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International Trade Theory With Endogenous Factor Supply

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INTERNATIONAL TRADE THEORY

WITH ENDOGENOUS FACTOR SUPPLY

by

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ABSTRACT

Traditional models of international trade theory typically assume that factors of production are in fixed supply. The objective of the thesis is to examine the effects upon a number of well-known comparative statics results when the levels of supply of some factors of production are allowed to adjust endogenously. A dual trade model is used to incorporate endogenous factor supply into a general m-good, n-factor trade model, to allow for as general an analysis as possible. The traditional assumptions of constant returns to scale in production and perfect competition in all markets are retained so as to concentrate on the effects of variable factor supply. It is assumed that there exists a single representative consumer who owns all factors of production and has preferences over produced goods and some of the factors of production. The effects of an exogenous world price shock on output supply, output demand, and net exports in a small open economy where some factors are endogenously supplied are derived and compared to the comparable effects in an economy where all factor supplies are fixed. Conditions are found under which the effects of a world price shock on output supply, output demand, and net exports are magnified when some factors are endogenously supplied. The effect of variable factor supply on the probability of factor price equalization is described. The effect of an output price shock on input prices, and of an endowment change on output supplies, is examined when factor supplies are variable. The model is modified to incorporate the presence of trade and factor taxes. Conditions are described under which a given trade tax change has a larger welfare effect in a small open economy when factor supplies are variable. When some factors are taxed, variable factor supply is sufficient to imply the existence of optimal second-best trade taxes in a small open economy. All theoretical results are repeated using a simple numerical general equilibrium model, to illustrate quantitatively how

the results of the comparative statics experiments change as factor supply elasticities varies.

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1. INTRODUCTION

The traditional two-country, two-good, two-factor model of international trade theory has been used to establish a number of results relating to the pattern of trade and the equalization of factor prices between countries, the effects of changes in endowments and output prices upon output levels and factor rewards, and changes in the pattern and volume of trade due to endowment and price changes, among many others. It was typically assumed that production sectors used identical constant returns to scale technology in both countries, conditions of perfect competition prevailed in both countries, and no externalities or distortions such as taxes were present. The theory has been advanced to a large extent by considering the implications for these and other results of the relaxation of these assumptions. The application in recent years of duality theory to the study of international trade has facilitated the analysis of how many of these results relate to or change in higher dimensional general equilibrium trade models. The relevance and robustness of many of these results has been considered when the traditional assumptions of perfect competition and constant returns to scale technology are relaxed to allow for monopoly and increasing returns to scale in production. The effects of trade, factor, and output taxes in traditional trade models are also relatively well known.

Considerably less attention has been paid to the assumption that a nation always fully employs a fixed and exogenously given supply of factors of production. The effects of changes in factor supplies are considered only to the extent that these factor supply changes are exogenous, as in the Rybczynski theorem. It would be much more appropriate to allow factor supplies to adjust endogenously to price changes and other exogenous shocks, so that the levels of factor supplies are ultimately consistent with the optimizing behaviour of the economic agent who owns these factors.

The appropriate place to begin any discussion of the effects of endogenous factor supply in a model of international trade is Kemp and Jones (1962). A simple two-good, two-factor Heckscher-Ohlin model is altered to reflect the fact that the supply of one factor (labour) is variable, while the supply of the second factor (land) is held fixed throughout the analysis. Total supply of labour is the difference between the total fixed stock of leisure in the economy and the demand for leisure in the economy. Leisure is expressed as a reduced-form function of the relative price of the two goods, the real wage, and the value of the total fixed stock of leisure. Enough structure is placed upon consumption technology so that leisure demand is a homogeneous function, and the price of good 1 is chosen as numeraire. Given this labour supply function, the model is completely described by a linearly homogeneous production function for each good, conditions reflecting the fact that each factor is paid the value of its marginal product in each industry, and a full employment condition for each factor.

An expression is derived for the effect of a change in the supply of output of goods 1 and 2 in response to a change in relative commodity prices. A necessary and sufficient condition is obtained such that an increase in the relative price of good 2 results in an increase in the supply of good 1 and a fall in the supply of good 2. The authors demonstrate that the condition for Hicksian stability in the labour market is not sufficient to rule out the possibility of observing these perverse output effects. Even if the labour supply elasticity with respect to the price of labour is zero, the possibility of substitution between leisure and the two goods implies that perverse output effects cannot be ruled out.

An offer curve is described by the difference between output demand functions, which depend upon the relative price of the two outputs and total income, and the output supply functions. It is shown that the presence of variable labour supply results in the possibility of the offer curve bending backward on itself, so that an increase in the price of the imported commodity leads to an increase in imports. It is shown that a backward-bending labour supply curve is a necessary condition to observe this result.

It is curious that, while the leisure demand function depends upon output prices, the price of labour, and total income, output demand functions depend only upon output prices and income. This assumption about preferences results in the fact that the offer curve will be backward-bending only if the labour supply curve is also backward-bending. It would be preferable for preferences to reflect the effect of changes in the price of labour upon the demand for outputs, since output price changes will necessarily imply changes in input prices. As well, it should be noted that sufficient conditions to rule out perverse output effects or backward-bending offers curves are not derived.

In Frenkel and Razin (1975), the effect of variable factor supplies upon the shape of the production possibility frontier is examined. The same two-good, two-factor production model as was used in Kemp and Jones (1962) is employed, but now both factors of production are in variable supply. The variable factor supply functions in this case are reduced form functions of the own-factor price only. The percentage change in factor supply is equal to the factor supply elasticity multiplied by the percentage change in the own-factor price. The Le Chatelier principle is applied to show that the production possibility frontier will always be concave to the origin, and that the substitution elasticity between factors as well as the production elasticities will be larger, the larger are the factor supply elasticities.

The implications of negative factor supply elasticities (ie: backward-bending factor supply functions) are not considered. As well, the indirect effects of output price changes on factor supplies are zero, since preferences are such that the supply of a factor is a function only of its own price. Income effects on factor supplies are also ignored.

A geometric examination of the effects of endogenous factor supply upon the shape of the production possibilities set is given in Martin (1976). The production side of the model is essentially the same as that in the two papers described above. Two goods are produced using two factors of production. Production technology is linearly homogeneous, and there is perfect competition and no externalities. The production possibilities frontier is derived for a small open economy, so world prices are given. Preferences over factors of production are not described by an explicit factor supply function, but it is implicitly assumed that factor supplies respond to own-price changes only.

The production possibility set is derived by considering the change in factor rewards implied by a given terms-of-trade change through the Stolper-Samuelson relationship between input and output prices (assuming both goods continue to be produced). The input price change implies a change in factor supply dependent only upon the own-price factor supply elasticity. This factor supply change results in an output supply change according to the Rybcyznski relationship between factor supplies and output supplies (at constant output prices). As a result, two changes in output supply are derived. The change in production due to the output price change at constant factor supplies, and the change in production due to the change in factor supplies at constant prices. It is shown that perverse output effects are possible if the factor supply responses are perverse. That is, if an increase in the real return to a factor causes factor supply to fall, it may be that an increase in the world output price causes output of that good to fall. It is also noted that actual output supply at any given terms-of-trade is dependent not upon factor endowments but upon factor supplies, which are now endogenous. It is also shown that the likelihood of complete specialization increases, and therefore the likelihood of factor-price equalization decreases, as factor supply elasticities increase.

It is important to mention again that the implicit assumption about preferences over the endogenously supplied factors implies that there is no possibility for substitution in consumption between factors and goods. As well, all income effects upon factor supplies are aggregated into a simple reduced-form factor supply elasticity.

The issue of perverse output effects resulting from the presence of endogenous factor supplies is considered from an empirical point of view in Martin and Neary (1980). The model used is the Kemp and Jones (1962) model where two goods are produced using two factors, one of which (labour) is endogenously supplied. The aggregate labour supply function is dependent upon output prices of the two goods, the return to labour, and total unearned income. As in Kemp and Jones (1962), it is demonstrated that perverse output effects are possible if the labour supply curve is backward-bending. It is also noted that the likelihood of factor-price equalization is reduced when factor supplies are variable. The objective is to examine whether perverse output effects can be generated using simulations based upon literature estimates of labour supply elasticities and cross-price elasticities.

The simulations are carried out in two stages. Estimates of the overall labour supply elasticity are generated, based upon literature estimates of the own-price and cross-price labour supply elasticities. It is established that a backward-bending supply curve (i.e. a negative overall labour supply elasticity) is consistent with a number of combinations of parameter estimates. Output responses are then simulated for given output price changes, using different estimates of the overall labour supply elasticity, for values of the factor substitution elasticity in output production ranging from 0 to 1. Only one combination of parameters is found to be consistent with the existence of perverse output effects and stability in the labour market.

A more general treatment of endogenous factor supplies in international trade theory is given in Woodland (1985). In an m-good, n-factor general equilibrium trade model, it is assumed that a subset of the factors of production are endogenously supplied. Preferences over endogenous factors and the m produced goods are represented by a continuously differentiable utility function. The consumption sector first chooses expenditure-minimizing levels of consumption of the m goods, for given output prices and given levels of supply of the endogenous factors, which yield a given level of utility. In the second stage of the consumers problem, the level of supply of endogenous factors is chosen. The production sector maximizes total revenue given supplies of both the endogenous and exogenous factors. In equilibrium, the consumption sectors reservation wages for the endogenous factor will equal the production sectors shadow prices for the endogenous factors.

Output supply and net export responses to a given output price shock are derived. The potential for perverse output effects is illustrated with a simple two-good, two-factor example when one factor is endogenously supplied. A necessary condition for an increase in the price of imports to cause an increase in imports is demonstrated for the same two-good, two-factor model.

The most recent treatment of variable labour supply in international trade theory is given in a working paper by Mayer (1988). The model is essentially the same as that described by Kemp and Jones (1962). Two goods are produced using two factors, capital and labour. Capital is in fixed supply. Labour supply is modelled as being variable in one of two ways. In the first scenario, a constant number of homogeneous workers choose number of hours worked per day, which essentially reduces to the familiar labour/leisure choice made by a representative consumer. Preferences are represented by a utility function in which leisure demand is weakly separable from demand for goods, eliminating the possibility that goods and leisure are complements. It is also assumed throughout that labour is a normal good, so that an increase in income causes an increase in the demand for leisure, implying a fall in labour supply. Since all perverse cross-price effects on labour supply are assumed away due to the special form of the utility function, perverse output effects (due to own-price changes) are possible only due to very strong income effects on labour supply, which are dependent wholly upon whether the good whose price is changing is imported or exported. If the good is imported (exported), income in equilibrium will fall (rise) as the price of the good rises, so that the income change will cause labour supply to fall (rise).

The analysis continues by examining the imposition of an import tariff in the neighbourhood of free trade. Income effects are zero since tariff revenue is zero at free trade. Along with the other assumptions of normality of labour supply and the absence of complementarity between labour and other goods, this leads to the conclusion that output supply effects of the imposition of a tariff in a small open economy are magnified if labour supply is variable.

The second scenario used to examine variable labour supply assumes that hours worked per employee are constant, but that workers make the discrete choice whether or not to work. In order to arrive at conclusions similar to those derived in the first scenario, a number of very strong assumptions are imposed (i.e.: Cobb-Douglas utility function, so that the income effect disappears).

The assumption made throughout that labour is normal and is never a complement for goods is very strong, especially when considering the fact that (as was pointed out in Kemp and Jones (1962, among others)) backward-bending labour supply curves cannot be ruled out even if income effects on labour supply are zero when consumers can freely substitute between labour and goods.

Our objective is to examine the effects of the presence of endogenously supplied factors in a general equilibrium trade model. In Section 2, we describe a small open economy producing m goods using n factors of production in equilibrium, where a subset of the factors are endogenously supplied. The model is similar to that described in Woodland (1982), except that in our case, the consumption sector chooses levels of consumption of goods and levels of sup .y of the endogenous factors to maximize utility, given output prices, returns to the endogenous factors, and total fixed factor income. In the Woodland model, the level of supply of the endogenous factors appears as a parameter in both the production sector's problem and in the consumption sector's problem due to the specification of the consumer's problem as a two-stage maximization problem. The model is analogous to one where factor supplies are fixed, but there exist non-traded goods. Instead of a factor being endogenously supplied, there exists an extra production sector whereby one unit of a factor (for example, labour) is converted into one unit of a consumable factor (for example, leisure), and this good is non-traded. It is felt that our approach lends itself more easily and provides for a more transparent analysis of the fundamental difference between a general equilibrium trade model where factor supplies are endogenous rather than perfectly inelastic, which is the separation of the production sector's problem from the consumption sector's problem.

To illustrate, consider a small open economy at free trade equilibrium, producing two goods using two factors of production, and suppose both factors are in fixed supply. It is well known that, given a change in world output prices, the behaviour of the production sector will imply a change in input prices, independent of the behaviour or response of the consumption sector to the change in output prices (the Stolper-Samuelson theorem), as long as both goods continue to be produced. The change in production of both goods implied by the output price change can also be determined independent of the consumption sector. If one of the factors is endogenously supplied, then as long as both goods continue to be produced, the same output price change will still imply the same input price change. However, since factor supply will now adjust to the output price change, the change in output supply can no longer be determined without explicitly considering the response of the consumption sector to the price change.

It is also well known that if the small open economy produces fewer goods in equilibrium than there are factors of production, then the change in input prices due to a given output price change when all factors are in fixed supply will generally depend upon the vector of fixed factor supplies. However, the input price change is still determined independent of the behaviour of the consumption sector. This separation of the production sector equilibrium from the consumption sector equilibrium breaks down when some factor supplies are endogenous.

In Section 3, we derive the effect of a given output price change on equilibrium output supply, output demand, and net exports, in an economy where some factors are endogenously supplied. Unlike most previous analyses, which focus primarily on identifying the possibility of observing perverse output supply and net export effects, we pay particular attention to the consumption sector as well. Attempts are made wherever possible to derive sufficient conditions which ensure that perverse results can not occur, and to identify the necessary conditions which must hold for perverse results to occur. We also derive the corresponding output supply, output demand, and net export effects due to the same output price change in the traditional model, when all factor supplies are fixed, and derive sufficient conditions such that output supply, output demand, and net export responses due to a given world price change are larger when factor supplies are variable. Results are derived in a general m-good, n-factor model. We consider separately the cases where the number of factors is equal to the number of goods and where there are more factors than goods. We also examine the two-good, two-factor case, and describe results when all production and consumption technology can be represented by Cobb-Douglas functions. The latter example is included to provide a more intuitive understanding of the effects of variable factor supply, since results tend to get complicated in the general model.

The presence of endogenous factor supply essentially implies that the endowment point in the traditional trade model is endogenous. As a result, the presence of endogenously supplied factors will affect the results of the factor-price equalization theorem. While the actual statement of the factor-price equalization theorem is unaffected by variable factor supplies, the probability of factor-price equalization may be increased or decreased when some factors are endogenously supplied. In Section 4, we illustrate this result with a simple two-good, two-factor numerical example, where production technology is represented by Cobb-Douglas production functions, and one of the two factors of production is endogenously supplied. Preferences over goods and the endogenous factor are represented by a Cobb-Douglas utility function. We can then show how changes in preferences over the endogenously supplied factors affect the shape of the diversification cone, thereby affecting the probability of factor-price equalization.

The presence of endogenous factor supply upon the effect of output price changes on input price changes (the Stolper-Samuelson effect) is examined in Section 5. As noted above, when the number of goods equals the number of factors, the effect of output prices on input prices is the same whether the factors are endogenously supplied or not as long as the same goods continue to be produced in equilibrium. This is not the case when there are more factors than goods. We also consider the effect of endogenous factor supply on the magnification effect of output prices on input prices predicted by the specific factors model. In Section 6, the dual Rybczynski effect is also analyzed when factor supplies are variable, in the case where the number of factors is equal to the number of goods, when there are more factors than goods, and in the specific factors model. An example is given to illustrate the results in the two-good, two-factor case, where one factor is endogenously supplied, and the production functions and the utility function are all Cobb-Douglas.

We proceed to consider the effect of endogenous factor supply in a small open economy where trade is distorted due to the existence of trade taxes. Sufficient conditions are derived such that a given trade tax change has a larger effect on welfare and trade tax revenue when factor supplies are allowed to adjust to the trade tax change than when factor supplies are held fixed. We also consider the optimal tariff problem for a small open economy, and provide a proof that a trade tax vector equal to the null vector is globally optimal for the small open economy, when some factors are endogenously supplied. We also consider the optimal tariff problem for a large economy able to affect world terms-of-trade, and give conditions under which the nation's optimal tariff is smaller when factors are endogenously supplied.

The model is modified to allow for the presence of taxes on the endogenously supplied factors. Of course, factor taxes in this model will generally result in factor supply changes. We show that in the neighbourhood of zero trade and factor taxes, an increase in the tax on an endogenously supplied factor causes an unambiguous fall in factor supply and domestic welfare in a small open economy. We then describe conditions under which a trade tax increase will improve domestic welfare, by offsetting the factor-market distortion caused by the tax on factor supply.

In the final section we construct a numerical general equilibrium model of the Canadian economy to illustrate the results described in the previous sections. The 1982 world benchmark equilibrium data set of Nguyen, Perroni and Wigle (1989) is modified so as to represent Canada as a small open economy producing two goods using endogenously supplied labour and inelastically supplied capital. A single representative consumer owns all factors and consumes the two goods and leisure. Production technology and preferences over goods and leisure are represented by flexible functional forms so that the initial equilibrium is calibrated to specified substitution matrices. The quantitative effects of variable factor supply are illustrated using this model by comparing the results of a given counterfactual experiment for different specifications of the leisure demand elasticity. A given world price change will result in different output supply, output demand, and net export changes as the leisure demand elasticity is changed. We also examine the effect of different specifications of the leisure demand elasticity upon the output supply response to a given endowment change. These results are all compared to the corresponding results derived in the earlier theoretical sections, and are meant to serve as quantitative examples to complement the earlier qualitative results. The same numerical model is revised to compare optimal tariffs for an economy with some market power, given different specifications of the leisure demand elasticity. Concluding comments follow in Section 9.

2. DUAL TRADE MODEL WITH ENDOGENOUS FACTOR SUPPLY

The standard m-good n-factor version of the Heckscher-Ohlin trade model is revised to accomodate the presence of endogenously supplied factors. The maximum value functions of the production and consumption sectors are derived. It is shown how the envelope properties of these functions yield standard output supply, output demand, and net export functions. Also, factor supply and factor demand functions are derived for those factors of production which are endogenously supplied.

2.1 Production Sector

Consider an economy with m perfectly competitive industries producing outputs $y' = (y_1, y_2, \ldots, y_m)$. Each industry uses inputs $a^{i'} = (a_1^i, \ldots, a_n^i)$, $i = 1, \ldots, m$, and constant returns to scale technology summarized by the twice continuously differentiable, strictly quasi-concave production functions $f^i(a^i)$, $i = 1, \ldots, m$. Suppose that a subset $e \leq n$ of the factors of production are endogenously supplied, while the remaining $n - e$ factors are exogenously supplied. The endowment vector can be written as:

$$(2.1) \qquad (v', \bar{v}') = (v_1, v_2, \ldots, v_e, \bar{v}_{e+1}, \bar{v}_{e+2}, \ldots, \bar{v}_n) \in R_+^n,$$

with $v \in R_+^e, \bar{v} \in R_+^{n-e}$. We correspondingly decompose the factor price vector into $w \in R_{++}^e, \bar{w} \in R_{++}^{n-e}$. There is no joint production and no intermediate inputs in production.[1] We let $p' = (p_1, \ldots, p_m) \in R_+^m$ be the vector of domestic output prices corresponding to y. For any given $\bar{v}$, the feasible production set is a subset of R^{m+e}, as given in equation (2.2), since industries will choose output levels and demand for

[1] For a demonstration of how joint production and intermediate inputs in production are modelled in this trade model with duality, see Woodland (1982), ch.5.

inputs which are endogenously supplied.

$$
(2.2) \qquad \begin{aligned} Y(\bar{v}) = \{(y,v) \in R_{+}^{m+e} \mid\ & y_i \leq f^i(a^i);\ a_j^i \geq 0\ \ \forall i,j;\ y \geq 0; \\ & \sum_{i=1}^{m} a_j^i \leq \bar{v}_j,\ j = e+1,\dots,m;\ \sum_{i=1}^{m} a_j^i \leq \tilde{v}_j,\ j = 1,\dots,e\} \end{aligned}
$$

Total use of the j^{th} endogenously supplied factor must be less than $\tilde{v}_j$, the total available supply of the j^{th} factor in the economy. The problem of maximizing GNP now consists of choosing a vector $(y', v')' \in R_{+}^{m+e}$ to maximize the difference between total revenue and total payments to the endogenously supplied factors, for given input and output prices w and p and endowment vector $\bar{v}$, subject to feasibility of the production process. This problem is summarized by the modified GNP function (2.3):

$$
(2.3) \qquad G(p,w,\bar{v}) = \max_{y,v} \{p'y - w'v \mid (y,v) \in Y(\bar{v})\}.
$$

Note that the modified GNP function maps from a parameter space which is a subset of R_{+}^{m+n}. This is also the case in the traditional version of the Heckscher-Ohlin model when all factors of production are exogenously supplied. However, in that model the vector of fixed factor endowments is an argument in the GNP function, while in the current version of the model with endogenously supplied factors, the price vector of the endogenously supplied factors, w, is an argument in the GNP function. This factor price vector will ultimately be determined from the interaction of the factor markets for endogenously supplied factors, in general as a function of output prices and fixed factor endowments, as will be explained after consideration of the demand side of the economy.

Note also that this modified GNP function does not actually measure GNP, but that, in equilibrium, it will equal fixed factor income, $m_f = \bar{w}'\bar{v}$.[2]

[2] When $e = n$, all factors are endogenously supplied. In this case, the modified GNP

The modified GNP function will be:

continuous and differentiable

non-decreasing, homogenous of degree one, strictly concave in $\bar{v}$

(2.4) homogenous of degree one in p and w

non-decreasing, strictly convex in p

non-increasing, strictly convex in w.

An increase in the price of any one endogenous factor causes GNP to fall linearly, cet. par. Substitution in production will result in a smaller fall in GNP as producers substitute away from the factor whose price has risen, so the GNP function will be convex in w. See Appendix, p.128, for proof.

Application of Hotelling's lemma will give:

$$G_p(p,w,\bar{v}) = y(p,w,\bar{v}) \tag{2.5}$$

$$G_w(p,w,\bar{v}) = -v(p,w,\bar{v}) \tag{2.6}$$

$$G_{\bar{v}}(p,w,\bar{v}) = \bar{w}. \tag{2.7}$$

The dual problem to maximizing GNP in this model is summarized by equation (2.8):

$$\min_{v,\bar{w}}\{w'v + \bar{w}'\bar{v} \mid c^i(w,\bar{w}) \geq p_i, i = 1,\ldots,m;\ w,\bar{w} \geq 0\}, \tag{2.8}$$

where $c^i(w,\bar{w})$ is the unit cost function for industry i. Since the production functions are all homogenous of degree one, we get $c^i(w,\bar{w})y_i = C^i(w,\bar{w},y_i),\ i = 1,\ldots,m,$ where $C^i(w,\bar{w},y_i)$ is the minimum cost function for firm i, $i = 1,\ldots,m$, given by equation (2.9):

$$C^i(w,\bar{w},y_i) = \min_{a^i}\{(w',\bar{w}')a^i \mid f^i(a^i) \geq y_i, a^i_j \geq 0, \forall i,j\}. \tag{2.9}$$

function becomes a profit function, and given the assumption of constant returns to scale production technology, will equal either zero or infinity in equilibrium. Given this somewhat undesireable characteristic, we assume $e < n$ unless otherwise noted.

Define $c(w, \bar{w})' \equiv (c^1(w, \bar{w}), \ldots, c^m(w, \bar{w}))$. Then we know that each $c^i(w, \bar{w})$ is:[3]

(2.10) continuous and differentiable

non-decreasing, homogenous of degree one and strictly concave in $w, \bar{w}$.

The Kuhn-Tucker necessary conditions for the industry cost minimization problem are given by equations (2.10) and (2.11).

$$(2.10) \qquad p - c(w, \bar{w}) \leq 0 \leq y$$

$$(2.11') \qquad c_w^i(w, \bar{w})y_i - v_i(p, w, m_f) \leq 0 \leq w_i \qquad i = 1, \ldots, e,$$

$$(2.11'') \qquad c_{\bar{w}}^i(w, \bar{w})y_i - \bar{v}_i \leq 0 \leq \bar{w}_i \qquad i = e+1, \ldots, n.$$

This notation is the same as that used in Woodland, 1982, p.13, so that $p - c(w, \bar{w}) \leq 0 \leq y$ is equivalent to $p_i - c^i(w, \bar{w}) \leq 0$; $y_i \geq 0$; and $y_i(p_i - c^i(w, \bar{w})) = 0$; $i = 1, \ldots, m$. The cost minimization problem is dual to the GNP problem in the sense that all information about the production technology in the economy can be retrieved from either the GNP function or the cost function. Note that equation (2.11′) reflects the first-order conditions for the endogenously supplied factors, whose supply is now determined by the consumption sector.

2.2 Consumption Sector

Suppose the economy is populated by ℓ individuals with identical and homothetic preferences. The consumption vector for the economy is represented by the vector $z' = (z_1, \ldots, z_m) \in R_+^m$. In order to incorporate the presence of endogenously supplied factors into the demand side of the economy, we assume that there exists a twice continuously differentiable strictly quasi-concave utility function which represents preferences over consumption and factor supply vectors, $U(z, v)$. The utility

[3] see Woodland, 1982, pp.25-26, or Varian, 1984, pp. 44-45.

function U is non-decreasing in z and non-increasing in v. The demand side of the economy can be summarized by the expenditure function:

$$(2.12) \qquad E(p, w, \mu) = \min_{z,v} \{p'z - w'v \mid U(z, v) \geq \mu;\ z, v \geq 0\}.$$

The expenditure function will be:

(2.13)

continuous and differentiable

homogenous of degree one in p and w

non-decreasing, strictly concave in p

non-increasing, strictly concave in w.

An increase in the price of any one endogenously supplied factor will cause expenditures to decrease linearly, cet. par. Consumers would then be able to substitute towards supply of the factor whose price has increased (for example, an increase in the price of labour is an increase in the price of leisure, so consumers would prefer to demand less leisure or supply more labour), and maintain the same level of utility. Thus the expenditure function will be strictly concave in w. See Appendix, p.128, for proof.

Application of Shepherd's lemma to equation (2.12) implies that (2.14) and (2.15) will hold in equilibrium.

$$(2.14) \qquad E_p(p, w, \mu) = z(p, w, \mu)$$

$$(2.15) \qquad E_w(p, w, \mu) = -v(p, w, \mu)$$

The dual problem of utility maximization is summarized by the indirect utility function in equation (2.16):

$$(2.16) \qquad V(p, w, m_f) = \max_{z,v} \{U(z, v) \mid p'z - w'v \leq m_f;\ z, v \geq 0\}.$$

The indirect utility function will be:[4]

$$
(2.17)\qquad
\begin{array}{l}
\text{continuous and differentiable} \\
\text{homogenous of degree zero in } p,\ w, \text{ and } m_f \\
\text{non-increasing, strictly quasi-convex in } p \\
\text{non-decreasing, strictly quasi-convex in } w \\
\text{non-decreasing in } m_f.
\end{array}
$$

Roy's identity ensures that (2.18) and (2.19) will hold in equilibrium:

$$
(2.18)\qquad V_p(p,w,m_f)/V_{m_f}(p,w,m_f) \;=\; -z(p,w,m_f)
$$

$$
(2.19)\qquad V_w(p,w,m_f)/V_{m_f}(p,w,m_f) \;=\; v(p,w,m_f).
$$

2.3 General Equilibrium and Trade

The balance of trade function for this economy with endogenous factor supplies is given by the difference between the modified GNP function and the expenditure function, in equation (2.20):

$$
(2.20)\qquad S(p,w,\bar{v}) \;=\; G(p,w,\bar{v}) \;-\; E(p,w,V(p,w,G(p,w,\bar{v}))).
$$

Since the GNP function is strictly concave in p and w, and since the expenditure function is strictly convex in p and w, the balance of trade function must be strictly concave in p and w. The equilibrium conditions for goods and the endogenously supplied factors can be obtained directly by differentiating the balance of trade function. If we assume that some of all goods is produced and consumed, the net export function for this economy will be given by the output price derivative of the balance of

[4] See Appendix, p.129, for the proof that $V(p,w,m_f)$ is non-decreasing, quasi-convex in w.

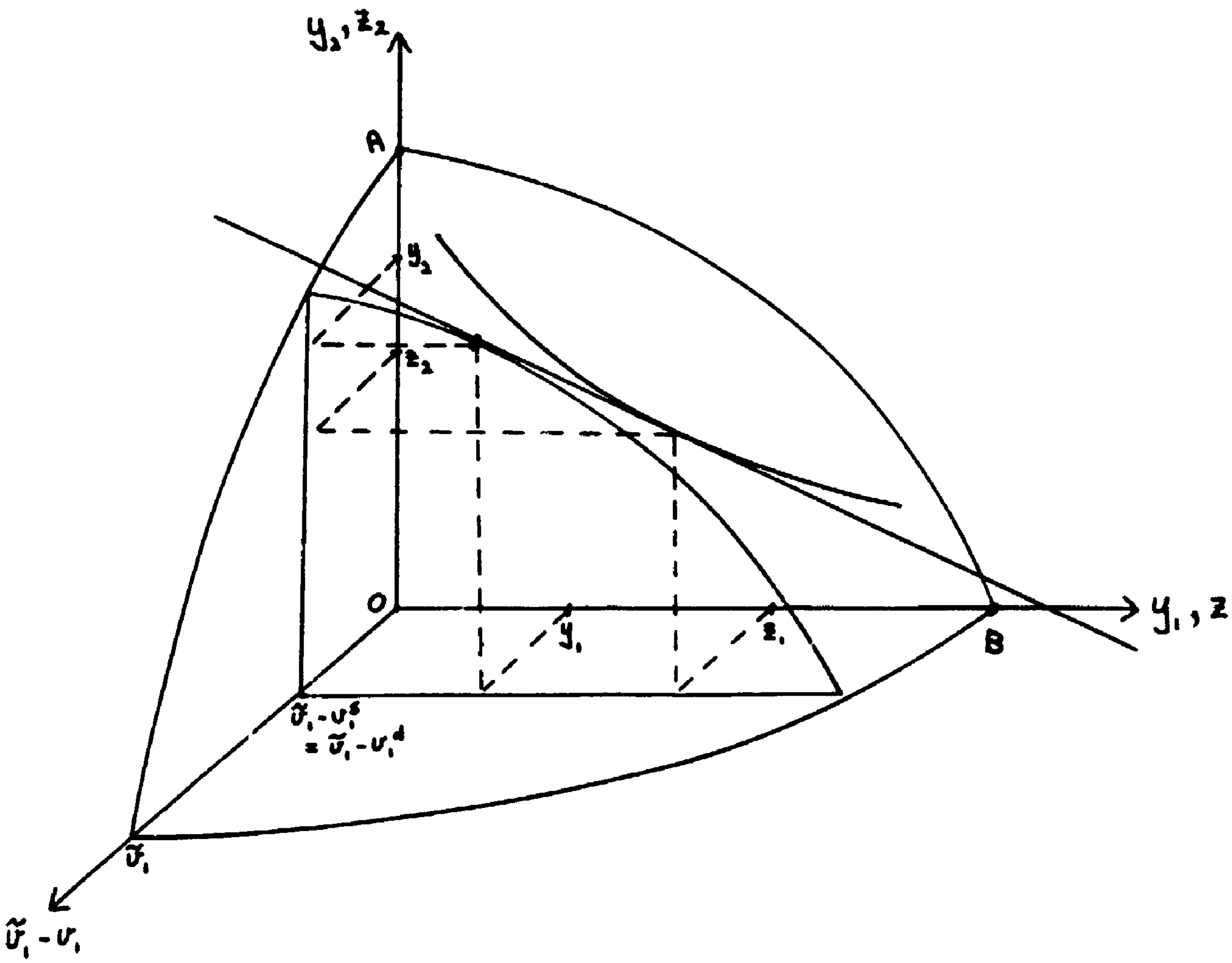


Figure 2.1

trade function:

$$
\begin{aligned}
x(p, w, \bar{v}) &= S_p(p, w, \bar{v}) \\
&= G_p(p, w, \bar{v}) - E_p(p, w, V(p, w, G(p, w, \bar{v}))).
\end{aligned}
\tag{2.21}
$$

The condition that factor demands equal factor supplies is given by the input price derivative of the balance of trade function:

$$
\begin{aligned}
0 &= S_w(p, w, \bar{v}) \\
&= G_w(p, w, \bar{v}) - E_w(p, w, V(p, w, G(p, w, \bar{v}))).
\end{aligned}
\tag{2.22}
$$

This equilibrium is illustrated in Fig. 2.1 for the case where $m = n = 2$, $e = 1$, so that one of the two factors of production are endogenously supplied.

The total available supply to the economy of the endogenously supplied factor v_1 is $\tilde{v}_1$. When all of this factor is consumed, none is supplied to the production sector, and the relevant production possibility frontier is the point $\bar{v}_1$, where none of goods 1 or 2 is produced. When all of factor v_1 is supplied to the production sector, we get the "maximal" production possibility frontier OAB. Given output prices p_1, p_2, and some endowment of the exogenously supplied factor $\bar{v}_2$, equilibrium output is (y_1, y_2), equilibrium consumption is z_1, z_2, and supply of factor v_1 equals demand for factor v_1.

3. OUTPUT SUPPLY, OUTPUT DEMAND, AND NET EXPORTS

The effect of an output price change upon output supply, output demand, and net export functions in this model when all factors are exogenously supplied is very well known. For example, given the assumptions made thus far, if $e = 0$, then output supply functions will slope upwards, or the supply of output of a good will rise when the price of that good rises. The objective of this section is to examine how robust these sorts of results are to the presence of endogenously supplied factors. We also seek to determine whether, for a given output price change, the output supply, output demand, and net export response is larger (in absolute value) when factors are endogenously supplied compared to when factor supplies are perfectly inelastic. Note that we will not be comparing these comparative statics results in the endogenous factor supply model to the corresponding results in a Heckscher-Ohlin model, since in the latter model, factors are never consumed. Factor supplies do not enter the utility function of agents in the Heckscher-Ohlin model, so that the two models are not directly comparable. Instead, we will proceed by considering some initial equilibrium where some amount of the endogenously supplied factors is consumed by the representative agents, and the remaining amount of the endogenous factors is supplied to the production sector. The effect of a price shock on the equilibrium values of output supply, output demand, and net exports will then be evaluated and compared when the supply of the endogenous factors is held fixed (inelastic supply) or when that supply is allowed to respond to the price shock.

We can find the effect of an output price change dp on output supply, output demand, and net exports by differentiating the output supply, output demand, and net export functions of Section 2 as follows. First totally differentiate the output demand function (2.14):

$$\begin{aligned} z &= E_p(p,w,\mu) \\ &= E_p(p,w,V(p,w,G(p,w,\bar{v}))). \end{aligned}$$

$$\begin{aligned} dz &= [E_{pp} + E_{p\mu}(V_p' + V_{m_f}G_p')]dp + [E_{pw} + E_{p\mu}(V_w' + V_{m_f}G_w')]dw \\ &= [E_{pp} + E_{p\mu}V_{m_f}(V_p'/V_{m_f} + G_p')]dp + [E_{pw} + E_{p\mu}V_{m_f}(V_w'/V_{m_f} + G_w')]dw. \end{aligned}$$

Now rewrite the output demand functions as follows, in order to identify the income effect on output demand:

$$z(p,w,m_f) = E_p(p,w,V(p,w,m_f))$$

$$\frac{\partial z}{\partial m_f} = z_{m_f} = E_{p\mu}V_{m_f}.$$

Using the envelope properties summarized by equations (2.18) and (2.5), we can write:

$$V_p'/V_{m_f} + G_p' = -z' + y' = x'.$$

Likewise, we can use equations (2.19) and (2.6) to get:

$$V_w'/V_{m_f} + G_w' = v_s' - v_d' = 0'.$$

Substituting into the expression for dz above yields:

$$dz = [E_{pp} + z_{m_f}x']dp + E_{pw}dw.$$

Totally differentiating the output supply functions (2.5) and net export functions (2.21) gives:

$$\begin{aligned} y &= G_p(p,w,\bar{v}) \\ dy &= G_{pp}dp + G_{pw}dw \\ x &= S_p(p,w,\bar{v}) = y(p,w,\bar{v}) - z(p,w,\bar{v}) \\ dx &= S_{pp}dp + S_{pw}dw = dy - dz. \end{aligned}$$

Using all of this information, the relevant differentials of net export, output supply, and output demand functions are given by equations (3.1) to (3.3), respectively:

$$\begin{aligned} dx &= S_{pp}dp + S_{pw}dw \\ &= [G_{pp} - E_{pp} - z_{m_f}x']dp + [G_{pw} - E_{pw}]dw \qquad (3.1) \end{aligned}$$

$$dy = G_{pp}dp + G_{pw}dw \qquad (3.2)$$

$$dz = [E_{pp} + z_{m_f}x']dp + E_{pw}dw, \qquad (3.3)$$

where z_{m_f} is the vector of income effects on consumption. The comparative statics results now take explicit account of how changes in the prices of the endogenously supplied factors change net exports, output supply and demand. In general, the change in the prices of the endogenously supplied factors can be found by differentiating the equilibrium condition for the endogenously supplied factors, equation (2.22), with respect to output prices and the returns to the endogenous factors, since at the new output prices, it must still be true that factor supplies equal factor demands:

$$\begin{aligned} G_{wp}dp + G_{ww}dw &= \{E_{wp} + E_{w\mu}[V_p' + V_{m_f}G_p']\}dp \\ &\quad +\{E_{ww} + E_{w\mu}[V_w' + V_{m_f}G_w']\}dw. \end{aligned}$$

We know from equations (2.18) and (2.5) that $[V_p' + V_{m_f}G_p'] = V_{m_f}[-z' + y'] = V_{m_f}x'$, and from equations (2.19) and (2.6) that $[V_w' + V_{m_f}G_w'] = V_{m_f}[v_s' - v_d'] = 0'$. Rearranging to solve for dw as a function of dp gives:

$$dw = -[G_{ww} - E_{ww}]^{-1}[G_{wp} - E_{wp} - v_{m_f}x']dp, \qquad (3.4)$$

where $v_{m_f} = E_{w\mu}V_{m_f}$ is the vector of income effects on the endogenously supplied factors. At this point we could substitute equation (3.4) into the equations for dx, dy, and dz above, in order to determine whether the presence of endogenously supplied factors still leads to the result in the traditional model with exogenous factor supplies that output supplies, for example, are positively associated with own price changes.

However the presence of income effects and substitution effects between goods and factor supplies precludes our arriving at any definite answer to this question at this point. It is possible that the income and substitution effects are such that the supply of output of a good may fall in response to a rise in its price.

3.1 Number of Goods Equals Number of Factors

Before we continue, it is important to note the difference in how input prices are determined when there are more goods than factors or when there are more factors than goods. First, suppose $m > n$, so that there are more goods than factors at the initial equilibrium. The equilibrium output supply vector $y(p, w, \bar{v})$ will no longer be unique, since with more goods than factors, there may exist multiple output supply vectors which could be consistent with equilibrium. The output supply function is now properly an output supply correspondence, written as $y(p, w, \bar{v}) \in G_p(p, w, \bar{v})$, and the comparative statics results below would not apply. As a result, in this section we will consider the case where the number of goods produced in equilibrium equals the number of factors of production. In order to avoid further differentiability problems, we will also assume throughout that the same m goods are still produced in the equilibrium after the comparative statics change as were produced in the initial equilibrium. That is, the economy stays within the cone of diversification formed by the same n first-order conditions given by equation (2.11′) and (2.11″).

