Sensitivity Analysis of Minimum Variance Portfolios

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Graduate Program in Applied Mathematics
A thesis submitted in partial fulfillment of the requirements for the degree in Master of Science
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SENSITIVITY ANALYSIS OF MINIMUM VARIANCE PORTFOLIOS
(Thesis format: Monograph)

by

Xiaohu Ji

Graduate Program in Applied Mathematics

A thesis submitted in partial fulfillment
of the requirements for the degree of
Masters of Science

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Abstract

The purpose of this thesis is to conduct a Best-Grauer style sensitivity analysis of the investment allocation decisions made, not within a modern portfolio theory (MPT), but within a capital asset pricing model (CAPM) framework. For analytic tractability, we made the simplification (of some current practical interest) that investors have the objective of minimizing the variance of their portfolios without reference to the expected returns to be obtained from these portfolios. Our analytic results reveal how the minimum variance portfolio composition, expected return and risk would change with respect to the changes of the underlying asset correlations and volatilities. We give the investors instructions on how to build the minimum variance portfolio and keep the portfolio risk minimized with variable market data. We also specifically discuss the two-asset portfolio, which is analytically tractable and we find many interesting results. Finally, we analyze the risk that is not covered when the investor makes estimation errors about the market data using our model. We show the portfolio minimum variance is stable.

**Keywords:** Capital Asset Pricing Model, Minimum Variance Problem, Modern Portfolio Theory, Sensitivity Analysis, Estimate Error Risk
Contents

Abstract ii
List of Figures v
List of Tables vii

1 Introduction 1
  1.1 Modern Portfolio Theory ................................. 2
  1.2 Capital Asset Pricing Model ............................. 3
  1.3 Brief Introduction to Best and Grauer’s work ................. 4
  1.4 Analysis of the Single-Constraint Mean-Variance Portfolio Problem ........ 5
  1.5 Combination of MPT and CAPM ............................ 6
  1.6 Risk of Portfolio Estimation Error and Main Result .......... 7

2 Minimum Variance Portfolio Theory 8
  2.1 Completely Risk Averse CAPM Case ...................... 8
  2.2 Verification with Monte Carlo Simulation .................. 12
  2.3 Reductions to 2 asset portfolios ........................ 13
  2.4 Sensitivity to Beta .................................. 17

3 Sensitivity Analysis of Portfolio’s Composition 20
  3.1 Dependence on Volatilities .............................. 20
  3.2 Dependence on Correlations ............................. 25
  3.3 Two-Asset Case .................................. 33
    3.3.1 Volatility Sensitivity ............................. 34
    3.3.2 Correlation Sensitivity ............................ 41
  3.4 An Interesting Relationship between Weights, Betas, Correlations and Volatilities 43

4 Sensitivity Analysis of Portfolio’s Expected Rate of Return 47
  4.1 Dependence on Volatilities .............................. 47
  4.2 Dependence on Correlation .............................. 52
  4.3 Two-Asset Portfolio ................................ 57
    4.3.1 Volatility Sensitivity ............................. 58
    4.3.2 Correlation Sensitivity ............................ 60

5 Sensitivity Analysis of Minimum Variance Portfolio Variance and Error Risk Analysis 66
5.1 Dependence on Volatilities .......................... 66
5.2 Dependence on Correlations .......................... 69

6  Error Risk Analysis ................................. 73

7  Summary ........................................... 77

Bibliography .......................................... 78

Curriculum Vitae .................................... 78
3.1 Asset Weights vs Perturbation of $\sigma_1$. $t$ is the difference between the perturbed value of $\sigma_1$ and its original value. $X_1$ decreases from 1 to 0 for $\sigma_1 \in (0, (1 + (n - 2)\rho^2)/(\rho^2A_1))$. As shown in Remark 3.1.1, assets with higher volatilities have smaller weights. ................................................................. 21

3.2 Asset Weights vs Perturbation of $\rho_1$. $t$ is the difference between its perturbed value and its original value. All assets are ordered as the original values of their correlations to the market. As shown in Remark 3.2.4, $X_1$ is strictly decreasing. $X_1$ passes 0 when $\rho_1 = [A + B - 2(n - 2)]/(A - B)$. As shown in Remark 3.2.5, when $\rho_j$ is large enough, the assets with smaller correlation to the market have more weights. As shown in Remark 3.2.6, the left end (also the highest point) of $X_1$ is below 1/2 and the right end (also the lowest end) of $X_j$ is above −1/2. ... 26

3.3 Optimal Weights vs. Ratio of SDs (Positively Correlated). $\lambda = \sigma_1/\sigma_2$ is the ratio of the two assets’ standard deviations, and the correlation $\rho = 1/2$. As $\lambda$ increases from 0 to $\infty$, the optimal weights of asset 1 $X_1$ will first increase above 1, then decrease below 0, and finally increase to an asymptotic of 0. Both curves of $X_1$ and $X_2$ will pass the point $E (1, 1/2)$. .................. 36

3.4 Optimal Weights vs. Log (Ratio of SDs) (Positively Correlated). Taking log($\lambda$) as the x-axis, the curves of the optimal weights are symmetric to the y-axis. As log($\lambda$) grows from $-\infty$ to $\infty$, $X_1$ will first increase above 1, then decrease down below 0, and finally approximately increase to 0. Both curves of $X_1$ and $X_2$ will pass the point (0, 1/2). .................. 37

3.5 Optimal Weight vs. Ratio of SDs (Negatively Correlated). $X_1(0) = 1$, $X_1(\infty) = 0$, and $X_1(\lambda)$ is monotone decreasing. .................. 40

3.6 Optimal Weight versus Ratio of Volatilities. The two dotted curves represent $\rho = 1$ and $\rho = -1$, respectively. All curves with $\rho \in (0, 1)$ are bounded by the two curves, and they only join at the point (1, 1/2). When $\rho$ is very close to 1, there are two peak points on the curve. As $\rho$ goes down to -1, the curve will become flatter and flatter. .................. 42

3.7 10 Asset Weights vs Perturbation of $\rho_1$. $\rho_1$ is perturbed. The curves of asset weights other than $X_1$ change slowly and in parallel. Asset 7 and asset 8 have quite close sensitivities to the market excess return ($\beta_7 = 1.3160$ and $\beta_8 = 1.3173$). But asset 7 has higher correlation and lower volatility. We can see that $X_8 < X_7 < 0$. Asset 5 and asset 6 have close sensitivities to the market excess return ($\beta_5 = 0.6750$ and $\beta_6 = 0.6898$). But asset 6 has lower volatility and higher correlation to the market. We can see that $X_6 > X_5 > 0$. ........... 45
4.1 \(\mu_p\) with same Value of Correlation. The portfolio contains 10 assets which have the same correlation to the market. Take the asset 1 as an example. When \(\beta_1\) grows from 0 (i.e. \(t = -\tilde{\beta}_1\)) to \(\beta_{\text{high}}\), the portfolio’s expected return \(\mu_p\) increases from \(r_f\). When \(\beta_1\) is larger than \(\beta_{\text{high}}, \mu_p\) decreases and converges to \(\mu_p(\infty)\). 48

4.2 \(\mu_p\) with same Value of Volatility. The 10 assets have the same volatility. Their label numbers are ordered by their correlations to the market. \(t\) is the difference between the perturbed asset correlation and its original value. At the point \(t = 0\), it is clear that assets with lower correlations change more rapidly. The shape of the curves indicates that as asset correlation varying from -1 to 1 \(\mu_p\) increases to its upper-bound then decreases. The left ends of all curves are below \(r_f\) and the right ends are higher than \(r_f\). 63

4.3 2-Asset Portfolio Variance vs Perturbation of \(\sigma_1\). \(t\) is the difference between the perturbed value of \(\sigma_1\) and its original value. It is clear that \(\mu_p\) increases from \(r_f\) and then decreases as \(\sigma_1\) grows. 64

4.4 \(\mu_p\) with different Values of \(q\). The portfolio contains two assets. The portfolio’s expected return \(\mu_p\) first increases and then decreases as \(\beta_1\) grows in its range. On the other hand, \(\mu_p\) strictly increases as \(\beta_2\) grows in its range. 65

5.1 Portfolio Variance vs Perturbation of \(\sigma_1\). \(t\) is the difference between the perturbed value of \(\sigma_1\) and its original value. The portfolio variance achieves its highest point at \(\sigma_1 = (1 + (n - 2)s^2)/(\beta^2A_1)\). \(\sigma_{p,n-1}^2\) is the variance of the \((n - 1)\)-asset portfolio which contains all assets of the \(n\)-asset portfolio but asset \(j\). 67

5.2 Portfolio Variance vs Perturbation of \(\rho_1\). \(t\) is the difference between the perturbed value of \(\rho_1\) and its original value. The portfolio variance \(\sigma_p^2\) reaches its highest point at \(\rho_j = (A + B - 2(n - 2))/(A - B)\). \(\sigma_{p,n-1}^2\) is the variance of the \((n - 1)\)-asset portfolio which contains all assets of the \(n\)-asset portfolio but asset \(j\). 70

6.1 Portfolio Variance vs. Asset Correlation Estimation Errors. \(t = \tilde{\rho}_1 - \rho_1\) is the estimation error of asset 1’s correlation to the market. The full curve \(X(t)^T V(t)X(t)\) represents the portfolio variance when the investor chooses the optimal portfolio weight \(X(t)\) with updating covariance matrix \(V(t)\). The dotted curve \(X^T V(t)X\) represents the portfolio variance when the investor doesn’t change his portfolio weights and the covariance matrix \(V(t)\) is no longer \(V\). The dashed curve \(X(t)^T VX(t)\) represent the actual portfolio variance when the investor chooses the “optimal” weights \(V(t)\) with the covariance matrix \(V(t)\) which is wrong but should be \(V\). The point \(X^T VX\) is the minimum point of \(X(t)^T VX(t)\). \(X^T V(t)X\) is higher than \(X(t)^T V(t)X(t)\) for all \(t \in (-1 - \rho_1, 0) \cup (0, 1 - \rho_1)\) and the two curves are tangent at the point \(X^T VX\). 76
List of Tables

1.1 Comparison of Best and Grauer’s Work and Our Work ............... 7
2.1 Matlab Code for Monte Carlo Simulation .......................... 14
2.2 SPTSX Top Ten Stocks ............................................. 15
2.3 SPTSX Top Ten Stocks 1 Year Data of 2003/12/5 to 2004/12/3. Statistics of correlations and volatilities is calculated using daily returns. Betas are calculated using formula (1.5) ............................................. 15
3.1 SPTSX Top Ten Stocks 1 Year Data of 2003/12/5 to 2004/12/3. ........ 44
Chapter 1

Introduction

A standard problem in quantitative finance describes how an investor should allocate funds between investments in \( n \) different assets. The earliest comprehensive answer to this problem was given by Markowitz’s Modern Portfolio Theory (MPT) [5] [6], which assumed a one period investment horizon. Over the investment horizon, the simple return of each asset was assumed to be jointly normally distributed. Markowitz assumed that investors looked for the largest possible expected return for a given level of risk, and measured risk by the variance of portfolio returns around this expectation. It turned out to be equivalent to state this problem as minimizing portfolio variance subject to a given target expected return. The solution of this problem is obtained by solving a quadratic program.

While conceptually very elegant and full of financial insights, MPT suffered from three main flaws, two of which will be discussed in this thesis. First, it was restricted to a single investment horizon. Merton [8] provided a continuous time extension of these ideas, but in this thesis we too will consider only a single period setting. Second, MPT did not address the practical problem of how to actually determine the parameters of the portfolio’s constituent assets. Not only is the number of underlying parameters large, they are also difficult to estimate and portfolio weights, returns and volatilities are remarkably sensitive to input parameters. This point was first made by Best and Grauer [1]. See also DeMiguel, Garlappi and Uppal [2] for a recent take on the same topic.

The third problem with MPT is economic in nature. It is a prescriptive, not a descriptive, theory of portfolio optimization. It does not impose any relation between the return and volatility of an asset. For instance, it suggests that in the same market, a stock with extremely low variance and extremely high mean return and a stock with an extremely high variance and an extremely low, or even negative, mean return might co-exist. In such a setting it, rather sensibly, suggests that an investor should sell as much as possible of the “bad” asset to buy as much as possible of the “good” asset. But if all investors do this, the price of the “good” assets will be driven up and of the “bad” assets driven down, raising the return of the “bad” asset and reducing the return of the “good” one. The Capital Asset Pricing Model (CAPM) of Sharpe [9] was in part developed in an effort to address this issue, and provides a relationship between the mean-variance properties of stocks.

The purpose of this thesis is to conduct a Best-Grauer style sensitivity analysis of the investment allocation decisions made, not within a MPT, but within a CAPM framework. For analytic tractability, the further simplification (of some current practical interest) is made that
investors have the objective of minimizing the variance of their portfolios without reference to the expected returns to be obtained from these portfolios.

1.1 Modern Portfolio Theory

In modern portfolio theory (MPT), it is assumed an investor has a \( n \)-asset portfolio, in which the stock simple return rate of the \( i \)-th asset is assumed to be a random variable \( R_i \), and is calculated using the “simple return” formula:

\[
\text{Simple Return} = \frac{\text{Total Proceeds}}{\text{Total Buying Costs}} - 1.
\] (1.1)

For example, suppose the share price of a stock was 100 yesterday, and it is 110 today. Then the daily simple return rate is \( \frac{110-100}{100} \times 100\% = 10\% \). The uncertainty of the return rate \( R_i \) in the future is the risk of buying this asset, and is measured by the volatility of \( R_i \). Suppose the portfolio has associated expected return rates \( \mu_i, i = 1, \cdots, n \), and covariance matrix \( V = (\text{Cov}(R_i, R_j))_{n \times n} \), which summarizes risk. Let \( X_i \) be the \( i \)-th asset allocation in the diversified portfolio, for \( i = 1, \cdots, n \), where \( \sum_i X_i = 1 \). Then the portfolio expected return rate is \( \sum_i X_i \mu_i \) and the portfolio variance is \( \sum_{i,j} X_i \text{Cov}(R_i, R_j) X_j \). The investor attempts to maximize portfolio expected return rate for a given amount of portfolio risk, or equivalently to minimize risk for a given level of expected return rate, by carefully choosing the proportions of various assets. If the risk is interpreted as portfolio variance this problem reduces to the solution of a quadratic programming problem.

The key idea is optimizing the utility function which represents the investor’s style, under some constraint conditions. The investor’s willingness to accept higher risk or volatility in exchange for higher potential returns is measured by his risk tolerance. The investors with high risk tolerance prefer aggressive investment strategies, while those with low risk tolerance, also called risk-averse, prefer conservative ones. For example, suppose an investor has tolerance \( T \). He could buy or sell any amount of assets freely, but the total amount of investment is fixed. Then the corresponding mean-variance (MV) problem is:

\[
\max \{ T \mu^T X - \frac{1}{2} X^T V X \mid \iota^T X = 1 \},
\] (1.2)

where \( \iota = (1, \cdots, 1)^T_{1 \times n} \), \( T \) is the investor’s risk tolerance, \( \mu = (\mu_1, \cdots, \mu_n)^T \) are the expected rates of return, \( X = (X_1, \cdots, X_n)^T \) are the portfolio weights, \( V \) is the covariance matrix of the portfolio asset returns, and only the budget constraint is active. A larger risk tolerance parameter \( T \) implies the investor pays more attention to return and endures a higher level of risk. A smaller \( T \) implies the investor pays more attention to risk and wishes the lowest level of risk.

In recent years, investors disappointed with the correlation breakdown observed during the 1999 and 2008 financial crises have become increasingly interested in selecting portfolios simply to minimize risk [4]. The thinking is presumably that the return “will be what it will be”, but at least we can control risk to some extent. For an academic discussion of some of these ideas see Scherer (2010) [10]. The approach of minimizing portfolio variance will be taken in the remainder of this thesis.
In this thesis we assume that the investor is extremely risk-averse. In other words, the investor is cautious and methodical about his investment strategies, and wants the portfolio’s risk to be as low as possible. In the idealized limiting case taken here, we assume his risk tolerance obeys $T = 0$. In this case (1.2) is equivalent to the minimum variance problem

$$\min \{ \frac{1}{2} X^T V X \mid \iota^T X = 1 \}.$$  

We will give the explicit solution to this simple MV problem in the CAPM framework and study sensitivities of the corresponding optimal portfolio with respect to variables such as volatilities, correlations and betas.

### 1.2 Capital Asset Pricing Model

Recall that the Capital Asset Pricing Model (CAPM) implies an equilibrium relationship between the asset risk and return, while MPT doesn’t. In CAPM, the simple return rate of asset $i$ (named as $R_i$) is modeled as a random variable, and the mean return of this security is given by:

$$\mu_i := E[R_i] = r_f + \beta_i (E[R_m] - r_f),$$  

where $r_f$ and $R_m$ are the risk-free return rate and the return of the entire market, respectively. Note that $R_i$ is often taken to be normal, but any random variable with finite mean and variance will work, provided that the investors are still happy to characterize the resulting portfolio returns by their mean and variance alone. The parameter $\beta_i$ is the sensitivity of the expected excess asset returns to the expected excess market returns, defined to be:

$$\beta_i = \frac{\text{Cov}(R_i, R_m)}{\text{Var}(R_m)} = \frac{\rho_{im} \sigma_i}{\sigma_m},$$  

where $\sigma_i$ and $\sigma_m$ are the standard deviations of asset $i$’s return and the market return respectively, while $\rho_{im}$ is the correlation between the return of the asset $i$ and the market return. Hence the four parameters ($\beta_i, \sigma_m, \sigma_i, \rho_i$) in CAPM are restricted to three free parameters, which means the value of each parameter can be determined by the values of the other three through (1.5).

Suppose the excess return of the overall market is known and denoted by $p = E[R_m] - r_f$, where $p$ is called the equity risk premium. Then the market return is a random variable:

$$R_m = r_f + p + \sigma_m Z_m,$$

where $Z_m$ is a random variable with zero mean and unity standard deviation.

If we decompose the random returns of other stocks into a component which scales with the market return and an idiosyncratic component of the market return, we can write the return of stock $i$ as:

$$R_i = r_f + \beta_i (p + \sigma_m Z_m) + \sqrt{\sigma_i^2 - \sigma_m^2 \beta_i^2} W_i,$$

where $W_i$ and $Z_m$ are independent random variables with zero mean and unity standard deviation. Subsequently we use the standard notation $W_i \perp Z_m$ to mean $W_i$ is independent of $Z_m$. The square root in (1.7) is well-defined since from (1.5) we have

$$\sigma_i^2 - \sigma_m^2 \beta_i^2 \geq \sigma_i^2 \rho_{im}^2 - \sigma_m^2 \beta_i^2 = 0.$$
It is straightforward to check that, from the definition of (1.7), the random variable $R_i$ has mean $r_f + \beta_i p$ and standard deviation $\sigma_i$, just as expected. Furthermore, noticing that $W_i \perp Z_m$, we have

$$\text{Cov}[R_i, R_m] = \rho_{im} \sigma_i \sigma_m = \sigma_m^2 \beta_i, \quad \text{(1.9)}$$

and

$$\text{Cov}[R_i, R_j] = \sigma_m^2 \beta_i \beta_j = \rho_{jm} \sigma_i \sigma_j. \quad \text{(1.10)}$$

**Proof of (1.9) and (1.10)** We only check (1.9) as the other results are even more straightforward to show:

\[
\text{Cov}[R_i, R_m] = E[(R_i - E[R_i])(R_m - E[R_m])]
= E[(\beta_i \sigma_m Z_m + \sqrt{\sigma_i^2 - \sigma_m^2 \beta_i^2 W_i}) \sigma_m Z_m]
= E[\beta_i \sigma_m^2 Z_m^2] + E[\sqrt{\sigma_i^2 - \sigma_m^2 \beta_i^2 W_i} \sigma_m Z_m]
= \beta_i \sigma_m^2 E[Z_m^2] + \sigma_m \sqrt{\sigma_i^2 - \sigma_m^2 \beta_i^2} E[W_i Z_m]
= \beta_i \sigma_m^2 = \rho_{im} \sigma_i \sigma_m = \rho_{im} \sigma_i \sigma_m. \quad \text{(1.11)}
\]

**Remark 1.2.1** In the remainder of the thesis we will write $\rho_i$ instead of $\rho_{im}$ for short.

### 1.3 Brief Introduction to Best and Grauer’s work

This article is inspired by the previous work of Best and Grauer in 1991 ([1]). They assume an investor is bullish or bearish on some security while another investor is not, which means they have different beliefs of the asset mean return. Then their question is: Are the resulting portfolios of the two investors slightly or radically different? They investigates the sensitivity of MV-efficient portfolios to changes in the mean of the individual assets. Their analysis indicates that when only a budget constraint is imposed on the problem all three variables (the portfolio’s mean, weights and variance) can be extremely sensitive to changes in the asset means. The key point of this conclusion is that they take the variables as functions of the inverse of the portfolio’s covariance matrix, which might have very large elements.

The MV problem with no risk-free asset subject to general linear constraints:

$$\max_{\mu^T X - \frac{1}{2} X^T V X \mid AX \leq b}, \quad \text{(1.12)}$$

where $\mu$ is the expected rates of return ($n$-vector), $X$ is portfolio weights ($n$-vector), $V$ is an $(n, n)$-positive definite covariance matrix, $A$ is an $(m, n)$-constraint matrix, and $b$ is an $m$-vector.

**Remark 1.3.1** Throughout the thesis we assume that the $n$ assets chosen are irreducible in the sense that none of them may be expressed as linear combinations of the others. In other words, the covariance matrix $V$ is positive definite.

Two ways of interpreting (1.12):
1.4 Analysis of the Single-Constraint Mean-Variance Portfolio Problem

- Parameter quadratic programming (PQP) problem with parameter $t$
- $t$ is an MV investors risk tolerance parameter

Hence for some fixed positive value of $t$, say $t = T$, the solution to (1.12) yields the MV-efficient portfolio for the investor with risk tolerance parameter $T$.

To study the sensitivity problem, they consider a corresponding PQP problem

$$\max \{ T(\mu + tq)^T X - \frac{1}{2} X^T VX \mid AX \leq b \},$$

(1.13)

where $tq$ captures the change in $\mu$, ie $\mu(t) = \mu + tq$. Then the optimal portfolio’s weight, mean and variance are functions of $t$, $q$, $\mu$, and $V$.

When only a budget constraint is active (looking forward to the next section, (1.12) and (1.13) are reduced to (1.14) and (1.22)), the closed forms of the optimal portfolio’s weights, mean and variance can be derived, as well as their upper bounds. Their analysis shows that the change of portfolio weights could be very sensitive since elements of the inverse covariance matrix $V^{-1}$ could be very large.

In the remainder of [1] they use a computational methodology to show change rates of portfolio weights are extremely sensitive to changes in the asset means with and without non-negativity constraints. As the number of stocks in the portfolio increases, the average change rates of portfolio returns are small (and decrease) and that of portfolio weights are large (and increase). They also examine the robustness of these results in the ways such as examining the results with allowance of borrowing or lending at the risk-less rate, examining the results of different covariance structures, and examining the effect of simultaneously increasing or decreasing all of the asset means by the same percent.

Both their analytical and computational comparative statics results indicate that the investors should buy or sell assets that they feel under- or overpriced in large amounts, although usually active portfolio managers tend to hold the market portfolio and to buy or sell assets that they feel under- or overpriced in small amounts.

The comparison of our work and that of Best and Grauer will be given in Remark 1.6.1.

1.4 Analysis of the Single-Constraint Mean-Variance Portfolio Problem

Consider the simplest mean-variance portfolio problem

$$\max \{ T\mu^T X - \frac{1}{2} X^T VX \mid \iota^T X = 1 \},$$

(1.14)

where $T$ is a risk tolerance parameter, and only the budget constraint is active ($\iota = (1, \cdots, 1)^T_{1\times n}$).

Since the equality constraint $\iota^T X = 1$ is imposed, by using the Lagrange multipliers, solving (1.14) is equivalent to solving the first-order conditions

$$VX + \iota \lambda = T \mu,$$

(1.15)

$$\iota^T X = 1.$$

(1.16)
Since the covariance matrix $V$ is positive definite, we can multiply its inverse on both sides of (1.15)

$$X = -\lambda V^{-1}t + TV^{-1}\mu.$$  \hspace{1cm} (1.17)

Using the equality constraint, we get

$$1 = t^TX = -\lambda t^TV^{-1}t + Tt^TV^{-1}\mu.$$  \hspace{1cm} (1.18)

So the Lagrange multiplier $\lambda$ is

$$\lambda = (Tt^TV^{-1}\mu - 1)/t^TV^{-1}t.$$  \hspace{1cm} (1.19)

Then the solution to the problem (1.14) is

$$X = V^{-1}t/t^TV^{-1}t + T[V^{-1}(\mu - u^TV^{-1}\mu/t^TV^{-1}t)].$$  \hspace{1cm} (1.20)

Particularly, if the investor is completely risk-averse, i.e. the risk tolerance $T = 0$, then the solution becomes

$$X = V^{-1}t/t^TV^{-1}t.$$  \hspace{1cm} (1.21)

Furthermore, suppose the investor with risk tolerance $T$ wishes to analyze the sensitivity of the optimal portfolio’s weights, expected return and variance to changes in the asset mean $\mu$. It is performed by solving the related parameter quadratic programming (PQP) problem:

$$\max \{T(\mu + t\eta)^TX - \frac{1}{2}X^TVX | t^TX = 1\},$$  \hspace{1cm} (1.22)

which is the PQP problem corresponding to (1.14).

## 1.5 Combination of MPT and CAPM

MPT and CAPM provide two related perspectives of the microeconomics of capital markets. The MPT considers how an optimizing investor would behave while that of CAPM is concerned with economic equilibrium assuming all investors would use MPT to optimize their investment. Compared to the previous work of Best and Grauer [1], the introduction of CAPM in our model incorporates the relationship between these parameters (see (1.4)). It allows us to write the portfolio’s expected return rate, weights and variance as functions of the asset variance and correlation to the market (or variance and beta).

In the classical MPT framework we need to estimate $n(n+1)/2 + n$ parameters from the market data to create a $n$-asset portfolio, where the covariance matrix contains $n(n+1)/2$ parameters, i.e. $\text{Var}(R_i)$ for $i = 1, \cdots, n$ and $\text{Cov}(R_i, R_j)$ for $i, j = 1, \cdots, n$, $i < j$, and the asset expected return rates contain $n$ parameters. On the other hand, if we use CAPM to express the covariance matrix (see (2.20)) and model the asset expected return rates (see (1.4)), then we need only $2n + 2$ parameters, which contain $n$ asset-variances, $n$ asset-correlations to the market, the market excess rate and the market volatility. Obviously, when $n > 2$, we have $n(n+1)/2 + n > 2n + 2$.

