August 2013

On degeneracies in the family of fibres of a complex analytic mapping

Hadi Seyedinejad
The University of Western Ontario

Supervisor
Janusz Adamus
The University of Western Ontario

Graduate Program in Mathematics

A thesis submitted in partial fulfillment of the requirements for the degree in Doctor of Philosophy

On degeneracies in the family of fibres of a complex analytic mapping

(Thesis format: Monograph)

by

S. M. Hadi Seyedinejad

Graduate Program in Mathematics

A thesis submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy

The School of Graduate and Postdoctoral Studies
The University of Western Ontario
London, Ontario, Canada

Abstract

We study (local) properties of complex analytic mappings (and modules over them) by analyzing their underlying family of fibres. Two important properties of such mappings, namely openness and flatness, are our main object of interest. Failure of either of these properties manifests itself as some sort of *degeneracy* in the family of fibres. The first goal of this thesis is to develop criteria that allow one to effectively (i.e., computationally) detect these degeneracies, and in addition, that can be applied to the case of mappings with *singular* targets. This is the subject of Chapters 2 and 3. Particularly regarding flatness, no such algorithms that work in the general setting of singular targets were known before.

We prove that a mapping germ $\varphi : X \to Y$ (under some assumptions) is flat (resp. open) if and only if after pulling φ back by the blowing-up of the origin in Y, the special fibre does not contain an irreducible component (resp. an isolated irreducible component). Algebraically, this criterion is equivalent to a test for a specific zero-divisor in the local ring (resp. reduced local ring) of the pullback. Also, as a generalization of previous flatness criteria of Auslander's type to the case of singular bases, we prove that an analytic R-module F is flat if and only if the analytic tensor product $\underbrace{F \tilde{\otimes}_R \cdots \tilde{\otimes}_R F}_{n \text{ times}} \tilde{\otimes}_R S$ has no vertical components, where R is a complex analytic algebra which is an integral domain of dimension n, and S is the local ring of a desingularization of $\mathrm{Specan}(R)$.

Our second goal is to characterize different modes of the above mentioned degeneracies. This is the topic of Chapter 4. We study the verticality index of a mapping, defined as the highest fibred power in which no verical components emerge. This is a gauge which measures the level of non-openness of mappings. We obtain some results about verticality index, especially on its behaviour over singular targets.

Keywords: analytic module, analytic tensor product, complex analytic set, fibred product, flatness descent, flatness testing, local geometry, openness testing, singular base, torsion-free, vertical component, zero-divisor.

*I dedicate my thesis to **Saeideh**, a peerless wife, for her patience and backing me remotely up, during the long and extremely exhausting months of staying apart, when I was preparing this thesis and the Canadian Embassy did not issue a visa for her in order to join me in Canada.*

Acknowledgements

At this moment, before I LaTeX my dissertation for the last time, I express my deepest gratitude to three persons, to whom I genuinely owe my doctoral success:

> my **mother** and **father**, for being so precious, with the heartfelt compassion and unaffected support of whom I was able to come to Canada, stay abroad for years, and do my PhD; and,
>
> **Janusz Adamus**, for being beyond a supervisor, who just showed me a true meaning of mathematical research, academic performance, and professional discipline.

I am so proud of graduating from the wonderful *University of Western Ontario*, with its great community, where I spent the best years of my whole student life. I should be highly grateful to Western, for a full support of every kind that provided me with. This includes, especially, a complete financial support, so that I accomplished my degree without any need for external resources. I see all the quality of my research as a direct outcome of such condition. In this regard, my special acknowledgement should go to the *Mathematics Department*, for being an exceptional academic home of mine for four years. Also, I greatly appreciate the *Faculty of Science*, for their prestigious scholarship that partially funded my study.

I would like to thank dear members of my advisory committee and the Thesis Examination Board, *Professor Rasul Shafikov*, for everything that I learnt from him during these years, and *Professor André Boivin*, for all his valuable advice and kind support. Finally, I give a special thank to the External Examiner of the thesis, *Professor Wiesław Pawłucki*, for having the honour of his review and evaluation, and also, to *Professor Greg Reid*, for accepting to be the University Examiner of the thesis.

Hadi Seyedinejad, August 2013

List of symbols

$\mathbb{N}$	$\{0, 1, 2, \dots\}$	page 2
$\mathbb{R}$	field of real numbers	31
$\mathbb{R}^+$	positive real numbers	25
$\mathbb{C}$	field of complex numbers	2
$\mathbb{C}^n_x$	nth Cartesian power of $\mathbb{C}$ with coordinates $x = (x_1, \dots, x_n)$	2
$\mathbb{C}\{x\}$	convergent power series over $\mathbb{C}$ in multivariable x	6
R^p	p-fold direct sum of the ring or module R	8
$R[x]$	polynomials over the ring R in multivariable x	32
$R[[x]]$	formal power series over the ring R in multivariable x	7
$[r]$	largest integer less than or equal to the real number r	41
$\lvert(a_1, \dots, a_n)\rvert$	$a_1 + \cdots + a_n$	32
$\lvert A\rvert$	size of the set A	32
$\lvert x\rvert$	absolute value of the real or complex number x	32
ϕ_{vert}	verticality index	40
ϕ_{s}	verticality index for smooth target case	41
$\phi_{\_}$	lower bound for verticality index	53
$\mathfrak{N}_L$	diagram of initial exponents with respect to the ordering L	31
$\exp_L$	initial exponent with respect to the ordering L	31
Ass_R	associated primes over the ring R	10
reg	regular locus	42
sng	singular locus	42
dim	dimension	12
$\dim_\xi$	dimension at the point ξ	12
codim_X	codimension with respect to the ambient space X	56
fbd	fibre dimension	41
fbd_ξ	fibre dimension at the point ξ	27
$\times_Y$	fibred product over the space Y	4
$\varphi^{\{i\}}$	i-fold fibred power of the mapping φ	6
$X^{\{i\}}$	i-fold fibred power of the space X	6
$\otimes_R$	tensor product over the ring R	8
$\tilde{\otimes}_R$	analytic tensor product over the ring R	7
$F^{\tilde{\otimes}^i_R}$	i-fold analytic tensor power of the module F over the ring R	19
$A\rvert_B$	set or mapping A restricted to the set B	3
sup	supremum	12
min	minimum	31

max	maximum	32
s.t.	such that	43
$/$	quotient	3
$\setminus$	subtraction of sets	25
$\sqrt{I}$	radical of the ideal I	27
ord_x	order with respect to multivariable x	34
supp	support	31
Spec	prime spectrum	61
Specan	analytic spectrum	7
$\mathcal{O}_X$	structure sheaf of the ringed space on X	3
$\mathcal{O}_{X,\xi}$	stalk of the sheaf $\mathcal{O}_X$ at ξ	3
X_ξ	germ of the set X at ξ	4
φ^*	underlying sheaf morphism of the ringed space morphism φ	3
φ_ξ	germ of the mapping φ at ξ	4
φ^*_ξ	induced morphism on stalk at ξ by the mapping φ	4
f_ξ	germ of the sheaf section f at ξ	4
$\varphi_*\mathcal{O}$	direct image of the sheaf $\mathcal{O}$ by the mapping φ	3
id_X	identity mapping $X \to X$	13
$\varnothing$	empty set	32
$\simeq$	is isomorphic to	5
$\equiv \pmod{}$	is congruent to (modulo)	35
$:=$	is defined as	3
$=:$	defines	36
$\square$	end of proof	18
$\triangle$	end of example, remark, etc.	3

Contents

Chapter 1

Introduction

1.1 Overview

The present thesis concerns essentially the (local) properties of holomorphic mappings of complex analytic sets, and thus belongs to the realm of complex analytic geometry. We study the behaviour of such mappings by analyzing the family of their fibres. The motivation of this research is two-fold. On the one hand, studying the family of fibres of a mapping can help one to understand the behaviour of the mapping. On the other hand, understanding the behaviour of a mapping helps to study its family of fibres—viewed as a parametrized family of complex spaces. Broadly speaking, the former is an approach in complex analysis, and the latter in the theory of singularities. In addition, due to the strong interaction with commutative algebra (thanks to the algebraically closed $\mathbb{C}$), our study often produces analogous results in the algebraic category (see Appendix), which can be used for effective computation with the aid of computer algebra.

Let us mention some classical examples of how the family of fibres encodes information about the properties of a mapping. Let $\varphi : X \to Y$ be a holomorphic mapping of analytic sets. It is not true, in general, that the image $\varphi(X)$ is an analytic set, which means that $\varphi(X)$ might not be an object of our category. Remmert gives a sufficient condition for $\varphi(X)$ to be an analytic set; namely, that the fibres of φ be all of the same dimension (Remmert's Rank Theorem). Remmert's Open Mapping Theorem is another instance, where openness of φ is characterized in terms of a simple continuity condition on the topological dimension of fibres. Flatness, the algebraic sister of openness, has also a tight relation with the continuity of fibres. In a simple case, for a mapping φ with finite fibres, flatness turns out to be equivalent to the condition that multiplicities of the fibres are locally constant.

Two important properties of a holomorphic mapping $\varphi : X \to Y$, namely openness and flatness, are our main object of interest. Failure of either of these properties manifests itself as some sort of *discontinuity* or *degeneracy* in the family of fibres of φ. This fact is expressed in the classical criteria of Remmert (for openness) and Hironaka (for flatness). Based on these criteria, the first goal of our thesis is to develop algorithms to effectively (i.e., computationally) detect such degeneracies, even for mappings with singular targets. This is done in Chapters 2 and 3. The next goal is to characterize different modes of such degeneracies, leading to a classification of non-open or non-flat mappings. This is the topic of Chapter 4.

1.2 Preliminaries

Throughout, $\mathbb{C}$ denotes the field of complex numbers, and $\mathbb{N}$ denotes the set of natural numbers $\{0, 1, 2, \dots\}$. For $n \in \mathbb{N}$, the n-fold Cartesian power of $\mathbb{C}$ is denoted by $\mathbb{C}^n$, and we may use the notation $\mathbb{C}^n_x$ to indicate that a coordinate system $x = (x_1, \dots, x_n)$ is chosen for $\mathbb{C}^n$ endowed with the canonical structure of a $\mathbb{C}$-vector space. Topology is always the Euclidean one.

Complex spaces and mappings

Let U be an open subset of $\mathbb{C}^n$, where $n \in \mathbb{N}$. A subset $X \subseteq U$ is called an *analytic subset* of U if, for every $\xi \in U$, there exist an open neighbourhood V of ξ in U and finitely many holomorphic functions $f_1, \ldots, f_r : V \to \mathbb{C}$ such that $X \cap V = \{x \in V \mid f_1(x) = \cdots = f_r(x) = 0\}$.

Consider $\mathcal{O}_U$, the sheaf of holomorphic functions on an open set U in $\mathbb{C}^n$. Let $\mathcal{I}$ be a coherent ideal of $\mathcal{O}_U$, and consider the quotient $\mathcal{O}_U/\mathcal{I}$ with the support $A := \{x \in U \mid (\mathcal{O}_U/\mathcal{I})_x \neq 0\}$. We obtain a $\mathbb{C}$-ringed space[1] $(A, (\mathcal{O}_U/\mathcal{I})|_A)$, which is called a *(complex) local model*. We call $\mathcal{I}$ the *defining ideal* of the local model. In fact, A is easily seen to be an analytic subset of U (see e.g. [F, § 0.13]), and we call $\mathcal{O}_A = (\mathcal{O}_U/\mathcal{I})|_A$ the sheaf of holomorphic functions on A.

By gluing together local models, one obtains a complex analytic space.

Definition 1.1. A *complex analytic space* is a $\mathbb{C}$-ringed space $(X, \mathcal{O}_X)$ such that the topological space X is Hausdorff and for every $\xi \in X$, there exists an open neighbourhood $U \subseteq X$ with the restriction $(U, \mathcal{O}_X|_U)$ isomorphic (as a $\mathbb{C}$-ringed space) to some local model. △

By a *complex space*, or simply a *space*, we always mean a complex analytic space. By a *complex mapping* (or a *complex morphism*), or simply a *mapping*, we mean a morphism $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of complex spaces, which is a $\mathbb{C}$-ringed space morphism (φ, φ^*), where $\varphi : X \to Y$ is a continuous mapping of topological spaces, $\varphi^* : \mathcal{O}_Y \to \varphi_*\mathcal{O}_X$ is a $\mathbb{C}$-algebra homomorphism, and $\varphi_*\mathcal{O}_X$ is the direct image of $\mathcal{O}_X$ under φ. We use the same notation for a complex space (or mapping) and its underlying topological space (or mapping), but we do clarify whenever necessary. For instance, we denote the space $(X, \mathcal{O}_X)$ simply by X, and the mapping $(\varphi, \varphi^*) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$

[1] A $\mathbb{C}$-*ringed space* is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X and a sheaf of $\mathbb{C}$-algebras $\mathcal{O}_X$ (the *structure sheaf* on X), such that for every $\xi \in X$, the stalk $\mathcal{O}_{X,\xi}$ is a local ring with the maximal ideal $\mathfrak{m}_{X,\xi}$ and with the residue field $\mathcal{O}_{X,\xi}/\mathfrak{m}_{X,\xi}$ isomorphic as a $\mathbb{C}$-algebra to $\mathbb{C}$.

by $\varphi : X \to Y$. Note that in the case that X and Y are local models, the underlying topological mapping $\varphi : X \to Y$ is just a holomorphic mapping in the classical sense.

We reserve $\mathcal{O}$ to mean always the structure sheaf as defined above.

Given a complex space X, the *complex germ* of X at ξ is the pair $(X_\xi, \mathcal{O}_{X,\xi})$ (may be denoted simply by X_ξ), where the first element is the germ at ξ of the topological space X, and the second is the stalk at ξ of the sheaf $\mathcal{O}_X$. We call $\mathcal{O}_{X,\xi}$ also the *local ring of X at ξ*. A morphism $\varphi_\xi : X_\xi \to Y_\eta$ of complex germs is the germ at $\xi \in X$ of a complex morphism $(\varphi, \varphi^*) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$, where $\varphi(\xi) = \eta$.

Let $\varphi_\xi : X_\xi \to Y_\eta$ be a morphism of complex germs. The induced homomorphism $\varphi_\xi^* : \mathcal{O}_{Y,\eta} \to \mathcal{O}_{X,\xi}$ of local rings makes $\mathcal{O}_{X,\xi}$ a module over $\mathcal{O}_{Y,\eta}$. This is what we mean every time by the module structure of the local ring of the source over the local ring of the target of a mapping. We would like to recall how the homomorphism φ_ξ^* acts. Suppose X_ξ is a reduced germ (i.e., $\mathcal{O}_{X,\xi}$ has no nilpotent elements). We can assume that $X \subseteq \mathbb{C}^m$ and $Y \subseteq \mathbb{C}^n$ are already local models, where $m, n \in \mathbb{N}$. Then for every germ $f_\eta \in \mathcal{O}_{Y,\eta}$ of a holomorphic function f defined on a neighbourhood of η in Y, we have $\varphi_\xi^*(f_\eta) = (f \circ \varphi)_\xi$, for some suitable representative $\varphi : X \to Y$ such that $\varphi(X)$ is contained in the domain of f. In case X_ξ is not reduced, we have $\varphi_\xi^*(f_\eta) = \overline{(\tilde{f} \circ \tilde{\varphi})_\xi}$, where $\tilde{f}$ and $\tilde{\varphi}$ are respectively holomorphic extensions of some representatives f and φ to some neighbourhoods of η in $\mathbb{C}^n$ and ξ in $\mathbb{C}^m$, and $\overline{(\tilde{f} \circ \tilde{\varphi})_\xi} \in \mathcal{O}_{X,\xi}$ is the class of $(\tilde{f} \circ \tilde{\varphi})_\xi$ (as a power series about ξ) modulo the ideal defining X_ξ in $\mathbb{C}^m_\xi$.

Given mappings $\varphi_1 : X_1 \to Y$ and $\varphi_2 : X_2 \to Y$, it can be shown that the fibred product $X_2 \times_Y X_1$ in the category of complex spaces exists (see [F, § 0.32]).[2] The underlying topological space of the fibred product of X_1 and

[2] In any category, the *fibred product* of morphisms $\varphi_1 : X_1 \to Y$ and $\varphi_2 : X_2 \to Y$ is an object, denoted by $X_2 \times_Y X_1$, together with morphisms $\varphi_1' : X_2 \times_Y X_1 \to X_2$ and $\varphi_2' : X_2 \times_Y X_1 \to X_1$, such that we have $\varphi_2 \circ \varphi_1' = \varphi_1 \circ \varphi_2'$ and the following universal property holds: for every object X and morphisms $\psi_2 : X \to X_2$ and $\psi_1 : X \to X_1$

X_2 is the same as the fibred product of their underlying topological spaces; that is, it coincides with the set $\{(\xi_2, \xi_1) \in X_2 \times X_1 \mid \varphi_2(\xi_2) = \varphi_1(\xi_1)\}$. The underlying topological mapping of the pullback $\varphi_1' : X_2 \times_Y X_1 \to X_2$ of φ_1 by φ_2 is just a projection, and so is φ_2', the pullback of φ_2 by φ_1. We will mention about the sheaf structure of the fibred product soon.

Remark 1.2 (pasting Cartesian squares)**.** Consider the following diagram of morphisms (in any category):

$$\begin{array}{ccccc} X_3 \times_{X_2} (X_2 \times_Y X_1) & \longrightarrow & X_2 \times_Y X_1 & \longrightarrow & X_1 \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ X_3 & \longrightarrow & X_2 & \longrightarrow & Y \end{array}$$

Two internal squares are Cartesian by definition. Then using the universal property of fibred product, one can show that the outer rectangle is also a Cartesian square. In other words, $X_3 \times_{X_2} (X_2 \times_Y X_1) \simeq X_3 \times_Y X_1$. By multiple pasting of Cartesian squares, one can easily obtain the following corollaries about fibred product:

(i) *Associativity.* Given morphisms $X_1 \to Y$, $X_2 \to Y$, and $X_3 \to Y$, we have

$$(X_1 \times_Y X_2) \times_Y X_3 \simeq X_1 \times_Y (X_2 \times_Y X_3).$$

(ii) *Commutativity with base change.* Given morphisms $X_1 \to Y$, $X_2 \to Y$, and $Z \to Y$, we have

---

satisfying $\varphi_1 \circ \psi_1 = \varphi_2 \circ \psi_2$, there exists a unique morphism $\psi : X \to X_2 \times_Y X_1$ such that $\varphi_1' \circ \psi = \psi_2$ and $\varphi_2' \circ \psi = \psi_1$. In this case, φ_1' is called the *pullback* of φ_1 by φ_2, φ_2' is called the pullback of φ_2 by φ_1, and the (commutative) diagram

$$\begin{array}{ccc} X_2 \times_Y X_1 & \xrightarrow{\varphi_2'} & X_1 \\ \big\downarrow{\scriptstyle \varphi_1'} & & \big\downarrow{\scriptstyle \varphi_1} \\ X_2 & \xrightarrow{\varphi_2} & Y \end{array}$$

is a called a *Cartesian square.* The mapping $\varphi_2 \circ \varphi_1'$ (which is equal to $\varphi_1 \circ \varphi_2'$) is called the fibred product of φ_1 and φ_2, and is denoted by $\varphi_2 \times_Y \varphi_1 : X_2 \times_Y X_1 \to Y$.

$$(X_1 \times_Y X_2) \times_Y Z \simeq (X_1 \times_Y Z) \times_Z (X_2 \times_Y Z).$$

$\triangle$

The i-fold fibred power of a mapping $\varphi : X \to Y$, $i \geq 1$, will be denoted by $\varphi^{\{i\}} : X^{\{i\}} \to Y$, where $X^{\{i\}}$ is the i-fold fibred power of the space X (over Y). (Of course, $X^{\{1\}} := X$ and $\varphi^{\{1\}} := \varphi$.)

For more on complex spaces and morphisms, see [F] or [GPR]. For a brief review of basic categorical tools we need, we suggest [La, Chapter I].

Complex analytic algebras and modules

Consider a complex space X and a point $\xi \in X$. Let a local model for X at ξ be given about the origin of some $\mathbb{C}^n_x$ by the holomorphic functions $f_1, \dots, f_r$ defined on an open subset $U \subseteq \mathbb{C}^n_x$ containing the origin. That is, the defining ideal of the local model is given by $\mathcal{I} := f_1\mathcal{O}_U + \dots + f_r\mathcal{O}_U$. By definition,

$$\mathcal{O}_{X,\xi} \simeq (\frac{\mathcal{O}_U}{\mathcal{I}})_0 = \frac{\mathbb{C}\{x\}}{(f_{1,0}, \dots, f_{r,0})},$$

where $\mathbb{C}\{x\}$ denotes the ring of convergent power series in variables $x = (x_1, \dots, x_n)$ and with coefficients in $\mathbb{C}$, and where $f_{1,0}, \dots, f_{r,0}$ are the power series expansions of $f_1, \dots, f_r$ about the origin. We call $\mathcal{I}_0 = (f_{1,0}, \dots, f_{r,0})$ the *defining ideal* of the germ X_ξ in $\mathbb{C}^n_x$.

