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# Symplectomorphism Groups of Weighted Projective Spaces and Related Embedding Spaces

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# Symplectomorphism Groups of Weighted Projective Spaces and Related Embedding Spaces

(Thesis format: Monograph)

by

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in
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# Abstract

In this thesis, we study 4-dimensional weighted projective spaces and homotopy properties of their symplectomorphism groups. Using these computations, we also investigate some homotopy theoretic properties of a few associated embedding spaces. In the classical case of the complex projective plane, Gromov observed that its symplectomorphism group is homotopy equivalent to its subgroup of Kahler isometries. We find that in the case of one singularity, the symplectomorphism group is weakly homotopy equivalent to the Kahler isometry group of a certain Hirzebruch surface, which corresponds to the resolution of the singularity. In the case of multiple singularities, the symplectomorphism groups are weakly equivalent to tori. These computations then allow us to investigate some properties of related embedding spaces.

**Keywords**: symplectic orbifold, weighted projective space, symplectmorphism group, Hirzebruch surface, toric orbifold, toric manifold, symplectic cutting, symplectic blow-up, Hirzebruch-Jung resolution

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**To my wife,**
**Nargess Hojati Kermani**

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# Chapter 1

# Introduction

Consider the weighted projective space $\mathbb{C}P^2_{a,b,c}$ for $a, b, c$ relatively prime. This is the space $\mathbb{C}^3 \setminus \{0\} / \sim$, where the equivalence relation $\sim$ is given by

$$(z_0, z_1, z_2) \sim (w_0, w_1, w_2) \iff (w_0, w_1, w_2) = (\lambda^a z_0, \lambda^b z_1, \lambda^c z_2)$$

for $\lambda \in \mathbb{C}^*$. Let's write the equivalence class of a point $(z_0, z_1, z_2)$ in homogeneous coordinates as $[z_0 : z_1 : z_2] \in \mathbb{C}P^2_{a,b,c}$ without any subscripts that identify the weights $a, b, c$. Then $\mathbb{C}P^2_{a,b,c}$ is a 4 dimensional orbifold, and we can put a symplectic form on it using symplectic reduction; the same way we do for the *manifold* $\mathbb{C}P^2$. The symplectic form on $\mathbb{C}P^2_{a,b,c}$ depends on the weights, so we will call it $\omega_{a,b,c}$. The purpose of this thesis is to investigate the homotopy type of the group of symplectomorphisms of the symplectic orbifold $(\mathbb{C}P^2_{a,b,c}, \omega_{a,b,c})$ and to use this to probe various embedding spaces. There is a lot of recent history to the investigation of symplectomorphism groups, at least for manifolds. It seems that symplectic orbifolds have been discriminated against and that's just sad.

Some early results about the topology of symplectomorphism groups are by McDuff in [34] where she uses the *Moser fibration*

$$\mathrm{Symp}(M, \omega) \cap \mathrm{Diff}_0(M) \longrightarrow \mathrm{Diff}_0(M) \longrightarrow \mathcal{S}_{[\omega]}$$

to detect differences in the identity component $\mathrm{Diff}_0(M)$ of the diffeomorphism group of a manifold $M$ and its subgroup of symplectomorphisms. Here, $\mathcal{S}_{[\omega]}$ is the space of

symplectic forms on $M$ that are isotopic to $\omega$ and we require that the symplectic forms be standard outside some compact set. The two main results in this paper are:

**Theorem** ([34]). *Let $(M, \omega)$ be the 1-point blow up of $\mathbb{C}P^2$ with its standard Kahler form. Then $\pi_1 \mathrm{Symp}(M, \omega)$ does not surject onto $\pi_1 \mathrm{Diff}_0(M)$.*

**Theorem** ([34]). *Let $M = \mathbb{C}^2 \backslash \{0\}$ with its standard symplectic form $\omega$. Then $\mathrm{Symp}(M, \omega)$ is not connected.*

These results are proved using cellular decomposition methods, and came before the machinery of Gromov's theory of $J$-holomorphic curves was introduced in [18]. The techniques developed by Gromov proved to be extremely useful in the study of symplectomorphism groups. The following results form the foundation for much of the work that followed:

**Theorem** ([18]). *Let $\sigma \oplus \sigma$ be the standard split symplectic form on $S^2 \times S^2$, where $\sigma$ gives area 1 to each sphere. Then $\mathrm{Symp}(S^2 \times S^2, \sigma \oplus \sigma)$ is homotopy equivalent to the subgroup $(\mathrm{SO}(3) \times \mathrm{SO}(3)) \rtimes \mathbb{Z}_2$ of Kahler isometries.*

**Theorem** ([18]). *Consider $\mathbb{R}^4$ with its standard symplectic form $\omega_0$. Then the group $\mathrm{Symp}_c(\mathbb{R}^4, \omega_0)$ of compactly supported symplectomorphisms is contractible.*

**Theorem** ([18]). *Consider $\mathbb{C}P^2$ with the standard Kahler form $\omega_{FS}$. Then $\mathrm{Symp}(\mathbb{C}P^2, \omega_{FS})$ is homotopy equivalent to the subgroup $\mathrm{PU}(3)$ of Kahler isometries.*

In a sequel to Gromov's paper, Abreu [1] extended some of Gromov's results by considering the group

$$G_\mu := \mathrm{Symp}(S^2 \times S^2, \mu\sigma \oplus \sigma), \quad 1 < \mu \le 2.$$

In fact, it was Gromov [18] who warned that the topology of $G_\mu$ changes if the spheres are allowed to have different areas, but he didn't pursue the details. This task fell to

Abreu and many of the techniques used in [1] have now become standard. In particular, they were used to compute

$$\pi_1(G_\mu / \mathrm{SO}(3) \times \mathrm{SO}(3)) \quad \text{and} \quad H^*(G_\mu / \mathrm{SO}(3) \times \mathrm{SO}(3); \mathbb{R}).$$

Then, in a sequel to this sequel, Abreu and McDuff [4] extended these results some more by considering the symplectomorphism groups of the manifolds $(S^2 \times S^2, \mu\sigma \oplus \sigma)$ for $\mu > 2$, and also the non-trivial bundle

$$(\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2, \omega_\mu),$$

where $\omega_\mu$ gives area $\mu > 0$ to the exceptional divisor and area 1 to each fibre. It was found that the topology of these symplectomorphism groups changes whenever $\mu$ crosses integer values, but we will not explain the details here.

These early works are only the tip of the iceberg and we will not give a complete survey here as this would just take too long. Here are some highlights though:

- Anjos [3] extended Abreu's results and computed the full homotopy type of the group $\mathrm{Symp}(S^2 \times S^2, \mu\sigma \oplus \sigma)$ for $1 < \mu \leq 2$. She also computed its homology group with $\mathbb{Z}_2$-coefficients. These results have since been extended further by various people.

- Lalonde-Pinsonnault [25] and Pinsonnault [40] extended the results of Abreu-McDuff to the 1-point blow ups of these manifolds. These are the manifolds $(S^2 \times S^2) \# \overline{\mathbb{C}P}^2 \cong \mathbb{C}P^2 \# 2\overline{\mathbb{C}P}^2$ with induced symplectic forms from before. They found, as expected, that the topology of these groups changes as $\mu$ passes integer values. These results were then used to analyze various embedding spaces.

- The previous results were extended again by Anjos-Pinsonnault [6] to the manifold $\mathbb{C}P^2 \# 3\overline{\mathbb{C}P}^2$ with the induced symplectic form from before.

- Seidel [42] computed the homotopy type of the group $\mathrm{Symp}_c(T^*S^2)$ of compactly supported symplectomorphisms of $T^*S^2$ with its canonical symplectic form.

- Evans [14] used the previous result of Seidel to compute the homotopy type of the group $\mathrm{Symp}_c(T^*\mathbb{R}P^2)$ of compactly supported symplectomorphisms of $T^*\mathbb{R}P^2$. He also considered the symplectomorphism groups of the 3, 4, and 5-point blow ups of $\mathbb{C}P^2$ with their monotone symplectic forms, and computed their homotopy groups.

- Evans, in the same paper [14] also considered the algebraic variety given as the solution to the equation $x^2 + y^2 + z^n = 1$ with a Kahler form induced from $\mathbb{C}^3$. He proved that its symplectomorphism group is homotopy equivalent to its group of components, and that its group of components injects into the braid group of $n$-strands on the disc.

- In a recent work, Hind-Pinsonnault-Wu [21] considered the symplectization $s(S^3 / \mathbb{Z}_n)$ of the lens space $S^3 / \mathbb{Z}_n$ and computed the homotopy type of the corresponding group of compactly supported symplectomorphisms. This point of view was then used the investigate the space of embeddings of a singular ball into a bigger singular ball.

- Lastly, we should mention another work by McDuff [31] that in many ways inspired our approach to this problem. In order to construct certain 6-dimensional symplectic manifolds with $S^1$-action, she must consider the reduced spaces by the $S^1$-action, which are symplectic orbifolds. To establish some uniqueness properties, she must prove that these reduced spaces are "rigid" ([31]-Definition 2.13). One of these properties of rigidity is the connectedness of an orbifold symplectomorphism group. This paper contains the only results that we know of about orbifold symplectomorphism groups.

We want to investigate orbifolds, not manifolds. But, the useful thing in our case is that we can resolve the singularities (get rid of them) and then use well-known techniques

to investigate the symplectomorphism group of the resolution. The idea is to compare a certain subgroup of the symplectomorphism group in question to a subgroup of the symplectomorphism group of the resolution.

There is a problem though: Given a symplectic orbifold $(\mathcal{O}, \omega)$, then how do we define its symplectomorphism group? This is easy in the case of a symplectic manifold; a symplectomorphism is just a diffeomorphism that preserves the symplectic form. Maps between orbifolds become more complicated though. In fact, there seem to be 4 distinct notions of orbifold map [9], and each of these involve remembering different parts of the data that we need to define maps on orbifolds in the first place. In this thesis, we really only care about the weakest possible notion of orbifold map; these are the *reduced* orbifold maps as defined in [8] (see also [9]). The focus in [8] is on 2 types of orbifold diffeomorphism group. First, the group $\mathrm{Diff}^{\mathrm{orb}}(\mathcal{O})$ is defined as a space of maps of the form $(f, \{f_x\})$, where $f$ is a continuous map of the underlying topological space, and $\{f_x\}$ is a set of lifts of $f$ to uniformizing charts; these lifts being parametrized according to a natural stratification of the orbifold. The group $\mathrm{Diff}^{\mathrm{red}}(\mathcal{O})$ is defined by "forgetting the lifts"; that is, by defining a stricter notion of equivalence on the space of orbifold diffeomorphisms. Thus, the group $\mathrm{Diff}^{\mathrm{red}}(\mathcal{O})$ is naturally viewed as the quotient of $\mathrm{Diff}^{\mathrm{orb}}(\mathcal{O})$ by a subgroup that consists of all lifts of the identity map. We choose to follow the conventions in [8] when defining orbifold symplectomorphism groups: We first define the group $\mathrm{Symp}^{\mathrm{orb}}(\mathcal{O})$ to be the subgroup of $\mathrm{Diff}^{\mathrm{orb}}(\mathcal{O})$ whose lifts preserve the symplectic forms on all uniformizing charts. Then we define the group $\mathrm{Symp}^{\mathrm{red}}(\mathcal{O})$ to be the quotient of $\mathrm{Symp}^{\mathrm{orb}}(\mathcal{O})$ by the subgroup of all lifts of the identity map.

Our main focus is on the group $\mathrm{Symp}^{\mathrm{red}}(\mathcal{O})$, where $\mathcal{O}$ is the weighted projective space $\mathbb{C}P^2_{a,b,c}$ equipped with its natural symplectic form $\omega_{a,b,c}$. We call this group $\mathrm{Symp}^{\mathrm{red}}_{a,b,c}$. It's important that the weights $a, b, c$ (the orders of the singularities) be relatively prime, otherwise the singularities would not be isolated. In fact, we are only able to get results when $a = 1$; thus we consider the groups $\mathrm{Symp}^{\mathrm{red}}_{1,b,c}$ for $1 < b < c$ with $b$ and $c$ relatively prime. We are confident that we know how to prove the more general

result for $\mathrm{Symp}^{\mathrm{red}}_{a,b,c}$, but this is not included in the thesis due to lack of time. The work is done in stages, reflecting our general approach to the problem. We prove the following results about the symplectomorphism groups:

**Theorem 1.** *The group* $\mathrm{Symp}^{\mathrm{red}}_{1,1,c}$ *is weakly homotopy equivalent to* $\mathrm{U}(2)\,/\,\mathbb{Z}_c$, *where* $c > 1$.

Here, there is a natural linear action of $\mathrm{U}(2)\,/\,\mathbb{Z}_c$ on $\mathbb{C}P^2_{1,1,c}$ given by

$$A \cdot [z_0 : z_1 : z_2] = [\alpha z_0 + \beta z_1 : \gamma z_0 + \delta z_1 : z_2],$$

where $A$ is the matrix with entries $\alpha, \beta, \gamma, \delta$. We should note that the group $\mathrm{U}(c)\,/\,\mathbb{Z}_c$ can be interpreted as the Kahler isometry group of the Hirzebruch surface $W_c$, since blowing up the singularity of $\mathbb{C}P^2_{1,1,c}$ results in the Hirzebruch surface $W_c$, and there is a biholomorphism

$$\mathbb{C}P^2_{1,1,c} \setminus p_c \cong W_c \setminus zero\ section,$$

where $p_c \in \mathbb{C}P^2_{1,1,c}$ is the singularity. We should note that this identification can be made symplectic around arbitrarily small neighbourhoods ([38]-Theorem 2). We prove Theorem 1 in Section 4.2 by using exactly this idea; that is, we resolve the singularity of $\mathbb{C}P^2_{1,1,c}$ and show that the subgroup of symplectomorphisms acting as the identity near $p_c$ can be identified, up to weak homotopy equivalence, with the subgroup of symplectomorphisms of $W_c$ acting as the identity near the zero section. We then use known results about the subgroup of symplectomorphisms of $W_c$ acting as the identity near the zero section (see [11]-Proposition 3.2 and [20]-Lemma 9.1).

The next step in our investigation involves the groups $\mathrm{Symp}^{\mathrm{red}}_{1,b,c}$, where $c = bk + 1$ and $k \geq 1$ is an integer. To investigate the symplectomorphism group in this case, we first find the resolution and identify it symplectically with a $b$-fold blow up of the Hirzebruch surface $W_k$. This process is arduous and seems unnecessary in light of recent discoveries by us. Nevertheless, it is still included in Section 3.2, perhaps for cultural reasons, and also because we didn't have enough time to re-organize it. Section 3.2 is still informative

though, because we go through the process of constructing an explicit resolution of the toric model and then making the proper identifications with the non-toric resolution in Section 3.1. It is perhaps redundant, though enlightening and informative to see how the resolutions match up from two different points of view.

The bulk of the computations for the groups $\mathrm{Symp}^{\mathrm{red}}_{1,b,c}$ are done in Sections 4.3 and 4.4, where we prove the following results:

**Theorem 2a.** $\mathrm{Symp}^{\mathrm{red}}_{1,b,c}$ *is weakly homotopy equivalent to* $\mathrm{Aut}(T_{p_c}) \simeq \mathrm{T}^2 / \mathbb{Z}_c$ *when* $c = bk + 1$, *where* $\mathrm{Aut}(T_{p_c})$ *is the group of automorphisms of the uniformized tangent space at* $p_c$.

**Theorem 2b.** $\mathrm{Symp}^{\mathrm{red}}_{1,b,c}$ *is weakly homotopy equivalent to either* $\mathrm{Aut}(T_{p_b})$ *or* $\mathrm{Aut}(T_{p_c})$ *when* $1 < b < c$. *In this more general case, it turns out that both uniformized tangent spaces are isomorphic. Up to homotopy, these automorphism groups are just* $\mathrm{T}^2$.

We should note that Theorem 2a is a special case of Theorem 2b. The proof of Theorem 2a is given in Section 4.3 and the proof of the latter is given in Section 4.4. Actually, most of the work is contained in Section 4.3, and then we realized that we could prove the more general result using similar methods, so we decided to include a less detailed version of this argument in Section 4.4. Let us outline the general approach to the proof. For the case $\mathrm{Symp}^{\mathrm{red}}_{1,b,c}$ with $c = bk + 1$, we already described in the last paragraph that we identify the resolution $R$ symplectically with a $b$-fold blow up of the Hirzebruch surface $W_k$. The resolution creates a chain of embedded symplectic spheres whose self-intersection numbers are given by the continued fraction expansion of $\frac{bk+1}{b}$. Let $\mathrm{Symp}(R)$ be the symplectomorphism group of the resolution, where $R$ is equipped with a natural symplectic form that comes from the resolution process. If $\Gamma$ is the configuration of embedded symplectic spheres created from the resolution, then we are interested in the group $\mathrm{Symp}^{cpt}(R \setminus \Gamma)$ of symplectomorphisms that are compactly supported away from the configuration $\Gamma$. This group turns out to be homotopy equivalent to the kernel $\mathcal{K}$ in the fibration

$$\mathcal{K} \longrightarrow \mathrm{Symp}^{\mathrm{red}}_{1,b,c} \longrightarrow \mathrm{Aut}(T_{p_b}) \times \mathrm{Aut}(T_{p_c}),$$

so the main focus is on computing the weak homotopy type of the group $\mathrm{Symp}^{cpt}(R \setminus \Gamma)$. The techniques that we use are standard and are variations on the techniques used in many previous works; see for instance [1], [14], [26] and [21].

The next chapter (Chapter 5) involves applying the techniques from the previous chapter to various embedding spaces. We follow the general framework from [26] and [40]. The key idea is to recognize that the groups $\mathrm{Symp}^{\mathrm{red}}_{1,1,c}$ and $\mathrm{Symp}^{\mathrm{red}}_{1,b,c}$ act transitively on certain spaces of embedded singular or smooth balls. The first result in this chapter is in Section 5.1. If $\Im\mathrm{Emb}^{\epsilon}_{1,1,c}$ is the space of singular balls of size $\epsilon < 1$ in $\mathbb{C}P^2_{1,1,c}$ modulo reparametrization, then we have

**Theorem 3a.** $\Im\mathrm{Emb}^{\epsilon}_{1,1,c}$ *is weakly contractible.*

There is a corollary of this result that we mention in Section 5.1 as well. This is about the corresponding *unparametrized* space of symplectic embeddings; the space $\mathrm{Emb}^{\epsilon}_{1,1,c}$. The relation between the space $\mathrm{Emb}^{\epsilon}_{1,1,c}$ and the space $\Im\mathrm{Emb}^{\epsilon}_{1,1,c}$ is just that the latter is the quotient of the former by the group $\mathrm{Symp}^{\mathrm{red}}(B_c(\epsilon))$ of reduced symplectomorphisms of the orbi-ball (singular ball) $B_c(\epsilon)$. We then have the following:

**Theorem 3b.** $\mathrm{Emb}^{\epsilon}_{1,1,c}$ *is weakly homotopy equivalent to* $\mathrm{U}(2) / \mathbb{Z}_c$.

When $(M, \omega)$ is a symplectic manifold, let $\Im\mathrm{Emb}(B(\lambda), M)$ be the space of unparametrized embeddings of balls $B(\lambda)$ of capacity $\lambda$ in $M$. This space carries information about symplectic blow ups; for instance, if the space $\Im\mathrm{Emb}(B(\lambda), M)$ is connected, then two symplectic blow ups of $(M, \omega)$ of the same size are isotopic ([35]-Proposition 7.18). It was proved in ([30]-Corollary 1.5) that the space $\Im\mathrm{Emb}(B(\lambda), M)$ is connected when $M$ has *nonsimple type*. Naturally, these embedding spaces should also carry information about the symplectic orbifolds in question, but it's not clear to what extent because there aren't many general results about symplectic orbifolds.

In Section 5.2, we consider the space $\Im^{\infty}\mathrm{Emb}^{\delta}_{1,1,c}$ of *smooth* symplectic balls of capacity $\delta < 1$ embedded into the weighted projective space $\mathbb{C}P^2_{1,1,c}$. The following result is proved:

**Theorem 4.** *Let $p_c$ be the singular point of $\mathbb{C}P^2_{1,1,c}$. Then $\Im^\infty\mathrm{Emb}^\delta_{1,1,c}$ is weakly homotopy equivalent to $\mathbb{C}P^1 \simeq \mathbb{C}P^2_{1,1,c} \setminus p_c$.*

This result is an analogue of Theorem 1.10(1) in [40], where it is proved that the corresponding unparametrized embedding space of balls into the manifold $\mathbb{C}P^2$ is weakly homotopy equivalent to $\mathbb{C}P^2$ itself; essentially meaning that balls of capacity less than 1 behave like points, homotopically. In the same theorem, Pinsonnault also proves a corresponding result for spaces of two disjoint balls in $\mathbb{C}P^2$, showing that this space is weakly equivalent to the space of ordered configurations of two points in $\mathbb{C}P^2$. It's possible that a similar result holds in our case, but we haven't investigated this yet.

# Chapter 2

# Preliminaries

## 2.1 Symplectic Orbifolds

Let's just start with the definition of uniformizing chart. Let $X$ be a Hausdorff space. A $C^r$-**uniformizing chart** on $X$ is a triple $(\widetilde{U}, G, \pi)$, where

- $\widetilde{U}$ is a connected open subset of the origin in $\mathbb{R}^n$.

- $G$ is a finite group acting on $\widetilde{U}$ by $C^r$-diffeomorphisms and fixing $0$.

- $\pi : \widetilde{U} \to X$ is a continuous map inducing a homeomorphism $\widetilde{U} / G \cong U$ onto an open set $U \subset X$.

Note that the map $\pi$ should be $G$-invariant. We will also assume that $G$ acts effectively on $\widetilde{U}$.

**Definition 2.1.1.** *A $C^r$-**orbifold atlas** on $X$ is a family $\mathcal{U}$ of $C^r$-uniformizing charts on $X$ such that for each $x \in X$ and neighbourhood $U$ of $x$, there is an element $(\widetilde{U}_x, G_x, \pi_x)$ in $\mathcal{U}$ with $\pi_x$ inducing a homeomorphism $\widetilde{U}_x / G_x$ onto an open neighbourhood $U_x \subset U$ with $x \in U_x$. We also want $\pi_x$ to map the origin $0$ to $x$. The atlas $\mathcal{U}$ should satisfy the following local compatibility conditions:*

- *For any neighbourhood $U_z \subset U_x$ and corresponding uniformizing chart $(\widetilde{U}_z, G_z, \pi_z)$ in $\mathcal{U}$, there is a $C^r$-embedding $\lambda : \widetilde{U}_z \to \widetilde{U}_x$ and an injective group homomorphism*

$\theta : G_z \to G_x$ *such that* $\lambda$ *is* $\theta$*-equivariant and the following diagram commutes:*

$$
\begin{array}{ccc}
\widetilde{U}_z & \xrightarrow{\ \lambda\ } & \widetilde{U}_x \\
\downarrow & & \downarrow \\
\widetilde{U}_z \,/\, G_z & \xrightarrow{\ \overline{\lambda}\ } & \widetilde{U}_x \,/\, \theta(G_z) \\
\downarrow{\scriptstyle \pi_z} & & \downarrow{\scriptstyle \pi_x} \\
U_z & \xrightarrow{\ \subset\ } & U_x
\end{array}
$$

**Remarks**:

(1) If $g_x \in G_x$, then $g_x \cdot \lambda : \widetilde{U}_z \to \widetilde{U}_x$ is also a $C^r$-embedding that descends to the same map as $\lambda$ and is equivariant with respect to the injective homomorphism $\overline{\theta}(g_z) = g_x \cdot \theta(g_z) \cdot g_x^{-1}$ for $g_z \in G_z$. For this reason, we regard $\lambda$ as being defined only up to composition with elements of $G_x$ and $\theta$ defined only up to conjugation by elements of $G_x$.

(2) We regard two atlases $\mathcal{U}$ and $\mathcal{V}$ as equivalent if they can be combined to give a larger atlas still satisfying the above definition of being locally compatible.

**Definition 2.1.2.** *A $C^r$-**orbifold** $\mathcal{O}$ is a pair $(X_{\mathcal{O}}, [\mathcal{U}])$, where $X_{\mathcal{O}}$ is a paracompact Hausdorff space (called the underlying space) and $[\mathcal{U}]$ is an equivalence class of $C^r$-orbifold atlases.*

Given any point $x$ in an orbifold $\mathcal{O}$, by definition there is a neighbourhood $U_x$ of $x$ and a homeomorphism $U_x \cong \widetilde{U}_x \,/\, G_x$, where $\widetilde{U}_x$ is a neighbourhood of the origin in $\mathbb{R}^n$. It is possible to show that the germ of this action in a neighbourhood of $0 \in \mathbb{R}^n$ is unique. We say that $G_x$ is the **isotropy group** of $x$. The **singular set** $\mathrm{Sing}(\mathcal{O})$ of the orbifold $\mathcal{O}$ is the set of points $x \in \mathcal{O}$ with $G_x \neq \{\mathrm{Id}\}$. We want to move quickly into symplectic territory, so in analogy with $C^r$-uniformzing chart let us define what we mean by symplectic uniformizing chart,

**Definition 2.1.3.** *Let $X$ be a Hausdorff space and let $(\widetilde{U}, G, \pi)$ be a $C^\infty$-uniformizing chart on $X$. Suppose $\widetilde{U}$ comes equipped with a symplectic form $\widetilde{\omega}$ that is $G$-invariant. Then we call $(\widetilde{U}, \widetilde{\omega}, G, \pi)$ a **symplectic uniformizing chart**.*

Now, a **symplectic orbifold** $(\mathcal{O}, \omega)$ is just an orbifold $\mathcal{O}$ with a covering by open sets such that for each $U$ in the covering, there is a symplectic uniformizing chart $(\widetilde{U}, \widetilde{\omega}, G, \pi)$ such that $\widetilde{\omega}$ descends to $\omega$ on $U$. Moreover, the symplectic unformizing charts should satisfy compatibility conditions analogous to those in Definition 2.1.1.

**Defintion 2.1.4.** *Let $\mathcal{O}$ be an $n$-dimensional smooth ($C^\infty$) orbifold. The **tangent or-bibundle**, $p : T\mathcal{O} \to \mathcal{O}$, of $\mathcal{O}$ is defined as follows: If $(\widetilde{U}_x, G_x)$ is a uniformizing chart above $x \in \mathcal{O}$ then $p^{-1}(U_x) \cong (\widetilde{U}_x \times \mathbb{R}^n) / G_x$, where $G_x$ acts on $\widetilde{U}_x \times \mathbb{R}^n$ by $g \cdot (\widetilde{y}, \widetilde{v}) = (g \cdot \widetilde{y}, dg_{\widetilde{y}}(\widetilde{v}))$. The fibre $p^{-1}(x)$ over $x \in \mathcal{O}$ is called the **uniformized tangent space** at $x$ and it is denoted by $T_x\mathcal{O}$.*

It's possible to show that $T\mathcal{O}$ is itself a smooth orbifold with uniformizing charts that are just lifts of the charts on the base; ie. they have the form $(T\widetilde{U}_x, G_x)$ with $(\widetilde{U}_x, G_x)$ a uniformizing chart for $\mathcal{O}$ (see [2]-Section 1.3).

If $\mathcal{S}$ is a suborbifold of $\mathcal{O}$ (as defined in [8]-Definition 16), then we can define the **normal orbibundle** to $\mathcal{S}$ in $\mathcal{O}$ as follows: If $s \in \mathcal{S}$, we view the uniformized tangent space $T_s\mathcal{S}$ as a subspace of $T_s\mathcal{O}$. The **normal space** at $s$ is defined as the quotient $T_s\mathcal{O} / T_s\mathcal{S}$. The normal orbibundle is then

$$\nu\mathcal{S} := \{(s, v) \,|\, s \in \mathcal{S} \,,\, v \in T_s\mathcal{O} / T_s\mathcal{S}\}.$$

## 2.2   Weighted Projective Spaces

Let $a, b, c$ be positive integers that are pairwise relatively prime. From a symplectic point of view, the most natural way to view the weighted projective space $\mathbb{C}P^2_{a,b,c}$ is

via symplectic reduction. Consider the symplectic manifold $(\mathbb{C}^3, \omega_{std})$ with its standard symplectic form $\omega_{std} = \sqrt{-1} \sum_{j=1}^{3} dz_j \wedge d\bar{z}_j$. Let $S^1$ act on $\mathbb{C}^3$ with weights $(a, b, c)$:

$$\lambda \cdot (z_0, z_1, z_2) = (\lambda^a z_0, \lambda^b z_1, \lambda^c z_2). \tag{2.1}$$

This action is Hamiltonian with moment map

$$\mathbb{C}^3 \xrightarrow{H} \mathbb{R}, \quad H(z_0, z_1, z_2) = a|z_0|^2 + b|z_1|^2 + c|z_2|^2.$$

All non-zero real numbers are regular values of $H$. Thus, $H^{-1}(abc)$ is a submanifold of $\mathbb{C}^3$; in fact it is the boundary of the ellipsoid

$$E(bc, ac, ab) := \left\{ \frac{|z_0|^2}{bc} + \frac{|z_1|^2}{ac} + \frac{|z_2|^2}{ab} \leq 1 \right\}.$$

A well-known result of Alan Weinstein (see [45]) provides the reduced space $H^{-1}(abc) \, / \, S^1$ with a symplectic form $\omega_{a,b,c}$ induced from $\omega_{std}$ and gives $(H^{-1}(abc) \, / \, S^1, \omega_{a,b,c})$ the structure of a symplectic orbifold (back in the day, they called orbifolds "V-manifolds", until Thurston came along and changed it in one of his classes). Our policy will be to take this reduced space as the definition of $(\mathbb{C}P^2_{a,b,c}, \omega_{a,b,c})$.

The symplectic orbifold $(\mathbb{C}P^2_{a,b,c}, \omega_{a,b,c})$ also comes with a natural toric structure. To see this, consider the standard $\mathrm{T}^3$-action on $\mathbb{C}^3$

$$(t_0, t_1, t_2) \cdot (z_0, z_1, z_2) = (t_0 z_0, t_1 z_1, t_2 z_2).$$

with corresponding moment map $\mu_{\mathrm{T}^3}(z_0, z_1, z_2) = (|z_0|^2, |z_1|^2, |z_2|^2)$. This $\mathrm{T}^3$-action commutes with the weighted $S^1$-action, so there is an induced $\mathrm{T}^3$-action on the quotient $H^{-1}(abc) \, / \, S^1$. This action is not effective, but the action of $\mathrm{T}^3 \, / \, i(S^1)$ induced by the inclusion

$$i : S^1 \hookrightarrow \mathrm{T}^3, \quad \lambda \mapsto (\lambda^a, \lambda^b, \lambda^c)$$

is effective. Standard results about symplectic orbifolds (see [28]) show that this new $T^2$-action makes $H^{-1}(abc)/S^1 = \mathbb{C}P^2_{a,b,c}$ into a toric orbifold, whose moment polytope is given by

$$\{i^*(x,y,z) = abc\} \cap \mathbb{R}^3_{\geq 0},$$

where $i^* : \mathbb{R}^3 \to \mathbb{R}$ is dual to the linearization of the inclusion $i$. Being a linear map, $i^*$ is just the matrix $[a\ b\ c]^T$, so that the moment polytope is just given by the intersection of the hyperplane $ax + by + cz = abc$ with the positive orthant in $\mathbb{R}^3$.

