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Modelling Credit Value Adjustment Using Defaultable Options Approach

Sidita Zhabjaku
The University of Western Ontario

Supervisor
Matt Davison
The University of Western Ontario

Graduate Program in Statistics and Actuarial Sciences

A thesis submitted in partial fulfillment of the requirements for the degree in Master of Science

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by

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The University of Western Ontario

London, Ontario, Canada

Abstract

This thesis calculates Credit Value Adjustment on defaultable options. The prices of defaultable European options are computed through analytical, quadrature approximation and Monte Carlo simulations under the assumption of a constant rate of default. Subsequently, we propose to inversely relate the company's instantaneous rate of default to its underlying stock price, resulting in a non-constant rate of default. This allows for a new approach to estimate the default of company different from previous work where default is calculated through historical data. The rationale behind this idea relies on the fact that price of the stock plunges before the event of default. For a given set of option parameters, we show that it is possible to find an optimal intensity, which produces the same prices of European options under a simpler framework. However, this intensity fluctuates with changes in other parameters. Implementation details and analysis of the results are provided.

Keywords: Defaultable Options, Credit Value Adjustment, Poisson Process

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Chapter 1

Introduction and Motivation

1.1 Introduction to Credit Value Adjustment

Since the recent financial crisis that started in 2007, there has been an increased interest in finding the root cause of that and other similar collapses. It became evident that new policies are needed to control similar future events (Gregory, 2010). An important area of interest in the financial industry today is counterparty credit risk. For instance, in 2008 Lehman Brothers reported a 4 billion dollar loss in the market, which were passed on to all other financial institutions and immediately affected their financial statements. Preparations for such a crisis were not in place, as it was simply not expected.

Counterparty credit risk refers to the risk of one party losing money due to the default of the other party to one or more agreements. Default in this context is the failure of a given company to meet a contractual obligation. This means that the issuer fails to make the payments that they have promised on their contracts. Default may be seen as a rare event when considering the large number of corporations that issue fixed income securities. However, all companies have a positive default probability that should not go unnoticed. Delianedis and Geske (1998) state that "default probability should change continuously with changes in the firm's stock price and thus its leverage". Although the above mentioned authors never put their theory to practice, we will test such an idea as one of our proposed models of this thesis.

Calculation of counterparty credit risk is visible in four specific areas of the finance world, as defined by Ruiz et al. (2013):

- **Pricing** - Counterparty credit risk needs to be considered when adjusting the pricing of financial instruments. This is where Credit Value Adjustment calculations become important, which can be defined as the difference between a default free portfolio and a defaultable portfolio. For pricing purposes it is needed to find the true value of an option contract providing that its counterparty has a positive chance of default.

- **Capital Calculation** - In this area financial institutions use two capital models. The first one is called economic capital, which is the bank's own calculation of capital needed to cover losses at a certain confidence level. The second model is called regulatory capital, which is a government driven capital requirement that a financial institution must respect to be considered stable.

- **Exposure Management** - A way to manage the credit risk for a financial institution is to set credit exposure limits. These limits are set by Potential Future Exposure (PFE) value at a required confidence level (Gregory, 2010). A counterparty might default well into the future and therefore credit risk metrics must also have a time component which extends to the maturity date of the underlying instrument. If the PFE for a counterparty reaches the set limit, such a breach is investigated and the trading desk will need to fix it by either unwinding trades or purchasing new offsetting contracts.

- **Initial Margin** - When two companies enter into a contract, one of them, usually the one with better credit, will demand some form of collateral from the other. Sometimes both demand collateral from one another. Collateral can be defined as an easily sold asset that is put aside in case one company does not fulfil its contract requirements. Such collateralisation is performed by derivative dealers or by central clearing houses. These assets are referred to as Initial Margin, which is calculated by finding the maximum exposure at a set confidence level.

Before the crisis, credit risk was already supervised by not allowing trades with larger default probabilities. After 2007 many large and apparently safe companies including Lehman Brothers, with high credit ratings, also went bankrupt (Gregory, 2010). Feng and Volkmer (2012) state that counterparty credit risk is inherited through trading and that the cost of such risk needs to be reported appropriately. Before the 2007 crisis, the measure of the effects one company's failure would have on the other were not sufficient. This is where Credit Value Adjustment (CVA) comes in. CVA is a way to measure the dollar value of the risk one is taking in the contract with a counterparty involved. By having a dollar value of such expectations one can be more prepared in case of a counterparty's default.

1.2 Background on CVA and its Change from Basel II to III

We will now go back in time and review how CVA appeared in the financial markets. Going back to 1998, after the Asian crises occurred, there was an increased interest in counterparty credit risk. From 1999 to 2007, many larger financial institutions started using CVA to assess counterparty credit risk, where CVA was calculated with accountant type assumptions but not taken very seriously (Gregory, 2010). It was then widely believed that companies with excellent credit ratings should be trusted. Not until Lehman Brothers fell did it became evident that even larger and stable firms may fail. Many policies were already in place before the crisis in regards to counterparty credit risk but were insufficient. It was reported that two thirds of the credit losses during the financial crisis were CVA losses (Ruiz, et al. 2013). Soon after 2007 many new ideas and regulator changes emerged. The most important of these changes were strongly related to counterparty risk.

To control the financial markets worldwide, the Basel Committee on Banking Supervision (BCBS) was established by the Group of Ten (G10) wealthy countries in 1974. This committee has no formal authority, but merely sets standards for the relevant countries which use such guidelines to shape their regulations accordingly (Gregory, 2011). To date, three Basel Accords have been published. In 1988, BCBS introduced Basel I, then in 1999 Basel II was published. Appendix A provides a list of the countries in the G10 and the countries involved in the Basel Committee today. Basel III has now been introduced and is being implemented over the 2011-2018 period, in hopes that it will correct the damage caused by the credit crisis. Basel II and III are the two that focus on counterparty credit risk (CCR). Basel III has imposed stronger regulations on CCR, especially on calculations of CVA. Basel III mentions Credit Value Adjustment in the proposals regarding capital section, which proposes new rules to increase capital requirements for different types of transactions (BCBS, 2011). CVA can be seen as the risk value that one might lose in an investment and therefore should increase capital to cover such losses.

Recent changes in accounting regulations have also made CVA more significant for corporate financial statements. As summarized by Stein and Lee (2010), accounting for counterparty risk is mandatory in regulations set by the Financial Accounting Standard Board (FASB, United States); in 2007 the regulations stated that:

- Risk averse market participants generally seek compensation for bearing the uncertainty inherent in the cash flows of an asset or liability (risk premium). A fair value measurement should include a risk premium reflecting the amount market participants would demand because of the risk (uncertainty) in the cash flows. (Financial Accounting Standards Board, 2007)

Since the 2007 FASB rules have progressed to focus on the accountability of credit risk. The following results were seen in the 2009 rule release:

- A fair value measurement should include a risk premium reflecting the amount market participants would demand because of the risk (uncertainty) in the cash flows. Otherwise, the measurement does not faithfully represent fair value. In some cases, determining the appropriate risk premium might be difficult. However, the degree of difficulty alone is not sufficient basis on which to exclude a risk adjustment. (Financial Accounting Standards Board 2009)

The need for new regulations became apparent during the financial crises as also seen from the difference in accounting regulations that were stated above from pre to post crises. The motivation for this thesis lies in the emphasized regulations of CVA and its calculations that are in demand within the financial industry. The following chapters will provide a unique approach to evaluating CVA and its impact on option pricing.

In credit risk, the contract payoff and the counterparty's credit worthiness are interconnected, and making the assumption of independence could drive a strong miscalculation of credit risk. Two types of credit risk generally revealed in the market were first introduced by Levy (1999). There are right-way and wrong-way risks, which we will call directional risks.

Iacono (2011) states that wrong-way risk occurs when one parties' exposure to another counterparty is highly correlated with the counterparty's credit quality. We can see this in the value of a derivative contract which increases or decreases depending on the creditworthiness of the counterparty. On the other hand, right way risk as defined by Gregory (2011) is a beneficial relationship, meaning that the exposure to the counterparty will reduce risk.

To be more specific, wrong way risk occurs when exposure increases with a rise in default probability of the counterparty. In contrast, right-way risk defines the opposite situation. Right-way risk exists in any swap contracts between two counterparties which are similar. If one company has wrong-way risk then the other will have right-way risk (Gregory, 2010). It is evident that one type of risk always appears in most transactions and therefore very important to quantify (Pykhtin, 2011). We will refer to these two types of risks as directional way risks. One simple example of how wrong-way risk appears in the market is when a company has issued put options on their own stock. As the companies default risk increases their stock price will fall, increasing the value of the put option they have issued. However, the ability of the counterparty to realize this value also decreases as the company might be going bankrupt in the near future.

Many problems arise with wrong/right way risk. But even though they are difficult to model, some type of awareness on these issues must arise since they are an important factor of credit risk. As stated by Stein and Lee (2012), the biggest problem with wrong-way risk is finding the right calculations and hedging the correlation. Correlation inputs can only be found by looking at historical data, which is not always a good indication of what the future correlations will be. Another issue with wrong-way risk is in estimating the recovery rate. Recovery rate can be defined as the percentage of money you will get back from the contract in case the other party defaults. Although recovery rates are needed for calculations, they aren't actually known until after the company has gone bankrupt.

Wrong way risk is the more common directional way risk in the market, and right-way risk is normally ignored. It is very important to realize that both do exist in the market. Directional-way risk appears in four different types of assets as defined by Ruiz et al. (2013):

- **Equity** - Wrong way risk is represented through for example a put option written on the stock of the issuer, where the value will increase with the probability of default. Similarly right way risk is represented by the corresponding call option where its value would be approaching zero when the probability of default increases.

- **Foreign Exchange** - The following example illustrates the risks that appear in this asset class. Suppose a company enters a cross currency agreement with an institution in an emerging market. This contract delivers the mature economy's currency and pays the developing economy's currency. When the counterparty default probability rises, its currency will be downgraded which would result in the value of the transaction for the solid institution to increase. As a result wrong-way risk appears through foreign exchange agreements.

- **Commodities** - In commodities we will see directional-risk. As an example of right-way risk consider an oil producing company. If the oil producer was to short oil futures, on the other side the dealer will be longing these futures. If we had a collapse in the price of oil, the producer's default chances will increase, but on the dealers side of counterparty credit risk this is a good thing.

- **Credit** - This type of asset will involve wrong-way risk. Let's say we buy protection on company A from company B, that has similar business and economic path as A. Therefore, if counterparty B has an increase in default rate the contract will be in-the-money and the potential loss will be very high.

Another classification of risk is to look at general and specific directional risk. A transaction has specific directional risk when transaction and default dependency is absolute. The classic example here is buying a put option from the company of the underlying stock. On the other hand, general directional risk is when this dependency comes from economic factors. An example of this general case can be seen when one buys a put option from a company whose underlying stock price is only correlated to the companies stock. Therefore, default might drive the option price up or down (Ruiz et al.,2013).

In this thesis we will be analyzing Credit Value Adjustment by computing the price of defaultable European call and put options. Among the above classified asset classes this would fall under equity. Our options' calculations will have specific directional risk. This thesis will include wrong-way risk for the put and right-way risk for the call option. These defaultable options will be modelled through two methods. The first case will consist of a constant default value that will drive the price of the option over time. The second, more complicated case, will be examined at a non-constant default arrival rate, which is dependent on the underlying stock price time series. First we look at analytic calculations of pricing a defaultable option, then we move on to simulation techniques and results. We will analyze the the constant and non-constant default rate methods that we have suggested in pricing a defaultable option and compare their results.

1.3 Thesis at a Glance

This thesis will involve analysis of CVA using different methods. We specialize our discussion to the CVA of options. In Chapter 2 we provide a background review supporting the path that will follow for the rest of the thesis. We discuss CVA by presenting its definitions and analyzing its common calculation techniques, based on work of other researchers in this area. We describe the types of calculations used for CVA in the literature and we also define structural models and reduced form models. The novel contributions that this thesis makes are focused in Chapters 3 and 4.

The CVA throughout this thesis will be calculated as the difference between a non-defaultable and a defaultable option price. The risk free option is priced with an otherwise similar Black-Scholes value. Hence, the main focus will be to price defaultable options through the help of Poisson processes. Initially in Chapter 3 we focus on calculating defaultable European options through a simple method of a constant default rate accompanied by a homogeneous Poisson process. We evaluate these defaultable option prices using three methods:

1. Monte Carlo Simulations
2. Analytical Approximation
3. Quadrature Approximation

These three methods are analyzed in detail by comparing error and computational performance. The analytical approximation was performed through Taylor series and its results are very similar to the Monte Carlo simulations with only minor discrepancies. On the other hand, quadrature approximation gave the best results where the prices were exactly the same as given by the simulations but with quicker computational time.

In Chapter 4, we present two main ideas:

1. A new default model is proposed where we have a non-constant default rate inversely proportional to the underlying stock price.

2. Non-constant default rates are adjusted to represent a similar constant rate.

The first idea defined above is based on the knowledge that if a company's stock price decreases their profits appear as weaker and their chances of bankruptcy are higher. Therefore, the non-constant default rate is used to represent the intuition that a company's default should be portrayed by how strong their finances are, in this case we used its stock value as an indicator. The prices of a defaultable options in this scenario are evaluated through Monte Carlo simulations. We also found a method of transforming the non-constant default rate to a constant rate, which gave similar results. The reason for such conversion is that one could compare the constant and non-constant default models centred around a similar hazard rate. The results will show that the constant and non-constant default rate models will be similar to one another at lower volatilities, but further apart at higher volatility rates of the the stock price. In Chapter 5 the thesis is brought together by explaining results through a wider lens and discussing their financial modelling implications.

Chapter 2

Background Review

2.1 Counterparty Credit Risk

The importance of evaluating counterparty risk in over the counter products has become increasingly recognized, especially with the new Basel III requirements as mentioned in Chapter 1. Over the counter (OTC) products are traded between two or more companies without any regulatory supervision or control. Once these transactions became popular it was hard to keep track of the risk involved in each and every one of them. During the 2007 market crash it was evident that if there had been a higher control over these OTC products then the crash impact might have been less staggering to the financial markets. It is crucial for a company to report in their books a dollar value of how much they are risking while getting involved in OTC products. This is how the idea of evaluation credit risk arises.

2.2 CVA Equation

In Chapter 1 we defined and explained CVA by examining its history, background and related literature; now we need to explore its quantitative side. A way to evaluate the counterparty credit risk in an OTC product is called a Credit Value Adjustment. The CVA formula calculates the difference between a contract with a perfectly default free counterparty and that of a similar contract with a defaultable counterparty (Hoffman, 2011). This difference is the CVA; in plain terms, the value that we are risking in our portfolio if a default to the counterparty were to happen. As defined in Kjaer (2011) the CVA formula for a portfolio is represented by:

$$V(t) = \hat{V}(t) + \psi(t). \tag{2.1}$$

Here $V(t)$ is the value of the portfolio assuming that no defaults occur during the maturity of the issued option and $\hat{V}(t)$ is the value of the portfolio if the party can default. CVA is represented in the above formula as $\psi(t)$. Therefore, to calculate the value of a portfolio where there is a risk of default, $\hat{V}(t)$, one needs to compute $V(t) - \psi(t)$.

The two most frequent methods to calculate CVAs are called unilateral and bilateral approaches. For unilateral CVA we only consider the default of one counterparty, but in bilateral CVA we consider the fact that both parties involved in the contract can default, issuer and buyer alike. This means that when a company is computing the value of a contract they need to consider both their own default rate and that of the counterparty. To understand the CVA formula we first assume we have entered a contract where we are the issuer B and our counterparty is C, the purchaser. The default times of the two entities are denoted by τ_B and τ_C respectively. R is the recovery rate and its subscript corresponds to the recovery rate of each party involved in the contract. Derivations of the CVA formula make two of the following assumptions as defined by Kjaer (2011):

1. The company that defaults gets its money back from the surviving company if $V(\tau_B) < 0$.

2. The company that defaults will pay back as much as it can to the surviving company when $V(\tau_C) > 0$, this is where the recovery rate comes in.

Under the above assumptions, let $V^{+}(t) = \max(V(t), 0)$ and $V^{-}(t) = \min(V(t), 0)$. Also let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathcal{P})$ be a filtered probability space, where $\mathcal{F}_t = \sigma(\mathcal{G}_t \cup \mathcal{H}_t)$. $\mathcal{G}_t$ represents the market information up to time t and $\mathcal{H}_t$ holds the information about default times that we know up to time t. $\hat{N}(t)$ is the numéraire and $\frac{\hat{N}(t)}{\hat{N}(\tau)}$ adjusts for the distribution of time and therefore, it scales calculations of CVA to the random default time τ. The bilateral CVA formula is then given by (Kjaer, 2011):

$$\psi(t) = (1 - R_B)\mathbb{E}[V^{-}(\tau)\mathcal{I}_{\tau=\tau_B,\tau\leq T}\frac{\hat{N}(t)}{\hat{N}(\tau)}|\mathcal{F}_t] + (1 - R_C)\mathbb{E}[V^{+}(\tau)\mathcal{I}_{\tau=\tau_C,\tau\leq T}\frac{\hat{N}(t)}{\hat{N}(\tau)}|\mathcal{F}_t] \tag{2.2}$$

In equation (2.2) the issuer, party B, only considers the case when they owe money to company C, i.e. $V(\tau) < 0$, and therefore we are interested in $V^{-}(\tau)$. Meanwhile company C, if defaulting, needs to pay back $V^{+}(\tau)$. If we only consider the default of company C and not the issuer B, then we have unilateral CVA, given by:

$$\phi(t) = (1 - R_C)\mathbb{E}[V^{+}(\tau)\mathcal{I}_{\tau_C \leq T}\frac{\hat{N}(t)}{\hat{N}(\tau)}|\mathcal{F}_t]. \tag{2.3}$$

Before we proceed we must explain the CVA equation in detail. The CVAs given in equations (2.2), and (2.3) are in general terms and as stated by Kjaer (2011), they hold for any credit and market models. To make the CVA equations more specific, we need to define the distributions of the default times and $V(t)$. As a consequence, interest in our own default is not as high. In this thesis we will specify distributions of τ_C, which we can replace with just τ since we will only be interested in unilateral CVA and therefore only one default time of the counterparty is important to us.

Bilateral CVA given in equation (2.2) can get difficult to approximate and usually it is derived by taking the difference between the investor's CVA to the counterparty and vice versa. This approximation can have high errors because it does not take into account the relationship between the two parties. Investors favour bilateral CVA since it reduces CVA charges. This arises from the apparent paradox that if the investor's default rate increases more than the counterparties it can drive the option price up. Another problem with bilateral CVA calculations is that the price of a derivative needs to be associated with a hedge but with bilateral the investors cannot hedge their own default risk (Stein & Lee, 2011). Although bilateral CVA is a valuable concept, because of the above mentioned problems it seems more intuitive to use unilateral CVA. Therefore, in this paper we will focus on unilateral CVA as we assume that we are the other party in the contract and we evaluate CVA for us.

The τ in equation (2.3) is the default time which is only visible if $\tau < T$, the exercise time. So when τ appears before time to maturity then the indicator function $\mathcal{I}_{\tau_C \leq T}$ will jump to 1 and equation (2.3) will be calculated but when τ falls outside of the time to maturity, the indicator function will return a 0. If τ falls within the exercise period then we evaluate our option value at that certain τ with a recovery rate added to it. If τ does not fall within T we simply use the

Black-Scholes equation to evaluate our options price at T. This will be further explained in Chapter 3.

2.3 CVA Calculations in the Literature

As shown in the previous section the CVA is represented by the difference in the market value of a portfolio and the value of portfolio when accounting for counterparty risk. Much of the literature in this area mostly look at pricing CVA not through the difference of these two portfolios but through the CVA formula (Gregory, 2010). Let us look at a simple and practical way to calculate the portfolio CVA. In this simple CVA formula we will ignore wrong way risk for now and assume that we have independence between default probability and exposure. As given by Gregory (2010) the unilateral CVA formula without the wrong-way risk can be simply calculated through:

$$CVA \approx (1-\delta)\sum_{j=1}^{m} B(t_j)EE(t_j)q(t_{j-1},t_j). \tag{2.4}$$

The expressions in the above formula are defined as (Gregory, 2010):

- Loss given default is the expression $(1-\delta)$, which is the fraction of loss in case of default, and δ is the recovery rate of that counterparty.
- The discount factor is the expression $B(t_j)$, which gives the the risk free discount factor at the given time. This is important since any loss expected in the future should be discounted back to today's time.
- Expected exposure, denoted by EE, which is the expected exposure: the amount put at risk if the counterparty defaults.
- Default Probability, given by the expression $q(t_{j-i},t_j)$ is the marginal default probability between times t_{j-i} and t_j.

Since equation (2.3) given in Section 2.2 was a general CVA formula we can show that equation (2.4) can be derived the same way. Equation (2.3) can be rewritten as:

$$= (1-R_C)\mathbb{E}\left[\mathbb{E}[V^+(\tau)\mathcal{I}_{\tau_C\leq T}\frac{\hat{N}(t)}{\hat{N}(\tau)}|\mathcal{F}_t]\right] \tag{2.5}$$

$$= (1 - R_C) \int_t^T \mathbb{E}[V^+(u) \frac{\hat{N}(t)}{\hat{N}(u)} | \mathcal{F}_t] f_\tau(u) du, \tag{2.6}$$

where

$$\mathbb{E}[V^+(u) \frac{\hat{N}(t)}{\hat{N}(u)} | \mathcal{F}_t] \text{ is the expected exposure,}$$

and

$$f_\tau(u) \text{ is the density of default time } \tau.$$

Therefore, we can see that the general CVA equation can be transformed easily to the no wrong way risk CVA formula in (2.4). If now we decide to bring in wrong-way risk, things do get a lot more complicated. Just by intuition one can see that wrong-way risk will actually increase the value of the CVA being calculated. There are two ways of calculating CVA with the presence of wrong-way risk as defined by Gregory (2010):

- The first approach is to calculate the exposure and default of a counterparty together and then find the economic relationship between the two. This is the right approach but very hard to compute.

- The second approach uses the simpler formula (2.4) that does not account for wrong way risk but there would be adjustment made to the exposure or default probability so it goes upwards to show the presence of wrong-way risk. In his book Counterparty Credit Risk, Gregory (2010) states that, "wrong-way risk may be subtle and not revealed via any historical data analysis. It may be a result of a casualty: a-cause-and-effect-type relationship between two events." This means that correlation is not the same as finding the dependence when calculating CVA.

To further define Expected Exposure (EE) calculated in the industry, we will start by giving the exposure of a simple portfolio under the normal distribution assumption. Let V represent the value of such portfolio which is given by:

$$V = \mu + \sigma Z, \tag{2.7}$$

where Z is a standard normal random variable. Now the exposure of such portfolio is given by (Gregory, 2010):

$$E = \max(V, 0) = \max(\mu + \sigma Z, 0) \tag{2.8}$$

Now the EE represents the expected value of only positive Mark-to-Market value in the future. The formula for expected exposure is given by:

$$EE = \int_{-\frac{\mu}{\sigma}}^{\infty} (\mu + \sigma x)\varphi(x)dx = \mu N\left(\frac{\mu}{\sigma}\right) + \sigma\varphi\left(\frac{\mu}{\sigma}\right), \tag{2.9}$$

where $\varphi(.)$ represents the normal density function and $N(.)$ represents the cumulative normal distribution function. Expected exposure depends on the mean and standard deviation and as they increase so will EE (Gregory, 2010).

The next expression that was used in the general formula of CVA, equation (2.4) is the default probability, $q(t_{j-i}, t_j)$. In general terms to define a default probability one has to use survival probability $S(u)$, which represents the probability of no default up to a certain time u. Similarly one can represent the cumulative probability of default up to time u, conditioned on the fact that there were no defaults up to the current time. We then can construct the marginal probability of default (Gregory, 2010):

$$q(u_1, u_2) = S(u_1) - S(u_2) = F(u_2) - F(u_1) \qquad u_1 \leq u_2. \tag{2.10}$$

Calculation of the likelihood of default can be determined through historical data. Using default information about the past we can predict defaults for the future. These type of calculations can be easily correlated to the credit rating of a company as given by firms such as Moody's or Standard & Poor's. If a company is Triple-A rated then their default probability is low as expected but does increase depending on the time of maturity their products hold. In particular, a Triple-A company as rated by Moody's has 0% chance of default within one year but 0.52% chance of default within ten year (Gregory, 2010).

2.4 Black-Scholes Equation

Since we will be analyzing defaultable European options we must first understand the pricing of such options in a riskless world. The Black-Scholes equation, introduced by Black and Scholes (1973), is a method of pricing options. To build the Black-Scholes equation first a few assumptions need to be stated (Andricopoulos, 2002, Hull, 2008):

- The underlying stock price follows a geometric Brownian process with constant drift and volatility.
- The risk-free interest rate is constant both over time and across maturity.
- There are no opportunities for arbitrage.
- Both short selling and purchasing of the underlying stock is permitted and can be done continuously without commissions.

The calculation of a Black-Scholes equation begins with a random variable V as a function of S, stock price and t, time. We then take Π to represent a portfolio which consists of taking a long position in an option and shorting a Δ position of the underlying, this gives us:

$$\Pi = V - \Delta S. \tag{2.11}$$

The change in the value of the portfolio is given by:

$$d\Pi = dV - \Delta dS. \tag{2.12}$$

A stock price model dS, under real world measure, can be written as:

$$dS = \mu S\, dt + \sigma S\, dX, \tag{2.13}$$

where dX is a random variable taken from a normal distribution with mean zero and variance dt, this is called a Wiener process (Merton, 1990). In Black-Scholes we assume the underlying stock follows a log-normal random walk. We can calculate dV by applying a Taylor Series expansion. In this expansion we will ignore all higher terms that go beyond the second term, since the values will be small and so negligible. We then use the result that

$$\Delta X^2 \Rightarrow \Delta t \;\; as \quad \Delta t \Rightarrow 0. \tag{2.14}$$

Now we will use a Taylor expansion for ΔV which holds two uncorrelated variables S and t:

$$\Delta V = \frac{\partial V}{\partial S}\Delta S + \frac{\partial V}{\partial t}\Delta t + \frac{\partial^2 V}{\partial S\,\partial t}\Delta S\,\Delta t + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\Delta S^2 + \frac{1}{2}\frac{\partial^2 V}{\partial t^2}\Delta t^2 + ... \tag{2.15}$$

As we previously stated above, the stock process in (2.13) can now be inputed in ΔS of (1.3.6). As Δt tends to zero, $(\Delta S)^2$ tends to $S^2\sigma^2 dt$ and ignoring all terms with higher order than Δt, following results are obvious:

$$dV = \frac{\partial V}{\partial S}dS + \frac{\partial V}{\partial t}dt + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}dt. \tag{2.16}$$

Then,

$$d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}\right)dt + \left(\frac{\partial V}{\partial S} - \Delta\right)dS. \tag{2.17}$$

Now if we observe equation (2.17) we can see that there are two types of terms, one under dt which is deterministic and the other one under dS which is not deterministic. If we choose $\Delta = \frac{\partial V}{\partial S}$ then the dS term disappears and removes any ties to the drift term μ. This process of eliminating risk is called delta hedging (Andicopoulos, 2002). This implies that:

$$d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}\right)dt. \tag{2.18}$$

Now note that as stated above, the change in Π is completely riskless due to the delta hedging we performed which results in the return of the portfolio being just the risk free rate interest, r. This gives:

$$d\Pi = r\Pi dt. \tag{2.19}$$

The Black-Scholes equation is then obtained by combining equations (2.11), (2.18) and (2.19) yielding the following results:

$$d\Pi = \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0. \tag{2.20}$$

Equation (2.20) is a second order linear partial differential equation which can be solved for an analytical solution of a European option. It turns out that the solution of this is equivalent to pricing an option using:

$$\mathbb{E}^Q[e^{-r(T-t)}(S_T - K)^+], \tag{2.21}$$

where Q represents a risk neutral measure, μ becomes r explaining the absence of μ in (2.20). Hence, we will be able to use Monte Carlo methods in our subsequent results.