In general, for any ideal $I \subseteq \mathbb{C}\{x\}$ (which by Noetherianity is finitely generated, and hence is the defining ideal of a complex germ), a $\mathbb{C}$-algebra of the form $\mathbb{C}\{x\}/I$ is called an *analytic* $\mathbb{C}$*-algebra*. It is a local ring with the maximal ideal equal to $(x) \cdot \mathbb{C}\{x\}/I$.

Taking $\mathbb{C}$-algebra homomorphisms as morphisms, analytic $\mathbb{C}$-algebras form a category, which is dually equivalent to the category of complex germs. This fact is called *Anti-Equivalence Principle* (see e.g. [F, § 0.21]). To every analytic $\mathbb{C}$-algebra $\mathbb{C}\{x\}/I$, where $x = (x_1, \dots, x_n)$, we assign a unique (isomorphism class of a) complex germ denoted by $\text{Specan}(\mathbb{C}\{x\}/I)$, and represented by $(X_0, \mathbb{C}\{x\}/I)$, where X_0 is the germ at the origin of the zero set of I in $\mathbb{C}^n_x$. Any morphism $R \to A$ of analytic $\mathbb{C}$-algebras corresponds to a unique mapping germ $\text{Specan}A \to \text{Specan}R$.

Let $R = \mathbb{C}\{y\}/J$ be an analytic $\mathbb{C}$-algebra, where J is an ideal of $\mathbb{C}\{y\}$. We define $R\{x\} \coloneqq \mathbb{C}\{y,x\}/(J \cdot \mathbb{C}\{y,x\})$, which can be regarded naturally as an R-subalgebra of $R[[x]]$ (formal power series with coefficients in R). For any ideal $I \subseteq R\{x\}$, the quotient $R\{x\}/I$ is called an *analytic R-algebra*.

Taking the morphisms as R-algebra homomorphisms, analytic R-algebras form a category. The coproduct in this category exists and is called the *analytic tensor product*. We denote it by $\tilde{\otimes}_R$.

Let $R\{x\}/I$ and $R\{x'\}/I'$ be two analytic R-algebras. It can be shown (cf. [GR]) that

$$\frac{R\{x\}}{I} \tilde{\otimes}_R \frac{R\{x'\}}{I'} = \frac{R\{x,x'\}}{I\tilde{\otimes}_R 1 + 1\tilde{\otimes}_R I'}\ , \tag{1.1}$$

in which by $I\tilde{\otimes}_R 1$ and $1\tilde{\otimes}_R I'$ we mean respectively the ideals in $R\{x,x'\}$ generated by the images of the canonical homomorphisms $I \to R\{x,x'\}$ and $I' \to R\{x,x'\}$. Identifying I and I' with their generators, we may denote the right-hand side of (1.1) simply by $R\{x,x'\}/(I+I')$.

Using the universal property and uniqueness of fibred product and coproduct, one proves that analytic tensor product of algebras is dual to the fibred product of mapping germs: Given mappings $X \to Y$ and $X' \to Y$, respectively with $\xi \mapsto \eta$ and $\xi' \mapsto \eta$, we have

$$\mathcal{O}_{X' \times_Y X, (\xi', \xi)} \simeq \mathcal{O}_{X,\xi} \tilde{\otimes}_{\mathcal{O}_{Y,\eta}} \mathcal{O}_{X',\xi'},$$

or

$$\text{Specan}(\mathcal{O}_{X,\xi} \tilde{\otimes}_{\mathcal{O}_{Y,\eta}} \mathcal{O}_{X',\xi'}) \simeq (X' \times_Y X)_{(\xi',\xi)}.$$

Let R be an analytic $\mathbb{C}$-algebra. A module F over R is called an *analytic*

R-module[3] if it is finitely generated over some analytic R-algebra A (called its *witness* ring). In this case, we can represent F by an A-module isomorphism as $F \simeq A^p/M$, for some $p \in \mathbb{N}$ and some A-submodule M of A^p, where A^p is the p-fold direct sum of A. Note that, as every analytic R-algebra $R\{x\}/I$ is a finitely generated module over $R\{x\}$, we can always choose a witness ring of the form $R\{x\}$.

Analytic tensor product of two analytic R-modules F and G is defined as follows. First, let A and B be witness rings for F and G respectively. Then,

$$F\tilde{\otimes}_R G \coloneqq (F \otimes_A (A\tilde{\otimes}_R B)) \otimes_{A\tilde{\otimes}_R B} ((A\tilde{\otimes}_R B) \otimes_B G).$$

This definition is independent of the chosen witness rings, up to isomorphism of R-modules. For analytic R-modules $F = R\{x\}^p/M$ and $G = R\{x'\}^q/M'$, we have

$$F\tilde{\otimes}_R G = \frac{R\{x,x'\}^{pq}}{M\tilde{\otimes}_R 1 + 1\tilde{\otimes}_R M'}, \tag{1.2}$$

in which by $M\tilde{\otimes}_R 1$ and $1\tilde{\otimes}_R M'$ we mean respectively the $R\{x,x'\}$-submodules of $R\{x,x'\}^{pq}$ generated by the images of the canonical homomorphisms $M^q \to (R\{x,x'\}^p)^q$ and $M'^p \to (R\{x,x'\}^q)^p$. We denote the right-hand side of (1.2) simply by $R\{x,x'\}^{pq}/(M^q + M'^p)$.

Vertical components

Let $(X_\xi, \mathcal{O}_{X,\xi})$ be a complex germ. A (closed) subgerm $(X_{\iota,\xi}, \mathcal{O}_{X_\iota,\xi})$, where $\mathcal{O}_{X_\iota,\xi} = \mathcal{O}_{X,\xi}/\mathcal{I}_{\iota,\xi}$ for some coherent ideal $\mathcal{I}_\iota$ of $\mathcal{O}_X$, is called an *irreducible component* of $(X_\xi, \mathcal{O}_{X,\xi})$ if $\mathcal{I}_{\iota,\xi}$ is an associated prime of the $\mathcal{O}_{X,\xi}$-module $\mathcal{O}_{X,\xi}$. Let $\{(X_{\iota,\xi}, \mathcal{O}_{X,\xi}/\mathcal{I}_{\iota,\xi})\}_\iota$ be the set of all irreducible components of $(X_\xi, \mathcal{O}_{X,\xi})$. A maximal element of $\{X_{\iota,\xi}\}_\iota$ (with respect to inclusion) is called an *isolated* irreducible component (whose defining ideal is then an isolated associated prime of the $\mathcal{O}_{X,\xi}$-module $\mathcal{O}_{X,\xi}$); but if $X_{\iota_1,\xi} \subseteq X_{\iota_2,\xi}$ for some

[3] We chose this terminology from [GR]. Sometimes, following [GK], such a module is called an *almost finitely generated* R-module.

ι_1 and ι_2, then $X_{\iota_1,\xi}$ is called an *embedded* irreducible component (whose defining ideal is then an embedded associated prime of the $\mathcal{O}_{X,\xi}$-module $\mathcal{O}_{X,\xi}$).

For a complex space X, irreducible components of its germs can be glued together to obtain the *global* irreducible components. However, we shall never need the details of this beyond just the fact that the germ $X_{\iota,\xi}$ at $\xi \in X$ of every irreducible component X_ι of X is decomposed into some irreducible components of X_ξ of the same dimension as X_ι, and conversely, given X and $\xi \in X$, every irreducible component of X_ξ has a representative which is dominantly (i.e., with a non-empty interior) contained in an irreducible component of X. We refer the interested reader to [ST] and [K].

Definition 1.3 ([K]). Let $\varphi : X \to Y$ be a mapping. An irreducible component Σ of X (passing through ξ) is called a *vertical component* of φ (at ξ)—or, when the mapping φ is clear, a vertical component (at ξ) over Y—if $\varphi(\Sigma)$ has no interior points in Y.[4] $\triangle$

For a mapping germ $\varphi_\xi : X_\xi \to Y_\eta$, an (isolated or embedded) irreducible component $X_{\iota,\xi}$ of X_ξ is called an (isolated or embedded) vertical component if there exist representatives X_ι and $\varphi : X \to Y$, with $X_\iota \subseteq X$, such that $\varphi(X_\iota)$ has empty interior in Y. As the dimension of the image of an irreducible space is equal to the dimension of the image of every of its non-empty open subsets,[5] it follows that a mapping $\varphi : X \to Y$ has a vertical component at $\xi \in X$ if and only if $\varphi_\xi : X_\xi \to Y_\eta$ has a vertical component in the above sense. In fact, given $\varphi : X \to Y$, any vertical component of its germ $\varphi_\xi : X_\xi \to Y_\eta$ at a point $\xi \in X$ has a representative which dominantly

[4] Authors sometimes define verticality by requiring that $\varphi(\Sigma)$ be a nowhere-dense subset of Y, instead of having no interior points. These two definitions are equivalent, because the image of an analytic set by a holomorphic mapping is a locally finite union of manifolds.

[5] For a mapping $\varphi : X \to Y$ with X irreducible, the (complex) dimension of the image set $\varphi(X)$ is given by the *Dimension Formula* as $\dim \varphi(X) = \dim X - \lambda$, where λ is the minimal fibre dimension of φ. Note that the locus of points at which the fibre dimension is λ is open and dense in X. So, from the Dimension Formula, one can conclude that $\dim \varphi(U) = \dim \varphi(X)$ for every non-empty open subset $U \subseteq X$. See [L, § V.3] for details.

(i.e., with a non-empty interior) embeds in a vertical component of φ.

More generally, one can define the notion of *verticality* for an analytic module ([GK, Definition 4.4]) as follows. Let F be an analytic module over an analytic $\mathbb{C}$-algebra R. Let A be an analytic R-algebra over which F is finitely generated, and let $\tau : \mathrm{Specan}A \to \mathrm{Specan}R$ be the mapping germ induced by the canonical homomorphism $R \to A$. Consider $\mathrm{Ass}_A F$, the set of all associated primes of the A-module F. Every $\mathfrak{p} \in \mathrm{Ass}_A F$ defines a closed subgerm $\mathrm{Specan}(A/\mathfrak{p})$ (of $\mathrm{Specan}A$) which we are going to call an irreducible component of the A-module F. Now, we say that the A-module F has a vertical component over R if there exists some $\mathfrak{p} \in \mathrm{Ass}_A F$ such that the irreducible germ $\mathrm{Specan}(A/\mathfrak{p})$ has a representative which is mapped by a representative of τ into a subset with empty interior in the target. It is not difficult to show (see [GK, Proposition 3.6]) that if the A-module F has a vertical component over R, then it does so for any other choice of a witness ring A. Therefore, we can safely speak of F having a vertical component over R or not, without mentioning a witness ring at all.

In particular, a mapping germ $\varphi_\xi : X_\xi \to Y_\eta$ has no vertical components if and only if the module $\mathcal{O}_{X,\xi}$ has no vertical components over $\mathcal{O}_{Y,\eta}$.

The notion of vertical component can be viewed as a geometric generalization of the zero-divisor. In fact, the emergence of some special types of vertical components is equivalent to the presence of zero-divisors. The following remark elaborates on this.

Remark 1.4. Let $\varphi_\xi : X_\xi \to Y_\eta$ be a mapping germ, with Y_η irreducible (i.e., $\mathcal{O}_{Y,\eta}$ is an integral domain). Let Σ_ξ be an irreducible component of X_ξ defined by an associated prime $\mathfrak{p}$ of the $\mathcal{O}_{X,\xi}$-module $\mathcal{O}_{X,\xi}$. Suppose there exist representatives $\varphi : X \to Y$ and $\Sigma \subseteq X$, such that the image germ of Σ is contained in a proper complex subgerm of the target; that is, there exists a (nonzero) germ $f_\eta \in \mathcal{O}_{Y,\eta}$ such that $\varphi(\Sigma)_\eta \subseteq \mathrm{Specan}(\mathcal{O}_{Y,\eta}/(f_\eta)) \subsetneqq Y_\eta$. Of course, Σ_ξ is then a vertical component of a special type, which is named *algebraic vertical component* by Adamus [A2]. Now, by the action of φ_ξ^* explained before, it follows that $\varphi_\xi^*(f_\eta) \in \mathfrak{p}$. Recall that the union of associated primes of a module is equal to the set of all zero-divisors (together with zero). Thus, $\varphi_\xi^*(f_\eta)$ is a zero-divisor in the $\mathcal{O}_{X,\xi}$-module $\mathcal{O}_{X,\xi}$, and hence f_η is a zero-divisor in the $\mathcal{O}_{Y,\eta}$-module $\mathcal{O}_{X,\xi}$. Conversely, one can similarly see that if $f_\eta \in \mathcal{O}_{Y,\eta}$ is a zero-divisor in $\mathcal{O}_{X,\xi}$, then there exist representatives $\varphi : X \to Y$ of φ_ξ and Σ of an irreducible component of X_ξ such that $\varphi(\Sigma)_\eta$ is contained in the hypersurface germ $\mathrm{Specan}(\mathcal{O}_{Y,\eta}/(f_\eta)) \subsetneqq Y_\eta$.

Similarly for an analytic R-module F (with R an integral domain), F has a vertical component admitting a representative whose image germ is contained in $\mathrm{Specan}(R/(f)) \subsetneqq \mathrm{Specan} R$, if and only if f is a zero-divisor in F (where $f \in R$). $\triangle$

Some other definitions and conventions

We will say that a mapping $\varphi : X \to Y$ is *dominant* if $\varphi(X)$ has a non-empty interior in Y.[6] A mapping germ $\varphi_\xi : X_\xi \to Y_\eta$ is called dominant if there exists an open subset $U \subseteq X$ containing ξ such that $\varphi|_V : V \to Y$ is dominant for every open subset $V \subseteq U$ containing ξ; in other words, every sufficiently small representative of φ_ξ is dominant.

Throughout our study, we will need to assume that the target space of our mapping is locally irreducible (and connected); that is, every local ring of the space is an integral domain. We require this naturally in order for the relation between the desired properties of a dominant mapping and the continuity of its family of fibres to exist. Indeed, for example, consider the canonical embedding $\varphi : \mathbb{C}_{x_1} \to \{x_1 x_2 = 0\} \subseteq \mathbb{C}^2_{x_1,x_2}$, whose fibres form a trivial family of singletons, but the mapping is not even open for a vacuous reason. Sometimes, the weaker assumption of irreducibility at only the special point would suffice (e.g., in Theorem 3.5).

[6] We remark that this is not the common definition for dominance in algebraic geometry, where a mapping is defined to be dominant if the image is dense in the target. In fact, by taking Zariski topology and over irreducible targets, these definitions turn out to be equivalent.

A mapping $\varphi : X \to Y$ is *finite at a point* $\xi \in X$ if the germ of its fibre passing through ξ is the singleton $(\varphi^{-1}(\varphi(\xi)))_\xi = \{\xi\}$. In this case, it can be shown that there exist an open neighbourhood U of ξ in X and an open subset $V \subseteq Y$ with $\varphi(U) \subseteq V$, such that the restriction $\varphi|_U : U \to V$ is finite as a topological mapping (i.e., a continuous mapping which is closed and has only finite fibres (see e.g. [GPR, § I.8])). We say that $\varphi : X \to Y$ is finite if it is finite at every $\xi \in X$.

We emphasize that the topology of a complex space X is the Euclidean topology induced by the local model spaces. For a subset $A \subseteq X$, the dimension at $\xi \in X$, denoted by $\dim_\xi A$, is defined as the largest dimension of a (complex) manifold which is contained in A and is adherent to ξ. Dimension of A is defined as $\dim A \coloneqq \sup_{\xi \in X}\{\dim_\xi A\}$, which is in fact the supremum dimension of all manifolds contained in A.

A complex mapping $\varphi : X \to Y$ is said to be *open* if it is open as a morphism between topological spaces; that is, if it maps every open subset of X onto an open subset of Y. We say that a mapping $\varphi : X \to Y$ is *open at* $\xi \in X$, or $\varphi_\xi : X_\xi \to Y_{\varphi(\xi)}$ is an *open morphism of germs*, if there exists a representative $\varphi : X \to Y$ which is an open mapping.

A complex mapping $\varphi : X \to Y$ is said to be *flat at* $\xi \in X$, or $\varphi_\xi : X_\xi \to Y_{\varphi(\xi)}$ is said to be a *flat morphism of germs*, if $\mathcal{O}_{X,\xi}$ is a flat module over $\mathcal{O}_{Y,\varphi(\xi)}$.[7] (Recall that the module structure of $\mathcal{O}_{X,\xi}$ over $\mathcal{O}_{Y,\varphi(\xi)}$ is by default the one induced by φ^*_ξ.) A mapping $\varphi : X \to Y$ is said to be *flat* if it is flat at every point of X. More generally, given a mapping $\varphi : X \to Y$, a coherent $\mathcal{O}_X$-module $\mathcal{F}$ is said to be flat over $\mathcal{O}_Y$ at $\xi \in X$ if the stalk $\mathcal{F}_\xi$ is a flat module over $\mathcal{O}_{Y,\varphi(\xi)}$.

A mapping is flat at a point, only if it is open at that point ([D], or see e.g. [F, § 3.19]).

[7] An R-module M is called a *flat module* over R if, for any R-modules N_1 and N_2 such that $N_1 \subseteq N_2$, the canonical homomorphism $N_1 \otimes_R M \to N_2 \otimes_R M$ is injective. (See e.g. [La].)

Remark 1.5. Up to an isomorphism of the source, every mapping can be regarded as a projection. Indeed, let $\varphi : X \to Y$ be a mapping, and consider its graph $\Gamma_\varphi \subseteq Y \times X$, defined as the analytic inverse image of the diagonal space $\mathrm{D}_Y \subseteq Y \times Y$ under the mapping $\mathrm{id}_Y \times \varphi : Y \times X \to Y \times Y$; that is, $\Gamma_\varphi = D_Y \times_{Y \times Y} (Y \times X)$. Then we have a canonical isomorphism $\Psi : X \to \Gamma_\varphi$, so that by considering the projection $\pi : Y \times X \to Y$, we get $\pi \circ \Psi = \varphi$. (It is said that φ and $\pi|_{\Gamma_\varphi}$ are *right-equivalent*.) In this case, if we assume that $Y \subseteq \mathbb{C}^n_y$, then for $\xi \in X$ with the image $\varphi(\xi) = 0$, we can write

$$\mathcal{O}_{X,\xi} \simeq \mathcal{O}_{\Gamma_\varphi,(0,\xi)} \simeq \frac{\mathcal{O}_{X,\xi}\{y\}}{(y_1 - \varphi_{1,\xi}, \dots, y_n - \varphi_{\xi,n})},$$

where $\varphi_1, \dots, \varphi_n \in \mathcal{O}_X(X)$ are the n coordinate components of φ. If we assume additionally that $X \subseteq \mathbb{C}^m_x$, then for $0 \in X$ with the image $\varphi(0) = 0$, we can write

$$\begin{aligned}\mathcal{O}_{X,0} \simeq \mathcal{O}_{\Gamma_\varphi,(0,0)} &\simeq \frac{\mathbb{C}\{y, x\}}{\mathcal{I}_0 + (y_1 - \tilde{\varphi}_{1,0}, \dots, y_n - \tilde{\varphi}_{n,0})} \\ &\simeq \frac{\mathcal{O}_{Y,0}\{x\}}{\mathcal{I}_0 + (y_1 - \tilde{\varphi}_{1,0}, \dots, y_n - \tilde{\varphi}_{n,0})},\end{aligned}$$

where $\tilde{\varphi}$ is an extension of φ to an open neighbourhood $U \subseteq \mathbb{C}^m$ of X, $\tilde{\varphi}_1, \dots, \tilde{\varphi}_n \in \mathcal{O}_{\mathbb{C}^m}(U)$ are the n coordinate components of $\tilde{\varphi}$, and $\mathcal{I}$ is the ideal sheaf defining X in $\mathbb{C}^m$. The last isomorphism is due to the fact that the ideal $\mathcal{I}_0 + (y_1 - \tilde{\varphi}_{1,0}, \dots, y_n - \tilde{\varphi}_{n,0})$ already contains the ideal defining Y_0 in $\mathbb{C}^n_0$. $\triangle$

Let R be a (commutative) ring, and let M be an R-module. A nonzero element $r \in R$ is called a *zero-divisor* in M if $r \cdot m = 0$ for some nonzero $m \in M$. In this case, m is also called a zero-divisor of the module M over R.

