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Analysis of Re-advanceable Mortgages

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Graduate Program in Applied Mathematics

A thesis submitted in partial fulfillment of the requirements for the degree in Doctor of Philosophy

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ANALYSIS OF RE-ADVANCEABLE MORTGAGES
(Thesis format: Monograph)

by

Almas Naseem

Graduate Program in Applied Mathematics

A thesis submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy

The School of Graduate and Postdoctoral Studies
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London, Ontario, Canada

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Abstract

According to Canadian tax law the interest payments on loans used for investment purposes are tax deductible while interest on personal mortgage loans is not. One way of transforming from non-tax deductible to tax deductible interest expenses is to borrow against home equity to make investments. This can be achieved through a re-advanceable mortgage and has been promoted by personal financial planners as a way of significantly decreasing the time required to pay off a mortgage and the associated total interest cost. However, the notion of risk associated with the investment holdings is not emphasized. Using simulation we study the risk associated with the re-advanceable mortgage strategy to provide a better description of the mortgagor’s position. We assume that the mortgagor invests the entire proceeds from the line of credit into a single risky asset (e.g., stock or mutual fund) whose evolution is described by geometric Brownian motion (GBM). We find that this strategy reduces both the average mortgage payoff time and the total interest cost. However, there is considerable variation in the payoff times with a significant probability of a payoff time exceeding the mortgage term. Furthermore the higher the marginal tax rate, the more the average payoff time and interest cost are reduced implying that this strategy is more beneficial to high-wage earners. Using a simple stochastic model for job status we also investigate the effect of job loss on the payoff time distribution. In the event of job loss, the investment portfolio protects the homeowner from default as part of the investment portfolio can be sold to fund mortgage payments.

Variable rate mortgages are also considered in our study. The mortgage rate models we consider are the mean reverting rate model without a diffusion term and the CIR process. There is not a big difference compared with the fixed-rate mortgage in the average payoff time for the mean reverting rate model with no diffusion. However, once the diffusion term is included, the average payoff time and the volatility of payoff time increase according to the volatility of CIR process. We also incorporate a housing price model in the re-advanceable scheme and see that it can decrease the average payoff time and average total cost, however, this comes at the cost of increased standard deviations of the payoff time and total cost distributions.

Results of this study could be of interest to policy makers as they continue to adjust mortgage rules to induce desired behavior, such as reducing personal debt burdens. The modelling framework provided here can be adjusted to analyze the effect of potential policy actions on homeowners who borrow against home equity to invest.

Keywords: Re-advanceable mortgage, fixed-rate mortgage, variable-rate mortgage, CIR process, simulation, geometric Brownian motion, stochastic differential equation
Dedication

To my parents and husband for their endless support and unconditional love.
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Nomenclature

\[ \Delta t \] time step
\[ \gamma(\cdot, \cdot) \] the lower incomplete gamma function
\[ \kappa \] mean reversion speed of \( r_t \) process
\[ \mathcal{I}(\cdot) \] an indicator function which is one if event \( \cdot \) occurs and zero otherwise
\[ \mathbb{E}[\cdot] \] expected value of \( \cdot \)
\[ \text{Var}[\cdot] \] variance of \( \cdot \)
\[ \mu \] growth rate of the stock price process
\[ \phi \] growth rate of housing price process
\[ \rho_{\cdot,-} \] instantaneous correlation between \( \cdot \) and -
\[ \sigma \] volatility of the stock price process
\[ \sigma_H \] volatility of housing price process
\[ \sigma_r \] volatility of interest rate process
\[ \tau \] mortgage payoff time
\[ \tau_x \] \( x \)-years worth of mortgage payments in the investment portfolio
\[ \theta \] mean reversion level of interest rate process
\[ B_t \] time-\( t \) value of outstanding original mortgage principal balance
\[ c \] tax rate
\[ C_t \] time-\( t \) value of the line of credit balance
\[ \text{cov}(\cdot, -) \] covariance between \( \cdot \) and -
\[ H_t \] time-\( t \) housing price
\[ i \] line of credit rate
\( I_m \)  total interest cost of a traditional mortgage
\( I_t \)  time-\( t \) value of investment capital
\( I_r \)  total cost of the re-advanceable mortgage scheme
\( L \)  original mortgage principal
\( m \)  amortization term
\( M_t \)  time-\( t \) running maximum housing price
\( N_t \)  number of units of stock bought at time \( t \)
\( P \)  mortgage payment rate
\( p \)  probability of a homeowner to maintain the status of employed
\( Pr(\cdot) \)  probability that the event \( \cdot \) occurs
\( q \)  probability of a homeowner to maintain the status of unemployed
\( r \)  annual nominal percentage mortgage rate
\( r_t \)  time-\( t \) variable mortgage rate
\( S_t \)  time-\( t \) stock price
\( S D[\cdot] \)  standard deviation of \( \cdot \)
\( t \)  time
\( V_t \)  time-\( t \) value of the risky investment portfolio
\( W \)  Wiener process
Chapter 1

