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Centralizer Of A Semisimple Element On A Reductive Algebraic Monoid

Marjoie Eileen Hull

CENTRALIZER OF A SEMISIMPLE ELEMENT ON A REDUCTIVE ALGEBRAIC MONOID

by

M. Eileen Hull

Department of Mathematics

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ABSTRACT

Let M be a reductive linear algebraic monoid with unit group G and let the derived group of G be simply connected. The purpose of this thesis is to study the centralizer in M of a semisimple element of G. We call this set M_0.

We use a combination of the theories of algebraic geometry, linear algebraic groups and linear algebraic monoids in our study. One of our main tools is Renner's analogue of the classical Bruhat decomposition for reductive algebraic monoids. Our principal result establishes an analogue of the Bruhat decomposition for M_0. This is a more general result than Renner's decomposition for the centralizer of a torus on a reductive algebraic monoid.

Early research by M. S. Putcha and L. E. Renner presents the basic notation and general theory of algebraic monoids and is mainly descriptive. Later interest centres around the theory of reductive algebraic monoids which, by definition, are always irreducible. In this thesis we investigate the irreducibility of M_0. After proving propositions about the structural properties of M_0, we give a characterization of the irreducibility of M_0.

Finally, we give examples of algebraic monoids that have only irreducible centralizers and one in which the centralizer M_0 is reducible.

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CHAPTER 1
INTRODUCTION

A linear algebraic monoid is an affine algebraic variety M, defined over the algebraically closed field k, along with an associative morphism $m : M \times M \to M$ and a two sided identify $1 \in M$ for m.

The theory of linear algebraic monoids owes its development largely to M. S. Putcha and L. E. Renner. Early research presents the basic notation and general theory of algebraic monoids and is mainly descriptive ([P1]-[P3] and [R1]). Later interest centres around the theory of reductive algebraic monoids which are always irreducible.

In this thesis we are concerned with the fixed point set of a semisimple element on a reductive linear algebraic monoid. We find that this set is an algebraic monoid that is not necessarily irreducible.

Throughout the dissertation M denotes a reductive linear algebraic monoid over the algebraically closed field k, G its group of units, T a maximal torus of G and B a Borel subgroup of G containing T. We assume that the derived group of G is simply connected.

In Chapter 2 we summarize the background information from algebraic geometry, linear algebraic groups and linear algebraic monoids that is needed in the development of this thesis.

Chapter 3 is the main part of this work. In the first section we introduce Renner's orbit monoid, $\mathfrak{R} = \overline{N_G(T)}/T$, and several preliminary results relating to it.

In particular, we note that Renner obtains an analogue of the classical Bruhat decomposition for reductive algebraic monoids in [R3]. He also proves a Bruhat type decomposition for the centralizer of a torus on a reductive algebraic monoid in [R5]. The principal result of this thesis establishes a similar decomposition for the fixed point set of a semisimple element on a reductive algebraic monoid. To state our result, which is more general than Renner's, we need the following notation. Let $s \in G$ be semisimple. Let

$$M_0 = \{x \in M | xs = sx\} = C_M(s),$$

$$B_0 = \{b \in B | bs = sb\} = C_B(s), \text{ and}$$

$$\Re_0 = \{r \in \overline{N_G(T)} | (rT)_0 \neq \emptyset\}/T, \text{ where } (rT)_0 = \{rt \in rT | rts = srt\}.$$

Theorem: $M_0 = \bigcup_{r \in \Re_0} B_0 r B_0$, and the union is disjoint.

In Chapter 4 we explore further the monoids $\Re_0$ and M_0 that were presented in the previous chapter. In particular we prove that $\Re_0$ is a finite inverse monoid and M_0 is a regular algebraic monoid. In addition we note that $M_0 = C_M(s)$ and $\overline{C_G(s)}$ are not necessarily the same. Our final result in this section establishes conditions for the equivalence of these two structures.

Finally, in Chapter 5 we illustrate our principal results with concrete examples. In Example 5.1.1, in which M is the reductive linear algebraic monoid $M_3(k)$, we use Theorem 4.2.1 to show that $M_0 = \overline{C_G(s)}$. In Example 5.1.2, we let

$$G_1 = \{A \oplus (A^{-1})^t | A \in Sl_3(k)\}$$

$$G = k^* G_1 \text{ and } M = \overline{kG_1} \subseteq M_6(k).$$

Then M is a reductive linear algebraic monoid with group of units G. Although M is a more complicated monoid than that in the first example, we are surprised to find that the centralizer of each s in M is also an irreducible monoid. Finally, Example 5.2.1 indicates that $M_0 = C_M(s)$ is not necessarily an irreducible monoid. In this case we let $\rho : Sl_2(k)) \times Sl_2(k) \longrightarrow Gl_6(k)$ be a representation defined by

$$\rho(A, B) = \begin{pmatrix} A \otimes (B^{-1})^t & 0 \\ 0 & B \end{pmatrix}.$$

Let $G_1 = \rho(Sl_2(k) \times Sl_2(k))$. Now $T_1 = \{\rho(A, B) | A, B \in D_2^*(k),\ \det A = \det B = 1\}$ is a maximal torus of G_1. Let $T = k^* T_1$ and $G = k^* C_1$. Then $M = \overline{kG_1} \subseteq M_6(k)$ is a reductive algebraic monoid with group of units G and T is a maximal torus of G. To show that $C_M(s)$ is not always irreducible, take $s = (1, -1, -1, 1, i, -i)$.

CHAPTER 2
BACKGROUND

This chapter provides a summary of the fundamental concepts and results from algebraic geometry, algebraic groups and algebraic monoids that are required in this thesis. Although we include some proofs, usually we give explicit references and no proofs for known results.

2.1 ALGEBRAIC GEOMETRY

In this section we list the prerequisites from algebraic geometry. References for this material include [Ho],[Hu],[N] and [Sp].

Let k be an algebraically closed field. The set k^n is called *affine n-space*. A subset $X \subseteq k^n$ is *closed* if it is the set of common zeros of a collection of polynomials in $k[x_1, \ldots, x_n]$. Let $k[X] = k[x_1, \ldots, x_n]/I$ where $I = \{f \in k[x_1, \ldots, x_n] | f(X) = 0\}$.

If I is any ideal in $k[x_1, \ldots, x_n]$, let $\mathcal{V}(I)$ be the set of its common zeros in k^n. For a subset $X \subseteq k^n$, denote by $\mathcal{I}(X)$ the collection of all polynomials vanishing on X. Recall that the *radical* $\sqrt{I}$ of an ideal I is $\{f \in k[x_1, \ldots, x_n] | f^r \in I$ for some $r \geq 1\}$. $\sqrt{I}$ is an ideal. A *radical ideal* is one that is equal to its radical.

Theorem 2.1.1 (Hilbert's Nullstellensatz) [Sp; Theorem 1.1.2]. *If I is any ideal in $k[x_1, \ldots, x_n]$, then $\sqrt{I} = \mathcal{I}(\mathcal{V}(I))$.*

Theorem 2.1.1 implies that there is a 1-1 correspondence between the radical ideals in $k[x_1, \ldots, x_n]$ and the closed subsets of k^n.

The topology on k^n is called the ***Zariski topology***. It has the properties that points are closed and every open cover of $X \subseteq k^n$ has a finite subcover.

A closed subset X of k^n is *irreducible* if it cannot be written as the union of two proper, non-empty, closed subsets.

Let X be a topological space. Suppose that for each non-empty open set U of X, there is associated a subalgebra $\mathcal{O}(U)$ of the k-algebra of k-valued functions on U, subject to the following conditions. (We agree that $\mathcal{O}(\emptyset) = \{0\}$.)

(1) If U and V are non-empty open sets with $U \subseteq V$ and $f \in \mathcal{O}(V)$, then $f|_U \in \mathcal{O}(U)$.

(2) If U is a non-empty open set with an open covering $\{U_\alpha, \alpha \in A\}$ and if there is a function $f : U \to k$ for which $f|_{U_\alpha} \in \mathcal{O}(U_\alpha)$ for all $\alpha \in A$, then $f \in \mathcal{O}(U)$.

Then $\mathcal{O} = \mathcal{O}_X$ is an example of a *sheaf of functions* on X. A pair $(X, \mathcal{O}_X)$ of a topological space and a sheaf of functions is called a *ringed space*.

Proposition 2.1.2. *Let $X \subseteq k^n$ be a closed set. For each non-empty open subset U of X, let $\mathcal{O}(U) = \mathcal{O}_X(U) = \{\alpha : U \to k|$ there exists an open cover $\{U_\gamma\}$ of U such that $\alpha|_{U_\gamma} = f/g$ for some $f, g \in k[X]$ and g is non-zero on $U_\gamma\}$. Then $X = (X, \mathcal{O})$ is a ringed space.*

Proof. It is clear that $\mathcal{O}(U)$ is a subalgebra of k-valued functions. Suppose U and V are non-empty open sets with $U \subseteq V$ and $\alpha \in \mathcal{O}(V)$. Then we have $\alpha|_U \in \mathcal{O}(U)$. Next let U be a non-empty open set with an open covering $\{U_\alpha, \alpha \in A\}$, and let $\beta : U \to k$ be a function with $\beta|_{U_\alpha} \in \mathcal{O}(U_\alpha)$. For each α, we have an open covering $U_{\alpha\gamma}$ and $\beta|_{U_{\alpha\gamma}} = f/g$. The set of all $U_{\alpha\gamma}$'s is an open covering of U and $\beta|_{U_{\alpha\gamma}} =$

$(\beta|_{U_a})|_{U_{a\gamma}} = f/g$, proving the proposition. □

The ringed spaces $(X, \mathcal{O}_X)$ of Proposition 2.1.2 are the *affine algebraic varieties* over k.

Proposition 2.1.3 [Hu; Section 1.5], [Sp; Proposition 1.3.3]. *Let X be an affine variety. There is a bijection between the points of X and the maximal ideals of $k[X]$.*

Theorem 2.1.4 [Sp; Theorem 1.4.5]. *Let $X = (X, \mathcal{O}_X)$ be an affine algebraic variety. There is an isomorphism $\phi : k[X] \to \mathcal{O}_X(X)$, namely the identity map on functions.*

We call $k[X]$ the *affine algebra* of X. If X and Y are varieties then a *morphism of affine varieties* from X to Y is a map $\phi : X \to Y$ such that for every open set V of Y and $f \in k[V]$, $U = \phi^{-1}(V)$ is open in X and $f \circ \phi \in k[U]$.