If the number of goods equals the number of factors, then the m price-equals-marginal-cost conditions in equation (2.10) will uniquely determine the n input prices, given output prices, as long as the economy is fully diversified in production (assuming that the m equations in (2.10) are linearly independent). That is, given the initial equilibrium and the change in output prices dp, we can solve for the change in input prices dw independently of the consumption sector. As a result, the consumption sector treats the returns to the endogenously supplied factors as given, and solves for the equilibrium supply of the endogenous factors. In equilibrium, the production sector stands ready to use all of this supply of the endogenous factors. In a sense, the returns to the endogenously supplied factors are determined by the production sector, and the levels of supply of the endogenous factors are determined by the consumption sector. While this does not affect our formulation of the representative consumer's problem, we must now re-specify the producer's problem, since the production sector now treats the supply of endogenous factors as given. The GNP function needs to be rewritten as follows:

$$\hat{G}(p, v, \bar{v}) = \max_y \{p'y \mid y \in \hat{Y}(v, \bar{v})\}, \tag{2.3'}$$

where the production possibility set $\hat{Y}(v, \bar{v})$ is suitably redefined to incorporate the supply of the endogenous factors v determined by the consumption sector. Of course, equations (2.5) and (2.7) continue to apply, so that $\hat{G}_p = y$ and $\hat{G}_{\bar{v}} = \bar{w}$. However, the returns to the endogenously supplied factors are now given by:

$$w = \hat{G}_v(p, v, \bar{v}). \tag{2.6'}$$

The relevant expression for the change in the returns to the endogenously supplied factors will not be equation (3.4), but rather

$$\begin{aligned} dw &= \hat{G}_{vp}dp + \hat{G}_{vv}dv \\ &= \hat{G}_{vp}dp, \end{aligned} \tag{3.4'}$$

since $\hat{G}_{vv} = 0$ by assumption. Since we assume that all goods continue to be produced, a factor supply change will not change input prices, so that $\hat{G}_{v_i v_j} = \partial w_i / \partial v_j = 0,\ i, j, = 1, \ldots, e$.

Let's now consider how the output supply functions respond to a given terms-of-trade shock. Since the output sector solves for the change in input prices, given dp, and the consumption sector solves for the change in endogenous factor supplies, given the change in input prices, we need to rewrite equation (3.2) as:

$$(3.2') \qquad dy = \hat{G}_{pp}dp + \hat{G}_{pv}dv.$$

The matrix $\hat{G}_{pv}$ gives the change in output supplies (recall that $\hat{G}_p = y$ in equilibrium) given the change in the supply of the endogenous factors dv. The change in endogenous factor supplies is found by differentiating the equilibrium factor supply function (2.15) with respect to output and input prices:

$$(2.15) \qquad \begin{aligned} v(p,w,\bar{v}) &= -E_w(p,w,V(p,w,\hat{G}(p,v,\bar{v}) - w'v)) \\ dv &= -[E_{wp} + E_{w\mu}(V_p' + V_{m_f}\hat{G}_p')]dp \\ &\quad -[E_{ww} + E_{w\mu}(V_w' - V_{m_f}v')]dw \\ &\quad -E_{w\mu}V_{\cdot\cdot} \text{[illegible]}_v' - w')dv \end{aligned}$$

In equilibrium, $E_{w\mu}(V_w' - V_{m_f}v') = E_{w\mu}V_{m_f}(V_w'/V_{m_f} - v') = 0'$, since $V_w'/V_{m_f} = v'$. As well, $\hat{G}_v' = w'$, and as we saw earlier, $E_{w\mu}(V_p' + V_{m_f}\hat{G}_p') = -v_{m_f}x'$,so that we can rewrite this expression as:

$$(3.5) \qquad dv = -[E_{wp} - v_{m_f}x']dp - E_{ww}dw.$$

The change in the returns to the endogenously supplied factors dw is determined by the production sector, and is given by equation (3.4$'$). Substituting (3.4$'$) and (3.5) into (3.2$'$) and premultiplying by dp' gives:

$$(3.6) \qquad \begin{aligned} dp'dy &= dp'\hat{G}_{pp}dp - dp'\hat{G}_{pv}E_{ww}\hat{G}_{vp}dp \\ &\quad -dp'\hat{G}_{pv}[E_{wp} - v_{m_f}x']dp. \end{aligned}$$

The GNP function is convex in output prices and the expenditure function is concave in input prices, so the first two quadratic forms on the right-hand side of (3.6) are both positive. Thus we can write:

$$(3.7) \qquad dp'dy > -dp'\hat{G}_{pv}[E_{wp} - v_{m_f}x']dp.$$

If the right-hand side of (3.7) were non-negative, then we could say that output supply curves slope upwards when some factors of production are endogenously supplied. Unfortunately, we cannot sign this last term in general.

Consider the special case where the representative consumer's utility function takes the additively separable form $U(z,v) = U_1(z) + U_2(v)$, where $U_1(z)$ is linearly homogenous and $U_2(v)$ is strictly concave. Since the utility function $U_1(z)$ is linearly homogenous, we can use a two-stage budgeting procedure to solve the consumer's maximization problem. The agent first minimizes expenditures on output demands as follows:

$$\min_{z} \{p'z \mid U_1(z) \geq z^*\}.$$

If $\tilde{z} \in R^m$ solves the representative agents problem, then $U_1(\tilde{z}) = z^*$. Define $E^*(p)$ as the minimum expenditure needed to buy one unit of z^* at prices p. Then the second stage of the consumers problem is summarized by:

$$\max_{v,z^*} \{z^*(p) + U_2(v) \mid E^*(p)z^*(p) - w'v \leq m_f\}.$$

The first-order necessary conditions for this problem are:

$$1 - \lambda E^*(p) = 0$$

$$U_{2i} - \lambda w_i = 0, \quad i = 1, \ldots, e$$

$$m_f - E^*(p)z^*(p) + w'v = 0$$

We can choose a numeraire such that output prices lie on the unit simplex.[5] In this case, $E^*(p) = 1 = \lambda$. Since $U_2(v)$ is strictly concave, we can solve the e first-order conditions over factor supplies, and the appropriate expression for the factor supply functions will now be:

$$v_e = v_e(w). \tag{3.8}$$

[5] The choice of numeraire is arbitrary. We choose that output prices lie on the unit simplex only for simplicity, to illustrate the subsequent results.

That is, factor supply functions will now be a function only of factor prices, and will be independent of output prices and fixed factor income. The matrix $[E_{wp} - v_{m_f}x']$ will be a null matrix, since factor supplies are independent of output prices or fixed factor income. In this case, we can conclude that $dp'dy > 0$ in equation (3.7), so that output supply functions will be upward sloping.

Now consider the case where $m = n = 2$, and $e = 1$, and we no longer use the additively separable utility function. That is, consider the simple two good, two factor trade model where one of the two factors is endogenously supplied. We can now rewrite equation (3.7) as follows:

(3.9)

$$
\begin{aligned}
&-dp'\hat{G}_{pv}[E_{wp} - v_{m_f}x']dp \\
&= \begin{pmatrix} dp_1 & dp_2 \end{pmatrix} \begin{pmatrix} (\partial y_1/\partial v_1) \\ (\partial y_2/\partial v_1) \end{pmatrix} \begin{pmatrix} (\partial v_1/\partial p_1) + v_{m_f}x_1 & (\partial v_1/\partial p_2) + v_{m_f}x_2 \end{pmatrix} \begin{pmatrix} dp_1 \\ dp_2 \end{pmatrix}, \\
&= \begin{pmatrix} dp_1 & dp_2 \end{pmatrix} \begin{pmatrix} \frac{\partial y_1}{\partial v_1}\frac{\partial v_1}{\partial p_1} + \frac{\partial y_1}{\partial v_1}v_{m_f}x_1 & \frac{\partial y_1}{\partial v_1}\frac{\partial v_1}{\partial p_2} + \frac{\partial y_1}{\partial v_1}v_{m_f}x_2 \\ \frac{\partial y_2}{\partial v_1}\frac{\partial v_1}{\partial p_1} + \frac{\partial y_2}{\partial v_1}v_{m_f}x_1 & \frac{\partial y_2}{\partial v_1}\frac{\partial v_1}{\partial p_2} + \frac{\partial y_2}{\partial v_1}v_{m_f}x_2 \end{pmatrix} \begin{pmatrix} dp_1 \\ dp_2 \end{pmatrix}
\end{aligned}
$$

In order to find conditions under which the supply of output of a good responds positively to changes in its own price, suppose that the price of only good i increases, so that $dp_i > 0, dp_j = 0\ \forall j \neq i$. A sufficient condition to observe upward-sloping output supply curves ($dp_i dy_i > 0$) would be that the diagonal terms of the matrix $\hat{G}_{pv}[E_{wp} - v_{m_f}x']$ be negative. Consider the term $\hat{G}_{p_iv} = \partial y_i/\partial v$. If good y_i uses the endogenously supplied factor intensively, then $\hat{G}_{p_iv} > 0$, else $\hat{G}_{p_iv} < 0$.

Now consider the term $E_{wp_i} - v_{m_f}x_i = -\partial v^s/\partial p_i$. Suppose that good i and the endogenously supplied factor (leisure, for example) are substitutes in consumption. Then an increase in the price of good i will cause an increase in the demand for leisure if leisure (the endogenously supplied factor) and good i are gross substitutes. Since $\partial v^s/\partial p_i$ is the change in supply of the endogenous factor due to an output price change, $-\partial v^s/\partial p_i$ reflects the change in demand for the endogenous factor by the consumption sector (ie: change in leisure demand). That is, $E_{wp_i} - v_{m_f} x_i = -\partial v^s/\partial p_i$ will be positive if the endogenous factor (demand for leisure) and good i are gross substitutes, else $E_{wp_i} - v_{m_f} x_i < 0$. As a result, we can state sufficient conditions for $dp_i dy_i > 0$ as follows: An increase in the price of good i will cause output of good i to rise if good i uses the endogenously supplied factor intensivel   d good i and the endogenously supplied factor are gross complements in consumption, or if good i does not use the endogenously supplied factor intensively and good i and the endogenously supplied factor are gross substitutes in consumption.

Alternatively, we could evaluate equation (3.9) in the neighbourhood of autarky, so that x' is a null vector and (3.9) reduces to $-dp'\hat{G}_{pv}E_{wp}dp$. A small open economy takes world prices as given. If we evaluate this expression in the neighbourhood of autarky, we are implying that at given world prices p, the small open economy does not trade, or that world terms of trade are such that the small open economy is not trading in equilibrium. In this case, the sufficient conditions above can be weakened, so that output supply functions slope upwards if good i uses the endogenous factor intensively and good i and the endogenous factor are net complements, or if good i does not use the endogenously supplied factor intensively and good i and the endogenously supplied factor are net substitutes.

At this point we can also describe necessary conditions to observe perverse output effects. For the two good, two factor example above, if an increase in the price of good i is to cause a fall in output of good i, the^n it is necessary that the diagonal elements of the matrix in equation (3.9) be negative. That is, if good i uses the endogenously supplied factor j intensively, it is necessary that good i and the endogenously supplied factor be gross complements in consumption. Similarly, if good i does not use factor

j intensively, then it is necessary that good i and the endogenously supplied factor be gross substitutes in consumption. Of course, these are necessary conditions, so that even if these necessary conditions hold, output supply functions may very well still be upward sloping. However, given these necessary conditions, we cannot rule out the possibility of the existence of perverse output effects.

Now compare the slope of the output supply function when some factors are endogenously supplied to the slope of the output supply function when factor supplies are inelastic. In the latter case, we can solve for the change in output supply $d\tilde{y}$ for a given price change dp using equation (3.2′) and noting that $dv = 0$:

$$d\tilde{y} = \hat{G}_{pp}dp. \tag{3.2″}$$

If we premultiply equation (3.2″) by dp' and subtract the resulting expression from equation (3.6) we get:

$$\begin{aligned} dp'(dy - d\tilde{y}) &= -dp'\hat{G}_{pv}E_{ww}\hat{G}_{vp}dp - dp'\hat{G}_{pv}[E_{wp} - v_{m_f}x']dp \\ &> -dp'\hat{G}_{pv}[E_{wp} - v_{m_f}x']dp. \end{aligned} \tag{3.10}$$

Note that the right-hand side of equation (3.10) is identical to the right-hand side of equation (3.7). This illustrates that the conditions which are sufficient to ensure that equation (3.7) holds (output supply functions are upward sloping when factors are endogenously supplied) are also sufficient to ensure that output supply responses to a given output price change are greater when factors are endogenously supplied than when factors supplies are all perfectly inelastic. Note also that if we use our additively separable utility function, then we can also conclude that the output supply response to a given output price change would be larger when factor supplies are endogenous rather than perfectly inelastic, since $[E_{wp} - v_{m_f}x']$ would be a null matrix.

Now let's turn our attention to the consumption sector. The representative consumer takes the change in input prices due to the output price change as given, so the relevant expression for output demand changes due to the output price change is equation (3.3). The input price change is given in equation (3.4′). Substituting this into equation (3.3) and premultiplying by dp' gives:

$$dp'dz \;=\; dp'[E_{pp} + z_{m_f}x']dp + dp'E_{pw}\hat{G}_{vp}dp. \tag{3.11}$$

As in equation (3.7) above, the presence of income and substitution effects precludes our signing the expression $dp'dz$ in general. For the moment, let's simplify the problem by evaluating equation (3.11) in the neighbourhood of autarky, so that $x = 0$. The term $dp'E_{pp}dp$ is strictly negative since the expenditure function is strictly concave in output prices. As a result, we can write:

$$dp'dz \;<\; dp'E_{pw}\hat{G}_{vp}dp. \tag{3.12}$$

Consider the vector E_{pw}. The Hessian of the expenditure function is symmetric, so that $E_{pw} = E'_{wp}$. In the special separable utility function which we looked at above, we found that the matrix E_{wp} was a null matrix. (Recall that E_w is the vector of endogenously supplied factors. With the additively separable utility function used above, we found that the factor supply functions were a function of only the prices of the endogenous factors. Thus E_{wp} was a null matrix.) Thus if we evaluate equation (3.12) using the same additively separable utility function as above, we conclude that output demand functions will slope downwards as long as the income effect $z_{m_f}x'$ does not outweigh the substitution effect E_{ww}.

Now let's go back to equation (3.11) and consider the case where $dp_i > 0$, $dp_j = 0 \;\; \forall j \neq i$, so that only the price of good i is changing. Equation (3.11) now becomes:

$$dp_i dz_i \;=\; dp_i[E_{p_ip_i} + z_{im_f}x_i]dp_i + dp_iE_{p_iw}\hat{G}_{vp_i}dp_i, \tag{3.11′}$$

the effect of a change in the price of good i on the demand for good i. The term $E_{p_ip_i}$ is negative since the expenditure function is strictly concave in output prices. As noted above, with our additively separable utility function, the term E_{p_iw} is equal to zero. In this case, a sufficient condition for consumption of good i to fall in response to an increase in its price is for the term $(\partial z_i/\partial m_f)x_i$ to be negative. That is, if good i is inferior and is exported, or if good i is normal and is imported, demand curves will necessarily be downward sloping. If the income effect term $(\partial z_i/\partial m_f)x_i$ is positive, then the demand curve for good i will still be downward sloping as long as $|E_{p_ip_i}| > |(\partial z_i/\partial m_f)x_i|$, so that the substitution effect dominates the income effect.

Now consider the two good, two factor example, where one factor is endogenously supplied. The Hessian of the GNP function is also symmetric, so $\hat{G}_{vp} = \hat{G}'_{pv}$. Suppose again that the income effect $z_{m_f}x'$ does not outweigh the substitution effect. Then we are again left with equation (3.12). Notice that the right-hand side of this expression can be rewritten as $dp'[\hat{G}_{pv}E_{wp}]'dp$. This is the negative of the first of the two terms on the right-hand side of equation (3.7). As a result, the conditions which were sufficient to ensure that output supply functions were upward-sloping in the two-good, two-factor model above are sufficient to ensure that output-demand functions are downward-sloping. That is, if the good whose price is changing uses the endogenously supplied factor intensively, then the demand function for that good will be downward-sloping if that good and the endogenously supplied factor are gross complements. If that good does not use the endogenously supplied factor intensively, then the sufficient condition is that the good and the endogenously supplied factor be gross substitutes.

Suppose now that all factors of production are inelastically supplied. In this case the consumers problem must be changed so as to reflect the extra constraint that the total supply of the endogenously supplied factors is fixed. Consequently, the expenditure function is rewritten as follows:

$$E(p,w,\mu) \;=\; \min_{z}\{p'z - w'\hat{v} \mid U(z,\hat{v}) \geq \mu\},$$

where the constraint that the total supply of endogenous factors is fixed is reflected by the fact that $v = \hat{v} \in R^{e}_{+}$ has been substituted into the consumers minimization problem. The matrix E_{wp} will now be a null matrix, since $E_w = -\hat{v}$, and $d\hat{v} = 0$. Equation (3.11) can be written as:

$$dp'd\tilde{z} \;=\; dp'[E_{pp} + z_{m_f}x']dp, \tag{3.13}$$

where $d\tilde{z}$ is the change in output demand for the given output price change when the supply of all factors of production is perfectly inelastic. We can subtract (3.13) from (3.11) to get:

$$dp'(dz - d\tilde{z}) \;=\; dp'E_{pw}\hat{G}_{vp}dp. \tag{3.14}$$

If equation (3.14) is equal to zero, then the output demand effects of a price change are the same when factors are endogenously supplied as when factor supplies are all perfectly inelastic. If equation (3.14) is negative, then the output demand response is greater when factor supplies are allowed to respond to output price changes than when factors are all inelastically supplied.

Now recall that when we used our additively separable utility function, the matrix $E_{pw}\hat{G}_{vp}$ was a null matrix. That is, when preferences can be represented by our separable utility function, the slope of the output demand functions when some factors are endogenously supplied is the same as the slope of the demand functions when the supply of all factors is perfectly inelastic.

The sufficient conditions which ensured that output demand functions were downward sloping in the two-good, two-factor model above imply that the right-hand side of equation (3.14) is negative. That is, when the good which uses the endogenously supplied factor intensively is a complement for the endogenously supplied factor, or when the good that does not use the endogenously supplied factor intensively is a substitute for the endogenously supplied factor, the output demand response for a given output price change is larger when the factor is endogenously supplied than when the supply of all factors is perfectly inelastic.

To examine the effects of the presence of endogenously supplied factors on net exports, we need only subtract equation (3.11) from equation (3.6):

$$\begin{aligned} dp'dx &= dp'dy - dp'dz \\ &= dp'[\hat{G}_{pp} - E_{pp} - z_{m_f}x']dp \\ &-dp'\hat{G}_{pv}E_{ww}\hat{G}_{vp}dp \\ &-dp'\hat{G}_{pv}[E_{wp} - v_{m_f}x']dp \\ &-dp'E_{pw}\hat{G}_{vp}dp. \end{aligned} \tag{3.15}$$

We know that the GNP function is strictly convex in output prices, and that the expenditure function is strictly concave in input and output prices. Thus we can rewrite equation (3.15) as:

$$\begin{aligned} dp'dx &> -dp'(E_{pp} + z_{m_f}x')dp + dp'\hat{G}_{pv}v_{m_f}x'dp \\ &\quad -dp'\hat{G}_{pv}E_{wp}dp - dp'E_{pw}\hat{G}_{vp}dp. \end{aligned} \tag{3.16}$$

As was the case when we were evaluating the effect of endogenous factor supply on output supply and output demand functions, the presence of income and substitution effects in equation (3.16) does not allow us to sign the expression for $dp'dx$ in general. However, if the representative consumer's preferences can be represented by our additively separable utility function, $E_{wp} = E_{pw}$ are both null matrices, and the vector of income effects v_{m_f} would be a null vector. Equation (3.16) could then be rewritten as:

$$(3.16') \qquad dp'dx \; > \; -dp'(E_{pp} + z_{m_f}x')dp.$$

If we evaluate this expression in the neighbourhood of autarky, x is a null vector. Equation (3.16′) would then be strictly positive, and we could conclude that changes in net exports are positively associated with output price changes.

Now suppose that the price of only good i changes. If the income effect for good i, $z_{im_f}x_i$ is negative, then equation (3.16′) will be strictly positive, and net exports will be positively associated with own price changes. That is, imports of good i will tend to fall as the price of good i rises, and imports of good i will tend to rise as the price of that good falls. The sufficient conditions for this result are the same as those which ensure that own-price changes result in a decrease in demand, namely that good i be normal (inferior) if it is imported (exported). Of course, these conditions are by no means necessary. In fact, even if the income effect above is positive, this result will still obtain as long as $E_{p_ip_i} + z_{im_f}x_i$ is negative (recall that the expenditure function is strictly concave in output prices, so the substitution effect for good i, $E_{p_ip_i}$, is strictly negative). As well, these results are not any different from those in a standard trade model without endogenous factor supplies. A strong enough perverse income effect $z_{im_f}x_i$ can lead to perverse net export effects even in a traditional trade model, resulting in a backward-bending offer curve.

Now consider the two-good, two-factor model, where one of the two factors is endogenously supplied. We saw above that the sufficient conditions to ensure that output supply functions sloped upwards in this model were also sufficient to ensure that output demand functions were downward sloping (assuming that the income effect $v_{m_f}x'$ did not outweigh the substitution effect E_{ww}). As a result, the same conditions will also be sufficient to ensure that net export functions are upward sloping. That is, if good i is exported, an increase in p_i will cause exports of good i to increase if good i uses the endogenously supplied factor intensively and if good i and the endogenously supplied factor are gross complements in consumption. If good i is imported, then a decrease in p_i will cause imports of good i to increase if good i uses the fixed factor intensively and if good i and the endogenous factor are gross substitutes in consumption. On the other hand, the same conditions which were necessary in order to observe perverse output effects are also necessary to observe perverse net export effects. For example, in order to observe exports of a good falling as the price of that good rises, it is necessary that that good be a substitute for the endogenously supplied factor if it uses the endogenously supplied factor intensively, and so on.

We are now in a position to compare the relative effects of a given output price change on net exports when factors are endogenously supplied to when factors are all in perfectly inelastic supply. In the latter case, we can solve for the change in net exports $d\tilde{x}$ as a function of the output price change as follows:

$$d\tilde{x} = dp'[\hat{G}_{pp} - E_{pp} - z_{m_f}x']dp. \tag{3.17}$$

If we subtract equation (3.17) from equation (3.15), we get:

$$\begin{aligned} dp'(dx - d\tilde{x}) &= -dp'\hat{G}_{pv}E_{ww}\hat{G}_{vp}dp \\ &\quad -dp'\hat{G}_{pv}[E_{wp} - v_{m_f}x']dp \\ &\quad -dp'E_{pw}\hat{G}_{vp}dp. \end{aligned} \tag{3.18}$$

The expenditure function is strictly concave in input prices, so that the first term on the right-hand side of equation (3.18) is a quadratic form about a positive definite matrix. Thus we can rewrite (3.18) as:

$$\begin{aligned} dp'(dx - d\tilde{x}) &> -dp'\hat{G}_{pv}[E_{wp} - v_{m_f}x']dp \\ &\quad -dp'E_{pw}\hat{G}_{vp}dp. \end{aligned} \tag{3.19}$$

In the case where preferences can be represented by our additively separable utility function, the right-hand side of equation (3.19) is equal to zero. That is, when the supply functions for the endogenous factors are independent of output prices and income, net export responses to a given output price change will be greater (in absolute value) when factors are endogenously supplied than when factor supplies are all perfectly inelastic. Since the right-hand side of equation (3.19) is the same as the right-hand side of equation (3.16) above (except that the term on the income effect $z_{m_I}x'$ does not appear in equation (3.19)), the same result will also hold in the two-good, two-factor model when the sufficient conditions to ensure that output supply curves are upward sloping hold.

To examine the effect of the presence of endogenously supplied factors on a nation's pattern of trade, starting from autarky, evaluate equation (3.16) at the point $x = 0$, so that we get:

$$dp'dx \ > \ -dp'\hat{G}_{pv}E_{wp}dp - dp'E_{pw}\hat{G}_{vp}dp.$$

If we begin at autarky, and the free trade price of a good is higher than its autarky price, so that $dp_i > 0$, then if $dp'dx > 0$, the nation would export that good whose price is rising. Therefore we can interpret the condition $dp'dx > 0$ as implying that a nation tends to export those goods whose prices rise as the nation moves to free trade, and to import those goods whose prices fall. Since the two terms on the right-hand side of the term $dp'dx$ above are transposes of each other, a sufficient condition for $dp'dx > 0$ is that $dp'\hat{G}_{pv}E_{wp}dp > 0$. That is, if the good whose price is increasing uses the endogenously supplied factor intensively, then that good should be a complement for the endogenously supplied factor, and vice versa if the good whose price is changing does not use the endogenously supplied factor intensively. However, if the good whose price is changing uses the endogenously supplied factor intensively and is a substitute in consumption for that factor, then as the price of that good rises, demand for that good would fall, demand for the endogenous factor would rise since it is a substitute, and thus supply of the endogenous factor would fall. Since the good whose price is rising uses the endogenously supplied factor intensively, output of that good may fall in response to an increase in its price, and the nation may actually end up importing the good when its price rises from autarky if the decrease in output supply outweighs the decrease in output demand.

3.2 More Factors than Goods

Now consider the case where the total number of factors exceeds the number of goods. The modified GNP function is convex in input prices. The expenditure function is concave in input prices. As a result, $[G_{ww} - E_{ww}]^{-1}$ is a positive definite matrix. Substituting (3.4) into equation (3.1) and premultiplying by dp' gives:

$$
\begin{aligned}
dp'dx \; &= dp'[G_{pp} - E_{pp} - z_{m_f}x']dp \\
&\quad -dp'[G_{pw} - E_{pw}][G_{ww} - E_{ww}]^{-1}[G_{wp} - E_{wp}]dp \qquad (3.20) \\
&\quad -dp'[G_{pw} - E_{pw}][G_{ww} - E_{ww}]^{-1}[v_{m_f}x']dp.
\end{aligned}
$$

Suppose that we evaluate this expression in the neighbourhood of autarky, so that $x = 0$. This assumption is sufficient to ensure that net export functions slope upwards in the traditional trade model with exogenous factor supplies.[6] Here we see that the right-hand side of (3.20) is the difference between two quadratic forms, both positive. This implies that even when preferences are homothetic and the economy is not trading, net export functions may be backward bending, when some factors are endogenously supplied. We cannot rule out the case where (3.20) is a negative scalar

[6] see Woodland (1982), p.209. In the neighbourhood of autarky, this expression should be interpreted as: "A nation tends to export (import) those goods whose free trade prices are higher (lower) than their autarky prices."

when evaluated in the neighbourhood of autarky, and as a result it is possible that a country could export (import) those goods whose free trade prices are lower (higher) than their autarky prices.

We can derive similar expressions for output supply and demand changes, by substituting out for input price changes in equations (3.2) and (3.3) as follows:

$$\begin{aligned} dp'dy &= dp'G_{pp}dp \\ &\quad -dp'G_{pw}[G_{ww} - E_{ww}]^{-1}G_{wp}dp \\ &\quad -dp'G_{pw}[G_{ww} - E_{ww}]^{-1}[-E_{wp} + v_{m_f}x']dp \end{aligned} \tag{3.21}$$

$$\begin{aligned} dp'dz &= dp'[E_{pp} + z_{m_f}x']dp \\ &\quad +dp'[-E_{pw}][G_{ww} - E_{ww}]^{-1}[-E_{wp}]dp \\ &\quad +dp'[-E_{pw}][G_{ww} - E_{ww}]^{-1}[G_{wp} + v_{m_f}x']dp \end{aligned} \tag{3.22}$$

The first two terms on the right-hand side of (3.21) constitute the difference between positive scalars, and the last term is indefinite. Clearly we can describe cases where output supply curves slope downwards, since it is no longer true in general that $dp'dy$ in equation (3.21) is positive.

The first two terms on the right-hand side of (3.22) are negative and positive scalars, respectively, and the last term is indefinite. Therefore, even if we assume that the economy is not trading, we cannot say that output demand curves will generally be downward sloping.

Another objective of this section is to compare comparative statics results like equations (3.20) - (3.22) to their analogues in the case when factors of production are not allowed to respond to input or output price changes. Consider the problem of determining input price changes as a function of output price changes, while holding the amount of the endogenously supplied factors constant. Demand for endogenously supplied factors is given by:

$$(3.23) \qquad -v_d \;=\; G_w(p, w, \bar{v}).$$

The extra constraint is found by differentiating equation (3.23) with respect to p and w, substituting $dv_d = 0$, and solving for dw:

$$(3.4'') \qquad dw \;=\; -[G_{ww}]^{-1}G_{wp}dp.$$

This is the change in input prices required to ensure that the change in aggregate demand for endogenously supplied factors by the production sector is zero. Alternatively we can think of equation (3.4″) as giving the change in the prices of the endogenously supplied factors for a given price change dp when the supply of these factors is perfectly inelastic. The assumption here that $dv_d = 0$ is used so that the following results cn be interpreted in the same sense as those in the standard Heckscher-Ohlin model. Total usage of factors of production is fixed, so that the matrix E_{wp} is a null matrix. (Recall that $E_w = -v^s$. If factor supplies are held constant, $E_{wi} = 0, \;\; \forall i \neq w$.) Since $E'_{pw} = E_{wp}$, the matrix E_{pw} will also be a null matrix. The following results can then be compared to the corresponding results derived above when factor supplies were endogenous.

If we substitute (3.4″) into (3.1) - (3.3) and premultiply throughout by dp', using $E_{pw} = 0$, we get:

$$(3.20') \qquad \begin{aligned} dp'd\tilde{x} \;&= dp'[G_{pp} - E_{pp} - z_{m_I}x']dp \\ &\quad -dp'G_{pw}[G_{ww}]^{-1}G_{wp}dp, \end{aligned}$$

$$(3.21') \qquad \begin{aligned} dp'd\tilde{y} \;&= dp'G_{pp}dp \\ &\quad -dp'G_{pw}[G_{ww}]^{-1}G_{wp}dp, \end{aligned}$$

$$dp'd\tilde{z} = dp'[E_{pp} + z_{m_f}x']dp \tag{3.22'}$$

These are the changes in net exports, output supply, and output demand, respectively, due to a change in output prices, when input prices are allowed to change according to equation (3.4″), and the supply of the endogenously supplied factors is not allowed to change.

We can now compare the effect of a price change on net exports, output supply, and output demand when some factors are endogenously supplied with the corresponding effects when those factors are inelastically supplied. Subtract (3.20′) from (3.20), and likewise for (3.21) and (3.22) to get:

$$\begin{aligned} dp'(dx - d\tilde{x}) &= dp'[G_{pp} - E_{pp} - z_{m_f}x']dp \\ &\quad -dp'[G_{pw} - E_{pw}][G_{ww} - E_{ww}]^{-1}[G_{wp} - E_{wp} + v_{m_f}x']dp \\ &\quad -dp'[G_{pp} - E_{pp} - z_{m_f}x']dp \\ &\quad +dp'G_{pw}[G_{ww}]^{-1}G_{wp}dp, \end{aligned}$$

$$\begin{aligned} dp'(dy - d\tilde{y}) &= dp'G_{pp}dp \\ &\quad -dp'G_{pw}[G_{ww} - E_{ww}]^{-1}G_{wp}dp \\ &\quad -dp'G_{pw}[G_{ww} - E_{ww}]^{-1}[-E_{wp} + v_{m_f}x']dp \\ &\quad -dp'G_{pp}dp \\ &\quad +dp'G_{pw}[G_{ww}]^{-1}G_{wp}dp, \end{aligned}$$

$$\begin{aligned} dp'(dz - d\tilde{z}) &= dp'[E_{pp} + z_{m_f}x']dp \\ &\quad +dp'[-E_{pw}][G_{ww} - E_{ww}]^{-1}[-E_{wp}]dp \\ &\quad +dp'[-E_{pw}][G_{ww} - E_{ww}]^{-1}[G_{wp} + v_{m_f}x']dp \\ &\quad -dp'[E_{pp} + z_{m_f}x']dp \end{aligned}$$

We can cancel out terms and rewrite these expressions as follows:

$$\begin{aligned}
dp'(dx - d\tilde{x}) &= -dp'G_{pw}[G_{ww} - E_{ww}]^{-1}G_{wp}dp \\
&\quad + dp'G_{pw}[G_{ww}]^{-1}G_{wp}dp \\
&\quad - dp'[G_{pw} - E_{pw}][G_{ww} - E_{ww}]^{-1}[-E_{wp} + v_{m_f}x']dp \\
&\quad - dp'[-E_{pw}][G_{ww} - E_{ww}]^{-1}[G_{wp}]dp \\
dp'(dy - d\tilde{y}) &= -dp'G_{pw}[G_{ww} - E_{ww}]^{-1}G_{wp}dp \\
&\quad + dp'G_{pw}[G_{ww}]^{-1}G_{wp}dp \\
&\quad - dp'G_{pw}[G_{ww} - E_{ww}]^{-1}[-E_{wp} + v_{m_f}x']dp \\
dp'(dz - d\tilde{z}) &= dp'[-E_{pw}][G_{ww} - E_{ww}]^{-1}[-E_{wp}]dp \\
&\quad + dp'[-E_{pw}][G_{ww} - E_{ww}]^{-1}[v_{m_f}x']dp \\
&\quad + dp'[-E_{pw}][G_{ww} - E_{ww}]^{-1}G_{wp}dp
\end{aligned}$$

Define the following matrices: $[G_{ww}] = B$, $[-E_{ww}] = C$, and $[G_{ww} - E_{ww}] = B + C = A$. B and C are both positive definite matrices, so A is also a positive definite matrix, as are B^{-1} and C^{-1}. Combining the fact that a quadratic form of a positive definite matrix is a positive scalar with the fact that, if B and C are positive definite matrices and $A = B + C$, then $x'A^{-1}x < x'B^{-1}x$ for any vector x (see Graybill, 1983, pp. 409-411, or the Appendix, p.130), we can write:

(3.24)
$$\begin{aligned}
dp'(dx - d\tilde{x}) &> dp'[G_{pw} - E_{pw}][G_{ww} - E_{ww}]^{-1}[-E_{wp} + v_{m_f}x']dp \\
&\quad - dp'[-E_{pw}][G_{ww}]^{-1}[G_{wp}]dp
\end{aligned}$$

(3.25)
$$dp'(dy - d\tilde{y}) > -dp'G_{pw}[G_{ww} - E_{ww}]^{-1}[-E_{wp} + v_{m_f}x']dp$$

(3.26)
$$dp'(dz - d\tilde{z}) = dp'[-E_{pw}][G_{ww} - E_{ww}]^{-1}[G_{wp} - E_{wp} + v_{m_f}x']dp.$$

In general, the complexity of the income and substitution effects in the above equations precludes our signing these terms. We will therefore proceed by considering the case where the representative consumer's preferences can be represented by the additively separable utility function of Section 3.1. As we saw before, the matrix $[E_{wp} - v_{m_f} x']$ will be a null matrix, as will the matrix E_{pw}. Substituting these terms into equations (3.24) - (3.26) above yields the following results. From equation (3.26), $dp'(dz - d\tilde{z}) = 0$, so that the response of output demand to a given change in the vector of output prices dp is the same whether factor supplies respond to this output price change or not. From equation (3.25), $dp'(dy - d\tilde{y}) > 0$, so that output supply responses are larger when factor supplies are endogenous. Similarly, since $dp'(dx - d\tilde{x}) > 0$ in equation (3.24), we can also conclude that the net export response to a given output price shock is larger when factors are endogenously supplied. Output supply functions and net export functions are more price elastic when factor supplies are allowed to adjust to output price shocks than when factor supplies are perfectly inelastic. Note also that we can now conclude that output supply and net export functions are upward sloping when factor supplies are endogenous, and that output demand functions are downward sloping when factors are endogenously supplied.

Let us now consider a two good, three factor version of this model, where goods 1 and 2 are produced using capital, K, land, H, and labour, L. The prices of K, H, and L are r, h, and w, respectively, and assume for example that labour is endogenously supplied and that capital and land are in fixed supply. In this case $[G_{ww} - E_{ww}]^{-1}$ will be a positive scalar. We can now write:

$$
\begin{aligned}
(3.27) \qquad & -G_{pw}[G_{ww} - E_{ww}]^{-1}[-E_{wp} + v_{m_f} x'] \\
&= [s] \begin{pmatrix} \frac{\partial y_1}{\partial w}(L^s_{m_f} x_1 + \frac{\partial L^s}{\partial p_1}) & \frac{\partial y_1}{\partial w}(L^s_{m_f} x_2 + \frac{\partial L^s}{\partial p_2}) \\ \frac{\partial y_2}{\partial w}(L^s_{m_f} x_1 + \frac{\partial L^s}{\partial p_1}) & \frac{\partial y_2}{\partial w}(L^s_{m_f} x_2 + \frac{\partial L^s}{\partial p_2}) \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
(3.28) \qquad & -E_{pw}[G_{ww} - E_{ww}]^{-1}[-E_{wp} + v_{m_f} x'] \\
&= [s] \begin{pmatrix} \frac{\partial z_1}{\partial w}(L^s_{m_f} x_1 + \frac{\partial L^s}{\partial p_1}) & \frac{\partial z_1}{\partial w}(L^s_{m_f} x_2 + \frac{\partial L^s}{\partial p_2}) \\ \frac{\partial z_2}{\partial w}(L^s_{m_f} x_1 + \frac{\partial L^s}{\partial p_1}) & \frac{\partial z_2}{\partial w}(L^s_{m_f} x_2 + \frac{\partial L^s}{\partial p_2}) \end{pmatrix}
\end{aligned}
$$

$$(3.29) \qquad \begin{aligned} &-E_{pw}[G_{ww} - E_{ww}]^{-1}G_{wp} \\ &= [s]\begin{pmatrix} -\frac{\partial L^d}{\partial p_1}\frac{\partial x_1}{\partial w} & -\frac{\partial L^d}{\partial p_2}\frac{\partial x_1}{\partial w} \\ -\frac{\partial L^d}{\partial p_1}\frac{\partial x_2}{\partial w} & -\frac{\partial L^d}{\partial p_2}\frac{\partial x_2}{\partial w} \end{pmatrix}, \end{aligned}$$

where $[s] = -[G_{ww} - E_{ww}]^{-1}$ is a negative scalar.

If we pre-and post-multiply equation (3.27) by dp' and dp respectively, we would get the right-hand side of equation (3.25). If the quadratic form on the right-hand side of equation (3.25) is non-negative, we could conclude that output supply functions are more elastic when labour is endogenously supplied. Since $[s]$ is a negative scalar, a sufficient condition for this result would be that the symmetric version of the matrix $[G_{pw}][-E_{wp}+v_{m_f}x']$ in equation (3.27) be negative semi-definite. (The matrix must be symmetric if we are to use information about its definiteness to sign the quadratic form in equation (3.25)). Note that the matrix in (3.27) is of the form ab', where a and b are both 2x1 vectors. The off-diagonal elements of ab' are replaced with $(a_ib_j + a_jb_i)/2$, $\forall i, j, i \neq j$, in order to get a symmetric version of the matrix ab' without changing the value of the quadratic form $dp'[ab']dp$. However, we can show (see Appendix, p.131) that the symmetric version of any matrix of the form ab', $a, b \in R^m$, $m \geq 2$, is indefinite. As a result, we cannot determine the definiteness of the matrix in equation (3.27) in general, and therefore the right-hand side of equation (3.25) may be either positive or negative. Note that even if the right-hand side of (3.25) is negative, it may still be true that output supply functions are more elastic when labour is endogenously supplied.