As first derived by Black and Scholes (1973), the price of a call option at time t, notice we now can replace the V the random variable, with C representing a call, with strike price K and spot price S is:

$$C(S,t) = SN(d_1) - Ke^{-r(T-t)}N(d_2), \tag{2.22}$$

where

$$d_1 = \frac{\ln(\frac{S}{K}) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{(T-t)}}, \tag{2.23}$$

and

$$d_2 = d_1 - \sigma\sqrt{(T-t)}. \tag{2.24}$$

The put option with the same parameters is given by:

$$P(S,t) = Ke^{-r(T-t)}N(-d_2) - SN(-d_1). \tag{2.25}$$

In the above put and call pricing formulas, K is the strike price at maturity and S is the spot price of the underlying asset. $N(.)$ is the cumulative distribution function of a standard normal. r is the risk free rate and σ is the volatility of the return. The above equation represents a portfolio which is perfectly hedged, and μ, the expected rate of return on the stock price, has disappeared from our calculations.

In this thesis we will use Black-Scholes equation to evaluate the option price with and without default. This way we can compare the staggering effects that a probability of default can cause to to the option price. In the further works of this thesis we will bring together two methods of pricing an option that is defaultable. These defaultable options will be price using a Black-Scholes equation, however the simple formula above is no longer sufficient; we also have to include a default probability into the equations.

2.5 Structural Models

Structural models look at economic information about the defaultable company. They model the likelihood of default by including parameters like the company's debt and equity ratio, the price of their stock and corporate balance sheets. The default models are seen as a company's financial capability to repay its debt. We will define two well known structural models below, and compare their differences.

2.5.1 Merton Model

In 1974 Merton introduced a new structural model. He priced defaultable options by considering the stock price in a company as a call option on the assets with the exercise price as the value of the debt obligations. The stock at time T can be represented by:

$$S_T = \max(0, V_T - M), \tag{2.26}$$

this condition looks the same as a call option. Here M is the discounted debt face value which matures at T. Therefore, the stock of a company is priced as a call option which behaves the same as Black-Scholes pricing option. This model states that if the stock of a company falls below a threshold then the value of the company will be \$0. Similarly in a call options if the stock price goes below the strike price then the value of the call is \$0. The current stock price given the boundary condition is then represented by the known Black-Scholes call option formula which we derived in equation (2.22). Therefore, the current stock price of the company would be given by the Black-Scholes formula if the following parameters are replaced:

S = current stock price; would replace $C(S, t)$ in equation (2.22).

V = current firm value; would be replacing the stock price S in equation (2.22).

M = debt at face value; strike price K in equation (2.22).

r_F = risk-free interest rate, this stays the same in both equations.

σ_v = the variance on the return of the company's assets; would replace σ, volatility of the stock price in equation (2.22).

t = current time.

T = debt maturity date; would replace the time to maturity of the contract in a call option.

$N(.)$ = cumulative normal distribution function.

In the Merton model the risk-neutral probability of default (RNPD_M) on the debt obligation is given by:

$$RNPD_M = 1 - N(\hat{d}_2), \tag{2.27}$$

where $\hat{d}_2$ is the given in equation (2.24) but in this model different parameters are used which were provided above. The firm value diffusion in Merton's model is given by:

$$dV_t = \mu V_t dt + \sigma V_t dW_t, \tag{2.28}$$

where $V(0) = V_0$ and W is a Brownian Motion. Also μ is the mean rate return and σ is the volatility of the company's assets. Note that the asset process does not have the same mean and variance as the stock process. In this model default can only occur at time T, and it is triggered if the company's debt is higher than its current value V_T (Hoffman, 2011).

2.5.2 Black Cox Model

The Black Cox model introduced by Black and Cox (1976), on the other hand, defines the default time to be when the firm's value hits a certain threshold. This threshold may be some type of legal safety that forces the company into early default. In the Black Cox (BC) model, the company has to pay their debt when V_t hits a predefined, time dependent barrier H_t. The default time is then given by:

$$\tau = \inf\{t > 0 : V_t < H_t\}. \tag{2.29}$$

Now if M, the face value of debt was less then H_t then the bond holder will always be covered which is not realistic in time of default. In other words, if we make $H_T \leq M$ then we have a more consistent definition of default under the BC model. One choice for the safety barrier is to take an increasing time dependent model as for example, $H_t = K_0 e^{kt}$. This model is already more realistic than Merton's model (Hoffman, 2011).

2.6 Reduced Form Models

The modelling of credit events can be divided into two main approaches: structural and reduced form approaches (Jeanblanc & Le Cam, 2007). Reduced form models are the type of models that involve a hazard rate. These models are interconnected with the structural models described in Section 2.5. Those structural models assumed that the default rate is dependent on some financial parameters of the company. However, reduced form models assume that the default event is a surprise, with an unpredictable stopping time. Reduced form models focus more on the conditional probability of a default occurring.

One assumption that reduced form models make is that the occurrence of a default at time t has no impact on the future events (Jeanblanc & Le Cam, 2007). Simple examples of such models are homogeneous and non-homogeneous Poisson processes, which are further explained in Chapters 3 and 4, respectively. In this thesis we use a hybrid of reduced form models with the ideas taken from the structural models. The theory behind Merton and Black-Cox was that default risk should increase as the value of the company's shares decreases. We use this idea and apply it to the reduced form model in Chapter 4.

2.7 Conclusion

In this chapter we reviewed Credit Value Adjustment. First, we introduced the simple idea behind calculating CVA in two methods, bilateral and unilateral. We then followed by describing how CVA is calculated in the literature and the main problems with them in the industry. To build the background for the rest of this thesis we reviewed the details of the Black-Scholes option pricing methodology. We also defined structural models and reduced form models of credit risk. In these models we saw that the company's economic state is important in determining the default probability. This theory will be followed in the thesis when we relate the stock price to the default rate of that given company. In this chapter we build a base for the following analysis that will be done throughout the following chapter.

Chapter 3

Detailed Analysis of Constant Default Rate Case

3.1 Introduction to Constant Default Arrivals

In this section we will analyze the changes in option prices based on the default probability of that option. We adopt two related, but distinct modelling approaches.

1. The first method will use a constant default rate, meaning the default probability does not change during the time of the option.

2. The second method will use a non-constant default rate where the probability of default will rise or fall during the time that the certain position is held (shown in Chapter 4).

In this thesis we have decided to calculate the price of a defaultable option and not the payoff. Although the payoff would also be interesting to analyze, since that is what many companies will consider to be the end value of their contract. We decided not to further pursue this alternative since looking at the defaultable payoff of a contract can get very complicated. Think of this scenario, company A owns a call option from company B, with maturity date in one year. After three months the stock is out of the money and company B goes bankrupt, but still can pay a recovery rate to company A. If we only look at payoff then since the call option was out of the money, company A gets nothing, but if we looked at price of the option company A is still able to recover some of its money back. Results would be frequent zero payoff for options which retain considerable time value to the detriment of the option holder. Therefore, in this thesis we decided to look at option prices and not their payoffs.

To further explain what we mean by default rate we will need to go into detail on how the probability of default works and its relation to the default arrival times, τ. First we consider a Poisson Process that maps out random arrival times. We are interested in only one arrival time since intuitively if our counterparty defaults once, that's it; a default in the market cannot happen more than once to the same company. The Poisson process gives us the number of arrival times within one period. There are two types of Poisson processes, one is homogeneous, meaning the default arrival rate, is constant. The other, with a non-constant default rate is called inhomogeneous. We will further explain inhomogeneous Poisson processes in Chapter 4 where we examine non-constant arrival rates.

3.2 Homogeneous Poisson Processes

A homogeneous Poisson process is characterized by a hazard rate λ. The number of events in a time interval $(t, t+h)$ has a Poisson distribution with parameter λh. The definition of a Poisson process as given by Mikosch (1998) is as follows. A continuous-time stochastic process $(\mathcal{N}(t) : t \geq 0)$ is a Poisson Process with rate λ if it satisfies the following conditions:

1. $\mathcal{N}(0) = 0$.
2. $\mathcal{N}(t)$ represents the number of events that happened up to time t.
3. The times between events are independent and identically distributed with an Exponential distribution of rate λ.

Therefore, the probability of k events happening within the given time interval above is:

$$P[\mathcal{N}(t+\tau) - \mathcal{N}(t) = k|\mathcal{F}_t] = \frac{e^{-\lambda\tau}(\lambda\tau)^k}{k!}. \tag{3.1}$$

Now we will explain the last property of the Poisson process in more detail. Since we will be using it in the later formulas on option pricing, it is crucial to understand its derivation from the homogeneous Poisson process. Consider a Poisson process $N(t)$ and let T_k be the time of the k^{th} arrival. As stated for example by Tankov (2005) the number of arrivals before t is less than k only if the time until the kth arrival is bigger than t. Therefore in symbols we have:

$$P(T_k > t) = P(\mathcal{N}(t) < k). \tag{3.2}$$

This is a general case, we need to consider waiting time until the first arrival, since we are only interested in the first default that will happen. Therefore the distribution of the first arrival time T_1 is:

$$\begin{aligned}
P(T_1 > t) = P[(\mathcal{N}(t) - \mathcal{N}(0)) = 0] &= \frac{e^{-\lambda t}(\lambda t)^0}{0!} \\
P(T_1 \leq t) &= 1 - P(T_1 > t) \\
P(T_1 \leq t) &= 1 - \frac{e^{-\lambda t}(\lambda t)^0}{0!} \\
P(T_1 \leq t) &= 1 - e^{-\lambda t} \\
\frac{d}{dt}P(T_1 \leq t) &= \lambda e^{-\lambda t}.
\end{aligned} \tag{3.3}$$

The above formula results in the exponential distribution function. We will use this distribution to evaluate the arrival time of the first default, τ.

3.3 Monte Carlo Simulations

We will now use a Monte Carlo method to approximate the option value for a vanilla put and call. This Monte Carlo method will also incorporate a default function. We have done this procedure by simulating random τ values through an inverse continuous distribution function of an exponential with a hazard rate λ as defined in Section 3.2. Once we have used a Monte Carlo method for such a process we then compare the results with the true Black-Scholes value of an option. The smaller this hazard rate the less chance of a default we have. We can interpret this theory intuitively by looking at the expectation function of an exponential distribution which is $\frac{1}{\lambda}$. As one can see when default rate increases our mean arrival rates of τ decrease, giving a higher probability of default happening within the option period.

In this section we have used a range of constant default rate values. We will be using

European call and put options. European options can only be exercised at maturity date which is pre-determined at the start of the contract. In the Monte Carlo simulations we have evaluated an average defaultable option price by using the part of CVA formula defined in Chapter 2:

$$V(t) = \hat{V}(t) + \phi, \tag{3.4}$$

$\hat{V}(t)$ was the price of a defaultable option, which we will estimate assuming a risk neutral world and $V(t)$ was the riskless option price and ϕ represented CVA which was given by:

$$\phi = (1 - R_C)\mathbb{E}[V^+(\tau)1_{\tau_C \leq T}\frac{\hat{N}(t)}{\hat{N}(\tau)}|\mathcal{F}_t]. \tag{3.5}$$

CVA is the value we might lose in the event of default. In the simulations we will be calculating the price of a defaultable options, where CVA can then be easily extracted. We have discretized the time to maturity of the option and then used Monte Carlo methods to simulate τ. As mentioned above the arrival of the first default time is exponentially distributed, so our default time τ can be simulated through the inverse cumulative distribution function (CDF) of an exponential:

$$P(\tau \leq T) = 1 - e^{-\lambda\tau}. \tag{3.6}$$

The inverse CDF used to simulate the default time τ is given by:

$$\tau = -\frac{1}{\lambda}\ln(U_{[0,1]}), \tag{3.7}$$

where λ is already a given parameter in our simulations that we specify and $U_{[0,1]}$ is the random variable simulated from uniform distribution on an interval from 0 to 1. Once τ is simulated we then determine if it falls in the time before our time to maturity. In that event the code stops and evaluates the Black-Scholes equation with a τ time to maturity instead of the regular T, time to maturity, it also multiplies this result with R, the recovery rate that we can expect at default. When the simulated default time does not fall within the specific time frame of the option then the simulations continue to the next time step in the discretization path and it simulates a new τ. If the default time τ never falls within the option maturity time period then the final result will just be the non defaultable option price, which is given by solving:

$$C(T, S_T) = e^{-rT}\mathbb{E}[S_T - K]^+, \tag{3.8}$$

for a call option and

$$P(T, S_T) = e^{-rT}\mathbb{E}[K - S_T]^+, \tag{3.9}$$

for a put option, where the stock path is simulated based on a Geometric Brownian Motion model (GBM), which is calculated at every discretized point given that default did not occur at that period. The GBM model we use is:

$$dS_t = rS_t dt + \sigma S_t dW_t \tag{3.10}$$

and

$$S_{k+1} = S_k e^{(r-\frac{1}{2}\sigma^2)\Delta t + \sigma\sqrt{\Delta t}Z_t} \tag{3.11}$$

where Δt is $t_{k+1} - t_k$ and Z_t is a standard normal random variable, Z_t with mean 0 and standard deviation 1.

Continuing on to the simulations of our defaultable option price, in every time step that we recalculate τ, the exponential CDF inverse that is used does not change. Although intuitively one would think it would through conditional probability, since we know that we survived at the previous point. But the exponential distribution, see for example Tankov (2005), has a memoryless property:

$$P(T > s + \tau | T > s) = P(T > \tau). \tag{3.12}$$

Therefore, in the Homogeneous Poisson process we do not have to worry about what happens previously since our distribution is guaranteed to be memoryless. The default time τ is driven by the hazard rate λ. We expect that as the default rate λ increases the defaultable option value decreases. Returning to our simulations we replicate our function 10000 times and take the expectation value to get the best estimate for the option price based on a default probability. As seen in Figure 3.1 the error bars of the Monte Carlo simulation are very small and smooth, therefore we can see that the number of simulations we used are sufficient for pricing these options. We have simulated these results for a put and call option with different values of hazard rates. There will be a selected range of λ's throughout the results of this thesis. This will easily depict how the defaultable option behaves at such default values. The parameters used are $S_0 = 35$, $K = 35$, $r = 0\%$, $R = 70\%$ and $\sigma = 18\%$. These parameters were picked because we wanted to model a stable looking company that is selling its options at-the-money. Figure 3.1 shows the result of a defaultable call and put option in comparison to the Black-

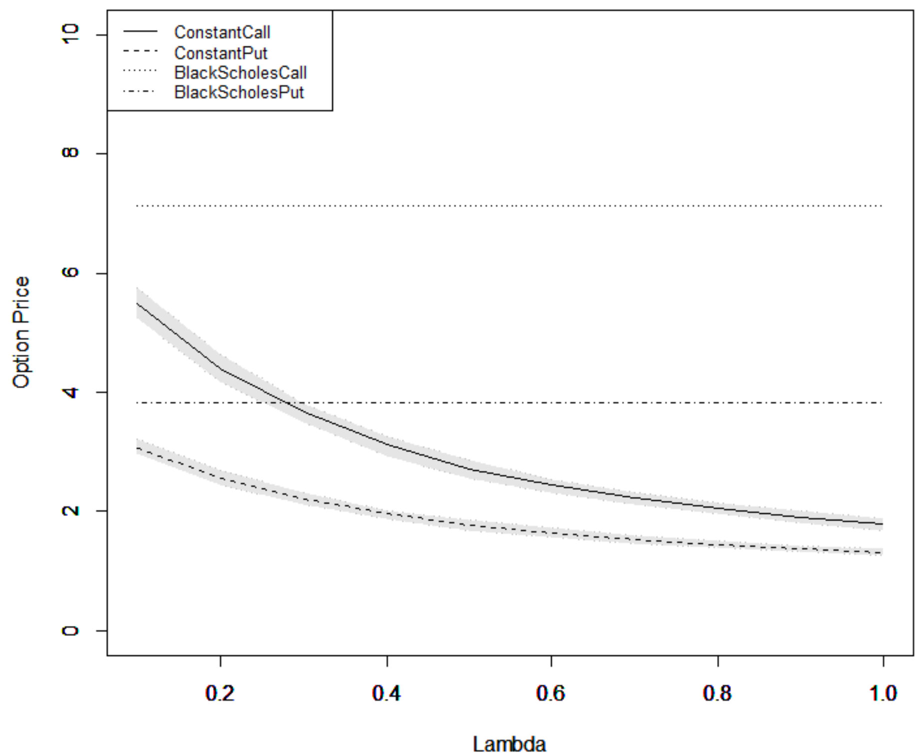

Figure 3.1: Defaultable call and put option prices against Black-Scholes option prices with $r = 2\%$, $S = K = \$35$, $R = 70\%$, $\sigma = 18\%$, $T = 5$, with a 99% confidence level.

Scholes results of the two non defaultable options. An increase in the default rate causes the option price to drop dramatically. What we can also notice is when the default rate is closer to zero, the defaultable options almost reach the Black-Scholes risk neutral prices. This will be further investigated in Chapter 4. Now in Figure 3.2 with the same results and simulation steps, we have calculated the put and call option prices with risk free interest rate $r = 0$, which we will need in the later analytical calculations for comparison. As expected the results between put

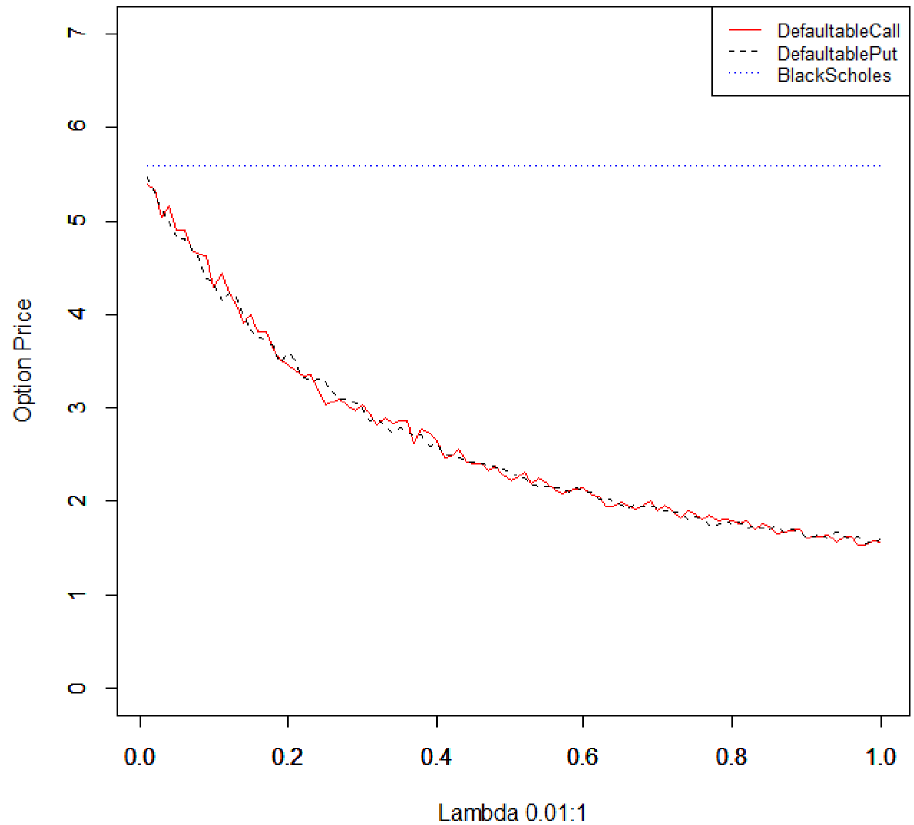

Figure 3.2: Defaultable call and put option prices with $r = 0\%$, $S = K = \$35$, $\sigma = 18\%$, $R = 70\%$, $T = 5$, for error bars of the simulations refer to Appendix C, Figure C.1.

and call option prices are essentially the same except some minor differences due to simulation error. It can be seen that the Put-Call parity holds well here even for defaultable option.

3.4 Analytical Approximation for Defaultable Option Prices

To see how accurate our simulations are we have decided to further proceed in finding an analytical solution to our simulations. There are always errors in using Monte Carlo methods so if we can find an analytical solution of our defaultable option we can then compare the results. In this chapter we propose an analytical approximation to pricing defaultable options which we have not seen in the literature. A closed form solution consists of solving the following formula:

$$e^{-rT}\mathbb{E}[f(S_T)\mathcal{I}_{\tau\leq T}] + e^{-rT}\mathbb{E}[f(S_T)\mathcal{I}_{\tau>T}]. \tag{3.13}$$

In the equation (3.13), $f(S_T)$ is simply an option function evaluated at T, time to maturity and the indicator function $\mathcal{I}$ represents a 1 if default happens within T or a 0 otherwise. r is the risk free interest rate with which we discount values to find the option price at $t = 0$. The formula can then be expanded to the option function of a call. When $\tau \leq T$ then:

$$f(S_T) = R[(S_\tau - K)^+ e^{r(T-\tau)}]. \tag{3.14}$$

When $\tau > T$ then:

$$f(S_T) = (S_T - K)^+. \tag{3.15}$$

Now we can bring together the two cases shown in equation (3.14) and (3.15) above, and plug them into equation (3.13). The results will give us the following:

$$e^{-r(T)}\mathbb{E}[R(S_\tau - K)^+ e^{r(T-\tau)}\mathcal{I}_{\tau\leq T}] + e^{-rT}\mathbb{E}[(S_T - K)^+\mathcal{I}_{\tau>T}]$$

$$= \mathbb{E}[R(S_\tau - K)^+ e^{-r\tau}\mathcal{I}_{\tau\leq T}] + e^{-rT}\mathbb{E}[(S_T - K)^+\mathcal{I}_{\tau>T}]$$

$$= R\mathbb{E}[\mathbb{E}[R(S_\tau - K)^+ e^{-r\tau}|\tau = t_1, t_1 \leq T]] + e^{-rT}\mathbb{E}[\mathbb{E}[(S_T - K)^+|\tau = t_1, t_1 > T]]. \tag{3.16}$$

Let $C_{BS}(T)$ and $C_{BS}(\tau)$ represent the call option function calculated at T and τ, respectively. So now once we substitute in (3.16) the actual option function of Black Scholes we get:

$$R\mathbb{E}[C_{BS}(\tau)|\tau = t_1, t_1 \leq T]] + e^{-rT}\mathbb{E}[(S_T - K)^+]\mathbb{E}[\mathcal{I}|\tau = t_1, t_1 > T]. \tag{3.17}$$

Because of independence in the second part of the equation, we were able to separate the expectation into two parts allowing us to use the basic Black-Scholes formula at time T for part of the evaluation. This will be denoted by $C_{BS}(T)$, which is the Black-Scholes formula for a call option in a risk neutral world. Now we need to replace the indicator functions with actual default probabilities and as stated above the waiting time until the first arrival of a default is represented by an exponential distribution which will change formula (3.17) in the following way:

$$R\int_0^T C_{BS}(\tau)P_\lambda(\tau)d\tau + C_{BS}(T)\int_{\tau=T}^{\infty} P_\lambda(\tau)d\tau, \tag{3.18}$$

where:

$$P_\lambda(\tau) = \lambda e^{-\lambda\tau}.$$

To solve equation (3.18), we first looked at a simple case of using a Taylor series approximation to expand the standard normal distribution that is in the Black-Scholes formula. During these works we realized that there are a few different ways to get such solutions depending on how versatile we would like our analytical results to be. These results will be explained and compared in the following sections.

3.4.1 Taylor Series Approximation Method for Standard Normal

First for simplicity we will number two parts of equation (3.18), so we can simply identify them:

$$R\int_0^T C_{BS}(\tau)P_\lambda(\tau)d\tau, \tag{3.19}$$

$$C_{BS}(T)\int_{\tau=T}^{\infty} P_\lambda(\tau)d\tau \tag{3.20}$$

Before we proceed further into solving with Taylor Series approximation notice that equation (3.20) is a simple integral which we can solve in closed form. Equation (3.20) will give us the following:

$$\begin{aligned} & C_{BS}(T)\int_{\tau=T}^{\infty} P_\lambda(\tau)d\tau \\ &= C_{BS}(T)\int_{\tau=T}^{\infty} \lambda e^{-\lambda\tau}d\tau \\ &= C_{BS}(T)e^{-\lambda T}. \end{aligned} \tag{3.21}$$

In equation (3.21) notice how our result is simply a Black-Scholes formula multiplied by the probability that the default did not happen in the intervals 0 and T. Now we move to the more complex case which is shown in equation (3.19):

$$R\int_0^T C_{BS}(\tau)P_\lambda(\tau)d\tau$$

$$= R\int_0^T C_{BS}(\tau)\lambda e^{-\lambda\tau}d\tau$$

$$= R \int_0^T [S N(d_1(\tau)) - N(d_2(\tau))Ke^{-r\tau}]\lambda e^{-\lambda\tau} d\tau. \tag{3.22}$$

If we let $r = 0\%$ and $S_0 = K$ it will help us in the approximation since in this setting (2.23) and (2.24) reduce to:

$$d_1(\tau) = \frac{1}{2}\sigma\sqrt{T}, \tag{3.23}$$

$$d_2(\tau) = -d_1(\tau). \tag{3.24}$$

We then use a Taylor Series Expansion for the standard normal distributions that we have in our Black Scholes formula $N(.)$:

$$N(x) = N(0) + N'(0)x + N''(0)\frac{x^2}{2} + O(x^3). \tag{3.25}$$

We apply results of (3.23) and (3.24) to our Taylor Series in (3.25). The call option at τ can now be represented through a Taylor Series approximation in this way (Notice: many cancellations have happened here due to making $S = Ke^{-r(T-t)}$):

$$C_{BS}(\tau) = S[N(d_1(\tau)) - N(-d_1(\tau))]$$

$$= S[N(0) + N'(0)d_1(\tau) + N''(0)\frac{d_2(\tau)^2}{2} - N(0) + N'(0)(-d_1(\tau)) + N''(0)\frac{d_2(\tau)^2}{2}]$$

$$= S[2N'(0)d_1(\tau)]$$

$$= S[N'(0)\sigma\sqrt{\tau}],$$

where

$$N'(0) = \frac{1}{\sqrt{2\pi}}.$$

Let $Q = N'(0)$ for simplicity, since it will be carried through our calculations. Now:

$$C_{BS}(\tau) = S Q\sigma\sqrt{\tau}. \tag{3.26}$$

We can now calculate the integral represented in equation (3.19) by inputting the equation (3.26) which results in:

$$R\int_0^T C_{BS}(\tau)P_\lambda(\tau)d\tau = R\sigma\lambda S\,Q\int_0^T \sqrt{\tau}e^{-\lambda\tau}d\tau.$$

By changing variables, letting some variable $x = \sqrt{\tau}$ it results in the following integral:

$$R\sigma\lambda S\,Q\int_0^{\sqrt{T}} 2x^2\lambda e^{-\lambda x^2}dx.$$

Then, we apply integration by parts where we let:

$$v = x,$$

$$du = -2x\lambda e^{-\lambda x^2}dx.$$

Solving the integration by parts gives us the following results:

$$-R\sigma S\,Q[\sqrt{T}e^{-\lambda T} - \int_0^{\sqrt{T}} e^{-\lambda x^2}dx]. \tag{3.27}$$

Since we are looking at a closed form solution, notice how the integral $\int_0^{\sqrt{T}} e^{-\lambda x^2}dx$ is quite similar to the error function. The error function is:

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt.$$

Error function can also be represented through the cumulative distribution, given by $N(.)$, which is the integral of the standard normal distribution, the relationship is:

$$N(x) = \frac{1}{2} + \frac{1}{2}\mathrm{erf}\left(\frac{x}{\sqrt{2}}\right). \tag{3.28}$$

When we manipulate $\int_0^{\sqrt{T}} e^{-\lambda x^2} dx$ by changing variable, we can get a result by using the error function, which is simple to get an answer for through a computer language:

$$\int_0^{\sqrt{T}} e^{-\lambda x^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{\lambda}}\,\mathrm{erf}(\sqrt{\lambda T}).$$

So now looking back at equation (3.27), with the error function, we finally have a solution to equation (3.19) at the beginning of this section, the result is:

$$-R\sigma S\,Q[\sqrt{T}e^{-\lambda T} - \frac{1}{2}\sqrt{\frac{\pi}{\lambda}}\,\mathrm{erf}(\sqrt{\lambda T})]. \tag{3.29}$$

Our approximate solution can be brought together by adding equation (3.29) and (3.21):

$$-R\sigma S\,Q[\sqrt{T}e^{-\lambda T} - \frac{1}{2}\sqrt{\frac{\pi}{\lambda}}\,\mathrm{erf}(\sqrt{\lambda T})] + C_{BS}(T)e^{-\lambda T}. \tag{3.30}$$

The analytical solution works well for short times to maturity. Because we used a Taylor series, we assumed that the function is centred around 0. This will hold well for small values of T and σ, the time to maturity and its volatility, but otherwise it will fail. In order to make this analytical approximation more accurate, so we can have closed form expression that will approximately hold for all cases, we have decided to expand the Taylor approximation by using a different centering than 0, since our time to maturity and volatility should not be required to be small.