Finally, by the dimension of a local ring R we mean its Krull dimension, that is the supremum of all $n \in \mathbb{N}$ for which there is a chain of prime ideals $\mathfrak{p}_0 \subsetneqq \cdots \subsetneqq \mathfrak{p}_n$ in R. For the local ring of a space X at $\xi \in X$, recall that $\dim \mathcal{O}_{X,\xi} = \dim_\xi X$ (see e.g. [L, § IV.4.3]).

1.3 Review of the main results

This thesis is based on, and is a continuation of, the approach initiated in analytic geometry by Kwieciński et al. [KT, K, GK]. The idea is that by raising the mapping to its fibred powers, the degeneracies in the family of fibres will eventually grow to the extent that the exceptional fibres themselves form irreducible components of the source—which are vertical components (see Definition 1.3).

In [KT], the authors show that for a complex mapping $\varphi : X \to Y$ with Y locally irreducible and of dimension n, φ is open if and only if the n-fold fibred power $\varphi^{\{n\}} : X^{\{n\}} \to Y$ has no isolated vertical components. This was later improved by Adamus [A2] to the following: For a mapping $\varphi : X \to Y$ with pure-dimensional X and locally irreducible Y of dimension n, φ is open at $\xi \in X$ if and only if the n-fold fibred power $\varphi^{\{n\}} : X^{\{n\}} \to Y$ has no isolated algebraic vertical components (see Remark 1.4) at the diagonal point in $X^{\{n\}}$ corresponding to ξ. The latter is equivalent to the lack of zero-divisors in the reduct (i.e., the quotient by the nilradical) of the n-fold analytic tensor power of $\mathcal{O}_{X,\xi}$ over $\mathcal{O}_{Y,\varphi(\xi)}$.

Our main result on testing for openness is Theorem 3.1. This is an improvement with respect to previous works in several ways. *First*, it involves only one fibred product rather than possibly a high fibred power of the mapping. *Second*, taking fibred product with the blowing-up of a point has less complexity, in general, compared with the fibred product of a mapping with itself. Indeed, the defining equations of the blowing-up are simply quadratic polynomials. These two factors make for a considerable reduction of computational expense overall. And *third*, our criterion explicitly determines a location (namely the special fibre) where an (isolated) irreducible component is produced in the pullback of a non-open mapping. This has turned the detection algorithm into a test for a specific zero-divisor (see Corollaries 3.2 and 3.3), versus a search for general zero-divisors.

The first flatness criterion in terms of fibred powers appears in [K]. It states that a mapping $\varphi : X \to Y$ with Y locally irreducible is flat if and only if, for every $i \geq 1$, the fibred power $\varphi^{\{i\}} : X^{\{i\}} \to Y$ has no vertical components. In [GK], the flatness of a mapping $\varphi : X \to Y$ with pure-dimensional X and n-dimensional, *smooth* Y is proved to be equivalent to the lack of vertical components in the n-fold fibred power $\varphi^{\{n\}} : X^{\{n\}} \to Y$. The latter is of course an improvement as to the effectiveness, but at the expense of losing the generality of allowing the target to be singular. This criterion is generalized in [ABM] to the case of coherent modules (still over smooth bases) as follows: Consider a mapping $\varphi : X \to Y$ with Y smooth and of dimension n, and let $\mathcal{F}$ be a coherent module over X. Then $\mathcal{F}$ is flat over $\mathcal{O}_Y$ at $\xi \in X$ if and only if the n-fold analytic tensor power of $\mathcal{F}_\xi$ has no vertical components over $\mathcal{O}_{Y,\varphi(\xi)}$. In [A1], an upper bound for fibred powers in the criterion [K] is given which works in the general case of singular targets, but it is equal to the (not easy to determine) minimal number of generators of the flattener ideal. This type of flatness criterion, in fact, originates in the algebraic work of Auslander [Au], who proved that a finitely generated module over an unramified regular local ring R is free (equivalently, flat, in this case) if and only if M has no zero-divisors over R.[8]

Our main result on testing for flatness is Theorem 3.5. This is an improvement with respect to previous works in several ways. The comparison regarding the first two factors is similar to the ones mentioned above about our openness criterion. Now *third*, our criterion explicitly determines a location (namely the special fibre) where an irreducible component is produced in the pullback of a non-flat mapping. This has turned the detection algorithm into a test for a specific zero-divisor (see Corollaries 3.8 and 3.9). Prior to this, all criteria in their general forms are in terms of only vertical components (which do not produce zero-divisors in general). And *fourth*, our

[8] There have been also efforts to obtain generalized flatness criteria of Auslander's type in the algebraic setting. Over regular bases, the best result we know of is given by [AI]. Over singular bases, various results exist but only in (very) special cases (see e.g. [HW]).

criterion applies to the case of singular bases (targets) as well.

In Chapter 2, we present a flatness testing method (Theorem 2.5) that was developed by us with the aim of extending the previous flatness criteria of Auslander's type, as in [GK] or [ABM], to the case of singular bases. Of course, Theorem 2.5 has a lesser computational value compared to the highly efficient results in section 3.2 (which were developed later). Also, alongside the proof of Theorem 2.5, we observe that *flatness descent* always holds in the complex analytic category (Proposition 2.4), which is independently a new and interesting result.

Chapter 4 opens a new direction of research, in which we go beyond just the detection and try to characterize different modes of non-openness. We take the index introduced in [KT] as a gauge for this purpose, and concentrate on its behaviour and the calculation of it. The main results of this chapter are Propositions 4.6, 4.9, and 4.12.

All numbered results of this dissertation are new (except for Lemma 4.3, which is of course well-known) and were obtained during the doctoral study of the author. Chapters 2 and 3 contain joint work (see [AS1, AS2]) with Janusz Adamus, the supervisor of the thesis.

Chapter 2

Flatness and fibred powers, a generalization to the singular target case

The aim of this chapter is to extend the best previously known flatness criterion of Auslander's type, so that it is applicable to the modules (or mappings) with singular bases (targets). We need first to develop some tools, among which the flatness descent is an independently interesting result in the category of analytic $\mathbb{C}$-algebras.

2.1 Associativity of analytic tensor product over different bases

Lemma 2.1. *Let R be an analytic $\mathbb{C}$-algebra, and let S be an analytic R-algebra.*

(i) Let A be an analytic R-algebra, and let B and C be analytic S-algebras. Then $(A\tilde{\otimes}_R B)\tilde{\otimes}_S C \simeq A\tilde{\otimes}_R(B\tilde{\otimes}_S C)$ (as R- or S-algebras).

(ii) Let F be an analytic R-module, and let G and H be analytic S-modules.

Then $(F\tilde{\otimes}_R G)\tilde{\otimes}_S H \simeq F\tilde{\otimes}_R(G\tilde{\otimes}_S H)$ *(as R- or S-modules).*

Proof. (i) By definition, we have $S = R\{y\}/I$, $A = R\{x\}/J$, $B = S\{x'\}/K$, and $C = S\{x''\}/L$, for some ideals I, J, K, and L. Then, using (1.1) we write

$$\begin{aligned}
(A\tilde{\otimes}_R B)\tilde{\otimes}_S C &= \Big(\frac{R\{x\}}{J}\tilde{\otimes}_R \frac{\frac{R\{y\}}{I}\{x'\}}{K}\Big)\tilde{\otimes}_S \frac{S\{x''\}}{L} \\
&\simeq \Big(\frac{R\{x\}}{J}\tilde{\otimes}_R \frac{R\{y,x'\}}{I+K}\Big)\tilde{\otimes}_S \frac{S\{x''\}}{L} \\
&= \frac{R\{x,y,x'\}}{J+I+K} \qquad \tilde{\otimes}_S \frac{S\{x''\}}{L} \\
&\simeq \frac{S\{x,x'\}}{J+K} \qquad \tilde{\otimes}_S \frac{S\{x''\}}{L} \\
&= \frac{S\{x,x',x''\}}{J+K+L} \\
&\simeq \frac{R\{y,x,x',x''\}}{I+J+K+L}.
\end{aligned}$$

Similarly, one obtains that

$$A\tilde{\otimes}_R(B\tilde{\otimes}_S C) \simeq \frac{R\{y,x,x',x''\}}{I+J+K+L}.$$

All isomorphisms are clearly R- or S-algebra isomorphisms.

(ii) Set $S = R\{y\}/I$ for some ideal I, and take presentations $F \simeq R\{x\}^p/M$, $G \simeq S\{x'\}^q/N$, and $H \simeq S\{x''\}^r/P$, for some submodules M, N, and P, and some $p, q, r \in \mathbb{N}$. Then using (1.2), carry out a similar calculation as in part (i) to see that both $(F\tilde{\otimes}_R G)\tilde{\otimes}_S H$ and $F\tilde{\otimes}_R(G\tilde{\otimes}_S H)$ are isomorphic (as R- or S-modules) with

$$\frac{R\{y,x,x',x''\}^{pqr}}{I^{pqr}+M^{qr}+N^{pr}+P^{pq}}.$$

□

Corollary 2.2 (cf. [AS1, Lemma 3.1])**.** *Let R be an analytic $\mathbb{C}$-algebra, and let S be an analytic R-algebra. Let F and G be analytic R-modules. Then*

$$(F\tilde{\otimes}_R G)\tilde{\otimes}_R S \simeq (F\tilde{\otimes}_R S)\tilde{\otimes}_S(G\tilde{\otimes}_R S) \qquad \textit{(as R- or S-modules).}$$

In particular, analytic tensor power commutes with the analytic base change. That is, for every $i \geq 1$, we have

$$(F^{\tilde{\otimes}^i_R})\tilde{\otimes}_R S \simeq (F\tilde{\otimes}_R S)^{\tilde{\otimes}^i_S} \qquad \text{(as } R\text{- or } S\text{-modules).}$$

Proof. Write

$$\begin{aligned}(F\tilde{\otimes}_R G)\tilde{\otimes}_R S &\simeq F\tilde{\otimes}_R(G\tilde{\otimes}_R S)\\ &\simeq F\tilde{\otimes}_R(S\tilde{\otimes}_S(G\tilde{\otimes}_R S))\\ &\simeq (F\tilde{\otimes}_R S)\tilde{\otimes}_S(G\tilde{\otimes}_R S).\end{aligned}$$

The first isomorphism is by associativity of analytic tensor product over all the same bases, which is concluded from the dual fact in Remark 1.2. The second isomorphism is obvious (because the action of $\tilde{\otimes}_S S$ is an isomorphism, as it is so with the coproduct in any category). The last isomorphism is by Lemma 2.1.

All isomorphisms are over S, and hence over R, too. □

2.2 Analytic flatness descent

By definition, a ring homomorphism $R \to S$ descends flatness if, for any R-module F, flatness of $F \otimes_R S$ (as an S-module) implies flatness of F (as an R-module). In general, flatness descent is a rare luxury, as opposed to the converse fact, namely, that flatness is always preserved by base change. However, as we show below, it does hold for analytic modules over integral domains and analytic $\mathbb{C}$-algebra homomorphisms inducing dominant morphisms of complex germs. Upon realizing this, we would like to set a terminology for flatness descent in the analytic category.

Definition 2.3. We will say that a homomorphism $R \to S$ of analytic $\mathbb{C}$-algebras *analytically descends flatness* if S-flatness of $F\tilde{\otimes}_R S$ implies R-flatness of F for every analytic R-module F. △

We shall now prove what we claimed about flatness descent in the complex analytic category.

Proposition 2.4 ([AS1, Proposition 2.1])**.** *Let $\tau : R \to S$ be a homomorphism of analytic $\mathbb{C}$-algebras, where R is an integral domain. If the induced morphism* $\mathrm{Specan}S \to \mathrm{Specan}R$ *of complex germs is dominant, then τ analytically descends flatness.*

Proof. For a proof by contradiction, suppose the morphism $\mathrm{Specan}S \to \mathrm{Specan}R$ is dominant and there exists a non-flat analytic R-module F such that $F\tilde{\otimes}_R S$ is flat over S. According to Hironaka's local flattener theorem (see e.g. [BM1, Theorem 7.12]), there exists a unique nonzero ideal P in R such that $F\tilde{\otimes}_R R/P$ is R/P-flat and, for every morphism of complex germs $\varphi : T \to \mathrm{Specan}R$, if $F\tilde{\otimes}_R \mathcal{O}_T$ is $\mathcal{O}_T$-flat then φ factors as

$$T \to \mathrm{Specan}R/P \hookrightarrow \mathrm{Specan}R.$$

Since $\mathrm{Specan}R$ is irreducible, it follows that $\mathrm{Specan}R/P$ has empty interior in $\mathrm{Specan}R$. Setting $T \coloneqq \mathrm{Specan}S$, we get a contradiction with the dominance of $\mathrm{Specan}S \to \mathrm{Specan}R$. $\square$

2.3 Flatness testing by desingularizing the target

We will now state and prove the main result of this chapter, which is a generalization to the singular target case of the following criterion:

> [ABM, Theorem 1.9] Let F be an analytic module over an analytic $\mathbb{C}$-algebra R, where R is n-dimensional and a regular ring. Then, F is a flat module over R if and only if the n-fold analytic tensor power $F\tilde{\otimes}_R \cdots \tilde{\otimes}_R F$ has no vertical components over R.

Theorem 2.5 ([AS1, Theorem 1.6]). *Let F be an analytic module over an analytic $\mathbb{C}$-algebra R, where R is an integral domain of dimension n. Consider any analytic R-algebra S, regular and of dimension n, such that the induced morphism* $\mathrm{Specan}S \to \mathrm{Specan}R$ *of complex germs is dominant. The following statements are equivalent:*

(i) F is a flat module over R,

(ii) the analytic module $\underbrace{F\tilde{\otimes}_R \ldots \tilde{\otimes}_R F}_{n\ times} \tilde{\otimes}_R S$ has no vertical components over S (or, equivalently, over R).

Proof. By Proposition 2.4 and since flatness is preserved by analytic base change (see [H], or for sketch of a proof, see [GPR, § II.2]), it follows that F is a flat R-module if and only if $F\tilde{\otimes}_R S$ is a flat S-module. As S is regular, it follows from [ABM, Theorem 1.9] (above) that F is R-flat if and only if the n-fold tensor power $(F\tilde{\otimes}_R S)^{\tilde{\otimes}^n_S}$ has no vertical components over S. By Corollary 2.2, this is equivalent to saying that $(F^{\tilde{\otimes}^n_R})\tilde{\otimes}_R S$ has no vertical components over S.

It remains only to observe that an analytic S-module M has a vertical component over S if and only if it does so over R (with the R-module structure of M as induced by S). To see this, let M be such a module. Let A be an analytic S-algebra over which M is finitely generated. Now, A is a witness ring for both the S- and R-module structure of M.

Let $\sigma : Z \to Y$, $\sigma(0) = 0$, be a dominant representative of the mapping germ $\mathrm{Specan}S \to \mathrm{Specan}R$, where Z and Y are n-dimensional irreducible spaces with $\mathcal{O}_{Z,0} = S$ and $\mathcal{O}_{Y,0} = R$. Consider $\mathrm{Specan}(A/\mathfrak{p})$, an irreducible component of M over A, where $\mathfrak{p}$ is an associated prime of the A-module M. Let Σ be a representative of the germ $\mathrm{Specan}(A/\mathfrak{p})$, with an image W under a representative of $\mathrm{Specan}A \to \mathrm{Specan}S$. Take Σ small enough so that $W \subseteq Z$. As σ is a dominant mapping between irreducible spaces of the same dimension, it is elementary to see that $\sigma(W)$ has no interior points if and only if W has no interior points. Hence, the result follows from the definition of vertical component of an analytic module (see page 10). $\square$

Remark 2.6. Such an S as required in Theorem 2.5 always exists for any reduced analytic $\mathbb{C}$-algebra R. Indeed, we can take the local ring of a desingularization of Specan$R$ (see e.g. [BM2], and cf. e.g. [Ha1, Ha2] on algebraic varieties). $\triangle$

The geometric version of Theorem 2.5 reads as follows.

Corollary 2.7 ([AS1, Corollary 1.9])**.** *Consider $\varphi : X \to Y$, where Y is locally irreducible and of dimension n. Let $\sigma : Z \to Y$ be a complex mapping which is dominant, with Z smooth and of dimension n. (E.g., σ can be a desingularization of Y.) Let*

$$\varphi' : Z \times_Y \underbrace{X \times_Y \cdots \times_Y X}_{n \text{ times}} \to Z$$

be the pullback of $\varphi^{\{n\}} : \underbrace{X \times_Y \cdots \times_Y X}_{n \text{ times}} \to Y$ by σ. Take a point $\xi \in X$ and a point $\xi' \in \sigma^{-1}(\varphi(\xi))$. The following statements are equivalent:

(i) φ is a flat mapping at $\xi \in X$,

(ii) φ' has no vertical components at $(\xi', \xi, \ldots, \xi)$,

(iii) $\sigma \circ \varphi'$ has no vertical components at $(\xi', \xi, \ldots, \xi)$.

Proof. Immediate; apply Theorem 2.5 with $R = \mathcal{O}_{Y,\varphi(\xi)}$, $S = \mathcal{O}_{Z,\xi'}$, and $F = \mathcal{O}_{X,\xi}$. $\square$

We now show by an example that tensoring with S in Theorem 2.5 is in general necessary in order to test for flatness. This will show also that the result [ABM, Theorem 1.9] that we extended, would not work in general over a non-regular base.

Example 2.8 (cf. [AS1, Example 4.5])**.** Let Y be the curve in $\mathbb{C}^2_{y_1,y_2}$ defined by $y_1^3 - y_2^2 = 0$. Define the parametrization mapping $\sigma : Z = \mathbb{C}_t \to Y$, by $t \mapsto (t^2, t^3)$. Then, in fact, σ is a desingularization of Y.

Set $R \coloneqq \mathcal{O}_{Y,0} = \mathbb{C}\{y_1, y_2\}/(y_1^3 - y_2^2)$, and $S \coloneqq \mathcal{O}_{Z,0}$. By Remark 1.5, we have $S \simeq \mathbb{C}\{y_1, y_2, t\}/(y_1 - t^2, y_2 - t^3)$. Consider an R-module $F \coloneqq S$. We want to investigate flatness of F over R.

As $\dim R = 1$, in light of Theorem 2.5 we should compute

$$F\tilde{\otimes}_R S \simeq \frac{\mathbb{C}\{y_1, y_2, t, t'\}}{(y_1 - t^2, y_2 - t^3, y_1 - t'^2, y_2 - t'^3)},$$

in which we immediately find that $y_1 \cdot (\overline{t - t'}) = 0$, where $\overline{t - t'}$ is the class of $t - t'$ modulo the ideal $(y_1 - t^2, y_2 - t^3, y_1 - t'^2, y_2 - t'^3)$. Of course, $\overline{t - t'}$ is a nonzero element of $F\tilde{\otimes}_R S$ (as $t - t'$ does not belong to the ideal, which can be seen by looking at the orders of the generators with respect to t and t'). Even more obvious is that y_1 is not a zero element of R. Thus, $F\tilde{\otimes}_R S$ has a zero-divisor over R, and hence, by Remark 1.4, a vertical component over R. We conclude, by Theorem 2.5, that F is not a flat module over R. Non-flatness of F over R can also be independently verified, easily, by means of the characterization of flatness in terms of multiplicities (see e.g. [F, § 3.13]).

It remains to notice that F itself does not have any vertical components over R. This is geometrically clear, as the mapping $\sigma : Z \to Y$ does not have any vertical components (passing through $0 \in Z$). $\triangle$

Chapter 3

A fast method of detecting degeneracy in the family of fibres

By passing to fibred powers of a mapping, all exceptional fibres get amplified, so that they eventually stand out in the form of vertical components. This is what we have seen so far. In this chapter, we introduce a new method of openness and flatness testing. Its development was triggered when we were trying to find a *test map* which would pick out the exceptionality of the special fibre alone. Consequently, this would make for a greatly efficient method of detection. We show that pulling a mapping back by the blowing-up of the special point in the target leads to the formation of a vertical component precisely when the special fibre is exceptional. Note that blowing-up of a point is an isomorphism outside of only a single fibre, which is the reason why it proves to be the right tool for our purpose.

Throughout this chapter, if not mentioned otherwise, we consider a setup as follows.