Furthermore, the expected return rates of the assets in the portfolio could be calculated using (1.4), which don’t require the market information of the historical mean return rates. Therefore using the same parameters (betas and volatilities) we could estimate the expected return rate of the optimal portfolio.
1.6 Risk of Portfolio Estimation Error and Main Result

Investors might have estimation errors of the market information when they create the “optimal” portfolios with their own risk tolerances and constraint conditions. For example, suppose the investor is risk-averse. The portfolio weights should be compatible with the portfolio covariance matrix such that the portfolio variance is minimized. Otherwise, the actual portfolio risk level must be higher than the investor thought. Therefore, when the investor has estimation errors, he will face uncovered portfolio risk.

Our main result is:

Main Result 1 The uncovered portfolio risk the risk-averse investor faces is not high when the estimation errors are small. In other words, the MPT system with CAPM framework is stable.

The analysis of the main result refers to Chapter 6. Chapter 2 through Chapter 5 provide a lot of financial intuition and all of the mathematical apparatus needed to address Chapter 6, in which the Best-Grauer style sensitivity analysis of a CAPM-MPT blend is investigated. Readers interested in the main economic insight of the thesis are encouraged to skim Chapter 6 first to keep the end in view.

Remark 1.6.1 Table 1.1 compares and contrasts our work with that of Best and Grauer.

<table>
<thead>
<tr>
<th></th>
<th>Best and Grauer’s Work</th>
<th>Our Work</th>
</tr>
</thead>
<tbody>
<tr>
<td>Model</td>
<td>MPT</td>
<td>MPT with CAPM</td>
</tr>
<tr>
<td>MV Problem</td>
<td>max$(T\mu'X - \frac{1}{2}X'VX \mid AX \leq b)$</td>
<td>min${\frac{1}{2}X'VX \mid t'X = 1}$</td>
</tr>
<tr>
<td>Investor</td>
<td>Risk Tolerance $T \geq 0$</td>
<td>Completely Risk-Averse</td>
</tr>
<tr>
<td>Constraint</td>
<td>$AX \leq b$</td>
<td>$t'X = 1$</td>
</tr>
<tr>
<td>Perturbation</td>
<td>Asset Expected Return</td>
<td>Asset Volatility/Correlation</td>
</tr>
<tr>
<td>Key Method</td>
<td>Computational</td>
<td>Analytical</td>
</tr>
<tr>
<td>Main Conclusion</td>
<td>Portfolio weights are quite sensitive to changes in asset expected returns</td>
<td>The investor’s uncovered portfolio risk is not high if the estimate errors are small</td>
</tr>
</tbody>
</table>

Table 1.1: Comparison of Best and Grauer’s Work and Our Work
Chapter 2

Minimum Variance Portfolio Theory

In this chapter we will (i) give the explicit analytic solution to the MV problem (1.3), which is a special case of the MV problem (1.2), under the CAPM assumption (Section 2.1); (ii) use a Monte Carlo simulation to verify our result (Section 2.2); (iii) reduce the number of assets to 2 and re-write the solution formula in an insightful way (Section 2.3); and (iv) discuss intuition arisen from the sensitivity analysis of the optimal portfolio (Section 2.4).

2.1 Completely Risk Averse CAPM Case

In this section we will (i) give novel and explicit expressions of the optimal portfolio’s weight (2.4), return (2.6) and variance (2.7), as a special case \((T = 0)\) of the simple MV problem (1.2) which has been studied in Best and Grauer [1] (see Section 1.4); (ii) introduce notations of \(f_i\) and \(g_i\) to simplify the formula expressions (see Remark 2.1.1); (iii) show the portfolio variance naturally has an upper bound (see Remark 2.1.3); and (iv) give analytic proofs of the optimal portfolio’s weights, return and variance. (ii) to (iv) are dedicated to deriving the results in (i). Readers interested in more financial aspects of the problem can move ahead to the next section.

The completely risk averse MV problem is

\[
\min \left\{ \frac{1}{2} X^T V X \mid \iota^T X = 1 \right\}, \tag{2.1}
\]

where \(V\) is the covariance matrix and \(X\) is the portfolio weight. Hence \(X^T V X\) is the portfolio variance and the sum of elements of \(X\) must be 1.

The solution to problem (1.3) (or (2.1)) is

\[
X_j = \sum_i V_{ij}^{-1} / c, \quad j = 1, \ldots, n, \tag{2.2}
\]

where

\[
c = \iota^T V^{-1} \iota = \sum_{i,j} V_{ij}^{-1} \tag{2.3}
\]

is also the inverse of the minimum portfolio variance (see (2.35)).

**Proof of (2.2)** See Section 1.4, setting \(T = 0\) in (1.20).
2.1. Completely Risk Averse CAPM Case

Under the assumption of CAPM, the allocation to asset $i$ is

$$X_i = \left[ \frac{f_i + g_i}{\sigma_i} (2n - 2 - \sum_k \sigma_k(f_k + g_k)) + (f_i - g_i) \sum_k (f_k - g_k) \right] / \hat{c},$$

where

$$f_i := \frac{1}{\sigma_i - \sigma_m \beta_i} = \frac{1}{\sigma_i(1 - \rho_i)}, \quad g_i := \frac{1}{\sigma_i + \sigma_m \beta_i} = \frac{1}{\sigma_i(1 + \rho_i)}.$$  

(2.5)

The portfolio’s return is

$$\mu_p = r_f - \frac{2 \sum_i (f_i - g_i)}{\sigma_m \hat{c}} (E[R_m] - r_f),$$

and the portfolio’s variance is

$$\sigma_p^2 = \frac{4n - 2 \sum_k \sigma_k (f_k + g_k)}{\hat{c}},$$

where

$$\hat{c} = \left( 2n - 2 - \sum_k \sigma_k (f_k + g_k) \right) \sum_k \frac{f_k + g_k}{\sigma_k} + \left( \sum_k (f_k - g_k) \right)^2.$$  

(2.8)

$f_i$, $g_i$ and $\hat{c}$ are defined to simplify the expressions.

Remark 2.1.1 Although $f_i$ and $g_i$ are introduced to simplify mathematical expressions, they do admit the development of some intuition. For example, by definition, we know that:

$$\begin{cases} 
    f_i > \frac{1}{\sigma_i} > g_i, & \text{when } \rho_i > 0, \\
    f_i = g_i = \frac{1}{\sigma_i}, & \text{when } \rho_i = 0, \\
    f_i < \frac{1}{\sigma_i} < g_i, & \text{when } \rho_i < 0.
\end{cases}$$

(2.9)

In addition we have

$$\lim_{\rho_i \to 1} \frac{f_i \pm g_i}{f_i} = 1,$$

which means we can neglect the effect of $g_i$ (or $f_i$) when the correlation $\rho_i$ is positively (or negatively) high. Hence in our formulas, $f_i$ and $g_i$ are the smallest terms that connect the asset volatility $\sigma_i$ and correlation to the market $\rho_i$.

Remark 2.1.2 We will prove later in many cases that the term $\sum_k (f_k - g_k)$ is positive and the term $\hat{c}$ is negative and therefore $\mu_p$ is larger than $r_f$.

Remark 2.1.3 The portfolio’s variance $\sigma_p^2$ is the minimum value of the perturbed MV problem

$$\min \{ \frac{1}{2} X^T VX \mid t^T X = 1 \}.$$  

(2.11)

Hence it must satisfy

$$\sigma_p^2 = X^T VX = \min \{ X^T VX \mid t^T X = 1 \} \leq \min_i \sigma_i^2.$$  

(2.12)

In fact, the optimal portfolio of the variance minimization problem must have smaller variance than that of any of its assets. Otherwise, it can’t be the optimal choice.
To calculate $X$, $\mu_p$, and $\sigma_p^2$, we need the following facts.

**Remark 2.1.4** For any positive definite diagonal matrix $A$ and any vector $B$, the matrix $A + BB^T$ is positive definite.

**Proof of Remark 2.1.4** Suppose

$$A = \begin{pmatrix} a_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_n \end{pmatrix}, a_i > 0, i = 1, \cdots , n, B = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}. \tag{2.13}$$

Then for any non-zero vector $X = (x_1, \cdots , x_n)^T$, we have

$$X^T (A + BB^T) X = \sum a_i x_i^2 + \left( \sum b_i x_i \right)^2 > 0. \tag{2.14}$$

**Remark 2.1.5** Let $A$ be a symmetric positive semi-definite matrix, then

$$(A + U_1 U_2^T)^{-1} = A^{-1} - \frac{A^{-1} U_1 U_2^T A^{-1}}{1 + U_2^T A^{-1} U_1}, \tag{2.15}$$

for arbitrary $n$-vectors $U_1$ and $U_2$.

**Proof of Remark 2.1.5**

$$= I - \frac{A^{-1} U_1 U_2^T}{1 + U_2^T A^{-1} U_1} + A^{-1} U_1 U_2 - \frac{A^{-1} U_1 U_2^T A^{-1} U_1 U_2}{1 + U_2^T A^{-1} U_1} \tag{2.16}$$

Similarly, we also have

$$(A + U_1 U_2^T)(A^{-1} - \frac{A^{-1} U_1 U_2^T A^{-1}}{1 + U_2^T A^{-1} U_1}) = I.$$

**Proof of (2.4), (2.6), (2.7)** Recall that CAPM assumes that

$$\mu_i = r_f + \beta_i (E[R_m] - r_f), \tag{2.17}$$

where

$$\beta_i \sigma_m = \rho_{im} \sigma_i. \tag{2.18}$$

From (1.9) and (1.10), the covariance matrix of the portfolio’s assets $V$ satisfies:

$$V_{ij} = \begin{cases} \sigma_i^2, & i = j \\ \rho_{im} \rho_{jm} \sigma_i \sigma_j = \sigma_m^2 \beta_i \beta_j, & i \neq j \end{cases} \tag{2.19}$$
or equivalently
\[ V_{ij} = \delta_{ij}(1 - \rho_{im}^2)\sigma_i^2 + \rho_{im}\rho_{jm}\sigma_i\sigma_j = \delta_{ij}(\sigma_i^2 - \sigma_{m}^2\beta_i^2) + \sigma_{m}^2\beta_i\beta_j, \] (2.20)
where the Kronecker symbol is given by
\[ \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \] (2.21)
i.e.,
\[ V = A + BB^T \] (2.22)
where
\[
A = \begin{pmatrix}
(1 - \rho_{im}^2)\sigma_i^2 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & (1 - \rho_{mm}^2)\sigma_m^2
\end{pmatrix} = \begin{pmatrix}
\sigma_i^2 - \sigma_{m}^2\beta_i^2 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \sigma_m^2 - \sigma_{m}^2\beta_m^2
\end{pmatrix},
\]
and
\[
B = \begin{pmatrix}
\rho_{1m}\sigma_1 \\
\vdots \\
\rho_{nm}\sigma_n
\end{pmatrix} = \sigma_m \begin{pmatrix}
\beta_1 \\
\vdots \\
\beta_n
\end{pmatrix}.
\] (2.24)
From Remark 2.1.4, we know \( V \) is positive definite. Using Remark 2.1.5, the inverse is
\[
V_{ij}^{-1} = \frac{\delta_{ij}}{\sigma_i^2 - \sigma_{m}^2\beta_i^2} - \frac{\sigma_{m}^2}{1 + \sigma_m^2 \sum_k \frac{\beta_k^2}{\sigma_k^2 - \sigma_{m}^2\beta_k^2} \left( \frac{\beta_i}{\sigma_i^2 - \sigma_{m}^2\beta_i^2} \right) \left( \frac{\beta_j}{\sigma_j^2 - \sigma_{m}^2\beta_j^2} \right)}.
\] (2.25)
Note \( V_{ij}^{-1} \) is the \((i, j)\) element of the inverse matrix \( V^{-1} \), not the reciprocal of \( V_{ij} \).
Consider the minimum variance problem (1.3). The solution is
\[
X_i = \frac{V^{-1}t}{t^TV^{-1}t} = \sum_j V_{ij}^{-1}/c,
\] (2.26)
where
\[
c = t^TV^{-1}t = \sum_{ij} V_{ij}^{-1}
\] (2.27)
\[
= \sum_i \frac{1}{\sigma_i^2 - \sigma_{m}^2\beta_i^2} \left( \sum_k \frac{\beta_k^2}{\sigma_k^2 - \sigma_{m}^2\beta_k^2} \right) \left( \frac{\beta_i}{\sigma_i^2 - \sigma_{m}^2\beta_i^2} \right)^2
\] (2.28)
Substituting into (2.26), we obtain the optimal weight (2.4). Notice that
\[
\frac{1}{\sigma_i^2 - \sigma_{m}^2\beta_i^2} = \frac{1}{2\sigma_i} \left( \frac{1}{\sigma_i - \sigma_{m}\beta_i} + \frac{1}{\sigma_i + \sigma_{m}\beta_i} \right) = \frac{1}{2\sigma_i} (f_i + g_i),
\] (2.28)
and
\[
\frac{\sigma_{m}\beta_k^2}{\sigma_k^2 - \sigma_{m}^2\beta_k^2} = \frac{\sigma_k}{2} \left( \frac{1}{\sigma_k - \sigma_{m}\beta_k} + \frac{1}{\sigma_k + \sigma_{m}\beta_k} \right) - 1 = \frac{\sigma_k}{2} (f_k + g_k) - 1,
\] (2.29)
and
\[ \frac{\sigma_m \beta_k}{\sigma_k^2 - \sigma_m^2 \beta_k^2} = \frac{1}{2} \left( \frac{1}{\sigma_k - \sigma_m \beta_k} - \frac{1}{\sigma_k + \sigma_m \beta_k} \right) = \frac{1}{2} (f_k - g_k). \] (2.30)

The portfolio’s return is
\[ \mu_p = X^T \mu = \sum X_i \mu_i = \sum_i X_i [r_f + \beta_i (E[R_m] - r_f)] = r_f \sum_i X_i + p \sum_i X_i \beta_i = r_f + p \sum_i X_i \beta_i. \] (2.31)

Hence we only need to calculate the term \( \sum_i X_i \beta_i \). In fact, notice that
\[ f_i \beta_i = \frac{\beta_i}{\sigma_i - \sigma_m \beta_i} = - \frac{1}{\sigma_m} + \frac{\sigma_i}{\sigma_m} f_i, \] (2.32)
\[ g_i \beta_i = \frac{1}{\sigma_m} - \frac{\sigma_i}{\sigma_m} g_i. \] (2.33)

Then we have
\[ \hat{\sigma} \sum_i X_i \beta_i = \sum \left[ \frac{f_i g_i}{\sigma_i} (2n - 2 - \sum_k \sigma_k (f_k + g_k)) + (f_i - g_i) \sum_k (f_k - g_k) \right] \beta_i = (2n - 2 - \sum_k (\sigma_k f_k + \sigma_k g_k)) \sum_i \frac{1}{\sigma_i} (f_i + g_i) + \sum_k (f_k - g_k) \sum_i \frac{1}{\sigma_i} (- \frac{1}{\sigma_m} + \frac{\sigma_i}{\sigma_m} f_i + \frac{1}{\sigma_m} - \frac{\sigma_i}{\sigma_m} g_i) + \sum_k (f_k - g_k) \sum_i (- \frac{1}{\sigma_m} + \frac{\sigma_i}{\sigma_m} f_i + \frac{1}{\sigma_m} + \frac{\sigma_i}{\sigma_m} g_i) + \frac{1}{\sigma_m} (2n - 2 \sum_k (f_k + g_k) \sum_i (f_i - g_i) + \frac{1}{\sigma_m} \sum_i (f_i + g_i) \sum_k (f_k - g_k) = - \frac{2}{\sigma_m} \sum_i (f_i - g_i). \] (2.34)

Therefore, substituting (2.34) into (2.31) and we obtain (2.6).

The portfolio’s variance
\[ \sigma_p^2 = X^T VX = \frac{(V^{-1} \ell)^T VV^{-1} \ell}{c^2} = \frac{\ell^T V^{-1} \ell}{c^2} = \frac{1}{c}. \] (2.35)

Substituting \( f_i \) and \( g_i \), we get (2.7).

### 2.2 Verification with Monte Carlo Simulation

In this section we will (i) outline a Monte Carlo algorithm that can be used to verify our formulas (2.6) and (2.7); and (ii) give a computational example in the end of this section.

**Step 1.** Estimate 1-year risk-free rate \( r_f \), the market’s excess return \( p \) and volatility \( \sigma_m \). Find \( n \) stocks’ daily returns’ 1-year volatility \( \sigma_i \) and sensitivities \( \beta_i \).

**Step 2.** Use the formula (2.4) to calculate the optimal weights \( X_i \) of the portfolio which contains the \( n \) stocks.

**Step 3.** Calculate the portfolio’s expected return \( \mu_p \) and variance, using the formulas (2.6) and (2.7).
2.3. Reductions to 2 asset portfolios

**Step 4.** Use CAPM model to simulate the $n$ stocks’ daily return $R_i$, which satisfies (1.7). Calculate the portfolio’s daily return

$$R_p = \sum_i X_i R_i.$$  \hspace{1cm} (2.36)

**Step 5.** Redo Step 4 $N$ times. Calculate the mean value and variance of the portfolio’s daily return.

**Step 6.** Compare the results of Step 3 with those of Step 5.

For example, consider a portfolio which contains the ten sector sub-indices of the SP/TSX index as in Table 2.2. We collect 1 year of data from Dec 5, 2003 to Dec 3, 2004. See Table 2.3 for summary statistics. We approximately take the risk-rate as 1.46% (1-year LIBOR of Dec, 2003), the market’s excess return as 11.11% and the market’s volatility as 11.37%.

**Remark 2.2.1** In this example, we assume the situation that some of the simple returns are less than -1 never happens, since the volatilities are small. Therefore, we assume the market return rate and the asset expected return rates are normal random variables in our simulation. See the Matlab Code in Table 2.1.

After some calculations (see the MATLAB Code in Table 2.1), the optimal weights of the variance minimization problem are

$$X = (30.45\%, 16.02\%, 21.72\%, -3.45\%, 18.38\%, 9.48\%, -3.07\%,$$

$$-1.77\%, -2.80\%, 15.04\%)^T.$$ \hspace{1cm} (2.37)

The portfolio’s expected return and variance in Step 3 are

$$\mu_p = 7.51\%, \sigma_p^2 = 0.77\%.$$ \hspace{1cm} (2.38)

Implementing the Monte Carlo algorithm above in Step 5 with 500,000 replications, we get the sample’s mean and variance

$$\hat{\mu}_p = 7.52\%, \hat{\sigma}_p^2 = 0.77\%.$$ \hspace{1cm} (2.39)

2.3 Reductions to 2 asset portfolios

In this section we will (i) discuss some interesting special cases, the solution of which may be reduced to a 2-asset portfolio; (ii) give the explicit formulas of the optimal 2-asset portfolio’s weights and return; (iii) show the two asset allocations is actually determined by only two terms: the ratio of the two asset volatilities and the correlation between the two assets, although the MPT-CAPM framework requires more parameters to estimate than the classical MPT framework.

Recall that the classic MPT framework requires $n(n + 1)/2 + n$ (or $n(n + 1)/2$ if the investor is risk-averse) parameters to estimate from the market information and the MPT-CAPM framework requires $2n + 2$ parameters (see Section 1.5). It is easy to check that when $n \geq 3$, we have $n(n + 1)/2 + n > 2n + 2$, which means the introduction of the CAPM reduces the number of parameters we need to estimate in the classic MPT problem. But for $n = 1, 2$, the classical
% market data
>> r_f = 0.0146;
>> p = 0.1111;
>> beta = [0.5306, 0.6237, 0.6898, 1.3737, 0.6573,
           0.6750, 1.3173, 1.3160, 1.3222, 0.6397];
>> sigma = [0.1295, 0.1609, 0.1392, 0.2838, 0.1499,
           0.1932, 0.2694, 0.3300, 0.2805, 0.1635];
>> sigma_m = 0.1137;

% the optimal weights to the risk aversion MV problem
>> f = 1. / (sigma - sigma_m * beta);
>> g = 1. / (sigma + sigma_m * beta);
>> n = length(betta);
>> c = (2 * n - 2 - sum(sigma .* (f + g))) * (sum((f + g) ./ sigma)) + (sum((f - g)) * .^ 2);
>> x = (2 * n - 2 - sum(sigma .* (f + g))) * ((f + g) ./ sigma) + (sum((f - g)) * (f - g)) / c;

% portfolio’s expected return and variance
>> mu_p = r_f - 2 * p * sum(f - g) / (sigma_m * c);
>> var_p = 2 * (2 * n - 2 - sum(sigma .* (f + g))) / c;

% Monte Carlo Simulation
>> N = 500000;
>> r_p = [1:N];
>> for i = 1:N,
       w = normrnd(r_f + beta * p, sqrt(sigma.^2 - sigma_m^2 * beta.^2));
       r_m = randn;
       R = r_m * sigma_m * beta + w;
       R_p(i) = sum(x * R);
   end
>> mean(R_p)
>> var(R_p)

Table 2.1: Matlab Code for Monte Carlo Simulation
2.3. Reductions to 2 asset portfolios

<table>
<thead>
<tr>
<th>Stocks</th>
<th>Full Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>RY</td>
<td>Royal Bank of Canada</td>
</tr>
<tr>
<td>TD</td>
<td>Toronto-Dominion Bank</td>
</tr>
<tr>
<td>BNS</td>
<td>Bank of Nova Scotia</td>
</tr>
<tr>
<td>SU</td>
<td>Suncor Energy Inc.</td>
</tr>
<tr>
<td>BMO</td>
<td>Bank of Montreal</td>
</tr>
<tr>
<td>CNR</td>
<td>Canadian National Railway Company</td>
</tr>
<tr>
<td>ABX</td>
<td>Barrick Gold Corporation</td>
</tr>
<tr>
<td>G</td>
<td>Goldcorp Inc.</td>
</tr>
<tr>
<td>POT</td>
<td>Potash Corporation of Saskatchewan Inc.</td>
</tr>
<tr>
<td>BCE</td>
<td>BCE Inc.</td>
</tr>
</tbody>
</table>

Table 2.2: SPTSX Top Ten Stocks

<table>
<thead>
<tr>
<th>Stocks</th>
<th>Beta</th>
<th>Volatility</th>
<th>Correlation</th>
</tr>
</thead>
<tbody>
<tr>
<td>RY</td>
<td>0.5306</td>
<td>12.95%</td>
<td>46.59%</td>
</tr>
<tr>
<td>TD</td>
<td>0.6237</td>
<td>16.09%</td>
<td>44.07%</td>
</tr>
<tr>
<td>BNS</td>
<td>0.6898</td>
<td>13.92%</td>
<td>56.34%</td>
</tr>
<tr>
<td>SU</td>
<td>1.3737</td>
<td>28.38%</td>
<td>55.04%</td>
</tr>
<tr>
<td>BMO</td>
<td>0.6573</td>
<td>14.99%</td>
<td>49.86%</td>
</tr>
<tr>
<td>CNR</td>
<td>0.6570</td>
<td>19.32%</td>
<td>39.72%</td>
</tr>
<tr>
<td>ABX</td>
<td>1.3173</td>
<td>26.94%</td>
<td>55.60%</td>
</tr>
<tr>
<td>G</td>
<td>1.3160</td>
<td>33%</td>
<td>45.34%</td>
</tr>
<tr>
<td>POT</td>
<td>1.3222</td>
<td>28.05%</td>
<td>53.60%</td>
</tr>
<tr>
<td>BCE</td>
<td>0.6397</td>
<td>16.35%</td>
<td>44.49%</td>
</tr>
</tbody>
</table>

Table 2.3: SPTSX Top Ten Stocks 1 Year Data of 2003/12/5 to 2004/12/3. Statistics of correlations and volatilities is calculated using daily returns. Betas are calculated using formula (1.5).
model requires fewer parameters. The $n = 1$ case is vacuous, as no allocations can be made. So we need only discuss two cases: $n = 2$, and $n > 2$.

Consider a portfolio which contains two types of assets: assets of each particular type have the same correlation to the market and the same variance. If the investor is only interested in minimizing the risk (so we are dealing with problem (1.3)), the portfolio can be replaced by a new portfolio containing only two assets: one with correlation $\rho_1$ and standard deviation $\sigma_1$, the other with $\rho_2$ and $\sigma_2$. Then it is equivalent to solving the optimal problem (1.3) when $n = 2$.

We will show that the portfolio is actually defined by two terms: $\sigma_1 / \sigma_2$ (if $\sigma_2 \neq 0$) and $\rho_1 \rho_2$, where $\sigma_1 / \sigma_2$ is the ratio of the two asset volatilities, and $\rho_1 \rho_2$ can be replaced by $\rho = \rho_1 \rho_2$ as the correlation between the two assets. Specializing (2.2) to the present setting, the optimal weights could be rewritten as

$$X_1 = \frac{1 - \rho \sigma_1}{(\sigma_1)^2 - 2 \rho \sigma_1 + 1}, \quad X_2 = 1 - X_1.$$ (2.40)

Equivalently we can write

$$X_1 = \frac{\sigma_2^2 - \sigma_m^2 \beta_1 \beta_2}{\sigma_1^2 + \sigma_2^2 - 2 \sigma_m^2 \beta_1 \beta_2}, \quad X_2 = 1 - X_1.$$ (2.41)

From CAPM relation (2.17), the expected return of the minimum variance portfolio is

$$\mu_p = r_f + p \left( \frac{\sigma_1^2 - \sigma_m^2 \beta_1 \beta_2}{\sigma_1^2 + \sigma_2^2 - 2 \sigma_m^2 \beta_1 \beta_2} \beta_1 + \frac{\sigma_1^2 - \sigma_2^2 \beta_1 \beta_2}{\sigma_1^2 + \sigma_2^2 - 2 \sigma_m^2 \beta_1 \beta_2} \beta_2 \right).$$ (2.42)

**Proof of (2.40)** Suppose the portfolio only contains two assets, i.e. $n = 2$. The covariance matrix is

$$V = \begin{pmatrix} \sigma_1^2 & \rho_1 \rho_2 \sigma_1 \sigma_2 \\ \rho_1 \rho_2 \sigma_1 \sigma_2 & \sigma_2^2 \end{pmatrix},$$ (2.43)

where we write $\rho_1$ and $\rho_2$ to replace $\rho_{1m}$ and $\rho_{2m}$.