3.4.2 Discretization of Taylor Series Approximation, Second Term

In this section we will follow the same steps as Section 3.4.1 but improve our Taylor Series by using a different expansion point:

$$N(x) = N(a) + N'(a)(x-a) + N''(a)\frac{(x-a)^2}{2} + O(x^3). \tag{3.31}$$

Now we are differentiating in the neighbourhood of a. Therefore, we need to specify the value of a, that would make this equation most versatile. If we choose $a = t_i$, where t_i would be the discretization steps into which we divide $[0, T]$, then we can integrate the defaultable call option

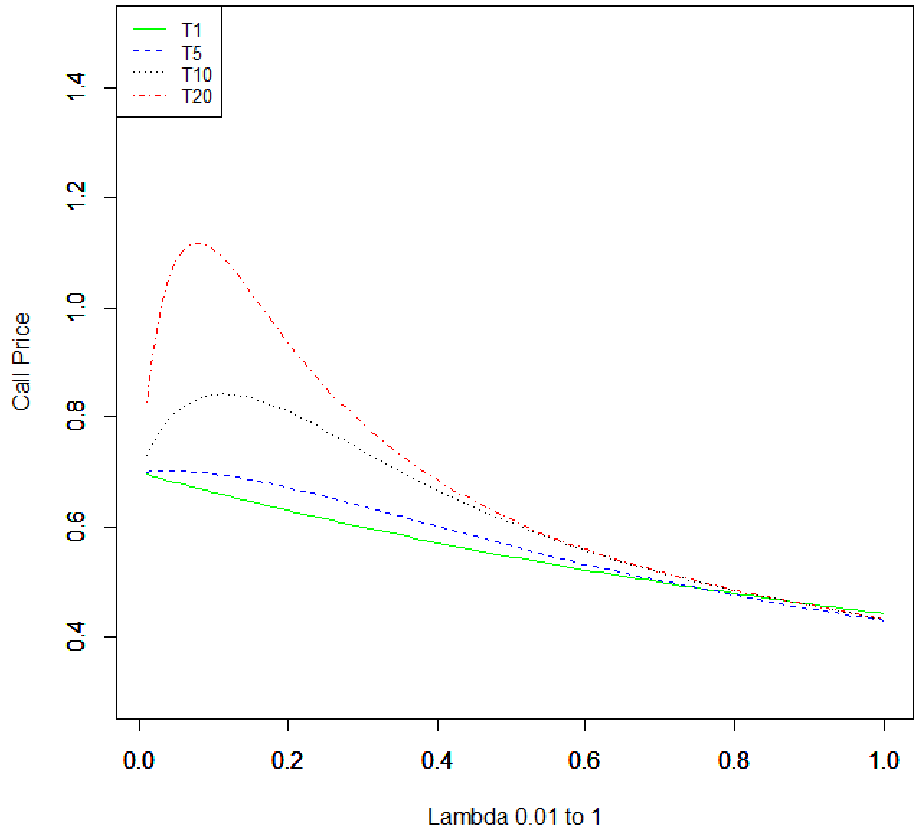

Figure 3.3: Analytical approximation of equation (3.30) with different maturity dates, T=1, 5, 10, 20, $\sigma = 18\%$, $r = 0\%$, $R = 70\%$ and $S = K = \$35$.

similarly to Section 3.4.1 but the integral would be from t_i to t_{i+1}. Afterwards, the integrals of small step sizes would need to be summed. We now will redo the Taylor Approximation up to the second term for the Standard Normal, which will not produce such nice cancellation as it did in the previous section. Hence, the new results for calculating $C_{BS}(\tau)$ centered at t_i will be:

$$= S\left[N(a) + N'(a)[d_1(\tau) - a] + N''(a)\frac{[d_1(\tau) - a]^2}{2} - N(a) - N'(a)[-d_1(\tau) - a] - N''(a)\frac{[-d_1(\tau) - a]^2}{2}\right].$$

After some cancellation we obtain the following simplified formula:

$$C_{BS}(\tau) = S\sigma\sqrt{\tau}[N'(a) - N''(a)a]$$

where,

$$N'(a) = \frac{1}{\sqrt{2\pi}}e^{\frac{-a^2}{2}},$$

and

$$N''(a) = -\frac{a}{\sqrt{2\pi}} e^{\frac{-a^2}{2}}.$$

Therefore, we have:

$$[N'(a) - N''(a)a] = [1 + a^2]\frac{e^{\frac{-a^2}{2}}}{\sqrt{2\pi}},$$

resulting in

$$C_{BS}(\tau) \approx S\sigma\sqrt{\tau}[1 + a^2]\frac{e^{\frac{-a^2}{2}}}{\sqrt{2\pi}}. \tag{3.32}$$

Now we need to resolve equation stated in (3.19) using the above $C_{BS}(\tau)$ but the second part of the equation which is (3.20), will not change in this section. Hence, the new analytical results of equation (3.19) will be:

$$LHS = R\int_0^T C_{BS}(\tau)P_\lambda(\tau)d\tau$$

$$LHS = R\int_0^T S\sigma\sqrt{\tau}[1 + a^2]\frac{e^{\frac{-a^2}{2}}}{\sqrt{2\pi}}\lambda e^{-\lambda\tau}d\tau. \tag{3.33}$$

Finally, we replace a by t_i and sum the integrals from t_i to t_{i+1}, the results will look like this:

$$LHS = R\sum_{t_i=0}^{n-1}\int_{t_i}^{t_{i+1}} S\sigma\sqrt{\tau}[1 + t_i^2]\frac{e^{\frac{-t_i^2}{2}}}{\sqrt{2\pi}}\lambda e^{-\lambda\tau}d\tau. \tag{3.34}$$

By applying a change of variable method and integration by parts, we take similar steps as in Section 3.4.1; for more detailed calculation refer to Appendix B. The analytical approximation obtained is:

$$LHS = R\sum_{i=0}^{n-1} S\sigma\sqrt{\tau}[1 + t_i^2]\frac{e^{\frac{-t_i^2}{2}}}{\sqrt{2\pi}}\left[\sqrt{t_{i+1}}e^{-\lambda t_{i+1}} - \sqrt{t_i}e^{-\lambda t_i} - \frac{1}{2}\sqrt{\frac{\pi}{\lambda}}[\mathrm{erf}[\sqrt{t_{i+1}\lambda}] - \mathrm{erf}[\sqrt{t_i\lambda}]]\right]. \tag{3.35}$$

Adding equation (3.20), whose solution was given in equation (3.21), our final approximate solution, to 3.18 is:

$$R\sum_{i=0}^{n-1} S\sigma\sqrt{\tau}[1 + t_i^2]\frac{e^{\frac{-t_i^2}{2}}}{\sqrt{2\pi}}\left[\sqrt{t_{i+1}}e^{-\lambda t_{i+1}} - \sqrt{t_i}e^{-\lambda t_i} - \frac{1}{2}\sqrt{\frac{\pi}{\lambda}}[\mathrm{erf}[\sqrt{t_{i+1}\lambda}] - \mathrm{erf}[\sqrt{t_i\lambda}]]\right] + C_{BS}(T)e^{-\lambda T}. \tag{3.36}$$

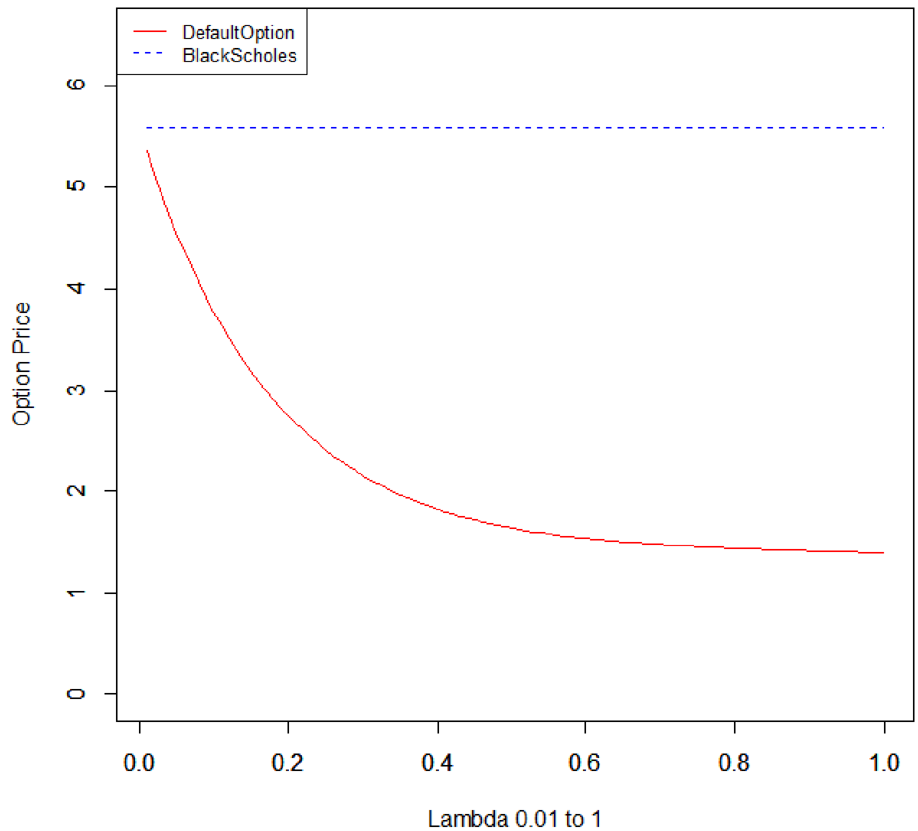

Figure 3.4: Analytical approximation of equation (3.36) using second term Taylor expansion. Parameters: $S_0 = K = \$35, \sigma = 18\%, r = 0\%, T = 5, R = 70\%, n = 1000$.

As seen in Figure 3.4, this approximation does allow us to get results for larger values of T, the time to maturity. This is in contrast to the previous method in Section 3.4.1 where we had severe restrictions to our analytical results. Also, it can simply be checked that equation (3.36) can be simplified to (3.30) by letting $t_i = 0$ and $t_{i+1} = 1$.

We have truncated the Taylor series to the second term hoping that it will be a good approximation of our solution. In the next section we will take the Taylor Series to the third term, this way we can compare solutions and see if we are losing value to our answers and if so how much. The following section will provide more details to answering such questions.

3.4.3 Discretization of Taylor Series Approximation, Third Term

In this section we want to improve the results given in Section 3.4.2 by adding a third term to the Taylor Series. Similar steps have to be followed and again it is the first part, equation (3.19) that we need to expand and the second part of it (3.20) remains the same. The Taylor series expansion up the third term is:

$$N(x) = N(a) + N'(a)(x-a) + N''(a)\frac{(x-a)^2}{2} + N'''(a)\frac{(x-a)^3}{6}. \tag{3.37}$$

Then we apply equation (3.37) to our option evaluation to get the following results:

$$N'''(a)\frac{(x-a)^3}{6} = \frac{1}{6}(N'''(a)(d_1(\tau)-a)^3 - N'''(a)(-d_1(\tau)-a)^3). \tag{3.38}$$

After expanding the brackets the above equation simplifies to:

$$= \frac{N'''(a)}{6}[2d_1(\tau)^3 + 6a^2 d_1(\tau)]. \tag{3.39}$$

Once we have plugged in the values for d_1 we are ready to discretize as in the previous section and sum everything up. The equation that we must solve is given by:

$$\sum_{i=0}^{n-1} \frac{e^{\frac{-t_i^2}{2}}}{6\sqrt{2\pi}}[t_i^2 - 1]\sigma\left[\frac{\sigma^2}{4}\int_{t_i}^{t_i+1} \tau^{\frac{3}{2}}\lambda e^{-\lambda\tau}d\tau + 3t_i^2\int_{t_i}^{t_i+1} \lambda\sqrt{\tau}e^{-\lambda\tau}d\tau\right]. \tag{3.40}$$

After completing some onerous calculations the third term in the Taylor Series to our defaultable option price and solution to equation 3.18 is:

$$\begin{aligned}&\sum_{i=0}^{n-1} \frac{e^{\frac{-t_i^2}{2}}}{6\sqrt{2\pi}}[t_i^2 \\ &- 1]\sigma\left[\frac{\sigma^2}{4}\left(t_i^{\frac{3}{2}}e^{-\lambda t_i} - t_{i+1}^{\frac{3}{2}}e^{-\lambda t_{i+1}} + \frac{3}{2}(\sqrt{t_{i+1}}e^{-\lambda t_{i+1}} - \sqrt{t_i}e^{-\lambda t_i} - \frac{1}{2}\sqrt{\frac{\pi}{\lambda}}[\mathrm{erf}[\sqrt{t_{i+1}\lambda}] - \mathrm{erf}[\sqrt{t_i\lambda}]])\right)\right. \\ &\left. + 3t_i^2[\sqrt{t_{i+1}}e^{-\lambda t_{i+1}} - \sqrt{t_i}e^{-\lambda t_i} - \frac{1}{2}\sqrt{\frac{\pi}{\lambda}}[\mathrm{erf}[\sqrt{t_{i+1}\lambda}] - \mathrm{erf}[\sqrt{t_i\lambda}]]]\right].\end{aligned} \tag{3.41}$$

We can now add up (3.41) with the first and second term which are calculated in Section 3.4.2. Figure 3.5 shows that our third term is overestimating the option price and in fact it is higher than Black-Scholes price when λ is small. This is a problem with the Taylor series method, which we will show details for in the next section.

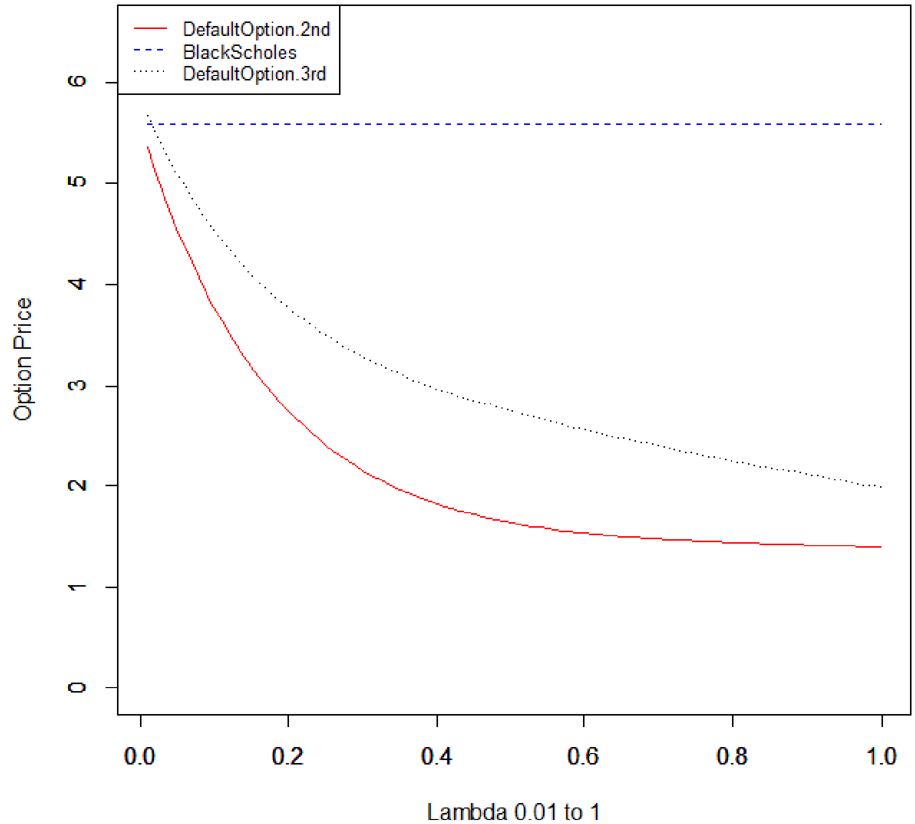

Figure 3.5: Analytical approximation with the second and third term of the Taylor Series, comparing it to Black-Scholes. Parameters: $S_0 = K = \$35$, $\sigma = 18\%$, $r = 0\%$, $T = 5$, $R = 70\%$, $n = 1000$.

3.5 Problems with Taylor Series Approximation and Errors

As seen in Figure 3.5, our approximate solution where the Taylor series was expanded up to the third term has caused an over approximation of the defaultable option value. More importantly, we can consider the third term Taylor expansion as the upper boundary and the second term as the lower boundary of our analytical approximation results. The reason that the second and third term can be considered as boundaries is that the series for $N(.)$ converges uniformly and so it can be integrated term by term, hence by its alternating signs will converge. Figure 3.5 shows that when the hazard rate is close to zero, the defaultable option value exceeds the Black-Scholes option price. This cannot be right, since it is evident that when an option price has no probability of default the maximum price it can reach is given by the Black-Scholes equation. This lets us consider the terms of the Taylor series and its errors.

First we notice that the first term in the Taylor series is positive and the second term is negative, our first term is:

$$N'(a), \tag{3.42}$$

second term is:

$$-N''(a)a, \tag{3.43}$$

and the third term is given by:

$$\frac{N'''(a)}{6}[2d_1(\tau)^3 + 6a^2 d_1(\tau)]. \tag{3.44}$$

Since each of the parameters τ, σ, S must be positive, we have alternating terms in our series starting with first term being positive. What we see in equations (3.42), (3.43) and (3.44) is that all even terms of the Taylor Series in our case will be negative and the odd terms will be positive, therefore, our Taylor series has alternating signs. This can really affect our results depending at what term our Taylor Series stops at. This clarifies why in Figure 3.5 the approximation with only the first two terms of the Taylor expansion are not exceeding Black-Scholes. The sum of even terms are under estimating and the odd terms are over estimating. To further analyze this problem we will find the absolute value of the errors on the first three terms.

Using Taylor's Theorem we can find the remainder of the Taylor series approximation. What we are interested in is the remainder term of the Taylor approximation that we computed previously. Taylor Theorem (Burden & Faires, 1989) states that $f \in C^n[a, b]$ and $f^{(n+1)}$ exists on $[a, b]$. Let $x_0 \in [a, b]$. Then for every $x \in [a, b]$, there exists ξ_x between x_0 and x with:

$$f(x) = P_n(x) + R_n(x), \tag{3.45}$$

where

$$P_n(x) = \sum_{k=0}^{n} \frac{f^k(x_0)}{k!}(x - x_0)^k, \tag{3.46}$$

and

$$R_n(x) = \frac{f^{(n+1)}(\xi_x)}{(n+1)!}(x - x_0)^{n+1}. \tag{3.47}$$

Where $P_n(x)$ is called the nth Taylor polynomial for f about x_0 and $R_n(x)$ is called the remainder term associated with $P_n(x)$. By using this theorem we will approximate the error that we have in the first three terms of our Taylor approximation which are calculated through the following formulas:

$$R_1(x) \leq S \sum_{i=1}^{n} \int_{t_i}^{t_i+1} \frac{N^2(\xi)}{2!}(0.5\sigma\sqrt{\tau} - t_i)^2 \lambda e^{-\lambda\tau} d\tau, \tag{3.48}$$

$$R_2(x) \leq S \sum_{i=1}^{n} \int_{t_i}^{t_i+1} \frac{N^3(\xi)}{3!}(0.5\sigma\sqrt{\tau} - t_i)^3 \lambda e^{-\lambda\tau} d\tau, \tag{3.49}$$

$$R_3(x) \leq S \sum_{i=1}^{n} \int_{t_i}^{t_i+1} \frac{N^4(\xi)}{4!}(0.5\sigma\sqrt{\tau} - t_i)^4 \lambda e^{-\lambda\tau} d\tau. \tag{3.50}$$

The error have been plotted in the graph below for different λ's:

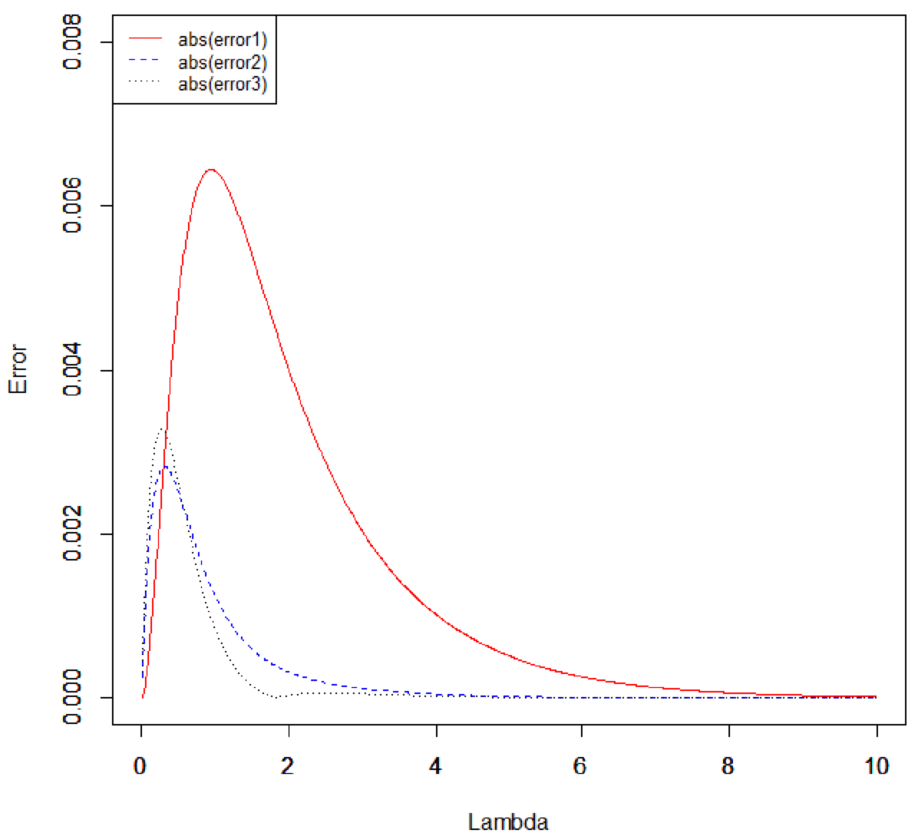

Figure 3.6: Errors of the first three terms of the Taylor expansion with λ from 0.01 to 10.

As one can see in Figure 3.6 as expected the error is converging but we do have a problem when λ, the default rate, is small. In each error term we can see the same pattern. In the next section we will compare our approximate solution with an adaptive quadrature result to further investigate this error.

3.6 Quadrature Solution to Defaultable Option with Constant Default Arrival

3.6.1 Risk Free Interest Rate of Zero

So far we have looked into finding an analytical result of a defaultable option which to make calculations simpler we set the risk free interest rate at 0%. By doing this we were able to have some cancellations in the Black-Scholes formula. To test our analytical results with that of quadrature solution, we will apply an adaptive quadrature method to the formula:

$$R \int_0^T C_{BS}(\tau) P_\lambda(\tau) d\tau \tag{3.51}$$

and the second part

$$C_{BS}(T) \int_{\tau=T}^{\infty} P_\lambda(\tau) d\tau, \tag{3.52}$$

which will not change since as previously mentioned it is a closed form solution so it will be added in the end. We used a Gaussian quadrature method, as seen for example in Numerical Analysis, Burden & Faires (1989), which was implemented using the R language. The tolerance used by this quadrature method was set to $2.2 \cdot 10^{-4}$. The integral was approximated from 0 to time to maturity, $T = 5$, on 10 subintervals, which gave us correct results. This will be shown in Section 3.6.2 when we compare these results to Monte Carlo simulations. The results of comparing the analytical solution up to the second Taylor expansion, with the quadrature results are shown in the graph below. A similar problem encountered in the Taylor error also appears here, we have a discrepancy in the comparison of the two results mainly where the default rate is small. For all other values of λ the analytic solutions appear to be a great approximation. As seen in Figure 3.7 the quadrature results appear to be accurate and with low error. We will further use this approximation in the next section to be able to compare the put and call differences when defaulting. Thus far we only looked at results when risk free rate,

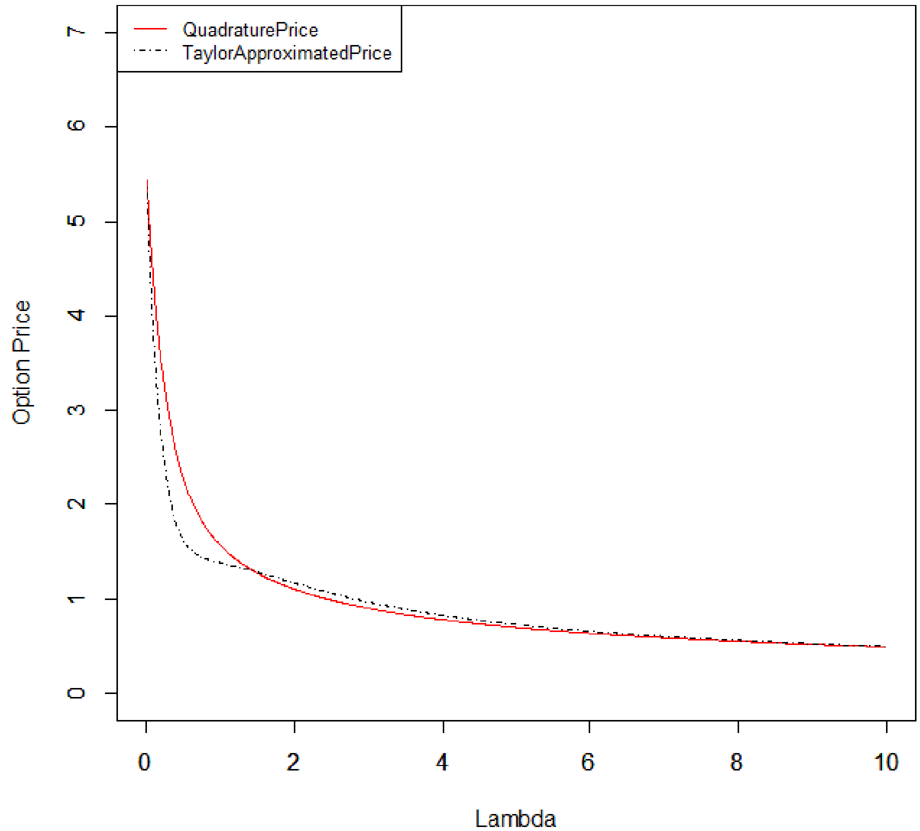

Figure 3.7: Comparison between a quadrature solution of equations (3.51) added with (3.52) and the analytical solution that we derived in Section 3.4, with λ from 0.01 to 10, $\sigma = 18\%$, $r = 0\%$, $T = 5$, $R = 70\%$, $S = K = \$35$.

$r = 0\%$.