Thesis Roadmap

This thesis has been organized in three main chapters and an additional chapter of conclusions and future work extensions of this thesis. Chapter 2 provides an introduction to the Smith Manoeuvre which is a re-advanceable mortgage scheme that enables homeowners in Canada to transform their non-tax deductible interest expenses on home mortgages into tax deductible. A re-advanceable mortgage scheme is a traditional mortgage coupled with a home equity line of credit. The principal associated with the line of credit increases at the time of each mortgage payment and by an amount equivalent to the principal component of each payment. We assume that the entire proceeds from the home equity line of credit are invested in a single risky asset. According to Canadian tax law the interest cost on the borrowings which are used for investment is tax deductible. This effectively transforms the non-tax deductible interest payments into tax deductible payments since the whole mortgage principal is transferred into the line of credit principal and is used for investment. We derive a continuous time model for the investment portfolio value. By simulating sample paths of the investment portfolio we generate corresponding values of the mortgage payoff time and the total cost of the strategy. We compare the mortgage payoff time and total cost distributions with the amortization term and the total interest cost, respectively, of the traditional mortgage for different sets of parameter values. Based on this comparison we discuss if the re-advanceable mortgage scheme is effective in reducing the mortgage payoff time and the total interest cost. In our study we also incorporated the scenario of the homeowner who losses his (her) job. Whether the re-advanceable mortgage scheme protects the homeowner from defaulting on the mortgage or not, it has been discussed in Chapter 2. See [22] for the published version of the work presented in Chapter 2.

The ability of the re-advanceable mortgage to improve a homeowner’s position depends on a number of parameters. A list of parameters that affect the ability of the re-advanceable mortgage scheme includes the time, the mortgage rate, the line of credit interest rate, growth rate and volatility of the risky asset. In Chapter 3 we perform a sensitivity analysis of the investment portfolio value with respect to these parameters by investigating the rate of change in investment portfolio mean and standard deviation. The chapter is divided in two sections. The first section discusses the sensitivity of investment capital to different values of mortgage rates and line of credit rates. The second section discusses the derivatives of the investment portfolio value mean and standard deviation with respect to the parameters mentioned above.

Chapters 2 and 3 focus on the problem of re-advanceable mortgage scheme with fixed mortgage and line of credit rates. In Chapter 4 we extend our study to a variable rate mortgage. Two
models for the mortgage rate are used. First we consider a deterministic mean reverting rate model. We assume that the line of credit also follows this model. Secondly we consider the Cox-Ingersoll-Ross model and we assume that the line of credit rate also follows this model. In Chapter 4 we also incorporate a housing price model in the re-advanceable mortgage scheme, considering all three scenarios of the mortgage rate; fixed rate, deterministic mean reverting rate and stochastic rate. Without the incorporation of the housing price model the total debt outstanding for a homeowner was fixed at the original mortgage principal. When we incorporate the housing price model, the total borrowings change according to movements in housing price and this effectively increases the line of credit borrowing limit. We assume that the homeowner can re-borrow 80% of the increase in the housing price in the line of credit and there is no new borrowing if the house price falls. At any time the total debt outstanding is equal to the sum of the original mortgage principal and 80% of the increase in the housing price at that time. The thesis concludes with a summary and brief discussion of future work.
Chapter 2

Analysis of Re-advanceable Mortgage

Re-advanceable mortgage strategy is a way of transforming non-tax deductible interest expenses on a house mortgage into a tax deductible one. This chapter provides the introduction of the strategy and its mechanism. Financial planners advertise the strategy in market in a way that poses an impression to the homeowners that they would be able to pay off the mortgage earlier if they implement this strategy. Another attractive point being advertised is the reduction in the total cost of the mortgage while implementing the re-advanceable mortgage scheme. The sort of information made available to the homeowners while selling this strategy is not complete in the sense that it does not talk about the risk involved with this scheme. In fact some of the advantages of the strategy are also not mentioned in the promotional material available for the strategy. One of them is the benefit a homeowner can enjoy in the event of job loss with the re-advanceable mortgage strategy. In this chapter we assess the risk associated with this strategy and also investigate how the strategy behaves in the event of job loss.

2.1 Introduction

In order to understand a re-advanceable mortgage it is necessary to be familiar with a traditional mortgage. We start with some general features of a traditional mortgage and some basic definitions. A mortgage is a loan that uses a property as security to ensure that the debt is repaid. The borrower is referred to as the mortgagor, the lender as the mortgagee. Typically a mortgage is a loan used to purchase a house. Like any other loan, a mortgage payment is usually made up of principal and interest. Principal is the total amount of money a mortgagor borrows from the lender (after a down payment has been made). Interest or mortgage rate is the cost of the loan.

2.1.1 General Features of a Traditional Home Mortgage

The mortgage term is the period of time over which specified conditions of a mortgage are in force. In Canada mortgage terms range from six months to 10 years. The amortization period is the number of years it would take to repay the entire mortgage amount based on a set of fixed payments. The longer the amortization, the more interest is paid over the life of the mortgage. Therefore, when choosing the amortization period, careful planning should be done to meet
one’s cash flow needs. In Canada the shortest amortization period is usually 5 years, and the longest is 30 years. Currently very few lenders will agree to an amortization longer than 25 years. It is to the homeowner’s advantage to choose the shortest amortization that one can afford as it can save thousands of dollars in interest on a mortgage.