The simplified inverse matrix is

$$V^{-1} = \frac{1}{1 - \rho_1^2 \rho_2^2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \frac{1}{1 + \rho_1^2 \rho_2^2} \begin{pmatrix} \rho_1^2 \sigma_1^2 & 0 \\ 0 & \rho_2^2 \sigma_2^2 \end{pmatrix} - \frac{1}{1 + \rho_1^2 \rho_2^2} \begin{pmatrix} 1 & \rho_1 \rho_2 \sigma_1 \sigma_2 \\ \rho_1 \rho_2 \sigma_1 \sigma_2 & 1 \end{pmatrix}$$ (2.44)

Hence we get

$$c = \frac{1}{1 - \rho_1^2 \rho_2^2} \left( \frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} - \frac{2 \rho_1 \rho_2}{\sigma_1 \sigma_2} \right).$$ (2.45)

We have

$$V^{-1}_t = \frac{1}{1 - \rho_1^2 \rho_2^2} \begin{pmatrix} \frac{1}{\sigma_1^2} - \rho_1 \rho_2 \sigma_1 \sigma_2 & \rho_1 \rho_2 \sigma_1 \sigma_2 \\ \rho_1 \rho_2 \sigma_1 \sigma_2 & 1 \end{pmatrix}$$ (2.46)
2.4 Sensitivity to Beta

By (2.2), the solution to (1.3) is

\[ X_1 = \frac{\frac{1}{\sigma_1^2} - \frac{\rho_1 \rho_2}{\sigma_1 \sigma_2}}{\frac{1}{\sigma_1^2} - 2 \frac{\rho_1 \rho_2}{\sigma_1 \sigma_2} + \frac{1}{\sigma_2^2}} = \frac{\sigma_2^2 - \rho_1 \rho_2 \sigma_1 \sigma_2}{\sigma_1^2 - 2 \rho_1 \rho_2 \sigma_1 \sigma_2 + \sigma_2^2}, \]  

(2.47)

\[ X_2 = \frac{\frac{1}{\sigma_2^2} - \frac{\rho_1 \rho_2}{\sigma_1 \sigma_2}}{\frac{1}{\sigma_1^2} - 2 \frac{\rho_1 \rho_2}{\sigma_1 \sigma_2} + \frac{1}{\sigma_2^2}} = \frac{\sigma_1^2 - \rho_1 \rho_2 \sigma_1 \sigma_2}{\sigma_1^2 - 2 \rho_1 \rho_2 \sigma_1 \sigma_2 + \sigma_2^2}. \]  

(2.48)

Remark 2.3.1 Obviously, for \( \rho_1 \rho_2 \in [-1, 1] \), we have

\[ \sigma_1^2 - 2 \rho_1 \rho_2 \sigma_1 \sigma_2 + \sigma_2^2 \geq (\sigma_1 - \sigma_2)^2 \geq 0, \]

where the two equalities hold only while the two assets are perfect-positive-linearly correlated (i.e. the two assets’ correlation \( \rho_1 \rho_2 = 1 \)) and share the same volatility, which normally is not true. Hence we can always assume that

\[ \sigma_1^2 - 2 \rho_1 \rho_2 \sigma_1 \sigma_2 + \sigma_2^2 > 0. \]

What’s more, it is easy to check that \( X_1 + X_2 = 1 \).

2.4 Sensitivity to Beta

In this section we (i) discuss why and how we study the sensitivities of the optimal portfolio to the parameters; and (ii) introduce two kinds of perturbations and give their different changing ranges.

The optimal portfolio is chosen by the investor with the best market information he or his agent could obtain. There are several reasons why we need to consider the sensitivities of the portfolio to its parameters. First of all, the parameters such as the correlations between assets and the market and the volatilities of the assets are not constant forever. In fact, volatilities may change rapidly. Accidents such as wars, market crashes, and nation-wide natural disasters often lead to financial crises during which all assets tend to become more highly correlated to the market. Second, the investors don’t have the same information at the same time. Some investors such as insiders may also be better informed than others. Furthermore, using different sources of data also gives investors different parameter estimates. For example, suppose an investor constructs the minimum-variance portfolio using the data in Table 2.3, but that the Royal Bank (RY) correlation subsequently rises. What are the consequences for the return and volatility of the investor’s portfolio? How should he adjust the allocations of his investment to keep the portfolio variance minimized?

We study six main kinds of sensitivities in this thesis:

1. the sensitivity of the portfolio’s proportions to the volatilities,
2. the sensitivity of the portfolio’s proportions to the correlations between assets and the market,

3. the sensitivity of the portfolio’s expected return rate to the volatilities,

4. the sensitivity of the portfolio’s expected return rate to the correlations,

5. the sensitivity of the portfolio’s minimum variance to the volatilities, and

6. the sensitivity of the portfolio’s minimum variance to the correlations.

In particular we also study a special case of the two-asset portfolio.

Now let’s consider two ways of perturbing the important parameter $\beta$.

**Remark 2.4.1** In this thesis, if not mentioned specifically, we will use the term $tq$ to represent the change of any perturbed parameter, where $q = (q_1, \ldots, q_n)$ is a unit-length vector. Hence $t$ denotes the magnitude of the perturbation and $q$ is the direction. For example, we can write

$$\tilde{\beta} = \beta + tq.$$  \hfill (2.49)

This notation is used in the figures of this article, where $t$ is usually the label of x-axis to represent how far the perturbed value of the parameter (such as $\beta$) is away from the original value (when $t = 0$).

First, we assume that the volatilities $\sigma_i$ are fixed. From (2.18), it implies that in fact the correlations $\rho_{im} = \frac{\sigma_m}{\sigma_i} \beta_i$ are perturbed. This kind of perturbation naturally gives us the range of possible values of $\beta_i$. See (2.50).

**Remark 2.4.2** Notice that from (2.18), we have

$$|\beta_i| = \left| \frac{\rho_{im} \sigma_i}{\sigma_m} \right| \leq \frac{\sigma_i}{\sigma_m}. \hfill (2.50)$$

Hence the perturbation must also satisfy

$$|\tilde{\beta}_i| \leq \frac{\sigma_i}{\sigma_m}. \hfill (2.51)$$

Suppose $q_i > 0$, then

$$\left( -\frac{\sigma_i}{\sigma_m} - \beta_i \right) / q_i \leq t \leq \left( \frac{\sigma_i}{\sigma_m} - \beta_i \right) / q_i. \hfill (2.52)$$

The two end points of $t$ are just the zero points of $f_i$ and $g_i$.

It is easy to check that if the range of $t$ is taken to be

$$\left( -\frac{\sigma_i}{\sigma_m} - \beta_i \right) / q_i < t < \left( \frac{\sigma_i}{\sigma_m} - \beta_i \right) / q_i, \hfill (2.53)$$

then

$$f_i(t) > 0, \text{ and } g_i(t) > 0. \hfill (2.54)$$
2.4. Sensitivity to Beta

Second, we assume that the correlations to the market $\rho_i$’s are fixed. From (2.18), that implies that in fact the volatilities $\sigma_i = \frac{\sigma_m}{\rho_i} \cdot \beta_i$ are perturbed. If $\rho_i > 0$, then the range of $\beta_i$ is $[0, \infty)$, where $\beta_i = 0$ means the asset $i$ is risk-free and $\beta_i \to \infty$ means the asset $i$ is quite risky. We will discuss a special case when all correlations to the market are the same.
Chapter 3

Sensitivity Analysis of Portfolio’s Composition

In this chapter we study the sensitivity of the portfolio’s weights to changes in volatilities (Section 3.1) and correlations (Section 3.2). In order to avoid confounding effects we will simply assume all the volatilities are the same while we study the effect of correlation, and vice versa. For the special 2-asset case (Section 3.3), we study the effect of the ratio of volatilities and the correlation between the two assets. The explicit formulas give us intuition about how changes in these parameters will result in changes to optimal portfolio weights. In the end of this chapter, we also give an interesting discussion about the relationship between the minimum variance portfolio’s weights, betas, correlations and volatilities (Section 3.4).

3.1 Dependence on Volatilities

We begin with an example. Suppose the portfolio contains 10 assets. Using the same example as in Section 2.2, the volatilities are

$$\sigma = (0.1295, 0.1392, 0.1499, 0.1609, 0.1635, 0.1932, 0.2694, 0.2805, 0.2838, 0.3300)^T,$$

which are ordered by their values. The assets have the same correlation to the market, which is 20%. The market volatility is 11.37%. The market excess return is 11.11%. The risk-free rate is 1.46%. We perturb the volatility of asset 1. See Figure 3.1.

In Figure 3.1, \(t\) is the difference between the perturbed value of \(\sigma_1\) and its original value. Assets with higher volatilities have smaller weights, which will be shown in Remark 3.1.1. \(X_1\) decreases from 1 to 0 for \(\sigma_1\) varying from 0 to some point, which will be shown in Remark 3.1.3. When \(\sigma_1\) is large, all asset weights change very slowly. The remainder of this section is dedicated to understanding the behavior observed above in this figure.

In the remainder of this section, we will (i) give the solution formula (3.3) of MV-problem (1.3) under the assumption that all assets in the portfolio have the same correlation with the market; (ii) show in this case the investor’s magnitude of position depends on the reciprocals of asset volatilities and discuss conditions under which the investor should take a long or short position in some asset (see Remark 3.1.1); (iii) show the investor should invest less money in
3.1. Dependence on Volatilities

Figure 3.1: Asset Weights vs Perturbation of $\sigma_1$. $t$ is the difference between the perturbed value of $\sigma_1$ and its original value. $X_1$ decreases from 1 to 0 for $\sigma_1 \in (0, (1 + (n - 2)\rho^2)/(\rho^2 A_1))$. As shown in Remark 3.1.1, assets with higher volatilities have smaller weights.

We now suppose that all the assets share the same correlation to the market, i.e.

$$\rho_1 = \cdots = \rho_n = \rho. \tag{3.2}$$

The solution to problem (1.3) is then:

$$X_j = \frac{1}{(1 - \rho^2)c} \left( \frac{1}{\sigma_j^2} - \frac{\rho^2 \sum_i \frac{1}{\sigma_i}}{1 + (n - 1)\rho^2 \sigma_j} \right) = \frac{1}{\sigma_j} \sum_i \frac{1}{\sigma_i} \left( \frac{1}{\sigma_j} - \frac{\rho^2}{1 + (n - 1)\rho^2} \right). \tag{3.3}$$

**Proof of (3.3)** In fact, the covariance matrix becomes

$$V_{ij} = \delta_{ij}(1 - \rho^2)\sigma_i^2 + \rho^2 \sigma_i \sigma_j, \tag{3.4}$$
and the portfolio inverse matrix is
\[
V^{-1}_{ij} = \frac{1}{1 - \rho^2} \left( \frac{\delta_{ij}}{\sigma_i^2} - \frac{\rho^2}{1 + (n-1)\rho^2} \frac{1}{\sigma_i \sigma_j} \right). \tag{3.5}
\]

Hence we have
\[
c = \frac{1}{1 - \rho^2} \left( \sum_i \frac{1}{\sigma_i^2} - \frac{\rho^2}{1 + (n-1)\rho^2} \sum_{i,j} \frac{1}{\sigma_i \sigma_j} \right). \tag{3.6}
\]

(3.3) now follows from (2.2).

\[\text{Intuition 3.1.1} \quad \text{It is quite clear that in this case the investor’s magnitude of position (long or short) in asset } j \text{ depends on the ratio } \frac{1}{\sigma_j} (\sum_i \sigma_i^{-1}). \text{ If the volatility of asset } j \text{ is small, such that}
\begin{align*}
\frac{1}{\sigma_j} > \frac{\rho^2}{1 + (n-1)\rho^2},
\end{align*}
\text{then } X_j > 0 \text{ which means the investor should take a long position in asset } j. \text{ This makes sense as investors prize low volatility assets.}
\]

\[\text{If the volatility of asset } j \text{ is large, such that}
\begin{align*}
\frac{1}{\sigma_j} < \frac{\rho^2}{1 + (n-1)\rho^2},
\end{align*}
\text{then } X_j < 0 \text{ which means the investor should take a short position in asset } j. \text{ This makes sense as investors do not like highly risky assets.}
\]

\[\text{If the volatility of asset } j \text{ exactly satisfies}
\begin{align*}
\frac{1}{\sigma_j} = \frac{\rho^2}{1 + (n-1)\rho^2},
\end{align*}
\text{then } X_j = 0 \text{ which means the investor should delete asset } j \text{ from the portfolio.}
\]

\[\text{Notice that}
\begin{align*}
\frac{\rho^2}{1 + (n-1)\rho^2} < \frac{1}{n},
\end{align*}
\text{So it is not possible for the investor to take short positions in all assets. After all, all the money must be invested.}
\]

\[\text{Remark 3.1.1} \quad \text{Suppose the volatilities of asset } k \text{ and asset } l \text{ satisfy } \sigma_k < \sigma_l \text{ and}
\begin{align*}
\frac{1}{\sigma_k} + \frac{1}{\sigma_l} > \frac{\rho^2}{1 + (n-1)\rho^2},
\end{align*}
\text{We have } X_k > X_l.
\]
### Proof of Remark 3.1.1
Comparing the two weights, we have
\[
X_k - X_l = \frac{1}{(1-\rho^2)c} \left[ \left( \frac{1}{\sigma_k^2} - \frac{1}{\sigma_l^2} \right) - \rho^2 \frac{\sum_i \frac{1}{\sigma_i^2}}{1+(n-1)\rho^2} \left( \frac{1}{\sigma_k} - \frac{1}{\sigma_l} \right) \right]
\]
\[
= \frac{1}{(1-\rho^2)c} \left( \frac{1}{\sigma_k} - \frac{1}{\sigma_l} \right) \left( \frac{1}{\sigma_k} + \frac{1}{\sigma_l} - \rho^2 \frac{\sum_i \frac{1}{\sigma_i^2}}{1+(n-1)\rho^2} \right). \tag{3.12}
\]
Since \(\sigma_k < \sigma_l\) and \(\frac{1}{\sigma_k} + \frac{1}{\sigma_l} > \frac{\rho^2}{1+(n-1)\rho^2}\), we can show
\[
X_k < X_l. \tag{3.13}
\]

### Intuition 3.1.2
In Remark 3.1.1 case, the investor should invest less money in assets with higher volatilities. See Figure 3.1.

### Remark 3.1.2
Check (3.3) for the special case when all the assets are independent of the market, i.e. \(\rho = 0\). Then from above we have
\[
X_j = \frac{1}{\sum_i \frac{1}{\sigma_i^2}} \tag{3.14}
\]

### Intuition 3.1.3
From (3.14) it is clear that to minimize the risk, the investor should buy more lower variance assets and fewer higher variance assets.

### Remark 3.1.3
Suppose the investor has two portfolios. Portfolio 1 contains \(n\) assets while portfolio 2 contains \(n-1\) assets. Suppose asset \(j\) is the only asset included in portfolio 1 but not in portfolio 2. If the volatility of asset \(j\) satisfies
\[
\sigma_j = \frac{1 + (n - 2)\rho^2}{\rho^2 A_1}, \tag{3.15}
\]
where \(A_1 = \sum_{k \neq j} \frac{1}{\sigma_k}\), then portfolio 1 becomes portfolio 2.

### Proof of Remark 3.1.3
From Intuition 3.1.1, we know
\[
X_j = 0 \tag{3.16}
\]
if and only if
\[
\frac{1}{\sigma_j} \frac{\sigma_j}{A_1 + \frac{1}{\sigma_j}} = \frac{\rho^2}{1+(n-1)\rho^2}, \tag{3.17}
\]
i.e.
\[
\sigma_j = \frac{1 + (n - 2)\rho^2}{\rho^2 A_1}. \tag{3.18}
\]
For the other \(n - 1\) assets in portfolio 1, letting \(i \neq j\), we have
\[
X_i \bigg|_{\sigma_j = \frac{1 + (n - 2)\rho^2}{\rho^2 A_1}} = \frac{1}{(1-\rho^2)c} \left( \frac{1}{\sigma_i^2} - \rho^2 \frac{(A_1 + \frac{1}{\sigma_j})}{1+(n-1)\rho^2} \frac{1}{\sigma_i} \right) \bigg|_{\sigma_j = \frac{1 + (n - 2)\rho^2}{\rho^2 A_1}}
\]
\[
= \frac{1}{(1-\rho^2)c_{n-1}} \left( \frac{1}{\sigma_i^2} - \rho^2 A_1 \frac{1}{1+(n-2)\rho^2} \frac{1}{\sigma_i} \right), \tag{3.19}
\]
where
\[ c_{n-1} := \frac{1}{1 - \rho^2} \left( B_1 - \frac{\rho^2}{1 + (n-2)\rho^2} \sum_{i,j} A_{i,j}^2 \right) = c_{\sigma_j = 1 + (n-2)\rho^2}, \] (3.20)
and \( B_1 = \sum_{k+j} \frac{1}{\sigma_i^2} \). It is easy to check that \( X_i \) is just the optimal weight of asset \( i \) in portfolio 2.

**Intuition 3.1.4** We have proved that only when \( \sigma_j \) takes some special value does the optimal \( n \)-asset portfolio actually have \( n-1 \) assets. Recall Figure 3.1, we can see when \( \sigma_1 \) is large, the weights of the portfolio change very slowly. In other words, they all “look like” the \( (n-1) \)-asset portfolio in Remark 3.1.3. See Figure 3.1.

**Remark 3.1.4** If \( \sigma_j = \infty \), then the optimal portfolio is “like” a portfolio which only contains the other \( n-1 \) assets.

**Proof of 3.1.4** Using (3.6), we have
\[
c_{\infty} := \lim_{\sigma_j \to \infty} c = \frac{1}{1 - \rho^2} \left[ \sum_{i \neq j} \frac{1}{\sigma_i^2} - \frac{\rho^2}{1 + (n-1)\rho^2} \left( \sum_{i \neq j} \frac{1}{\sigma_i} \right)^2 \right], \] (3.21)
which implies
\[
\sigma_{p,\infty}^2 = \frac{1}{c_{\infty}} = \frac{1 - \rho^2}{\sum_{i \neq j} \frac{1}{\sigma_i^2} - \frac{\rho^2}{1 + (n-1)\rho^2} \left( \sum_{i \neq j} \frac{1}{\sigma_i} \right)^2}. \] (3.22)

and for \( k \neq j \),
\[
X_{j,\infty} := \lim_{\sigma_j \to \infty} X_j = 0, \] (3.23)
\[
X_{k,\infty} := \lim_{\sigma_j \to \infty} X_k = \frac{\sigma_{p,\infty}^2}{1 - \rho^2} \left( \frac{1}{\sigma_k^2} - \frac{\rho^2 \sum_{i \neq j} \frac{1}{\sigma_j} \left( \frac{1}{\sigma_i} \right) \frac{1}{\sigma_k}}{1 + (n-1)\rho^2 \sigma_k^2} \right). \] (3.24)

It is easy to check that
\[
\sum_k X_{k,\infty} = 1. \] (3.25)

On the other hand, suppose the investor has another portfolio which contains the remaining \( n-1 \) assets. Then the optimal portfolio’s variance is
\[
\sigma_{p,n-1}^2 = \frac{1 - \rho^2}{\sum_{i \neq j} \frac{1}{\sigma_i^2} - \frac{\rho^2}{1 + (n-2)\rho^2} \sum_{i \neq j} \left( \frac{1}{\sigma_i} \right)^2} > \sigma_{p,\infty}^2. \] (3.26)

And its weights are
\[
X_{k,n-1} = \frac{V_{p,n-1}^2 \left( \frac{1}{\sigma_k^2} - \frac{\rho^2 \sum_{i \neq j} \frac{1}{\sigma_j} \left( \frac{1}{\sigma_i} \right) \frac{1}{\sigma_k}}{1 + (n-2)\rho^2 \sigma_k^2} \right)}{1 - \rho^2} \neq X_{k,\infty}. \] (3.27)

Therefore, the two portfolios have different weights. This makes sense since they have different covariance matrices.
3.2 Dependence on Correlations

**Intuition 3.1.5** Remark 3.1.4 shows the investor should invest a little in the risky assets to minimize the portfolio risk.

**Remark 3.1.5** In the proof of Remark 3.1.4, we know when $\sigma_j \rightarrow \infty$, $X_j \rightarrow 0$. But the rest $X_k$ ($k \neq j$) are not exactly the same as the optimal weights of the $(n-1)$-asset portfolio, because this case will never happen in real life. In fact, from the definition of the minimum-variance problem, we know that the more assets the portfolio contains, the lower risk level it can achieve.

### 3.2 Dependence on Correlations

We begin with an example. Suppose a portfolio contains 10 assets. The original asset correlations to the market are

$$\rho = \begin{pmatrix} 30.16\%, 35.46\%, 36.37\%, 37.38\%, 38.37\%, 39.22\%, 74.81\%, \\ 74.89\%, 75.17\%, 78.09\% \end{pmatrix}^T,$$

which are ordered by their values. All assets have the same volatility $\sigma$. The value of $\sigma$ does not affect asset weights. To test it, we take $\sigma = 0.2, 0.5$ as two examples, and they have the same output. See Figure 3.2.

In Figure 3.2, we perturb $\rho_1$, the correlation of asset 1. $i$ is the difference between its perturbed value and its original value. All assets are ordered as the original values of their correlations to the market. $X_1$ is strictly decreasing, which will be shown in Remark 3.2.4. $X_1$ passes 0 when $\rho_1 = [A + B - 2(n - 2)] / (A - B)$. When $\rho_j$ is large enough, the assets with smaller correlation to the market have more weights, which will be shown in Remark 3.2.5. The left end (also the highest point) of $X_1$ is below 1/2 and the right end (also the lowest end) of $X_j$ is above $-1/2$, which will be shown in Remark 3.2.6. The remainder of this section is dedicated to understand the behavior observed above in this figure.

In the remainder of this section, we will (i) give the solution formula (3.31) of MV-problem (1.3) under the assumption that all assets in the portfolio have the same volatilities, which follows the intuition that in this case the portfolio composition depends solely on asset correlations; (ii) discuss when the position (long or short) of one asset would change (see Remark 3.2.3 and Intuition 3.2.1); (iii) show investors should invest less money on assets with high correlations to the market (see Remark 3.2.4, Remark 3.2.5 and Intuition 3.2.2); (iv) show the weight of each asset should be between $-1/2$ and $1/2$ (see Remark 3.2.6); and (v) give the special two-type case when some assets have the same correlation to the market and others are unrelated to the market (see Remark 3.2.7).

To begin, suppose that the $n$ assets in the portfolio have the same volatility, i.e.

$$\sigma_1 = \cdots = \sigma_n = \sigma.$$  \hspace{1cm} (3.29)

Then we have

$$\beta_k = \frac{\sigma_k \rho_k}{\sigma_m} = \frac{\sigma}{\sigma_m} \cdot \rho_k.$$  \hspace{1cm} (3.30)

Therefore in this case variability of Beta is arisen solely from variability of the correlations.
Figure 3.2: Asset Weights vs Perturbation of $\rho_1$. $t$ is the difference between its perturbed value and its original value. All assets are ordered as the original values of their correlations to the market. As shown in Remark 3.2.4, $X_1$ is strictly decreasing. $X_1$ passes 0 when $\rho_1 = \frac{A + B - 2(n - 2)}{(A - B)}$. As shown in Remark 3.2.5, when $\rho_j$ is large enough, the assets with smaller correlation to the market have more weights. As shown in Remark 3.2.6, the left end (also the highest point) of $X_1$ is below $1/2$ and the right end (also the lowest end) of $X_j$ is above $-1/2$. 
Then the solution to MV-problem (1.3), i.e. the composition of the optimal portfolio, is

\[
X_j = \frac{\bar{B} - n + 1}{2\bar{A}\bar{B} - (n - 1)(\bar{A} + \bar{B})} \frac{1}{1 - \rho_j} + \frac{\bar{A} - n + 1}{2\bar{A}\bar{B} - (n - 1)(\bar{A} + \bar{B})} \frac{1}{1 + \rho_j} = \frac{(\bar{A} + \bar{B} - 2n + 2) - (\bar{A} - \bar{B})\rho_j}{2\bar{A}\bar{B} - (n - 1)(\bar{A} + \bar{B})} \frac{1}{1 - \rho_j^2},
\]

where

\[
\bar{A} := \sum_k \frac{1}{1 - \rho_k}, \quad \bar{B} := \sum_k \frac{1}{1 + \rho_k}.
\]

**Proof of (3.31)** In fact,

\[
f_k = \frac{1}{\sigma} \frac{1}{1 - \rho_k}, \quad g_k = \frac{1}{\sigma} \frac{1}{1 + \rho_k}.
\]

\[
\hat{c} = \frac{(2n - 2) - \sum_k (f_k + g_k)}{\sigma} \sum_k (f_k + g_k) = \frac{(2n - 2) - \sum_k (f_k - g_k)^2}{\sigma}.
\]

Then for \( j = 1, \cdots, n \), we have

\[
X_j = \frac{\frac{2n - 2 - 2 \sum_k g_k}{\sigma} f_j + \frac{2n - 2 - 2 \sum_k f_k}{\sigma} g_j}{\sum_k \frac{1}{1 - \rho_k} - (n - 1) \frac{1}{1 + \rho_j}} + \frac{\sum_k \frac{1}{1 - \rho_k} - (n - 1) \frac{1}{1 + \rho_j}}{\sum_k \frac{1}{1 - \rho_k} - (n - 1) \frac{1}{1 + \rho_j}} \frac{1}{1 - \rho_j}.
\]

**Remark 3.2.1** The closed form (3.31) of optimal weights \( X \) is well-defined, since the denominator \( 2\bar{A}\bar{B} - (n - 1)(\bar{A} + \bar{B}) \) is positive.

**Proof of Remark 3.2.1** Using the inequality property that harmonic mean \( \leq \) arithmetic mean, we have

\[
\frac{n}{\sum_k \frac{1}{1 - \rho_k}} + \frac{n}{\sum_k \frac{1}{1 + \rho_k}} \leq \frac{\sum_k (1 - \rho_k)}{n} + \frac{\sum_k (1 + \rho_k)}{n} = 2,
\]

which implies

\[
\frac{1}{\sum_k \frac{1}{1 - \rho_k}} + \frac{1}{\sum_k \frac{1}{1 + \rho_k}} \leq \frac{2}{n} < \frac{2}{n - 1}
\]

i.e.

\[
2 \sum_k \frac{1}{1 - \rho_k} \sum_k \frac{1}{1 + \rho_k} - (n - 1) \sum_k \left( \frac{1}{1 - \rho_k} + \frac{1}{1 + \rho_k} \right) > 0.
\]

**Remark 3.2.2** We can also prove that

\[
\bar{A} + \bar{B} - 2n + 2 > \bar{A} + \bar{B} - 2n \geq 0.
\]
Proof of Remark 3.2.2 Using the inequality property that harmonic mean is less than or equal to arithmetical mean, we have
\[
\frac{2n}{\bar{A} + \bar{B}} = \frac{2n}{\sum_k \left( \frac{1}{1 - \rho_k} + \frac{1}{1 + \rho_k} \right)} \leq \frac{\sum_k (1 - \rho_k + 1 + \rho_k)}{2n} = 1, \quad (3.40)
\]
which implies (3.39). □

Remark 3.2.3 If all assets have the same volatilities, \(X_j\), the weight of asset \(j\), is 0, if and only if the correlation of asset \(j\) satisfies
\[
\rho_j = \frac{A + B - 2(n - 2)}{A - B}, \quad (3.41)
\]
where
\[
A = \sum_{k \neq j} \frac{1}{1 - \rho_k}, \quad B = \sum_{k \neq j} \frac{1}{1 + \rho_k}. \quad (3.42)
\]
In this case the \(n\)-asset portfolio is the same as an \((n - 1)\)-asset portfolio which contains the remaining \(n - 1\) assets.