The orbifold structure of $\mathbb{C}P^2_{a,b,c}$ can be explicitly described as follows. Let

$$
\begin{aligned}
U_a &:= \{[z_0 : z_1 : z_2] \in \mathbb{C}P^2_{a,b,c} \mid z_0 \neq 0\} \\
U_b &:= \{[z_0 : z_1 : z_2] \in \mathbb{C}P^2_{a,b,c} \mid z_1 \neq 0\} \\
U_c &:= \{[z_0 : z_1 : z_2] \in \mathbb{C}P^2_{a,b,c} \mid z_2 \neq 0\}.
\end{aligned}
$$

Then $\mathbb{C}P^2_{a,b,c}$ is covered by these three open sets. Take, for instance, a point $[z_0 : z_1 : z_2] \in U_a$. Pick an $a$-th root of $z_0$ and put $\lambda := 1/z_0^{1/a}$. Then

$$[z_0 : z_1 : z_2] = [\lambda^a z_0 : \lambda^b z_1 : \lambda^c z_2] = \left[1 : \frac{z_1}{z_0^{b/a}} : \frac{z_2}{z_0^{c/a}}\right].$$

Letting $\lambda$ vary over all $a$ roots of $z_0$ gives us a homeomorphism

$$U_a \longrightarrow \mathbb{C}^2/\mathbb{Z}_a, \quad [z_0 : z_1 : z_2] \mapsto \left[\frac{z_1}{z_0^{b/a}}, \frac{z_2}{z_0^{c/a}}\right],$$

with $\lambda$ acting on $\mathbb{C}^2$ as

$$\lambda \cdot (z_1, z_2) = (\lambda^b z_1, \lambda^c z_2). \tag{2.2}$$

Similar computations apply to the other neighbourhoods $U_b$ and $U_c$. Thus, $\mathbb{C}P^2_{a,b,c}$ has an orbifold structure where all singularities have cyclic structure groups. We should note that $\mathbb{C}P^2_{a,b,c}$ (as an orbifold) is **not** a global quotient in the following sense: There is a

holomorphic map

$$\mathbb{C}P^2 \longrightarrow \mathbb{C}P^2_{a,b,c}, \quad [z_0 : z_1 : z_2] \mapsto [z_0^a : z_1^b : z_2^c]$$

that is invariant under the $\mathbb{Z}_a \times \mathbb{Z}_b \times \mathbb{Z}_c \cong \mathbb{Z}_{abc}$-coordinatewise action on $\mathbb{C}P^2$. Thus, as algebraic varieties there is an isomorphism

$$\mathbb{C}P^2 / \mathbb{Z}_{abc} \cong \mathbb{C}P^2_{a,b,c},$$

but they cannot be isomorphic as orbifolds since their singular sets do not coincide. The question of what it means to be an isomorphism in the orbifold category will be discussed in Section 4.1.

## 2.3 Hirzebruch Surfaces

Hirzebruch surfaces are complex, rational, ruled surfaces and symplectic forms on them have been classified by Lalonde-McDuff in [25]. They are classified by their cohomology class (any two cohomologous symplectic forms are diffeomorphic) and, after rescaling, any symplectic rational ruled 4-manifold is symplectomorphic to one of the following:

- $(S^2 \times S^2, \mu\sigma_1 \oplus \sigma_2)$, the trivial bundle, where $\sigma_1$ and $\sigma_2$ give area 1 to each sphere.

- $(\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2, \omega_\mu)$, the non-trivial bundle, where the symplectic area of the exceptional divisor is $\mu > 0$ and the area of each fibre is 1.

At the homology level, we will work with the basis $\{B, F\}$ of $H_2(S^2 \times S^2; \mathbb{Z})$ and the basis $\{B^*, F^*\}$ of $H_2(\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2; \mathbb{Z})$, where $B = [S^2 \times \star]$ and $F = [\star \times S^2]$. Also, $B^*$ is the homology class of a section in $\mathbb{C}P^1 \# \overline{\mathbb{C}P}^2$ of self-intersection -1 and area $\mu$, while $F^*$ is the homology class of a typical fibre. The $c^{th}$ Hirzebruch surface is

$$W_c = \{([a : b], [z_0 : z_1 : z_2]) \in \mathbb{C}P^1 \times \mathbb{C}P^2 \,|\, a^c z_1 = b^c z_0\},$$

where $c$ is a positive integer. We give it a symplectic form by restricting the following form to $W_c$

$$\Omega_{\mu,c} := \begin{cases} \left(\mu - \frac{c}{2}\right)\omega_{\mathbb{C}P1} \oplus \omega_{\mathbb{C}P2} & \text{if c is even and } \mu > \frac{c}{2} \\ \left(\mu - \left(\frac{c-1}{2}\right)\right)\omega_{\mathbb{C}P1} \oplus \omega_{\mathbb{C}P2} & \text{if c is odd and } \mu > \frac{c-1}{2} \end{cases}$$

where $\omega_{\mathbb{C}P1}, \omega_{\mathbb{C}P2}$ are, respectively, the standard Kahler forms on $\mathbb{C}P^1$ and $\mathbb{C}P^2$ normalized so that the areas of the embedded $\mathbb{C}P^1$'s are equal to 1. The restriction of the projection $\mathbb{C}P^1 \times \mathbb{C}P^2 \to \mathbb{C}P^1$ makes $W_c$ a $\mathbb{C}P^1$-bundle over $\mathbb{C}P^1$ which is, topologically, $S^2 \times S^2$ when $c$ is even, and $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$ when $c$ is odd. The zero section is

$$\mathcal{Z}_0 := \{([a:b], [0:0:1])\}.$$

It corresponds to a section of self-intersection $-c$ and represents the class $B - \frac{c}{2}F$ if $c$ is even and $B^* - \left(\frac{c-1}{2}\right)F^*$ if $c$ is odd. The section at infinity is

$$\mathcal{Z}_\infty := \{([a:b], [a^c : b^c : 0])\},$$

and it has self intersection $+c$, representing the class $B + \frac{c}{2}F$ if $c$ is even and $B^* + \left(\frac{c-1}{2}\right)F^*$ if $c$ is odd.

The relationship between $W_c$ and $\mathbb{C}P^2_{1,1,c}$ is given by the following proposition (for a proof, see [16]-Section 4).

**Proposition 2.3.1** ([16]). *Let $V_c$ be the subvariety of $\mathbb{C}P^1 \times \mathbb{C}P^2_{1,1,c}$ defined as*

$$V_c = \{([a:b], [z_0 : z_1 : z_2]_{1,1,c}) \in \mathbb{C}P^1 \times \mathbb{C}P^2_{1,1,c} \,|\, az_1 = bz_0\}.$$

*Then $V_c$ is biholomorphic to $W_c$.*

Thus, $V_c$ is the (complex) blow-up of $\mathbb{C}P^2_{1,1,c}$ at the singular point $p_c$. We put a symplectic form on $V_c$ in an analogous way; define

$$\widetilde{\Omega}_{\mu,c} := \begin{cases} \left(\mu - \frac{c}{2}\right)\omega_{\mathbb{C}P1} \oplus \omega_{1,1,c} & \text{if c is even and } \mu > \frac{c}{2} \\ \left(\mu - \left(\frac{c-1}{2}\right)\right)\omega_{\mathbb{C}P1} \oplus \omega_{1,1,c} & \text{if c is odd and } \mu > \frac{c-1}{2} \end{cases}$$

and restrict it to $V_c$. The zero section is the set $\mathcal{Z}'_0 := \{([a:b], [0:0:1]_{1,1,c})\}$, and the infinity section is now

$$\mathcal{Z}'_\infty := \{([a:b], [a:b:0]_{1,1,c})\}.$$

## 2.3.1   Toric Models

The Hirzebruch surfaces $(W_k, \Omega_{\mu,k})$ are symplectic toric manifolds, and a very nice property of these manifolds is that they are determined, up to equivariant symplectomorphism, by their moment polytopes (see [12]). Let the torus $\mathrm{T}^2$ act on $\mathbb{C}P^1 \times \mathbb{C}P^2$ by

$$(t_1, t_2) \cdot ([a:b], [z_0:z_1:z_2]) = ([t_1 a : b], [t_1^k z_0 : z_1 : t_2 z_2]),$$

and restrict the action to $W_k$. Then the quotient $W_k / \mathrm{T}^2$ appears in Figure 2.1 or Figure 2.2, for $k$ even or odd.[1] These are also the images of $W_k$ under the moment map

$$\Phi([a:b], [z_0:z_1:z_2]) = \left(\frac{|a|^2}{|a|^2 + |b|^2} + \frac{k|z_0|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2}, \frac{|z_2|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2}\right).$$

The outward normal vector to the slanted edge is $(1, k)$, so that this edge has slope $-\frac{1}{k}$. The image of the zero section $\mathcal{Z}_0$ is the top horizontal edge, the image of the infinity section $\mathcal{Z}_\infty$ is the bottom horizontal edge, and the image of the fibre $\mathcal{F}$ is the slanted edge. These edges are labelled by their homology classes in $H_2(W_k; \mathbb{Z})$, and they encode the symplectic areas and self-intersection numbers of the spheres $\mathcal{Z}_0, \mathcal{Z}_\infty,$

---

1. The only difference between the two pictures is the labelling of homology classes
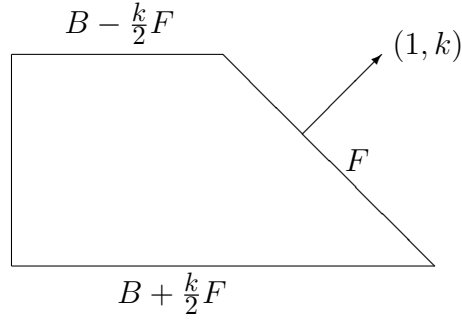
Figure 2.1: The Hirzebruch trapezoid ($k$ even).
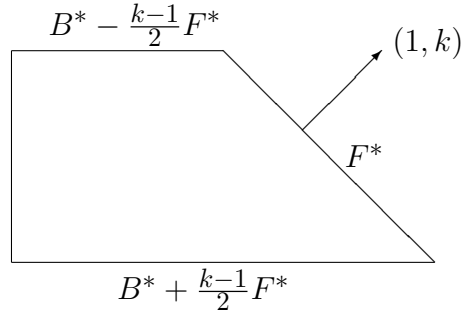


Figure 2.2: The Hirzebruch trapezoid ($k$ odd).

and $\mathcal{F}$. We should mention a convention we are going to use throughout the rest of this thesis. In subsequent sections, we will be talking a lot about various blow ups of the Hirzebruch surfaces $W_k$ and their resulting homology classes. Since these classes are different depending on whether $k$ is even or odd, it would be annoying to have to repeat our arguments for two separate cases, and it turns out that this distinction is not so important. In fact, blowing up $S^2 \times S^2$ or $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$ leads to diffeomorphic smooth manifolds (see [13]-page 13). This diffeomorphism produces an isomorphism in homology $H_2\big( (S^2 \times S^2) \# \overline{\mathbb{C}P}^2; \mathbb{Z}\big) \cong H_2(\mathbb{C}P^2 \# 2\overline{\mathbb{C}P}^2; \mathbb{Z})$ with the following identification of basis

elements:

$$F \quad \leftrightarrow \quad F^* \tag{2.3}$$

$$B \quad \leftrightarrow \quad B^* + F^* - E_1^*$$

$$E_1 \quad \leftrightarrow \quad F^* - E_1^*$$

$$B - E_1 \quad \leftrightarrow \quad B^*$$

$$F - E_1 \quad \leftrightarrow \quad E_1^*$$

where $\{B, F, E_1\}$ is a basis for $H_2((S^2 \times S^2) \# \overline{\mathbb{C}P}^2; \mathbb{Z})$ and $\{B^*, F^*, E_1^*\}$ is a basis for $H_2(\mathbb{C}P^2 \# 2\overline{\mathbb{C}P}^2; \mathbb{Z})$. Here, the classes $E_1, E_1^*$ are exceptional classes whose homological self-intersection is -1. Thus, in subsequent sections when we speak about "blowing up the manifold $W_k$", our arguments will be carried out fully for the case $k$ is even and we will be careful to point out that the case "$k$ odd" follows with similar arguments by swapping the homology classes in the above fashion.

We will now recall some facts about toric geometry (see [23]-Section 2). A polygon $\Delta \subset \mathbb{R}^2$ is called a **Delzant polygon** if for each vertex $p$ of $\Delta$, the edges emanating from $p$ have the form $p + tv_i$, $t \geq 0$, where $v_i \in \mathbb{Z}^2$, and the $v_i$ $(i = 1, 2)$ can be chosen to be a $\mathbb{Z}$-basis of the lattice $\mathbb{Z}^2$ (this last condition is called being **smooth**). Let $e$ be an edge of $\Delta$ with rational slope. The **rational length** of $e$ is the largest positive number $\ell$ such that $\frac{1}{\ell} \cdot e$ has its endpoints on the lattice $\mathbb{Z}^2$. Let $e_1, e_2, e_3$ be three consecutive edges in a Delzant polygon, ordered anti-clockwise, and let $n_1, n_2, n_3$ be outward primitive normal vectors to these edges, respectively. Then each of $\{n_1, n_2\}, \{n_2, n_3\}$ is an oriented $\mathbb{Z}$-basis for $\mathbb{Z}^2$. Thus, there is an integer $m$ such that $n_1 + n_3 = mn_2$. Define the **combinatorial self-intersection number** of $e_2$ to be $-m$.

**Propostion 2.3.1.1** ([23])**.** *Let $(M, \omega)$ be a compact connected symplectic toric 4-manifold. Let $\Phi : M \to \mathbb{R}^2$ be the moment map for a toric action, and let $\Delta = \Phi(M)$.*

(1) *If $e$ is an edge of $\Delta$ of rational length $\ell$, then the pre-image $\Phi^{-1}(e)$ is a symplectically embedded 2-sphere in $M$, invariant under the torus action, and with symplectic*

*area*

$$\int_{\Phi^{-1}(e)} \omega = 2\pi\ell$$

(2) *If $e$ is an edge of $\Delta$ and $S = \Phi^{-1}(e)$ is its pre-image in $M$, then the combinatorial self-intersection number of $e$ is equal to the self-intersection of $S$ in $M$.*

(3) *The pre-images of the edges of $\Delta$ generate the second homology group of $M$. The number of vertices of $\Delta$ is equal to $\dim H_2(M;\mathbb{Z}) + 2$.*

## 2.4   $J$-Holomorphic Spheres

At a few crucial points in this thesis, our arguments explicitly use $J$-holomorphic spheres (though implicitly our whole house of cards would collapse without them). We will briefly mention what they are and a few results about them without getting bogged down in all the analysis. Some good references are [5], [25], and [36]. All maps are $C^\infty$-smooth and spaces of maps have the $C^\infty$-topology.

An almost complex structure on a manifold $M$ is an automorphism $J : TM \to TM$ such that $J^2 = -\text{Id}$. The almost complex structure is **tamed** by a symplectic form $\omega$ if

$$\omega(v, Jv) > 0 \text{ whenever } v \neq 0.$$

If $\omega$ is also $J$-invariant, then $J$ is said to be **compatible** with $\omega$. The spaces of all compatible with $\omega$, respectively tamed by $\omega$, almost complex structures on $M$ are both contractible spaces ([35]-Chapter 2.5), but it's often more convenient to work with the bigger space of tamed ones because this space is open in the space of all almost complex structures on $M$.

For a fixed symplectic manifold $(M, \omega)$, let $\mathcal{J}$ be the space of all almost complex structures $J$ on $M$ that are tamed by $\omega$. A (parametrized) $J$-holomorphic sphere in $M$ is a map $u : (\mathbb{C}P^1, j) \to (M, J)$ that is a solution of the *generalized Cauchy-Riemann*

equations

$$du \circ j = J \circ du.$$

It is **simple** if it can't be factored through a branch covering of $\mathbb{C}P^1$. An **embedded** $J$**-sphere** $C \subset M$ is the image of a $J$-holomorphic embedding. Note that $C$ must be a symplectic submanifold because the restriction of $\omega$ to $TC$ is non-degenerate by the taming condition. If $C$ is an embedded $J$-sphere, then we will usually just say that $C$ is $J$-holomorphic, or that $C$ is a $J$-sphere.

Let $A \in H_2(M; \mathbb{Z})$ be a homology class. We say that a $J$-sphere $C$ is represented by $A$ if $u_*[\mathbb{C}P^1] = A$, where $u$ is a parametrization of $C$. We should emphasize that all of our almost complex structures $J$ come from the space $\mathcal{J} = \mathcal{J}(\omega)$ consisting of those that are tamed by a fixed symplectic form $\omega$. Here are some nice properties of $J$-holomorphic spheres that will be important in the work we do:

- *Positivity of area*: Write $[\omega] \cdot A$ for the cohomology-homology pairing. If $A \in H_2(M; \mathbb{Z})$ can be represented by a $J$-holomorphic sphere for some $J \in \mathcal{J}$, then

$$[\omega] \cdot A = \int_{u(\mathbb{C}P^1)} \omega = \int_{\mathbb{C}P^1} u^*\omega > 0.$$

- *Positivity of intersections* (only true in dimension 4): Let $A, B$ be homology classes in $H_2(M^4; \mathbb{Z})$ that are represented by distinct simple $J$-holomorphic spheres for $J \in \mathcal{J}$. Write $A \cdot B$ for their homological intersection number. Then $A \cdot B \geq 0$. Furthermore, if $C_A, C_B$ are distinct $J$-holomorphic representatives of the classes $A$, respectively $B$, then $A \cdot B = 1$ if and only if $C_A$ and $C_B$ intersect exactly once transversally. Also, $A \cdot B = 0$ if and only if $C_A$ and $C_B$ are disjoint.

- *Adjunction formula* (only true in dimension 4): Let $[c_1(TM)] \in H^2(M; \mathbb{Z})$ be the first Chern class of the complex vector bundle $(TM, J)$ for any $J \in \mathcal{J}$. It is a fact that $[c_1(TM)]$ is independent of $J$. Let $A$ be a class in $H_2(M; \mathbb{Z})$. We give the

number

$$g_v(A) := 1 + \frac{1}{2}(A \cdot A - [c_1(TM)] \cdot A)$$

a special name. It's called the **virtual genus** of $A$. Then for any $J \in \mathcal{J}$, if $A \in H_2(M)$ is represented by a simple $J$-sphere $C_A$, we have $g_v(A) \geq 0$ with equality if and only if $C$ is embedded.

Now let's focus on the case where $(M^4, \omega)$ is a symplectic 4-manifold. We say that a homology class $E \in H_2(M; \mathbb{Z})$ is **exceptional** if it is represented by an embedded symplectic sphere with self-intersection -1. If $C$ is a $J$-holomorphic sphere that represents an exceptional homology class, then $C$ is unique by positivity of intersections. Here are some facts about exceptional homology classes that will also be important in the work that we do (see [41]-Lemma 2.1):

- Let $\mathcal{J}_E \subset \mathcal{J}$ be the space of $\omega$-tame $J$ for which there exists an embedded $J$-holomorphic sphere in class $E$. Then $\mathcal{J}_E$ is open, dense, and path-connected in $\mathcal{J}$.

- *Corollary of Gromov compactness*: If $J \in \mathcal{J}$, then any exceptional class $E$ is represented by either an embedded $J$-holomorphic sphere or a connected union of possibly multiply-covered $J$-spheres (called cusp-curves) of the form

$$C = m_1 C_1 \cup \ldots \cup m_n C_n, \quad n \geq 2$$

where $m_i C_i$ stands for a multiply covered (ie., non-simple) $J$-sphere with multiplicity $m_i$.

- Any two exceptional classes intersect non-negatively.

## 2.5 Quotient Singularities and Continued Fractions

This section will describe the Hirzebruch-Jung method for resolving singularities. Some good references for this material are ([10]-Section 2) and ([15]-Chapters 2.2 and 2.6). We should start by describing cyclic quotient singularities and their local toric models. Cyclic quotient singularities are just the special type of orbifold singularities that we care about in this thesis. Suppose a cyclic group $\mathbb{Z}_c$ acts on $\mathbb{C}^2$ as follows

$$\xi \cdot (z_0, z_1) = (\xi z_0, \xi^b z_1), \quad 0 < b < c \tag{2.4}$$

with $b, c$ relatively prime. Then the quotient is an orbifold with an isolated singularity of order $c$ at the origin. If we put a $\mathbb{Z}_c$-invariant symplectic form on $\mathbb{C}^2$, then this form descends to the quotient $\mathbb{C}^2 / \mathbb{Z}_c$ which naturally becomes symplectic. Let's consider the standard $\mathrm{T}^2$-action on $\mathbb{C}^2$ given by $(z_0, z_1) \mapsto (t_0 z_0, t_1 z_1)$. The image under the moment map

$$(z_0, z_1) \mapsto (|z_0|^2, |z_1|^2)$$

is the first quadrant in $\mathbb{R}^2$. Since the $\mathrm{T}^2$-action commutes with the $\mathbb{Z}_c$-action on $\mathbb{C}^2$, there is an induced $\mathrm{T}^2$-action on $\mathbb{C}^2 / \mathbb{Z}_c$ that we get by composing with an isomorphism $\mathrm{T}^2 \stackrel{\cong}{\to} \mathrm{T}^2 / \mathbb{Z}_c$. We want to describe the moment map and its image. Consider the surjective homomorphism

$$
\begin{aligned}
\mathrm{T}^2 &\longrightarrow \mathrm{T}^2 \\
(t_0, t_1) &\mapsto (t_0^c, t_0^{-b} t_1).
\end{aligned} \tag{2.5}
$$

Its kernel is isomorphic to $\mathbb{Z}_c \hookrightarrow \mathrm{T}^2$ viewed as the inclusion $\xi \mapsto (\xi, \xi^b)$ , so we have an isomorphism

$$\mathrm{T}^2 / \mathbb{Z}_c \xrightarrow{\cong} \mathrm{T}^2$$

via the map (2.5). The inverse map is given by

$$\mathrm{T}^2 \longrightarrow \mathrm{T}^2 / \mathbb{Z}_c, \quad (t_0, t_1) \mapsto (t_0^{1/c}, t_0^{b/c} t_1).$$

Therefore, $\mathrm{T}^2$ acts on $\mathbb{C}^2 / \mathbb{Z}_c$ via this map, and the corresponding moment map is given by

$$(z_0, z_1) \mapsto (\frac{|z_0|^2}{c} + \frac{b|z_1|^2}{c}, |z_1|^2).$$

Its image is the convex subset of $\mathbb{R}^2$ spanned by the vectors $(1, 0)$ and $(b, c)$. If we go back to the image of the moment map for the standard $\mathrm{T}^2$-action on $\mathbb{C}^2$, then this new picture transforms the old one by the matrix

$$\begin{bmatrix} \frac{1}{c} & \frac{b}{c} \\ 0 & 1 \end{bmatrix}.$$

This is a local toric model for the order $c$ singularity given by the action (2.4). The vertex $v_c$ in this picture corresponds to the singularity of order $c$. In ([15]-Chapter 2.6), Fulton describes how to resolve such a singularity using Hirzebruch-Jung continued fractions and by adding rays to a convex cone. We prefer to view this process as "corner cutting" at a vertex using co-normal vectors because this is the symplectic way of doing things, though in this section we won't specify the sizes of the cuts.

The co-normals to the edges with vertex $v_c$ are $(0, -1)$ and $(-c, b)$. If we put the tails of these vectors together at the origin, they will span a convex cone. To make our picture correspond to Fulton's picture on ([15]-page 45) we have to rotate this cone by 180 degrees, so just use the matrix

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Then the cone spanned by $(0, -1)$ and $(-c, b)$ is sent to the cone spanned by $(0, 1)$ and

$(c, -b)$.

The Hirzebruch-Jung continued fraction expansion of $\frac{c}{b}$ is computed as follows: Let $a_1 = \lceil \frac{c}{b} \rceil$ be the least integer bigger than or equal to $\frac{c}{b}$. If $b = 1$, then $a_1 = c$ so we stop. Otherwise, there are positive integers $k_1 < m_1$ such that

$$\frac{c}{b} = a_1 - \frac{k_1}{m_1} = a_1 - \frac{1}{\frac{m_1}{k_1}},$$

and we set $a_2 = \lceil \frac{m_1}{k_1} \rceil$. If $k_1 = 1$, then stop. Otherwise, we can write $\frac{m_1}{k_1} = a_2 - \frac{k_2}{m_2}$ for positive integers $k_2 < m_2$. Thus,

$$\frac{c}{b} = a_1 - \frac{1}{a_2 - \frac{1}{\frac{m_2}{k_2}}}$$

and so on. This process will eventually stop. In the literature, this type of continued fraction is often written as $\frac{c}{b} = [a_1, a_2, \ldots, a_k]$. To resolve the singularity corresponding to the vertex $v_c$, set $\vec{n}_0 = (0, -1)$ and $\vec{n}_1 = (-1, 0)$. Now recursively define

$$\vec{n}_{i+1} = a_i \vec{n}_i - \vec{n}_{i-1}$$

for $i = 1, \ldots, k$. The normals $\vec{n}_1, \ldots, \vec{n}_k$ specify $k$ new edges in the toric picture originally determined by the edges with co-normals $(0, -1)$ and $(-c, b)$. These new edges correspond to a chain of embedded spheres $C_1 \cup \ldots \cup C_k$ intersecting positively and transversely with self-intersection numbers $C_i \cdot C_i = -a_i$. Note that since $a_i \geq 2$ this resolution is minimal in the sense that it contains no (-1)-spheres.

# Chapter 3

# Resolving Singularities

## 3.1 Blowing up Orbifold Singularities

Recall that the blow up of a symplectic $2n$-manifold at a point $x$ is obtained by removing an embedded ball around this point and then squishing the boundary (which is an $S^{2n-1}$) along the fibres of the Hopf fibration. A similar situation happens in the orbifold case, except it now involves removing a singular orbi-ball and similarly squishing its boundary. A more general approach is the *weighted blow up* (see [17]) which involves removing an embedded ellipsoid and collapsing the boundary. The approach we describe here involves symplectic cutting, a technique developed by Lerman in [27]. For a good reference on how to use symplectic cutting in the orbifold case, see [38].

### 3.1.1 Resolving $\mathbb{C}P^2_{1,1,c}$

Let us start with the simplest possible case: The single isolated singularity $p_c$ of the weighted projective space $(\mathbb{C}P^2_{1,1,c}, \omega_{1,1,c})$. This singularity is modelled by the following $\mathbb{Z}_c$-action on the uniformizing chart $\widetilde{U}_c$:

$$\xi \cdot (z, w) = (\xi z, \xi w), \quad \xi \in \mathbb{Z}_c.$$

Define an $S^1$-action on $\mathbb{C}^2$ by $\lambda \cdot (z, w) = (\lambda z, \lambda w)$. This $S^1$-action commutes with the $\mathbb{Z}_c$-action, so there is an induced action on $\mathbb{C}^2 / \mathbb{Z}_c$ though this action is not effective

(it has a global $\mathbb{Z}_c$ stabilizer). We can fix this by taking the quotient by $\mathbb{Z}_c$ and defining a new $S_c^1 := S^1 / \mathbb{Z}_c$-action on $\mathbb{C}^2 / \mathbb{Z}_c$

$$\mu \cdot [z, w] = [\lambda \cdot (z, w)] = [\lambda z, \lambda w] \quad \text{for } \mu \in S_c^1,$$

where $\mu = \lambda^c$ for some $\lambda \in S^1$. This action is Hamiltonian with corresponding Hamiltonian function

$$H_1 : \mathbb{C}^2 / \mathbb{Z}_c \longrightarrow \mathbb{R}, \quad [z, w] \mapsto |z|^2 + |w|^2.$$

Now perform a symplectic cut with respect to this $S_c^1$-action: Take the product, $((\mathbb{C}^2 / \mathbb{Z}_c) \times \mathbb{C}, \omega_{1,1,c} \oplus -idw' \wedge d\overline{w}')$, with the effective $S_c^1$-action

$$\mu \cdot ([z, w], w') = (\mu \cdot [z, w], \mu^{-1} w') = ([\lambda z, \lambda w], \lambda^{-c} w'), \quad \lambda^c = \mu. \tag{3.1}$$

This action is also Hamiltonian; it's Hamiltonian function is

$$\begin{aligned} H_2([z, w], w') &= |z|^2 + |w|^2 - c|w'|^2 \\ &= H_1([z, w]) - c|w'|^2. \end{aligned}$$

Let $\epsilon > 0$ be a regular value of $H_2$. Then

$$\begin{aligned} H_2^{-1}(\epsilon) &= \{([z, w], w') \in \mathbb{C}^2 / \mathbb{Z}_c \times \mathbb{C} \mid H_1([z, w]) - c|w'|^2 = \epsilon\} \\ &= \{([z, w], 0) \mid H_1([z, w]) = \epsilon\} \\ &\quad \bigsqcup \left\{([z, w], w') \mid H_1([z, w]) > \epsilon, |w'|^2 = \frac{H_1([z, w]) - \epsilon}{c}\right\} \\ &\cong H_1^{-1}(\epsilon) \sqcup (H_1^{-1}(\epsilon, \infty) \times S_c^1). \end{aligned}$$

The symplectic quotient via the action (3.1) is $H_1^{-1}(\epsilon) / S_c^1 \sqcup \{H_1 > \epsilon\}$. The manifold $\{H_1 > \epsilon\}$ embeds into $H_2^{-1}(\epsilon) / S_c^1$ as an open dense symplectic submanifold, and the remaining set $H_1^{-1}(\epsilon) / S_c^1$ is called the exceptional divisor and has codimension 2. A

priori, the exceptional divisor may not be smooth, but in this case it is because $S_c^1$ acts freely on $H_1^{-1}(\epsilon) = \{([z,w],0) \mid H_1([z,w]) = \epsilon\}$.