Setup

Let $\varphi : X \to Y$ be a morphism between complex analytic spaces X and Y, with $\dim Y \geq 1$. We exclude the obvious case of $\dim Y = 0$, as every mapping over a singleton has only a single fibre. Since all our discussion concerns only local properties, without loss of generality, we will assume that $X \subseteq \mathbb{C}^m_x$ and $Y \subseteq \mathbb{C}^n_y$, where $m, n \in \mathbb{N}$, and assume that $0 \in X$, $0 \in Y$, and $\varphi(0) = 0$.

We set $\beta : \mathbb{C}^n_z \to \mathbb{C}^n_y$ to be the mapping defined by

$$(z_1, \dots, z_n) \mapsto (z_1 z_n, \dots, z_{n-1} z_n, z_n),$$

which, in fact, is the blowing-up of $\mathbb{C}^n_y$ with center the origin, restricted to an affine coordinate chart $\mathbb{C}^n_z$. We denote the strict transform of Y under β by Y_{st}, which is (by definition) the smallest closed subspace of $\mathbb{C}^n_z$ that contains $\beta^{-1}(Y) \backslash \{z_n = 0\}$. The underlying topological space of Y_{st} is the closure of $\beta^{-1}(Y) \backslash \{z_n = 0\}$ in $\mathbb{C}^n_z$, and it can be shown that if $J \subseteq \mathbb{C}\{y\}$ is the defining ideal of the germ Y_0 in $\mathbb{C}^n_y$, then the defining ideal of $(Y_{\mathrm{st}})_0$ in $\mathbb{C}^n_z$ will be $J_{\mathrm{st}} \coloneqq \{f \in \mathbb{C}\{z\} \mid z_n^k \cdot f \in \beta_0^*(J) \cdot \mathbb{C}\{z\} \text{ for some } k \in \mathbb{N}\}$ (cf. e.g. [Ha1, Ha2] on algebraic varieties).

We will need to have that Y_{st} passes through the origin of $\mathbb{C}^n_z$ to ensure that the germ $(Y_{\mathrm{st}})_0$ would not be empty, and also, equivalently, the element y_n would not belong to J. This is always possible up to a (linear) change of coordinates in $\mathbb{C}^n_y$. Indeed, if the vector $(0, \dots, 0, 1) \in \mathbb{C}^n_y$ belongs to the tangent cone of Y at the origin, then the strict transform of Y under β will contain $0 \in \mathbb{C}^n_z$.[9] We always assume that such a coordinate system is set on

[9]The *tangent cone* of $Y \subseteq \mathbb{C}^n$ at 0 is the set of all tangent vectors to Y at 0. A vector $v \in \mathbb{C}^n$ is a *tangent vector* to Y at 0 if there exist sequences $\{s_i\}_{i=1}^{\infty} \subseteq Y$ and $\{\lambda_i\}_{i=1}^{\infty} \subseteq \mathbb{R}^+$, such that $s_i \to 0$ and $\lambda_i s_i \to v$ when $i \to \infty$. It is not difficult to see that the set of all (complex) lines through the origin lying inside the tangent cone of Y minus the lines inside $\{y_n = 0\}$ is in one-to-one correspondence with the set of limit points of Y_{st} inside the exceptional divisor $\beta^{-1}(0) = \mathbb{C}^{n-1}$. The correspondence is established by sending a tangent vector $v = (v_1, \dots, v_n) \notin \{y_n = 0\}$ to the point $(v_1/v_n, \dots, v_{n-1}/v_n) \in \beta^{-1}(0) = \mathbb{C}^{n-1}$. See also [W].

Now, it is clear that if the y_n-axis is tangent (in the above sense) to Y at 0, then the origin in $\beta^{-1}(0)$ will be attained by Y_{st} as required.

the target of β. Of course, if we happen to need a change of coordinates in $\mathbb{C}_y^n$ again (e.g., in the proof of Theorem 3.5), we shall choose only one that fixes y_n. However, at the end of this chapter, we will give a coordinate-free version of our testing criteria.

Now, the mapping $\beta|_{Y_{\mathrm{st}}} : Y_{\mathrm{st}} \to Y$ is going to be our test map for two kinds of degeneracies in the family of fibres of the mapping $\varphi : X \to Y$. In section 3.1, we study degeneracy in the sense of Remmert's Open Mapping Theorem, and we derive an openness testing method. In section 3.2, we study degeneracy in the sense of Hironaka's flatness criterion, and we derive a flatness testing method.

3.1 Openness testing by a single blowing-up

Here, we present our openness testing method. We state and prove it first in the geometric form.

Theorem 3.1 (cf. [AS2, Theorem 4.1]). *Consider $\varphi : X \to Y$ and $\beta|_{Y_{\mathrm{st}}} : Y_{\mathrm{st}} \to Y$ as above, with X of pure dimension and Y locally irreducible. Let*

$$\varphi' : Y_{\mathrm{st}} \times_Y X \to Y_{\mathrm{st}}$$

be the pullback of $\varphi : X \to Y$ by $\beta|_{Y_{\mathrm{st}}} : Y_{\mathrm{st}} \to Y$. The following statements are equivalent:

(i) φ is an open mapping at $0 \in X$,

(ii) no isolated irreducible component of $Y_{\mathrm{st}} \times_Y X$ passing through $(0,0)$ is mapped to the singleton zero by $(\beta|_{Y_{\mathrm{st}}}) \circ \varphi'$.

Proof. For (i) $\Rightarrow$ (ii), suppose φ is open at $0 \in X$. Then, by Lemma 4.3 (below), φ' is open in a neighbourhood of $(0,0)$. This implies that no isolated irreducible component of $Y_{\mathrm{st}} \times_Y X$ through $(0,0)$ can be mapped by φ' into the set $(\beta|_{Y_{\mathrm{st}}})^{-1}(0)$ (which has empty interior in Y_{st}), and hence none can be mapped by $(\beta|_{Y_{\mathrm{st}}}) \circ \varphi'$ into $\{0\} \subseteq Y$.

To prove (ii) $\Rightarrow$ (i), suppose that φ is not open at $0 \in X$. Then, by Remmert's Open Mapping Theorem[10] and upper semi-continuity of fibre dimension, we get $\mathrm{fbd}_0\varphi > \dim X - \dim_0 Y$, or

$$\dim X \leq \dim_0 Y - 1 + \mathrm{fbd}_0\varphi, \tag{3.1}$$

where $\mathrm{fbd}_0\varphi := \dim_0 \varphi^{-1}(\varphi(0))$. Since $\beta|_{Y_{\mathrm{st}}}$ is a biholomorphism outside $(\beta|_{Y_{\mathrm{st}}})^{-1}(0)$, we can write $Y_{\mathrm{st}} \times_Y X = T \cup T'$, where $T' = (\beta|_{Y_{\mathrm{st}}})^{-1}(0) \times \varphi^{-1}(0)$, and T is biholomorphic with $\varphi^{-1}(Y \setminus \{y_n = 0\})$. As T embeds in X, we readily have that $\dim T \leq \dim X$. Also, $\dim_0 (\beta|_{Y_{\mathrm{st}}})^{-1}(0) = \dim_0 Y - 1$. Now, by (3.1),

$$\dim T \leq \dim_0 Y - 1 + \mathrm{fbd}_0\varphi = \dim_{(0,0)} T'.$$

It follows that $\dim_{(0,0)} T' = \dim_{(0,0)}(Y_{\mathrm{st}} \times_Y X)$, and hence T' must contain an isolated irreducible component of $Y_{\mathrm{st}} \times_Y X$ through $(0,0)$. By definition of T', such a component is mapped to $0 \in Y$ by $(\beta|_{Y_{\mathrm{st}}}) \circ \varphi'$. $\square$

The algebraic version of Theorem 3.1 now follows.

Corollary 3.2. *Consider $\varphi : X \to Y$ and $\beta|_{Y_{\mathrm{st}}} : Y_{\mathrm{st}} \to Y$ as above, with X of pure dimension and Y locally irreducible. We have $\mathcal{O}_{Y,0} = \mathbb{C}\{y\}/J$, where J is the defining ideal of Y at 0 in $\mathbb{C}^n_y$. By Remark 1.5, we have $\mathcal{O}_{Y_{\mathrm{st}},0} \simeq \mathbb{C}\{y,z\}/J_{\mathrm{st}}$ and $\mathcal{O}_{X,0} \simeq \mathbb{C}\{y,x\}/I$, for some ideals I and J_{st} (containing J). The following statements are equivalent:*

(i) φ is an open mapping at $0 \in X$,

(ii) y_n (or, equivalently, z_n) is not a zero-divisor in $\dfrac{\mathbb{C}\{y,x,z\}}{\sqrt{I + J_{\mathrm{st}}}}$.

[10]**Remmert's Open Mapping Theorem.** Let $\varphi : X \to Y$ be a complex mapping, with Y locally irreducible. Then, φ is an open mapping if and only if $\mathrm{fbd}_\xi\varphi = \dim_\xi X - \dim_{\varphi(\xi)} Y$ for every $\xi \in X$. (See e.g. [GPR, § II.1], [L, § V.6], or [F, § 3.9].)

Proof. Let $\varphi' : Y_{\text{st}} \times_Y X \to Y_{\text{st}}$ be the pullback of φ by $\beta|_{Y_{\text{st}}}$. We have $\mathcal{O}_{Y_{\text{st}} \times_Y X} \simeq (\mathbb{C}\{y,x\}/I)\tilde{\otimes}_{\mathbb{C}\{y\}/J}(\mathbb{C}\{y,z\}/J_{\text{st}}) = \mathbb{C}\{y,x,z\}/(I + J_{\text{st}})$, and the induced homomorphism on local rings $\big((\beta|_{Y_{\text{st}}}) \circ \varphi'\big)_0^*$ is the canonical homomorphism $\mathbb{C}\{y\}/J \to \mathbb{C}\{y,x,z\}/(I + J_{\text{st}})$.

It suffices to show that assertion (ii) is equivalent to assertion (ii) of Theorem 3.1. First note that $\overline{y_n}$, the class of y_n in $\mathbb{C}\{y\}/J$, is nonzero by the Setup (on page 25). So, assertion (ii) is true if and only if $\overline{y_n}$ is not a zero-divisor in $\mathbb{C}\{y,x,z\}/\sqrt{I + J_{\text{st}}}$. By Remark 1.4, the latter is equivalent to the statement that $(\beta|_{Y_{\text{st}}}) \circ \varphi'$ has no isolated irreducible components which pass through $(0,0)$ and are mapped into $\{(y_1,\dots,y_n) \in Y \mid y_n = 0\}$. And this statement is equivalent to assertion (ii) of Theorem 3.1, since $\{(y_1,\dots,y_n) \in Y \mid y_n = 0\}$ intersects the image of $\beta|_{Y_{\text{st}}}$ only at 0.

Finally, notice that $y_n - z_n \in J_{\text{st}}$, so y_n and z_n can be divisors of zero in $\mathbb{C}\{y,x,z\}/\sqrt{I + J_{\text{st}}}$ only at the same time. □

The criterion becomes even simpler if the target is smooth, as the following corollary shows.

Corollary 3.3. *Consider $\varphi : X \to Y$ as above, with X of pure dimension and Y smooth. We can write $\mathcal{O}_{Y,0} = \mathbb{C}\{y\}$. By Remark 1.5, we have $\mathcal{O}_{X,0} \simeq \mathbb{C}\{y,x\}/I$, for an ideal I. Let $\tilde{I}$ be the ideal obtained from I by substituting $y_i y_n$ for y_i, $i = 1,\dots,n-1$. The following statements are equivalent:*

(i) φ is an open mapping at $0 \in X$,

(ii) y_n is not a zero-divisor in $\dfrac{\mathbb{C}\{y,x\}}{\sqrt{\tilde{I}}}$.

Proof. With Y being smooth, we can set $J = 0$ in Corollary 3.2, and $J_{\text{st}} = (y_1 - z_1 z_n, \dots, y_{n-1} - z_{n-1} z_n, y_n - z_n)$. Then we obtain a ($\mathbb{C}$-algebra) isomorphism

$$\frac{\mathbb{C}\{y,x,z\}}{\sqrt{I + J_{\text{st}}}} \to \frac{\mathbb{C}\{y,x\}}{\sqrt{\tilde{I}}},$$

defined by $y_i \mapsto y_i y_n$, $y_n \mapsto y_n$, and $z_j \mapsto y_j$, where $i = 1, \dots, n-1$ and $j = 1, \dots, n$. Now, assertion (ii) of Corollary 3.2 gets converted to assertion (ii) above. $\square$

In Theorem 3.1 and its corollaries, the assumption of pure-dimensionality for the source space is necessary. In the following example, we give a non-open mapping with a source space which is not pure-dimensional, such that no vertical components exist in the mapping or its pullback by our blowing-up. This example will prove to be even more interesting later in the context of our flatness criterion, Theorem 3.5.

Example 3.4 ([AS2, Example 4.5])**.** Let $X \subseteq \mathbb{C}^9 = \mathbb{C}^3_t \times \mathbb{C}^6_x$ be defined as $X = X_1 \cup X_2$, where

$$X_1 = \{(t,x) \in \mathbb{C}^9 \mid t_1x_1 + t_2x_2 + t_3x_3 = t_2x_1 + t_1x_2 = x_4 = x_5 = x_6 = 0\},$$
$$X_2 = \{(t,x) \in \mathbb{C}^9 \mid t_1 = t_2 = t_3 = 0\}.$$

Clearly, X_2 is irreducible, of dimension 6. We claim that X_1 is of pure dimension 4. To see this, set $A = \{(t,x) \in X_1 \mid \det \begin{bmatrix} t_1 & t_2 \\ t_2 & t_1 \end{bmatrix} = 0\}$. In $X_1 \backslash A$, one can solve the first two defining equations of X_1 for x_1 and x_2, hence $X_1 \backslash A$ is a 4-dimensional manifold. On the other hand, it is easy to compute that $\dim A = 3$. Since X_1 is defined by 5 equations in $\mathbb{C}^9$, it follows that $\dim_\xi X_1 \geq 4$ for every $\xi \in X_1$. Therefore, A is nowhere-dense in X_1, and so $X_1 = \overline{X_1 \backslash A}$ is of pure dimension 4.

Define $\varphi : X \to Y = \mathbb{C}^3$ as

$$(t,x) \to (t_1 + x_4, t_2 + x_5, t_3 + x_6).$$

We claim that φ is not open at 0. Suppose otherwise that $\varphi|_U$ is open for some open subset $U \subseteq X$ at 0, which implies that $\varphi|_{U \cap (X_1 \backslash X_2)}$ is open. Then, by Remmert's Open Mapping Theorem, for every $\xi \in U \cap (X_1 \backslash X_2)$, we should have

$$\operatorname{fbd}_\xi \varphi|_{U \cap (X_1 \backslash X_2)} = \dim \big(U \cap (X_1 \backslash X_2)\big) - \dim Y = 4 - 3 = 1. \tag{3.2}$$

But consider the set $W = \{(t,x) \in X_1 \mid t_3 = 0, t_1 = t_2 \neq 0\}$, in which for every ξ, it is easy to see that $\mathrm{fbd}_\xi \varphi|_{U\cap(X_1\backslash X_2)} = \mathrm{fbd}_\xi\varphi = 2$. Note that W is a subset of $X_1\backslash X_2$ and is adherent to 0, hence $W \cap \big(U \cap (X_1\backslash X_2)\big)$ is not empty. So we get points inside $U \cap (X_1\backslash X_2)$ at which fibre dimension is 2, and this contradicts (3.2).

Now, consider $\beta : Z = \mathbb{C}^3_z \to Y$, given as $\beta(z_1, z_2, z_3) = (z_1z_3, z_2z_3, z_3)$. Consider the pullback $\varphi' : Z \times_Y X \to Z$. We shall show that $\beta \circ \varphi'$ has no isolated irreducible components mapped to $0 \in Y$. Suppose otherwise that Σ is such a component. Then, of course, Σ lies in the fibre of $\beta \circ \varphi'$ above $0 \in Y$, which is just the space $\mathbb{C}^2_{z_1,z_2} \times \mathbb{C}^3_{x_1,x_2,x_3}$. So by irreducibility, $\Sigma = \mathbb{C}^2_{z_1,z_2} \times \mathbb{C}^3_{x_1,x_2,x_3}$. But on the other hand, $\mathbb{C}^2_{z_1,z_2} \times \mathbb{C}^3_{x_1,x_2,x_3}$ is the fibre of the mapping $\beta \circ \psi'$ above $0 \in Y$, where $\psi' : Z \times_Y X_2 \to Z$ is the pullback by β of the open mapping $\varphi|_{X_2}$. By Theorem 3.1, it follows that the fibre over $0 \in Y$ of $\beta \circ \psi'$ contains no isolated irreducible components (through $(0,0)$), which implies that Σ cannot be an isolated irreducible component of $Z \times_Y X_2$; while it actually is, as we have an embedding $Z \times_Y X_2 \hookrightarrow Z \times_Y X$. Therefore, such Σ cannot exist. $\triangle$

3.2 Flatness testing by a single blowing-up

In this section, we present our flatness testing method analogous to Theorem 3.1 above. Although the geometric proof of this openness criterion was the main motivation for us to develop a similar criterion for flatness, the proof of the flatness criterion is totally different and is based on the characterization of flatness by Hironaka in terms of algebro-combinatorial properties of the algebra of power series. First, let us recall the required formalism.

Hironaka's diagram of initial exponents

Let $R = \mathbb{C}\{y\}/J$ be an analytic $\mathbb{C}$-algebra with the maximal ideal $\mathfrak{m}$, where $y = (y_1, \ldots, y_n)$ and J is a proper ideal of $\mathbb{C}\{y\}$. Set $x = (x_1, \ldots, x_m)$.

Given $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m$ and a positive integer p, we will denote by x^α the monomial $x_1^{\alpha_1} \cdots x_m^{\alpha_m}$, and by $x^{\alpha,j}$, for $j = 1, \ldots, p$, the p-tuple $(0, \ldots, 0, x^\alpha, 0, \ldots, 0)$ with x^α in the j'th place. Then, a p-tuple of formal power series $f = (f_1, \ldots, f_p) \in R\{x\}^p$ (recall that $R\{x\}^p \subseteq R[[x]]^p$) can be written as $f = \sum_{\alpha,j} c_{\alpha,j} x^{\alpha,j}$, where $c_{\alpha,j} \in R$ and the indices (α, j) belong to $\mathbb{N}^m \times \{1, \ldots, p\}$.

The mapping $R \to \mathbb{C}$, $g \mapsto g(0)$, of evaluation at zero (defined as evaluation at 0 of a representative of g (modulo J), or, defined canonically by identifying the target as $\mathbb{C} \simeq R/\mathfrak{m}$) induces the evaluation mapping $R\{x\}^p \to \mathbb{C}\{x\}^p$ defined as $f = \sum_{\alpha,j} c_{\beta,j} x^{\alpha,j} \mapsto f(0,x) = \sum_{\alpha,j} c_{\alpha,j}(0) x^{\alpha,j}$. For an $R\{x\}$-submodule M of $R\{x\}^p$, we will denote by $M(0,x)$ the image of M under the evaluation mapping (of variables y at 0).

Let L be any positive linear form on $\mathbb{R}^m$, $L(\alpha) = \sum_{i=1}^m \lambda_i \alpha_i$ (where $\lambda_i > 0$). We define a total ordering on $\mathbb{N}^m \times \{1, \ldots, p\}$ (denoted by L again) by lexicographic ordering of the $(m+2)$-tuples $(L(\alpha), j, \alpha_1, \ldots, \alpha_m)$, where $(\alpha, j) \in \mathbb{N}^m \times \{1, \ldots, p\}$, and $\alpha = (\alpha_1, \ldots, \alpha_m)$. For a p-tuple $f = \sum_{\alpha,j} c_{\alpha,j} x^{\alpha,j} \in R\{x\}^p$, define the *support* of f as

$$\operatorname{supp}(f) \coloneqq \{(\alpha, j) \in \mathbb{N}^m \times \{1, \ldots, p\} \mid c_{\alpha,j} \neq 0\},$$

and if f is a nonzero element, the *initial exponent* of f (with respect to the total ordering L) as $\exp_L(f) \coloneqq \min_L(\operatorname{supp}(f))$, where $\min_L$ denotes the minimum with respect to the total ordering L.

For an $R\{x\}$-submodule M of $R\{x\}^p$, the *diagram of initial exponents* of M is defined as

$$\mathfrak{N}_L(M) \coloneqq \{\exp_L(f) \mid f \in M \setminus \{0\}\} \subseteq \mathbb{N}^m \times \{1, \ldots, p\}.$$

Note that $\mathfrak{N}_L(M) + \mathbb{N}^m = \mathfrak{N}_L(M)$, since M is an $R\{x\}$-module. Indeed, $\exp_L(x^\gamma \cdot f) = \exp_L(f) + (\gamma, 0)$ for any $f \in R\{x\}^p$ and $\gamma \in \mathbb{N}^m$.