Suppose now that the price of only good i is changing, $i = 1, 2$. In this case, equation (3.27) reduces to:

$$(3.27') \qquad [s](\partial y_i/\partial w)(L^s_{m_f}x_i + \partial L^s/\partial p_i).$$

We will say that the endogenously supplied factor (labour) is a normal factor in production if an increase in the price of labour causes output to fall, ceteris paribus. That is, labour is normal in production if $\partial y_i/\partial w < 0$, and is inferior if $\partial y_i/\partial w > 0$.

In general, the sign on the term $\partial y_i/\partial w$ can be either positive or negative, as can be demonstrated by applying Euler's theorem to the equation for y_i: $0 = (\partial y_i/\partial p_1)dp_1 + (\partial y_i/\partial p_2)dp_2 + (\partial y_i/\partial w)dw$. Output supply functions are homogenous of degree zero in (p, w), and $(\partial y_i/\partial p_i) > 0$, but this information is not sufficient to sign $(\partial y_i/\partial w)$. Let us assume that labour is normal in production. Recall that $[s]$ is a negative scalar. Then a sufficient condition for the output supply function for good i to be more elastic when labour supply is endogenous $(dp_i(dy_i - d\tilde{y}_i) > 0)$ is that the term $(L^s_{m_f}x_i + \partial L^s/\partial p_i) = -E_{wp_i} + v_{m_f}x_i$ be positive. In the previous section (see p. 29, following equation (3.9)) we established that $E_{wp_i} - v_{m_f}x_i = -\partial L^s/\partial p_i$ was positive if good i and the endogenously supplied factor (labour) were gross substitutes. As a result, we can state that the term $-E_{wp_i} + v_{m_f}x_i$ will be positive if good i and labour are gross complements in consumption. If labour is a normal factor in production, then a sufficient condition for the output supply response of good i to a given own-price change to be greater when labour supply is endogenous is that labour and good i be gross complements in consumption.

We can apply the same analysis to equations (3.28) and (3.29) to find sufficient conditions under which output demand functions are more elastic. In this case we would want the right-hand side of equation (3.26) to be negative, so that the output demand response to a given output price change would be larger (in absolute value) when labour is endogenously supplied . However, the matrices in equations (3.28) and (3.29) are also of the form ab', $a, b \in R^2$, so that they are also indefinite. So suppose as we did above that the price of only good i is changing. Sufficient conditions for output demand functions to be more elastic when labour is endogenously supplied are that the diagonal elements of the matrices in (3.28) and (3.29) are positive. That is, we would want to observe:

$$\frac{\partial z_i}{\partial w}\left\{-\frac{\partial L^d}{\partial p_i}+\frac{\partial L^s}{\partial p_i}+L^s_{m_f}x_i\right\} > 0.$$

We know that $\partial z_i/\partial w = E_{p_iw} = E_{wp_i} = -\partial L^s/\partial p_i$, so that the term $(\partial z_i/\partial w)(\partial L^s/\partial p_i)$ is always negative. As before, the term $-\partial L^d/\partial p_i = G_{wp_i} = G_{p_iw} = \partial y_i/\partial w$ can be either positive or negative, so assume that labour is normal in production of good i. Then an increase in the price of labour will cause output of good i to fall. As a result, $\partial y_i/\partial w = -\partial L^d/\partial p_i < 0$. To simplify this expression, evaluate it in the neighbourhood of autarky. Then if good i and labour are net complements, E_{wp_i} will be negative, or $-\partial z_i/\partial w = -E_{p_iw} = -E_{wp_i} = \partial L^s/\partial p_i$ will be positive. Hence we need to impose a further restriction to ensure that output demand functions are more elastic when labour is endogenously supplied, since the labour supply effect of a change in the price of good i and the labour demand effect work in opposite directions. The extra condition that we need is that the labour demand effect is larger in absolute value than the (uncompensated) labour supply effect, so that $|\partial L^d/\partial p_i| > |\partial L^s/\partial p_i - L^s_{m_f}x_i|$.

At this point we could apply the same analysis to the right-hand side of equation (3.24) to find sufficient conditions under which the net export response to a given price change is greater when labour is endogenously supplied. However, since the matrices in the resultant expressions would also be indefininte, and since we know that $dx = dy - dz$ and $d\tilde{x} = d\tilde{y} - d\tilde{z}$, we can say directly that when the price of only good i changes, the change in net exports of good i will be greater when labour is endogenously supplied as long as the sufficient conditions described above are satisfied. So to summarize, the own-price output supply, output demand, and net export elasticities for good i will all be larger (in absolute value) when the endogenously supplied factor is normal in production of good i if good i and the endogenous factor are gross complements in consumption, and when the labour demand effect of a change in the price of labour is larger in absolute value than the (uncompensated) labour supply effect. This result applies as well to the m-good case, as long as there are more factors than goods and only one factor is endogenously supplied ($n > m,\ e = 1$).

3.3 Cobb-Douglas Examples

In this Section we illustrate the effects of endogenous factor supply in a small open economy where two goods, x and y, are produced using capital (K) and labour (L). Labour is endogenously supplied. There is a single representative consumer in the economy who owns all capital and labour. His preferences over consumption of the two goods and labour/leisure are represented by the Cobb-Douglas utility function $U = x^a y^b (\bar{L} - L)^c,\ a + b + c = 1$, where $\bar{L}$ is the consumer's total endowment of labour, and $\bar{L} - L$ is his consumption of leisure. Production technology is summarized by Cobb-Douglas production functions $x = K_x^{\alpha_x} L_x^{\beta_x},\ y = K_y^{\alpha_y} L_y^{\beta_y},\ \alpha_i + \beta_i = 1,\ i = x, y$. In equilibrium, factors are fully employed, so that $K_x + K_y = \bar{K},\ L_x + L_y = L$. The small open economy takes output prices p_x and p_y as given. We will assume throughout that both goods are produced in equilibrium, so that the prices r and w of capital and labour, respectively, are uniquely determined by the production sector independent of the consumption sector (as long as factor intensities differ between industries in equilibrium, so that $\alpha_x/\beta_x \neq \alpha_y/\beta_y$).

The producer's revenue maximization problem can be posed as follows:

$$\max_{x,y} \quad p_x \cdot x + p_y \cdot y \qquad \text{s.t.:} \qquad \begin{aligned} x &= K_x^{\alpha_x} L_x^{\beta_x} \\ y &= K_y^{\alpha_y} L_y^{\beta_y} \\ \bar{K} &= K_x + K_y \\ L &= L_x + L_y \end{aligned}$$

In the Appendix (see p. 135) we solve this problem and derive the output supply functions $x^s(p_x, p_y, \bar{K}, L)$ and $y^s(p_x, p_y, \bar{K}, L)$, where L is the endogenously determined supply of labour which the production sector takes as given.

$$(3.30) \quad \begin{pmatrix} x^s \\ y^s \end{pmatrix} = \left(\frac{-\alpha_x \beta_y}{q}\right)\left(\frac{\alpha_x}{\alpha_y}\right)^{\frac{-\alpha_y \beta_x}{q}}\left(\frac{\beta_x}{\beta_y}\right)^{\frac{\alpha_y \beta_x}{q}} \begin{pmatrix} \left(\frac{\beta_x}{\beta_y}\frac{p_x}{p_y}\right)^{\frac{-\beta_x}{q}} \cdot \bar{K} - \left(\frac{\alpha_x}{\alpha_y}\frac{p_x}{p_y}\right)^{\frac{\alpha_x}{q}} \cdot L \\ -\left(\frac{\beta_x}{\beta_y}\frac{p_x}{p_y}\right)^{\frac{-\beta_y}{q}} \cdot \bar{K} + \left(\frac{\alpha_x}{\alpha_y}\frac{p_x}{p_y}\right)^{\frac{\alpha_y}{q}} \cdot L \end{pmatrix}$$

$$\begin{aligned} \text{where:} \quad q &= \alpha_y \beta_x - \alpha_x \beta_y \\ &= \frac{\beta_x \beta_y}{\beta_x \beta_y}(\alpha_y \beta_x - \alpha_x \beta_y) \\ &= \beta_x \beta_y \left(\frac{\alpha_y}{\beta_y} - \frac{\alpha_x}{\beta_x}\right). \end{aligned}$$

If good x is relatively labour-intensive, good y is relatively capital-intensive, then $\frac{\alpha_y}{\beta_y} > \frac{\alpha_x}{\beta_x}$, and $q > 0$.

We can state the representative consumer's utility maximization problem as follows:

$$\max_{x,y,L} \quad U(x,\ y\ , L) \ = \ x^a y^b (\bar{L} - L)^c$$

$$\text{s.t.} \qquad p_x \cdot x + p_y \cdot y \le wL + r\bar{K}$$

We solve for the output demand functions $x^d(p_x, p_y, w, m_f)$ and $y^d(p_x, p_y, w, m_f)$, and the labour supply function $L^s(p_x, p_y, w, m_f)$ in the Appendix (see p. 138).

$$(3.31) \qquad \begin{pmatrix} x^d \\ y^d \\ L^s \end{pmatrix} = \begin{pmatrix} \left(\frac{a}{p_x}\right)(w\bar{L} + m_f) \\ \left(\frac{b}{p_y}\right)(w\bar{L} + m_f) \\ \left(\frac{-c}{w}\right)(w\bar{L} + m_f) + \bar{L} \end{pmatrix},$$

where $m_f = r\bar{K}$ is total fixed factor income.

Now consider the effects of an increase in the world price of good x, $dp_x > 0$, $dp_y = 0$. Since we have assumed that the small open economy continues to produce both goods in equilibrium, we can solve for the equilibrium input prices as a function of the given output prices (see Appendix, p. 133):

$$(3.32) \qquad \begin{pmatrix} w \\ r \end{pmatrix} = \begin{pmatrix} p_x^{\frac{\alpha_y}{q}} p_y^{\frac{-\alpha_x}{q}} \left(\frac{\alpha_x}{\alpha_y}\right)^{\frac{\alpha_x \alpha_y}{q}} \beta_x^{\frac{\alpha_y \beta_x}{q}} \beta_y^{\frac{-\alpha_x \beta_y}{q}} \\ p_x^{\frac{-\beta_y}{q}} p_y^{\frac{\beta_x}{q}} \alpha_x^{\frac{-\beta_y \alpha_x}{q}} \alpha_y^{\frac{\beta_x \alpha_y}{q}} \left(\frac{\beta_x}{\beta_y}\right)^{\frac{-\beta_x \beta_y}{q}} \end{pmatrix}$$

If we differentiate the expression for the equilibrium wage rate with respect to the price of good x, we get:

$$\begin{aligned} \frac{\partial w}{\partial p_x} &= \frac{\alpha_y}{q} p_x^{\frac{\alpha_y - q}{q}} p_y^{\frac{-\alpha_x}{q}} \left(\frac{\alpha_x}{\alpha_y}\right)^{\frac{\alpha_x \alpha_y}{q}} \beta_x^{\frac{\alpha_y \beta_x}{q}} \beta_y^{\frac{-\alpha_x \beta_y}{q}} \\ &= \frac{\alpha_y}{q} \left(\frac{p_x}{p_y}\right)^{\frac{\alpha_x}{q}} \left(\frac{\alpha_x}{\alpha_y}\right)^{\frac{\alpha_x \alpha_y}{q}} \beta_x^{\frac{\alpha_y \beta_x}{q}} \beta_y^{\frac{-\alpha_x \beta_y}{q}} \end{aligned}$$

Then an increase in the price of x holding p_y constant causes the return to labour to rise when good x is labour-intensive ($q > 0$), and causes the return to labour to fall when x is capital-intensive ($q < 0$).

From the expression for the equilibrium labour supply function from the solution to the consumer's problem in equation (3.31), we can solve for the equilibrium change in labour supply due to a change in the return to labour:

$$\begin{aligned} L^s &= \frac{-c}{w}(w\bar{L} + m_f) + \bar{L} \qquad\qquad m_f = r\bar{K} = p_x \cdot x + p_y \cdot y - wL \\ L^s &= \frac{-c}{w}\Big(p_x \cdot x + p_y \cdot y + w(\bar{L} - L)\Big) + \bar{L} \\ \frac{\partial L^s}{\partial w} &= \frac{c}{w}\left(\frac{p_x \cdot x + p_y \cdot y + w(\bar{L} - L)}{w}\right) - \frac{c}{w}(\bar{L} - L) \\ \frac{\partial L^s}{\partial w} &= \frac{c}{w}\left(\frac{p_x \cdot x + p_y \cdot y}{w}\right) > 0 \end{aligned}$$

Then we can solve for the change in labour supply due to a change in the price of good x as follows:

$$\begin{aligned} \frac{\partial L^s}{\partial p_x} &= \frac{\partial L^s}{\partial p_x}\Big|_{dw=0} + \frac{\partial L^s}{\partial w}\frac{\partial w}{\partial p_x} \\ \frac{\partial L^s}{\partial p_x} &= \frac{-c}{w} \cdot x + \frac{c}{w}\left(\frac{p_x \cdot x + p_y \cdot y}{w}\right)\frac{\partial w}{\partial p_x} \end{aligned}$$

Due to the specification of the utility function, leisure is a normal good, so that an increase in the price of leisure ($dw > 0$) causes leisure demand to fall, labour supply to rise, and $\frac{\partial L^s}{\partial w} > 0$. The sign on the term $\frac{\partial L^s}{\partial w}\frac{\partial w}{\partial p_x}$ is therefore totally dependent upon whether good x is labour-intensive or capital-intensive. In the former case, $q > 0$ and $\frac{\partial w}{\partial p_x} > 0$. If x is capital-intensive, then a rise in the price of good x causes w to fall, so that labour supply falls.

If we now consider the direct effect of the output-price change on labour supply, we see that $\frac{\partial L^s}{\partial p_x}|_{dw=0}$ is always negative.

Due to the specification of the utility function, goods are always substitutes, so that an increase in the price of good x causes demand for leisure to rise, supply of labour falls, and $\frac{\partial L^s}{\partial p_x}|_{dw=0} < 0$. Then if good x is capital-intensive, an increase in the price of good x causes an unambiguous fall in the supply of labour. However, if x is labour-intensive, the substitution effect $\frac{\partial L^s}{\partial p_x}|_{dw=0} < 0$ works in opposite direction of the term $\frac{\partial L^s}{\partial x}\frac{\partial w}{\partial p_x}$, so that it is not possible to rule out the perverse case of an increase in the price of the labour-intensive good leading to a fall in the supply of labour.

Now note that we can rewrite the expression for labour supply changes due to output price changes as follows:

$$\frac{\partial L^s}{\partial p_x} = \frac{cx}{w}\left[\frac{\partial w}{\partial p_x} \cdot \frac{p_x}{w} - 1\right] + \frac{c}{w}\left(\frac{p_y \cdot y}{w}\right)\frac{\partial w}{\partial p_x} \tag{3.33}$$

If x is capital-intensive, then the wage elasticity with respect to the price of good x is negative, so as we saw above, $q < 0 \Rightarrow \frac{\partial L^s}{\partial p_x} < 0$. If x is labour-intensive and the wage elasticity with respect to the price of good x is greater than one, then $q > 0 \Rightarrow \frac{\partial L^s}{\partial p_x} > 0$, ruling out the perverse effect of an increase in the price of the labour-intensive good leading to a fall in the supply of labour. Therefore a necessary condition to observe this perverse effect is that the wage elasticity with respect to the price of x be between zero and one.

Now solve for the change in the output of good x due to a change in the price of x by differentiating the equation for x^e in (3.30) with respect to p_x. Recall that since labour supply is endogenous, output of good x will change indirectly due to the change in labour supply caused by the output-price change.

$$\begin{aligned} \frac{\partial x^e}{\partial p_x} = \left(\frac{-\alpha_x\beta_y}{q}\right)\left(\frac{\alpha_x}{\alpha_y}\right)^{\frac{-\alpha_y\beta_x}{q}}\left(\frac{\beta_x}{\beta_y}\right)^{\frac{\alpha_y\beta_x}{q}} \Big[& -\left(\frac{\beta_x}{q}\right)\left(\frac{\beta_x}{\beta_y}\right)^{\frac{-\beta_x}{q}}\left(\frac{p_x}{p_y}\right)^{\frac{-\beta_x-q}{q}}\left(\frac{1}{p_y}\right)\cdot\bar{K} \\ & -\left(\frac{\alpha_x}{q}\right)\left(\frac{\alpha_x}{\alpha_y}\right)^{\frac{\alpha_x}{q}}\left(\frac{p_x}{p_y}\right)^{\frac{\alpha_x-q}{q}}\left(\frac{1}{p_y}\right)\cdot L \\ & -\left(\frac{\alpha_x}{\alpha_y}\frac{p_x}{p_y}\right)^{\frac{\alpha_x}{q}}\left(\frac{\partial L}{\partial p_x}\right)\Big] \end{aligned} \tag{3.34}$$

Of these three terms, the first two will always be positive, since only q has an indeterminate sign, and the first two terms are a function of q^2, which is always positive. The sign of the last term is given by the sign on $\frac{\partial L^e}{\partial p_x}\cdot\frac{1}{q}$. As we saw above, if good x is capital-intensive, $q < 0$, and $\frac{\partial L^e}{\partial p_x} < 0$, so that $\frac{\partial x^e}{\partial p_x} > 0$. If good x is labour-intensive and the wage elasticity with respect to the price of x is greater than unity, then $q > 0$ and $\frac{\partial L^e}{\partial p_x} > 0$, and again $\frac{\partial x^e}{\partial p_x} > 0$.

If good x is labour-intensive and the wage elasticity with respect to p_x is between zero and one, then it is possible that $\frac{\partial L^e}{\partial p_x} < 0$. In this case, if $\frac{\partial L^e}{\partial p_x}$ is negative, and if the third term outweighs the first two, we would find that an increase in the price of good x would lead to a fall in supply of x. That is, if the wage elasticity with respect to the price of good x is between zero and one, we cannot rule out the possibility of observing perverse output effects.

In the traditional two good, two factor model without endogenous labour supply, we would get the same expression for a change in output of good x due to the change in the price of good x as in equation (3.34), except that the term $\frac{\partial L^e}{\partial p_x}$ would equal zero, since labour supply is fixed. If we call the change in output of good x in the model with fixed labour supply $\frac{\partial \tilde{x}^e}{\partial p_x}$, then we can solve for the expression $\frac{\partial x^e}{\partial p_x} - \frac{\partial \tilde{x}^e}{\partial p_x}$, which is analagous to equation (3.10) where we solved for the expression $dp'(dy - d\tilde{y})$. In equation (3.10), a given output price change dp caused a change in output supply when some factor supplies were endogenous (dy) and a corresponding output supply change when all factor supplies were fixed. In our Cobb-Douglas example, we get:

$$\frac{\partial x^s}{\partial p_x} - \frac{\partial \bar{x}^s}{\partial p_x} = \frac{\alpha_y \beta_x}{q} \left(\frac{\alpha_x}{\alpha_y} \right)^{\frac{-\alpha_y \beta_x}{q}} \left(\frac{\beta_x}{\beta_y} \right)^{\frac{\alpha_y \beta_x}{q}} \left(\frac{\alpha_x}{\alpha_y} \frac{p_x}{p_y} \right)^{\frac{\alpha_x}{q}} \frac{\partial L^s}{\partial p_x} \tag{3.35}$$

If x is labour-intensive, and if the wage elasticity with respect to the price of good x is greater than one, then $q > 0$, $\frac{\partial L^s}{\partial p_x} > 0$, and $\frac{\partial x^s}{\partial p_x} - \frac{\partial \bar{x}^s}{\partial p_x} > 0$. If x is capital-intensive, $q < 0$, $\frac{\partial L^s}{\partial p_x} < 0$, and $\frac{\partial x^s}{\partial p_x} - \frac{\partial \bar{x}^s}{\partial p_x} > 0$. As a result, we can conclude that a given output price change always causes a larger increase in output supply when one of the factors is endogenously supplied, in this two good, two factor Cobb-Douglas economy, as long as the wage elasticity with respect to the price of good x is not between zero and one.

Now consider the effect of a change in the price of good x on demand for good x when labour is endogenously supplied in our simple two good, two factor, Cobb-Douglas economy. We differentiate the output demand function for good x in equation (3.31) with respect to p_x as follows:

$$x^d = \frac{a}{p_x}(w\bar{L} + m_f)$$

$$x^d = \frac{a}{p_x}(p_x \cdot x + p_y \cdot y + w(\bar{L} - L))$$

$$\begin{aligned} \frac{\partial x^d}{\partial p_x} &= \frac{\partial x^d}{\partial p_x}\Big|_{dw=0} + \frac{\partial x^d}{\partial w}\frac{\partial w}{\partial p_x} \\ \frac{\partial x^d}{\partial p_x} &= -\frac{a}{p_x}\left(\frac{p_x \cdot x + p_y \cdot y + w(\bar{L}-L)}{p_x}\right) + \frac{a}{p_x} \cdot x + \frac{a}{p_x}(\bar{L} - L)\frac{\partial w}{\partial p_x} \\ \frac{\partial x^d}{\partial p_x} &= -\frac{a}{p_x}\left(\frac{p_y \cdot y + w(\bar{L}-L)}{p_x} - (\bar{L} - L)\frac{\partial w}{\partial p_x}\right) \\ \frac{\partial x^d}{\partial p_x} &= -\frac{a}{p_x}\frac{p_y \cdot y}{p_x} - \frac{a}{p_x}\frac{w(\bar{L}-L)}{p_x}\left[1 - \frac{\partial w}{\partial p_x}\frac{p_x}{w}\right] \end{aligned} \tag{3.36}$$

Given the specification of the utility function, x must be a normal good, so that an increase in p_x at constant input prices must lead to a fall in demand for x. This is shown by the term:

$$\frac{\partial x^d}{\partial p_x}\Big|_{dw=0} = -\frac{a}{p_x}\left(\frac{p_y \cdot y + w(\bar{L}-L)}{p_x}\right) < 0.$$

As well, leisure and good x must be substitutes, as demonstrated by the term $\frac{\partial x^d}{\partial w} = \frac{a}{p_x}(\bar{L} - L) > 0$. An increase in the price of leisure causes demand for good x to rise.

If good x is capital-intensive, then an increase in p_x causes the return to labour to fall. In this case, $\frac{\partial w}{\partial p_x} < 0$, so that an increase in p_x leads to an unambiguous fall in demand for good x. If x is labour-intensive, and the wage elasticity with respect to p_x is greater than unity, then the second term in the expression above for $\frac{\partial x^d}{\partial p_x}$ is positive, and if the second term outweighs the first term, the demand for x may rise in response to an increase in p_x, even though x is a normal good. If the wage elasticity with respect to the price of good x is less than unity, then the demand curve for good x will be downward-sloping.

Now consider the effect of a change in p_x on demand for good x in the same model where labour supply is fixed. In this case, the term $\frac{\partial x^d}{\partial w} = E_{p_x w} = E_{w p_x} = \frac{-\partial L^s}{\partial p_x} = 0$. Since labour supply is fixed, there can be no substitution between labour and good x. If we write the change in demand for x in this case as $\frac{\partial \bar{x}^d}{\partial p_x}$, we can show that:

$$\frac{\partial x^d}{\partial p_x} - \frac{\partial \bar{x}^d}{\partial p_x} = \frac{a}{p_x}(\bar{L} - L)\frac{\partial w}{\partial p_x}. \tag{3.37}$$

Since $\frac{\partial w}{\partial p_x}$ is positive when x is labour-intensive and negative when x is capital-intensive, we can conclude that an increase in the price of good x causes a larger fall in demand for x when labour is endogenous if x is capital-intensive. If x is labour-intensive, the demand response when labour is endogenously supplied will be weaker, and may even be in the opposite direction, then when labour supply is fixed.

Using the equations derived above, we can determine the effect of endogenous factor supply on net exports in our two good, two factor, Cobb-Douglas economy. The change in exports of good x is simply the difference in the change in output of good x

in equation (3.34) and the change in demand for good x in equation (3.36). We will consider the difference between net exports of x when labour supply is endogenous and when labour supply is fixed by looking at the difference between equations (3.35) and (3.37):

$$(3.38) \qquad \frac{\partial(x^s-x^d)}{\partial p_x} - \frac{\partial(\bar{x}^s-\bar{x}^d)}{\partial p_x} = \frac{\alpha_x\beta_x}{q}\left(\frac{a_x}{\alpha_y}\right)^{\frac{-\alpha_y\beta_x}{q}}\left(\frac{\beta_x}{\beta_y}\right)^{\frac{\alpha_y\beta_x}{q}}\left(\frac{a_x}{\alpha_y}\,\frac{p_x}{p_y}\right)^{\frac{a_x}{q}}\frac{\partial L^s}{\partial p_x} - \frac{a}{p_x}(\bar{L}-L)\frac{\partial w}{\partial p_x}$$

where the change in labour supply due to the change in p_x is given by equation (3.33) as:

$$\frac{\partial L^s}{\partial p_x} = \frac{cx}{w}\left[\frac{\partial w}{\partial p_x}\frac{p_x}{w} - 1\right] + \frac{c}{w}\,\frac{p_x \cdot Y}{w}\,\frac{\partial w}{\partial p_x}.$$

Suppose x is capital-intensive, so that $q < 0$. In this case, $\frac{\partial w}{\partial p_x} < 0$ since an increase in p_x causes the return to capital to rise and the return to labour to fall. As well, the term $\frac{\partial x^s}{\partial p_x}$ is also negative. As a result, the change in net exports of good x is unambiguously larger when labour is endogenously supplied when good x is capital-intensive.

If x is labour-intensive, then $q > 0$. Suppose that the elasticity of the wage rate with respect to p_x is greater than unity. Then since $\frac{\partial w}{\partial p_x} > 0$ when x is labour-intensive, $\frac{\partial L^s}{\partial p_x} > 0$. If the first term in equation (3.38) outweighs the second, an increase in the price of good x causes a larger change in net exports of x when labour supply is endogenous. However, if the wage elasticity with respect to p_x is between zero and one, then it is possible for an increase in p_x to have a perverse effect on labour supply, so that $\frac{\partial L}{\partial p_x} < 0$. If perverse result actually does obtain, then the change in net exports would be smaller when labour supply is endogenous.

4. FACTOR PRICE EQUALIZATION

It seems appropriate at this stage to reconsider the question of factor price equalization. This problem has been explored in great detail in Heckscher-Ohlin models even when the number of goods and factors is allowed to be greater than two.[7] In such models without trade distortions, if trade equalizes output prices and both countries produce the same m goods with identical production technology, input prices for the n endowed factors will be equalized between countries if each country's endowment vector is in the interior of the same cone of diversification formed by n first-order conditions of the form (2.11″). The issue of the relative number of goods and factors is important since if, for example, $n > m$, then the diversification cone is degenerate and the event that a nation's endowment vector is in the interior of the cone occurs with probability zero. If on the other hand $m > n$, then the equilibrium input price vector $\bar{w}$ will depend upon which m goods are produced in equilibrium. The diversification cone will not be unique, and which goods are produced will depend upon the location of the endowment point.

The critical condition for the number of factors versus the number of goods is only slightly more complicated when some factors are endogenously supplied. To begin with, consider the case where $m = n$, so that the number of goods produced in equilibrium equals the total number of factors of production. Given output prices and the technology specified by the m production functions $f^i(a^i)$, the n first-order conditions (2.10) will uniquely determine the returns to both the exogenously and endogenously supplied factors of production (assuming that the equations in (2.10) are all linearly independent). The consumption sector then solves for the levels of

[7] See Woodland (1982), pp.70-77, or Dixit and Norman (1980), pp.110-125, among others.

supply of the endogenous factors, given output prices and income, which is determined by the inner product of the input price vector from the production sector and the endowment vector. In this special case the separation of the determination of input prices from the activities of the consumption sector still applies. The problem of maximizing GNP is summarized by equation (2.3′), in Section 3.1. All input prices are determined by the production sector alone, and the consumption sector solves for the levels of supply of the endogenous factors. The production sector stands ready to buy up whatever amounts of the endogenously supplied factors are sold by the consumption sector at the determined input prices $(w, \bar{w})$. Note that the assumption of constant returns to scale production technology is crucial here.

Now suppose that there are more factors than goods. This case is analogous to the situation where $n > m$ in the standard Heckscher-Ohlin model where all factors are exogenously supplied.[8] The returns to the factors of production $(w, \bar{w})$ will now depend upon endowments $\bar{v}$ as well as output prices p. To illustrate, suppose that $m = n - e$, so that the number of goods is equal to the number of exogenously supplied factors. If the m price-equals-marginal cost equations in (2.10) are linearly independent, then we can think of three blocks of equations which are all interdependant. The equilibrium input price vector $(w, \bar{w})$ must be such that (i) we observe zero profits in the m industries such that the equations in (2.10) are all satisfied, (ii) factor supplies equal factor demands for the e endogenously supplied factors such that the e equations in (2.11′) are satisfied, and (iii) the $n - e$ exogenously supplied factors are all fully employed, such that the $n - e$ equations in (2.11″) are all satisfied.

We are now in a position to determine the effects of the presence of endogenously supplied factors on the factor price equalization theorem. Suppose that there are

[8] See Woodland (1982), p.75, and Dixit and Norman (1980), p.111.

more goods than exogenously supplied factors. As noted above, factor prices will now depend upon which m goods are produced. If we consider the special case where the number of goods equals the total number of factors in the economy ($m = n$), then the usual statement of the factor-price equalization theorem still applies. That is, even if a subset of the n factors of production in an economy are endogenously supplied, as long as both countries produce the same m goods in equilibrium with the same constant returns to scale production technology, and if trade equalizes the prices of these m goods, then if both country's equilibrium endowment point is within the cone of diversification, factor prices will be equalized. However, it is important to note that even in this special case, things are not completely the same as they were in the model without endogenously supplied factors, since the location of the endowment point is not exogenous even if only one factor is endogenously supplied. Thus the probability of factor price equalization is dependant on the actions of the consumption sector, even when factor prices are determined solely by the actions of the production sector.

To show how the probability of Factor Price Equalization is affected by the presence of endogenous factor supplies, consider the following simple Numerical General Equilibrium (NGE) example. Two countries, H and F, produce identical outputs x and y with identical constant returns to scale production technology. Outputs are produced using inputs of capital (K) and labour (L) according to the Cobb-Douglas production functions $x = K_x^{\alpha_x} L_x^{\beta_x}$, $y = K_y^{\alpha_y} L_y^{\beta_y}$, $\alpha_i + \beta_i = 1$, $i = x, y$. Suppose for example that y is capital-intensive, so that $\alpha_y/\beta_y > \alpha_x/\beta_x$. There is an identical representative consumer in both H and F who consumes x, y, and L according to the Cobb-Douglas utility function $U = x^a y^b (\bar{L} - L)^c$, $a + b + c = 1$. $\bar{L}$ is the total endowment of labour, corresponding to $\bar{v}_1$ in Fig. 2.1, p.20, so that $(\bar{L} - L)$ represents consumption of leisure. Suppose that H is relatively well-endowed with labour, so that $\bar{K}_H/\bar{L}_H < \bar{K}_F/\bar{L}_F$. The producers' problem is to minimize factor payments subject to the technology constraints given by the Cobb-Douglas production functions. The solution will be un:. factor demand functions for industries x and y in H and F as follows (see Appendix, p. 132):

$$K_i(w,r) = \left(\frac{\alpha_i}{\beta_i}\frac{w}{r}\right)^{\beta_i} \qquad i = x, y \tag{4.1}$$

$$L_i(w,r) = \left(\frac{\beta_i}{\alpha_i}\frac{r}{w}\right)^{\alpha_i} \qquad i = x, y. \tag{4.2}$$

The representative consumer in H and F maximizes utility subject to the constraint that net spending $p_x x + p_y y - wL$ is less than or equal to fixed factor income $m_f = r\bar{K}$. The equilibrium output demand and labour supply functions are given by:

$$x(p_x, p_y, \bar{L}, m_f) = \left(\frac{a}{a+b+c}\frac{m_f + w\bar{L}}{p_x}\right) \tag{4.3}$$

$$y(p_x, p_y, \bar{L}, m_f) = \left(\frac{b}{a+b+c}\frac{m_f + w\bar{L}}{p_y}\right) \tag{4.4}$$

$$L(p_x, p_y, \bar{L}, m_f) = \left(\frac{-c}{a+b+c}\frac{m_f + w\bar{L}}{w}\right) + \bar{L}. \tag{4.5}$$

The labour supply function (4.5) is probably more familiar when rewritten as a demand for leisure function, if we subtract $\bar{L}$ from both sides and multiply through by -1.

We will describe the initial equilibrium numerically by assigning parameters and endowments as follows:

$$\alpha_x = 0.4 \qquad \alpha_y = 0.6 \qquad (\bar{K}_H, \bar{L}_H) = (9{,}21)$$

$$\beta_x = 0.6 \qquad \beta_y = 0.4 \qquad (\bar{K}_F, \bar{L}_F) = (11{,}19),$$

and $a = b = c = 1/3$. The equilibrium prices in this situation (p_x, p_y, w, r) are all equal to unity. In the analysis that follows, we assume that output prices remain fixed.

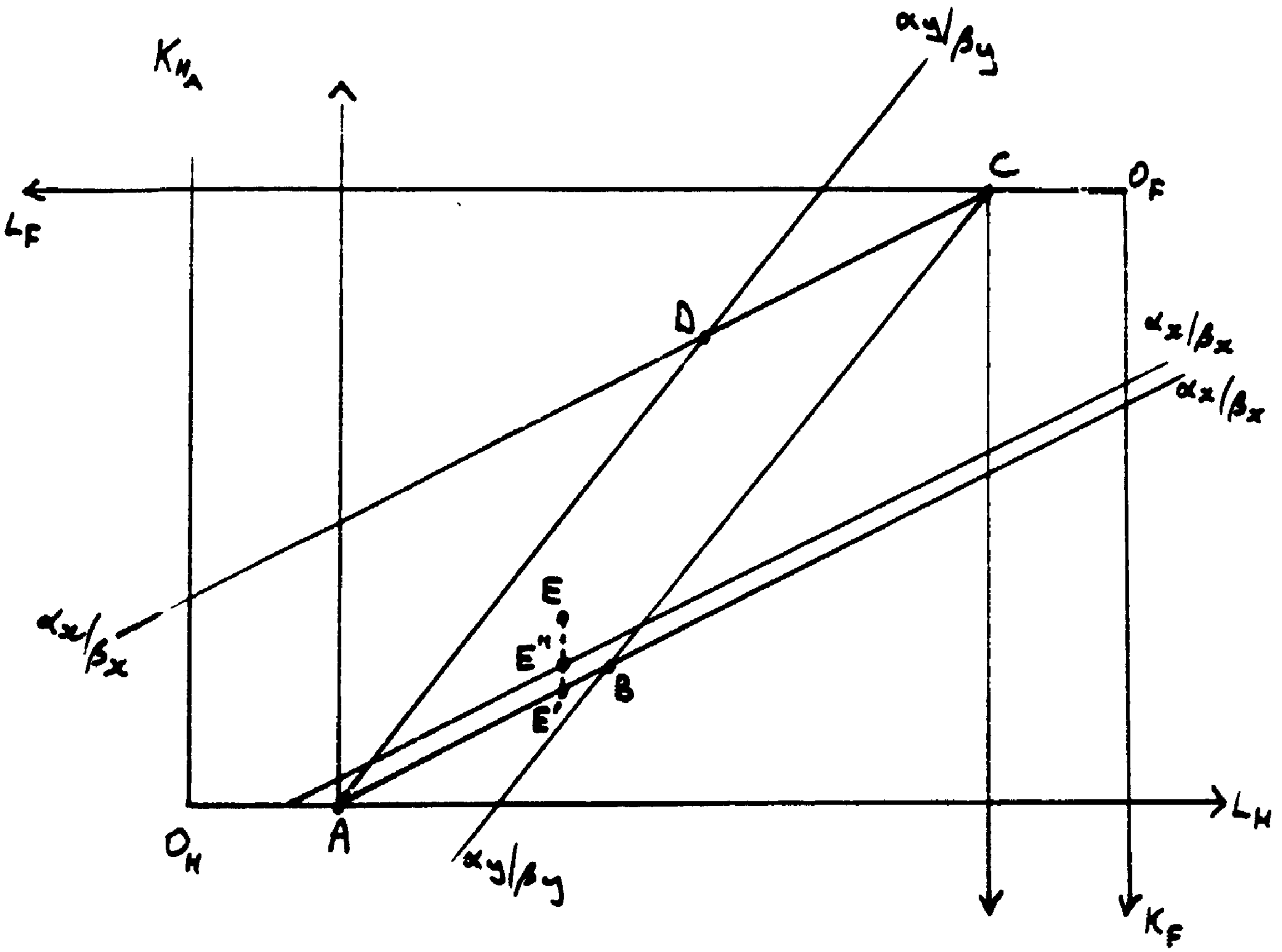


Figure 4.1

The world equilibrium for this example is illustrated graphically in Fig. 4.1. Home and foreign outputs are $(x_H, y_H) = (15, 5)$ and $(x_F, y_F) = (5, 15)$, respectively, and the domestic and foreign representative agents each consume 10 units of x, y, and leisure. Home consumption of leisure is thus represented by the distance $O_H A$ in Fig. 4.1, and foreign consumption of leisure by $O_F C$. Because all prices in equilibrium are equal to unity, the capital/labour ratio in industry i is given by the ratio α_i/β_i, as can be seen by dividing equation (4.1) by (4.2) and substituting $w = r = 1$. These boundaries define the cone of diversification $ABCD$ in Fig. 4.1. The endowment point E is in the interior of the cone, so we observe factor price equalization. In this example, factor price equalization will always obtain as long as $\alpha_x/\beta_x < \bar{K}_H/L_H < \bar{K}_F/L_F < \alpha_y/\beta_y$. Notice that it is total labour supplied to the production sector that appears in this expression, and not the total labour endowment of the economy, since now some of $\bar{L}$ is consumed as leisure.

Our objective here is to demonstrate how factor price equalization is affected by the presence of an endogenously supplied factor. We know very well how factor price equalization works in the usual Heckscher-Ohlin model where all factors are exogenously supplied. The problem is that there is no leisure consumption in the usual formulation of the Heckscher-Ohlin model. As was noted in Section 3, we cannot directly compare results in our model with an endogenously supplied factor to results of the usual Heckscher-Ohlin model since the utility functions in each model are inherently different. To get around this problem, we'll again compare results in our model with an endogenously supplied factor to results in the same model where the supply of the endogenous factor is not allowed to adjust to any changes in the system. If the change to our system is an exodus of capital from the home country to the foreign country, for example, then in the usual Heckscher-Ohlin model, total labour supplied to the production sector does not change. To replicate this, we'll suppose that total leisure demand stays constant at $O_H A$ in the home country and at $O_F C$ in the foreign country. Then we know that if we move more than 5/3 units of capital from H to F, we move past point E' in Fig. 4.1, the home country specializes in x-production, and factor price equalization fails to hold.

Let's now repeat the same experiment, allowing leisure demand to adjust to the shift of capital away from the home country. Leisure is a normal good in our example, due to the way in which the consumption technology is specified. If we suppose that ownership of the capital which moves from H to F is transferred along with the factor being moved, then income in H declines as we move south of point E in Fig. 4.1, and total leisure demand will fall as a result. The boundary of the diversification cone shifts, and we find that we need move only 1.36 units of capital to the foreign country before the home country specializes in production of x and factor price equalization breaks down at point E''.