Another way to look at the above construction is as follows. The map

$$\varphi : (S^3 / \mathbb{Z}_c) \times \mathbb{C} \longrightarrow H_2^{-1}(\epsilon)$$
$$(x, w') \mapsto ((\epsilon + c|w'|^2)^{1/2}x, w')$$

is an $S_c^1$-equivariant diffeomorphism. Hence,

$$\left( (S^3 / \mathbb{Z}_c) \times \mathbb{C} \right) / S_c^1 \cong H_2^{-1}(\epsilon) / S_c^1.$$

Let $\tau = \omega_{1,1,c}|_{U_c} \oplus -idw' \wedge d\overline{w}'$. Observe that, away from the origin, $\omega_{1,1,c}|_{U_c} = \frac{i}{2}(dz \wedge d\overline{z} + dw \wedge d\overline{w})$ is standard. A simple computation shows that

$$\varphi^* \tau |_{S^3 / \mathbb{Z}_c \times \{0\}} = \epsilon \cdot \frac{i}{2}(dx_1 \wedge d\overline{x}_1 + dx_2 \wedge d\overline{x}_2)|_{S^3 / \mathbb{Z}_c}$$

so the restriction of this form to the exceptional divisor is $\epsilon$ times the standard form on $\mathbb{C}P^1$.

### 3.1.2 Resolving $\mathbb{C}P^2_{1,b,c}$ for $c = bk + 1$

Consider first the order $c$ singularity $p_c \in \mathbb{C}P^2_{1,b,c}$. As explained in Section 2.2, it is locally modelled by the $\mathbb{Z}_c$-action

$$\xi \cdot (z, w) = (\xi z, \xi^b w), \quad \xi \in \mathbb{Z}_c. \tag{3.2}$$

As in the previous section, let $S_c^1 := S^1 / \mathbb{Z}_c$ act on $\mathbb{C}^2 / \mathbb{Z}_c$ in the same way as (3.2). The action is Hamiltonian with moment map

$$H_1 : \mathbb{C}^2 / \mathbb{Z}_c \longrightarrow \mathbb{R}, \quad [z, w] \mapsto |z|^2 + b|w|^2.$$

Now perform the symplectic cut with this $S_c^1$-action. The product $(\mathbb{C}^2/\mathbb{Z}_c) \times \mathbb{C}$ admits the effective $S_c^1$-action

$$\mu \cdot ([z, w], w') = (\mu \cdot [z, w], \mu^{-1}w') = ([\lambda z, \lambda^b w], \lambda^{-c}w'), \quad \lambda^c = \mu$$

that is Hamiltonian with moment map

$$\begin{aligned} H_2([z, w], w') &= |z|^2 + b|w|^2 - c|w'|^2 \\ &= H_1([z, w]) - c|w'|^2. \end{aligned}$$

Letting $\alpha_1 > 0$ be a regular value of $H_2$, we have

$$H_2^{-1}(\alpha_1)/S_c^1 \cong H_1^{-1}(\alpha_1)/S_c^1 \sqcup H_1^{-1}(\alpha_1, \infty) \cong \mathbb{C}P_{1,b}^1 \sqcup \{H_1 > \alpha_1\},$$

so this time the exceptional divisor is not smooth, but is the weighted projective space $\mathbb{C}P_{1,b}^1$. Thus, we've removed a neighbourhood of $p_c$ and replaced it with $\mathbb{C}P_{1,b}^1$, hence reducing the order $c$ singularity to an order $b$ singularity. Give this new singularity the designation $q_b$.

We would like to compute the cohomology class of the resulting symplectic form on the exceptional divisor. This is done by Godinho in [17] (see the very end of the paper), so we will explain her computation. Put $\Sigma_{\alpha_1} := H_1^{-1}(\alpha_1)/S_c^1$ and let $\omega_{\alpha_1}$ be the form obtained by symplectic reduction. Observe that $\Sigma_{\alpha_1}$ is the quotient of the ellipsoid boundary $\{|z|^2 + b|w|^2 = \alpha_1\}$ by the weighted $S_c^1$-action. This is a *weighted blow up* in the context of [17]. After quotienting, there is a residual $S^1$-action on the exceptional divisor whose moment map is given by projecting to the vertical coordinate. This can be seen by looking at the local toric picture in Section 2.5. The local toric model for the singularity given by the action (3.2) is the open convex subset of $\mathbb{R}^2$ generated by the edge vectors $(1, 0)$ and $(b, c)$. Making the symplectic cut at level $\alpha_1$ adds a new edge with $x$-coordinate $\alpha_1$ and co-normal vector $(-1, 0)$. This $S^1$-action has two fixed points:

The singularity $q_b$, and a smooth point that we'll call $p$. Let $H_{\alpha_1}$ be the Hamiltonian for this new $S^1$-action; thus, $H_{\alpha_1}$ is just projection to the vertical coordinate. Let $\psi_s$ be the corresponding Hamiltonian flow. Also, let $\gamma_t$ be a smooth path from $q_b$ to $p$. Consider the function

$$
[0, 2\pi] \times [0, 1] \xrightarrow{\ f\ } H_2^{-1}(\alpha_1) \, / \, S_c^1
$$
$$
(s, t) \ \mapsto \ \psi_s^{-1}(\gamma_t).
$$

Then we have

$$
\begin{aligned}
[\omega_{\alpha_1}](\Sigma_{\alpha_1}) &= \frac{1}{2\pi} \int_{\Sigma_{\alpha_1}} \omega_{\alpha_1} \\
&= \frac{1}{2\pi} \int_{[0,2\pi] \times [0,1]} f^* \omega_{\alpha_1} \\
&= \frac{1}{2\pi} \int_{[0,2\pi] \times [0,1]} \omega_{\alpha_1}(\dot{\psi}_s, \dot{\gamma}_t) \, ds \wedge dt \\
&= \frac{1}{2\pi} \int_{[0,2\pi] \times [0,1]} dH_{\alpha_1}(\dot{\gamma}_t) \, ds \wedge dt \\
&= H_{\alpha_1}(q_b) - H_{\alpha_1}(p) = \frac{c\,\alpha_1}{b}.
\end{aligned}
$$

Hence, the symplectic area of the exceptional divisor is $\frac{c\,\alpha_1}{b}$.

Points of the form $([0, w], 0) \in H_2^{-1}(\alpha_1)$ have stabilizer $\mathbb{Z}_b$, so they collapse to the order $b$ singularity in the quotient $H_2^{-1}(\alpha_1) \, / \, S_c^1$. Let $q$ be a point in the $S_c^1$-orbit of $([0, w], 0)$. By the orbifold slice theorem ([28]-Proposition 2.2), an $S_c^1$-invariant neighbourhood of the orbit

$$
S_c^1 \cdot q := \{([0, \lambda^b w], 0) \mid \lambda \in S_c^1\} \cong S_c^1 \, / \, \mathbb{Z}_b
$$

is equivariantly diffeomorphic to a neighbourhood of the 0-section in the associated orbi-

bundle

$$S_c^1 \times_{\mathbb{Z}_b} (\nu_q / \mathbb{Z}_b),$$

where $\nu_q$ is normal to the uniformized tangent space at $q$. The normal direction to the orbit $S_c^1 \cdot q$ is

$$\{([z,0], w') \mid (z, w') \in (\mathbb{C} / \mathbb{Z}_c) \times \mathbb{C}\},$$

and is equipped with the $\mathbb{Z}_b$-action $\xi \cdot ([z,0], w') = ([\xi z, 0], \xi^{-c} w')$. This provides a new orbifold chart around the singularity $q_b \in H_2^{-1}(\alpha_1) / S_c^1$. Note that

$$-c = b - 1 - b(k+1) \overset{\mathrm{mod}\, b}{\equiv} b - 1.$$

Thus, our new singularity can be locally modelled by a neighbourhood of the origin in $\mathbb{C}^2$ with the $\mathbb{Z}_b$-action $\xi \cdot (z, w) = (\xi z, \xi^{b-1} w)$, and we can repeat the same process as above. There is an $S^1$-action that commutes with this $\mathbb{Z}_b$-action, and so again we have an induced effective action of $S_b^1 := S^1 / \mathbb{Z}_b$ on $\mathbb{C}^2 / \mathbb{Z}_b$. This action is Hamiltonian with moment map $J_1(z, w) = |z|^2 + (b-1)|w|^2$. Perform another symplectic cut: $S_b^1$ acts on the product $\mathbb{C}^2 / \mathbb{Z}_b \times \mathbb{C}$ by

$$\mu \cdot ([z, w], w') = (\mu \cdot [z, w], \mu^{-1} w') = ([\lambda z, \lambda^{b-1} w], \lambda^{-b} w'), \;\; \lambda^b = \mu \qquad (3.3)$$

with moment map $J_2([z, w], w') = |z|^2 + (b-1)|w|^2 - b|w'|^2 = J_1([z, w]) - b|w'|^2$. Choose a regular value $\alpha_2$ of $J_2$. This time, the reduced space via the action (3.3) is decomposed as

$$J_2^{-1}(\alpha_2) / S_b^1 \cong J_1^{-1}(\alpha_2) / S_b^1 \sqcup \{J_1 > \alpha_2\}.$$

Put $\Sigma_{\alpha_2} := J_1^{-1}(\alpha_2) / S_b^1$ and note that $\Sigma_{\alpha_2}$ is isomorphic to the weighted projective space $\mathbb{C}P_{1,b-1}^1$. Also, let $\widehat{\Sigma}_{\alpha_1}$ be the proper transform of the earlier exceptional divisor $\Sigma_{\alpha_1}$. If $\omega_{J_2, \alpha_2}$ is the induced symplectic form on $J_2^{-1}(\alpha_2) / S_b^1$, then a similar

computation to that above shows that $[\omega_{J_2,\alpha_2}]([\Sigma_{\alpha_2}]) = \frac{b\,\alpha_2}{b-1}$, and so

$$[\omega_{J_2,\alpha_2}]([\widehat{\Sigma}_{\alpha_1}]) = [\omega_{J_2,\alpha_2}]([\Sigma_{\alpha_1}] - [\Sigma_{\alpha_2}]) = \frac{c\,\alpha_1}{b} - \frac{b\,\alpha_2}{b-1} =: \ell_b(\alpha_1, \alpha_2). \qquad (3.4)$$

Using the slice theorem again, we can produce a new orbifold chart with $\mathbb{Z}_{b-1}$ acting as $(z, w) \mapsto (\xi z, \xi^{-b} w)$. Noting that

$$-b = b - 2 - (b-1) \cdot 2 \overset{\mod b - 1}{\equiv} b - 2,$$

we see that the new singularity can be modelled with $\mathbb{Z}_{b-1}$ acting as $(z, w) \mapsto (\xi z, \xi^{b-2})$. Yes, there is a pattern here. The reader who has looked at Section 2.5 should realize that each singularity reduction is governed by the continued fraction expansion of $\frac{bk+1}{b} = [k+1, 2, \ldots, 2]$ where the number of 2's in the string is $b - 1$. More details about this will be explained in the next section.

Thus, we can resolve the singularity $p_c$ with $b$ symplectic cuts at levels $\alpha_1, \ldots, \alpha_b$ with sizes $\frac{c\,\alpha_1}{b}$ and

$$\frac{\left(b - (i-2)\right)\alpha_i}{b - (i-1)} \quad \text{for } i = 2, \ldots, b.$$

This produces a chain of embedded symplectic spheres $\widehat{\Sigma}_{\alpha_1} \cup \widehat{\Sigma}_{\alpha_2} \cup \ldots \widehat{\Sigma}_{\alpha_{b-1}} \cup \Sigma_{\alpha_b}$ with respective sizes $\ell_b(\alpha_1, \alpha_2), \ell_b(\alpha_2, \alpha_3), \ldots, \ell_b(\alpha_{b-1}, \alpha_b), 2\alpha_b$, where $\ell_b(\alpha_1, \alpha_2)$ is given in (3.4) and

$$\ell_b(\alpha_i, \alpha_{i+1}) = \frac{\left(b - (i-2)\right)\alpha_i}{b - (i-1)} - \frac{\left(b - (i-1)\right)\alpha_{i+1}}{b - i}, \quad i = 2, \ldots, b - 1. \qquad (3.5)$$

A similar procedure shows that we can resolve the order $b$ singularity $p_b \in \mathbb{C}P^2_{1,b,c}$ by performing only one symplectic cut, at level $\alpha_{b+1}$, and resulting in a smooth exceptional divisor $\Sigma_{b+1}$. Let's call the resulting symplectic form on the resolution $\widetilde{\omega}_{\alpha_1, \ldots, \alpha_{b+1}}$.

## 3.2  Toric Models

Recall from Section 2.2 that $\mathbb{C}P^2_{a,b,c}$ is a toric orbifold whose moment polygon is given by the intersection of the hyperplane $ax + by + cz = abc$ with the positive orthant in $\mathbb{R}^3$. Now assume that $a = 1$, so that the vertex $(bc, 0, 0)$ corresponds to a smooth point in $\mathbb{C}P^2_{1,b,c}$. Then $\mathcal{P} := \{x + by + cz = bc\} \cap \mathbb{R}^3_{\geq 0}$ intersects the coordinate axes at $(bc, 0, 0), (0, c, 0), (0, 0, b)$. We want to identify this moment polygon with the one in Figure 3.1.
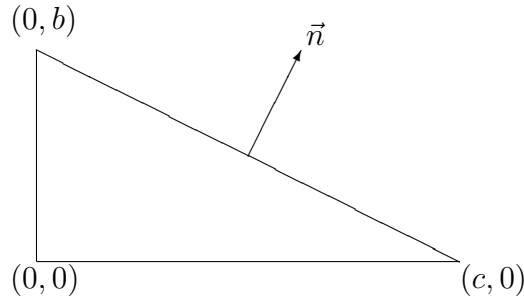


Figure 3.1: The moment polygon $\Delta_{b,c}$. Note that $\vec{n} = (b, c)$.

Consider the matrix

$$A = \begin{bmatrix} b & -1 & 0 \\ c & 0 & -1 \end{bmatrix}$$

as a map $A : \mathbb{R}^3 \to \mathbb{R}^2$. This matrix comes from the Delzant construction. If we let $\overline{A} : \mathbb{R}^2 \to \mathbb{R}^3$ be the affine map

$$\overline{A}(x, y) = A^T(x, y) - (bc, 0, 0),$$

then $\overline{A}$ is an affine embedding, so is a bijection onto its image. It is then easy to check that $\overline{A}(\Delta_{b,c}) = -\mathcal{P}$, and this allows us to identify $\mathcal{P}$ with the polygon $\Delta_{b,c}$ from Figure

3.1, up to a change of sign. The moment map that gives the polygon $\Delta_{b,c}$ is

$$[z_0 : z_1 : z_2] \mapsto \frac{bc}{|z_0|^2 + b|z_1|^2 + c|z_2|^2} (|z_1|^2, |z_2|^2).$$

In fact, this polygon determines $(\mathbb{C}P^2_{1,b,c}, \omega_{1,b,c})$ up to equivariant symplectomorphism:

**Theorem 3.2.1** ([28])**.** *Compact symplectic toric orbifolds are classified by convex rational simple polytopes with a positive integer label attached to each facet.*

In dimension 2, a convex polygon is always simple (2 edges meeting at each vertex). It is **rational** if the edges emanating from $p$ have the form $p + tv_i, t \geq 0$, where $v_i \in \mathbb{Z}^2$. Unlike Delzant polygons though (see Section 2.3.1), the smoothness condition is not satisfied. Instead, we have the following: For each vertex $p$, the $v_i$ $(i = 1, 2)$ can be chosen to be a $\mathbb{Q}$-basis for the lattice $\mathbb{Z}^2$. Let $\Delta$ be a rational polygon in $\mathbb{R}^2$. For any vertex $p \in \Delta$, let $\vec{m} = (m_1, m_2), \vec{n} = (n_1, n_2)$ be the primitive outward pointing co-normals to the edges emanating from $p$, oriented anti-clockwise. If $\vec{m}$ and $\vec{n}$ are a $\mathbb{Z}$-basis of $\mathbb{Z}^2$, then the matrix having these vectors as rows is an element of $\mathrm{GL}(2, \mathbb{Z})$. Thus, we have that $p$ is smooth if and only if

$$\det \begin{bmatrix} m_1 & m_2 \\ n_1 & n_1 \end{bmatrix} = \pm 1.$$

Otherwise, $p$ corresponds to an orbifold singularity of order the absolute value of this determinant. In the Lerman-Tolman classification theorem, the positive integer labels attached to each facet in our picture should be 1, since only the vertices correspond to non-smooth points; hence we can just omit the labels. In Figure 3.1, we have

$$\det \begin{bmatrix} 0 & -1 \\ b & c \end{bmatrix} = b \quad and \quad \det \begin{bmatrix} b & c \\ -1 & 0 \end{bmatrix} = c$$

so the vertex $(c, 0)$ corresponds to an orbifold point of order $b$ and $(0, b)$ corresponds to

an orbifold point of order $c$ (this can be somewhat confusing). Obviously, the origin is a smooth point.

We will now describe the resolutions of $(\mathbb{C}P^2_{1,b,bk+1}, \omega_{1,b,bk+1})$ in terms of their toric models (for $b \geq 2$ and $k \geq 1$) and show that the resolution $(R_{1,b,bk+1}, \widetilde{\omega}_{\alpha_1,\ldots,\alpha_{b+1}})$ is symplectomorphic to a manifold obtained by blowing up a certain Hirzebruch surface $b$ times. Recall from Section 3.1.2 that the symplectic form $\widetilde{\omega}_{\alpha_1,\ldots,\alpha_{b+1}}$ on the resolution is obtained from making $b+1$ symplectic cuts (blow ups): The singularity $p_{bk+1} \in \mathbb{C}P^2_{1,b,bk+1}$ is resolved by $b$ consecutive symplectic cuts at levels $\alpha_1, \ldots, \alpha_b$ and the singularity $p_b \in \mathbb{C}P^2_{1,b,bk+1}$ is resolved by making 1 symplectic cut at level $\alpha_{b+1}$.

Let's start with the case of $(\mathbb{C}P^2_{1,2,2k+1}, \omega_{1,2,2k+1})$. Recall from Section 2.5 how we use continued fractions to resolve singularities. The singularity $p_{2k+1} \in \mathbb{C}P^2_{1,2,2k+1}$ corresponding to the vertex with co-normals $(-1,0), (2, 2k+1)$ is resolved by making corner cuts determined by the continued fraction expansion of $\frac{2k+1}{2}$. Observe that

$$\frac{2k+1}{2} = k+1 - \frac{1}{2} = [k+1, 2],$$

so $p_{2k+1}$ is resolved by a chain of two spheres $C_1, C_2$ such that $[C_1] \cdot [C_1] = -(k+1)$ and $[C_2] \cdot [C_2] = -2$ and $C_1, C_2$ intersect once transversely. Set $\vec{n}_0 = (-1, 0)$ and $\vec{n}_1 = (0, 1)$. Define

$$
\begin{aligned}
\vec{n}_2 &= (k+1)\vec{n}_1 - \vec{n}_0 = (1, k+1) \\
\vec{n}_3 &= 2\vec{n}_2 - \vec{n}_1 = (2, 2k+1).
\end{aligned}
$$

Then the moment polygon for the resolution of the singularity $p_{2k+1}$ appears in Figure 3.2. Observe that all the vertices are smooth, except the one with co-normals $(0, -1)$ and $\vec{n}_3$. This corresponds to an orbifold singularity of order 2 because

$$
\det \begin{bmatrix} 0 & -1 \\ 2 & 2k+1 \end{bmatrix} = 2 \,.
$$
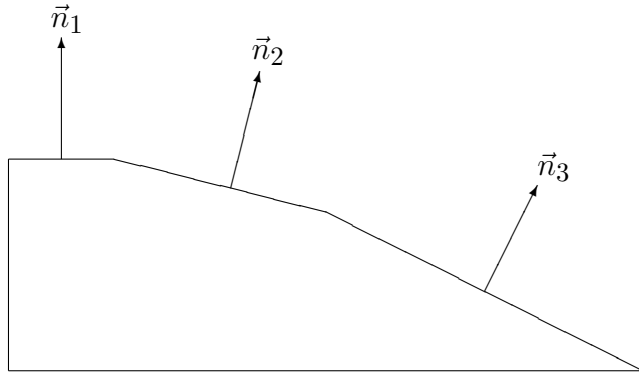
Figure 3.2: The resolution of the singularity $p_{2k+1}$.
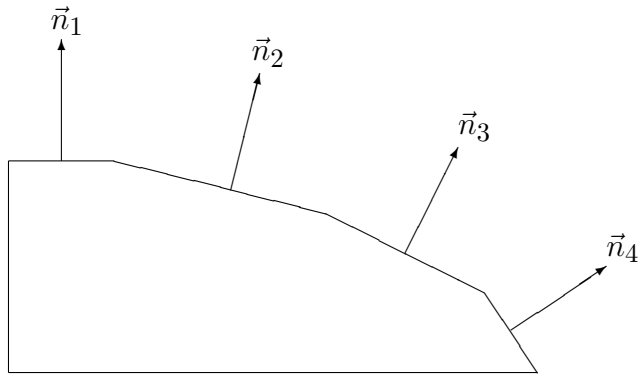


Figure 3.3: The full resolution of $\Delta_{2,2k+1}$.

Observe also that the edge with co-normal $\vec{n}_3 = (2, 2k+1)$ is what remains after making two cuts to the polygon $\Delta_{2,2k+1}$ in Figure 3.1.

The remaining singularity can be resolved by cutting the vertex labelled 2 with co-normal $\vec{n}_4 = (1, k)$ and it is easy to check that this results in a smooth polygon (Figure 3.3), hence it corresponds to a smooth symplectic manifold which is the resolution. We also have

$$(0, -1) + \vec{n}_3 = 2\vec{n}_4,$$

so the new edge with co-normal $\vec{n}_4$ corresponds to an embedded symplectic sphere $C_4$ with self-intersection $-2$. It is also easy to check that the edge with co-normal $\vec{n}_3$ corresponds to a sphere $C_3$ with self-intersection $-1$. To make things coherent with Section 3.1.2, we will be identifying the chain $C_1 \cup C_2$ with $\widehat{\Sigma}_{\alpha_1} \cup \Sigma_{\alpha_2}$ and the sphere $C_4$ with $\Sigma_{\alpha_3}$ so that

$$\begin{aligned}
[\widetilde{\omega}_{\alpha_1,\alpha_2,\alpha_3}](C_1) &= \ell_2(\alpha_1,\alpha_2) = \frac{c\,\alpha_1}{2} - 2\alpha_2 \\
[\widetilde{\omega}_{\alpha_1,\alpha_2,\alpha_3}](C_2) &= 2\alpha_2 \\
[\widetilde{\omega}_{\alpha_1,\alpha_2,\alpha_3}](C_4) &= 2\alpha_3.
\end{aligned}$$

In terms of cutting the polygon, this means that we resolve the vertex labelled $2k+1$ so that our first corner cut has size $\frac{c\,\alpha_1}{2}$ and the next corner cut has size $2\alpha_2$. Similarly, the vertex labelled $2$ should be cut with size $2\alpha_3$. The remaining sphere $C_3$ corresponds to an unnamed symplectic sphere from Section 3.1.2. Since the diagonal vertex in Figure 3.1 has rational length[1] equal to one, the remaining sphere $C_3$ must satisfy

$$[\widetilde{\omega}_{\alpha_1,\alpha_2,\alpha_3}](C_3) = 1 - \left( \frac{c\,\alpha_1}{2} + 2\alpha_2 + 2\alpha_3 \right).$$

We now show how to resolve the singularities of $(\mathbb{C}P^2_{1,b,bk+1}, \omega_{1,b,bk+1})$ using toric models. The arguments are completely analogous to the previous case. The singularity $p_{bk+1} \in \mathbb{C}P^2_{1,b,bk+1}$ corresponding to the vertex with co-normals $(-1,0), (b, bk+1)$ is resolved by making corner cuts determined by the Hirzebruch-Jung continued fraction

---

1. See the end of section 2.3.1 for the meaning of *rational length*.

expansion of $\frac{bk+1}{b}$. We have

$$
\begin{aligned}
\frac{bk+1}{b} &= k+1-\left(\frac{b-1}{b}\right) \\
&= k+1-\cfrac{1}{2-\frac{b-2}{b-1}} \\
&= k+1-\cfrac{1}{2-\cfrac{1}{2-\frac{b-3}{b-2}}} \\
&= k+1-\cfrac{1}{2-\cfrac{1}{2-\cdots-\frac{1}{2}}}
\end{aligned}
$$

so the continued fraction expansion is given by the string $[k+1, 2, 2, \ldots, 2]$, where the number of 2's in the string is $b-1$. This tells us that the resolution of $p_{bk+1}$ produces a chain of embedded spheres $C_1, C_2, \ldots, C_b$ such that $[C_1] \cdot [C_1] = -(k+1)$ and $[C_i] \cdot [C_i] = -2$ for $i = 2, \ldots, b$. Moreover, $[C_i] \cdot [C_j] = 1$ if $|i-j| = 1$. Set $\vec{n}_0 = (-1, 0)$ and $\vec{n}_1 = (0, 1)$. Define $\vec{n}_2 = (k+1)\vec{n}_1 - \vec{n}_0 = (1, k+1)$ and

$$
\vec{n}_{i+1} = 2\vec{n}_i - \vec{n}_{i-1} = (i, ik+1) \text{ for } i = 2, \ldots, b. \tag{3.6}
$$

The moment polygon for the resolution is a generalization of that in Figure 3.2 with $b$ new co-normals $\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_b$. The edge with co-normal $\vec{n}_{b+1} = (b, bk+1)$ is what remains after making $b$ cuts to the polygon $\Delta_{b,bk+1}$. It is easy to check that all vertices are smooth except the one with co-normals $(0, -1)$ and $\vec{n}_{b+1}$ which corresponds to the remaining order $b$ singularity. This one resolved by making a cut with co-normal $\vec{n}_{b+2} = (1, k)$ and this new edge corresponds to a smooth symplectic sphere $C_{b+2}$ satisfying $[C_{b+2}] \cdot [C_{b+2}] = -b$. Finally, the edge that corresponds to what remains of the diagonal in Figure 3.1 corresponds to a smooth symplectic sphere $C_{b+1}$ such that $[C_{b+1}] \cdot [C_{b+1}] = -1$. The new polygon is a generalization of that in Figure 3.3; it has $b+4$ edges and corresponds to a smooth symplectic manifold, which is the resolution $R_{1,b,bk+1}$. Again, to make things coherent with Section 3.1.2, we identify the chain of spheres $C_1 \cup \ldots \cup C_{b-1} \cup C_b$ with

$\widehat{\Sigma}_{\alpha_1} \cup \ldots \cup \widehat{\Sigma}_{\alpha_{b-1}} \cup \Sigma_{\alpha_b}$ and the sphere $C_{b+2}$ with $\Sigma_{\alpha_{b+1}}$ so that

$$
\begin{aligned}
[\widetilde{\omega}_{\alpha_1,\ldots,\alpha_{b+1}}](C_i) &= \ell_b(\alpha_i, \alpha_{i+1}) \quad i = 1, \ldots, b-1 \qquad (3.7) \\
[\widetilde{\omega}_{\alpha_1,\ldots,\alpha_{b+1}}](C_b) &= 2\alpha_b \\
[\widetilde{\omega}_{\alpha_1,\ldots,\alpha_{b+1}}](C_{b+1}) &= 1 - \left( \frac{c\,\alpha_1}{b} + \sum_{i=2}^{b} \frac{(b - (i-2))\alpha_i}{b - (i-1)} + b\,\alpha_{b+1} \right) \\
[\widetilde{\omega}_{\alpha_1,\ldots,\alpha_{b+1}}](C_{b+2}) &= b\,\alpha_{b+1}.
\end{aligned}
$$

Recall from (3.5) that $\ell_b(\alpha_i, \alpha_{i+1}) = \frac{(b-(i-2))\alpha_i}{b-(i-1)} - \frac{(b-(i-1))\alpha_{i+1}}{b-i}$.

**Lemma 3.2.2.** *There exists $\varepsilon_1 > \varepsilon_2 > \ldots > \varepsilon_b > 0$ (depending on $\alpha_b, \ldots, \alpha_{b+1}$) and a symplectic form $\Omega_{\mu,k,\varepsilon_1,\ldots,\varepsilon_b}$ on $W_k \# b\overline{\mathbb{C}P}^2$ such that $(R_{1,b,bk+1}, \widetilde{\omega}_{\alpha_1,\ldots,\alpha_{b+1}})$ is symplectomorphic to $(W_k \# b\overline{\mathbb{C}P}^2, \Omega_{\mu,k,\varepsilon_1,\ldots,\varepsilon_b})$, where the symplectic form on $W_k \# b\overline{\mathbb{C}P}^2$ comes from the form $\Omega_{\mu,k}$ on $W_k$ by blowing up with sizes $\varepsilon_1, \ldots, \varepsilon_b$.*

*Proof.* We will first establish that $R_{1,b,2k+1}$ and $W_k \# b\overline{\mathbb{C}P}^2$ are isomorphic as *toric varieties*. Then we'll see how to put a symplectic form $\Omega_{\mu,k,\varepsilon_1,\ldots,\varepsilon_b}$ on $W_k \# b\overline{\mathbb{C}P}^2$ so that $[\Omega_{\mu,k,\varepsilon_1,\ldots,\varepsilon_b}] = [\widetilde{\omega}_{\alpha_1,\ldots,\alpha_{b+1}}]$. By ([30]-Corollary 1.3), any two blow up forms in the same cohomology class must be diffeomorphic. Hence, this will prove Lemma 3.2.2.

**Step 1.** $R_{1,b,bk+1}$ *and* $W_k \# b\overline{\mathbb{C}P}^2$ *have the same fan.*

A rational polygon in $\mathbb{R}^2$ determines a fan by its primitive co-normal vectors. This fan determines a toric variety. Note that the co-normal vectors do not encode the sizes of their respective edges, which means that they cannot determine the symplectic form on the resulting toric variety. The fan corresponding to $R_{1,b,bk+1}$ is determined by the co-normals $\vec{n}_1, \ldots, \vec{n}_{b+2}$ described previously, in addition to $(0, -1)$ and $(-1, 0)$. We'll show that the moment polygon of $W_k \# b\overline{\mathbb{C}P}^2$ has the same co-normal vectors.

To see this, go back to Figure 2.1 or 2.2 in Section 2.3.1. The co-normal to the diagonal edge in the Hirzebruch trapezoid is $\vec{n}_{b+2} = (1, k)$ and the co-normal to the top horizontal edge is $\vec{n}_1 = (0, 1)$. To get the moment polygon for $W_k \# b\overline{\mathbb{C}P}^2$ we make

$b$ consecutive corner cuts, starting at the vertex meeting at the edges with co-normals $\vec{n}_1, \vec{n}_{b+2}$. The first cut produces a new co-normal $\vec{n}_2^*$ satisfying

$$\vec{n}_2^* = \vec{n}_1 + \vec{n}_{b+2} = (1, k+1).$$

Therefore, $\vec{n}_2^* = \vec{n}_2$ above. Next we cut at the vertex with co-normals $\vec{n}_2$ and $\vec{n}_{b+2}$, producing a new co-normal $\vec{n}_3^*$ such that

$$\vec{n}_3^* = \vec{n}_2 + \vec{n}_{b+2} = (2, 2k+1),$$

so that $\vec{n}_3^* = \vec{n}_3$ in (3.6) above. In general, the $i$th cut is made at the vertex with co-normals $\vec{n}_i$ and $\vec{n}_{b+2}$, and makes a new co-normal $\vec{n}_{i+1}^*$ with

$$\vec{n}_{i+1}^* = \vec{n}_i + \vec{n}_{b+2} = (i, ik+1), \quad i = 1, \ldots, b.$$

Comparing this with (3.6), it's easy to see that $R_{1,b,bk+1}$ and $W_k \# b\overline{\mathbb{C}P}^2$ have the same fan, hence they are isomorphic as toric varieties.

