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Essays on Portfolio Optimization, Simulation and Option Pricing

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ESSAYS ON PORTFOLIO OPTIMIZATION, SIMULATION AND OPTION PRICING

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Abstract

This thesis consists of three papers which cover the efficient Monte Carlo simulation in option pricing, the application of realized volatility in trading strategies and geometrical analysis of a four asset mean variance portfolio optimization problem. The first paper studies different efficient simulation methods to price options with different characters such as moneyness and maturity times. The incomplete market environments are also been considered. The second paper uses realized volatility based on high frequency data to improve the volatility trading strategy. The performance is compared with that using the implied volatility. The last paper re-examines the Markowitz's portfolio optimization problem using a general case. It also extends the problem to four assets, it describes the exact mean variance efficient fronter in the weight space and studies the frontier in the mean variance space. The thesis may serve to help our understanding of how to apply numerical and analytical methods to solve financial problems.

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Chapter 1

Introduction

The dissertation consists of three papers dealing with efficient Monte Carlo simulation strategies for option pricing, the use of realized volatility in high frequency volatility trading strategies and a geometrical analysis of Markovitz's four asset problem.

The first paper compares different efficient Monte Carlo simulation methods for the purpose of pricing derivatives under incomplete market environments. Option prices were simulated based on three incomplete option price models: stochastic volatility model, jump diffusion model, and stochastic volatility with concurrent jumps in the stock price and variance process model. Using the simulated option prices as well as the option prices based on S&P 500 index returns, we tested and compared the performance of the standard Monte Carlo simulation and five other efficient simulation methods including Antithetic

Variables, Control Variates, Stratified Sampling (SS), Importance Sampling, and Quasi-Monte Carlo (QMC). The comparison was made on different moneyness and maturity times. According to Root Mean Squared Error, QMC is the best choice for the out-of-the-money options. For in-the-money options, there was no clear winners as the performance of the methods changed with the option pricing model. Considering the standard error, QMC and SS did the best and much better than the other methods. The study may serve to improve the speed and accuracy of Monte Carlo methods for option pricing under incomplete environments.

The second paper focuses on applying realized volatility in the high frequency volatility trading strategies. The implied stochastic volatility regression method has commonly been used to predict the conditional volatility of stock prices. However, implied volatility has proven to be a biased predictor of the realized volatility across asset markets. With the increasing emphasis on computer-assisted techniques, high frequency data can be applied to process realized volatility. This paper investigates the Delta Neutral strategy, with the realized volatility forecasting based on high frequency data. A comparison between the effectiveness of applying realized volatility to the trading strategies and that of the implied volatility is conducted.This study showed that each of the two types of volatility performed well in different settings, but the advantage of the realized volatility lies in that it is much quicker to obtain the results than that of implied volatility, and this would be important in practice because the application of the realized volatility improves the calculation efficiency.

The third paper re-examines the mean variance efficient problem in Markowitz (2005) by adding up the non-negativity constraints of the asset weights. It also examines the problem in a general case without specifying values for the means and variances. Furthermore, the problem is extended to four assets so that the weights can be described in a three-dimension space as some important features of many securities portfolio optimization can be analyzed in the four assets problem. In this paper, I calculated the solution of four important portfolios including the minimum variance portfolio, the maximum return portfolio and two corner portfolios at turning points. So the tedious algebra shows that, in the weight space, the efficient line started from the point of minimum variance inside the tetrahedron and always hit the plane where the lowest return asset was equal to zero. Then the efficient line would hit the plane where the second lowest return was equal to zero. This leads to the result that with the increase of the given expected portfolio return, the efficient portfolio always drops off the asset with a lower return first. By mapping the efficient portfolio from weight space to mean variance space, we prove that there is no kink at the corner points in mean variance space (i.e. the efficient frontier is continuous). The result is consistent with Dybvig(1984) etc. We also show that in some conditions, the mean variance efficient frontier can be described as a few parabolas tangent at the corner points. The solution was tested on a specific example of four assets with eight years daily stock prices. Monte Carlo simulation was also used in this study to test a wider dataset and the results matched well. This research may help us develop a deeper understanding of the efficient portfolio. The analysis in weight space may also be extended to deal with more constraints on the portfolio weights.

Chapter 2

Empirical Performance of Efficient Monte Carlo Simulations for Option Pricing in Incomplete Markets

2.1 Introduction

The benchmark model for option pricing is the Black-Scholes(B-S) model, which assumes that the market is complete. For instance, an investor can borrow as much as he needs at a constant risk-free interest rate; there is no transaction cost in the market; the underlying asset prices follow the Geometric Brownian Motion(GBM) process with constant drift and volatility; it is free of short selling constraints in the market, etc. The B-S model uses no-arbitrage theory and martingale methods to get a closed form of option price in the complete market.

The Black-Scholes model is considered one of the most popular models because it can bring out the main features of the option price. However, it has been criticized for its normal distribution and the complete market assumptions. For example, the empirical data show conflicts in this model, such as leptokurosis with the assets return having a higher peak and heavier tails than the normal distribution; the ”smile” in volatility can not be explained by the Black-Scholes model. Also, the volatility of the underlying asset return can not maintain constancy. The Black-Scholes option pricing model comes from replicating portfolios to cover the risk totally. However, it needs to be recognized that the market is significantly incomplete and the perfect replication is impossible. There are many factors contributing to the incompleteness of the market. Factors such as transaction costs, portfolio constraints, insufficient assets for investing and the volatility in B-S model can not be perfectly estimated. In this paper, the models used to describe the incompleteness are stochastic volatility and mixed jump diffusion processes which can better match the empirical data than in B-S model.

The computation in financial theory and practice is complex. There is no analytical solution to it some time, so the numerical methods have become necessary. Boyle(1977) recommended the use of Monte Carlo simulation to price the options and other derivatives. Monte Carlo methods have become especially useful with the development of computing power. The technique has many advantages compared to other numerical methods. It is easy to apply to complicated problems, and with it people can simulate the paths and estimate the expectations in most cases. The convergence speed does not rely on the dimension number of the problem. Moreover, the Monte Carlo estimate can provide more accurate confidence intervals.

In order to get more accurate results with the Monte Carlo simulation method, a large number of replications are needed. Thus, efficient strategies are almost compulsory in order to reduce the variance of the estimator and improve the accuracy. The popular variance techniques include antithetic variates, control variates, moment matching, stratificaiton and Latin hypercube sampling, importance sampling, repricing-matching-weights, conditional Monte Carlo and Quasi-Monte Carlo simulation.

Much research has been done in the field of applying Monte Carlo simulation to pricing the American style option and path dependent option, such as Asian options. However, how to apply the efficient simulation in the environment of incomplete market has not drawn much attention. The goal of this paper is to apply and compare the various efficient simulation strategies in pricing the European style options in the incomplete financial markets.

The rest of the paper is organized as follows. Section 2.2 demonstrates the option pricing models in the incomplete markets and the Monte Carlo method to price options. Section 2.3 presents the efficient simulation methods such as four variance reduction techniques and Quasi-Monte Carlo method. Section 2.4 describes the data used in this research including generated data and real data. Section 2.5 provides the numerical results and Section 2.6 draws conclusions.

2.2 Option Pricing

2.2.1 Black Scholes Model in complete markets

It is assumed that the stock prices follow a (continuous time) geometric Brownian motion process:

$$dS = \phi S dt + \sigma S dW \tag{2.1}$$

where,

$$\begin{aligned} S &= \text{the current stock price} \\ \phi &= \text{the expected return} \\ \sigma &= \text{volatility of the stock return} \\ W &= \text{Brownian Motion process} \\ dW &= \epsilon(dt)^{0.5}, \epsilon \text{ is the standard normal distributed random variable} \end{aligned}$$

Together with other strict assumptions such as that the transactions do not incur any fees or costs, it is possible to buy and sell any amount of stock, and it is possible to borrow and lend cash at the risk-free interest rate, the Black-Scholes European call option price can be obtained as:

$$f^{BS} = SN(d_1) - Ke^{-rT}N(d_2) \tag{2.2}$$

Where,

S = current stock price

K = option strike price

r = annual risk-free interest rate

T = time to expiration, current time is set to zero, T should be annualized since the annual interest rate is used

N = the cumulative normal density function

$$d_1 = \frac{ln(S/K) + (r + \sigma^2/2)T}{\sigma T^{1/2}}$$

$$d_2 = \frac{ln(S/K) + (r - \sigma^2/2)T}{\sigma T^{1/2}} = d_1 - \sigma T^{1/2}$$

2.2.2 Incomplete Market Models

A lot of evidence has shown that the complete market assumptions can not be satisfied, and the incompleteness can be presented in the models by stochastic volatility and jumps in volatility and underlying stock prices. In order to capture the main features of option prices, we use the stochastic volatility model(SV), the pure jump diffusion model(Jump) and the stochastic volatility with concurrent jumps model(SVCJ).

Stochastic Volatility Model (SV)

In the incomplete market, the underlying asset return volatility is stochastic rather than constant. We use the Hull and White (1987) model to describe the option prices process. If the stock price is S_t and its instantaneous variance is V_t, the asset price can be described in the following stochastic processes,

$$dS = S(\phi dt + \sigma dw) \tag{2.3}$$

$$dV = V(\mu dt + \xi dz) \tag{2.4}$$

Where $V = \sigma^2$ follows a geometric Brownian Motion. dw and dz are correlated Brownian motion process with correlation coefficient ρ. ϕ is the expected return of the share, μ is the drift (expected growth rate) of the variance, σ is the volatility and ξ is the volatility of volatility. The security price $f(S_t, \sigma_t^2, t)$ is the present value of the expected terminal value of f discounted at the risk free rate, thus the closed form of the option price is:

$$f(S_t, \sigma_t^2, t) = e^{-r(T-t)} \int f(S_T, \sigma_T^2, T) p(S_T | S_t, \sigma_t^2) dS_T \tag{2.5}$$

Where T is the time at which the option matures, S_t is the security price at time t, σ_t is the instantaneous standard deviation at time t, and $p(S_T | S_t, \sigma_t)$ is the conditional density function of S_T given the security price and variance at time t. $\overline{V} = \frac{1}{T-t} \int_t^T \sigma_\tau^2 d\tau$ denotes the mean of variance over the life of the derivation security, and the price can be written as

$$f(S_t, \sigma_t^2, t) = \int \left[e^{-r(T-t)} \int f(S_T) g(S_T | \overline{V}) dS_T \right] h(\overline{V} | \sigma_t^2) d\overline{V} \tag{2.6}$$

where $h(\cdot)$ denotes the conditional distribution of $\overline{V}$. The inner integral produces the Black-Scholes price.

If we assume that the correlation $\rho = 0$ and μ and ξ are independent of $S(t)$, the Hull and White price can be seen as the integral of the Black-Scholes price over the conditional distribution of mean variance $\overline{V}$ and in Hull and White(1987) model:

$$f^{HW}(S_t, \sigma_t^2) = \int f^{BS}(\overline{V})h(\overline{V}|\sigma_t^2)d\overline{V} \tag{2.7}$$

f^{BS} is the Black-Scholes European option price defined in previous section. By expanding Black-Scholes price $f^{BS}(\overline{V})$ from its expected average variance $E(\overline{V})$ in a Taylor series, Hull-White also propose a power series approximation technique to get the option price f^{HW} as:

$$\begin{aligned} f^{HW}(S_t, \sigma_t^2) &= f^{BS}(E(\overline{V})) + \frac{1}{2}\frac{\partial^2 f^{BS}(E(\overline{V}))}{\partial \overline{V}^2}E(\overline{V}^2) \\ &+ \frac{1}{6}\frac{\partial^3 f^{BS}(E(\overline{V}))}{\partial \overline{V}^3}E(\overline{V}^3) + \ldots \end{aligned} \tag{2.8}$$

Where $E(\overline{V}^2)$ and $E(\overline{V}^3)$ are the second and third central moments of $\overline{V}$.

Jump-Diffusion Model(Jump)

In this model, the market incompleteness comes from the jumps of the security price. Merton (1976) option pricing formula is that the basic model takes into consideration the jump diffusion which can lead to the leptokurtic and implied volatility smile. Merton(1976) assumes that the underlying stock price follows

Brownian motion as in Black-Scholes(1973) model, together with jumps which are modeled with a compound Poisson process. The dynamics of the stock prices is described as

$$dS/S = (\phi - \lambda\overline{\mu})dt + \hat{\sigma}dW_t + dq_t \tag{2.9}$$

Where ϕ is the instantaneous expected return of the asset, λ is the mean number of arrival events in unit time, $\overline{\mu}$ is the mean jump size. $\hat{\sigma}^2$ is the instantaneous variance of the return when the Poisson event does not occur. W_t is a standard Brownian motion. q_t is the independent Poisson process. And the price of an European call option in Jump-Diffusion model is given by

$$f^J = \sum_{i=0}^{\infty} \frac{e^{-\lambda T}(\lambda T)^i}{i!} f^{BS}(S, K, T, r, \sigma_i) \tag{2.10}$$

where T is the time to expiration; K is the strike price; r is the annual risk free interest rate; $f^{BS}(S, K, T, r, \lambda_i)$ is the Black-Scholes pricing formula for an European Call option, and

$$\sigma_i = \sqrt{z^2 + \delta^2(i/T)},$$

where

$$z^2 = \sigma^2 - \lambda\delta^2, \delta^2 = \frac{\gamma\sigma^2}{\lambda}$$

σ is the total volatility including jumps, λ is the expected yearly number of jumps and γ is the percentage of total volatility due to the jumps.

Stochastic Volatility with Concurrent Jumps in the Stock Price and the Variance Process(SVCJ)

There is strong empirical evidence of stochastic volatility and jumps in financial markets. We follow the SVCJ model in Duffie et al.(2000) which is based on the dynamics of the underlying stock price and variance,

$$\begin{aligned} dS_t &= (\phi - \lambda\overline{\mu})S_t dt + \sqrt{V_t}S_t\left[\rho dW_t^{(1)} + \sqrt{1-\rho^2}dW_t^{(2)}\right] \\ &\quad +(J^s - 1)dN_t \end{aligned} \tag{2.11}$$

$$dV_t = k(\theta - V_t)dt + \sigma_v\sqrt{V_t}dW_t^{(1)} + J^v dN_t \tag{2.12}$$

where S_t is the stock price at time t, ϕ is the interest rate, $\sqrt{V_t}$ is the volatility, θ is the long-run mean of variance, k is the speed of mean reversion, σ_v determines the volatility of the variance process, $W_t^{(1)}$ and $W_t^{(2)}$ are independent Brownian motion processes, and ρ is the instantaneous correlation between the return process and the volatility process.

N_t denotes a Poisson process independent of the Brownian motions with constant intensity λ, J^s is the relative jump size of the stock price and J^v is the jump size of the variance. If a jump occurs at time t, we have

$$S_{t^+} = S_{t^-}J^s$$

$$V_{t^+} = V_{t^-}J^v$$

The jumps in stock price occur concurrently with that in the variance and the correlation is determined by ρ_J. J^v follows exponential distribution with mean

μ_v and J^s follows lognormal distribution with mean $(\mu_s + \rho_J J^v)$ and variance σ_s^2. And the parameters μ_s and $\overline{\mu}$ are related as $\mu_s = log[(1+\overline{\mu})(1-\rho_J \mu_v)] - 0.5\sigma_s^2$. There is no closed form solution for the option price in SVCJ model and it only has numerical solution.

2.2.3 Monte Carlo(MC) Simulation Approach for Option Pricing

The Monte Carlo simulation method was first used in option pricing by Boyle and it has proved to be a powerful tool in finance. There has been a lot of research on the MC simulation in American style options and path depended options such as Asian options, but not much attention has been given to the incomplete market environment. There is a lot of work to do on the improvement of the algorithm of MC approach in this field. This paper focuses on how to choose the efficient strategies of MC simulation in the incomplete market environment. Following is the basic MC approach which simulates the process of how an option is priced.

The payoff for an European call option with strike price K at expiry time T is $f_{call}(S,T,K) = max\{S_T - K, 0\}$ where S_T is the point stock price. Monte Carlo simulation method generates m paths of stock prices, calculates option pay off for each path and takes average to get

$$\bar{f}_{call}(S,T,K) = \frac{1}{m}\sum_{i=1}^{m} f_{call}^{(i)}(S,T,K) \tag{2.13}$$

The approximation of the present time option price is obtained by discounting the approximate future price by e^{-rT}, where r is the risk free interest rate.

$$f_{fair}(S,0,K) = e^{-rT}\bar{f}_{call}(S,T,K) \tag{2.14}$$

Following one of the assumptions of Black-Scholes model, we simulate the underlying stock prices whose natural logarithm follow a geometric Brownian motion process. Same as equation (2.1), the stock prices dynamic is described as the SDE:

$$dS = \phi S dt + \sigma S dW \tag{2.15}$$

By Ito's Lemma,

$$S_T = S_t exp\{(\phi - 0.5\sigma^2)(T-t) + \sigma\sqrt{T-t}\epsilon\} \tag{2.16}$$

This is the continuous time model of the underlying stock price at maturity time T. Accepting the risk neural assumption, stock return ϕ is equal to the risk-free interest rate r_f. However, ϕ can also denote the cost of carry rate which is the cost of interest plus additional costs such as the cost of paying dividends.

In practice, we can only observe the stock prices discretely such as every 5

minutes. In order to simulate the stock prices, we separate 1 year into n periods. Assuming that there are N days in a year and D periods every day, we have $n = N * D$ and the maturity time $T(days)$ is scaled to $T^* = T * D$ periods. The asset return per period is $\phi^* = \frac{\phi_{year}}{n}$. The asset volatility per period is $\sigma^* = \frac{\sigma_{year}}{\sqrt{n}}$. The stock prices process follows lognormal distribution. If current time is $t = 0$, stock price at scaled maturity T^* is:

$$S(T^*) = S(0)exp\{\sum_{i=1}^{T^*} Z_i\} \tag{2.17}$$

where Z_i follows the normal distribution with mean $\mu = \phi^* - \frac{1}{2}(\sigma^*)^2$ and volatility σ^*. Equation (2.17) can also be written as

$$S(T^*) = S(0)exp\{\mu T^* + \sigma^* \sqrt{T^*} \epsilon_i\} \tag{2.18}$$

ϵ_i is drawn from a standard normal distribution. If we have simulated the stock price at maturity, the present-time fair option price can be obtained by discounting the payoff to the factor e^{-rT} as

$$f(S_0) = \frac{1}{m} e^{-rT} \sum_{i=1}^{m} [max\{S_T - K, 0\}] \tag{2.19}$$

The number of replications m must be set large enough, such as 10^4, to get an accurate result. This computer intensive approach is the main drawback of Monte Carlo simulation method, thus the efficient simulation strategies are significant in applying the Monte Carlo method to improve the performance.

2.3 Efficient Monte Carlo Simulation Methods

2.3.1 Variance Reduction Techniques

The convergence speed of Monte Carlo simulation is $N^{-\frac{1}{2}}$, but there are several variance reduction methods which can improve the accuracy of the simulation process. I use four techniques in this paper to compare their performance in the incomplete market.

Antithetic Variables(Anti-V)

Anti-V uses pairs of random variables that follow the same probability distribution but with negative correlation. The average of N pairs of antithetic variables has smaller variance than that of 2N independent variables. If we want to estimate $E(h(\mathbf{U}))$ where $\mathbf{U}$ is uniformly distributed on $[0,1]^N$. We can get the antithetic variate of U,

$$1 - \mathbf{U} = (1 - U_1, 1 - U_2, ..., 1 - U_N)$$

Now we can estimate h by

$$\bar{h} = \frac{1}{N}\sum_{i=1}^{N} h(U_i)$$

and

$$\bar{h}^A = \frac{1}{N}\sum_{i=1}^{N} h(1 - U_i)$$

The Antithetic estimator is $(\bar{h} + \bar{h}^A)/2$. The variance of this estimator is

$$\frac{Var(h(\mathbf{U}))}{2N}(1 + \rho)$$

where ρ is the correlation between $h(U_1)$ and $h(U_1^A)$. The variance is reduced if $\rho < 0$, and it is always the case if $Var(h(\mathbf{U}))$ is monotonic in $\mathbf{U}$. The idea is that h can be decomposed to symmetric part $(h(U) + h(1-U))/2$ and antisymmetric part $(h(U) - h(1-U))/2$. Because the Antithetic version of estimator has only the symmetric part of h, the variance is reduced.

In this study, option prices are simulated based on normal random variable Z_i and $-Z_i$. Option prices are

$$f_i = e^{-rT} max\{0, S_T^{(i)} - K\}, \text{ where } S_T^{(i)} \text{ is simulated based on } Z_i \quad (2.20)$$

$$\tilde{f}_i = e^{-rT} max\{0, \tilde{S}_T^{(i)} - K\}, \text{ where } \tilde{S}_T^{(i)} \text{ is simulated based on } -Z_i \quad (2.21)$$

and an unbiased estimator of the option price is

$$f_{AntiV} = \frac{1}{N} \sum_{i=1}^{N} \frac{f_i + \tilde{f}_i}{2} \quad (2.22)$$

Control Variates(CV)

The control variates method adjusts the outputs of Monte Carlo simulation directly. It uses the known errors of the estimator which contains the information of the unknown error of the interesting estimator, for example, in the case of estimating $E(h(X))$ or $E(h(X_1, ..., X_T))$. Suppose we know $E(f(X))$ and the error of $E(f(X))$ and we also have $\rho(f(X), h(X)) \neq 0$, then the estimation errors of these two expectations are correlated. We can use the standard linear estimation to reduce the variance of $E(h(x))$.

Now let

$$\begin{aligned}
\bar{h} &= \frac{1}{N}\sum_{i=1}^{N} h(X_i) \\
\bar{f} &= \frac{1}{N}\sum_{i=1}^{N} f(X_i) \\
\sigma_h^2 &= Var(h(X)) \\
\sigma_f^2 &= Var(f(X))
\end{aligned}$$

Construct new estimator

$$\bar{h}_\alpha = \bar{h} + \alpha(E(f(X)) - f)$$

Since $E(\bar{h}_\alpha) = E(h(X))$, the new estimator is still unbiased. However, the variance for the new estimator is

$$Var(\bar{h}_\alpha) = \frac{1}{N}(\sigma_h^2 + 2\alpha\sigma_h\sigma_f\rho(h(X), f(X)))$$

Given the variances and correlation, it is obvious that

$$\hat{\alpha} = argmin_\alpha Var(\bar{h}_\alpha) = -(\sigma_h/\sigma_f)\rho(h(X), f(X))$$

and $min_\alpha = \sigma_h^2(1-\rho^2)/N$. The more $h(X)$ and $f(X)$ are correlated, the more the variance is reduced.

Following Broadie and Glasserman(1996), we use the terminal asset price as the control variate and let C be the unbiased simulation estimator of option price and $c = E[C]$ where c is the true value of the option. We let S_T be the simulated terminal price of the underlying stock at time T. By the Black-Scholes assumption, we have the expected value of terminal asset price $E[S_T] = S_0 e^{rT}$. Now we can construct a control variates estimator of the option price as

$$C^{CV} = C + \alpha(e^{rT} S_0 - S_T) \tag{2.23}$$

since the variance of the new estimator is

$$Var[C^{CV}] = Var[C] + \alpha^2 Var[S_T] - 2\alpha Cov[C, S_T] \tag{2.24}$$

α is chosen to minimize $E[C^{CV} - c]^2$ and the variance-minimizing α is

$$\alpha^* = \frac{Cov(C, S_T)}{Var(S_T)} \tag{2.25}$$

This problem can be solved by a linear regression of C on S_T.

Stratified sampling(SS)

Random variables X_i are sampled in a way that a specified number of samples selected from each stratum. Thus, the whole domain can be covered. It is useful if there is a good approximation for the average over small subdomain. The stratification should be chosen so that the subdomains have equal probability associated. Consider $h(U1, ..., U_d)$, the standard stratified sampling is to divide the sample space of U_1 into equiprobable strata $[0, 1/N], ..., [(N-1)/N, 1]$ and the stratified estimator can be described as

$$\frac{1}{N} \sum_{i=1}^{N} h\left(\frac{i-1+U_1^{(i)}}{N}, U_2^{(i)}, ..., U_d^{(i)}\right) \tag{2.26}$$

Note that $(i-1+U_1^{(i)})/N$ falls between $(i-1)$th and ith with probability $1/N$.

In general, if we want to sample from a mix of N distributions in which the ith distribution has probability p_i, mean μ_i and variance σ_i^2. Thus, the mixed distribution has mean

$$\sum_{i=1}^{N} p_i \mu_i$$

and variance

$$\sum_{i=1}^{N} p_i(\mu_i^2 + \sigma_i^2) - \left(\sum_{i=1}^{N} p_i \mu_i\right)^2$$

Applying the stratified sampliing, the variance of the new stratified estimate is $\sum_{i=1}^{N} p_i \sigma_i^2$, and the variance reduction is

$$\sum_{i=1}^{N} p_i(\mu_i^2) - \left(\sum_{i=1}^{N} p_i \mu_i\right)^2$$

The SS removes the variance of conditional expectation of the outcome given the information being stratified.

In our option pricing case, the payoff depends mainly on the terminal stock price S_T which is assumed to follow a Brownian motion process W. If we want to generate 10^5 times standard normal distributed number, we can apply SS process to improve the simulation. For example, separate the whole field to 10^3 straddles and do 10^2 independent simulations in each straddle. The random number in each straddle is

$$z_i^j = \Phi^{-1}\left(\frac{i-1+U_i}{10^3}\right) \quad i = 1, ..., 10^3 \tag{2.27}$$

Where Φ^{-1} is the inverse cumulative distribution function of a standard normal, U_i is drawn from $Unif(0,1)$. Then in the ith straddle, we simulate random number z_i^j 10^2 times. Note that $\frac{i-1+U_i}{10^3}$ falls between the $(i-1)$th and ith percentiles of the uniform distribution with equally probability.

Importance sampling (IS)

In Monte Carlo simulation process, importance sampling is applied to change the measure for obtaining a new estimator with lower variance. Random variables X_i's are selected according to a different probability measure Q. The probability measure Q is viewed as a way to control the choice of X_i's in order to consider the underlying structure of value function h. We use the likelihood ratios w_i's to remove the bias due to sampling from measure Q. It also can be viewed as an indirect way to bias the sampling towards the "important" samples. In finance, importance sampling is mostly used to ensure that all samples are drawn in the regions where the function is nonzero, for example, pricing the out-of-money option. The standard process of generating paths will lead to many zero payoffs. The idea of using IS as variance reduction technique is that the estimate under new measure has less variance than that under the initial probability measure. For example, if the payoff h can be obtained by simulating many paths of $X_1, ..., X_m$ and take average. This process is the same as to estimate the integral

$$\int h(x)g(x)dx = \int \left(\frac{hg}{\tilde{g}}\right)(x)\tilde{g}(x)dx \tag{2.28}$$

where $\tilde{g}$ is nonzero. Now the payoff is $\tilde{h} = \frac{hg}{\tilde{g}}$ under new measure. The importance sampling method chooses $\tilde{g}$ such that the payoff $\tilde{h}$ has less variance under the new measure. The ideal way is to choose $\tilde{g} = \frac{hg}{\mu}$ to make the new payoff have zero variance. But constant $\mu = \int h(x)g(x)dx$ is unknown. Thus the goal of importance sampling is to choose density $\tilde{g}$ proportional to hg.

The ideal importance sampling can construct a zero-variance estimator by sampling S_T from the density,

$$f(x) = c^{-1}max\{x - K, 0\}e^{-rT}g(x) \tag{2.29}$$

where $g()$ is the log-normal density of S_T; c is a constant which normalizes the integration of density function f to 1. Here, c is just the current time option price. It is not applicable in practice.

Following Boyle(1997), we apply importance sampling in pricing the European style call option. We need to price the option by $e^{-rT}E[max\{S_T - K, 0\}]$. The standard approach is to generate samples of the terminal prices S_T in (2.16) with Brownian Motion having drift r and volatility σ. However, we can also generate S_T with any other drift μ and adjust the expectation with the likelihood ratio. We use higher drift in importance sampling to obtain higher percentage of sample paths with positive payoffs.

$$E_r[max\{S_T - K, 0\}] = E_\mu[max\{S_T - K, 0\}L] \tag{2.30}$$

where L is the likelihood ratio of the log-normal densities with parameters μ and r defined as

$$L = \left(\frac{S_T}{S_0}\right)^{(r-\mu)/\sigma^2} exp\left(\frac{(\mu^2 - r^2)T}{2\sigma^2}\right) \tag{2.31}$$

2.3.2 Quasi-Monte Carlo (QMC)

Quasi-Monte Carlo simulation, which uses the Low-discrepancy sequences and is also called Quasi-random sequences, can provide a convergence of $O(N^{-1}(log(N))^d)$, where d is the dimension number of the integration. The standard Monte Carlo offers convergence as $O(1/\sqrt{N})$. Thus, QMC sequence improves the convergence when the dimension d is small. QMC uses pre-selected deterministic points rather than random samplings to evaluate the integral. The accuracy of this approach depends on how the deterministic points are dispersed throughout the domain of integration.

There are two main approaches to construct QMC which are randomized QMC (RQMC) and effective dimension. We use the RQMC based on Lemieux and L'Ecuyer(2001). First, we use lattice rules, Korobov rules specifically, to create the low-discrepancy point set. For sample size n and dimension d, we choose an integer $a \in \{1, ..., n-1\}$ and let $a^j = a^{j-1} \; mod \; n \;, for \; j = 1, ..., d$. The lattice point set P_n in d dimensions is described as

$$P_n = \{\frac{i}{n}(1, a, a^2, ..., a^{d-1}) \; mod \; 1, i = 0, ..., n-1\} \tag{2.32}$$

Second, get the randomize QMC point sets. We randomly generate a vector Δ in $[0,1]^d$ and add it to each point of P_n with modulo 1. i.e. RQMC point set $\tilde{P}_n$is

$$\tilde{P}_n = (P_n + \Delta)\ mod\ 1 \tag{2.33}$$

To apply QMC in estimating the call option price, we take U_i's from RQMC sequence rather than from the uniformly distributed variables in MC sequence.