Lemma 2.1.5 [Sp; Section 1.4.7]. *A morphism $\phi : X \to Y$ of affine varieties induces a homomorphism $\phi^* : k[Y] \to k[X]$ defined by $\phi^*(f) = f \circ \phi$. Conversely, if $\psi : k[Y] \to k[X]$ is an algebra homomorphism, there exists a morphism $\psi' : X \to Y$ with $(\psi')^* = \psi$.*

Theorem 2.1.6 [N; Section 2.6, Theorem 21]. *Let X and Y be affine varieties. There is a natural bijection between the morphisms $\phi : X \to Y$ and the homomorphisms $\phi^* : k[Y] \to k[X]$ of k-algebras.*

We say that ϕ is an *isomorphism* of X on to Y if it is a bijection and $\phi^{-1} : Y \to X$ is also a morphism.

Proposition 2.1.7 [Sp; p.12]. *A morphism of varieties $\phi : X \to Y$ is an isomorphism if and only if the algebra homomorphism $\phi^* : k[Y] \to k[X]$ is an isomorphism.*

Proposition 2.1.8 [Ho; Chapter IX, Proposition 2.2], [Sp; Section 1.5], [Hu: Proposition 2.4]. *Let X and Y be affine varieties. Then $X \times Y$ is an affine variety and $k[X \times Y] \cong k[X] \otimes_k k[Y]$.*

Proposition 2.1.9 [Ho; Chapter IX, Proposition 2.1], [Hu; p.23]. *If X and Y are varieties and $\phi,\psi : X \to Y$ are morphisms, then the set of points x in X such that $\phi(x) = \psi(x)$ is closed in X.*

Proposition 2.1.10 [Sp; Lemma 1.9.1]. *Let X and Y be affine varieties and let $\phi : X \to Y$ be a morphism. If X is irreducible, then $\overline{\phi(X)}$ is also irreducible.*

Proposition 2.1.11 [Ho; p.16], [N; Section 3.1, Theorem 2]. *Let X be an irreducible variety. If U is a non-empty open subset of X, then $\overline{U} = X$.*

Proposition 2.1.12 [Ho; p.125-126], [N; Section 3.2, Theorem 16]. *Let X and Y be irreducible affine varieties. Then $X \times Y$ is an irreducible variety.*

2.2 Linear Algebraic Groups

The basic reference for the theory of algebraic groups is [Hu]. (See also [B],[N],[Sp] and [St].)

A *linear algebraic group* is an affine algebraic variety G along with morphisms $m : G \times G \to G$, where $m(g,h)=gh$, and $i : G \to G$, where $i(g)=g^{-1}$, such that G is also a group with respect to m and i. We denote the identity element of G by 1. If $H \subseteq G$, then $N_G(H)=\{g \in G | g^{-1}Hg=H\}$ is the *normalizer of H in G* and

$C_G(H)=\{g \in G | gh=hg$ for all $h \in H\}$ is the *centralizer of H in G*. Two subsets A and B of G are *conjugate* if $g^{-1}Ag = B$ for some $g \in G$. A *connected* (irreducible) group G is an algebraic group whose underlying variety is irreducible. G is *unipotent* if it is isomorphic to a closed subgroup of $U(n)$, the set of matrices of $Gl_n(k)$ with $a_{ij} = 0$ if $i > j$ and $a_{ii} = 1$. G is *reductive* if it is connected and every unipotent normal subgroup of G is trivial. A closed connected subgroup T of G is a *torus* if it is isomorphic to $D_n^*(k)$ for some n. A maximal, closed, connected solvable subgroup B of G is called a *Borel subgroup*.

Theorem 2.2.1 [Hu; Section 12.1], [Sp; Section 5.2]. *Let H be a closed normal subgroup of the linear algebraic group G. Then*

(1) *G/H is a linear algebraic group.*

(2) *There is a canonical morphism $\pi : G \to G/H$.*

Theorem 2.2.2 [Hu; Theorem 21.3 and Corollary 21.3A]. *Let G be a connected algebraic group and let B be any Borel subgroup of G. Then*

(1) *All other Borel subgroups of G are conjugate to B.*

(2) *The maximal tori of G are the maximal tori of the Borel subgroups of G, and they are all conjugate.*

Proposition 2.2.3 [Hu; Corollary 21.3C], [Sp; Corollary 7.2.7]. *Let T be a torus in the algebraic group G. Then the image of T under any morphism is also a torus.*

Proposition 2.2.4 [Hu; Section 7.6], [N; Section 5.3]. *The additive group* *of k, denoted by G_a, has affine algebra $k[G_a] = k[X]$, the polynomial ring in one variable. The group G_a^n has $k[G_a^n] = k[X_1, \ldots, X_n]$, the polynomial ring in n variables. The multiplicative group of k, k^*, sometimes denoted by G_m, has affine algebra $k[G_m] = k[Y, Y^{-1}]$, where Y denotes an indeterminate. The group G_m^n has $k[G_m^n] = k[Y_1, Y_1^{-1}, Y_2, Y_2^{-1}, \ldots, Y_n, Y_n^{-1}]$, where $Y_1, Y_2, \ldots, Y_n$ are distinct indeterminates.*

Theorem 2.2.5 (Lie-Kolchin) [Hu; Theorem 17.6], [Sp; Theorem 6.7]. *If G is a closed connected solvable subgroup of $Gl_n(k)$, then G is conjugate to a subgroup of $T_n^*(k)$, where $T_n^*(k)=\{A \in M_n(k) | A$ is upper triangular and det $A \neq 0\}$. If G is a torus in $Gl_n(k)$, then it is conjugate to a subgroup of $D_n^*(k)$, where $D_n^*(k)=\{A \in M_n(k) | A$ is diagonal and det $A \neq 0\}$.*

Let $x \in \mathrm{End}(V)$, where V is a finite dimensional vector space over the algebraically closed field k. Then x is *nilpotent* if $x^n = 0$ for some n. If $x \in Gl(V)$, we say that x is *unipotent* if it is the sum of the identity and a nilpotent endomorphism.

Theorem 2.2.6 [B; Chapter III, Theorem 10.6], [Hu; Theorem 19.3]. *Let G be a connected solvable algebraic group and let $U = G_u=\{g \in G | g$ is unipotent$\}$. Then*

(1) *U is a closed connected normal subgroup of G.*

(2) *If T is a maximal torus of G, then $G = TU$.*

Let G be a connected algebraic group and let T be a maximal torus of G. Then $W = W(G) = N_G(T)/C_G(T)$ is the *Weyl group* of G.

Theorem 2.2.7 [B; Chapter IV, Section 11.19]. *Let T be a maximal torus of the connected algebraic group G. Then W is a finite group.*

Theorem 2.2.8 [Hu; Corollary 26.2A]. *Let T be a maximal torus of the reductive algebraic group G. Then $W = N_G(T)/T$.*

Theorem 2.2.9 (The Bruhat Decomposition) [Hu; Theorem 28.3]. *Let G be a reductive algebraic group, T a maximal torus of G and let B be a Borel subgroup of G containing T. Then $G = \bigcup_{\sigma \in W} B\sigma B$, and the union is disjoint.*

A homomorphism $\chi : G \to k^*$ is called a *character* of the algebraic group G. Let $X(G)$ denote the set of characters of G. Assume G is reductive and let $T \subseteq B$, where T is a maximal torus of G and B is a Borel subgroup of G containing T. If $\rho : G \to Gl(V)$ is a rational representation, then $\chi \in X(T)$ is a *weight* of ρ if $V_\chi = \{v \in V | \rho(t)v = \chi(t)v \text{ for all } t \in T\} \neq \{0\}$, and we call V_χ the *weight space* of χ. Recall that the tangent space of G at the identity forms a Lie algebra $\mathfrak{g}$. Let $Ad : G \to Gl(\mathfrak{g})$ be the *adjoint representation* [Hu; Chapter III]. The nonzero weights of $Ad : G \to Gl(\mathfrak{g})$ are called the *roots* of G and we denote them by Φ. Then $Ad|_T$ determines a direct sum decomposition $\mathfrak{g} = \mathfrak{h} \oplus \coprod_{\alpha \in \Phi} \mathfrak{g}_\alpha$, where $\dim \mathfrak{g}_\alpha = 1$, for each $\alpha \in \Phi \subseteq X(T)$ [Hu; Corollary 26.2B]. A *base* $\Delta \subseteq \Phi$ is a basis of $\mathbf{R} \otimes X(T)$ such that for all $\beta \in \Phi$, $\beta = \sum m_\alpha \alpha$, where the m_α are integers all of the same sign. Bases exist and there is a 1-1 correspondence between bases and Borel subgroups containing T [Hu; Section 27.3].

Theorem 2.2.10 [Hu; Theorem 26.3]. *Let G be a reductive algebraic group, T*

a maximal torus of G and $B \supseteq T$ a Borel subgroup of G. Let $\alpha \in \Phi$. Then

(1) *There is a unique connected T-stable subgroup U_α of G with Lie algebra $\mathfrak{g}_\alpha$.*

(2) *There exists an isomorphism $\epsilon_\alpha : G_a \to U_\alpha$ such that for all $t \in T$, $x \in G_a$, $t\epsilon_\alpha(x)t^{-1} = \epsilon_\alpha(\alpha(t) \cdot x)$ and $\dim U_\alpha = 1$.*

(3) *G is generated by the groups $U_\alpha (\alpha \in \Phi)$ and T.*

We define the *positive roots* Φ^+ to be those roots $\alpha \in \Phi$ for which $U_\alpha \subseteq B$.

Proposition 2.2.11 [Sp; Proposition 10.1.1]. *Let G be a reductive algebraic group, T a maximal torus of G and $B \supseteq T$ a Borel subgroup of G. Let $U = B_u = \{b \in B | b \text{ is unipotent }\}$. Then U is generated by the U_α with $\alpha \in \Phi^+$.*

We say that an algebraic group H is *directly spanned* by its closed subgroups $H_1, \ldots, H_n$ in the given order if the product morphism $\phi : H_1 \times \cdots \times H_n \to H$ is bijective.

Proposition 2.2.12 [Hu; Proposition 28.1]. *Suppose G is a reductive algebraic group, T a maximal torus of G and $B \supseteq T$ a Borel subgroup of G. Let $U = B_u$. Let H be a closed, T-stable subgroup of U. Then H is connected and directly spanned by those U_α's it contains (in any order).*

Let V be a finite dimensional vector space over the algebraically closed field k. We call $x \in End(V)$ *semisimple* if x is diagonalizable over k.

Proposition 2.2.13 [Hu; Sections 19.3 and 22.2]. *Let G be a connected algebraic group. Then each semisimple element of G lies in a maximal torus.*

Proposition 2.2.14 [Hu; p.125]. *Let G be a connected solvable group. Let $s \in G$ be semisimple. Then $C_G(s)$ is connected.*

We call a connected algebraic group of positive dimension *semisimple* if every closed connected abelian normal subgroup is trivial. A semisimple algebraic group G is said to be *simply connected* if the fundamental group of G is trivial.