**Step 2.** *Finding a suitable cohomology class.*

We will start with $(R_{1,b,bk+1}, \omega_{\alpha_1, \ldots, \alpha_{b+1}})$ and show how to blow down $b$ times with specific sizes in order to obtain a symplectic form on $W_k$ in the same cohomology class as $\Omega_{\mu,k}$ from Section 2.3.1. Let's assume that $k$ is even; we will explain later how to modify the argument for the odd case. Let $\{B, F, E_1, \ldots, E_b\}$ be the natural basis of $H_2(W_k \# b\overline{\mathbb{C}P}^2; \mathbb{Z})$. The embedded spheres $C_1, \ldots, C_{b+2}$ obtained from the resolution process each represent homology classes in $H_2(R_{1,b,bk+1}; \mathbb{Z}) \cong H_2(W_k \# b\overline{\mathbb{C}P}^2; \mathbb{Z})$ and

we will make the following identifications

$$
\begin{aligned}
[C_1] &\longleftrightarrow B - \frac{k}{2}F - E_1 \\
[C_i] &\longleftrightarrow E_{i-1} - E_i \ (i = 2, \ldots b) \\
[C_{b+1}] &\longleftrightarrow E_b \\
[C_{b+2}] &\longleftrightarrow F - \sum_{i=1}^{b} E_i.
\end{aligned}
$$

The sizes of these spheres are given in (3.7). Set

$$
\varepsilon'_b := 1 - \left( \frac{c\,\alpha_1}{b} + \sum_{i=2}^{b} \frac{(b - (i-2))\alpha_i}{b - (i-1)} + b\,\alpha_{b+1} \right).
$$

Now blow down the sphere $C_{b+1}$. This cuts out a neighbourhood of $C_{b+1}$ and glues in a 4-ball. Set

$$
\begin{aligned}
\varepsilon'_{b-1} &:= [\omega_{\alpha_1,\ldots,\alpha_{b+1}}](C_b) + \varepsilon'_b \\
&= 1 - \left( \frac{c\,\alpha_1}{b} + \sum_{i \neq b} \frac{(b - (i-2))\alpha_i}{b - (i-1)} + b\,\alpha_{b+1} \right),
\end{aligned}
$$

and note that $\varepsilon'_{b-1} > \varepsilon'_b$. The blow down process transforms $C_{b+1}$ into a sphere of size $\varepsilon'_{b-1}$. In general, for $j = 2, \ldots, b - 1$, we put

$$
\begin{aligned}
\varepsilon'_j &:= [\omega_{\alpha_1,\ldots,\alpha_{b+1}}](C_{j+1}) + \epsilon'_{j+1} \\
&= 1 - \left( \frac{c\,\alpha_1}{b} + \sum_{i \neq j+1} \frac{(b - (i-2))\alpha_i}{b - (i-1)} + b\,\alpha_{b+1} \right),
\end{aligned}
$$

producing a sequence $\varepsilon'_1 > \varepsilon'_2 > \ldots > \epsilon'_b > 0$. The sphere $C_{b+2}$ is sent by the $b$-fold anti-blow up to a sphere $\mathcal{F} \subset W_k$ of size

$$
b\,\alpha_{b+1} + \varepsilon'_1 + \cdots + \varepsilon'_b.
$$

The sphere $C_1$ is sent by the blow down to a sphere $\mathcal{Z}_0 \subset W_k$ of size $\eta := \frac{c\,\alpha_1}{b} - \frac{b\,\alpha_2}{b-1} + \epsilon'_1$.

Now we will scale the symplectic form; put

$$\mu := \frac{\eta}{b\,\alpha_{b+1} + \sum_{i=1}^{b} \varepsilon'_i} + \frac{k}{2} \quad \text{and} \quad \varepsilon_i := \frac{\varepsilon'_i}{b\,\alpha_{b+1} + \sum_{i=1}^{b} \varepsilon'_i}.$$

The symplectic form $\Omega_{\mu,k}$ on $W_k$ in cohomology class $\mathrm{PD}(B + \mu F)$ now satisfies

$$[\Omega_{\mu,k}](\mathcal{Z}_0) = \mu - \frac{k}{2} \quad \text{and} \quad [\Omega_{\mu,k}](\mathcal{F}) = 1.$$

Therefore, by blowing up $(W_k, \Omega_{\mu,k})$ consecutively with sizes $\varepsilon_1 > \cdots > \epsilon_b$, we get a symplectic form $\Omega_{\mu,k,\varepsilon_1,\ldots,\varepsilon_b}$ in cohomology class $\mathrm{PD}(B + \mu F - \sum_{i=1}^{b} \varepsilon_i E_i)$. Scaling this form then gives a form in class $[\widetilde{\omega}_{\alpha_1,\ldots,\alpha_{b+1}}]$. This proves Lemma 3.2.2. $\qquad\square$

**Remark**: When $k$ is odd, we start with the basis $\{B^*, F^*\}$ of $H_2(W_k; \mathbb{Z})$ given in Section 2.3. Then we let $\{B^*, F^*, E_1^*, \ldots, E_b^*\}$ be the corresponding basis for $H_2(W_k \# b\overline{\mathbb{CP}}^2; \mathbb{Z})$, where $E_1^*, \ldots, E_b^*$ are the classes of the exceptional divisors. Make the swaps in (2.3) of Section 2.3.1 combined with the swaps $E_i^* \leftrightarrow E_i$ for $i = 2, \ldots, b$ which allows us to make the following identifications:

$$
\begin{aligned}
[C_1] &\longleftrightarrow B - \left(\frac{k+1}{2}\right) F \\
[C_2] &\longleftrightarrow F - E_1 - E_2 \\
[C_i] &\longleftrightarrow E_{i-1} - E_i \ (i = 3, \ldots, b) \\
[C_{b+1}] &\longleftrightarrow E_b \\
[C_{b+2}] &\longleftrightarrow E_1 - E_2 - \cdots - E_b.
\end{aligned}
$$

Again, the sizes of these spheres are given in (3.7). Now set

$$
\begin{aligned}
\varepsilon'_b &:= [\widetilde{\omega}_{\alpha_1,\ldots,\alpha_{b+1}}](C_{b+1}) \\
\varepsilon'_j &:= [\widetilde{\omega}_{\alpha_1,\ldots,\alpha_{b+1}}](C_{j+1}) + \varepsilon'_{j+1}
\end{aligned}
$$

for $j = 1, \ldots, b - 1$ just like before. This time the spheres will be blown down in a different order, but a similar computation will go through.

Before finishing up this chapter, there is an important notion about homology classes that needs to be discussed. Let $X_{b+1} = \mathbb{C}P^2 \,\#\, (b+1)\overline{\mathbb{C}P}^2$. Then $X_{b+1}$ is diffeomorphic to both $R_{1,b,bk+1}$ and $W_k \,\#\, b\overline{\mathbb{C}P}^2$. Scale the Fubini-Study form on $\mathbb{C}P^2$ so that the symplectic area of $\mathbb{C}P^1 \subset \mathbb{C}P^2$ is 1. Now blow up $\mathbb{C}P^2$ $b+1$ times symplectically with sizes $\delta_1, \ldots, \delta_{b+1}$ and call the resulting symplectic form $\omega_{\delta_1, \ldots, \delta_{b+1}}$. Then we have

$$\mathrm{PD}[\omega_{\delta_1, \ldots, \delta_{b+1}}] = L - \sum_{i=1}^{b+1} \delta_i V_i \,,$$

where $\{L, V_1, \ldots, V_{b+1}\}$ is the standard basis of $H_2(X_{b+1}; \mathbb{Z})$. Since $X_{b+1}$ is diffeomorphic to the $b$-fold blow up of $S^2 \times S^2$, we get an isomorphism in homology that acts on basis elements as follows

$$
\begin{array}{ccll}
H_2(X_{b+1}; \mathbb{Z}) & \longrightarrow & H_2\big((S^2 \times S^2) \,\#\, b\overline{\mathbb{C}P}^2; \mathbb{Z}\big) & \qquad (3.8) \\[4pt]
L & \mapsto & B + F - E_1 & \\[2pt]
V_1 & \mapsto & B - E_1 & \\[2pt]
V_2 & \mapsto & F - E_1 & \\[2pt]
V_3 & \mapsto & E_2 & \\[2pt]
\vdots & & \vdots & \\[2pt]
V_{b+1} & \mapsto & E_b \,. &
\end{array}
$$

Again by [30], two blow up forms in the same cohomology class are diffeomorphic, so by scaling and comparing cohomology classes, we see that $(X_{b+1}, \omega_{\delta_1, \ldots, \delta_{b+1}})$ is symplectomorphic to the $b$-fold blow up of $(S^2 \times S^2, \nu\sigma_1 + \sigma_2)$ with sizes $\gamma_1, \ldots, \gamma_b$ such that

$$\nu = \frac{1 - \delta_2}{1 - \delta_1} \,, \quad \gamma_1 = \frac{1 - \delta_1 - \delta_2}{1 - \delta_1} \,, \quad \gamma_i = \frac{\delta_{i+1}}{1 - \delta_1}$$

for $i = 1, \ldots, b$.

**Definition 3.2.3.** *We say that a homology class $A = a_0 L - \sum_i a_i V_i$ is **reduced** with respect to the basis $\{L, V_1, \ldots, V_{b+1}\}$ if $a_1 \geq a_2 \geq \ldots \geq a_{b+1} \geq 0$ and $a_0 \geq a_1 + a_2 + a_3$.*

We should check what the conditions are for a homology class to be reduced in the new basis $\{B, F, E_1, \ldots, E_b\}$. To do this, reverse the isomorphism (3.8). Now the ordered basis $\{B, F, E_1, \ldots, E_b\}$ is sent to the ordered basis $\{L - V_2, L - V_1, L - V_1 - V_2, V_3, \ldots, V_{b+1}\}$. Writing $B + \nu F - \sum_i \gamma_i E_i$ in terms of the other basis, we get

$$(1 + \nu - \gamma_1)L - (\nu - \gamma_1)V_1 - (1 - \gamma_1)V_2 - \sum_{i=2}^{b} \gamma_i V_{i+1}\,.$$

The conditions for this homology class to be reduced are then

$$\nu - \gamma_1 \;\geq\; 1 - \gamma_1 \;\geq\; \gamma_2 \;\geq\; \ldots \;\geq\; \gamma_b$$
$$1 + \nu - \gamma_1 \;\geq\; (\nu - \gamma_1) \;+\; (1 - \gamma_1) \;+\; \gamma_2\,.$$

Putting these together gives $\nu \geq 1 \geq \gamma_1 + \gamma_2 > \gamma_1 \geq \ldots \geq \gamma_b$. It is now easy to check the following (this will be important in Section 4.3)

**Lemma 3.2.4.** *The Poincare dual of the cohomology class $[\Omega_{\mu, k, \varepsilon_1, \ldots, \varepsilon_b}]$ from Lemma 3.2.2 is reduced with respect to the basis $\{B, F, E_1, \ldots, E_b\}$.*

# Chapter 4

# The Symplectomorphism Groups of $\mathbb{C}P^2_{a,b,c}$

## 4.1 Orbifold Diffeomorphisms and Symplectomorphisms

We begin by discussing orbifold maps and reduced orbifold maps, as defined in Borzellino and Brunsden's paper [8]. Let $\mathcal{O}$ be a $C^r$-orbifold ($r \geq 0$) with isolated singular points. Recall that this means that $\mathcal{O}$ is a Hausdorff space such that for each $x \in \mathcal{O}$, there is a $C^r$-uniformizing chart around $x$ and satisfying certain compatibility conditions. Let $\mathrm{Sing}(\mathcal{O})$ be the singular set of $\mathcal{O}$ and $\mathrm{Reg}(\mathcal{O})$ the complement of the singular set. Note that $\mathrm{Reg}(\mathcal{O})$ is open and dense in $\mathcal{O}$. We should mention that in [8], they are dealing with more general orbifolds where the singular points are not necessarily isolated, so our definitions differs from theirs in some small details.

**Definition 4.1.1.** *Let $\mathcal{O}$ be a $C^r$-orbifold with isolated singular points. Then $\mathcal{O}$ comes equipped with a natural partition*

$$\mathcal{O} = \mathrm{Reg}(\mathcal{O}) \sqcup \mathrm{Sing}_{i_1} \sqcup \mathrm{Sing}_{i_2} \sqcup \ldots \sqcup \mathrm{Sing}_{i_n},$$

*where $\mathrm{Sing}_{i_k}$ consists of all singularities whose local groups have a fixed isomorphism type and $\mathrm{Sing}_{i_1} \sqcup \ldots \sqcup \mathrm{Sing}_{i_n} = \mathrm{Sing}(\mathcal{O})$. For any $x \in \mathcal{O}$, let $\mathcal{P}_x$ be the piece of the partition containing $x$.*

**Definition 4.1.2.** *Let $\mathcal{O}_1, \mathcal{O}_2$ be $C^r$-orbifolds. A $\mathbf{C^0}$-**orbifold map** $(f, \{\widetilde{f_x}\})$ from $\mathcal{O}_1$ to $\mathcal{O}_2$ consists of the following:*

(1) *A continuous map $f : X_{\mathcal{O}_1} \longrightarrow X_{\mathcal{O}_2}$ of the underlying topological spaces.*

(2) *For each $y \in \mathcal{P}_x$, there are uniformizing charts $(\widetilde{U}_y, G_y, \pi_y)$ around $y$ and $(\widetilde{V}_{f(y)}, G_{f(y)}, \pi_{f(y)})$ around $f(y)$ with $f(\widetilde{U}_y) \subset \widetilde{V}_{f(y)}$, along with a group homomorphism $\Theta_{f,y} : G_y \to G_{f(y)}$ such that the following diagram commutes:*

$$
\begin{array}{ccc}
\widetilde{U}_y & \xrightarrow{\;\widetilde{f}_y\;} & \widetilde{V}_{f(y)} \\
\downarrow & & \downarrow \\
\widetilde{U}_y \,/\, G_y & \longrightarrow & \widetilde{V}_{f(y)} \,/\, \Theta_{f,y}(G_y) \\
\downarrow{\scriptstyle \pi_y} & & \downarrow{\scriptstyle \pi_{f(y)}} \\
U_y & \xrightarrow{\;\;f\;\;} & V_{f(y)}
\end{array}
$$

(3) *Each local lift $\widetilde{f}_y$ is required to be $\Theta_{f,y}$-equivariant.*

(4) *Two orbifold maps $(f, \{\widetilde{f}_x\})$ and $(g, \{\widetilde{g}_x\})$ are considered equivalent if for each $x \in \mathcal{O}_1$, there exists a uniformizing chart $(\widetilde{U}_x, G_x)$ around $x$ such that $\widetilde{f}_x|_{\widetilde{U}_x} = \widetilde{g}_x|_{\widetilde{U}_x}$. Note that this implies that $f = g$.*

It is a fact (see [8]-Lemma 23) that a local lift $\widetilde{f}_x$ chosen on a particular uniformizing chart around $x$ uniquely specifies a local lift on any other chart around $x$. Thus, the $\widetilde{f}_x$'s, once chosen, are independent of the choice of local charts.

We say that an orbifold map $(f, \{\widetilde{f}_x\})$ is **$C^r$-smooth** if each $\widetilde{f}_x$ can be chosen to be $C^r$-differentiable. The set of $C^r$-smooth orbifold maps from $\mathcal{O}_1$ to $\mathcal{O}_2$ is topologized as in ([8]-Section 4), and we denote this space by $C_r^{orb}(\mathcal{O}_1, \mathcal{O}_2)$. Now put $\mathcal{O} = \mathcal{O}_1 = \mathcal{O}_2$, so that $C_r^{orb}(\mathcal{O})$ is the space of $C^r$-orbifold maps from $\mathcal{O}$ to itself.

**Definition 4.1.3.** *Let $\mathcal{O}$ be a $C^r$-orbifold. We define the following subspaces of $C_r^{orb}(\mathcal{O})$:*

- $\mathrm{Diff}_r^{\mathrm{orb}}(\mathcal{O}) := \{(f, \{\widetilde{f}_x\}) \in C_r^{orb}(\mathcal{O}) \mid f^{-1} \in C_r^{orb}(\mathcal{O})\}$

- $\mathrm{Diff}^{\mathrm{orb}}(\mathcal{O}) := \mathrm{Diff}_\infty^{\mathrm{orb}}(\mathcal{O})$

**Theorem 4.1.4** ([8])**.** $\mathrm{Diff}^{\mathrm{orb}}(\mathcal{O})$ *is a Fréchet manifold.*

In fact, $\mathrm{Diff}^{\mathrm{orb}}(\mathcal{O})$ is a Fréchet Lie group where the group operation is composition: $(f, \{\widetilde{f}_x\}) \circ (g, \{\widetilde{g}_x\}) = (f \circ g, \{\widetilde{f}_x \circ \widetilde{g}_x\})$, but for our purposes we only care that it's a topological group. Consider the following subgroup of $\mathrm{Diff}^{\mathrm{orb}}(\mathcal{O})$:

$$\mathcal{I}(\mathcal{O}) := \{(f, \{\widetilde{f}_x\}) \in \mathrm{Diff}^{\mathrm{orb}}(\mathcal{O}) \,|\, f = \mathrm{Id}\}.$$

This is the subgroup consisting of all lifts of the identity map. It follows from the definition of orbifold map that this subgroup is finite if $\mathcal{O}$ is compact.

**Definition 4.1.5.** *The quotient group* $\mathrm{Diff}^{\mathrm{orb}}(\mathcal{O}) \,/\, \mathcal{I}(\mathcal{O})$ *is called the group of* ***reduced orbifold diffeomorphisms*** *of* $\mathcal{O}$*, and we denote it by* $\mathrm{Diff}^{\mathrm{red}}(\mathcal{O})$*. Note that* $\mathrm{Diff}^{\mathrm{red}}(\mathcal{O})$ *inherits a topological group structure from* $\mathrm{Diff}^{\mathrm{orb}}(\mathcal{O})$*.*

Two elements $(f, \{\widetilde{f}_x\}), (g, \{\widetilde{g}_x\})$ lie in the same coset of $\mathcal{I}(\mathcal{O})$ if and only if $f = g$ and $\widetilde{f}_x = \widetilde{I}_x \circ \widetilde{g}_x$, where $\widetilde{I}_x$ is some lift of the identity over $x$. Thus, the images of $(f, \{\widetilde{f}_x\}), (g, \{\widetilde{g}_x\})$ are equal in the quotient if and only if $f = g$ and their lifts are related by composition with elements from $\mathcal{I}(\mathcal{O})$. For this reason, we denote the image of $(f, \{\widetilde{f}_x\}) \in \mathrm{Diff}^{\mathrm{orb}}(\mathcal{O})$ in the quotient simply by $f$, where it should be understood that $f : \mathrm{Reg}(\mathcal{O}) \to \mathrm{Reg}(\mathcal{O})$ is a diffeomorphism and for each $x \in \mathrm{Sing}(\mathcal{O})$, there are uniformizing charts, along with a group homomorphism and suitable lifts (as in Definition 4.1.2) making a commutative square. Note that we have a short exact sequence

$$1 \to \mathcal{I}(\mathcal{O}) \to \mathrm{Diff}^{\mathrm{orb}}(\mathcal{O}) \to \mathrm{Diff}^{\mathrm{red}}(\mathcal{O}) \to 1.$$

Now let $(\mathcal{O}, \omega)$ be a symplectic orbifold. Recall, this means that each local uniformizing chart $(\widetilde{U}, G, \pi)$ comes equipped with a $G$-invariant symplectic form $\widetilde{\omega}$ that descends to $\omega$ on $U \cong \widetilde{U}/G$ and transforms correctly under overlapping maps. The quadruple $(\widetilde{U}, G, \pi, \widetilde{\omega})$ is a called symplectic uniformizing chart. We often simply denote it by $(\widetilde{U}, \widetilde{\omega})$.

**Definition 4.1.6.** *Let $(\mathcal{O}, \omega)$ be a symplectic orbifold and $(f, \{\widetilde{f}_x\}) \in \mathrm{Diff}^{\mathrm{orb}}(\mathcal{O})$ a $C^\infty$-orbifold diffeomorphism. We call $(f, \{\widetilde{f}_x\})$ an **orbifold symplectomorphism** if the following holds:*

- *For each $y \in \mathcal{P}_x$, there are symplectic uniformizing charts $(\widetilde{U}_y, \widetilde{\omega}_y)$ around $y$ and $(\widetilde{V}_{f(y)}, \widetilde{\omega}_{f(y)})$ around $f(y)$ such that $\widetilde{f}_y^* \widetilde{\omega}_{f(y)} = \widetilde{\omega}_y$. There should also be a group homomorphism and a commutative diagram similar to that in Definition 4.1.2. Note that this implies that $f^* \omega = \omega$ on $\mathrm{Reg}(\mathcal{O})$.*

For a symplectic orbifold $(\mathcal{O}, \omega)$, let $\mathrm{Symp}^{\mathrm{orb}}(\mathcal{O})$ be the subgroup of $\mathrm{Diff}^{\mathrm{orb}}(\mathcal{O})$ consisting of orbifold symplectomorphisms. Similarly, let $\mathrm{Symp}^{\mathrm{red}}(\mathcal{O})$ be the quotient group $\mathrm{Symp}^{\mathrm{orb}}(\mathcal{O}) / \mathcal{I}(\mathcal{O})$. Note that both $\mathrm{Symp}^{\mathrm{orb}}(\mathcal{O})$ and $\mathrm{Symp}^{\mathrm{red}}(\mathcal{O})$ are topological groups.

Now consider the weighted projective spaces $(\mathbb{C}P^2_{a,b,c}, \omega_{a,b,c})$, where $a, b, c \geq 1$, and they are pairwise relatively prime. Recall that $\mathrm{Sing}(\mathbb{C}P^2_{a,b,c}) = \{p_a, p_b, p_c\}$, where $p_a = [1:0:0], p_b = [0:1:0], p_c = [0:0:1]$.

**Definition 4.1.7.** *Let $\mathrm{Symp}^{\mathrm{orb}}_{a,b,c}$ be the group of orbifold symplectomorphisms of $\mathbb{C}P^2_{a,b,c}$ with the symplectic form $\omega_{a,b,c}$. Similarly, we use $\mathrm{Symp}^{\mathrm{red}}_{a,b,c}$ to denote the group of reduced orbifold symplectomorphisms of $(\mathbb{C}P^2_{a,b,c}, \omega_{a,b,c})$.*

Elements of $\mathrm{Symp}^{\mathrm{orb}}_{a,b,c}$ have the form $(f, \widetilde{f}_a, \widetilde{f}_b, \widetilde{f}_c)$, where $\widetilde{f}_a, \widetilde{f}_b, \widetilde{f}_c$ fit into equivariant diagrams

$$
\begin{array}{ccc}
\widetilde{V}_a \xrightarrow{\widetilde{f}_a} \widetilde{U}_a & \quad \widetilde{V}_b \xrightarrow{\widetilde{f}_b} \widetilde{U}_b & \quad \widetilde{V}_c \xrightarrow{\widetilde{f}_c} \widetilde{U}_c \\
\downarrow \quad\quad \downarrow & \quad \downarrow \quad\quad \downarrow & \quad \downarrow \quad\quad \downarrow \\
V_a \xrightarrow{f} U_a & \quad V_b \xrightarrow{f} U_b & \quad V_c \xrightarrow{f} U_c
\end{array}
$$

where $\widetilde{V}_a, \widetilde{V}_b, \widetilde{V}_c$ are, respectively, uniformizing charts above the singular points $p_a, p_b, p_c$ and the lifts of course preserve the corresponding symplectic forms (that we have not written). Here it should be noted that $f$ fixes each of the points $p_a, p_b, p_c$ since we are assuming that these singular points have different order. As before, we will denote a reduced symplectomorphism simply by $f$, with the understanding that around each

singular point there exist diagrams like those above where the lifts are defined only up to composition by elements from $\mathcal{I}(\mathbb{C}P^2_{a,b,c}) \cong \mathbb{Z}_a \times \mathbb{Z}_b \times \mathbb{Z}_c \cong \mathbb{Z}_{abc}$.

## 4.2 The Groups $\mathrm{Symp}^{\mathrm{red}}_{1,1,c}$

The goal of this section is to prove the following theorem:

**Theorem 4.2.1.** $\mathrm{Symp}^{\mathrm{red}}_{1,1,c}$ *is weakly homotopy equivalent to* $\mathrm{U}(2)\,/\,\mathbb{Z}_c$ *for any positive integer* $c$.

Start by considering the map

$$\Psi : \mathrm{Symp}^{\mathrm{orb}}_{1,1,c} \longrightarrow \mathrm{Aut}^{\mathbb{Z}_c}(T_0\widetilde{U}_c),$$
$$(f, \widetilde{f_c}) \mapsto d\widetilde{f_c}(0).$$

where $\mathrm{Aut}^{\mathbb{Z}_c}(T_0\widetilde{U}_c)$ is the group of linear $\mathbb{Z}_c$-equivariant automorphisms of the tangent space $T_0\widetilde{U}_c$. This map is a well-defined group homomorphism. Also, it is easy to see that the induced $\mathbb{Z}_c$-action on $T_0\widetilde{U}_c$ is the same as the $\mathbb{Z}_c$-action on $\widetilde{U}_c$, namely the diagonal action: $(z, w) \mapsto (\xi z, \xi w)$. It follows that any linear automorphism of $T_0\widetilde{U}_c$ is equivariant under this action. We therefore have

$$\mathrm{Aut}^{\mathbb{Z}_c}(T_0\widetilde{U}_c) \cong \mathrm{Aut}(\mathbb{C}^2) = \mathrm{Sp}(4)^{\mathbb{Z}_c} \simeq \mathrm{U}(2),$$

where the last relation is a homotopy equivalence since $\mathrm{Sp}(4)^{\mathbb{Z}_c}$ retracts onto $\mathrm{U}(2)$[1] Let $\mathrm{K}_\Psi := \ker \Psi$, so that we have an exact sequence of topological groups

$$1 \longrightarrow \mathrm{K}_\Psi \longrightarrow \mathrm{Symp}^{\mathrm{orb}}_{1,1,c} \longrightarrow \mathrm{Sp}(4)^{\mathbb{Z}_c} \longrightarrow 1. \tag{4.1}$$

We claim that $\Psi$ is a locally trivial fibration. To establish this, we make use of a result of Richard Palais (see [39]-Theorem A),

---

1. This can be proved using the same argument as the Claim right after Theorem 4.3.1.

**Proposition 4.2.2** ([39]). *If $G$ is a topological group and $X$ is a $G$-space admitting local sections, then any equivariant map of a $G$-space into $X$ is locally trivial.*

Recall that if $x_0$ is an element of a $G$-space $X$, then a local section for $X$ at $x_0$ is a map $\sigma : U \to G$ ($U$ a neighbourhood of $x_0$) such that $\sigma(u) \cdot x_0 = u$ for all $u \in U$. Observe that $\mathrm{Sp}(4)^{\mathbb{Z}_c}$ becomes a $\mathrm{Symp}^{\mathrm{orb}}_{1,1,c}$-space under the action

$$(f, \widetilde{f_c}) \cdot A = d\widetilde{f_c}(0)A \,, \tag{4.2}$$

where $A \in \mathrm{Sp}(4)^{\mathbb{Z}_c}$ and the action is by matrix multiplication. Also, $\mathrm{Symp}^{\mathrm{orb}}_{1,1,c}$ acts on itself (on the left) by composition

$$(g, \widetilde{g_c}) \cdot (f, \widetilde{f_c}) = (g \circ f, \widetilde{g_c} \circ \widetilde{f_c}),$$

and it's easy to see that the map $\Psi : \mathrm{Symp}^{\mathrm{orb}}_{1,1,c} \to \mathrm{Sp}(4)^{\mathbb{Z}_c}$ is equivariant with respect to both these actions. Thus, by Palais' result, to prove that $\Psi$ is a locally trivial fibration it suffices to find a local section over any element $A_0 \in \mathrm{Sp}(4)^{\mathbb{Z}_c}$. In fact, it suffices to find local sections in a neighbourhood of $\mathrm{Id} \in \mathrm{Sp}(4)^{\mathbb{Z}_c}$, since $\mathrm{Sp}(4)^{\mathbb{Z}_c}$ is a topological group and we can get to any other neighbourhood by conjugation.

**Lemma 4.2.3.** *Given $\mathrm{Id} \in \mathrm{Sp}(4)^{\mathbb{Z}_c}$, there is a continuous map*

$$\sigma : \mathcal{N}_{\mathrm{Id}} \to \mathrm{Symp}^{\mathrm{orb}}_{1,1,c}$$

*such that $\sigma(A) \cdot \mathrm{Id} = A$ for all $A \in \mathcal{N}_{\mathrm{Id}}$, where $\mathcal{N}_{\mathrm{Id}}$ is a contractible neighbourhood of the identity in $\mathrm{Sp}(4)^{\mathbb{Z}_c}$.*

*Proof.* Let $\mathfrak{sp}(4)^c$ be the Lie algebra of $\mathrm{Sp}(4)^{\mathbb{Z}_c}$ and consider the exponential map

$$\exp : \mathfrak{sp}(4)^c \longrightarrow \mathrm{Sp}(4)^{\mathbb{Z}_c},$$

which is a local diffeomorphism from a neighbourhood $\mathcal{U}_0$ of the origin in $\mathfrak{sp}(4)^c$ onto a neighbourhood $\mathcal{N}_{\mathrm{Id}}$ of $\mathrm{Id} \in \mathrm{Sp}(4)^{\mathbb{Z}_c}$. Thus, we can define a local inverse $\log : \mathcal{N}_{\mathrm{Id}} \to \mathcal{U}_0$ which gives a deformation retraction

$$\mathcal{N}_{\mathrm{Id}} \longrightarrow \mathcal{N}_{\mathrm{Id}}, \quad A \mapsto \exp(t \cdot \log(A))$$

that defines a canonical (equivariant) path $A_t$ from any $A \in \mathcal{N}_{\mathrm{Id}}$ to the identity. The vector field

$$X = \frac{d}{dt} A_t = \log(A) A_t.$$

must be invariant under the $\mathbb{Z}_c$-action because the path $A_t$ is equivariant. Since $\widetilde{U}_c$ is contractible, all 1-forms on $\widetilde{U}_c$ are exact, hence there exists a smooth Hamiltonian $H : \widetilde{U}_c \to \mathbb{R}$ such that $\imath_X \widetilde{\omega}_c|_0 = dH|_0$, and the functions $H$ must be invariant under the group action. Now let $\rho : \widetilde{U}_c \to \mathbb{R}$ be a smooth bump function satisfying

- $\mathrm{supp}(\rho) \subset \widetilde{U}_c$.