Furthermore, \(X_j > 0\) if
\[
\rho_j < \frac{A + B - 2(n - 2)}{A - B}, \quad (3.43)
\]
and \(X_j < 0\) if
\[
\rho_j > \frac{A + B - 2(n - 2)}{A - B}. \quad (3.44)
\]

Proof of Remark 3.2.3 From (3.31), we know that \(X_j = 0\) if and only if
\[
\rho_j = \frac{\bar{A} + \bar{B} - 2n + 2}{\bar{A} - \bar{B}}. \quad (3.45)
\]
Substituting \(A\) and \(B\), we have equation
\[
\rho_j = \frac{A + B + \frac{1}{1 - \rho_j} + \frac{1}{1 + \rho_j} - 2n + 2}{A - B + \frac{1}{1 - \rho_j} - \frac{1}{1 + \rho_j}}. \quad (3.46)
\]
Solving equation (3.46), we get (3.41).

For the other assets, for \(i \neq j\), we have
\[
X_i \big|_{\rho_j = \frac{A + B - 2(n - 2)}{A - B}} = \frac{(A + B + \frac{1}{1 - \rho_j} + \frac{1}{1 + \rho_j} - 2n + 2) - (A - B + \frac{1}{1 - \rho_j} - \frac{1}{1 + \rho_j}) \rho_j - 1}{2(A + \frac{1}{1 - \rho_j})(B + \frac{1}{1 + \rho_j}) - (n - 1)(A + B + \frac{1}{1 - \rho_j} + \frac{1}{1 + \rho_j}) \rho_j - 1} \bigg|_{\rho_j = \frac{A + B - 2(n - 2)}{A - B}} \quad (3.47)
\]
which is just the optimal weight of asset \(i\) in the \((n - 1)\)-portfolio.

Furthermore, we can show that
\[
\rho_j < \frac{A + B - 2(n - 2)}{A - B} \quad (3.48)
\]
is equivalent to
\[ \rho_j < \frac{\bar{A} + \bar{B} - 2n + 2}{\bar{A} - \bar{B}}, \]  
which implies \( X_j > 0 \), and
\[ \rho_j > \frac{A + B - 2(n - 2)}{A - B} \]  
is equivalent to
\[ \rho_j > \frac{\bar{A} + \bar{B} - 2n + 2}{\bar{A} - \bar{B}}, \]
which implies that \( X_j < 0 \).

**Intuition 3.2.1** We can see clearly that in this case the value of \( X_j \) depends only on correlations. This makes sense since all assets share the same volatility.

Furthermore, suppose all assets other than asset \( j \) satisfy \( A > B \) (for example, the \( n - 1 \) assets are positively correlated to the market, i.e. \( \rho_k > 0 \), for \( k \neq j \)). Then the investor should take a long position in asset \( j \) (i.e. \( X_j > 0 \)) provided that
\[ \rho_j < \frac{A + B - 2(n - 2)}{A - B} = 1 + \frac{2(B - n + 2)}{A - B}. \]  
So investors should invest in low correlation assets as much as possible.

If the correlations to the market of all assets are very small, such that
\[ B \geq n - 2, \]
then the inequality (3.52) will automatically hold for all \( j = 1, \ldots, n \), which implies the investor should take long positions in one asset if the other assets have sufficiently small correlations.

On the contrary, if
\[ B < n - 2, \]
and
\[ \rho_j > \frac{A + B - 2(n - 2)}{A - B}, \]
then the investor should take a negative position in (i.e. short sell) asset \( j \) (i.e. \( X_j < 0 \)).

If
\[ \rho_j = \frac{A + B - 2(n - 2)}{A - B}, \]
then the investor should delete asset \( j \) from the portfolio (\( X_j = 0 \)).

In addition, the investor should long or short asset \( j \) no more than half of the total investment.

**Remark 3.2.4** Suppose all assets but asset \( j \) are positively correlated to the market and \( B < n - 2 \). As a function of \( \rho_j \), the weight \( X_j \) is decreasing while \( \rho_j \) varying from \(-1\) to \(1\).
Proof of Remark 3.2.4 Writing $X_j$ as a function of $\rho_j$, we have

$$X_j(\rho_j) = \frac{A + B + \frac{2}{1 - \rho_j^2} - 2n + 2 - \left( A - B + \frac{2}{1 - \rho_j^2} \right) \rho_j}{2 + \left( A - B + \frac{2}{1 - \rho_j^2} \right) (n-1) \left( A + B + \frac{2}{1 - \rho_j^2} \right)} - \frac{\rho_j}{\bar{A} - B}.$$  \hspace{1cm} (3.57)

Therefore, the derivative of $X$ with $\rho_j$ is

$$\frac{\partial}{\partial \rho_j} X_j(\rho_j) = \frac{-A - B (2AB - (n-1)(A+B)) \rho_j^2 + 2(A+B-2(n-2)) [2AB - (n-1)(A+B)] \rho_j - (A-B) [2AB - (n-1)(A+B-4)]}{(2AB - (n-1)(A+B))^2} \frac{1}{\bar{A} - B}.$$ \hspace{1cm} (3.58)

where

$$F(\rho_j) := \rho_j^2 - \frac{2[A + B - 2(n-2)]}{A - B} \rho_j + 1 + \frac{4(n - 1)}{2AB - (n-1)(A + B)}.$$  \hspace{1cm} (3.59)

Since $\rho_i > 0$ for all $i \neq j$, it is easy to prove that

$$A = \sum_{i \neq j} \frac{1}{1 - \rho_i} > n - 1 > B = \sum_{i \neq j} \frac{1}{1 + \rho_i}.$$ \hspace{1cm} (3.60)

Using Remark 3.2.1, we know for $\rho_j \in (-1, 1)$ the sign of $\frac{\partial}{\partial \rho_j} X_j(\rho_j)$ depends on the sign of $F(\rho_j)$.

Since $B < n - 2$, we can prove

$$0 < \frac{A + B - 2(n-2)}{A - B} < 1.$$ \hspace{1cm} (3.61)

Therefore, we have

$$F(\rho_j) = \left( \rho_j - \frac{A + B - 2(n-2)}{A - B} \right)^2 + 1 + \frac{4(n - 1)}{2AB - (n-1)(A + B)} - \left( \frac{A + B - 2(n-2)}{A - B} \right)^2.$$ \hspace{1cm} (3.62)

Hence $\frac{\partial}{\partial \rho_j} X_j(\rho_j) < 0$ for all $\rho_j \in (-1, 1)$, which implies $X_j$ is decreasing. \vspace{0.5cm}

Remark 3.2.5 Suppose all asset correlations satisfy $\bar{A} > n > n - 1 > \bar{B}$. If $\rho_k > \rho_l$, then $X_k < X_l$.

Proof of Remark 3.2.5 Consider the following function

$$G(x) := \frac{1}{2\bar{A} \bar{B} - (n-1)(\bar{A} + \bar{B})} \frac{(\bar{A} + \bar{B} - 2n + 2) - (\bar{A} - \bar{B})x}{1 - x^2}.$$ \hspace{1cm} (3.63)

It is easy to check that

$$X_k = G(\rho_k),$$ \hspace{1cm} (3.64)
for \( k = 1, \cdots, n \). We will show that \( G(x) \) is decreasing for \( x \in (-1, 1) \).

In fact, its derivative with respect to \( x \) is

\[
\frac{\partial}{\partial x} G(x) = \frac{1}{2AB - (n-1)(A+B)} \frac{-(\bar{A}-\bar{B})x^2 + 2(A+\bar{B}-2n+2)x - (\bar{A}-\bar{B})}{(1-x^2)^2} \left[ x^2 - \frac{2(A+\bar{B}-2n+2)}{A-B} x + 1 \right]
\]

\( (3.65) \)

Since \( \bar{B} < n - 1 \), we can show that

\[
0 < \frac{\bar{A} + \bar{B} - 2n + 2}{\bar{A} - \bar{B}} < 1,
\]

which implies \( \frac{\partial}{\partial x} G(x) < 0 \).

Therefore, if \( \rho_k < \rho_l \), then we must have

\[
X_k = G(\rho_k) > G(\rho_l) = X_l.
\]

\( (3.67) \)

**Intuition 3.2.2** From Remark 3.2.4 and Remark 3.2.5, we know the investor should invest less money on assets with high correlations to the market. See Figure 3.2.

**Remark 3.2.6** Suppose all asset correlations but \( \rho_j \) are positive and satisfy \( B < n - 2 \). Then the weight \( X_j \) could neither be higher than \( 1/2 \) nor lower than \(-1/2\). In addition we have

\[
\lim_{\rho_j \to -1} \frac{X_j}{X_i} = (1-\rho_i)(A-n+2) < 1,
\]

\( (3.68) \) and

\[
\lim_{\rho_j \to -1} \frac{X_j}{X_i} = (1+\rho_i)(B-n+2) < 0.
\]

\( (3.69) \)

**Proof of Remark 3.2.6** Here we need only verify the case when \( \rho_j \) approaches \(-1\). Notice that

\[
A = \sum_{k \neq j} \frac{1}{1-\rho_k} > \frac{n-1}{2} > \frac{n-2}{2},
\]

which implies

\[
2A + 2 - n > 0.
\]

\( (3.70) \)

Similarly, we can show

\[
2B + 2 - n > 0.
\]

\( (3.71) \)

Using (3.57), we have

\[
\lim_{\rho_j \to -1} X_j = \frac{A + B - 2(n-2) - (A - B)\rho_j}{2AB - (n-1)(A+B)(A-B)\rho_j^2 - 2(A-B)\rho_j + [2(A+1)(B+1) - (n-1)(A+B+2)]}
\]

\[
= \frac{A - n + 2}{2A - n + 2} < \frac{1}{2}.
\]

\( (3.73) \)
Similar to (3.47), we have

\[
\lim_{\rho_j \to -1} X_i = \lim_{\rho_j \to -1} \frac{A + B + \frac{1}{1 + \rho_i^2} + \frac{1}{1 + \rho_j^2} - 2n + 2 - (A - B + \frac{1}{1 + \rho_i^2} - \frac{1}{1 + \rho_j^2})}{2(A + B + \frac{1}{1 + \rho_i^2} + \frac{1}{1 + \rho_j^2})} \frac{1}{1 - \rho_i^2} + \frac{1}{1 - \rho_j^2} \\
= \lim_{\rho_j \to -1} \frac{1}{\rho_j \to -1} \frac{1}{2(A + B + \frac{1}{1 + \rho_i^2} - 2n + 2 - (A - B + \frac{1}{1 + \rho_i^2} + \frac{1}{1 + \rho_j^2}) \frac{1}{1 - \rho_i^2}} < \frac{1}{(1 - \rho_i)n}.
\]

(3.74)

Using the same method, we can show

\[
\lim_{\rho_j \to 1} X_j = \frac{B - n + 2}{2B - n + 2} > \frac{n + 2}{2(n - 2) - n + 2} = \frac{1}{2},
\]

(3.75)

and

\[
\lim_{\rho_j \to 1} X_i = \frac{1}{(1 - \rho_i)(2B - n + 2)} > \frac{1}{(1 - \rho_i)(n - 2)}.
\]

(3.76)

Furthermore, if \( \rho_k > 0, k \neq j \), then we have

\[
\sum_{k \neq i,j} \frac{1}{1 - \rho_k} > n - 2,
\]

(3.77)

and hence

\[
\lim_{\rho_j \to 1} \frac{X_j}{X_i} = (1 - \rho_i)(A - n + 2) = 1 + (1 - \rho_i)(\sum_{k \neq i,j} \frac{1}{1 - \rho_k} - n + 2) > 1.
\]

(3.78)

From Remark 3.2.4, \( X_j \) is decreasing. Therefore, from monotonicity, we have

\[
-\frac{1}{2} < X_j < \frac{1}{2}.
\]

(3.79)

Remark 3.2.7 Furthermore, suppose that of the above \( n \) assets, \( k (k = 0, 1, \ldots, n) \) assets which share the same correlation to the market, while the remaining \( n - k \) are uncorrelated to the market, i.e.

\[
\rho_{1m} = \rho_{2m} = \cdots = \rho_{km} = \rho, \rho_{(k+1)m} = \cdots = \rho_{nm} = 0.
\]

(3.80)

Then we have for \( j = 1, \ldots, k \),

\[
X_i = \left( \frac{1}{1 + (1 - \frac{k}{n})(k - 1)\rho^2} \right) \frac{1}{n},
\]

(3.81)

and for \( i = k + 1, \ldots, n \),

\[
X_i = \left( \frac{1 + (k - 1)\rho^2}{1 + (1 - \frac{k}{n})(k - 1)\rho^2} \right) \frac{1}{n}.
\]

(3.82)
3.3. Two-Asset Case

**Proof of Remark 3.2.7** Substituting (3.80), we have

\[
\tilde{A} = \frac{k}{1 - \rho} + n - k, \quad \tilde{B} = \frac{k}{1 + \rho} + n - k.
\]

(3.83)

\[
\tilde{A} - n + 1 = \frac{k}{1 - \rho} - k + 1, \quad \tilde{B} - n + 1 = \frac{k}{1 + \rho} - k + 1,
\]

(3.84)

\[
2\tilde{A}\tilde{B} - (n - 1)(\tilde{A} + \tilde{B}) = \frac{2k(n - k + 1)}{1 - \rho^2} - 2(n - k)(k - 1).
\]

(3.85)

Therefore, for \( j = 1, \cdots, k, \)

\[
X_j = \left( \frac{\frac{k}{1 - \rho} - k + 1}{1 - \rho} + \left( \frac{\frac{k}{1 + \rho} - k + 1}{1 + \rho} \right) \frac{1}{1 - \rho} \right) \frac{1}{n},
\]

(3.86)

and for \( i = k + 1, \cdots, n, \)

\[
X_i = \left( \frac{\frac{k}{1 - \rho} - k + 1 + \frac{k}{1 + \rho} - k + 1}{2k(n - k + 1)/1 - \rho^2} - 2(n - k)(k - 1) \right) \left( \frac{1 + (k - 1)\rho^2}{1 + (1 - \frac{k}{n})(k - 1)\rho^2} \right) \frac{1}{n}.
\]

(3.87)

It is easy to verify that \( \sum_j X_j = 1. \)

**Intuition 3.2.3** In Remark 3.2.7 case, when \( k > 1, \) to minimize the risk, the investor should take long positions in all assets and invest more weights on the assets that are uncorrelated to the market. In particular, if \( k = 1, \) then it is easy to see that

\[
X_1 = X_2 = \cdots = X_n = \frac{1}{n},
\]

(3.88)

which is equivalent to the case when \( \rho = 0. \) This result makes sense because in both cases the portfolio contains \( n \) assets uncorrelated to each other and sharing the same risk. Hence the optimal way to minimize the risk is to weight the assets equally.

## 3.3 Two-Asset Case

In this section we focus on the two-asset portfolio. This special portfolio is analytically tractable that we find many interesting results. We start this section with some interesting observations (3.89) and (3.90). In Remark 3.3.1 we discuss the resulting intuition and in Remark 3.3.1 we try to give economic explanations. We find that the two-asset portfolio’s composition depends on two parameters, the ratio of asset volatilities and the correlation between assets. In Section 3.3.1 we discuss the shape of the Proportion vs. Volatility Ratio curve. We study two cases with different signs of correlation. In section 3.3.2 we discuss the Proportion vs. Correlation curve. In each case we give the boundary and asymptotic properties of the Proportion curve.
Suppose now there is a portfolio which contains two assets. The closed form solution to the corresponding MV problem (1.3) is given by (2.40) (or (2.41)). From (2.40), we have

\[ X_1 - X_2 = \frac{1 - \left(\frac{\sigma_1}{\sigma_2}\right)^2}{\left(\frac{\sigma_1}{\sigma_2}\right)^2 - 2\rho\frac{\sigma_1}{\sigma_2} + 1}, \tag{3.89} \]

\[ \frac{\partial X_1}{\partial \rho} = \frac{\frac{\sigma_1}{\sigma_2} \left[1 - \left(\frac{\sigma_1}{\sigma_2}\right)^2\right]}{\left[\left(\frac{\sigma_1}{\sigma_2}\right)^2 - 2\rho\frac{\sigma_1}{\sigma_2} + 1\right]^2} = -\frac{\partial X_2}{\partial \rho}. \tag{3.90} \]

**Remark 3.3.1** First, if \( \sigma_1 = \sigma_2 \), from (3.89) we get \( X_1 = X_2 = \frac{1}{2} \). Intuition of this is discussed in Intuition 3.3.1. Furthermore, from (3.90)

\[ \frac{\partial X_1}{\partial \rho} = \frac{\partial X_2}{\partial \rho} = 0, \tag{3.91} \]

i.e. the correlation \( \rho \) has no influence on the portfolio weights.

Second, if \( \sigma_1 > \sigma_2 \), then \( X_1 < \frac{1}{2} < X_2 \), and the reverse is also true. We can conclude that whatever the correlations of these two assets are, the asset with less risk should have more weight and the other with more risk should have less weight.

More particularly, if \( \sigma_1 = 0 \) and \( \sigma_2 > 0 \), then using (2.40) we get \( X_1 = 1 \) and \( X_2 = 0 \). This means that if the asset 1 has no risk the completely risk-averse investor should invest all money into it. On the other hand, if \( \sigma_1 = \infty \) and \( \sigma_2 < \infty \), then we get \( X_1 = 0 \) and \( X_2 = 1 \), which means that asset 1 is so risky that the investor should invest all his/her money in the other one.

**Intuition 3.3.1** When the two assets have the same volatility, from the above discussion, we know that no matter the correlation between them, the best way for the investor to reduce the risk of the portfolio is to invest the same amount of money in the two. Notice the two-asset mean-variance problem can be looked at as a hedging problem with constraint condition of fixed total investment. To briefly interpret this interesting result, let’s see some special cases of the correlation.

First, the two assets are positively correlated, i.e. \( \rho > 0 \). If there is no constraint on the total investment, then the investor would want to short one asset to hedge the risk of another. But in our case, selling one asset will make him buy more of another, which makes it impossible to hedge the total risk perfectly, since they have the same risk. So he has to take long positions in both of the two assets.

Second, the two assets are uncorrelated, i.e. \( \rho = 0 \). Then the investor surely should invest more money in the asset with less risk.

Third, the two assets are negatively correlated, i.e. \( \rho < 0 \). To hedge the risk of buying one asset, the investor has to invest some money in the other one.

Therefore, in all cases, the investor should buy the two assets. Since the two assets have the same risk, he must split his investment equally between them.

### 3.3.1 Volatility Sensitivity

From (2.40) we notice that the values of \( X_1 \) and \( X_2 \) depends only on two parameters: the ratio of two volatilities \( \frac{\sigma_1}{\sigma_2} \) and the correlation between the two assets \( \rho_1 \rho_2 \). Therefore, making the
natural assumption that $\sigma_2 > 0$, we set

$$\lambda := \frac{\sigma_1}{\sigma_2},$$  \hspace{1cm} (3.92)

and

$$\rho := \rho_1 \rho_2.$$  \hspace{1cm} (3.93)

Then (2.40) can be rewritten as

$$X_1 = \frac{1 - \frac{\rho\lambda}{\lambda^2 - 2\rho\lambda + 1}}{\lambda^2 - 2\rho\lambda + 1}, \quad X_2 = 1 - X_1.$$  \hspace{1cm} (3.94)

We already obtained that

$$\begin{cases} X_1 = 1, X_2 = 0 & \text{when } \lambda = 0 \\ X_1 = X_2 = \frac{1}{2} & \text{when } \lambda = 1 \\ X_1 = 0, X_2 = 1 & \text{when } \lambda = \infty \end{cases},$$  \hspace{1cm} (3.95)

which might suggest the intuition that $X_1$ is decreasing while $\lambda$ is increasing. This intuition is not true. To check this, let’s take the partial derivatives of $X$ with respect to $\lambda$, which is

$$\frac{\partial X_1}{\partial \lambda} = -\frac{\partial X_2}{\partial \lambda} = \frac{(\lambda^2 + 1)[\rho - \frac{2\lambda}{\lambda^2 + 1}]}{[\lambda^2 - 2\rho\lambda + 1]^2},$$  \hspace{1cm} (3.96)

**Proof of (3.96)** Consider the partial derivatives of the weights with respect to the ratio of standard deviations. We get

$$\frac{\partial}{\partial \lambda} X_1 = \frac{-\rho(\lambda^2 - 2\rho\lambda + 1)(\lambda - \rho)}{[\lambda^2 - 2\rho\lambda + 1]^2} = \frac{\lambda^2 - 2\rho\lambda + 1 + (\lambda^2 + 1)[\rho - \frac{2\lambda}{\lambda^2 + 1}]}{[\lambda^2 - 2\rho\lambda + 1]^2},$$  \hspace{1cm} (3.97)

and

$$\frac{\partial}{\partial \lambda} X_2 = \frac{\partial}{\partial \lambda}(1 - X_1) = -\frac{\partial X_1}{\partial \lambda}.$$  \hspace{1cm} (3.98)

**Remark 3.3.2** Here we find the very interesting fact that whether $X_1$ (or $X_2$) increases or decreases as $\lambda$ increases depends on the sign of the term

$$\rho - \frac{2\lambda}{\lambda^2 + 1}.$$  \hspace{1cm} (3.99)

- **When $\rho$ is positive**, (3.99) could be zero for some $\lambda$. Since

$$\lambda^2 + 1 - 2\lambda = (\lambda - 1)^2 \geq 0,$$  \hspace{1cm} (3.100)

we have

$$0 \leq \frac{2\lambda}{\lambda^2 + 1} \leq 1,$$  \hspace{1cm} (3.101)

where the first equality holds when $\lambda$ is zero, and the second equality holds when $\lambda = 1$. 
Figure 3.3: Optimal Weights vs. Ratio of SDs (Positively Correlated). $\lambda = \sigma_1/\sigma_2$ is the ratio of the two assets’ standard deviations, and the correlation $\rho = 1/2$. As $\lambda$ increases from 0 to $\infty$, the optimal weights of asset 1 $X_1$ will first increase above 1, then decrease below 0, and finally increase to an asymptotic of 0. Both curves of $X_1$ and $X_2$ will pass the point $E$ (1, 1/2).

- When $\rho$ is negative, (3.99) will always be negative for any $\lambda \geq 0$.

We will discuss the two cases in the rest of this section.

**Case 1** Suppose the two assets are positively correlated (i.e. $\rho > 0$). See Figure 3.3 and Figure 3.4, where $\rho = 1/2$ and $\lambda$ is perturbed.

In Figure 3.3, as $\lambda$ increases from 0 to $\infty$, the optimal weights of asset 1 $X_1$ will first increase above 1, then decrease below 0, and finally increase to an asymptote of 0. We will explain this behaviour in the remainder of this subsection.

**Remark 3.3.3** As $\lambda$ increases from 0 to $\infty$, $X_1$ will first increase above 1, then decrease below 0, and finally increase to a horizontal asymptote of 0.
3.3. Two-Asset Case

Figure 3.4: Optimal Weights vs. Log (Ratio of SDs) (Positively Correlated). Taking log(λ) as the x-axis, the curves of the optimal weights are symmetric to the y-axis. As log(λ) grows from −∞ to ∞, X₁ will first increase above 1, then decrease down below 0, and finally approximately increase to 0. Both curves of X₁ and X₂ will pass the point (0, 1/2).

Proof of Remark 3.3.3 For fixed ρ ∈ (0, 1) and σ₂ > 0, the following equation w.r.t. λ

\[ \rho \lambda^2 - 2\lambda + \rho = 0 \]  

has two non-negative solutions, namely

\[ \lambda = \frac{1 \pm \sqrt{1 - \rho^2}}{\rho}. \]
Hence
\[
\begin{align*}
\frac{\partial X_1}{\partial \lambda} > 0, & \text{ when } 0 < \lambda < \frac{1 - \sqrt{1 - \rho^2}}{\rho} \\
\frac{\partial X_1}{\partial \lambda} = 0, & \text{ when } \lambda = \frac{1 - \sqrt{1 - \rho^2}}{\rho} \\
\frac{\partial X_1}{\partial \lambda} < 0, & \text{ when } \frac{1 - \sqrt{1 - \rho^2}}{\rho} < \lambda < \frac{1 + \sqrt{1 - \rho^2}}{\rho} \\
\frac{\partial X_1}{\partial \lambda} = 0, & \text{ when } \lambda = \frac{1 + \sqrt{1 - \rho^2}}{\rho} \\
\frac{\partial X_1}{\partial \lambda} > 0, & \text{ when } \lambda > \frac{1 + \sqrt{1 - \rho^2}}{\rho}
\end{align*}
\] (3.104)

Therefore, as \( \lambda \) increases from 0 to \( \infty \), \( X_1 \) will first increase above 1, then decrease below 0, and finally increase to a horizontal asymptote of 0.

**Remark 3.3.4** Noticing that when \( \sigma_1 = \sigma_2/\rho \), or \( \sigma_1 = \rho \sigma_2 \), although neither of \( \sigma_1 \) and \( \sigma_2 \) is zero or infinity, we still have \( X_1 = 0 \) or 1 respectively. That means besides the two ends, we have 5 other special points:

- **A**: \( X_1 = \frac{\frac{\rho^2}{2 \sqrt{1 - \rho^2}(1 - \sqrt{1 - \rho^2})}}{\lambda^2 - \lambda + 1} \) achieves its highest point, when \( \lambda = \frac{1 - \sqrt{1 - \rho^2}}{\rho} \)
- **B**: \( X_1 = -\frac{\rho^2}{2 \sqrt{1 - \rho^2}(1 + \sqrt{1 - \rho^2})} \) achieves its lowest point, when \( \lambda = \frac{1 + \sqrt{1 - \rho^2}}{\rho} \)
- **C**: \( X_1 = 1 \), when \( \lambda = \rho \)
- **D**: \( X_1 = 0 \), when \( \lambda = 1/\rho \)
- **E**: \( X_1 = 0.5 \), when \( \lambda = 1 \)

**Intuition 3.3.2** To get an intuitive image, assume that \( \rho = 1/2 \). Then
\[
X_1 = \frac{1 - \frac{1}{2} \lambda}{\lambda^2 - \lambda + 1}, \quad X_2 = 1 - X_1, \quad \frac{\partial X_1}{\partial \lambda} = \frac{1}{2} \lambda^2 - 2 \lambda + \frac{1}{2} \frac{(\lambda^2 - \lambda + 1)^2}{(\lambda^2 - \lambda + 1)^2}.
\] (3.105)

See Figure 3.3 and Figure 3.4. Figure 3.3 clearly depicts that the shape of \( X_1 \) varies just as discussed above. The curve of \( X_2 \) has a same shape but as a mirror reflection of \( X_1 \).