2.4 Data

2.4.1 Real Data

For the real data, we use the European type call options on the S&P 500 index(SPX) because this is one of the most actively traded options in the world. The daily dividend distributions of the index are available. Furthermore, there has been a lot of research based on the SPX.

Several filters are used on the data. First, we wish to use the options with maturity time ranging from 10 days to 360 days. Second, the price is larger than $0.05. Third, the implied volatility is less than 70%. The average of bid and ask prices are used as option price. The option data are divided into several categories in accordance with the maturity time and the monyness which is strike stock ratio K/S. Based on the time to expiration, the options are classified as short-term($<$ 60 days); medium-term(60-180 days) and long-term($>$180 days). The options are also classified as in-the-money($K/S \leq 0.97$); at-the-money($K/S \in (0.97, 1.03)$) and out-the-money($K/S \geq 1.03$).

2.4.2 Simulated Data

It is useful to test the efficient simulation strategies on the simulated data since we will be able to examine their performances under exact models. We use the SV, Jump-diffusion and SVCJ models in this study. In the SV model and Jump-diffusion model, closed form option prices are taken as ”true” price. There is no closed form price for SVCJ model and I use the almost exact simulation methods discussed in Alexander & Antoon (2008) to generate the ”true” price.

In all the three models we assume that the market has 250 trading days a year. The first two models have closed form of option price and the daily price can be exactly obtained. For the SVCJ model, we simulate option prices for every 5 minutes. The market is usually open at 9:00 in the morning and closed at 5:00 at the afternoon. By generating the stock prices every 5 minutes, there are 96 prices observed for each day and the last one is taken as the daily price. In order to show the performances under different maturity times and strike stock ratios, we consider three maturity times: 30 days as a short term, 90 days as a medium term and 180 days as a long term. Although there can be longer term options such as maturity time of over years in markets, in this study, we only simulate 180 days to compare with the short term options. Strike stock ratios are 0.8 as in-the-money, 1.0 as at-the-money and 1.2 as out-of-the-money in this study.

Option Price in Stochastic Volatility(SV) model

In SV model, we use the option price form of Hull and White (1987) in equation (2.8) in which the Black-Scholes price is obtained from equation (2.2). The result depends on the parameters μ and ξ. Assuming μ is zero and by the moments for the distribution of $\overline{V}$, the Hull-White option price can be described as:

$$\begin{aligned} f^{HW}(S,\sigma^2) &= f^{BS}(\sigma^2) \\ &+ \frac{1}{2}\frac{S\sqrt{T-t}N'(d_1)(d_1d_2-1)}{4\sigma^3} \times \left[\frac{2\sigma^4(e^k-k-1)}{k^2} - \sigma^4\right] \\ &+ \frac{1}{6}\frac{S\sqrt{T-t}N'(d_1)[(d_1d_2-3)(d_1d_2-1)-(d_1^2+d_2^2)]}{8\sigma^5} \\ &\times \sigma^6\left[\frac{e^{3k}-(9+18k)e^k+(8+24k+18k^2+6k^3)}{3k^3}\right] + ..., \end{aligned} \qquad \mathbf{(2.34)}$$

where $f^{BS}(\sigma^2)$ is the Black-Scholes price and $\sigma^2 = E[\bar{V}] = V_0$. $k = \xi^2(T-t)$ which is sufficiently small and ξ is from 1 to 4. From Hull and White(1987), $\xi = 1$ leads to the least bias when pricing the options with stochastic volatilities.

Because it is difficult to get the analytical solution for the SDE of stock price dynamics in Hull-White model, Monte Carlo simulation can be used to get the numerical solution according to the following equations:

$$S_i = S_{i-1} exp\{(\phi - \frac{V_{i-1}}{2})\Delta t + u_i \sqrt{V_{i-1}\Delta t}\} \tag{2.35}$$

$$V_i = V_{i-1} exp\{(\mu - \frac{\xi^2}{2})\Delta t + v_i \xi \sqrt{\Delta t}\} \tag{2.36}$$

The annualized interest rate ϕ is set to 0.07 and μ is set to 0. i is the index where $1 \leq i \leq n$. u_i and v_i are sampled from independent standard normal distributions. V_0 can be obtained from $V_0 = \sigma_0^2$ where σ_0 is obtained from the S&P 500 index option. In Hull and White (1987) model, the correlation ρ between stock price and variance is assumed to be zero to get closed form option price. I keep the assumption here.

In order to simulate one year's daily option prices, we need to simulate the stock prices first. The time interval $t^* - t = 1$ is separated to n subintervals and $\Delta t = (t^* - t)/n$ where t is set to zero. I simulate 96 observations each day and apply the last one in the Black-Scholes formula to obtain the option price of that day. In this case, $n = 96 * 250$. The stock prices are taken to calculate the daily option prices are at index $i = 96h$, where h is the date number. When I have the closing time stock price for day h, the Hull-White option price f_h^{HW} is obtained from equation (2.34). Replicating this process m times independently, and the simulated option price at day h is described as

$$\overline{f}_h^{HW} = \frac{1}{m} \sum_{m}^{j=1} f_h^{HW} \tag{2.37}$$

Replication number m is set to $10,000$ in this process.

Option Price in Jump-diffusion model

Jump-diffusion model has closed form for option price as described equation (2.10). In order to sum from 0 to ∞ in the price equation, a stopping rule is set for the iteration.

Simulated Option Price in SVCJ model

Although Broadie and Kaya(2006) have given an exact simulation for the SV model, the process is slow and can barely be used in practice. In this study, I use the direct interpolation combined with the Quadratic Exponential scheme in Andersen(2007) and martingale correction in Andersen and Piterbbarg (2007) to obtain an efficient simulation process.

At time t, given S_u and V_u, for $u < t$, the dynamics of stock price S_t and variance V_t are described as

$$
\begin{aligned}
S_t &= S_u exp[(\phi - \lambda\overline{\mu})(t-u) - 0.5\int_u^t V_s ds \\
&\quad +\rho \int_u^t \sqrt{V_s} dW_s^{(1)} + \sqrt{1-\rho^2}\int_u^t \sqrt{V_s} dW_s^{(2)}] \\
&\quad \times \prod_{i=N_u+1}^{N_t} J_i^s
\end{aligned}
\tag{2.38}
$$

and the variance is

$$
\begin{aligned}
V_t &= V_u + k\theta(t-u) - k\int_u^t V_s ds + \sigma_v \int_u^t \sqrt{V_s} dW_s^{(1)} \\
&\quad + \prod_{i=N_u+1}^{N_t} J_i^v
\end{aligned}
\tag{2.39}
$$

In this study, we follow the option price simulation algorithm in Broadie and Kaya(2006), but use the alternative efficient simulation rather than the exact simulation in the second step. The time horizon is divided according to the jumps and the variance, and stock prices are simulated at each jump. The algorithm for simulation option price based on SVCJ model is described as followings:

Step 1

Simulate a Poisson process with intensity λ to determine the time for the jumps. If the maturity is T, the expected jump times during this time horizon is $\lambda * T$. Also, the time of next jump t_j is set to T if $t_j > T$. For the property of a Poisson process, time between two jumps R_j has an exponential distribution $Exp(\lambda)$ with mean $\frac{1}{\lambda}$. The steps to simulate the jump time t_j are described as followings

$$1 \quad , \quad \text{generate } R_j \text{ from exponential distribution } Exp(\lambda),$$
$$i.e. E(R_j) = \frac{1}{\lambda}$$
$$2 \quad , \quad t_j = t_{j-1} + R_j$$

Step 2

During the time interval $t_j - t_0$, we ignore the jump process and simulate the stock price S_{t_j} and variance V_{t_j} according to the SV model. The time grid is set to 5 minutes, i.e. the time interval $t_j - t_0$ is parted as $0 = t^0 < t^1 < .... < t^M = t_j$.

Where $M = \frac{t_j - t_0}{\Delta t}$ and $\Delta t = 5\ minutes$. The stock price and variance process are

$$\frac{dS_t}{S_t} = \phi dt + \sqrt{v_t} dW_t^s \tag{2.40}$$

$$dv_t = k(\theta - v_t)dt + \sigma_v \sqrt{v_t} dW_t^v \tag{2.41}$$

where $dW_t^s dW_t^v = \rho dt$. The exact solution of (2.40) is

$$S_t = S_s exp\left[\int_s^t [\phi - 0.5 v_u] du + \int_s^t \sqrt{v_u} dW_u^s\right] \tag{2.42}$$

Using Ito's Lemma and Cholesky decomposition, we have

$$\begin{aligned} log(S_t) &= log(S_s) - 0.5 \int_s^t v_u du + \rho \int_s^t \sqrt{v_u} dW_u^v \\ &\quad + \sqrt{1-\rho^2} \int_s^t \sqrt{v_u} dW_u \end{aligned} \tag{2.43}$$

By integrating the variance process (2.41), the variance can be described as

$$v_t = v_s + \int_s^t k(\theta - v_u)du + \sigma_v \int_s^t \sqrt{v_u} dW_u^v \tag{2.44}$$

$$or$$

$$\int_s^t \sqrt{v_u} dW_u^v = \frac{1}{\sigma^v}\left[v_t - v_s - k\theta\Delta t + k\int_s^t v_u du\right] \tag{2.45}$$

Plugging (2.45) into (2.43), the logarithmic asset price is

$$\begin{aligned} log(S_t) &= log(S_s) + \frac{k\rho}{\sigma_v}\int_s^t v_u du - 0.5\int_s^t v_u du + \frac{\rho}{\sigma_v}(v_t - v_s - k\theta\Delta t) \\ &+ \sqrt{1-\rho^2}\int_s^t \sqrt{v_u} dW_u \end{aligned} \tag{2.46}$$

Drift interpolation

The simple drift interpolation scheme is defined as

$$\int_s^t v_u du | v_s, v_t \approx \gamma_1 v_s + \gamma_2 v_t \ , \quad \gamma_1 = \gamma_2 = 0.5 \tag{2.47}$$

Applying the drift interpolation into equation (2.46), we have the approximate logarithmic stock price as

$$log(S_t) = log(S_s) + \phi\Delta t + K_0 + K_1 v_s + K_2 v_t + \sqrt{K_3 v_s + K_4 v_t} Z_s \tag{2.48}$$

where Z_s is drawn from a standard normal distribution, and

$$\begin{aligned} K_0 &= -\frac{\rho k\theta}{\sigma_v}\Delta t, \quad K_1 = \gamma_1\Delta t\left(\frac{k\rho}{\sigma_v} - 0.5\right) - \frac{\rho}{\sigma_v}, \quad K_2 = \gamma_2\Delta t\left(\frac{k\rho}{\sigma_v} - 0.5\right) + \frac{\rho}{\sigma_v} \\ K_3 &= \gamma_1\Delta t(1-\rho^2), \quad K_4 = \gamma_2\Delta t(1-\rho^2) \end{aligned}$$

Quadratic Exponential(QE) Scheme for Variance Process

Given v_s, compute

$$\begin{aligned} m &= \theta + (v_s - \theta)e^{-k\Delta t} \\ s^2 &= \frac{v_s\sigma_v^2 e^{-k\Delta t}}{k}(1 - e^{-k\Delta t}) + \frac{\theta\sigma_v^2}{2k}(1 - e^{-k\Delta t})^2 \\ \psi &= \frac{m^2}{s^2} \end{aligned}$$

Let $\psi_c = 1.5$. **(a)** If $\psi \leq \psi_c$,

$$\begin{aligned} b^2 &= 2\psi^{-1} - 1 + \sqrt{2\psi^{-1}}\sqrt{2\psi^{-1} - 1} \\ a &= \frac{m}{1 + b^2} \\ v_t &= a(b + Z_v)^2 \end{aligned}$$

where Z_v is drawn from a standard normal distribution. **(b)** If $\psi > \psi_c$,

$$\begin{aligned} p &= \frac{\psi - 1}{\psi + 1} \\ \beta &= \frac{1 - p}{m} = \frac{2}{m(\psi + 1)} \\ v_t &= L^{-1}(U_v) \end{aligned}$$

where U_v is drawn from an uniform distribution and L^{-1} is defined as

$$L^{-1}(u) = \begin{cases} 0 & 0 \leq u \leq p \\ \beta^{-1} log(\frac{1-p}{1-u}) & p \leq u \leq 1 \end{cases}$$

Martingale Correction

Since the discretized stock price from equation (2.48) does not satisfy the martingale condition under the risk-neutral measure, we apply the martingale correction scheme in Anderson(2007) in the QE process. The method is to replace K_0 in equation (2.48) by modified parameter K_0^* which is described as

$$K_0^* = \begin{cases} -\frac{Ab^2 a}{1-2Aa} + 0.5log(1-2Aa) - (K_1 + 0.5K_3)v_s & \psi \leq \psi_c \\ -log(p + \frac{\beta(1-p)}{\beta - A}) - (K_1 + 0.5K_3)v_s & \psi > \psi_c \end{cases}$$

where $A = K_2 + 0.5K_4$.

Simulation Algorithm

Given v_0, $\psi_c = 1.5$, $\gamma_1 = \gamma_2 = 0.5$,

1, Use QE scheme to sample v_t.

2, Calculate the parameter K_0^* using the martingale correction method.

3, Generate the stock price S_t from equation (2.48).

Step 3

If the next jump time t_j is equal or larger than T, this jump is skipped and the stock price at maturity is S_T. Otherwise, we simulate the jump ξ^v for volatility at time t_j. The jump size is sampled from exponential distribution with mean μ_v. The variance when jump occurs is updated as $\tilde{V}_{t_j} = V_{t_j} + \xi^v$.

Step 4

The jump of stock price ξ^s is also simulated at time t_j. The jump size ξ^s is sampled from a lognormal distribution with mean $\mu_s + \rho_J \xi^v$ and variance σ_s^2. The stock price at jump is set to $\tilde{S}_{t_j} = S_{t_j}\xi^s$.

Step 5

Now the new stock price and variance are updated as $S_0 = \tilde{S}_{t_j}, V_0 = \tilde{V}_{t_j}, t_0 = t_j$, and repeat from **step 1** to get next jump until we reach the maturity time T.

The payoff of the option is simulated by taking average on enough number of paths and discount to the factor e^{-rT} to get present-time fair option price.

2.5 Numerical Results

2.5.1 Experiment Using Simulation Data

We use the same replication number and sample size in all efficient strategies and compare the results with standard error and root mean squared error (RMSE). When doing experiments on the simulation data, we compare their performances on different stock/strike ratios and time to expiration. We use the same parameters in all the simulations. The same seeds are used to generate random number for different efficient simulation strategies during the experiments.

For each option price model(SV, Jump, SVCJ), we do the experiment 500 times. In each experiment, one year's daily option prices are simulated according to specific model and are taken as the 'true' prices. Monte Carlo with different efficient simulation strategies is used to estimate the option price for everyday. The simulations number in each experiment is from 500 to 5000. The estimated price is compared with the 'true' price. We use the standard error and the root mean squared error to measure the accuracy. Each Monte Carlo simulation takes replications number m, which is important for the accuracy. In each experiment we change the values of m and change the expiration time and moneyness. The standard error of the mean(SE) and Root Mean square error(RMSE) are defined as

$$SE = \frac{s}{\sqrt{N}}, \text{where } s = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(f_i - \bar{f}_i)^2} \quad (2.49)$$

$$RMSE = \sqrt{\frac{\sum_{i=1}^{N}(f_i - f_i^*)^2}{N}} \quad (2.50)$$

where

s the sample standard deviation

N the sample size

f_i option prices obtained by MC simulation

$\bar{f}_i$ average of option prices obtained by MC simulation

f_i^* realized option prices, either from real data or from simulated data

The standard error(SE) measures the standard deviation of the estimates' sampling distribution. The root mean squared error (RMSE) defined in this research measures the distance between the estimated prices and the true prices. The results of RMSE are in Table 2.2 and Figure 2.1 to Figure 2.3. The results of standard errors are in Table 2.3 and Figure 2.4 to 2.6. From the numerical results, we can have some interesting findings as follows:

For the results of the RMSE, it is clear that the the effects are different based on different option pricing models, strike stock ratios, and maturity times.

Firstly, the results under option pricing models are: For SV model, IS performs the best for in-the-money option and QMC performs the best for most cases in at-the-money options and out-the-money options. For Jump model, CV performs the best for in-the-money option, but QMC does the best for the at-the-money options and out-the-money options. For SVCJ model, SS does the best for in-the-money option and QMC performs the best for the at-the-money and out-the-money options.

Secondly, the results based on times to maturity are: For the short term option(30 days), QMC performs the best in at-the-money and out-the-money options. But for in-the-money options, IS performs the best in SV model, while CV performs the best in Jump model and SS does the best in SVCJ model. For the medium term options(90 days), QMC does the best in at-the-money option and out-the-money options. The other different methods perform the best for in-the-money options. It is the same for long term options(180 days); QMC does the best for at-the-money and out-the-money options, while the other methods beat QMC for the in-the-money options.

Thirdly, considering the strike stock ratios, different models and maturity times show different results. For in-the-money options, IS performs the best for all maturity times in SV model. CV and AntiV both do the best in the Jump model. SS and QMC both do the best in the SVCJ model. For at-the-money options, QMC does the best in most cases except that IS does the best for the medium term option based on SV model. Also SS and AntiV work as well as QMC does in in-the-money options. For out-the-money options, QMC does the best and much better than the other methods. SS also performs better than the other methods except QMC method.

Thus, according to the RMSE, QMC performs the best in most cases, and SS's performance is close to QMC. For the medium term and long term options, QMC beats the other models. The out-of-money options are usually difficult to be priced and QMC is proved to be a good tool in this case. During the experiments, it is noticeable that the importance sampling method does not work well in the out-of-money cases. It was expected to reduce the chance of zero payoff and offer better results in this case. The reason may be that, we only chose the simplest algorithm in this study rather than the complex time consuming optimal algorithm. Also, in some cases, not all the efficient simulations can produce better results than the standard Monte Carlo; some even performed worse.The accuracy doesn't improve significantly with increasing the simulations number. RMSE measures the distance between the true option prices and the prices obtained by Monte Carlo simulation. It is even more important in practice than the standard error. In order to get more accurate prices in the incomplete market, we should choose right efficient simulation strategies and better pricing algorithms as well.

If we think about the standard error and the methods' capacity to reduce the variance only, all efficient simulations can work much better than the standard Monte Carlo process. For the three incomplete market option pricing models, the performances of QMC and SS are very close and they do the best in all the cases. Since most variance reduction methods can be combined with QMC, it is worth studying the effects of the combined methods. Also, the term of maturity and the strike stock ratio have little influence on how to choose effective simulation strategies based on the standard error.

2.5.2 Experiment Using Real Data

In this research, we use the European style call options on S&P 500 index of a specific day. The options are filtered and categorized as described in section 2.4.1. We compared the standard error (SE) and the root mean squared error (RMSE) based on different simulation times. The SE and RMSE based on CPU time in seconds used in simulation are also counted and compared. The experiment results are close to that based on the generated data from models but different to some extent. Experiments results are listed in Figure 2.7, 2.8, 2.9 and 2.10.

According to standard error of simulations on the in-the-money options, all efficient methods do much better than the standard Monte Carlo simulation. The stratified sampling and Control variate did the best, and the quasi Monte Carlo also did well. Importance Sampling did better than standard method but not as good as the other efficient methods. For at-the-money options, Stratified Sampling did the best,and QMC is the second best. For out-the-money options, results are different from the above: the Anti Variates method did not do better than the standard method. The QMC did best for out-the-money options.

From the stated results, we can see that, the time to maturity has little influence on how to choose the simulation methods and the moneyness is the key. Stratified sampling can be applied to all the moneyness, and QMC is good for the out-the-money options. On the other hand, Anti Variates method should not be used to simulate on the out-the-money options.

In this experiment, different simulation methods do not have big differences in time consumed. By comparing the CPU time spent on the simulation, the performances of the efficient methods are almost identical according to either simulation numbers or time consumed. The main reason may be that the sample size of this experiment is not big enough to tell the difference in time consumed. Further experiment should be done regarding this.

According to RMSE, increasing the simulation times can not reduce the error efficiently. Also, the time to maturity has less influence than the moneyness on how to choose the simulation method. For in-the-money option, CV did the best. For the at-the-money option, QMC and CV did the best. For the out-the-money option, QMC did the best, and the Anti Variates did not do better than the standard Monte Carlo method.

2.6 Conclusion

The Monte Carlo simulation method is popular in the financial field, especially for the purpose of pricing the derivatives. In order to apply this simulation method better, under the incomplete market environment, it is necessary to compare different efficient simulation methods which can reduce the variance, speed the simulation and get more accurate results.

In this study, we simulated option prices based on three incomplete option price models: stochastic volatility model, jump diffusion model and stochastic volatility with concurrent jumps in the stock price and the variance process model. Under the three pricing models and based on different maturity times and different strike stock ratios, we tested and compared the performances of standard Monte Carlo simulation and other five efficient simulation methods, which are Antithetic Variables(Anti-V), Control Variates(CV), Stratified sampling(SS), Importance sampling (IS) and Quasi-Monte Carlo (QMC). The results are obvious. For RMSE, QMC is the best choice for out-the-money option. It is also the best choice for medium term and long term options. But for in-the-money option, the performance of the methods depends on different option pricing models. For standard error, QMC and SS do the best and much better than the other methods.

We also did the same experiments on the S&P 500 index option of a specific day. The results are close to that based on the generated data. Moneyness plays a crucial role in choosing the efficient methods. The time consumed for one simulation is close to that in all methods. The trends under either time consumed or simulation numbers show similarities. For in-the-money option, Stratified Sampling(SS) and Control Variates(CV) are the best according to both SE and RMSE. For at-the-money option, QMC is the best according to both SE and RMSE. Moreover, Antithetic Variables(Anti-V) should not be used in out-the-money option pricing because it can not beat the standard method. It is worth noting that there could be differences between the results obtained with the use of real data and simulation data. However, the results from the simulation lends an insight into real operation and therefore, they are helpful for practitioners.

Maturity time is not a key factor for choosing the simulation method in this research because all the options chosen are shorter than 1 year. Options with longer maturity times should be considered in the future work. QMC has the limitations of working effectively in integration with low number of dimensions, and we have used only one dimension integration in this study. In this regard, the high-dimension situation should be given attention. The future research can also verify the performance of other simulation methods and the combination of different efficient methods. Also, it is important to improve the algorithm of pricing since from the study we can see that increasing the simulation paths can reduce the standard error but can not reduce the RMSE which measures the distance between the true prices and the ones obtained by simulations.

Appendix

S_0	initial stock price at $t = 0$	\$100
r_f	annualized risk-free interest rate	0.0319%
K/S	strike/stock ratios	0.8 to 1.2
σ	annualized asset volatility	29%
λ	jump intensity	0.47
V_0	starting volatility in SVCJ model	0.007569
k	speed of mean reversion	3.46
θ	long-run mean variance	0.008
σ_v	volatility of the variance	0.14
ρ	correlation between the return and volatility process	-0.82
$\bar{\mu}$	mean of jump in stock price	-0.1
σ_s	volatility of jump in stock price	0.0001
μ_v	mean of exponential process for jump in volatility	0.05
ρ_J	correlation between jump in stock price and jump in volatility	-0.38

Note: The fitted parameters are for S&P 500 on a particular day.

Table 2.1: Parameters used in simulation data

	T=30days			T=90days			T=180days		
K/S	0.8	1.0	1.2	0.8	1.0	1.2	0.8	1.0	1.2
SV model price									
MC	1.9668	3.0349	3.4107	2.5607	3.3894	3.1855	3.1142	3.9520	3.2846
AntiV	1.9582	3.0327	3.4061	2.5557	3.3861	3.1807	3.1107	3.9466	3.2783
CV	1.9583	3.0342	3.4278	2.5543	3.3876	3.2809	3.1130	3.9473	3.4664
SS	2.4903	4.4459	3.4648	3.8738	4.0484	3.0319	5.5746	4.0423	2.7013
IS	1.7694*	2.4962*	3.2196	2.1440*	3.0021*	3.0241*	2.3830*	3.7288*	3.1664
QMC	2.3616	4.2127	2.6435*	3.6979	3.3778	2.3132	5.3829	3.0840	2.0624*
Jump model price									
MC	0.1356	1.3968	2.5132	0.3965	2.1123	2.3963	0.8702	2.9050	2.5763
AntiV	0.0369	1.3532	2.4851	0.3250	2.0761	2.3710	0.8117	2.8722	2.5518
CV	0.0352*	1.3496	2.5039	0.3245*	2.0735	2.4687	0.8072*	2.8694	2.7376
SS	0.4973	1.4613	1.7210	0.9955	1.6272	1.5099	1.6552	2.0078	1.3552
IS	0.2403	1.9144	2.9259	0.7473	2.5702	2.7632	1.5476	3.3861	2.9277
QMC	0.3684	1.3181*	0.9987*	0.8195	1.0955*	0.8779*	1.4634	1.1652*	0.7934*
SVCJ model price									
MC	0.6903	2.6176	3.3008	1.4753	3.0986	3.0887	2.5459	3.8238	3.2029
AntiV	0.6958	2.6115	3.2940	1.4702	3.0924	3.0820	2.5374	3.8159	3.1950
CV	0.6950	2.6091	3.3136	1.4710	3.0908	3.1801	2.5345	3.8140	3.3810
SS	0.3991*	0.7316	1.2189	0.5157*	1.0371	1.0671	0.6302*	1.4220	0.9527
IS	0.8538	3.1521	3.7227	1.8595	3.5698	3.4637	3.2457	4.3157	3.5614
QMC	0.4343	0.6828*	0.5908*	0.5632	0.6233*	0.5174*	0.6581	0.6892*	0.4642*

MC means standard Monte Carlo without variance reduction; * denotes the lowest RMSE

Table 2.2: RMSE of the Estimates (10000 simulation paths)

	T=30days			T=90days			T=180days		
K/S	0.8	1.0	1.2	0.8	1.0	1.2	0.8	1.0	1.2
SV model price									
MC	0.0358	0.0511	0.0459	0.0490	0.0501	0.0430	0.0603	0.0528	0.0431
AntiV	0.0051	0.0152	0.0180	0.0099	0.0172	0.0176	0.0160	0.0206	0.0187
CV	0.0040	0.0098	0.0137	0.0062	0.0120	0.0148	0.0093	0.0147	0.0166
SS	0.0050	0.0038	0.0025	0.0046	0.0034	0.0022	0.0044	0.0030	0.0020
IS	0.0224	0.0294	0.0271	0.0289	0.0296	0.0258	0.0330	0.0311	0.0259
QMC	0.0035*	0.0025*	0.0016*	0.0033*	0.0021*	0.0014*	0.0032*	0.0019*	0.0013*
Jump model price									
MC	0.0367	0.0516	0.0462	0.0495	0.0505	0.0432	0.0610	0.0532	0.0433
AntiV	0.0058	0.0159	0.0184	0.0107	0.0177	0.0180	0.0169	0.0210	0.0191
CV	0.0048	0.0105	0.0142	0.0070	0.0126	0.0152	0.0102	0.0152	0.0170
SS	0.0052	0.0045	0.0029	0.0056	0.0039	0.0026	0.0053	0.0034	0.0023
IS	0.0232	0.0297	0.0273	0.0294	0.0299	0.0259	0.0333	0.0313	0.0260
QMC	0.0044*	0.0035*	0.0022*	0.0046*	0.0030*	0.0019*	0.0045*	0.0026*	0.0017*
SVCJ model price									
MC	0.0353	0.0511	0.0460	0.0490	0.0502	0.0431	0.0604	0.0529	0.0431
AntiV	0.0075	0.0163	0.0188	0.0117	0.0182	0.0183	0.0175	0.0214	0.0194
CV	0.0070	0.0113	0.0147	0.0085	0.0133	0.0157	0.0112	0.0159	0.0174
SS	0.0078	0.0055	0.0035	0.0068	0.0047	0.0031	0.0066	0.0041	0.0027
IS	0.0225	0.0295	0.0272	0.0291	0.0298	0.0258	0.0331	0.0312	0.0259
QMC	0.0067*	0.0045*	0.0028*	0.0057*	0.0038*	0.0025*	0.0057*	0.0033*	0.0022*

MC means standard Monte Carlo without variance reduction; * denotes the lowest SE

Table 2.3: Standard Errors of the Estimates (10000 simulation paths)