3.6.2 Non Zero Risk Free Interest Rate

In this section we will focus more on a quadrature solution instead of our analytical results. The reason for that is that the analytical approximation was created on an assumption that our interest rate is 0%. In that case a defaultable call and put will have the exact same values. We are interested in seeing how a call behaves differently from a put in case of a constant default arrival. We also did this with a Monte Carlo simulation method, so we can now compare the quadrature solution versus simulation results. What we see in Figure 3.8 is that the Monte Carlo simulations for a call and put give the same results as the quadrature solution. When the hazard rate increases to a large value, meaning that our default probability has increased then

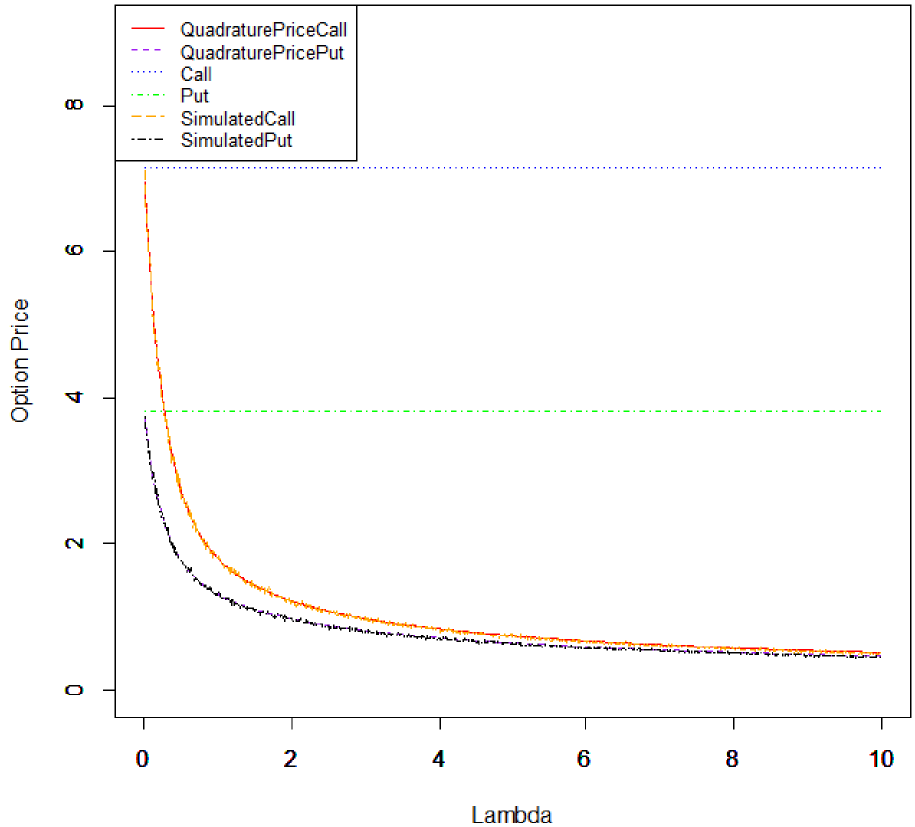

Figure 3.8: Comparison between quadrature and simulated results, with λ from 0.01 to 10, $r = 2\%$, $\sigma = 18\%$, $R = 70\%$, $S = K = \$35$ and $T = 5$.

both options approach each other at the neighbourhood of zero. In reality, when a company has a large chance of default we can expect its option values to approach zero. Since the quadrature and simulation results are the same in this case it would be better to use a quadrature solution for computational time purposes.

3.7 Conclusion

To conclude this section, we initially started with solving a defaultable option, with exponential default arrival times, that were driven by λ, the default rate. The expression of interest was:

$$R \int_0^T C_{BS}(\tau) P_\lambda(\tau) d\tau + C_{BS}(T) \int_{\tau=T}^{\infty} P_\lambda(\tau) d\tau. \tag{3.53}$$

We were able to solve this in three ways. Our analytical approximation had some errors since we used a Taylor series expansion on it. We then developed a Monte Carlo method to evaluate these integrals, which gave us really good results but with a big downfall of computational time. We then used a quadrature method to solve the integrals which gave us a nice and clean solution in a very short time. The intuition behind a constant default rate was that if we know the chances of default in a company are stable then we can easily compute the CVA. But what is often seen in the market is that the default probability of a company can change abruptly. We now have to come up with a different way to model a changing default rate. One way which will be shown in Chapter 4 is to relate the default rate to the stock price. Based on the intuition that the stock of a company is a good indicator of how they are performing as a company. This will give us a default variable that depends on the stock price and therefore not constant throughout the survival period.

Chapter 4

Pricing Results for Variable Default Rate Case

4.1 Non-Constant Default Rates

In Chapter 3 we did some detailed analysis on defaultable options based on a constant default arrival rate. In reality the default rate varies through time and is highly dependent on the economic status of the company and the market. We know the default rate of many companies can change over time depending on different drivers. The most well known way to evaluate a companies' credibility is through the credit rating systems as offered by companies such as Standard & Poor's or Moody's. In this chapter we propose a model where we observe a defaultable option in which the rate of failure depends on the underlying stock price over time. This is supported by the intuition that if the stock of a company is increasing, they look as a more profitable company, so then its default will be decreasing and vice-versa.

4.1.1 Inhomogeneous Poisson Process

For the purpose of a more realistic result we will develop and use a failure model with a time dependent hazard rate $\lambda(t)$. This is a Poisson process with time dependency, better known as inhomogeneous Poisson process. This leads to the default times being dependent on another variable. In other words we want a default rate to be applied to the financial markets that changes over time, as in reality the probabilities of a company going bankrupt changes over time and does not remain constant. These non-constant default rates can be modelled through an inhomogeneous Poisson process. To define this process, lets start with a counting process $\{\mathcal{N}(t) : t \geq 0\}$ which is said to be an inhomogeneous process with an intensity $\lambda(t)$ where $t \geq 0$ if:

1. $\mathcal{N}(0) = 0$.

2. For all $t > 0$, $\mathcal{N}(t)$ has a Poisson distribution with mean $\Lambda(t) = \int_0^t \lambda(s)ds$.

3. For each $0 \leq t_1 < t_2 < .... < t_m$, $\mathcal{N}(t_1), \mathcal{N}(t_2) - \mathcal{N}(t_1) ... \mathcal{N}(t_m) - \mathcal{N}(t_{m-1})$ are independent random variables.

Also it can be seen that once the intensity rate $\lambda(t) = \lambda$ for each $t \geq 0$ then we return to the homogeneous process that we initially defined in Chapter 3. The mean value function of the inhomogeneous Poisson process is $m(t)$ which is defined in (ii). The process increments are Poisson distributed and also independent but not stationary.

The probability mass function of a inhomogeneous Poisson process that follows the definitions above is (see for example Casarin (2005)):

$$\mathbb{P}\{(\mathcal{N}_t - \mathcal{N}_s) = k|\mathcal{F}\} = \mathbb{P}\{(\mathcal{N}_t - \mathcal{N}_s) = k\} = \frac{(\Lambda(t) - \Lambda(s))^k}{k!} e^{-(\Lambda(t) - \Lambda(s))}, \tag{4.1}$$

with $k = 0, 1, 2....$ where k represents the number of jumps in the process and the intensity measure or cumulative hazard function is:

$$\Lambda(t) = \int_0^t \lambda(u)du. \tag{4.2}$$

Although, this process maps out many default jumps, we are only interested in the time to the first default, which is at $k = 1$ and $s = 0$, the first jump starting from time 0 to t. This results in the following probability mass function:

$$\mathbb{P}\{(\mathcal{N}_t) = 1\} = (\Lambda(t))e^{-\Lambda(t)}. \tag{4.3}$$

Now that we have defined the probability mass function of the first default from time 0 to t, we need to make it comparable to what we had in Chapter 3. When the default rate was constant we had a homogeneous Poisson process, where the distribution of the waiting time until the first default arrival was given by an exponential function with constant intensity λ . To be able to compare a constant versus a non-constant default rate, we must make them globally similar to one another.

4.2 Constant and Non-Constant Default Rates

In this section Monte Carlo simulations are performed for the non constant default rate. These results are also then compared to the constant default rates and their behaviours are analyzed. First we need to define the non-constant default rate λ_t:

$$\lambda_t = \frac{C}{S_t}, \tag{4.4}$$

where C is a constant variable that we give to λ_t. We have chosen a non-constant default rate to represent the probability of default so that the rate will also vary with time. The probability of default now depends on the underlying stock price. As seen in equation (4.4) we have C as just some constant value and S_t the stock price at time t. The reason why this relationship was picked was mostly by intuition. We wanted to create a default probability that changes through time, depending on some type of information about the company. The idea of having a default related to the stock value of a company also comes from the structural models that we mentioned in Chapter 2 of this thesis.

The Merton model has a boundary based on the stock price, and if the company went below this boundary they would be considered as bankrupt. Similarly here we use the stock price to determine the default probability as well. Our model though is continuously dependent on the stock price, a slight change in the stock price immediately affects the default rate. What is represented in equation (4.4) is a probability of default that rises when the stock price decreases and vice-versa. So what we are claiming in this model is that the stock price holds information on how the company's finances stand and should be a good indication of default of that company. To find comparable default rates of constant and non-constant intensities we propose two methods which shall further explore below.

4.2.1 Method 1

The results that were concluded in Chapter 3 focused on a constant default rate. We would like to compare these results with the new model proposed here. This method shows a way to compare similar default rates by finding an average of the non-constant λ_t model and inserting it in the constant model, so we can see discrepancies between the two results. This average default rate, which will be denoted by $\bar{\lambda}$ and is given by:

$$\bar{\lambda} = E\left[\frac{1}{T}\int_0^T \frac{C}{S_t}dt\right]. \tag{4.5}$$

Using Fubini's Theorem as seen for example in Calculus on Manifolds, Spivak (1965), equation (4.5) becomes:

$$\bar{\lambda} = \frac{1}{T}\int_0^T E\left[\frac{C}{S_t}\right]dt. \tag{4.6}$$

As we previously defined our stock follows a Geometric Brownian motion given by:

$$S_t = S_0 e^{(r-\frac{\sigma^2}{2})t+\sigma W_t}, \tag{4.7}$$

where W_t is a Brownian motion with mean 0 and variance Δt. If we substitute the stock value into the $\bar{\lambda}$ which is given in equation (4.5), the following results can be seen:

$$\begin{aligned} \bar{\lambda} &= \frac{1}{T}\int_0^T E\left[\frac{C}{S_0 e^{(\mu-\frac{\sigma^2}{2})t+\sigma W_t}}\right]dt \\ &= \frac{C}{TS_0}\int_0^T E\left[\frac{1}{e^{(\mu-\frac{\sigma^2}{2})t+\sigma W_t}}\right]dt \\ &= \frac{C}{TS_0}\int_0^T E\left[e^{-(\mu-\frac{\sigma^2}{2})t-\sigma W_t}\right]dt. \end{aligned} \tag{4.8}$$

We know that $W_t = -W_t$ in distribution so, it is important to understand all equalities as equalities in distribution, therefore the results are the following:

$$\begin{aligned}\bar{\lambda} &= \frac{C}{TS_0}\int_0^T E\left[e^{-(\mu-\frac{\sigma^2}{2})t+\sigma W_t}\right]dt \\ &= \frac{C}{TS_0}\int_0^T e^{-(\mu-\frac{\sigma^2}{2})t}E[e^{\sigma W_t}]dt. \end{aligned} \tag{4.9}$$

To be able to calculate $E[e^{\sigma W_t}]$ we can use the moment generating function of a Normal distribution with mean μ and variance σ^2, since Brownian motion is also normally distributed with mean 0 and variance t. The moment generating function is:

$$E[e^{tX}] = e^{\mu t+\frac{1}{2}\sigma^2 t^2}, \tag{4.10}$$

where X is normally distributed with parameters μ and σ^2. In Brownian motion we have W_t instead of X, with parameters $\mu = 0$ and $\sigma^2 = t$. Therefore,

$$E[e^{\sigma W_t}] = e^{\frac{1}{2}\sigma^2 t}. \tag{4.11}$$

Now we can proceed to the calculations in equation (4.9):

$$\begin{aligned}\bar{\lambda} &= \frac{C}{TS_0}\int_0^T e^{-(\mu-\frac{\sigma^2}{2})t}e^{\frac{1}{2}\sigma^2 t}dt \\ &= \frac{C}{TS_0}\int_0^T e^{-(\mu-\sigma^2)t}dt. \end{aligned} \tag{4.12}$$

In equation (4.12) we have calculated $\bar{\lambda}$ for a certain time t. We now need the sum of all such events from time 0 to maturity, which in our case is T. This can be done through:

$$\begin{aligned}\bar{\lambda}_{[0,T]} &= \frac{C}{TS_0}\int_0^T e^{-(\mu-\sigma^2)t}dt \\ &= \frac{C}{TS_0}\frac{1}{\mu-\sigma^2}[1-e^{-(\mu-\sigma^2)T}].\end{aligned} \tag{4.13}$$

Once the given parameters that have been used so far are inputted into equation (4.13), the results give a constant value that will only change with C, a value that we choose depending on the state that we would like the default to be at.

We now have a way to compare the defaultable options with two different hazard rate models. We expect that if the stock price is not volatile then the results between a constant and non-constant default rate should be similar to one another. However, once we add a volatility to the stock price, differences should appear. These difference will be through the effect that these two models will have on the time of default that will be simulated through a Monte Carlo method. For a non-constant default rate, τ the default time will be calculated through:

$$\tau = -\frac{1}{\lambda_t}\ln(U_{[0,1]}), \tag{4.14}$$

where $U_{[0,1]}$ is a random uniform distribution with parameters 0 and 1. Also, λ_t was defined in equation (4.4) and is dependent on the stock price at time t, which is constantly changing depending on the drift and volatility. On the other hand we will simulate a new τ, to find the constant time of default by applying the results of $\bar{\lambda}$ given in equation (4.13), which will result in the following:

$$\bar{\tau} = -\frac{1}{\bar{\lambda}}\ln(U_{[0,1]}). \tag{4.15}$$

4.2.2 Method 2

In this method we will try a more sophisticated calculation of the average non constant default rate. First we know that if we would like to find the probability of default, in an exponential function with parameter λ we can take the integral in interval of time that we desire:

$$P(\tau < T) = \int_0^T \hat{\lambda} e^{-\hat{\lambda} x} dx, \tag{4.16}$$

where $P(\tau < T)$ is the probability of default which we can calculate through a Monte Carlo simulation. We will find the probability by using a non-constant default rate since we are trying to find $\hat{\lambda}$ that corresponds to λ_t. The solution of the integral in equation (4.16) is:

$$P(\tau < T) = 1 - e^{-\hat{\lambda} T}. \tag{4.17}$$

We now solve for the average non-constant default rate $\hat{\lambda}$, which results in the following:

$$\hat{\lambda} = -\frac{\ln(1 - P(\tau < T))}{T}. \tag{4.18}$$

This method will be compared to Method 1, so we can determine which one will be more appropriate, the calculation of $\bar{\lambda}$ or $\hat{\lambda}$. We can continue this method analytically to get an idea of how this $\hat{\lambda}$ will look like through an inhomogeneous Poisson process that has been defined in Section 4.1.1:

$$P(\tau \leq T) = P(N_T \geq 1) \tag{4.19}$$

$$= 1 - P(\tau > T)$$

$$= 1 - E[e^{-\Lambda(t)}], \tag{4.20}$$

where

$$\Lambda(t) = \int_0^T \frac{C}{S_t} dt. \tag{4.21}$$

We now must substitute equation (4.21) into (4.20) to get the following:

$$= 1 - E[e^{-\int_0^T \frac{C}{S_t} dt}]. \tag{4.22}$$

The $P(\tau < T)$ can now be input into equation (4.18), which results in:

$$\hat{\lambda} = \frac{-\ln(E[e^{-\int_0^T \frac{C}{S_t}dt}])}{T}. \tag{4.23}$$

The results of both methods that we have suggested will be shown and interpreted in the following sub section.

4.2.3 Method 1 Results

In this sub section, Method 1 results will be analyzed for validation purposes. We will incorporate the calculations of the average default time in our results. Once τ and $\bar{\tau}$ values are produced, then the option value is evaluated depending on when and if the default time happens. If the time of default appears within the Δt time step then the option is evaluated at that certain time and discounted. If no default happens then essentially we are just evaluating a regular Black-Scholes pricing model, through a crude Monte Carlo method.

Notice in Figure 4.1 the top two lines on the graph representing the call option prices are quite close to one another, with the highest one being the non-constant default. The put options have given the exact same results for constant and non-constant intensity models. Similarly in the second portion of the graph we have plotted the time of default of the two scenarios that we have created and again the results are not easily distinguished from one another. We will also show the two results through a relative difference to Black-Scholes options that have no default risk.

In both Figures 4.1 and 4.2 the most important result that one can notice is that as the default probability increases the option value is always decreasing. In our case the call value is decreasing faster than the put value. Similarly looking at Figure 4.2 the relative difference to the Black-Scholes is increasing with the default rate going up. The one benefit of using Method 1 for calculating the average of a non-constant default rate was that the result was a constant value only depending on C, which was a given parameter chosen at our preference. Therefore, when inputting $\bar{\lambda}$ there was no need for any intense use of computer memory space in comparison to Monte Carlo simulations. This makes the process of comparing the constant against non-constant default rate much more efficient.

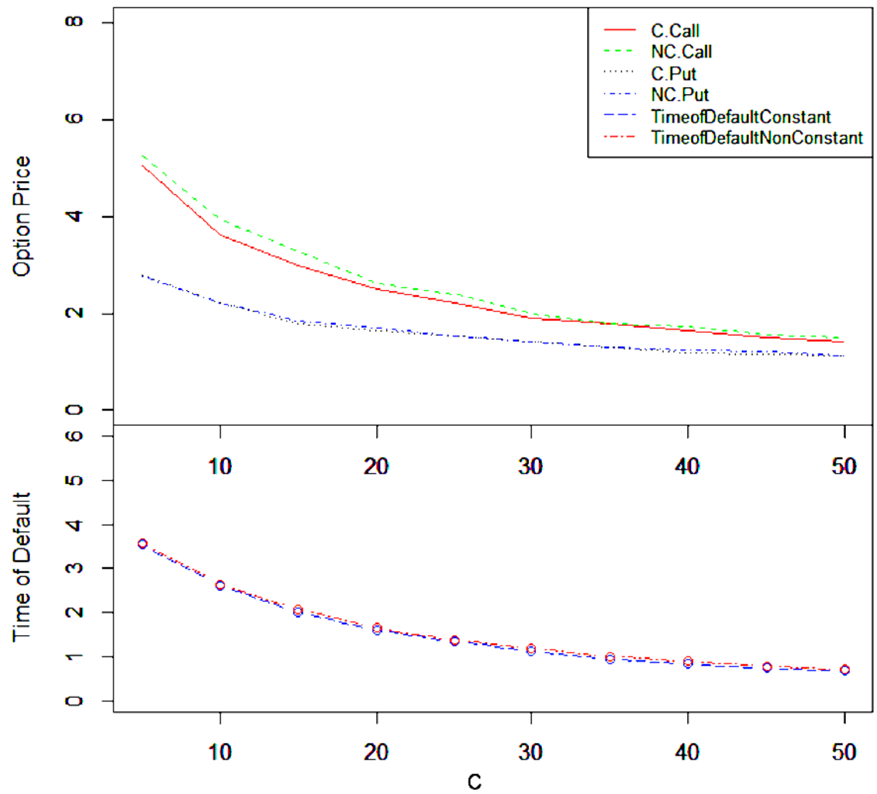

Figure 4.1: Method 1 comparison between constant and non-constant default rates of a call and put option, through Monte Carlo simulations, with C from 5 to 50, $r = 2\%$, $\sigma = 18\%$, $S = K = \$35$, $R = 70\%$ and $T = 5$. For error bars of the options and default times refer to Appendix C, Figure C.2, and Figure C.3.

4.2.4 Method 2 Results

In this method one crucial calculation that we needed was finding the probability of default that a non-constant rate of default gives and finding a corresponding constant rate of default to then be able to compare them in an equal matter. In Figure 4.3 one can see that the results do not change much compared with Method 1 results. In fact they are quite similar. Both methods clearly hold the same pattern as expected that when default rate is increasing the option value is decreasing. The default time in both methods will be identical and that's why they are not shown in the sections.

Looking at Figure 4.4 one can see that as the default rate increases the option value is moving further from the risk free value of such option in the Black-Scholes world. The conclusion here from the comparison of Method 1 and 2 is that they are both performing the same way,

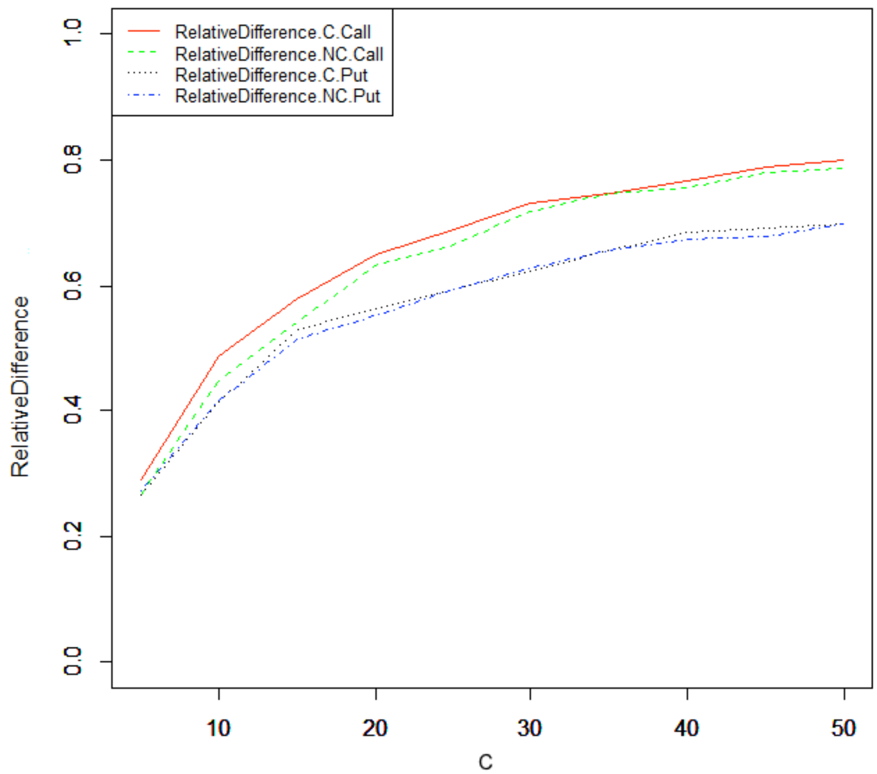

Figure 4.2: Method 1 relative difference between constant and non-constant default rates of a call and put option with the Black-Scholes pricing of such options, with parameters C from 5 to 50, $r = 2\%$, $\sigma = 18\%$, $S = K = \$35$, $R = 70\%$ and $T = 5$.

Method 1 is simpler and easier to calculate, instead Method 2 takes longer and Monte Carlo simulations are needed, to only result in the same output. Our suggestion here is that Method 1 will be a better estimation, saves time and no information is lost. For further investigation of our analysis between constant and non-constant default rates, Method 1 will be used.

4.3 Model Validation

To validate what has been done thus far with the constant and non-constant default rates through a Monte Carlo simulation, we have looked into a few different methods. One way for us to see that the model is in fact working is the observation that when default rates are very small the option value is approaching the Black-Scholes equation. We would like to perform other methods of validating our models.

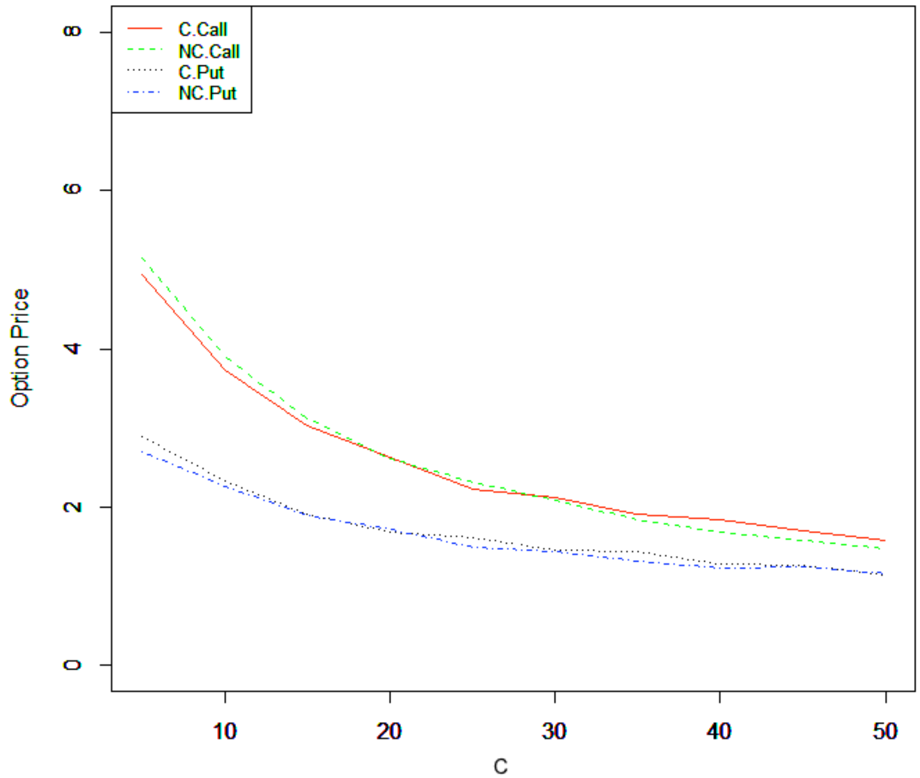

Figure 4.3: Method 2 results on the comparison between constant and non-constant default rates of a call and put option, through Monte Carlo simulations, with C from 5 to 50, $r = 2\%$, $\sigma = 18\%$, $S = K = \$35$, $R = 70\%$ and $T = 5$.

First we will test out our default times output by the Monte Carlo simulations we have performed. To test these default times we use the financial intuition that if we used the failure times toward pricing a simple default free Black-Scholes option, it should result in higher values than our defaultable option. Lets assume the failure times are denoted by some $\hat{t}$. What we are testing essentially is:

$$\frac{1}{N}\sum_{0}^{N} e^{-rT} C_i(S_i, \sigma, r, K, i\Delta t) < C_{BS}(S_t, \sigma, r, K, \hat{t}), \tag{4.24}$$

where C_{BS} is the call value of a risk free option that matures at $\hat{t}$ and C_i represents the defaultable call option value that we have simulated. Similar to the call option we have done the same check for the put. This check was then followed through for both cases, constant and non constant default rate. In Figure 4.5 it is seen that the theory of equation (4.24) holds

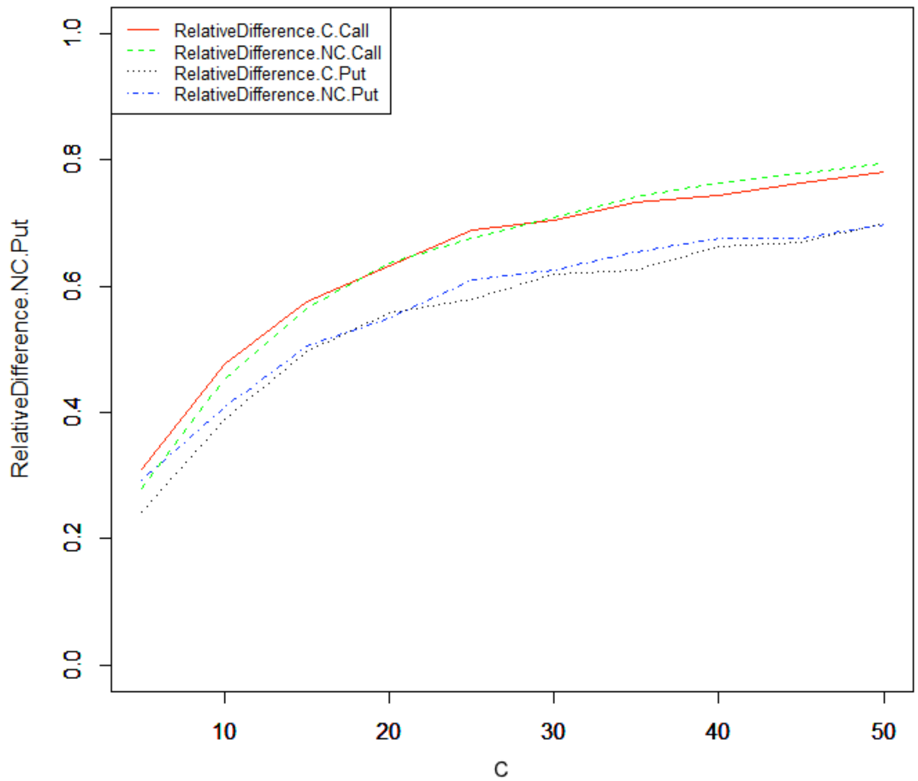

Figure 4.4: Method 2 results on the comparison of relative difference between constant and non-constant default rates of a call and put option with the Black-Scholes pricing of a call and put, with parameters C from 5 to 50, $r = 2\%$, $\sigma = 18\%$, $S = K = \$35$, $R = 70\%$ and$T = 5$.

very nicely as we expected and the first check to validate our results has been completed. The Black-Scholes options in both cases assessed at the default times do give higher values than the unstable Monte Carlo results of the defaultable options at the same default times.