- $\rho \equiv 1$ on a smaller neighbourhood $\widetilde{U}_c' \subset \widetilde{U}_c$ containing 0.

By averaging we can make $\rho$ invariant under the $\mathbb{Z}_c$-action. Now define $G : \widetilde{U}_c \to \mathbb{R}$ by $G(x) = \rho(x)H(x)$. Again, this function remains invariant under the group action. Define a vector fields $Y$ by $\imath_Y \widetilde{\omega}_c = dG$, and let $\widetilde{g}_t : \widetilde{U}_c \to \widetilde{U}_c$ be the corresponding Hamiltonian isotopy. Then $\widetilde{g}_1$ satisfies $d\widetilde{g}_1(0) = A$. Since $\widetilde{g}_1$ is equivariant, it descends to to a symplectic map $g : U_c \to U_c$ which extends by the identity to give a global symplectomorphism on $\mathbb{C}P^2_{1,1,c}$ having $\widetilde{g}_1$ as a local lift over the singularity $p_c$. We can now define a local section by $\sigma(A) := (g, \widetilde{g}_1)$. Via the action (4.2), it's easy to see that it satisfies the requirements of the lemma. $\qquad\square$

Recall that the group $\mathrm{Symp}^{\mathrm{red}}_{1,1,c}$ is the quotient of $\mathrm{Symp}^{\mathrm{orb}}_{1,1,c}$ by its subgroup consisting of lifts of the identity map. This subgroup is isomorphic to $\mathbb{Z}_c$. Thus, we have an exact sequence

$$1 \longrightarrow \mathbb{Z}_c \longrightarrow \mathrm{Symp}^{\mathrm{orb}}_{1,1,c} \longrightarrow \mathrm{Symp}^{\mathrm{red}}_{1,1,c} \longrightarrow 1,$$

and we denote the image of an element $(f, \widetilde{f}_c) \in \mathrm{Symp}^{\mathrm{orb}}_{1,1,c}$ in the quotient simply by $f$. Notice that we have another map

$$\mathrm{Symp}^{\mathrm{red}}_{1,1,c} \xrightarrow{\Phi} \mathrm{Sp}(4)^{\mathbb{Z}_c} / \mathbb{Z}_c$$
$$f \mapsto [d\widetilde{f}_c(0)].$$

This map is well-defined because above the singular point $p_c$, any two local lifts of $f$ are related via an action of $\mathbb{Z}_c$. Thus, all local lifts above $p_c$ are equivalent in the quotient. Let $\mathrm{K}_\Phi := \ker \Phi$, so that we have another exact sequence of groups

$$1 \longrightarrow \mathrm{K}_\Phi \longrightarrow \mathrm{Symp}^{\mathrm{red}}_{1,1,c} \longrightarrow \mathrm{Sp}(4)^{\mathbb{Z}_c} / \mathbb{Z}_c \longrightarrow 1. \tag{4.3}$$

In fact, the two sequences (4.1) and (4.3) fit nicely into a diagram where everything commutes:

$$\begin{array}{ccccc} \mathrm{K}_\Psi & \longrightarrow & \mathrm{Symp}^{\mathrm{orb}}_{1,1,c} & \xrightarrow{\Psi} & \mathrm{Sp}(4)^{\mathbb{Z}_c} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{K}_\Phi & \longrightarrow & \mathrm{Symp}^{\mathrm{red}}_{1,1,c} & \xrightarrow{\Phi} & \mathrm{Sp}(4)^{\mathbb{Z}_c} / \mathbb{Z}_c \end{array} \tag{4.4}$$

**Lemma 4.2.4.** *The map $\Phi$ is also locally trivial.*

*Proof.* This is also a consequence of Proposition 4.2.2. Again, let $\mathcal{N}_{\mathrm{Id}}$ be a contractible neighbourhood of the identity in $\mathrm{Sp}(4)^{\mathbb{Z}_c}$ with $A \in \mathcal{N}_{\mathrm{Id}}$. By the previous lemma, a local section for $\Psi$ over $A$ is given by

$$\sigma : \mathcal{N}_{\mathrm{Id}} \longrightarrow \mathrm{Symp}^{\mathrm{orb}}_{1,1,c}$$
$$A \mapsto (g, \widetilde{g}_1),$$

where $(g, \widetilde{g}_1)$ depends continuously on $A$ from the previous construction. Let $Q$ be the quotient map given by the $\mathbb{Z}_c$-action. Then $Q(g, \widetilde{g}_1) = g$. Let $[A]$ be the image of $A$

under the $\mathbb{Z}_c$-action. We want to find a map

$$\tau : \mathcal{N}_{\mathrm{Id}} \, / \, \mathbb{Z}_c \longrightarrow \mathrm{Symp}^{\mathrm{red}}_{1,1,c}$$

such that $\tau[A] \cdot [\mathrm{Id}] = [A]$ for any $[A] \in \mathcal{N}_{\mathrm{Id}} \, / \, \mathbb{Z}_c$. It seems reasonable to define $\tau[A] :=$ $Q(\sigma(A))$, but we must check that this is independent of the representative $A$. This follows from the following

**Claim.** *For any $A \in \mathcal{N}_{\mathrm{Id}}$ and $\xi \in \mathbb{Z}_c$, if $\sigma(A) = (g, \widetilde{g}_1)$, then*

$$\sigma(\xi A) = (g, \xi \cdot \widetilde{g}_1)$$

*Proof.* Go back to the proof of Lemma 4.2.3, replace the path $A_t$ with $\xi A_t$ and just carry everything through. Notice that the neighbourhood $\mathcal{N}_{\mathrm{Id}}$ is replaced by $\mathcal{N}_{\xi \cdot \mathrm{Id}}$. Also notice that $\imath_{(\xi X)} \widetilde{\omega}_c = d(\xi H)$, and we can use the same partition of unity. $\qquad \square$

Back to the original proof. Let $A'$ be another representative of $[A]$. Then $A' = \xi A$ for some $\xi \in \mathbb{Z}_c$, thus we have $\sigma(A') = \sigma(\xi A) = (g, \xi \cdot \widetilde{g}_1)$. Hence, $Q(\sigma(A')) = Q(\sigma(A)) = g$, so we have found a local section

$$\tau : \mathcal{N}_{\mathrm{Id}} \, / \, \mathbb{Z}_c \longrightarrow \mathrm{Symp}^{\mathrm{red}}_{1,1,c}.$$

By Proposition 4.2.2, the map $\Phi$ is locally trivial. $\qquad \square$

Now we want to understand the fibration

$$\mathrm{K}_\Phi \longrightarrow \mathrm{Symp}^{\mathrm{red}}_{1,1,c} \longrightarrow \mathrm{Sp}(4)^{\mathbb{Z}_c} \, / \, \mathbb{Z}_c.$$

To do this, we first consider the other kernel, $\mathrm{K}_\Psi$, in the top sequence of (4.4), and the following subspace of $\mathrm{K}_\Psi$:

$$\mathrm{K}^*_\Psi := \{(f, \widetilde{f}_c) \, | \, \widetilde{f}_c = \mathrm{Id} \text{ near } 0\}.$$

**Lemma 4.2.5.** *The inclusion* $K_\Psi^* \hookrightarrow K_\Psi$ *is a weak homotopy equivalence.*

*Proof.* See Section 6.1. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Let $K_\Phi^* := Q(K_\Psi^*)$, and consider the following extension of diagram (4.4):

$$
\begin{array}{ccccccc}
K_\Psi^* & \overset{i}{\hookrightarrow} & K_\Psi & \longrightarrow & \mathrm{Symp}_{1,1,c}^{\mathrm{orb}} & \longrightarrow & \mathrm{Sp}(4)^{\mathbb{Z}_c} \\
\downarrow{\scriptstyle Q} & & \downarrow{\scriptstyle Q} & & \downarrow & & \downarrow \\
K_\Phi^* & \overset{j}{\hookrightarrow} & K_\Phi & \longrightarrow & \mathrm{Symp}_{1,1,c}^{\mathrm{red}} & \longrightarrow & \mathrm{Sp}(4)^{\mathbb{Z}_c} / \mathbb{Z}_c
\end{array}
\tag{4.5}
$$

The fact that $i$ is a weak homotopy equivalence implies the same for the map $j$. Also, it's clear that if $(f, \widetilde{f_c}) \in K_\Psi^*$, then $f = \mathrm{Id}$ near $p_c$. This means that

$$
K_\Phi^* = \{f \in \mathrm{Symp}_{1,1,c}^{\mathrm{red}} \,|\, f = \mathrm{Id} \text{ near } p_c\}.
$$

**Lemma 4.2.6.** $K_\Phi^*$ *is weakly contractible.*

Recall from Section 2.3 that blowing up $\mathbb{C}P_{1,1,c}^2$ at the singularity $p_c$ gives a variety $V_c = \{([a:b], [z_0:z_1:z_2]_{1,1,c}) \in \mathbb{C}P^1 \times \mathbb{C}P_{1,1,c}^2 \,|\, az_1 = bz_0\}$ that can be identified symplectically with the Hirzebruch surface

$$
W_c = \{([a:b], [z_0:z_1:z_2]) \in \mathbb{C}P^1 \times \mathbb{C}P^2 \,|\, a^c z_1 = b^c z_0\}.
$$

Let $\mathrm{Symp}(V_c)$ denote the group of symplectomorphisms of $V_c$ (with the form from Section 2.3) acting as the identity on homology. Let $\mathcal{S}_0(V_c)$ be the subgroup of $\mathrm{Symp}(V_c)$ consisting of those $f \in \mathrm{Symp}(V_c)$ for which $f = \mathrm{Id}$ near the zero section, $\mathcal{Z}_0$. There is another lemma we need before proving Lemma 4.2.6.

**Lemma 4.2.7.** $\mathcal{S}_0(V_c)$ *and* $K_\Phi^*$ *are weakly homotopy equivalent.*

*Proof.* Recall that the symplectic blow up operation removes a ball and collapses its boundary along the Hopf fibration. In the case of $\mathbb{C}P_{1,1,c}^2$, symplectically blowing up at

$p_c$ amounts to removing an orbi-ball (singular ball) centred at $p_c$ and similarly collapsing its boundary, which is now a lens space $\partial(B^4 / \mathbb{Z}_c) \cong S^3 / \mathbb{Z}_c$.

Let $f_\lambda$, $\lambda \in S$, be a compact family of symplectomorphisms in $\mathrm{K}^*_\Phi$ that smoothly vary with $\lambda$. For each fixed $\lambda_0 \in S$, there is an open ball $B_{\lambda_0}$ containing $p_c$ such that

$$f_{\lambda_0}|_{B_{\lambda_0}} = \mathrm{Id}.$$

Consider the function $S \to \mathbb{R}$, $\lambda \mapsto \mathrm{Vol}(B_\lambda)$. It is smooth because $f_\lambda$ varies smoothly with $\lambda$. Since $B_\lambda$ is parametrized by a compact set, the function Vol must have a minimum that is non-zero. Therefore, there exists $B_{min}$ such that

$$B_{min} \subseteq B_\lambda \quad \text{for all } \lambda \in S.$$

The point is that we want to blow up with a small enough ball so that it is contained in $B_{min}$; then the compact family $f_\lambda$ lifts to a compact family $\widetilde{f}_\lambda : V_c \to V_c$ such that $\widetilde{f}_\lambda = \mathrm{Id}$ on a neighbourhood $\mathcal{N}_{min}$ of the zero-section in $V_c$. So we have a commutative diagram

$$
\begin{array}{ccc}
V_c \setminus \mathcal{N}_{min} & \xrightarrow{\widetilde{f}_\lambda} & V_c \setminus \mathcal{N}_{min} \\
\beta \downarrow & & \downarrow \beta \\
\mathbb{C}P^2_{1,1,c} \setminus B_{min} & \xrightarrow{f_\lambda} & \mathbb{C}P^2_{1,1,c} \setminus B_{min} \,,
\end{array}
$$

where $f_\lambda$ and $\widetilde{f}_\lambda$ restrict to the identity on the respective neighbourhoods. Theorem 2 in [38] guarantees that the blow down map $\beta$ is a symplectomorphism for arbitrarily small neighbourhoods. Thus, the correspondence $f_\lambda \mapsto \beta^{-1} \circ f_\lambda \circ \beta$ sends compact families of symplectomorphisms in $\mathrm{K}^*_\Phi$ to compact families in $\mathcal{S}_0(V_c)$. Similarly, any compact family in $\mathcal{S}_0(V_c)$ will descend to an compact family in $\mathrm{K}^*_\Phi$. This proves Lemma 4.2.7. $\qquad\square$

To finish the proof of Lemma 4.2.6, we'll show that the space $\mathcal{S}_0(V_c)$ is contractible. This follows from ([20]-Lemma 9.1) because $V_c$ is a ruled symplectic 4-manifold. We will briefly sketch the argument.

Let $\mathcal{Z}_\infty$ be the infinity section in $V_c$ and let $A := [Z_\infty] \in H_2(V_c; \mathbb{Z})$ be its homology class. We define a space of symplectic spheres in $V_c$ on which $\mathcal{S}_0(V_c)$ acts: Let $\mathcal{C}^A(V_c \setminus \mathcal{Z}_0)$ be the space of symplectic spheres in $V_c$ representing the homology class $A$ and disjoint from $\mathcal{Z}_0$. It follows from ([20]-Theorem 1.2, see also Theorem 8.1) that the set $\mathcal{C}^A(V_c \setminus \mathcal{Z}_0)$ is contractible. Observe that $\mathcal{S}_0(V_c)$ acts on the space $\mathcal{C}^A(V_c \setminus \mathcal{Z}_0)$. It also follows from ([20]-Theorem 8.1) that this action is transitive. Let $\mathrm{Stab}(\Sigma)$ be the stabilizer of a sphere $\Sigma \in \mathcal{C}^A(V_c \setminus \mathcal{Z}_0)$ under this action. Then we have a fibration

$$\mathrm{Stab}(\Sigma) \longrightarrow \mathcal{S}_0(V_c) \longrightarrow \mathcal{C}^A(V_c \setminus \mathcal{Z}_0),$$

so $\mathrm{Stab}(\Sigma)$ is the subgroup of $\mathcal{S}_0(V_c)$ consisting of symplectomorphisms that leave $\Sigma$ invariant. It follows from ([11]-Propostion 3.2) that this stabilizer is contractible. Hence, $\mathcal{S}_0(V_c)$ is contractible as well.

Now Theorem 4.2.1 follows easily. The fibration

$$\mathrm{K}_\Phi^* \simeq \mathrm{K}_\Phi \longrightarrow \mathrm{Symp}_{1,1,c}^{\mathrm{red}} \longrightarrow \mathrm{Sp}(4)^{\mathbb{Z}_c} / \mathbb{Z}_c \simeq \mathrm{U}(2) / \mathbb{Z}_c$$

with $\mathrm{K}_\Phi^*$ weakly contractible gives the result.

## 4.3 The Groups $\mathrm{Symp}_{1,b,c}^{\mathrm{red}}$ for $c = bk + 1$

In this section we prove

**Theorem 4.3.1.** $\mathrm{Symp}_{1,b,c}^{\mathrm{red}}$ is homotopy equivalent to $\mathrm{Aut}(T_{p_c}) \simeq \mathrm{T}^2 / \mathbb{Z}_c$ when $c = bk + 1$. Here, $\mathrm{Aut}(T_{p_c})$ is the linear automorphism group of the uniformized tangent space at $p_c \in \mathbb{C}P_{1,b,c}^2$ and $\mathrm{T}^2$ is the diagonal torus inside $\mathrm{U}(2)$.

Start by considering the map

$$\Psi : \mathrm{Symp}_{1,b,c}^{\mathrm{orb}} \longrightarrow \mathrm{Aut}^{\mathbb{Z}_b}(T_0 \widetilde{U}_b) \times \mathrm{Aut}^{\mathbb{Z}_c}(T_0 \widetilde{U}_c) \tag{4.6}$$

given by $\Psi(f, \widetilde{f_b}, \widetilde{f_c}) = (d\widetilde{f_b}(0), d\widetilde{f_c}(0))$ where $\mathrm{Aut}^{\mathbb{Z}_b}$, respectively, $\mathrm{Aut}^{\mathbb{Z}_c}$ denote $\mathbb{Z}_b, \mathbb{Z}_c$-equivariant linear automorphisms. This is a well-defined group homomorphism, and using the techniques of the previous section it follows that this map is a locally trivial fibration. Note that we have $\mathrm{Aut}^{\mathbb{Z}_b}(T_0\widetilde{U}_b) \cong \mathrm{Aut}^{\mathbb{Z}_b}(\mathbb{C}^2) = \mathrm{Sp}(4)^{\mathbb{Z}_b}$.

**Claim.** $\mathrm{Aut}^{\mathbb{Z}_c}(T_0\widetilde{U}_c) \cong \mathrm{Aut}^{\mathbb{Z}_c}(\mathbb{C}^2)$ *retracts onto* $\mathrm{T}^2$, *the diagonal torus inside* $\mathrm{U}(2)$.

*Proof.* In the non-equivariant case, we know that $\mathrm{Aut}(\mathbb{C}^2) = \mathrm{Sp}(4)$ retracts onto $\mathrm{U}(2)$. This retraction is given by the polar decomposition: Let $\mathcal{P}$ be the space of symmetric positive definite matrices. Then for every $A \in \mathrm{Sp}(4)$ there is a unique $U \in \mathrm{U}(2)$ and $P \in \mathcal{P}$ such that $A = UP$; just let $P = (AA^T)^{1/2}$ and $U = A(AA^T)^{-1/2}$. Then we have a diffeomorphism

$$\mathrm{Sp}(4) \longrightarrow \mathrm{U}(2) \times \mathcal{P}, \quad A \mapsto UP,$$

and the map $\Theta_t : A \mapsto A(AA^T)^{-t/2}$ is a deformation retraction of $\mathrm{Sp}(4)$ onto $\mathrm{U}(2)$. Let $D_\xi$ be the image of the diagonal matrix $\mathrm{diag}(\xi, \xi^b)$ in $\mathrm{Sp}(4)$, ie. write

$$\mathrm{diag}(\xi, \xi^b) = R_\xi + iI_\xi,$$

where $R_\xi$ and $I_\xi$ are the diagonal matrices consisting of real, respectively imaginary parts of $\xi, \xi^b \in \mathbb{Z}_c$. Then $D_\xi$ is the block matrix

$$D_\xi = \begin{bmatrix} R_\xi & -I_\xi \\ I_\xi & R_\xi \end{bmatrix}.$$

Let $\mathrm{Sp}(4)^{\mathbb{Z}_c}$ be the subspace of $\mathrm{Sp}(4)$ whose elements commute with $D_\xi$ (this is the subspace that is equivariant under the $\mathbb{Z}_c$-action). If $A \in \mathrm{Sp}(4)^{\mathbb{Z}_c}$, then we want to see that $\Theta_t(A)D_\xi = D_\xi\Theta_t(A)$. This follows because:

- Since $D_\xi$ commutes with $A$, then we have (since $D_\xi$ is orthogonal) that $D_\xi$ commutes with $AA^T$.

- Since $D_\xi$ and $AA^T$ commute, they can be simultaneously diagonalized, where we are considering them as operators on $\mathbb{C}^4$. From this it's easy to check that $(AA^T)^{-t/2}D_\xi = D_\xi(AA^T)^{-t/2}$.

Therefore, $\Theta_t$ also retracts $\mathrm{Sp}(4)^{\mathbb{Z}_c}$ onto the equivariant subspace of U(2). Now identify U(2) $\subset$ Sp(4) with $2 \times 2$ unitary matrices and check that a matrix $U \in$ U(2) is $\mathbb{Z}_c$-equivariant if and only if $U$ is a diagonal matrix, ie. iff $U \in \mathrm{T}^2$. This proves the claim. $\qquad\square$

Now consider the map

$$\Phi : \mathrm{Symp}^{\mathrm{red}}_{1,b,c} \longrightarrow \mathrm{Aut}^{\mathbb{Z}_b}(T_0\widetilde{U}_b)\,/\,\mathbb{Z}_b \times \mathrm{Aut}^{\mathbb{Z}_c}(T_0\widetilde{U}_c)\,/\,\mathbb{Z}_c \qquad (4.7)$$

given by $\Phi(f) = ([\,d\widetilde{f}_b(0)\,],[\,d\widetilde{f}_c(0)\,])$. Again, as follows from the previous section, this map is a locally trivial fibration. Up to homotopy, (4.7) becomes

$$\mathrm{Symp}^{\mathrm{red}}_{1,b,c} \longrightarrow \mathrm{U}(2)\,/\,\mathbb{Z}_b \times \mathrm{T}^2\,/\,\mathbb{Z}_c\,,$$

Let $\mathrm{K}_\Phi$ be the kernel of the map $\Phi$. Then $\mathrm{K}_\Phi$ is weakly homotopy equivalent to its subspace

$$\mathrm{K}_\Phi^* = \{f \in \mathrm{K}_\Phi \mid f = \mathrm{Id} \text{ near } p_b \text{ and } f = \mathrm{Id} \text{ near } p_c\},$$

so that the we have the homotopy fibration

$$\mathrm{K}_\Phi^* \overset{\simeq}{\hookrightarrow} \mathrm{K}_\Phi \longrightarrow \mathrm{Symp}^{\mathrm{red}}_{1,b,c} \longrightarrow \mathrm{U}(2)\,/\,\mathbb{Z}_b \times \mathrm{T}^2\,/\,\mathbb{Z}_c.$$

Let $(R_{1,b,c}, \widetilde{\omega}_{\alpha_1,\ldots,\alpha_{b+1}})$, $c = bk + 1$, be the resolution of $\mathbb{C}P^2_{1,b,c}$ as described in Sections 3.1.2 and 3.2. For the rest of this section, we'll refer to $R_{1,b,c}$ simply as $R$. The resolution creates a chain of embedded symplectic spheres $\Gamma := C_1 \cup \ldots \cup C_{b+2}$, and the symplectomorphism $R \cong W_k \# b\overline{\mathbb{C}P}^2$ from Lemma 3.2.2 produces an isomorphism $H_2(R;\mathbb{Z}) \cong H_2(W_k \# b\overline{\mathbb{C}P}^2;\mathbb{Z})$ such that for $k$ even

- $[C_1] \longleftrightarrow B - \frac{k}{2}F - E_1$

- $[C_i] \longleftrightarrow E_{i-1} - E_i \ \ (i = 2, \ldots, b)$

- $[C_{b+1}] \longleftrightarrow E_b$

- $[C_{b+2}] \longleftrightarrow F - \sum_{i=1}^{b} E_i$

and for $k$ odd (see (2.3) in Section 2.3.1)

- $[C_1] \longleftrightarrow B - (\frac{k+1}{2})F$

- $[C_2] \longleftrightarrow F - E_1 - E_2$

- $[C_i] \longleftrightarrow E_{i-1} - E_i \ \ (i = 3, \ldots, b)$

- $[C_{b+1}] \longleftrightarrow E_b$

- $[C_{b+2}] \longleftrightarrow E_1 - E_2 - \cdots - E_b$

As mentioned before, we focus on the case where $k$ is even. The odd case is analogous and gives the same answer. Let $\Gamma_{[b+1]} := \Gamma \setminus C_{b+1}$ and let $\mathrm{Symp}^{cpt}(R \setminus \Gamma_{[b+1]})$ be the subgroup of $\mathrm{Symp}(R)$ whose symplectomorphisms are compactly supported away from $\Gamma_{[b+1]}$. An argument similar to the proof of Lemma 4.2.7 gives the following:

**Lemma 4.3.2.** $\mathrm{K}_\Phi^*$ *is weakly equivalent to* $\mathrm{Symp}^{cpt}(R \setminus \Gamma_{[b+1]})$.

So we will focus our efforts on the group $\mathrm{Symp}^{cpt}(R \setminus \Gamma_{[b+1]})$. Most of the remaining work in this section is aimed at proving the following:

**Lemma 4.3.3.** $\mathrm{Symp}^{cpt}(R \setminus \Gamma_{[b+1]})$ *is weakly equivalent to* $\Omega(\mathrm{U}(2) \,/\, \mathbb{Z}_b)$, *the loopspace of* $\mathrm{U}(2) \,/\, \mathbb{Z}_b$.

We now focus on proving Lemma 4.3.3. Let $\mathrm{Symp}(R, \Gamma_{[b+1]})$ be the subgroup of $\mathrm{Symp}(R)$ that leaves each sphere in $\Gamma_{[b+1]}$ invariant, but not necessarily pointwise. Let $\mathcal{J}$ be the space of $\widetilde{\omega}_{\alpha_1,\ldots,\alpha_{b+1}}$-tame almost complex structures on $R$. We define a space of symplectic spheres on which $\mathrm{Symp}(R, \Gamma_{[b+1]})$ acts

- Let $\mathcal{C}_{b,b+2}^{\perp}[C_{b+1}]$ denote the space of embedded symplectic spheres in class $[C_{b+1}]$ that satisfy the following properties:

  - Any $S \in \mathcal{C}_{b,b+2}^{\perp}[C_{b+1}]$ intersects $C_b$ exactly once and $C_{b+2}$ exactly once. Also, we require these intersections to be symplectically orthogonal.

  - If $S \in \mathcal{C}_{b,b+2}^{\perp}[C_{b+1}]$, then $S$ must be disjoint from each sphere in the set $\{C_1, C_2, \ldots, C_{b-1}\}$.

  - For each $S \in \mathcal{C}_{b,b+2}^{\perp}[C_{b+1}]$, there is a $J \in \mathcal{J}$ making $C_1, \ldots, C_b, S, C_{b+2}$ simultaneously $J$-holomorphic.

**Lemma 4.3.4.** $\mathrm{Symp}(R, \Gamma_{[b+1]})$ *acts transitively on* $\mathcal{C}_{b,b+2}^{\perp}[C_{b+1}]$.

*Proof.* We will see later that the space $\mathcal{C}_{b,b+2}^{\perp}[C_{b+1}]$ is contractible, hence it is path-connected. If $S \in \mathcal{C}_{b,b+2}^{\perp}[C_{b+1}]$, then it is easy to see that $f(S) \in \mathcal{C}_{b,b+2}^{\perp}[C_{b+1}]$: First of all, there exists $J \in \mathcal{J}$ such that $C_1, \ldots, C_b, S, C_{b+2}$ are all $J$-holomorphic. Let $J_f := df \circ J \circ (df)^{-1}$; then $f(S)$ is $J_f$-holomorphic. Further, $C_1, \ldots, C_b, C_{b+2}$ are $J_f$-holomorphic as well because $f$ leaves these spheres invariant. The fact that $f(S)$ is disjoint from all the spheres $C_1, \ldots, C_{b-1}$ is a consequence of positivity of intersections for $J_f$-holomorphic spheres (see Section 2.4). Hence, there is a well-defined action. Let $S_0, S_1$ be any two elements of $\mathcal{C}_{b,b+2}^{\perp}[C_{b+1}]$ with $S_t$ a path connecting them. Put

$$\widehat{S}_t := C_1 \cup \cdots \cup C_b \cup S_t \cup C_{b+2}.$$

By the symplectic neighbourhood theorem, the isotopy $\widehat{S}_t$ extends to an isotopy $\phi_t : \mathcal{N}_0 \to \mathcal{N}_t$ where $\mathcal{N}_t$ is a small neighbourhood of $\widehat{S}_t$. Since $\widehat{S}_t$ leaves the other spheres invariant, so will the isotopy $\phi_t$. We can choose the neighbourhoods $\mathcal{N}_t$ so that they retract onto $\widehat{S}_t$ for each $t$. Then $H^2(R, \widehat{S}_t; \mathbb{R}) = 0$, so $\phi_t$ extends to $R$ by Banyaga's isotopy extension theorem ([35]-Theorem 3.19). The time 1-map of this extension sends $S_0$ to $S_1$, proving the lemma. $\qquad\square$

The stabilizer (of $C_{b+1}$) of the action of $\mathrm{Symp}(R, \Gamma_{[b+1]})$ on $\mathcal{C}^{\perp}_{b,b+2}[C_{b+1}]$ is the subgroup $\mathrm{Symp}(R, \Gamma) \subset \mathrm{Symp}(R)$ leaving each sphere in the configuration $\Gamma$ invariant. Hence, we have a fibration

$$\mathrm{Symp}(R, \Gamma) \longrightarrow \mathrm{Symp}(R, \Gamma_{[b+1]}) \longrightarrow \mathcal{C}^{\perp}_{b,b+2}[C_{b+1}]. \qquad (4.8)$$

Let $\mathcal{C}^{\pitchfork}_{b,b+2}[C_{b+1}]$ be the space of embedded symplectic spheres in class $[C_{b+1}]$ satisfying exactly the same properties as those in $\mathcal{C}^{\perp}_{b,b+2}[C_{b+1}]$ except now we require that any $S \in \mathcal{C}^{\pitchfork}_{b,b+2}[C_{b+1}]$ intersects $C_b$ and $C_{b+2}$ once transversely and positively.

**Lemma 4.3.5.** $\mathcal{C}^{\pitchfork}_{b,b+2}[C_{b+1}]$ *is weakly homotopy equivalent to* $\mathcal{C}^{\perp}_{b,b+2}[C_{b+1}]$.

*Proof.* See Section 6.2. □

**Lemma 4.3.6.** $\mathcal{C}^{\pitchfork}_{b,b+2}[C_{b+1}]$ *is weakly contractible.*

Let $\mathcal{J}_{1...b,b+2} \subseteq \mathcal{J}$ be the subset of $J$'s for which the spheres $C_1, \ldots, C_b, C_{b+2}$ are simultaneously $J$-holomorphic. We will define a map

$$\pi : \mathcal{J}_{1,...b,b+2} \longrightarrow \mathcal{C}^{\pitchfork}_{b,b+2}[C_{b+1}]$$

and show that it is a weak homotopy equivalence. Note that $\mathcal{J}_{1...b,b+2}$ is weakly contractible by ([14]-Appendix A), so the lemma will follow from this.

**Claim 1.** *For every* $J \in \mathcal{J}_{1...b,b+2}$, *there is a unique embedded $J$-holomorphic sphere in class* $E_b = [C_{b+1}]$.