A couple of points require further examination here. If we suppose that the ownership of the capital being transferred from H to F does not leave the home country, then domestic income does not change, leisure demand does not respond to the movement of capital (output prices aren't changing in this example) and the results of our experiment are no different than in the case where leisure demand is not allowed to respond to the shock. However, what is important to note here is that if we consider an endowment point between E'' and E' in Fig. 4.1, then if labour is inelastically supplied (in our example at 11 units in H, 9 units in F), we will observe factor price equalization. If on the other hand labour supply is truly endogenous, then factor prices will not be equalized if the endowment point is between E'' and E' because the home country will specialize in production of good x. What we need to do now to complete this example is to find out how the presence of an endogenously supplied factor affects the dimensions of the diversification cone over the entire area of the Edgeworth box in Fig. 4.1, not only at the starting point E which we considered in our example. That is, the diversification cone should be described not by the ratio of capital to labour supplied to the production sector, since total labour supplied to the production sector is endogenous. The diversification cone should be described by the ratio of capital to the total labour endowment.

To see how a nation's diversification cone is affected by the presence of endogenously supplied factors, focus on the home nation. To isolate the effect of endogenous factor supply, suppose that H faces fixed output prices $p_x = p_y = 1$. The boundaries of the diversification cone are defined by the capital/labour ratios in industries x and y when the home country specializes in either x-production or y-production. For example, if H produces only good i, then the capital/labour ratio in industry i is given by:

$$\left(\frac{\bar{K}}{L}\right)_i = \frac{\alpha_i}{\beta_i}\frac{w}{r}, \quad i = x, y. \tag{4.6}$$

At any point on the boundary of the diversification cone, all of capital $\bar{K}$ and the total amount of labour supplied to the production sector L is used to produce good i, for i equal to either x or y. However, as was just noted, L is endogenous. Given output prices p_x and p_y, we do not want a diversification cone that is endogenous. The expression we are looking for is $(\bar{K}/\bar{L})_i$, not $(\bar{K}/L)_i$ in equation (4.6) above.

To proceed, we can solve for the representative consumer's labour supply function as a function of $\bar{K}$, $\bar{L}$, r, and w, by substituting the equilibrium constraint $m_f = r\bar{K}$ into the labour supply function:

$$\begin{aligned} L &= -c\left(\frac{r\bar{K} + w\bar{L}}{w}\right) + \bar{L} \\ &= \frac{-cr}{w}\bar{K} + (1-c)\bar{L}. \end{aligned} \tag{4.7}$$

Substituting this into the capital/ labour ratio in equation (4.6) above yields:

$$\bar{K} = \frac{\alpha_i}{\beta_i}\frac{w}{r}\left[\frac{-cr}{w}\bar{K} + (1-c)\bar{L}\right].$$

Collecting and cancelling terms allows us to rewrite this expression as:

$$\left[1 + \frac{\alpha_i}{\beta_i}c\right]\bar{K} = (1-c)\frac{\alpha_i}{\beta_i}\frac{w}{r}\bar{L}.$$

We can now solve for the capital/labour ratio:

$$\left(\frac{\bar{K}}{\bar{L}}\right)_i = \frac{\alpha_i(1-c)}{\beta_i + \alpha_i c}\frac{w}{r}. \tag{4.8}$$

Recall that equation (4.8) gives the capital/labour ratio in industry i when the home country just begins to specialize in production of good i, so that all of capital and

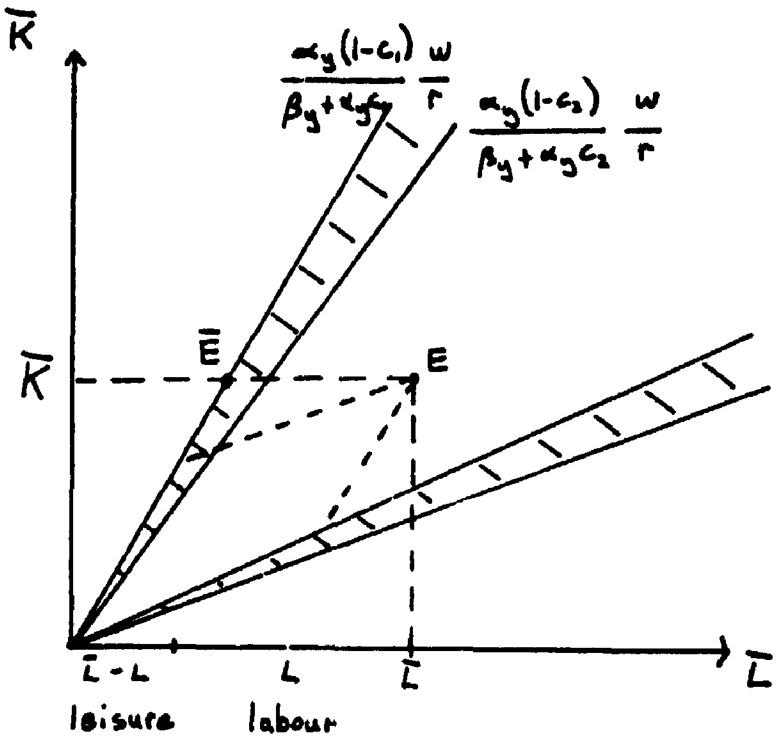


Figure 4.2

labour available to the production sector at the given output prices p_x and p_y is used to produce only good i. When evaluated for $i = x, y$, the two equations in equation (4.8) give a region in $\bar{K} - \bar{L}$ -space. When the nation's endowment vector can be formed as a strictly convex combination of these two equations, then the nation will diversify and produce both x and y. This situation is illustrated in Fig. 4.2. If the nation's endowment point $(\bar{K}, \bar{L})$ is at point $\bar{E}$ in Fig. 4.2, then H specializes in production of good y, but factor-price equalization still obtains as long as F also has an endowment point within the same diversification cone.

The important point to note is that the diversification cone for a nation is now a function of the parameter c, which in this model demonstrates the representative consumer's preferences over leisure demand or labour supply.

Notice first that if we substitute $c = 0$ into equation (4.8), then the capital/labour ratios reduce to the familiar:

$$\left(\frac{\bar{K}}{\bar{L}}\right)_i = \frac{\alpha_i}{\beta_i}\frac{w}{r}.$$

In this case, the representative agent does not consume the endogenously supplied factor, so that we are left with the standard Heckscher-Ohlin model.

Now suppose $c \neq 0$. Once we know what the foreign country looks like, we can find equations like (4.8) for the foreign country as well, and we would then be able to construct the diversification cone for the world economy. If the endowment point is located within this cone, then we would observe factor-price equalization.

How does the diversification cone for a country change as the parameter c changes? Of course, different values of c will in general imply different equilibrium solutions of all endogenous variables. To isolate the effect of the parameter from the consumption side of the economy, focus on the home country, and suppose that H faces some fixed output prices p_x and p_y. If tastes in H change so that the parameter c changes, then as long as both goods are produced in the initial equilibrium before c changes and in the final equilibrium after c has changed, input prices will not change. If we differentiate equation (4.8) with respect to c, we get:

$$\frac{\partial(\bar{K}/\bar{L})_i}{\partial c} = \frac{-\alpha_i}{(\beta_i + \alpha_i c)^2}\frac{w}{r} < 0. \tag{4.9}$$

As we increase the parameter c, the slopes of the rays which form the boundaries of the diversification cone for country H fall. This situation is illustrated in Fig. 4.2, where the endowment point for H is initially E. This endowment point is in the interior of the initial diversification cone when $c = c_1$, and is in the final diversification cone when $c = c_2 > c_1$. The country would be producing both x and y when $c = 1$ and when $c = 2$. Then if the foreign country had the same production technology and also produced both x and y at a trading equilibrium, and if trade equalized the prices of x and y between H and F, then we would observe factor price equalization. However, if the home country's endowment point were in the shaded area to the left of the endowment point E in Fig. 4.2, then as tastes in H changed so that the parameter c rose from c_1 to c_2, H would specialize in production of y, and even given the conditions noted above about production in F and trade equalizing output prices, factor price equalization would no longer obtain.

This simple example serves to illustrate how the probability of factor price equalization is affected by the presence of endogenously supplied factors. Recall that we used very simple production and consumption technology, represented by Cobb-Douglas production functions. Also, since we had a two-good, two-factor model, the returns to capital and labour were uniquely determined by output prices and production technology alone, independent of the consumption sector, as long as both x and y were produced. However, when we started increasing the parameter c, this reflected a relative increase in demand for leisure by the representative consumer, ceteris paribus. Supply of labour to the production sector fell, and as a result, H specialized in production of the capital-intensive good. Of course, if we had decreased the parameter c, the home country would ultimately have specialized in production of the labour-intensive good.

Note also that the technical statement of the factor price equalization theorem remains unchanged. If two countries produce the same goods with the same inputs and the same production technology, then trade in goods which equalizes output prices between the two countries will equalize factor prices. What this analysis does illustrate is the relationship between the goods which a nation produces in equilibrium and preferences over an endogenously supplied factor. In a sense, we can think of a nation with a relatively strong preference for leisure as producing more of the labour-intensive non-traded good leisure in equilibrium and thereby specializing in production of a single more capital-intensive good.

5. REVISED STOLPER-SAMUELSON EFFECTS

The effect of output price changes on input prices in the traditional trade model with fixed factor supplies is well known. In the two-good, two-factor model, the Stolper-Samuelson theorem predicts that the return to the factor used intensively in production of the good whose price is rising will rise, and the return to the other factor will fall. When there are more factors than goods, the effect of an output price change on some input prices is indeterminate. When some factors are endogenously supplied, we need to take account of the fact that an exogenous output price change may not affect the returns to the endowed factors in the same way as it affects the returns to the endogenously supplied factors. The former effect can be obtained by differentiating equation (2.7), holding exogenous factor supplies constant:

$$(5.1) \qquad d\bar{w} = G_{\bar{v}p}dp + G_{\bar{v}w}dw.$$

In general, the relationship between output prices and the returns to the endogenously supplied factors is given by:

$$(5.2) \qquad dw = -[G_{ww} - E_{ww}]^{-1}[G_{wp} - E_{wp} - v_{m_I}x']dp.$$

This equation is derived by differentiating the equilibrium condition for the endogenously supplied factors, equation (2.22), since at the new output prices, it must still be true that factor supplies equal factor demands.

5.1 Number of Goods Equals Number of Factors

The simplest case to consider is when the number of goods is at least as large as the total number of factors, so that $m \geq n$. If $m > n$ at the initial equilibrium, then the output price change dp would cause production in $n - m$ industries to cease to be profitable, so it is sufficient here to consider the case $m = n$. As was shown in

Section 4, all input prices will be determined as a function of output prices according to the m price-equals-marginal cost equations (2.10), so that the usual version of the Stolper-Samuelson theorem will continue to hold. That is, for the case $m = n = 2$, the return to the factor used intensively in the industry whose price has increased will rise, and the return to the other factor will fall, independent of whether one of the factors is endogenously supplied or not.

5.2 More Factors than Goods

If on the other hand we begin with an equilibrium where the number of goods is fewer than the total number of factors, the behavior of the consumption sector will affect the determination of the returns to the endogenously supplied factors according to equation (5.2) above. To illustrate, consider the case where $m = 2$, $n = 3$, and $e = 1$. Then (5.1) can be written as:

$$(5.3) \qquad \begin{pmatrix} d\bar{w}_2 \\ d\bar{w}_3 \end{pmatrix} = \begin{pmatrix} \frac{\partial \bar{w}_2}{\partial p_1} dp_1 + \frac{\partial \bar{w}_2}{\partial p_2} dp_2 + \frac{\partial \bar{w}_2}{\partial w_1} dw_1 \\ \frac{\partial \bar{w}_3}{\partial p_1} dp_1 + \frac{\partial \bar{w}_3}{\partial p_2} dp_2 + \frac{\partial \bar{w}_3}{\partial w_1} dw_1 \end{pmatrix}.$$

Since we now have more than two factors, our usual notion of factor intensity becomes ill-defined. To proceed, we use the notion of "generalized factor intensities" (see Dixit and Woodland, 1982, p.209, among others). Good i is defined as using factor j intensively if $\partial y_i / \partial v_j > 0$.

If we ignore the effect of the change in the return to the endogenously supplied factor for the moment, we can see that (5.3) gives the usual Stolper-Samuelson result. Note first that since the Hessian of the modified GNP function is symmetric, $\frac{\partial \bar{w}_i}{\partial p_j} = \frac{\partial y_j}{\partial v_i}$, $j = 1, 2$, $i = 2, 3$. If good 1 uses factor 2 intensively, then $\frac{\partial y_1}{\partial v_2} > 0$ and $\frac{\partial y_1}{\partial v_3} < 0$. Equation (5.3) says that if we hold the price of good 2 constant and suppose that $dp_1 > 0$, the return to the factor used intensively in the production of good 1 will rise, and the return to the other factor will fall.

We must now consider the effect of the change in the return to the endogenously supplied factor, dw_1. In our example, we can write equation (5.2) as:

(5.4)

$$dw_1 = -(G_{w_1 w_1} - E_{w_1 w_1})^{-1} \left(-\frac{\partial v_1^d}{\partial p_1} + \frac{\partial v_1^s}{\partial p_1} - v_{1m_f} x_1 \quad -\frac{\partial v_1^d}{\partial p_2} + \frac{\partial v_1^s}{\partial p_2} - v_{1m_f} x_2 \right) \begin{pmatrix} dp_1 \\ dp_2 \end{pmatrix}.$$

We know that $(G_{w_1 w_1} - E_{w_1 w_1})^{-1}$ is a positive scalar, but in general the sign on dw_1 is indeterminate. In fact, it is also impossible in general to sign the term $\partial \bar{w}_i / \partial w_1$ in equation (5.3). However, if we borrow the notion of "cooperativeness" of factors of production introduced in Svensson (1984),[9] we can say that $\partial \bar{w}_i / \partial w_1 > 0$ (< 0) if the exogenously supplied factor i is cooperative (non-cooperative) with the endogenously supplied factor.

Suppose that $\bar{v}_2$ and v_1 are cooperative (so that $\partial \bar{w}_2 / \partial w_1 > 0$), and suppose that good 1 uses factor $\bar{v}_2$ intensively. Then the usual Stolper-Samuelson effect (an increase in the return to factor $\bar{v}_2$) is magnified if $dw_1 > 0$. Recall that in Section 3, we defined a factor as being normal in production if $\partial y_1 / \partial w_1 = G_{p_1 w_1} = G_{w_1 p_1} = -\partial v_1^d / \partial p_1$ was negative, so that an increase in the return to the factor v_1 caused output of good y_1 to fall, ceteris paribus. We also saw that if good i and the endogenous factor v_1 were gross substitutes, then $E_{w_1 p_1} - v_{1m_f} x_1 = -\partial v_1^s / \partial p_1 + v_{1m_f} x_1$ was positive. Of course, this implies that $-E_{w_1 p_1} + v_{1m_f} x_1 = \partial v_1^s / \partial p_1 - v_{1m_f} x_1$ will be negative if good 1 and factor v_1 are substitutes in consumption. If these conditions hold, then an increase in the price of good 1 holding the price of good 2 constant will result in an increase in the return to factor v_1. Thus we can state that a sufficient condition for the usual Stolper-Samuelson effect to be magnified is that the endogenously supplied factor be normal in production and that good 1 and the endogenously supplied factor be gross substitutes in consumption.

[9] see Svensson (1984), pp.373-374

It is also important to note at this point that it is possible to construct examples where the usual Stolper-Samuelson results are dampened, and that we cannot rule out the theoretical possibility of the usual Stolper-Samuelson results being reversed. That is, even in the simple model with two goods, two exogenously supplied factors, and one endogenously supplied factor, we cannot rule out the case where an increase in the price of one good reduces the return to the exogenously supplied factor used intensively in production of the good whose price has increased.

5.3 Specific Factors Model

The interesting case to consider when the number of factors exceed the number of goods is the specific factors model, so that some factors may only be employed to produce specific goods (see Jones (1971), among others). Suppose that $m = n - e = 2$ and $e = 1$ as in the model above, but that factor $\bar{v}_2$ is used only in production of good y_1, and that factor $\bar{v}_3$ is specific to industry y_2. This gives us more information about the terms in equation (5.3). Specifically, it is true that $\partial\bar{w}_2/\partial p_1 > 0$, $\partial\bar{w}_2/\partial p_2 < 0$, since factor $\bar{v}_2$ is specific to industry 1. Likewise, $\partial\bar{w}_3/\partial p_1 < 0$, and $\partial\bar{w}_3/\partial p_2 > 0$. Of course, the return to the mobile factor v_1 is still given by equation (5.4).

In the Jones version of this model, with factor v_1 mobile between industries y_1 and y_2 but still in fixed supply, the result of a relative increase in the price of good y_1 is the so-called magnification effect:[10]

$$d\bar{w}_3 < dp_2 < dw_1 < dp_1 < d\bar{w}_2.$$

But again dw_1 is generally indeterminate. If the price of good 1 rises and the price of good 2 does not change, then the results of Jones' model above say that the returns to both factors used in industry 1 rise, and the return to the factor specific to industry

[10] see Jones (1971), p.9.

2 falls.

Consider the same example in the model where the mobile factor is endogenously supplied. The return to the mobile factor will rise or fall according to equation (5.4). Specifically, with $dp_1 > 0$, $dp_2 = 0$, then the return to the mobile factor will rise if good 1 and the mobile factor are substitutes in consumption ($\partial v^s/\partial p_1 < 0$), and if the mobile factor is normal in production ($\partial v^d/\partial p_1 > 0$). Of course, similar conditions could be found such that the return to the mobile factor falls when the price of good 1 rises and the price of good 2 does not change.

Now consider the returns to the specific factors. Ignoring the effect of the change in dw for the moment, the result of our example will be an increase in the return to the factor specific to production of good 1 ($d\bar{w}_2 > 0$) and a decrease in the return to the other specific factor ($d\bar{w}_3 < 0$). With $dw_1 > 0$, this result will be magnified if the factor specific to good 1 is cooperative with the mobile factor ($\partial\bar{w}_2/\partial w_1 > 0$) and if the other specific factor is not cooperative with the mobile factor ($\partial\bar{w}_3/\partial w_1 < 0$). However, if this relative cooperativeness were reversed so that $\bar{v}_3$ was cooperative with the mobile factor and $\bar{v}_2$ was not, then the increase in the return to the factor specific to industry 1 would be dampened and may even be reversed, and the decrease in the return to the factor specific to industry 2 would be dampened and may even be reversed. That is, in our example, if the increase in the price of good 1 causes such a large increase in supply of the mobile factor, and if this mobile factor is cooperative with the good whose relative price has fallen and non-cooperative with the good whose price has increased, it could be that the return to the factor specific to the good whose relative price has increased would fall, and that the return to the other specific factor would increase.

5.4 Cobb-Douglas Examples

Consider the effect of a given change in world prices dp on input prices in our small open economy in which two goods are produced using capital and labour, where labour is endogenously supplied, and production and consumption technology is represented by Cobb-Douglas production and utility functions, respectively. As was shown in Section 5.1 above, as long as the number of goods produced in equilibirum is equal to the number of factors of production, input prices are determined solely by the production sector according to the price-equals-marginal-cost equations, independent of the consumption sectors. We solve for the percentage change in input prices as a function of the percentage change in output prices in the Appendix (see p. 133):

$$\begin{pmatrix} \hat{w} \\ \hat{r} \end{pmatrix} = \begin{pmatrix} \frac{\alpha_y}{q}\hat{p}_x - \frac{\alpha_x}{q}\hat{p}_y \\ \frac{-\beta_y}{q}\hat{p}_x + \frac{\beta_x}{q}\hat{p}_y \end{pmatrix},$$

where $q = \beta_x\beta_y\left(\frac{\alpha_y}{\beta_y} - \frac{\alpha_x}{\beta_x}\right)$. If good x is relatively labour intensive, good y is relatively capital-intensive, then $\frac{\alpha_y}{\beta_y} > \frac{\alpha_x}{\beta_x}$, and $q > 0$. Then an increase in the price of x holding p_y constant causes the return to labour to rise ($\hat{w} = \frac{\alpha_y}{q}\hat{p}_x$) and the return to capital to fall ($\hat{r} = \frac{-\beta_y}{q}\hat{p}_x$).

While the presence of an endogenously supplied factor has no effect on the determination of input prices as a function of output prices in this example where the number of factors equals the number of goods, we should still note that knowledge of this input price change is necessary to determine the effect of the exogenous world price shock on the supply of labour, as was illustrated in Section 3.3.

6. REVISED RYBCZYNSKI EFFECTS

In the traditional two-good, two-factor trade model with fixed factor supplies, the Rybczynski theorem predicts that at constant output prices, an increase in the endowment of a factor causes output of the good that uses the factor intensively to rise, and output of the other good to fall. With endogenous factor supply, the endowment change also has the effect of changing factor supply, since income changes as the endowment changes. In order to incorporate the effects of the presence of endogenously supplied factors on the Rybczynski theorem, first differentiate the output supply function (2.5):

$$dy = G_{pw}dw + G_{p\bar{v}}d\bar{v}, \tag{6.1}$$

holding output prices fixed.

Suppose that the change $d\bar{v}$ is such that the same m goods continue to be produced. Since the change in endowments will in general result in a change in the supply of endogenous factors, there will in general be a change in the returns to the endogenously supplied factors, even though output prices are assumed not to change. In fact, since the return to the endogenously supplied factors will change, the $n - e$-dimensional cone of diversification will be different due to the change in endowments, so that the returns to the exogenously supplied factors will also change.

6.1 Number of Goods Equals Number of Factors

Suppose the number of goods produced in equilibrium before the factor endowment change equals the total number of factors, $m = n$. In this case, input prices will still be determined solely as a function of output prices as long as the economy does not move out of the original diversification cone. The change in endowments $d\bar{v}$ will cause

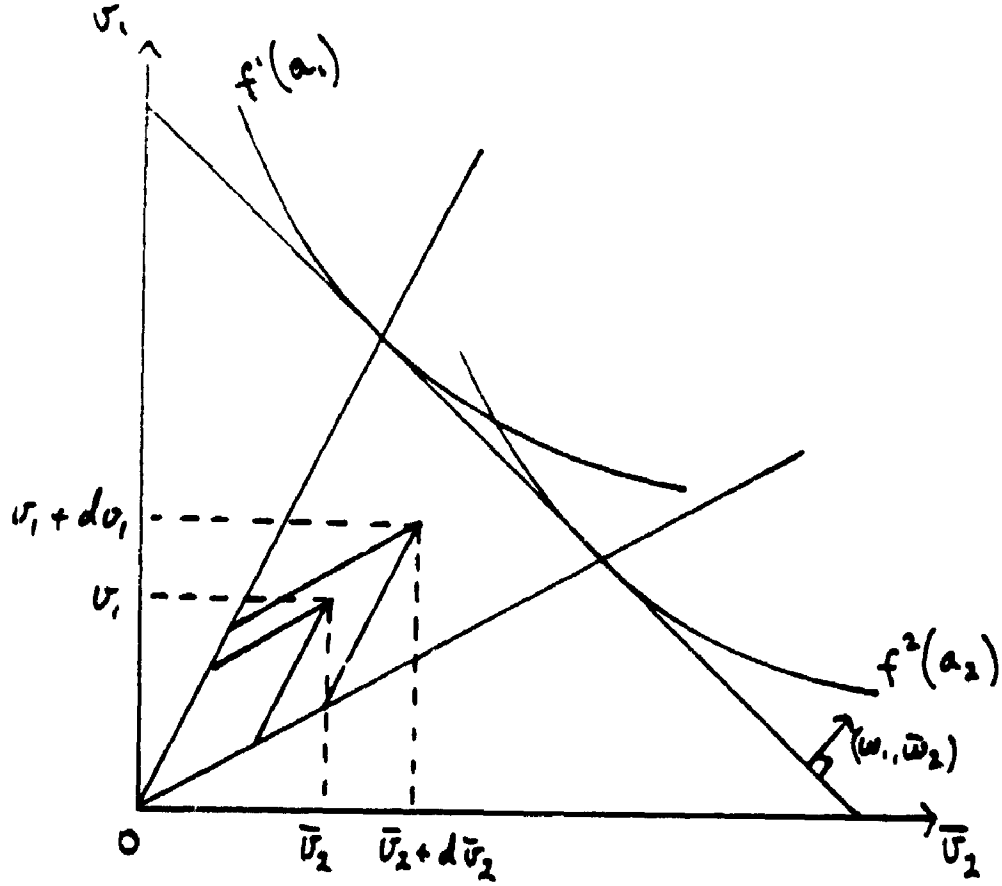


Figure 6.1

a change in fixed factor income $dm_f = \bar{w}'d\bar{v}$, which will generally lead to a change in the supply of the endogenous factors. Suppose that the movement in the endowment point does not shift the economy out of the initial diversification cone, so that there will be no change in input prices. This situation is illustrated in Fig. 6.1, for the case $m = n = 2$, $e = 1$.

The change in outputs due to the change in the endowments $d\bar{v}$ is now given by:

(6.2) $$dy = \hat{G}_{p\bar{v}}d\bar{v} + \hat{G}_{pv}dv.$$

The second term in equation (6.2) is the change in output supplies due to the change in the supply of the endogenous factor, where dv is given by:

(6.3) $$dv = v_{m_f}dm_f = v_{m_f}[\bar{w}'d\bar{v}].$$

The income effect on the supply of endogenous factors v_{m_f} is generally indeterminate. Consider the special case illustrated in Fig. 6.1 above. Equation (6.2) can be written as:

$$\text{(6.4)} \qquad \begin{pmatrix} dy_1 \\ dy_2 \end{pmatrix} = \begin{pmatrix} (\partial y_1/\partial \bar{v}_2)d\bar{v}_2 + (\partial y_1/\partial v_1)(\partial v_1/\partial m_f)dm_f \\ (\partial y_2/\partial \bar{v}_2)d\bar{v}_2 + (\partial y_2/\partial v_1)(\partial v_1/\partial m_f)dm_f \end{pmatrix}.$$

Suppose the supply of factor $\bar{v}_2$ increases. If good 1 uses factor $\bar{v}_2$ intensively, then $(\partial y_1/\partial \bar{v}_2) > 0$ and $(\partial y_1/\partial v_1) < 0$. Since an endowment is increasing while prices are constant, fixed factor income rises, so that $dm_f > 0$. Thus the usual Rybczynski relationship will hold when the supply of the endogenous factor decreases as income increases, or when the endogenously supplied factor is normal.

Note also that if the supply elasticity of the endogenous factor were zero, $dv_1 = 0$. Now equation (6.4) is simply the usual statement of the Rybczynski theorem. That is, an increase in the endowment of factor $\bar{v}_2$ holding output prices fixed will increase output of the good which uses factor $\bar{v}_2$ intensively, and decrease the output of the other good. If the supply of factor v_1 adjusts to the change in the endowment of factor $\bar{v}_2$, this Rybczynski relationship is strengthened (weakened) if the endogenously supplied factor is normal (inferior).

We can illustrate this result using our simple numerical general equilibrium model of Section 4. First we'll generate the usual Rybczynski result by assuming that the supply of the endogenous factor (labour) to the production sector is fixed at 11 units in the home country and 9 units in the foreign country. In this example, equilibrium prices are all equal to unity. Recall from Section 4 that in this example, factor price equalization did obtain, home and foreign outputs was $(x_H, y_H) = (15, 5)$ and $(x_F, y_F) = (5, 15)$, respectively, and the representative agent in each country consumed 10 units of x, y, and leisure. The home country was relatively well endowed with labour, and x was the relatively labour-intensive good.

Suppose we move one unit of capital from the home country to the foreign country. The Rybczynski theorem tells us that domestic production of x should rise, domestic production of y should fall, and the reverse should happen in the foreign country. For brevity, we'll consider only changes in the domestic production of x. In this example, the transfer of capital from H to F causes domestic production of x to rise by $13.\bar{3}$ percent to 17 units.

Now consider the same experiment, but allow the supply of labour to the production sector to respond to the shift in capital. If we imagine that the ownership of capital is transferred away from H as the capital itself is moved from H to F, then in our simple example, the income effect causes the supply of labour in H to rise (demand for leisure falls as income falls). The resulting increase in x-production in the home country is 20 percent, to 18 units. The effect on x-production in the home country of the movement of capital is 50 percent greater when labour is endogenously supplied.

Obviously these results are completely dependant upon the functional forms chosen in this simple example. The Cobb-Douglas utility function implies that labour is a normal good. If we had chosen a utility function such that labour were an inferior good, then the effect of the movement of capital from H to F on domestic production of good x in our experiment would be dampened. In the extreme case, if this income effect on labour supply were sufficiently large, we could construct examples where the domestic production of the labour-intensive good x actually falls when capital is moved out of the domestic country. If we had described the representative agent's preferences by a quasi-homothetic utility function where the consumer had a subsistence consumption of leisure, then the effect of our experiment on domestic

x-production is almost twice as large when labour is endogenous compared to the case when labour supply in perfectly inelastic.

6.2 More Factors than Goods

Now suppose that the number of factors exceeds the number of goods. Consider the simple model with two goods ($m = 2$), three factors ($n = 3$), and suppose one of the three factors is endogenously supplied ($e = 1$). Since output prices are assumed to remain fixed, we can reduce the equation system (6.1) to:

$$\begin{pmatrix} dy_1 \\ dy_2 \end{pmatrix} = \begin{pmatrix} \frac{\partial y_1}{\partial \bar{v}_2} d\bar{v}_2 + \frac{\partial y_1}{\partial \bar{v}_3} d\bar{v}_3 + \frac{\partial y_1}{\partial w_1} dw_1 \\ \frac{\partial y_2}{\partial \bar{v}_2} d\bar{v}_2 + \frac{\partial y_2}{\partial \bar{v}_3} d\bar{v}_3 + \frac{\partial y_2}{\partial w_1} dw_1 \end{pmatrix}. \tag{6.5}$$

As was the case for the Stolper-Samuelson experiment above, suppose first that the price of the endogenously supplied factor does not change. Equation (6.5) then gives the usual Rybczynski result, that an increase in the supply of a factor of production (say $d\bar{v}_2 > 0$) holding the supply of the other factor constant ($d\bar{v}_3 = 0$) will increase output of the good that uses intensively the factor whose supply has risen (say $dy_1 > 0$), and reduces output of the other good.

Now consider the effect of the change in the price of the endogenously supplied factor. As we saw above, this term depends upon output responses, substitutability in consumption, and the income effect upon the supply of the endogenous factor, which is indeterminate in general. Suppose that $d\bar{v}_2 > 0$, $d\bar{v}_3 = 0$, good 1 uses factor $\bar{v}_2$ intensively, and output of good 1 rises as the return to the endogenously supplied factor rises ($\partial y_1/\partial w_1 > 0$). Then the usual Rybczynski result will be magnified if an increase in the supply of factor $\bar{v}_2$ causes the price of the endogenously supplied factor to rise. This effect is derived by differentiating the equilibrium condition for the endogenously supplied factor (2.22) with respect to the vector of exogenous factor supplies:

$$(6.6) \qquad dw = -[G_{ww} - E_{ww}]^{-1}[G_{w\bar{v}} - E_{w\bar{v}} - v_{m_f}\bar{w}']d\bar{v},$$

or in this particular example:

(6.7)

$$dw_1 = -(G_{w_1w_1} - E_{w_1w_1})^{-1}\left(-\frac{\partial v_1^d}{\partial \bar{v}_2} + \frac{\partial v_1^s}{\partial \bar{v}_2} - v_{1m_f}\bar{w}_2 \quad -\frac{\partial v_1^d}{\partial \bar{v}_3} + \frac{\partial v_1^s}{\partial \bar{v}_3} - v_{1m_f}\bar{w}_3 \right)\begin{pmatrix} d\bar{v}_2 \\ d\bar{v}_3 \end{pmatrix}.$$

As above for the Stolper-Samuelson experiment, we can construct sufficient conditions such that the usual Rybczynski results are magnified. If the endogenously supplied factor and factor $\bar{v}_2$ are not cooperative in production, then $\partial v_1^d/\partial \bar{v}_2 < 0$. If an increase in the exogenous supply of factor $\bar{v}_2$ causes a gross decrease in demand for the endogenously supplied factor, then gross supply of factor v_1 to the production sector will rise, so that $[\partial v_1^s/\partial \bar{v}_2 - v_{1m_f}\bar{w}_2] < 0$. These conditions are sufficient for an increase in the exogenous supply of factor $\bar{v}_2$ to result in an increase in the return to the endogenously supplied factor, so that the Rybczynski result will be magnified. However, we must also note that it is possible to construct cases where the presence of only one endogenously supplied factor in this simple model causes the usual Rybczynski result to be dampened, and that we cannot rule out the possibility that the usual Rybczynski result may be reversed, so that output of good y_1, for example, would actually fall in response to an increase in the supply of the factor used intensively in production of good 1. For example, if the endogenously supplied factor is normal in production of good 1, then if the increase in the supply of factor $\bar{v}_2$ results in an increase in the return to the endogenous factor, then the term $(\partial y_1/\partial dw_1)dw_1$ will work in the opposite direction to the term $(\partial y_1/\partial d\bar{v}_2)d\bar{v}_2$, and the usual Rybczynski result will be dampened. The return to the endogenous factor will fall in response to an increase in the exogenous supply of factor $\bar{v}_i$ if $\bar{v}_i$ and the endogenous factor are cooperative in production and if the increase in the supply of factor $\bar{v}_i$ causes a fall in gross demand for the endogenous factor.

6.3 Specific Factors Model

Let us again consider the specific factors model. In particular, suppose as in Section 5 that the exogenously supplied factors $\bar{v}_2$ and $\bar{v}_3$ are specific to industries y_1 and y_2 respectively. In the traditional version of this model, factor v_1 is in fixed supply and mobile between industries y_1 and y_2. The effects of an increase in the supply of one of the specific factors is given in Jones (1971), p.11. Since the return to the mobile factor is a function of endowments as well as output prices when there are more factors than goods, an increase in $\bar{v}_2$, for example, causes the return to the mobile factor v_1 to rise, and the resulting changes in factor usage ratios cause a dampening of the magnification effect seen in the two-good, two-factor model. For example, an increase in the endowment of $\bar{v}_2$ causes output of good y_1 to rise and of good y_2 to fall, but the percentage increase in y_1 is smaller than that in the supply of factor $\bar{v}_2$.

In our version of the specific factors model, factor v_1 is mobile and endogenously supplied. Its return will also change as the supply of an exogenous factor changes, given by equation (6.7). The output supply changes due to the change in the endowment of the fixed factors is still given by equation (6.5). Since factor $\bar{v}_2$ is specific to industry 1, we know that $\partial y_1/\partial \bar{v}_2 > 0$, and $\partial y_2/\partial \bar{v}_2 < 0$. Likewise, since factor $\bar{v}_3$ is specific to industry 2, $\partial y_1/\partial \bar{v}_3 < 0$, and $\partial y_2/\partial \bar{v}_3 > 0$. If we ignore the effect of the change in the return to the mobile factor, an exogenous increase in the endowment of factor $\bar{v}_2$ holding the endowment of the other factor fixed will cause output of good 1 to rise and of good 2 to fall.

The effect of the change in the endowment of $\bar{v}_2$ on the return to the mobile endogerous factor is given by equation (6.7). If the mobile factor is normal $v_{1m_f} < 0$, then the income effect of an endowment increase will always be to reduce the return to the mobile factor. However, since $\bar{v}_2$ and the mobile factor may be cooperative $(\partial v_1^d/\partial \bar{v}_2 > 0)$ or not, and since the endowment change may increase of decrease the supply of the mobile factor $(\partial v_1^s/\partial \bar{v}_2 >$ or $< 0)$, the return to the mobile factor may increase of decrease due to the increase in the endowment of the factor specific to industry 1.

The result in Jones (1971) that the percentage change in output of good y_1 is smaller than the percentage change in the supply of the factor specific to good y_1 was driven by the fact that as the supply of factor $\bar{v}_2$ rose, the return to the mobile factor v_1 rose, serving to reduce the ratio of mobile to specific factor used to produce y_1. In our model with v_1 endogenously supplied, we can construct examples whereby the change in the supply of the exogenous factor $\bar{v}_2$ causes changes in the supply of the endogenous factor v_1 such that the return to v_1 actually falls. Suppose $dw_1 < 0$. If industry 1 does not use the mobile factor intensively and industry 2 does, then the output supply changes noted above would be magnified. Of course, the opposite result is also possible, and it may even be that the change in supply of the mobile factor due to the endowment change has such a strong perverse effect upon output supplies that an increase in the endowment of the factor specific to industry 1 actually causes a reduction in output supply on that industry and an increase in output supply in the other industry.

6.4 Cobb-Douglas Examples

Now examine the effect of endogenous factor supply on the Rybczynski effect in the two good, two factor Cobb-Douglas economy where labour is endogenously supplied. Now an exogenous increase in the endowment of capital $d\bar{k}$ will cause a change in the equilibrium supply of labour in the small open economy. World output prices are assumed constant, and we assume that both goods continue to be produced, so that input prices are also constant. We solve for the equilibrium labour supply function in the Appendix (see p. 140):

$$L^s = \frac{-c}{w}(w\bar{L} + r\bar{K}) \cdot \bar{L}.$$

The change in labour supply due to the change in the endowment of capital is given by:

$$\frac{\partial L^s}{\partial \bar{K}} = \frac{-cr}{w} < 0.$$

An increase in the endowment of capital, holding output prices constant, will cause an increase in fixed factor income, leading to an unambiguous increase in leisure demand, or an unambiguous fall in labour supply.

We solve for the equilibrium output supply functions in the Appendix (see p. 138):

$$\begin{pmatrix} x^s \\ y^s \end{pmatrix} = \left(\frac{-\alpha_x\beta_y}{q}\right)\left(\frac{\alpha_x}{\alpha_y}\right)^{\frac{-\alpha_y\beta_x}{q}}\left(\frac{\beta_x}{\beta_y}\right)^{\frac{\alpha_y\beta_x}{q}} \begin{pmatrix} \left(\frac{\beta_x}{\beta_y}\frac{p_x}{p_y}\right)^{\frac{-\beta_x}{q}} \cdot \bar{K} - \left(\frac{\alpha_x}{\alpha_y}\frac{p_x}{p_y}\right)^{\frac{\alpha_x}{q}} \cdot L \\ -\left(\frac{\beta_x}{\beta_y}\frac{p_x}{p_y}\right)^{\frac{-\beta_y}{q}} \cdot \bar{K} + \left(\frac{\alpha_x}{\alpha_y}\frac{p_x}{p_y}\right)^{\frac{\alpha_y}{q}} \cdot L \end{pmatrix}$$

where $q = \alpha_y\beta_x - \alpha_x\beta_y$. The Rybczynski effect of the change in the endowment of capital, $d\bar{K}$, on output of good x, for example, is found simply by differentiating the output supply function $x^s(p_x, p_y, L, \bar{K})$ with respect to $\bar{K}$, noting that the supply of labour also changes when $\bar{K}$ changes.

$$\begin{aligned} \frac{\partial x^s}{\partial \bar{K}} = & -\left(\frac{\alpha_x\beta_y}{q}\right)\left(\frac{\alpha_x}{\alpha_y}\right)^{\frac{-\alpha_y\beta_x}{q}}\left(\frac{\beta_x}{\beta_y}\right)^{\frac{-\alpha_y\beta_x}{q}}\left(\frac{\beta_x}{\beta_y}\frac{p_x}{p_y}\right)^{\frac{-\beta_x}{q}} \\ & + \left(\frac{\alpha_x\beta_y}{q}\right)\left(\frac{\alpha_x}{\alpha_y}\right)^{\frac{-\alpha_y\beta_x}{q}}\left(\frac{\beta_x}{\beta_y}\right)^{\frac{-\alpha_y\beta_x}{q}}\left(\frac{\alpha_x}{\alpha_y}\frac{p_x}{p_y}\right)^{\frac{\alpha_x}{q}}\frac{\partial L^s}{\partial \bar{K}} \end{aligned}$$

where we saw above that $\frac{\partial L^s}{\partial \bar{K}} = \frac{-cr}{w} < 0$. If good x is labour-intensive, so that $q > 0$, then $\frac{\partial x^s}{\partial \bar{K}} < 0$. If good x is capital-intensive, so that $q < 0$, then $\frac{\partial x^s}{\partial \bar{K}} > 0$. Therefore we can conclude that an increase in the endowment of capital holding output prices constant will always increase (decrease) output of the capital-(labour-) intensive good.