The curve of \( \frac{\partial}{\partial \lambda} X_1 \) always starts from \( \rho \) (take \( \lambda = 0 \)) and is negative for \( \lambda \in \left( \frac{1 - \sqrt{1 - \rho^2}}{\rho}, \frac{1 + \sqrt{1 - \rho^2}}{\rho} \right) \), then it is positive but quite close to zero. Hence we can expect to find its boundary.

**Remark 3.3.5** An upper-bound of \( \frac{\partial}{\partial \lambda} X_1 \) is
\[
\left| \frac{\partial X_1}{\partial \lambda} \right| \leq \frac{\frac{3}{2}(1 - \rho^2)(\rho + \sqrt{3 + \rho^2})}{\left[ -\frac{4}{3} \rho (\rho + \sqrt{3 + \rho^2}) + \frac{1}{3} \right]^2}.
\] (3.106)
3.3. Two-Asset Case

Proof of Remark 3.3.5 Calculating the derivative of $X_1$ with $\lambda$, we have

$$\frac{\partial X_1}{\partial \lambda} = \frac{\rho (\lambda^2 - 2\lambda + \rho)}{(\lambda^2 - 2\rho \lambda + 1)^2}$$

$$= \frac{\rho}{\lambda^2 - 2\rho \lambda + 1} - \frac{2(1 - \rho^2)\lambda}{(\lambda^2 - 2\rho \lambda + 1)^2}$$

$$\leq \max_{\lambda > 0} \left\{ \rho g(\lambda) - 2(1 - \rho^2) f(\lambda), 2(1 - \rho^2) f(\lambda) - \rho g(\lambda) \right\}$$

$$\leq \max \left\{ \rho \max_{\lambda > 0} g(\lambda) - 2(1 - \rho^2) \min_{\lambda > 0} f(\lambda), 2(1 - \rho^2) \max_{\lambda > 0} f(\lambda) - \rho \min_{\lambda > 0} g(\lambda) \right\},$$

where

$$f(\lambda) := \frac{\lambda}{(\lambda^2 - 2\rho \lambda + 1)^2}, \quad g(\lambda) := \frac{1}{\lambda^2 - 2\rho \lambda + 1}. \quad (3.107)$$

Then we need only to find the upper-bounds and lower-bounds of $f(\lambda)$ and $g(\lambda)$. To do this, check the derivative of $f(\lambda)$

$$\frac{d}{d\lambda} f(\lambda) = -\frac{3\lambda^2 + 2\rho \lambda + 1}{(\lambda^2 - 2\rho \lambda + 1)^3}. \quad (3.109)$$

To find its extrema, we must solve the quadratic equation

$$f_1(\lambda) := -3\lambda^2 + 2\rho \lambda + 1 = 0. \quad (3.110)$$

The solutions are

$$\lambda_1 = \frac{1}{3}(\rho + \sqrt{3 + \rho^2}), \quad \lambda_2 = \frac{1}{3}(\rho - \sqrt{3 + \rho^2}) \quad (3.111)$$

where apparently $\lambda_1 > 0$ is the only root of $f_1(\lambda)$ lying inside $(0, \infty)$, which can be used to find the maximum point of $f(\lambda)$ for $\lambda > 0$. Notice that $f_1(\lambda)$ is concave, which means that it is positive for $0 < \lambda < \lambda_1$ and negative for $\lambda > \lambda_1$, and so is $f'(\lambda)$. Hence $\lambda_1$ is the local maximum of $f(\lambda)$. At this point, we have

$$f(\lambda_1) = \frac{\frac{1}{3}(\rho + \sqrt{3 + \rho^2})}{\frac{1}{3}(\rho + \sqrt{3 + \rho^2})^2 - 2\frac{1}{3}(\rho + \sqrt{3 + \rho^2}) + 1}$$

$$= \frac{\frac{1}{3}(\rho + \sqrt{3 + \rho^2})}{\frac{1}{3}(\rho + \sqrt{3 + \rho^2})^2 - \frac{2}{3}(\rho + \sqrt{3 + \rho^2}) + 1}. \quad (3.112)$$

Since $f(0) = f(\infty) = 0$, using monotone properties we know that $f(\lambda)$ achieves its maximum $f(\lambda_1)$ at $\lambda_1$ and its minimum 0 at 0 and $\infty$.

For $g(\lambda)$ we have

$$0 < g(\lambda) = \frac{1}{(\lambda - \rho)^2 + 1 - \rho^2} \leq \frac{1}{1 - \rho^2}. \quad (3.113)$$

And we know

$$\lim_{\lambda \to \infty} g(\lambda) = 0. \quad (3.114)$$

Therefore,

$$\min_{\lambda > 0} g(\lambda) = 0, \quad \max_{\lambda > 0} g(\lambda) = \frac{1}{1 - \rho^2}. \quad (3.115)$$

So we have

$$\left| \frac{\partial X_1}{\partial \lambda} \right| < \max \left\{ \frac{\rho}{1 - \rho^2}, \left[ -\frac{4}{3} \rho (\rho + \sqrt{3 + \rho^2}) + \frac{4}{3} \right]^2 \right\} = \frac{2}{3} (1 - \rho^2) (\rho + \sqrt{3 + \rho^2}) \left[ -\frac{4}{3} \rho (\rho + \sqrt{3 + \rho^2}) + \frac{4}{3} \right]^2 \quad (3.116)$$

The second equality holds for $\rho \in (0, 1)$. \[\square\]
Case 2 Suppose now the two assets are negatively correlated (i.e. \(-1 \leq \rho < 0\)). See Figure 3.5, where \(\rho = -1/2\) and \(\lambda\) is perturbed.

In Figure 3.5. It is clear that \(X_1(0) = 1, X_1(\infty) = 0\), and \(X_1(\lambda)\) is monotone decreasing. We will explain this behaviour in the remainder of this subsection.

![Figure 3.5: Optimal Weight vs. Ratio of SDs (Negatively Correlated).](image)

Figure 3.5: Optimal Weight vs. Ratio of SDs (Negatively Correlated). \(X_1(0) = 1, X_1(\infty) = 0\), and \(X_1(\lambda)\) is monotone decreasing.

**Remark 3.3.6** \(X_1\) decreases asymptotically to 0 and \(X_2\) increases asymptotically to 1 as \(\lambda\) increases from 0 to \(\infty\).

**Proof of Remark 3.3.6** For all \(\lambda \geq 0\), we always have

\[
\frac{\partial X_1}{\partial \lambda} = -\frac{\partial X_2}{\partial \lambda} = \frac{(\lambda^2 + 1)[\rho - \frac{2\lambda}{\mu + 1}]}{[(\lambda^2 - 2\rho\lambda + 1)^2]} < 0.
\] (3.117)

Hence \(X_1\) decreases and \(X_2\) increases as \(\lambda\) increases from 0. Noticing that \(X_1(0) = 1, X_1(\infty) = 0\), and \(X_1(\lambda)\) is monotone decreasing, we know \(X_1 \in (0, 1)\) for \(\lambda > 0\), and so does \(X_2\).
3.3. Two-Asset Case

**Intuition 3.3.3** It is clear from Figure 3.5 that $X_1(0) = 1$, $X_1(\infty) = 0$, and $X_1(\lambda)$ is monotone decreasing.

**Remark 3.3.7** Since $\frac{\partial}{\partial \lambda} X_1 = \rho$ when $\lambda = 0$ and $\lim_{\lambda \to \infty} \frac{\partial}{\partial \lambda} X_1 = 0$, we can find its boundary for $\lambda \in [0, \infty)$, which is

$$\left| \frac{\partial X_1}{\partial \lambda} \right| < \frac{\frac{2}{3}(1 - \rho^2)(\rho + \sqrt{3 + \rho^2})}{\left[ \frac{4}{3} - \frac{4}{3}(\rho + \sqrt{3 + \rho^2}) \right]^2} - \frac{\rho}{1 + \rho^2}. \quad (3.118)$$

**Proof of 3.3.7** Using almost the same steps as in Case 1, we obtain

$$\left| \frac{\partial X_1}{\partial \lambda} \right| = \frac{\rho \lambda^2 - 2 \lambda \rho + \rho}{(\lambda^2 - 2 \rho \lambda + 1)^2} - \frac{2(1 - \rho^2) \lambda}{(\lambda^2 - 2 \rho \lambda + 1)^2}$$

$$= \frac{-\rho + \frac{2(1 - \rho^2)}{1 - \rho^2} \lambda^2}{(\lambda^2 - 2 \rho \lambda + 1)^2} \leq -\rho \max_{\lambda > 0} g(\lambda) + 2(1 - \rho^2) \max_{\lambda > 0} f(\lambda)$$

$$= -\rho \max_{\lambda > 0} \left\{ \frac{\lambda(1 - \lambda)(1 + \lambda)}{(\lambda + 1)^3}, \frac{\lambda(1 + \lambda)}{(1 - \lambda)^3} \right\}.$$  

The equality holds when $\lambda = \lambda_1 = \frac{1}{3}(\rho + \sqrt{3 + \rho^2})$.  

3.3.2 Correlation Sensitivity

Now consider the sensitivity of the optimal weight to correlation $\rho$. See Figure 3.6 as an example. The two dashed curves represent $\rho = 1$ and $\rho = -1$, respectively. The full curves represent $\rho = -0.1, 0.9, 0.999$, respectively.

In Figure 3.6, all curves are disjoint from each other for $0 < \lambda < 1$ and $\lambda > 1$. The full curves are bounded by the two dashed curves. The curve of $\rho = 1$ is meaningless when $\lambda = 1$. The curve of $\rho = 0.999$ has two sharp peaks for some values of $\lambda$ not far from 1. The curves with smaller values of $\rho$ are flatter. We will explain the behaviour observed in the remainder of this subsection.

To explain the behaviour observed above in Figure 3.6, we need to study the derivative of weight $X_1$ to the correlation $\rho$.

**Remark 3.3.8** The derivative of weight $X_1$ to the correlation $\rho$ is

$$\frac{\partial X_1}{\partial \rho} = \frac{\lambda(1 - \lambda)(1 + \lambda)}{(\lambda^2 - 2 \rho \lambda + 1)^2}. \quad (3.120)$$

And we have

$$\min \left\{ \frac{\lambda(1 - \lambda)}{(\lambda + 1)^3}, \frac{\lambda(1 + \lambda)}{(1 - \lambda)^3} \right\} < \frac{\partial X_1}{\partial \rho} < \max \left\{ \frac{\lambda(1 - \lambda)}{(\lambda + 1)^3}, \frac{\lambda(1 + \lambda)}{(1 - \lambda)^3} \right\}. \quad (3.121)$$

**Proof of Remark 3.3.8** It is much easier to find upper and lower bounds for this expression. Noticing that for $-1 < \rho < 1$, we have

$$\lambda(1 - \lambda)^2 < \lambda^2 - 2 \rho \lambda + 1 < (\lambda + 1)^2. \quad (3.122)$$
Figure 3.6: Optimal Weight versus Ratio of Volatilities. The two dotted curves represent $\rho = 1$ and $\rho = -1$, respectively. All curves with $\rho \in (0, 1)$ are bounded by the two curves, and they only join at the point $\left(1, \frac{1}{2}\right)$. When $\rho$ is very close to 1, there are two peak points on the curve. As $\rho$ goes down to -1, the curve will become flatter and flatter.

Therefore, if $0 \leq \lambda < 1$, 

$$0 < \frac{\lambda(1 - \lambda)}{(\lambda + 1)^3} < \frac{\partial X_1}{\partial \rho} < \frac{\lambda(1 + \lambda)}{(1 - \lambda)^3}.$$  (3.123)

If $\lambda = 1$, then 

$$\frac{\partial X_1}{\partial \rho} = 0.$$  (3.124)

If $\lambda > 1$, then 

$$\frac{\lambda(1 + \lambda)}{(1 - \lambda)^3} < \frac{\partial X_1}{\partial \rho} < \frac{\lambda(1 - \lambda)}{(\lambda + 1)^3} < 0.$$  (3.125)

**Remark 3.3.9** It is shown that when $0 \leq \lambda < 1$, $\frac{\partial}{\partial \rho} X_1 > 0$, i.e. for any $0 < \rho_1 < \rho_2 < 1$, for all $\lambda \in [0, 1)$, 

$$X_1|_{\rho_1} < X_1|_{\rho_2}.$$  (3.126)
Hence the two curves which represent the optimal weights with different correlations are disjoint from each other for $\lambda \in [0, 1)$. That implies that all the possible optimal weights will be bounded by the two special cases when $\rho = 1$ and $\rho = 0$. See Figure 3.6.

The curve of $\rho = 1$ is meaningless when $\lambda = 1$. In fact, in that case, the two assets are positively linearly correlated and have the same risk. Hence the optimal weights problem is meaningless, as any allocation is equivalent to any other.

**Intuition 3.3.4** We can see that when the correlation of the two assets is quite close to 1 (e.g. 0.999), there will be two sharp peaks on the curve for some $\lambda$ not far from 1. To avoid highly sensitive cases such as these, the investor should build his/her portfolio with assets not too alike.

### 3.4 An Interesting Relationship between Weights, Betas, Correlations and Volatilities

Recall Figure 3.2 and Figure 3.1. It can be observed that as $\rho_1$ and $\sigma_1$ varying the curves of assets 2 to asset 10 change slowly and “in parallel”. We will discuss this observation and give a relationship between the minimum variance portfolio’s weights, betas, correlations and volatilities (see Remark 3.4.1). We will show that if two different assets have the same Beta, the risk-averse investor, when faced with the choice between the low volatility asset or the low correlation asset, should always opt for the low volatility asset (see Remark 3.4.2).

We begin with an example in Figure 3.7, where the original data is shown in Table 3.1. The correlation of asset 1 is perturbed, which means the betas of other assets are constant. Therefore, in Figure 3.7 the curves of assets weights other than $X_1$ change slowly and in parallel.

We can see that asset 7 and asset 8 have quite close sensitivities to the market excess return ($\beta_7 = 1.3160$ and $\beta_8 = 1.3173$). But asset 7 has higher correlation and lower volatility. It is quite clear that $X_8 > X_7 > 0$. Asset 5 and asset 6 have close sensitivities to the market excess return ($\beta_5 = 0.6750$ and $\beta_6 = 0.6898$). But asset 6 has lower volatility and higher correlation to the market. In Figure 3.7 it is quite clear that $X_5 > X_5 > 0$.

**Remark 3.4.1** Suppose $X = (X_1, \cdots, X_N)$ is the solution of the MV problem (1.3). For each $k \neq j$, we have

\[
\sum_i \rho_i \sigma_i X_i = \frac{(1 - \rho_k^2)\sigma_k^2 X_k - (1 - \rho_j^2)\sigma_j^2 X_j}{\rho_j \sigma_j - \rho_k \sigma_k}, \tag{3.127}
\]

or we can write it in the other forms

\[
\sum_i \beta_i X_i = \frac\left(\frac{(\sigma_i}{\sigma_m})^2 - \beta_i^2\right) X_k - \frac\left(\frac{(\sigma_i}{\sigma_m})^2 - \beta_j^2\right) X_j = \frac{\left(\frac{1}{\rho_i^2} - 1\right)}{\beta_j - \beta_k} \beta_k^2 X_k - \frac{\left(\frac{1}{\rho_j^2} - 1\right)}{\beta_j - \beta_k} \beta_j^2 X_j. \tag{3.128}
\]

**Proof of Remark 3.4.1** Using the definitions of $f_i$ and $g_i$, we have

\[
(1 - \rho_k^2)\sigma_k(f_k + g_k) = (1 - \rho_k^2)\sigma_k \left(\frac{1}{\sigma_k(1 - \rho_k)} + \frac{1}{\sigma_k(1 + \rho_k)}\right) = 2, \tag{3.129}
\]
Table 3.1: SPTSX Top Ten Stocks 1 Year Data of 2003/12/5 to 2004/12/3.

<table>
<thead>
<tr>
<th>Assets</th>
<th>Correlation</th>
<th>Volatility</th>
<th>Beta</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>46.59%</td>
<td>12.95%</td>
<td>0.5306</td>
</tr>
<tr>
<td>2</td>
<td>44.07%</td>
<td>16.09%</td>
<td>0.6237</td>
</tr>
<tr>
<td>3</td>
<td>44.49%</td>
<td>16.35%</td>
<td>0.6397</td>
</tr>
<tr>
<td>4</td>
<td>49.86%</td>
<td>14.99%</td>
<td>0.6573</td>
</tr>
<tr>
<td>5</td>
<td>39.72%</td>
<td>19.32%</td>
<td>0.6750</td>
</tr>
<tr>
<td>6</td>
<td>56.34%</td>
<td>13.92%</td>
<td>0.6898</td>
</tr>
<tr>
<td>7</td>
<td>45.34%</td>
<td>33%</td>
<td>1.3160</td>
</tr>
<tr>
<td>8</td>
<td>55.60%</td>
<td>28.05%</td>
<td>1.3222</td>
</tr>
<tr>
<td>9</td>
<td>53.60%</td>
<td>26.94%</td>
<td>1.3173</td>
</tr>
<tr>
<td>10</td>
<td>55.04%</td>
<td>28.38%</td>
<td>1.3737</td>
</tr>
</tbody>
</table>

\[
(1 - \rho_k^2)\sigma_k^2(f_k - g_k) = (1 - \rho_k^2)\sigma_k^2\left(\frac{1}{\sigma_k(1 - \rho_k)} - \frac{1}{\sigma_k(1 + \rho_k)}\right) = 2\sigma_k\rho_k, \quad (3.130)
\]

\[
\sum_i \rho_i \sigma_i \left(\frac{f_i + g_i}{\sigma_i}\right) = \sum_i \frac{2\rho_i}{\sigma_i(1 - \rho_i^2)} = \sum_i (f_i - g_i), \quad (3.131)
\]

and

\[
\sum_i \rho_i \sigma_i (f_i - g_i) = \sum_i \frac{2\rho_i^2}{1 - \rho_i^2} = -2n + \sum_i \sigma_i (f_i + g_i). \quad (3.132)
\]

Therefore, substituting formulas (2.4) into both sides of (3.127), we have

\[
\text{Left Side} = \sum_i \rho_i \sigma_i \left[\frac{f_i + g_i}{\sigma_i}\right] (2n - 2 - \sum_i \sigma_i (f_i + g_i)) + (f_i - g_i) \sum_i (f_i - g_i) \bigg/ \hat{c}
\]

\[
= \left[ (2n - 2 - \sum_i \sigma_i (f_i + g_i)) \sum_i \rho_i \sigma_i \left[\frac{f_i + g_i}{\sigma_i}\right] + \sum_i (f_i - g_i) \sum_i \rho_i \sigma_i (f_i - g_i) \right] \bigg/ \hat{c} \quad (3.133)
\]

\[
= \left[ (2n - 2 - \sum_i \sigma_i (f_i + g_i)) \sum_i (f_i - g_i) + \sum_i (f_i - g_i) (-2n + \sum_i \sigma_i (f_i + g_i)) \right] \bigg/ \hat{c}
\]

\[
= -2 \sum_i (f_i - g_i) \bigg/ \hat{c},
\]

and

\[
(\sigma_j \rho_j - \sigma_k \rho_k) \cdot \text{Right Side}
\]

\[
= (1 - \rho_j^2)\sigma_j^2 \left[\frac{f_i + g_i}{\sigma_j}\right] (2n - 2 - \sum_i \sigma_i (f_i + g_i)) + (f_j - g_j) \sum_i (f_i - g_i) \bigg/ \hat{c}
\]

\[
- (1 - \rho_j^2)\sigma_j^2 \left[\frac{f_i + g_i}{\sigma_j}\right] (2n - 2 - \sum_i \sigma_i (f_i + g_i)) + (f_j - g_j) \sum_i (f_i - g_i) \bigg/ \hat{c}
\]

\[
= (2n - 2 - \sum_i \sigma_i (f_i + g_i)) \left[ (1 - \rho_j^2)\sigma_j (f_k + g_k) - (1 - \rho_j^2)\sigma_j (f_j + g_j) \right] \bigg/ \hat{c} + \sum_i (f_i - g_i) \left[ (1 - \rho_j^2)\sigma_j^2 (f_k - g_k) - (1 - \rho_j^2)\sigma_j^2 (f_j - g_j) \right] \bigg/ \hat{c}
\]

\[
= (2n - 2 - \sum_i \sigma_i (f_i + g_i)) (2 - 2) \bigg/ \hat{c} + \sum_i (f_i - g_i) (2\sigma_k \rho_k - 2\sigma_j \rho_j) \bigg/ \hat{c}
\]

\[
= 2(\sigma_k \rho_k - \sigma_j \rho_j) \sum_i (f_i - g_i) \bigg/ \hat{c}
\]

Hence Left Side = Right Side, which verifies equation (3.127).

We can get many interesting results from (3.127).
3.4. An Interesting Relationship between Weights, Betas, Correlations and Volatilities

Figure 3.7: 10 Asset Weights vs Perturbation of $\rho_1$. $\rho_1$ is perturbed. The curves of asset weights other than $X_1$ change slowly and in parallel. Asset 7 and asset 8 have quite close sensitivities to the market excess return ($\beta_7 = 1.3160$ and $\beta_8 = 1.3173$). But asset 7 has higher correlation and lower volatility. We can see that $X_8 < X_7 < 0$. Asset 5 and asset 6 have close sensitivities to the market excess return ($\beta_5 = 0.6750$ and $\beta_6 = 0.6898$). But asset 6 has lower volatility and higher correlation to the market. We can see that $X_6 > X_5 > 0$.

Remark 3.4.2 If asset $k$ and asset $j$ have the same sensitivity to the excess market return, i.e.

$$\beta_k = \beta_j = \beta,$$

then either (i) $\sigma_k > \sigma_j$ and $\rho_k < \rho_j$, or (ii) $\sigma_k < \sigma_j$ and $\rho_k > \rho_j$. If (i) happens, then $0 < X_k < X_j$ or $X_j < X_k < 0$. If (ii) happens, then $0 < X_j < X_k$ or $X_k < X_j < 0$. In other words, the investor must effectively choose between the asset with low volatility or that with low correlation and should always choose the one with low volatility.

Proof of Remark 3.4.2 We need only verify case (i). Since $\beta_k = \beta_j = \beta$, using (3.128) we obtain

$$\left(\left(\frac{\sigma_k}{\sigma_m}\right)^2 - \beta^2\right)X_k - \left(\left(\frac{\sigma_j}{\sigma_m}\right)^2 - \beta^2\right)X_j = 0.$$

(3.136)
Therefore, if \( \sigma_k > \sigma_j \), then \( 0 < X_k < X_j \) or \( X_j < X_k < 0 \), and the ratio of \( X_k \) and \( X_j \) is

\[
\frac{X_k}{X_j} = \left( \frac{\sigma_j}{\sigma_m} \right)^2 - \beta^2 \left( \frac{\sigma_k}{\sigma_m} \right)^2 - \beta^2 < 1.
\]

(3.137)

Also, we have

\[
\left( \frac{1}{\rho_k^2} - 1 \right) \beta^2 X_k - \left( \frac{1}{\rho_j^2} - 1 \right) \beta^2 X_j = 0.
\]

(3.138)

If \( \rho_k < \rho_j \), then \( 0 < X_k < X_j \) or \( X_j < X_k < 0 \), and the ratio is

\[
\frac{X_k}{X_j} = \frac{1}{\rho_j^2} - 1 - \frac{1}{\rho_k^2} - 1 < 1.
\]

(3.139)

Notice that \( \beta_k = \beta_j \) implies \( \rho_k \sigma_k = \rho_j \sigma_j \).

**Intuition 3.4.1** Remark 3.4.2 shows that if two assets have the same sensitivity to the market excess return, the investor should invest more money in the asset with lower risk and higher correlation to the market as opposed to the asset with higher risk and lower correlation to the market.
Chapter 4

Sensitivity Analysis of Portfolio’s Expected Rate of Return

In this chapter we study the sensitivity of the portfolio’s expected return rate to the changes of two parameters: volatility (beta) (Section 4.1) and correlation (Section 4.2). After assuming values of volatilities/correlations are the same to simplify the problem, we try to describe the change of the portfolio’s expected return rate as a function of the perturbing term $t$. We show how the return rate would change while the variable volatility/correlation changes. We also study the special 2-asset portfolio in the end of this chapter (Section 4.3).

4.1 Dependence on Volatilities

We begin with an example. We assume that the ten assets of the portfolio have the same correlation of $\rho = 0.5$ to the market, and the original values of the sensitivities $\bar{\beta}$ are given by the same values of Table 2.3. The 1 year risk-free rate, the market’s excess return rate and volatility are taken as 1.46%, 11.11% and 11.37% respectively. Here we perturb $\beta_1$. See Figure 4.1.

In Figure 4.1, when $\beta_1$ grows from 0 to some point $\beta_{\text{high}}$, the portfolio’s expected return $\mu_p$ increases from $r_f$. When $\beta_1$ is larger than $\beta_{\text{high}}$, $\mu_p$ decreases and converges to $\mu_p(\infty)$. The remainder of this section is dedicated to understanding the behaviour observed above in this figure.

In the remainder of this section, we will (i) give the expected return rate of the optimal portfolio (4.8) under the assumption that all assets have the same correlation to the market; (ii) show the portfolio expected return has an upper bound for all possible values of beta (see Remark 4.1.2); and (iii) discuss the asymptotic properties of the portfolio expected return with respect to beta (see Remark 4.1.3 and Remark 4.1.4), etc.