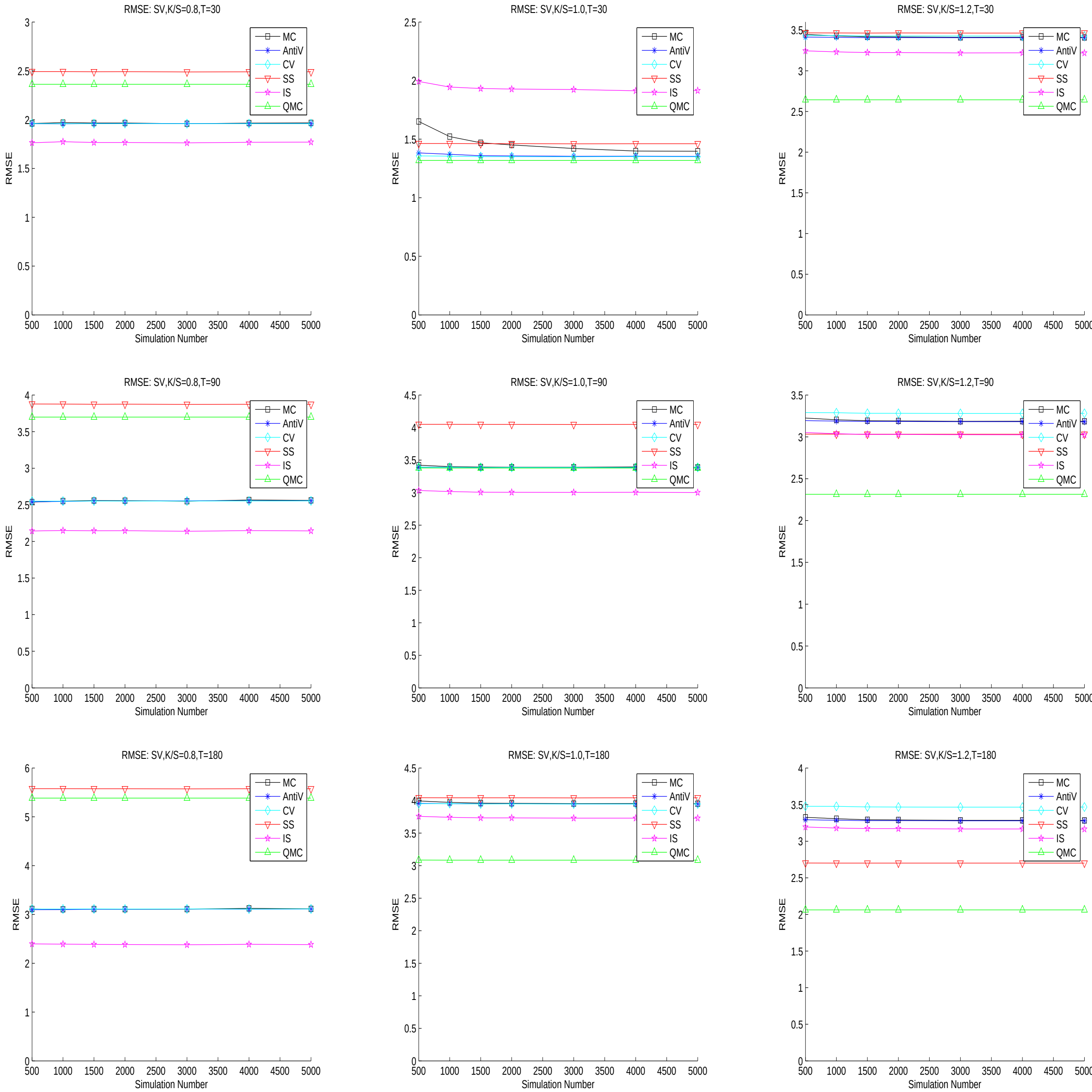
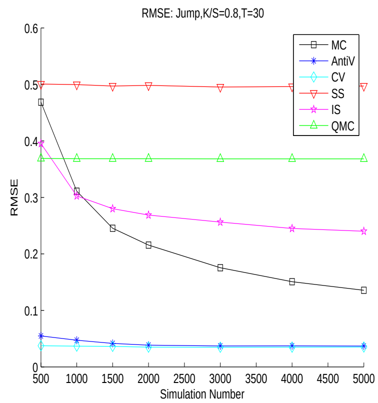
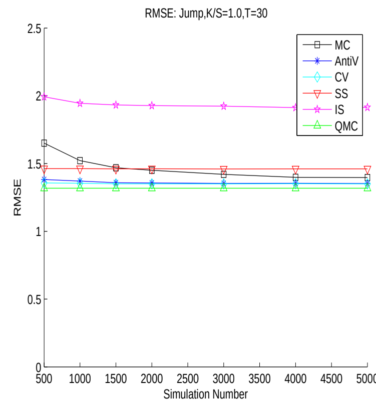


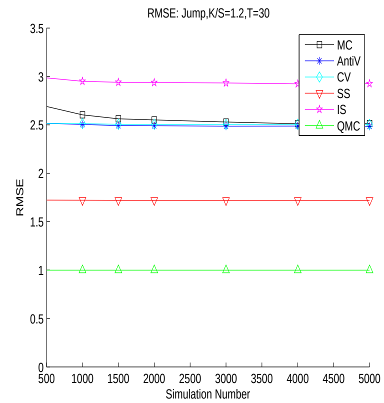
Figure 2.1: RMSE of Estimates based on generated data by SV model



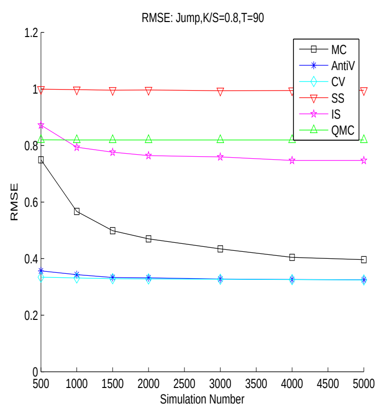
(a) Short Term, In-the-money



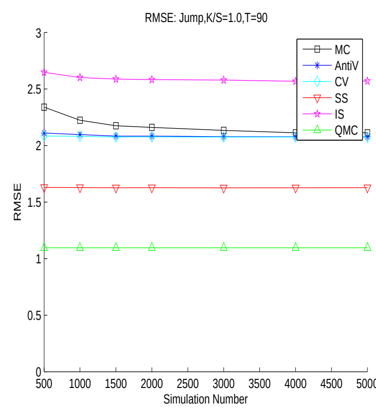
(b) Short Term, At-the-money



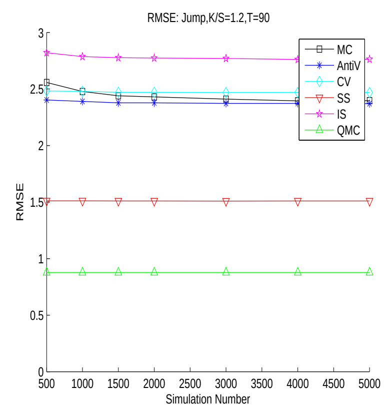
(c) Short Term, Out-the-money



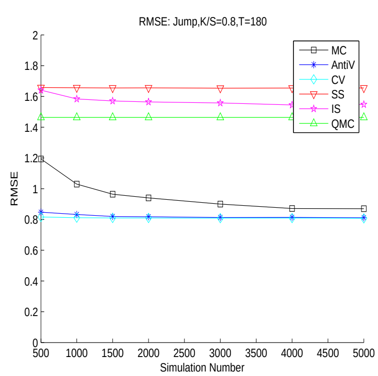
(d) Medium Term, In-the-money



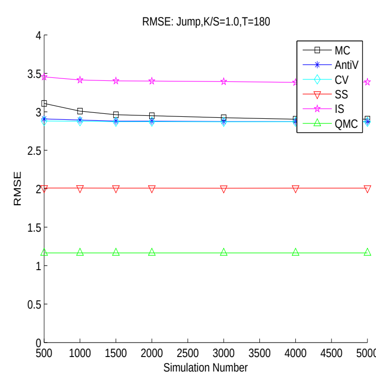
(e) Medium Term, At-the-money



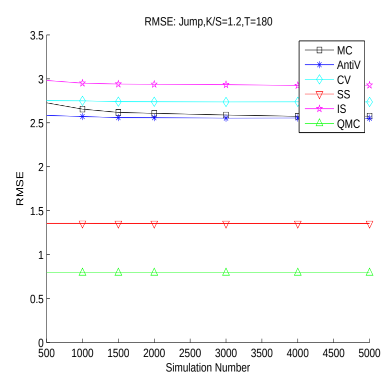
(f) Medium Term, Out-the-money



(g) Long Term, In-the-money

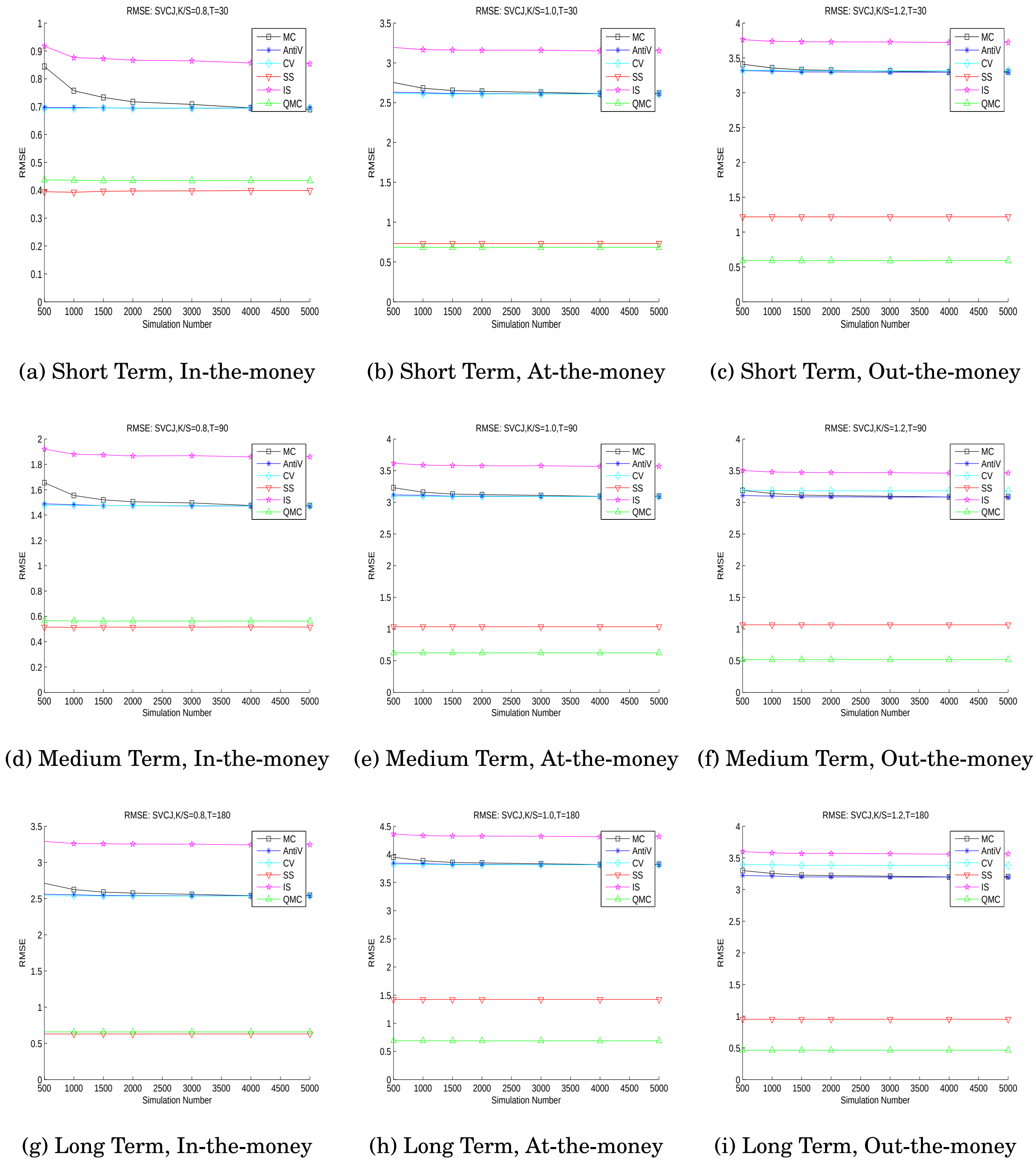


(h) Long Term, At-the-money



(i) Long Term, Out-the-money

Figure 2.2: RMSE of Estimates based on generated data by Jump model

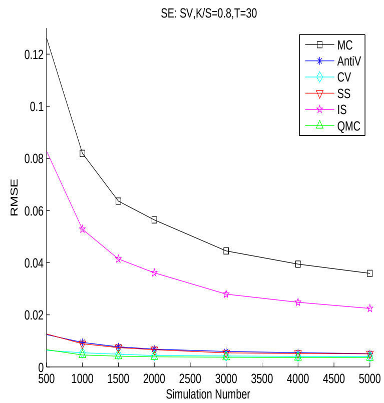


(a) Short Term, In-the-money (b) Short Term, At-the-money (c) Short Term, Out-the-money

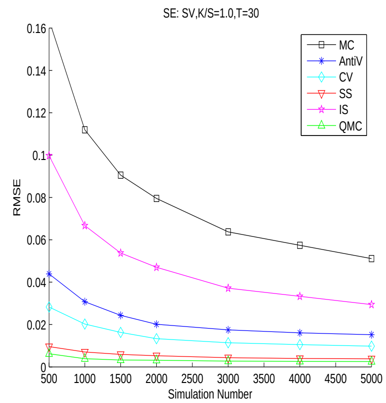
(d) Medium Term, In-the-money (e) Medium Term, At-the-money (f) Medium Term, Out-the-money

(g) Long Term, In-the-money (h) Long Term, At-the-money (i) Long Term, Out-the-money

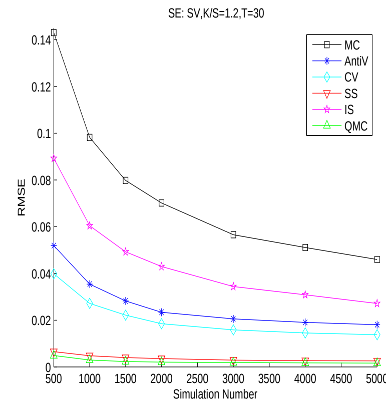
Figure 2.3: RMSE of Estimates based on generated data by SVCJ model



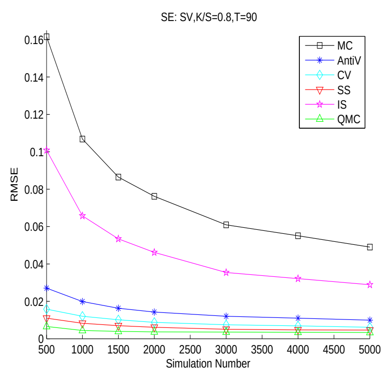
(a) Short Term, In-the-money



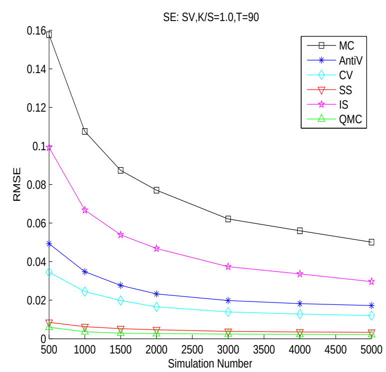
(b) Short Term, At-the-money



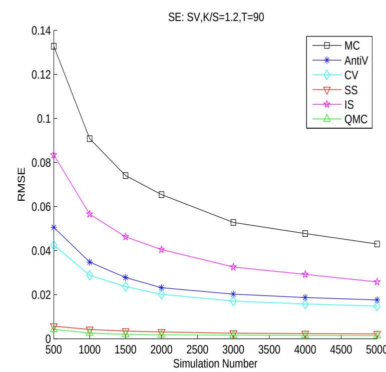
(c) Short Term, Out-the-money



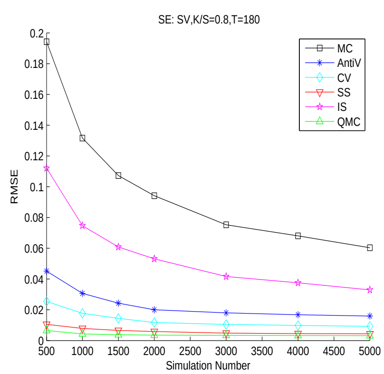
(d) Medium Term, In-the-money



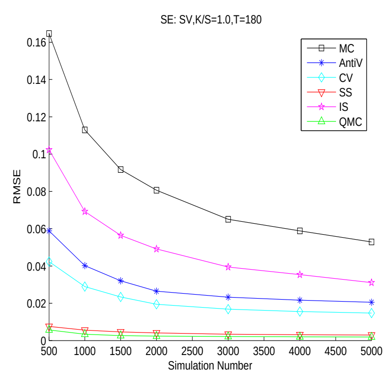
(e) Medium Term, At-the-money



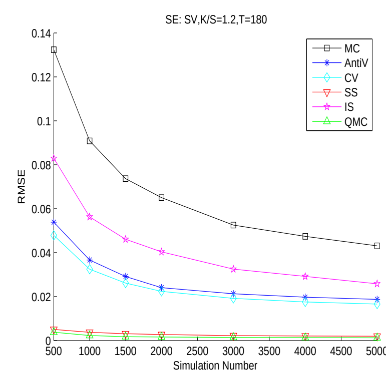
(f) Medium Term, Out-the-money



(g) Long Term, In-the-money

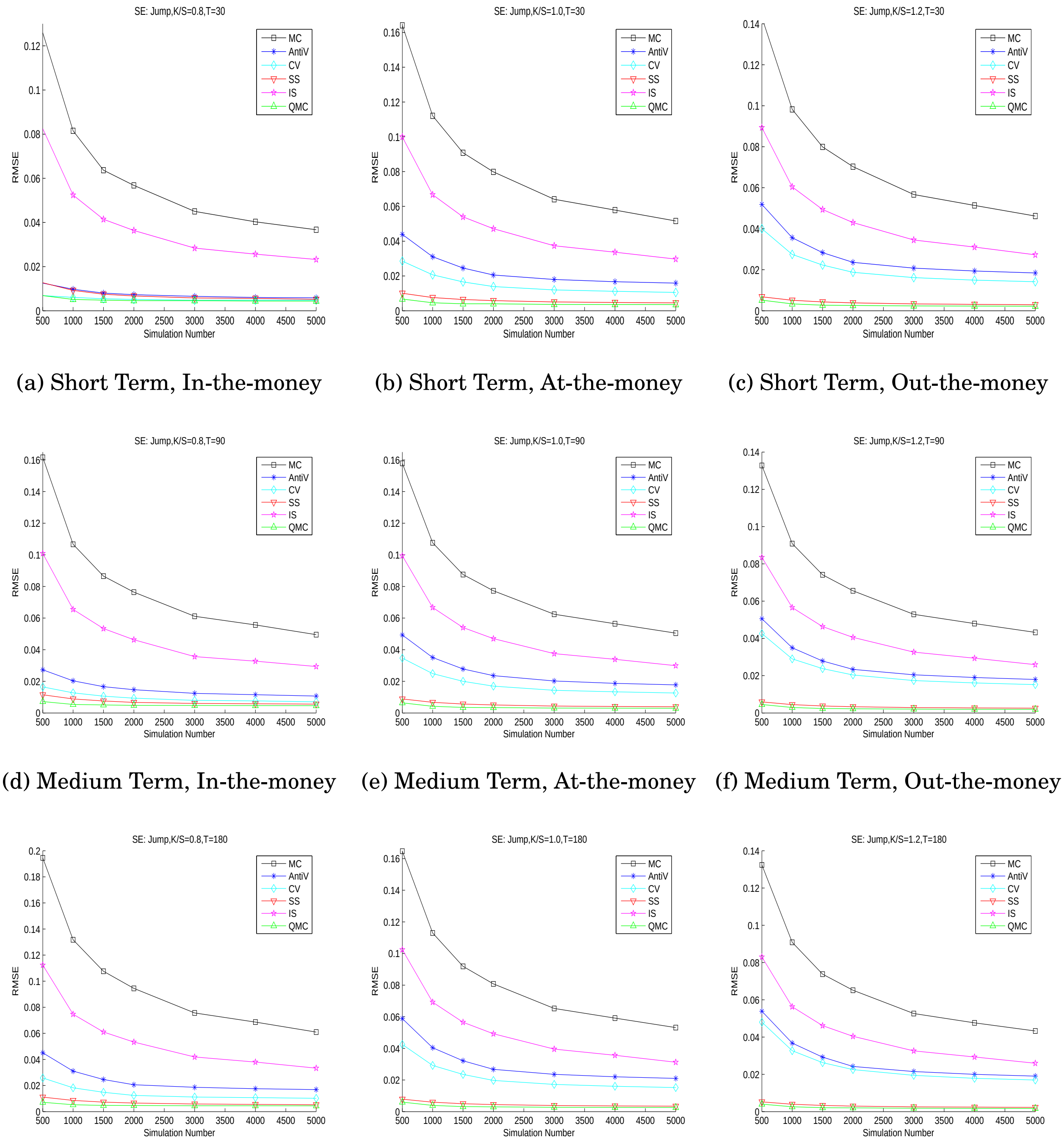


(h) Long Term, At-the-money



(i) Long Term, Out-the-money

Figure 2.4: SE of Estimates based on generated data by SV model

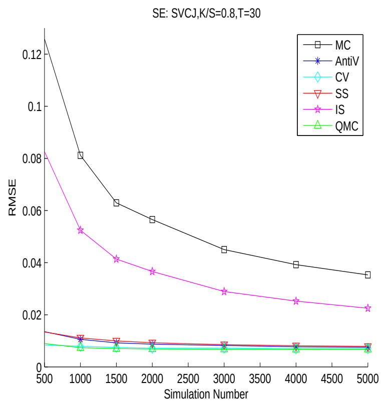


(a) Short Term, In-the-money (b) Short Term, At-the-money (c) Short Term, Out-the-money

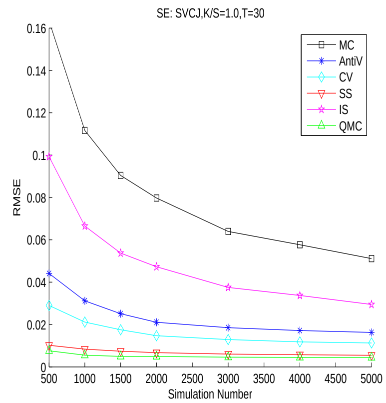
(d) Medium Term, In-the-money (e) Medium Term, At-the-money (f) Medium Term, Out-the-money

(g) Long Term, In-the-money (h) Long Term, At-the-money (i) Long Term, Out-the-money

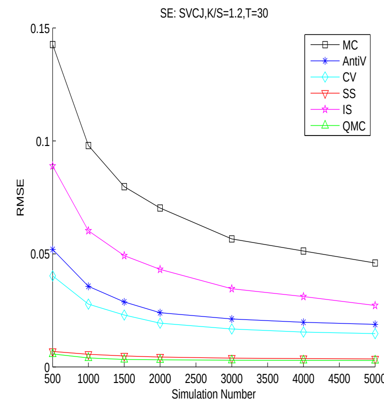
Figure 2.5: SE of Estimates based on generated data by Jump model



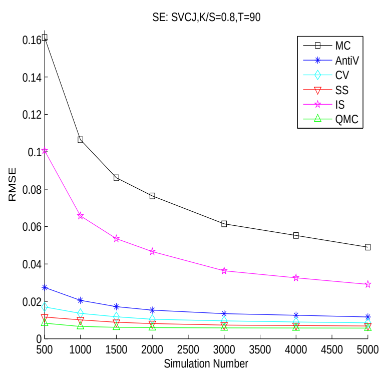
(a) Short Term, In-the-money



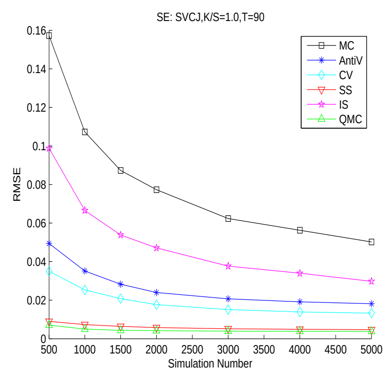
(b) Short Term, At-the-money



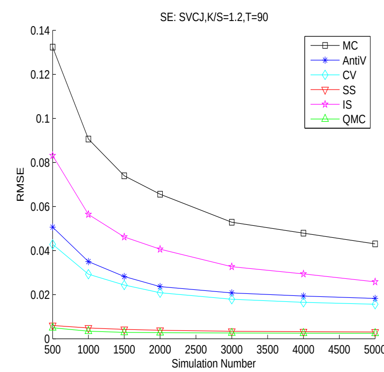
(c) Short Term, Out-the-money



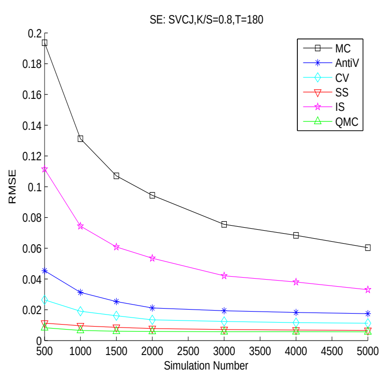
(d) Medium Term, In-the-money



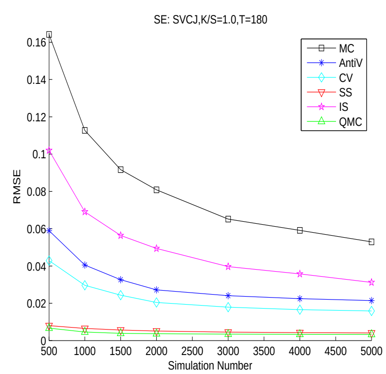
(e) Medium Term, At-the-money



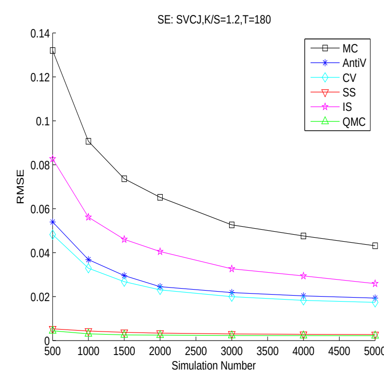
(f) Medium Term, Out-the-money



(g) Long Term, In-the-money

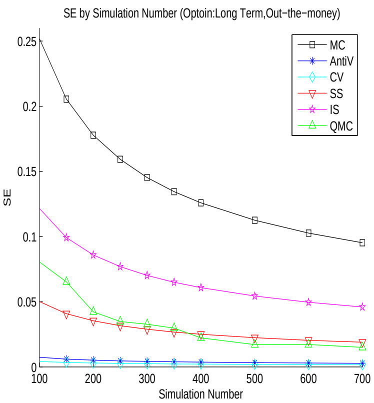


(h) Long Term, At-the-money

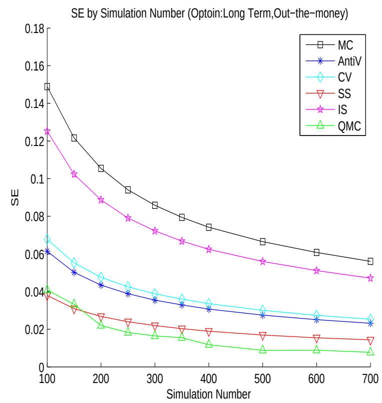


(i) Long Term, Out-the-money

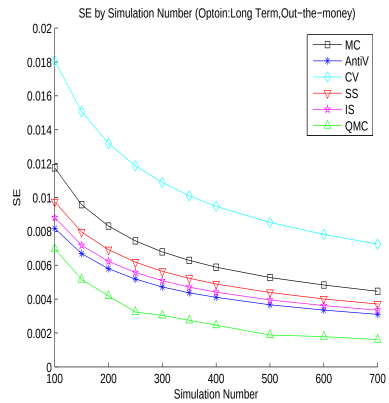
Figure 2.6: SE of Estimates based on generated data by SVCJ model