Now note that if labour were not endogenously supplied, then the term $\frac{\partial L^e}{\partial K}$ would equal zero. Denoting the change in output of good x due to the exogenous increase in the endowment of capital when labour supply is not endogenous by $\frac{\partial \bar{x}^e}{\partial K}$, we get:

$$\frac{\partial x^e}{\partial K} - \frac{\partial \bar{x}^e}{\partial K} = -\left(\frac{cr}{w}\right)\left(\frac{\alpha_x \beta_x}{q}\right)\left(\frac{\alpha_x}{\alpha_y}\right)^{\frac{-\alpha_y \beta_x}{q}}\left(\frac{\beta_x}{\beta_y}\right)^{\frac{\alpha_y \beta_x}{q}}\left(\frac{\alpha_x}{\alpha_y}\,\frac{p_x}{p_y}\right)^{\frac{\alpha_x}{q}}$$

Suppose x is capital-intensive, so that $q < 0$. Then the increase in the endowment of capital will cause a larger increase in the output of x when labour is endogenously supplied than when labour supply is perfectly inelastic. If x is labour-intensive, then $q > 0$, so that an increase in the endowment of capital would cause a larger fall in production of good x when labour is endogenously supplied than when labour supply is fixed.

7. INTERNATIONAL TRADE WITH TRADE AND FACTOR TAXES

The results of the previous chapters can now be used to consider some normative effects of endogeneous factor supply in international trade models. To begin with, assume that a nation enacts trade policy by imposing taxes (either positive or negative) on international trade, and for present purposes we assume that no other distortionary policies (quotas, VER's, etc.) are used. The effects of changes in tariff/subsidy policies in the simple trade model where all factor supplies are exogenous are well known. The specific objective of this section is to try to determine how the effects of trade tax policy changes are affected by the presence of endogeneously supplied factors.

We begin by revising the model of Chapter 2 to incorporate the presence of trade taxes. The only trade policy instrument available to a nation will be an ad valorem tax or subsidy on exports or imports, summarized by the vector $t' = (t_1, t_2, \ldots, t_m)$. Define the vector $T' \equiv (1 + t_1, 1 + t_2, \ldots, 1 + t_m)$, and describe the vector of world output prices as $p^{*\prime} = (p_1^*, p_2^*, \ldots, p_m^*)$. The vector of domestic prices can then be expressed as:

$$(7.1) \qquad p' = T'D(p^*),$$

where $D(\cdot)$ denotes a diagonal matrix with the elements of $(\cdot)$ on the main diagonal. Total trade tax revenue in this economy will be given by the expression:

$$(7.2) \qquad TR = -t'D(p^*)x.$$

To illustrate, suppose that good i is imported, so that $x_i < 0$. Then a tariff on good i ($t_i > 0$) causes TR to be positive, while an import subsidy ($t_i < 0$) causes $TR < 0$. We require $t_i \in (-1, \infty)$ in order to rule out negative prices. If good i is exported, $x_i > 0$. The tax wedge is now reflected by the relationship $(1 + s_i)p_i = p_i^*$, where s_i is the tax rate on exported good i. In the notation of equation (7.1), $t_i = -s_i/1+s_i$.[11] If the exports of good i are subsidized ($s_i < 0$), then $t_i > 0$ and tax revenue is reduced, while if the exported good is taxed ($s_i > 0$), $t_i < 0$ and we get $TR > 0$. As above, we require $s_i \in (-1, \infty)$ to avoid negative prices.

It is now assumed that this tax revenue is costlessly collected and redistributed to the representative consumer. The consumption sector will now maximize utility subject to the constraint:

$$(7.3) \qquad m_f = G(p, w, \bar{v}) + TR.$$

Recall that the indirect utility function which summarized the consumption sector's maximization problem was given by equation (2.16) as $V(p, w, m_f)$. Substituting equations (7.1) to (7.3) into this equation allows us to express the economy's indirect utility function in terms of the vector of world prices, the vector of domestic trade taxes, and the vector of fixed factor endowments.

$$(7.4) \quad V(t, p^*, w(t, p^*)) = V(T'D(p^*), w(t, p^*), G(T'D(p^*), w(t, p^*)) - t'D(p^*)x),$$

In the analysis that follows, we suppose that the supply of the endowed factors does not change ($d\bar{v} = 0$) so that the variable $\bar{v}$ is suppressed throughout. Note that we can also use equations (7.1) to (7.3) to write the net export function in equation (2.21) as $x(t, p^*, w(t, p^*), \bar{v})$.

In order to close the model, we need to describe the nation's trading partner. The domestic country we have been considering will trade with a single foreign country (the rest-of-the-world). The foreign country is treated the same as the domestic country, so that all assumptions which characterize behavior in the domestic country

[11] This is the convention used by Woodland (1982), p.299, to describe export taxes/subsidies.

will also be made with respect to the foreign country. For the purposes of this section, we can describe behavior in the foreign country with the indirect utility function:

(7.4*)

$$V(t^*, p^*, w^*(t^*, p^*)) = V(T^{*\prime} D(p^*), w^*(t^*, p^*), G^*(T^{*\prime} D(p^*), w^*(t^*, p^*)) - t^{*\prime} D(p^*) x^*),$$

where $x^* = x^*(t^*, p^*, w^*(t^*, p^*))$, and the * superscript denotes foreign country variables. Note that the vector of fixed foreign factor endowments $\bar{v}^*$ has been suppressed.

To find the welfare effects of a change in the nation's trade taxes, differentiate the indirect utility function (7.4) with respect to the vector t:

$$\begin{aligned} dV &= V_p'[D(T)M(\partial p^*/\partial t)dt + D(p^*)dt] + V_w' dw \\ &+ V_{m_f}[G_p'[(D(T)M(\partial p^*/\partial t)dt + D(p^*)dt] + G_w' dw] \\ &- V_{m_f}[x' D(p^*)dt + x' D(t) M(\partial p^*/\partial t)dt + t' D(p^*) M(\partial x/\partial t)dt], \end{aligned}$$

where $M(\cdot)$ denotes a matrix comprised of elements $(\cdot)$. To simplify this expression, first divide through by the marginal utility of income V_{m_f}, and then use equations (2.5), (2.6), (2.18), and (2.19) to get:

$$\begin{aligned} dV/V_{m_f} &= [y' - z'][D(T)M(\partial p^*/\partial t)dt + D(p^*)dt] + [v^{d\prime} - v^{s\prime}]dw \\ &\quad -x'[D(t)M(\partial p^*/\partial t)dt + D(p^*)dt] - t' D(p^*) M(\partial x/\partial t)dt. \end{aligned}$$

By definition, $y' - z' = x'$, and $v^{d\prime} = v^{s\prime}$, so that:

$$\begin{aligned} dV/V_{m_f} &= x'[D(T) - D(t)]M(\partial p^*/\partial t)dt - t' D(p^*) M(\partial x/\partial t)dt \\ &= x' M(\partial p^*/\partial t)dt - t' D(p^*) M(\partial x/\partial t)dt. \end{aligned} \tag{7.5}$$

This is the familiar decomposition of the welfare effects of a change in trade taxes into a terms of trade effect $(x' M(\partial p^*/\partial t)dt)$ and a volume of trade effect $(-t' D(p^*) M(\partial x/\partial t)dt)$. [12]

[12] Equation (7.5) is similar to equation (54) in Woodland (1982), p.314, although the derivation is slightly different. An analogous expression is derived in Markusen and Wigle (1989) in the presence of scale economies and imperfect competition.

7.1 Trade Tax Changes and Welfare in a Small Open Economy

For a small open price-taking economy (SOPTE), the model is closed by noting that the economy cannot affect its terms-of-trade, but can buy or sell outputs on the world market at prices p^*. As a result, the matrix $M(\partial p^*/\partial t)$ in equation (7.5) is a null matrix, so we get:

$$\frac{dV}{V_{m_f}} = -t'D(p^*)M(\partial x/\partial t)dt. \tag{7.6}$$

To illustrate, consider the two-good, two-factor version of this model. Suppose that the domestic country imports good 1, exports good 2, and taxes imports of good 1, so that $t' = (t_1, 0)$, $t_1 > 0$. If an increase in the tariff on good 1 causes a decrease in imports of good 1, so that $\partial x_1/\partial t_1 > 0$, then a tariff increase in the domestic country ($dt_1 > 0$) would reduce domestic welfare ($dV < 0$). (Recall that V_{m_f} is the marginal utility of fixed factor income, and is positive.) In Section 3, we derived conditions under which the domestic net export elasticity was larger when some factors were endogenously supplied. We now proceed to consider the implications of this fact for welfare changes in a small open economy, for given fixed world prices p^*, as relative prices within the small open economy change due to trade tax changes.

Let us look more closely at equation (7.6). First consider the case where the number of factors equals the number of goods. In Section (3.1), we showed that if the representative agent in the domestic country had preferences which could be represented by the additively separable utility function:

$$U(z,v) = U_1(z) + U_2(v),$$

where $U_1(z)$ is linearly homogenous and $U_2(v)$ is strictly concave, then net exports would be positively associated with own-price changes. If this own-price change is caused by a trade tax change, then an increase in the tariff on an import ($dt_i > 0$) would cause the relative price of the imported good to rise, so that net imports would fall, and domestic welfare would decrease. As well, an increase in the export tax on an exported good ($dt_i < 0$) would cause the relative price of that good to the domestic economy to fall, net exports of that good would fall, and domestic welfare would fall. That is, any change in trade taxes which increases trade taxes in absolute value is welfare reducing, while any decrease in trade taxes in absolute value is welfare increasing.

We also saw in Section (3.1) that net export responses to a given output price change are larger (in absolute value) when factors are endogenously supplied than when factor supplies are perfectly inelastic (see p.35), when preferences can be represented by our separable utility function. As a result, if we consider a SOPTE at some initial equilibrium where t is not a null vector, and then change trade taxes, the resulting change in domestic output prices will have a larger effect on net exports (in absolute value) when factors are endogenously supplied than when factor supplies are perfectly inelastic. A given trade tax change will therefore cause a greater change in domestic welfare if factor supplies are allowed to respond to the price changes entailed by the trade tax change. For example, a decrease in the tariff charged on an imported good will be more welfare improving if factors are endogenously supplied.

We also found conditions in Section 3.1 under which net exports were positively related to output price changes in the two-good, two-factor version of this model, when one factor is endogenously supplied. For example, if the imported good uses the endogenously supplied factor intensively, and if the imported good is a substitute for the endogenously supplied factor, then an increase in the relative price of the imported good will cause imports to fall, so that a decrease in the import tariff would be welfare improving. In general, trade tax decreases (in absolute value) would be welfare improving when the conditions at the end of Section 3.1 are satisfied.

The conditions which ensured that net exports were positively related to output price changes were also sufficient to ensure that a given relative price change had a larger effect on net exports (in absolute value) when one factor was endogenously supplied than when both factors were in fixed supply. As a result, these conditions are also sufficient to ensure that decreases in trade taxes (in absolute value) are more welfare improving when one factor is endogenously supplied than when the supply of both factors is perfectly inelastic.

Now consider the model described in Section 3.2 where there are more factors than goods, and a subset of those factors are endogenously supplied. It was shown that net export functions were positively sloped when the representative consumer's preferences could be represented by the additively separable utility function described above. This implies that an increase in an import tariff would cause the relative price of that import to rise, so that imports of that good would fall ($\partial x_i/\partial t_i > 0$). Equation (7.6) then implies that in this model, a tariff increase ($dt_i > 0$) would be welfare reducing. Of course, we can apply the same argument to all trade taxes, so that we can conclude that an increase in trade taxes in absolute value will be welfare reducing, and a decrease in trade taxes will be welfare improving.

We also saw that when we could use the additively separable utility function noted above, net export functions were more elastic when some factors were endogenously supplied than when all factors were perfectly inelastically supplied. A given trade tax change will therefore have a larger effect on net exports (in absolute value) when factors are endogenously supplied. This combined with the previous result implies that a given reduction in trade taxes (in absolute value) will be more welfare improving if some factor supplies are endogenous when the representative consumers' preferences can be represented by our additively separable utility function.

In the two-good, three-factor version of this model when one factor is endogenously supplied, we found sufficient conditions at the end of Section 3.2 under which net export changes were positively associated with own-price changes, and that these changes were larger (in absolute value) when there was an endogenously supplied factor rather than when all factors were inelastically supplied. When we apply this result to equation (7.6), we see that any decrease in the absolute value of any trade tax in the SOPTE will be welfare improving in this model when one factor is endogenously supplied. As well, the same trade tax change will be more welfare improving than it would have been had all factors of production been in perfectly inelastic supply.

7.2 Trade Tax Changes and Trade Tax Revenue

Now consider the effect in the SOPTE of a change in the domestic vector of trade taxes on trade tax revenue, TR when some factors are endogenously supplied. If we differentiate equation (7.2), we get:

$$(7.7) \qquad dTR = -p^* x(t, p^*, w(t, p^*)) - t' D(p^*) M(\partial x / \partial t) dt.$$

If we imagine the same small open economy facing the same world prices p^*, with the same trade tax vector, the same endowment of fixed factors, and a supply of the endogenous factors fixed at the level $v^s = v^d$, then the corresponding change in tariff revenue due to the same trade tax change in equation (7.7) would be:

$$(7.7') \qquad d\tilde{TR} = -p^* \tilde{x}(t, p^*, w(t, p^*)) - t' D(p^*) M(\partial \tilde{x} / \partial t) dt.$$

Of course, the initial level of net exports in the SOPTE is the same in both cases, since fixing factor supplies at the level equal to the equilibrium value of endogenous factor supplies in the model where some factor supplies are endogenous implies that the initial equilibrium in each model is identical. Since $x = \tilde{x}$, we can subtract equation (7.7′) from equation (7.7) to get:

$$(7.8) \qquad dTR - d\tilde{TR} \;=\; -t'D(p^*)[M(\partial x/\partial t) - (\partial\tilde{x}/\partial t)]dt.$$

If factor supplies are held fixed so that all factor supplies are perfectly inelastic, then the change in trade tax revenue $d\tilde{TR}$ due to some given trade tax change dt will not be the same as the change in trade tax revenue dTR when some factors are endogenously supplied.

We can apply the results of Section 3 just as we did in the previous section. We know that when the preferences of the representative consumer in the SOPTE can be summarized by the additively separable utility function above, changes in net exports are always positively associated with own-price changes, and the own net export response due to a given price change is always larger (in absolute value) when some factors are endogenous supplied than when all factors are in fixed supply. We also saw that this result holds independently of whether there are more factors than goods or the same number of goods and factors. As a result, we can conclude that an increase in a tariff on good i will cause a larger increase in tariff revenue when some factors are endogenously supplied than when all factor supplies are perfectly inelastic, when our separable utility function can be used to represent preferences in the SOPTE.

This result will hold whenever net export functions are more elastic when factor supplies are endogenous. Therefore, the sufficient conditions which we found at the end of Sections 3.1 and 3.2 which ensured that net export functions were more elastic in the two-good, two-factor model and the two-good, three-factor model will also be sufficient to ensure that trade tax changes will have a larger effect (in absolute value) on trade tax revenue when factor supplies are endogenous than when factor supplies do not respond to trade tax changes.

7.3 Optimal Trade Taxes With Endogenous Factor Supply

In Section 3 we derived conditions under which the presence of endogenous factor supply resulted in a higher net export elasticity. We now consider the implications of this result for the optimal tariff problem. The optimal tariff problem dates back at least to Bickerdike (1906), and the standard "modern" reference is Graaff (1949). The optimal vector of trade taxes is the set of trade taxes such that welfare is maximized, for any given choice of trade taxes by the nation's trading partner(s). This conditionality upon a fixed vector of foreign trade taxes has led many economists to model the optimal tariff problem as a non-cooperative game between the trading countries, the equilibrium being characterized by a set of Nash-equilibrium trade taxes (see Markusen and Wigle (1989), Mayer (1981), pp.139-140, or McMillan (1985), pp.24-28, among others). We will denote the vector of optimal or Nash-equilibrium trade taxes by t_N.

The vector of optimal trade taxes for the SOPTE will be the vector t_N such that any differential change in trade taxes causes a decrease in welfare. We can solve for t_N by setting equation (7.6) equal to zero, after noting that the SOPTE cannot affect world terms-of-trade.

$$0 = -t'D(p^*)M(\partial x/\partial t)dt. \tag{7.9}$$

Of course, setting $t_N = 0$ solves equation (7.9), so as shown in the proof above, the null vector will still be the globally optimal trade tax vector. To prove this statement, let p^f and p^t be the world prices at free-trade and when the trade-tax vector is not a null vector, respectively. Let y^f and v^f be the optimal output supply and endogenous factor demand vectors at free-trade prices p^f and w^f, respectively, and let y^t and v^t be the optimal output supply and endogenous factor demand vectors at taxed prices p^t and w^t, respectively. Then the following inequality must hold:

$$p^{f\prime}y^f - w^{f\prime}v^f \;\geq\; p^{f\prime}y^t - w^{f\prime}v^t,$$

since (y^f, v^f) was optimal at (p^f, w^f). By adding and subtracting $p^{t\prime}y^t$ to the right-hand side, we can rearrange the inequality to yield:

$$p^{f\prime}y^f - w^{f\prime}v^f \;\geq\; p^{t\prime}y^t - w^{f\prime}v^t + (p^f - p^t)'y^t.$$

Now note that in equilibrium, $p^{f\prime}y^f - w^{f\prime}v^f \;=\; G(p^f, w^f, \bar{v}) = E(p^f, w^f, \mu^f)$, so that we can rewrite the last inequality as:

$$E(p^f, w^f, \mu^f) \;\geq\; p^{t\prime}y^t - w^{f\prime}v^t + (p^f - p^t)'y^t.$$

Let (z^t, v^t) be the vector of optimal output demands and endogenous factor supplies at prices p^t, w^t. According to the balance of payments constraint, the value of production must equal the value of consumption, so that $p^{t\prime}y^t = p^{t\prime}z^t$. If we substitute this constraint into the right-hand side of the last equation, and add and subtract the term $p^{f\prime}z^t$, we get:

$$E(p^f, w^f, \mu^f) \;\geq\; p^{f\prime}z^t - w^{f\prime}v^t + (p^f - p^t)'(y^t - z^t).$$

It must be true that:

$$p^{f\prime}z^t - w^{f\prime}v^t \;\geq\; E(p^f, w^f, \mu^t),$$

since $E(p^f, w^f, \mu^t)$ is the minimum expenditure required to attain the level of utility μ^t at prices (p^f, w^f). As a result, we can write:

$$E(p^f, w^f, \mu^f) \;\geq\; E(p^f, w^f, \mu^t) + (p^f - p^t)'(y^t - z^t).$$

For a small open economy, $(p^f - p^t)$ must be a null vector, since the small open economy cannot affect world prices. Thus we are left with:

$$E(p^f, w^f, \mu^f) \geq E(p^f, w^f, \mu^t),$$

which states that at prices (p^f, w^f), expenditure needed to reach the level of utility μ^f is higher than that required to reach μ^t. Since the expenditure function is increasing in the level of utility, it must be true that the free trade level of utility is higher than any other, so that free trade is globally optimal for a small open economy.

For a large economy able to affect world terms-of-trade, the model is closed by noting that the change in domestic net exports must equal the negative of the change in rest-of-world net exports, $dx = -dx^*$. Substituting this constraint into equation (7.5) yields:

$$\text{(7.10)} \qquad dV/V_{m_f} = -x^{*\prime} M(\partial p^*/\partial t)dt + t' D(p^*) M(\partial x^*/\partial t)dt.$$

For a large economy the appropriate decomposition of the volume-of-trade effect is as follows:

$$\text{(7.11)} \qquad M(\partial x^*/\partial t) = M(\partial x^*/\partial p^*)M(\partial p^*/\partial t) + M(\partial x^*/\partial t^*)M(\partial t^*/\partial t),$$

since $x^* = x^*(t^*, p^*, w^*(t^*, p^*))$. Note that the matrix capturing the elasticity of the net export functions $M(\partial x^*/\partial p^*)$ also contains terms reflecting how the changes in the prices of the endogenously supplied factors due to the terms-of-trade change affect net exports.

The vector of optimal tariffs for a large economy is the vector t which maximizes equation (7.10) for any given vector of foreign tariffs t^*. We solve for t_N by setting (7.10) equal to zero, substituting in equation (7.11), and noting that the matrix reflecting foreign tariff retaliation in equation (7.11), $M(\partial t^*/\partial t)$, is a null matrix. Thus we get:

$$t'_N = x^{*'}M(\partial x^*/\partial p^*)^{-1}D(p^*)^{-1}, \tag{7.12}$$

which says that the optimal tariff vector for a large open economy is given by the inverse of the foreign country's vector of net export elasticities. Then we can conclude that if the representative consumer's preferences in the large economy can be represented by the additively separable utility function $U(z,v) = U_1(z) + U_2(v)$, the vector of optimal trade taxes will be smaller when factors are endogenously supplied.

7.4 Taxes on Endogenous Factor Supply

Now consider the effects of taxes on factors. Suppose all trade taxes are zero, and that the only taxes present are applied to the level of usage of the endogenously supplied factors. In order to simplify the analysis, suppose that the number of factors equals the number of goods in equilibrium when there are factor taxes, and that the same goods would continue to be produced in the undistorted equilibrium with no taxes at all. In this case, the gross return to factors in the small open economy facing given world prices p^* will be the same, whether or not there are factor taxes. However, the net wage received by or paid to the representative consumer for supply of the endogenous factor will vary according to the factor tax.

The tax on the endogenously supplied factors will be paid by the production sector, and total tax revenue will be redistributed back to the representative consumer. Since the number of goods equals the total number of factors, the gross wage w paid to the endogenously supplied factors is determined solely by the production sector according to equation (2.6′) on p. 26:

$$w = \hat{G}_v(p, v, \bar{v}), \tag{2.6′}$$

where $v(p, \tilde{w}, m_f)$ is the level of supply of the endogenous factors determined by the consumption sector, $\tilde{w} = w - f$ is the net-of-tax return received by the representative consumer per unit supplied of the endogenous factor, and f is the vector of per-unit factor taxes. Recall that in this example, the production sector treats v as given and solves for the gross equilibrium input price vector w, while the consumption sector treats the after-tax input price vector $\tilde{w}$ as given and solves for the equilibrium endogenous factor supply vector v. We will consider only an ad valorem tax, so that total factor tax revenue collected per unit of the i'th endogenously supplied factor is $f_i w_i$, and:

$$FTR = f'D(w)\, v(p, \tilde{w}, m_f) ,$$

where FTR now represents total factor tax revenue. Otherwise the notation is exactly the same as in equation (7.2) above.

For given world output prices p^*, a given endowment of the factors in fixed supply $\bar{v}$, and the level of supply of the endogenous factors $v(p, \tilde{w}, m_f)$, the gross return to the endogenous factors is w. For each unit of the i'th endogenous factor employed, $f_i w_i$ is paid by the production sector as a tax and $(1 - f_i)w_i = \tilde{w}_i$ is paid to the owner of the factor. Net factor rewards for the endogenous factors received by the consumption sector are given by:

$$\tilde{w}' = (1 - f)'\, D(w) .$$

For given world prices, the gross wage w is fixed as long as all goods continue to be produced. As a result, any change in the vector of factor taxes df will result in a change in factor rewards received by the representative consumer per unit of supply of the endogenous factors according to:

$$d\tilde{w} = -D(w)\, df .$$

We can solve for the change in endogenous factor supply caused by a given change in factor taxes df by differentiating the equation for endogenous factor supplies as follows:

$$\begin{aligned} v(p^*,\tilde{w},m_f) &= -E_w\ (p^*,(1-f')D(w),V(p^*,(1-f')D(w),m_f)) \\ dv(p^*,\tilde{w},m_f) &= E_{ww}\ D(w)df - E_{w\mu}[-V_w'D(w)df + V_{m_f}dm_f] \\ dv(p^*,\tilde{w},m_f) &= E_{ww}\ D(w)df - v_{m_f}[v'D(w)df - dm_f]\ , \end{aligned} \tag{7.13}$$

where $v_{m_f} = -E_{w_\mu}V_{m_f}$ is the income effect on endogenous factor supply, and dm_f is the change in fixed factor income due to the factor tax change df. Fixed factor income in this model is equal to total revenue earned by the fixed factors ($\bar{w}'\bar{v}$) and total factor tax revenue. In equilibrium, $\bar{w}'\bar{v} = \hat{G}(p^*,v,\bar{v}) - w'v$, so that we can write total fixed factor income as follows:

$$\begin{aligned} m_f &= \hat{G}(p^*,v,\bar{v}) - w'v + FTR \\ &= \hat{G}(p^*,v,\bar{v}) - w'v + f'D(w)v. \end{aligned}$$

To solve for the change in fixed factor income (dm_f) due to a change in the vector of factor taxes (df), differentiate this equation with respect to f, holding output prices p^* and the endowment of fixed factors $\bar{v}$ constant. No 'hat if p^* is fixed, then as long as all goods continue to be produced, gross input prices will not change, so that $dw = 0$. However, the change in factor taxes df will cause a change in factor supply dv:

$$\begin{aligned} dm_f &= \hat{G}_v'(p^*,v,\bar{v}) - w'dv + f'D(w)dv + v'D(w)df \\ dm_f &= f'D(w)dv + v'D(w)df\ , \end{aligned} \tag{7.14}$$

since $\hat{G}_v' = w'$ in equilibrium. Substituting this expression for dm_f into equation (7.13) above yields:

$$\begin{aligned} dv(p^*,\tilde{w},m_f) &= E_{ww}D(w)df - v_{m_f}[v'D(w)df - f'D(w)dv - v'D(w)df] \\ &= E_{ww}D(w)df + v_{m_f}[f'D(w)dv]. \end{aligned} \tag{7.15}$$

If we premultiply both sides of this expression by $df'D(w)$, we get:

$$df'D(w)dv = df'D(w)E_{ww}D(w)df + df'D(w)v_{m_f}[f'D(w)dv]\,. \tag{7.16}$$

The term $df'D(w) = -d\tilde{w}'$ is the vector of changes in per unit factor tax revenue due to the given factor tax change df. The first term on the right-hand side is a quadratic form about a negative definite matrix (recall that the expenditure function is strictly concave in w), so that $df'D(w)E_{ww}D(w)df < 0$. This is the direct effect of the factor tax change on endogenous factor supply. Essentially, an increase in factor taxes lowers factor returns and therefore lowers endogenous factor supply.

The second term represents the income effect of a factor tax change on the supply of the endogenous factors. When evaluated in the neighbourhood of zero factor taxes, the vector f is a null vector, so that:

$$df'D(w)dv = df'D(w)E_{ww}D(w)df < 0.$$

In the neighbourhood of zero factor taxes, an increase in the tax on any endogenously supplied factor causes endogenous factor supply to fall. As well, an increase in the subsidy on any endogenously supplied factor ($df_i < 0$) causes an unambiguous increase in the supply of the endogenous factor.

Now consider the effect of a factor tax change on output supply functions. In our model where the number of factors equals the number of goods produced in equilibrium:

$$y = \hat{G}_p(p, v, \bar{v})$$

$$dy = \hat{G}_{pv}dv \tag{7.17}$$

$$\text{where:} \qquad dv = E_{ww}D(w)df + v_{m_f}[f'D(w)dv].$$

The matrix $\hat{G}_{pv} = [\frac{\partial y_i}{\partial v_j}]$ is the generalized Rybczynski matrix with respect to the endogenously supplied factors. Of course, it is impossible to determine the effect of a factor tax change on output supplies in general, since any output will be produced using some of the endogenously supplied factors intensively ($\frac{\partial u}{\partial v_j} > 0$) and not using some other endogenously supplied factors intensively ($\frac{\partial u}{\partial v_j} < 0$). In order to interpret the effect of a change in factor taxes on output supply, suppose that of the n factors of production, only the j'th factor is endogenously supplied, and suppose that there are originally no taxes on factors. In this case, df_j is a scalar, and E_{ww} is a negative scalar. Thus we can write:

$$df_j D(w) dy \;=\; \hat{G}_{pv} df_j D(w) E_{ww} D(w) df_j, \tag{7.18}$$

where $\hat{G}_{pv}$ is now an mx1 vector and $df_j D(w) E_{ww} D(w) df_j$ is a negative scalar.

In this case, an increase in the tax on the endogenously supplied factor will result in a fall in endogenous factor supply. Equation (7.18) then says that if good i uses (does not use) the endogenously supplied factor intensively, an increase in the tax on factor j causes output of good i to fall (rise) as supply of factor j falls.

The effect of a factor tax change on output demand can be determined by writing the output demand functions as follows:

$$\begin{aligned} z(p^*, \tilde{w}, m_f) &= E_p(p^*, (1-f)'D(w), V(p^*, (1-f)'D(w), m_f)) \\ dz &= -E_{pw} D(w) df + E_{pu}[-V_w' D(w) df + v_{m_f} dm_f] \\ dz &= -E_{pw} D(w) df + z_{m_f}[-v' D(w) df + d_{m_f}], \end{aligned}$$

where $z_{m_f} = E_{p\mu} v_{m_f}$ is the income effect on output demand, and we used the fact that in equilibrium, $\frac{-V_w'}{V_{m_f}} = -v'$. Substituting for the change in fixed factor income (dm_f) from equation (7.14) above and cancelling terms yields:

$$dz \;=\; -E_{pw} D(w) df + z_{m_f}[f' D(w) dv]. \tag{7.19}$$

As was the case for output supply functions above, we cannot solve for the effect of factor tax changes on output demands in general since for any given demand function

z_i, some factors may be substitutes ($E_{p_i w_j} = \frac{\partial z_i}{\partial w_j} > 0$) and some may be complements ($E_{p_i w_j} = \frac{\partial z_i}{\partial w_j} < 0$). So consider the case where only the j'th factor is endogenously supplied, and consider the effect of an increase in f_j from an initial equilibrium where all factor taxes are zero. In this example, we can rewrite (7.19) as:

$$dz = -E_{pw}D(w)df_j,$$

where $D(w)df_j$ is a positive (negative) scalar if the j'th factor is being taxed (subsidized). If good i is a substitute (complement) for the endogenously supplied factor j, then an increase in the tax on factor j causes an increase in the consumer's demand for factor j, and a decrease (increase) in demand for good i.

Since net exports are simply the difference between output supply and output demand in equilibrium, we can use the results to determine the effect of a factor tax change on net exports. As in the above examples, we will suppose that all factor taxes are originally zero, and that only the j'th factor is endogenously supplied. Since $x = y - z$, we can write:

$$\begin{aligned} dx &= dy - dz \\ dx &= [\hat{G}_{pv}E_{ww} - E_{pw}]D(w)df_j, \end{aligned} \tag{7.20}$$

where E_{ww} is a negative scalar, and $D(w)df_j$ is a positive (negative) scalar if factor j is being taxed (subsidized). If good i uses the endogenously supplied factor intensively ($\hat{G}_{p_i v_j} > 0$) and is a substitute in consumption for the endogenous factor ($E_{p_i w_j} > 0$), then an increase in the tax on factor j will cause net exports of good i to fall. On the other hand, if good i does not use the endogenously supplied factor intensively and is a complement in consumption for factor j, then an increase in f_j will lead to an increase in net exports of good i.

Now consider the effect of factor taxes on domestic welfare. We will consider a small open economy, and assume that world output prices p^* are fixed. To simplify the analysis, suppose that the number of goods produced in the initial equilibrium is equal to the total number of factors of production, and that all goods continue to be produced after the tax change. We will also suppose that there exist trade taxes at the initial equilibrium, using the model developed in Section 7.1. If the initial vector of trade taxes is t, and the initial vector of taxes on the endogenously supplied factors is f, then we can write the representative consumer's indirect utility function as:

$$V(p, \tilde{w}, m_f) = V(t'D(p^*), (1-f)'D(w), m_f),$$

where $m_f = \hat{G}(p, v, \bar{v}) - w'v - t'D(p^*)x + f'D(w)v$ is total fixed factor income in equilibrium. The effect of a change in trade taxes dt and factor taxes df on welfare is found by totally differentiating the indirect utility function, noting that a trade tax change will cause domestic output prices to change, which implies a change in input prices according to the Stolper-Samuelson relationship:

$$\begin{aligned} dV &= V_p'D(p^*)dt - V_w'D(w)df - V_w'D(f)dw + V_{m_f}dm_f \\ dV/V_{m_f} &= -z'D(p^*)dt - v'D(w)df - v'D(f)dw + dm_f, \end{aligned} \tag{7.21}$$

where we used the fact that $V_p'/V_{m_f} = -z'$ and $-V_w'/V_{m_f} = -v'$ in equilibrium. The change in fixed factor income is found by differentiating the expression for m_f above:

$$\begin{aligned} dm_f = {} & \hat{G}_p'D(p^*)dt + \hat{G}_v'dv - w'dv - v'dw - t'D(p^*)dx - x'D(p^*)dt \\ & + f'D(w)dv + v'D(w)df + v'D(f)dw. \end{aligned}$$

Using the fact that $\hat{G}_p' = y'$ and $\hat{G}_v' = w'$ in equilibrium and substituting into (7.21) for dm_f gives:

$$\begin{aligned} dV/V_{m_f} = {} & - z'D(p^*)dt - v'D(w)df - v'D(f)dw + y'D(p^*)dt \\ & + w'dv - w'dv - v'dw - t'D(p^*)dx - x'D(p^*)dt + f'D(w)dv \\ & + v'D(w)df + v'D(f)dw. \end{aligned}$$

In equilibrium, $y' - z' = x'$, sc that:

$$dV/V_{m_f} = -t'D(p^*)dx - v'dw + f'D(w)dv. \tag{7.22}$$

Since we have assumed that the number of goods produced in equilibrium is always equal to the total number of factors of production, the change in the vector of prices of the endogenously supplied factors dw can be written as a function of output price changes only. World prices are fixed, so domestic output prices change only due to trade tax changes.

$$w = \hat{G}_v(t'D(p^*), v, \bar{v})$$

$$dw = \hat{G}_{vp}D(p^*)dt$$

Using this expression for input price changes, we can rewrite (7.22) as:

$$dV/V_{m_f} = -t'D(p^*)dx - v'\hat{G}_{vp}D(p^*)dt + f'D(w)dv. \tag{7.23}$$

The first term in equation (7.23) is the familiar volume of trade effect due to the trade tax change. In the second term, note that $\hat{G}_{vp} = \hat{G}'_{pv}$, which is the generalized Rybczynski matrix. If good i uses (does not use) the endogenously supplied factor intensively, then $\hat{G}_{v_j p_i} = \hat{G}_{p_i v_j} > 0$ (< 0). An increase in the tax on a good which uses the endogenously supplied factor j intensively causes an increase in w_j. The final term is the income effect of the tax change on factor supply, since $f'D(w)$ is the vector of per unit factor taxes which are redistributed lump-sum to the representative consumer.

Suppose we evaluate equation (7.23) in the neighbourhood of zero trade and factor taxes. Consider the special case where there is only one endogenously supplied factor. Then (7.23) reduces to:

$$dV/V_{m_f} = -v'\hat{G}_{vp}D(p^*)dt,$$

where v' is now a scalar, and $\hat{G}_{vp_i}$ is a vector whose i'th element is positive (negative) if good i uses (does not use) the endogenously supplied factor intensively. If the trade tax on good i is increased so that the domestic price of good i rises ($dt_i > 0$), then the return to the endogenous factor will rise if i uses the endogenously supplied factor intensively. The price of consumption of the endogenous factor rises, so that welfare falls ($dV/V_{m_f} < 0$). If the trade tax is charged on the good which does not use the endogenously supplied factor intensively, then domestic welfare rises as the cost of consuming the endogenous factor falls.

Now consider an initial equilibrium where trade taxes are all zero, trade taxes are not changing, but there exists a tax on the single endogenously supplied factor. The first two terms in equation (7.23) are zero. The last term gives the effect of a change in the factor tax on welfare due to the factor supply change dv. Equation (7.16) above gives the effect of a change in factor taxes on the supply of endogenous factors. As long as the endogenously supplied factor is normal, an increase in the tax on the endogenous factor reduces its real return and causes factor supply to fall, thereby lowering welfare. This reduction in welfare due to an increase in the factor tax is a direct result of the fact that the tax causes the factor to be under-supplied in equilibrium due to the factor market distortion.

Suppose we begin at a distorted equilibrium where the tax $f_j > 0$ on the single endogenously supplied factor v_j has led the factor to be under-supplied. Is it possible to introduce a tax on trade which would offset the factor market distortion and lead to an increase in welfare? Starting from zero trade taxes, for a given vector of factor taxes, equation (7.22) can be written as:

$$dV/V_{m_f} = -v'dw + f'D(w)dv. \tag{7.24}$$

If the trade tax is increased on the good which uses the endogenously supplied factor intensively, then the increase in the return to the endogenous factor would cause factor supply to rise, $dv_j > 0$. This would cause an increase in welfare according to the second term in (7.24). However, increasing the return to the endogenous factor would increase the cost of consuming the endogenous factor, according to the first term in (7.24), resulting in a decrease in welfare. Note that we can rewrite (7.24) as follows:

$$\begin{aligned} dV/V_{m_f} &= -v_j dw_j \frac{w_j}{w_j} + f_j w_j dv_j \frac{v_j}{v_j} \\ &= v_j w_j (f_j \hat{v}_j - \hat{w}_j) \\ &= f_j v_j dw_j \left(\frac{\hat{v}_j}{\hat{w}_j} - \frac{1}{f_j} \right), \end{aligned}$$

where $\hat{v}_j$ and $\hat{w}_j$ denote the percentage change in factor supply and the price of the endogenously supplied factor, respectively, due to the trade tax change, and $\frac{\hat{v}_j}{\hat{w}_j}$ is the factor supply elasticity. A trade tax change which causes an increase in the return to the endogenous factor ($dw_j > 0$) will improve welfare as long as the elasticity of supply of the endogenous factor is larger than the reciprocal of the factor tax. The initial term $f_j v_j dw_j$ is the change in total factor tax revenue due to the factor price change dw_j caused by the trade tax change.

8. NUMERICAL GENERAL EQUILIBRIUM MODEL

Thus far we have been concerned with the theoretical implications of the presence of endogenously supplied factors on a number of well-known results of international trade theory. In a number of cases, the presence of endogenously supplied factors in the model precluded our ability to determine the results of some of the comparative statics experiments in general. The ability on the part of the representative consumer to substitute factor consumption for consumption of goods made the effect of variable factor supply ambiguous in many of the examples which we considered.

We now turn our attention to an empirical example of the effect of endogenous factor supplies upon the above-mentioned results. A numerical general equilibrium model will be the tool used to carry out this empirical analysis. The motivation for this analysis is twofold. The preceding analysis has enabled us to determine the qualitative effect of the presence of endogenously supplied factors on the trade theory results considered thus far in a number of cases. However, the theoretical analysis gives no indication about the quantitative significance of endogenous factor supply for the comparative statics experiments which have been examined to this point. In fact, the response of factor supplies to some of the comparative statics shocks precluded the derivation of any definite results in some cases. A numerical general equilibrium model is used to determine the quantitative effects of the comparative statics results which were analyzed in the theoretical model, with different assumptions made about the elasticity of labour supply in order to compare the comparative statics results as the labour supply elasticity changes from zero.