In the previous discussions, we assume that the volatilities $\sigma_i$’s are fixed, which implies that changes in $\beta_i$ must result from changes in $\rho_{im} = \beta_i \sigma_m / \sigma_i$. On the other hand, in the real world, sometimes the correlation $\rho_{im}$ doesn’t change very much compared to the volatility’s change. Or sometimes the estimation error doesn’t happen with the correlation but with the volatility. Therefore we assume now that the correlations are fixed and then perturbation $\tilde{\beta}_i = \beta_i + tq_i$ also
Figure 4.1: $\mu_p$ with same Value of Correlation. The portfolio contains 10 assets which have the same correlation to the market. Take the asset 1 as an example. When $\beta_1$ grows from 0 (i.e. $t = -\bar{\beta}_1$) to $\beta_{\text{high}}$, the portfolio’s expected return $\mu_p$ increases from $r_f$. When $\beta_1$ is larger than $\beta_{\text{high}}$, $\mu_p$ decreases and converges to $\mu_p(\infty)$.

implies

$$\tilde{\sigma}_i = \frac{\sigma_m}{\rho_{im}} \tilde{\beta}_i = \frac{\sigma_m}{\rho_{im}} \beta_i + t \frac{\sigma_m q_i}{\rho_{im}}. \tag{4.1}$$

Recall Remark 2.4.2, the range of the perturbation term $t$ is given by (2.52), i.e.

$$\left(-\frac{\sigma_i}{\sigma_m} - \beta_i\right) / q_i \leq t \leq \left(\frac{\sigma_i}{\sigma_m} - \beta_i\right) / q_i. \tag{4.2}$$

Consider a portfolio which consists of assets with the same correlation $\rho$ to the market. We have

$$\sigma_i = \frac{\sigma_m}{\rho} \cdot \beta_i, \tag{4.3}$$

which implies that the risk of each asset is proportional to its sensitivity to the market with the same ratio.
In order to explain the behaviour observed in Figure 4.1, we need to study the perturbation of asset \( j \)'s volatility and write \( \mu_p \) as a function of \( \beta_j \), which is

\[
\mu_p(\beta_j) = r_f + \frac{(1 - \rho^2)pA_1(\beta_j^2 + \frac{\beta_j}{A_1})}{[[1 + (n - 1)\rho^2]B_1 - \rho^2A_1^2\beta_j^2 - 2\rho^2A_1\beta_j + [1 + (n - 2)\rho^2]}} \tag{4.4}
\]

where

\[
A_1 := \sum_{k=1}^{\infty} \frac{1}{\beta_k^2}, \quad B_1 := \sum_{k=1}^{\infty} \frac{1}{\beta_k^2}. \tag{4.5}
\]

**Proof of 4.4** In fact, since the assets have the same correlation to the market, we have

\[
f_i = \frac{1}{\sigma_i - \sigma_m \beta_i} = \frac{\rho}{\sigma_m(1 - \rho) \beta_i}, \tag{4.6}
\]

\[
g_i = \frac{1}{\sigma_i + \sigma_m \beta_i} = \frac{\rho}{\sigma_m(1 + \rho) \beta_i}. \tag{4.7}
\]

Using (2.6), we get the portfolio expected return rate

\[
\mu_p = r_f + \frac{(1 - \rho^2)p \sum_k \frac{1}{\beta_k}}{[1 + (n - 1)\rho^2] \sum_k \frac{1}{\beta_k} - \rho^2 \left( \sum_k \frac{1}{\beta_k} \right)^2}. \tag{4.8}
\]

Hence, substituting \( A_1 \) and \( B_1 \) into (4.8) and rationalizing the denominator, we obtain (4.4).

**Remark 4.1.1** (4.4) is well-defined for \( \beta_j \in (0, \infty) \). Furthermore, \( \mu_p > r_f \) if \( \sum_k \frac{1}{\beta_k} > 0 \).

**Proof of Remark 4.1.1** Using inequality properties (arithmetic mean \( \leq \) quadratic mean), we know

\[
\frac{\sum_k \frac{1}{\beta_k}}{n} \leq \frac{\sum_k \frac{1}{\beta_k}}{n} \leq \sqrt{\frac{\sum_k \frac{1}{\beta_k}}{n}}, \tag{4.9}
\]

where the first equality holds if \( \beta_k > 0 \) for \( k = 1, \cdots, n \), and the second equality holds if \( \beta_k = 1 \) for \( k = 1, \cdots, n \). It implies

\[
n \sum_k \frac{1}{\beta_k^2} - \left( \sum_k \frac{1}{\beta_k} \right)^2 > 0. \tag{4.10}
\]

Therefore we have

\[
[1 + (n - 1)\rho^2] \sum_k \frac{1}{\beta_k} - \rho^2 \left( \sum_k \frac{1}{\beta_k} \right)^2 = \rho^2 \left[ n \sum_k \frac{1}{\beta_k} - \left( \sum_k \frac{1}{\beta_k} \right)^2 \right] + (1 - \rho^2) \sum_k \frac{1}{\beta_k} > \rho^2 \left[ n \sum_k \frac{1}{\beta_k} - \left( \sum_k \frac{1}{\beta_k} \right)^2 \right] > 0. \tag{4.11}
\]

Hence if \( \sum_k \frac{1}{\beta_k} > 0 \), using (4.8), we have

\[
\mu_p > r_f. \tag{4.12}
\]

Since (4.8) is well defined, (4.4) is well-defined too.
Remark 4.1.2 Suppose $\sum k \frac{1}{\beta_k} > 0$ and all assets have the same constant correlation $\rho$ to the market. As $\beta_j$ varying from 0 to $\infty$, $\mu_\rho$ first increases from $r_f$ then decreases to $r_f + \frac{(1-\rho^2)\rho A_1}{[1+(n-1)\rho^2]B_1-\rho^2 A_1^2}$. Hence $\mu_\rho$ has an upper bound for $\beta_j \in (0, \infty)$.

Proof of Remark 4.1.2 We only need to prove $\mu_\rho$ has an upper bound for $\beta_j \in (0, \infty)$.

\[
\dot{\hat{c}} = \frac{\partial}{\partial \beta_j} \left( 2n - 2 - \sum \frac{\sigma_k(f_k + g_k)}{\sigma_k} \right) \sum \frac{\frac{f_k+g_k}{\sigma_k} + \left( \sum (f_k - g_k) \right)^2}{\sigma_\beta(1-\rho^2)} \frac{1}{\beta_k^2} + \frac{\partial^2}{\partial \beta_j \partial \beta_j} \left[ \frac{1}{\beta_j^2} \right]
\]

\[
= \frac{4p^2}{\sigma_\beta(1-\rho^2)} \left[ \frac{1}{\beta_j^2} \left( \left( 1 + \frac{np^2}{1-\rho^2} \right) \frac{1}{\beta_j} - \frac{p^2}{1-\rho^2} \sum k \frac{1}{\beta_k} \right) \right] + \frac{8p^2}{\sigma_\beta(1-\rho^2)} \left[ \frac{1}{\beta_j^2} \left( 1 + \frac{np^2}{1-\rho^2} \right) \frac{1}{\beta_j} - \frac{p^2}{1-\rho^2} \sum k \frac{1}{\beta_k} \right] + \frac{4p^2}{\sigma_\beta(1-\rho^2)} \left[ \frac{1}{\beta_j^2} \left( \left( 1 + \frac{np^2}{1-\rho^2} \right) \frac{1}{\beta_j} - \frac{p^2}{1-\rho^2} \sum k \frac{1}{\beta_k} \right) \right] (4.13)
\]

The derivative of $\mu_\rho$ with $\beta_j$ is

\[
\frac{\partial}{\partial \beta_j} \mu_\rho = \frac{\partial}{\partial \beta_j} \left( r_f - \frac{2p}{\sigma_m} \sum (f_k - g_k) \right) - \frac{\partial}{\partial \beta_j} \left( \frac{4p^2}{\sigma_\beta(1-\rho^2)} \left[ \frac{1}{\beta_j^2} \left( \left( 1 + \frac{np^2}{1-\rho^2} \right) \frac{1}{\beta_j} - \frac{p^2}{1-\rho^2} \sum k \frac{1}{\beta_k} \right) \right] + \frac{8p^2}{\sigma_\beta(1-\rho^2)} \left[ \frac{1}{\beta_j^2} \left( 1 + \frac{np^2}{1-\rho^2} \right) \frac{1}{\beta_j} - \frac{p^2}{1-\rho^2} \sum k \frac{1}{\beta_k} \right] + \frac{4p^2}{\sigma_\beta(1-\rho^2)} \left[ \frac{1}{\beta_j^2} \left( \left( 1 + \frac{np^2}{1-\rho^2} \right) \frac{1}{\beta_j} - \frac{p^2}{1-\rho^2} \sum k \frac{1}{\beta_k} \right) \right] \right)
\]

\[
= \frac{16p^4}{\sigma_\beta^2(1-\rho^2)} \left[ \frac{1}{\beta_j^2} \left( \left( 1 + \frac{np^2}{1-\rho^2} \right) \frac{1}{\beta_j} - \frac{p^2}{1-\rho^2} \sum k \frac{1}{\beta_k} \right) \right]^2 - \frac{16p^4}{\sigma_\beta^2(1-\rho^2)} \left[ \frac{1}{\beta_j^2} \left( \left( 1 + \frac{np^2}{1-\rho^2} \right) \frac{1}{\beta_j} - \frac{p^2}{1-\rho^2} \sum k \frac{1}{\beta_k} \right) \right] \cdot \left( \frac{4p^2}{\sigma_\beta(1-\rho^2)} \left[ \frac{1}{\beta_j^2} \left( \left( 1 + \frac{np^2}{1-\rho^2} \right) \frac{1}{\beta_j} - \frac{p^2}{1-\rho^2} \sum k \frac{1}{\beta_k} \right) \right] \right)^2
\]

\[
= \frac{p}{\beta_j^2} \left[ \left( 1 + \frac{np^2}{1-\rho^2} \right) \frac{1}{\beta_j} - \frac{p^2}{1-\rho^2} \sum k \frac{1}{\beta_k} \right]^2 .
\]
Let
\[ F_1(\beta_j) := (1 + \frac{\mu_p^2}{1 - p^2}) \left( \frac{1}{\beta_j} \sum_k \frac{1}{\beta_k} - \sum_k \frac{1}{\beta_k} \right) - \frac{\rho^2}{1 - p^2} \left( \sum_k \frac{1}{\beta_k} \ soln}
= (1 + \frac{\mu_p^2}{1 - p^2}) \left( \frac{1}{\beta_j} + \sum_k \frac{1}{\beta_k} \right) - \frac{\rho^2}{1 - p^2} \left( \sum_k \frac{1}{\beta_k} \right)^2 - \left( 1 + \frac{\mu_p^2}{1 - p^2} \right) \left( \frac{1}{\beta_j} + \sum_k \frac{1}{\beta_k} \right).
\]

It is easy to see that \( \frac{\partial}{\partial \beta_j} \mu_p \) has the same sign as its numerator \( F_1(\beta_j) \), since its denominator is always positive.

Suppose \( \sum_k \frac{1}{\beta_k} > 0 \) and \( \beta_j > 0 \). We have
\[ F_1(\beta_j) > 0 \]
is equivalent to
\[ \beta_j < \bar{\beta}_j := \frac{1}{\sqrt{\frac{1 + (n-1)p^2}{1 + (n-2)p^2} (A_1^2 + B_1) - A_1}}. \]

Therefore, for \( 0 < \beta_j \leq \bar{\beta}_j \), \( \mu_p \) is increasing. For \( \beta_j > \bar{\beta}_j \), \( \mu_p \) is decreasing. \( \mu_p \) has an upper bound \( \mu_p(\bar{\beta}_j) \).

**Intuition 4.1.1** Suppose all assets have positive sensitivities to the market excess return, the optimal portfolio has expected return rate higher than the risk-free rate.

**Remark 4.1.3** When \( \beta_j = 0 \), which implies \( \sigma_j = 0 \), we have
\[ \lim_{\beta_j \to 0^+} \mu_p(\beta_j) = r_f, \]

**Intuition 4.1.2** If the portfolio contains a risk-free asset, to minimize the portfolio’s risk, the investor should invest all his money in the risk-free asset and therefore his return will be the risk-free rate \( r_f \).

**Remark 4.1.4** When \( \beta_j = \infty \), which implies \( \sigma_j = \infty \), we have
\[ \lim_{\beta_j \to \infty} \mu_p(\beta_j) = r_f + \frac{(1 - p^2)pA_1}{1 + (n - 1)p^2} B_1 - \rho^2 A_1^2. \]

Similar to Remark 4.1.1, we also have
\[ \left[ 1 + (n - 1)p^2 \right] B_1 - \rho^2 A_1^2 > 0, \]

**Intuition 4.1.3** If the portfolio contains an asset much risker than other assets, the investor should invest little in that risky asset and the portfolio is similar to a \((n-1)\)-asset-portfolio (see also Remark 3.1.4).
4.2 Dependence on Correlation

Begin with an example. We take the risk-free rate $r_f = 1.46\%$, the market excess return rate $p = 11.11\%$, the market volatility $\sigma_m = 11.37\%$, the same asset volatility $\sigma = 20\%$, and the correlations to the market

$$\rho = (30.16\%, 35.46\%, 36.37\%, 37.37\%, 38.37\%, 39.22\%, 74.81\%, 74.89\%, 75.17\%, 78.09\%).$$  

(4.22)

The 10 assets have the same volatility. Their label numbers are ordered by their correlations to the market. $t$ is the difference between the perturbed asset correlation and its original value. See Figure 4.2.

In Figure 4.2, it is clear that assets with lower correlations change more rapidly at the point $t = 0$. The shape of the curves indicates that as the asset correlation varies from from -1 to 1, $\mu_p$ increases to its upper-bound then decreases. The left ends of $\mu_p$ are below $r_f$ and the right ends are higher than $r_f$. The remainder of this section is dedicated to understanding the behaviour observed above in this figure.

In the remainder of this section we will (i) derive the portfolio expected return rate under the assumption that all the assets have the same volatility and write the portfolio expected return as a function of the $j$-th asset’s correlation to the market (see (4.25)); (ii) rewrite the portfolio expected return as a composition of two simple functions (see Remark 4.2.3) and find its upper bound for all possible correlation values (see Remark 4.2.5); (iii) show adding assets uncorrelated to the market in the minimum variance portfolio will drag the portfolio expected return down (see Remark 4.2.6), and adding assets positively highly correlated to the market will make the portfolio expected return higher than the risk-free rate (see Remark 4.2.7), while adding assets negatively highly correlated to the market could drag the portfolio expected return lower than the risk-free rate (see Remark 4.2.8), etc; and (iv) show the portfolio’s expected return is more sensitive to correlation changes of assets with lower correlation to the market (see Remark 4.2.10).

Suppose now the $n$ assets in the portfolio have the same volatility, i.e.

$$\sigma_1 = \cdots = \sigma_n = \sigma.$$  

(4.23)

Then we have

$$\beta_i = \frac{\sigma \rho_i}{\sigma_m} = \frac{\sigma}{\sigma_m} \cdot \rho_i.$$  

(4.24)

Therefore in this case the dependence of Beta is equivalent to the dependence of the correlations, which has been discussed in Section 3.2.

Suppose we perturb the $j$–th asset correlation to the market $\rho_j$. Then $\mu_p$ can be considered as a function of $\rho_j$.

$$\mu_p = r_f + p \cdot \frac{\sigma}{\sigma_m} \frac{1}{2AB-(n-1)(A+B)} \rho_j^2 + \frac{(A-B)\rho_j^2 - 2\rho_j - (A-B)}{(2A-B)(2A-B)+2(A-B)\rho_j - 2(A+1)(A+1)-(n-1)(A+B+2)}.$$  

(4.25)

where

$$A = \sum_{k \neq j} \frac{1}{1-\rho_k}, B = \sum_{k \neq j} \frac{1}{1+\rho_k}.$$  

(4.26)
4.2. Dependence on Correlation

Proof of (4.25) From (2.6) and (3.34), we have

\[ \mu_p = r_f - \frac{2p \sum (f_k - g_k)}{\sigma_m^2} \]

\[ = r_f + \frac{p \sigma}{\sigma_m} \sum \frac{\sigma_k + \sigma_{1-k} - \sum_k \rho_k}{A - B} \]

\[ = r_f + \frac{p \sigma}{\sigma_m} \sum \frac{1}{1 - \rho_k}, \quad B = \sum \frac{1}{1 + \rho_k} \tag{4.27} \]

where

\[ \bar{A} = \sum_k \frac{1}{1 - \rho_k}, \quad \bar{B} = \sum_k \frac{1}{1 + \rho_k} \tag{4.28} \]

as denoted in Section 3.2.

Furthermore, substituting A and B, we obtain (4.25) by rationalizing the denominator.

Remark 4.2.1 From Remark 4.2.4, we know the denominator of (4.27) is positive. Therefore, if \( \sum_k \frac{1}{1 - \rho_k} > \sum_k \frac{1}{1 + \rho_k} \) (for example, all the assets are positively correlated to the market, or \( \rho_k > 0 \) for \( k = 1, \ldots, n \)), then \( \mu_p > r_f \).

Intuition 4.2.1 If all assets are positive correlated to the market, the investor could expect his optimal portfolio’s expected return rate to be higher than the risk-free rate.

Furthermore, consider the shape of \( \mu_p(\rho_j) \) for \( \rho_j \in (-1, 1) \). We want to rewrite \( \mu_p(\rho_j) \) in the form of partial fraction functions, which can help us better understand its sensitivity properties.

Remark 4.2.2 As a function of \( \rho_j \), \( \mu_p \) is a sum with two denominators with powers of 1.

To prove Remark 4.2.2, we first discuss the coefficients of the denominator in (4.25). We will show that the quadratic polynomial in the denominator has two roots and is negative for \( \rho_j \in (-1, 1) \).

Remark 4.2.3 Similar to the inequality (3.38) in Remark 4.2.4, we get

\[ 2AB - (n - 1)(A + B) \geq 0 \tag{4.29} \]

and

\[ 2(A + 1)(B + 1) - (n - 1)(A + B + 2) > 2(A + 1)(B + 1) - n(A + B + 2) \geq 0 \tag{4.30} \]

where the equality holds if and only if \( \rho_k = 0 \) for all \( k \neq j \). Notice that A and B are sums of \( n - 1 \) terms and \( \bar{A} \) and \( \bar{B} \) in Remark 4.2.4 are sums of \( n \) terms. Also \( A + 1 \) and \( B + 1 \) are the special cases of \( \bar{A} \) and \( \bar{B} \) when \( \rho_j = 0 \).

Remark 4.2.4 Suppose \( A > B \) (for example, all assets other than asset j are positively correlated to the market). The quadratic polynomial in the denominator position of the last term in (4.25) has two roots with different signs, which can be solved using the quadratic formula to obtain the roots -\( E_1 \) and \( E_2 \), where \( E_1 > E_2 > 1 \).
Proof of Remark 4.2.4 Consider the quadratic polynomial

\[ 2AB - (n - 1)(A + B)]\rho_j^2 + 2(A - B)\rho_j - [2(A + 1)(B + 1) - (n - 1)(A + B + 2)]. \quad (4.31) \]

From Remark 4.2.3, its leading coefficient is positive and constant term is negative. Then the polynomial must have two rational roots with different signs. Using the quadratic formula we can obtain the two roots \( E_1 \) and \( E_2 \), where \( E_1 \) and \( E_2 \) are positive constants. Since the coefficient of 1st degree term is \( A - B > 0 \), then \( E_1 - E_2 > 0 \). Notice that for \( \rho_j \in (-1, 1) \), the denominator of (4.25) cannot be zero. From above discussion, we conclude that the polynomial is negative for \( \rho_j \in (-1, 1) \), i.e.

\[-E_1 < -1 < 1 < E_2.\] \quad (4.32)

Second, since the denominator can be written into two factors, \( \mu_p \) can be simplified by splitting into partial fraction functions.

Proof of Remark 4.2.2 \( \mu_p \) can be written as

\[
\mu_p = r_f + \frac{p\sigma(A-B)}{\sigma_n[2AB-(n-1)(A+B)]} \left( 1 + \frac{-2(2AB-(n-1)(A+B))}{\sigma_n[2AB-(n-1)(A+B)]} \right) \\
= r_f + \frac{p\sigma(A-B)}{\sigma_n[2AB-(n-1)(A+B)]} \left( 1 + \frac{(-E_1+E_2-\frac{2E_1E_2}{\rho_j-1})}{(\rho_j+E_1)(\rho_j-E_2)} \right) \\
= r_f + \frac{p\sigma(A-B)}{\sigma_n[2AB-(n-1)(A+B)]} \left( 1 + \frac{-E_1^2-2E_2}{E_1+E_2} + \frac{1}{\rho_j+E_1} + \frac{E_2^2-E_1}{E_1+E_2} \right). \tag{4.33} \]

Now let’s consider its shape and boundary.

Remark 4.2.5 As \( \rho_j \) varies from \(-1\) to \(-1\), \( \mu_p \) will first increase and then decrease. \( \mu_p \) will achieve its upper-bound in \((-1, 1)\) at some value of \( \rho_j \). In other words, for variable \( j \)-th asset correlation to the market, the optimal portfolio expected return has an upper-bound \( \mu_p(\bar{\rho}_j) \), where \( \bar{\rho}_j \) is the root in \((-1, 1)\) of the following quadratic function

\[
F(\rho_j) := \left( E_1 - E_2 + \frac{2}{A + B} \right) \rho_j^2 - 2(E_1E_2 - 1)\rho_j + E_1 - E_2 + \frac{2E_1E_2}{A + B}. \tag{4.34} \]

Roof of Remark 4.2.5

\[
\frac{\partial}{\partial \rho_j} \mu_p(\rho_j) = -\frac{p\sigma(A-B)}{\sigma_n[2AB-(n-1)(A+B)]} \left( \frac{-E_1^2+2E_2}{E_1+E_2} + \frac{1}{\rho_j+E_1} + \frac{E_2^2-E_1}{E_1+E_2} \right) = -\frac{p\sigma(A-B)}{\sigma_n[2AB-(n-1)(A+B)]} \left( E_2 - E_1 + \frac{2E_1E_2}{\rho_j+E_1} \right) \left( \rho_j + E_1 \right). \tag{4.35} \]

Let \( F(\rho_j) \) be defined as (4.34). We will show that there is (only one) \( \bar{\rho}_j \in (-1, 1) \), such that

\[ F(\bar{\rho}_j) = 0, \tag{4.36} \]
4.2. Dependence on Correlation

which implies \( \frac{\partial}{\partial \rho_j} \mu_p(\tilde{\rho}_j) = 0 \) as well. In fact, we need only to determine the signs of \( F(-1) \) and \( F(1) \). We have

\[
F(-1) = \left( E_1 - E_2 + \frac{2}{A + B} \right) + 2(E_1 E_2 - 1) + E_1 - E_2 + \frac{2E_1 E_2}{A + B} > 0, \tag{4.37}
\]

since \( E_1 > E_2 > 1 \) and \( A + B > 0 \).

Notice that \(-E_1 \) and \( E_2 \) are the two roots of quadratic polynomial (4.31), which implies

\[
\begin{align*}
    E_1 - E_2 &= \frac{2(A - B)}{2AB - (n - 1)(A + B)} \\
    E_1 E_2 &= 1 + \frac{2E_1 E_2}{2AB - (n - 1)(A + B)}.
\end{align*} \tag{4.38}
\]

Therefore,

\[
F(1) = \left( E_1 - E_2 + \frac{2}{A + B} \right) - 2(E_1 E_2 - 1) + E_1 - E_2 + \frac{2E_1 E_2}{A + B} \\
= 2 \left[ \frac{2(A - B)}{2AB - (n - 1)(A + B)} + \frac{1}{A + B} \left( 2 + \frac{2(A + B + 2 - n)}{2AB - (n - 1)(A + B)} - \frac{2(A + B + 2 - n)}{2AB - (n - 1)(A + B)} \right) \right] \\
= \frac{2(A - B)}{2AB - (n - 1)(A + B)} + \frac{1}{A + B} \left( 2 + \frac{2AB - (n - 1)(A + B)}{2AB - (n - 1)(A + B)} \right). \tag{4.39}
\]

Using the inequality (3.70) in Remark 4.2.7, we have

\[
2B^2 - n + 2 > 2 \left( \frac{n - 1}{2} \right)^2 - n + 2 = \frac{(n - 2)^2 + 1}{2} > 0. \tag{4.40}
\]

Hence

\[
F(1) < 0. \tag{4.41}
\]

Using the properties of quadratic polynomial functions, we know that in the interval \((-1, 1)\) \( F(\rho_j) \) must have and only have one root \( \tilde{\rho}_j \), and furthermore

\[
\begin{align*}
    F(\rho_j) &> 0, \quad -1 \leq \rho_j < \tilde{\rho}_j, \\
    F(\rho_j) &< 0, \quad \tilde{\rho}_j < \rho_j \leq 1.
\end{align*} \tag{4.42}
\]

Notice that \( F(\rho_j) \) and \( \frac{\partial}{\partial \rho_j} \mu_p(\rho_j) \) have the same sign. We conclude that \( \mu_p \) is increasing for \( \rho_j \in (-1, \tilde{\rho}_j) \) and decreasing for \( \rho_j \in (\tilde{\rho}_j, 1) \). Hence \( \mu_p \) achieves its maximum point in \((-1, 1)\) at \( \tilde{\rho}_j \).  

**Remark 4.2.6** Suppose the asset \( j \) is uncorrelated to the market \( (\rho_j = 0) \), then the portfolio’s expected return rate is

\[
\mu_p |_{\rho_j=0} = r_f + \frac{p\sigma(A - B)}{\sigma_m[2(A + 1)(B + 1) - (n - 1)(A + B + 2)]}. \tag{4.43}
\]

Consider another portfolio which contains the other \( n - 1 \) assets. From (4.27) we know its expected return should be

\[
\mu_{p,n-1} = r_f + \frac{p\sigma(A - B)}{\sigma_m[2AB - (n - 2)(A + B)]}. \tag{4.44}
\]

Then

\[
\mu_p |_{\rho_j=0} < \mu_{p,n-1}. \tag{4.45}
\]
Proof of Remark 4.2.6 We need only compare their denominators:

\[ [2(A + 1)(B + 1) - (n - 1)(A + B + 2)] - [2AB - (n - 2)(A + B)] = A + B - 2(n - 2) > 2 > 0. \]  

\[ (4.46) \]

Intuition 4.2.2 In Remark 4.2.6 case, if the investor deletes an asset which is uncorrelated to the market from his portfolio, he will get higher optimal portfolio expected return.

Remark 4.2.7 Suppose the asset \( j \) is perfectly positively correlated to the market (\( \rho_j = 1 \)), then the portfolio’s expected return rate is

\[ \lim_{\rho_j \to 1} \mu_p = r_f + \frac{p \sigma}{\sigma_m[2B + 2 - n]}. \]  

\[ (4.47) \]

In the proof of Remark 3.2.6, we have proved

\[ 2B + 2 - n > 0. \]  

\[ (4.48) \]

Therefore

\[ \lim_{\rho_j \to 1} \mu_p > r_f. \]  

\[ (4.49) \]

Intuition 4.2.3 In Remark 4.2.7 case, if the investor has an asset which is highly correlated to the market, no matter whether the other \( n - 1 \) assets are positively or negatively correlated to the market, the optimal portfolio expected return rate is higher than the risk-free rate.