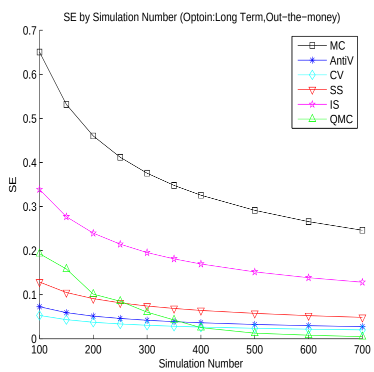
(a) Short Term, In-the-money



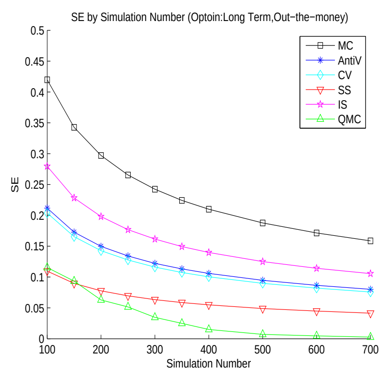
(b) Short Term, At-the-money



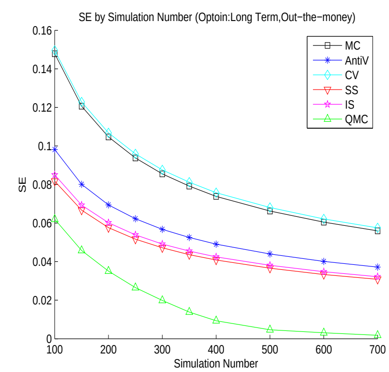
(c) Short Term, Out-the-money



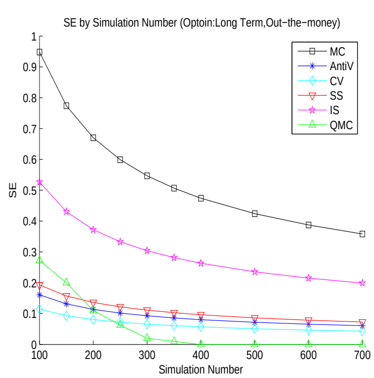
(d) Medium Term, In-the-money



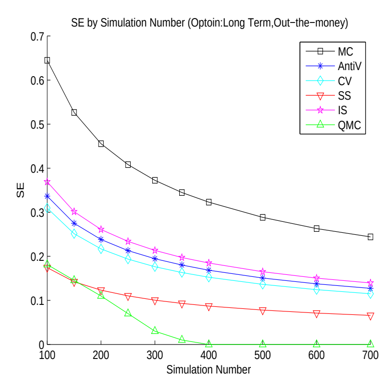
(e) Medium Term, At-the-money



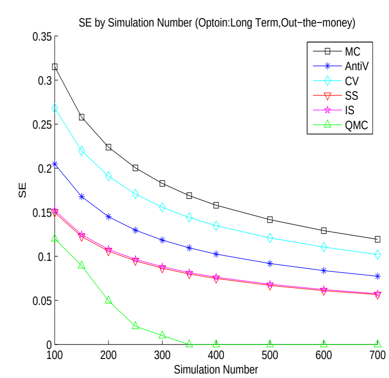
(f) Medium Term, Out-the-money



(g) Long Term, In-the-money

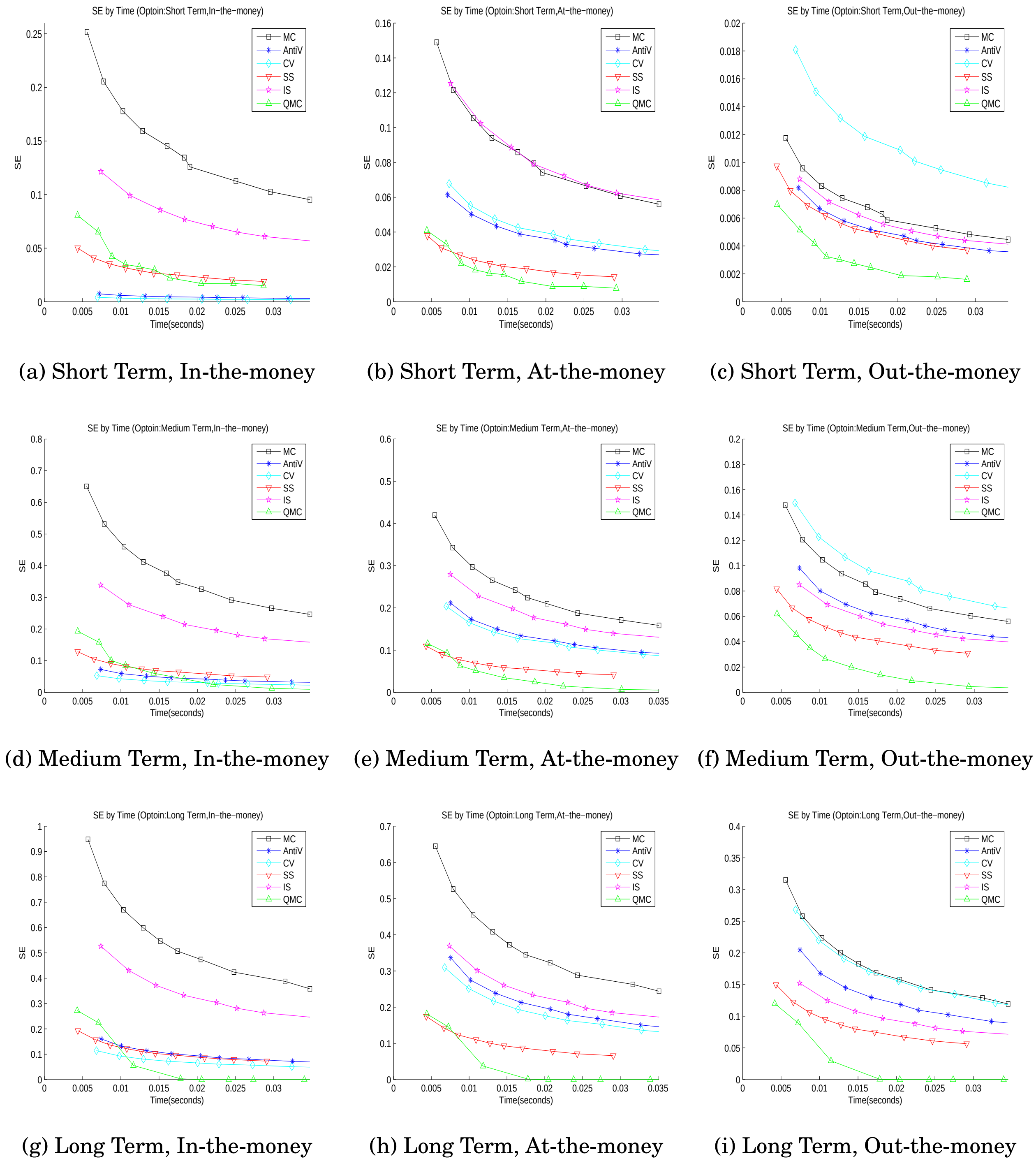


(h) Long Term, At-the-money



(i) Long Term, Out-the-money

Figure 2.7: SE by Simulation Number of Samples on Call Option of S&P500

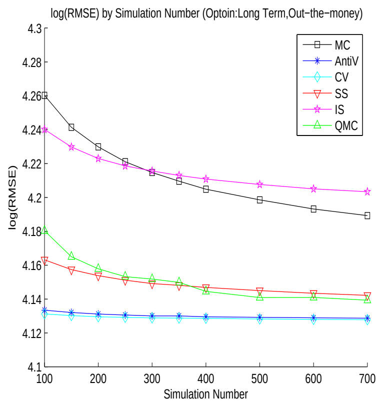


(a) Short Term, In-the-money (b) Short Term, At-the-money (c) Short Term, Out-the-money

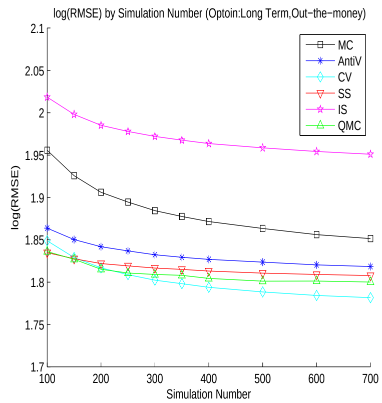
(d) Medium Term, In-the-money (e) Medium Term, At-the-money (f) Medium Term, Out-the-money

(g) Long Term, In-the-money (h) Long Term, At-the-money (i) Long Term, Out-the-money

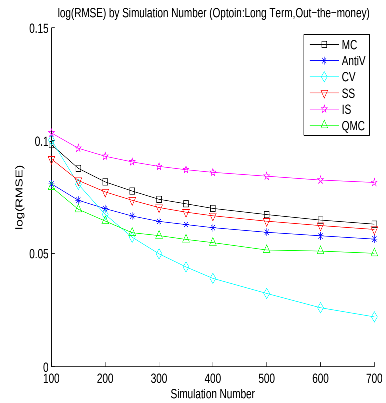
Figure 2.8: SE by Time on Call Option of S&P500



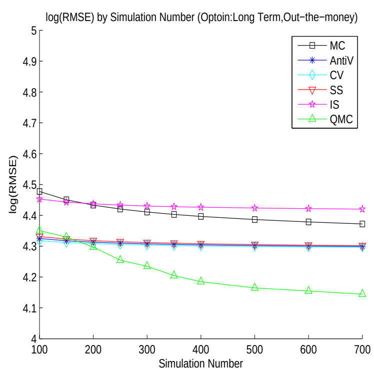
(a) Short Term, In-the-money



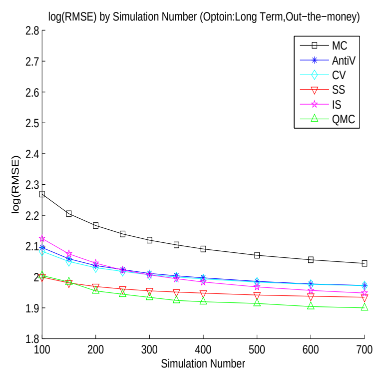
(b) Short Term, At-the-money



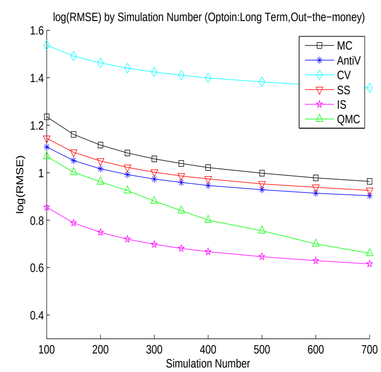
(c) Short Term, Out-the-money



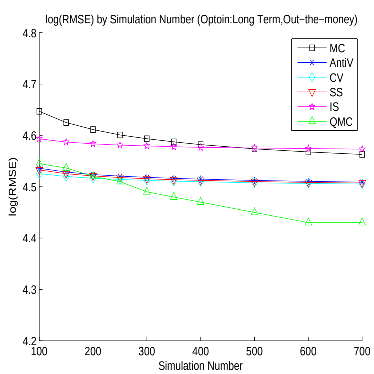
(d) Medium Term, In-the-money



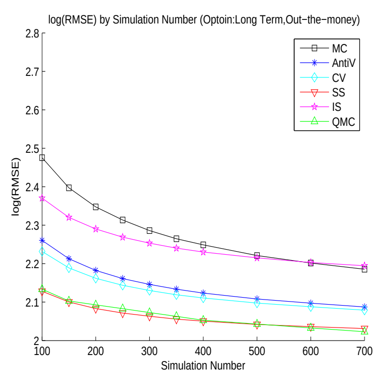
(e) Medium Term, At-the-money



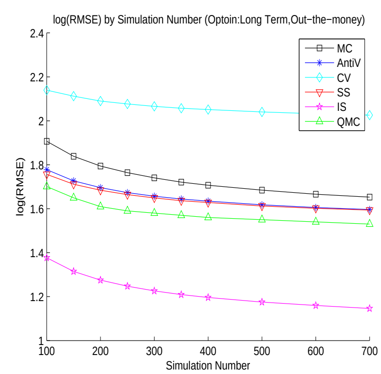
(f) Medium Term, Out-the-money



(g) Long Term, In-the-money

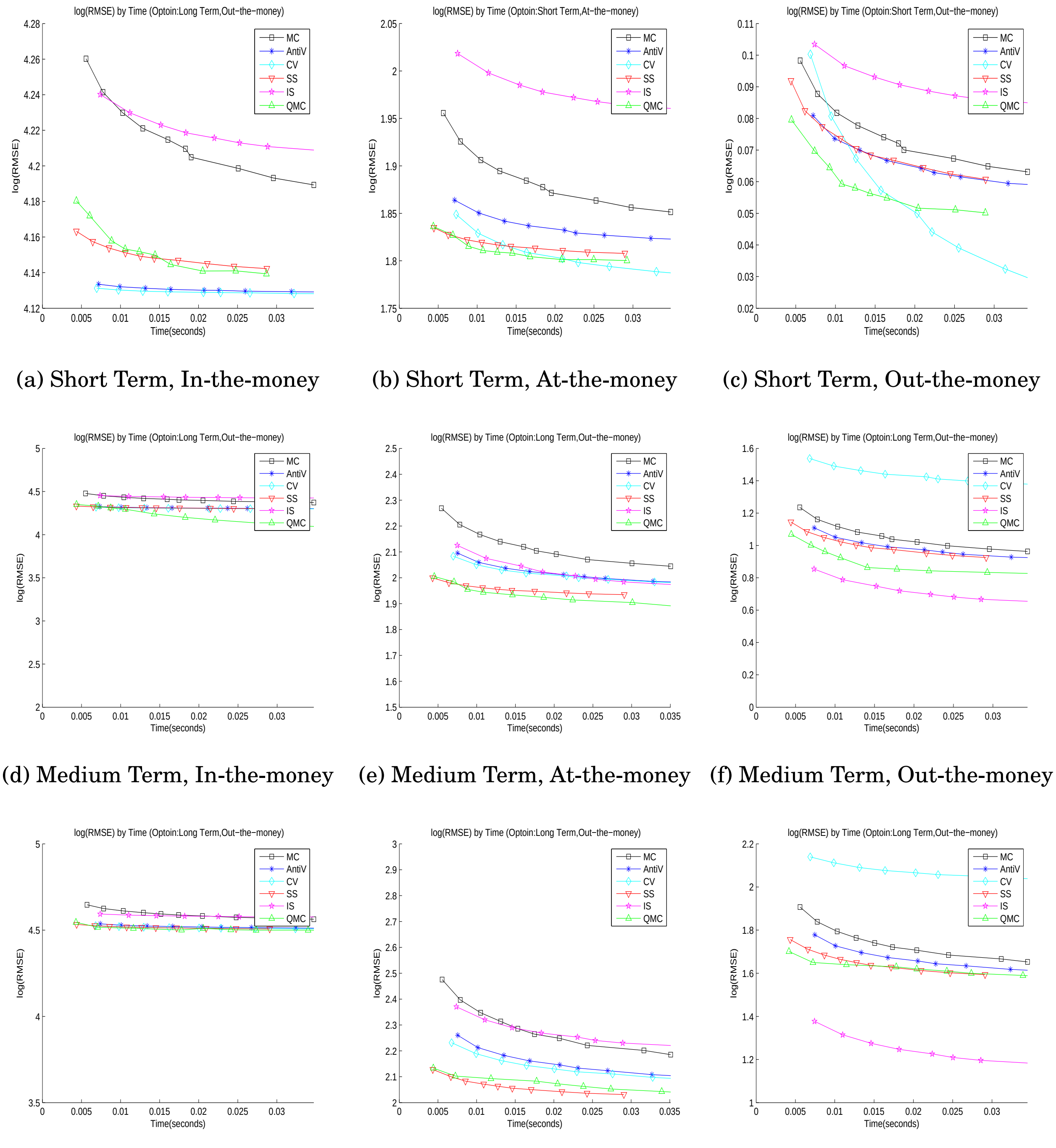


(h) Long Term, At-the-money



(i) Long Term, Out-the-money

Figure 2.9: RMSE by Simulation Number of Samples on Call Option of S&P500



(a) Short Term, In-the-money (b) Short Term, At-the-money (c) Short Term, Out-the-money

(d) Medium Term, In-the-money (e) Medium Term, At-the-money (f) Medium Term, Out-the-money

(g) Long Term, In-the-money (h) Long Term, At-the-money (i) Long Term, Out-the-money

Figure 2.10: RMSE by Time on Call Option of S&P500

Chapter 3

The effects of the Use of Realized Volatility on Volatility Trading Strategies

3.1 Introduction

Technical analysis has been through constant development and enhancement in recent years, with an increasing emphasis on computer-assisted techniques. In this study, we investigate the volatility trading strategy, in particular, the Delta Neutral strategy, with the realized volatility forecasting as well as option pricing models. Also, we compare the effectiveness of applying realized volatility to the trading strategy and that of the implied volatility. Through the study, we hope it will test the realized volatility performance in the volatility trading strategy and produce a comparison with that of implied volatility.

The Efficient Market Hypothesis (EMH) asserts that it is impossible to consistently outperform the market by using any information that the market already knows and there is no arbitrage. However, some studies have proved that this predictability is economically meaningful. For example, Engel, Kane and Noh(1994) calculated the accumulated profit/loss of each agent from the maturity of each traded options to enumerate the economic value of the volatility forecast algorithms used by the agent on the NYSE index options. If we need to prove that the volatility trading strategy used can in actuality lead to extra profits, the transaction costs must be taken into consideration. Studies that include the consideration of transaction costs yield different results. The study that Black-Scholes(1972), Galai(1976), Shastri & Tandon(1986) and Harvey & Whaley(1991) conducted shows that the inclusion of transaction costs does not cause the positive excess profits to be significantly different from zero. On the other hand, some other papers such as Guo's (1999) study, which used stochastic volatility forecasts on dynamic volatility trading strategies in the currency option market, concluded that with the use of Implied Stochastic Volatility Regression (ISVR) method, observed profits could be different from zero in specific trading strategies in consideration of transaction costs. The implied volatility has been shown to be a conditionally biased predictor of the realized volatility across asset markets. But research on the use of realized volatility in the options trading strategy is rare. Informed by the previous studies, we will compare the application of realized volatility with implied volatility on the same trading strategy in option market.

Volatility is one of the most important drivers of the securities prices. It was assumed to be a constant driver in studies such as Black-Scholes(1973) and Merton (1973), but by now it is well understood that the volatility of most financial returns is time-varying. The Generalized Autoregressive Conditional Heteroskedastic (GARCH) framework and continuous time Stochastic Volatility (SV) framework are extensions to the constant volatility framework which allows for time-varying volatility. Several models (e.g. Heston (1993); Hull-White(1987); Johnson-Shanno(1987)) can derive option pricing in a continuous time world. However, when applied in practice, only return data has been available and volatility is treated as an unobserved state variable.

The use of high frequency intraday data for estimating daily stock return volatility has drawn a lot of attention because they have certain advantages in capturing the stylized facts of asset returns compared to the use of lower frequency daily data. The underlying idea is to use sums of squared returns at high frequency to estimate volatility at a lower frequency. This idea was formalized as the Realized Volatility(RV). Volatility estimates are used as a risk measure in many asset-pricing models as Black-Scholes(1973) and its extensions, and volatility appears in option pricing formulas derived from these models. This study firstly calculates RV and IV. Secondly, it implements RV and IV forecast models. Thirdly, it applies the forecasted RV and IV to the pricing model for the purpose of obtaining the theoretical option prices.

Finally, the theoretical option prices can be applied to option trading strategies such as the delta-neutral or the straddles trading .

The increased availability of complete transaction and quote records from high frequency data strengthens the ability to obtain more additional information. There are, however some difficulties in practice. For example, the underlying asset price can not be continuously observed; they are observed occasionally instead. Thus the observation frequency is hard to choose but it's key to calculate the volatility. If the frequency is too high, many properties of volatility tend to disappear. On the other hand, if it is too low, the measurement is subject to big errors and the advantage of using high frequency data is lost. Hence, we use the 5 minute data in this study. Furthermore, the recorded prices do not reflect direct observations of a frictionless price process. A lot of components such as bid-ask spread, different prices quoted by different market makers due to heterogeneous beliefs and inventory positions, etc. are referred to as market microstructure effects. There is still a lot of work that can be done on how to reduce the microstructure noise to improve the model and the trading strategy.

The ideal data is from the options on stocks such as IBM or options on S&P 500 index or on the S&P 100 index, since these options have high volume and high open interest which mean high liquidity. If the market is not liquid, it is difficult to find a buyer to close the position, and the extra cost of illiquid market will reduce the profit. Also, applying RV in pricing model does not work well in illiquid market since the observation frequency is not high enough. In this study, without the ideal data, we simulate the option prices and the underlying stock prices according to the assumptions and presetting price model. We then use this simulated data to test and compare the realized volatility and implied volatility measure, prediction, option pricing and application in trading strategy.

We use the delta-neutral trading strategy which is one of the most popular dynamic volatility trading strategies. For the option pricing model, we use the Hull and White (1987a) rather than Black-Scholes(1973) since the latter assumes constant volatility. For the purpose of simplicity, we only use the European style call and put options.

This paper is structured as follows: Section 3.2 describes the option pricing model; Section 3.3 reviews how to measure and predict the implied volatility; Section 3.4 introduces the measure and prediction of realized volatility; Section 3.5 presents the ideal data and the process to simulate the data; Section 3.6 reveals the details of the trading strategy; and Section 3.7 concludes.

3.2 Option pricing model

Since the stock return volatility is stochastic rather than constant, the Black-scholes(1973) model, which assumes constant volatility, is not suitable. Recognizing the many already existing articles about the stochastic volatility models, for this study, I used the Hull and White (1987) model to compute the option prices.

The main assumptions in Hull-White(1987) model are: A1, The market is frictionless, and the trading is continuous in time. There are no transaction costs, taxes or short sale restrictions. In this study, we also consider the situation that transaction costs are included. A short sale is a sale of a security by an investor who does not own the asset. It is generally used to profit from an expected downward price movement, to provide liquidity in response to buyer's demand or to hedge the risk of a long position in the same security. This study fixes everyday's investment as $100 and avoids considering the short sale. A2, The stock price is instantaneously uncorrelated with the volatility. A3, The correlation between the instantaneous change rate of volatility and the change rate of aggregate consumption is constant and can be accommodated.

If the stock price is S_t and its instantaneous variance is V_t, under the above assumptions, the asset price can be described in the following stochastic processes which has been discussed in section 2.2.2.

$$dS = \phi S dt + \sigma S dw \tag{3.1}$$

$$dV = \mu V dt + \xi V dz \tag{3.2}$$

In Hull-White model, the security $f(S, \sigma^2, t)$ is the present value of the expected terminal value of f discounted at the risk free rate, thus the price of the option is:

$$f(S_t, \sigma_t^2, t) = e^{-r(T-t)} \int f(S_T, \sigma_T^2, T) p(S_T | S_t, \sigma_t^2) dS_T \tag{3.3}$$

Where T is the time at which the option matures, S_t is the security price at time t, σ_t is the instantaneous standard deviation at time t, and $p(S_T|S_t, \sigma_t)$ is the conditional distribution of S_T given the security price and variance at time t. $\overline{V} = \frac{1}{T-t} \int_t^T \sigma_\tau^2 d\tau$ denotes the mean variance over the life of the derivation security. And the price can be written as

$$f(S_t, \sigma_t^2, t) = \int \left[e^{-r(T-t)} \int f(S_T) g(S_T | \overline{V}) dS_T \right] h(\overline{V} | \sigma_t^2) d\overline{V} \tag{3.4}$$

where h is the conditional distribution of $\overline{V}$. The inner integral produces the Black-Scholes price.

It is assumed that the correlation $\rho = 0$ and μ and ξ are independent of $S(t)$, then the Hull and White price can be seen as the integral of the Black-Scholes price over the conditional distribution of mean variance $\overline{V}$ and in Hull and White(1987) model:

$$f^{HW}(S_t, \sigma_t^2) = \int f^{BS}(\overline{V}) h(\overline{V} | \sigma_t^2) d\overline{V} \tag{3.5}$$

The Black-Scholes European call and put prices are defined as:

$$f_{call}^{BS} = SN(d_1) - Xe^{-rT}N(d_2) \tag{3.6}$$

$$f_{put}^{BS} = Xe^{-rT}N(-d_2) - SN(-d_1) \tag{3.7}$$

Where

f_{call}^{BS} = price of the call option

f_{put}^{BS} = price of the put option

S = price of the underlying stock

X = option striking price

r = risk-free interest rate

T = current time until expiration, current time is set to zero

N = the cumulative normal density function

$d_1 = \frac{ln(S/X)+(r+\sigma^2/2)T}{\sigma T^{1/2}}$

$d_2 = d_1 - \sigma T^{1/2} = \frac{ln(S/X)+(r-\sigma^2/2)T}{\sigma T^{1/2}}$

σ = standard deviation of stock returns

By expanding Black-Scholes price $f^{BS}(\overline{V})$ from its expected average variance $E(\overline{V})$ in a Taylor series, Hull-White also proposes a power series approximation technique to get Hull-White price f^{HW} as:

$$\begin{aligned} f^{HW}(S_t, \sigma_t^2) &= f^{BS}(E(\overline{V})) + \frac{1}{2}\frac{\partial^2 f^{BS}(E(\overline{V}))}{\partial \overline{V}^2}E(\overline{V}^2) \\ &\quad + \frac{1}{6}\frac{\partial^3 f^{BS}(E(\overline{V}))}{\partial \overline{V}^3}E(\overline{V}^3) + ... \end{aligned} \tag{3.8}$$

Where $E(\overline{V}^2)$ and $E(\overline{V}^3)$ are the second and third central moments of $\overline{V}$. The result of equation (3.8) depends on the parameters μ and ξ. Assuming μ is zero and by the moments for the distribution of $\overline{V}$, the price is:

$$
\begin{aligned}
f^{HW}(S,\sigma^2) &= f^{BS}(\sigma^2) \\
&+ \frac{1}{2}\frac{S\sqrt{T-t}N'(d_1)(d_1d_2-1)}{4\sigma^3} \times \left[\frac{2\sigma^4(e^k-k-1)}{k^2} - \sigma^4\right] \\
&+ \frac{1}{6}\frac{S\sqrt{T-t}N'(d_1)[(d_1d_2-3)(d_1d_2-1)-(d_1^2+d_2^2)]}{8\sigma^5} \\
&\times \sigma^6\left[\frac{e^{3k}-(9+18k)e^k+(8+24k+18k^2+6k^3)}{3k^3}\right] + ...,
\end{aligned} \tag{3.9}
$$

where $k = \xi^2(T-t)$, the suggested value of μ is zero, k is sufficiently small and ξ is from 1 to 4. From Hull and White(1987), $\xi = 1$ leads to the least bias when pricing the options with stochastic volatilities. So we take $\xi = 1$ when generating the stock prices and option prices in this study.

3.3 Implied volatility and its prediction

Guo(1999) predicted the daily volatilities for the currency exchange rate with the Implied Stochastic Volatility Regression (ISVR) model and GARCH model and compared the effectiveness of using these two methods on the trading strategy. One of the conclusions in Guo(1999) is that using GARCH method on delta-neutral trading strategy can not get significant non-zero economic profits if the transaction cost is considered but the ISVR model can. In this study, in order to see the performance of applying the Realized Volatility on trading strategy, we follow the methods in Guo(1999) to measure and predict the implied volatility which is used as benchmark.

3.3.1 Implied Volatility Estimation

If the option market is informationally efficient, then the market prices of options should reflect the market expectation of future volatility. Rather than to guess the value of volatility parameter in option pricing model like the Black-Scholes model, the alternative approach is to insert the actual market option price into the pricing model and let the formula tell what the volatility should be. The volatility obtained in this way is the implied volatility.

Let $\widehat{\overline{V}}_t = E(\overline{V}|\Omega_t)$ denote the daily average variance which can be estimated by NLS(nonlinear least square), where V_t is σ_t^2 in the Black-Scholes model. In this study, I use the Black-scholes model described as equation(3.6) and Hull-White(1987) model described as equation(3.9) separately according to the methods of simulating the option prices. Therefore, $\widehat{\overline{V}}_t$ is to minimize the distance between the observed market price and the theoretical option price got from the Hull-White(1987) model as:

$$\min_{\widehat{\overline{V}}_t} SSE(\widehat{\overline{V}}_t) = \sum_i [f_{t,i} - f_{t,i}^{model}]^2 \tag{3.10}$$

The risk free rate, the strike price and spot price ratio $(X/S)_{t,i}$ (0.8 to 1.2) and the option's remaining maturity time T are all given. i is the index over observations in day t, $f_{t,i}$ is the observed option price from market, and $f_{t,i}^{model}$ is the theoretical option price either from the Black-Scholes or the Hull-White(1987) model.

3.3.2 Implied Volatility Regression

We predict one period ahead of implied volatility based on the implied stochastic volatility regression method. In this regression formula, the implied volatilities are regressed over lagged implied volatilities of put options and call options as well as two dummy variables for Monday and Friday which are important for the weekend effect. It is believed that the implied volatility is higher on Fridays than on Mondays because the market is closed over the weekend, which increases the uncertainty. It's obvious that in my simulated data, the weekend effect can be ignored, but it is important in the real data.

The following equations are used to predict the one-period ahead volatility for call option and put option:

$$\Delta V_{C,t} = \alpha_0 + \alpha_1 D_{t,1} + \alpha_2 D_{t,5} + \sum_{i=1}^{3} \beta_i \Delta V_{P,t-i} + \sum_{i=1}^{3} \gamma_i \Delta V_{C,t-i} + \epsilon_t \tag{3.11}$$

$$\Delta V_{P,t} = \alpha_0 + \alpha_1 D_{t,1} + \alpha_2 D_{t,5} + \sum_{i=1}^{3} \beta_i \Delta V_{P,t-i} + \sum_{i=1}^{3} \gamma_i \Delta V_{C,t-i} + \epsilon_t \tag{3.12}$$

Where $\Delta V_{C,t}$ and $\Delta V_{C,t}$ are the changes for one day call and put option implied volatilities. The regression uses first differences here because in many cases, the series can be transformed from nonstationary to stationary by taking the first difference. $D_{t,1}$ and $D_{t,5}$ are the dummy variables for weekend effects. In order to get the out of sample prediction, we use data of the previous year to estimate the parameters. The result shows that, for both call and put options, α_0 , α_1 and α_2 are not significant. For call/put option, the one period lagged implied volatility change of call/put option makes the greatest contribution. The parameter values obtained from 800 simulations are shown in Table 3.2 in appendix.

3.4 Realized volatility and its prediction

As research showed, implied volatility is a conditionally biased predictor of realized volatility across asset markets. For example, Neely(2004) explains the bias in the market for options on foreign exchange futures. The intuition is that using realized volatility for option pricing can avoid the bias created by the implied volatility. However, there is model free IV in the literature which can reduce the bias, and it will be interesting to compare the RV and the model free IV in the future research.

3.4.1 Realized volatility

Since volatility can not be directly observed, a lot of research has focused on how to estimate it. However, either the approach to get it with the statistical model such as ARCH or Stochastic Volatility, or the approach to link the information to the volatility of the underlying asset depend on the specific assumptions of the models. Thus, it has the advantage to use the model free measuring approach such as realized volatility which is the sample of variance of returns.

It is important to choose the observation frequency of the time series. If the frequency is too high, some properties of volatility tend to disappear such as the leverage effect and the volatility clustering. However, if it is too low, the measure is subject to errors. In this study, we choose the frequency as each 5 minutes.

I use the daily squared return as the indicator of volatility and calculate it with intraday high-frequency returns. Let $S_{n,t}$ denote the time $n \geq 0$ stock price at day t. The logarithmic returns with N observations per day are defined as

$$r_{n,t} = ln(S_{n,t}) - ln(S_{n-1,t}) \tag{3.13}$$

Where $n = 1, ..., N$ for N observations in one day and $t = 1, ..., T$ for T days. We use the logarithm here because it is closer to normality than the series in levels. Also log transformation of realized volatility is preferred to the raw version of RV because of its superior finite sample properties, such as, the skewness of log transformed statistic is smaller than that of the raw form. We assume that returns have mean zero and to be uncorrelated; the variance and covariances of squared returns exist and are finite. The assumptions are specified as:

$$E[r_{n,t}] = 0 \ \ (A1) \tag{3.14}$$

$$E[r_{n,t}r_{m,s}] = 0 \ \forall n, m, s, t \ but \ not \ n = m \ and \ s = t \ \ (A2) \tag{3.15}$$

$$E[r_{n,t}^2 r_{m,s}^2] < \infty \ \forall n, m, s, t \ \ (A3) \tag{3.16}$$

Based on the assumptions, it has been shown in Heiko Ebens(1999) that an estimator of the daily return volatility is the sum of intraday squared returns, that is, the realized volatility is

$$RV_t^2 = \sum_{n=1}^{N} r_{n,t}^2 = \sum_{n=1}^{N} (ln(S_{n,t}) - ln(S_{n-1,t}))^2 \tag{3.17}$$

and this estimator is unbiased:

$$E[RV_t^2] = \sigma_t^2 \tag{3.18}$$

There are several stylized facts about realized volatility. First, there is a long memory in the data because the autocorrelation function is dying out at a hyperbolic rate rather than exponential. Second, the distribution of logarithm of realized volatility is close to Gaussian. Third, the distribution in levels is right skewed and leptokurtic.