*Proof.* The symplectic form $\widetilde{\omega}_{\alpha_1,...,\alpha_{b+1}}$ is diffeomorphic to the form $\Omega_{\mu,k,\varepsilon_1,...,\varepsilon_b}$ from Lemma 3.2.2 whose Poincaré dual $\mathrm{PD}(B + \mu F - \sum_i \varepsilon_i E_i)$ is a *reduced* homology class (see Definition 3.2.3). Therefore, by ([24]-Corollary 7.12), for every $J \in \mathcal{J}_{1...b,b+2}$ there exists an embedded $J$-holomorphic sphere in class $E_b$. This sphere is unique by positivity of intersections. □

**Claim 2.** *The map $\pi$ that sends $J \in \mathcal{J}_{1\ldots b, b+2}$ to the unique $J$-sphere in class $E_b$ is a fibration.*

*Proof.* First of all, for the map $\pi$ to even exist we need Claim 1 to be true. The image is unique because $J$-spheres intersect positively. By ([37]-Corollary 13), $\pi$ will be a fibration if: (i) It is a smooth submersion; and (ii) Its fibres are weakly contractible. To see (ii) is straightforward, since for $C_{b+1} \in \mathcal{C}^{\pitchfork}_{b,b+2}[C_{b+1}]$, the fibre $\pi^{-1}(C_{b+1})$ is the space of $J \in \mathcal{J}_{1\ldots b, b+2}$ such that $C_1, \ldots, C_{b+1}, C_{b+2}$ are simultaneously $J$-holomorphic, and this is weakly contractible by ([14]-Appendix A). To see (i), recall that $\mathcal{J}_{1\ldots b, b+2}$ and $\mathcal{C}^{\pitchfork}_{b,b+2}[C_{b+1}]$ are spaces of smooth maps, so that they are naturally infinite dimensional Fréchet manifolds. The tangent space $T_J \mathcal{J}_{1\ldots b, b+2}$ at $J \in \mathcal{J}_{1\ldots b, b+2}$ is the space of endomorphisms $A \in \mathrm{Aut}(TR)$ such that $AJ = -JA$. The space $\mathcal{C}^{\pitchfork}_{b,b+2}[C_{b+1}]$ is a subspace of the space $C^\infty(S^2, R) / \mathrm{Diff}(S^2)$, so the tangent space $T_S \mathcal{C}^{\pitchfork}_{b,b+2}[C_{b+1}]$ is a subspace of the space of sections of a pullback bundle, modulo reparametrization (See [7]-Section 1.2). We want to show that the derivative

$$d\pi_J : T_J \mathcal{J}_{1\ldots b, b+2} \longrightarrow T_{\pi(J)} \mathcal{C}^{\pitchfork}_{b,b+2}[C_{b+1}]$$

is surjective. Given $v \in T_S \mathcal{C}^{\pitchfork}_{b,b+2}[C_{b+1}]$, we can think of $v$ as an equivalence class of smooth curves

$$\begin{aligned} [0,1] &\longrightarrow \mathcal{C}^{\pitchfork}_{b,b+2}[C_{b+1}] \\ t &\mapsto S_t \end{aligned}$$

with $S_0 = S$. Then a representative $S_t$ generates an isotopy $\phi_t$ in $\mathrm{Symp}(R, \Gamma_{[b+1]})$. For $J \in \pi^{-1}(S)$, let

$$J_t := d\phi_t \circ J \circ (d\phi_t)^{-1}.$$

Since $J$ tames $\widetilde{\omega}_{\alpha_1, \ldots, \alpha_{b+1}}$, it is easily checked that $J_t$ tames $\phi_t^* \widetilde{\omega}_{\alpha_1, \ldots, \alpha_{b+1}} = \widetilde{\omega}_{\alpha_1, \ldots, \alpha_{b+1}}$. Also note that $C_1, \ldots, C_b, C_{b+2}$ are $J_t$-holomorphic, since $\phi_t$ leaves these spheres in-

variant; hence $J_t \in \mathcal{J}_{1...b,b+2}$. So, $J_t$ represents a vector $w \in T_J \mathcal{J}_{1...b,b+2}$ such that $d\pi_J(w) = v$. $\qquad\square$

**Lemma 4.3.7.** $\mathrm{Symp}(R,\Gamma)$ *is weakly equivalent to* $T^2$.

*Proof.* Recall that $\Gamma = C_1 \cup C_2 \cup \ldots \cup C_{b+2}$. Let $q_i$ be the unique point of intersection of $C_i$ and $C_{i+1}$ for $i = 1, \ldots, b+1$. Write $\mathrm{Symp}(C_1, q_1)$ and $\mathrm{Symp}(C_{b+2}, q_{b+1})$ for the symplectomorphism groups of $C_1$, respectively $C_{b+2}$ that fix the points $q_1, q_{b+1}$. Also write $\mathrm{Symp}(C_i, q_{i-1}, q_i)$ for the symplectomorphism group of $C_i$ fixing both $q_{i-1}, q_i$ for $i = 2, \ldots b+1$. The product of restriction maps,

$$\mathrm{Symp}(R,\Gamma) \longrightarrow \mathrm{Symp}(C_1, q_1) \times \left( \prod_{i=2}^{b+1} \mathrm{Symp}(C_i, q_{i-1}, q_i) \right) \times \mathrm{Symp}(C_{b+2}, q_{b+1})$$
$$f \mapsto (f|_{C_1}, f|_{C_2}, \ldots, f|_{C_{b+2}}) \tag{4.9}$$

is a fibration by the orbit-stabilizer theorem, since the restriction of $f$ to each sphere acts transitively. Since each factor in the base is homotopy equivalent to $S^1$ ([14]-Section 4.2), this means that the base is homotopy equivalent to $(S^1)^{b+2}$. The fibre over $(\mathrm{Id}, \ldots, \mathrm{Id})$ of the above map is the subgroup $\mathrm{Fix}(\Gamma) \subset \mathrm{Symp}(R,\Gamma)$ that fixes the entire configuration $\Gamma$ pointwise. Let $\mathcal{G}(C_1, q_1)$ and $\mathcal{G}(C_{b+2}, q_{b+1})$ be the symplectic gauge groups of the normal bundles of $C_1$, respectively $C_{b+2}$ that act as the identity over the points $q_1, q_{b+1}$. Also, let $\mathcal{G}(C_i, q_{i-1}, q_i)$ be the symplectic gauge group of the normal bundle of $C_i$ acting as the identity over both points $q_{i-1}, q_i$ for $i = 2, \ldots, b+1$. From ([14]-Section 4.1), we have $\mathcal{G}(C_1, q_1) \simeq \mathcal{G}(C_{b+2}, q_{b+1}) \simeq \star$ (both contractible), and $\mathcal{G}(C_i, q_{i-1}, q_i) \simeq \mathbb{Z}$ for $i = 2, \ldots, b+1$. Now consider the product of restrictions map to the gauge groups

$$\mathrm{Fix}(\Gamma) \longrightarrow \mathcal{G}(C_1, q_1) \times \left( \prod_{i=2}^{b+1} \mathcal{G}(C_i, q_{i-1}, q_i) \right) \times \mathcal{G}(C_{b+2}, q_{b+1}) \tag{4.10}$$
$$f \mapsto (df|_{\nu(C_1)}, df|_{\nu(C_2)}, \ldots, df|_{\nu(C_{b+2})}).$$

This map is a fibration (see [14]-Section 6.2). The fibre over $(\mathrm{Id}, \ldots, \mathrm{Id})$ of (4.10) is

weakly equivalent to the subgroup $\mathrm{Symp}^{cpt}(R \setminus \Gamma) \subset \mathrm{Fix}(\Gamma)$ of symplectomorphisms that are compactly supported away from $\Gamma$.

**Claim.** $\mathrm{Symp}^{cpt}(R \setminus \Gamma)$ *is contractible.*

This works by thinking of the toric picture. Recall from Section 3.2 that the moment polygon $\widetilde{\Delta}_{b,bk+1}$ of the resolution has $b + 4$ edges. The configuration $\Gamma = C_1 \cup \ldots \cup C_{b+2}$ is the moment map pre-image of the edges $e_1 \cup \ldots \cup e_{b+2}$ with respective co-normals $\vec{n}_1, \ldots, \vec{n}_{b+2}$ in the toric model. Hence, $R \setminus \Gamma$ is the moment map pre-image of $\Delta' := \widetilde{\Delta}_{b,bk+1} \setminus (e_1 \cup \ldots \cup e_{b+2})$, which is an open convex subset of $\mathbb{R}^2$. So, the open set $R \setminus \Gamma$ is contained in a larger Darboux ball $B^4(r)$, viewed as an equilateral triangle minus the diagonal edge in the toric picture. Let $m_{1-t} : B^4(r) \to B^4(r)$ be the map $m_{1-t}(z) = (1-t)z$ for $t \in [0,1)$. Then when $t$ is sufficiently close to 1, $m_{1-t}$ retracts $B^4(r)$ (and hence $R \setminus \Gamma$) onto a smaller ball $B^4(\epsilon)$ contained in the open set $R \setminus \Gamma$. This shows that $R \setminus \Gamma$ is symplectically star-shaped, therefore $\mathrm{Symp}^{cpt}(R \setminus \Gamma)$ is contractible by ([36]-Theorem 9.5.2). This finishes the proof of the claim.

Now that we know $\mathrm{Symp}^{cpt}(R \setminus \Gamma)$ is contractible, let's write

- $\mathrm{Symp}^{b+2}$ for the product of symplectomorphism groups in (4.9).

- $\mathcal{G}^{b+2}$ for the product of gauge groups in (4.10).

Then the fibration (4.10) tells us that $\mathrm{Fix}(\Gamma)$ is weakly equivalent to $\mathcal{G}^{b+2} \simeq \mathbb{Z}^b$, hence we have the fibration

$$\mathrm{Fix}(\Gamma) \longrightarrow \mathrm{Symp}(R, \Gamma) \longrightarrow \mathrm{Symp}^{b+2}$$

where the fibre is weakly equivalent to $\mathbb{Z}^b$ and the base is weakly equivalent to $(S^1)^{b+2}$. The long exact sequence of this fibration reduces to

$$0 \longrightarrow \pi_1 \mathrm{Symp}(R, \Gamma) \longrightarrow \mathbb{Z}^{b+2} \xrightarrow{\partial} \mathbb{Z}^b \longrightarrow \pi_0 \mathrm{Symp}(R, \Gamma) \longrightarrow 0,$$

so we want to understand the boundary map $\partial$. The boundary map comes from $\partial$ : $\pi_1\mathrm{Symp}^{b+2} \to \pi_0\mathrm{Fix}(\Gamma) \cong \pi_0\mathcal{G}^{b+2}$. Evans had a groovy idea ([14]-4.3, see also 6.3), which is to understand the composition

$$\mathbb{Z}^{b+2} \cong \pi_1\mathrm{Symp}^{b+2} \longrightarrow \pi_0\mathrm{Fix}(\Gamma) \longrightarrow \pi_0\mathcal{G}^{b+2} \cong \mathbb{Z}^b \qquad (4.11)$$

by thinking purely locally in a neighbourhood of $\Gamma$. There is a Hamiltonian circle action that rotates each sphere $C_i$ in the configuration $\Gamma$ around the intersection points. These generate loops in $\mathrm{Symp}(C_1, q_1), \mathrm{Symp}(C_{b+2}, q_{b+1})$ and $\mathrm{Symp}(C_i, q_{i-1}, q_i)$ for each $i$, hence they generate $\pi_1\mathrm{Symp}^{b+2}$. Let $\theta_1, \ldots, \theta_{b+2}$ be these generators. For each $\theta_i$, lift the $S^1$-action to a path $\gamma_i^t$ in the normal bundle $\nu(C_i)$. By the symplectic neighbourhood theorem, this is a local model for $R$ near $C_i$. The path $\gamma_i^t$ is generated by a Hamiltonian that we can cut off by a compactly supported bump function to get a symplectic isotopy $\phi_t^i$, $0 \leq t \leq 2\pi$, supported in a neighbourhood of $C_i$. Then $\phi_{2\pi}^i \in \mathrm{Fix}(\Gamma)$. These $\phi_{2\pi}^i$ represent the images of the $\theta_i \in \pi_1\mathrm{Symp}(C_i, *)$ under the boundary map above. The idea now is to identify generators for $\pi_0\mathcal{G}^{b+2}$ and determine the images of $[\phi_{2\pi}^i]$ under the map $\pi_0\mathrm{Fix}(\Gamma) \to \pi_0\mathcal{G}^{b+2}$. For each sphere $C_i$ $(i = 2, \ldots b+1)$ in $\Gamma$, there are evaluation fibrations

$$\begin{aligned}
\mathrm{ev}_{q_i} : \mathcal{G}(C_i, q_{i-1}) &\longrightarrow \mathrm{Sp}(2) \\
\mathrm{ev}_{q_{i-1}} : \mathcal{G}(C_i, q_i) &\longrightarrow \mathrm{Sp}(2)
\end{aligned}$$

with fibre over the identity equal to $\mathcal{G}(C_i, q_{i-1}, q_i)$ in each case. This gives two maps

$$\partial_{q_i}, \partial_{q_{i-1}} : \pi_1\mathrm{Sp}(2) \to \pi_0\mathcal{G}(C_i, q_{i-1}, q_i).$$

Let $g_{C_i}(q_i), g_{C_i}(q_{i-1})$ be the images of $1 \in \mathbb{Z} \cong \pi_1\mathrm{Sp}(2)$ under $\partial_{q_i}, \partial_{q_{i-1}}$ respectively. Both of these are generators of $\pi_0\mathcal{G}(C_i, q_{i-1}, q_i)$ for $i = 2, \ldots, b+1$, but they are not independent. By ([14]-Lemma 20, see also 6.3), the composition map (4.11) acts in the

following way

$$\theta_1 \mapsto g_{C_2}(q_1) \in \pi_0\mathcal{G}(C_2, q_1, q_2)$$

$$\theta_2 \mapsto (0, g_{C_3}(q_2)) \in \pi_0\mathcal{G}(C_1, q_1) \times \pi_0\mathcal{G}(C_3, q_2, q_3)$$

$$\theta_{b+1} \mapsto (g_{C_b}(q_b), 0) \in \pi_0\mathcal{G}(C_b, q_{b-1}, q_b) \times \pi_0\mathcal{G}(C_{b+2}, q_{b+1})$$

$$\theta_{b+2} \mapsto g_{C_{b+1}}(q_{b+1}) \in \pi_0\mathcal{G}(C_{b+1}, q_b, q_{b+1}).$$

Moreover, for $i = 3, \ldots, b$ we have

$$\theta_i \mapsto (g_{C_{i-1}}(q_{i-1}), g_{C_{i+1}}(q_i)) \in \pi_0\mathcal{G}(C_{i-1}, q_{i-2}, q_{i-1}) \times \pi_0\mathcal{G}(C_{i+1}, q_i, q_{i+1}).$$

Therefore, the map (4.11) is surjective. From this, it follows that

$$\pi_0\mathrm{Symp}(R, \Gamma) = 0 \quad \text{and} \quad \pi_1\mathrm{Symp}(R, \Gamma) \cong \mathbb{Z}^2,$$

while all the other homotopy groups vanish. So, we have a weak equivalence $\mathrm{T}^2 \xrightarrow{\simeq} \mathrm{Symp}(R, \Gamma)$, where $\mathrm{T}^2$ is the toric action on $R$. This finishes the proof of Lemma 4.3.7. $\qquad\square$

Recall that $\Gamma_{[b+1]} = \Gamma \setminus C_{b+1}$. Given the previous lemmas, we now conclude from the fibration (4.8) that $\mathrm{Symp}(R, \Gamma_{[b+1]})$ is weakly homotopy equivalent to $\mathrm{T}^2$. Let $\Gamma_{1\ldots b} := C_1 \cup \ldots \cup C_b$ and define the following subgroup of $\mathrm{Symp}(R)$:

- Let $\mathrm{Symp}^{cpt}(R \setminus \Gamma_{1\ldots b}, C_{b+2})$ be the subgroup of $\mathrm{Symp}(R)$ consisting of symplecto-morphisms that are compactly supported away from $\Gamma_{1\ldots b}$ **and** leave $C_{b+2}$ invariant.

**Lemma 4.3.8.** $\mathrm{Symp}^{cpt}(R \setminus \Gamma_{1\ldots b}, C_{b+2})$ *is weakly contractible.*

*Proof.* Let's start with $\mathrm{Symp}(R, \Gamma_{[b+1]}) \simeq \mathrm{T}^2$ and consider the fibration that results from restriction to the spheres $C_1, \ldots, C_b$

$$\mathrm{Fix}(\Gamma_{1\ldots b}) \longrightarrow \mathrm{Symp}(R, \Gamma_{[b+1]}) \longrightarrow \mathrm{Symp}(C_1, q_1) \times \cdots \times \mathrm{Symp}(C_b, q_b),$$

where $\mathrm{Fix}(\Gamma_{1\ldots b})$ is the subgroup fixing $\Gamma_{1\ldots b}$ pointwise. The long exact sequence of this fibration reduces to

$$0 \longrightarrow \pi_1\mathrm{Fix}(\Gamma_{1\ldots b}) \longrightarrow \mathbb{Z}^2 \overset{\rho}{\longrightarrow} \mathbb{Z}^b \longrightarrow \pi_0\mathrm{Fix}(\Gamma_{1\ldots b}) \longrightarrow 0,$$

where $\rho$ comes from the map $\pi_1\mathrm{Symp}(R, \Gamma_{[b+1]}) \to \pi_1\mathrm{Symp}(C_1, q_1)\times\cdots\times\pi_1\mathrm{Symp}(C_b, q_b)$. Clearly this map is injective, hence $\pi_1\mathrm{Fix}(\Gamma_{1\ldots b})$ is trivial and $\pi_0\mathrm{Fix}(\Gamma_{1\ldots b}) \cong \mathbb{Z}^{b-2}$. Therefore, $\mathrm{Fix}(\Gamma_{1\ldots b})$ is weakly equivalent to $\mathbb{Z}^{b-2}$. Next restrict to the normal bundles of the spheres $C_1, \ldots, C_b$

$$\mathrm{Fix}(\Gamma_{1\ldots b}) \longrightarrow \underbrace{\mathcal{G}(C_1, q_1) \times \left( \prod_{i=2}^{b-1} \mathcal{G}(C_i, q_{i-1}, q_i) \right) \times \mathcal{G}(C_b, q_{b-1})}_{\simeq\, \mathbb{Z}^{b-2}}.$$

It follows that the fibres are weakly contractible. But, the fibre over Id is the subgroup of $\mathrm{Fix}(\Gamma_{1\ldots b})$ whose derivatives are the identity on $\nu(C_1) \cup \ldots \cup \nu(C_b)$. This is weakly equivalent to $\mathrm{Symp}^{cpt}(R \setminus \Gamma_{1\ldots b}, C_{b+2})$, so we are done. $\qquad\square$

**Proof of Lemma 4.3.3**: Write $\mathrm{Aut}(\nu(C_{b+2}))$ for the group of automorphisms of $\nu(C_{b+2})$ that are symplectic, linear, and preserve the zero section $C_{b+2}$. The map

$$\begin{aligned} \mathrm{Symp}^{cpt}(R \setminus \Gamma_{1\ldots b}, C_{b+2}) &\longrightarrow \mathrm{Aut}(\nu(C_{b+2})) \\ f &\mapsto df|_{TC_{b+2}} \end{aligned}$$

is a surjective group homomorphism, and the kernel $\mathcal{K}$ consists of the symplectomorphisms whose derivatives act as the identity on $\nu(C_1) \cup \ldots \cup \nu(C_b) \cup \nu(C_{b+2})$; thus $\mathcal{K}$ is weakly equivalent to $\mathrm{Symp}^{cpt}(R \setminus \Gamma_{[b+1]})$. Since the total space is contractible, it follows that $\mathrm{Symp}^{cpt}(R \setminus \Gamma_{[b+1]})$ is weakly equivalent to the loopspace $\Omega\mathrm{Aut}(\nu(C_{b+2}))$ ([19]-Proposition 4.66). Also, since $C_{b+2}$ has self-intersection $-b$, its normal bundle is isomorphic to $\mathcal{O}(-b)$, the complex line bundle with Chern number -2. Therefore, by

([21]-Proposition 2.5), $\mathrm{Aut}(\nu(C_{b+2}))$ is isomorphic to the Kahler isometry group of the Hirzebruch surface $W_b$. In particular, we have

$$\mathrm{Aut}(\nu(C_{b+2})) \cong \mathrm{U}(2) \,/\, \mathbb{Z}_b,$$

so the proof is finished.

**Proof of Theorem 4.3.1**: Consider the map

$$\mathrm{Symp}^{\mathrm{red}}_{1,b,c} \longrightarrow \mathrm{Aut}(T_{p_c}), \quad f \mapsto [d\widetilde{f}_c(0)].$$

Letting $\mathcal{K}_{p_c}$ be its kernel, we want to show that $\mathcal{K}_{p_c}$ is weakly contractible. Evaluating at the other singularity, we get another fibration

$$\mathrm{K}_\Phi \longrightarrow \mathcal{K}_{p_c} \longrightarrow \mathrm{Aut}(T_{p_b}) \cong \mathrm{U}(2) \,/\, \mathbb{Z}_b, \tag{4.12}$$

whose kernel is exactly $\mathrm{K}_\Phi$ from (4.7). By Lemma 4.3.2, $\mathrm{K}_\Phi$ is weakly equivalent to $\mathrm{Symp}^{cpt}(R \setminus \Gamma_{[b+1]})$, which is in turn weakly equivalent to $\Omega(\mathrm{U}(2) \,/\, \mathbb{Z}_b)$ by Lemma 4.3.3. Therefore, $\mathrm{K}_\Phi$ is weakly equivalent to the loopspace of the base in (4.12). Since

$$\pi_i \Omega(\mathrm{U}(2) \,/\, \mathbb{Z}_b) \cong \pi_{i-1} \Omega(\mathrm{U}(2) \,/\, \mathbb{Z}_b) \quad \text{for all } i,$$

the homotopy long exact sequence of (4.12) implies that $\mathcal{K}_{p_c}$ is weakly contractible. This proves Theorem 4.3.1.

## 4.4   The Groups $\mathrm{Symp}_{1,b,c}$ for $1 < b < c$

Now we have the most general result:

**Theorem 4.4.1.** $\mathrm{Symp}^{\mathrm{red}}_{1,b,c}$ *is weakly homotopy equivalent to either* $\mathrm{Aut}(T_{p_b})$ *or* $\mathrm{Aut}(T_{p_c})$ *when* $1 < b < c$ *and* $b, c$ *are relatively prime.*

We will give a quick description of how the symplectomorphism group can be computed using the same process as the last chapter. Most of our arguments from the Section 4.3 go through almost identically, and it further seems that many of our constructions (especially from Section 3.2) can be greatly simplified. Write $c = bk + r$ for $k$ a positive integer and $0 < r < b$. Then the combinatorics of the polygon corresponding to the resolution $R_{1,b,bk+r}$ are favourable in the sense that the resolution creates a chain of embedded symplectic spheres with a $(-1)$-sphere in between. To see this, we must resolve both singularities according to the Hirzebruch-Jung continued fraction expansions. To resolve the first one, write

$$\frac{bk + r}{b} = [a_1, \ldots, a_m].$$

Then the resolution of $p_{bk+r}$ creates a chain of embedded spheres $C_1 \cup \ldots \cup C_m$ such that $[C_i] \cdot [C_i] = -a_i$ for $i = 1, \ldots, m$. We also create chain of $m$ new edges in the polygon $\Delta_{b,bk+1}$ with respective co-normals $\vec{n}_1, \ldots \vec{n}_m$ satisfying

$$\vec{n}_{i+1} = a_i \vec{n}_i - \vec{n}_{i-1}.$$

Now we must resolve the other singularity. This requires that we first transform the respective corner into the standard model from Section 2.5, do the corner cutting in this local model, and then transform it back. The vertex corresponding to the order $b$ singularity $p_b$ has co-normals $(b, bk + r)$ and $(0, -1)$. Consider the transformation

$$A = \begin{bmatrix} 1 & 0 \\ -k & 1 \end{bmatrix}.$$

Then $A(b, bk + r) = (b, r)$ and $A(0, -1) = (0, -1)$. After composing $A$ with a reflection, the co-normals are put into the local toric model of Section 2.5. We now do the corner cuts as described in that section. Write

$$\frac{b}{r} = [d_1, \ldots, d_n],$$

so that the resolution creates a chain of embedded spheres $S_1 \cup \ldots \cup S_n$ such that $[S_i] \cdot [S_i] = -d_i$ for $i = 1, \ldots n$. We also have $n$ new edges in the local toric model with respective co-normals $\vec{m}_1, \ldots, \vec{m}_n$ satisfying

$$\vec{m}_{i+1} = d_i \vec{m}_i - \vec{m}_{i-1}.$$

Now, reflect these co-normals back over the $y$-axis and compose with the matrix $A^{-1}$. These are exactly the co-normals we need to resolve the remaining corner in the polygon $\Delta_{b,bk+r}$. Hence, the resolution transforms the diagonal edge in $\Delta_{b,bk+r}$ into a chain of $m + 1 + n$ edges. These new edges correspond to a chain of smooth embedded symplectic spheres

$$C_1 \cup \ldots \cup C_m \cup \mathcal{E} \cup S_n \cup \ldots \cup S_1,$$

where $\mathcal{E}$ is the sphere corresponding to what remains of the diagonal edge after making the corner cuts. Let $R$ be the resolution of $\mathbb{C}P^2_{1,b,c}$.

**Claim.** $\mathcal{E}$ *is an exceptional sphere. Moreover, it's homology class has minimal area among all exceptional classes in* $H_2(R; \mathbb{Z})$.

*Proof.* The first statement follows from ([23]-Lemma 2.16(3)). This Lemma says that any Delzant polygon with 5 or more edges is $\mathrm{AGL}(2, \mathbb{Z})$-congruent to a Delzant polygon that comes from a Hirzebruch trapezoid by a sequence of smooth corner cuts. At each stage, these corner cuts add an edge with combinatorial self-intersection -1. Hence, $\Delta_{b,bk+r}$ must contain at least one edge of this type. Let $e_\mathcal{E}$ be the edge corresponding to $\mathcal{E}$. Then no other edge but $e_\mathcal{E}$ can have combinatorial self-intersection -1. The reason for this is straightforward: Recall from Section 2.5 that each resolution in the local toric model is minimal in the sense that it contains no (-1)-spheres. Hence, none of the added edges in the resolution can correspond to (-1)-spheres. The two remaining edges are the vertical and horizontal, and it is easy to check that these are not -1. This proves that $\mathcal{E}$ is an exceptional sphere.

To see that its homology class is minimal, we use ([24]-Theorem 1.5) and the considerations at the end of Section 3.2. Together, these imply that on $R$ it is possible to put a symplectic form $\omega_R$ such that

$$\mathrm{PD}[\omega_R] = B + \mu F - \sum_{i=1}^{N} \varepsilon_i E_i.$$

Furthermore, this class is *reduced* with respect to the basis $\{B, F, E_1, \ldots, E_N\}$ of $H_2((S^2 \times S^2) \# N\overline{\mathbb{C}P^2}; \mathbb{Z})$ in the sense of Lemma 3.2.4 (see also the discussion before the Lemma). This means that the homology class of $\mathcal{E}$ must be $E_N$, which is minimal by ([24]-Corollary 7.10).  $\square$

**Remark:** The above argument that shows the homology class of $\mathcal{E}$ has minimal area depends on a specific basis of $H_2(R; \mathbb{Z})$. There is another more intrinsic way to see this though. We know that any exceptional class $E$ with minimal area among all exceptional classes is always represented by a unique embedded $J$-sphere for any tame $J$, in particular any compatible $J$ ([41]-Lemma 1.2). Any compatible $J$ defines a metric via $g(v, w) = \omega(v, Jw)$ that we can average over the $\mathrm{T}^2$-action to make it invariant; this gives a new almost complex structure that we'll call $J_{inv}$. Associated to $J_{inv}$ is a sphere $C_{inv}$ that is invariant under the $\mathrm{T}^2$-action, and this sphere lies in the same minimal homology class $E$. Since $C_{inv}$ is $\mathrm{T}^2$-invariant, it must be the pre-image of an edge in the polygon $\Delta_{b,bk+r}$. It follows that $C_{inv}$ and $\mathcal{E}$ are the same sphere, since there is only one edge with combinatorial self-intersection -1. Therefore, $\mathcal{E}$ represents a homology class with minimal area.

Now that we have the required information about the resolution, we consider the group $\mathrm{Symp}_{1,b,c}^{\mathrm{red}}$ and the sequence

$$\mathrm{K}_\Phi \longrightarrow \mathrm{Symp}_{1,b,c}^{\mathrm{red}} \xrightarrow{\ \Phi\ } \mathrm{Aut}^{\mathbb{Z}_b}(\mathbb{C}^2) \, / \, \mathbb{Z}_b \times \mathrm{Aut}^{\mathbb{Z}_c}(\mathbb{C}^2) \, / \, \mathbb{Z}_c \qquad (4.13)$$

with kernel $\mathrm{K}_\Phi$. Again, this kernel will be weakly homotopy equivalent to a certain

subgroup of $\mathrm{Symp}(R)$. Let

$$
\begin{aligned}
\Gamma_C &= C_1 \cup \ldots \cup C_m \\
\Gamma_S &= S_n \cup \ldots \cup S_1 \\
\Gamma &= \Gamma_C \cup \Gamma_S \cup \mathcal{E}.
\end{aligned}
$$

The arguments in the last chapter can be used to show that

$$
\mathrm{T}^2 \simeq \mathrm{Symp}(R, \Gamma) \simeq \mathrm{Symp}(R, \Gamma \setminus \mathcal{E}),
$$

where these symplectomorphisms are the subgroups of $\mathrm{Symp}(R)$ that, respectively, preserve $\Gamma$ and $\Gamma \setminus \mathcal{E}$. The crucial thing that we need here is that the exceptional class $E = [\mathcal{E}]$ is always represented by a $J$-holomorphic sphere for every tame $J$. This is because the class $E$ has minimal area (see [41]-Lemma 1.2). Now let

$$
\mathrm{Symp}^{cpt}(R \setminus \Gamma_C, \Gamma_S)
$$

be the subgroup of symplectomorphisms that are compactly supported away from $\Gamma_C$ and preserve $\Gamma_S$. Then we have

**Lemma 4.4.2.** $\mathrm{Symp}^{cpt}(R \setminus \Gamma_C, \Gamma_S)$ *is weakly contractible.*

*Proof.* Same argument as the proof of Lemma 4.3.8. □

Let $\mathrm{Aut}(\nu(S_i))$, $i = 1, \ldots, n$ be the group of automorphisms of $\nu(S_i)$ that are linear, symplectic, and preserve the zero section. Let $p_i = S_i \cap S_{i+1}$ be the unique point of intersection of $S_i, S_{i+1}$, and define the following subset of $\mathrm{Aut}(\nu(S_1)) \times \ldots \times \mathrm{Aut}(\nu(S_n))$:

- Over an intersection point $p_i$, write the differential of an element $\phi_i \in \mathrm{Aut}(\nu(S_i))$ as a sum of its tangent and normal components: $d\phi_i|_{p_i} = (d\phi_i^T|_{p_i}, d\phi_i^N|_{p_i})$. Now let $\mathrm{Aut}(\nu(\Gamma_S))$ be the set of pairs $(\phi_1, \ldots, \phi_n) \in \mathrm{Aut}(\nu(S_1)) \times \ldots \times \mathrm{Aut}(\nu(S_n))$

such that

$$d\phi_i^T|_{p_i} = d\phi_{i+1}^N|_{p_i} \text{ and } d\phi_{i+1}^T|_{p_i} = d\phi_i^N|_{p_i}$$

Since the tangent and normal directions intertwine over the intersection points, the restriction to $\mathrm{Aut}(\Gamma_S)$ via the map $f \mapsto (df|_{TR|_{S_1}}, \ldots, df|_{TR|_{S_n}})$ gives a fibration

$$\mathcal{K} \longrightarrow \mathrm{Symp}^{cpt}(R \setminus \Gamma_C, \Gamma_S) \longrightarrow \mathrm{Aut}(\nu(\Gamma_S)). \qquad (4.14)$$

The kernel $\mathcal{K}$ above is the subgroup of $\mathrm{Symp}^{cpt}(R \setminus \Gamma_C, \Gamma_S)$ whose derivatives fix both the tangent and normal directions of each sphere in the configuration $\Gamma_S$. We therefore have a weak homotopy equivalence

$$\mathcal{K} \simeq \mathrm{Symp}^{cpt}(R \setminus (\Gamma_C \cup \Gamma_S)),$$

Let's now analyze the sequence (4.14).