Theorem 2.2.15 [St; Theorem 8.1]. *Let G be a simply connected semisimple algebraic group. Let σ be a semisimple automorphism of G and G_σ its group of fixed points. Then G_σ is connected and reductive.*

Corollary 2.2.16 [St; Corollary 8.5 and Remark 8.3(c)]. *Let G be a simply connected semisimple algebraic group. The centralizer of a semisimple element of G is connected and reductive.*

Remark 2.2.17. Although Corollary 2.2.16 is stated for simply connected semisimple algebraic groups, it is also valid for reductive algebraic groups, provided that G', the derived group of G, is simply connected.

Proof:. Let G be a reductive algebraic group and let $G' = (G, G)$, the derived group of G, be simply connected. By [B; Chapter IV, Proposition 14.2], G' is a semisimple algebraic group and G is the product of G' and $Z(G)^0$, the identity component of $Z(G)$. Suppose $s \in G$ is semisimple. Then $s = \beta y$ for some $\beta \in Z(G)^0$, $y \in G'$. Thus, $C_G(s) = C_G(y)$ since $\beta \in Z(G)^0$. By Corollary 2.2.16, $C_{G'}(y)$ is connected and $C_G(y)$ is the product of $Z(G)^0$ and $C_{G'}(y)$. Therefore, by Proposition 2.1.12 $C_G(y)$ is connected.

2.3 Linear Algebraic Monoids

The recent development of the theory of linear algebraic monoids is due mainly to Putcha [P1-P6] and Renner [R1-R5]. We use the definitions and notation of Renner.

A *linear algebraic monoid* is an affine algebraic variety M, defined over the algebraically closed field k, along with an associative morphism $m : M \times M \to M$ and a two-sided identity $1 \in M$ for m. If $x, y \in M$, then y is an *inverse* of x if $xyx = x$ and $yxy = y$. An element $x \in M$ is *regular* if x has an inverse in M. We call an element $x \in M$ a *unit* if x has a unique inverse, denoted x^{-1}, in M such that $xx^{-1} = x^{-1}x = 1$. $G = G(M)=\{x \in M | x^{-1} \in M\}$ is called the *group of units* of M. $E(M) = \{e \in M | e^2 = e\}$ is the set of *idempotents* of M. An algebraic monoid M is *irreducible* if the underlying closed set is an irreducible variety. An irreducible algebraic monoid M is *reductive* if $G = G(M)$ is a reductive group. A monoid M is *von Neumann regular* if for each $x \in M$ there exists $a \in M$ such that $xax = x$. M is *unit regular* if for each $x \in M$ there exists $g \in G = G(M)$ such that $x = xgx$. A monoid M is called an *inverse monoid* if for all $x \in M$ there exists a unique $x^* \in M$ such that $xx^*x = x, x^*xx^* = x^*$, and $(x^*)^* = x$.

Remark 2.3.1. Putcha shows that G, the group of units of M, is an affine algebraic group [P2; p. 458], [P3; p. 472].

Theorem 2.3.2 [DG; II, §2, Theorem 3.3], [P1; Theorem 3.15]. *Let M be a linear algebraic monoid. Then M is isomorphic to a closed submonoid of $M_n(k)$ for some $n \in \mathbb{Z}^+$.*

Corollary 2.3.3 [DG; II, §2, Corollary 3.6], [P2; Corollary 1.2]. *Let M be an algebraic monoid which is not a group. Then the nonunits of M form a closed prime ideal of M.*

Therefore, we see that $G \subsetneq M$ is an open subset in the Zariski topology [P4; p. 668].

Proposition 2.3.4 follows from Proposition 2.1.11 and the fact that the group of units G is open in M.

Proposition 2.3.4 [P4; p. 668]. *If an algebraic monoid M is irreducible, $M = \overline{G}$, where $G \subsetneq M$ is the group of units of M.*

Theorem 2.3.5 [P5; Theorem 13(5)]. *Let M be an irreducible monoid with group of units G, and let $x \in M$ be regular. Then there exists $u \in G$ such that $xux = x$.*

The following theorem follows from Theorem 2.3.5.

Theorem: 2.3.6. *Let M be an irreducible algebraic monoid. Then the following conditions are equivalent.*

(1) *M is von Neumann regular.*

(2) *M is unit regular.*

(3) *For all $x \in M$, there exists $g \in G = G(M)$ and $e \in E(M)$ such that $x = ge$.*

Theorem 2.3.7 [R2; Theorem 3.1]. *Reductive algebraic monoids are von Neumann regular.*

We define an element x in an irreducible algebraic monoid M to be *semisimple* if $\rho(x)$ is diagonalizable for every rational representation $\rho : M \to M_n(k)$.

Proposition 2.3.8 [R4; Theorem 3.5]. *Suppose that M is a reductive algebraic monoid, $G \subseteq M$ the group of units of M and let $x \in M$. Then the following are equivalent.*

(1) *x is semisimple.*

(2) *$x \in \overline{T}$, the Zariski closure of some maximal torus T of G.*

Let M be an algebraic monoid, $e \in E(M)$, and let $H_e = \{x \in M | \exists y \in M$ such that $xy = yx = e$, $ex = xe = x$ and $ey = ye = y\}$. We say that M is *completely regular* if M is the union of the H_e as e varies over all idempotents of M. (We note that H_e is the group of units of eMe and so H_e is an algebraic group. If M is irreducible, then so is H_e [P3; Lemma 1.1] and $\overline{H}_e = eMe$ by Proposition 2.3.4.)

Proposition 2.3.9 [P3; Section 3], [P6; Theorem 2.1]. *Let M be a reductive algebraic monoid, G the group of units of M and $T \subseteq G$ a maximal torus. Then $\overline{T}$ is completely regular.*

CHAPTER 3
CENTRALIZERS OF SEMISIMPLE ELEMENTS

In this chapter we establish an analogue of the Bruhat decomposition for the centralizer of a semisimple element on a reductive algebraic monoid. Renner ascertains a similar result for the centralizer of a torus on a reductive algebraic monoid in [R5; Lemma 6.1]. Our result is more general than Renner's since the centralizer of a torus coincides with the centralizer of some element of that torus [Hu; Proposition 16.4].

3.1 Notation And Preliminary Results

Throughout this chapter let M be a reductive linear algebraic monoid with group of units G. Fix a maximal torus T of G and let $B \subseteq G$ be a Borel subgroup with $T \subseteq B$. Let $R = \overline{N_G(T)} \subseteq M$. (Zariski closure)

Proposition 3.1.1 [P1; Proposition 11.1]. *$R = \overline{N_G(T)}$ is a unit regular inverse monoid with group of units $N_G(T)$ and idempotent set $E(\overline{T})$.*

Proposition 3.1.2 [P1; Theorem 11.12(ii)]. *$R = \{x \in M | Tx = xT\}$.*

Remark 3.1.3 [R3;p.309]. Renner showed that $\mathfrak{R} = R/T$ has the unique structure of a monoid such that $\pi : R \to \mathfrak{R}$, with $\pi(x) = xT$, is a morphism of monoids.

Let $E(\mathfrak{R})$ be the set of idempotents of $\mathfrak{R}$ and let $\mathfrak{R}^*$ denote the set of units of $\mathfrak{R}$.

Proposition 3.1.4 [R3; Theorem 3.2.1]. *$\mathfrak{R}$ is a finite inverse monoid with group of units $\mathfrak{R}^* = W = N_G(T)/T$ and $E(\mathfrak{R}) \cong E(\overline{T})$.*

Renner proves an analogue of the Bruhat decomposition for reductive algebraic monoids in the following theorem. In his computation, the Weyl group W is replaced by the finite inverse monoid $\mathfrak{R}$, with unit group W.

Theorem 3.1.5 [R3; Corollary 5.8]. *$M = \bigcup_{r \in \mathfrak{R}} BrB$, and the union is disjoint.*

Remark 3.1.6. The most familiar example of a reductive linear algebraic monoid is $M = M_n(k)$. In this case,

$$M = M_n(k), \qquad G = Gl_n(k) \subseteq M,$$

$$B = \{(a_{ij}) \in G | a_{ij} = 0 \text{ if } i > j\},$$

$$T = \{(a_{ij}) \in G | a_{ij} = 0 \text{ if } i \neq j\},$$

$$\overline{T} = \{(a_{ij}) \in M | a_{ij} = 0 \text{ if } i \neq j\}.$$

Then

$$R = \overline{N_G(T)} = \{(a_{ij}) \in M | a_{ij}a_{ik} = 0 \text{ if } j \neq k \text{ and } a_{ij}a_{lj} = 0 \text{ if } i \neq l\},$$

$$\mathfrak{R} = \overline{N_G(T)}/T \cong \{(a_{ij}) \in R | a_{ij} = 0 \text{ or } 1 \text{ for all } i, j\},$$

$$E(\mathfrak{R}) \cong E(\overline{T}) = \{(a_{ij}) \in \overline{T} | a_{ij} = 0 \text{ or } 1 \text{ for all } i, j\}.$$

3.2 AN ANALOGUE OF THE BRUHAT DECOMPOSITION

As mentioned at the beginning of this chapter, Renner extends the Bruhat decomposition for algebraic groups to the centralizer of a torus on a reductive algebraic monoid in the following theorem.

Theorem 3.2.1 [R5; Lemma 6.1]. *Let M be a reductive algebraic monoid with unit group G. Let H be a torus of G and let $M_0 = C_M(H)$. Then $M_0 = \bigcup_{r \in \mathfrak{R}_0} B_0 r B_0$,* *where $B_0 = C_B(H)$ and $\Re_0 = \{r \in R | rt = tr \text{ for } t \in H\}/T$. (Here T is a maximal torus of G and $B \subseteq G$ is a Borel subgroup with $T \subseteq B$.)*

The principal result of this thesis generalizes the Bruhat decomposition to the centralizer of a semisimple element on a reductive algebraic monoid. Throughout this work we will use the following notation.