3.4.2 Realized Volatility Forecast

Heterogenous Autoregressive Realized Volatility Model

A popular model is the Heterogenous Autoregressive Realized Volatility model (HAR) from Corsi(2004). The HAR model is a component model containing daily, weekly and monthly realized volatility components. Although HAR has a short memory, it is proved to have the most important properties. The HAR model is described as:

$$\sqrt{RV_t^d} = \alpha_0 + \alpha_1\sqrt{RV_{t-1}^d} + \alpha_2\sqrt{RV_{t-1}^w} + \alpha_3\sqrt{RV_{t-1}^m} + \epsilon_t \tag{3.19}$$

where RV_t^d is the daily realized variance, $RV_{t-1}^w = \frac{1}{5}\sum_{i=1}^{5} RV_{t-i}$ is the weekly realized variance, $RV_{t-1}^m = \frac{1}{22}\sum_{i=1}^{22} RV_{t-i}$ is the monthly realized variance, and in the simulated data of this study, we set the data in one month as 20 for calculating convenience. The error term ϵ_t is a white noise process.

The logarithmic version of HAR model proposed by Andersen et al.(2005) is

$$ln(RV_t^d) = \alpha_0 + \alpha_1 ln(RV_{t-1}^d) + \alpha_2 ln(RV_{t-1}^w) + \alpha_3 ln(RV_{t-1}^m) + \epsilon_t \tag{3.20}$$

For this study, we use the logarithmic version of HAR model and adopted OLS to estimate and forecast the realized volatility. The parameters estimated by the regression for 800 simulations are shown in appendix Table 3.2.

3.4.3 Using the Predicted Volatility to Price the Options

We used the models described above to predict the implied volatility and realized volatility separately and applied the Hull-White (1987) model to get the theoretical option prices.

The predicted prices of both calls and puts from realized and implied volatility are all on average higher than the simulated prices. I use the following model to find the difference between predicted price and observed market price.

$$ln(f_t) = \alpha_0 + \alpha_1 ln(\widehat{f_t}) + \epsilon_t \tag{3.21}$$

where f_t is the simulated option price for calls or puts, and $\widehat{f_t}$ is the theoretical prices obtained from Hull-White model with the use of implied volatility or realized volatility. We use the log form to improve the forecast evaluation. Table 3.3 in the appendix shows the mean squared error for the regressions for call and put options based on implied volatility model and realized volatility model after adjustment. On the basis of MSE, RV does better in predicting the call option price but IV does better in forecasting the put option.

From this table, we can see that, for the call option using implied volatility, α_1 is positive and less than one and α_0 is positive. This means that the model underpredicts low priced call options and overpredicts high priced call options. The situations are the same for call and put options priced using realized volatility. But for put options using IV , α_1 is bigger than one and α_0 is negative which means this model tends to overpredict high priced put options.

3.5 Data

3.5.1 Simulation of Stock and Option Price

Before the availability of ideal stock price and options prices, it is meaningful to simulate them according to the assumptions I made for the pricing models and trading strategies. In this study, we set the replications number as 800.

Stock Price Simulation

It is necessary to simulate the underlying stock price to determine the correct option price and to estimate the premium of an option. The stock price is dependent upon the drift rate which is the expected return of the stock, the variance of the stock price and the interest rate. One property is that the average holding period return on one stock tends to increase over time.

Simulating based on the Black-Scholes model's assumptions(constant volatility)

It is assumed that the stock prices follow a (continuous time) geometric Brownian motion:

$$dS = \phi S dt + \sigma S dW \tag{3.22}$$

where,

S = the current stock price

ϕ = the expected return

σ = the stock return volatility

W = Brownian Motion process

$dW = \epsilon(dt)^{0.5}$, ϵ is the standard normal distributed random variable, i.e, $\epsilon \sim N(0,1)$

To get the continuous time stock price, we can solve the SDE in equation (3.22). Let $g(t,S) = lnS$, by $It\hat{o}$'s lemma, we have

$$\begin{aligned} dg(t,S) &= \left[\frac{\partial g(t,S)}{\partial t} + \phi(t,S)\frac{\partial g(t,S)}{\partial S} + 0.5\sigma^2(t,S)\frac{\partial^2 g(t,S)}{\partial S^2}\right] dt \\ &\quad + \sigma(t,S)\frac{\partial g(t,S)}{\partial S} dW \end{aligned} \tag{3.23}$$

where $\phi(t,S) = \phi S$ and $\sigma(t,S) = \sigma S$, thus

$$d(lnS) = [\phi - 0.5\sigma^2]dt + \sigma dW \tag{3.24}$$

Integrate on both sides, we have

$$lnS(t) = lnS(0) + (\phi - 0.5\sigma^2)t + \sigma W(t) \tag{3.25}$$

now the stock price can be described as

$$S(T) = S(t)exp\{(\phi - 0.5\sigma^2)(T-t) + \sigma\sqrt{T-t}\epsilon\} \tag{3.26}$$

where $W(T) - W(t)$ is replaced by $\epsilon\sqrt{T-t}$, ϵ is the standard normal as defined above.

We used Monte Carlo method to simulate the random trials in the process. Through a Monte Carlo simulation, we obtain the stock price as a sample average. We simulate the stock prices with $20 as the primary value. We then discard the first 1000 periods of data in order to eliminate the impact of the value of the primary stock price. Assuming that the stock market opens 250 days each year, starting from 9:00 in the morning and closing at 5:00 in the afternoon, the number of trials for each day is 96. The $\triangle t$ in this equation is $1/(250 * 96)$ since the working days in one year is set to 250 days and we want to get the stock prices for every 5 minutes. We take the return of the IBM stock as a reference and set the annualized ϕ to 0.07 and σ to 0.29.

Simulating based on the Hull-White(1987) model's assumptions(stochastic volatility)

Rather than assuming the volatility is a constant, the Hull-White(1987) model takes the stochastic volatility which is described in equations (3.1) and (3.2). Because it is difficult to get the analytical solution for the SDE, Monte Carlo simulation can be used to get the numerical solution according to the following equations:

$$S_i = S_{i-1}exp(\phi - \frac{V_{i-1}}{2})\Delta t + u_i\sqrt{V_{i-1}\Delta t} \quad (3.27)$$

$$V_i = V_{i-1}exp(\mu - \frac{\xi^2}{2})\Delta t + v_i\xi\sqrt{\Delta t} \quad (3.28)$$

Where ϕ is the annualized interest rate which is set to 0.07 and μ is set to 0. The time interval $T - t$ is separated to n subintervals and $\Delta t = (T - t)/n$. i is the index where $1 \leq i \leq n$. u_i and v_i are sampled from independent standard normal distributions. V_0 can be obtained from $V_0 = \sigma_0^2$ where σ_0 is also set to 0.29 following the IBM stock. The other parameters are defined as in previous section. Five paths for one year's simulated stock prices are shown in Figure 3.7.

Option Price Simulation

We also used Monte Carlo methods to simulate the prices of an European option. At maturity time t^*, the strike price is K, a call option is worth:

$$C_{t^*} = max(0, S_{t^*} - K) \tag{3.29}$$

Where in the simulation, we randomly choose the strike/stock ratios (K/S) from 0.8 to 1.2. At any earlier time t, the option value is the expected present value:

$$C_t = E[PV(max(0, S_{t^*} - K))] \tag{3.30}$$

By taking the problem as the decision of a risk neutral trader, we can modify the expected return of the stock so that it earns the risk free rate. Then we have

$$C_t = e^{-r(t^*-t)} E^*[max(0, S_{t^*} - K)] \tag{3.31}$$

where E^* is a transformation of the original expectation. We need to simulate a large number of sample values of S_{t^*} by the assumed price process and find the estimated call price as the average of the simulated values.

Here we take the stock prices simulated in previous subsection as the real stock prices for each period. We set the maturity time t^* as 30 days for now. We simulate the stock prices $S^i_{t^*}$ based on the Black-Scholes or the Hull-White(1987) model described above separately. For the first case, a set of time-T stock prices can be got directly by the following equation

$$S^{(i)}_{t^*} = S_t\, exp\left((r - 0.5\sigma^2)(t^* - t) + \sigma\sqrt{t^* - t}\, x^{(i)}\right) \tag{3.32}$$

Where $i = 1, 2, ..., n$ and n is set 1000. S_t is the stock price at time t which we take from the simulated data described in section 3.5.1. With the set of observations, $S^1_{t^*}, S^2_{t^*}, S^3_{t^*} ... S^n_{t^*}$, we can use it to estimate $E^*[max(0, S_{t^*} - K)]$ as the average of option payoffs at maturity time t^*. With the average from n simulations. In each simulation, $S^i_{t^*}$ and $V^i_{t^*}$ can be obtained following the gradual process described in equations (3.27) and (3.28). When a set of $S^i_{t^*}$ (where $i = 1, 2, ..., n; n = 1000$)are simulated, we use the average value as the approximation to the expected stock price at time t^*, and gain the option price as well.

Thus, the simulated European call option is

$$\widehat{C}_t = e^{-r(t^*-t)}\left(\frac{1}{n}\sum_{i=1}^{n} max(0, S_{t^*,i} - K)\right) \tag{3.33}$$

By a similar process I can estimate an European put option as

$$\widehat{P}_t = e^{-r(t^*-t)} \left(\frac{1}{n} \sum_{i=1}^{n} max(0, K - S_{t^*,i}) \right) \tag{3.34}$$

One set of simulation for stock prices, call option price and put option price is shown in Figure 3.2.

3.6 The volatility trading strategy: Delta Neutral

Although the Efficient Market Hypothesis is dominant in academic circles, there are many traders using trading strategies in the market. Delta neutral trading is one of the most popular strategies used in option market. In this section, we investigate if the predictable volatility changes can make significant economic profit by utilizing the delta-neutral strategy. There are also other popular strategies such as Straddles Trading Strategy which can also be used in this kind of study.

For a financial instrument, the delta is the change in value of that instrument when the price of the underlying asset (stock or index) increases by one unit, and the other influences are held fixed. The value of delta can be positive or negative according to whether the value of the financial instrument(option) increases or decreases in response to one unit increase in the asset (stock) price. A delta-neutral portfolio is one where the net delta of all components of the portfolio is zero. The significance of delta-neutral is that a small change in the price of underlying asset (stock) will have essentially no effect on the net value of the portfolio, that is, a delta-neutral portfolio is insensitive to small changes in the value of the stock that governs its components. In this study, the delta-neutral portfolio consists of selling or buying options and taking positions on holding or selling stocks. If the hedging position can be adjusted frequently, the delta-neutral trading strategy works well for the Hull and White (1987) option pricing model. According to Efficient Market Theory, the strategy should yield no extra returns, thus this study will also test this hypothesis as a lot researches already have.

We assume that the stock agent can trade at the market prices that indicate deviations from the model prices. Also we assume that the assumptions for the Hull and White (1987) model, as I mentioned in section 3.2, hold.

There are a few steps in the experiment. Firstly, we use Monte Carlo simulation method to simulate the underlying stock prices. We simulate two series of stock prices separately according to the Black Scholes model assumptions and the Hull White model assumptions. The former assumes constant volatility and the latter assumes stochastic volatility. Then, based on these two series of underlying stock prices, the Monte Carlo method is used to simulate the option prices. We take these two series of underlying stock prices and options prices as observed prices and test the trading strategy on them. In this research, we take the observed prices equally to the market prices. If we can obtain real market underlying stock prices and option prices, we would have three series of observed prices for the experiment. Secondly, we use the observed underlying stock prices to calculate realized volatility as in equation (3.17). The implied volatility is estimated based on the Hull White option pricing model as in equation (3.10). Then we predict the one period ahead of implied volatility and realized volatility as described in section 3.3.2 and 3.4.2. Thirdly, plug in the predicted implied volatility or realized volatility to Hull White model to calculate model option prices. Comparing these model option prices with the observed option prices, we have an indicator to show that the observed option prices are over estimated or under estimated and adjust our trading strategy according to the indicator. With more details, the delta-hedging trading strategy process is described as follows:

- On day t, the agent can use the volatility prediction method (IV or RV) to forecast volatility and compute the theoretical option price. The agent can change the position daily by buying the option if it is undervalued or selling it if it is overvalued.

- We assume that $100 worth of options and stocks are always bought and sold, and the agent does not reinvest the profit the next day.

- By comparing the adjusted model prices got from equation(3.21) with the simulated option prices, we find that their trends match very well but the values have almost constant differences. Thus, the next period theoretical price C_{t+1}^{T} is compared with the actual market price C_t^M. If $C_{t+1}^{T} > C_t^M$ meaning the option is underpriced in period t, the agent will buy the option, and delta-hedges the position by buying or selling the stocks. At day $t+1$, the hedged position is liquidated and the agent obtains the return.

- If $C_{t+1}^{T} < C_t^M$, the option is overpriced, the agent should sell the options, hedge the position by buying or selling stock, and invest the left over money on a risk-free asset.

It does not make a significant difference to consider whether the delta neutral trading strategy can make abnormal economic profits without considering the transaction cost. We assume the transaction cost is 1% of the option prices which include a round-trip cost of one tick plus commissions. This number is used in many papers. 2% transaction cost is also applied in this study. As the development of computer system and automatic trading platform, the transaction cost is becoming less and less, giving more chances to get profit for the trading strategies.

The return formulas for various strategies is:

A, buy call option at price C_t^A and sell stocks at price S_t, the absolute return is

$$\begin{aligned} AR_{t+1} &= n_c \cdot [(C_{t+1}^A - C_t^A) - \delta_c(S_{t+1} - S_t) - r_T \cdot (C_{t+1}^A + C_t^A)] \\ &\quad + r_F \cdot max\{100 - n_c C_t^A + n_c \delta_c S_t, 0\} \end{aligned} \tag{3.35}$$

Where δ_c is the delta of an European call on stock, r_T is the transaction cost rate , r_F is the risk free interest rate. The last part of the equation is the surplus money which can be invested at the risk free interest rate. n_c is the amount of call options bought which can be calculated by solving equations:

$$n_c \delta_c - n_s = 0 \tag{3.36}$$

$$100 - n_c C_t^A + n_s S_t = 0 \tag{3.37}$$

where n_s is the amount of stock sold. Equation (3.36) comes from the delta-neutral hedging and equation (3.37) follows the assumption that dollar 100 are used each day. Then we can get

$$n_c = \left| \frac{100}{C_t^A - \delta_c S_t} \right|$$

B, sell call option at price C_t^A and buy stock at price S_t, the absolute return is

$$AR_{t+1} = n_c \cdot [-(C_{t+1}^A - C_t^A) + \delta_c(S_{t+1} - S_t) - r_T \cdot (C_{t+1}^A + C_t^A)] \tag{3.38}$$

where

$$n_c = \frac{100}{-C_t^A + \delta_c S_t}$$

this comes from solving the following equations:

$$-n_c\delta_c + n_s = 0 \tag{3.39}$$

$$100 + n_c C_t^A - n_s S_t = 0 \tag{3.40}$$

C, buy put option and buy stock, the absolute return is

$$AR_{t+1} = n_p \cdot [(P_{t+1}^A - P_t^A) - \delta_p(S_{t+1} - S_t) - r_T \cdot (P_{t+1}^A + P_t^A)] \tag{3.41}$$

where δ_p denotes the delta of an American put option on stock, and

$$n_p = \left|\frac{100}{P_t^A - \delta_p S_t}\right|$$

D, sell put option and sell stock, the absolute return is

$$AR_{t+1} = n_p \cdot [-(P_{t+1}^A - P_t^A) + \delta_p(S_{t+1} - S_t) - r_T \cdot (P_{t+1}^A + P_t^A)] + 200 \cdot r_F \tag{3.42}$$

where

$$n_p = \left|\frac{100}{-P_t^A + \delta_p S_t}\right|$$

We set δ_c to 0.5 and δ_p to -0.5 which denote the approximate deltas for the at-the-money call and put options. $200 \cdot r_F$ is the profit by selling \$100 put option and \$100 stock and investing the \$200 on the risk free asset. If we set the initial investment as \$100, the relative return at time $t+1$ is

$$RR_{t+1} = AR_{t+1}/100 \tag{3.43}$$

Considering the transaction costs, more transaction number means higher cost. Thus, the filters are used to verify whether the profit gained from the price deviations is large enough to outpace the transactions cost. The options are only traded when the predicted price deviation is larger than the filter value to reduce the transaction number and the total transaction cost. If the value of the filter increases, the number of trades decreases. The agent can invest in the risk-free asset on no trading days by increasing the filter value and reducing the transaction number. We test the filter values from 2% to 5%.

For the real data, to investigate the performance of this trading strategy based on the realized volatility, we will also compare this return with the return of the 1 year U.S. Treasury Bill and with that of the S&P 500 index and S&P 100 index.

We calculate the Sharp Ratio which is one of the most important risk/return measures. This ratio describes the excess return for the extra volatility. Higher value means better risk-return trade off, that is, lower market risk and higher returns. The Sharp Ratio is defined as

$$S(x) = (r_x - r_f)/StdDev(r_x) \tag{3.44}$$

where x is the investment, r_x is the average rate of return of x, r_f is the return rate of a risk-free security and $StdDev(r_x)$ is the standard deviation of r_x.

We replicate the trading process 800 times in this study. Table 3.5 to 3.7 show that when generating stock prices and option prices, and analyzing the data with the same price model, Hull-White(1987) model, IV does better than RV in most cases, but the difference is not big. It demonstrates that the implied volatility model can obtain more information from the data. Also,for both IV and RV models, when the transaction costs rise from 0 to 1%, 2% and 5%, the profit declines accordingly but not significantly. The profits also decline when the filter is raised and the transaction number is less.

However, when we generate the data with the basic Black-Scholes model but analyze them with the Hull-White model, there is a bias between the two models and the results are different. Tables 3.8 to 3.10 show the profits of the trading strategies based on implied volatility prediction and realized volatility forecasting. Without transaction costs, both IV model and RV model can make profits on all the filters from 0 to 5%. The profits based on both RV and IV models are very close. When trading on both call options and put options, RV model dominates IV model a little on all filters except 0.05 filter. Considering the Sharp Ratio, RV does better in most of the cases.

When taking 1% transaction cost,which is closer to the real trading than zero transaction cost, the situation is similar to no transaction cost. The benefits from both models are close.

For the 2% transaction cost, when trading on both call and put options, IV is slightly better except the 0.02 filter. Only when trading on the call options, do both models make negative profits, but RV model loses less than IV model. While trading on the put options only, RV model dominates IV model significantly. For all filters, IV model can not make profit but RV model makes positive profit and the differences are significant. Also the Sharp Ratios from RV model are better than that from IV model in all filters.

From the latter case, we can see that if there is a bias between the data and the pricing model, when working on the Delta-neutral trading strategy, RV forecasting model can do at least as good as the IV forecasting model, and it does better in more cases. In most cases, the Sharp Ratios created by RV model are better than that of IV model.

3.7 Conclusion

There has been a lot of literature on both theoretical and empirical work in the volatility modeling and forecasting. With the option pricing models such as Black-Scholes and Hull-White, this research can be utilized on the trading strategies. For example, Guo(1999) proposed that the use of the implied stochastic volatility regression prediction method on option pricing can dominate the GARCH method and make profit significantly different from zero in some situations. However, implied volatility is a biased predictor of the realized volatility across asset markets.

In this paper, we chose the Hull-White(1987) option pricing model since this model is based on the stochastic stock return volatility rather than constant volatility in Black-Scholes model. For the purpose of comparison, we modeled and predicted the implied volatility by regression. We used the heterogenous autoregressive realized volatility model to forecast the realized volatility. The Monte Carlo method was taken to simulate the stock price and European call and put options with the number of simulations set to 800. Based on the simulated data, we modeled and forecasted the implied volatility and realized volatility and obtained the model prices of the options with the Hull-White model. Using these predicted prices as indicators, we used the Delta-neutral strategy, which is one of the most popular volatility trading strategies, to verify the effect of the RV model.

The result of the trading strategy based on the simulated data shows that,when there is no bias between the data and the pricing model, IV does better than RV in most cases since it can obtain more information, but the difference is not large. If there is a bias between the data and the pricing model, both IV and RV can make profits when the transaction cost is less than

1%. Considering the 2% transaction cost, both RV and IV can make similar profits when trading on both call and put options. However, both RV and IV can not make profits when trading only on call options. When trading only on put options, only RV can make positive profits and the differences between two models are significant. The Sharp Ratio of the return on RV model dominates that of the IV model in most cases. This denotes that, when there is bias between data and the pricing model, using realized volatility model on dynamic volatility strategy works at least as good as implied volatility model and does much better in some cases. Since there is no model that can describe the real option prices accurately, and the bias between real data and pricing model is bigger than that in this study, RV may do better than IV in that case. It may serve as a useful tool for the technical trading analysis.

However, because this study is based on the simulated data, the actual result of utilizing the market data bears further investigation. There is model free implied volatility now, so it would be worthwhile comparing RV to this model free IV. Moreover, in this paper, I only verify the Delta-Neutral trading strategy. Thus, more volatility trading strategies should be tested in the future work.

Appendix

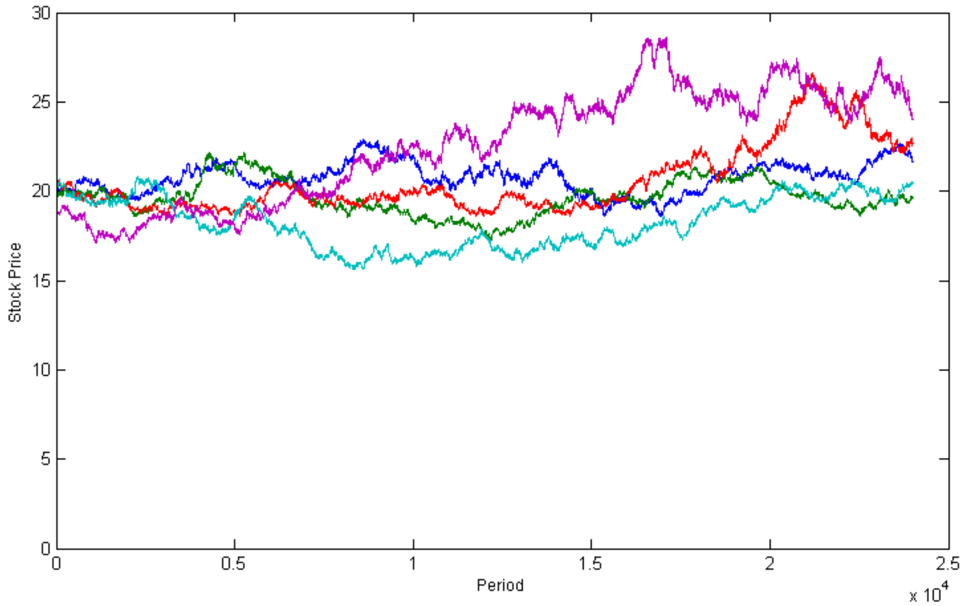


Figure 3.1: Five paths of simulated stock prices for one year

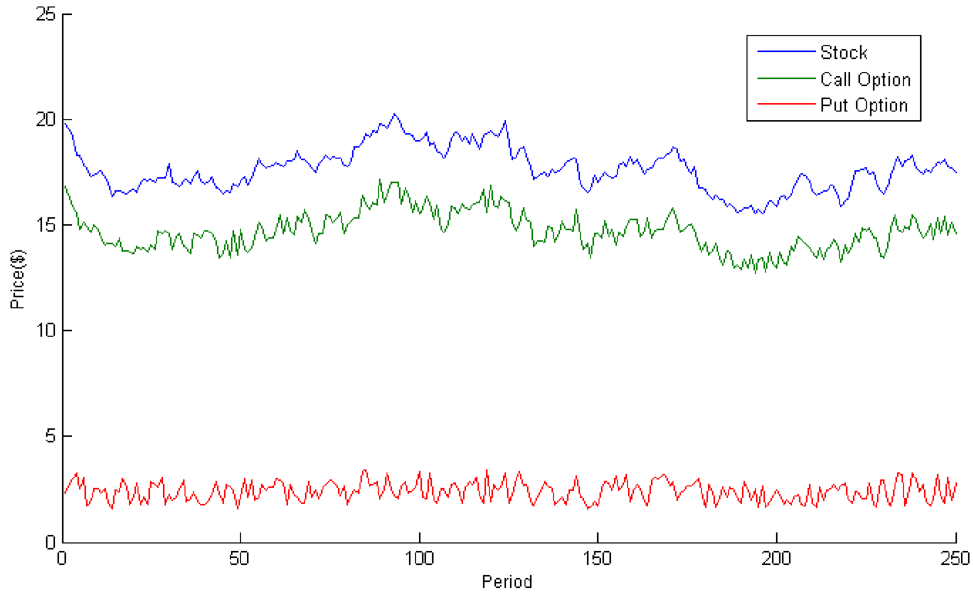


Figure 3.2: One simulation of call option and put option prices for one year

	α_0	α_1	α_2	β_1	β_2	β_3	γ_1	γ_2
Call	0.0003	-0.0022	0.0008	-0.7443	-0.4937	-0.4960	-12.6144	-31.6943
	(0.0006)	(0.0027)	(0.0013)	(0.0023)	(0.0023)	(0.0020)	(23.5406)	(37.5052)
Put	0.0000	-0.0000	-0.0000	-0.0005	-0.0007	-0.0010	-0.7353	-0.4830
	(0.0000)	(0.0000)	(0.0000)	(0.0011)	(0.0009)	(0.0007)	(0.0030)	(0.0029)

Table 3.1: Parameters values for the Implied Volatility Regression

Note:Based on 800 simulations; The standard errors are given in parenthesis;

parameter	α_0	α_1	α_2	α_3
estimate result	-0.4278	0.0500	0.3875	0.4080
	(0.0116)	(0.0026)	(0.0057)	(0.0059)

Table 3.2: Estimate result for the HAR model

Note:Based on 800 simulations; The standard errors are given in parenthesis;

	MSE	
	call	put
IV	0.1685	7.6680e-005
RV	0.1310	0.1673

Table 3.3: Mean Squared Errors in Pricing the Options by IV and RV

Note: Based on 800 simulations; Generating data with BS model and analysis with HW model

	α_0	α_1
call option, IV	0.5415(0.0434)	0.9555(0.0036)
put option, IV	-0.0022(0.0013)	1.0047(0.0004)
call option, RV	4.2096(0.0350)	0.7549(0.0036)
put option, RV	1.4737(0.0071)	0.6404(0.0043)

Table 3.4: Pricing the Option

Note: Based on 800 simulations; The standard errors are given in parenthesis; Generating data with BS model and analysis with HW model

Filter	Volatility	Return(%) for All	Sharp Ratio	Return(%) for Call	Sharp Ratio	Return(%) For Put	Sharp Ratio
0	IV	6.7091	1.4373	6.9209	1.2048	6.9191	1.2050
	RV	5.4572	1.2091	4.9829	0.8027	4.3870	1.1942
0.01	IV	6.7089	1.4372	6.9206	1.2047	6.9178	1.2049
	RV	5.4341	1.2055	4.9696	0.8007	4.3661	1.1876
0.02	IV	6.7084	1.4370	6.9200	1.2045	6.9172	1.2047
	RV	5.4071	1.2025	4.9504	0.7999	4.3445	1.1833
0.05	IV	6.7023	1.4345	6.9136	1.2025	6.9108	1.2027
	RV	5.2101	1.1610	4.7738	0.7712	4.2211	1.1402

Table 3.5: Daily Profits for Delta-Neutral Trading Strategy
(Generate and analysis data with HW model, No Transaction Cost)

Note: The returns are average in percentage; $\delta_c = 0.5$, $\delta_p = -0.5$; 'All' means trade both the call options and put options, 'Call' means only trade on call options and 'Put' means only trade on put options

Filter	Volatility	Return(%) for All	Sharp Ratio	Return(%) for Call	Sharp Ratio	Return(%) For Put	Sharp Ratio
0	IV	6.5218	1.4140	6.6058	1.1551	6.6033	1.1553
	RV	5.2808	1.1814	4.6678	0.7573	4.2479	1.1603
0.01	IV	6.5217	1.4139	6.6056	1.1550	6.6031	1.1552
	RV	5.2607	1.1789	4.6585	0.7503	4.2289	1.1547
0.02	IV	6.5214	1.4138	6.6053	1.1549	6.6027	1.1551
	RV	5.2382	1.1775	4.6458	0.7570	4.2101	1.1521
0.05	IV	6.5166	1.4121	6.6006	1.1536	6.5980	1.1538
	RV	5.0593	1.1420	4.4956	0.7338	4.0983	1.1153

Table 3.6: Daily Profits for Delta-Neutral Trading Strategy

(Generate and analysis data with HW model, 1% Transaction Cost)

Note: The returns are average in percentage; $\delta_c = 0.5$, $\delta_p = -0.5$; 'All' means trade both the call options and put options, 'Call' means only trade on call options and 'Put' means only trade on put options

Filter	Volatility	Return(%) for All	Sharp Ratio	Return(%) for Call	Sharp Ratio	Return(%) For Put	Sharp Ratio
0	IV	6.3345	1.3896	6.2907	1.1045	6.2884	1.1047
	RV	5.1045	1.1529	4.3529	0.7114	4.1088	1.1259
0.01	IV	6.3345	1.3896	6.2906	1.1045	6.2883	1.1047
	RV	5.0873	1.1515	4.3476	0.7115	4.0916	1.1214
0.02	IV	6.3344	1.3896	6.2905	1.1045	6.2882	1.1047
	RV	5.0693	1.1518	4.3412	0.7136	4.0757	1.1203
0.05	IV	6.3308	1.3887	6.2876	1.1039	6.2853	1.1041
	RV	4.9085	1.1224	4.2173	0.6957	3.9756	1.0897

Table 3.7: Daily Profits for Delta-Neutral Trading Strategy

(Generate and analysis data with HW model, 2% Transaction Cost)

Note: The returns are average in percentage; $\delta_c = 0.5$, $\delta_p = -0.5$; 'All' means trade both the call options and put options, 'Call' means only trade on call options and 'Put' means only trade on put options