**Lemma 4.4.3.** $\mathrm{Aut}(\nu(\Gamma_S))$ *is weakly equivalent to* $\mathrm{T}^2$.

*Proof.* First consider the restriction map

$$\begin{aligned}
\mathrm{Aut}(\nu(\Gamma_S)) &\longrightarrow \mathrm{Aut}(\nu(S_1)) \\
(\phi_1, \ldots, \phi_n) &\mapsto \phi_1.
\end{aligned}$$

If $\mathcal{K}_1$ is the kernel, then we'll show that $\mathcal{K}_1$ is contractible and that $\mathrm{Aut}(\nu(S_1))$ is homotopy equivalent to $\mathrm{T}^2$. To see the latter statement, let $\phi \in \mathrm{Aut}(\nu(S_1)$ be a generator. Since $\phi$ preserves the zero-section $S_1$, the restriction $\phi|_{S_1}$ generates a symplectomorphism that fixes the intersection point $p_1 := S_1 \cap S_2$. Since $\mathrm{Symp}(S_1, p_1) \simeq S^1$, homotopically $\phi$ will generate this $S^1$-action on $S_1$. The fibre over the identity of the map $\phi \mapsto \phi|_{S_1}$ consists of bundle maps $\nu(S_1) \to \nu(S_1)$ that cover Id; hence, the fibre is the gauge group $\mathcal{G}(S_1) \simeq S^1$, so $\phi$ generates an $S^1 \times S^1$, homotopically. This shows that $\mathrm{Aut}(\nu(S_1)) / \mathrm{T}^2$ is contractible, hence $\mathrm{Aut}(\nu(S_1)) \simeq \mathrm{T}^2$.

Now we show that $\mathcal{K}_1$ is weakly contractible. Note that $\mathcal{K}_1$ is the subgroup of $\mathrm{Aut}(\nu(\Gamma_S))$ that acts as the identity on $\nu(S_1)$, so it consists of $n$-tuples $\phi = (\mathrm{Id}, \phi_2, \ldots, \phi_n)$ whose tangent and normal components intertwine at the intersection points. An element $\phi \in \mathcal{K}_1$ will have $\phi_2|_{S_2} \in \mathrm{Symp}(S_2, p_1)$ with $d\phi_2^T(p_1) = \mathrm{Id}$. We can thus perturb $\phi_2|_{S_2}$ so that it is the identity near $p_1$, so homotopically it will generate an element of $\mathrm{Symp}^{cpt}(S_2 \setminus p_1)$, and these are just the symplectomorphisms of the disk $D^2$ that are the identity near the boundary. This group is contractible by Smale's result ([44]-Theorem B). The fibre over the identity of the map $\phi_2 \mapsto \phi|_{S_2}$ is the group of gauge transformations that act as the identity over $p_1$, i.e. the group $\mathcal{G}(S_2, p_1)$. This is contractible by ([14]-Section 4). Hence, $\mathcal{K}_1$ fibres over a contractible space with kernel $\mathcal{K}_2$:

$$\mathcal{K}_2 \longrightarrow \mathcal{K}_1 \longrightarrow \star,$$

where $\mathcal{K}_2$ is the subgroup that acts as the identity on $\nu(S_2) \cup \nu(S_1)$. Similarly, we can show that $\mathcal{K}_2$ fibres over a contractible space and so on, until we get to the very last fibration

$$\mathcal{K}_n \longrightarrow \mathcal{K}_{n-1} \longrightarrow \star,$$

where $\mathcal{K}_n$ is the subgroup that acts as the identity everywhere, so $\mathcal{K}_n = \{\mathrm{Id}\}$. Thus, if we work backward through the fibrations we see that $\mathcal{K}_1$ must be contractible. $\qquad\square$

Good, now go back to the fibration (4.14). Since the total space is contractible, and also by Lemma 4.4.3, we have weak equivalences

$$\mathcal{K} \simeq \Omega\mathrm{Aut}(\nu(S_2)) \simeq \Omega\mathrm{T}^2,$$

and we also know that $\mathcal{K} \simeq \mathrm{Symp}^{cpt}(R \setminus (\Gamma_C \cup \Gamma_S))$. This last group fits into the homotopy fibration

$$\mathrm{Symp}^{cpt}(R \setminus (\Gamma_C \cup \Gamma_S)) \longrightarrow \mathrm{Symp}_{1,b,c}^{\mathrm{red}} \longrightarrow \mathrm{Aut}(T_{p_b}) \times \mathrm{Aut}(T_{pc})$$

that comes from the fibration (4.13). Now Theorem 4.4.1 follows by restricting to each of $\mathrm{Aut}(T_{p_b})$ and $\mathrm{Aut}(T_{p_c})$ one at a time. The argument is the same as the very last step in the proof of Theorem 4.3.1 at the end of Section 4.3, except now it doesn't matter which automorphism group we restrict to first.

# Chapter 5

# Embedding Spaces

## 5.1 Embedding Singular Balls into $\mathbb{C}P^2_{1,1,c}$

Consider the standard orbi-ball $B_c(\epsilon) := B^4(\epsilon) / \mathbb{Z}_c$, where $B^4(\epsilon) \subset \mathbb{C}^2$ is the standard (smooth) 4-ball of capacity $\epsilon$ containing the origin and $\mathbb{Z}_c$ acts diagonally. The symplectic form on $B^4(\epsilon)$ is the restriction of the standard form $\omega_0$ on $\mathbb{C}^2$. This form is $\mathbb{Z}_c$-invariant, so it descends to the quotient $B_c(\epsilon)$. Let $\text{Emb}^\epsilon_{1,1,c}$ be the space of reduced symplectic embeddings of $B_c(\epsilon)$ into the weighted projective space $\mathbb{C}P^2_{1,1,c}$. Thus, $f$ is in $\text{Emb}^\epsilon_{1,1,c}$ if and only if $f : B_c(\epsilon) \to f(B_c(\epsilon)) \subset \mathbb{C}P^2_{1,1,c}$ is a reduced orbifold diffeomorphism in the sense of Definition 4.1.5, and $f$ pulls back $\omega_{1,1,c}$ to the symplectic form on $B_c(\epsilon)$. We define the space of unparametrized symplectic embeddings $\Im\text{Emb}^\epsilon_{1,1,c}$ as the quotient

$$\Im\text{Emb}^\epsilon_{1,1,c} := \text{Emb}^\epsilon_{1,1,c} / \text{Symp}^{\text{red}}(B_c(\epsilon)).$$

Our goal is to use the general framework developed in [26] to study the homotopy type of the space $\Im\text{Emb}^\epsilon_{1,1,c}$ and $\text{Emb}^\epsilon_{1,1,c}$ based on the correspondence between embeddings of balls and symplectic blowups. The main results are

**Theorem 5.1.1.** $\Im\text{Emb}^\epsilon_{1,1,c}$ *is contractilble.*

**Corollary 5.1.2.** $\text{Emb}^\epsilon_{1,1,c}$ *is homotopy equivalent to* $\text{U}(2) / \mathbb{Z}_c$.

Note that in order to deduce this corollary, we need information about the group $\text{Symp}^{\text{red}}(B_c(\epsilon))$, ie. that it is homotopy equivalent to $\text{U}(2) / \mathbb{Z}_c$; this will be proved in a forthcoming lemma. In [26], they show that in the smooth case the symplectomorphism

group acts transitively on the space of embeddings and use the resulting fibration to glean information about the embedding space. In [40], Pinsonnault uses this general framework to find information about the embedding space of balls in $\mathbb{C}P^2$. It is natural then, to try to generalize this approach to weighted projective spaces.

**Lemma 5.1.3.** $\text{Symp}_{1,1,c}^{\text{red}}$ *acts transitively on* $\Im\text{Emb}_{1,1,c}^{\epsilon}$.

The proof requires a few preliminary steps. What we first need to show is that any embedded ball can be isotoped to be disjoint from the line at infinity

$$S_\infty = \{[z_1 : z_2 : 0] \in \mathbb{C}P^2_{1,1,c}\},$$

as this will allow us to work in a single orbifold chart.

**Claim 1.** *Let* $L_1, L_2 \subset \mathbb{C}P^2_{1,1,c}$ *be any two embedded symplectic spheres in homology class* $[\mathbb{C}P^1]$. *Then* $L_1$ *and* $L_2$ *are isotopic. Thus, swapping* $S_\infty$ *with a symplectic sphere disjoint from a given orbi-ball* $B^c$ *will allow us to work in a single orbifold chart.*

*Proof.* First note that given any embedded orbi-ball $B^c \subset \mathbb{C}P^2_{1,1,c}$, there exists an embedded symplectic sphere in homology class $[\mathbb{C}P^1]$ that is disjoint from $B^c$; this is easily seen by passing to the blowup, which is a Hirzebruch surface $W_c$. We will sketch the argument of why $L_1$ and $L_2$ are isotopic.. Since these spheres have the same self-intersection numbers (+c), by the symplectic neighbourhood theorem we can find neighbourhoods $\mathcal{U}_1 \supset L_1, \mathcal{U}_2 \supset L_2$ and a reduced diffeomorphism $f : \mathbb{C}P^2_{1,1,c} \to \mathbb{C}P^2_{1,1,c}$ such that $f : \mathcal{U}_1 \to \mathcal{U}_2$ is a symplectomorphism. The pullback form $f^*\omega_{1,1,c}$ is then equal to $\omega_{1,1,c}$ near the boundary of $\mathbb{C}P^2_{1,1,c} \setminus L_1 \cong B^c$, which is an orbi-ball centred at the singular point $p_c$. We now assert that there is a diffeomorphism $\psi : B^c \to B^c$ that is the identity near the boundary and is such that $\psi^*(f^*\omega_{1,1,c}) = \omega_{1,1,c}$. For this it is sufficient to find a $\mathbb{Z}_c$-equivariant lift $\widetilde{f_c} : \widetilde{B} \to \widetilde{B}$ of $f$ and having the same properties on the smooth ball $\widetilde{B}$. This follows from a $\mathbb{Z}_c$-invariant version of Gromov's theorem about compactly supported diffeomorphisms of the ball (see, for instance [33]-Lemma 2.4; the proof can

be made $\mathbb{Z}_c$-equivariant). Given the existence of this diffeomorphism $f$, the composition $f \circ \psi$ is a symplectomorphism sending $L_1$ to $L_2$. What this shows is that $\mathrm{Symp}^{\mathrm{red}}_{1,1,c}$ acts transitively on the space of embedded non-singular symplectic spheres in class $[\mathbb{C}P^1]$. Since $\mathrm{Symp}^{\mathrm{red}}_{1,1,c}$ is path-connected, it follows that the same is true for the space it acts transitively on. Hence, any $L_1, L_2$ in this space are always isotopic. $\qquad\square$

Given this, we can assume that any $B^c \in \Im\mathrm{Emb}^\epsilon_{1,1,c}$ lies in a single orbifold chart. Our result will now follow from the following:

**Claim 2.** *The space of reduced symplectic embeddings of $B_c(\epsilon)$ into the open unit orbi-ball $B_c(1) \subset \mathbb{C}^2 / \mathbb{Z}_c$ is path-connected.*

*Proof.* This is the orbi-ball analogue of McDuff's result ([33]-Theorem 1.1). The way she proves it is by noticing that this statement about embeddings is equivalent to a statement about uniqueness up to diffeomorphism of certain symplectic forms on the space $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$, which is the smooth blow up of $\mathbb{C}P^2$. In our case, we are dealing with an embedding of a singular ball $B_c(\epsilon)$, so the statement now becomes equivalent to a certain uniqueness property of symplectic forms on the $c^{th}$ Hirzebruch surface $W_c$. In particular, we have bijective correspondences between the following sets ([33]-Proposition 1.4):

(1) The set of isotopy classes of reduced symplectic embeddings $B_c(\epsilon) \hookrightarrow B_c(1)$.

(2) The set of equivalence classes of symplectic forms $\omega$ on $W_c$ such that each $\omega$ gives area $\epsilon$ to the zero section $\mathcal{Z}_0$ and area 1 to the infinity section $\mathcal{Z}_\infty$. Moreover, we should assume that $\mathcal{Z}_0$ and $\mathcal{Z}_\infty$ are $\omega$-symplectic.

Let $\mathrm{Diff}(W_c, \mathcal{Z}_0, \mathcal{Z}_\infty)$ be the group of all diffeomorphisms of $W_c$ that stabilize $\mathcal{Z}_0$ and $\mathcal{Z}_\infty$. In the statement (2) above, two forms $\omega_1, \omega_2$ are *equivalent* if there exists $f \in \mathrm{Diff}(W_c, \mathcal{Z}_0, \mathcal{Z}_\infty)$ such that $f^*\omega_2 = \omega_1$. The equivalence of statements (1) and (2) follows from the blow up construction in ([35]-Proposition 7.17). Specifically, to any symplectic embedding $g : B_c(\epsilon) \hookrightarrow B_c(1)$, it's possible to put a symplectic form $\omega_g$ on $W_c$, and this form depends on the embedding $g$. We claim that in the statement (2)

above, there is only one such equivalence class of symplectic forms on $W_c$; this follows from the following points:

- By the Lalonde-McDuff classification theorem [25], any two cohomologous symplectic forms are diffeomorphic. Hence for any two forms $\omega_1, \omega_2$ in the same cohomology class, there exists a diffeomorphism $f : W_c \to W_c$ such that $f^*\omega_2 = \omega_1$.

- By [43] (see Section 1.4.2), the space of smooth holomorphic curves representing the class $[\mathcal{Z}_\infty]$ is connected. In this case, we can take an isotopy of $J_t$-holomorphic curves where each $J_t$ is compatible with $\omega_1$. Hence, we can find a Hamiltonian isotopy whose time 1-map $\phi$ preserves $\omega_1$ and satisfies $\phi(\mathcal{Z}_\infty) = f^{-1}(\mathcal{Z}_\infty)$. It follows that $f \circ \phi$ preserves $\mathcal{Z}_\infty$.

- By ([20]-Theorem 8.1), the space of $\omega$-positive embedded symplectic spheres in class $[\mathcal{Z}_0]$ (and disjoint from $\mathcal{Z}_\infty$) is contractible. This means that we can find another Hamiltonian isotopy whose time 1-map $\psi$ preserves $\omega_2$, fixes $\mathcal{Z}_\infty$, and satisfies $\psi^{-1}(\mathcal{Z}_0) = f \circ \phi(\mathcal{Z}_0)$.

From these three points, it follows that $\psi \circ f \circ \phi \in \mathrm{Diff}(W_c, \mathcal{Z}_0, \mathcal{Z}_\infty)$ and also pulls back $\omega_2$ to $\omega_1$. So the forms $\omega_1$ and $\omega_2$ are equivalent in the sense describe above. We are now done, because the equivalence of statements (1) and (2) above implies that there is only one isotopy class of symplectic embeddings $B_c(\epsilon) \hookrightarrow B_c(1)$. $\square$

**Proof of Lemma 5.1.3**: Let $B_0^c, B_1^c \in \mathfrak{S}\mathrm{Emb}_{1,1,c}^\epsilon$. By Claim 1, we can assume that $B_0^c$ and $B_0^c$ are contained in an orbi-ball of size 1. Choose parametrizations $g_0, g_1 : B_c(\epsilon) \to B_c(1)$. By Claim 2, there is a 1-parameter family $g_t : B_c(\epsilon) \to B_c(1)$ of reduced symplectic embeddings connecting $g_0$ and $g_1$. Now lift the family $g_t$ to a uniformizing chart, which is just a smooth ball $B(1)$

$$
\begin{array}{ccc}
B(\epsilon) & \xrightarrow{\ \widetilde{g_t}\ } & B(1) \\
\downarrow & & \downarrow \\
B_c(\epsilon) & \xrightarrow{\ g_t\ } & B_c(1)
\end{array}
$$

The family $\widetilde{g}_t$ is generated by a $\mathbb{Z}_c$-invariant vector field, which in turn generates a $\mathbb{Z}_c$-invariant Hamiltonian $\widetilde{H} : B(\epsilon) \to \mathbb{R}$ that we can extend to $B(1) \cong \widetilde{U}_c$ using an invariant bump function. The corresponding isotopy is equivariant and has as its time 1-map a symplectomorphism $\widetilde{\phi}_c : \widetilde{U}_c \to \widetilde{U}_c$ supported in a neighbourhood of $B(\epsilon)$. Hence, $\widetilde{\phi}_c$ descends to a symplectomorphism $\phi \in \mathrm{Symp}^{\mathrm{red}}_{1,1,c}$ that sends $g_0(B_c(\epsilon)) = B_0^c$ to $g_1(B_c(\epsilon)) = B_1^c$, proving that the action is transitive.

Moving on. The stabilizer of an element $B^c \in \Im\mathrm{Emb}^{\epsilon}_{1,1,c}$ under the action of $\mathrm{Symp}^{\mathrm{red}}_{1,1,c}$ is the subgroup $\mathrm{Stab}(B^c)$ consisting of those $f \in \mathrm{Symp}^{\mathrm{red}}_{1,1,c}$ that leave invariant the orbi-ball $B^c$, where $B^c$ is the image of a symplectic embedding $B_c(\epsilon) \to \mathbb{C}P^2_{1,1,c}$. We therefore have a fibration

$$\mathrm{Stab}(B^c) \longrightarrow \mathrm{Symp}^{\mathrm{red}}_{1,1,c} \longrightarrow \Im\mathrm{Emb}^{\epsilon}_{1,1,c}, \tag{5.1}$$

and restricting $\mathrm{Stab}(B^c)$ to the orbi-ball gives another fibration

$$\mathrm{Fix}(B^c) \longrightarrow \mathrm{Stab}(B^c) \longrightarrow \mathrm{Symp}^{\mathrm{red}}(B^c). \tag{5.2}$$

**Lemma 5.1.4.** $\mathrm{Fix}(B^c)$ *is contractible.*

*Proof.* Here, $\mathrm{Fix}(B^c)$ are the reduced symplectomorphisms that are the identity on $B^c$. If we blowup the singular point $p_c$ with a size that is smaller than the capacity of $B^c$, then $\mathrm{Fix}(B^c)$ can be identified with the group of symplectomorphisms of the Hirzebruch surface $W_c$ that fix a neighbourhood of the zero section $\mathcal{Z}_0$. This is contractible by ([20]-Lemma 9.1). $\square$

**Lemma 5.1.5.** $\mathrm{Symp}^{\mathrm{red}}(B^c)$ *is homotopy equivalent to* $\mathrm{U}(2)\,/\,\mathbb{Z}_c$.

*Proof.* Just evaluate the derivative at the singularity to get the fibration

$$\mathcal{K} \longrightarrow \mathrm{Symp}^{\mathrm{red}}(B^c) \longrightarrow \mathrm{Aut}(T_{p_c}) \simeq \mathrm{U}(2)\,/\,\mathbb{Z}_c.$$

Hence, we want to show that the kernel $\mathcal{K}$ is contractible. Note that $\mathcal{K}$ is weakly equivalent to the subgroup of symplectomorphisms of the orbi-ball that are the identity near the singularity. If we blow up the singular point $p_c \in B^c$, then the resulting space is a disk bundle inside the complex line bundle $\mathcal{O}(-c)$ that can be equipped with a standard Kahler form. A compact family $f_\lambda$ of symplectomorphisms that are the identity near $p_c$ will lift to a compact family $\widetilde{f}_\lambda : \mathcal{O}(-c) \to \mathcal{O}(-c)$ that are the identity near the zero section. In ([11]-Lemma 3.3), Coffey uses symplectic cutting to show how to compactify a disk bundle into a symplectic sphere bundle while preserving the areas of the fibres[1]. Let $\mathrm{Symp}(\mathcal{O}(-c), [\mathcal{Z}_0])$ be the group of symplectomorphisms of the unit disk bundle inside $\mathcal{O}(-c)$ that are the identity near $\mathcal{Z}_0$. Then Coffey's construction gives a homeomorphism

$$\mathrm{Symp}(\mathcal{O}(-c), [\mathcal{Z}_0]) \cong \mathrm{Symp}^{cpt}(W_c \setminus \mathcal{Z}_0, \mathcal{Z}_\infty),$$

where the latter group consists of symplectomorphisms of the Hirzebruch surface $W_c$ that are compactly supported away from $\mathcal{Z}_0$ and stabilize $\mathcal{Z}_\infty$. This latter group is contractible by ([11]-Proposition 3.2). $\qquad\square$

**Proof of Theorem 5.1.1:** The previous two lemmas and the fibration (5.2) all imply that $\mathrm{Stab}(B^c)$ is weakly homotopy equivalent to $\mathrm{U}(2) / \mathbb{Z}_c$. Therefore, the long exact homotopy sequence of (5.1) implies that $\Im\mathrm{Emb}^\epsilon_{1,1,c}$ is (weakly) contractible. Done.

**Proof of Corollary 5.1.2:** Consider the evaluation map from $\mathrm{Emb}^\epsilon_{1,1,c}$ to $\Im\mathrm{Emb}^\epsilon_{1,1,c}$ that sends an embedding $g$ onto its image $g(B_c(\epsilon))$. The fibre over an element $B^c \in \Im\mathrm{Emb}^\epsilon_{1,1,c}$ is the reparametrization group $\mathrm{Symp}^{\mathrm{red}}(B_c(\epsilon))$. Now the fibration

$$\mathrm{Symp}^{\mathrm{red}}(B_c(\epsilon)) \longrightarrow \mathrm{Emb}^\epsilon_{1,1,c} \longrightarrow \Im\mathrm{Emb}^\epsilon_{1,1,c}$$

implies the result.

---

1. There can be some confusion here: Coffey's compactification adds a section of self-intersection $-c$ that he calls $\mathcal{Z}_\infty$, but our convention has always been to declare the zero section $\mathcal{Z}_0$ to be of self intersection $-c$.

## 5.2 Embedding Smooth Balls into $\mathbb{C}P^2_{1,1,c}$

Now consider the smooth 4-ball $B(\delta) \subset \mathbb{C}^2$ of capacity $\delta < 1$ equipped with the restriction of the standard form on $\mathbb{C}^2$. Let $\mathrm{SEmb}^\delta_{1,1,c}$ be the space of smooth symplectic embeddings of $B(\delta)$ into $\mathbb{C}P^2_{1,1,c}$ equipped with the $C^\infty$-topology. Note that the embeddings are required to miss the singular point $p_c$, so really $\mathrm{SEmb}^\delta_{1,1,c}$ is the space of symplectic embeddings of $B(\delta)$ into $\mathbb{C}P^2_{1,1,c} \setminus p_c$. We define the space of unparametrized smooth symplectic embeddings as

$$\Im^\infty \mathrm{Emb}^\delta_{1,1,c} := \mathrm{SEmb}^\delta_{1,1,c} / \mathrm{Symp}(B(\delta))$$

The main result of this section is

**Theorem 5.2.1.** $\Im^\infty \mathrm{Emb}^\delta_{1,1,c}$ *is homotopy equivalent to* $\mathbb{C}P^1 \simeq \mathbb{C}P^2_{1,1,c} \setminus p_c$.

The proof is a bit more complicated compared to the last section, but the overall approach is quite similar to what we've been doing throughout this thesis. An argument that mimics the proof of Lemma 5.1.3 (but easier) can be used to show that $\mathrm{Symp}^{\mathrm{red}}_{1,1,c}$ acts transitively on $\Im^\infty \mathrm{Emb}^\delta_{1,1,c}$. Let $p_1 \in \mathbb{C}P^2_{1,1,c}$ be the smooth point $[0 : 1 : 0]$, and let $B_\delta = B_\delta(p_1) \in \Im^\infty \mathrm{Emb}^\delta_{1,1,c}$ be an embedded ball centred at this point. We will consider the stabilizer $\mathrm{Stab}(B_\delta)$ of this ball under the action of $\mathrm{Symp}^{\mathrm{red}}_{1,1,c}$. Then we have the fibration

$$\mathrm{Stab}(B_\delta) \longrightarrow \mathrm{Symp}^{\mathrm{red}}_{1,1,c} \longrightarrow \Im^\infty \mathrm{Emb}^\delta_{1,1,c}. \tag{5.3}$$

Note that there is no loss of generality in assuming that $B_\delta$ is centred at $p_1$ because all the fibres are homotopy equivalent. We will eventually conclude that $\mathrm{Stab}(B_\delta)$ is homotopy equivalent to $\mathrm{T}^2$.

Let's now blow up $\mathbb{C}P^2_{1,1,c}$ at the two points $p_c$ and $p_1$. This two-point blow up of $\mathbb{C}P^2_{1,1,c}$ is diffeomorphic to the one-point blow up $\widetilde{W}_c$ of the Hirzebruch surface $W_c$ at the point of intersection of the infinity section and fibre. We can equip $\widetilde{W}_c$ with the

symplectic form $\Omega_{\mu,c,\varepsilon_1}$ described in Lemma 3.2.2 from Section 3.2. Recall that

$$[\Omega_{\mu,c,\varepsilon_1}] = \mathrm{PD}(B + \mu F - \varepsilon_1 E_1).$$

The embedded singular sphere $\{[0 : z_1 : z_2]\} \subset \mathbb{C}P^2_{1,1,c}$ is sent via the blow up to a configuration of smooth spheres $\Gamma_{1,2,3} := C_1 \cup C_2 \cup C_3$ in $\widetilde{W}_c$, such that for $c$ even[2]

$$
\begin{aligned}
{[C_1]} &= B - \frac{c}{2}F \\
{[C_2]} &= F - E_1 \\
{[C_3]} &= E_1 \,.
\end{aligned}
$$

Put $\Gamma_{1,3} := C_1 \cup C_3$, and let $\mathrm{Symp}(\widetilde{W}_c, \Gamma_{1,3})$ be the subgroup of $\mathrm{Symp}(\widetilde{W}_c)$ that stabilizes $\Gamma_{1,3}$. In the same way, let $\mathrm{Symp}(\widetilde{W}_c, \Gamma_{1,2,3})$ be the subgroup that stabilizes $\Gamma_{1,2,3}$. The following lemma shouldn't be very surprising because we've seen the same phenomenon in Section 4.3.

**Lemma 5.2.1.** $\mathrm{Symp}(\widetilde{W}_c, \Gamma_{1,3})$ *is weakly equivalent to* $\mathrm{Symp}(\widetilde{W}_c, \Gamma_{1,2,3})$, *which is in turn weakly equivalent to* $\mathrm{T}^2$.

*Proof.* The fact that $\mathrm{Symp}(\widetilde{W}_c, \Gamma_{1,2,3})$ is weakly homotopy equivalent to $\mathrm{T}^2$ follows from Lemma 4.3.7 in Section 4.3, so we will work on proving the first statement. Let $\mathcal{J}$ be the space of $\Omega_{\mu,c,\varepsilon_1}$-tame almost complex structures on $\widetilde{W}_c$ and let $\mathcal{J}_{1,3} \subset \mathcal{J}$ be the subset of $J$'s for which $C_1$ and $C_3$ are $J$-holomorphic. For any $J \in \mathcal{J}_{1,3}$, we'll see that there is a unique embedded $J$-holomorphic sphere in class $F - E_1$, and this will imply the first statement of the lemma.

Specifically, let $\mathcal{C}^{\perp}_{1,3}[C_2]$ be the space of embedded symplectic spheres in homology class $[C_2] = F - E_1$ that intersect $C_1, C_3$ once in a symplectically orthogonal way. We should also assume that for every $S \in \mathcal{C}^{\perp}_{1,3}[C_2]$, there exists $J \in \mathcal{J}$ such that $C_1, S$, and

---

2. Again, the odd case is analogous. See Section 2.3.1.

$C_3$ are $J$-holomorphic. Then $\mathrm{Symp}(\widetilde{W}_c, \Gamma_{1,3})$ acts transitively on this space, giving a fibration

$$\mathrm{Symp}(\widetilde{W}_c, \Gamma_{1,2,3}) \longrightarrow \mathrm{Symp}(\widetilde{W}_c, \Gamma_{1,3}) \longrightarrow \mathcal{C}_{1,3}^{\perp}[C_2],$$

so we should prove, as before, that the base is contractible. Let $\mathcal{C}_{1,3}^{\pitchfork}[C_2] \supset \mathcal{C}_{1,3}^{\perp}[C_2]$ be the bigger space of embedded symplectic spheres in class $[C_2]$ that now only intersect $C_1, C_3$ transversely and also satisfy the same property with respect to $J$-holomorphic spheres.

**Claim.** *For every $J \in \mathcal{J}_{1,3}$, there is a unique embedded $J$-sphere in class $[C_2] = F - E_1$.*

*Proof.* To see this, recall from Section 2.4 that the subset of $\mathcal{J}$ for which the exceptional class $F - E_1$ is represented by an embedded $J$-sphere is open and dense in $\mathcal{J}$. By the corollary of Gromov compactness (Section 2.4), $F - E_1$ is either represented by an embedded $J$-sphere or a cusp-curve. We'll show that it can't degenerate into a cusp-curve. Write

$$F - E_1 = \sum_{i=1}^{n}(p_i B + q_i F - r_i E_1), \tag{5.4}$$

where each of the classes $B, F, E_1$ have simple representatives. It follows that the $p_i$ must sum to zero. By the adjunction formula (Section 2.4),

$$2p_i q_i - r_i^2 + 2 \geq 2p_i + 2q_i - r_i.$$

Rearranging things, we have $2g_v = 2(p_i - 1)(q_i - 1) - r_i(r_i - 1) \geq 0$. By positivity of area,

$$[\Omega_{\mu,c,\varepsilon_1}](p_i B + q_i F - r_i E_1) = \mu p_i + q_i - \varepsilon_1 r_i > 0,$$

and from this it follows that $q_i - 1 > \varepsilon_1 r_i - \mu p_i - 1$. Now we claim that these conditions force $p_i \geq 0$. Let's mimic the proof in ([6]-Lemma 2.4). Assume, for a contradiction,

that $p_i < 0$. Then $p_i < \frac{1}{2}$, which implies that $-2(p_i - 1) > 1$. So, we have

$$
\begin{aligned}
-2g_v & = -2(p_i - 1)(q_i - 1) + r_i(r_i - 1) \\
& > q_i - 1 + r_i(r_i - 1) \\
& > \varepsilon_1 - \mu p_i - 1 + r_i(r_i - 1) \\
& > \varepsilon_1 r_i + r_i(r_i - 1) \\
& = r_i(\varepsilon_1 + r_i - 1) \geq 0.
\end{aligned}
\tag{5.5}
$$

Here, the last inequality is because $r_i$ is an integer and the inequality (5.5) is because both $\mu > 1$ and $p_i < 0$. From all this we conclude that $g_v < 0$, which is a contradiction. So, our assumption that $p_i < 0$ was incorrect, which means that $p_i \geq 0$.