The next step in our validation was to see how the survival probability behaves. The survival probability is calculated through Monte Carlo simulation by finding how many of our defaultable options survived until the time to maturity T. The non-constant default rate will fluctuate more through the volatile nature of the stock path. In contrast, the constant default will be more stable and depended on the volatility of stock price. With the parameters that we have chosen to consistently use through this thesis, drift being 2% and volatility being 18%, then we can assume on average our stock price fluctuates with volatility. If the stock price is increasing then the non-constant default rate is decreasing resulting in a higher survival rate. These survival rates will be the same for both call and put options. What we see in this result is

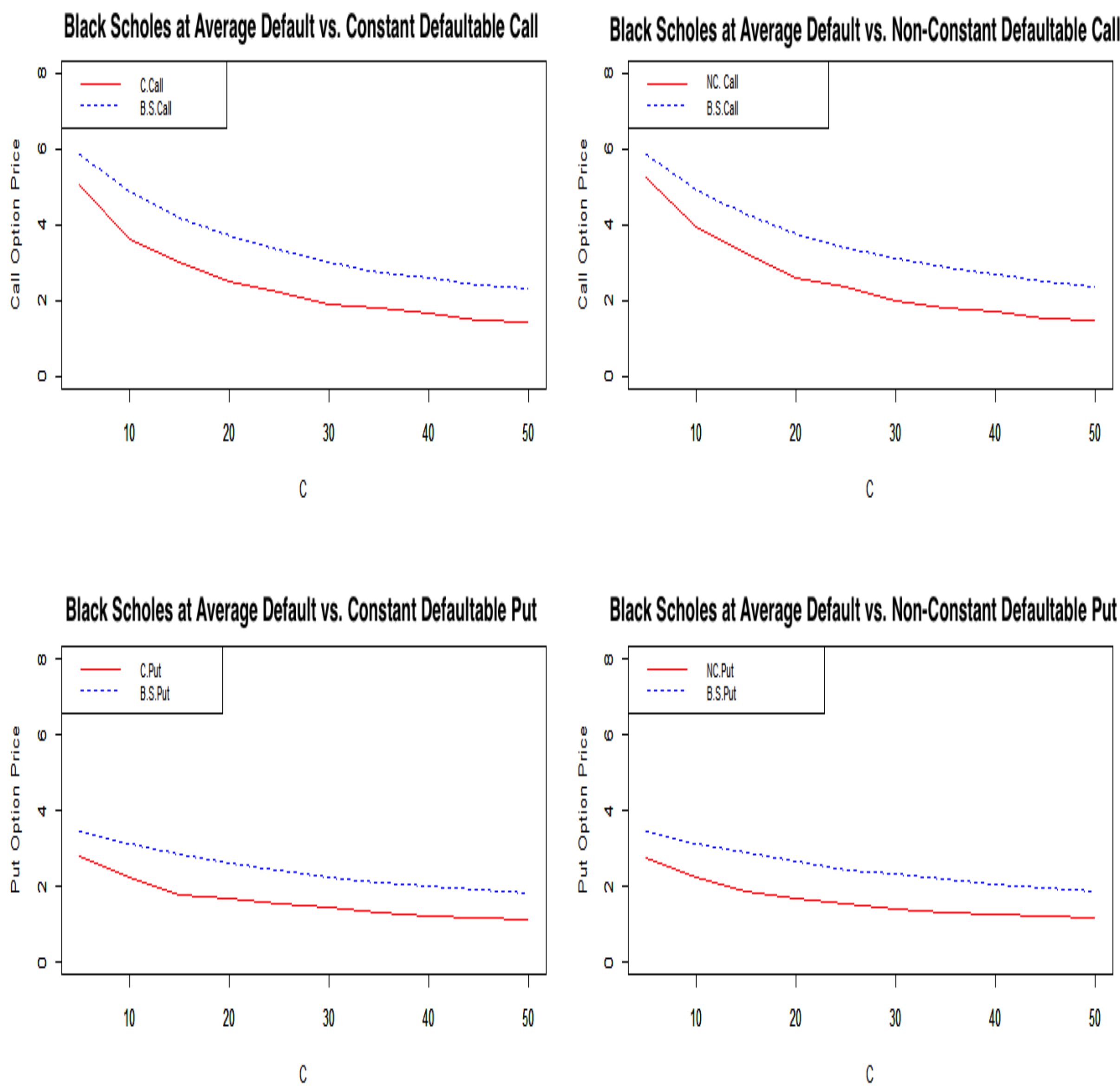

Figure 4.5: Comparison of formula (4.24) with parameters C from 5 to 50, $r = 2\%$, $\sigma = 18\%$, $S = K = \$35$ and $T = 5$.

that the survival probabilities are again similar but the non-constant results have slightly higher survival rates. At this point since results are so close to each other we wonder if a volatility of 18% is not high enough to see differentiation between the two methods of defaultable option pricing. So at this point we will calculate the survival rates with a higher volatility to analyze their difference in behaviour. What can be seen in Figure 4.7 is that the survival probabilities are dependent on the volatility of the stock. When σ was increased from 18% to 35% the non-

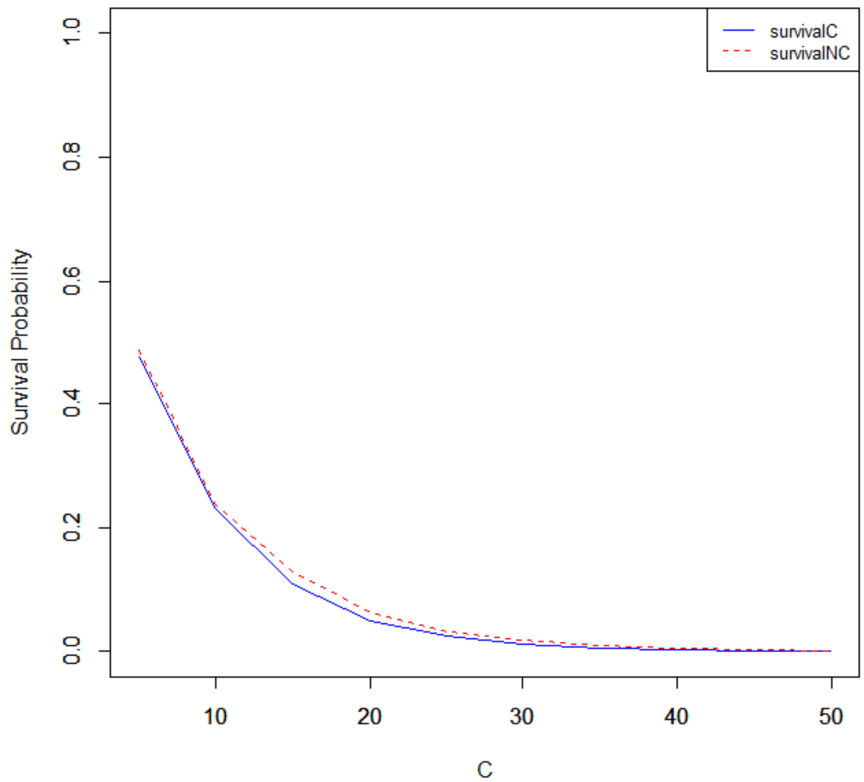

Figure 4.6: Survival probability with the constant and non-constant default rate with parameters C from 5 to 50, $r = 2\%$, $\sigma = 18\%$, $S = K = \$35$ and $T = 5$. For error bars refer to Appendix C, Figure C.4.

constant survival rate tends to separate further from the constant survival rate. Although when σ is low we do not see a significant difference in the survival probabilities, they increase with volatility.

Another area we considered to further validate our results thus far was to look at how these defaultable options perform when tested against Put-Call Parity, defined as:

$$C(t) - P(t) = S(t) - Ke^{-r(T)}. \tag{4.25}$$

In our results we will use τ instead of T, which represents the average time of default depending on the probability of default. Also, $C(t)$ and $P(t)$ represent the option prices when default is implied. $S(t)$ is our initial stock price which we have at \$35. Let's rearrange the Put-Call Parity

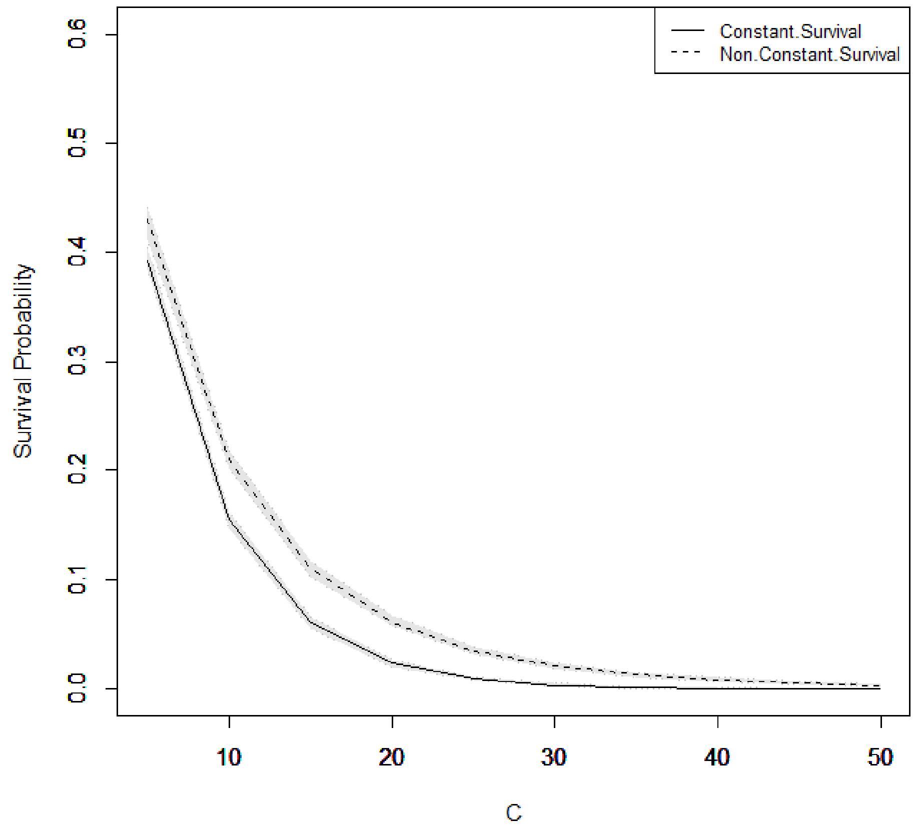

Figure 4.7: Survival probability with the constant and non-constant default rate with parameters C from 5 to 50, $r = 2\%$, $\sigma = 35\%$, $S = K = \$35$ and $T = 5$, with error bars at a 99% confidence level.

formula for the purpose of graphing our results:

$$0 = C(t) - P(t) - S(t) + Ke^{-r(T)}. \tag{4.26}$$

We then will graph this difference against an increasing hazard rate. In a Black-Scholes riskless world the difference should always be 0. In Figure 4.8 we have graphed the difference as defined in equation (4.26). As expected the parity does not hold since our world is no longer riskless but what we see is that with in increase in the default rate the difference in (4.26) increases and therefore we move further away from the Black-Scholes world. The non-constant default rate results in larger differences than the constant model. We can expect this in the sense

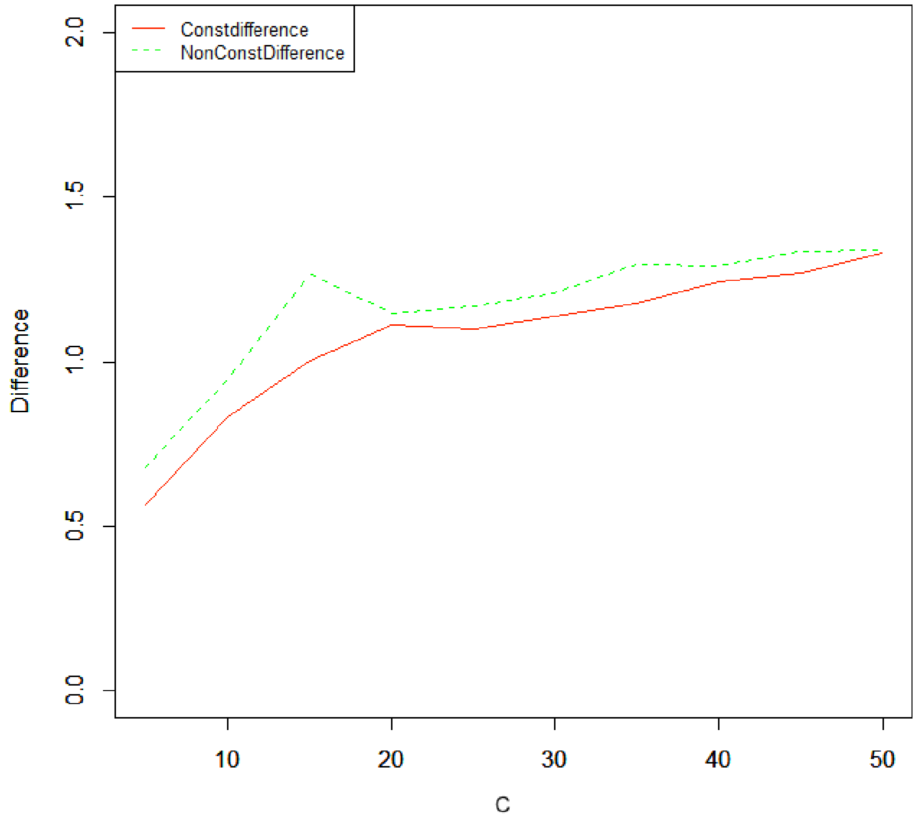

Figure 4.8: Put-Call Parity check holds for the defaultable options with C from 5 to 50, $r = 2\%$, $\sigma = 35\%$, $S = K = \$35$, $R = 70\%$ and $T = 5$.

that the non-constant model holds higher risk since it depends on other variables that drive the stock prices.

Thus far we have not seen a large difference between our two models. We have chosen a stock path that is mostly stable but when the underlying company is heading for bankruptcy their stock path, meaning their drift and volatility will change drastically and our models will not perform similarly anymore. In the further works of this thesis we will also analyze how the default options perform with different rates of volatility.

4.4 Non-Constant Default Against Constant Results and Comparison

After much validation and model checking in the previous section the reader should now be convinced with our results and should be much in agreement when we do the following analysis. We will now review how these option prices related to the CVA concept which is reviewed at the start of this thesis. Recall that CVA was a calculation done not through option pricing alone but through looking at expected exposure and its chances of default. These calculations were then to be subtracted from the risk-free option pricing to receive a fair value of the option that is at risk of default.

This far we priced defaultable options and now CVA could be calculated simply as the difference between the risk-free option and our defaultable option. Throughout the paper we have chosen two methods to price these options to see how they can compare to one another and which method would be more accurate or efficient. These analysis will be done in this section by looking at different views of our results. In the previous section we analyzed the two methods briefly but now we will also look at how our models perform with different parameters.

First, we are interested at looking at how the options will perform in both cases, not at different default rates, but at different strike prices. All parameters will be the same and only the strike price will vary with a fixed default rate. We will see how the option behaves in comparison to Black-Scholes pricing model. The following results will be based on a set value of $C = 15$ which follows that the probability rate of default will be:

$$\lambda(t) = \frac{15}{S_t}. \tag{4.27}$$

As one can see in Figure 4.9 the defaultable options follow the same path as Black-Scholes option pricing but on a lower value. The Black-Scholes options are at the money when strike price $K = \$38.68$, through the Put-Call parity formula. For the defaultable options this strike price is lower. This leads to the assumption that when at-the-money strike price is lower, it can be seen as an option with a shorter maturity, if the recovery rate was 100%. Hence, our defaultable option has a lower maturity than $T = 5$.

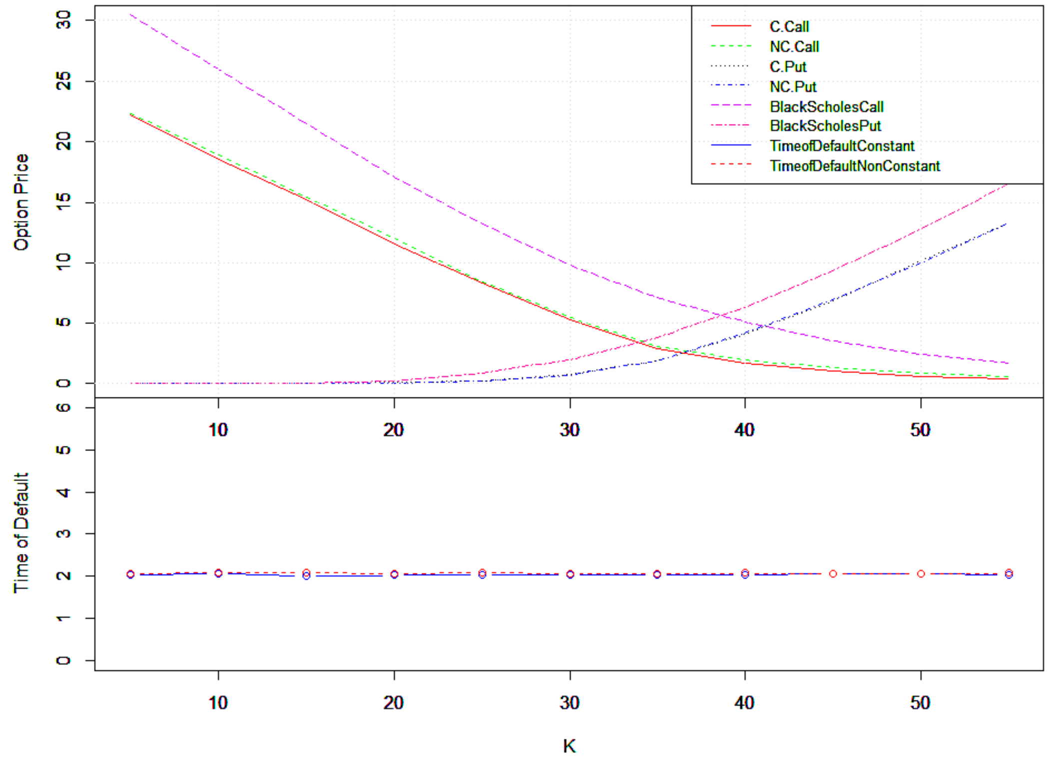

Figure 4.9: Comparison of default and non default option pricing with a range of different strike prices denoted by K, where $C = 15$, $r = 2\%$, $\sigma = 18\%$, $S = \$35$, $R = 70\%$ and $T = 5$.

Now we will proceed to seeing how the defaultable options behave when the recovery rate is full, 100% and when the recovery rate is 0%. In the market, calculating the correct recovery rate is a whole other issue; knowing what the underlying company will pay when they cannot complete their contracts and have filed for bankruptcy is a new area that we have not considered in this paper. Figure 4.10 shows the results of the defaultable options with a recovery rate of 100%. We can see that the option price does not fall as quickly. One can expect that if the recovery rate is 100% then we should not care about the default of an option because all money will be given back. What actually is still driving the option price down is the increase in C which drives the default probability up. Since default is still occurring our model is evaluating that option price based on a lower time of maturity given by τ. Given that all parameters stay the same, in Black-Scholes world an option costs less when its time to maturity is lower than that of a similar option with a later exercise date. This explains the fact that even though the recovery rate is 100% the option price is still decreasing since our option is being evaluated at a shorter time to maturity. Even though there is a default, all money will be collected, but some-

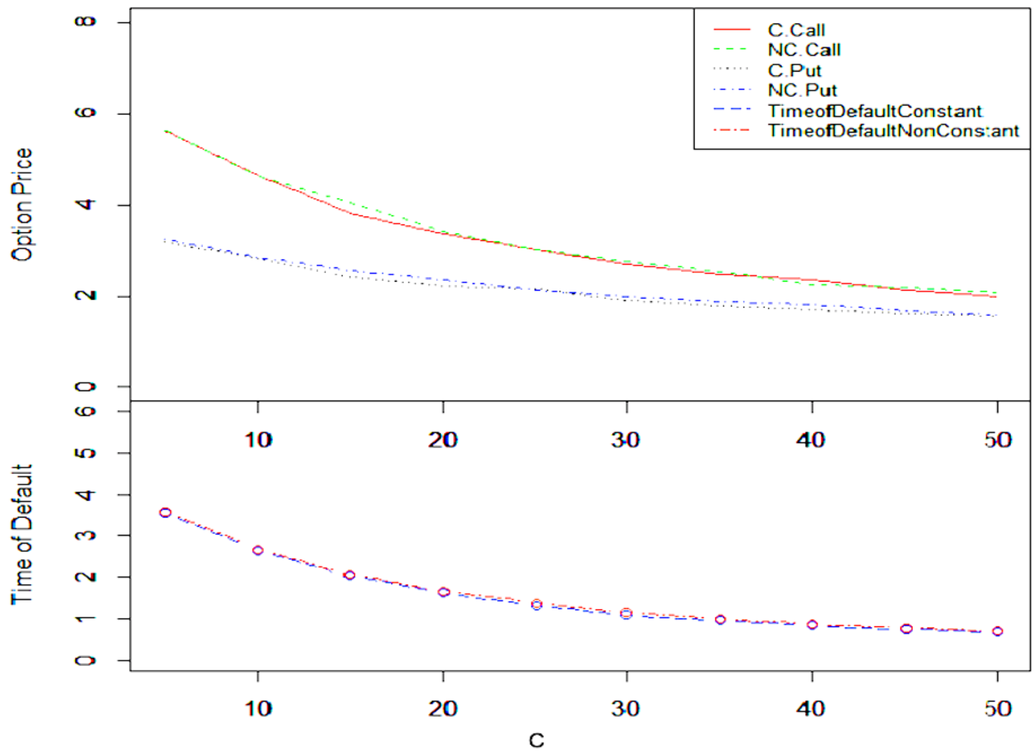

Figure 4.10: Comparison of constant default and non-constant default rate option pricing with a range of different C values, where $R = 100\%$, $r = 2\%$, $\sigma = 18\%$, $S = K = \$35$ and $T = 5$. For the error bars refer to Appendix C, Figure C.5.

times the default is happening right away which drives the option price down no matter what the recovery rate is. Hence, this is why the defaultable options are still decreasing but with a lower speed. In Figure 4.11 we can see that the option prices in both cases are approaching zero very quickly. Here we are basically only including the surviving options that make it up to maturity. As C increases the fewer survivors we have, as expected. The main things noticed in this scenario are that the non-constant against the constant default rate behave differently. The non-constant rate results in slightly higher option values than the constant default rate. Since the recovery rate is 0% and the non-constant default results in higher option pricing, we can see that on average they are surviving more than the constant rate. This was also supported when we looked at survival probability of both default methods, the non-constant rate had higher survival probability.

This far we have seen that volatility has a large influence in separating these models from one another. To further analyze this we have calculated the prices for a call and put option, at

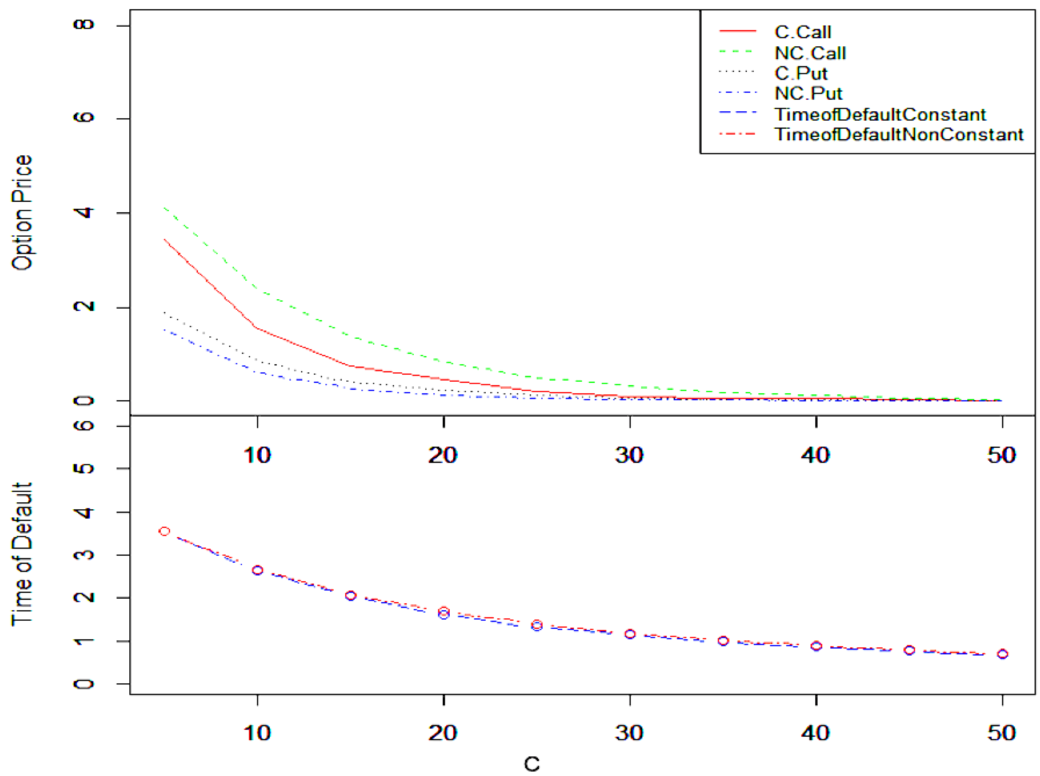

Figure 4.11: Comparison of default and non default option pricing with a range of different C values, where $R = 0\%$, $r = 2\%$, $\sigma = 18\%$, $S = K = \$35$ and $T = 5$.

different volatility values, for both cases of constant and non-constant default rates. Figure 4.12 shows defaultable call option prices with a non-constant rate of default at different volatility rates, where we have taken four values of C, which is the leading parameter of the default rate.

Looking at Figure 4.12 we can see that as C rises, the option pricing falls which is expected. Similarly, when the volatility increases, the option prices rise. We expect this since we know when the volatility of a stock price gets larger, the option price does too. When a stock is more volatile there are higher chances of profit, the demand for this option increases resulting in higher prices. Also we notice that as volatility increases, the gaps between the different defaults is increasing. When C is increasing the default rate is as well, and therefore the option prices are lower but still increasing relative to the volatility. This essentially is explained by the fact that the default rate is also depending on the stock price, which is driven by the volatility. Similar results will also be seen in in Figure 4.13, which is the defaultable put option.