Filter	Volatility	Return(%) for All	Sharp Ratio	Return(%) for Call	Sharp Ratio	Return(%) For Put	Sharp Ratio
0	IV	7.7868*	1.1740*	15.0281	1.1203*	15.0149	1.1205*
	RV	8.2809*	1.2047*	14.9382	1.1503*	4.8484	1.3212*
0.01	IV	7.7772*	1.1708*	15.0146	1.1175*	15.0014	1.1177*
	RV	8.2541*	1.1991*	14.9015	1.4449*	4.8458	1.3201*
0.02	IV	7.7271*	1.1548*	14.9413	1.1024*	14.9281	1.1027*
	RV	8.1062*	1.1664*	14.6929	1.1126*	4.8265	1.3085*
0.05	IV	7.2472	1.0722	13.4060	0.9874	13.3947	0.9877*
	RV	6.6778	0.9986	11.8211	0.8913	4.6021	1.1829*

Table 3.8: Daily Profits for Delta-Neutral Trading Strategy

(Generate data with BS model and analysis with HW model, No Transaction Cost)

Note:The returns are average in percentage; $\delta_c = 0.5$, $\delta_p = -0.5$; 'All' means trade both the call options and put options, 'Call' means only trade on call options and 'Put' means only trade on put options; the $*$ on the data means the case that RV performs better than IV

Filter	Volatility	Return(%) for All	Sharp Ratio	Return(%) for Call	Sharp Ratio	Return(%) For Put	Sharp Ratio
0	IV	6.3188*	1.1320*	6.2555	0.3918*	6.2530	0.3928*
	RV	6.5251*	1.1345*	6.1655	0.4071*	4.3782	1.1933*
0.01	IV	6.3337*	1.1428*	6.2877	0.3970*	6.2852	0.3980*
	RV	6.5656*	1.1520*	6.2499	0.4172*	4.3822	1.1968*
0.02	IV	6.3456*	1.1519*	6.3496	0.4045*	6.3470	0.4055*
	RV	6.5621*	1.1614*	6.3149	0.4305*	4.3801	1.1977*
0.05	IV	6.1493	1.0962	5.8425	0.3948*	5.8405	0.3958*
	RV	5.7716	1.0303	5.2434	0.3954*	4.2415	1.1278*

Table 3.9: Daily Profits for Delta-Neutral Trading Strategy

(Generate data with BS model and analysis with HW model, 1% Transaction Cost)

Note: The returns are average in percentage; $\delta_c = 0.5$, $\delta_p = -0.5$; 'All' means trade both the call options and put options, 'Call' means only trade on call options and 'Put' means only trade on put options; the $*$ on the data means the case that RV performs better than IV

Filter	Volatility	Return(%) for All	Sharp Ratio	Return(%) for Call	Sharp Ratio	Return(%) For Put	Sharp Ratio
0	IV	4.8509	0.9479	-2.5172	-0.3707^*	-2.5089^*	-0.3689^*
	RV	4.7692	0.8816	-2.6072	-0.3685^*	3.9081^*	1.0649^*
0.01	IV	4.8901	0.9761	-2.4391^*	-0.3619^*	-2.4309^*	-0.3601^*
	RV	4.8772	0.9256	-2.4016^*	-0.3528^*	3.9186^*	1.0723^*
0.02	IV	4.9641^*	1.0262^*	-2.2421^*	-0.3434^*	-2.2341^*	-0.3416^*
	RV	5.0180^*	1.0335^*	-2.0630^*	-0.3186^*	3.9337^*	1.0838^*
0.05	IV	5.0515	1.0826	-1.7210^*	-0.2943	-1.7138^*	-0.2925^*
	RV	4.8653	1.0298	-1.3343^*	-0.2417	3.8809^*	1.0667^*

Table 3.10: Daily Profits for Delta-Neutral Trading Strategy

(Generate data with BS model and analysis with HW model, 2% Transaction Cost)

Note: The returns are average in percentage; $\delta_c = 0.5$, $\delta_p = -0.5$; 'All' means trade both the call options and put options, 'Call' means only trade on call options and 'Put' means only trade on put options; the $*$ on the data means the case that RV performs better than IV

Filter	Volatility	Return(%) for All	Sharp Ratio	Return(%) for Call	Sharp Ratio	Return(%) For Put	Sharp Ratio
0	IV	6.7132	1.4424	6.9402	1.2106	6.9362	1.2107
	RV	7.0061*	1.2302	6.9441*	1.2083	1.0937	0.2461
0.01	IV	6.7132	1.4424	6.9403	1.2106	6.9363	1.2107
	RV	6.9995*	1.2300	6.9387*	1.2084	1.0624	0.2395
0.02	IV	6.7103	1.4424	6.9403	1.2106	6.9363	1.2106
	RV	6.9853*	1.2264	6.9253	1.2050	1.0366	0.2335
0.05	IV	6.7079	1.4404	6.9339	1.2086	6.9299	1.2087
	RV	6.8849*	1.1944	6.8269	1.1739	0.9451	0.2105

Table 3.11: Daily Profits for Delta-Neutral Trading Strategy with adjusted RV (Generate and analysis data with HW model, No Transaction Cost)

Note: The returns are average in percentage; $\delta_c = 0.5$, $\delta_p = -0.5$; 'All' means trade both the call options and put options, 'Call' means only trade on call options and 'Put' means only trade on put options

Filter	Volatility	Return(%) for All	Sharp Ratio	Return(%) for Call	Sharp Ratio	Return(%) For Put	Sharp Ratio
0	IV	6.5270	1.4192	6.6249	1.1608	6.6212	1.1609
	RV	6.6958*	1.1810	6.6288*	1.1587	0.9550	0.2204
0.01	IV	6.5270	1.4193	6.6251	1.1608	6.6214	1.1609
	RV	6.6977*	1.1844	6.6319*	1.1624*	0.9271	0.2144
0.02	IV	6.5207	1.4194	6.6253	1.1609	6.6216	1.1610
	RV	6.6919*	1.1844	6.6269*	1.1626*	0.9017	0.2089
0.05	IV	6.5231	1.4181	6.6206	1.1596	6.6109	1.1597
	RV	6.6167*	1.1626	6.5538	1.1415	0.8235	0.1935

Table 3.12: Daily Profits for Delta-Neutral Trading Strategy with adjusted RV (Generate and analysis data with HW model, 1% Transaction Cost)

Note: The returns are average in percentage; $\delta_c = 0.5$, $\delta_p = -0.5$; 'All' means trade both the call options and put options, 'Call' means only trade on call options and 'Put' means only trade on put options

Filter	Volatility	Return(%) for All	Sharp Ratio	Return(%) for Call	Sharp Ratio	Return(%) For Put	Sharp Ratio
0	IV	6.3407	1.3951	6.3095	1.1101	6.3061	1.1103
	RV	6.3855*	1.1310	6.3134*	1.1082	0.8164	0.1947
0.01	IV	6.3408	1.3952	6.3099	1.1102	6.3065	1.1103
	RV	6.3959*	1.1378	6.3251*	1.1154*	0.7918	0.1892
0.02	IV	6.3410	1.3954	6.3103	1.1103	6.3069	1.1105
	RV	6.3985*	1.1413	6.3286*	1.1190*	0.0671	0.1842
0.05	IV	6.3383	1.3949	6.3072	1.1097	6.3038	1.1098
	RV	6.3484*	1.1295	6.2807	1.1079	0.7018	0.1703

Table 3.13: Daily Profits for Delta-Neutral Trading Strategy with adjusted RV (Generate and analysis data with HW model, 2% Transaction Cost)

Note: The returns are average in percentage; $\delta_c = 0.5$, $\delta_p = -0.5$; 'All' means trade both the call options and put options, 'Call' means only trade on call options and 'Put' means only trade on put options

Chapter 4

Markovitz's Four Asset Problem, A Geometrical Analysis

4.1 Introduction

In Markowitz(2005), the three asset mean-variance portfolio optimization problem has been examined in the weight space. The paper has an example with specific values of means and variances and shows that the efficient line is limited in certain area. We want to know if it works generally and we re-examine the problem by using a general case without specific values. As there are many assets in one portfolio in reality, we extend the problem to four assets thus the weight space becomes a three-dimensional coordinate. The purpose of this research is to find the exact solution of the four-asset portfolio optimization problem, obtain the four important portfolios in this case including the minimum variance portfolio, the maximum return portfolio and two corner portfolios. Then we can describe the efficient line exactly in the weight space. Dybvig(1984) shows that with non-negativity constraints, corner portfolios are only non-differentiable when all assets have the same mean. We obtain the same result by showing that there is no kink on the efficient frontier through analysing the efficient line in weight space. Furthermore, by mapping the efficient frontier from the weight space to the mean variance space, we will know how the portfolio weights change while the efficient portfolio moves along the mean variance efficient frontier. In this study, to simplify the problem and get analytical solution, we assume that there is no correlation between assets in one portfolio. There is a lot of published work focusing on the numerical solution of the portfolio optimization problem. However, algebraic analysis can give us a better understanding of how the portfolio changes. Also we can extend the problem to study how constraints can affect the optimized portfolio.

The paper is organized as follows: Section 4.2 generalizes the three asset problem in Markowitz (2005). Section 4.3 extends the problem to four asset and obtain the exact solution. Section 4.4 presents the experiment results by Monte Carlo simulation methods. Section 4.5 concludes.

4.2 Markowitz's Three Asset Problem

We re-examine the standard Mean-Variance problem with both adding up constraints and non-negativity of the portfolio weights. Let Ω be the covariance matrix of the assets returns. We choose ω to

$$\min_{\omega} \frac{1}{\lambda}\omega'\Omega\omega - \mu'\omega \tag{4.1}$$

or

$$\min_{\omega} \omega'\Omega\omega - \lambda\mu'\omega \tag{4.2}$$

subject to

$$I'\omega = 1 \tag{4.3}$$

$$\omega_i \geq 0 \text{ for all } i \tag{4.4}$$

Our purpose is to find an explicit representation for the efficient frontier connecting the minimum variance to the maximum return portfolio and passing through the corner portfolios. To keep the problem simple, we consider a portfolio of three risky uncorrelated assets where their means and variances are such that

$$0 < \mu_1 < \mu_2 < \mu_3$$

and

$$0 < \sigma_1^2 < \sigma_2^2 < \sigma_3^2$$

In particular, we let μ_2 and μ_3 be defined as

$$\mu_2 = \mu_1 + \eta_1,\ \eta_1 > 0 \tag{4.5}$$

$$\mu_3 = \mu_1 + \eta_1 + \eta_2,\ \eta_1 > 0, \eta_2 > 0 \tag{4.6}$$

Markowitz(2005) examined this problem via a worked specific example. Our approach is to examine the problem algebraically without having to specify specific values for the means and variances. However, like Markowitz(2005), we examine the problem in "weight-space". That is we examine the two dimensional space spanned by two of the three assets weights. The weight of the third asset is found from the adding up constraint. As Markowitz(2005) pointed out, all feasible portfolios satisfying both the non-negative and the adding up constraints are contained on and within a triangle, in (ω_1, ω_2) space, with verticies $(0, 0)$, $(1, 0)$ and $(0, 1)$. Figure 4.1 illustrates this triangle.

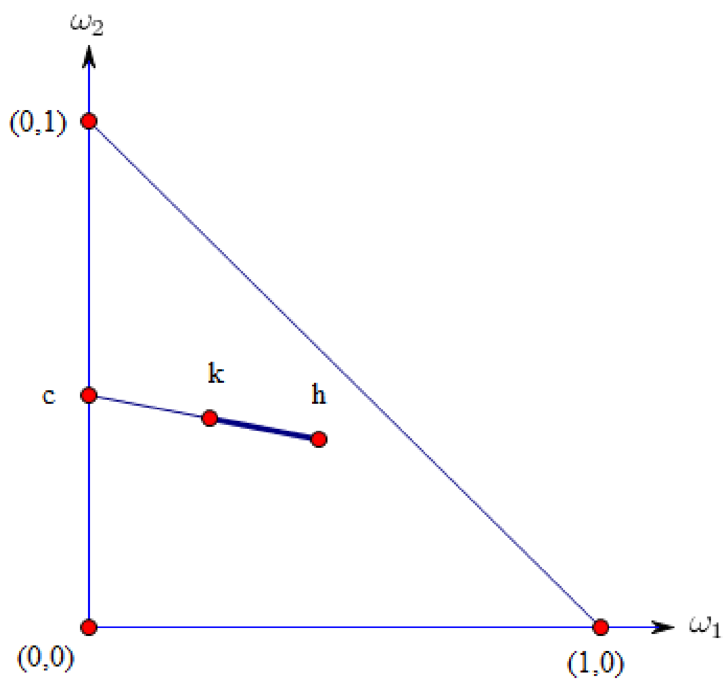


Figure 4.1

Within and on this triangle all efficient portfolios will lie. In particular, the minimum variance portfolio, the maximum return portfolio and the corner portfolio between these two. Where in particular, these portfolios are located is solely determined by the values for the means and variances. However, what we show is that the corner portfolios will only lie on the vertical axis, strictly between $(0,0)$ and $(0,1)$.

The maximum return portfolio is clearly the one where we hold 100% in asset 3, the one with the highest mean. This is located on the diagram at the point $(0,0)$.

The minimum variance portfolio will lie somewhere within the triangle with co-ordinates $(\omega_1, \omega_2) = (\frac{1/\sigma_1^2}{\sum 1/\sigma_j^2}, \frac{1/\sigma_2^2}{\sum 1/\sigma_j^2})$ (See Appendix 4.6.1). Its position within the triangle is determined purely by the σ_j^2's. We label this point h in the figure with co-ordinates (ω_1^h, ω_2^h). The weight for the third asset is clearly $\omega_3^h = 1 - \omega_1^h - \omega_2^h = \frac{1/\sigma_3^2}{\sum 1/\sigma_j^2}$. Also, within the triangle will be another point that minimizes portfolio variance for various levels of portfolio expected return. As in Markowitz(2005), this portfolio has the weights $\omega_i = \frac{\mu_j/\sigma_j^2}{\sum \mu_j/\sigma_j^2},\ j = 1,2,3$. We label this point k in the diagram with co-ordinates (ω_1^k, ω_2^k) (See Appendix 4.6.2). The straight line that passes through the two points (ω_1^h, ω_2^h) and (ω_1^k, ω_2^k) defines all portfolios with the minimum variance for various levels of expected return. However, the only efficient portfolios are those on the line starting from h and moving towards k and beyond. Once this line crosses over the sides of the triangle, the portfolios become infeasible since the non-negativity constraint is violated.

4.2.1 Remark

While the point h can be located anywhere within the triangle, the point k will always be such that $\omega_1^k < \omega_1^h$ while ω_2^k may be smaller, larger or equal to ω_2^h. In fact, as we will show, the point k can only fall strictly within the triangle defined by the verticies (ω_1^h, ω_2^h), $(0, 1)$ and $(0, 0)$. This also establishes the fact that the corner portfolio can only lie on the horizontal axis.

As a means of proving the claims made in the above remark, we now show the following:

(a) If we move along the line connecting h and k but move away from k and past h. Then the portfolios on that part of the line are not efficient.

Consider the equation of the line connecting h and k. That is, the straight line between (ω_1^h, ω_2^h) and (ω_1^k, ω_2^k) is given by

$$x_2 - \omega_2^h = \left(\frac{\omega_2^k - \omega_2^h}{\omega_1^k - \omega_1^h}\right)(x_1 - \omega_1^h) \tag{4.7}$$

where x_1, x_2 is any point on the line.

The slope of this line, given by $\left(\frac{\omega_2^k - \omega_2^h}{\omega_1^k - \omega_1^h}\right)$ could be positive, negative or zero. Substituting the expressions for the weights, given earlier, the slope can be shown to be given by

$$\begin{aligned} s &= \frac{(\mu_3 - \mu_2)\sigma_1^2 - (\mu_2 - \mu_1)\sigma_3^2}{(\mu_2 - \mu_1)\sigma_3^2 + (\mu_3 - \mu_1)\sigma_2^2} \\ &= \frac{\eta_2\sigma_1^2 - \eta_1\sigma_3^2}{\eta_1\sigma_3^2 + (\eta_1 + \eta_2)\sigma_2^2} \end{aligned} \tag{4.8}$$

From the expression for s, we have immediately that the sign of the slope is determined by the numerator. Thus, if $\frac{\sigma_3^2}{\sigma_1^2} > \frac{\eta_2}{\eta_1}$, the slope is negative, if $\frac{\sigma_3^2}{\sigma_1^2} = \frac{\eta_2}{\eta_1}$, the slope is zero and finally if $\frac{\sigma_3^2}{\sigma_1^2} < \frac{\eta_2}{\eta_1}$, the slope will be positive.

Now, let the equation of the line be given by

$$x_2 = \omega_2^h + s(x_1 - \omega_1^h) \tag{4.9}$$

We can now easily show that points on the line beyond h and in the opposite direction to k are not efficient. Let

$$x_1 = \omega_1^h + \varepsilon, 0 < \varepsilon < 1$$

then

$$x_2 = \omega_2^h + s\varepsilon$$

with

$$x_3 = 1 - x_1 - x_2$$

The mean of this portfolio is given by

$$\begin{aligned}
\mu_p^x &= \mu' x = \mu_1 x_1 + \mu_2 x_2 + \mu_3 x_3 \\
&= \mu_3 - x_1(\mu_3 - \mu_1) - x_2(\mu_3 - \mu_2) \\
&= \mu_3 - x_1(\eta_1 + \eta_2) - x_2 \eta_2 \\
&= \mu_3 - (\omega_1^h + \varepsilon)(\eta_1 + \eta_2) - (\omega_2^h + s\varepsilon)\eta_2 \qquad (4.10)
\end{aligned}$$

For the portfolio given by the point h, the minimum variance portfolio, we have

$$\mu_p^h = \mu_3 - \omega_1^h(\eta_1 + \eta_2) - \omega_2^h \eta_2 \tag{4.11}$$

Thus,

$$\mu_p^h - \mu_p^x = \varepsilon\eta_1 + (1+s)\varepsilon\eta_2 \tag{4.12}$$

and since $(1+s) > 0$, we have that the portfolio at x is inefficient. Similarly we can show that any points on the line to the left of h result in higher portfolio mean and variance.

(b) The slope of the line connecting h and k is bounded within the slopes of the two rays connecting (ω_1^h, ω_2^h) with $(0,1)$ and $(0,0)$. Here we ask the question: Can either of the points $(0,0)$ or $(0,1)$ lie on the line connecting h and k?

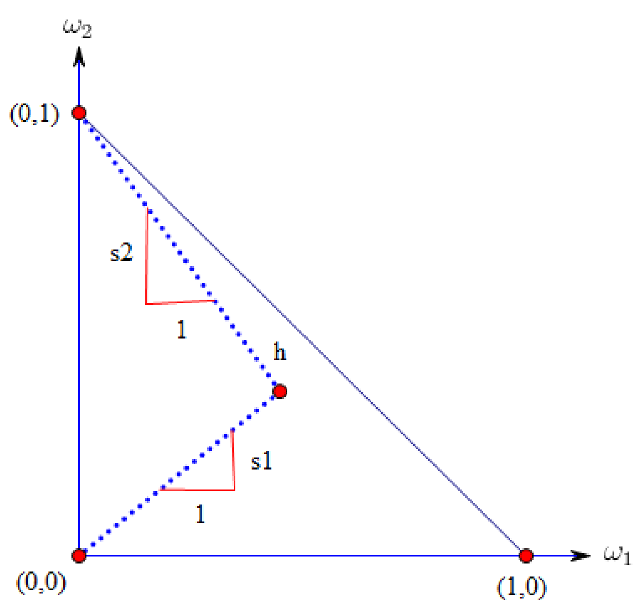


Figure 4.2

Consider the line connecting $(0, 0)$ and h as in Figure 3, the slope is

$$s_1 = \frac{0 - \omega_2^h}{0 - \omega_1^h} = \frac{\sigma_1^2}{\sigma_2^2} \tag{4.13}$$

The question is can s be equal to or bigger than s_1? Suppose $s \le s_1$ holds, we have

$$\frac{\eta_2\sigma_1^2 - \eta_1\sigma_3^2}{\eta_1\sigma_3^2 + (\eta_1 + \eta_2)\sigma_2^2} \le \frac{\sigma_1^2}{\sigma_2^2} \tag{4.14}$$

or

$$\sigma_1^2\sigma_3^2 + \sigma_1^2\sigma_2^2 + \sigma_2^2\sigma_3^2 \le 0 \tag{4.15}$$

It is a conflict as $\sigma_i^2 > 0$ for all i. Thus, we have $s < s_1$. Now consider the line connecting $(0, 1)$ and h as shown in Figure 3, the slope is

$$s_2 = \frac{1 - \omega_2^h}{-\omega_1^h} = \frac{\omega_1^h + \omega_2^h}{-\omega_1^h} = -\left(1 + \frac{\sigma_1^2}{\sigma_3^2}\right) \tag{4.16}$$

The question is can s be equal to or smaller than s_2? Suppose $s \le s_2$ holds, we have

$$\frac{\eta_2\sigma_1^2 - \eta_1\sigma_3^2}{\eta_1\sigma_3^2 + (\eta_1 + \eta_2)\sigma_2^2} \ge -\left(1 + \frac{\sigma_1^2}{\sigma_3^2}\right) \tag{4.17}$$

or

$$\eta_2\sigma_1^2\sigma_3^2 + \eta_1\sigma_1^2\sigma_3^2 + (\eta_1 + \eta_2)(\sigma_2^2\sigma_3^2 + \sigma_1^2\sigma_2^2) \le 0 \tag{4.18}$$

It is a conflict as $\sigma_i^2 > 0$ for all i. Thus, we have $s > s_2$. Now the line connecting h and k is strictly bounded inside the rays connecting (ω_1^h, ω_2^h) with $(0, 1)$ and $(0, 0)$.

We have thus established that irrespective of where the point h is located within the triangle all efficient portfolios will be defined by points to its left and lying on a straight line strictly within the triangle with verticies (ω_1^h, ω_2^h), $(0,1)$ and $(0,0)$. The straight line will eventually cross the vertical axis with all points beyond the crossing point defining infeasible portfolios. However, the point of crossing and all points below on the vertical axis until we reach the maximum return portfolio at the point $(0,0)$ will all define efficient portfolios. The point of crossing is known as a turning point and it defines what is known as a corner portfolio. This point is readily found using the straight line

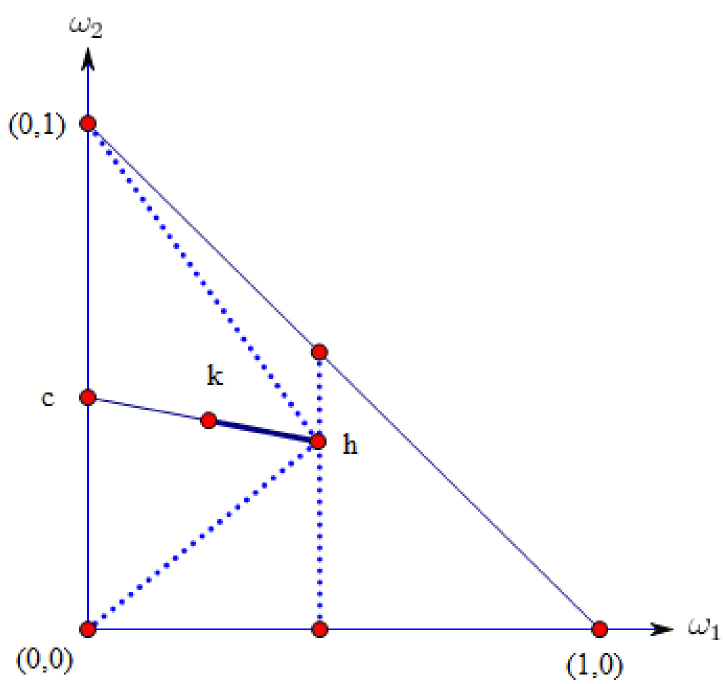


Figure 4.3

$$x_2 = \omega_2^h + s(x_1 - \omega_1^h) \tag{4.19}$$

at the point of crossing $x_1 = 0$ and thus letting this point be c we have

$$\omega_2^c = \omega_2^h - s\omega_1^h \tag{4.20}$$

with $\omega_3^c = 1 - \omega_2^c$

At this point, the portfolio's mean is given by

$$\begin{aligned} \mu_p^c &= \mu_3 - \omega_2^c(\mu_3 - \mu_2) \\ &= \mu_3 - (\omega_2^h - s\omega_1^h)\eta_2 \end{aligned} \tag{4.21}$$

Again, since

$$\mu_p^h - \mu_p^c = -\omega_1^h\eta_1 - \omega_1^h\eta_2(1+s) < 0 \tag{4.22}$$

The point c defines an efficient portfolio as do all points below c including (Appendix 4.6.3), of course, the maximum return portfolio at the point $(0, 0)$.

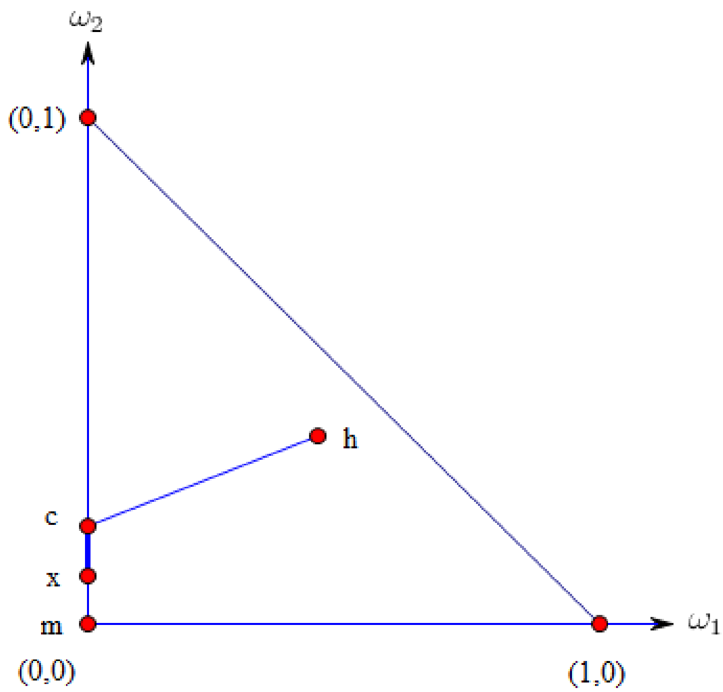


Figure 4.4

4.2.2 Efficient Frontier

We now consider the means and variances of the three important portfolios given by the points h, c and maximum return m in Figure 4.5. After some straightforward but tedious algebra in Appendix(4.6.4) we can readily derive explicit expressions for the portfolio means and variances which are given by

$$\mu_p^h = \frac{1}{B}(\mu_1\sigma_2^2\sigma_3^2 + \mu_2\sigma_1^2\sigma_3^2 + \mu_3\sigma_1^2\sigma_2^2) \quad (4.23)$$

$$(\sigma_p^h)^2 = \frac{\sigma_1^2\sigma_2^2\sigma_3^2}{B} \quad (4.24)$$

where

$$B = \sigma_1^2\sigma_2^2 + \sigma_1^2\sigma_3^2 + \sigma_2^2\sigma_3^2$$

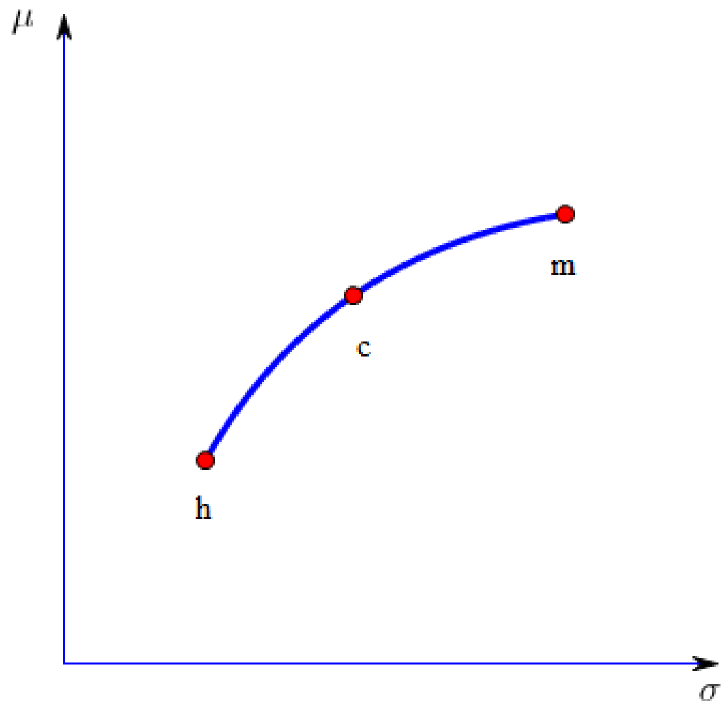


Figure 4.5

Also, in weight space, the functions of portfolio mean and variance to the asset weights are constructed. Then at both sides of the corner point c on the efficient frontier, the derivative of portfolio mean to variance can be calculated by the chain rule.

$$\frac{\partial \mu_p}{\partial \sigma_p^2}|_{left} = \frac{\partial \mu_p}{\partial x_2}\frac{\partial x_2}{\partial \sigma_p^2} = \frac{-(\eta_1 + \eta_2 + s\eta_2)}{2s\sigma 2^2 x_2 - 2(1+s)\sigma_3^2(1-x_2)} \tag{4.25}$$

$$\frac{\partial \mu_p}{\partial \sigma_p^2}|_{right} = \frac{\partial \mu_p}{\partial x_2}\frac{\partial x_2}{\partial \sigma_p^2} = \frac{-\eta_2}{2x_2(\sigma_2^2 + \sigma_3^2) - 2\sigma_3^2} \tag{4.26}$$

By plugging in the variables, it can be proved that the first order derivative from both sides of corner point c are equal. This leads to the conclusion that there is no kink at the corner point, i.e. the mean variance frontier is continuous. This result is consistent with that of Markowitz(1959) and Dybvig(1984) who both showed that with non-negativity constraints, corner portfolios are only non-differentiable when all assets have the same mean.