Now, we are ready to recall **Hironaka's flatness criterion** (see e.g. [BM1, Theorem 7.9]):

Let M be an $R\{x\}$-submodule of $R\{x\}^p$. Then, $R\{x\}^p/M$ is a flat module over R if and only if there exists a positive linear form L such that for every nonzero element $f \in M$, $\mathrm{supp}(f) \cap \mathfrak{N}_L(M(0,x)) \neq \varnothing$.

Our main flatness criterion

Theorem 3.5 ([AS2, Theorem 1.8]). *Consider $\varphi : X \to Y$ and $\beta|_{Y_{\mathrm{st}}} : Y_{\mathrm{st}} \to Y$ as above, with Y irreducible at $0 \in Y$. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then, $\mathcal{F}$ is a flat module over $\mathcal{O}_Y$ at $0 \in X$ if and only if y_n is not a zero-divisor in the module $\mathcal{F}_0 \tilde{\otimes}_{\mathcal{O}_{Y,0}} \mathcal{O}_{Y_{\mathrm{st},0}}$.*

To prove Theorem 3.5, we will need the following lemma.

Lemma 3.6 ([AS2, Lemma 3.1]). *Let $n \geq 2$, and let $h(y) = \sum_{|\alpha|=d} h_\alpha y^\alpha \in \mathbb{C}[y_1, \ldots, y_n]$ be a homogeneous polynomial of degree $d \geq 2$. There exist nonzero $c_1, \ldots, c_{n-1} \in \mathbb{C}$, such that after substituting $y_j + c_j y_n$ for y_j, $j = 1, \ldots, n-1$, $h(y)$ will contain a monomial cy_n^d for some nonzero $c \in \mathbb{C}$.*

Proof. Set $E_h := \{\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n \mid h_\alpha \neq 0\}$, and set $D := |E_h|$. If $D = 1$, then the lemma holds with $c_1 = \cdots = c_{n-1} = 1$. Suppose then that $D \geq 2$. Let α^* be the maximal element of E_h with respect to the lexicographic ordering of the n-tuples $(\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$. Set $M := \max\{|h_\alpha|/|h_{\alpha^*}| \mid \alpha \in E_h\}$; then $M \geq 1$. Define

$$c_1 = (DM)^{d^2(n-1)}, \quad c_2 = (DM)^{d^2(n-2)}, \quad \ldots, \quad c_{n-1} = (DM)^{d^2},$$

and, for $\alpha = (\alpha_1, \ldots, \alpha_n)$, set $p(\alpha) := d^{2(n-1)}\alpha_1 + d^{2(n-2)}\alpha_2 + \cdots + d^2\alpha_{n-1}$.

Now, after the substitutions $y_j \mapsto y_j + c_j y_n$, $j = 1, \ldots, n-1$, every term $h_\alpha y^\alpha$ of h gets transformed into a finite sum of terms, of which precisely one depends only on the variable y_n. This term is of the form $h_\alpha c_1^{\alpha_1} c_2^{\alpha_2} \cdots c_{n-1}^{\alpha_{n-1}} y_n^d$, which is $h_\alpha D^{p(\alpha)} M^{p(\alpha)} y_n^d$. Hence, $h(y)$ contains a term cy_n^d, where $c = \sum_{\alpha \in E_h} h_\alpha D^{p(\alpha)} M^{p(\alpha)}$. Therefore, to prove the lemma (i.e., to prove that $c \neq 0$) it suffices to show that

$$|h_{\alpha^*} D^{p(\alpha^*)} M^{p(\alpha^*)}| > \sum_{\alpha \in E_h \setminus \{\alpha^*\}} |h_\alpha D^{p(\alpha)} M^{p(\alpha)}|.$$

Given $\alpha = (\alpha_1, \ldots, \alpha_n) \in E_h \setminus \{\alpha^*\}$, there exists $0 \leq j_0 < n$ such that $\alpha_j = \alpha_j^*$ for all $1 \leq j \leq j_0$, but $\alpha_{j_0+1} < \alpha_{j_0+1}^*$. Note that, since $|\alpha| = |\alpha^*|$ and α^* is the unique maximal element of E_h, we actually have $j_0 \leq n-2$. It follows that

$$\begin{aligned} p(\alpha) &\leq d^{2(n-1)}\alpha_1^* + \cdots + d^{2(n-j_0)}\alpha_{j_0}^* + d^{2(n-j_0-1)}(\alpha_{j_0+1}^* - 1) + d^{2(n-j_0-2)}d \\ &= d^{2(n-1)}\alpha_1^* + \cdots + d^{2(n-j_0)}\alpha_{j_0}^* + d^{2(n-j_0-1)}\alpha_{j_0+1}^* - d^{2n-2j_0-3}(d-1) \\ &\leq p(\alpha^*) - d(d-1). \end{aligned}$$

Hence, for every $\alpha \in E_h \setminus \{\alpha^*\}$,

$$|h_\alpha D^{p(\alpha)} M^{p(\alpha)}| = \frac{|h_\alpha|}{|h_{\alpha^*}|} \cdot |h_{\alpha^*}| D^{p(\alpha)} M^{p(\alpha)} \leq M |h_{\alpha^*}| (DM)^{p(\alpha^*) - d(d-1)}.$$

Consequently

$$\begin{aligned} \sum_{\alpha \in E_h \setminus \{\alpha^*\}} |h_\alpha D^{p(\alpha)} M^{p(\alpha)}| &\leq DM |h_{\alpha^*}| (DM)^{p(\alpha^*) - d(d-1)} \\ &= |h_{\alpha^*}| (DM)^{p(\alpha^*) - d(d-1) + 1} < |h_{\alpha^*}| (DM)^{p(\alpha^*)}, \end{aligned}$$

because $d(d-1) \geq 2$ (as $d \geq 2$) and $DM > 1$. □

Proof of Theorem 3.5. As $\mathcal{F}_0$ is an analytic module over $\mathcal{O}_{Y,0}$, we can write $\mathcal{F}_0 = \mathcal{O}_{Y,0}\{x\}^p / M$ for some $p \in \mathbb{N}$, where $x = (x_1, \ldots, x_m)$ and M is an $\mathcal{O}_{Y,0}\{x\}$-submodule of $\mathcal{O}_{Y,0}\{x\}^p$. Write $\mathcal{O}_{Y,0} = \mathbb{C}\{y\}/J$, where J is the defining ideal of Y in $\mathbb{C}_y^n$ about the origin. Then write $\mathcal{O}_{Y_{\mathrm{st},0}} = \mathcal{O}_{Y,0}\{z\}/J_{\mathrm{st}}$, by Remark 1.5. Note that

$$(y_1 - z_1 z_n, \ldots, y_{n-1} - z_{n-1} z_n, y_n - z_n) \cdot \mathcal{O}_{Y,0}\{z\} \subseteq J_{\mathrm{st}}. \tag{3.3}$$

Suppose $\mathcal{F}_0$ is flat over $\mathcal{O}_{Y,0}$. Since flatness is preserved by analytic base change, $\mathcal{F}_0 \tilde{\otimes}_{\mathcal{O}_{Y,0}} \mathcal{O}_{Y_{\mathrm{st},0}}$ is flat and hence (by basic properties of flat modules) torsion-free over $\mathcal{O}_{Y_{\mathrm{st},0}}$.[11] Now, as the exceptional divisor (germ) $\{z_n = 0\}_0$ never contains an irreducible component of the strict transform (germ) $(Y_{\mathrm{st}})_0$, it is evident that the class of z_n in $\mathcal{O}_{Y_{\mathrm{st},0}}$ is non-zero and is not a zero-divisor (in the ring $\mathcal{O}_{Y_{\mathrm{st},0}}$). Therefore, z_n cannot be a zero-divisor in the module $\mathcal{F}_0 \tilde{\otimes}_{\mathcal{O}_{Y,0}} \mathcal{O}_{Y_{\mathrm{st},0}}$. As $y_n - z_n \in J_{\mathrm{st}}$, it follows that the same is true for y_n.

Conversely, suppose $\mathcal{F}_0$ is not flat over $\mathcal{O}_{Y,0}$. Then, by the criterion of Hironaka (see page 31), there exists some (p-tuple of formal power series) $f \in M \backslash \{0\}$ such that

$$\operatorname{supp}(f) \subseteq (\mathbb{N}^m \times \{1, \ldots, p\}) \backslash \mathfrak{N}_L(M(0,x)) \tag{3.4}$$

(for some arbitrary positive linear form L on $\mathbb{R}^m$), where by $M(0,x)$ we mean the module obtained from M by evaluation at 0 of variables y.

Write $f = \sum_{\substack{\alpha \in \mathbb{N}^n \\ 1 \le j \le p}} \overline{a_{\alpha,j}} x^{\alpha,j}$, where $\overline{a_{\alpha,j}} \in \mathcal{O}_{Y,0}$ denotes the class of $a_{\alpha,j} = a_{\alpha,j}(y) \in \mathbb{C}\{y\}$ modulo J. Set $d \coloneqq \operatorname{ord}_y f$ (i.e., the maximum power of the maximal ideal $(y) \cdot \mathcal{O}_{Y,0}$ that contains all the coefficients $\overline{a_{\alpha,j}}$ of f). Choose a monomial $\overline{a_{\alpha^*,j^*}} x^{\alpha^*,j^*}$ of f, with $\operatorname{ord}_y a_{\alpha^*,j^*} = d$. Write $a_{\alpha^*,j^*} = \sum_{\nu \ge d} a^{(\nu)}_{\alpha^*,j^*}$, where for every ν, $a^{(\nu)}_{\alpha^*,j^*} \in \mathbb{C}[y]$ is the homogeneous part of a_{α^*,j^*} of degree ν. By a change of coordinates as in Lemma 3.6, we can assume that $a^{(d)}_{\alpha^*,j^*}$ contains a monomial $c^* y_n^d$, for some $c^* \in \mathbb{C} \backslash \{0\}$.

By evaluating y at 0, we get $f(0,x) = 0$. This is because $f(0,x) \in M(0,x)$, while on the other hand, by (3.4), we have

$$\operatorname{supp}(f(0,x)) \cap \mathfrak{N}_L(M(0,x)) \subseteq \operatorname{supp}(f) \cap \mathfrak{N}_L(M(0,x)) = \varnothing.$$

It follows that $f \in (y) \cdot \mathcal{O}_{Y,0}\{x\}^p$, so that if we consider $\tilde{f} \coloneqq f(z_1 z_n, \ldots, z_{n-1} z_n, z_n, x) \in \mathcal{O}_{Y,0}\{x, z\}^p$, then $\tilde{f}$ will be divisible by z_n, and in fact divisible by z_n^d. Define the element $g \in \mathcal{O}_{Y,0}\{x, z\}^p$ as $g = z_n^{-d} \cdot \tilde{f}$. Notice that g contains the monomial

$$z_n^{-d} \cdot \overline{a_{\alpha^*,j^*}(z_1 z_n, \ldots, z_{n-1} z_n, z_n)} x^{\alpha^*,j^*}.$$

[11] We recall that an element m of an R-module M is called a *torsion element* if m is annihilated by a non-zero element $r \in R$ which is not a zero-divisor in the ring R.

This implies that after evaluating y and z at zero, we should get a nonzero element $g(0,x,0) \in \mathbb{C}\{x\}^p$, as it will contain the monomial

$$\left(z_n^{-d} \cdot a_{\alpha^*,j^*}(z_1 z_n, \ldots, z_{n-1}z_n, z_n)x^{\alpha^*,j^*}\right)|_{y=z=0} = c^* x^{\alpha^*,j^*}.$$

Next, consider

$$\mathcal{F}_0 \tilde{\otimes}_{\mathcal{O}_{Y,0}} \mathcal{O}_{Y_{\mathrm{st}},0} = \frac{\mathcal{O}_{Y,0}\{x,z\}^p}{M + J_{\mathrm{st}} \cdot \mathcal{O}_{Y,0}\{x,z\}^p},$$

in which we denote the class of g by $\overline{g}$.

We need to show $\overline{g} \neq 0$. Suppose otherwise that $g \in M + J_{\mathrm{st}} \cdot \mathcal{O}_{Y,0}\{x,z\}^p$. Then, evaluating variables y and z at zero gives $g(0,x,0) \in M(0,x) + J_{\mathrm{st}}(0) \cdot \mathbb{C}\{x\}^p$. We have $J_{\mathrm{st}}(0) = 0$, as the germ $(Y_{\mathrm{st}})_0$ is non-empty by the Setup. Hence $g(0,x,0) \in M(0,x)$. On the other hand, $\mathrm{supp}(g(0,x,0)) \subseteq \mathrm{supp}(f)$; thus by (3.4), it can only be that $g(0,x,0) = 0$ which is not the case as we observed above.

Now, by (3.3), we readily have

$$z_n^d \cdot g = \tilde{f} \equiv f \pmod{J_{\mathrm{st}} \cdot \mathcal{O}_{Y,0}\{x,z\}^p}.$$

Thus $z_n^d \cdot \overline{g} = 0$. Let d_0 be the minimal power for which we have $z_n^{d_0} \cdot \overline{g} = 0$. Since $\overline{g} \neq 0$, we have $d_0 \geq 1$. Then $z_n^{d_0-1} \cdot \overline{g} \neq 0$, while $z_n \cdot (z_n^{d_0-1} \cdot \overline{g}) = 0$. This means that z_n is a zero-divisor in $\mathcal{F}_0 \tilde{\otimes}_{\mathcal{O}_{Y,0}} \mathcal{O}_{Y_{\mathrm{st}},0}$. As $y_n - z_n \in J_{\mathrm{st}}$, it follows that the same is true for y_n. $\square$

The geometric version of Theorem 3.5, stated for mappings, reads as follows.

Corollary 3.7. *Consider $\varphi : X \to Y$ and $\beta|_{Y_{\mathrm{st}}} : Y_{\mathrm{st}} \to Y$ as above, and with Y irreducible at $0 \in Y$. Let*

$$\varphi' : Y_{\mathrm{st}} \times_Y X \to Y_{\mathrm{st}}$$

be the pullback of $\varphi : X \to Y$ by $\beta|_{Y_{\mathrm{st}}} : Y_{\mathrm{st}} \to Y$. The following statements are equivalent:

(i) φ *is a flat mapping at* $0 \in X$*,*

(ii) no irreducible component of $Y_{\mathrm{st}} \times_Y X$ *passing through* $(0,0)$ *is mapped to the singleton zero by* $(\beta|_{Y_{\mathrm{st}}}) \circ \varphi'$*.*

Proof. Let $\varphi' : Y_{\mathrm{st}} \times_Y X \to Y_{\mathrm{st}}$ be the pullback of φ by $\beta|_{Y_{\mathrm{st}}}$. Using Remark 1.5, we have

$$\mathcal{O}_{Y_{\mathrm{st}} \times_Y X} \simeq \frac{\mathbb{C}\{y,x\}}{I} \tilde{\otimes}_{\frac{\mathbb{C}\{y\}}{J}} \frac{\mathbb{C}\{y,z\}}{J_{\mathrm{st}}} = \frac{\mathbb{C}\{y,x,z\}}{I + J_{\mathrm{st}}},$$

for some ideals I and J_{st}, and the induced homomorphism on local rings $\left((\beta|_{Y_{\mathrm{st}}}) \circ \varphi'\right)_0^*$ is the canonical homomorphism $\mathbb{C}\{y\}/J \to \mathbb{C}\{y,x,z\}/(I + J_{\mathrm{st}})$.

Putting $\mathcal{F} \coloneqq \mathcal{O}_X$ in Theorem 3.5, it follows that flatness of φ at $0 \in X$ is equivalent to the fact that y_n is not a zero-divisor in $\mathbb{C}\{y,x,z\}/(I + J_{\mathrm{st}})$, which is equivalent to the fact that the class of y_n in $\mathbb{C}\{y\}/J$ (which is nonzero by the Setup) is not a zero-divisor in $\mathbb{C}\{y,x,z\}/(I + J_{\mathrm{st}})$. By Remark 1.4, the latter is equivalent to the statement that $(\beta|_{Y_{\mathrm{st}}}) \circ \varphi'$ has no irreducible components which pass through $(0,0)$ and are mapped into $\{(y_1,\dots,y_n) \in Y \mid y_n = 0\}$, and this itself is equivalent to the fact that $(\beta|_{Y_{\mathrm{st}}}) \circ \varphi'$ has no irreducible components which pass through $(0,0)$ and are mapped into $0 \in Y$. Indeed, $\{(y_1,\dots,y_n) \in Y \mid y_n = 0\}$ intersects the image of $\beta|_{Y_{\mathrm{st}}}$ only at 0. $\square$

The following version of Theorem 3.5 is suitable for computational purposes. Its formulation allows one also to have a comparison with the previous criteria of Auslander's type.

Corollary 3.8 (cf. [AS2, Theorem 1.3])**.** *Let* F *be an analytic module over the analytic* $\mathbb{C}$*-algebra and domain* $R = \mathbb{C}\{y\}/J$*; say* $F = R\{x\}^p/M$*, for some* $p \in \mathbb{N}$*, and some* $R\{x\}$*-submodule* M *of* $R\{x\}^p$*. Let* $Y \subseteq \mathbb{C}^n_y$ *be a representative of the complex germ* $\mathrm{Specan} R$ *(at* $0 \in Y$*), and consider* $\beta|_{Y_{\mathrm{st}}} : Y_{\mathrm{st}} \to Y$ *as above. By Remark 1.5, we have* $\mathcal{O}_{Y_{\mathrm{st}},0} \simeq R\{z\}/J_{\mathrm{st}} =: S$*, for some ideal* J_{st}*. Then we have* $F \tilde{\otimes}_R S = \dfrac{R\{x,z\}^p}{M + {J_{\mathrm{st}}}^p}$*, and the following statements are equivalent:*

(i) F is a flat module over R,

(ii) y_n (or, equivalently, z_n) is not a zero-divisor in the module $F\tilde{\otimes}_R S$.

Proof. Immediate; in Theorem 3.5, let $X \subseteq \mathbb{C}^m_x$ be a representative of the complex germ $\mathrm{Specan}(R\{x\})$ (at $0 \in X$) and let $\mathcal{F}$ be a coherent module on X such that $\mathcal{F}_0 = F$. (Also $y_n - z_n \in J_{\mathrm{st}}$, so y_n and z_n can be divisors of zero in $R\{x,z\}^p/(M + J_{\mathrm{st}}{}^p)$ only at the same time.) □

And again, the criterion becomes even simpler if the base is smooth, as the following corollary shows.

Corollary 3.9 ([AS2, Theorem 1.2])**.** *Let F be an analytic module over the (regular) n-dimensional analytic $\mathbb{C}$-algebra $R = \mathbb{C}\{y\}$; say $F = \mathbb{C}\{y,x\}^p/M$ for some $p \in \mathbb{N}$ and some $\mathbb{C}\{y,x\}$-submodule M of $\mathbb{C}\{y,x\}^p$. Let $\tilde{M}$ be the module obtained from M by substituting $y_i y_n$ for y_i, $i = 1,\dots,n-1$. The following statements are equivalent:*

(i) F is a flat R-module,

(ii) y_n is not a zero-divisor in the module $\dfrac{\mathbb{C}\{y,x\}^p}{\tilde{M}}$.

Proof. In Corollary 3.8, set $J = 0$, $R = \mathbb{C}\{y\}$, and get $J_{\mathrm{st}} = (y_1 - z_1 z_n, \dots, y_{n-1} - z_{n-1} z_n, y_n - z_n)$. Then we obtain an ($R$-module) isomorphism

$$\frac{R\{x,z\}^p}{M + J_{\mathrm{st}}{}^p} \to \frac{\mathbb{C}\{y,x\}^p}{\tilde{M}},$$

defined by $y_i \mapsto y_i y_n$, $y_n \mapsto y_n$, and $z_j \mapsto y_j$, where $i = 1,\dots,n-1$ and $j = 1,\dots,n$. Now, assertion (ii) of Corollary 3.8 gets converted to assertion (ii) above. □

Example 3.10 ([AS2, Example 5.2])**.** Consider φ as in Example 3.4. Being not open at 0, φ is not flat at 0 ([D], or see e.g. [F, § 3.19]). We want to verify non-flatness of φ at 0 directly, by using Corollary 3.9.