As well, the use of a numerical general equilibrium model to examine the effect of the presence of endogenously supplied factors allows us to establish clearly the effect of factor supply changes in each individual market in a general equilibrium model, under different assumptions about substitutability and complementarity in the production and consumption sectors. The importance of these assumptions in affecting results was clearly established in the preceding theoretical analysis. However, the effect of substitution, especially between consumption of goods and of the endogenous factors, tended to complicate the results in the theoretical model, sometimes to the point where no definite result could be established. The use of the numerical general equilibrium model will give us a better intuitive understanding of the effect of substitution in production and consumption. This should lead to a better understanding of how the presence of endogenous factor supplies affects the various comparative statics results considered thus far by examining the way in which factor supply responses affect each individual market in a general equilibrium model.

Numerical general equilibrium models have been used extensively to examine a number of economic problems since the implementation and computation of these models became feasible several years ago with the derivation of algorithms which could be implemented on computers and used to solve such general equilibrium models. Studies have used numerical general equilibrium models to examine the effects of changes in tax policy, trade tax reform, and customs union formation, among many others, and the results of such models have influenced government policy. Our objective here is not nearly so bold. In fact, the motivation for using a numerical general equilibrium model is subtly but distinctly different in our case. A "typical" general equilibrium model will begin with an initial equilibrium of some degree of complication and disaggregation. A particular policy change is examined, a new equilibrium generated, and the two equilibria are compared. Usually the same counterfactual experiment is repeated many times, starting from the same initial equilibrium, using different values for certain key parameters in the model, especially substitution elasticities, so as to generate a range of new equilibria and to examine how sensitive those equilibria are to various parameter specifications.

The objective of the current study is not so much to examine what the results of a particular counterfactual experiment are, but to determine how sensitive the results of any particular experiment are to various specifications of certain key parameters, especially the elasticity of labour supply. The emphasis in our model will be on the sensitivity analysis surrounding various results, not on the results themselves. Of course, this has been an emphasis throughout the theoretical analysis of the preceding chapters. For example, we sought to determine not only whether endogenous factor supplies cause net exports to respond positively to own-price changes, but whether net export changes to a given terms-of-trade shock were larger or smaller when some factors were endogenously supplied.

Since all theoretical results were derived in a general equilibrium model, it is imperative that the empirical results be derived in the same setting, in order for those results to have any relevance to those derived in the theoretical analysis. To derive the empirical results that we are looking for using an econometric model would require a considerable amount of data. The data requirements needed to implement numerical general equilibrium models are significantly less. Because of the cost (especially time) associated with gathering the data necessary to build a consistent econometric model, it was felt that the numerical general equilibrium model was the more appropriate analytical tool for our purposes.

The relative advantages of econometric models over numerical general equilibrium models are well-known and have been pointed out by numerous authors. Econometric estimation of the underlying parameters of a given model is always a preferable way of deriving these parameters when econometric estimation is possible, as opposed to the calibration method of determining these same parameters in a numerical general equilibrium model based only upon a single initial equilibrium data set. To compensate, the results of numerical general equilibrium models are usually derived for a number of different specifications of the model's parameters, to determine how sensitive the results are to changes in these parameters.

This particular feature of numerical general equilibrium modelling is seen as an advantage for the purposes of the present analysis. What we are most interested in is the sensitivity of certain results (e.g. the change in net exports due to a given world price shock) to different specifications of certain independent parameters of our model (e.g. the elasticity of labour supply). It is relatively easy to carry out this sensitivity analysis using a numerical general equilibrium model. This is another reason why a numerical general equilibrium model seemed more appropriate to examine the problems we are interested in than an econometric model.

8.1. Benchmark Equilibrium Data Set

In order to be consistent with general equilibrium in a perfectly competitive small open economy, the initial equilibrium from which our numerical general equilibrium begins must be micro-consistent. That is, the initial equilibrium must be such that excess demands and profits in all sectors equal zero. The initial Benchmark Equilibrium Data Set used in this model is a modified version of the data set developed by Nguyen, Perroni, and Wigle (1989) for a world trading equilibrium in the year 1982. A complete description of their data set is given in Nguyen, Perroni and Wigle (1989), "1982 World Benchmark Data Set: Sources and Methods". The model consists of ten regions, each producing nine goods and an investment good, using labour and capital as primary inputs. All goods and capital are traded, and there is cross-hauling in all goods and in capital. There are production taxes on all ten production sectors in each region. Domestic tax revenue is distributed to the single representative consumer in each region, who consumes some of all nine goods. There is balanced trade in all traded goods and in traded capital. Production in each region less demand by the respective representative consumer equals net exports in each region. There are zero profits in each sector in each region.

The data set required significant modification in order for it to be consistent with the assumptions made in our theoretical model. To begin with, all results derived in the theoretical analysis were for a small open economy. As a result, it is sufficient in the numerical general equilibrium model to consider a single region. Canada was chosen as the representative small open economy, so that the data upon which the subsequent analysis is based are representative of equilibrium in the Canadian economy in 1982. All data in the benchmark are in billions of U.S. dollars.

The model was reduced from one with ten goods to one with two goods. The main reason for this aggregation was to keep the model as simple and transparent as possible, so that in the subsequent counterfactual experiments we can identify the effect of the presence of endogenous factor supply as clearly as possible. All goods are both exported and imported in the initial equilibrium. In order to be consistent with the theoretical model, we must modify the data to reflect the assumption that all goods are perfect substitutes, so that there is no cross-hauling in equilibrium. We will consider only net trades in goods. As a result, we will aggregate together all goods where exports are larger than imports into a single good called exports (EXP), and all goods where imports are larger than exports into a single good called imports (IMP). The existence of a single representative consumer is retained. The initial equilibrium is displayed in the social accounting matrix in Table 8.1. As is evident from the data, the exported good is relatively capital-intensive, since $(\frac{CAP}{LAB})_{EXP} = 0.61$, $(\frac{CAP}{LAB})_{IMP} = 0.32$.

Table 8.1:

	Exports	Imports	Cons'n	Exports	Imports	Total
EXP	97.008	66.773	50.483	12.651	0	226.915
IMP	60.335	225.927	228.292	0	12.651	501.903
LAB	41.623	144.386	186.009			
CAP	25.425	46.312				
TXS	2.526	18.505				
TTL	226.916	501.903				

On the inputs side, the initial benchmark equilibrium was characterized by the existence of two factors of production, capital (CAP) and labour (LAB). In the initial equilibrium, Canada was a net importer of capital. This feature of the initial data set complicates matters since throughout our analysis thus far we have implicitly assumed that there is no trade in factors of production. In order to reflect this implicit assumption, we modify the data set by increasing Canada's endowment of capital by the amount of imported capital at the initial equilibrium. Once this change is made, the balance of trade and the income-equals-expenditure equilibrium conditions will no longer be satisfied. It is necessary to decrease exports by an amount equal to total capital imports, and to increase consumption of the exported good by the representative consumer by an amount equal to total capital imports. These adjustments ensure that the modified equilibrium data set is still micro-consistent.

While these modifications obviously imply a departure from the initial benchmark equilibrium data set, they are necessary in order to make the results derived from the numerical general equilibrium model consistent and comparable to those derived in the theoretical sections. The adjustments can be thought of as reflecting the existence of a foreign agent who sells capital to Canada in return for Canadian exported goods, but who lives in and consumes the goods in Canada. His preferences are identical to those of the representative Canadian consumer, and these agents are then aggregated together into a single representative Canadian consumer. This interpretation would imply that 13.1% of the capital in Canada was foreign-owned.

The other factor of production is labour, which we will assume is the endogenously supplied factor. At the initial benchmark equilibrium, the representative consumer has a given total endowment of labour. We will assume that half of this total endowment of labour is supplied to the production sector and half is consumed as leisure, at the initial equilibrium. When we examine some given comparative statics experiment, say an increase in the world price of exports, in the model with fixed factor supplies, total labour supply is held fixed in the numerical general equilibrium model through the use of rationing constraints. For example, if the effect of the comparative statics experiment upon labour supply would be to increase the amount of labour that the consumer would want to supply (due, for example, to an increase in the world price of the exported good), then a tax is charged on labour supply that would just induce the consumer to keep labour supply fixed. Likewise, labour supply is subsidized if the comparative statics shock would induce the consumer to decrease labour supply. In equilibrium after the comparative statics shock, total labour supply is the same as it was at the initial equilibrium, and the labour tax/subsidy is zero. When we want to consider the effect of endogenous factor supply, we repeat the same comparative statics experiment without the rationing constraint.

There were no trade taxes incorporated in the Nguyen, Perroni, and Wigle data set. Instead, taxes were reported as a share of total production. In order to discuss the theoretical results about trade taxes, we surveyed the literature to determine an estimate of average import tariffs between Canada and the U.S. (in particular, see Cameron (1986)). An average tariff of 4.6% on goods imported into Canada was used.

We can anticipate some of the results at this stage, based upon the information in the initial data set in Table 8.1. Since EXP is capital-intensive, an increase in the world price of the exported good should cause the return to labour to fall, making it relatively less expensive to consume leisure. Labour supply should fall in equilibrium, leading to a magnification of the effect of the output price change on output supplies through the Rybczynski effect.

8.2. Modelling Issues and Parameter Specification

The procedure for solving the initial benchmark equilibrium and the subsequent counterfactual experiments in practice in a general equilibrium model is completely consistent with the description of a general equilibrium in our theoretical general equilibrium trade model. A single representative consumer owns all factors of production in the economy. The consumer's preferences are represented by a continuously differentiable, linearly homogeneous utility function. The consumer maximizes utility given his endowment and all prices. Production technology is summarized by a continuously differentiable, linearly homogeneous production function in each production sector. Taking output prices as given, since we are dealing with a small open economy, the production sector maximizes total revenue, subject to the total availability of factors of production. In full general equilibrium, the consumer's total income equals total expenditure, each production sector earns zero profits, each factor of production is fully employed, and there is balance of payments in the trade sector.

The numerical general equilibrium model begins with an initial benchmark equilibrium data set which is completely micro-consistent, in the sense that all equilibrium conditions described above are satisfied at the initial equilibrium. The modeller then works backwards from this initial equilibrium to an explicit description of the consumption and production technology which is consistent with the initial data set. For example, suppose the modeller describes the representative consumer's preferences by the Cobb-Douglas utility function:

$$U = \prod_{i=1}^{n} x_i^{\alpha_i}, \qquad \sum_{i=1}^{n} \alpha_i = 1.$$

The modeller works backwards from the initial set of endowments, prices and levels of demand to find the parameters α_i, $i = 1, \ldots n$ in the utility function which are implied by the inital equilibrium. The same procedure is followed for the production sector, given some choice of functional form representative of the production technology. This "benchmarking" process results in a complete description of the production and consumption technology which is consistent with the initial micro-consistent equilibrium data set.

Once the initial data set has been benchmarked, the modeller can move on to consider a counterfactual experiment. Suppose, for example, we choose to examine the effect of an increase in the world price of good i. Taking all of the parameters derived in the benchmarking process as given, the initial data set will of course no longer be consistent with full general equilibrium, since the price of good i has increased. Some sectors will be earning positive profits, and there will be an excess demand for those goods whose relative price has fallen. The numerical general equilibrium model will be solved to determine the new set of output supplies and output demands consistent with full general equilibrium, subject to the specification of technology implied by the benchmarking process. In a theoretical model, the result of this sort of experiment might be the conclusion that output supply curves slope upwards. In the numerical general equilibrium model, the physical increase in the supply of good i is derived subject to the modeller's description of technology summarized in the benchmarking procedure.

Of course, the numerical general equilibrium model can be made more or less complicated, depending upon the modeller's choice of functional forms to represent production and consumption technology. For example, implicit in the use of the Cobb-Douglas utility function above was that the substitution elasticity between goods in consumption was unity. The modeller could, for example, have chosen to use a CES utility function to represent the consumer's preferences. In that case the benchmarking process would have included the independent specification of the substitution elasticities in consumption.

It is important for our purposes to be able to specify different substitution elasticity matrices in the numerical general equilibrium model, so that we can independently specify parameters like the labour supply elasticity and the substitution elasticity between goods and the endogenously supplied factor For this reason we use the method of flexible functional forms described in Perroni and Rutherford (1989) to benchmark the initial data set to particular substitution matrices. To illustrate, consider the consumer's preferences over EXP, IMP, and LAB. We want to benchmark the initial data set to a particular substitution matrix, represented by equation (8.1):

$$\sigma_{con} = \begin{pmatrix} \sigma_{ee} & \sigma_{ei} & \sigma_{el} \\ \sigma_{ie} & \sigma_{ii} & \sigma_{il} \\ \sigma_{le} & \sigma_{li} & \sigma_{ll} \end{pmatrix}, \tag{8.1}$$

where σ_{jk} is the substitution elasticity in consumption between goods j and k, for $j,k = e,i,l$. The consumer's preferences are represented by two-level nested CES functions. Three aggregate consumption goods are combined in the top-level CES function, according to equation (8.2):

$$(8.2) \qquad U = \left\{\sum_{i=1}^{3} a_i AGG_i^{\frac{\sigma-1}{\sigma}}\right\}^{\frac{\sigma}{\sigma-1}}.$$

Each aggregate consumption good is a function of the three consumption goods EXP, IMP, and LAB, described by the lower-level CES function in equation (8.3):

$$(8.3) \qquad AGG_i = \{b_i EXP^{\frac{\sigma_i-1}{\sigma_i}} + c_i IMP^{\frac{\sigma_i-1}{\sigma_i}} + d_i LAB^{\frac{\sigma_i-1}{\sigma_i}}\}^{\frac{\sigma_i}{\sigma_i-1}} \qquad i = 1, ..., 3.$$

We define the share of each consumption good entering aggregate k, for $k = 1,2,3$, as s_{jk}, for $j = EXP,\ IMP,\ LAB$, where $\sum_{j=e,i,l} s_{jk} = 1,\ k = 1,2,3$. Then the desired substitution elasticity between goods j and k in the Slutsky matrix in equation (8.1), σ_{jk}, for $j,k = e,i,l$, can be expressed as a function of the share of each consumption good entering each aggregate good, the three lower-level substitution elasticities, and the single top-level substitution elasticity, according to equation (8.4):

$$(8.4) \qquad \sigma_{jk} = \sigma + \sum_{p=1}^{3}(\sigma_p - \sigma)\left(s_{jp}s_{kp} / \sum_{q=e,i,l} \theta_q s_{qp}\right),$$

where σ is the elasticity of substitution in equation (8.2), and σ_p are the substitution elasticities in equation (8.3), for $p = 1,2,3$. The term θ_q, for $q = EXP, IMP, LAB$, is the share of consumption of each good out of total income. There are generally more independent variables (the substitution elasticities σ, σ_p, and σ_{jk}, and the shares θ) than there are dependent variables (the shares s_{jp}), so that there is no unique way of recovering the shares from equation (8.4). We use the lower-triangular mapping described in Perroni and Rutherford (1989) to derive the shares s_{jk} consistent with the Slutsky matrix in equation (8.1). Using this method, the initial data set is benchmarked not to a particular functional form of the utility function but to a particular Slutsky substitution matrix. To illustrate, suppose the goods EXP and IMP are substitutes in consumption ($\sigma_{ei} > 0$) and labour and imports are to be complements in consumption ($\sigma_{il} < 0$). Equation (8.4) would imply that one aggregate (say $AGG1$) have a large share of IMP and LAB and a relatively low elasticity of substitution (σ_1 relatively small) compared to the top-level substitution elasticity. Then a fall in the price of IMP, for example, would result in a very small change in the IMP/LAB ratio in $AGG1$, since σ_1 is small. The share of IMP in $AGG1$ is relatively large, so consumption of $AGG1$ would tend to rise as the price of IMP fell. With low substitution in $AGG1$, consumption of both IMP and LAB would rise together, reflecting the complementarity of IMP and LAB in consumption.

Of course the matrix of substitution elasticities in equation (8.1) must satisfy a number of conditions. The demand functions must be homogeneous of degree zero in the price of EXP, IMP, and LAB, so that $\sum_i \theta_i \sigma_{ij} = 0$, for $j = E$, I, and L in equation (8.1), where θ_i is the share of consumption of good i out of total income, $i = E$, I, L. The substitution matrix must also be symmetric. As a result, in the three good case, there are three elements of the substitution matrix which are independent.

In order for the utility function to be well-behaved, the substitution matrix must be an element of the set of substitution matrices such that the utility function is "regularly flexible", as described in Perroni and Rutherford (1989), pp. 7-8. Three substitution matrices were chosen as "representative" of the consumers preferences, given by σ^1_{con}, σ^2_{con}, and σ^3_{con} in equation (8.5) below. The independently specified elements are underlined.

Notice that $\sigma_{el} = \sigma_{il}$ in each case. Once the leisure demand elasticity is specified, the elasticity of substitution between exports and leisure and between imports and leisure is implied by the expression $\theta_e \sigma_{el} + \theta_i \sigma_{il} + \theta_l \sigma_{ll} = 0$, subject to the constraint $\sigma_{el} = \sigma_{il}$. This constraint is imposed to reflect the fact that at the initial equilibrium, leisure and the two goods are separable in consumption.

$$\sigma^1_{con} = \begin{pmatrix} \sigma_{ee} & \sigma_{ei} & \sigma_{el} \\ \sigma_{ie} & \sigma_{ii} & \sigma_{il} \\ \sigma_{le} & \sigma_{li} & \sigma_{ll} \end{pmatrix} = \begin{pmatrix} -0.6980645 & 0.10 & 0.0667237 \\ 0.10 & -0.0764788 & 0.0667237 \\ 0.0667237 & 0.0667237 & -0.1 \end{pmatrix}$$

$$(8.5) \quad \sigma^2_{con} = \begin{pmatrix} \sigma_{ee} & \sigma_{ei} & \sigma_{el} \\ \sigma_{ie} & \sigma_{ii} & \sigma_{il} \\ \sigma_{le} & \sigma_{li} & \sigma_{ll} \end{pmatrix} = \begin{pmatrix} -1.8483466 & 0.3 & 0.1334474 \\ 0.3 & -0.1750711 & 0.1334474 \\ 0.1334474 & 0.1334474 & -0.2 \end{pmatrix}$$

$$\sigma^3_{con} = \begin{pmatrix} \sigma_{ee} & \sigma_{ei} & \sigma_{el} \\ \sigma_{ie} & \sigma_{ii} & \sigma_{il} \\ \sigma_{le} & \sigma_{li} & \sigma_{ll} \end{pmatrix} = \begin{pmatrix} -3.2444773 & 0.50 & 0.2668949 \\ 0.50 & -0.3280289 & 0.2668949 \\ 0.2668949 & 0.2668949 & -0.4 \end{pmatrix}.$$

The results described in Section 8.3 below for Case 1 through Case 3 correspond to these three respective substitution matrices. Case 0 corresponds to the case where the consumer cannot vary his consumption of labour. A survey of the literature on leisure demand elasticities in Martin and Neary (1980) yielded an average estimate of between -0.1 and -0.2. The value of $\sigma_{ll} = -0.4$ in case 3 is meant to serve as an example of a "high" leisure demand elasticity in consumption.

The same procedure is followed in benchmarking each production sector to a desired matrix of substitution elasticities. Each production function uses inputs of labour, capital, and intermediate inputs of the two produced goods. While this involves a departure from the assumption in the theoretical model that there were no intermediate inputs, it is necessary here due to the structure of the data set used. To represent substitutability in production, we follow the same process outlined above for the consumption sector. The production function for each good is summarized by a Cobb-Douglas function given in equation (8.6):

$$(8.6) \qquad X_i = \alpha_i \, EXP_i^{\beta_i^1} \, AGG_i^{\beta_i^2} \; \sum_{j=1}^{2} \beta_i^j = 1, \quad i = EXP, IMP \,,$$

where EXP_i is intermediate use of EXP in production of good i. The good AGG_i is a top-level CES aggregate given by equation (8.7):

$$(8.7)\quad AGG_i = \gamma_i\left\{\delta_i^1 AGG1_i^{-\rho_i} + \delta_i^2 AGG2_i^{-\rho_i} + \delta_i^3 AGG3_i^{-\rho_i}\right\}^{\frac{-1}{\rho_i}} \qquad i = EXP,\ IMP.$$

Each of the aggregates $AGG1_i$, $AGG2_i$, and $AGG3_i$ are functions of the remaining inputs LAB, CAP, and IMP, given by the lower-level CES function in equation (8.8):

$$(8.8)\quad AGGI_i = \gamma_{Ii}\left\{\delta_{Ii}^1 LAB^{-\rho_{Ii}} + \delta_{Ii}^2 CAP^{-\rho_{Ii}} + \delta_{Ii}^3 IMP^{-\rho_{Ii}}\right\}^{\frac{-1}{\rho_{Ii}}} \qquad I = 1,\ 2,\ 3.$$

In the same way as described above for the consumption sector, the substitution elasticities ρ_i, ρ_{1i}, ρ_{2i}, and ρ_{3i}, and the shares of LAB, CAP, and IMP in each of the aggregates $AGG1_i$, $AGG2_i$, and $AGG3_i$, are chosen so as to represent the substitution matrix for the production sector i, $i = EXP, IMP$.

Just as was the case in the consumption sector, the substitution matrices for the production sector must satisfy certain aggregation and regularity conditions. Representative elasticity matrices for each production sector are given in equations (8.9) and (8.10).

$$(8.9)\quad \sigma_{EXP} = \begin{pmatrix} \sigma_{ll} & \sigma_{lc} & \sigma_{li} \\ \sigma_{cl} & \sigma_{cc} & \sigma_{ci} \\ \sigma_{il} & \sigma_{ic} & \sigma_{ii} \end{pmatrix} = \begin{pmatrix} -1.3355956 & 1.0 & 0.5 \\ 1.0 & -2.8237299 & 0.5 \\ 0.5 & 0.5 & -0.5556228 \end{pmatrix}$$

$$(8.10)\quad \sigma_{IMP} = \begin{pmatrix} \sigma_{ll} & \sigma_{lc} & \sigma_{li} \\ \sigma_{cl} & \sigma_{cc} & \sigma_{ci} \\ \sigma_{il} & \sigma_{ic} & \sigma_{ii} \end{pmatrix} = \begin{pmatrix} -1.1031227 & 1.0 & 0.5 \\ 1.0 & -5.5568654 & 0.5 \\ 0.5 & 0.5 & -0.4220346 \end{pmatrix}.$$

These are the substitution matrices of each production sector used in generating the results in Section 8.3. Note that the substitution elasticity between capital and labour in each sector is equal to one.

Once the functional forms and the independent substitution elasticities have been specified, the remaining parameters (a_i, b_i, c_i, and d_i in the consumption sector and α_i, β_i, γ_i, and δ_i in the production sector) are calibrated, given the initial benchmark equilibrium data set, so that all the micro-consistency conditions are satisfied. The modelling system MPS/GE was used to do this initial calibration, and to generate all subsequent comparative statics results.[13]

8.3. Comparative Statics Results

The ultimate objective of this section is to generate comparative statics results analogous to those derived in the theoretical section of the paper. To this end, we consider the effect of an exogenous change in the world price of one of the produced goods, and examine the changes in output supplies, output demands, and net exports. We also examine the effect of an exogenous increase in the total endowment of capital in the economy upon the supply of outputs. These results are derived in the standard model where the total demand for leisure is held constant, and are repeated under different specifications of labour supply elasticities in the consumption sector. Results reported under "Case 0" are those for the standard model with fixed leisure demand, and those reported as "Case 1" through "Case 3" correspond to the three different matrices of substitution elasticities for the consumption sector given in equation (8.5) above.

There are certain characteristics of the comparative statics results generated using the numerical general equilibrium model which need to be pointed out. To begin with, results are generated for discrete changes of some exogenous variable. For example, output supply responses reported in Graph 1 in the Appendix, p. 141, are

[13] See "General Equilibrium Modelling with MPS/GE" (1989) by Thomas Rutherford for a complete description of the MPS/GE modelling system.

due to output price changes ranging from 0.5% to 5%. The results described in the theoretical sections were all changes due to differential changes in some independent variable, and were all derived using local analysis. This difference must be kept in mind when comparing results of the numerical general equilibrium model to the analogous results of the theoretical section.

It should also be noted that as the leisure demand elasticity is changed, for example, from -0.1 to -0.2, the preferences of the representative consumer are changing. The total endowment of the consumer and all world prices in the benchmark case are held constant as this leisure demand elasticity changes, and the model is benchmarked to the same initial data set for every different value of the leisure demand elasticity. Of course, we cannot change only the leisure demand elasticity, since then the substitution matrix would no longer satisfy the aggregation condition implied by the fact that consumption demand functions are homogeneous of degree zero in prices. As a result, all elements of the substitution matrices σ_{con} need to be adjusted as the leisure demand elasticity is changed.

We can now proceed to an examination of the various comparative statics results. The results are reported graphically in the Appendix, pp. 141-151. Consider first the change in output supply, output demand, and net exports, due to an increase in the relative price of the composite good EXP (see Graph 1 to Graph 6 in the Appendix, pp. 141-146). A subset of the graphical results are given in Table 8.2 for a 2.5% increase and in Table 8.3 for a 5% increase in the world price of EXP. As noted previously, Case 0 corresponds to the situation where leisure demand (and therefore total labour supply) is held constant, while Cases 1 through 3 correspond to the substitution matrices for a leisure demand elasticity equal to -0.1, -0.2, and -0.4, respectively (see equation (8.5) above).

Table 8.2: Effect of a 2.5% increase in World Price of EXP

	Production of EXP	Demand for EXP	Exports of EXP	Production of IMP	Demand for IMP	Imports of IMP
Case 0	2.09867	1.0165091	14.3983	0.601188	1.0247891	14.7579
Case 1	2.17059	1.0147700	15.3459	0.561931	1.0147700	15.7292
Case 2	2.20176	1.0081343	15.7801	0.544916	1.0118353	16.1742
Case 3	2.24843	0.9992464	16.4259	0.519444	1.0071775	16.8362

Table 8.3: Effect of a 5% increase in World Price of EXP

	Production of EXP	Demand for EXP	Exports of EXP	Production of IMP	Demand for IMP	Imports of IMP
Case 0	3.21316	1.0480312	27.9269	0.1761840	1.0652933	14.7579
Case 1	3.41166	1.0414400	30.5325	0.0685703	1.0414400	15.7292
Case 2	3.47900	1.0285478	31.4589	0.0320660	1.0359448	16.1742
Case 3	3.52483	1.0114604	32.1590	0.0000013	1.0271699	16.8362

In all cases, the initial values before the world price change are equal to unity, so that, for example, in case 0, demand for the exported good rises by 1.65% due to a 2.5% increase in the world price of EXP. The production possibility frontier is very flat in the neighbourhood of the initial equilibrium, so that small price changes cause correspondingly large output supply changes.

Of the two produced goods, EXP is relatively capital-intensive. In all cases, an increase in the world price of exports causes the return to labour to fall. This results in a decrease in the price of leisure. Whenever the leisure demand elasticity is non-zero (i.e. in Cases 1-3), the increase in the world price of exports causes leisure demand to rise (see Graph 7 in the Appendix, p. 147). Output supply and net exports of EXP respond positively to the increase in the relativc price of EXP, and output supply and net exports of IMP fall due to the decrease in the relative price of IMP. Total labour supplied to the production sector falls by more as the leisure demand elasticity rises, for any given change in the world price of exports. As a result, output of exports rises by more and output of imports falls by more as the leisure demand elasticity rises, for any given increase in the world price of exports. Higher leisure demand elasticities have the same effect on net exports. Net exports of the exported good rise by more and net exports of the imported good fall by more the higher is the leisure demand elasticity. These results are all consistent with the theoretical results derived in Section 3 above, where we showed that when the consumer's preferences could be represented by our additively separable utility function, output supply and net export responses to a given terms-of-trade shock are larger when some factors are endogenously supplied.

The effect of variable labour supply on output demand is also consistent with the results derived in Section 3 above. As the leisure demand elasticity gets larger, the substitution elasticity between goods and leisure gets larger, so that demand increases by smaller amounts as the leisure demand elasticity rises. In fact, in one case the qualitative change in output demand is changed due to an increase in the leisure demand elasticity. In Cases 0-2, an increase in the price of exports always leads to an increase in export demand. However, in Case 3, when the leisure demand elasticity is highest, export demand falls for increases in the price of exports between 0.5% and 2.5%.

In all cases, both goods continue to be produced as the world price of exports rises. As a result, there is a unique change in input prices implied by the Stolper-Samuelson relationship between input and output prices, since the small open economy always remains in the initial cone of diversification. The exported good is relatively capital-intensive, so that a given increase in the world price of EXP always implies the same increase in the return to capital and the same decrease in the return to labour, independent of the leisure demand elasticity.

Now consider the effect of endogenous factor supply on the Rybczynski effect. We examine the effect of an increase in the endowment of capital of .5% to 5%, and report the changes in output of exports and imports in Graph 8 and Graph 9 in the Appendix, pp. 148-149. Results are given in Table 8.4, for a 2.5% and a 5% increase in the endowment of capital.

Table 8.4:

	Effect of a 2.5% increase in Endowment of CAP		Effect of a 5% increase in Endowment of CAP	
	Production of EXP	Production of IMP	Production of EXP	Production of IMP
Case 0	1.14849	0.957193	1.29709	0.914358
Case 1	1.17484	0.942733	1.34977	0.885435
Case 2	1.17484	0.942733	1.34977	0.885435
Case 3	1.17484	0.942733	1.34977	0.885435

In all cases, both goods continue to be produced, so since output prices are held constant, the returns to labour and capital do not change. The only exogenous change is to total fixed factor income, due to the change in endowment. We find that results are also consistent with those found in Section 6 above. That is, the increase in output of the capital-intensive good (EXP) is larger and the decrease in output of the labour-intensive good (IMP) is larger when leisure demand is allowed to adjust to the endowment change. There is no change as the leisure demand elasticity is changed from -0.1 to -0.2 to -0.4 since output prices and input prices are held constant. The only effect on leisure demand is an income effect, which is independent of the leisure demand elasticity.

We now modify the numerical general equilibrium model to illustrate the effects of the presence of variable factor supply on optimal tariffs. We include a trading partner called the rest-of-the-world (*ROW*) in the model, and assume that *ROW* is ten times the size of Canada. This assumption is intended to reflect the fact that Canada is roughly one tenth the size of its dominant trading partner, the U.S. We derive optimal tariffs in this simple two-good, two-factor model by finding the import tariff which yields the highest level of utility in Canada, given zero trade taxes in *ROW*. Results are given in Graph 10 in the Appendix, p.150. The experiment is conducted for case 0 and case 3, so that the curve labelled case 0 in Graph 10 corresponds to the level of utility in Canada for different levels of the domestic import tariff when labour supply is held constant, while the curve labelled case 3 gives those results when the domestic leisure demand elasticity is -0.4, using the Slutsky matrix σ^3_{con} in equation (8.4) above. The graph indic[illegible] that the optimal tariff is lower (approximately 3.8%) when the leisure demand elasticity is -0.4 in case 3, compared to the case when leisure demand is held constant in case 0 (approximately 6.2%). This result is due entirely to the fact that leisure demand is variable in *CAN*, since the specification of *ROW* is identical in case 0 and case 3. The fact that leisure demand is variable results in higher net export elasticities in Canada, which in turn implies that the optimal tariff for Canada is lower. This is consistent with the theoretical result derived in Chapter 7.

The final example illustrates the effect of variable labour supply on optimal trade taxes in a small open economy when there exists a tax on the endogenously supplied factor. The model is re-calibrated so that there exists a tax on labour supply of 15.1% at the initial benchmark equilibrium. This figure was chosen based upon income tax payments reported in the income tax tables published by Statistics Canada for 1982. Optimal tariffs in the small open economy are derived in the case where leisure demand is held fixed, and in the case where leisure demand is variable and the leisure demand elasticity is -0.4. Results are reported in Graph 11 in the Appendix, p.151.

We find that the optimal tariff in case 0 when labour supply is held fixed is equal to zero, as it should since with fixed labour supply, a factor tax in this model should have no effect, so that the optimal tariff for the small open economy should still be zero. In case 3 when the leisure demand elasticity is -0.4, the optimal tariff is marginally higher at 0.3%. The tariff in each case is lower that the 4.6% tariff in the initial data set, which implies that the price of the labour-intensive good IMP is relatively lower when the optimal tariff is charged. Therefore, the return to labour falls as the import tariff falls, so that in case 3, more leisure is consumed and less labour is supplied at the optimal tariff. The effect of variable labour supply on the level of the optimal tariff is relatively small since the tax on labour supply is relatively small. The distortionary effect of the tax on the endogenously supplied factor is relatively small.

9. CONCLUSION

By using duality theory to incorporate the presence of endogenous factor supply in a general equilibrium model of international trade, we have demonstrated conditions under which output supply, output demand, and net export responses to a given terms-of-trade shock in a small open economy are magnified when some factors are endogenously supplied. It was also shown that the ability of consumers to change the level of supply of a factor of production affects the probability that the equalization of output prices between different economies leads to the equalization of factor prices. A given change in endowments in an economy will cause a change in domestic income, which will cause a change in demand for an endogenously supplied factor by the consumer. Ultimately a nation's endowment vector must be viewed as being endogenously determined. Even in a model where the number of factors equals the number of goods, output supply changes due to an exogenous shock can no longer be determined independently of the consumption sector.

The model was modified to incorporate the presence of trade taxes and factor taxes. We find conditions under which a given trade tax change leads to larger changes in domestic welfare in a small open economy when some factors are endogenously supplied. We also find conditions under which optimal tariffs in a large economy with some market power are lower. The model also enables us to consider the effects of taxes on factors. Without variable factors, factor taxes can have no effect on output supply. We find that in the neighbourhood of zero taxes, taxes on factors cause an unambiguous fall in factor supply and welfare. We also provide an argument for a second-best trade tax when there exists taxes on factors.

A simple two-good, two-factor numerical general equilibrium model is constructed and used to evaluate the effects of variable factor supply as the representative consumer's elasticity of demand for the variable factor changes. We consider the effects of a change in the world price of outputs and the endowment of a factor in fixed supply. The resulting changes in factor supply have effects on output supply, output demand, and net exports which are consistent with those derived in the theoretical model. A given world price shock has larger effects on output supply and net exports as the factor supply elasticity is increased. Output demand changes are dampened when the factor supply elasticity is increased since the representative consumer substitutes consumption of the factor for output consumption. The Rybczynski effect of a change in the endowment of the factor in fixed supply is magnified when the supply of the other factor is variable. We also conduct a simple optimal tariff experiment for a large economy with some market power and show that the optimal tariff is lower when one factor is endogenously supplied.

10. APPENDIX

The modified GNP function $G(p, w, \bar{v})$ is non-increasing and convex in w:

Proof: If $w \geq \tilde{w}$, then

$$G(p, \tilde{w}, \bar{v}) = p'\tilde{y} - \tilde{w}'\tilde{v}$$

$$\geq p'y - \tilde{w}'v$$

$$w \geq \tilde{w} \Rightarrow w'v \geq \tilde{w}'v$$

$$\Rightarrow p'y - \tilde{w}'v \geq p'y - w'v = G(p, w, \bar{v})$$

so that: $w \geq \tilde{w} \Rightarrow G(p, \tilde{w}, \bar{v}) \geq G(p, w, \bar{v})$, so $G(p, w, \bar{v})$ is non-increasing in w.

$$G(p, w, \bar{v}) = p'y - w'v$$

$$G(p, \tilde{w}, \bar{v}) = p'\tilde{y} - \tilde{w}'\tilde{v}$$

$$G(p, \theta w + (1-\theta)\tilde{w}, \bar{v}) = p'\hat{y} - \hat{w}'\hat{v} \quad \text{where } \hat{w} = \theta w + (1-\theta)\tilde{w}$$

Then for all $\theta \in (0, 1)$:

$$\theta G(p, w, \bar{v}) = \theta p'y - \theta w'v \geq \theta p'\hat{y} - \theta w'\hat{v}$$

$$(1-\theta)G(p, \tilde{w}, \bar{v}) = (1-\theta)p'\tilde{y} - (1-\theta)\tilde{w}'\tilde{v} \geq (1-\theta)p'\hat{y} - (1-\theta)\tilde{w}'\hat{v}$$

$$\theta G(p, w, \bar{v}) + (1-\theta)G(p, \tilde{w}, \bar{v}) \geq p'\hat{y} - (\theta w'\hat{v} + (1-\theta)\tilde{w}'\hat{v})$$

$$= G(p, \theta w + (1-\theta)\tilde{w}, \bar{v})$$

which implies that $G(p, w, \bar{v})$ is convex in w. ∎

The expenditure function $E(p, w, \mu)$ is non-increasing and concave in w:

Proof: If $w \geq \tilde{w}$, then

$$E(p, w, \mu) = p'z - w'v$$

$$\leq p'\tilde{z} - w'\tilde{v}$$

$$w \geq \tilde{w} \Rightarrow w'\tilde{v} \geq \tilde{w}'\tilde{v}$$

$$\Rightarrow p'\tilde{z} - w'\tilde{v} \leq p'\tilde{z} - \tilde{w}'\tilde{v} = E(p, \tilde{w}, \mu)$$

so that: $w \geq \tilde{w} \Rightarrow E(p, w, \mu) \leq E(p, \tilde{w}, \mu)$, so $E(p, w, \mu)$ is non-increasing in w.

$$\begin{aligned} E(p, w, \mu) &= p'z - w'v \\ E(p, \tilde{w}, \mu) &= p'\tilde{z} - \tilde{w}'\tilde{v} \\ E(p, \theta w + (1-\theta)\tilde{w}, \mu) &= p'\hat{z} - \hat{w}'\hat{v} \quad \text{where } \hat{w} = \theta w + (1-\theta)\tilde{w} \end{aligned}$$

Then for all $\theta \in (0, 1)$:

$$\begin{aligned} \theta E(p, w, \mu) &= \theta p'z - \theta w'v \leq \theta p'\hat{z} - \theta w'\hat{v} \\ (1-\theta)E(p, \tilde{w}, \mu) &= (1-\theta)p'\tilde{z} - (1-\theta)\tilde{w}'\tilde{v} \leq (1-\theta)p'\hat{z} - (1-\theta)\tilde{w}'\hat{v} \end{aligned}$$

$$\begin{aligned} \theta E(p, w, \mu) + (1-\theta)E(p, \tilde{w}, \mu) &\leq p'\hat{z} - (\theta w'\hat{v} + (1-\theta)\tilde{w}'\hat{v}) \\ &= E(p, \theta w + (1-\theta)\tilde{w}, \mu) \end{aligned}$$

which implies that $E(p, w, \mu)$ is concave in w. ∎

The indirect utility function $V(p, w, m_f)$ is non-decreasing, quasi-convex in w:

Proof: (This proof follows that of Varian, 1984, p.121)

$$\begin{aligned} \text{Define the sets} \quad B &\equiv \{(z, v) \mid p'z - w'v \leq m_f\} \\ \tilde{B} &\equiv \{(z, v) \mid p'z - \tilde{w}'v \leq m_f\} \end{aligned}$$

for $\tilde{w} \geq w$. Then B is contained in $\tilde{B}$. Then the maximum of $U(z, v)$ over the set $\tilde{B}$ is as least as large as that over B, and thus $\tilde{w} \geq w$ implies $V(p, \tilde{w}, m_f) \geq V(p, w, m_f)$.

$$\begin{aligned} \text{Now let} \quad V(p, w, m_f) &\leq k \\ V(p, \tilde{w}, m_f) &\leq k \end{aligned}$$

for some scalar k. We want to show that, for any $\theta \in (0,1)$, $V(p,\hat{w},m_f) \leq k$, where $\hat{w} = \theta w + (1-\theta)\tilde{w}$.

$$\begin{aligned} \text{Define the sets } B &\equiv \{(z,v) \mid p'z - w'v \leq m_f\} \\ \tilde{B} &\equiv \{(z,v) \mid p'z - \tilde{w}'v \leq m_f\} \\ \hat{B} &\equiv \{(z,v) \mid p'z - \hat{w}'v \leq m_f\} \end{aligned}$$

We want to show that any $(z,v) \in \hat{B}$ must be an element of either B or $\tilde{B}$. Suppose not: Then $\exists\ (z,v) \in \hat{B}$ such that:

$$p'z - (\theta w'v + (1-\theta)\tilde{w}'v) \leq m_f \quad \text{and}$$

$$p'z - w'v > m_f \quad \text{or} \quad \theta p'z - \theta w'v > \theta m_f$$

$$p'z - \tilde{w}'v > m_f \quad \text{or} \quad (1-\theta)p'z - (1-\theta)\tilde{w}'v > (1-\theta)m_f$$

Summing the last two expressions yields $p'z - (\theta w'v + (1-\theta)\tilde{w}'v) > m_f$, which contradicts. Thus we can state the following:

$$\begin{aligned} V(p,\hat{w},m_f) &= \max U(z.v) \ \text{ subject to } \ (z,v) \in \hat{B} \\ &\leq \max U(z,v) \ \text{ subject to } \ (z,v) \in B \text{ or } (z,v) \in \tilde{B} \text{ since } \hat{B} \in B \cup \tilde{B} \\ &\leq k \end{aligned}$$

Thus $V(p,w,m_f)$ is quasi-convex in w. ∎

Let B and C be positive definite matrices. Then $A = B + C$ is also positive definite, and:

$$x'[B+C]^{-1}x = x'A^{-1}x < x'B^{-1}x \qquad \text{for all vectors } x.$$

Proof: (This proof follows that of Graybill, 1983, pp.409-411)

$A - B = C$ is a positive definite matrix. Let Q be a non-singular matrix such that :

$$Q'[A-B]Q = Q'AQ - Q'BQ = I.$$

Let P be an orthogonal matrix which diagonalizes $Q'AQ$:

$$P'[Q'AQ]P = D_1; \qquad P' = P^{-1}.$$

$$\begin{aligned} D_2 &= P'[Q'BQ]P = P'[Q'AQ - I]P \\ &= D_1 - I. \end{aligned}$$

Then D_2 is also a diagonal matrix.

$$\begin{aligned} D_1 - D_2 = I &\Rightarrow d^1_{ii} > d^2_{ii} \quad i = 1, \ldots, n \\ &\Rightarrow \frac{1}{d^2_{ii}} > \frac{1}{d^1_{ii}} \quad i = 1, \ldots, n. \end{aligned}$$

Hence $D_2^{-1} - D_1^{-1} = D_3$ is a positive definite matrix.

$$\begin{aligned} D_3 &= [P'Q'BQP]^{-1} - [P'Q'AQP]^{-1} \\ &= P'Q^{-1}B(Q^{-1})'P - P'Q^{-1}A(Q^{-1})'P \end{aligned}$$

Define $M = P'Q'$. Then since $D_2^{-1} - D_1^{-1}$ is positive definite, $M'[D_2^{-1} - D_1^{-1}]M$ is also positive definite.

$$\begin{aligned} M'[D_2^{-1} - D_1^{-1}]M &= QPP'Q^{-1}B^{-1}(Q^{-1})'PP'Q' - QPP'Q^{-1}A^{-1}(Q^{-1})'PP'Q' \\ &= B^{-1} - A^{-1}. \end{aligned}$$

As a result, $x'[B^{-1} - A^{-1}]x > 0$ for all vectors x, so:

$$x'B^{-1}x > x'A^{-1}x = x'[B + C]^{-1}x. \qquad \blacksquare$$

The symmetric version of the matrix ab', $a, b \in R^m$, is indefinite for $m \geq 2$.