4.3 Markowitz's Four Asset Problem

Now we extend the problem to optimize the portfolio of four risky uncorrelated assets where their means and variances are ranked as

$$0 < \mu_1 < \mu_2 < \mu_3 < \mu_4$$

and

$$0 < \sigma_1^2 < \sigma_2^2 < \sigma_3^2 < \sigma_4^2$$

In particular, we let μ_2, μ_3 and μ_4 be defined as

$$\mu_2 = \mu_1 + \eta_1, \ \eta_1 > 0 \tag{4.27}$$

$$\mu_3 = \mu_1 + \eta_1 + \eta_2, \ \eta_1 > 0, \eta_2 > 0 \tag{4.28}$$

$$\mu_4 = \mu_1 + \eta_1 + \eta_2 + \eta_3, \ \eta_1 > 0, \eta_2 > 0, \eta_3 > 0 \tag{4.29}$$

All feasible portfolios satisfying both the budget constraint and the non-negative constraint are on and inside the three-dimensional tetrahedron $(\omega_1, \omega_2, \omega_3)$ by verticies $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ as shown in Figure 4.6.

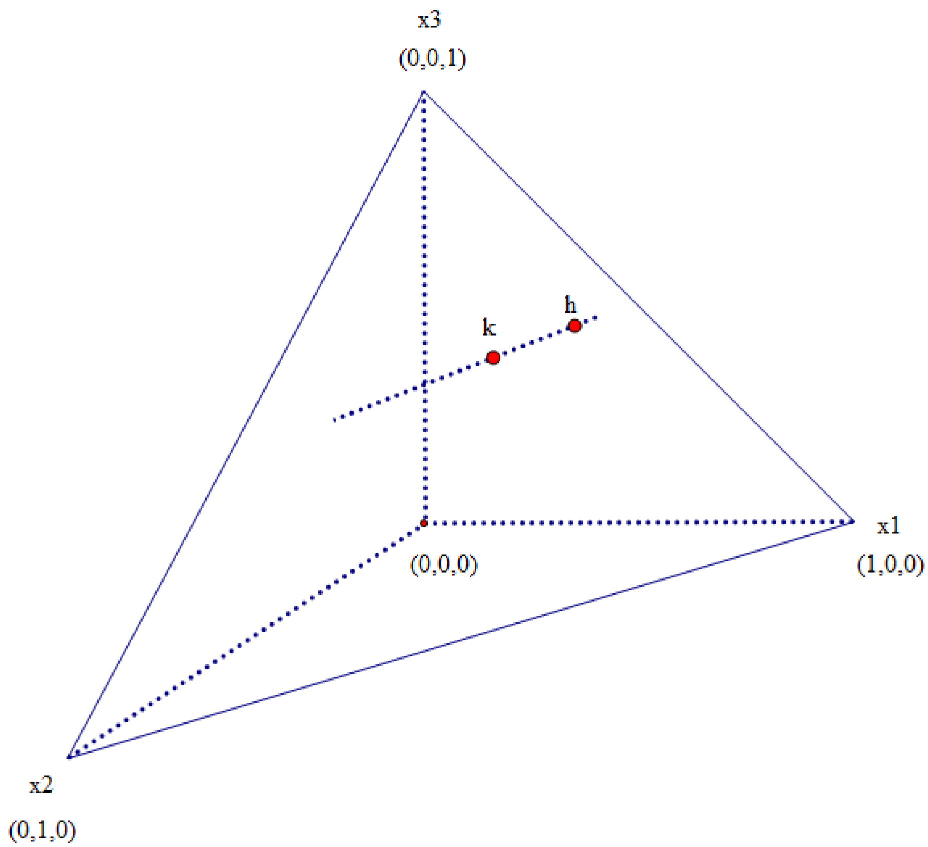


Figure 4.6

4.3.1 The Efficient Portfolio

The maximum return portfolio is the one which holds 100 % in asset 4 which is at point $(0,0,0)$. The minimum variance portfolio h is within the tetrahedron and with co-cordinates

$$(\omega_1^h, \omega_2^h, \omega_3^h) = \left(\frac{1/\sigma_1^2}{\sum 1/\sigma_j^2}, \frac{1/\sigma_2^2}{\sum 1/\sigma_j^2}, \frac{1/\sigma_3^2}{\sum 1/\sigma_j^2} \right) \text{ where } j = 1,2,3,4 \tag{4.30}$$

and the weight of the fourth asset is obtained by

$$\omega_4^h = \frac{1/\sigma_4^2}{\sum 1/\sigma_j^2} = 1 - \omega_1^h - \omega_2^h - \omega_3^h \tag{4.31}$$

We can find another portfolio k which minimizes the variance for various levels of given portfolio expected return.

$$\omega_i^k = \frac{\mu_j/\sigma_j^2}{\sum \mu_j/\sigma_j^2}, \text{ where } j = 1,2,3,4 \tag{4.32}$$

The straight line which passes through the two points $(\omega_1^h, \omega_2^h, \omega_3^h)$ and $(\omega_1^k, \omega_2^k, \omega_3^k)$ defines all portfolios with the minimum variance for various level of expected return.

Remark 1

Now we want to show that if we move along the line connecting points h and k but move away from k and past h, the portfolio on that part of the line are not efficient. Any point (x_1, x_2, x_3) on the line connecting these two points can be described by equation

$$\frac{x_1 - \omega_1^h}{\omega_1^k - \omega_1^h} = \frac{x_2 - \omega_2^h}{\omega_2^k - \omega_2^h} = \frac{x_3 - \omega_1^h}{\omega_1^k - \omega_1^h} \tag{4.33}$$

Also, the direction vector from point h to point k is defined as $\mathbf{S} = \{l, m, n\}$. Thus the point (x_1, x_2, x_3) can also be defined as

$$\frac{x_1 - \omega_1^h}{l} = \frac{x_2 - \omega_2^h}{m} = \frac{x_3 - \omega_3^h}{n} = t \tag{4.34}$$

where t is any number and

$$l = \omega_1^k - \omega_1^h$$

$$m = \omega_2^k - \omega_2^h$$

$$n = \omega_3^k - \omega_3^h$$

Plugging in the values of ω_j^h and ω_j^k, we have

$$\begin{aligned} l &= \frac{\mu_1/\sigma_1^2}{\sum \mu_j/\sigma_j^2} - \frac{1/\sigma_1^2}{\sum 1/\sigma_j^2} \\ &= \frac{1}{\sigma_1^2(\sum \mu_j/\sigma_j^2)(\sum 1/\sigma_j^2)} \left(\frac{-\eta_1}{\sigma_2^2} + \frac{-\eta_1 - \eta_2}{\sigma_3^2} + \frac{-\eta_1 - \eta_2 - \eta_3}{\sigma_4^2} \right) \end{aligned} \quad (4.35)$$

By the definition of $\eta_i > 0, i = 1, 2, 3$ we know that $l < 0$. Since

$$\begin{aligned} m &= \frac{\mu_2/\sigma_2^2}{\sum \mu_j/\sigma_j^2} - \frac{1/\sigma_2^2}{\sum 1/\sigma_j^2} \\ &= \frac{1}{\sigma_2^2(\sum \mu_j/\sigma_j^2)(\sum 1/\sigma_j^2)} \left(\frac{\eta_1}{\sigma_1^2} + \frac{-\eta_2}{\sigma_3^2} + \frac{-\eta_2 - \eta_3}{\sigma_4^2} \right) \end{aligned} \quad (4.36)$$

m can be positive, negative or zero. Also,

$$\begin{aligned} n &= \frac{\mu_3/\sigma_3^2}{\sum \mu_j/\sigma_j^2} - \frac{1/\sigma_3^2}{\sum 1/\sigma_j^2} \\ &= \frac{1}{\sigma_3^2(\sum \mu_j/\sigma_j^2)(\sum 1/\sigma_j^2)} \left(\frac{\eta_1 + \eta_2}{\sigma_1^2} + \frac{\eta_2}{\sigma_2^2} + \frac{-\eta_3}{\sigma_4^2} \right) \end{aligned} \quad (4.37)$$

n can also be positive, negative or zero.

Now the line connecting points h and k can be described as

$$
\begin{aligned}
x_1 &= \omega_1^h + lt \\
x_2 &= \omega_2^h + mt \\
x_3 &= \omega_3^h + nt
\end{aligned}
$$

Any points on this line beyond h and in the opposite direction to k can be obtained by letting $-1 < t < 0$. And the corresponding portfolio has mean return

$$
\begin{aligned}
\mu_p^x &= \mu^T x = (\mu_1, \mu_2, \mu_3, \mu_4)(x_1, x_2, x_3, 1 - x_1 - x_2 - x_3)^T \\
&= \mu_4 - (\omega_1^h + lt)(\eta_1 + \eta_2 + \eta_3) - (\omega_2^h + mt)(\eta_2 + \eta_3) - (\omega_3^h + \eta t)\eta_3 \quad (4.38)
\end{aligned}
$$

At point h, the corresponding portfolio has mean return

$$
\begin{aligned}
\mu_p^h &= (\mu_1, \mu_2, \mu_3, \mu_4)(x_1^h, x_2^h, x_3^h, 1 - x_1^h - x_2^h - x_3^h)^T \\
&= \mu_4 - \omega_1^h(\eta_1 + \eta_2 + \eta_3) - \omega_2^h(\eta_2 + \eta_3) - \omega_3^h(\eta_3) \quad (4.39)
\end{aligned}
$$

Now, by plugging in the values of l, m and n

$$
\begin{aligned}
\mu_p^h - \mu_p^x &= (\eta_1 + \eta_2 + \eta_3)lt + (\eta_2 + \eta_3)mt + \eta_3 nt \\
&= (t)\frac{1}{(\sum \mu_j/\sigma_j^2)(\sum 1/\sigma_j^2)}\Big[\frac{-\eta_2}{\sigma_1^2\sigma_2^2} + \frac{-\eta_1^2 - 2\eta_1\eta_2 - \eta_2^2}{\sigma_1^2\sigma_3^2} \\
&+ \frac{-\eta_1^2 - \eta_2^2 - \eta_3^2 - 2\eta_1\eta_2 - 2\eta_1\eta_3 - 2\eta_2\eta_3}{\sigma_1^2\sigma_4^2} \\
&+ \frac{-\eta_2^2}{\sigma_2^2\sigma_3^2} + \frac{-\eta_2^2 - 2\eta_2\eta_3 - \eta_3^2}{\sigma_2^2\sigma_4^2} + \frac{-\eta_3^2}{\sigma_3^2\sigma_4^2}\Big] \\
&> 0 \quad (4.40)
\end{aligned}
$$

since $-1 < t < 0$ and the parts in the parentheses are negative. Also the portfolio at point h has minimum variance, the portfolio at point x is inefficient. This shows the result.

Remark 2: $\vec{hk}$ **can not hit surface** p_3

Now we know that only the points on the line connecting points h and k and between h and k or passing k are efficient. We need to find the point c where this line hits the surface of the tetrahedron. Notice that $l < 0$, i.e. when the points from h to k and passing k, the value of x_1 can only be smaller. As in Fig 4.7, we want to show that the line $\vec{hk}$ can only hit the surface of $x_1 = 0$ and the line is contained in the tetrahedron with verticies $h(\omega_1^h, \omega_2^h, \omega_3^h), a(0,1,0), b(0,0,1)$ and $o(0,0,0)$.

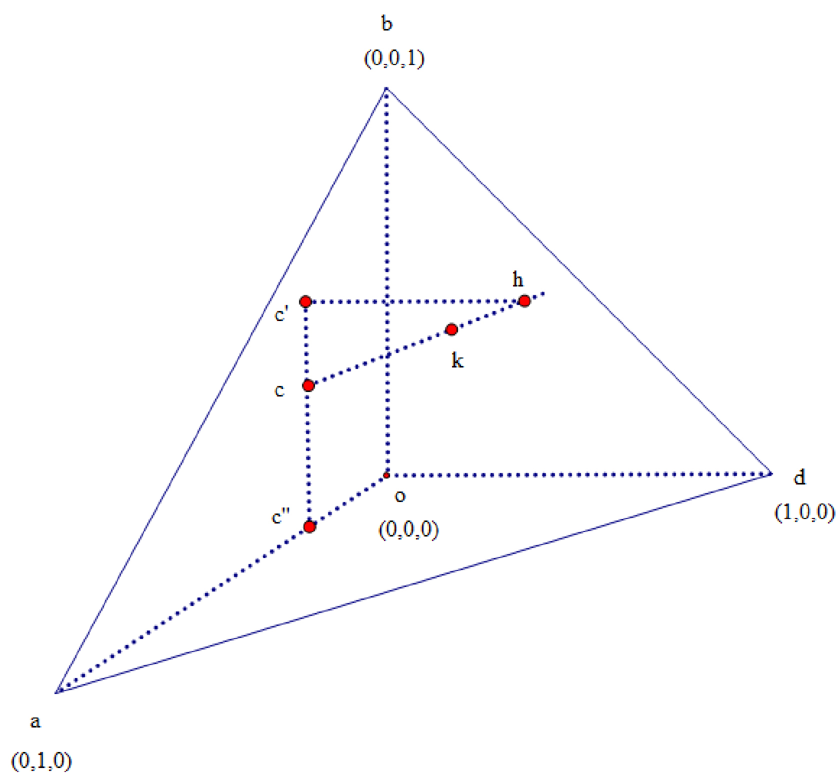


Figure 4.7

The direction vector of the line $\vec{ho}$ is $\{-\omega_1^h, -\omega_2^h, -\omega_3^h\}$. The plane through three points h, a and o is defined by the following equation

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ \omega_1^h & \omega_2^h & \omega_3^h \\ 0 & 1 & 0 \end{vmatrix} = 0 \tag{4.41}$$

where (x_1, x_2, x_3) is any point on this plane. We can have $x_1(-\omega_3^h) + x_3(\omega_1^h) = 0$. Let the four surfaces of the tetrahedron be p_1, p_2, p_3, p_4. p_1 is the plane where $x_1 = 0$, p_2 is the plane where $x_2 = 0$, p_3 is the plane where $x_3 = 0$ and p_4 is the plane through points $(1,0,0), (0,1,0), (0,0,1)$. The normal vector of plane p_1 is $\{A, B, C\} = \{-\omega_3^h, 0, \omega_1^h\}$. Let point c be where the line $\vec{hk}$ hits the plane p_1. Let point c' be such that it is on the plane p_1 and the line connecting points h and c' is vertical to plane p_1. Point c'' is the intersection of line connecting points c and c' and the line connecting points a and o. As shown in Fig 4.8, the angle between $\vec{hc'}$ and $\vec{hc}$ is α_1 and the angle between $\vec{hc'}$ and $\vec{hc''}$ is α_2.

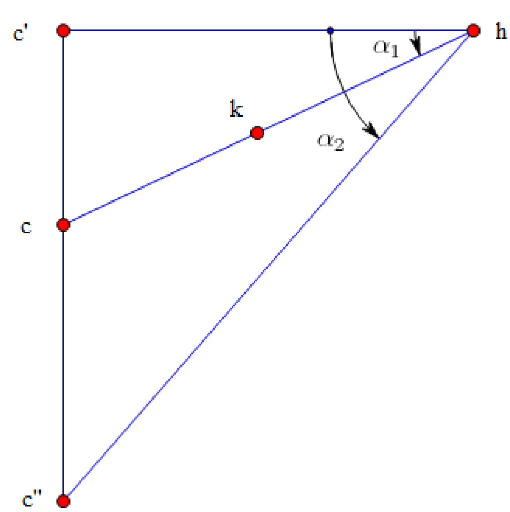


Figure 4.8

The coordinate of point c' is $(0, \omega_2^h, \omega_3^h)$, the coordinate of point c is defined as $(0, \omega_2^c, \omega_3^c)$. To get the coordinate of point c'', notice that c'' is where the line of axis x_2 crossing the plane through points h, c', k. The equation for this plane is

$$\begin{vmatrix} x_1 & x_2 - \omega_2^h & x_3 - \omega_3^h \\ \omega_1^h & 0 & 0 \\ \omega_1^k & \omega_2^k - \omega_2^h & \omega_3^k - \omega_3^h \end{vmatrix} = 0 \tag{4.42}$$

Plugging in the line of axis x_2 which is $x_1 = 0, x_3 = 0$, we have

$$x_2 = \omega_2^h - \frac{\omega_3^h(\omega_2^k - \omega_2^h)}{(\omega_3^k - \omega_3^h)}. \tag{4.43}$$

And the coordinate of c'' can be defined. The direction vector of $\vec{hc''}$ is $\{l_2, m_2, n_2\} = \{-\omega_1^h, -\omega_3^h \frac{m}{n}, -\omega_3^h\}$.

Assume that $\vec{hk}$ hits the plane p_3 at point c and c is not on the intersection line of plane p_1 and p_3. The following three conditions must be satisfied.

(C1) $0 < \alpha_2 < \alpha_1 < 90$

(C2) $\omega_3^c = 0$

(C3) For the direction vector $\{l, m, n\}$ of line $\vec{hk}$ or $\vec{hc}$, we have $n < 0$ because $n = 0 - \omega_3^h$ and $\omega_3^h > 0$

If we can find a conflict among these conditions, we can prove that line $\vec{hk}$ can not hit plane p_3. For angle α_1,

$$\begin{aligned} cos(\alpha_1) &= \frac{|l_1 l + m_1 m + n_1 n|}{\sqrt{l_1^2 + m_1^2 + n_1^2}\sqrt{l^2 + m^2 + n^2}} \\ &= \frac{l}{\sqrt{l^2 + m^2 + n^2}} \end{aligned} \tag{4.44}$$

or

$$\frac{1}{cos^2(\alpha_1)} = 1 + \frac{m^2 + n^2}{l^2} \tag{4.45}$$

The same, for angle α_2,

$$cos(\alpha_2) = \frac{\omega_1^h}{\sqrt{(\omega_1^h)^2 + \frac{m^2+n^2}{n^2}(\omega_3^h)^2}} \tag{4.46}$$

or

$$\frac{1}{cos^2(\alpha_2)} = 1 + \frac{m^2+n^2}{n^2}\left(\frac{\omega_3^h}{\omega_1^h}\right)^2 \tag{4.47}$$

From condition (C1), we have $cos(\alpha_2) > cos(\alpha_1)$ or

$$\frac{1}{cos^2(\alpha_2)} < \frac{1}{cos^2(\alpha_1)}$$

or

$$\frac{n^2}{l^2} > \left(\frac{\omega_3^h}{\omega_1^h}\right)^2 \tag{4.48}$$

Now let $Q \equiv \sum \mu_j/\sigma_j^2$, $R \equiv \sum 1/\sigma_j^2$ where $j = 1, 2, 3, 4$, we have

$$l = \omega_1^k - \omega_1^h = \frac{1}{\sigma_1^2}(\mu_1/Q - 1/R) \tag{4.49}$$

$$m = \omega_2^k - \omega_2^h = \frac{1}{\sigma_2^2}(\mu_2/Q - 1/R) \tag{4.50}$$

$$n = \omega_3^k - \omega_3^h = \frac{1}{\sigma_3^2}(\mu_3/Q - 1/R) \tag{4.51}$$

We have proved that $l < 0$, thus $R\mu_1 - Q < 0$.

From condition (C3), $n < 0$, thus $R\mu_3 - Q < 0$.

Also

$$\frac{n^2}{l^2} = \frac{\sigma_1^4}{\sigma_3^4}\frac{(R\mu_3 - Q)^2}{(R\mu_1 - Q)^2} \tag{4.52}$$

and

$$\frac{(\omega_3^h)^2}{(\omega_1^h)^2} = \frac{\sigma_1^4}{\sigma_3^4} \tag{4.53}$$

Plugging in (4.52) and (4.53) to (4.48), we have $(R\mu_3 - Q)^2 > (R\mu_1 - Q)^2$. Since $R\mu_1 - Q < 0$ and $R\mu_3 - Q < 0$, we have $R\mu_3 - Q < R\mu_1 - Q$, or $\mu_3 < \mu_1$. There is a conflict. Thus, the three conditions can not be satisfied. i.e. the line $\vec{hk}$ does not hit plane p_3 except the intersection line of plane p_1 and p_3.
Similarly, we can show that $\vec{hk}$ can not hit plane p_2 except the intersection line of plane p_1 and p_2. In summary, the line $\vec{hk}$ always hits the surface p_1.

Remark 3: $\vec{hk}$ can not hit surface p_4

Now we want to show that the efficient line can not hit surface p_4. Assume that it hits surface p_4 at point c as shown in Figure 4.9.

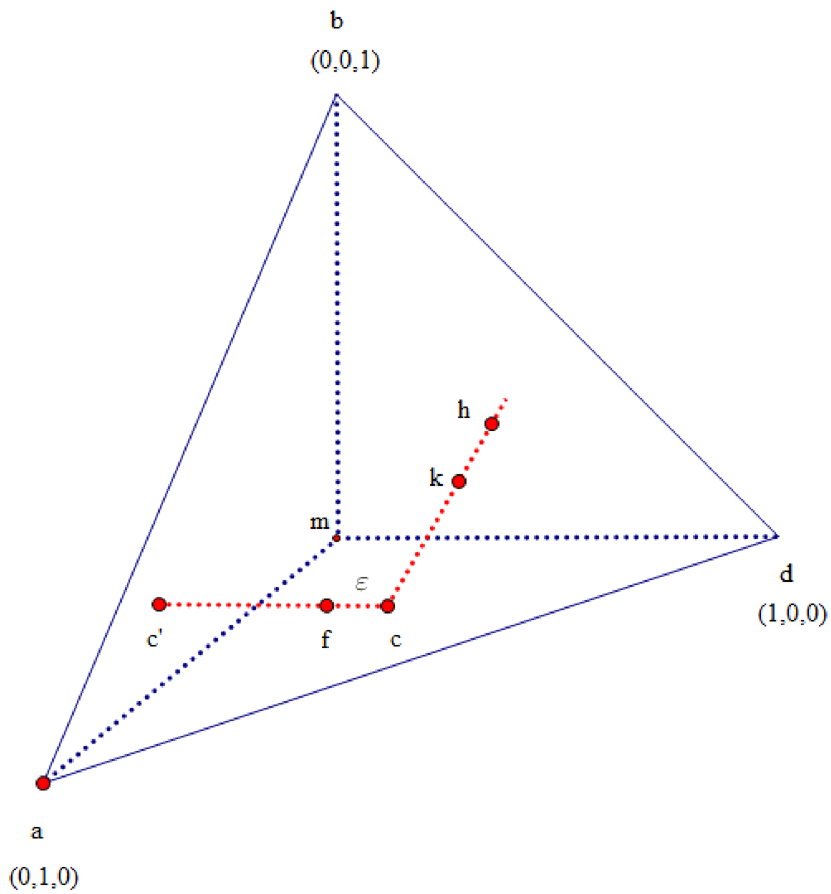


Figure 4.9

If c is on the surface, the following equations are satisfied.

$$x_1^c + x_2^c + x_3^c = 1 \tag{4.54}$$

i.e.

$$x_4^c = 0 \tag{4.55}$$

For the portfolio at point c, the weight of asset 4 is zero. Let point c' be the projection of c on plane p_1. There is a point f on the line $\vec{cc'}$, and the distance between f and c is ε. Point f is inside the tetrahedron and it is between c and c'. Thus, $0 < \varepsilon < x_1^c$.

The mean of the portfolio at point c is given by

$$\begin{aligned} \mu_p^c &= [x_1^c \; x_2^c \; x_3^c] \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix} \\ &= x_1^c \mu_1 + x_2^c \mu_2 + x_2^c \mu_2 \end{aligned} \tag{4.56}$$

And the variance of the portfolio can be obtained by

$$\begin{aligned} (\sigma_p^c)^2 &= [x_1^c \; x_2^c \; x_3^c] \begin{bmatrix} \sigma_1^2 \; 0 \; 0 \\ 0 \; \sigma_2^2 \; 0 \\ 0 \; 0 \; \sigma_3^2 \end{bmatrix} \begin{bmatrix} x_1^c \\ x_2^c \\ x_3^c \end{bmatrix} \\ &= (x_1^c)^2 \sigma_1^2 + (x_2^c)^2 \sigma_2^2 + (x_3^c)^2 \sigma_3^2 \end{aligned} \tag{4.57}$$

The coordinate of point f can be defined as

$$\begin{cases} x_1^f &= x_1^c - \varepsilon \\ x_2^f &= x_2^c \\ x_3^f &= x_3^c \\ x_4^f &= \varepsilon \end{cases} \tag{4.58}$$

Comparing the mean of the portfolio return at point f with that at point c we have

$$\begin{aligned}
\mu_p^f &= [x_1^f,\ x_2^f,\ x_3^f,\ x_4^f] \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{bmatrix} \\
&= [x_1^c - \varepsilon,\ x_2^f,\ x_3^f,\ \varepsilon] \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{bmatrix} \\
&= x_1^c\mu_1 + x_2^c\mu_2 + x_2^c\mu_2 + (\mu_4 - \mu_1)\varepsilon \\
&= \mu_p^c + (\mu_4 - \mu_1)\varepsilon \\
&> \mu_p^c
\end{aligned} \tag{4.59}$$

Looking at the variance of portfolio at point f we have

$$\begin{aligned}
(\sigma_p^f)^2 &= [x_1^f\ x_2^f\ x_3^f\ x_4^f] \begin{bmatrix} \sigma_1^2\ 0\ 0\ 0 \\ 0\ \sigma_2^2\ 0\ 0 \\ 0\ 0\ \sigma_3^2\ 0 \\ 0\ 0\ 0\ \sigma_3^2 \end{bmatrix} \begin{bmatrix} x_1^f \\ x_2^f \\ x_3^f \\ x_4^f \end{bmatrix} \\
&= (\sigma_p^c)^2 + (\sigma_1^2 + \sigma_4^2)\varepsilon - 2x_1^c\varepsilon\sigma_1^2
\end{aligned} \tag{4.60}$$

Considering the part of equation (4.60), $\Delta = (\sigma_1^2 + \sigma_4^2)\varepsilon - 2x_1^c\varepsilon\sigma_1^2$. The question is that find ε such that

$$(\sigma_1^2 + \sigma_4^2)\varepsilon - 2x_1^c\varepsilon\sigma_1^2 < 0 \tag{4.61}$$

or

$$\varepsilon < x_1^c \frac{2\sigma_1^2}{\sigma_1^2 + \sigma_4^2} \tag{4.62}$$

As $\sigma_1^2 < \sigma_4^2$, from equation (4.62) is, we have $0 < \varepsilon < x_1^c$. It exists such a ε that the portfolio at point f has higher mean return and lower variance than the portfolio at point c. Portfolio at c is not efficient which is a conflict. Thus, the efficient line $\vec{hk}$ can not hit plane p_4.

Remark 4

Since we have shown that, the line $\vec{hk}$ always hits the plane p_1 where $\omega_1 = 0$, the problem is reduced to a three-asset problem. i.e. there are $\omega_2, \omega_3, \omega_4$ in the efficient portfolio. As in Figure 4.10, the efficient line hits axis x_3 at point g and goes to the maximum return portfolio o through the axis. To obtain the coordinate of point c, notice that it is on the line $\vec{hk}$ with direction vector $\{l, m, n\}$, thus

$$\frac{\omega_1^c - \omega_1^h}{l} = \frac{\omega_2^c - \omega_2^h}{m} = \frac{\omega_3^c - \omega_3^h}{n} = t \tag{4.63}$$

Plugging in $\omega_1^c = 0$, we have $t = -\omega_1^h / l$, and

$$\omega_2^c = tm + \omega_2^h = \omega_2^h - \omega_1^h \frac{m}{l} \tag{4.64}$$

$$\omega_3^c = tn + \omega_3^h = \omega_3^h - \omega_1^h \frac{n}{l} \tag{4.65}$$

On the plane p_3, by dropping off asset 1, the problem is reduced to a three asset problem which is discussed in previous section. The efficient line goes to point g from c. The point g is on the axis x_3 and the coordinate can be obtained by $\omega_1^g = 0$, $\omega_2^g = 0$,

$$\omega_3^g = \frac{\sigma_4^2(\mu_3 - \mu_2)}{(\mu_3 - \mu_2)\sigma_4^2 + (\mu_4 - \mu_2)\sigma_3^2}$$

and $\omega_4^g = 1 - \omega_3^g$

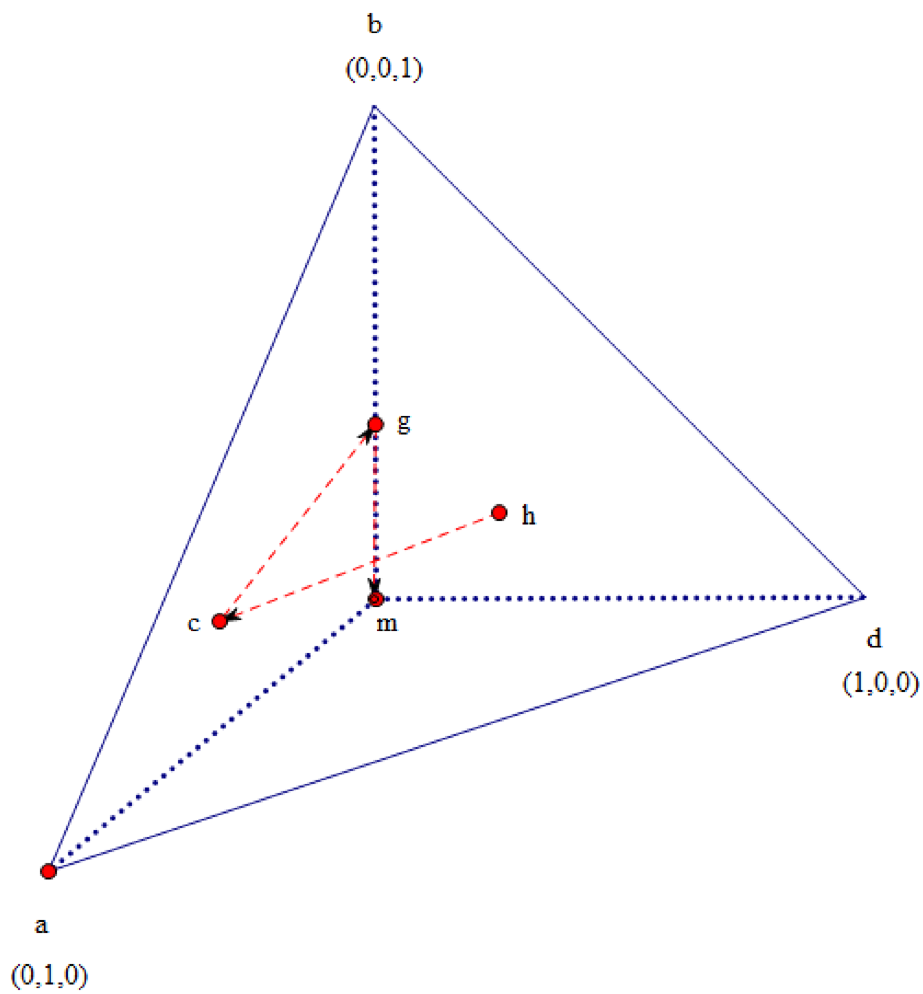


Figure 4.10

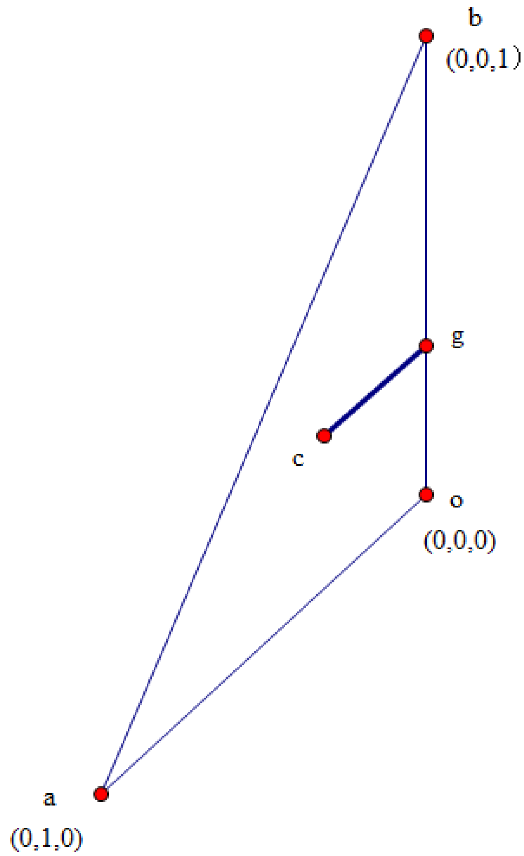


Figure 4.11

4.3.2 Efficient Frontier

We now consider the means and variances of the four important portfolios in the four assets problem, given by the points h,c,g and m in Figure 4.12.