Now go back to the decomposition (5.4). Since $\sum_i p_i = 0$ and each $p_i \geq 0$, the only possibility is that $p_i = 0$. Therefore, $F - E_1$ decomposes as

$$
F - E_1 = \sum_{i=1}^{n}(q_i F - r_i E_1).
\tag{5.6}
$$

For $J \in \mathcal{J}_{1,3}$, both the classes $[C_1] = B - \frac{c}{2}F$ and $[C_3] = E_1$ are represented by embedded $J$-spheres, so it follows from positivity of intersections that

$$
\begin{aligned}
(q_i F - r_i E_1) \cdot E_1 & = r_i \geq 0 \\
(q_i F - r_i E_1) \cdot (B - \frac{c}{2}F) & = q_i \geq 0,
\end{aligned}
$$

and from the adjunction inequality combined with $q_i \geq 0$, we get

$$
0 \leq 2q_i \leq 2 - r_i(r_i - 1).
$$

- If $r_i > 2$, then $q_i < 0$ which is a contradiction.

- If $r_i = 2$, then $q_i = 0$, and this contradicts positivity of area.

- If $r_i = 0$, then $q_i = 1$, and (5.6) gives a decomposition of $F - E_1$ into a bunch of $F$-spheres, which is not possible.

The only remaining case is $r_i = 1$. Then $q_i = 1$ as well, and this is the only possibility that doesn't lead to conflicting information. It follows that the class $F - E_1$ cannot degenerate, and this proves the claim. $\qquad\square$

To complete the proof of Lemma 5.2.1, we proceed in the same way as before. The space $\mathcal{J}_{1,3}$ is weakly contractible by ([14]-Appendix A) and the obvious map $\mathcal{J}_{1,3} \to \mathcal{C}_{1,3}^{\pitchfork}[C_2]$ is a fibration with contractible fibres. Therefore, $\mathcal{C}_{1,3}^{\pitchfork}[C_2]$ is contractible, and $\mathrm{Symp}(\widetilde{W}_c, \Gamma_{1,2,3})$ is weakly equivalent to $\mathrm{Symp}(\widetilde{W}_c, \Gamma_{1,3})$. $\qquad\square$

**Lemma 5.2.2.** $\mathrm{Symp}(\widetilde{W}_c, \Gamma_{1,3})$ *is weakly equivalent to its subgroup consisting of symplectomorphisms that act* $\mathrm{U}(2)$*-linearly near* $C_1$ *and* $C_3$.

*Proof.* Let $\mathrm{Aut}(\nu(\Gamma_i))$, $i = 1, 3$, be the group of linear symplectic automorphisms of the normal bundle $\nu(\Gamma_i)$ that preserve the zero section. Then we have a fibration

$$\mathcal{K} \longrightarrow \mathrm{Symp}(\widetilde{W}_c, \Gamma_{1,3}) \longrightarrow \mathrm{Aut}(\nu(\Gamma_1)) \times \mathrm{Aut}(\nu(\Gamma_3))$$

that we get by evaluating the derivative on $T\widetilde{W}_c|_{\Gamma_i}$ for $i = 1, 3$. Let $\mathrm{Symp}(\widetilde{W}_c, \Gamma_{1,3}^{\mathrm{U}(2)})$ be the other subgroup in the statement of the lemma. Since elements in this group act linearly near $\Gamma_{1,3}$, we can restrict to each sphere to get another fibration

$$\mathcal{K}^* \longrightarrow \mathrm{Symp}(\widetilde{W}_c, \Gamma_{1,3}^{\mathrm{U}(2)}) \longrightarrow \mathrm{U}(2) \times \mathrm{U}(2),$$

where $\mathcal{K}^*$ consists of symplectomorphisms acting as the identity near $\Gamma_{1,3}$. Putting these together gives a map of fibrations, and we know that $\mathcal{K}$ is weakly equivalent to $\mathcal{K}^*$ from Section 6.1. So, to prove the lemma it suffices to show that $\mathrm{Aut}(\nu(\Gamma_i)) \cong \mathrm{U}(2)$ for $i = 1, 3$. Actually, this follows from ([21]-Proposition 2.5) which says that these automorphisms groups are each homotopy equivalent to the Kahler isometry groups of $W_1, W_3$ respectively, so they are homotopy equivalent to $\mathrm{U}(2)$ in each case. $\qquad\square$

Via the symplectic blowdown map, a neighbourhood of $C_1$ is sent to a singular ball $B_\epsilon(p_c) \subset \mathbb{C}P^2_{1,1,c}$ centred at the point $p_c$, and a neighbourhood of $C_3$ is sent to the smooth ball $B_\delta$ centred at $p_1$. Any symplectomorphism $\widetilde{f} \in \mathrm{Symp}(\widetilde{W}_c, \Gamma_{1,3}^{\mathrm{U}(2)})$ descends to a symplectomorphism $f : \mathbb{C}P^2_{1,1,c} \to \mathbb{C}P^2_{1,1,c}$ that acts $\mathrm{U}(2)/\mathbb{Z}_c$-linearly near $B_\epsilon(p_c)$ and $\mathrm{U}(2)$-linearly near $B_\delta$. Give this latter subgroup the beastly designation

$$\mathrm{Symp}^{\mathrm{red}}_{1,1,c}(B_\epsilon(p_c)^{\mathrm{U}(2)c}, B_\delta^{\mathrm{U}(2)}).$$

We see, conversely, that any symplectomorphism in $\mathrm{Symp}^{\mathrm{red}}_{1,1,c}(B_\epsilon(p_c)^{\mathrm{U}(2)c}, B_\delta^{\mathrm{U}(2)})$ will lift to a symplectomorphism in $\mathrm{Symp}(\widetilde{W}_c, \Gamma_{1,3}^{\mathrm{U}(2)})$. This shows that

$$\mathrm{Symp}(\widetilde{W}_c, \Gamma_{1,3}^{\mathrm{U}(2)}) \text{ is homeomorphic to } \mathrm{Symp}^{\mathrm{red}}_{1,1,c}(B_\epsilon(p_c)^{\mathrm{U}(2)c}, B_\delta^{\mathrm{U}(2)}).$$

Now recall the group $\mathrm{Stab}(B_\delta)$ from the fibration (5.3). It is the subgroup of $\mathrm{Symp}^{\mathrm{red}}_{1,1,c}$ that stabilizes $B_\delta$. Here is the final lemma in this section

**Lemma 5.2.3.** $\mathrm{Stab}(B_\delta)$ *is weakly homotopy equivalent to* $\mathrm{Symp}^{\mathrm{red}}_{1,1,c}(B_\epsilon(p_c)^{\mathrm{U}(2)c}, B_\delta^{\mathrm{U}(2)})$.

*Proof.* First, it's possible to show that $\mathrm{Stab}(B_\delta)$ is weakly equivalent to its subgroup that acts linearly near an orbi-ball $B_\epsilon(p_c)$ centred at $p_c$; call this group $\mathrm{Stab}(B_\delta, p_c^{\mathrm{U}(2)c})$. Now consider the composition of fibrations

$$\mathrm{Stab}(B_\delta, p_c^{\mathrm{U}(2)c}) \longrightarrow \mathrm{Symp}(B_\delta) \longrightarrow \mathrm{Symp}(B_\delta)/\mathrm{U}(2).$$

Since the base is contractible, this shows that $\mathrm{Stab}(B_\delta, p_c^{\mathrm{U}(2)c})$ is weakly equivalent to its subgroup $\mathrm{Symp}^{\mathrm{red}}_{1,1,c}(B_{\delta*}^{\mathrm{U}(2)}, p_c^{\mathrm{U}(2)c})$ that acts linearly near a slightly smaller ball $B_{\delta*} \subset B_\delta$. This is fine, since we can always blow up using a slightly smaller ball. Observe that the group

$$\mathrm{Symp}^{\mathrm{red}}_{1,1,c}(B_{\delta*}^{\mathrm{U}(2)}, p_c^{\mathrm{U}(2)c})$$

consist of symplectomorphisms that, in particular, act linearly near an orbi-ball centred

at $p_c$. This can be identified with $\mathrm{Symp}^{\mathrm{red}}_{1,1,c}(B_\epsilon(p_c)^{\mathrm{U}(2)c}, B_\delta^{\mathrm{U}(2)})$ in the statement of the lemma by possibly varying the sizes of our blow ups, ie. we can show that they are each weakly equivalent to $\mathrm{T}^2$ by blowing up with slightly different sizes. $\qquad\square$

**Proof of Theorem 5.2.1:** Consider the action of $\mathrm{U}(2)/\mathbb{Z}_c$ on the subset $S_\infty := \{[z_0 : z_1 : 0] \in \mathbb{C}P^2_{1,1,c}\}$ given by $A \cdot [z_0 : z_1 : 0] = [az_0 + bz_1 : cz_0 + dz_1 : 0]$, where

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Since $S_\infty \cong \mathbb{C}P^1$, this is the same as the action of $\mathrm{U}(2)/\mathbb{Z}_c$ on $\mathbb{C}P^1$, so it is a transitive action; the reason being that $\mathrm{U}(2)$ already acts transitively on $\mathbb{C}P^1$ and $\mathbb{Z}_c \subset \mathrm{U}(2)$ is contained in the stabilizer of any point. The stabilizer of the action of $\mathrm{U}(2)/\mathbb{Z}_c$ is the torus of diagonal matrices $\mathrm{T}^2_\Delta := \{\mathrm{diag}(a,d) \mid |a| = |d| = 1\}$. Then we have a diagram of fibrations

$$
\begin{array}{ccccc}
\mathrm{Stab}(B_\delta) & \longrightarrow & \mathrm{Symp}^{\mathrm{red}}_{1,1,c} & \longrightarrow & \Im^\infty\mathrm{Emb}^\delta_{1,1,c} \\
\simeq\uparrow & & \simeq\uparrow & & \uparrow \\
\mathrm{T}^2_\Delta & \longrightarrow & \mathrm{U}(2)/\mathbb{Z}_c & \longrightarrow & \mathbb{C}P^1,
\end{array}
$$

where the vertical maps are inclusions. The group $\mathrm{U}(2)/\mathbb{Z}_c$ acts effectively (and symplectically) on $\mathbb{C}P^2_{1,1,c}$ while fixing the singular point $p_c$, so there is a natural inclusion $\mathrm{U}(2)/\mathbb{Z}_c \hookrightarrow \mathrm{Symp}^{\mathrm{red}}_{1,1,c}$ inducing the weak homotopy equivalence. Of course, $\mathrm{T}_\Delta$ acts on $\mathbb{C}P^2_{1,1,c}$ by restriction, so we just need to check that the action stabilizes embedded balls centred at $p_1 = [0 : 1 : 0]$. If $B_\delta$ is centred at $p_1$ with $\delta < 1$, then we can assume that $B_\delta \subset U_1$, where $U_1$ is the smooth chart $\{[z_0 : z_1 : z_2] \mid z_1 \neq 0\}$. In this chart, we have

$$B_\delta = \{[w_0 : 1 : w_2] \mid |w_0|^2 + |w_2|^2 \leq \delta\},$$

where $w_0 = \frac{z_0}{z_1}$ and $w_2 = \frac{z_2}{z_1^c}$, so it should be clear that the $\mathrm{T}_\Delta$-action leaves $B_\delta$ invariant, thus we also have a natural inclusion $\mathrm{T}_\Delta \hookrightarrow \mathrm{Stab}(B_\delta)$.

# Chapter 6

# Some postponed proofs

## 6.1 Proof of Lemma 3.2.6

Recall that we had the locally trivial fibration

$$\mathrm{Symp}_{1,1,c}^{\mathrm{red}} \xrightarrow{\Psi} \mathrm{U}(2), \quad (f, \widetilde{f}_c) \mapsto d\widetilde{f}_c(0)$$

with $\mathrm{K}_\Psi = \ker \Psi$. We defined the subgroup $\mathrm{K}_\Psi^* \subset \mathrm{K}_\Psi$ as

$$\mathrm{K}_\Psi^* = \{(f, \widetilde{f}_c) \in \mathrm{Symp}_{1,1,c}^{\mathrm{red}} \,|\, \widetilde{f}_c = \mathrm{Id} \text{ near } 0\}.$$

Our goal is to prove that the inclusion $i : \mathrm{K}_\Psi^* \hookrightarrow \mathrm{K}_\Psi$ is a weak homotopy equivalence.

**Step 1:** Let $(f, \widetilde{f}_c) \in \mathrm{K}_\Psi$. Then $d\widetilde{f}_c(0) = \mathrm{Id}$. We will first show how $\widetilde{f}_c$ can be isotoped to the identity near 0; then we will extend this argument to compact families.

We have $\widetilde{f}_c : \widetilde{U}_c \to \widetilde{V}_c$, where $\widetilde{U}_c$ and $\widetilde{V}_c$ are $\mathbb{Z}_c$-invariant neighbourhoods of the origin in $\mathbb{C}^2$ with $\widetilde{f}_c(\widetilde{U}_c) \subset \widetilde{V}_c$. Since $d\widetilde{f}_c(0) = \mathrm{Id}$, this means that $\Gamma = \mathrm{graph}(\widetilde{f}_c)$ is tangent to the diagonal $\Delta \subset \widetilde{U}_c \times \widetilde{V}_c$ at $0 \in \mathbb{C}^4$. Thus, in a neighbourhood of the origin $\Gamma$ appears as the graph of a function $F$ over $\Delta$. Choose a smooth bump function $\rho : \Delta \to \mathbb{R}$ such that $\rho$ vanishes near the origin and $\rho = 1$ outside of a neighbourhood of the origin. Now, average this function to make it $\mathbb{Z}_c$-invariant. If we multiply $F$ by $\rho$, then in this neighbourhood, the graph of $\rho \cdot F$ corresponds to the graph of a diffeomorphism $g : \widetilde{U}_c \to \widetilde{V}_c$ such that $g = \mathrm{Id}$ near 0 and $g = \widetilde{f}_c$ outside of some larger neighbourhood.

Now set

$$\omega_t = (1 - t)\widetilde{\omega}_c + tg^*\widetilde{\omega}_c.$$

The forms $\omega_t$ must be cohomologous since $H^2(\widetilde{U}_c; \mathbb{R}) = 0$. Since $g = \mathrm{Id}$ near 0 and $g = \widetilde{f}_c$ outside of a larger neighbourhood, the only place where it may fail to be symplectic is in the region where $g$ can be made $C^1$-small (by suitably bounding the derivative of $\rho$). Since non-degeneracy is an open condition, we can assume that the path $\omega_t$ is non-degenerate on a small enough neighbourhood.

Now apply Moser's argument to the family $\omega_t$. It follows that there is a smooth family of diffeomorphisms $\psi_t$ such that $\psi_0 = \mathrm{Id}$ and $\psi_t^*\omega_t = \widetilde{\omega}_c$. Moreover, since $\omega_t = \omega$ near 0, the isotopy will be the identity in this region. Note that by averaging the generating vector field for $\psi_t$, this argument becomes equivariant. It follows that $g \circ \psi_1 : \widetilde{U}_c \to \widetilde{V}_c$ is a symplectomorphism that is the identity near 0 and interpolates to $\widetilde{f}_c$ outside of this neighbourhood. It is also $\mathbb{Z}_c$-equivariant.

**Step 2:** Now consider a family of symplectomorphisms $(f_\lambda, \widetilde{f}_{c,\lambda}) \in \mathrm{K}_\Psi$ that is parametrized by a compact set $S$. Then $d\widetilde{f}_{c,\lambda} = \mathrm{Id}$ for each $\lambda \in S$. By Step 1, for each fixed $\lambda_0 \in S$ we can modify $\widetilde{f}_{c,\lambda_0}$ to a diffeomorphism $g_{\lambda_0}$ such that $g_{\lambda_0}|_{B_{\lambda_0}} = \mathrm{Id}$ for some open ball $B_{\lambda_0}$ containing the origin, and all the choices of parameters in Step 1 are made in contractible spaces. Thus the function

$$S \to \mathbb{R}_{>0}, \quad \lambda \mapsto \mathrm{Vol}(B_\lambda)$$

is continuous, because the functions $g_\lambda$ can be made to continuously depend on $\lambda$. Since $B_\lambda$ is parametrized by a compact set, the function Vol must have a minimum that is non-zero. Hence, there exists $B_{min}$ such that $B_{min} \subset B_\lambda$ for all $\lambda \in S$. So, we have

$$g_\lambda|_{B_{min}} = \mathrm{Id} \quad \text{for all } \lambda \in S.$$

As before, we can define $\omega_t = (1-t)\widetilde{\omega}_c + tg_\lambda^*\widetilde{\omega}_c$, and now the Moser argument works for all $\lambda \in S$ to give a diffeomorphism parametrized by $\lambda$ such that its composition with

$g_\lambda$ is a symplectomorphism, and is the identity on $B_{min}$ for all $\lambda \in S$. Thus, we have shown that compact families in $K_\Phi$ can be isotoped to compact families in $K_\Phi^*$. This proves that the spaces are weakly homotopy equivalent.

## 6.2 Making Transverse Intersections Orthogonal

In this section, we describe a standard construction that is used at various points in this thesis. This construction essentially mimics the one in ([32]-Lemma 3.11) at the most crucial points. Another nice construction along these lines is given in ([20]-Section 6). Let $C$ be a fixed embedded symplectic sphere in a symplectic 4-manifold $(M, \omega)$ and let $q$ be a point in $C$. Consider the space of all embedded symplectic spheres in $M$ that intersect $C$ transversely and positively at $q$; let's call this space $\mathcal{C}_q^{\pitchfork}$. Also consider the space $\mathcal{C}_q^\perp \subset \mathcal{C}_q^{\pitchfork}$, where $\mathcal{C}_q^\perp$ is the space of all embedded symplectic spheres in $M$ whose intersections with $C$ are symplectically orthogonal. These spaces are topologized as quotients of $C^\infty(S^2, M)$ modulo reparametrization. We want to show that these spaces are weakly homotopy equivalent. To do this, we should construct a symplectic isotopy that deforms a sphere $S \in \mathcal{C}_q^{\pitchfork}$ into one that intersects $\omega$-orthogonally at q; then we will describe how this construction can be extended to compact families.

Since we only care what happens at the point $q$, it suffices to choose a Darboux chart $(U_q \cong \mathbb{R}^2 \times \mathbb{R}^2)$ and work in a neighbourhood of $q$. Choosing coordinates $(x_1, x_2, y_1, y_2)$ in this neighbourhood, we have

$$\omega|_{U_q} = dx_1 \wedge dx_2 + dy_1 \wedge dy_2.$$

Let $S \in \mathcal{C}_q^{\pitchfork}$, so that $S$ intersects the fixed symplectic sphere $C$ transversely and positively at $q$. We will assume that the $(y_1, y_2)$-plane is orthogonal to $T_q C$. In a possibly smaller neighbourhood $U_q' \subset U_q$, we can modify $S$ to a sphere $S'$ so that it coincides with $T_q S$ in this region, and this can be done symplectically (see Section 6.1). Therefore, in this

neighbourhood $S'$ appears as a graph of a matrix over the $(y_1, y_2)$-plane

$$
\begin{aligned}
S' \cap U'_q &= \{(y_1, y_2, A(y_1, y_2)) \mid \det A > -1\} \\
&= \{(y_1, y_2, ay_1 + by_2, cy_1 + dy_2) \mid ad - bc > -1\},
\end{aligned}
$$

where $A$ is the matrix with entries $a, b, c, d$ that are smooth functions of $x_1, x_2$, and the condition $\det A > -1$ guarantees that $S'$ is symplectic. We want to dropkick $S' \cap U'_q$ so that it coincides with the $(y_1, y_2)$-plane in this neighbourhood, but do it symplectically. Let $r$ be the radial coordinate on the $(y_1, y_2)$-plane: $r^2 = y_1^2 + y_2^2$. The projection of $U'_q$ to the $(y_1, y_2)$-plane is given by $\{r \leq \varepsilon\}$ for a suitable $\varepsilon > 0$. Choose an increasing function $\alpha : \mathbb{R} \to \mathbb{R}$ such that

- $\alpha(r) \leq 1$ and $\alpha(r) = 1$ for $r \geq \varepsilon$.

- $\alpha(r) = 1$ for $r$ near $\varepsilon$.

- $\alpha(r) = 0$ for $r \leq \varepsilon_0$ where $\varepsilon_0 \in (0, \varepsilon)$.

- $\alpha'(r) \leq \frac{\delta}{r}$, where $\delta > 0$ satisfies $(1 + \delta) \det A > -1$.

Now, let $S'_{\alpha(r)}$ be the image of the map

$$
(y_1, y_2) \mapsto (y_1, y_2, \alpha(r)(ay_1 + by_2), \alpha(r)(cy_1 + dy_2)), \quad r \leq \varepsilon.
$$

Then $S'_{\alpha(r)}$ fits together smoothly with $S'$ when $r$ is near $\varepsilon$ and it coincides with the $(y_1, y_2)$-plane when $r \leq \varepsilon_0$. We should check that it is symplectic. A somewhat tedious computation shows that

$$
\omega|_{S'_{\alpha(r)}} = \left(1 + \left(\alpha^2(r) + r\alpha(r)\alpha'(r)\right) \det A\right) dy_1 \wedge dy_2, \tag{6.1}
$$

so this form is symplectic if and only if $\left(1 + \left(\alpha^2(r) + r\alpha(r)\alpha'(r)\right) \det A\right) > 0$. Since

$\alpha'(r) \leq \frac{\delta}{r}$ and $\alpha(r) \leq 1$, it follows that

$$\alpha^2(r) + r\alpha(r)\alpha'(r) \leq 1 + \delta.$$

If $\det A$ is positive, then it's clear that (6.1) is symplectic. Otherwise, we have

$$\left(\alpha^2(r) + r\alpha(r)\alpha'(r)\right) \det A \geq (1 + \delta) \det A > -1,$$

showing that (6.1) is still positive. We have therefore shown that we can deform the original sphere $S \in \mathcal{C}_q^{\pitchfork}$ in a symplectic way so that it coincides with the $(y_1, y_2)$-plane near $q$.

Extending the above argument to compact families is equivalent to proving the homotopy lifting property over compact sets for the following map

$$\mathcal{C}_q^{\pitchfork} \longrightarrow \mathrm{Gr}_2(T_qM) \setminus T_qC$$

that picks out the tangent plane at $q$. Here, $\mathrm{Gr}_2(T_qM)$ is the Grassmannian of all symplectic 2-planes in the tangent space $T_qM$. The construction above shows that this map is surjective, and since all the choices of parameters come from contractible spaces, the above construction can be made to depend continuously on a compact family of parameters.. Hence, this map is a fibration and the fibre over the orthogonal plane at $q$ is the space $\mathcal{C}_q^{\perp}$. It's not hard to see that the base is contractible: Since $\mathrm{Sp}(4)$ acts transitively on $\mathrm{Gr}_2(T_qM)$ with stabilizer $\mathrm{Sp}(2) \times \mathrm{Sp}(2)$, we can write $\mathrm{Gr}_2(T_qM)$ as a homogeneous space

$$\mathrm{Gr}_2(T_qM) \cong \mathrm{Sp}(4) / \mathrm{Sp}(2) \times \mathrm{Sp}(2).$$

But the latter space is homotopy equivalent to

$$\mathrm{U}(2) / \mathrm{U}(1) \times \mathrm{U}(1) \cong \mathrm{SU}(2) / \mathrm{U}(1) \cong \mathbb{C}P^1.$$

Therefore, if we remove a point from $\mathrm{Gr}_2(T_qM)$, it becomes contractible. We conclude that $\mathcal{C}_q^\perp$ is weakly equivalent to $\mathcal{C}_q^{\pitchfork}$.

## 6.3   Orbifold Restriction Maps are Fibrations

Suppose we have an embedded orbi ball $B(p_c) \subset \mathbb{C}P^2_{1,1,c}$ centred at the singular point $p_c \in \mathbb{C}P^2_{1,1,c}$. Let $\mathrm{Symp}^{\mathrm{red}}_{1,1,c}(B(p_c))$ be the subgroup of $\mathrm{Symp}^{\mathrm{red}}_{1,1,c}$ that leaves $B(p_c)$ invariant. The following holds, just as in the smooth case:

**Proposition 6.3.1.** *The restriction map*

$$\mathrm{Symp}^{\mathrm{red}}_{1,1,c}(B(p_c)) \longrightarrow \mathrm{Symp}^{\mathrm{red}}(B(p_c))$$

*is a locally trivial fibration.*

*Proof.* We will use Palais' result ([39]-Theorem A) and find local sections for the restriction map. We need to show that for any $f \in \mathrm{Symp}^{\mathrm{red}}(B(p_c))$, there is a neighbourhood $\mathcal{U}_f$ of $f$ and a local section $\sigma : \mathcal{U}_f \to \mathrm{Symp}^{\mathrm{red}}_{1,1,c}(B(p_c))$ such that $\sigma(u) \circ f = u$ for all $u \in \mathcal{U}_f$. In fact, it suffices to find local sections in a neighbourhood of $\mathrm{Id} \in \mathrm{Symp}^{\mathrm{red}}(B(p_c))$, since we can get to any other neighbourhood by conjugation ($\mathrm{Symp}^{\mathrm{red}}(B(p_c))$ being a topological group). The identity map $\mathrm{Id} \in \mathrm{Symp}^{\mathrm{red}}(B(p_c))$ has a local lift $\widetilde{I}_c$ (defined up to an action of $\mathbb{Z}_c$) that fits into the commutative equivariant diagram

$$
\begin{array}{ccc}
B_0 & \xrightarrow{\ \widetilde{I}_c\ } & B_0 \\
\downarrow & & \downarrow \\
B(p_c) & \xrightarrow{\ \mathrm{Id}\ } & B(p_c),
\end{array}
$$

where $B_0 \subset \widetilde{U}_c$ is a smooth ball centred at $0 \in \mathbb{C}^2$ in the uniformizing chart $\widetilde{U}_c$, and $B_0 / \mathbb{Z}_c \cong B(p_c)$. It's easy to see that $\widetilde{I}_c$ must be an element of the local $\mathbb{Z}_c$-action, so we have

$$\widetilde{I}_c \in \mathbb{Z}_c \subset \mathrm{Symp}^{\mathbb{Z}_c}(B_0).$$

Observe that the group $\mathrm{Symp}^{\mathbb{Z}_c}(B_0)$ is locally contractible because a neighbourhood of the identity is homeomorphic to a neighbourhood of the origin in the space of equivariant closed 1-forms (this follows from an equivariant version of Weinstein's Lagrangian neighbourhood theorem). Thus, there is a neighbourhood $\mathcal{U}_{I_c} \subset \mathrm{Symp}^{\mathbb{Z}_c}(B_0)$ of $\widetilde{I}_c$ that retracts onto it. If we fix a deformation retraction $r_t$, then for any $\widetilde{f}_c \in \mathrm{Symp}^{\mathbb{Z}_c}(B_0)$, $r_t$ defines a canonical (equivariant) path taking $\widetilde{f}_c$ to $\widetilde{I}_c$. This path is generated by a $\mathbb{Z}_c$-invariant Hamiltonian $H : B_0 \to \mathbb{R}$. Extend $H$ by a bump function that vanishes outside of a neighbourhood of $B_0$. The corresponding Hamiltonian isotopy $\widetilde{\phi}_t : \widetilde{U}_c \to \widetilde{U}_c$ is $\mathbb{Z}_c$-equivariant, supported in a neighbourhood of $B_0$, and its time 1 map restricts to $\widetilde{f}_c$ on $B_0$. Since $\widetilde{\phi}_1 : \widetilde{U}_c \to \widetilde{U}_c$ is equivariant, it descends to a symplectomorphism

$$\phi_1 : U_c \to U_c$$

that is supported in a neighbourhood of $B(p_c)$. Extend it by the identity (still calling it $\phi_1$) to get a global symplectomorphism preserving $B(p_c)$, i.e. $\phi_1 \in \mathrm{Symp}^{\mathrm{red}}_{1,1,c}(B(p_c))$. Note that $\widetilde{f}_c : B_0 \to B_0$ descends to a symplectomorphism $f \in \mathrm{Symp}^{\mathrm{red}}(B(p_c))$ and $\phi_1$ is an extension of $f$. Hence, the above construction produces a local section $\sigma : \mathcal{U}_{\mathrm{Id}} \to \mathrm{Symp}^{\mathrm{red}}_{1,1,c}(B(p_c))$ by defining $\sigma(f) := \phi_1$. $\qquad\square$

# Chapter 7

# Concluding Remarks

In this thesis, we've primarily been concerned with the weighted projective spaces $\mathbb{C}P^2_{1,b,c}$ and their reduced symplectomorphism groups $\mathrm{Symp}^{\mathrm{red}}_{1,b,c}$. From this, we were able to probe some embeddings spaces of balls into these orbifolds. This begs the question: What about the case $\mathrm{Symp}^{\mathrm{red}}_{a,b,c}$ when $a \neq 1$? Well, we expect it to be homotopy equivalent to $\mathrm{T}^2$. Initially, our opinion was that in order to probe the more general group $\mathrm{Symp}^{\mathrm{red}}_{a,b,c}$ we had to resolve all three singularities and then try to understand the subgroup of $\mathrm{Symp}(R_{a,b,c})$ acting as the identity near each configuration of curves resulting from the resolution process. This is a more difficult problem because:

(1) The complement of this configuration of curves is no longer a nice symplectically convex domain.

(2) More importantly though, understanding which exceptional curves in the full resolution are $J$-holomorphic for all tame $J$ poses a more difficult problem.

But it turns out that this may not be necessary. In fact, it should be sufficient to resolve only two of the singularities because then the complement of the resulting configuration in the resolution is a symplectically convex set that can be retracted into an orbi-ball. But we now know that compactly supported symplectomorphisms of the orbi-ball form a contractible space [21]. So it seems that this approach will work, but the details haven't been worked out yet.

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