Proof:

The second principal minor of the matrix ab' is:

$$\begin{vmatrix} a_1b_1 & a_1b_2 \\ a_2b_1 & a_2b_2 \end{vmatrix} = \begin{vmatrix} q & r \\ s & t \end{vmatrix} = 0$$

The second principal minor of the symmetrix version of the matrix ab' is:

$$\begin{vmatrix} a_1b_1 & \frac{a_1b_2+a_2b_1}{2} \\ \frac{a_1b_2+a_2b_1}{2} & a_2b_2 \end{vmatrix} = \begin{vmatrix} q & u \\ u & t \end{vmatrix} = 0$$

A necessary condition for the symmetric version of ab' to be either positive or negative definite or semi-definite is that $qt - u^2 \geq 0$.

$$\begin{aligned} qt - u^2 &= qt - (\frac{r+s}{2})^2 \\ &= qt - (\frac{r^2}{4} + \frac{rs}{2} + \frac{s^2}{4}) \\ &= qt - rs - (\frac{r^2}{4} - \frac{rs}{2} + \frac{s^2}{4}) \\ &= -(\frac{r}{2} - \frac{s}{2})^2 < 0 \quad \blacksquare \end{aligned}$$

COST MINIMIZATION

$$\min_{K_x, K_y, L_x, L_y} \quad w(L_x + L_y) + r(K_x + K_y)$$

$$\text{s.t.:} \quad 1 = K_x^{\alpha_x} L_x^{\beta_x} \qquad \alpha_x + \beta_x = 1$$

$$1 = K_y^{\alpha_y} L_y^{\beta_y} \qquad \alpha_y + \beta_y = 1$$

f.o.c.

$$\begin{aligned} r - \lambda_1 \alpha_x K_x^{\alpha_x - 1} L_x^{\beta_x} &= 0 \\ w - \lambda_1 \beta_x K_x^{\alpha_x} L_x^{\beta_x - 1} &= 0 \\ r - \lambda_2 \alpha_y K_y^{\alpha_y - 1} L_y^{\beta_y} &= 0 \\ w - \lambda_2 \beta_y K_y^{\alpha_y} L_y^{\beta_y - 1} &= 0 \\ 1 - K_x^{\alpha_x} L_x &= 0 \\ 1 - K_y^{\alpha_y} L_y^{\beta_y} &= 0 \end{aligned}$$

We can rearrange the first two first-order conditions to get:

$$\frac{r}{w} = \frac{\alpha_x}{\beta_x} \frac{L_x}{K_x} \quad \Rightarrow \quad L_x = \frac{r}{w} \frac{\beta_x}{\alpha_x} K_x$$

Substituting into the production function for good x:

$$K_x^{\alpha_x} \left(\frac{r}{w} \frac{\beta_x}{\alpha_x} K_x \right)^{\beta_x} = 1$$

$$K_x^{\alpha_x+\beta_x} \left(\frac{r}{w} \right)^{\beta_x} \left(\frac{\beta_x}{\alpha_x} \right)^{\beta_x} = 1$$

Therefore: $$K_x = \left(\frac{\alpha_x}{\beta_x} \frac{w}{r} \right)^{\beta_x}$$

Likewise we can solve for the remaining input demand functions:

$K_x = \left(\frac{\alpha_x}{\beta_x} \frac{w}{r} \right)^{\beta_x}$ $\qquad$ $L_x = \left(\frac{\beta_x}{\alpha_x} \frac{r}{w} \right)^{\alpha_x}$

$K_y = \left(\frac{\alpha_y}{\beta_y} \frac{w}{r} \right)^{\beta_y}$ $\qquad$ $L_y = \left(\frac{\beta_y}{\alpha_y} \frac{r}{w} \right)^{\alpha_y}$

STOLPER-SAMUELSON

$K_x = \left(\frac{\alpha_x}{\beta_x} \frac{w}{r} \right)^{\beta_x}$ $\qquad$ $L_x = \left(\frac{\beta_x}{\alpha_x} \frac{r}{w} \right)^{\alpha_x}$

Since: $\quad p_x \cdot x = wL_x + rK_x \quad$ and $\quad x = 1$,

$$\begin{aligned}
p_x &= w \left(\frac{\beta_x}{\alpha_x} \frac{r}{w} \right)^{\alpha_x} + r \left(\frac{\alpha_x}{\beta_x} \frac{w}{r} \right)^{\beta_x} \\
&= w^{\beta_x} r^{\alpha_x} \left(\frac{\beta_x}{\alpha_x} \right)^{\alpha_x} + r^{\alpha_x} w^{\beta_x} \left(\frac{\alpha_x}{\beta_x} \right)^{\beta_x} \\
&= w^{\beta_x} r^{\alpha_x} \left[\left(\frac{\beta_x}{\alpha_x} \right)^{\alpha_x} + \left(\frac{\beta_x}{\alpha_x} \right)^{-\beta_x} \right] \\
&= w^{\beta_x} r^{\alpha_x} \left(\frac{\beta_x}{\alpha_x} \right)^{\alpha_x} \left[1 + \left(\frac{\beta_x}{\alpha_x} \right)^{-\beta_x-\alpha_x} \right] \\
&= w^{\beta_x} r^{\alpha_x} \left(\frac{\beta_x}{\alpha_x} \right)^{\alpha_x} \left(\frac{\beta_x+\alpha_x}{\beta_x} \right)
\end{aligned}$$

Therefore: $\quad p_x = w^{\beta_x} r^{\alpha_x} \alpha_x^{-\alpha_x} \beta_x^{-\beta_x}$

Similarly: $\quad p_y = w^{\beta_y} r^{\alpha_y} \alpha_y^{-\alpha_y} \beta_y^{-\beta_y}$

Take logs and rearrange to get:

$$\begin{aligned}\beta_x lnw + \alpha_x lnr &= lnp_x + \alpha_x ln\alpha_x + \beta_x ln\beta_x \\ \beta_y lnw + \alpha_y lnr &= lnp_y + \alpha_y ln\alpha_y + \beta_y ln\beta_y\end{aligned}$$

$$\begin{pmatrix} \beta_x & \alpha_x \\ \beta_y & \alpha_y \end{pmatrix}\begin{pmatrix} lnw \\ lnr \end{pmatrix} = \begin{pmatrix} lnp_x + \alpha_x ln\alpha_x + \beta_x ln\beta_x \\ lnp_y + \alpha_y ln\alpha_y + \beta_y ln\beta_y \end{pmatrix}$$

Let $\alpha_y\beta_x - \alpha_x\beta_y = q$. Therefore:

$$\begin{pmatrix} lnw \\ lnr \end{pmatrix} = \frac{1}{q}\begin{pmatrix} \alpha_y & -\alpha_x \\ -\beta_y & \beta_x \end{pmatrix}\begin{pmatrix} lnp_x + \alpha_x ln\alpha_x + \beta_x ln\beta_x \\ lnp_y + \alpha_y ln\alpha_y + \beta_y ln\beta_y \end{pmatrix}$$

$$\begin{pmatrix} lnw \\ lnr \end{pmatrix} = \frac{1}{q}\begin{pmatrix} \alpha_y lnp_x - \alpha_x lnp_y + \alpha_x\alpha_y(ln\alpha_x - ln\alpha_y) + \alpha_y\beta_x ln\beta_x - \alpha_x\beta_y ln\beta_y \\ -\beta_y lnp_x + \beta_x lnp_y - \beta_y\alpha_x ln\alpha_x + \beta_x\alpha_y ln\alpha_y + \beta_x\beta_y(-ln\beta_x + ln\beta_y) \end{pmatrix}$$

$$\begin{pmatrix} w \\ r \end{pmatrix} = \begin{pmatrix} p_x^{\frac{\alpha_y}{q}} p_y^{\frac{-\alpha_x}{q}} \left(\frac{\alpha_x}{\alpha_y}\right)^{\frac{\alpha_x\alpha_y}{q}} \beta_x^{\frac{\alpha_y\beta_x}{q}} \beta_y^{\frac{-\alpha_x\beta_y}{q}} \\ p_x^{\frac{-\beta_y}{q}} p_y^{\frac{\beta_x}{q}} \alpha_x^{\frac{-\beta_y\alpha_x}{q}} \alpha_y^{\frac{\beta_x\alpha_y}{q}} \left(\frac{\beta_x}{\beta_y}\right)^{\frac{-\beta_x\beta_y}{q}} \end{pmatrix}$$

$$\begin{pmatrix} \hat{w} \\ \hat{r} \end{pmatrix} = \begin{pmatrix} \frac{\alpha_y}{q}\hat{p}_x - \frac{\alpha_x}{q}\hat{p}_y \\ \frac{-\beta_y}{q}\hat{p}_x + \frac{\beta_x}{q}\hat{p}_y \end{pmatrix}$$

Note that we can rewrite q as follows:

$$\begin{aligned} q &= \alpha_y\beta_x - \alpha_x\beta_y \\ &= \frac{\beta_x\beta_y}{\beta_x\beta_y}(\alpha_y\beta_x - \alpha_x\beta_y) \\ &= \beta_x\beta_y\left(\frac{\alpha_y}{\beta_y} - \frac{\alpha_x}{\beta_x}\right)\end{aligned}$$

If good x is relatively labour-intensive, good y is relatively capital-intensive, then $\frac{\alpha_y}{\beta_y} > \frac{\alpha_x}{\beta_x}$, and $q > 0$. Then an increase in the price of x holding p_y constant causes the return to labour to rise ($\hat{w} = \frac{\alpha_y}{q}\hat{p}_x$) and the return to capital to fall ($\hat{r} = \frac{-\beta_y}{q}\hat{p}_x$).

OUTPUT SUPPLY

The producers' revenue maximization problem is posed as follows:

$$\max_{x,y} \quad p_x \cdot x + p_y \cdot y \qquad \text{s.t.:} \qquad \begin{aligned} x &= K_x^{\alpha_x} L_x^{\beta_x} & \alpha_x + \beta_x = 1 \\ y &= K_y^{\alpha_y} L_y^{\beta_y} & \alpha_y + \beta_y = 1 \\ \bar{K} &= K_x + K_y & \\ L &= L_x + L_y & \end{aligned}$$

where $\bar{K}$ is the fixed endowment of capital, and L is the supply of labour from the consumption sector which the production sector takes as given.

From the solution to the producer's cost minimization problem:

$$K_x = \left(\frac{\alpha_x}{\beta_x} \frac{w}{r} \right)^{\beta_x} \qquad K_y = \left(\frac{\alpha_y}{\beta_y} \frac{w}{r} \right)^{\beta_y}$$

$$L_x = \left(\frac{\beta_x}{\alpha_x} \frac{r}{w} \right)^{\alpha_x} \qquad L_y = \left(\frac{\beta_y}{\alpha_y} \frac{r}{w} \right)^{\alpha_y}$$

As long as both goods are produced, input prices are given as a unique function of output prices, according to the Siolper-Samuelson results:

$$\begin{pmatrix} w \\ r \end{pmatrix} = \begin{pmatrix} p_x^{\frac{\alpha_y}{q}} p_y^{\frac{-\alpha_x}{q}} \left(\frac{\alpha_x}{\alpha_y} \right)^{\frac{\alpha_x \alpha_y}{q}} \beta_x^{\frac{\alpha_y \beta_x}{q}} \beta_y^{\frac{-\alpha_x \beta_y}{q}} \\ p_x^{\frac{-\beta_y}{q}} p_y^{\frac{\beta_x}{q}} \alpha_x^{\frac{-\beta_y \alpha_x}{q}} \alpha_y^{\frac{\beta_x \alpha_y}{q}} \left(\frac{\beta_x}{\beta_y} \right)^{\frac{-\beta_x \beta_y}{q}} \end{pmatrix}$$

$$\frac{w}{r} = p_x^{\frac{\alpha_y + \beta_y}{q}} p_y^{\frac{-\alpha_x - \beta_x}{q}} \alpha_x^{\frac{\alpha_x \alpha_y + \beta_y \alpha_x}{q}} \alpha_y^{\frac{-\alpha_x \alpha_y - \beta_x \alpha_y}{q}} \beta_x^{\frac{\alpha_y \beta_x + \beta_x \beta_y}{q}} \beta_y^{\frac{-\alpha_x \beta_y - \beta_x \beta_y}{q}}$$

$$\frac{w}{r} = p_x^{\frac{1}{q}} p_y^{\frac{-1}{q}} \alpha_x^{\frac{\alpha_x}{q}} \alpha_y^{\frac{-\alpha_y}{q}} \beta_x^{\frac{\beta_x}{q}} \beta_y^{\frac{-\beta_y}{q}}$$

Substituting for the wage/rent ratio in the output demand functions:

$$K_x = \left(\frac{\alpha_x}{\beta_x}\right)^{\beta_x}\left(\frac{p_x}{p_y}\right)^{\frac{\beta_x}{q}} \alpha_x^{\frac{\alpha_x\beta_x}{q}} \alpha_y^{\frac{-\alpha_y\beta_x}{q}} \beta_x^{\frac{\beta_x\beta_x}{q}} \beta_y^{\frac{-\beta_x\beta_y}{q}}$$

$$K_y = \left(\frac{\alpha_y}{\beta_y}\right)^{\beta_y}\left(\frac{p_x}{p_y}\right)^{\frac{\beta_y}{q}} \alpha_x^{\frac{\alpha_x\beta_y}{q}} \alpha_y^{\frac{-\alpha_y\beta_y}{q}} \beta_x^{\frac{\beta_x\beta_y}{q}} \beta_y^{\frac{-\beta_y\beta_y}{q}}$$

$$L_x = \left(\frac{\beta_x}{\alpha_x}\right)^{\alpha_x}\left(\frac{p_x}{p_y}\right)^{\frac{-\alpha_x}{q}} \alpha_x^{\frac{-\alpha_x\alpha_x}{q}} \alpha_y^{\frac{\alpha_x\alpha_y}{q}} \beta_x^{\frac{-\alpha_x\beta_x}{q}} \beta_y^{\frac{\alpha_x\beta_y}{q}}$$

$$L_y = \left(\frac{\beta_y}{\alpha_y}\right)^{\alpha_y}\left(\frac{p_x}{p_y}\right)^{\frac{-\alpha_y}{q}} \alpha_x^{\frac{-\alpha_x\alpha_y}{q}} \alpha_y^{\frac{\alpha_y\alpha_y}{q}} \beta_x^{\frac{-\alpha_y\beta_x}{q}} \beta_y^{\frac{\alpha_y\beta_y}{q}}$$

Collecting terms, we get:

$$K_x = \left(\frac{p_x}{p_y}\right)^{\frac{\beta_x}{q}} \alpha_x^{\frac{\beta_x q+\alpha_x\beta_x}{q}} \alpha_y^{\frac{-\alpha_y\beta_x}{q}} \beta_x^{\frac{-\beta_x q+\beta_x\beta_x}{q}} \beta_y^{\frac{-\beta_x\beta_y}{q}}$$

$$K_y = \left(\frac{p_x}{p_y}\right)^{\frac{\beta_y}{q}} \alpha_x^{\frac{\alpha_x\beta_y}{q}} \alpha_y^{\frac{\beta_y q-\alpha_y\beta_y}{q}} \beta_x^{\frac{\beta_x\beta_y}{q}} \beta_y^{\frac{-\beta_y q-\beta_y\beta_y}{q}}$$

$$L_x = \left(\frac{p_x}{p_y}\right)^{\frac{-\alpha_x}{q}} \alpha_x^{\frac{-\alpha_x q-\alpha_x\alpha_x}{q}} \alpha_y^{\frac{\alpha_x\alpha_y}{q}} \beta_x^{\frac{\alpha_x q-\alpha_x\beta_x}{q}} \beta_y^{\frac{\alpha_x\beta_y}{q}}$$

$$L_y = \left(\frac{p_x}{p_y}\right)^{\frac{-\alpha_y}{q}} \alpha_x^{\frac{-\alpha_x\alpha_y}{q}} \alpha_y^{\frac{-\alpha_y q+\alpha_y\alpha_y}{q}} \beta_x^{\frac{-\alpha_y\beta_x}{q}} \beta_y^{\frac{\alpha_y q+\alpha_y\beta_y}{q}}$$

We can simplify the exponents to show that:

$$K_x = \left(\frac{p_x}{p_y}\right)^{\frac{\beta_x}{q}} \left(\frac{\alpha_x}{\alpha_y}\right)^{\frac{\alpha_y\beta_x}{q}} \left(\frac{\beta_x}{\beta_y}\right)^{\frac{\beta_x\beta_y}{q}}$$

$$K_y = \left(\frac{p_x}{p_y}\right)^{\frac{\beta_y}{q}} \left(\frac{\alpha_x}{\alpha_y}\right)^{\frac{\alpha_x\beta_y}{q}} \left(\frac{\beta_x}{\beta_y}\right)^{\frac{\beta_x\beta_y}{q}}$$

$$L_x = \left(\frac{p_x}{p_y}\right)^{\frac{-\alpha_x}{q}} \left(\frac{\alpha_x}{\alpha_y}\right)^{\frac{-\alpha_x\alpha_y}{q}} \left(\frac{\beta_x}{\beta_y}\right)^{\frac{-\alpha_x\beta_y}{q}}$$

$$L_y = \left(\frac{p_x}{p_y}\right)^{\frac{-\alpha_y}{q}} \left(\frac{\alpha_x}{\alpha_y}\right)^{\frac{-\alpha_x\alpha_y}{q}} \left(\frac{\beta_x}{\beta_y}\right)^{\frac{-\alpha_y\beta_x}{q}}$$

Using full employment conditions:

$$\begin{pmatrix} K_x & K_y \\ L_x & L_y \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \bar{K} \\ L \end{pmatrix}$$

Inverting full employment conditions:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{|A|} \begin{pmatrix} L_y & -K_y \\ -L_x & K_x \end{pmatrix} \begin{pmatrix} \bar{K} \\ L \end{pmatrix}$$

$$= \frac{1}{|A|} \begin{pmatrix} L_y\bar{K} - K_yL \\ -L_x\bar{K} + K_xL \end{pmatrix}$$

$$= \frac{1}{|A|} \begin{pmatrix} (\frac{p_x}{p_y})^{\frac{-\alpha_y}{q}} (\frac{\alpha_x}{\alpha_y})^{\frac{-\alpha_x\alpha_y}{q}} (\frac{\beta_x}{\beta_y})^{\frac{-\alpha_y\beta_x}{q}} \cdot \bar{K} - (\frac{p_x}{p_y})^{\frac{\beta_y}{q}} (\frac{\alpha_x}{\alpha_y})^{\frac{\alpha_x\beta_y}{q}} (\frac{\beta_x}{\beta_y})^{\frac{\beta_x\beta_y}{q}} \cdot L \\ -(\frac{p_x}{p_y})^{\frac{-\alpha_x}{q}} (\frac{\alpha_x}{\alpha_y})^{\frac{-\alpha_x\alpha_y}{q}} (\frac{\beta_x}{\beta_y})^{\frac{-\alpha_x\beta_y}{q}} \cdot \bar{K} + (\frac{p_x}{p_y})^{\frac{\beta_x}{q}} (\frac{\alpha_x}{\alpha_y})^{\frac{\alpha_y\beta_x}{q}} (\frac{\beta_x}{\beta_y})^{\frac{\beta_x\beta_y}{q}} \cdot L \end{pmatrix}$$

where:

$$|A| = K_xL_y - K_yL_x$$

$$= (\frac{p_x}{p_y})^{\frac{\beta_x-\alpha_y}{q}} (\frac{\alpha_x}{\alpha_y})^{\frac{\alpha_y\beta_x-\alpha_x\alpha_y}{q}} (\frac{\beta_x}{\beta_y})^{\frac{\beta_x\beta_y-\alpha_y\beta_x}{q}} - (\frac{p_x}{p_y})^{\frac{\beta_y-\alpha_x}{q}} (\frac{\alpha_x}{\alpha_y})^{\frac{\alpha_x\beta_y-\alpha_x\alpha_y}{q}} (\frac{\beta_x}{\beta_y})^{\frac{\beta_x\beta_y-\alpha_x\beta_y}{q}}$$

$$= a_1a_2(\frac{p_x}{p_y})^{\frac{\beta_x-\alpha_y}{q}} - a_3a_4(\frac{p_x}{p_y})^{\frac{\beta_y-\alpha_x}{q}}$$

$$= (\frac{p_x}{p_y})^{\frac{\beta_x-\alpha_y}{q}} \left[a_1a_2 - a_3a_4(\frac{p_x}{p_y})^{\frac{\beta_y-\alpha_x-\beta_x+\alpha_y}{q}}\right]$$

$$= (\frac{p_x}{p_y})^{\frac{\beta_x-\alpha_y}{q}} [a_1a_2 - a_3a_4]$$

Solving for $a_1a_2 - a_3a_4$ gives:

$$a_1a_2 - a_3a_4 = (\frac{\alpha_x}{\alpha_y})^{\frac{\alpha_y\beta_x-\alpha_x\alpha_y}{q}} (\frac{\beta_x}{\beta_y})^{\frac{\beta_x\beta_y-\alpha_y\beta_x}{q}} - (\frac{\alpha_x}{\alpha_y})^{\frac{\alpha_x\beta_y-\alpha_x\alpha_y}{q}} (\frac{\beta_x}{\beta_y})^{\frac{\beta_x\beta_y-\alpha_x\beta_y}{q}}$$

$$= (\frac{\alpha_x}{\alpha_y})^{\frac{\alpha_y\beta_x-\alpha_x\alpha_y}{q}} \left[(\frac{\beta_x}{\beta_y})^{\frac{\beta_x\beta_y-\alpha_y\beta_x}{q}} - (\frac{\alpha_x}{\alpha_y})^{\frac{-\alpha_y\beta_x+\alpha_x\beta_y}{q}} (\frac{\beta_x}{\beta_y})^{\frac{\beta_x\beta_y-\alpha_x\beta_y}{q}}\right]$$

$$= (\frac{\alpha_x}{\alpha_y})^{\frac{\alpha_y\beta_x-\alpha_x\alpha_y}{q}} (\frac{\beta_x}{\beta_y})^{\frac{\beta_x\beta_y-\alpha_y\beta_x}{q}} \left[1 - (\frac{\alpha_x}{\alpha_y})^{-1} (\frac{\beta_x}{\beta_y})^{\frac{\alpha_y\beta_x-\alpha_x\beta_y}{q}}\right]$$

$$= (\frac{\alpha_x}{\alpha_y})^{\frac{\alpha_y\beta_x-\alpha_x\alpha_y}{q}} (\frac{\beta_x}{\beta_y})^{\frac{\beta_x\beta_y-\alpha_y\beta_x}{q}} (1 - \frac{\alpha_y}{\alpha_x} \frac{\beta_x}{\beta_y})$$

$$= (\frac{\alpha_x}{\alpha_y})^{\frac{\alpha_y\beta_x-\alpha_x\alpha_y}{q}} (\frac{\beta_x}{\beta_y})^{\frac{\beta_x\beta_y-\alpha_y\beta_x}{q}} (\frac{-q}{\alpha_x\beta_y})$$

Therefore: $|A| = (\frac{-q}{\alpha_x\beta_y})(\frac{\alpha_x}{\alpha_y})^{\frac{\alpha_y\beta_x-\alpha_x\alpha_y}{q}} (\frac{\beta_x}{\beta_y})^{\frac{\beta_x\beta_y-\alpha_y\beta_x}{q}} (\frac{p_x}{p_y})^{\frac{\beta_x-\alpha_y}{q}}$

Substituting for $\frac{1}{|A|}$ in the output supply functions gives:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \left(\frac{-\alpha_x\beta_y}{q} \right) \left(\frac{\alpha_x}{\alpha_y} \right)^{\frac{\alpha_x\alpha_y - \alpha_y\beta_x}{q}} \left(\frac{\beta_x}{\beta_y} \right)^{\frac{\alpha_y\beta_x - \beta_x\beta_y}{q}}$$

$$\cdot \begin{pmatrix} \left(\frac{p_x}{p_y} \right)^{\frac{-\beta_x}{q}} \left(\frac{\alpha_x}{\alpha_y} \right)^{\frac{-\alpha_x\alpha_y}{q}} \left(\frac{\beta_x}{\beta_y} \right)^{\frac{-\alpha_y\beta_x}{q}} \cdot \bar{K} - \left(\frac{p_x}{p_y} \right)^{\frac{\alpha_x}{q}} \left(\frac{\alpha_x}{\alpha_y} \right)^{\frac{\alpha_x\beta_y}{q}} \left(\frac{\beta_x}{\beta_y} \right)^{\frac{\beta_x\beta_y}{q}} \cdot L \\ -\left(\frac{p_x}{p_y} \right)^{\frac{-\beta_y}{q}} \left(\frac{\alpha_x}{\alpha_y} \right)^{\frac{-\alpha_x\alpha_y}{q}} \left(\frac{\beta_x}{\beta_y} \right)^{\frac{-\alpha_x\beta_y}{q}} \cdot \bar{K} + \left(\frac{p_x}{p_y} \right)^{\frac{\alpha_y}{q}} \left(\frac{\alpha_x}{\alpha_y} \right)^{\frac{\alpha_y\beta_x}{q}} \left(\frac{\beta_x}{\beta_y} \right)^{\frac{\beta_x\beta_y}{q}} \cdot L \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \left(\frac{-\alpha_x\beta_y}{q} \right)$$

$$\cdot \begin{pmatrix} \left(\frac{p_x}{p_y} \right)^{\frac{-\beta_x}{q}} \left(\frac{\alpha_x}{\alpha_y} \right)^{\frac{-\alpha_y\beta_x}{q}} \left(\frac{\beta_x}{\beta_y} \right)^{\frac{-\beta_x\beta_y}{q}} \cdot \bar{K} - \left(\frac{p_x}{p_y} \right)^{\frac{\alpha_x}{q}} \left(\frac{\alpha_x}{\alpha_y} \right)^{\frac{\alpha_x\alpha_y + \alpha_x\beta_y - \alpha_y\beta_x}{q}} \left(\frac{\beta_x}{\beta_y} \right)^{\frac{\alpha_y\beta_x}{q}} \cdot L \\ -\left(\frac{p_x}{p_y} \right)^{\frac{-\beta_y}{q}} \left(\frac{\alpha_x}{\alpha_y} \right)^{\frac{-\alpha_y\beta_x}{q}} \left(\frac{\beta_x}{\beta_y} \right)^{\frac{\alpha_y\beta_x - \beta_x\beta_y - \alpha_x\beta_y}{q}} \cdot \bar{K} + \left(\frac{p_x}{p_y} \right)^{\frac{\alpha_y}{q}} \left(\frac{\alpha_x}{\alpha_y} \right)^{\frac{\alpha_x\alpha_y}{q}} \left(\frac{\beta_x}{\beta_y} \right)^{\frac{\alpha_y\beta_x}{q}} \cdot L \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \left(\frac{-\alpha_x\beta_y}{q} \right)$$

$$\cdot \begin{pmatrix} \left(\frac{p_x}{p_y} \right)^{\frac{-\beta_x}{q}} \left(\frac{\alpha_x}{\alpha_y} \right)^{\frac{-\alpha_y\beta_x}{q}} \left(\frac{\beta_x}{\beta_y} \right)^{\frac{-\beta_x\beta_y}{q}} \cdot \bar{K} - \left(\frac{p_x}{p_y} \right)^{\frac{\alpha_x}{q}} \left(\frac{\alpha_x}{{}_y} \right)^{\frac{\alpha_x - \alpha_y\beta_x}{q}} \left(\frac{\beta_x}{\beta_y} \right)^{\frac{\alpha_y\beta_x}{q}} \cdot L \\ -\left(\frac{p_x}{p_y} \right)^{\frac{-\beta_y}{q}} \left(\frac{\alpha_x}{\alpha_y} \right)^{\frac{-\alpha_y\beta_x}{q}} \left(\frac{\beta_x}{\beta_y} \right)^{\frac{\alpha_y\beta_x - \beta_y}{q}} \cdot \bar{K} + \left(\frac{p_x}{p_y} \right)^{\frac{\alpha_y}{q}} \left(\frac{\alpha_x}{\alpha_y} \right)^{\frac{\alpha_x\alpha_y}{q}} \left(\frac{\beta_x}{\beta_y} \right)^{\frac{\alpha_y\beta_x}{q}} \cdot L \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \left(\frac{-\alpha_x\beta_y}{q} \right) \left(\frac{\alpha_x}{\alpha_y} \right)^{\frac{-\alpha_y\beta_x}{q}} \left(\frac{\beta_x}{\beta_y} \right)^{\frac{\alpha_y\beta_x}{q}} \begin{pmatrix} \left(\frac{\beta_x}{\beta_y} \frac{p_x}{p_y} \right)^{\frac{-\beta_x}{q}} \cdot \bar{K} - \left(\frac{\alpha_x}{\alpha_y} \frac{p_x}{p_y} \right)^{\frac{\alpha_x}{q}} \cdot L \\ -\left(\frac{\beta_x}{\beta_y} \frac{p_x}{p_y} \right)^{\frac{-\beta_y}{q}} \cdot \bar{K} + \left(\frac{\alpha_x}{\alpha_y} \frac{p_x}{p_y} \right)^{\frac{\alpha_y}{q}} \cdot L \end{pmatrix}$$

OUTPUT DEMAND

$$\max_{x,y,L} \quad U(x,\ y\ ,L) \ = \ x^a y^b (\bar{L} - L)^c \qquad a + b + c = 1$$

$$\text{s.t.} \quad p_x \cdot x + p_y \cdot y \leq wL + rK$$

$$Z \ = \ x^a y^b (\bar{L} - L)^c + \lambda \left(wL + rK - p_x \cdot x - p_y \cdot y \right)$$

f.o.c.

$$\begin{aligned}
ax^{a-1}y^b(\bar{L}-L)^c - \lambda p_x &= 0 \\
bx^a y^{b-1}(\bar{L}-L)^c - \lambda p_y &= 0 \\
-cx^a y^b(\bar{L}-L)^{c-1} + \lambda w &= 0 \\
wL + r\bar{K} - p_x \cdot x - p_y \cdot y &= 0
\end{aligned}$$

Therefore:

$$\begin{aligned}
\tfrac{ax^a y^b(\bar{L}-L)^c}{x} &= \lambda p_x & \Rightarrow \quad ax^a y^b(\bar{L}-L)^c &= \lambda p_x \cdot x \\
\tfrac{bx^a y^b(\bar{L}-L)^c}{y} &= \lambda p_y & \Rightarrow \quad bx^a y^b(\bar{L}-L)^c &= \lambda p_y \cdot y \\
\tfrac{-cx^a y^b(\bar{L}-L)^c}{\bar{L}-L} &= -\lambda w & \Rightarrow \quad -cx^a y^b(\bar{L}-L)^c &= -\lambda w(\bar{L}-L)
\end{aligned}$$

$$wL + r\bar{K} - p_x \cdot x - p_y \cdot y = 0$$

Substituting out for λ gives:

$$p_x \cdot x = (\tfrac{a}{c})\, w(\bar{L}-L)$$

$$p_y \cdot y = (\tfrac{b}{c})\, w(\bar{L}-L)$$

$$\Rightarrow \qquad wL + r\bar{K} - (\tfrac{a}{c})\, w(\bar{L}-L) - (\tfrac{b}{c})\, w(\bar{L}-L) = 0$$

$$\Rightarrow \qquad cwL + cr\bar{K} - aw\bar{L} + awL - bw\bar{L} + bwL = 0$$

$$\Rightarrow \qquad wL(a+b+c) = (a+b)w\bar{L} - cr\bar{K}$$

Add and subtract $cw\bar{L}$ to the right-hand side to get:

$$wL(a+b+c) = (a+b+c)w\bar{L} - c(w\bar{L} + r\bar{K})$$

$$\Rightarrow \qquad L = \bar{L} - (\tfrac{c}{a+b+c})(\tfrac{w\bar{L}+r\bar{K}}{w})$$

$$\Rightarrow \qquad wL = w\bar{L} - (\tfrac{c}{a+b+c})(w\bar{L} + r\bar{K})$$

From the budget constraint, we can write:

$$w\bar{L} - (\tfrac{c}{a+b+c})(w\bar{L} + r\bar{K}) + r\bar{K} - p_x \cdot x - p_y \cdot y = 0$$

$$\Rightarrow \qquad (\tfrac{a+b}{a+b+c})(w\bar{L} + r\bar{K}) - p_x \cdot x - p_y \cdot y = 0$$

We have already solved for the leisure demand function as:

$$(\bar{L} - L) = (\tfrac{c}{a+b+c})(\tfrac{w\bar{L}+r\bar{K}}{w})$$

We also showed above that:

$$\begin{aligned}
p_x \cdot x &= \tfrac{a}{c} w(\bar{L} - L) \\
&= (\tfrac{a}{a+b+c})(w\bar{L} + r\bar{K}) \\
x &= (\tfrac{a}{a+b+c})(\tfrac{w\bar{L}+r\bar{K}}{p_x}) \\
p_y \cdot y &= \tfrac{b}{c} w(\bar{L} - L) \\
&= (\tfrac{b}{a+b+c})(w\bar{L} + r\bar{K}) \\
y &= (\tfrac{b}{a+b+c})(\tfrac{w\bar{L}+r\bar{K}}{p_y})
\end{aligned}$$

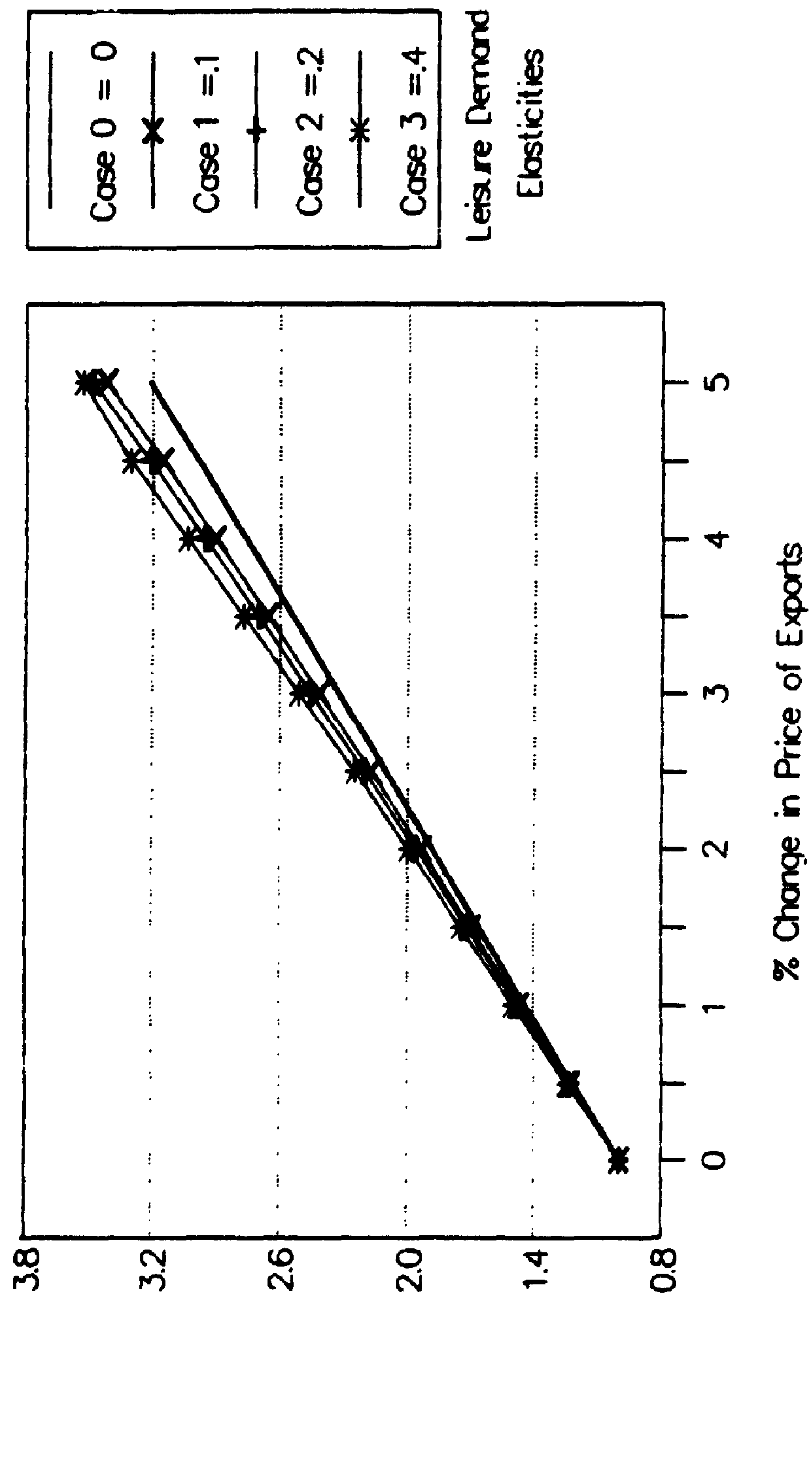
Therefore:

$$\begin{pmatrix} x \\ y \\ L \end{pmatrix} = \begin{pmatrix} (\tfrac{a}{p_x})(w\bar{L} + m_f) \\ (\tfrac{b}{p_y})(w\bar{L} + m_f) \\ (\tfrac{-c}{w})(w\bar{L} + m_f) + \bar{L} \end{pmatrix}$$