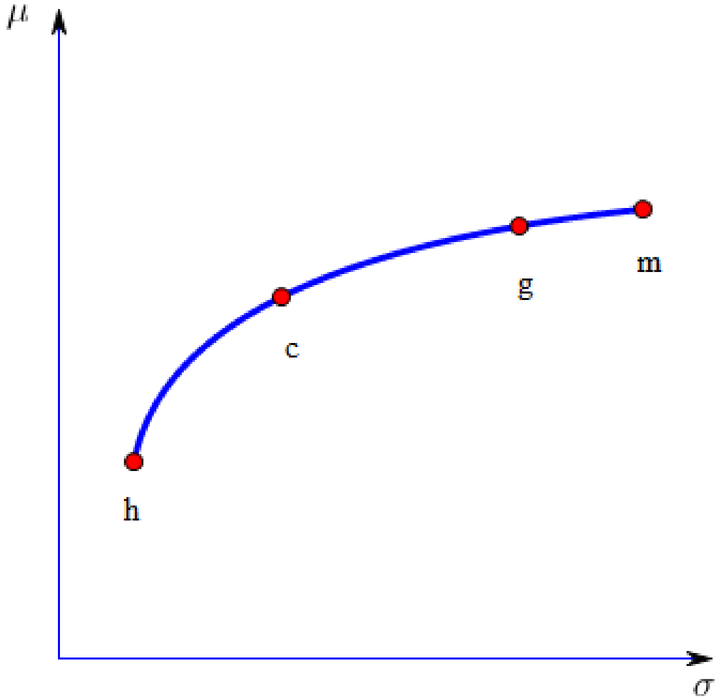


Figure 4.12

After some straightforward but tedious algebra in Appendix(4.6.5), we can derive explicit expressions for the portfolio means and variances.

1, For the minimum variance portfolio at point h, the expression are given by

$$\mu_p^h = \frac{\mu_1\sigma_2^2\sigma_3^2\sigma_4^2 + \mu_2\sigma_1^2\sigma_3^2\sigma_4^2 + \mu_3\sigma_1^2\sigma_2^2\sigma_4^2 + \mu_4\sigma_1^2\sigma_2^2\sigma_3^2}{D} \tag{4.66}$$

$$(\sigma_p^h)^2 = \frac{1}{R} = \frac{\sigma_1^2\sigma_2^2\sigma_3^2\sigma_4^2}{D} \tag{4.67}$$

where

$$D \equiv \sigma_2^2\sigma_3^2\sigma_4^2 + \sigma_1^2\sigma_3^2\sigma_4^2 + \sigma_1^2\sigma_2^2\sigma_4^2 + \sigma_1^2\sigma_2^2\sigma_3^2 \tag{4.68}$$

2, For the first corner portfolio at point c where the efficient line hits plane p_1, the expression are given by

$$\mu_p^c = \frac{1}{E}\left[\mu_1(\frac{\mu_2-\mu_1}{\sigma_2^2}) + \mu_3(\frac{\mu_3-\mu_1}{\sigma_3^2}) + \mu_4(\frac{\mu_4-\mu_1}{\sigma_4^2})\right] \quad (4.69)$$

$$(\sigma_p^c)^2 = \frac{1}{E^2}\left[\frac{(\mu_2-\mu_1)^2}{\sigma_2^2} + \frac{(\mu_3-\mu_1)^2}{\sigma_3^2} + \frac{(\mu_4-\mu_1)^2}{\sigma_4^2}\right] \quad (4.70)$$

where

$$E \equiv \frac{\mu_2-\mu_1}{\sigma_2^2} + \frac{\mu_3-\mu_1}{\sigma_3^2} + \frac{\mu_4-\mu_1}{\sigma_4^2} \quad (4.71)$$

Notice that the relationship between the portfolio mean and the portfolio volatility

$$\mu_p^c = (\sigma_p^c)^2 E + \mu_1 \quad (4.72)$$

Further, since

$$\frac{\partial \mu_p^c}{\partial \sigma_p^c} = 2\sigma_p^c E \quad (4.73)$$

We know that the frontier is differentiable at the corner portfolio c.

3, For the second corner portfolio at point g where the efficient line hits the plan p_2, the expression are given by

$$\mu_p^g = \frac{1}{F}\left[\mu_3\sigma_4^2(\mu_3-\mu_2) + \mu_4\sigma_3^2(\mu_4-\mu_2)\right] \quad (4.74)$$

$$(\sigma_p^g)^2 = \frac{\sigma_3^2\sigma_4^2}{F^2}\left[\sigma_4^2(\mu_3-\mu_2)^2 + \sigma_3^2(\mu_4-\mu_2)^2\right] \quad (4.75)$$

where

$$F = \sigma_4^2(\mu_3 - \mu_2) + \sigma_3^2(\mu_4 - \mu_2) \tag{4.76}$$

The relationship between the portfolio mean and volatility is given by

$$\mu_p^g = \frac{(\sigma_p^g)^2 F}{\sigma_3^2\sigma_4^2} + \mu_2 \tag{4.77}$$

Further, since

$$\frac{\partial(\mu_p)}{\partial(\sigma_p^g)} = 2\frac{\sigma_p^g F}{\sigma_3^2\sigma_4^2} \tag{4.78}$$

We have that the frontier is differentiable at the corner portfolio g.

4, Finally for the maximum return portfolio at point m, the result is straightforward and we have

$$\mu_p^m = \mu_4 \tag{4.79}$$

$$\sigma_p^m = \sigma_4 \tag{4.80}$$

With the same approach as for the three asset problem, we can prove that in the mean variance space, there is no kink at the corner points c and g, i.e. the mean variance frontier is continuous.

4.4 Experimental Results

4.4.1 Experiment

Data

The data used for this study are four stock prices including SPX Index, IBM, RIMM and Ford. The data are daily stock prices from August 18, 2004 to August 31, 2012. The prices are fixed to 260 days for one year. The daily stock prices are used to obtain rolling annual return from August 17, 2005 to August 31, 2012. Based on the rolling annual return, we calculate and rank the mean and volatility for the four assets as followings

	μ	σ
SPX Index	0.0356	0.1917
IBM	0.1362	0.1955
RIMM	0.1745	0.8109
Ford	0.2422	1.1260

Table 4.1

Experiment and Results

From this study, the portfolio at four important points h, c, g and m are calculated. The weights of the corner portfolios are calculated by the analytical solution and the results are

	w1	w2	w3	w4
h	0.4887	0.4699	0.0273	0.0142
c	0	0.8755	0.0703	0.0542
g	0	0	0.4106	0.5894
m	0	0	0	1

Table 4.2

At the same time, we use the numerical approach to solve the problem (4.104). By increasing the value of given portfolio return μ_p and solve the quadratic problem, we can also obtain the weights for the corner portfolios as follows

	w1	w2	w3	w4
h	0.4881	0.4710	0.0271	0.0137
c	0	0.8636	0.0749	0.0615
g	0	0	0.3931	0.6069
m	0	0	0.0091	0.9909

Table 4.3

We can see that the results match well. Then we can solve the curve equation which fits the four points and can obtain any point on the curve. The result should also match the numerical solution.

4.4.2 Monte Carlo Simulation

In previous section, we test the result on the example of four assets. However, only doing the experiment based on one dataset is not enough. Thus we use Monte Carlo method to repeat the experiment for more simulated dataset. We assume the return of assets falls into the interval $(0, 2)$ and the volatility are in the interval $(0, 3)$.

Firstly, we randomly generate four asset returns with the rank of from small to large and four volatilities as well. We combine the return and volatility in the same order to create four assets. Then we use both the numerical method and the analytical method in this study to calculate the mean and variance of the four important portfolios . The difference between the portfolio variance are recorded. Repeat this process 500 times, then taking the average of the difference we get the results in Table 4.4 as followings. We can see that the results match well.

	h	c	g	m
Average Difference	1.2228e-06	2.1430e-04	2.9391e-06	-0.0015

Table 4.4

4.5 Conclusion

In this study, we re-examine the mean variance portfolio optimization problem in Markowitz (2005). We examine the problem in a general case without specifying values for the means and variances. Furthermore, we extend the problem to four assets where the weights can be described in three dimension space. We find the analytical solution of four important portfolios including the minimum variance portfolio h, the maximum return portfolio m and two corner portfolios c and g. With tedious algebra, we show that, in the weight space, the efficient line starts from the point of minimum variance inside the tetrahedron and always hits the plane where the lowest return asset is equal to zero. Then the efficient line would hit the plane where the second lowest return is equal to zero. This leads to the result that with the increase of the given expected portfolio return, the efficient portfolio always drops off the asset with a lower return first. By mapping the efficient portfolio from weight space to mean variance space, we prove that there is no kink at the corner points in mean variance space i.e. the efficient frontier is continuous. We test the solution on the example of four assets with eight years daily stock prices. Monte Carlo simulation method is also used in this study to test wider dataset and the results match well. This research may help us to develop a deeper understanding of the efficient portfolio. The analysis in weight space may also be extended to deal with more constraints on the portfolio weights in future research.

4.6 Appendix

4.6.1 Minimum Variance Portfolio

The question is

$$\min_{\omega} \frac{1}{2} \omega' \Omega \omega \tag{4.81}$$

subject to

$$I'\omega = 1 \; where \; I' = [1, 1, ...] \tag{4.82}$$

$$\omega_j \geq 0 \text{ for all } j \tag{4.83}$$

Now we have

$$L = \frac{1}{2}\omega'\Omega\omega - \theta(I'\omega - 1) \tag{4.84}$$

$$\frac{\partial L}{\partial \omega} = \Omega\omega - \theta I = 0 \tag{4.85}$$

$$\frac{\partial L}{\partial \theta} = \omega' I - 1 = 0 \tag{4.86}$$

from (4.85)

$$\omega = \theta\Omega^{-1} I \tag{4.87}$$

from (4.86)

$$\omega' I = 1 \tag{4.88}$$

using (4.88) in (4.87)

$$1 = \omega' I = \theta I'\Omega^{-1} I \tag{4.89}$$

or

$$\theta = \frac{1}{I'\Omega^{-1} I} \tag{4.90}$$

plug (4.90) into (4.87)

$$\omega = \frac{\Omega^{-1} I}{I'\Omega^{-1} I} \tag{4.91}$$

since

$$\Omega = \begin{pmatrix} \sigma_1^2 & 0 & 0 \\ 0 & \sigma_2^2 & 0 \\ 0 & 0 & \sigma_3^2 \end{pmatrix} \tag{4.92}$$

and

$$\Omega^{-1} = \begin{pmatrix} 1/\sigma_1^2 & 0 & 0 \\ 0 & 1/\sigma_2^2 & 0 \\ 0 & 0 & 1/\sigma_3^2 \end{pmatrix} \tag{4.93}$$

plug Ω^{-1} into (4.91), we have

$$\omega = \begin{pmatrix} 1/\sigma_1^2 \\ 1/\sigma_2^2 \\ 1/\sigma_3^2 \end{pmatrix} \frac{1}{\sum 1/\sigma_j^2} \tag{4.94}$$

that is, the minimize variance portfolio has assets weights

$$\omega_j = \frac{1/\sigma_j^2}{\sum 1/\sigma_j^2} \tag{4.95}$$

4.6.2 Efficient Portfolio with Given Expected Return

The problem is to minimize the portfolio variance based on the given expected portfolio return and budget constraints. Now the problem is

$$min \left\{ \frac{1}{2} \omega^T \Omega \omega \right\} \tag{4.96}$$

$$s.t.\ A^T \omega = B \tag{4.97}$$

where ω is the weight of n assets in portfolio. $\mu = \begin{bmatrix} \mu_1 \\ ... \\ \mu_n \end{bmatrix}$ is the return of n assets. $I = \begin{bmatrix} 1 \\ ... \\ 1 \end{bmatrix}$, $A = \begin{bmatrix} \mu & I \end{bmatrix}$ and $B = \begin{bmatrix} \mu_p \\ 1 \end{bmatrix}$ where μ_p is the given portfolio return. So the constraints can also be written as $\mu^T \omega = \mu_p$ and $I^T \omega = 1$.

This Mean-Variance problem has only equality constraints and can be solved analytically. Let the Lagrange be

$$L = \frac{1}{2}\omega^T \Omega \omega + \theta^T (B - A^T \omega) \tag{4.98}$$

The first order conditions are

$$\Omega\omega - A\theta = 0 \tag{4.99}$$

$$A^T \omega = B \tag{4.100}$$

And the solution is

$$\omega = \Omega^{-1} A \theta \tag{4.101}$$

$$\theta = (A^T \Omega^{-1} A)^{-1} B \tag{4.102}$$

In this study, to simplify the question, we assume that there is no correlation between different asset. Thus by applying equation (4.101), the portfolio weights at point k is given by

$$\omega_i^h = \frac{\mu_j/\sigma_j^2}{\sum_{i=1}^4 \mu_i/\sigma_i^2} \tag{4.103}$$

Note that with the change of the given expected portfolio return μ_p, the solution moves along the efficient line. And the corresponding portfolio mean and variance also moves along the efficient frontier. If we add inequality constraints to the (4.97) such as the no short selling constraints $\omega_i \geq 0$, the problem is described as

$$min \left\{ \frac{1}{2}\omega^T \Omega \omega \right\} \tag{4.104}$$

$$s.t.\ A^T \omega = B$$

$$\omega_i \geq 0$$

There will be no analytical solution to the equation (4.104) and the numerical methods have to be used to solve the quadratic problem.

4.6.3 Efficient Portfolio Area Beneath Point c

Points beneath c on the vertical axis are efficient. Figure 4.4. Suppose $x_2 = \omega_2^c - \varepsilon$ where $0 < \varepsilon < 1$. The portfolio return at point x is

$$\mu_p^x = \mu_3 - \eta_2(\omega_2^c - \varepsilon) \tag{4.105}$$

and

$$\mu_p^c - \mu_p^x = -\eta_3\varepsilon < 0 \tag{4.106}$$

Thus, x is an efficient point.

4.6.4 Explicit Expressions for the Portfolio Mean and Variance in Three Asset Problem

1, The minimum variance portfolio h. Since we have $\omega_i^h = \frac{1/\sigma_i^2}{\sum 1/\sigma_j^2}$, we have

$$\begin{aligned} \mu_p^h &= \begin{pmatrix} \mu_1^h & \mu_2^h & \mu_3^h \end{pmatrix} \begin{pmatrix} \omega_1^h \\ \omega_2^h \\ \omega_3^h \end{pmatrix} \\ &= \mu_1 \frac{1/\sigma_1^2}{\sum 1/\sigma_j^2} + \mu_2 \frac{1/\sigma_2^2}{\sum 1/\sigma_j^2} + \mu_3 \frac{1/\sigma_3^2}{\sum 1/\sigma_j^2} \\ &= \frac{\mu_1\sigma_2^2\sigma_3^2 + \mu_2\sigma_1^2\sigma_2^2 + \mu_3\sigma_1^2\sigma_2^2}{B} \end{aligned} \tag{4.107}$$

where

$$B = \sigma_2^2\sigma_3^2 + \sigma_1^2\sigma_2^2 + \sigma_1^2\sigma_3^2 \tag{4.108}$$

Also, the portfolio variance is

$$
\begin{aligned}
(\sigma_p^h)^2 &= \begin{pmatrix} \omega_1^h & \omega_2^h & \omega_3^h \end{pmatrix} \begin{pmatrix} \sigma_1^2 & 0 & 0 \\ 0 & \sigma_2^2 & 0 \\ 0 & 0 & \sigma_3^2 \end{pmatrix} \begin{pmatrix} \omega_1^h \\ \omega_2^h \\ \omega_3^h \end{pmatrix} \\
&= (\omega_1^h)^2\sigma_1^2 + (\omega_2^h)^2\sigma_2^2 + (\omega_3^h)^2\sigma_3^2 \\
&= \frac{1/\sigma_1^2}{\sum 1/\sigma_j^2} + \frac{1/\sigma_2^2}{\sum 1/\sigma_j^2} + \frac{1/\sigma_3^2}{\sum 1/\sigma_j^2} \\
&= \frac{1}{\sigma_j^2} = \frac{\sigma_1^2\sigma_2^2\sigma_3^2}{B} \qquad (4.109)
\end{aligned}
$$

2, The corner portfolio c. We have

$$\omega_1^c = 0 \qquad (4.110)$$

$$
\begin{aligned}
\omega_2^c &= \omega_2^h - s\omega_1^h \\
&= \frac{1/\sigma_2^2}{\sum 1/\sigma_j^2} - \left(\frac{\eta_2\sigma_1^2 - \eta_1\sigma_3^2}{\eta_1\sigma_3^2 + (\eta_1+\eta_2)\sigma_2^2}\right)\left(\frac{1/\sigma_1^2}{\sum 1/\sigma_j^2}\right) \\
&= \frac{(\mu_2-\mu_1)\sigma_3^2}{(\mu_2-\mu_1)\sigma_3^2 + (\mu_3-\mu_1)\sigma_2^2} \qquad (4.111)
\end{aligned}
$$

$$\omega_3^c = 1 - \omega_1^c - \omega_2^c = \frac{(\mu_3-\mu_1)\sigma_2^2}{(\mu_2-\mu_1)\sigma_3^2 + (\mu_3-\mu_1)\sigma_2^2} \qquad (4.112)$$

thus,

$$
\begin{aligned}
\mu_p^c &= \begin{pmatrix} \mu_1^c & \mu_2^c & \mu_3^c \end{pmatrix} \begin{pmatrix} \omega_1^c \\ \omega_2^c \\ \omega_3^c \end{pmatrix} \\
&= \frac{\mu_2(\mu_2-\mu_1)\sigma_3^2 + \mu_3(\mu_3-\mu_1)\sigma_2^2}{A} \qquad (4.113)
\end{aligned}
$$

where

$$A = (\mu_2-\mu_1)\sigma_3^2 + (\mu_3-\mu_1)\sigma_2^2 \qquad (4.114)$$

and the portfolio variance is

$$\begin{aligned}
(\sigma_p^c)^2 &= (\omega_1^c)^2\sigma_1^2 + (\omega_2^c)^2\sigma_2^2 + (\omega_3^c)^2\sigma_3^2 \\
&= \frac{\sigma_2^2\sigma_3^2}{A^2}\left(\sigma_3^2(\mu_2-\mu_1)^2 + \sigma_2^2(\mu_3-\mu_1)^2\right) \\
&= \frac{\sigma_2^2\sigma_3^2}{A^2}\left(\sigma_3^2(\mu_2-\mu_1)\mu_2 + \sigma_2^2(\mu_3-\mu_1)\mu_3 - \mu_1 A\right) \\
&= \frac{\sigma_2^2\sigma_3^2}{A^2}(A\mu_p^c - A\mu_1) \\
&= \frac{\sigma_2^2\sigma_3^2}{A}(\mu_p^c - \mu_1) \qquad (4.115)
\end{aligned}$$

3, For the maximum return portfolio m,

$$\mu_p^m = \mu_3 \qquad (4.116)$$

$$(\sigma_p^m)^2 = \sigma_3^2 \qquad (4.117)$$

4.6.5 Explicit Expressions for the Portfolio Mean and Variance in Four Asset Problem

1, For the minimum variance portfolio at point h, the weights for the four assets are given by $\omega_i^h = \frac{1/\sigma_i^2}{R}$ where $R \equiv \sum_{i=1}^4 1/\sigma_i^2$ and the portfolio mean and variance can be calculated as

$$
\begin{aligned}
\mu_p^h &= \begin{bmatrix}\mu_1 \ \mu_2 \ \mu_3 \ \mu_4\end{bmatrix} \begin{bmatrix}\omega_1^h \\ \omega_2^h \\ \omega_3^h \\ \omega_4^h\end{bmatrix} \\
&= \frac{\mu_1 \frac{1}{\sigma_1^2} + \mu_2 \frac{1}{\sigma_2^2} + \mu_3 \frac{1}{\sigma_3^2} + \mu_4 \frac{1}{\sigma_4^2}}{R} \\
&= \frac{\mu_1 \sigma_2^2 \sigma_3^2 \sigma_4^2 + \mu_2 \sigma_1^2 \sigma_3^2 \sigma_4^2 + \mu_3 \sigma_1^2 \sigma_2^2 \sigma_4^2 + \mu_4 \sigma_1^2 \sigma_2^2 \sigma_3^2}{D} \qquad (4.118) \\
(\sigma_p^h)^2 &= \begin{bmatrix}\omega_1^h \ \omega_2^h \ \omega_3^h \ \omega_4^h\end{bmatrix} \begin{bmatrix}\sigma_1^2 \ 0 \ 0 \ 0 \\ 0 \ \sigma_2^2 \ 0 \ 0 \\ 0 \ 0 \ \sigma_3^2 \ 0 \\ 0 \ 0 \ 0 \ \sigma_4^2\end{bmatrix} \begin{bmatrix}\omega_1^h \\ \omega_2^h \\ \omega_3^h \\ \omega_4^h\end{bmatrix} \\
&= (\omega_1^h)^2 \sigma_1^2 + (\omega_2^h)^2 \sigma_2^2 + (\omega_3^h)^2 \sigma_3^2 + (\omega_4^h)^2 \sigma_4^2 \\
&= \frac{1}{R} = \frac{\sigma_1^2 \sigma_2^2 \sigma_3^2 \sigma_4^2}{D} \qquad (4.119)
\end{aligned}
$$

where

$$D \equiv \sigma_2^2 \sigma_3^2 \sigma_4^2 + \sigma_1^2 \sigma_3^2 \sigma_4^2 + \sigma_1^2 \sigma_2^2 \sigma_4^2 + \sigma_1^2 \sigma_2^2 \sigma_3^2 \qquad (4.120)$$

2, For the first corner portfolio at point c where the efficient line hits plane $p1$. The efficient line $\overrightarrow{hk}$ has equation

$$\frac{x_1 - \omega_1^h}{l} = \frac{x_2 - \omega_2^h}{m} = \frac{x_3 - \omega_3^h}{n} = t \qquad (4.121)$$

where l, m, n has been defined in previous section. Since the corner portfolio c is at where the efficient line hits plane $p1$ where $\omega_1^c = 0$. Plug in this to the equation (4.121), we have

$$t = -\frac{\omega_1^h}{l} = \frac{\omega_1^h}{\omega_1^h - \omega_1^k} = \frac{Q}{Q - R\mu_1} \qquad (4.122)$$

then we have

$$\omega_2^c \equiv x_2^c = tm + \omega_2^h = \frac{\mu_2 - \mu_1}{\sigma_2^2(Q - R\mu_1)} \tag{4.123}$$

$$\omega_3^c \equiv x_3^c = tn + \omega_3^h = \frac{\mu_3 - \mu_1}{\sigma_3^2(Q - R\mu_1)} \tag{4.124}$$

$$\omega_4^c \equiv x_4^c = 1 - x_2^c - x_3^c = \frac{(\mu_2 - \mu_1)/\sigma_2^2}{E} \tag{4.125}$$

the portfolio mean and variance can be calculated as

$$\begin{aligned} \mu_p^c &= \mu_2\omega_2^c + \mu_3\omega_3^c + \mu_4\omega_4^c \\ &= \frac{1}{E}\left[\mu_2(\frac{\mu_2 - \mu_1}{\sigma_2^2}) + \mu_3(\frac{\mu_3 - \mu_1}{\sigma_3^2}) + \mu_4(\frac{\mu_4 - \mu_1}{\sigma_4^2})\right] \qquad (4.126) \\ (\sigma_p^c)^2 &= (\omega_2^c)^2\sigma_2^2 + (\omega_3^c)^2\sigma_3^2 + (\omega_4^c)^2\sigma_4^2 \\ &= \frac{1}{E^2}\left[\frac{(\mu_2 - \mu_1)^2}{\sigma_2^2} + \frac{(\mu_3 - \mu_1)^2}{\sigma_3^2} + \frac{(\mu_4 - \mu_1)^2}{\sigma_4^2}\right] \qquad (4.127) \end{aligned}$$

where

$$E \equiv \frac{\mu_2 - \mu_1}{\sigma_2^2} + \frac{\mu_3 - \mu_1}{\sigma_3^2} + \frac{\mu_4 - \mu_1}{\sigma_4^2} \tag{4.128}$$

From the expression of the portfolio variance, we know

$$\begin{aligned} (\sigma_p^c)^2 &= \frac{1}{E^2}[\frac{\mu_2(\mu_2 - \mu_1)}{\sigma_2^2} + \frac{\mu_3(\mu_3 - \mu_1)}{\sigma_3^2} + \frac{\mu_4(\mu_4 - \mu_1)}{\sigma_4^2} \\ &\quad -\mu_1\left(\frac{(\mu_2 - \mu_1)}{\sigma_2^2} + \frac{(\mu_3 - \mu_1)}{\sigma_3^2} + \frac{(\mu_4 - \mu_1)}{\sigma_4^2}\right)] \\ &= \frac{1}{E}(\mu_p^c - \mu_1) \qquad (4.129) \end{aligned}$$

then we have

$$\mu_p^c = (\sigma_p^c)^2 E + \mu_1 \tag{4.130}$$

Further, since

$$\frac{\partial \mu_p^c}{\partial \sigma_p^c} = 2\sigma_p^c E \tag{4.131}$$

We know that the frontier is differentiable at the corner portfolio c.

3, For the first corner portfolio at point g where the efficient line hits plane $p2$. Since we have proved that $\omega_1^g = 0$ and $\omega_2^g = 0$ and the question has been reduced to three asset problem. From the results in section 4.3.1, we have $\omega_3^g = (\sigma_4^2(\mu_3 - \mu_2))/F$ and $\omega_4^g = 1 - \omega_3^g$ and the portfolio mean and variance are calculated as

$$\mu_p^g = \frac{1}{F}\left[\mu_3\sigma_4^2(\mu_3 - \mu_2) + \mu_4\sigma_3^2(\mu_4 - \mu_2)\right] \tag{4.132}$$

$$(\sigma_p^g)^2 = \frac{\sigma_3^2\sigma_4^2}{F^2}\left[\sigma_4^2(\mu_3 - \mu_2)^2 + \sigma_3^2(\mu_4 - \mu_2)^2\right] \tag{4.133}$$

where

$$F = \sigma_4^2(\mu_3 - \mu_2) + \sigma_3^2(\mu_4 - \mu_2) \tag{4.134}$$

The relationship between the portfolio mean and volatility is given by

$$\begin{aligned}(\sigma_p^g)^2 &= \frac{\sigma_3^2\sigma_4^2}{F^2}\left[\sigma_4^2\mu_3(\mu_3 - \mu_2) + \sigma_3^2\mu_4(\mu_4 - \mu_2) + \mu_2\frac{\sigma_4^2(\mu_2 - \mu_3) + \sigma_3^2(\mu_2 - \mu_4)}{A}\right] \\ &= \frac{\sigma_3^2\sigma_4^2}{F^2}(\mu_p^g - \mu_2)\end{aligned} \tag{4.135}$$

i.e

$$\mu_p^g = \mu_2 + \frac{(\sigma_p^g)^2 F}{\sigma_3^2\sigma_p^g} \tag{4.136}$$

Further, since

$$\frac{\partial(\mu_p)}{\partial(\sigma_p^g)} = 2\frac{\sigma_p^g F}{\sigma_3^2\sigma_4^2} \tag{4.137}$$

We have that the frontier is differentiable at the corner portfolio g.

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