Notation. *Let M be a reductive linear algebraic monoid with group of units G, where $G' = (G, G)$ is simply connected, and let $s \in G$ be semisimple. By Theorem 2.2.2 and Proposition 2.2.13, we may assume $s \in T \subseteq B$, where T is a maximal torus of G and B is a Borel subgroup of G containing T. Consider*

$$R = \overline{N_G(T)} \text{ and } \Re = \overline{N_G(T)}/T. \text{ Let}$$

$$\Re_0 = \{r \in R | (rT)_0 \neq \emptyset\}/T, \text{ where } (rT)_0 = \{rt \in rT | rts = srt\},$$

$$B_0 = \{b \in B | bs = sb\} = C_B(s),$$

$$G_0 = \{g \in G | gs = sg\} = C_G(s), \text{ and}$$

$$M_0 = \{x \in M | xs = sx\} = C_M(s).$$

The restriction that G' be simply connected is required so that G_0 is connected. Note that B_0 is connected by Proposition 2.2.14 and G_0 is connected and reductive by Corollary 2.2.16 and Remark 2.2.17.

This notation will be used throughout the thesis.

Statement of the Main Theorem.

$M_0 = \bigcup_{r \in \Re_0} B_0 r B_0$, and the union is disjoint.

To obtain this result, this author proves a series of lemmas and propositions.

Lemma 3.2.2. *Let $r \in R$. If $(rT)_0 \neq \emptyset$, then $(rT)_0 = rT$.*

Proof. Assume that $(rT)_0 \neq \emptyset$. Then $srts^{-1} = rt$ for some $t \in T$. Hence, $sr = rtst^{-1}$ and so $srt's^{-1} = rtst^{-1}t's^{-1} = rt'$ for all $t' \in T$ since T is commutative.

Lemma 3.2.3. *Let $r \in R$. Then $BrB \cong rT \times k^a$, for some $a \geq 0$.*

Proof. Let $V=\{u \in U | urB \subseteq rB\}$, where U is the unipotent part of B. To prove that V is a closed subgroup of U, we show that $V = V_1$, where V_1 is the closed subgroup of U defined by $V_1 = \{u \in U | u\overline{rB} \subseteq \overline{rB}\}$ ([Hu; Proposition 8.2] and [R1; Corollary 2.2.2]). Clearly $V \subseteq V_1$. On the other hand, let $u \in V_1$. Since V_1 is a group, $u^{-1} \in V_1$ and so $u^{-1}\overline{rB} \subseteq \overline{rB}$. Hence, $uu^{-1}\overline{rB} \subseteq u\overline{rB}$ or $\overline{rB} \subseteq u\overline{rB}$. Therefore, $V_1 = \{u \in U | u\overline{rB} = \overline{rB}\}$. Now $u\overline{rB} = \overline{rB}$ implies that $urB = rB$, since $urB \cap rB \neq \emptyset$ and two B-orbits are either equal or disjoint. Hence, $V_1 \subseteq V$. Thus we have proved that V is a closed subgroup of U and $V = \{u \in U | urB = rB\}$. To show $T \subseteq N_G(V)$, let $t \in T$ and $u \in V$. Then

$$
\begin{aligned}
tut^{-1}rB &= turB, \text{ since } Tr = rT \text{ by Proposition 3.1.2} \\
&= trB, \text{ because } u \in V \\
&= rB, \text{ since } Tr = rT.
\end{aligned}
$$

Thus, $T \subseteq N_G(V)$. Since V is a closed subgroup of U that is normalized by T, by Proposition 2.2.12, $V = \prod_{U_\alpha \subseteq V} U_\alpha$, where the U_α's are the connected T-stable subgroups of G defined in Theorem 2.2.10. Let $X = \prod_{U_\alpha \not\subseteq V} U_\alpha$. Now $X \times V \to U$, defined by $(x,v) \mapsto xv$, is an isomorphism. Then

$$\begin{aligned} BrB &= UTrB, \text{ since } B = UT \text{ by Theorem 2.2.6(2)} \\ &= UrTB \qquad (rT = Tr) \\ &= UrB \qquad (TB = B) \\ &= XVrB, \text{ since } XV = U \\ &= XrB, \text{ since } VrB = rB. \end{aligned}$$

Define $\phi : X \times rB \to BrB$ by $\phi(x,rb) = xrb$. We prove ϕ is injective. Suppose $xrb_1 = yrb_2$. Then $rb_1 = x^{-1}yrb_2$, and we have $rB = x^{-1}yrB$. Hence, $x^{-1}y \in V$ and so $x^{-1}y = v \in V$ or $y = xv$. So $yV = xV$. Now $X \times V \to U$ is an isomorphism with

$$(x, v) \mapsto xv,$$
$$(y, 1) \mapsto y \text{ and } y = xv.$$

Therefore, $x = y$ and ϕ is bijective. Thus,

$$X \times rB \cong BrB. \tag{1}$$

Let

$$\begin{aligned} Z &= \{u \in U | rTu = rT\}, \text{ where } r = \sigma e, \sigma \in W, e \in E(\overline{T}) \\ &= \{u \in U | eu = e\}. \end{aligned}$$

To prove $T \subsetneq N_G(Z)$, let $t \in T$, $u \in Z$. Then,

$$\begin{aligned} rTtut^{-1} &= rTut^{-1}, \\ &= rTt^{-1}, \text{ since } u \in Z \\ &= rT. \end{aligned}$$

So $T \subsetneq N_G(Z)$. Since Z is a closed subgroup of U and $T \subsetneq N_G(Z)$, by Proposition 2.2.12, $Z = \prod_{U_\alpha \subsetneq Z} U_\alpha$. Let $Y = \prod_{U_\alpha \not\subseteq Z} U_\alpha$. The morphism $Z \times Y \to U$, defined by $(z,y) \mapsto zy$, is an isomorphism. Then

$$\begin{aligned} rB &= rTU, \text{ since } B = TU \text{ by Theorem 2.2.6(2)} \\ &= rTeZY \quad (ZY = U \text{ and } rT = Tr = Tre = rTe) \\ &= rTeY \quad (Z = \{u \in U | eu = e\}) \\ &= rTY. \end{aligned}$$

Define $\gamma : rT \times Y \to rB$ by $\gamma(rt,y) = rty$. To show γ is injective, we let $rtx = rty$. Then $rt = rtyx^{-1}$ and $yx^{-1} \in Z$. So $yx^{-1} = z \in Z$ and $y = zx$. Thus, $Zy = Zx$. Now $Z \times Y \to U$ is an isomorphism with

$$(z, x) \mapsto zx,$$

$$(1, y) \mapsto y \text{ and } y = zx.$$

Therefore, $x = y$ and γ is injective. Hence,

$$(?) \qquad rT \times Y \cong rB.$$

Thus, we have

$$X \times rT \times Y \cong X \times rB \text{ by (2)},$$

$$\cong BrB \text{ by (1)}.$$

The isomorphism is given by $(x, a, y) \mapsto xay$. Since $U_\alpha \cong k$, $X = \prod_{\alpha \in A} U_\alpha$, where $A = \{\alpha | U_\alpha \not\subseteq V\}$ and $Y = \prod_{\alpha \in C} U_\alpha$, where $C = \{\alpha | U_\alpha \not\subseteq Z\}$, we have $X \cong k^{a_1}$, $a_1 = |A|$ and $Y \cong k^{a_2}$, $a_2 = |C|$. Let $a = a_1 + a_2$. Then

$$BrB \cong X \times rT \times Y$$

$$\cong rT \times k^a$$

for some $a \geq 0$. □

Lemma 3.2.4. *Let $r \in R$. Then $rT \cong k^* \times \cdots \times k^* = (k^*)^b$ for some $b \geq 0$.*

Proof. Let $S=\{t \in T | rt = r\}$. It is easily proved that S is a subgroup of T. That S is closed in T follows from Proposition 2.1.9. Because S is a subgroup of the abelian group T, S is a normal subgroup of T.

Thus, since S is a closed normal subgroup of T, by Theorem 2.2.1 we can form the algebraic group T/S and we have the canonical morphism, $\pi : T \to T/S$, given by $\pi(t) = [t] = tS$. Clearly π is surjective. Therefore, since T is a torus in G and $\pi(T) = T/S$, T/S is a torus by Proposition 2.2.3.

Define $\gamma : T/S \to rT$ by $\gamma([t]) = rt$. The map γ is a morphism of affine varieties. Suppose $rt_1 = rt_2$. Then $rt_1t_2^{-1} = r$ and $t_1t_2^{-1} \in S$. Therefore, $t_1S = t_2S$ and γ is injective. Given $rt \in rT$, we have $\gamma([t]) = rt$. Hence, γ is surjective. Thus, γ is an isomorphism. The isomorphism γ shows that rT is isomorphic, as a variety, to the torus T/S. Therefore, $rT \cong (k^*)^b$ for some $b \geq 0$. □

Fix $r \in R$. Let dim $rT = b$.

Lemma 3.2.5. *Let $H = k[BrB]$ be the coordinate ring of BrB and let $H^* = k[BrB]^*$ be the units of $k[BrB]$. Then $H^* \cong k^* \times \mathbf{Z}^b$.*

Proof. By Lemma 3.2.3 $BrB \cong rT \times k^a$, for some $a \geq 0$. Also, $rT \cong (k^*)^b$ by Lemma 3.2.4. Thus,

$$\begin{aligned} H &= k[rT \times k^a] \\ &= k[(k^*)^b \times k^a] \\ &= k[Y_1, Y_1^{-1}, \ldots, Y_b, Y_b^{-1}, X_1, \ldots, X_a], \text{ by Proposition 2.2.4.} \end{aligned}$$

Hence,

$$\begin{aligned} H^* &= \{\eta Y_1^{\alpha_1} \cdots Y_b^{\alpha_b} | \alpha_i \in \mathbf{Z}, \eta \in k^*\} \\ &\cong k^* \times \mathbf{Z}^b. \end{aligned}$$

The isomorphism is given by $\eta Y_1^{\alpha_1} \cdots Y_b^{\alpha_b} \mapsto (\eta, (\alpha_1, \cdots, \alpha_b))$. □

Lemma 3.2.6. *Let $H = k[BrB]$ and let H^* be its units. Let $k[H^*]$ be the k-algebra generated by H^* as a subalgebra of H. There is an algebraic variety L such that $H \supseteq k[H^*]{=}k[L]$. Then we get the commutative diagram*

$$\begin{array}{ccc} rT & \xrightarrow{s} & rT \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i} \\ BrB & \xrightarrow{s} & BrB \\ {\scriptstyle \psi}\downarrow & & \downarrow{\scriptstyle \psi} \\ L & \xrightarrow{s} & L \end{array}$$

where i is the inclusion map, s denotes the unique isomorphism from L to L induced by $s : BrB \to BrB$ and rT is isomorphic to L.