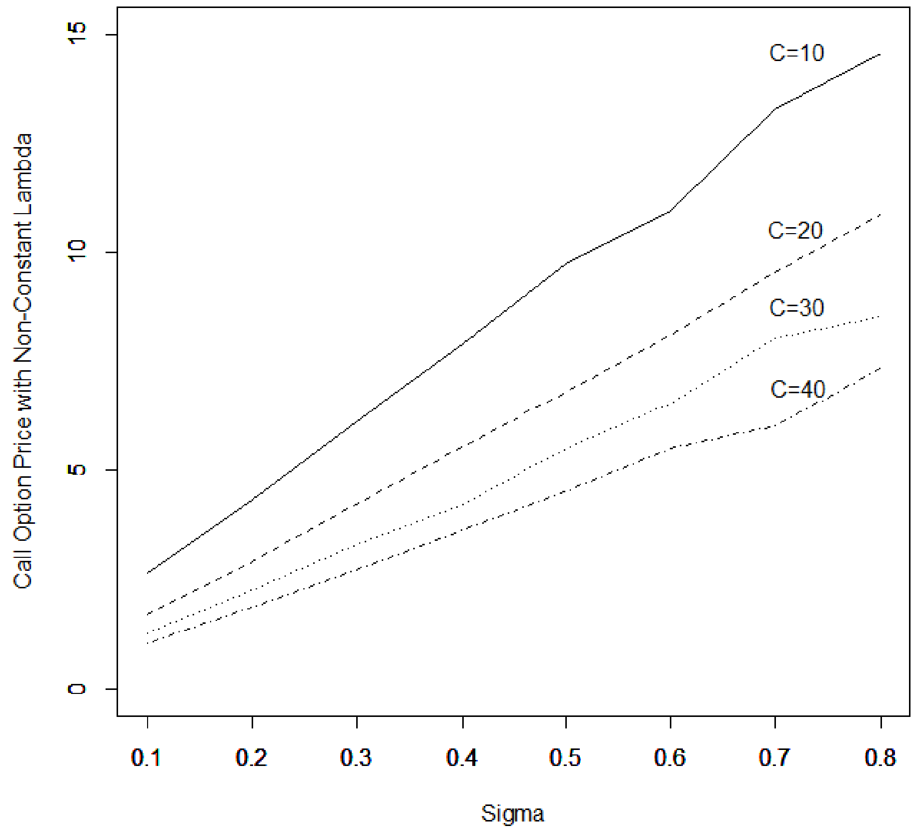

Figure 4.12: Call option pricing with non-constant default rate, at different values of σ and C, $R = 70\%$, $r = 2\%$, $S = K = \$35$ and $T = 5$. For the error bars refer to Appendix C, Figure C.6.

Now we repeat the same graphs as seen in Figures 4.12 and 4.13 but we will use a constant default rate by introducing $\bar{\lambda}$ from Method 1, instead of the non-constant rate, λ_t. What we see when using $\bar{\lambda}$ in Figure 4.14 is that they are no longer performing similar to the non-constant rate. Similar results will be given for a defaultable put option. Both Figures 4.14 and 4.15, show that when the volatility increases we eventually get a decreasing option value, which is not as intuitive. What is happening here is that we have a constant default rate that we calculate through the average of the non-constant rate. Once volatility is high, these averages are becoming so high that our constant default rate is killing the option price right away, resulting in a decline of the option price, when larger σ values occur.

In Figure 4.16 and 4.17 we have graphed the difference of constant and non-constant default

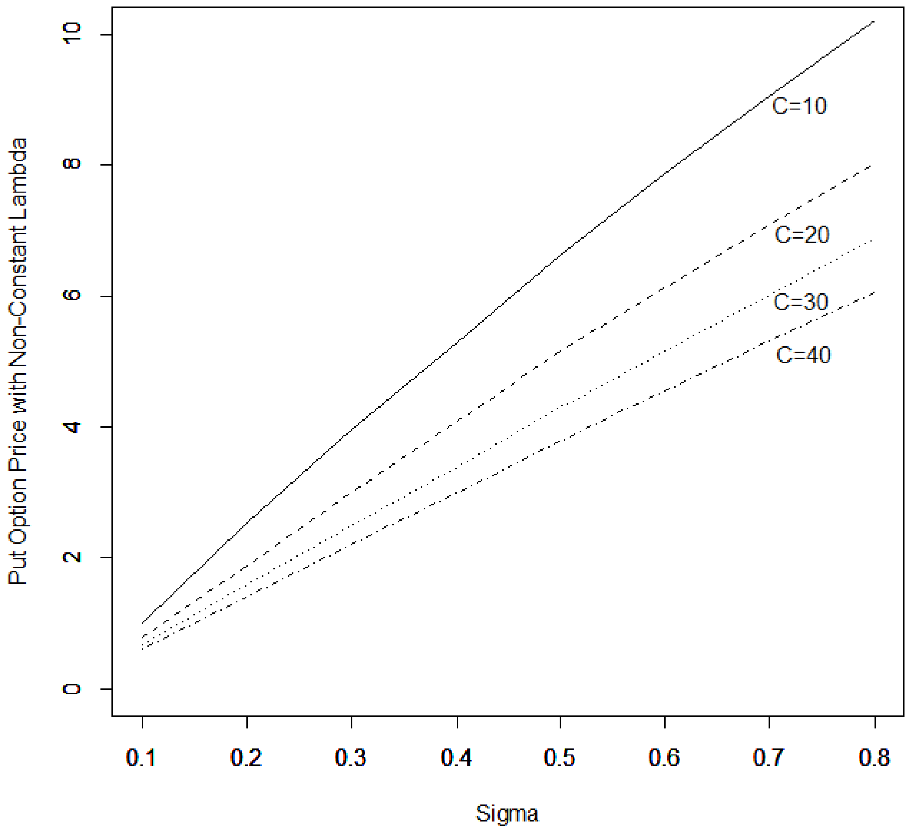

Figure 4.13: Put option pricing with non-constant default rate, at different values of σ and C, $R = 70\%$, $r = 2\%$, $S = K = \$35$ and $T = 5$. For the error bars refer to Appendix C, Figure C.7.

rates for a call and put option. We can see in both of these graphs that when the volatility is low they perform very similarly, but once the volatility increases then the two models split apart. The constant default rate model performs similar to the non-constant in cases of low volatility. Once the volatility passes beyond 50% we actually see the option value for both call and put decrease. This is not something that we expect to happen as we know when the volatility is high the option price needs to increase. This decrease is similar no matter what the C values are. The reason for this can be further explained when looking at our methods of making the constant default rate similar to the non-constant default rate. As mentioned in Section 4.2.1 we

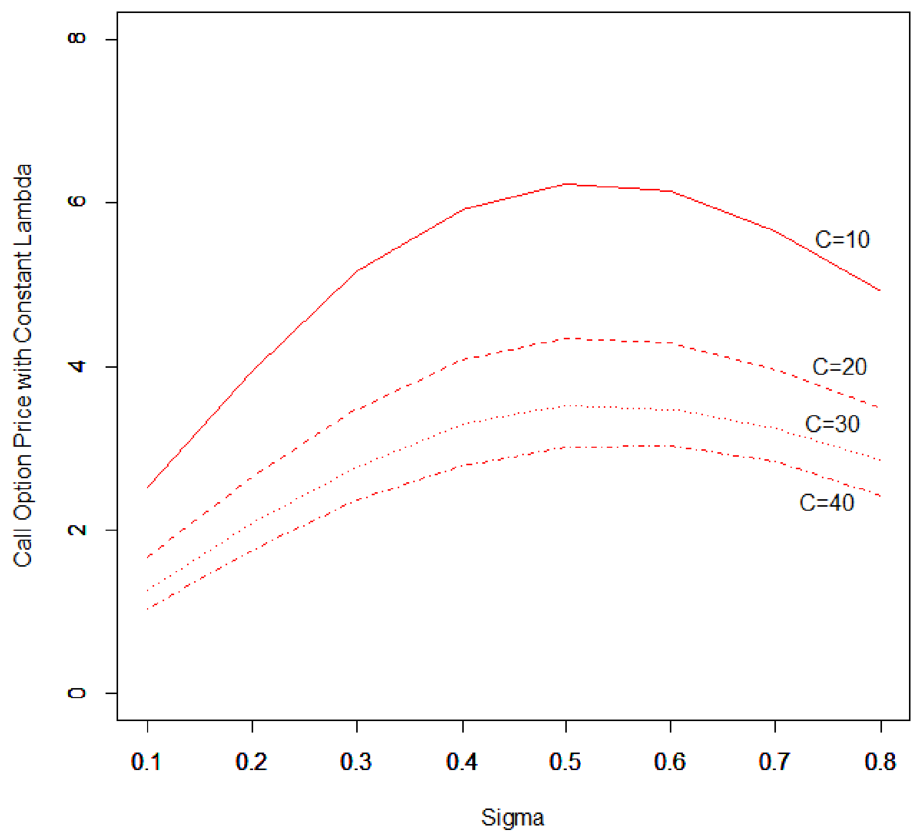

Figure 4.14: Call option pricing with constant default rate, at different values of σ and C, $R = 70\%$, $r = 2\%$, $S = K = \$35$ and $T = 5$.

found an average of the non-constant hazard rate given by:

$$\bar{\lambda}_{[0,T]} = \frac{C}{TS_0} \frac{1}{\mu - \sigma^2} [1 - e^{-(\mu - \sigma^2)T}]. \tag{4.28}$$

Equation 4.28 worked well when used in low volatility rates as we have thus far, but when the volatility does increase $\bar{\lambda}$ increases exponentially which results in a default occurring immediately. This causes the average option price to plunge at such high volatility values. Once $\bar{\lambda}$ reaches high values and is inputted into the option simulation, it defaults right away. At such high volatile cases, a fast default also means that the stock price has a higher chance of being below strike price. Once such results are averaged out we see a huge decline in option price as it was clear in Figure 4.14 and 4.15.

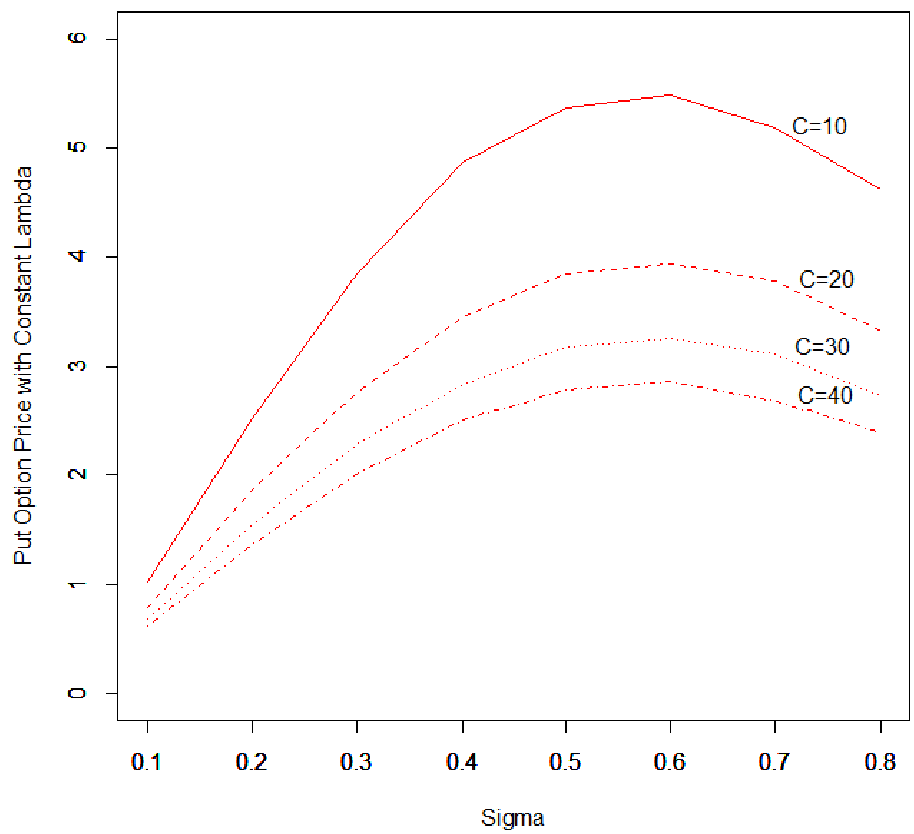

Figure 4.15: Put option pricing with constant default rate, at different values of σ and C, $R = 70\%$, $r = 2\%$, $S = K = \$35$ and $T = 5$.

The default rate that we have calculated through the help of equation 4.28 is a good approximation of the non-constant default rate but only in cases of low volatility. The reason for this is simply that when we calculate $\bar{\lambda}$ we assume that the default rate lasts all the way up to T, the maturity time. But with the non-constant default, λ does not always make it to the maturity time and instead it stops when τ, the default time occurs before T. As shown in Figure 4.18 the two default rates perform similar only at low volatility rates. To solve this issue we need to reconsider Method 1 of calculating $\bar{\lambda}$ which was defined in Section 4.2.1:

$$\bar{\lambda} = \frac{1}{T}\int_0^T E\left[\frac{C}{S_t}\right]dt. \tag{4.29}$$

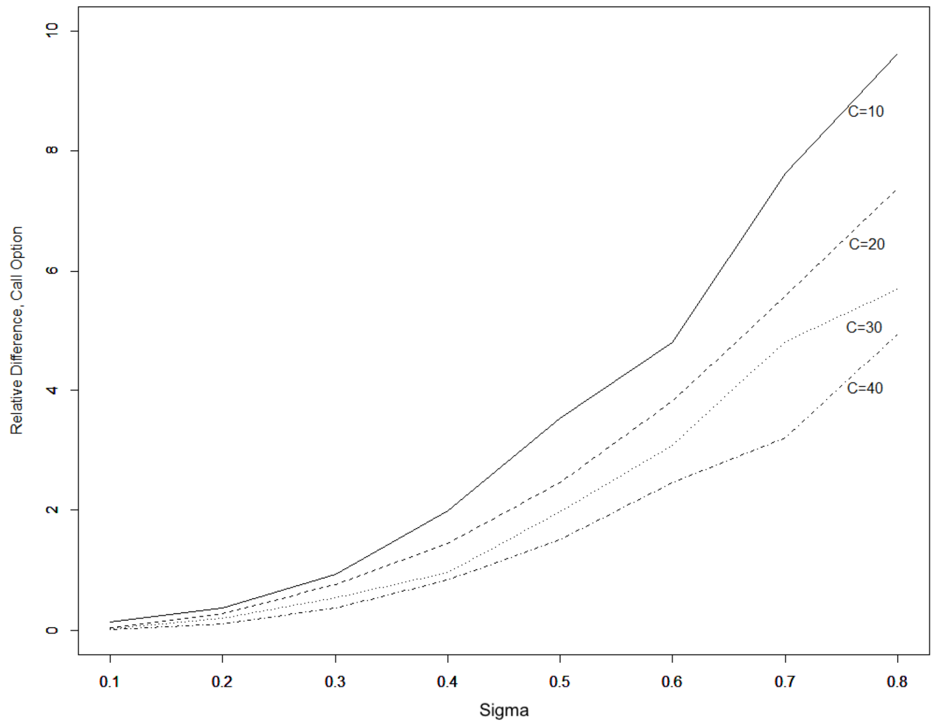

Figure 4.16: Absolute difference of call option pricing with constant and non-constant default rates, at different values of σ and C, $R = 70\%$, $r = 2\%$, $S = K = \$35$ and $T = 5$.

Now we need to consider τ the default time as a random variable to find $\bar{\lambda}$:

$$\bar{\lambda}^* = \int_0^T E\left[\frac{C}{S_\tau}\bar{\lambda}^* e^{-\bar{\lambda}^*\tau}\right] d\tau. \tag{4.30}$$

Following the same steps as in Section 4.2.1 we can calculate the expected value and we will get the following results:

$$\begin{aligned} \bar{\lambda}^* &= \frac{C}{S_0}\int_0^T \bar{\lambda}^* e^{-[(\mu-\sigma^2)+\bar{\lambda}^*]\tau} d\tau \\ &= \frac{C}{S_0}\frac{\bar{\lambda}^*}{(\mu-\sigma^2)+\bar{\lambda}^*}[1 - e^{-[(\mu-\sigma^2)+\bar{\lambda}^*]T}]. \end{aligned} \tag{4.31}$$

In equation 4.31 we assume that $\bar{\lambda}^*$ is independent of the stock price so we can take it out

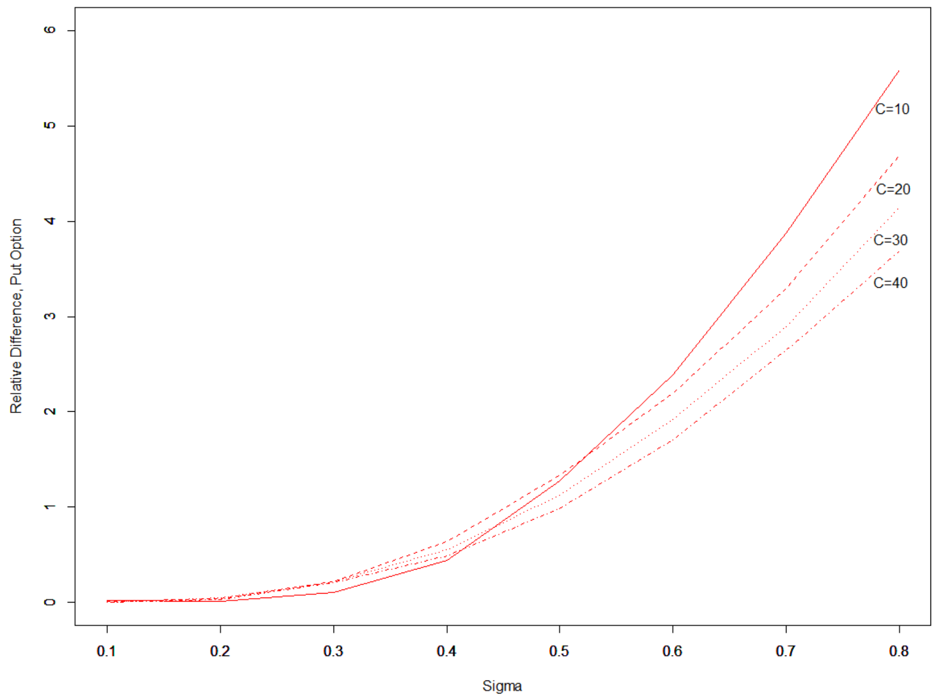

Figure 4.17: Absolute difference of put option pricing with constant and non-constant default rates, at different values of σ and C, $R = 70\%$, $r = 2\%$, $S = K = \$35$ and $T = 5$.

of the integral. By doing so we have the following results:

$$\begin{aligned} 1 &= \frac{C}{S_0}\frac{1}{(\mu-\sigma^2)+\bar{\lambda}^*}[1-e^{-[(\mu-\sigma^2)+\bar{\lambda}^*]T}] \\ 0 &= \frac{C}{S_0}\frac{1}{(\mu-\sigma^2)+\bar{\lambda}^*}[1-e^{-[(\mu-\sigma^2)+\bar{\lambda}^*]T}]-1. \end{aligned} \tag{4.32}$$

We can calculate a solution to (4.32) through simple numerical methods, for example Newton type methods. The results of this solution should converge closer to the non-constant default rate than the previous calculations of $\bar{\lambda}$ in Method 1. As shown in Figure 4.19 the new method calculated through $\bar{\lambda}^*$, does bring the default rate closer to the non-constant rate that we extract from the Monte Carlo simulations. The results are still not the same and $\bar{\lambda}^*$ is still overestimating the true non-constant default rate. The reason for such results can be explained by referring back to results shown in 4.31. For calculation simplicity we made an assumption while taking the integral, that $\bar{\lambda}^*$ was independent of τ the default time. In better words we treated $\bar{\lambda}^*$ as a constant variable. This made the integral calculation easier but in fact there is dependence

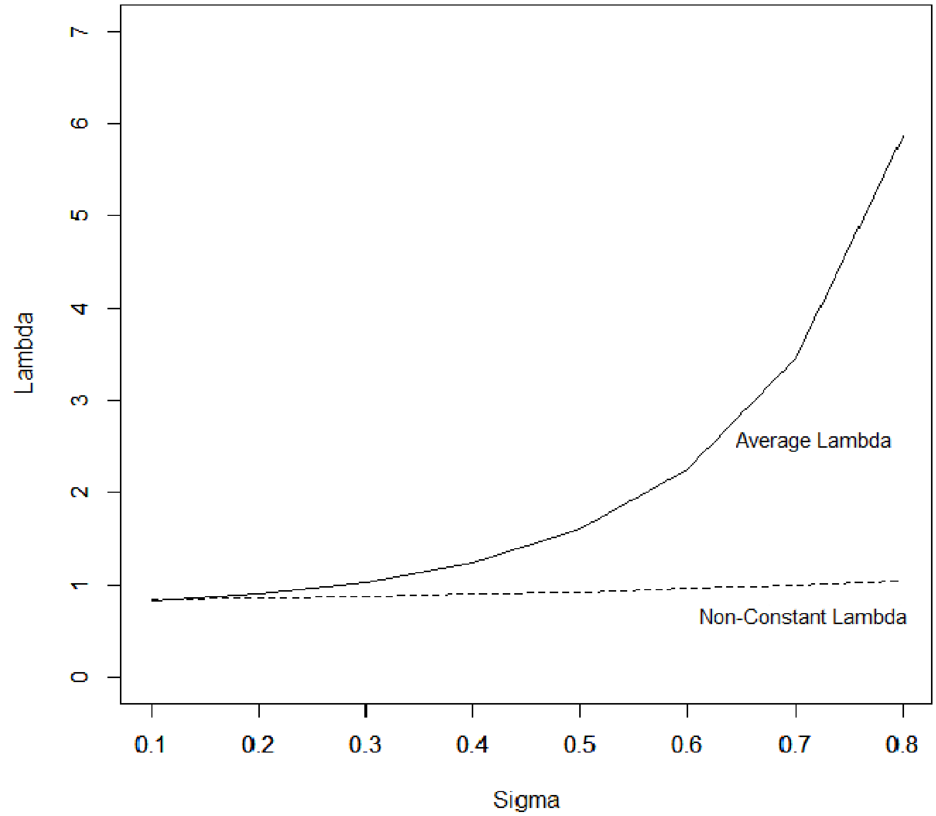

Figure 4.18: Comparison of the average default rate given in equation 4.28 and the non-constant default rate calculated through Monte Carlo Simulations, shown at different values of σs where $C = 30$, $R = 70\%$, $r = 2\%$, $S = K = \$35$ and $T = 5$.

between $\bar{\lambda}^*$ and the stock price since the default rate is inversely related to the stock path. This explains why we still have an overestimation although this new way of approximating an average non-constant default rate is more accurate than Method 1 results of Section 4.2.1. If we observe in Figure 4.18 at a volatility of 80% the average $\bar{\lambda}$ was six times larger then λ_t, now with a new method as shown in Figure 4.19, $\bar{\lambda}^*$ is only half larger.

After we applied a change to the average default rate that we are using as a constant rate, the results improved. We have applied this new default rate to the Monte Carlo simulations of the constant default option pricing algorithm. The results of these option prices are shown in Figures 4.20 and 4.21. The option prices are now performing better and not decreasing with the volatility increasing as we saw previously. We have been able to find an analytical method to average out the non-constant default rate to a constant. This method gave us very similar results to the non-constant simulations even at higher volatilities. With such results we are able

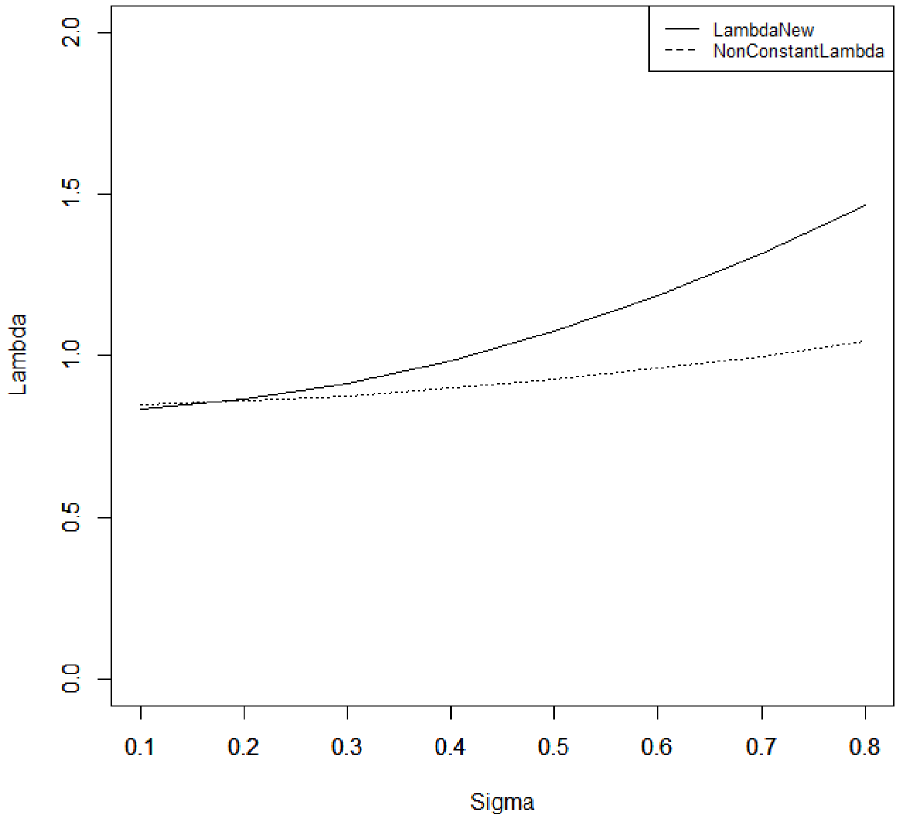

Figure 4.19: Comparison of the average default rate, $\bar{\lambda}^*$ called the new λ, given in equation 4.32 and the non-constant default rate calculated through Monte Carlo Simulations, shown at different values of σs where $C = 30$, $R = 70\%$, $r = 2\%$, $S = K = \$35$ and $T = 5$.

to compute the defaultable options price with a non-constant default rate more efficiently, by converting such rate into a constant one. This saves computational time while still holding the intuition of a non-constant default rate.