Remark 4.2.8 Suppose the asset \( j \) is linearly negatively correlated to the market (\( \rho_j = -1 \)), then the portfolio’s expected return rate is

\[ \lim_{\rho_j \to -1} \mu_p = r_f - \frac{p \sigma}{\sigma_m[2A + 2 - n]}. \]  

\[ (4.50) \]

In the proof of Remark 3.2.6, we have proved

\[ 2A + 2 - n > 0. \]  

\[ (4.51) \]

Therefore, we have in this case

\[ \lim_{\rho_j \to -1} \mu_p < r_f. \]  

\[ (4.52) \]

Intuition 4.2.4 From Remark 4.2.8 we know an asset which is highly negatively correlated to the market could make the portfolio’s expected return level lower than the risk-free rate. The investor should avoid investing in such securities. Since asset \( j \) is negatively correlated to the market, its expected return rate must be lower than the risk-free rate. So the portfolio’s expected return rate level is dragged down by asset \( j \). Recall Remark 3.2.6, we have shown that when \( \rho_j \) approaches \( -1 \), to minimize the portfolio variance the weight of \( X_j \) should be larger than that of any other asset. Therefore, the high weight of asset \( j \) makes the portfolio expected return rate level lower than the risk-free rate as well.
4.3. Two-Asset Portfolio

Remark 4.2.9 Suppose all assets are uncorrelated to the market ($\rho_k = 0$, for $k = 1, \cdots, n$), then

$$\mu_p = r_f.$$  (4.53)

Intuition 4.2.5 In Remark 4.2.9 case there is no market excess return gained. Therefore the portfolio is equivalent to a risk-free asset.

Remark 4.2.10 Suppose $\bar{A} > \bar{B}$ (for example $\rho_k > 0$ for $k = 1, \cdots, n$, $\rho_{j_1} < \rho_{j_2}$ and $\bar{B} < n - 1$, then $\mu_p$ changes more rapidly in the direction of $\rho_{j_1}$ than in that of $\rho_{j_2}$.

Proof of Remark 4.2.10 Take the derivative of $\mu_p$ to $\rho$, we have

$$\frac{\partial}{\partial \rho} \mu_p = \frac{p \sigma \left[ 2\bar{B}(\bar{B} - n + 1) \frac{1}{(1-\rho)^2} + 2\bar{A}(\bar{A} - n + 1) \frac{1}{(1+\rho)^2} \right]}{[2\bar{A} - (n-1)(\bar{A} + \bar{B})]^2}.$$  (4.54)

Now compare the derivatives with $\rho_{j_1}$ and $\rho_{j_2}$ at the same point. We have

$$\frac{\partial}{\partial \rho_{j_1}} \mu_p - \frac{\partial}{\partial \rho_{j_2}} \mu_p = \frac{p \sigma \left[ 2\bar{B}(\bar{B} - n + 1) \frac{1}{(1-\rho_{j_1})^2} + 2\bar{A}(\bar{A} - n + 1) \frac{1}{(1+\rho_{j_1})^2} \right]}{[2\bar{A} - (n-1)(\bar{A} + \bar{B})]^2}.$$  (4.55)

From Remark (3.2.2), we know

$$\bar{A} + \bar{B} > 2n.$$  (4.56)

Since $\bar{A} > \bar{B}$, it must be true that

$$\bar{A} > n > n - 1.$$  (4.57)

Therefore, if $\rho_{j_1} < \rho_{j_2}$ and $\bar{B} < n - 1$, then

$$\frac{\partial}{\partial \rho_{j_1}} \mu_p - \frac{\partial}{\partial \rho_{j_2}} \mu_p > 0.$$  (4.58)

It means at the point $\mu_p$ changes more rapidly in the direction of $\rho_{j_1}$ than in that of $\rho_{j_2}$.

Intuition 4.2.6 In Remark 4.2.10 case, the portfolio expected return rate is more sensitive to assets with lower correlation to the market.

See Figure 4.2, where the original values of $\rho_1, \cdots, \rho_n$ are ordered from low to high. And it is clear that at that point ($t = 0$), the asset with lower correlation will change more rapidly.

4.3. Two-Asset Portfolio

Suppose the portfolio contains just two different stocks. Their sensitivities to the market are $\beta_1$ and $\beta_2$. $\sigma_m$, $\sigma_1$, and $\sigma_2$ are the volatilities of the market, asset 1, and asset 2 respectively. Consider the mean variance problem of the portfolio, from previous discussion (2.47) and (2.48), we know the minimum variance portfolio weights for $n = 2$ are

$$X_1 = \frac{\sigma_2^2 - \sigma_m \beta_1 \beta_2}{\sigma_1^2 + \sigma_2^2 - 2\sigma_m \beta_1 \beta_2}, \quad X_2 = 1 - X_1.$$  (4.59)

We discuss the dependence of the portfolio expected return rate $\mu_p$ on asset volatilities (see Section 4.3.1) and correlations (see Section 4.3.2). We discuss the upper bound properties of the portfolio expected return for all possible values of variables in both two cases.
4.3.1 Volatility Sensitivity

We begin with an example. The portfolio contains two assets. Their correlations to the market are \( \rho = (49.27\%, 45.48\%) \) and their volatilities are \( \sigma = (30\%, 15\%) \). The market excess return is 11.11\% and the market volatility is 11.37\%. The risk-free rate is 1.46\%. We perturb \( \sigma_1 \). See Figure 4.3.

In Figure 4.3, it is clear that \( \mu_p \) increases from \( r_f \) and then decreases as \( \sigma_1 \) growing. We will explain this behavior in the remainder of this subsection.

Suppose that \( \sigma_2, \rho_1 \) and \( \rho_2 \) are fixed values, which implies that variability of Beta arises solely from the variability of the volatilities.

**Remark 4.3.1** As a function of asset 1 volatility \( \sigma_1 \), the 2-asset portfolio expected return \( \mu_p \) is increasing for \( \sigma_1 \in (0, \lambda_1 \sigma_2) \), and decreasing for \( \sigma_1 \in (\lambda_1 \sigma_2, \infty) \), where

\[
\lambda_1 = \frac{\rho_2(1 - \rho_1^2) + \sqrt{\rho_2^2(1 - \rho_1^2)^2 + \rho_1^2(1 - \rho_1^2)^2 + 2 \rho_1 \rho_2^2(1 - \rho_1^2)(1 - \rho_2^2)}}{\rho_1(1 - \rho_2^2) + 2 \rho_1 \rho_2^2(1 - \rho_1^2)}. \tag{4.60}
\]

**Proof of Remark 4.3.1** To simplify, let

\[
\lambda := \frac{\sigma_1}{\sigma_2}. \tag{4.61}
\]

Then the portfolio expected return rate is

\[
\mu_p = r_f + p \left( \frac{\sigma_1^2 - \sigma_1 \sigma_2 \rho_1 \rho_2}{\sigma_m^2} - \frac{\rho_1^2}{\sigma_m^2} + \frac{\sigma_2^2 - \sigma_1 \sigma_2 \rho_1 \rho_2}{\sigma_m^2} \right) = r_f + \frac{\rho_2^2}{\sigma_m^2} \left( 1 - \rho_1^2 \right) \lambda \left( 1 - \rho_1^2 \right) + \frac{\rho_2^2}{\sigma_m^2} \left( 1 - \rho_1^2 \right) \lambda \left( 1 - \rho_1^2 \right) \tag{4.62}
\]

Taking derivative of \( \mu_p \) with \( \lambda \), we have

\[
\frac{\partial}{\partial \lambda} \mu_p = \frac{\rho_2^2 - \rho_1(1 - \rho_1^2) + 2 \rho_1 \rho_2^2(1 - \rho_1^2) \lambda + 2 \rho_1(1 - \rho_1^2) \lambda^2 + 2 \rho_1(1 - \rho_1^2) \lambda^3}{\sigma_m^2 (1 - \rho_1^2)^2} \left( \lambda^2 - \frac{2 \rho_1(1 - \rho_1^2)}{\rho_1(1 - \rho_1^2) + 2 \rho_1 \rho_2^2(1 - \rho_1^2)} \lambda - \frac{\rho_1(1 - \rho_1^2)}{\rho_1(1 - \rho_1^2) + 2 \rho_1 \rho_2^2(1 - \rho_1^2)} \right). \tag{4.63}
\]

Hence the signs of \( \frac{\partial}{\partial \lambda} \mu_p \) depends on the signs of the quadratic polynomial

\[
G(\lambda) := \lambda^2 - \frac{2 \rho_1(1 - \rho_1^2)}{\rho_1(1 - \rho_1^2) + 2 \rho_1 \rho_2^2(1 - \rho_1^2)} \lambda - \frac{\rho_1(1 - \rho_1^2)}{\rho_1(1 - \rho_1^2) + 2 \rho_1 \rho_2^2(1 - \rho_1^2)}, \tag{4.64}
\]

which can be written as

\[
G(\lambda) = \left( \lambda - \frac{1}{A_3 + 2 \rho} \right)^2 - \frac{1 + A_3^2 + 2 A_3 \rho}{(A_3 + 2 \rho)^2}, \tag{4.65}
\]

where

\[
A_3 := \frac{\rho_1(1 - \rho_1^2)}{\rho_2(1 - \rho_1^2)}, \quad \rho := \rho_1 \rho_2. \tag{4.66}
\]
Suppose $\rho > 0$. Using the quadratic formula we obtain two roots of (4.65)

$$
\lambda_1 = \frac{1 + \sqrt{1 + A_3^2 + 2A_3\rho}}{A_3 + 2\rho}, \quad \lambda_2 = \frac{1 - \sqrt{1 + A_3^2 + 2A_3\rho}}{A_3 + 2\rho}.
$$
(4.67)

It is easy to see that only $\lambda_1$ is positive.

Therefore, for $\lambda \in (0, \lambda_1)$, $G(\lambda) < 0$ and $\mu_p$ is increasing since its derivative is positive. For $\lambda \in (\lambda_1, \infty)$, $G(\lambda) > 0$ and $\mu_p$ is decreasing since its derivative is negative.

**Remark 4.3.2** Supposing asset 1 is risk-free, we have

$$
\mu_p|_{\lambda=0} = r_f.
$$
(4.68)

Supposing the volatility of asset 1 approaches infinity, we have

$$
\lim_{\lambda \to \infty} \mu_p = r_f + (1 - \rho_1^2)\frac{\sigma_2\rho_2}{\sigma_m}.
$$
(4.69)

**Intuition 4.3.1** From Remark 4.3.2, we know if one asset is risk-free, to minimize the risk the investor would invest all his money in this risk-free asset. Therefore the portfolio expected return rate is the risk-free rate.

On the contrary, as one asset volatility grows to infinity, the weight of the risky asset is going to 0 but its return is going to infinity as well. Therefore, in this case the portfolio expected return would be less than the other asset return rate since from Remark 4.3.2 we know when the volatility of the risky asset is larger than some level, the investor should take a short position in it.

**Remark 4.3.3** Suppose asset 1 and asset 2 have the same volatility. We have

$$
\mu_p|_{\lambda=1} = r_f + \frac{\rho_1 + \rho_2}{2} \frac{p\sigma_2}{\sigma_m} = r_f + \frac{1}{2} \left( \frac{pp_1\sigma_1}{\sigma_m} + \frac{pp_2\sigma_2}{\sigma_m} \right).
$$
(4.70)

**Intuition 4.3.2** From Remark 3.3.1, we know if the two assets have the same volatility, no matter what correlations are, the investor should invest them equally. Remark 4.3.3 shows in this case the portfolio return rate is the average of their return rates.

**Remark 4.3.4**

$$
\mu_p|_{\lambda=p_1p_2} = r_f + \frac{p\sigma_2}{\sigma_m} \rho_1\rho_2 = r_f + \frac{pp_1\sigma_1}{\sigma_m},
$$
(4.71)

$$
\mu_p|_{\lambda=\frac{1}{p_1p_2}} = r_f + \frac{pp_2\sigma_2}{\sigma_m}.
$$
(4.72)

**Intuition 4.3.3** From Remark 3.3.4, we know if the ratio of asset volatilities is $\rho_1\rho_2$ or $\frac{1}{\rho_1\rho_2}$, the investor should invest all his money in only one asset. Remark 4.3.4 shows in this case the portfolio return is just the same as one of the assets.
4.3.2 Correlation Sensitivity

We begin with an example. Suppose a portfolio contains two stocks with volatilities and sensitivities to the market \( \sigma_1, \sigma_2, \beta_1, \) and \( \beta_2 \) respectively. The observed data is \( \sigma_1 = 0.3, \sigma_2 = 0.15, \beta_1 = 1.3, \) and \( \beta_2 = 0.6. \) The risk-free rate is 0.0146. The market excess return is 0.1111. The market volatility is 0.1137. Then the curves of perturbations of \( \beta_1 \) and \( \beta_2 \) are shown in Figure 4.4.

In Figure 4.4, the curve of \( \beta_1 \) first increases then decreases as \( \beta_1 \) grows in its region, while that of \( \beta_2 \) increases. We will explain the behavior in the remainder of this subsection.

Suppose that \( \beta_2, \sigma_1 \) and \( \sigma_2 \) are fixed values, which implies that variability of beta arises solely from variability of the correlation.

**Remark 4.3.5** As a function of \( \beta_1, \) \( \mu_p \) can be written as a sum of a linear function and a partial fraction with power 1 denominator. Using the properties of quadratic functions, we know that just three cases are possible:

- \( \mu_p(\beta_1) \) is increasing for \( \beta_1 \in (\frac{-\sigma_1}{\sigma_m}, \frac{\sigma_1}{\sigma_m}) \) if \( F(\frac{\sigma_1}{\sigma_m}) > 0, \)

- \( \mu_p(\beta_1) \) is decreasing for \( \beta_1 \in (\frac{-\sigma_1}{\sigma_m}, \frac{\sigma_1}{\sigma_m}) \) if \( F(\frac{-\sigma_1}{\sigma_m}) < 0, \)

- \( \mu_p(\beta_1) \) achieves its maximum point at \( \beta_1 = B_2 - \sqrt{A_2(B_2 - \beta_2)}, \) which is inside \( (\frac{-\sigma_1}{\sigma_m}, \frac{\sigma_1}{\sigma_m}) \), if \( F(\frac{-\sigma_1}{\sigma_m}) > 0 \) and \( F(\frac{\sigma_1}{\sigma_m}) < 0, \)

where

\[
A_2 := \frac{\sigma_1^2 - \sigma_2^2}{2\sigma_m^2 \beta_2}, \quad B_2 := \frac{\sigma_1^2 + \sigma_2^2}{2\sigma_m^2 \beta_2},
\]

\[ \text{(4.73)} \]

and

\[
F(\beta_1) := (\beta_1 - B_2)^2 - A_2(B_2 - \beta_2).
\]

\[ \text{(4.74)} \]

**Proof of Remark 4.3.5** From CAPM model (2.17), the return of the lowest risk portfolio should be

\[
\mu_p(\beta_1, \beta_2) = X_1 \mu_1 + X_2 \mu_2 = X_1 (r_f + p \beta_1) + X_2 (r_f + p \beta_2) = r_f + p \left( \frac{\sigma_1^2 - \sigma_2^2 \beta_1}{\sigma_1^2 + \sigma_2^2} \right) \beta_1 + \frac{\sigma_1^2 - \sigma_2^2 \beta_2}{\sigma_1^2 + \sigma_2^2} \beta_2.
\]

\[ \text{(4.75)} \]

Consider the derivative of \( \mu_p \) with \( \beta_1 \), we have

\[
\frac{\partial}{\partial \beta_1} \mu_p = p \cdot \frac{2 \sigma_1^4 \beta_2^2 \beta_1^2 - 2 \sigma_2^4 \beta_1^2 + \beta_2^2 \sigma_1^2 - \beta_2^2 \sigma_2^2 + \sigma_1^2 - \sigma_2^2 + \sigma_1^2 + \sigma_2^2}{\sigma_1^2 + \sigma_2^2} = p \left( \frac{\beta_1 - \frac{\sigma_1^2 + \sigma_2^2}{2 \sigma_m^2 \beta_2}}{\frac{\sigma_1^2 + \sigma_2^2}{2 \sigma_m^2 \beta_2}} \right) = p \left( \frac{\beta_1 - \frac{\sigma_1^2 + \sigma_2^2}{2 \sigma_m^2 \beta_2}}{\frac{\sigma_1^2 + \sigma_2^2}{2 \sigma_m^2 \beta_2}} \right)^2 = \frac{\beta_1 - \frac{\sigma_1^2 + \sigma_2^2}{2 \sigma_m^2 \beta_2}}{\sigma_1^2 + \sigma_2^2}.
\]

\[ \text{(4.76)} \]

Hence \( \frac{\partial}{\partial \beta_1} R_p \) and \( F(\beta_1) \) have the same signs.
We know that for all possible values of $\beta_1$, it is always true that
\[
\sigma_1^2 + \sigma_2^2 - 2\sigma_m^2\beta_1\beta_2 > 0,
\] (4.77)
which implies that
\[
\beta_1 < \frac{\sigma_1^2 + \sigma_2^2}{2\sigma_m^2\beta_2} = B_2,
\] (4.78)
when $\beta_2 \neq \frac{\sigma_1}{\sigma_m}$, or $\sigma_1 \neq \sigma_2$.

Using the quadratic function properties, we know that only the following 3 cases could happen:

- If and only if $F\left(-\frac{\sigma_1}{\sigma_m}\right) > 0$, and $F\left(\frac{\sigma_1}{\sigma_m}\right) < 0$, which means the left root $\beta_1 = B_2 - \sqrt{A_2(B_2 - \beta_2)}$ of the equation $F(\beta_1) = 0$ exists and lies in the range $(-\frac{\sigma_1}{\sigma_m}, \frac{\sigma_1}{\sigma_m})$, $F(\beta_1)$ is negative for $\beta_1 \in (-\frac{\sigma_1}{\sigma_m}, B_2 - \sqrt{A_2(B_2 - \beta_2)})$, positive for $\beta_1 \in (B_2 - \sqrt{A_2(B_2 - \beta_2)}, \frac{\sigma_1}{\sigma_m})$.

- If $F\left(-\frac{\sigma_1}{\beta_1}\right) < 0$, we know $F(\beta_1) < 0$ for $\beta_1 \in (-\frac{\sigma_1}{\sigma_m}, \frac{\sigma_1}{\sigma_m})$.

- If $F\left(\frac{\sigma_1}{\beta_1}\right) > 0$, we know $F(\beta_1) > 0$ for $\beta_1 \in (-\frac{\sigma_1}{\sigma_m}, \frac{\sigma_1}{\sigma_m})$.

Furthermore, by cancellation we have
\[
\frac{\partial}{\partial \beta_1} \mu_p = \frac{p}{2} \left(1 - \frac{A_2(B_2 - \beta_2)}{(\beta_1 - B_2)^2}\right).
\] (4.79)

After integration, we have a new expression of $R_p$, which is
\[
\mu_p(\beta_1) = \mu_p(0) + \frac{p}{2} \int_0^{\beta_1} \left(1 - \frac{A_2(B_2 - \beta_2)}{(\beta_1 - B_2)^2}\right) dt
= r_f + \frac{A_2(B_2 - \beta_2)}{2\sigma_m^2\beta_1} + \frac{p}{2} \left(\beta_1 + \frac{A_2(B_2 - \beta_2)}{\beta_1 - B_2}\right)
= r_f + \frac{A_2(B_2 - \beta_2)}{2\sigma_m^2\beta_1} + \frac{p}{2} \left(\beta_1 + \frac{A_2(B_2 - \beta_2)}{\beta_1 - B_2}\right)
\] (4.80)

**Remark 4.3.6** Suppose the volatility of asset 2 satisfies $\sigma_2 > \sqrt{2}\sigma_2\sigma_m$, i.e. the correlation of asset 2 satisfies $\rho_2 < 1/\sqrt{2}$, and the volatility of asset 1 satisfies $\sigma_1 < \sigma_2$. As $\beta_1$ varying from $-\frac{\sigma_1}{\sigma_m}$ to $\frac{\sigma_1}{\sigma_m}$, $\mu_p$ is increasing.

**Proof** If $\sigma_1 < \sigma_2$, and $\sigma_2 > \sqrt{2}\sigma_2\sigma_m$, then
\[
A_2 = \frac{\sigma_1^2 - \sigma_2^2}{2\sigma_m^2\beta_2} < 0,
\] (4.81)
and
\[
B_2 - \beta_2 = \frac{\sigma_1^2 + \sigma_2^2 - 2\sigma_m^2\beta_2^2}{2\sigma_m^2\beta_2} > \frac{\sigma_1^2}{2\sigma_m^2\beta_2} > 0.
\] (4.82)
Therefore, in this case $F > 0$ for $\beta_j \in (-\frac{\sigma_1}{\sigma_m}, \frac{\sigma_1}{\sigma_m})$. So $\mu_p$ is increasing.  

Remark 4.3.7 Suppose the volatility of asset 2 satisfies $\sigma_2 < \sqrt{2}\beta_2 \sigma_m$, i.e. the correlation of asset 2 satisfies $\rho_2 > 1/\sqrt{2}$, and the volatility of asset 1 satisfies $\sqrt{2}\sigma_1^2 \beta_2^2 - \sigma_2^2 < \sigma_1 < \sigma_2$. As $\beta_1$ varying from $-\frac{\sigma_1}{\sigma_m}$ to $\frac{\sigma_1}{\sigma_m}$, $\mu_p$ is increasing.

Proof of Remark 4.3.7 If $\sqrt{2}\sigma_1^2 \beta_2^2 - \sigma_2^2 < \sigma_1 < \sigma_2$, and $\sigma_2 \leq \sqrt{2}\sigma_2 \sigma_m$, then

$$A_2 = \frac{\sigma_1^2 - \sigma_2^2}{2\sigma_m^2 \beta_2} < 0, \quad (4.83)$$

and

$$B_2 - \beta_2 = \frac{\sigma_1^2 + \sigma_2^2 - 2\sigma_m^2 \beta_2^2}{2\sigma_m^2 \beta_2} > 0. \quad (4.84)$$

In this case, $F > 0$ for $\beta_j \in (-\frac{\sigma_1}{\sigma_m}, \frac{\sigma_1}{\sigma_m})$. So $\mu_p$ is increasing.

Intuition 4.3.4 Recall Figure 4.4. In fact, for the curve of $\beta_1$, it is easy to check that $A_2 > 0$ and $B_2 > \beta_2$. On the other hand, for the curve of $\beta_2$, the corresponding $A_2$ is negative and $B_2 - \beta_1$ is positive, which implies that $\frac{\partial}{\partial \beta} \mu_p > 0$, i.e. the curve is strictly increasing.

Remark 4.3.8 Suppose the two assets are positively linearly correlated to the market, i.e. $\beta_i = \frac{\sigma_i}{\sigma_m}$ for $i = 1, 2$. We have

$$\mu_p \Big|_{\beta_1 = \frac{\sigma_1}{\sigma_m}, \beta_2 = \frac{\sigma_2}{\sigma_m}} = r_f + \frac{p}{\sigma_1 \sigma_2 - \sigma_1^2 \sigma_2} \sigma_1 \sigma_2. \quad (4.85)$$

Intuition 4.3.5 From Remark 3.3.9, we know when the correlation between the two assets are close to 1, the weights would change quite sharply when the volatilities are getting close to each other (i.e. $\lambda = 1$). Remark 4.3.8 shows in this case the portfolio return would change sharply as well. Therefore the investor should diversify his investment to get a more stable portfolio.
Figure 4.2: $\mu_p$ with same Value of Volatility. The 10 assets have the same volatility. Their label numbers are ordered by their correlations to the market. $t$ is the difference between the perturbed asset correlation and its original value. At the point $t = 0$, it is clear that assets with lower correlations change more rapidly. The shape of the curves indicates that as asset correlation varying from -1 to 1 $\mu_p$ increases to its upper-bound then decreases. The left ends of all curves are below $r_f$ and the right ends are higher than $r_f$. 
Figure 4.3: 2-Asset Portfolio Variance vs Perturbation of $\sigma_1$. $t$ is the difference between the perturbed value of $\sigma_1$ and its original value. It is clear that $\mu_p$ increases from $r_f$ and then decreases as $\sigma_1$ grows.
4.3. Two-Asset Portfolio

Figure 4.4: $\mu_p$ with different Values of $q$. The portfolio contains two assets. The portfolio’s expected return $\mu_p$ first increases and then decreases as $\beta_1$ grows in its range. On the other hand, $\mu_p$ strictly increases as $\beta_2$ grows in its range.
Chapter 5

Sensitivity Analysis of Minimum Variance Portfolio Variance and Error Risk Analysis

In this chapter we study the sensitivity of the minimum portfolio variance with respect to asset volatilities (see Section 5.1) and correlations (see Section 5.2). In both cases we discuss the upper bound of the minimum variance and compare the \( n \)-asset portfolio variance and the corresponding \( (n - 1) \)-asset portfolio variance. In other words, we show the “worst” (maximal) \( n \)-asset portfolio minimum variance for all possible values of some asset variable (its correlation or volatility) is just the minimum variance of the \( (n - 1) \)-asset portfolio which doesn’t contain that asset.

5.1 Dependence on Volatilities

Recall the example given in Section 3.1. Suppose the portfolio contains 10 assets. The volatilities are

\[
\sigma = (0.1295, 0.1392, 0.1499, 0.1609, 0.1635, 0.1932, 0.2694, 0.2805, 0.2838, 0.3300)^T,
\]

which are ordered by their values. The assets have the same correlation to the market, which is 20%. The market volatility is 11.37%. The market excess return is 11.11%. The risk-free rate is 1.46%. We perturb the volatility of asset 1. See Figure 5.1.

In Figure 5.1, the portfolio variance achieves its highest point \( \sigma^2_{p,n-1} \), which is the variance of the \( (n - 1) \)-asset portfolio which contains all assets of the \( n \)-asset portfolio but asset \( j \). Recall Figure 3.1, the portfolio variance achieves its highest point when the weight of asset 1 is 0. We will explain this behavior in the remainder of this section.