Proof. By Proposition 2.1.3 and Theorem 2.1.6, there is a morphism $\psi : BrB \to L$, where L is identified with the set of maximal ideals of $k[H^*]$, $\Omega(k[H^*])$. Now $k[H^*]=k[Y_1,\dots,Y_b,Y_1^{-1},\dots,Y_b^{-1}]$ by Proposition 2.2.4. If m is a maximal ideal of $k[H^*]$, $m = (Y_1 - \alpha_1,\dots,Y_b - \alpha_b),\alpha_i \neq 0$. We have a bijection between $(k^*)^b$ and $\Omega(k[H^*])$, the set of maximal ideals of $k[H^*]$, given by $m \mapsto (\alpha_1,\dots,\alpha_b)$.

Since $BrB \cong rT \times k^a$ by Lemma 3.2.3 and $rT \cong (k^*)^b$ by Lemma 3.2.4, we have the diagram

$$\begin{array}{ccccc} rT & \xrightarrow{i} & rT \times k^a & \xrightarrow{\kappa} & (k^*)^b \times k^a \\ \Vert & & \delta\downarrow & & p\downarrow \\ rT & \xrightarrow{i} & BrB & \xrightarrow{\psi} & L \end{array}$$

where p is the projection of the first factor, i is the inclusion map and κ and δ are isomorphisms. It follows that there is an isomorphism γ from rT to L. Now

$$\begin{aligned} H^* &= k[BrB]^* \\ &= \{\eta Y_1^{\alpha_1} Y_2^{\alpha_2} \cdots Y_b^{\alpha_b} | \eta \in k^*, \alpha_i \in \mathbf{Z}\} \\ &= \{f \in k[BrB] | f(BrB) \not\ni 0\} \\ &\cong k^* \times \mathbf{Z}^b. \end{aligned}$$

Let $H_r^*=\{\chi \in H^* | \chi(r) = 1\}$. H_r^* is a subgroup of H^*. If $\chi,\mu \in H_r^*$, then $(\chi\mu)(r) = \chi(r)\mu(r) = 1$ and we have $\chi\mu \in H_r^*$. Also $\chi^{-1}(r) = \chi(r)^{-1} = 1^{-1} = 1$. Thus, $\chi^{-1} \in H_r^*$ whenever $\chi \in H_r^*$. Let $\chi \in H^*$. Then $\chi = tY_1^{\alpha_1} \cdots Y_b^{\alpha_b}$ and $\chi(r) = s \in k^*$. Hence, $(s^{-1}\chi)(r) = 1$ for some unique $s^{-1} \in k^*$. So for $\chi \in H^*$, there exists a unique $s \in k^*$ such that $s^{-1}\chi \in H_r^*$. Also, $H_r^* = \{Z_1^{\alpha_1} \cdots Z_b^{\alpha_b} | \alpha_i \in \mathbb{Z}, Z_i = \eta_i Y_i,$ for some $\eta_i \in k^*\}$ and H_r^* is isomorphic to $\mathbb{Z}^b$. The isomorphism $s^{-1} : H_r^* \to \mathbb{Z}^b$ is given by $s^{-1}\chi \mapsto (\alpha_1, \cdots, \alpha_b)$. Similarly $H_{rt}^* \cong \mathbb{Z}^b$ for all t. It follows that $k[H^*]$, $k[H_r^*]$, $k[H_{rt}^*]$, etc. yield the same subring of $H = k[BrB]$.

Let $s : BrB \to BrB$ be defined by $s(b_1 r b_2) = s b_1 r b_2 s^{-1}$. Conjugation is a morphism. The map s is an isomorphism with its inverse being $s^{-1} : BrB \to BrB$ given by $s^{-1}(b_1 r b_2) = s^{-1} b_1 r b_2 s$. By Lemma 2.1.5, there is a natural homomorphism $s^* : k[BrB] \to k[BrB]$. By Proposition 2.1.7, s^* is an isomorphism since $s : BrB \to BrB$ is an isomorphism.

Thus, s^* satisfies $s^*(H^*) = H^*$ where $H^* = k[BrB]^*$, and $s^*(k[H^*]) = k[H^*]$. Therefore, we have

$$\begin{array}{ccc} rT & \xrightarrow{\;s\;} & rT \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i} \\ BrB & \xrightarrow{\;s\;} & BrB \\ {\scriptstyle \psi}\downarrow & & \downarrow{\scriptstyle \psi} \\ L & \xrightarrow{\;s\;} & L \end{array}$$

where i is the inclusion map, s is the unique isomorphism from L to L induced by $s : BrB \to BrB$ and rT is isomorphic to L. □

Lemma 3.2.7. *Let* $(BrB)_0 = \{b_1 r b_2 \in BrB | s b_1 r b_2 s^{-1} = b_1 r b_2\}$. *Then* $(BrB)_0 \neq \emptyset \Leftrightarrow (rT)_0 \neq \emptyset$.

Proof. Assume $(BrB)_0 \neq \emptyset$. It follows that $\psi((BrB)_0) \neq \emptyset$. By Lemma 3.2.6, $\psi((BrB)_0) \subseteq L_0$. Therefore, $L_0 \neq \emptyset$. Since $rT \cong L$ by Lemma 3.2.6, we have $(rT)_0 \neq \emptyset$.

To prove the converse, suppose that $(rT)_0 \neq \emptyset$. Then $(rT)_0 = rT$ by Lemma 3.2.2. Because $T \subseteq B_0$, $B_0 \neq \emptyset$. Hence, $(BrB)_0 \supseteq B_0 r B_0 \neq \emptyset$. □

Proposition 3.2.8.

$$(BrB)_0 = \begin{cases} \emptyset, r \notin \Re_0, \\ B_0 r B_0, r \in \Re_0. \end{cases}$$

Proof. If $r \notin \Re_0$, then $(rT)_0 = \emptyset$ by definition and so $(BrB)_0 = \emptyset$ by Lemma 3.2.7.

Now let $r \in \Re_0$. We must show that $(BrB)_0 = B_0 r B_0$. Clearly, $B_0 r B_0 \subseteq (BrB)_0$. By the proof of Lemma 3.2.3, $BrB \cong X \times rT \times Y$, where $X = \prod_{U_\alpha \not\subseteq V} U_\alpha \subseteq B$ and $Y = \prod_{U_\alpha \not\subseteq Z} U_\alpha \subseteq B$. So

$$\begin{aligned} (BrB)_0 &\cong (X \times rT \times Y)_0 \\ &\cong X_0 \times (rT)_0 \times Y_0, \text{ since } sXs^{-1} = X, sYs^{-1} = Y, srTs^{-1} = rT. \end{aligned}$$

The isomorphism is given by $s(x, a, y)s^{-1} = (sxs^{-1}, sas^{-1}, sys^{-1})$. Its inverse is $s^{-1}(x', a', y')s = (s^{-1}x's, s^{-1}a's, s^{-1}y's)$. Since $(rT)_0 = rT$,

$$\begin{aligned} (BrB)_0 &\cong X_0 \times rT \times Y_0 \\ &\subseteq B_0 rT B_0, \text{ since } X \subseteq B \text{ and } Y \subseteq B \\ &= B_0 r B_0 \ (B = TB). \end{aligned}$$

This proves the proposition. □

We are now able to complete the proof of the main theorem.

Theorem 3.2.9. *$M_0 = \bigcup_{r \in \Re_0} B_0 r B_0$ and the union is disjoint.*

Proof. By Theorem 3.1.5,

$$M_0 = \Big(\bigcup_{r\in\mathcal{R}} BrB\Big)_0 \text{ and the union is disjoint}$$

$$= \bigcup_{r\in\mathcal{R}} (BrB)_0, \text{ since } sBrBs^{-1} = BrB$$

$$= \bigcup_{r\in\mathcal{R}_0} B_0 r B_0 \text{ by Proposition 3.2.8. } \square$$

CHAPTER 4
IRREDUCIBILITY OF CENTRALIZERS

In this chapter we examine in more detail the structures $\Re_0$ and M_0 that were introduced in Chapter 3. In particular, we observe that the set $\Re_0$ is a finite inverse monoid. Also, we note that the set M_0 is a regular algebraic monoid that is not necessarily irreducible and we give a characterization of M_0 when M_0 is irreducible.

Let M be a reductive linear algebraic monoid with group of units G, where G' is simply connected, and let $s \in G$ be semisimple. By Theorem 2.2.2 and Proposition 2.2.13, we may assume that $s \in T \subseteq B$, where T is a maximal torus of G and B is a Borel subgroup of G containing T. We recall and use the notation of Chapter 3. Thus, we have

$$R = \overline{N_G(T)} = \{x \in M | Tx = xT\}, \Re = \overline{N_G(T)}/T,$$

$$\Re_0 = \{r \in R | (rT)_0 \neq \emptyset\}/T, \text{where} (rT)_0 = \{rt \in rT | rts = srt\},$$

$$R_0 = \{x \in R | xs = sx\} = C_R(s),$$

$$B_0 = \{b \in B | bs = sb\} = C_B(s),$$

$$G_0 = \{g \in G | gs = sg\} = C_G(s), \text{ and}$$

$$M_0 = \{x \in M | xs = sx\} = C_M(s).$$

4.1 Structural Properties of M_0

In Section 4.1 the author establishes two significant structure propositions for M_0.

Proposition 4.1.1. *$\Re_0$ is a finite inverse monoid.*

Proof. Since $\Re$ is a finite monoid by Proposition 3.1.4, $\Re_0 \subseteq \Re$ is finite. Because $st = ts$ for all $t \in T, (1 \cdot T)_0 \neq \emptyset$ and so $1 \in \Re_0$. Also, if $x, y \in \Re_0$, then $xy \in \Re_0$ since $sxy = xsy = xys$. Hence, $\Re_0$ is a finite monoid.

By the above, $\Re_0$ is nonempty. Let $r \in \Re_0$. By Proposition 3.1.4, we may let r^* be the unique inverse of r in $\Re$. Since $r \in \Re_0, (rT)_0 \neq \emptyset$ and $srts^{-1} = rt$ for some $t \in T$. So $(srts^{-1})^{-1} = (rt)^{-1}$ or $st^{-1}r^*s^{-1} = t^{-1}r^*$. Thus, $(Tr^*)_0 = (r^*T)_0 \neq \emptyset$, proving that $r^* \in \Re_0$. □

Proposition 4.1.2. *M_0 is a regular algebraic monoid.*

Proof. By Theorem 3.2.9 we have

$$\begin{aligned} M_0 &= \{x \in M \mid sx = xs\} \\ &= \{x \in M \mid sxs^{-1} = x\} \\ &= \bigcup_{r \in \Re_0} B_0 r B_0, \text{ disjoint union .} \end{aligned}$$

It is clear that M_0 is a monoid since $1 \in M_0$ and if $x, y \in M_0$, then $sxy = xsy = xys$ implies that $xy \in M_0$. Also, $f : M \to M$ defined by $f(x) = sxs^{-1}$ and $g : M \to M$ defined by $g(x) = x$ are morphisms. Thus M_0 is a closed set by Proposition 2.1.9. Therefore, M_0 is a closed submonoid of the algebraic monoid M. Hence, M_0 is an algebraic monoid.