Since $a + b + c = 1$, $\quad r\bar{K} = m_f = p_x \cdot x + p_y \cdot y - wL.$

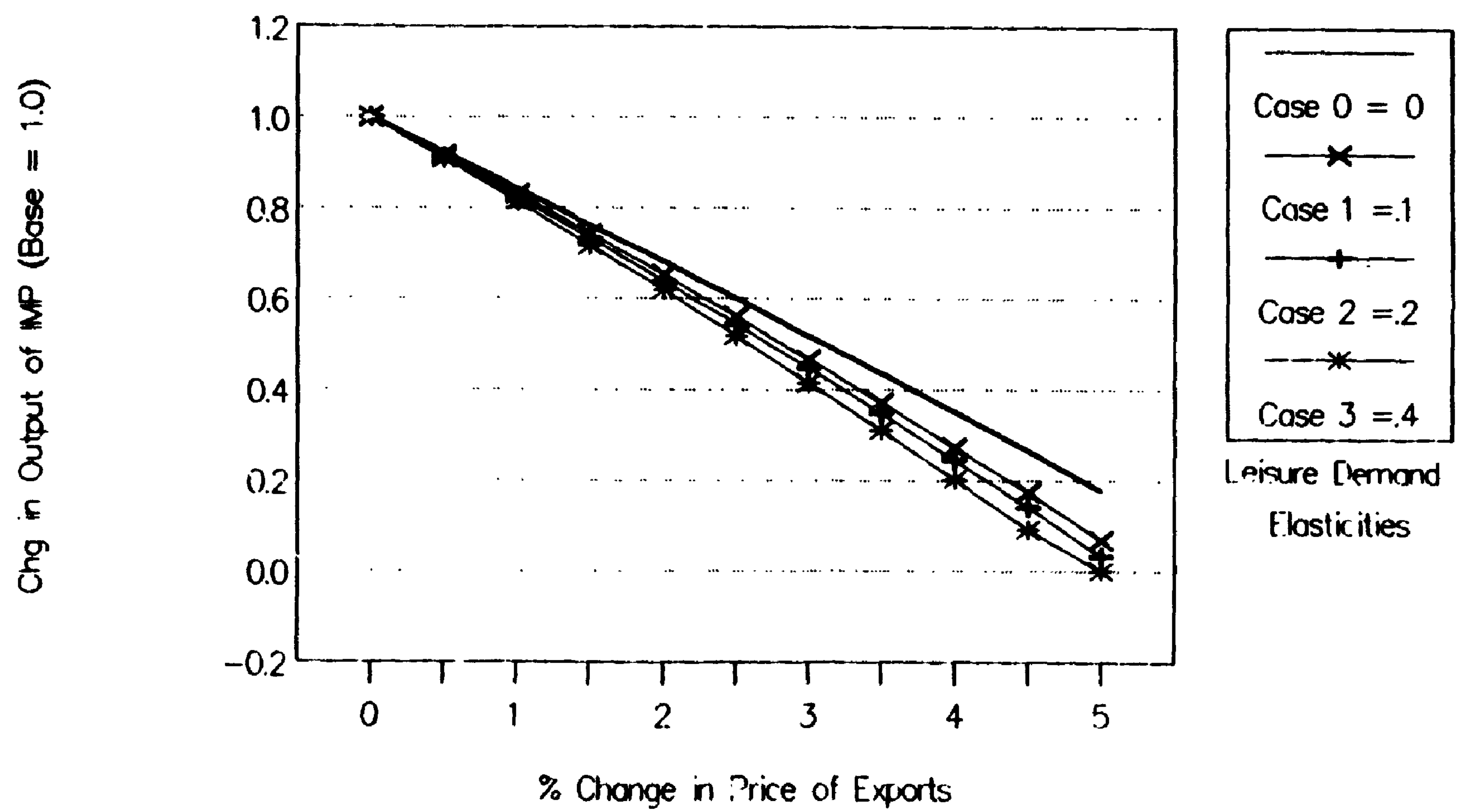
GRAPH 1

Response in Production of Exports to Change in World Relative Price

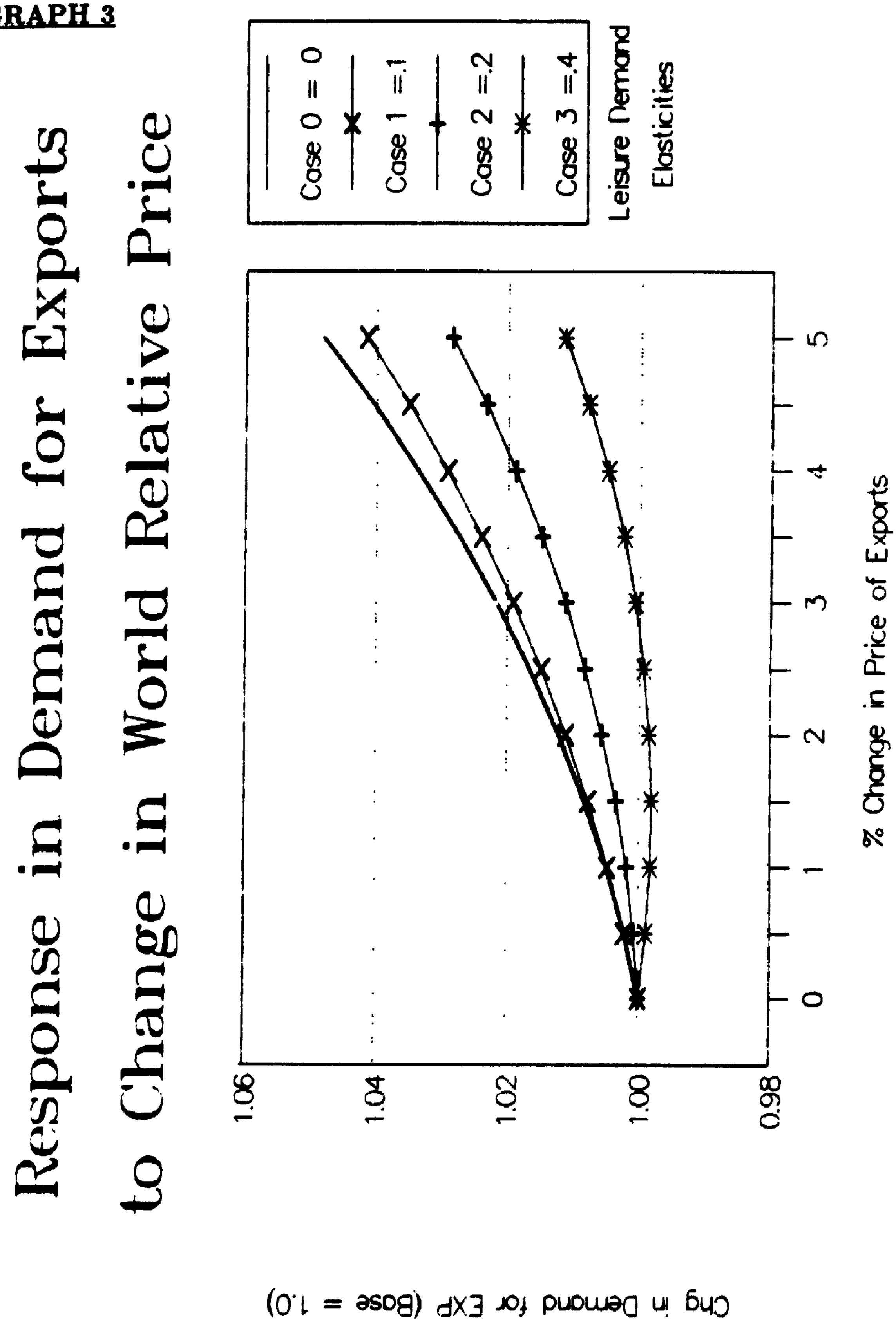


GRAPH 2

Response in Production of Imports to Change in World Relative Price

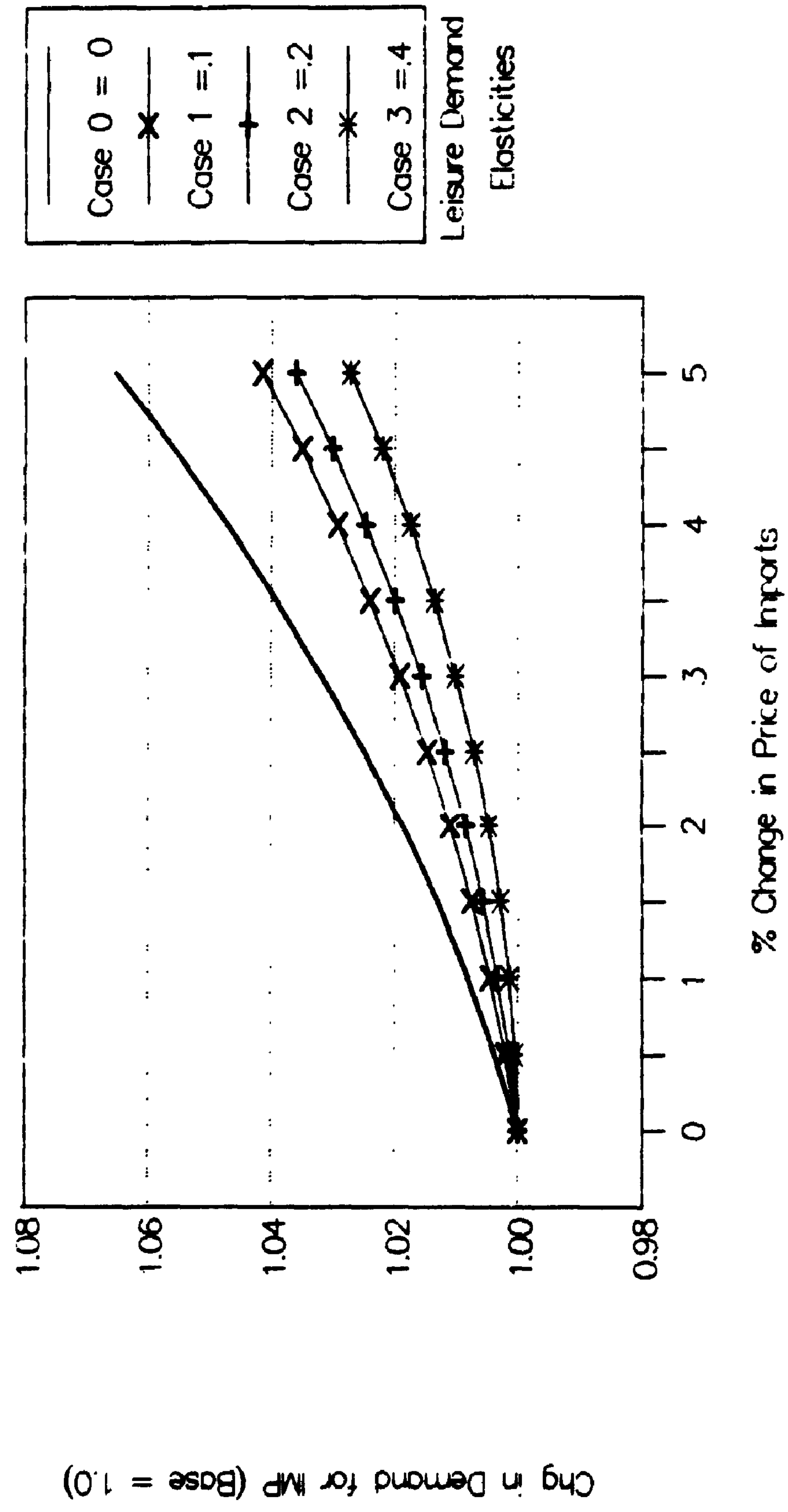


GRAPH 3


Response in Demand for Exports
to Change in World Relative Price
Case 0 = 0
Case 1 =.1
Case 2 =.2
Case 3 =.4
Leisure Demand Elasticities
1.06
1.04
1.02
1.00
0.98
0
1
2
3
4
5
% Change in Price of Exports
Chg in Demand for EXP (Base = 1.0)

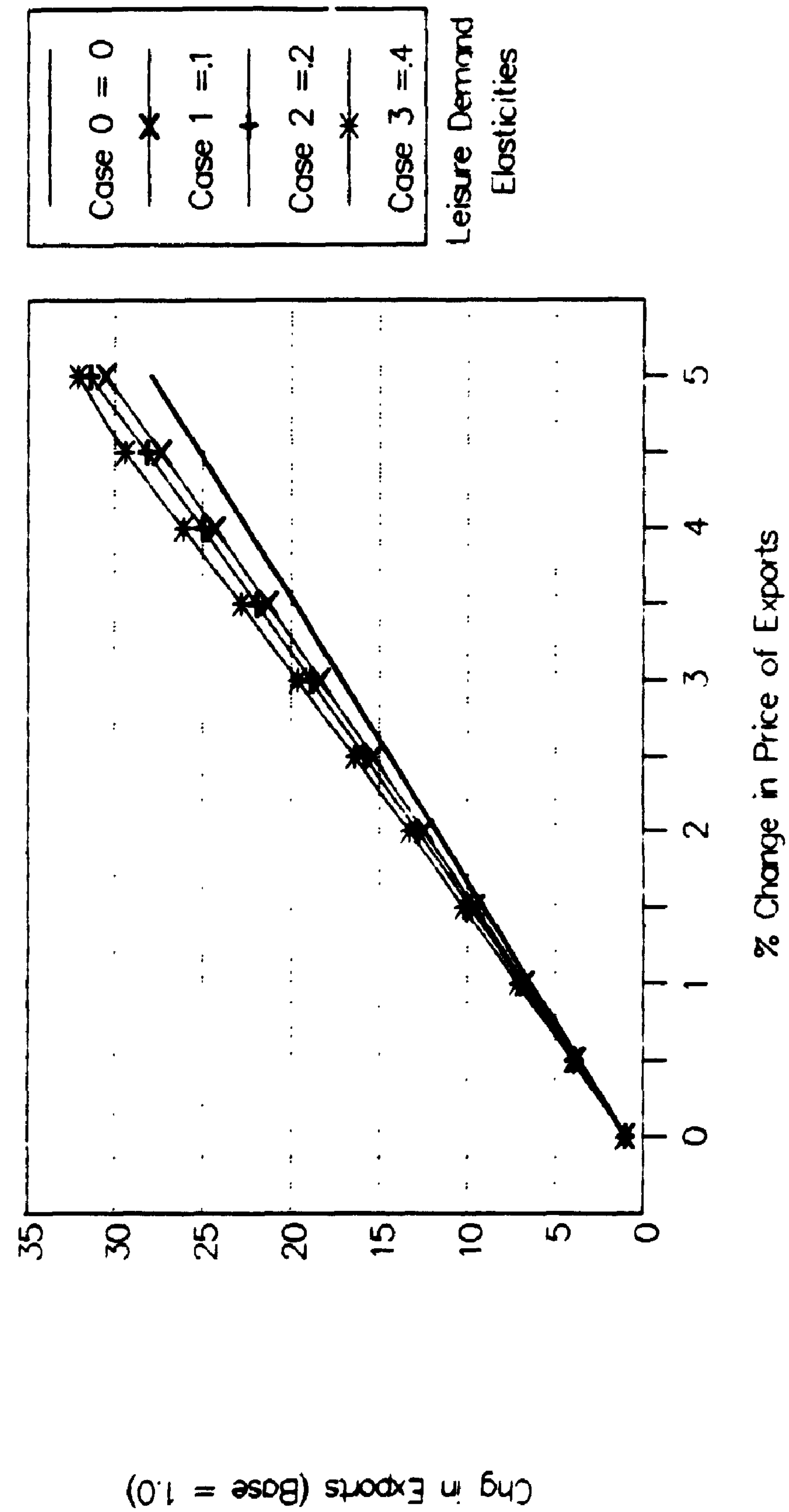
GRAPH 4

Response in Demand for Imports to Change in World Relative Price



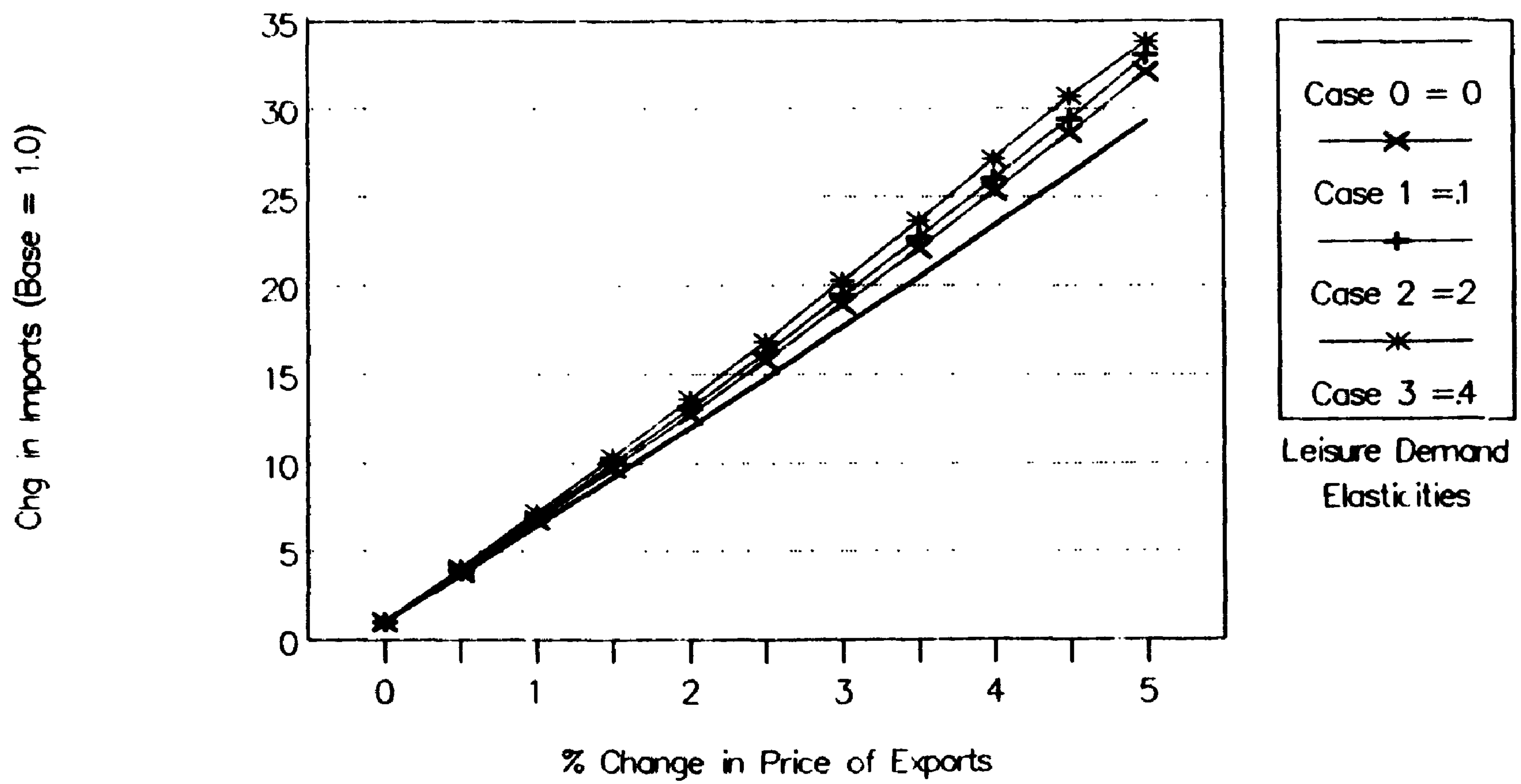
GRAPH 5

Response in Exports to Change in World Relative Price



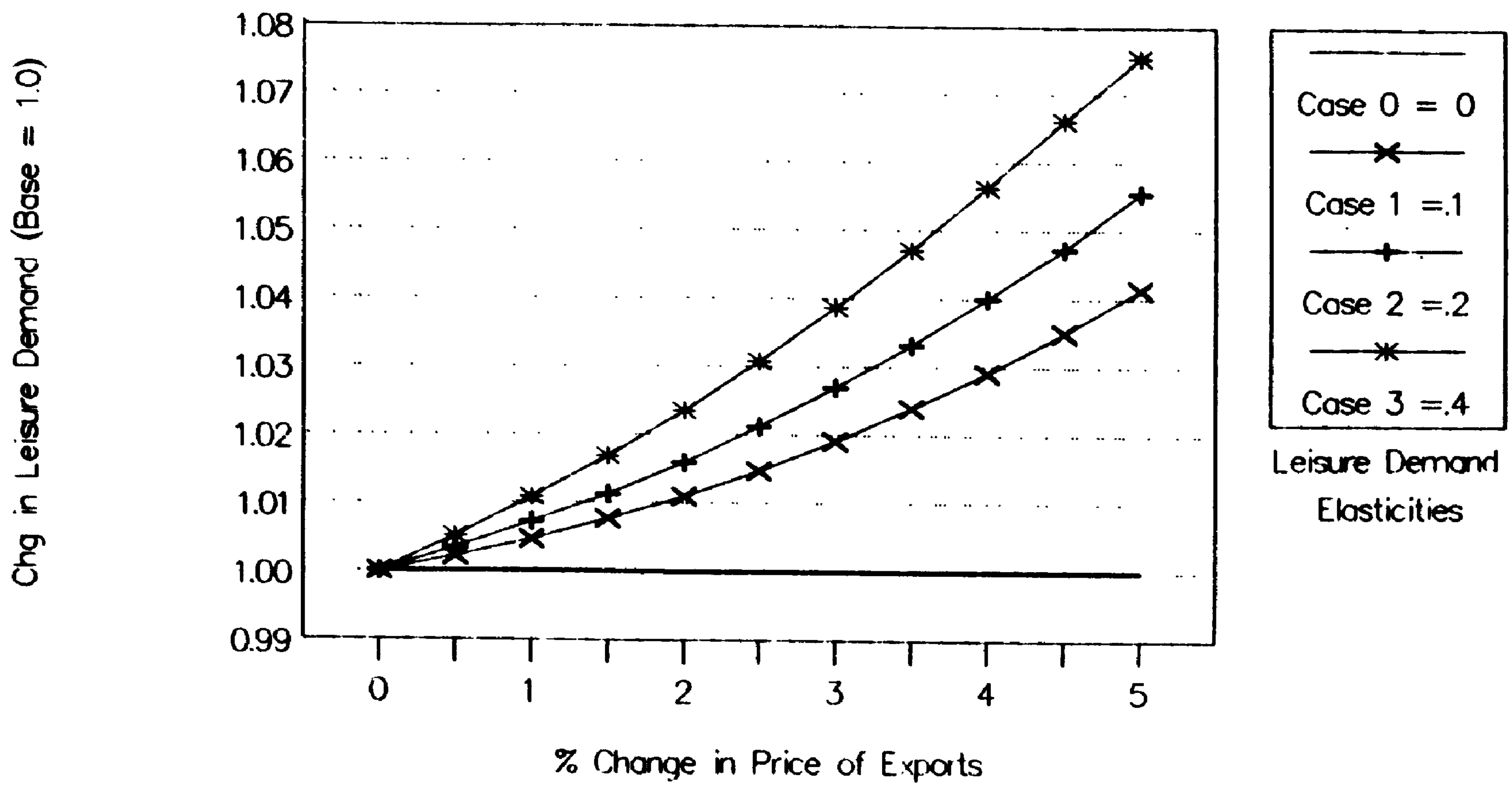
GRAPH 6

Response in Imports to Change in World Relative Price



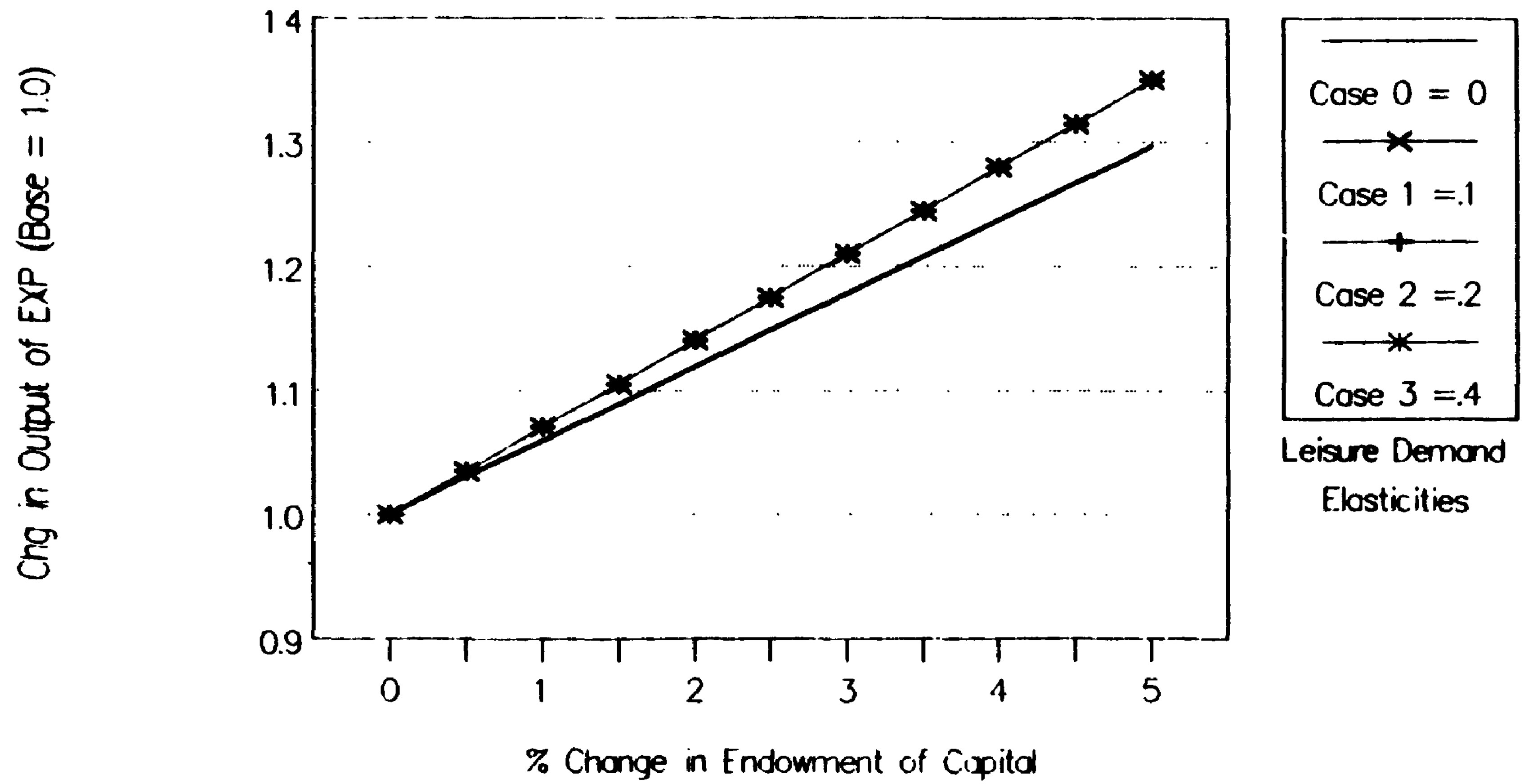
GRAPH 7

Response in Leisure Demand to Change in World Relative Price



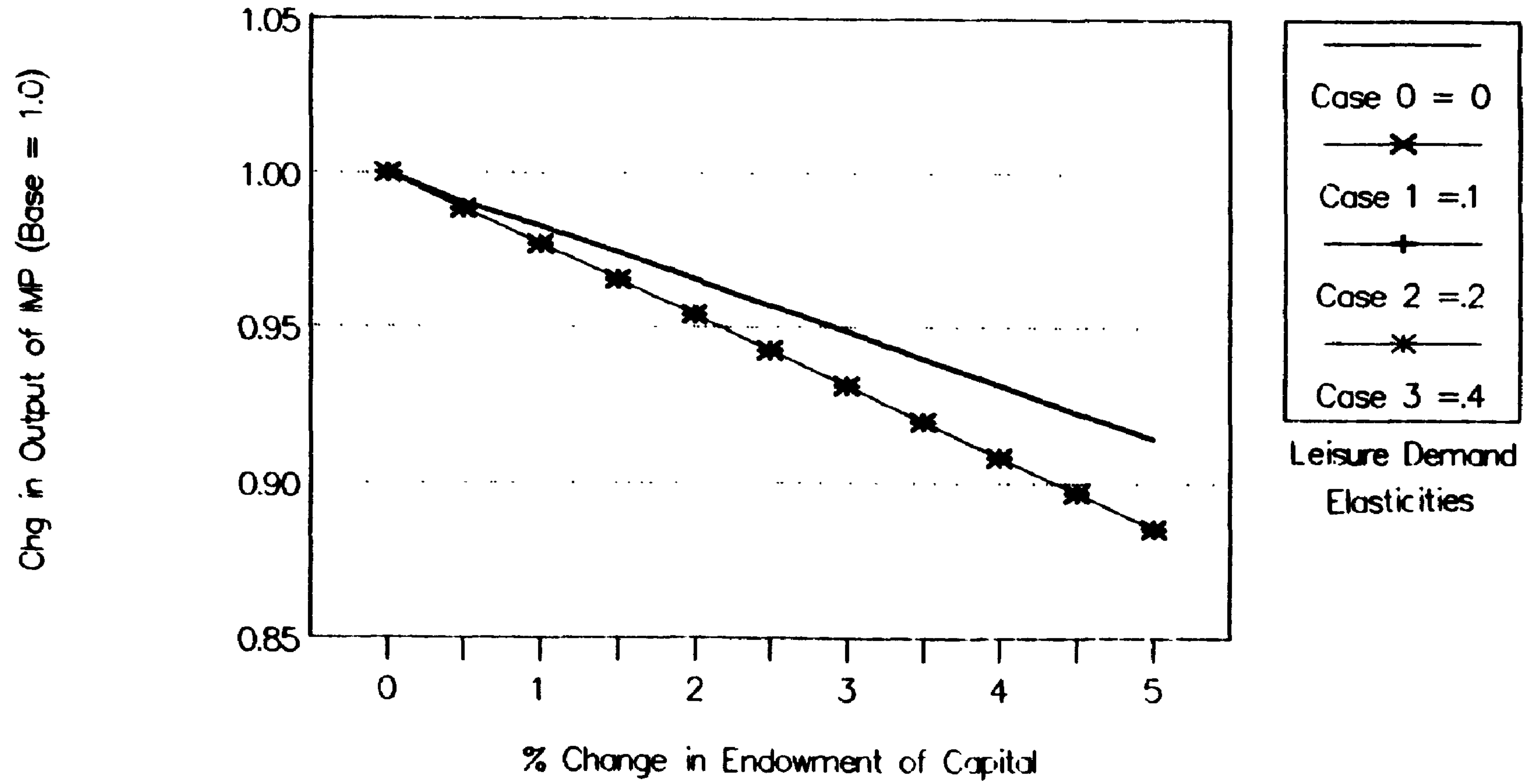
GRAPH 8

Response in Production of Exports to Change in Endowment of Capital

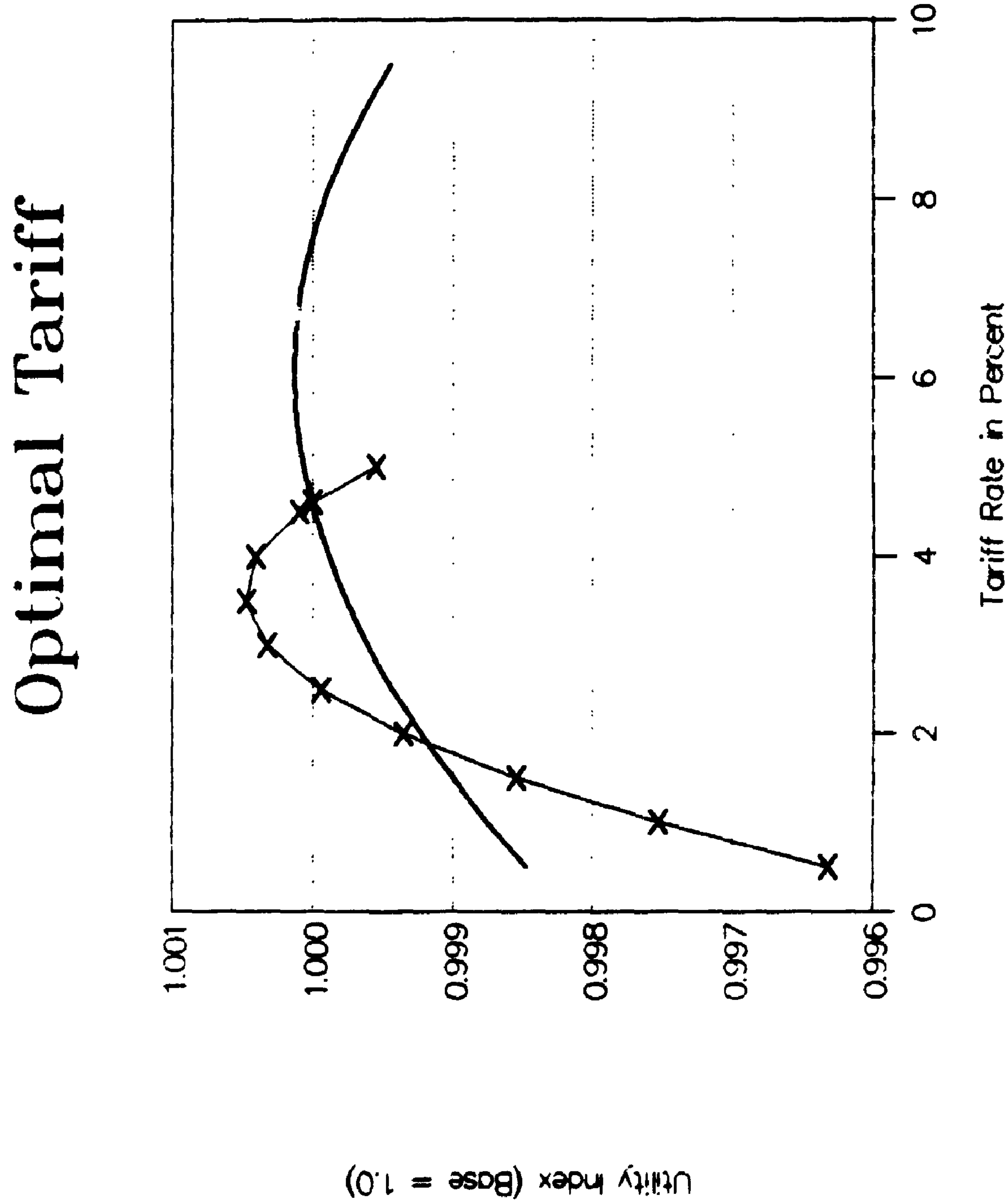


GRAPH 9

Response in Production of Imports to Change in Endowment of Capital

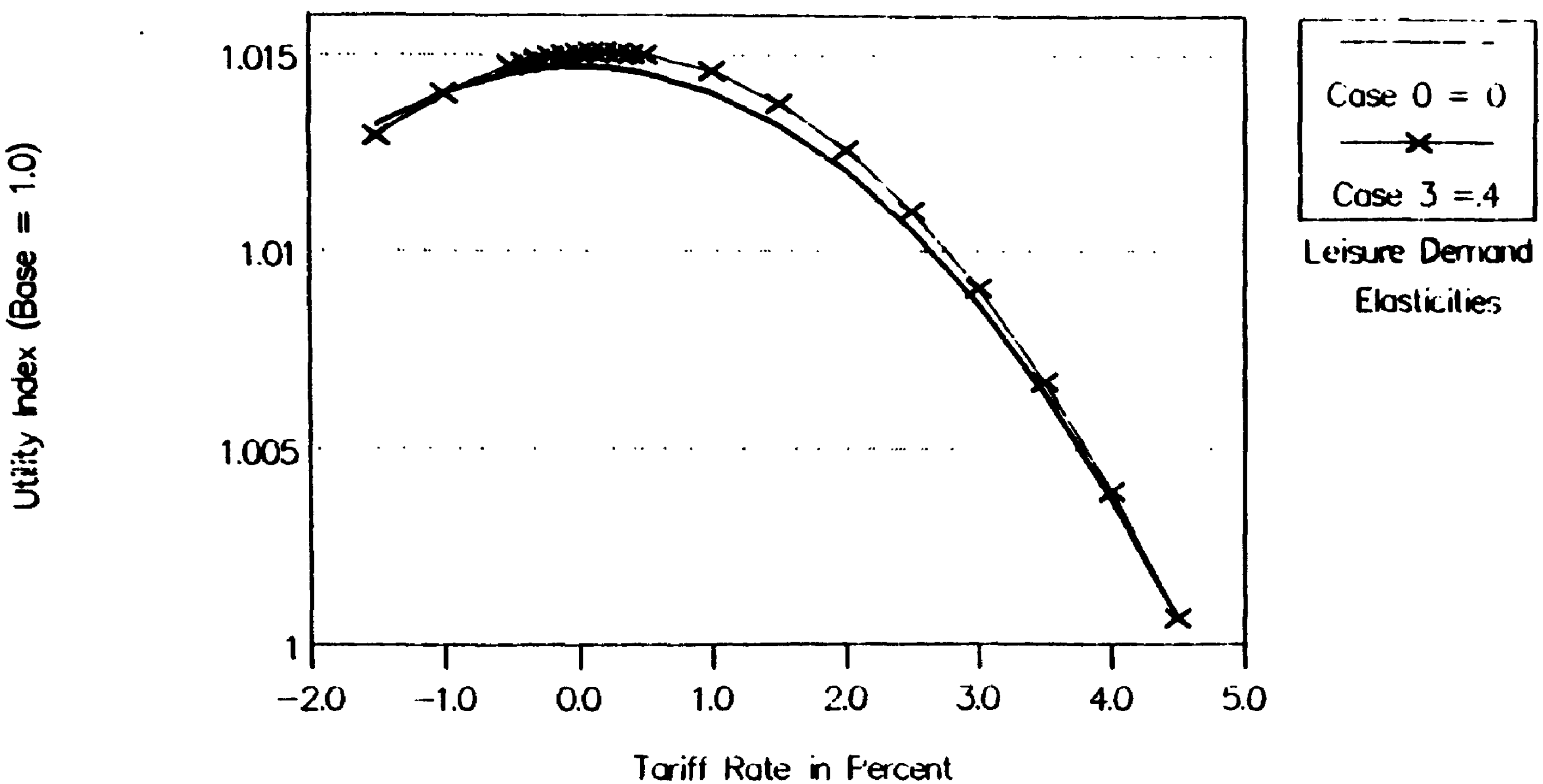


GRAPH 10


Optimal Tariff
Case 0 = 0
Case 3 =.4
Leisure Demand Elasticities
1.001
1.000
0.999
0.998
0.997
0.996
0
2
4
6
8
10
Tariff Rate in Percent
Utility Index (Base = 1.0)

GRAPH 11

Optimal Tariff with Factor Tax



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