4.5 Credit Value Adjustment

Credit value adjustment was defined in Chapters 1 and 2. Throughout this thesis we mainly focused on the calculations of defaultable European call and put options. As defined in Chapter 2, CVA is given through:

$$V(t) = \hat{V}(t) + \phi(t), \tag{4.33}$$

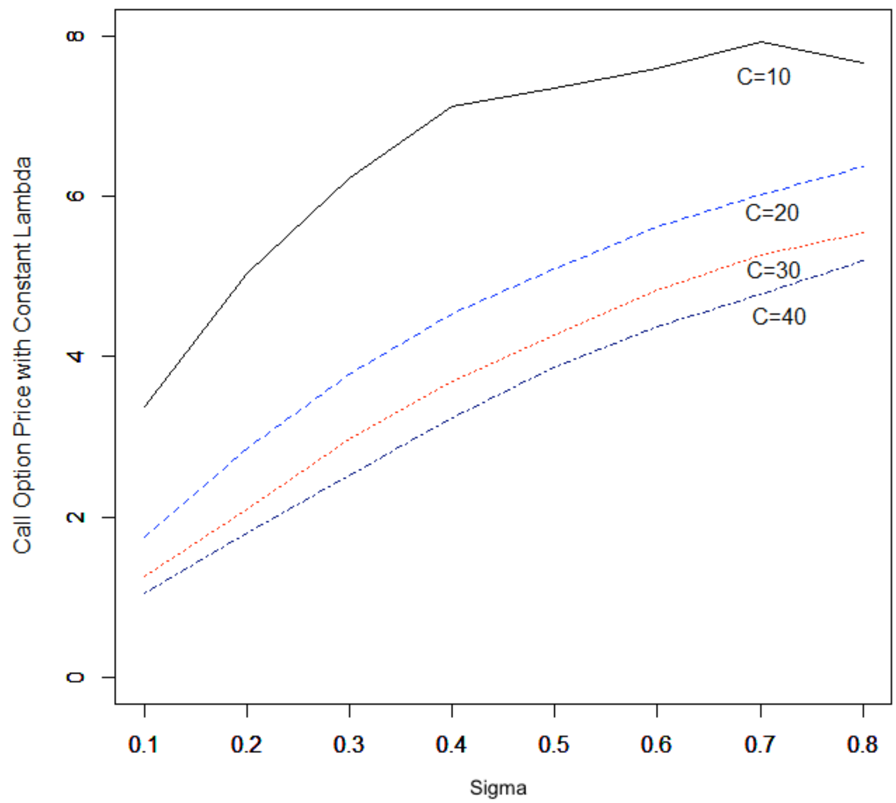

Figure 4.20: Call option pricing with constant default rate, $\bar{\lambda}^*$, at different values of σs where $R = 70\%$, $r = 2\%$, $S = K = \$35$ and $T = 5$. For the error bars refer to Appendix C, Figure C.8.

where $\phi(t)$ is the CVA, $V(t)$ is the risk free value of a portfolio containing a single option and $\hat{V}(t)$ represents the value of the same portfolio but with a chance of default. Since we have calculated the price of defaultable option and the risk free option price is simply given by Black-Scholes formula. The CVA can now be calculated through:

$$\phi(t) = V(t) - \hat{V}(t). \tag{4.34}$$

The way we have chosen to estimate CVA is through pricing defaultable options. This was the main challenge throughout the thesis. Therefore, the CVA is only a simple subtraction of defaultable option from the Black-Scholes price. The CVA value of a constant default rate and non constant default rate call option are shown in Figures 4.22 and 4.23; for put options see

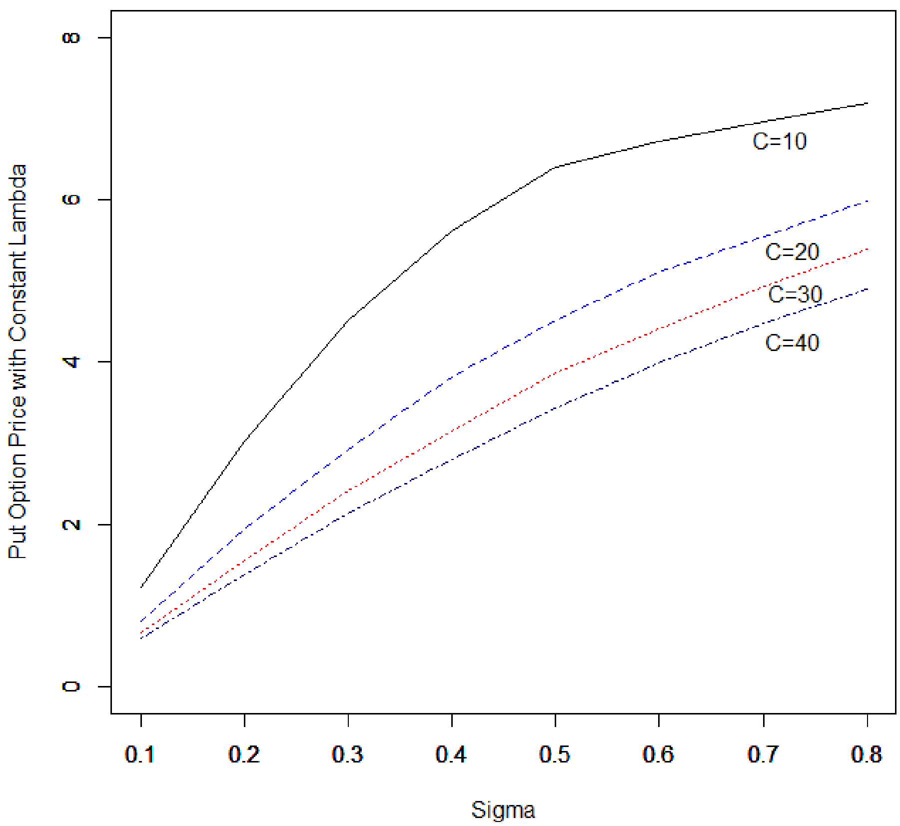

Figure 4.21: Put option pricing with constant default rate, $\bar{\lambda}^*$, at different values of σs where $R = 70\%$, $r = 2\%$, $S = K = \$35$ and $T = 5$. For the error bars refer to Appendix C, Figure C.9.

Figures 4.24 and 4.25.

As expected by our intuition, the CVA in all cases is increasing with volatility and with C, the constant parameter that drives the hazard rate. When C increases so is the default probability and hence CVA is also on the rise. This is explained when thinking about CVA as the value of the portfolio that we are putting at risk. We can think of this through the example of company A entering into a contract with company B. If the the default probability of company B is increasing, the CVA is too, this shows company A the maximum amount that they lose given the default of the other company. When consider the volatility, we know Black-Scholes option prices always increase with volatility. Similarly, when we evaluate the defaultable options, the same pattern follows.

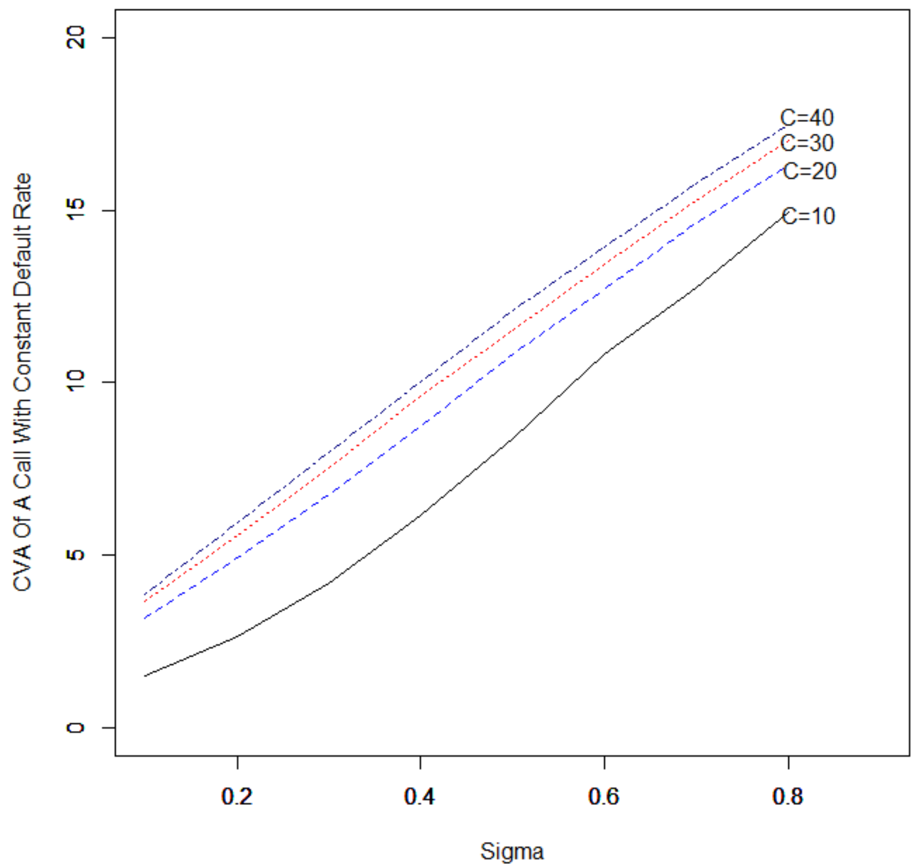

Figure 4.22: CVA of a call option with a constant default rate, $\bar{\lambda}^*$, at different values of σs where $R = 70\%$, $r = 2\%$, $S = K = \$35$ and $T = 5$.

When observing the constant and non constant default models notice that for both put and call options there is a discrepancy at higher volatility rates. The CVA for both models starts at the same values when the volatility is low but once the volatility increases notice that CVA of a constant default rate is higher than that of a non-constant default rate model. This was explained through the over estimation of $\bar{\lambda}^*$ as it was shown in Figure 4.19. A CVA that is falsely higher is not a desirable scenario in the financial world since this portrays the underlying companies' option value as actually undervalued. The method of an average non-constant hazard rate works well in terms of low volatility and is preferred because of lower computational time. Once the volatility increases in the stock price the non-constant default rate should be used since it it will give a better approximation of CVA. This will allow the

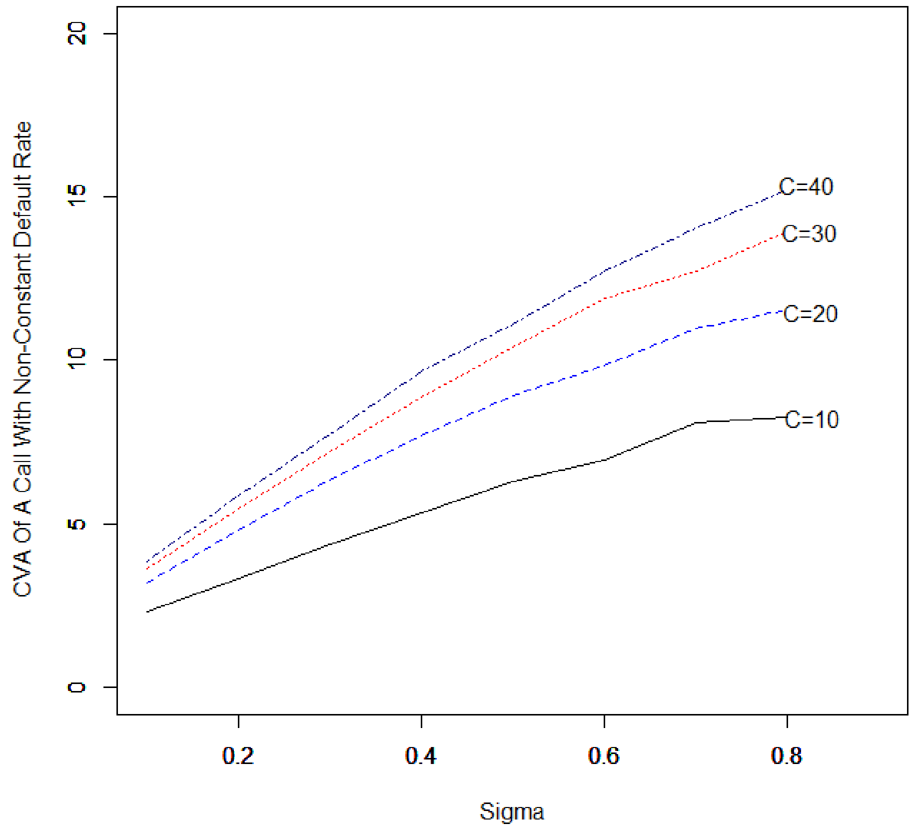

Figure 4.23: CVA of a call option with a non-constant default rate, λ_t, at different values of σs where $R = 70\%$, $r = 2\%$, $S = K = \$35$ and $T = 5$.

company that issues the contract to have a true valuation of their default rate. Hence, the company entering the contract will have less collateral requirement for such options.

4.6 Conclusion

This chapter focuses on modelling a defaultable option price through a non-constant default rate that is inversely related to the underlying stock price. We started by explaining an inhomogeneous Poisson process and how it relates to our model of a non-constant default rate. In Chapter 3 we had analytical and simulation results to defaultable options with a constant default rate. In this chapter we bring the constant and non-constant default rates together. We found two methods of converting the non-constant hazard rate into a comparable constant rate.

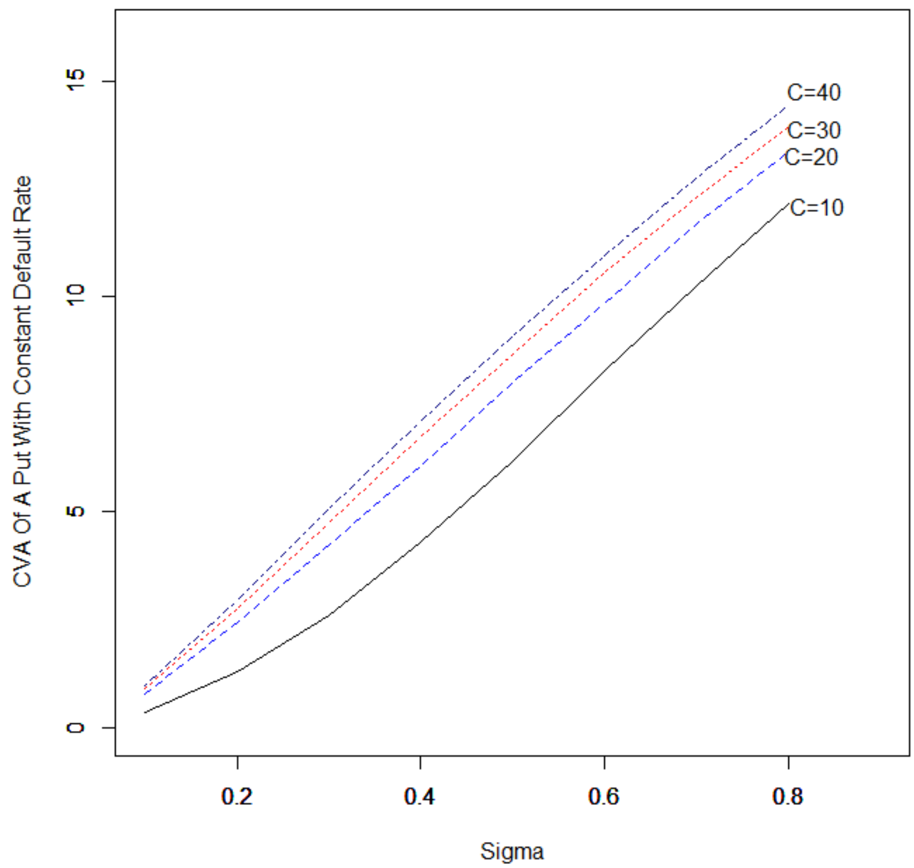

Figure 4.24: CVA of a put option with a constant default rate, $\bar{\lambda}^*$, at different values of σs where $R = 70\%$, $r = 2\%$, $S = K = \$35$ and $T = 5$.

After thorough analysis it was evident that one method outperformed the other through computational time.

Now that we had converted the non-constant default rate into a constant, we were able to graph and compare our option pricing results. What we found in this chapter is that when we use a default rate that depends on the stock price, we can estimate a constant rate that corresponds closely to the non-constant one. We were then able to proceed on calculating the option price which would take shorter computational time. The results showed that at low volatility our method of converting a non-constant rate of default into a constant one performed similar to one another. We also calculated the CVA through the simple process of subtracting the default option prices form the Black-Scholes pricing. The average constant default rate method

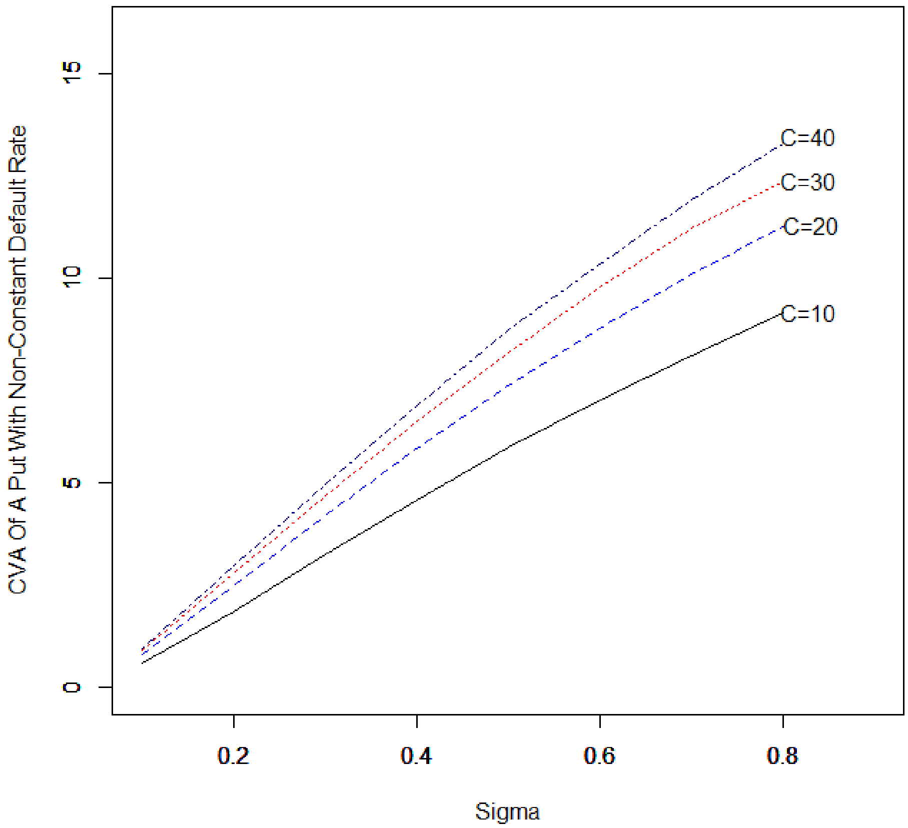

Figure 4.25: CVA of a put option with a non-constant default rate, λ_t, at different values of σs where $R = 70\%$, $r = 2\%$, $S = K = \$35$ and $T = 5$.

resulted in higher CVA values than that of the non-constant rate which is unlikable in the industry. In case of a counterparty defaulting, having estimated a higher CVA would be beneficial to the company because they would be more prepared for such downfall, but in reality the industry avoids higher CVA charges as it makes their accounting books look worst.

Chapter 5

Conclusion

5.1 Discussion

In this thesis, we focused on pricing defaultable options. This work was inspired by the Credit Value Adjustment concept in demand since the recent crisis. At first we explained in detail how CVA started, and the change in the Basels and Accounting Regulation requirements that made it increasingly popular. Counterparty credit risk was explained as the initial idea behind CVA calculations. This type of risk is directional which means having right or wrong way risk. Counterparty risk can be seen in different types of assets, in this thesis we only focused on over the counter vanilla call and put options. To further analyze defaultable options we defined a few building blocks of our research by explaining the Black-Scholes equation and two structural models, Merton and Black & Cox model.

At first we presented a detailed new method to our knowledge, on calculating the analytical results of defaultable options with a constant default rate. In Chapter 3 we attempted to solve for a closed form approximate solution, we did this by setting the strike price equal to the stock price at the initial time and by setting the risk free interest rate to zero. Controlling such parameters we were able to solve through the defaultable options equation with the help of the Black-Scholes formula. We first had to apply the Taylor series to solve the integrals and then used the error function to approximate the results. In the end our analytical results worked efficiently with small errors. Our results were then compared to Monte Carlo simulations, and a quadrature approximation. In conclusion to Chapter 3, the analytical solution we solved, appeared to be a great approximation of the defaultable options with only minor errors. Quadrature approximation, on the other hand, had even less error and performed exactly the same as Monte Carlo simulations, but this low dimensional model took a lot less computational time.

In Chapter 4 we proposed a new model where the default rate is non-constant and dependent on the stock price. We related the stock price to the default rate by assuming that when the stock price rises, the company's rate of default decreases and vice-versa. To compare how the constant and non-constant default rates affect the behaviour of such options we came up with two methods. One method involved finding an average of the non-constant default rate and using it as the constant rate. The second method was finding the constant default rate through the probability of default of the non-constant hazard rate. Both methods worked similarly and therefore we decided to use method one which was simpler to use with less computational time.

In the next step, by using method one we compared the results of a defaultable option with constant and non-constant rate being similar to one another. After much validation we were able to find an average of the non-constant default rate that best fit the true value. In the end our conclusion states that the constant default rate against the non-constant default rate gave the same results at low volatilities. Our method of converting the non-constant rate was then presented through calculations of the CVA. This was a simple step of taking the difference between the riskless option and the defaultable option value. The results showed that at higher volatility rates, CVA was larger for the constant default rate method than that of the non-constant method. From a companies point of view, the CVA being falsely higher is an unlikable aspect. Therefore, we recommend that when the stock is highly volatile one should directly use the non-constant default rate method. However, when the stock is less volatile then the constant default rate will give similar results with shorter computational time.

5.2 Financial Modelling Implications

Default rates in the market are not clear and are hard to model. They are very random events and it is hard to predict when a company is getting riskier. The best prediction one can do is to come up with a model that would warn you if a company might be struggling financially.

Credit scoring companies have been calculating default rates based on historical results. What we suggest in this thesis is that default rates can be modelled through a random Poisson jump process. This process will be driven by a default rate, which we suggest to have it dependent on the current stock price. This allows for a flexible default rate that captures current events of the company instead of historical.

In the financial world our non-constant default rate model is also easily transformed to the constant rate, making computational time shorter. Therefore, if a default rate is dependent on the stock price, it holds information about the company in the current time but it still can be calculated through a similar constant model. However, this relies on the process that our stock price follows and the parameters that drive it.

5.3 Future Work

For future work we recommend that one could find a more complex model to represent the default rate. In this thesis we used an inverse of a Geometric Brownian Motion stock price to represent the default rate, this made financial sense, but it is not adequate enough to represent reality. We suggest that:

$$\lambda_t = f(S_t), \tag{5.1}$$

where the default rate is still represented as some function of the stock price. Presumably, the non-constant λ_t could still be transformed into constant default rate, $\bar{\lambda}$. The conversion would make calculations of defaultable options efficient. The initial intuition still has to hold where if a company is doing well, hence their stock price is strong then the default rate should decrease.

In our model, small changes in the stock price are always being captured as an effect to the default rate, for future analysis one should consider that small stock price movements either up or down should not be as influential to the default price. Here one could also apply the idea of the Black Cox structural model, where a boundary is also considered in terms of the variability of the stock price. The work we have done in this area can also be applied and tested to real financial data. It is recommended that these models of estimating the default rate of a company can be further tested through historical data and evaluated. The usage of recovery rate in this thesis remained constant, in the industry this is a very important parameter that can also be estimated by looking a company's financial stability. This topic can be extended and improved in many of the above mentioned ways, to make it more applicable to real world scenarios.

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Appendix A

Basel Committee on Banking Supervision Countries

Group of 10 Countries: Canada, United States, Germany, France, Belgium, Italy, Japan, Netherlands, Sweden, Switzerland, United Kingdom.

Basel Committee on Banking Supervision Participating Members: Argentina, Australia, Belgium, Brazil, Canada, China, France, Germany, Hong Kong SAR, India, Indonesia, Italy, Japan, South Korea, Luxembourg, Mexico, Netherlands, Russia, Saudi Arabia, Singapore, South Africa, Spain, Sweden, Switzerland, Turkey, United Kingdom, United States.

Appendix B

Detailed Analytic Calculations for Defaultable Options

Here we will show detailed calculations of how we have derived an analytical solution to a defaultable call option. Taylor series expansion up to the second term is:

$$N(x) = N(a) + N'(a)(x-a) + N''(a)\frac{(x-a)^2}{2} + O(x^3). \tag{B.1}$$

We use the Taylor expansion centred at some value a to get an analytical solution of a call option, given by the Black-Scholes equation:

$$C(S,t) = SN(d_1) - Ke^{-r(T-t)}N(d_2). \tag{B.2}$$

As mentioned in Chapter 3, $S = K$ and $r = 0\%$. This results in $d_1 = \frac{1}{2}\sigma\sqrt{\tau}$ and $d_2 = -d_1$. Now we further proceed to input the Taylor expansion up to the second term in the call option pricing equation:

$$\begin{aligned} C_{BS}(\tau) &\approx S\left[N(a)+N'(a)[d_1-a]+N''(a)\frac{[d_1-a]^2}{2}-N(a)-N'(a)([-d_1-a]-N''(a)\frac{[-d_1-a]^2}{2}\right] \\ &= S\left[N'(a)(d_1-a)+N'(a)(d_1+a)+\frac{N''(a)}{2}[(d_1-a)^2-(d_1+a)^2]\right] \\ &= S\left[N'(a)(2d_1)+\frac{N''(a)}{2}(d_1^2-2ad_1+a^2-d_1^2-2ad_1-a^2)]\right] \\ &= S\left[N'(a)(2d_1)+\frac{N''(a)}{2}(-4ad_1)\right]. \end{aligned} \tag{B.3}$$

Now we input $d_1 = \frac{1}{2}\sigma\sqrt{\tau}$ in equation (B.3), which results in the following:

$$= S\sigma\sqrt{\tau}[N'(a) - N''(a)a] \tag{B.4}$$

where,

$$N'(a) = \frac{1}{\sqrt{2\pi}}e^{\frac{-a^2}{2}}, \tag{B.5}$$

and

$$N''(a) = -\frac{a}{\sqrt{2\pi}}e^{\frac{-a^2}{2}}. \tag{B.6}$$

Therefore,

$$[N'(a) - N''(a)a] = [1+a^2]\frac{e^{\frac{-a^2}{2}}}{\sqrt{2\pi}}, \tag{B.7}$$

$$C_{BS}(\tau) = S\sigma\sqrt{\tau}[1+a^2]\frac{e^{\frac{-a^2}{2}}}{\sqrt{2\pi}}. \tag{B.8}$$

Our analytical approximations of a defaultable call option is given by:

$$R\int_{\tau=0}^{T} C_{BS}(\tau)\lambda e^{-\lambda\tau}d\tau + C_{BS}(T)\int_{\tau=T}^{\infty}\lambda e^{-\lambda\tau}d\tau. \tag{B.9}$$

We are interested in analytically solving the first integral of (B.9), since the second integral was solved in Chapter 3. Now we have some results for $C_{BS}(\tau)$ in equation (B.8), which we will input and solve through in the following:

$$R\int_{\tau=0}^{T} C_{BS}(\tau)\lambda e^{-\lambda\tau}d\tau = R\int_{\tau=0}^{T}(S\sigma\sqrt{\tau}[1+a^2]\frac{e^{\frac{-a^2}{2}}}{\sqrt{2\pi}}P_\lambda(\tau))d\tau \tag{B.10}$$

$$= R\int_{\tau=0}^{T} S\sigma\sqrt{\tau}[1+a^2]\frac{e^{\frac{-a^2}{2}}}{\sqrt{2\pi}}\lambda e^{-\tau\lambda}d\tau.$$

Now we discretized T, and we let $a = t_i$, so the Taylor series is centred at t_i giving us the opportunity to evaluate this integral correctly at all different values between $[0, T]$. So we will take the integral in small t_i values and then sum these integrals together from $[0, T]$, which results in:

$$R\int_{\tau=0}^{T} C_{BS}(\tau)\lambda e^{-\lambda\tau}d\tau = R\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}(S\sigma\sqrt{\tau}[1+t_i^2]\frac{e^{\frac{-t_i^2}{2}}}{\sqrt{2\pi}}\lambda e^{-\tau\lambda}d\tau \tag{B.11}$$

$$= R\sigma S\sum_{i=0}^{n-1}[1+t_i^2]\frac{e^{\frac{-t_i^2}{2}}}{\sqrt{2\pi}}\int_{t_i}^{t_{i+1}}\sqrt{\tau}\lambda e^{-\tau\lambda}d\tau.$$

Now we apply change of variables:

$$x^2 = \tau, \qquad x = \sqrt{\tau}, \qquad 2xdx = d\tau. \tag{B.12}$$

Boundary conditions are:

$$\text{when: } \tau = t_i \Rightarrow x = \sqrt{t_i} \qquad \text{and when: } \tau = t_{i+1} \Rightarrow x = \sqrt{t_{i+1}} \tag{B.13}$$

Changing the variables results in the following:

$$R\sigma S\sum_{i=0}^{n-1}[1+t_i^2]\frac{e^{\frac{-t_i^2}{2}}}{\sqrt{2\pi}}\int_{\sqrt{t_i}}^{\sqrt{t_{i+1}}}2x^2\lambda e^{-x^2\lambda}dx.$$

Now we apply integration by parts:

$$\text{Let } u = x, \qquad du = dx. \tag{B.14}$$

$$\text{Let } dv = -2x\lambda e^{-\lambda x^2}, \qquad v = e^{-\lambda x^2}. \tag{B.15}$$

Therefore applying equations (B.14) and (B.15) through the integration by parts, we get the following results:

$$\begin{aligned} &= -R\sigma S \sum_{i=0}^{n-1}[1+t_i^2]\frac{e^{\frac{-t_i^2}{2}}}{\sqrt{2\pi}}\left[xe^{\lambda x^2}\mid_{\sqrt{t_i}}^{\sqrt{t_{i+1}}} - \int_{\sqrt{t_i}}^{\sqrt{t_{i+1}}} e^{-\lambda x^2}dx\right] \\ &= -R\sigma S \sum_{i=0}^{n-1}[1+t_i^2]\frac{e^{\frac{-t_i^2}{2}}}{\sqrt{2\pi}}\left[\sqrt{t_{i+1}}e^{\lambda t_{i+1}} - \sqrt{t_i}e^{\lambda t_i} - \int_{\sqrt{t_i}}^{\sqrt{t_{i+1}}} e^{-\lambda x^2}dx\right]. \end{aligned} \tag{B.16}$$