Let $x \in M_0$. Then by Theorem 3.2.9, $x = b_1 r b_2$, where $b_1, b_2 \in B_0$ and $r \in \Re_0$. Let $a = b_2^{-1} r^* b_1^{-1}$, where $b_2^{-1}, b_1^{-1} \in B_0$, and r^* is the unique inverse of r in $\Re$.

Therefore, by Proposition 4.1.1, $r^* \in \Re_0$. Then $a \in M_0$. Hence,

$$\begin{aligned} xax &= b_1 r b_2 b_2^{-1} r^* b_1^{-1} b_1 r b_2 \\ &= b_1 r r^* r b_2 \\ &= b_1 r b_2 \\ &= x \\ \text{and } axa &= b_2^{-1} r^* b_1^{-1} b_1 r b_2 b_2^{-1} r^* b_1^{-1} \\ &= b_2^{-1} r^* r r^* b_1^{-1} \\ &= b_2^{-1} r^* b_1^{-1} \\ &= a. \end{aligned}$$

Thus, for each $x \in M_0$, there exists an element $a \in M_0$ such that $xax = x$ and $axa = a$. Therefore, M_0 is a regular algebraic monoid. □

4.2 IRREDUCIBILITY OF M_0

The most important result the author obtains in Chapter 4 is the following characterization of the irreducibility of M_0.

Theorem 4.2.1. *The following conditions are equivalent.*

(1) *M_0 is irreducible.*

(2) *For all $r \in R_0$, there exists $\sigma \in N_{C_G(s)}(T)$ and $e \in E(\overline{T})$ such that $r = e\sigma$.*

(3) *R_0 is unit regular.*

Proof. (1) $\Rightarrow$ (2). Assume that M_0 is irreducible. Therefore, $M_0 = \overline{C_G(s)}$ by Proposition 2.3.4. Let $r \in R_0$. Then $rT = Tr$ by Proposition 3.1.2. Also, we have

$r \in M_0 = \overline{C_G(s)}$. Hence, $r \in \overline{N_{C_G(s)}(T)}$. Since $N_{C_G(s)}(T)/T$ is a finite group,

$$\begin{aligned}\overline{N_{C_G(s)}(T)} &= N_{C_G(s)}(T)\overline{T} \\ &= N_{C_G(s)}(T)TE(\overline{T}), \quad \text{by Proposition 2.3.9} \\ &= N_{C_G(s)}(T)E(\overline{T}), \quad \text{since } T \subseteq N_{C_G(s)}(T) \\ &= E(\overline{T})N_{C_G(s)}(T).\end{aligned}$$

Therefore, $r \in \overline{N_{C_G(s)}(T)}$ implies that $r = e\sigma$ for some $e \in E(\overline{T}), \sigma \in N_{C_G(s)}(T)$.

(2) $\Rightarrow$ (1). Assume that for all $r \in R_0$ there exists $\sigma \in N_{C_G(s)}(T)$ and $e \in E(\overline{T})$ such that $r = e\sigma$. We prove that M_0 is irreducible by showing that $M_0 = \overline{C_G(s)}$. Clearly $\overline{C_G(s)} \subseteq M_0$. We show $M_0 \subseteq \overline{C_G(s)}$. Let $x \in M_0$. Then

$$\begin{aligned}x &= b_1 r b_2, \quad b_1, b_2 \in B_0, r \in \Re_0, \text{ by Theorem 3.2.9} \\ &= b_1 e\sigma b_2, \quad e \in E(\overline{T}), \sigma \in N_{C_G(s)}(T), \text{ by assumption .}\end{aligned}$$

Now $\sigma \in C_G(s), b_1, b_2 \in C_G(s)$ and $e \in \overline{T}$ implies that $e \in \overline{C_G(s)}$. Hence, $x \in \overline{C_G(s)}$ and we have $M_0 \subseteq \overline{C_G(s)}$. Therefore, $M_0 = \overline{C_G(s)}$, proving that M_0 is irreducible.

(2) $\Rightarrow$ (3). Assume that for all $r \in R_0$, there exists $\sigma \in N_{C_G(s)}(T), e \in E(\overline{T})$ such that $r = e\sigma$. We prove R_0 is unit regular.

We claim that $E(\overline{T}) = E(R_0)$. Clearly, $E(R_0) \subseteq E(R)$ and $E(R) = E(\overline{T})$ by Proposition 3.1.1. Let $e \in E(\overline{T}) = E(R)$. Then $e \in \overline{T}$, which is commutative. Since $s \in T$, we have $e \in R_0$. Thus, $e \in E(R_0)$ and $E(\overline{T}) \subseteq E(R_0)$.

Let $r \in R_0$. By assumption $r = e\sigma$ for some $\sigma \in N_{C_G(s)}(T), e \in E(\overline{T})$. Since $E(R_0) = E(\overline{T})$, we have $r = e\sigma$ for some $\sigma \in N_{C_G(s)}(T), e \in E(R_0)$. Therefore, R_0 is unit regular.

(3) $\Rightarrow$ (2). Assume R_0 is unit regular. Let $r \in R_0$. Then $r = e\sigma$ for some $e \in E(R_0), \sigma \in N_{C_G(s)}(T)$. Since $E(R_0) = E(\overline{T})$ by the above claim, we have $r = e\sigma$ for $\sigma \in N_{C_G(s)}(T), e \in E(\overline{T})$ and the proof of the theorem is complete. $\square$

CHAPTER 5
EXAMPLES

In this chapter we present examples to illustrate the centralizer of a semisimple element on a reductive linear algebraic monoid.

We use the notation established in Chapter 3 throughout the discussion of these examples. Thus, we let M be a reductive linear algebraic monoid with unit group G, where G', the derived group, is simply connected, and we let $s \in G$ be semisimple. We may assume that $s \in T \subseteq B$, where T is a maximal torus of G and B is a Borel subgroup of G containing T. We have $R = \overline{N_G(T)}, \Re = \overline{N_G(T)}/T$ and $W = \Re^* = N_G(T)/T$, the Weyl group of G. Also $M_0 = C_M(s), R_0 = C_R(s) = \{r \in R | sr = rs\}$ and $\Re_0 = \{r \in R | (rT)_0 \neq \emptyset\}/T$, where $(rT)_0 = \{rt \in rT | srt = rts\}$. Recall that $(rT)_0 = rT$ if $(rT)_0 \neq \emptyset$.

In Examples 5.1.1 and 5.1.2 we observe that each $M_0 = C_M(s)$ is an irreducible monoid by Theorem 4.2.1. However, in Example 5.2.1 we note that $M_0 = C_M(s)$ is not necessarily irreducible, although $C_G(s)$ is always a connected group by Theorem 2.2.16.

5.1 Irreducible Centralizers

Example 5.1.1. Let $M = M_3(k), G = Gl_3(k), T = \{(a_{ij}) \in G | a_{ij} = 0 \text{ if } i \neq j\}$, and $\overline{T} = \{(a_{ij}) \in M | a_{ij} = 0 \text{ if } i \neq j\}$. Then $T \subseteq G$ is a maximal torus and $E(\overline{T}) = \{0, 1\} \cup \{e_i | i = 1, 2, \ldots, 6\}$, where $e_1 = \text{diag}(1,0,0), e_2 = \text{diag}(0,1,0), e_3 = \text{diag}(0,0,1), e_4 = \text{diag}(1,1,0), e_5 = \text{diag}(1,0,1)$ and $e_6 = \text{diag}(0,1,1)$. We have $W = \{w_1, w_2, \ldots, w_6\}$, where $w_1 =$ diag $(1,1,1)$,

$$w_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad w_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad w_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$w_5 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \text{and} \quad w_6 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Let $s \in T$ be a semisimple element of G. Then $s =$ diag (α, β, γ), where $\alpha, \beta, \gamma \in k^*$. We wish to find $\Re_0$. If $x \in \Re$, then $x = we$ for some $w \in W$ and $e \in E(\overline{T})$. From easy calculations we obtain all relations on the coordinates of $s =$ diag $(\alpha, \beta, \gamma) \in T$ that lead to maximal proper subsets of $\Re$ of the form $\Re_0$. The list follows.

Case 1. Suppose $\alpha = \beta = \gamma$. Then $\Re_0 = \Re$.

Case 2. Suppose $\alpha = \beta \neq \gamma$. Then $\Re_0 = E(\overline{T}) \cup w_2 E(\overline{T})$. The cases $\alpha = \gamma \neq \beta$ and $\beta = \gamma \neq \alpha$ give sets that are conjugate to $\Re_0$ by some $w \in W$.

Case 3. Suppose $\alpha \neq \beta \neq \gamma, \alpha \neq \gamma$. Then $\Re_0 = E(\overline{T})$.

In all three cases, we note that each $x \in \Re_0$ can be expressed as $x = \sigma' e'$ for some $\sigma' \in N_{C_G(s)}(T)/T, e' \in E(\overline{T})$. Since $\Re_0 = R_0/T$, for all $y \in R_0$ there exist $\sigma \in N_{C_G(s)}(T), e \in E(\overline{T})$ such that $y = e\sigma$. Therefore, $M_0 = C_M(s)$ is irreducible by Theorem 4.2.1. Hence, $M_0 = \overline{C_G(s)}$ by Proposition 2.3.4.