By Remark 1.5, we can write $\mathcal{O}_{X,0} = \mathbb{C}\{y,t,x\}/(I_1 + I_2)$, where

$$\begin{aligned} I_1 &= (y_1 - t_1 - x_4, y_2 - t_2 - x_5, y_3 - t_3 - x_6) \\ I_2 &= (t_1x_1 + t_2x_2 + t_3x_3, t_2x_1 + t_1x_2, x_4, x_5, x_6) \cap (t_1, t_2, t_3). \end{aligned}$$

We want to verify that $\mathcal{O}_{X,0} = \mathbb{C}\{y,t,x\}/(I_1 + I_2)$ is not a flat module over $\mathcal{O}_{Y,0} = \mathbb{C}\{y\}$.

Let $\tilde{I}_1$ and $\tilde{I}_2$ denote the ideals obtained from I_1 and I_2 by substituting y_1y_3 for y_1 and y_2y_3 for y_2; that is,

$$\begin{aligned} \tilde{I}_1 &= (y_1y_3 - t_1 - x_4, y_2y_3 - t_2 - x_5, y_3 - t_3 - x_6) \\ \tilde{I}_2 &= (t_1x_1 + t_2x_2 + t_3x_3, t_2x_1 + t_1x_2, x_4, x_5, x_6) \cap (t_1, t_2, t_3). \end{aligned}$$

With the help of a computer algebra system (Singular, see [GrPf]), we have found the element $x_6y_2 - x_5 \in \mathbb{C}\{y,t,x\}$ which does not belong to $\tilde{I}_1 + \tilde{I}_2$, but $y_3(x_6y_2 - x_5) \in \tilde{I}_1 + \tilde{I}_2$. To see this, on the one hand we have

$$\begin{aligned} y_3(x_6y_2 - x_5) &= x_6y_2y_3 - x_5y_3 \\ &\equiv x_6(t_2 + x_5) - x_5y_3 = t_2x_6 + x_5x_6 - x_5y_3 \pmod{\tilde{I}_1} \\ &\equiv 0 + x_5x_6 - x_5y_3 \pmod{\tilde{I}_2} \\ &\equiv x_5x_6 - x_5(t_3 + x_6) = -t_3x_5 \pmod{\tilde{I}_1} \\ &\equiv 0 \pmod{\tilde{I}_2}. \end{aligned}$$

On the other hand, suppose that $x_6y_2 - x_5 \in \tilde{I}_1 + \tilde{I}_2$. Then, after evaluating at zero the variables y_1, y_2, y_3, t_1, t_3, x_1, x_2, x_3, x_4, and x_6, we would get $-x_5 \in (t_2 + x_5, t_2x_5) \cdot \mathbb{C}\{t_2, x_5\}$, which is false. $\triangle$

We finish this chapter by presenting a coordinate-free version of our openness and flatness criteria.

Theorem 3.11. *Consider the mapping $\varphi : X \to Y$, with $\varphi(\xi) = \eta$ and Y locally irreducible. Let $\beta : \widetilde{Y} \to Y$ be the blowing-up of Y with center*

$\{\eta\}$. Pick $\xi' \in \beta^{-1}(\eta)$, and let the exceptional divisor around ξ' be defined by $f \in \mathcal{O}_{\widetilde{Y}}$. Let Y_{st} be the strict transform of Y under β. Then we have the following:

(i) The mapping φ is open at ξ if and only if $f_{\xi'}$ is not a zero-divisor in $(\mathcal{O}_{X,\xi}\tilde{\otimes}_{\mathcal{O}_{Y,\eta}}\mathcal{O}_{Y_{\mathrm{st}},\xi'})/\sqrt{(0)}$.

(ii) A coherent $\mathcal{O}_X$-module $\mathcal{F}$ is at $\xi \in X$ flat over $\mathcal{O}_Y$ if and only if $f_{\xi'}$ is not a zero-divisor in the module $\mathcal{F}_\xi\tilde{\otimes}_{\mathcal{O}_{Y,\eta}}\mathrm{O}_{Y_{\mathrm{st}},\xi'}$.

Proof. Let n be a natural number at least equal to the embedding dimension of Y at η, and choose local models about η for Y in $\mathbb{C}^n_y$, and about ξ' for $\widetilde{Y}$ in $\mathbb{C}^n_z$, according to the Setup at page 25. Then we can assume that $f_{\xi'} = z_n$. Part (i) now follows from Corollary 3.2, and part (ii) from Theorem 3.5. $\square$

Chapter 4

Characterizing non-open complex mappings

Following [KT], for a mapping $\varphi : X \to Y$, we define

$$\phi_{\mathrm{vert}}(\varphi) \coloneqq \sup\{i \geq 1 \mid \varphi^{\{i\}} \text{ has no (isolated) vertical components}\}$$

if φ has no (isolated) vertical components, and $\phi_{\mathrm{vert}}(\varphi) \coloneqq 0$ otherwise. We are going to call ϕ_{vert} the *(topological) verticality index* of φ.

In this chapter, we will study the behaviour of the topological verticality index. We will call it simply a verticality index, while we mean always the topological one. Likewise, as it is only a topological study of complex mappings, by a vertical component we will always mean an isolated one.[12]

First, let us summarize some known facts about this index. A mapping $\varphi : X \to Y$ with Y locally irreducible is open if $\phi_{\mathrm{vert}}(\varphi) = \infty$, or else $0 \leq \phi_{\mathrm{vert}}(\varphi) < \dim Y$ and it is not open ([KT, Theorem 3.2]). Based on this, it is natural to view ϕ_{vert} as an index that for a fixed n, partitions all non-open mappings with n-dimensional targets into n classes, where the i'th class, $i = 0, \ldots, n-1$, is identified with $\phi_{\mathrm{vert}}(\varphi) = i$.

[12] Of course, one can consider the study of verticality index by taking into account also the embedded vertical components, with the aim of leading to a characterization of non-flat mappings.

With regard to the family of fibres, the verticality index measures how much any fibre of the mapping can be *approximated* by means of the *general* fibres. This description is made precise in the following equivalent definition for the verticality index ([KT, Proposition 6.2]):

> For a mapping $\varphi : X \to Y$, the index $\phi_{\mathrm{vert}}(\varphi)$ is equal to the supremum of all i for which we have that, for every $x_1, \ldots, x_i$ taken on an arbitrary fibre $\varphi^{-1}(\varphi(\xi))$, and for every subset $B \subseteq Y$ with empty interior, there exist a sequence $(y_j)_j$ in $Y \backslash B$ with $y_j \to \varphi(\xi)$, and sequences $(x_{k,j})_j$, $k = 1, \ldots, i$, such that $x_{k,j} \in \varphi^{-1}(y_j)$ and $x_{k,j} \to x_k$.

For every $i \leq \phi_{\mathrm{vert}}(\varphi)$, $\varphi^{\{i\}}$ is *quasiopen*, meaning that it maps subsets of X with interior points to subsets of Y with interior points; this is a simple consequence of having no (isolated) vertical components.

4.1 Verticality index over smooth targets

When Y is smooth (and connected), a formula for the verticality index of a mapping $\varphi : X \to Y$ is given by

$$(4.1)\quad \phi_{\mathrm{vert}}(\varphi) = \min_p \{ \left[\frac{\dim Y - \dim \varphi(X_p) - 1}{\mathrm{fbd}\varphi|_{X_p} - (h_p - \dim Y)} \right] \ \mid \ \mathrm{fbd}\varphi|_{X_p} > h_p - \dim Y \},$$

where $h_p = \min\{\dim_\xi X \mid \xi \in X_p\}$, and $\{X_p\}_p$ is an *equidimensional partition* of X for φ ([KT, Theorem 3.5]). An equidimensional partition $\{X_p\}_p$ is a locally finite partition of X such that each X_p is a non-empty irreducible (locally closed) complex subspace of X, and such that the restriction $\varphi|_{X_p}$ is an equidimensional mapping (i.e., a mapping whose non-empty fibres are all of pure and the same dimension). One readily verifies that formula (4.1) is independent of the choice of an equidimensional partition.

We denote the right hand side of (4.1) by ϕ_{s}. We know (from the proof of [KT, Theorem 3.5]) that in the case of singular Y, one only has $\phi_{\mathrm{vert}}(\varphi) \leq \phi_{\mathrm{s}}(\varphi)$.

Let λ be the minimal, and κ the maximal fibre dimension of φ. For every j, $\lambda \leq j \leq \kappa$, define $X_\varphi^{(j)} := \{x \in X \mid \mathrm{fbd}_x\varphi = j\}$. Every $X_\varphi^{(j)}$ is a (locally closed) complex subspace of X, by the Cartan-Remmert Theorem (which states that fibre dimension is an upper semicontinuous function on X in the analytic Zariski topology; see e.g. [L, § V.3]). We obtain a partitioning $\{X_\varphi^{(j)}\}_{\lambda\leq j\leq\kappa}$ of X, which we are going to call the *fbd-partition* of X *with respect to* φ. Since they are easier to compute than the equidimensional partitions, we would like to use fbd-partitions in the study and calculation of ϕ_{vert}. The following remark is the first indication of how we do this. We give a simplified version of formula (4.1).

Remark 4.1. When X is of pure dimension m, and Y is locally irreducible (and connected) and of dimension n, then

$$\phi_{\mathrm{s}}(\varphi) = \min\{\left\lceil \frac{n - \dim\varphi(X_\varphi^{(j)}) - 1}{j - (m-n)} \right\rceil \mid \lambda \leq j \leq \kappa, j > m-n\}.$$

△

Proof. We want to construct an equidimensional partition out of the fbd-partition $\{X_\varphi^{(j)}\}_{\lambda\leq j\leq\kappa}$, and calculate ϕ_{s} by formula (4.1). First, one should observe that for any j, each fibre of $\varphi|_{X_\varphi^{(j)}}$ is of pure dimension j, and hence further restrictions to the regular locus $\mathrm{reg}(X_\varphi^{(j)})$ and its connected components will also have pure j-dimensional fibres. We put these components (which are mutually disjoint) as the initial members of our equidimensional partition. To complete this partition, we have to stratify the singular locus $\mathrm{sng}(X_\varphi^{(j)})$. Suppose $X_p \subseteq \mathrm{sng}(X_\varphi^{(j)})$ is to be a member of the equidimensional partition. We have $\mathrm{fbd}\varphi|_{X_p} \leq j$, and also $\dim\varphi(X_p) \leq \dim\varphi(X_\varphi^{(j)})$. So, as to formula (4.1), the partition members out of $\mathrm{sng}(X_\varphi^{(j)})$ will not count. Therefore, it suffices to consider only those (equidimensional) partition members we have obtained out of $\mathrm{reg}(X_\varphi^{(j)})$, which then gives the formula that we are after. □

The above formulas provide a way of explaining different cases of verticality (viewed as different modes of non-openness) for the mappings with smooth targets. Considering a mapping $\varphi : X \to Y$ as a family of fibres parametrized by the smooth space Y, φ shows verticality sooner if it possesses higher dimensional fibres (larger j) parametrized by higher dimensional subsets of Y (larger $\dim \varphi(X_\varphi^{(j)})$). Roughly speaking, a low verticality index means the stronger presence of exceptional fibres in the family. Now, we construct a family of examples which shows that for a fixed target dimension n, all the cases of verticality, namely $\phi_{\mathrm{vert}} = 0, \ldots, n-1$, can actually happen and so, our classification of non-open mappings is not void.

Example 4.2. Choose $n, \ell \in \mathbb{N}$, with $1 \leq \ell \leq n$. Let X be the analytic subset of $\mathbb{C}^{2n+1}$ with coordinates $(y_1, \ldots, y_n, x_1, \ldots, x_{n+1})$, defined by

$$y_1x_1 + \cdots + y_\ell x_\ell + x_{n+1}^2 = 0 \quad \text{and} \quad y_2x_1 + \cdots + y_\ell x_{\ell-1} + y_1x_\ell = 0.$$

(If $\ell = 1$, consider the second equation as $y_1x_1 = 0$.)

Set $Y = \mathbb{C}^n$, with coordinates $(y_1, \ldots, y_n)$. Define $\varphi : X \to Y$ as the projection. We claim that $\phi_{\mathrm{vert}}(\varphi) = \ell - 1$.

First, we justify that X is pure-dimensional. If $\ell = 1$, then X is just a union of two $(2n-1)$-dimensional planes (one of which is clearly vertical over Y). So suppose that $\ell > 1$.
For $y \in Y = \mathbb{C}^n$, define $D_y = \begin{bmatrix} y_1 & \cdots & y_{\ell-1} & y_\ell \\ y_2 & \cdots & y_\ell & y_1 \end{bmatrix}$, and set $A = \big\{(y, x) \in X \mid \mathrm{rank} D_y < 2\big\}$. We have

$$\begin{aligned} A &= \big\{(y,x) \in X \mid \exists\, c \in \mathbb{C}\backslash\{0\} \text{ s.t. } (y_1, \ldots, y_\ell) = c(y_2, \ldots, y_\ell, y_1)\big\} \\ &\quad \cup \big\{(y,x) \in X \mid y_1 = \cdots = y_\ell = 0\big\} \\ &= \big\{(y,x) \in X \mid \exists\, c \in \mathbb{C}\backslash\{0\} \text{ s.t. } y_1 = c^\ell y_1, y_2 = c^{\ell-1} y_1, \ldots, y_\ell = c y_1\big\} \\ &\quad \cup \big\{(y,x) \in X \mid y_1 = \cdots = y_\ell = 0\big\} \\ &= \bigcup_{\substack{c \in \mathbb{C} \\ c^\ell = 1}} \big\{(y,x) \in X \mid y_2 = c^{\ell-1} y_1, \ldots, y_\ell = c y_1\big\}. \end{aligned}$$

By considering the defining equations of X, we get

$$
\begin{aligned}
A = \bigcup_{c\in\mathbb{C},\, c^\ell=1} \Big\{(y,x)\in\mathbb{C}^{2n+1} \mid\ & y_2 = c^{\ell-1}y_1, \ldots, y_\ell = cy_1,\\
& y_1x_1 + c^{\ell-1}y_1x_2 + \cdots + cy_1x_\ell + x_{n+1}^2 = 0,\\
& c^{\ell-1}y_1x_1 + \cdots + cy_1x_{\ell-1} + y_1x_\ell = 0\Big\},
\end{aligned}
$$

in which multiplying the third equation by c gives us

$$
\begin{aligned}
A = \bigcup_{c\in\mathbb{C},\, c^\ell=1} \Big\{(y,x)\in\mathbb{C}^{2n+1} \mid\ & y_2 = c^{\ell-1}y_1, \ldots, y_\ell = cy_1,\\
& y_1x_1 + c^{\ell-1}y_1x_2 + \cdots + cy_1x_\ell + x_{n+1}^2 = 0,\\
& y_1x_1 + c^{\ell-1}y_1x_2 + \cdots + cy_1x_\ell = 0\Big\},
\end{aligned}
$$

which simplifies to

$$
\begin{aligned}
A = \bigcup_{c\in\mathbb{C},\, c^\ell=1} \Big\{(y,x)\in\mathbb{C}^{2n+1} \mid\ & y_2 = c^{\ell-1}y_1, \ldots, y_\ell = cy_1,\\
& x_{n+1} = 0,\\
& y_1(x_1 + c^{\ell-1}x_2 + \cdots + cx_\ell) = 0\Big\}.
\end{aligned}
\tag{4.2}
$$

It is now easily seen that $\dim A = 2n - \ell$. Write $X = A \cup (X \backslash A)$. Since X is defined by two equations, we have $\dim_\xi X \geq 2n+1-2 = 2n-1$, for every $\xi \in X$ (see footnote 13 on page 54). On the other hand, $\dim A = 2n - \ell < 2n - 1$, and hence $\dim A < \dim_\xi X$, for every $\xi \in X$. We conclude that A is a nowhere-dense subset of X. Then $X\backslash A$ is open and dense and hence for pure-dimensionality of X it will suffice to show $X\backslash A$ is pure-dimensional.

Take a point $(\eta, \xi) \in X\backslash A$, where $\eta \in \mathbb{C}^n_y$ and $\xi \in \mathbb{C}^{n+1}_x$. Since $\operatorname{rank} D_\eta = 2$, there is a nonsingular submatrix $\begin{bmatrix} \eta_i & \eta_j \\ \eta_{\sigma(i)} & \eta_{\sigma(j)} \end{bmatrix}$, for some i, j, $1 \leq i < j \leq \ell$, where σ is the permutation $\begin{pmatrix} 1 & 2 & \cdots & \ell \end{pmatrix}$. It follows that we can solve the defining equations of X in a neighbourhood of (η, ξ) in $X\backslash A$ for x_i and x_j. Hence $\dim_{(\eta,\xi)}(X\backslash A) = 2n+1-2 = 2n-1$, and in particular, $X\backslash A$ is of pure dimension.

Now, we are able to calculate $\phi_{\mathrm{vert}}(\varphi)$ by means of Remark 4.1. We need to find the fbd-partition of X with respect to φ. Take a point $\eta \in Y = \mathbb{C}^n$. If $\mathrm{rank} D_\eta = 2$, then from defining equations of X, we get $\dim \varphi^{-1}(\eta) = n-1$. If $\mathrm{rank} D_\eta < 2$ and $\eta_1 \neq 0$, then by (4.2), we get $\varphi^{-1}(\varphi(\eta)) = \{(\eta, x) \in \mathbb{C}^{2n+1} \mid x_{n+1} = 0, x_1 + c^{\ell-1}x_2 + \cdots + cx_\ell = 0\}$. So again, $\dim \varphi^{-1}(\eta) = n-1$. Finally assume $\mathrm{rank} D_\eta < 2$ and $\eta_1 = 0$, or equivalently by (4.2), $\eta_1 = \cdots = \eta_\ell = 0$. In this case, $\varphi^{-1}(\varphi(\eta)) = \{(\eta, x) \in \mathbb{C}^{2n+1} \mid x_{n+1} = 0\}$, and hence $\dim \varphi^{-1}(\eta) = n$. Thus, there is only one non-generic fibre locus to consider: $X_\varphi^{(n)}$, with $\varphi(X_\varphi^{(n)}) = \{\eta_1 = \cdots = \eta_\ell = 0\}$. One then calculates

$$\phi_{\mathrm{vert}}(\varphi) = \phi_{\mathrm{s}}(\varphi) = \left\lceil \frac{n-(n-\ell)-1}{n-(2n-1-n)} \right\rceil = \ell - 1.$$

$\triangle$

4.2 Behaviour of verticality index under pulling back

We first state and prove two simple lemmas that will come handy in our arguments.

