-Severi varieties.

Finally, as an application, we show the existence of tilting bundle on arithmetic toric varieties whose associated split toric varieties are obtained as successive projective bundles (Theorem 4.2.1). This result generalizes the result obtained by Costa and Miró-Roig in [CM1].

In a series of papers [K1, K2, K2, K4], Kapranov gave full strong exceptional collections, and hence tilting bundles, on Grassmann and flag varieties with base field has characteristic zero. And Kaneda [Ka] showed that Kapranov's construction works on the Grassmannian in sufficiently large positive characteristic. In addition, we also know the automorphisms of Grassmann varieties [Ch] and flag varieties [T]. Our first natural interesting question is that whether we can use these results and Galois descent theory to show the existence of tilting bundles on the generalized Brauer-Severi varieties. Indeed, this is true for the family of generalized Grassmann varieties whose associated split varieties are of form  $Gr(k, n)$  such that  $n \neq 2k$ .

Let  $\pi : X_L \rightarrow X_K$  be the canonical morphism of the finite Galois extension  $L/K$ . In cases of Brauer-Severi varieties (Theorem 4.1.12) and certain arithmetic varieties (Theorem 4.1.12), we can construct the tilting bundle on  $X_K$  as the as the pushforward of the known tilting bundle on  $X_L$ , as indicated in Remark 4.1.13. But in general, the statement that the pushforward bundle  $\pi_* \mathcal{T}$  is a tilting bundle on  $X_K$  provided  $\mathcal{T}$  is a tilting bundle on  $X_L$  may not be true. A counterexample is the Grassmannian variety  $Gr(k, n)$  with  $n = 2k$ . So another interesting question is to find or characterize the family of varieties such that the statement above is true.

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# Curriculum Vitae

Youlong Yan

## EDUCATION:

- **The University of Western Ontario:** 2010-2014.  
Ph.D. Candidate in Mathematics.
- **East China Normal University:** 2007-2010.  
M.Sc. in Mathematics, August 2010.
- **Wuhan University:** 2003-2007.  
B.Sc. in Mathematics, August 2007.

## TEACHING EXPERIENCE:

- **The University of Western Ontario:** 2010 - 2014.  
Teaching Assistant.