Example 5.1.2. Let $G_1 = \{A \oplus (A^{-1})^t | A \in Sl_3(k)\}$, and let

$$T_1 = \{A \oplus (A^{-1})^t | A \in D_3^*(k), \text{ det } A = 1\}$$

$$= \{ \text{ diag } (\alpha, \beta, \alpha^{-1}\beta^{-1}, \alpha^{-1}, \beta^{-1}, \alpha\beta) | \alpha, \beta \in k^* \}.$$

Clearly, T_1 is a maximal torus of G_1. Let $T = k^* T_1, G = k^* G_1$ and $M = \overline{kG_1} \subseteq$

$M_6(k)$. Then M is a reductive algebraic monoid with group of units G and T is a maximal torus of G. Putcha shows [P1; Example 8.6] that the idempotents of $\overline{T}, E(\overline{T}) = \{0,1\} \cup \{h_i \oplus h_j | i,j = 1,2,3, i \neq j\} \cup \{h_i \oplus 0 | i = 1,2,3\} \cup \{0 \oplus h_i | i = 1,2,3\}$, where $h_1 = \text{diag}\,(1,0,0), h_2 = \text{diag}\,(0,1,0)$, and $h_3 = \text{diag}\,(0,0,1)$. Let

$$e_1 = \text{diag}\,(1,0,0,0,1,0), e_2 = \text{diag}\,(0,1,0,1,0,0), e_3 = \text{diag}\,(1,0,0,0,0,1),$$

$$e_4 = \text{diag}\,(0,0,1,1,0,0), e_5 = \text{diag}\,(0,1,0,0,0,1), e_6 = \text{diag}\,(0,0,1,0,1,0),$$

$$e_7 = \text{diag}\,(1,0,0,0,0,0), e_8 = \text{diag}(0,1,0,0,0,0), e_9 = \text{diag}\,(0,0,1,0,0,0),$$

$$e_{10} = \text{diag}\,(0,0,0,1,0,0), e_{11} = \text{diag}\,(0,0,0,0,1,0), e_{12} = \text{diag}\,(0,0,0,0,0,1).$$

It follows that the partially ordered set of all regular $\mathcal{J}$-classes of $M, \mathcal{U}(M) = \{0, J_1, J_2, J_3, G\}$ with $J_3 > J_1, J_3 > J_2$. Also,

$$E(J_3) = \{e \in E(\overline{T}) |\ \text{rank}(e) = 2\},$$

$$E(J_2) = \{e \in E(\overline{T}) |\ \text{rank}\,(e) = 1 \text{ and } e = h_i \oplus 0, i = 1,2,3\}, \text{ and}$$

$$E(J_1) = \{e \in E(\overline{T}) |\ \text{rank}\,(e) = 1 \text{ and } e = 0 \oplus h_i, i = 1,2,3\}.$$

The Weyl group is $W = \{w_1, w_2, \ldots, w_6\}$, where $w_1 = \text{diag}\,(1,1,1,1,1,1)$,

$$w_2 = \begin{pmatrix} 1&0&0&0&0&0\\ 0&0&1&0&0&0\\ 0&1&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&1&0 \end{pmatrix}, \quad w_3 = \begin{pmatrix} 0&1&0&0&0&0\\ 1&0&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&1&0&0\\ 0&0&0&0&0&1 \end{pmatrix},$$

$$w_4 = \begin{pmatrix} 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 1&0&0&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ 0&0&0&1&0&0 \end{pmatrix}, \quad w_5 = \begin{pmatrix} 0&0&1&0&0&0\\ 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0 \end{pmatrix},$$

and

$$w_6 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Let $s \in T \subseteq G$ be semisimple. Then $s = a \operatorname{diag}(\alpha, \beta, \alpha^{-1}\beta^{-1}, \alpha^{-1}, \beta^{-1}, \alpha\beta)$, where $a, \alpha, \beta \in k^*$. We calculate $\Re_0$. Let $x \in \Re$. Then $x = we$ for some $w \in W$ and $e \in E(\overline{T})$. If $e = 0$, then $x = 0$ and $s \in T$ commutes with 0. If $w = w_1, x = w_1 e = e \in \overline{T}$ and $s \in T$ commutes with e. Thus, we consider the cases when $e \neq 0$ and $w \neq w_1$. We obtain all relations on the coordinates of $s = a \operatorname{diag}(\alpha, \beta, \alpha^{-1}\beta^{-1}, \alpha^{-1}, \beta^{-1}, \alpha\beta) \in T$ that lead to maximal proper subsets of $\Re$ of the form $\Re_0$.

A summary of the results follows.

Case 1. Suppose $\alpha = \beta = \alpha^{-1}\beta^{-1}$. This is the case $\alpha = \beta = 1, \omega$ or $\overline{\omega}$, where ω is a solution to $x^2 + x + 1 = 0$ and $\overline{\omega}$ is the complex conjugate of ω. Then $\Re_0 = \Re$.

Case 2. Suppose $\alpha = \beta \neq \alpha^{-1}\beta^{-1}$. This is the case $\alpha = \beta, \alpha \neq 1, \omega, \overline{\omega}$. Then $\Re_0 = E(\overline{T}) \cup w_3 E(\overline{T})$.

The cases $\alpha = \alpha^{-1}\beta^{-1} \neq \beta$ and $\alpha \neq \beta = \alpha^{-1}\beta^{-1}$ gives sets that are conjugate to $\Re_0$ by some $w \in W$.

Case 3. Suppose $\alpha \neq \beta \neq \alpha^{-1}\beta^{-1}, \alpha \neq \alpha^{-1}\beta^{-1}$. Then $\Re_0 = E(\overline{T})$.

In all cases we see that each $x \in \Re_0$ can be expressed as $x = \sigma' e'$ for some $\sigma' \in N_{C_G(s)}(T)/T, e' \in E(\overline{T})$. Since $\Re_0 = R_0/T$, for all $y \in R_0$ there exist $\sigma \in N_{C_G(s)}(T), e \in E(\overline{T})$ such that $y = e\sigma$. Therefore, $M_0 = C_M(s)$ is irreducible by

Theorem 4.2.1. Hence, by Proposition 2.3.4 $M_0 = \overline{C_G(s)}$ and we unexpectedly find that each centralizer of s in M is completely determined by the centralizer of s in G.

Some Calculations. For completeness we include calculations used to determine the subsets $\Re_0$ in Example 5.1.2. We observe that $sw_ie_js^{-1} = sw_is^{-1}e_j$ since $e_j, s^{-1} \in \overline{T}$.

$$sw_2s^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha\beta^2 & 0 & 0 & 0 \\ 0 & \alpha^{-1}\beta^{-2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha^{-1}\beta^{-2} \\ 0 & 0 & 0 & 0 & \alpha\beta^2 & 0 \end{pmatrix}$$

We see that $w_2e_j \in \Re_0$ for all α, β if $j = 7$ or 10, and that $w_2e_j \in \Re_0$ for all j if $\alpha^{-1} = \beta^2$.

$$sw_3s^{-1} = \begin{pmatrix} 0 & \alpha\beta^{-1} & 0 & 0 & 0 & 0 \\ \beta\alpha^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha^{-1}\beta & 0 \\ 0 & 0 & 0 & \alpha\beta^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Note that $w_3e_j \in \Re_0$ for all α, β if $j = 9$ or 12, and that $w_3e_j \in \Re_0$ for all j if $\alpha = \beta$.

$$sw_4s^{-1} = \begin{pmatrix} 0 & \alpha\beta^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha\beta^2 & 0 & 0 & 0 \\ \alpha^{-2}\beta^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha^{-1}\beta & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha^{-1}\beta^{-2} \\ 0 & 0 & 0 & \alpha^2\beta & 0 & 0 \end{pmatrix}$$

$$\text{Here }, w_4 e_j \in \Re_0 \text{ if } \begin{cases} \alpha = \beta, j = 8, 11 \\ \alpha^{-1} = \beta^2, j = 9, 12 \\ \beta^{-1} = \alpha^2, j = 7, 10 \\ \alpha = \beta \text{ and } \beta^{-1} = \alpha^2, j = 1, 2 \\ \alpha = \beta \text{ and } \alpha^{-1} = \beta^2, j = 5, 6 \\ \beta^{-1} = \alpha^2 \text{ and } \alpha^{-1} = \beta^2, j = 3, 4. \end{cases}$$

$$sw_5 s^{-1} = \begin{pmatrix} 0 & 0 & \alpha^2\beta & 0 & 0 & 0 \\ \alpha^{-1}\beta & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha^{-1}\beta^{-2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha^{-2}\beta^{-1} \\ 0 & 0 & 0 & \alpha\beta^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha\beta^2 & 0 \end{pmatrix}$$

$$\text{We see that } w_5 e_j \in \Re_0 \text{ if } \begin{cases} \alpha = \beta, j = 7, 10 \\ \alpha^{-1} = \beta^2, j = 8, 11 \\ \beta^{-1} = \alpha^2, j = 9, 12 \\ \alpha = \beta \text{ and } \beta^{-1} = \alpha^2, j = 3, 4 \\ \alpha = \beta \text{ and } \alpha^{-1} = \beta^2, j = 1, 2 \\ \beta^{-1} = \alpha^2 \text{ and } \alpha^{-1} = \beta^2, j = 5, 6. \end{cases}$$

$$sw_6 s^{-1} = \begin{pmatrix} 0 & 0 & \alpha^2\beta & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \alpha^{-2}\beta^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha^{-2}\beta^{-1} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \alpha^2\beta & 0 & 0 \end{pmatrix}.$$

Then $w_6 e_j \in \Re_0$ for all α, β if $j = 8$ or 11 and $w_6 e_j \in \Re_0$ if $\beta^{-1} = \alpha^2$ for all j.

Relations Observed and The Corresponding Fixed Points. The following is a list of the points in $\Re$ fixed by $s \in T \subseteq G$ for all possible relations on the coordinates of s.

(1) $\alpha = \beta$. Since $w_4e_8 = w_3e_8, w_4e_{11} = w_3e_{11}, w_5e_7 = w_3e_7$ and $w_5e_{10} = w_3e_{10}$, we have $x \in \Re_0$ if $x \in w_3E(\overline{T})$.

(2) $\alpha^{-1} = \beta^2$. Here $x \in \Re_0$ if $x \in w_2E(\overline{T})$, since $w_4e_9 = w_2e_9, w_4e_{12} = w_2e_{12}, w_5e_8 = w_2e_8$ and $w_5e_{11} = w_2e_{11}$.

(3) $\beta^{-1} = \alpha^2$. Then $x \in \Re_0$ if $x \in w_6E(\overline{T})$, since $w_4e_7 = w_6e_7, w_4e_{10} = w_6e_{10}, w_5e_9 = w_6e_9$ and $w_5e_{12} = w_6e_{12}$.

(4) α **is not related to** β. Now $x \in \Re_0$ if $x \in E(\overline{T})$ since $w_2e_7 = e_7, w_2e_{10} = e_{10}, w_3e_9 = e_9, w_3e_{12} = e_{12}, w_6e_8 = e_8$ and $w_6e_{11} = e_{11}$.

(5) $\alpha = \beta$ and $\beta^{-1} = \alpha^2$. (Here $\alpha = \beta = 1, \omega$ or $\overline{\omega}$.) In this case $x \in \Re_0$ if $x = w_4e_1, w_4e_2, w_5e_3$ or w_5e_4.

(6) $\alpha = \beta$ and $\alpha^{-1} = \beta^2$. ($\alpha = \beta = 1, \omega$ or $\overline{\omega}$) Then $x \in \Re_0$ if $x = w_4e_5, w_4e_6, w_5e_1$ or w_5e_2.