Lemma 4.3. *The pullback of every open mapping is open.*

Proof. By definition (page 12), it suffices to justify this in the category of topological spaces. So let $\varphi_1 : X_1 \to Y$ and $\varphi_2 : X_2 \to Y$ be continuous mappings of topological spaces, where φ_1 is open. We want to show the pullback $\varphi_1' : X_2 \times_Y X_1 \to X_2$ is open. Take a basis open set $(U_2 \times U_1) \cap (X_2 \times_Y X_1)$, for some basis open sets $U_2 \subseteq X_2$ and $U_1 \subseteq X_1$. It suffices to show that $\varphi_2'((U_2 \times U_1) \cap (X_2 \times_Y X_2))$ is an open subset of X_2. We have

$$\begin{aligned}\varphi_2'((U_2 \times U_1) \cap (X_2 \times_Y X_1)) &= \{\xi_2 \in U_2 \mid \varphi_2(\xi_2) = \varphi_1(\xi_1) \text{ for some } \xi_1 \in U_1\} \\ &= \varphi_2^{-1}(\varphi_1(U_1)) \cap U_2.\end{aligned}$$

But $\varphi_2^{-1}(\varphi_1(U_1))$ is open in X_2 by openness of φ_1 and continuity of φ_2. So the intersection $\varphi_2^{-1}(\varphi_1(U_1)) \cap U_2$ is an open subset of X_2. $\square$

Lemma 4.4. *Let $\varphi : X \to Y$ and $\sigma : Z \to Y$ be mappings, and let $\varphi' : Z \times_Y X \to Z$ be the pullback of φ by σ. If σ is surjective and has no vertical components, then $\phi_{\mathrm{vert}}(\varphi') \le \phi_{\mathrm{vert}}(\varphi)$.*

Proof. Set $i \coloneqq \phi_{\mathrm{vert}}(\varphi) + 1$, so that the i-fold fibred power $\varphi^{\{i\}} : X^{\{i\}} \to Y$ has a vertical component, say Σ. Let σ' be the pullback of σ by $\varphi^{\{i\}}$, and let $(\varphi^{\{i\}})'$ be the pullback of $\varphi^{\{i\}}$ by σ. We get the following Cartesian square:

$$\begin{array}{ccc} Z \times_Y X^{\{i\}} & \xrightarrow{\sigma'} & X^{\{i\}} \\ \big\downarrow{\scriptstyle (\varphi^{\{i\}})'} & & \big\downarrow{\scriptstyle \varphi^{\{i\}}} \\ Z & \xrightarrow{\sigma} & Y \end{array}$$

Suppose σ is surjective. Then the pullback σ' is surjective, and so $\sigma'^{-1}(\mathrm{reg}(\Sigma))$ is a non-empty open subset of $Z \times_Y X^{\{i\}}$. Let Σ' be an irreducible component of $Z \times_Y X^{\{i\}}$ with a non-empty intersection with $\sigma'^{-1}(\mathrm{reg}(\Sigma))$. Then $\mathrm{reg}(\Sigma') \cap \sigma'^{-1}(\mathrm{reg}(\Sigma))$ is a non-empty open subset of Σ', which is mapped into the set $\sigma^{-1}(\varphi^{\{i\}}(\Sigma))$ by $(\varphi^{\{i\}})'$. Suppose now σ has no vertical components, which implies that the inverse image of a set with empty interior by σ has empty interior. Therefore, $\sigma^{-1}(\varphi^{\{i\}}(\Sigma))$ has empty interior in Z, as by verticality of Σ, $\varphi^{\{i\}}(\Sigma)$ has empty interior in Y. Now that an open subset of Σ' has an image with empty interior in Z, we conclude (by footnote 5 on page 9) that the whole Σ' should have such image. So $(\varphi^{\{i\}})'$, which by Remark 1.2 is equivalent to $(\varphi')^{\{i\}}$, has a vertical component. By definition, we get $\phi_{\mathrm{vert}}(\varphi') < i$, and thus $\phi_{\mathrm{vert}}(\varphi') \le \phi_{\mathrm{vert}}(\varphi)$. $\square$

Remark 4.5. In Lemma 4.4, the surjectivity of the mapping that pulls back cannot be weakened, even to dominance. For instance, let $X \subseteq \mathbb{C}^3_x$ be defined by $x_1x_3 = x_2x_3 = 0$, and define the mapping $\varphi : X \to \mathbb{C}^2_y$ as $(x_1, x_2, x_3) \mapsto (x_1, x_2 + x_3)$. The mapping φ is not open around the origin, as the irreducible component $\mathbb{C}_{x_3}$ of X is mapped to a line in $\mathbb{C}^2_y$. But the pullback of φ by $\sigma : \mathbb{C}^2_z \to \mathbb{C}^2_y$, defined as $(z_1, z_2) \overset{\sigma}{\mapsto} (z_1, z_1z_2)$, is equivalent to the identity mapping on $\mathbb{C}^2$, which shows no verticality. Indeed, by the structure of the fibred product, the pullback of φ by σ is the same as the pullback of $\varphi|_{\varphi^{-1}(\sigma(\mathbb{C}^2))}$ by σ, but $\sigma(\mathbb{C}^2) = (\mathbb{C}^2_y \backslash \mathbb{C}_{y_2}) \cup \{0\}$, hence $\varphi^{-1}(\sigma(\mathbb{C}^2)) = (\mathbb{C}^2_{x_1,x_2} \backslash \mathbb{C}_{x_2}) \cup \{0\}$, and so $\varphi|_{\varphi^{-1}(\sigma(\mathbb{C}^2))}$ is identified with the identity mapping on $(\mathbb{C}^2 \backslash \mathbb{C}) \cup \{0\}$, which is then pulled back by σ to the identity on $\mathbb{C}^2$ with an infinite verticality index. $\triangle$

The pullback by an open, surjective mapping preserves the verticality index of every mapping.

Proposition 4.6. *Consider a mapping $\varphi : X \to Y$, and let $\varphi' : X' \to Z$ be its pullback by an open and surjective mapping $\sigma : Z \to Y$. Then, $\phi_{\mathrm{vert}}(\varphi) = \phi_{\mathrm{vert}}(\varphi')$.*

Proof. For every $i \geq 1$, form a diagram

$$\begin{array}{ccc} T & \xrightarrow{\sigma_i} & X^{\{i\}} \\ \downarrow{\scriptstyle \pi'_i} & & \downarrow{\scriptstyle \pi_i} \\ X' & \xrightarrow{\sigma_1} & X \\ \downarrow{\scriptstyle \varphi'} & & \downarrow{\scriptstyle \varphi} \\ Z & \xrightarrow{\sigma} & Y \end{array}$$

where $X^{\{i\}} = \underbrace{X \times_Y \cdots \times_Y X}_{i \text{ times}}$, π_i is a projection, σ_1 is the pullback of σ by φ, σ_i is the pullback of σ_1 by π_i, φ' is the pullback of φ by σ, and π'_i is the pullback of π_i by σ_1. Note that, by Remark 1.2, we have $T \simeq Z \times_Y X^{\{i\}} \simeq X'^{\{i\}} = \underbrace{X' \times_Z \cdots \times_Z X'}_{i \text{ times}}$, so let us set $T = X'^{\{i\}}$.

We have to show that $X^{\{i\}}$ has a vertical component over Y if and only if $X'^{\{i\}}$ has a vertical component over Z. Observe (by Lemma 4.3) that σ_1 and σ_i are open and surjective, for they are pullbacks of open and surjective mappings.

Suppose $X^{\{i\}}$ has a vertical component Σ over Y. Consider $\sigma_i^{-1}(\mathrm{reg}(\Sigma))$, which is an open subset of $X'^{\{i\}}$, and is non-empty by surjectivity of σ_i. Hence $\sigma_i^{-1}(\mathrm{reg}(\Sigma))$ intersects the regular locus of some irreducible component Σ' of

$X'^{\{i\}}$. Now, $\operatorname{reg}(\Sigma') \cap \sigma_i^{-1}(\operatorname{reg}(\Sigma))$ is a non-empty open subset of Σ' which is mapped to a set with empty interior in Y. Therefore (by footnote 5 on page 9), Σ' is a vertical component of $X'^{\{i\}}$ over Y; but since σ is quasiopen, it follows easily that Σ' is vertical over Z, too.

Conversely, suppose $X'^{\{i\}}$ has a vertical component Σ' over Z. There is some irreducible component Σ of $X^{\{i\}}$ such that $\sigma_i(\Sigma') \subseteq \Sigma$. We claim that Σ is a vertical component of $X^{\{i\}}$. Since $\sigma_i(\operatorname{reg}(\Sigma'))$ is a (non-empty) open subset of Σ, it suffices to show that $(\varphi\circ\pi_i)(\sigma_i(\operatorname{reg}(\Sigma'))$ is a subset of Y with no interior points. Suppose otherwise. Then there exists some non-empty open $U \subseteq \sigma_i(\operatorname{reg}(\Sigma'))$ with the image $V = (\varphi\circ\pi_i)(U)$ open in Y. We can shrink V and U to get an open mapping $(\varphi\circ\pi_i)|_U : U \to V$ (first shrink V so that it is smooth and connected, then shrink U to some smooth, connected, open subset on which fibre dimension is constant and whose image is of dimension $\dim V$; we get an open mapping, by Remmert's Open Mapping Theorem). Set $U' = \sigma_i^{-1}(U) \cap \operatorname{reg}(\Sigma')$ (which is non-empty by surjectivity of σ_i). We have

$$\sigma^{-1}(V) \times_V U = \big((\varphi'\circ\pi_i')^{-1}(\sigma^{-1}(V))\big) \cap \sigma_i^{-1}(U),$$

hence U' is an open subset of $\sigma^{-1}(V) \times_V U$. Since $(\varphi\circ\pi_i)|_U$ is open, its pullback $(\varphi'\circ\pi_i')|_{\sigma^{-1}(V)\times_V U} : \sigma^{-1}(V)\times_V U \to \sigma^{-1}(V)$ is open (by Lemma 4.3). Then, $(\varphi'\circ\pi_i')(U')$ will be an open subset of $\sigma^{-1}(V)$, which contradicts the verticality of Σ' over Z. $\square$

On the other extreme, there is the pullback by a blowing-up. We saw in Theorem 3.1 that blowing-up with center the origin decreases the verticality index of a mapping to zero in the pullback—unless the mapping has an infinite verticality index (that is precisely, when it is open).

One now may wonder about the existence of a general relation between the verticality index of a mapping and that of its pullback. Upon the existence of an effective relation, one might be able to obtain some sort of *test maps*, by which if a given mapping φ is pulled back, the verticality index of φ would be determined. We have seen already an instance of such a test map, namely,

locus of $\eta = (\eta_1, \eta_2, \eta_3) \in Y$	dimension	$\dim \varphi^{-1}(\eta)$
$\eta_1 = \eta_2,\ \eta_3 = 0$	1	1
$\eta_1 \neq \eta_2$	3	0

locus of $\eta = (\eta_1, \eta_2, \eta_3) \in Y$	dimension	$\dim(\varphi')^{-1}(\eta)$
$\eta_1 = \eta_3,\ \eta_2 = 0$	1	1
$\eta_1 \neq \eta_3$	3	0

Table 4.1: Analysis of fibres for φ and φ' (Example 4.7)

the blowing-up, which is able to test whether the verticality index is infinite or not (Theorem 3.1). In general however, such a luxury cannot exist. In Example 4.7, we show that the verticality index of the pullback might depend on factors that are not of interest.

Example 4.7. Define the space

$$X = \{(y, x) \in \mathbb{C}^3 \times \mathbb{C} \mid (y_1 - y_2)x + y_3 = 0\},$$

which is of pure dimension 3. Set $Y = \mathbb{C}^3_y$, and consider the projection $\varphi : X \to Y$. Consider $\varphi' : X \to Y$ defined as $\varphi' = \psi \circ \varphi$, where $\psi : Y \to Y$ is the biholomorphism defined by $(y_1, y_2, y_3) \mapsto (y_1, y_3, y_2)$.

Notice that φ and φ' are (left-) equivalent mappings (i.e., they are the same up to a biholomorphism on their targets). Hence, $\phi_{\text{vert}}(\varphi) = \phi_{\text{vert}}(\varphi')$. The goal is to show the following: $\phi_{\text{vert}}(\varphi \times_Y \varphi) = 0$, but $\phi_{\text{vert}}(\varphi \times_Y \varphi') = 1$.

The analysis of fibres for φ and φ' is summarized in Table 4.1, from which by Remark 4.1, we calculate

$$\phi_{\text{vert}}(\varphi) = \left[\frac{3 - 1 - 1}{1 - (3 - 3)}\right] = 1.$$

Hence, $\varphi \times_Y \varphi$ has a vertical component, and thus, $\phi_{\text{vert}}(\varphi \times_Y \varphi) = 0$.

Next, we compute $\phi_{\text{vert}}(\varphi \times_Y \varphi')$. Using Table 4.1, we have obtained the analysis of fibres for $\varphi \times_Y \varphi'$ in Table 4.2.

The fbd-partition of $X \times_Y X'$ with respect to $\varphi \times_Y \varphi'$ is of the form $\{X^{(0)}_\varphi, X^{(1)}_\varphi, X^{(2)}_\varphi\}$, with $\dim(\varphi \times_Y \varphi')(X^{(1)}_\varphi) = 1$, and $\dim(\varphi \times_Y \varphi')(X^{(2)}_\varphi) = 0$.

locus of $\eta = (\eta_1, \eta_2, \eta_3) \in Y$	dimension	$\dim(\varphi \times_Y \varphi')^{-1}(\eta)$
$\eta_1 = \eta_2 = \eta_3 = 0$	0	2
$\eta_1 = \eta_2 \neq 0,\ \eta_3 = 0$	1	1
$\eta_1 = \eta_3 \neq 0,\ \eta_2 = 0$	1	1
$\eta_1 \neq \eta_2,\ \eta_1 \neq \eta_3$	3	0

Table 4.2: Analysis of fibres for $\varphi \times_Y \varphi'$ (Example 4.7)

Then, by Remark 4.1, we calculate

$$\phi_{\text{vert}}(\varphi \times_Y \varphi') = \min\{\left\lceil \frac{3-1-1}{1-(3-3)} \right\rceil, \left\lceil \frac{3-0-1}{2-(3-3)} \right\rceil\} = 1.$$

Similar situation is happening for pullbacks. That is, if π is the pullback of φ by itself, and π' is the pullback of φ by φ', then $\phi_{\text{vert}}(\pi) = 0$, while $\phi_{\text{vert}}(\pi') = 1$. To see this, just observe that $\varphi \times_Y \varphi$ has a vertical component if and only if π does; likewise, this holds for $\varphi \times_Y \varphi'$ and π' (this is an easy consequence of the topology of φ and φ', that (as a result of continuity and $\phi_{\text{vert}} \neq 0$) send sets with empty interior to sets with empty interior and vice versa). It follows that $\phi_{\text{vert}}(\pi) = 0$, and $\phi_{\text{vert}}(\pi') \geq 1$. On the other hand by Lemma 4.4, $\phi_{\text{vert}}(\pi') \leq \phi_{\text{vert}}(\varphi) = 1$, hence $\phi_{\text{vert}}(\pi') = 1$. △

Remark 4.8. Example 4.7 shows that for a fixed mapping $\varphi : X \to Y$, the operator

$$\Phi_\varphi : (\psi : Z \to Y) \mapsto \phi_{\text{vert}}(\varphi \times_Y \psi)$$

is not *left-invariant*, meaning that, it is not well-defined on the classes of the left-equivalence relation defined on the set of mappings with targets equal to Y. △

4.3 Verticality index over singular targets

Recall that if the target of a mapping $\varphi : X \to Y$ is not smooth, then we have only $\phi_{\text{vert}}(\varphi) \leq \phi_{\text{s}}(\varphi)$. In this section, we will study the index ϕ_{vert} for mappings with singular targets. In particular, we will find some classes of mappings for which we do have $\phi_{\mathrm{vert}} = \phi_{\mathrm{s}}$.

Proposition 4.9. *Consider a mapping $\varphi : X \to Y$, with X of pure dimension and Y locally irreducible (and connected). If Y admits a desingularization which is a finite mapping (in particular, if $\dim Y = 1$), then $\phi_{\mathrm{vert}}(\varphi) = \phi_{\mathrm{s}}(\varphi)$.*

Proof. Let $\sigma : Z \to Y$ be a finite desigularization of Y. Let $\varphi' : Z \times_Y X \to Z$ be the pullback of φ by σ, and $\sigma' : Z \times_Y X \to X$ the pullback of σ by φ. Since σ is surjective and has no vertical components, by Lemma 4.4 we get $\phi_{\mathrm{vert}}(\varphi') \leq \phi_{\mathrm{vert}}(\varphi)$. We know that $\phi_{\mathrm{vert}}(\varphi) \leq \phi_{\mathrm{s}}(\varphi)$, and Z being smooth, we have $\phi_{\mathrm{s}}(\varphi') = \phi_{\mathrm{vert}}(\varphi')$. Altogether,

$$\phi_{\mathrm{s}}(\varphi') = \phi_{\mathrm{vert}}(\varphi') \leq \phi_{\mathrm{vert}}(\varphi) \leq \phi_{\mathrm{s}}(\varphi).$$

To conclude the result, it suffices to show $\phi_{\mathrm{s}}(\varphi') = \phi_{\mathrm{s}}(\varphi)$. Observe that, for every $\xi \in X$, the fibre of φ through ξ is isomorphic to the fibre of φ' through every $\xi' \in \sigma'^{-1}(\xi)$. This implies that if $\{X_\varphi^{(j)}\}_j$ is the fbd-partition of X with respect to φ, then $\{\sigma'^{-1}(X_\varphi^{(j)})\}_j$ is the fbd-partition of $Z \times_Y X$ with respect to φ'. To obtain the dimension of the image for each member of the partition, observe that σ' is surjective (for σ is so), and thus

$$\sigma(\varphi'\big(\sigma'^{-1}(X_\varphi^{(j)})\big)) = \varphi(X_\varphi^{(j)}).$$

Hence, as a result of finiteness of σ, $\dim \varphi'\big(\sigma'^{-1}(X_\varphi^{(j)})\big) = \dim \varphi(X_\varphi^{(j)})$. Now, if we have the pure-dimensionality of $Z \times_Y X$, then by Remark 4.1 we will conclude that $\phi_{\mathrm{s}}(\varphi') = \phi_{\mathrm{s}}(\varphi)$.

To show $Z \times_Y X$ is pure-dimensional, first observe that the finite mapping σ is an open mapping, by Remmert's Open Mapping Theorem, and surjective, by definition. It follows that its pullback σ' is an open (by Lemma 4.3), finite mapping onto the (pure-dimensional) space X. Pure-dimensionality of $Z \times_Y X$ follows. $\square$

Given only the (singular) target Y, it is interesting to know whether we are able to determine if $\phi_{\text{vert}} = \phi_{\text{s}}$ holds or not, for every mapping with target Y. The following two examples (taken from a preprint version of [KT]) show that this is not the case in general; that is, the equality depends on factors beyond the properties of the target space (except, of course, for special cases such as the one in Proposition 4.9).

Example 4.10. Set

$$\begin{aligned} Y &= \{y \in \mathbb{C}^4 \mid y_1y_4 - y_2y_3 = 0\}, \\ X &= \{(y,x) \in Y \times \mathbb{C} \mid y_1x + y_2 = 0, y_3x + y_4 = 0\}, \end{aligned}$$

and consider $\varphi : X \to Y$ as the projection.

Take a point $\eta \in Y$. If $\eta_1 \neq 0$ or $\eta_3 \neq 0$, then $\varphi^{-1}(\eta)$ is a singleton. If $\eta_1 = \eta_3 = 0$, but $\eta_2 \neq 0$ or $\eta_4 \neq 0$, then the fibre over η is empty. Finally if $\eta = 0$, then $\varphi^{-1}(\eta) = \mathbb{C}$. Thus, the fbd-partition is $\{X_\varphi^{(0)}, X_\varphi^{(1)}\}$, with $\dim \varphi(X_\varphi^{(1)}) = 0$.

Since X is defined in $\mathbb{C}^5$ by 3 equations, it is at least of local dimension 2 (everywhere). So $X_\varphi^{(1)} = \mathbb{C}$ is nowhere-dense in X, and hence $X_\varphi^{(0)} = X \backslash X_\varphi^{(1)}$ is dense. Also $X_\varphi^{(0)}$ is open in X and of pure dimension 3 (it is locally a graph of $x = -y_2/y_1$ or $x = -y_4/y_3$ over the open subset $\{y_1 \neq 0 \text{ or } y_3 \neq 0\}$ of Y), thus we conclude that X is of pure dimension 3.

Now, by Remark 4.1, calculate

$$\phi_{\text{s}}(\varphi) = \left\lceil \frac{3 - 0 - 1}{1 - (3 - 3)} \right\rceil = 2.$$

To find $\phi_{\text{vert}}(\varphi)$, notice that φ is a generically one-to-one mapping, and thus by the equivalent definition of the verticality index (see page 41), we should have $\phi_{\text{vert}}(\varphi) \leq 1$. Obviously, φ has no vertical components, so $\phi_{\text{vert}}(\varphi) = 1$. $\triangle$

Example 4.11. With Y as in Example 4.10, set

$$X = \{(y,x) \in Y \times \mathbb{C} \mid y_1x^2 + y_4x + y_2 - y_3 = 0\},$$

and consider $\varphi : X \to Y$ as the projection.

Take a point $\eta \in Y$. If $\eta_1 \neq 0$ or $\eta_4 \neq 0$, then $\varphi^{-1}(\eta)$ is either a singleton or a pair of points. If $\eta_1 = \eta_4 = 0$, then (by defining equations of Y) $\eta_2\eta_3 = 0$, and in order to have a non-empty fibre, we should have (by defining equations of X) $\eta_3 = \eta_4 = 0$, and so $\eta = 0$ and the fibre will be $\mathbb{C}_x$. Thus, the fbd-partition is $\{X_\varphi^{(0)}, X_\varphi^{(1)}\}$, with $\dim \varphi(X_\varphi^{(1)}) = 0$.

Since X is defined in $\mathbb{C}^5$ by 2 equations, it is at least of local dimension 3 (everywhere). So $X_\varphi^{(1)} = \mathbb{C}$ is nowhere-dense in X, and so $X \backslash X_\varphi^{(1)} = X_\varphi^{(0)}$ is open and dense. Notice that $\varphi|_{X_\varphi^{(0)}} : X_\varphi^{(0)} \to Y$ is a finite mapping with $\dim_\xi X_\varphi^{(0)} \geq 3$ for every $\xi \in X_\varphi^{(0)}$, and with $\dim Y = 3$. This implies that $X_\varphi^{(0)}$ is of pure dimension 3. So X is of pure dimension 3.