In this section, we will (i) give the portfolio minimum variance (5.3) under the assumption that all assets in the portfolio have the same correlation to the market, which follows the intuition that in this case the portfolio change solely depends on the asset volatility changes; and (ii) show that adding assets to the portfolio could reduce its minimum variance (see Remark 5.1.2).
5.1. Dependence on Volatilities

Figure 5.1: Portfolio Variance vs Perturbation of $\sigma_1$. $t$ is the difference between the perturbed value of $\sigma_1$ and its original value. The portfolio variance achieves its highest point at $\sigma_1 = (1 + (n - 2)p^2)/(\rho^2 A_1)$. $\sigma_{p,n-1}^2$ is the variance of the $(n - 1)$-asset portfolio which contains all assets of the $n$-asset portfolio but asset $j$.

Suppose the assets have the same correlation to the market $\rho$. From (2.35) the portfolio variance is

$$
\sigma_p^2 = \frac{1 - \rho^2}{\sum_i \frac{1}{\sigma_i^2} - \frac{\rho^2}{1+(n-1)p^2} \sum_{i,j} \frac{1}{\sigma_i \sigma_j}}.
$$

(5.2)

Consider $\sigma_p^2$ as a function of $\sigma_j$, we have

$$
\sigma_p^2(\sigma_j) = \frac{(1 - \rho^2)[1 + (n - 1)p^2]\sigma_j^2}{[B_1 + ((n - 1)B_1 - A_1^2)p^2]\sigma_j^2 - 2p^2 A_1 \sigma_j + 1 + (n - 2)p^2},
$$

(5.3)

where

$$
A_1 = \sum_{k \neq j} \frac{1}{\sigma_k}, \quad B_1 = \sum_{k \neq j} \frac{1}{\sigma_k^2}.
$$

(5.4)
Proof of (5.2) and (5.3) In fact, since the assets have the same correlation \( \rho \), from (3.6) we have

\[
c = \frac{1}{1 - \rho^2} \left( \sum_i \frac{1}{\sigma_i^2} - \frac{\rho^2}{1 + (n - 1)\rho^2} \sum_{i,j} \frac{1}{\sigma_i \sigma_j} \right). \tag{5.5}
\]

Using the fact that \( \sigma_p^2 = \frac{1}{c} \), we obtain (5.2).

Consider \( \sigma_p^2 \) as a function of \( \sigma_i \). Rationalizing the denominator, we have

\[
\sigma_p^2(\sigma_j) = \frac{(1-\rho^2)(1+(n-1)\rho^2)}{[1+(n-1)\rho^2](B_1 + \frac{\rho^2}{\sigma_j^2})} - \rho^2 \left( A_1 + \frac{1}{\sigma_j} \right)^2
\]

\[
= \frac{(1-\rho^2)(1+(n-1)\rho^2)}{[B_1 + (n-1)A_1^2\rho^2\sigma_j^2 - 2\rho^2A_1\sigma_j + 1+(n-2)\rho^2]} \cdot \tag{5.6}
\]

\[
\text{Remark 5.1.1} \ (5.2) \text{ and } (5.3) \text{ are well-defined.}
\]

Proof of Remark 5.1.1 Similar to (4.9) we have

\[
\sum_k \frac{1}{\sigma_k^2} \leq \sum_k \frac{1}{|\sigma_k|} \leq \sqrt{\sum_k \frac{1}{\sigma_k^2}}. \tag{5.7}
\]

Therefore, we can show the denominator of (5.2) satisfies

\[
\sum_i \frac{1}{\sigma_i^2} - \frac{\rho^2}{1+(n-1)\rho^2} \sum_{i,j} \frac{1}{\sigma_i \sigma_j} = \frac{1}{1+(n-1)\rho^2} \left[ (1+(n-1)\rho^2) \sum_k \frac{1}{\sigma_k^2} - \rho^2 \left( \sum_k \frac{1}{\sigma_k} \right)^2 \right]
\]

\[
= \frac{1}{1+(n-1)\rho^2} \left[ \rho^2 \left( n \sum_k \frac{1}{\sigma_k^2} - \left( \sum_k \frac{1}{\sigma_k} \right)^2 \right) + (1-\rho^2) \sum_k \frac{1}{\sigma_k} \rho^2 \right] > 0. \tag{5.8}
\]

Then \( \sigma_p^2 \) is positive for all \( \sigma_j > 0 \), if \(-1 < \rho < 1\).

Recall Remark 3.1.3, we know when \( \sigma_j = (1+(n-2)\rho^2)/(\rho^2A_1) \), the minimum variance \( n \)-asset portfolio is in fact a \( (n-1) \)-asset portfolio. We show in this case the portfolio minimum variance achieves its maximum for all possible values of \( \sigma_j \).

Remark 5.1.2 When \( \sigma_j = (1+(n-2)\rho^2)/(\rho^2A_1) \), the optimal portfolio achieves its highest variance for all \( \sigma_j > 0 \), which is the same as the variance of a \( (n-1) \)-asset portfolio.

Proof of Remark 5.1.2 Suppose portfolio 1 contains \( n \) assets and portfolio 2 contains all assets of portfolio 1 but asset \( j \).

Using (5.2), the optimal variance of portfolio 2 is

\[
\sigma_{p, n-1}^2 = \frac{1 - \rho^2}{B_1 - \frac{\rho^2}{1+(n-2)\rho^2}A_1^2}. \tag{5.9}
\]
Therefore, the difference between \( \sigma_{p,n-1}^2 \) and \( \sigma_p^2 \) is

\[
\sigma_{p,n-1}^2 - \sigma_p^2 = \frac{1 - \rho^2}{B_1 - \frac{\rho^2}{1+ \rho^2} A_1^2} - \frac{1 - \rho^2}{B_1 + \frac{1}{\sigma_j^2} - \frac{\rho^2}{1+ \rho^2} \left(A_1 + \frac{1}{\sigma_j^2}\right)^2} \\
= \frac{(1 - \rho^2) \left[1+ (n-2)\rho^2 \right]}{B_1 - \frac{\rho^2}{1+ \rho^2} A_1^2} \left(B_1 + \frac{1}{\sigma_j^2} - \frac{\rho^2}{1+ \rho^2} \left(A_1 + \frac{1}{\sigma_j^2}\right)^2\right) \\
= \frac{(1 - \rho^2) \left[1+ (n-2)\rho^2 \right]}{B_1 - \frac{\rho^2}{1+ \rho^2} A_1^2} \left(B_1 + \frac{1}{\sigma_j^2} - \frac{\rho^2}{1+ \rho^2} \left(A_1 + \frac{1}{\sigma_j^2}\right)^2\right) \\
\geq 0,
\]

where the equality holds if and only if

\[
\sigma_j = \frac{1 + (n-2)\rho^2}{A_1 \rho^2}.
\]

\[ (5.10) \]

Intuition 5.1.1 Remark 5.1.2 shows that adding assets in the portfolio can reduce the portfolio variance. Therefore, the investor should create a more diversified portfolio to minimize the risk he could have. See Figure 5.1.

Therefore, even if an asset is quite risky, most of the time the investor can add it into his portfolio to minimize the risk.

Intuition 5.1.2 Suppose the investor may select between two portfolios. Portfolio 1 contains \( n \) assets with the same correlation to the market. Portfolio 2 contains \( n - 1 \) assets which are taken from portfolio 1. Then the optimal portfolio 1 has lower variance than portfolio 2.

5.2 Dependence on Correlations

Recall the example given in Section 3.2. Suppose the portfolio contains 10 assets. The original asset correlations to the market are

\[
\rho = (30.16\%, 35.46\%, 36.37\%, 37.38\%, 38.37\%, 39.22\%, 74.81\%,
74.89\%, 75.17\%, 78.09\%)^T,
\]

which are ordered by their values. All assets have the same volatility \( \sigma \). The value of \( \sigma \) does not affect asset weights. To test it, we take \( \sigma = 0.2, 0.5 \) as two examples, and they have the same output. See Figure 5.2.

In Figure 5.2, the portfolio variance \( \sigma_p^2 \) reaches its highest point \( \sigma_{p,n-1}^2 \), which is the variance of the \((n-1)\)-asset portfolio which contains all assets of the \(n\)-asset portfolio but asset \(j\). Recall Figure 3.2, the portfolio variance achieves its highest point when the weight of asset 1 is 0. We will explain this behavior in the remainder of this section.
Chapter 5. Sensitivity Analysis of Minimum Variance Portfolio Variance and Error Risk Analysis

Figure 5.2: Portfolio Variance vs Perturbation of $\rho_1$. $t$ is the difference between the perturbed value of $\rho_1$ and its original value. The portfolio variance $\sigma_p^2$ reaches its highest point at $\rho_j = (A + B - 2(n - 2))/(A - B)$. $\sigma_{p,n-1}^2$ is the variance of the $(n - 1)$-asset portfolio which contains all assets of the $n$-asset portfolio but asset $j$.

In this section, we will (i) give the portfolio minimum variance (5.15) under the assumption that all assets in the portfolio have the same volatilities, which follows the intuition that in this case the portfolio changes solely depend on the asset correlation changes; and (ii) show that adding assets to the portfolio can reduce its minimum variance (see Remark 5.2.2).

Suppose the assets have the same volatility $\sigma$. From (2.35) the portfolio variance is

$$
\sigma_p^2 = \frac{\bar{A} + \bar{B} - 2(n - 1)}{2\bar{A}\bar{B} - (n - 1)(\bar{A} + \bar{B})}\sigma^2, 
$$

(5.13)

where

$$
\bar{A} := \sum_k \frac{1}{1 - \rho_k}, \quad \bar{B} := \sum_k \frac{1}{1 + \rho_k}.
$$

(5.14)

Suppose $\rho_j$ is the only changing variable. As a function of $\rho_j$, the portfolio expected return
Therefore, substituting into (2.35), we obtain
\[
\sigma_p^2(\rho_j) = \frac{[A + B - 2(n - 1)]\rho_j^2 - [A + B - 2(n - 2)]}{[2AB - (n - 1)(A + B)]\rho_j^2 + 2(A - B)\rho_j - [2(A + 1)(B + 1) - (n - 1)(A + B + 2)]} \sigma^2. \tag{5.15}
\]
where
\[
A := \sum_{k \neq j} \frac{1}{1 - \rho_k}, \quad B := \sum_{k \neq j} \frac{1}{1 + \rho_k}. \tag{5.16}
\]

**Proof of (5.13) and (5.15)** Since the assets have the same volatility, we have
\[
f_k = \frac{1}{\sigma(1 - \rho_k)}, \quad g_k = \frac{1}{\sigma(1 + \rho_k)}. \tag{5.17}
\]
Therefore, substituting into (2.35), we obtain
\[
\sigma_p^2 = \frac{\sum_{k} \sigma(f_k + g_k)^2 - 4n + 4}{\sum_{k} \sigma(f_k + g_k)^2 - 2n + 2} \sum_{k} \frac{1}{\phantom{\sum_{k} \sigma(f_k + g_k)^2 - 2n + 2} \sum_{k}} \left(\frac{1}{\phantom{\sum_{k} \sigma(f_k + g_k)^2 - 2n + 2} \sum_{k}} \right) \sigma^2. \tag{5.18}
\]
Furthermore, rationalizing the denominator, we have
\[
\sigma_p^2(\rho_j) = \frac{A + B + \frac{2}{\phantom{A + B + 1} \sum_{k} 1 - \rho_k^2} - 2(n - 1)}{2\left(\frac{A + B}{\phantom{A + B + 1} \sum_{k} 1 - \rho_k^2} + \frac{2}{\phantom{A + B + 1} \sum_{k} 1 - \rho_k^2}\right) \sigma^2} \tag{5.19}
\]
\[= \frac{[A + B - 2(n - 1)]\rho_j^2 + 2(A - B)\rho_j - [2(A + 1)(B + 1) - (n - 1)(A + B + 2)]}{[2AB - (n - 1)(A + B)]\rho_j^2 + 2(A - B)\rho_j - [2(A + 1)(B + 1) - (n - 1)(A + B + 2)]} \sigma^2. \]

**Remark 5.2.1** (5.13) and (5.15) are well-defined.

**Proof of Remark 5.2.1** From Remark 4.2.4, we have
\[
2\bar{A}\bar{B} - (n - 1)(\bar{A} + \bar{B}) > 0. \tag{5.20}
\]
From Remark 3.2.2, we have
\[
\bar{A} + \bar{B} - 2(n - 1) > 0. \tag{5.21}
\]
From Remark 4.2.4, for all \(\rho_j \in (-1, 1)\), we have
\[
[2AB - (n - 1)(A + B)]\rho_j^2 + 2(A - B)\rho_j - [2(A + 1)(B + 1) - (n - 1)(A + B + 2)] < 0 \tag{5.22}
\]
For all \(\rho_j \in (-1, 1)\), we have
\[
[A + B - 2(n - 1)]\rho_j^2 - [A + B - 2(n - 2)] < [A + B - 2(n - 1)] - [A + B - 2(n - 2)] = -2 < 0. \tag{5.23}
\]
Recall Remark 3.2.3, we know when \( \rho_j = \frac{A + B - 2(n-2)}{A - B} \), the minimum variance \( n \)-asset portfolio is in fact a \((n-1)\)-asset portfolio. We show in this case the portfolio minimum variance achieves its maximum for all possible values of \( \rho_j \).

**Remark 5.2.2** When \( \rho_j = \frac{A + B - 2(n-2)}{A - B} \), the optimal portfolio variance achieve its highest value for \( \rho_j \in (-1, 1) \), which is the same as the variance of a \((n-1)\)-asset portfolio.

**Proof of Remark 5.2.2** Suppose portfolio 1 contains \( n \) assets, and portfolio 2 contains \((n-1)\) assets. Portfolio 2 contains all assets of portfolio 1 but asset \( j \). From (5.13), we know the variance of portfolio 2 is

\[
\sigma^2_{p_{n-1}} = \frac{A + B - 2(n-2)}{2AB - (n-2)(A + B)} \sigma^2.
\]

(5.24)

For all \( \rho_j \in (-1, 1) \), we have

\[
\sigma^2_{p_{n-1}} - \sigma^2_p = \left( \frac{A + B - 2(n-2)}{2AB - (n-2)(A + B)} - \frac{[A + B - 2(n-1)] \rho^2_j - [A + B - 2(n-2)]}{[2AB - (n-1)(A + B)] \rho^2_j + 2(A - B) \rho_j - [2(A + 1)(B + 1) - (n-1)(A + B + 2)]} \right) \sigma^2
\]

\[
= \frac{(A - B)^2 \rho^2_j + 2(A - B) \rho_j - [2(A + 1)(B + 1) - (n-1)(A + B + 2)]}{[2AB - (n-2)(A + B)] [(2AB - (n-1)(A + B)] \rho^2_j + 2(A - B) \rho_j - [2(A + 1)(B + 1) - (n-1)(A + B + 2)]} \sigma^2
\]

\[
\geq 0,
\]

where the equality holds if and only if

\[
\rho_j = \frac{A + B - 2(n-2)}{A - B}.
\]

(5.25)

(5.26)

**Intuition 5.2.1** Remark 5.2.2 shows that adding assets in the portfolio can reduce the portfolio variance. Therefore, the investor should create a larger portfolio to minimize risk. See Figure 5.2.
Chapter 6

Error Risk Analysis

In the previous chapters we have studied the situation where the parameters (variances and correlations to the market) of the assets in the portfolio have changed. We discussed how the portfolio’s composition should be changed to keep its risk as low as possible (Chapter 3) and how the portfolio return (Chapter 4) and variance (Chapter 5) respond. In this chapter we will study two different stories.

Suppose that the investor has a portfolio with \( n \) assets. The true market values of the portfolio’s parameters are \( \sigma \) (volatilities), \( \rho \) (correlations to the market) and \( \beta \) (sensitivities of the expected excess asset returns to the expected excess market returns). But his estimations are \( \tilde{\sigma} \) and \( \tilde{\rho} \), which are different from the true values. Using formula (2.4) he would get the “proper” proportion \( \tilde{X} = X(\tilde{\rho}, \tilde{\sigma}) \) of his portfolio and be under the impression that the expected portfolio return rate and the portfolio’s risk should be \( \mu_{\tilde{\rho}}(\tilde{\rho}, \tilde{\sigma}) = \tilde{X}^T \tilde{\mu} \) and \( \sigma_{\tilde{\rho}}^2(\tilde{\rho}, \tilde{\sigma}) = \tilde{X}^T \tilde{V} \tilde{X} \), which are calculated by the use of formulas (2.6) and (2.7), where \( \tilde{\mu} = \mu(\tilde{\rho}, \tilde{\sigma}) \) are the corresponding asset expected returns and \( \tilde{V} = V(\tilde{\rho}, \tilde{\sigma}) \) is the corresponding portfolio covariance matrix. But since the asset expected returns and portfolio covariance matrix are in fact the true values \( \mu \) and \( V \), the portfolio expected return rate and variance the investor actually gets will be \( \mu'_{\rho} = \tilde{X}^T \mu \) and \( \sigma^2_{\rho} = \tilde{X}^T V \tilde{X} \). Therefore, the investor thought his portfolio had the minimum variance \( \sigma^2_{\rho}(\rho, \sigma) \) but in fact the minimum variance is \( \sigma^2_{\rho}(\rho, \sigma) \) and the actual variance of his portfolio is \( \sigma^2_{\rho} \). Because of estimation errors, the investor doesn’t successfully minimize the portfolio’s risk (see Remark 6.0.4).

On the other hand, suppose the investor keeps his portfolio weights as \( X \). He doesn’t realize the portfolio covariance matrix is no longer \( V \) but \( \tilde{V} \). In this case the theoretical minimum variance of his portfolio should be \( \tilde{X}^T \tilde{V} \tilde{X} \), where \( \tilde{X} \) is the theoretical minimum variance portfolio weights corresponding to the covariance matrix \( \tilde{V} \). But the actual portfolio variance he obtained is \( X^T \tilde{V} X \). Because of estimation errors, the investor doesn’t successfully minimize the portfolio’s risk either (see Remark 6.0.5).

Remark 6.0.3 In this chapter, if not mentioned specifically we will use \( \sigma = (\sigma_1, \cdots, \sigma_n) \), \( \rho = (\rho_1, \cdots, \rho_n) \) and \( \beta = (\beta_1, \cdots, \beta_n) \) to represent variables and \( \sigma_0 = (\sigma_{1,0}, \cdots, \sigma_{n,0}) \), \( \rho_0 = (\rho_{1,0}, \cdots, \rho_{n,0}) \) and \( \beta_0 = (\beta_{1,0}, \cdots, \beta_{n,0}) \) to represent the fixed true values. For example, if the investor made only an error in \( \sigma_j \), we have

\[
\sigma_j \neq \sigma_{j,0}, \quad \sigma_i = \sigma_{i,0}, \text{ for } i \neq j, \quad \text{and } \rho = \rho_0. \tag{6.1}
\]
Remark 6.0.4 Whatever the perturbation $\tilde{X}$ is, we must have
\[ \frac{\partial}{\partial t} \sigma_p^2(t) \bigg|_{t=0} = 0, \]
which implies
\[ (1 - \rho_k^2)\sigma_k^2 \tilde{X}_k - (1 - \rho_j^2)\sigma_j^2 \tilde{X}_j + (\sum_i \rho_{ik}\sigma_i \tilde{X}_i)(\rho_{kj}\sigma_k - \rho_{kj}\sigma_j) \bigg|_{t=0} = 0 \]
for all $k \neq j$.

Since $\sigma_p^2 = \sigma_p^2(0)$ is the minimum portfolio variance, by definition we know (6.2) must be true. To prove it analytically, we can use Remark 3.4.1, which reflects the relationship between the minimum variance portfolio weights and the assets correlations and volatilities.

**Proof of Remark 6.0.4** The perturbed portfolio variance’s derivative to the perturbation is
\[
\frac{\partial}{\partial t} \sigma_p^2(t) = \frac{\partial}{\partial t} \left[ \sum_k (1 - \rho_k^2)\sigma_k^2 \tilde{X}_k^2 + \left( \sum_k \rho_{ik}\sigma_i \tilde{X}_i \right)^2 \right] \\
= \frac{\partial}{\partial t} \left[ \sum_{k\neq j} (1 - \rho_k^2)\sigma_k^2 \tilde{X}_k^2 + (1 - \rho_j^2)\sigma_j^2 (1 - \sum_{k\neq j} \tilde{X}_k^2) + \left( \sum_{k\neq j} \rho_{ik}\sigma_k - \rho_{kj}\sigma_j \right) \tilde{X}_k + \rho_{kj}\sigma_j \tilde{X}_j \right] \\
= 2 \sum_{k\neq j} \left[ (1 - \rho_k^2)\sigma_k^2 \tilde{X}_k - (1 - \rho_j^2)\sigma_j^2 \tilde{X}_j + (\sum_i \rho_{ik}\sigma_i \tilde{X}_i)(\rho_{kj}\sigma_k - \rho_{kj}\sigma_j) \right] \frac{\partial}{\partial t} \tilde{X}_k.
\]
Notice the fact that for each \( j = 1, \cdots, N \), we have \( \tilde{X}_j = 1 - \sum_{k \neq j} X_k \).

Since \( X = \tilde{X} \mid_{t=0} \) is the optimal weight that minimizes the portfolio variance, using (3.127) we obtain (6.3), which implies (6.2).

Furthermore, let \( \tilde{V} = V(t) \) be any (differentiable) perturbation of the portfolio covariance matrix \( V \). Suppose \( X(t) \) is the minimum variance portfolio weights corresponding to \( V(t) \). Therefore \( V(0) = V \) and \( X(0) = X \).

**Remark 6.0.5** Whatever the perturbation \( V(t) \) is, we must have

\[
\left. \left( \frac{\partial}{\partial t} X^T V(t) X \right) \right|_{t=0} = \left. \left( \frac{\partial}{\partial t} (X(t))^T V(t) X(t) \right) \right|_{t=0}.
\]  

(6.5)

**Proof of Remark 6.0.5** Notice that

\[
\left. \left( \frac{\partial}{\partial t} X^T V(t) X \right) \right|_{t=0} = X^T \left. \frac{\partial}{\partial t} V(t) \right|_{t=0} X,
\]  

(6.6)

and

\[
\left. \left( \frac{\partial}{\partial t} (X(t))^T V(t) X(t) \right) \right|_{t=0} = X(t)^T \left. \frac{\partial}{\partial t} V(t) \right|_{t=0} X(t) + 2X(t)^T V(t) \left. \frac{\partial}{\partial t} X(t) \right|_{t=0}.
\]  

(6.7)

Since

\[
\left. X^T \frac{\partial}{\partial t} V(t) \right|_{t=0} X = X(t)^T \left. \frac{\partial}{\partial t} V(t) \right|_{t=0} X(t),
\]  

(6.8)

we only need to show

\[
X(t)^T V(t) \left. \frac{\partial}{\partial t} X(t) \right|_{t=0} = 0.
\]  

(6.9)

In fact, notice

\[
\left. \left( X(t)^T V(t) \frac{\partial}{\partial t} X(t) - X(t)^T V \frac{\partial}{\partial t} X(t) \right) \right|_{t=0} = 0.
\]  

(6.10)

Using Remark 6.0.5, we have

\[
\left. X(t)^T V(t) \frac{\partial}{\partial t} X(t) \right|_{t=0} = X(t)^T V \left. \frac{\partial}{\partial t} X(t) \right|_{t=0} = \frac{1}{2} \left. \frac{\partial}{\partial t} (X(t)^T V X(t)) \right|_{t=0} = 0,
\]  

(6.11)

which proves (6.5).

**Intuition 6.0.2** First, suppose the investor has error estimates of the portfolio covariance matrix. He uses the wrong covariance matrix to get the “optimal” minimum variance portfolio weights. From Remark 6.0.4, we know that when the error is not too large, the actual portfolio variance he faced is only a little higher than the theoretical true minimum variance.

Second, suppose the investor keeps his portfolio weights even though the true portfolio covariance matrix has changed. From Remark 6.0.5, we know that when the error is not too large, the actual portfolio variance he faced is only a little higher than the theoretical true minimum variance.
Figure 6.1: Portfolio Variance vs. Asset Correlation Estimation Errors. $t = \tilde{\rho}_1 - \rho_1$ is the estimation error of asset 1’s correlation to the market. The full curve $X(t)^T V(t) X(t)$ represents the portfolio variance when the investor chooses the optimal portfolio weight $X(t)$ with updating covariance matrix $V(t)$. The dotted curve $X^T V(t) X$ represents the portfolio variance when the investor doesn’t change his portfolio weights and the covariance matrix $V(t)$ is no longer $V$. The dashed curve $X(t)^T V(t) X(t)$ represent the actual portfolio variance when the investor chooses the “optimal” weights $V(t)$ with the covariance matrix $V(t)$ which is wrong but should be $V$. The point $X^T V X$ is the minimum point of $X(t)^T V X(t)$. $X^T V(t) X$ is higher than $X(t)^T V(t) X(t)$ for all $t \in (-1 - \rho_1, 0) \cup (0, 1 - \rho_1)$ and the two curves are tangent at the point $X^T V X$. 
Chapter 7

Summary

Our analytic results reveal how the minimum variance portfolio composition, expected return and risk would change with respect to changes in the underlying asset correlations and volatilities. We give the investors instructions on how to build the minimum variance portfolio and keep the portfolio risk minimized with variable market data. For example, if two assets have similar correlations, we show the investor should invest less money in the asset with higher volatility. If two assets have similar volatilities, the investor should invest less money in the asset with higher correlation to the market. If two assets have similar sensitivities to the market excess return, the investor should invest less money in the asset with higher risk or lower correlation to the market. We show the minimum variance portfolio expected return has upper bounds for variable asset volatilities and variable correlations to the market. We show adding assets negatively correlated or uncorrelated to the market in the minimum variance portfolio will drag the portfolio expected return down. If two assets have similar volatilities, the portfolio’s expected return is more sensitive to correlation changes of assets with lower correlation to the market. We show the larger the portfolio size is, the lower portfolio minimum variance the investor could get. In other words, if portfolio 1 contains all assets of portfolio 2, then the minimum variance of portfolio 1 is lower than (at least equal to) that of portfolio 2, no matter what the rest assets of portfolio 1 are.

We also discuss specifically the two-asset portfolio, which is analytically tractable, and we find many interesting results. We find that the two-asset portfolio’s composition depends on two parameters: the ratio of asset volatilities and the correlation between assets. We give the boundaries of the two-asset portfolio minimum variance and corresponding portfolio weights and expected return.

Finally, we analyze the risk that is not covered when the investor makes estimation errors about the market data using our model. We show if the estimation errors of the portfolio covariance matrix and portfolio weights are not too large, the actual portfolio variance the investor faced is not quite higher than the theoretical true minimum variance. In other words, the portfolio minimum variance is stable.

In the future, we will study more complicated Mean-Variance problems. For example, we will study MV problems with a fixed portfolio expected return constraint condition, which comes from the efficient frontier portfolio theory.
Bibliography


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