(7) $\beta^{-1} = \alpha^2$ and $\alpha^{-1} = \beta^2$. ($\alpha = \beta = 1, \omega$ or $\overline{\omega}$) Then $x \in \Re_0$ if $x = w_4e_3, w_4e_4, w_5e_5$ or w_5e_6.

5.2 Reducible Centralizers

Example 5.2.1. Let $\rho : Sl_2(k) \times Sl_2(k) \longrightarrow Gl_6(k)$ be a representation defined by

$$\rho(A,B) = \begin{pmatrix} A \otimes (B^{-1})^t & 0 \\ 0 & B \end{pmatrix}.$$

Let $G_1 = \rho(Sl_2(k) \times Sl_2(k))$. Now $T_1 = \{\rho(A,B) | A, B \in D_2^*(k)$, det $A =$ det $B = 1\}$ is a maximal torus of G_1. Let $T = k^*T_1$ and $G = k^*G_1$. Then

$M = \overline{kG_1} \subseteq M_6(k)$ is a reductive algebraic monoid with group of units G and T is a maximal torus of G. Clearly, $T = \{\text{ diag }(w,x,y,z,r,s) | wz = xy = rs, r^2 = xz, s^2 = wy, w,x,y,z,r,s \in k^*\}$. Hence, it is easily proved that $E(\overline{T}) = \{0,1\} \cup \{e_i | i = 1,2,\ldots,8\}$, where

$$e_1 = \text{ diag }(1,0,0,0,0,0), e_2 = \text{ diag }(0,1,0,0,0,0), e_3 = \text{ diag }(0,0,1,0,0,0),$$

$$e_4 = \text{ diag }(0,0,0,1,0,0), e_5 = \text{ diag }(1,1,0,0,0,0), e_6 = \text{ diag }(0,0,1,1,0,0),$$

$$e_7 = \text{ diag }(1,0,1,0,0,1) \text{ and } e_8 = \text{ diag }(0,1,0,1,1,0).$$

It follows that the partially ordered set of all regular $\mathcal{J}$-classes of M,$\mathcal{U}(M) = \{0, J_1, J_2, J_3, G\}$ with $J_3 > J_1, J_2 > J_1$. Also,

$$E(J_3) = \{e \in E(\overline{T}) | \text{ rank } (e) = 3\},$$

$$E(J_2) = \{e \in E(\overline{T}) | \text{ rank } (e) = 2\} \text{ and}$$

$$E(J_1) = \{e \in E(\overline{T}) | \text{ rank } (e) = 1\}.$$

The Weyl group of G is $W = \{w_1, w_2, w_3, w_4\}$, where $w_1 = \text{ diag }(1,1,1,1,1,1)$,

$$w_2 = \rho\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix},$$

$$w_3 = \rho\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix},$$

and

$$w_4 = \rho\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let $s \in T \subseteq G$ be semisimple. Then $s = a\rho(u,v)$ where $u =$ diag $(\alpha, \alpha^{-1}), v =$ diag (β, β^{-1}) and $a, \alpha, \beta \in k^*$. Now, $s = a$ diag $(\alpha\beta^{-1}, \alpha\beta, \alpha^{-1}\beta^{-1}, \alpha^{-1}\beta, \beta, \beta^{-1})$. We wish to find $\Re_0$. Let $x \in \Re$. Then $x = we$ for some $w \in W, e \in E(\overline{T})$. If $e = 0$, then $x = 0$ and $s \in T$ commutes with 0. If $w = w_1, x = w_1 e = e \in \overline{T}$ and $s \in T$ commutes with e. Hence, we must consider the cases when $e \neq 0$ and $w \neq w_1$. In order to obtain $\Re_0$ we need to find the $x = w_i e_j, i = 2,3,4, j = 1,2,\ldots,8$ and the case $e_j = 1$, for which $sw_ie_js^{-1} = w_ie_j$. We note that $sw_ie_js^{-1} = sw_is^{-1}e_j$ since $e_j, s^{-1} \in \overline{T}$.

Examples of calculations follow:

$$sw_2s^{-1} = \begin{pmatrix} 0 & (\beta^{-1})^2 & 0 & 0 & 0 & 0 \\ -\beta^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (\beta^{-1})^2 & 0 & 0 \\ 0 & 0 & -\beta^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta^2 \\ 0 & 0 & 0 & 0 & -(\beta^{-1})^2 & 0 \end{pmatrix}$$

We discover that, for each $j, sw_2e_js^{-1} = w_2e_j$ if and only if $\beta = \pm 1$.

$$sw_4s^{-1} = \begin{pmatrix} 0 & 0 & \alpha^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha^2 & 0 & 0 \\ -(\alpha^{-1})^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -(\alpha^{-1})^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

We find that, for each $j, sw_4e_js^{-1} = w_4e_j$ if and only if $\alpha = \pm 1$.

$$sw_3s^{-1} = \begin{pmatrix} 0 & 0 & 0 & (\alpha\beta^{-1})^2 & 0 & 0 \\ 0 & 0 & -(\alpha\beta)^2 & 0 & 0 & 0 \\ 0 & -(\alpha^{-1}\beta^{-1})^2 & 0 & 0 & 0 & 0 \\ (\alpha^{-1}\beta)^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta^2 \\ 0 & 0 & 0 & 0 & -(\beta^{-1})^2 & 0 \end{pmatrix}$$

Using this matrix we obtain the following results.

$$sw_3e_1s^{-1} = w_3e_1 \Leftrightarrow (\alpha^{-1}\beta)^2 = 1 \Leftrightarrow \beta = \pm\alpha.$$

$$sw_3e_2s^{-1} = w_3e_2 \Leftrightarrow -(\alpha^{-1}\beta^{-1})^2 = -1 \Leftrightarrow \beta = \pm\alpha^{-1}.$$

$$sw_3e_3s^{-1} = w_3e_3 \Leftrightarrow -(\alpha\beta)^2 = -1 \Leftrightarrow \beta = \pm\alpha^{-1}.$$

$$sw_3e_4s^{-1} = w_3e_4 \Leftrightarrow (\alpha\beta^{-1})^2 = 1 \Leftrightarrow \beta = \pm\alpha.$$

$$\begin{aligned} sw_3e_5s^{-1} = w_3e_5 &\Leftrightarrow (\alpha^{-1}\beta)^2 = 1 \text{ and } -(\alpha^{-1}\beta^{-1})^2 = -1 \\ &\Leftrightarrow \beta = \pm\alpha \text{ and } \beta = \pm\alpha^{-1} \\ &\Leftrightarrow \beta = \alpha = \pm1, \pm i \text{ or } \beta = -\alpha \text{ and } \alpha = \pm1, \pm i. \end{aligned}$$

$$\begin{aligned} sw_3e_6s^{-1} = w_3e_6 &\Leftrightarrow -(\alpha\beta)^2 = -1 \text{ and } (\alpha\beta^{-1})^2 = 1 \\ &\Leftrightarrow \beta = \pm\alpha \text{ and } \beta = \pm\alpha^{-1} \\ &\Leftrightarrow \beta = \alpha = \pm1, \pm i \text{ or } \beta = -\alpha \text{ and } \alpha = \pm1, \pm i. \end{aligned}$$

$$\begin{aligned} sw_3e_7s^{-1} = w_3e_7 &\Leftrightarrow (\alpha^{-1}\beta)^2 = 1 \text{ and } -(\alpha\beta)^2 = -1 \text{ and } \beta^2 = 1 \\ &\Leftrightarrow \beta = \pm\alpha \text{ and } \beta = \pm\alpha^{-1} \text{ and } \beta = \pm1 \\ &\Leftrightarrow \beta = \alpha = \pm1, \text{ or} \beta = -\alpha \text{ and } \alpha = \pm1. \end{aligned}$$

$$sw_3e_8s^{-1} = w_3e_8 \Leftrightarrow -(\alpha^{-1}\beta^{-1})^2 = -1, (\alpha\beta^{-1})^2 = 1 \text{ and } -(\beta^{-1})^2 = -1$$

$$\Leftrightarrow \beta = \pm\alpha^{-1}, \pm\alpha \text{ and } \beta = \pm 1$$

$$\Leftrightarrow \alpha = \beta = \pm 1 \text{ or } \beta = -\alpha \text{ and } \alpha = \pm 1.$$

$$sw_3s^{-1} = w_3 \Leftrightarrow \beta = \pm\alpha, \beta = \pm\alpha^{-1}, \beta^{-1} = \pm 1 \text{ and } \beta = \pm 1$$

$$\Leftrightarrow \alpha = \beta = \pm 1 \text{ or } \beta = -\alpha \text{ and } \alpha = \pm 1.$$

The following is a summary of the results.

Case 1. Suppose $\beta = \pm\alpha$ and $\alpha = \pm 1$. Then $\Re_0 = \Re$.

Case 2. Suppose $\beta = \pm\alpha$ and $\alpha = \pm i$. Then $\Re_0 = E(\overline{T}) \cup \{w_3e_j | j = 1, 2, \ldots, 6\}$.

Case 3. Suppose $\beta = \pm\alpha$ and $\alpha \neq \pm 1, \pm i$. Then $\Re_0 = E(\overline{T}) \cup \{w_3e_1, w_3e_4\}$.

The case $\beta = \pm\alpha^{-1}, \alpha \neq \pm 1, \pm i$ gives a set conjugate to $\Re_0$ in case 3.

Case 4. Suppose $\beta \neq \pm\alpha, \alpha = \pm 1$. Then $\Re_0 = E(\overline{T}) \cup w_4E(\overline{T})$.

The case $\beta \neq \pm\alpha, \beta = \pm 1$ gives a set conjugate to $\Re_0$ in case 4 .

Case 5. Suppose $\beta \neq \pm\alpha, \beta \neq \pm\alpha^{-1}, \alpha \neq \pm 1, \beta \neq \pm 1$. Then $\Re_0 = E(\overline{T})$.

In cases 1, 4 and 5 we observe that each $x \in \Re_0$ can be expressed as $x = \sigma'e'$ for some $\sigma' \in N_{C_G(s)}(T)/T, e' \in E(\overline{T})$. Thus each $r \in R_0$ can be written as $r = \sigma e$ for some $\sigma \in N_{C_G(s)}(T), e \in E(\overline{T})$. Hence, by Theorem 4.2.1 we have $M_0 = C_M(s) = \overline{C_G(s)}$ in these cases. However, $w_3 \in N_{C_G(s)}(T)/T$ if and only if $\beta = \pm\alpha$ and $\alpha = \pm 1$. Hence, by Theorem 4.2.1 M_0 is not irreducible in cases 2 and 3. Thus we have illustrated the fact that $C_M(s)$ is not always an irreducible monoid, although M is irreducible and $C_G(s)$ is always a connected group.

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