Now, by Remark 4.1, calculate

$$\phi_{\mathrm{s}}(\varphi) = \left[\frac{3-0-1}{1-(3-3)}\right] = 2.$$

To compute that $\phi_{\mathrm{vert}}(\varphi) = 2$, it is easier to first have Proposition 4.12. We will work out the computation later in Example 4.18. $\triangle$

Proposition 4.12. *Let Y be a locally irreducible space contained in a space Υ of local dimension at least N at every point, and let Ω be a space of pure dimension k. Let $X \subseteq Y \times \Omega$ be a space which can be defined in $\Upsilon \times \Omega$ locally by at most r holomorphic functions (i.e., every stalk of the coherent ideal of $\mathcal{O}_{\Upsilon\times\Omega}$ defining X admits r generators). Let $\varphi : X \to Y$ be the projection and suppose φ has no vertical components. Then*

$$\phi_{\mathrm{vert}}(\varphi) \geq \phi_{\_}(\varphi) := \min_{\lambda<j\leq\kappa} \left[\frac{N - \dim \varphi(X_\varphi^{(j)}) - 1}{j-(k-r)}\right],$$

where $\{X_\varphi^{(j)}\}_{\lambda\leq j\leq\kappa}$ is the fbd-partition of X with respect to φ, $\lambda = \min_{\xi\in X} \mathrm{fbd}_\xi\varphi$, and $\kappa = \max_{\xi\in X} \mathrm{fbd}_\xi\varphi$.

Proof. For every $\zeta \in X^{\{i\}}$, $i \geq 1$, the germ $X_\zeta^{\{i\}}$ will be defined in $(\Upsilon \times \Omega^i)_\zeta$ by at most ir equations, according to the underlying set of fibred product.

Hence, by estimating the dimension of intersection,[13] for every $\zeta \in X^{\{i\}}$, we can write

$$\dim X^{\{i\}}_\zeta \geq \dim(\Upsilon \times \Omega^i)_\zeta - ir \geq N + i(k-r). \tag{4.3}$$

By considering again the set structure of fibred product, write

$$X^{\{i\}} = \bigcup_{(j_1,\dots,j_i)} X^{(j_1)}_\varphi \times_Y \cdots \times_Y X^{(j_i)}_\varphi,$$

where the union is taken over all $(j_1,\dots,j_i) \in \mathbb{N}^i$, with $\lambda \leq j_1 \leq \kappa, \dots, \lambda \leq j_i \leq \kappa$. For $(j_1,\dots,j_i) \in \mathbb{N}^i$, denote by j_0 the maximum of $\{j_1,\dots,j_i\}$, and consider the projection $\pi : X^{(j_1)}_\varphi \times_Y \cdots \times_Y X^{(j_i)}_\varphi \to X^{(j_0)}_\varphi$. Since $\mathrm{fbd}_\zeta(\varphi\circ\pi) \leq ij_0$ for every $\zeta \in X^{(j_1)}_\varphi \times_Y \cdots \times_Y X^{(j_i)}_\varphi$, by Dimension Formula we have

$$\dim(X^{(j_1)}_\varphi \times_Y \cdots \times_Y X^{(j_i)}_\varphi) \leq ij_0 + \dim\varphi(X^{(j_0)}_\varphi). \tag{4.4}$$

Now suppose $i \geq 1$ is such that for every j, $\lambda < j \leq \kappa$, we have

$$ij + \dim\varphi(X^{(j)}_\varphi) < N + i(k-r). \tag{4.5}$$

By (4.3), (4.4), and (4.5), we get $\dim(X^{(j_1)}_\varphi \times_Y \cdots \times_Y X^{(j_i)}_\varphi) < \dim X^{\{i\}}_\zeta$, for any $(j_1,\dots,j_i) \neq (\lambda,\dots,\lambda)$, and every $\zeta \in X^{\{i\}}$. This implies that $(X^{(j_1)}_\varphi \times_Y \cdots \times_Y X^{(j_i)}_\varphi)_\zeta$ is a nowhere-dense subgerm of $X^{\{i\}}_\zeta$, for any $(j_1,\dots,j_i) \neq (\lambda,\dots,\lambda)$, and every $\zeta \in X^{\{i\}}$. Now take a point $\zeta \in X^{\{i\}}$ and write

$$X^{\{i\}}_\zeta = \Big((X^{(\lambda)}_\varphi)^{\{i\}}\Big)_\zeta \cup \Big(\bigcup_{(j_1,\dots,j_i)\neq(\lambda,\dots,\lambda)} X^{(j_1)}_\varphi \times_Y \cdots \times_Y X^{(j_i)}_\varphi\Big)_\zeta.$$

We get that $\Big((X^{(\lambda)}_\varphi)^{\{i\}}\Big)_\zeta$ is a dense subgerm of $X^{\{i\}}_\zeta$. On the other hand, $\varphi|_{X^{(\lambda)}_\varphi}$ is an open mapping (by Remmert's Rank Theorem[14] and the fact that

[13] For any analytic subsets A_1 and A_2 of a manifold U, we have $\dim_\xi(A_1 \cap A_2) \geq \dim_\xi A_1 + \dim_\xi A_2 - \dim U$ for every $\xi \in A_1 \cap A_2$ (see e.g. [L] or [GPR]). By induction on the number of equations, one obtains that if A_1 is defined by r equations in A_2, then $\dim_\xi A_1 \geq \dim_\xi A_2 - r$ for every $\xi \in A_1$.

[14] **Remmert's Rank Theorem.** Let $\varphi : X \to Y$ be a mapping with a constant $r \in \mathbb{N}$ such that $\mathrm{fbd}_\xi\varphi = r$ for every $\xi \in X$. Then, every point $\xi \in X$ admits an open neighbourhood $U \subseteq X$ such that the image $\varphi(U)$ is a locally analytic subset of Y of dimension $\dim_\xi X - r$. (See e.g. [L, § V.6].)

φ has no vertical components), and therefore by Lemma 4.3 and induction, it follows that $(\varphi|_{X_\varphi^{(\lambda)}})^{\{i\}} : (X_\varphi^{(\lambda)})^{\{i\}} \to Y$ is an open mapping. Thus, $X_\zeta^{\{i\}}$ cannot have any vertical components over Y. We thus showed that for any $i \geq 1$ such that (4.5) holds for every $\lambda < j \leq \kappa$, $X_\zeta^{\{i\}}$ has no vertical components for every $\zeta \in X^{\{i\}}$, and hence $X^{\{i\}}$ has no vertical components. But (4.5) is equivalent to

$$i \leq \frac{N - \dim \varphi(X_\varphi^{(j)}) - 1}{j - (k-r)},$$

for every $j > \lambda$. (Note that for every $j > \lambda$, we have $j > k - r$, since by estimation of codimension applied to a fibre, $\lambda \geq \dim \Omega - r = k - r$). By definition of ϕ_{vert}, the result follows. □

Remark 4.13. The lower bound obtained in Proposition 4.12 for verticality index of a mapping φ is denoted by $\phi_{\_}(\varphi)$. Note that $\phi_{\_}(\varphi)$ is not only a function of φ, but rather, it depends also on the parameters of the model in which we have set up the mapping φ according to the proposition. △

Remark 4.14. In Proposition 4.12, the assumption of φ having no vertical components is not really restrictive. If the mapping φ has a vertical component, then one already computes that $\phi_{\mathrm{s}}(\varphi) = 0$, and there will be no need for a lower bound. △

To study the verticality index of a general mapping (not necessarily modelled as a projection), one may apply Proposition 4.12 to the equivalent projection which can be obtained by the procedure from Remark 1.5. Though, it may not necessarily be a model leading to an efficient lower approximation for verticality index.

Corollary 4.15. *Let $\varphi : X \to Y$ be a mapping, with $Y \subseteq \mathbb{C}^N$ locally irreducible, and X of pure dimension m. Suppose φ has no vertical components. Then,*

$$\phi_{\mathrm{vert}}(\varphi) \geq \phi_{\_}(\varphi) = \min_{\lambda < j \leq \kappa} \left[\frac{N - \dim \varphi(X_\varphi^{(j)}) - 1}{j - (m-N)} \right],$$

where $\{X_{\varphi}^{(j)}\}_{\lambda \leq j \leq \kappa}$ *is the fbd-partition of* X *with respect to* φ*,* $\lambda = \min_{\xi \in X} \operatorname{fbd}_{\xi}\varphi$*, and* $\kappa = \max_{\xi \in X} \operatorname{fbd}_{\xi}\varphi$*.*

Proof. In Proposition 4.12, set $\Upsilon = \mathbb{C}^N$ and $\Omega = X$. Then the graph of φ (see Remark 1.5), namely $\Gamma_{\varphi} \subseteq Y \times X$, will be defined in $\mathbb{C}^N \times X$ by $r = N$ holomorphic functions (i.e., the N components of the mapping φ). So, for the projection $\Gamma_{\varphi} \to Y$, which is equivalent to our mapping φ, the estimate in Proposition 4.12 gets converted to the estimate that we are proving. $\square$

Corollary 4.16. *Let* Y *be a locally irreducible (and connected) space, and let* X *be a pure-dimensional space defined in* $Y \times \Omega$ *locally by* $r = \operatorname{codim}_{Y \times \Omega} X$ *holomorphic functions, where* Ω *is a space of pure dimension* k*. Then for* $\varphi : X \to Y$*, defined as the projection, we have* $\phi_{\mathrm{vert}}(\varphi) = \phi_{\mathrm{s}}(\varphi)$*.*

Proof. We can assume that φ has no vertical components, by Remark 4.14. In the context of Proposition 4.12, set $\Upsilon = Y$, and $n \coloneqq N = \dim Y$. Set $m \coloneqq \dim X$. Then, by the Dimension Formula, we have $\lambda = m - n$, and by assumption we have $k - r = m - n$. Now, compare the definition of ϕ_{-} and Remark 4.1, to conclude that $\phi_{-}(\varphi) = \phi_{\mathrm{s}}(\varphi)$. But $\phi_{-}(\varphi) \leq \phi_{\mathrm{vert}}(\varphi) \leq \phi_{\mathrm{s}}(\varphi)$, hence $\phi_{-}(\varphi) = \phi_{\mathrm{vert}}(\varphi) = \phi_{\mathrm{s}}(\varphi)$. $\square$

By means of our lower bound, we are now able to compute the verticality index for the mysterious examples we had before.

Example 4.17. Consider $\varphi : X \to Y$ from Example 4.10. We justified that X is of pure dimension $3 = \dim Y$, and φ being generically one-to-one, we get that φ has no vertical components. In the notation of Proposition 4.12, set $\Upsilon = Y$, $N = 3$, $\Omega = \mathbb{C}$, and $k = 1$, then using the fbd-partition we obtained before, calculate

$$\phi_{-}(\varphi) = \left\lceil \frac{3 - 0 - 1}{1 - (1 - 2)} \right\rceil = 1.$$

By Proposition 4.12, $\phi_{\mathrm{vert}}(\varphi) \geq \phi_{-}(\varphi) = 1$. But $\phi_{\mathrm{vert}}(\varphi) \leq 1$ (since φ is generically one-to-one). So $\phi_{\mathrm{vert}}(\varphi) = 1$. $\triangle$

Example 4.18. Consider $\varphi : X \to Y$ from Example 4.11. We justified that X is of pure dimension $3 = \dim Y$, and φ being generically finite, it follows that φ cannot have vertical components. We computed that $\phi_{\mathrm{s}}(\varphi) = 2$. Now, since X is defined in $Y \times \mathbb{C}$ by $1 = \operatorname{codim}_{Y\times\mathbb{C}} X$ equation, Corollary 4.16 implies that $\phi_{\mathrm{vert}}(\varphi) = 2$. $\triangle$

Bibliography

[A1] Adamus, J.; *Natural bound in Kwieciński's criterion for flatness*, Proc. Amer. Math. Soc. 130 (2002), no. 11, 3165–3170.

[A2] Adamus, J.; "Vertical components in fibre powers of analytic mappings," PhD thesis, University of Toronto (Canada), 2003, 58 pp.

[ABM] Adamus, J.; Bierstone, E.; Milman P. D.; *Geometric Auslander criterion for flatness*, Amer. J. Math. 135 (2013), no. 1, 125–142.

[AS1] Adamus, J.; Seyedinejad, H.; *Flatness testing over singular bases*, Ann. Polon. Math. 107 (2013), no. 1, 87–96.

[AS2] Adamus, J.; Seyedinejad, H.; *A fast flatness testing algorithm in characteristic zero*, submitted (2012), arXiv:1211.2834.

[AI] Avramov, L. L.; Iyengar, S. B.; *Detecting flatness over smooth bases*, J. Algebraic Geom. 22 (2013), no. 1, 35–47.

[Au] Auslander, M.; *Modules over unramified regular local rings*, Illinois J. Math. 5 (1961), 631–647.

[BM1] Bierstone, E.; Milman, P. D.; *The local geometry of analytic mappings*, Dottorato di Ricerca in Matematica, ETS Editrice, Pisa, 1988, iv+77 pp.

[BM2] Bierstone, E.; Milman, P. D.; *Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant*, Invent. Math. 128 (1997), no. 2, 207–302.

[D] Douady, A.; *Le problème des modules pour les sous-espaces analytiques compacts d'un espace analytique donné* (French), Ann. Inst. Fourier (Grenoble) 16 (1966) fasc. 1, 1–95.

[F] Fischer, G.; "Complex analytic geometry," Lecture Notes in Mathematics, Vol. 538, Springer-Verlag, Berlin-New York, 1976, vii+201 pp.

[GK] Galligo, A.; Kwieciński, M.; *Flatness and fibred powers over smooth varieties*, J. Algebra 232 (2000), no. 1, 48–63.

[GPR] Grauert, H.; Peternell, Th.; Remmert, R.; eds., "Several complex variables VII: sheaf-theoretical methods in complex analysis," Springer-Verlag, Berlin, New York, 1994, 369 pp.

[GR] Grauert, H.; Remmert, R.; "Analytische Stellenalgebren" (German), Band 176, Springer-Verlag, Berlin-New York, 1971, ix+240 pp.

[GrPf] Greuel, G.-M.; Pfister, G.; "A Singular introduction to commutative algebra," 2nd edition, Springer, Berlin, 2008, xx+689 pp.

[H] Hironaka, H; *Stratification and flatness*, Real and complex singularities (Proc. Ninth Nordic Summer School/NAVF Sympos. Math., Oslo, 1976), pp. 199–265.

[Ha1] Hauser, H.; *The Hironaka theorem on resolution of singularities (or: A proof we always wanted to understand)*, Bull. Amer. Math. Soc. (N.S.) 40 (2003), no. 3, 323–403.

[Ha2] Hauser, H.; *Seven short stories on blowups and resolutions*, Proceedings of Gökova Geometry-Topology Conference 2005, 1–48, Gökova Geometry/Topology Conference (GGT), Gökova, 2006.

[HW] Huneke, C.; Wiegand, R.; *Tensor products of modules and the rigidity of* Tor, Math. Ann. 299 (1994), no. 3, 449–476.

[K] Kwieciński, M.; *Flatness and fibred powers*, Manuscripta Math. 97 (1998), no. 2, 163–173.

[KT] Kwieciński, M.; Tworzewski, P.; *Fibres of analytic maps*, Bull. Polish Acad. Sci. Math. 47 (1999), no. 3, 245–255.

[L] Łojasiewicz, S.; "Introduction to complex analytic geometry," Birkhäuser Verlag, Basel, 1991, xiv+523 pp.

[La] Lang, S.; "Algebra," revised third edition, Graduate Texts in Mathematics 211, Springer-Verlag, New York, 2002, xvi+914 pp.

[ST] Siu, Y.-T.; Trautmann, G.; "Gap-sheaves and extension of coherent analytic subsheaves," Lecture Notes in Mathematics, Vol. 172 Springer-Verlag, Berlin-New York, 1971, v+172 pp.

[W] Whitney, H.; *Tangents to an analytic variety*, Ann. of Math. (2) 81 (1965), 496–549.

Appendix

Derived results in the algebraic setting

To further justify the regularity criteria of this thesis, we gathered here a sample list of analogous results in the algebraic category. The main interest in the following theorems comes from the fact that they allow one to verify flatness with a help of simple computer algebra algorithms. The results below are derived from their local analytic analogues according to the following general scheme: One uses standard faithful flatness arguments to transfer the statements from the analytic to the formal and then to the algebraic category. The space of coefficients is then generalized from $\mathbb{C}$ to an arbitrary zero-characteristic field $\Bbbk$ by means of the Tarski-Lefschetz Principle. We refer to [AS1, AS2] for details.

Theorem A.1 ([AS1, Theorem 4.1], cf. Theorem 2.5)**.** *Let $\Bbbk$ be a field of characteristic zero, and let R be an n-dimensional $\Bbbk$-algebra of finite type which is geometrically unibranch. Let A denote an R-algebra essentially of finite type, and let F denote a finitely generated A-module. Let S be any n-dimensional, regular R-algebra of finite type such that the induced morphism* $\mathrm{Spec} S \to \mathrm{Spec} R$ *is dominant. Then, F is R-flat if and only if the tensor* *product*

$$\underbrace{F \otimes_R \cdots \otimes_R F}_{n \text{ times}} \otimes_R S$$

is a torsion-free R-module (equivalently, a torsion-free S-module).

The following is an immediate corollary to Theorem A.1, which is a neat generalization of the classical flatness criterion of Auslander for a certain class of modules.

Corollary A.2 ([AS1, Corrolary 4.2])**.** *Let $\Bbbk$ be a field of characteristic zero, and let R be an n-dimensional $\Bbbk$-algebra of finite type which is geometrically unibranch. Let A be a regular, n-dimensional R-algebra of finite type such that the induced morphism* $\mathrm{Spec}A \to \mathrm{Spec}R$ *is dominant. Then A is R-flat if and only if $\underbrace{A \otimes_R \cdots \otimes_R A}_{n+1 \text{ times}}$ is a torsion-free R-module.*

Theorem A.3 ([AS2, Theorem 1.10], cf. Corollary 3.8)**.** *Let $\mathbb{K}$ denote $\mathbb{R}$ or $\mathbb{C}$. Set $R = \mathbb{K}[y_1, \ldots, y_n]/I$, where I is a proper ideal in $\mathbb{K}[y_1, \ldots, y_n]$. Let $A = R[x_1, \ldots, x_m]/Q$ be an R-algebra of finite type, and let F be a finitely generated A-module. Set $S = \mathbb{K}[z_1, \ldots, z_n]$, and let $\tau : \mathbb{K}[y_1, \ldots, y_n] \to S$ be the morphism defined as*

$$\tau(y_1) = z_1 z_n, \quad \ldots, \quad \tau(y_{n-1}) = z_{n-1} z_n, \quad \tau(y_n) = z_n.$$

Let I_{st} be the strict transform ideal of I under τ, and suppose that I_{st} is a proper ideal in S. Then, $F_{(x,y)}$ is a flat $R_{(y)}$-module if y_n is not a zero-divisor in $F_{(x,y)} \otimes_{R_{(y)}} (S/I_{\mathrm{st}})_{(z)}$.

Theorem A.4 ([AS2, Theorem 1.1], cf. Corollary 3.9)**.** *Let $\Bbbk$ be a field of characteristic zero, and set $R = \Bbbk[y_1, \ldots, y_n]$. Let F be a module finitely generated over an R-algebra of finite type, say $F = R[x]^q/M$, where $x = (x_1, \ldots, x_m)$ and M is a submodule of $R[x]^q$. Let $\tilde{M}$ be the module obtained from M by substituting $y_j y_n$ for y_j, $j = 1, \ldots, n-1$. Then $F_{(x,y)}$ is a flat $R_{(y)}$-module if and only if $\tilde{M} = \tilde{M} :_{R[x]^q} y_n$.*

Curriculum Vitae

Name:	Seyed Mohammad Hadi, Seyedinejad
Post-Secondary Education and Degrees:	Sharif University of Technology Tehran, Iran 2001–2009, BSc and MSc
	University of Western Ontario London, Canada 2009–2013, PhD
Honours and Awards:	Faculty of Science Graduate Student Teaching Award 2013
	Faculty of Science Scholarship 2009–2013
	Western Graduate Research Scholarship 2009–2013
Related Work Experience:	Teaching Assistant The University of Western Ontario 2009 - 2013

Publications related to PhD thesis:

- *A fast flatness testing algorithm in characteristic zero* (with J. Adamus), submitted (2012), arXiv:1211.2834.
- *Flatness testing over singular bases* (with J. Adamus), Ann. Polon. Math., **107** (2013), no. 1, 87–96.