Now we will focus on the integral:

$$\begin{aligned} \int_{\sqrt{t_i}}^{\sqrt{t_{i+1}}} e^{-\lambda x^2}dx &= \int_{\sqrt{t_i}}^{0} e^{-\lambda x^2}dx + \int_{0}^{\sqrt{t_{i+1}}} e^{-\lambda x^2}dx \\ &= -\int_{0}^{\sqrt{t_i}} e^{-\lambda x^2}dx + \int_{0}^{\sqrt{t_{i+1}}} e^{-\lambda x^2}dx \\ &= \int_{0}^{\sqrt{t_{i+1}}} e^{-\lambda x^2}dx - \int_{0}^{\sqrt{t_i}} e^{-\lambda x^2}dx. \end{aligned} \tag{B.17}$$

Notice in equation (B.17), the two integrals are similar to the error function which was defined in Chapter 3, here we will go through the details of the transformation.

$$-u^2 = -\lambda x^2, \qquad u^2 = \lambda x^2, \qquad u = x\sqrt{\lambda}, \qquad \frac{du}{dx} = \sqrt{\lambda}. \tag{B.18}$$

The boundary conditions change to:

$$\text{when } x \in [0, t_i] \text{ then } u \in [0, \sqrt{t_i\lambda}], \tag{B.19}$$

and

$$\text{when } x \in [0, t_{i+1}] \text{ then } u \in [0, \sqrt{t_{i+1}\lambda}]. \tag{B.20}$$

We can apply this to equation (B.17):

$$\int_{\sqrt{t_i}}^{\sqrt{t_{i+1}}} e^{-\lambda x^2} dx = \frac{1}{\sqrt{\lambda}} \int_0^{\sqrt{t_{i+1}\lambda}} e^{-u^2} du - \frac{1}{\sqrt{\lambda}} \int_0^{\sqrt{t_i\lambda}} e^{-u^2} du. \tag{B.21}$$

Now a simple transformation will make equation (B.22) into the error function:

$$\int_{\sqrt{t_i}}^{\sqrt{t_{i+1}}} e^{-\lambda x^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{\lambda}} erf(\sqrt{t_{i+1}\lambda}) - \frac{1}{2}\sqrt{\frac{\pi}{\lambda}} erf(\sqrt{t_i\lambda}). \tag{B.22}$$

Lets go back to equation (B.11) since the integral has been fully solved:

$$\begin{aligned} R\int_{\tau=0}^{T} C_{BS}(\tau)\lambda e^{-\lambda\tau} d\tau &= R\sigma S \sum_{i=0}^{n-1} [1 + t_i^2] \frac{e^{\frac{-t_i^2}{2}}}{\sqrt{2\pi}} \int_{t_i}^{t_{i+1}} \sqrt{\tau}\lambda e^{-\tau\lambda} d\tau \qquad \text{(B.23)} \\ &= R\sigma S \sum_{i=0}^{n-1} [1 + t_i^2] \frac{e^{\frac{-t_i^2}{2}}}{\sqrt{2\pi}} \left[\sqrt{t_{i+1}} e^{\lambda t_{i+1}} - \sqrt{t_i} e^{\lambda t_i} - \frac{1}{2}\sqrt{\frac{\pi}{\lambda}} \left(erf(\sqrt{t_{i+1}\lambda}) - erf(\sqrt{t_i\lambda}) \right) \right]. \end{aligned}$$

The original equation of a defaultable option that had to be solved was:

$$R\int_{\tau=0}^{T} C_{BS}(\tau)\lambda e^{-\lambda\tau} d\tau + C_{BS}(T)\int_{\tau=T}^{\infty} \lambda e^{-\lambda\tau} d\tau, \tag{B.24}$$

and now the complete analytical solution to equation (B.24) is given by:

$$= R\sigma S \sum_{i=0}^{n-1} [1 + t_i^2] \frac{e^{\frac{-t_i^2}{2}}}{\sqrt{2\pi}} [\sqrt{t_{i+1}} e^{\lambda t_{i+1}} - \sqrt{t_i} e^{\lambda t_i} - \frac{1}{2}\sqrt{\frac{\pi}{\lambda}} \left(erf(\sqrt{t_{i+1}\lambda}) - erf(\sqrt{t_i\lambda}) \right)] + C_{BS}(T) e^{-\lambda T}.$$

Appendix C

Monte Carlo Simulations with Errors

This appendix provides all the graphs that were presented in this thesis with the error bars. These bars are shown to support the correct usage of Monte Carlo simulations. All error bars in the following graphs are small and smooth, justifying that these simulations behaved well.

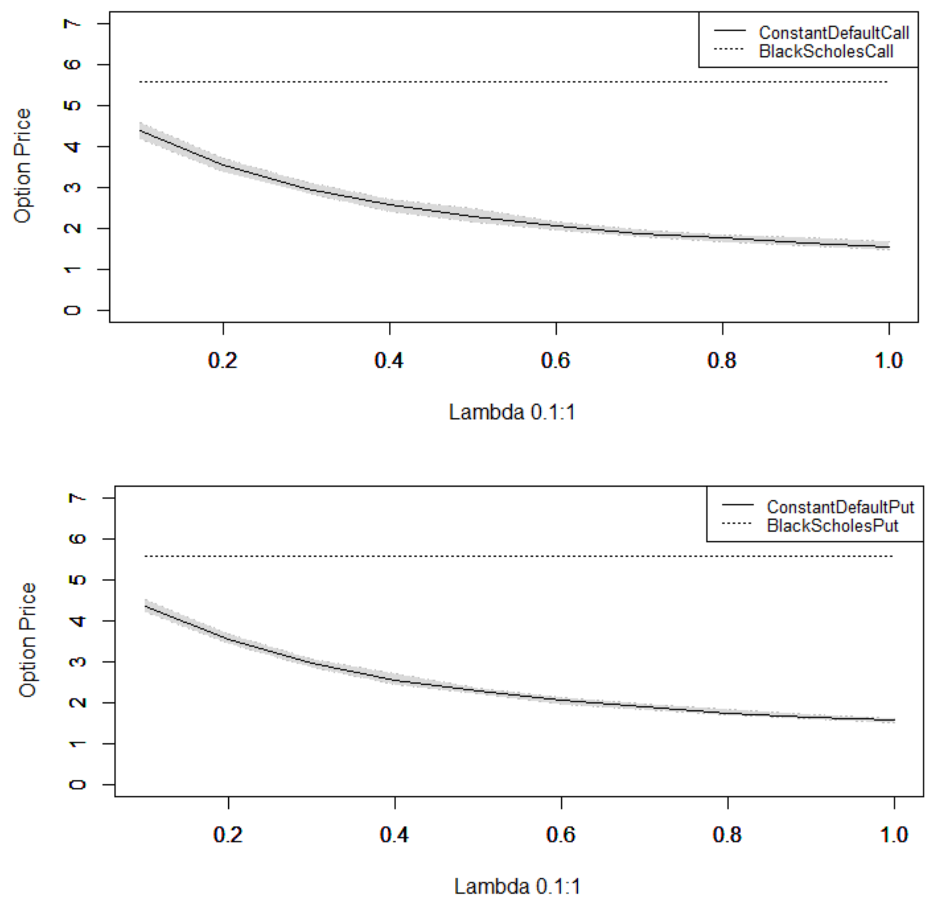

Figure C.1: Defaultable call and put option prices including error bars, with $r = 0\%$, $S = K =$ \$35, $\sigma = 18\%$, $R = 70\%$, $T = 5$, with 99% confidence level.

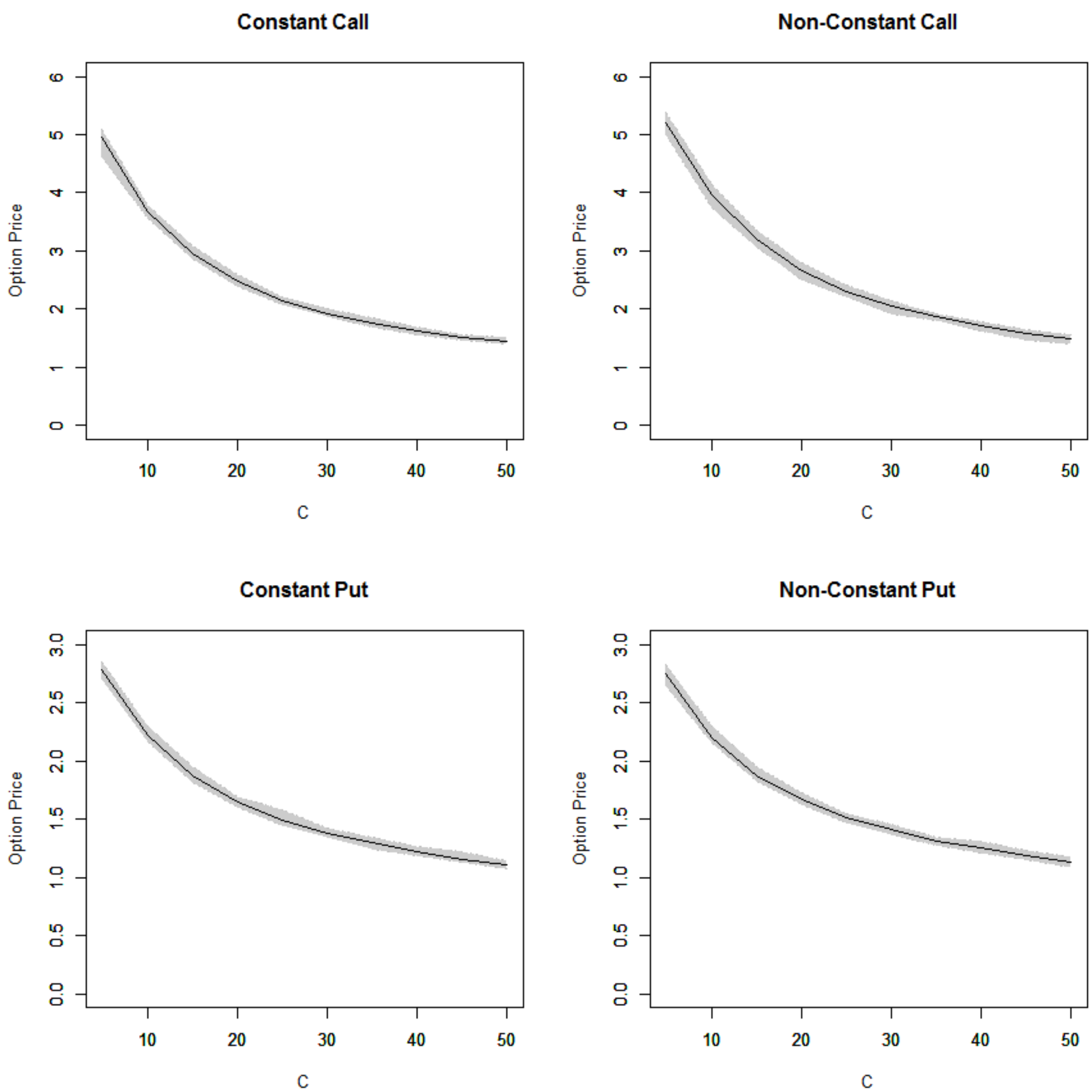

Figure C.2: Method 1 of defaultable call and put option prices including error bars, with $r = 2\%$, $S = K = \$35$, $\sigma = 18\%$, $R = 70\%$, $T = 5$, with a 99% confidence level.

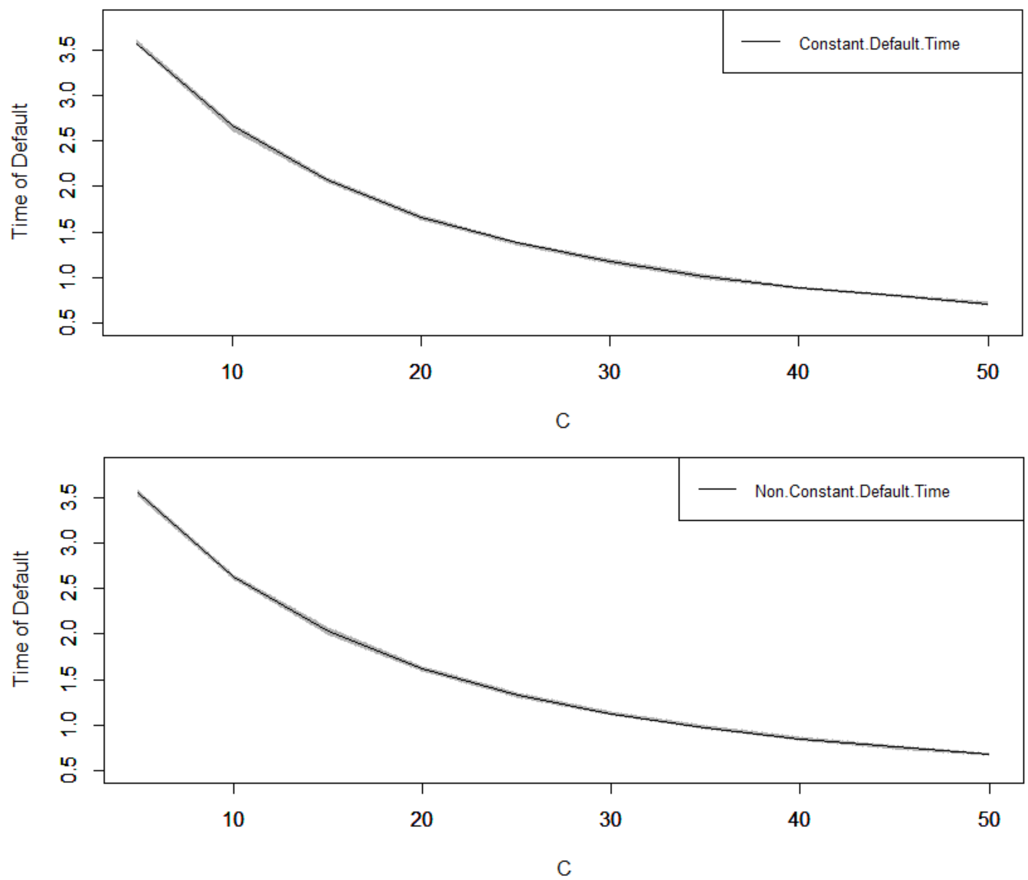

Figure C.3: Constant and non-constant time of default with error bars, with $r = 2\%$, $S = K =$ \$35, $\sigma = 18\%$, $T = 5$, with a 99% confidence level.

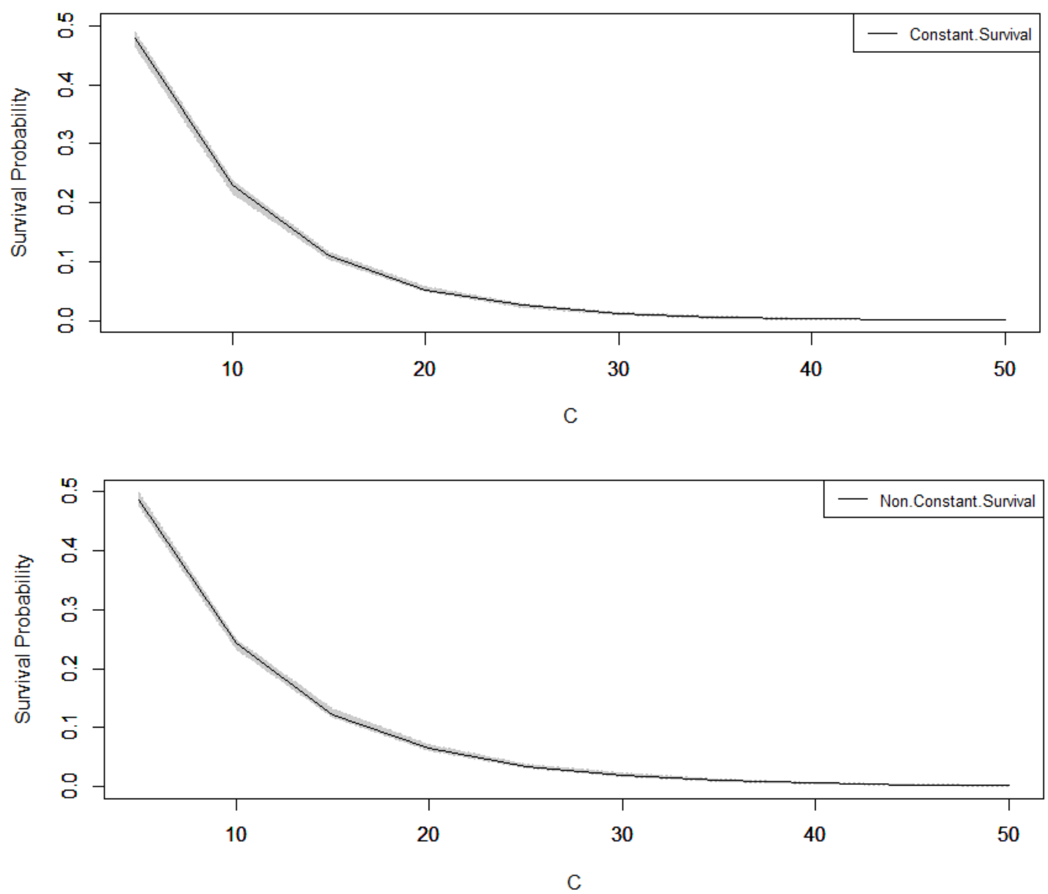

Figure C.4: Constant and non-constant survival probability with error bars, with $r = 2\%$, $S = K = \$35$, $\sigma = 18\%$, $T = 5$, with a 99% confidence level.

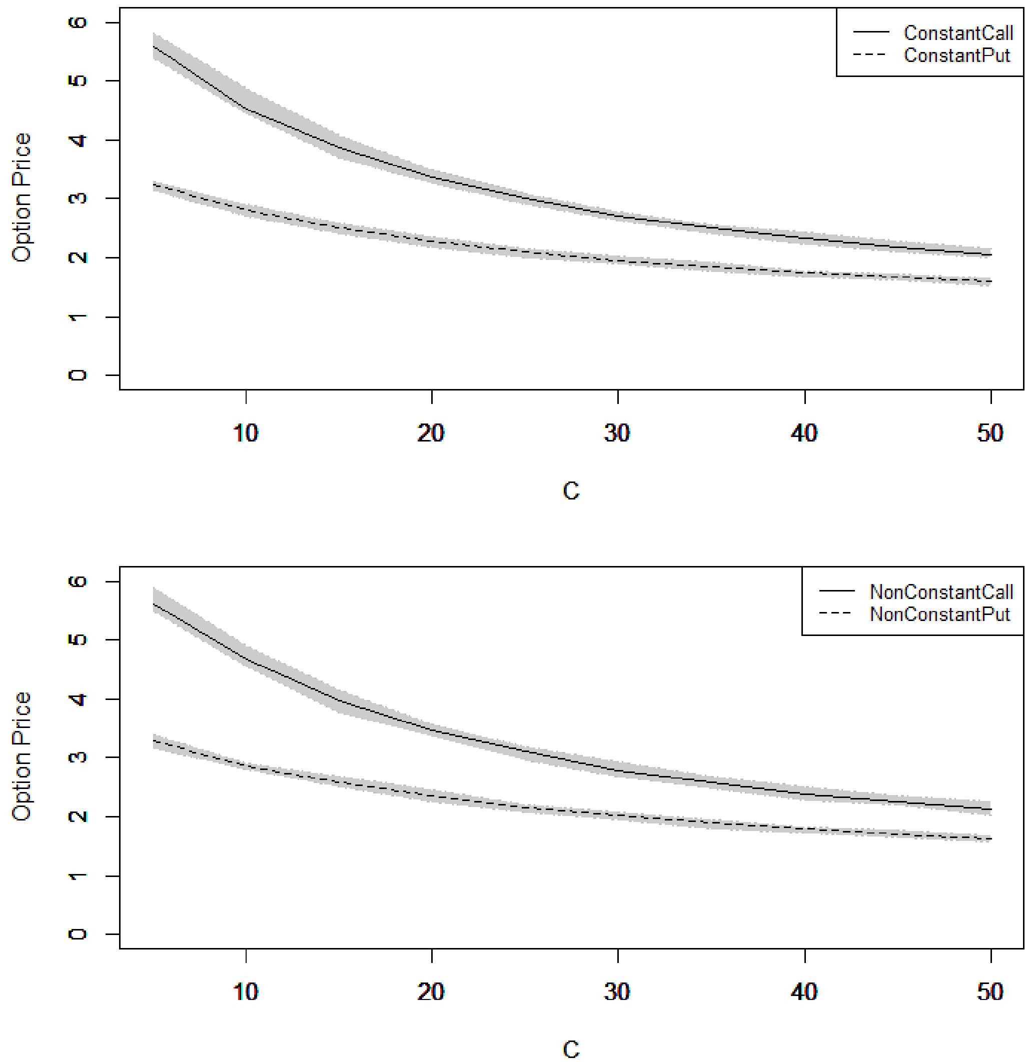

Figure C.5: Constant and non-constant defaultable option price at $R = 100\%$ with error bars, with $r = 2\%$, $S = K = \$35$, $\sigma = 18\%$, $T = 5$, with a 99% confidence level.

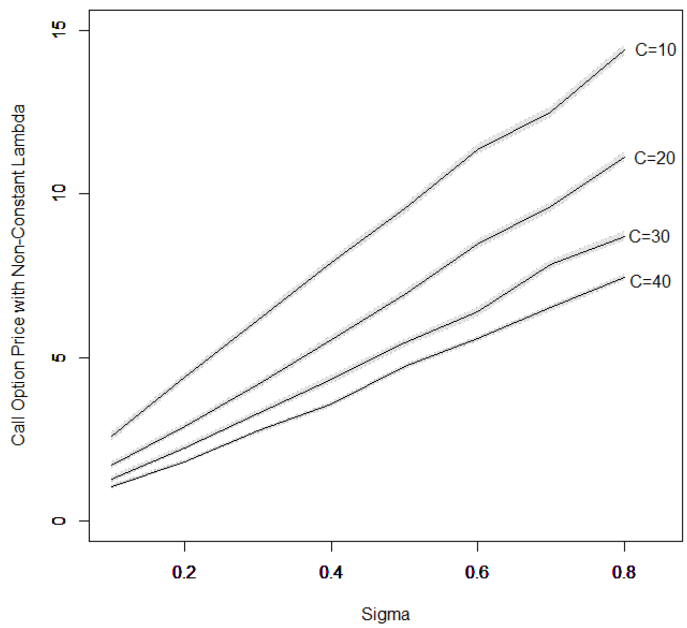

Figure C.6: Call option price with non-constant default rate including error bars, with $r = 2\%$, $S = K = \$35$, $T = 5$, $R = 70\%$, with a 99% confidence level.

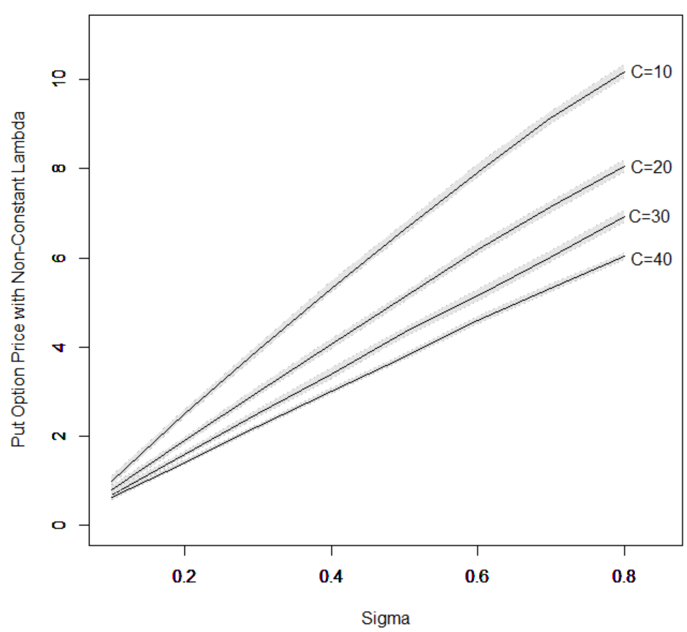

Figure C.7: Put option price with non-constant default rate including error bars, with $r = 2\%$, $S = K = \$35$, $T = 5$, $R = 70\%$, with a 99% confidence level.

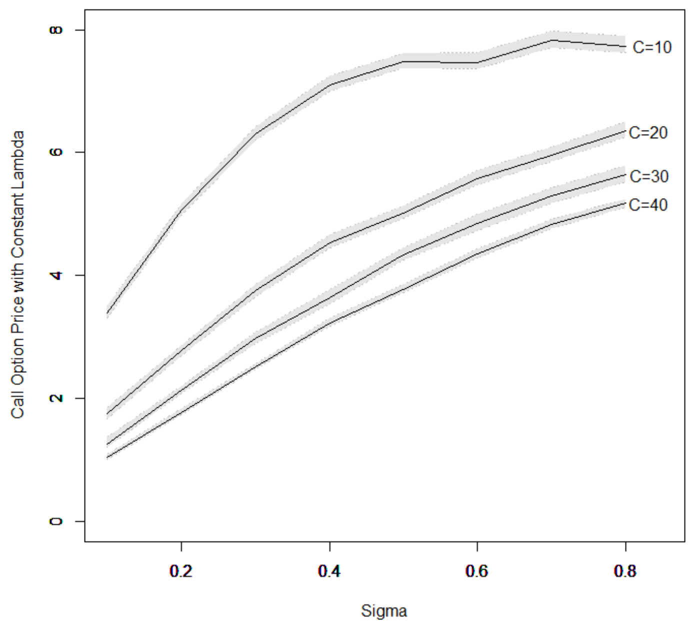

Figure C.8: Call option price with constant average default rate including error bars, with $r = 2\%$, $S = K = \$35$, $T = 5$, $R = 70\%$, with a 99% confidence level.

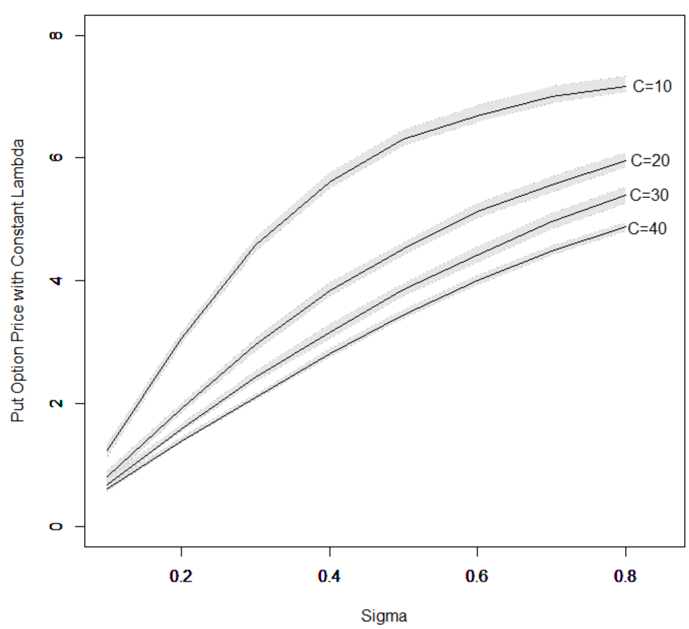

Figure C.9: Put option price with constant average default rate including error bars, with $r = 2\%$, $S = K = \$35$, $T = 5$, $R = 70\%$, with a 99% confidence level.

Curriculum Vitae

Name:	Sidita Zhabjaku
Post-Secondary Education and Degrees:	The University of Western University London, Ontario, Canada 2007- 2011 B.Sc, Financial Modelling
	The University of Western University London, Ontario, Canada 2011 - 2013 M.Sc, Financial Modelling
Related Work Experience:	Teaching Assistant The University of Western University 2011 - 2013 Financial Markets and Investments (2557B) Applied Probability and Statistics for Engineers (2141A)
	Business Analyst CIBC Toronto, Ontario, Canada Summer 2012