Mortgage interest rates are of two types; fixed rate and variable rate. A fixed-rate mortgage is one for which the interest rate is constant for the mortgage term. Typically periodic mortgage payments are constant throughout the mortgage term. These fixed mortgage payments can aid in a mortgagor’s budgeting. With a fixed rate mortgage it is known in advance how much principal remains at the end of the term. In low rate climates many mortgagors opt for longer term, fixed-rate mortgages for protection against upward movements in future rates.

A variable-rate or adjustable-rate mortgage is one for which the mortgage rate fluctuates according to market conditions. The rate is based on a mortgagee’s prime rate and is adjusted to reflect current rates. Typically, the mortgage payment remains constant, but the ratio between the principal and interest components of the payment fluctuates. When interest rates are low less of the mortgage payment goes to interest and more goes to principal. If rates are high more goes to interest and less to principal. If rates rise enough, the original payment may not cover both the interest and principal components and the homeowner will be asked to increase the mortgage payment. For this reason variable-rate mortgages are considered riskier than fixed-rate mortgages.

Mortgage products come with different terms and conditions. A mortgage can be open, closed or convertible. An open mortgage can be prepaid, renegotiated or refinanced without any penalty. An open mortgage allows the flexibility to pay off some or all of the mortgage balance at any time without penalty. Open mortgages are available in shorter terms, 6 months or 1 year only, and the interest rate is higher than that for closed mortgages. An open mortgage can be an attractive option for those expecting a large sum of money or with the ability to make extra payments. The downside is that interest rates are higher on open mortgages because of the prepayment flexibility.

On the other hand, a closed mortgage is one which cannot be prepaid, renegotiated or refinanced before maturity, except according to its terms. A closed mortgage offers the security of fixed payment for terms from 6 months to 10 years. Interest rates for closed mortgages are lower than rates for open mortgages. Closed mortgages typically allow some prepayment rights. For example, it is common to have a right to pay as much as 20% of the original principal annually. If one wanted to pay off the full mortgage or refinance prior to maturity, a penalty would be charged to break that mortgage. The penalty is usually 3 months interest or some interest rate differential. The interest rate differential is an interest rate that equals the difference between the original mortgage interest rate and the interest rate that the mortgagee can charge today when re-lending the funds for the remaining term of the mortgage. A convertible closed mortgage is similar to a closed mortgage but typically has a shorter mortgage term. Convertible mortgages can be converted to a longer term closed mortgage at then prevailing rate at any time without penalty.

In addition to the money required to cover the mortgage, obtaining a mortgage often requires a substantial amount of money for the down payment and closing costs. The down payment is the lump sum the mortgagor pays up front for the property and it reduces the amount of money that has to be financed. The homeowner can put as much money down as he/she wants, or the homeowner can sometimes pay as little as 3 to 5 percent of the purchase price. The
more money one puts down, the less one has to finance and the lower the financing costs. If
the down payment is equal to or greater than 20% of the price of the dwelling the mortgage is
called a conventional mortgage. Down payments of less than 20% of the purchase price result
in high-ratio mortgages which, in Canada, require the purchase of private mortgage insurance
(PMI). This insurance provides protection to the mortgagee in case the mortgagor defaults and
is sold by Canada Mortgage and Housing Corporation or Genworth Capital. The PMI premium
is also added to the mortgage principal, resulting in an additional cost to the mortgagor.

2.1.2 Smith Manoeuvre and the Re-advanceable Mortgage.

History

Canadian homeowners did not always have the opportunity to convert their non-tax-deductible
mortgages interest expenses into tax-deductible ones. The first case with this regard originated
in 1988. John R. Singleton [30], a partner in a law firm, had $300,000 in his business capital
account. In October, 1988, he withdrew this money and used it to purchase a house. On the
same day, he then mortgaged the house by borrowing $298,750 from a bank and deposited
the money back into his partnership account, along with $1250 of his own money. This way
Singleton managed to have a new house, a mortgage on a new house and refinanced his law
firm equity with borrowed money.

Singleton paid $3,688 mortgage interest in 1988 and $27,415 mortgage interest in 1989. He
deducted the interest on his tax return for those two years. His position was that the borrowed
funds, not the withdrawn funds, were used for investment purposes. Interest on loans used
for investment purposes are tax deductible according to Canadian tax law while interest on
mortgages is not. The Canada Revenue Agency (CRA) reassessed the case and denied the
interest deduction on the grounds that the borrowed money was used to finance the purchase
of the house and not as a business investment. Singleton appealed to the Tax Court of Canada.
The court began its analysis of whether the interest was deductible with an examination of
what the borrowed money was used for. The case was dismissed on October 27, 1988. The
conclusion of the case was: “On any realistic view of the matter it could not be said that the
money was used for the purpose of making a contribution of capital to the partnership. The
fundamental purpose was the purchase of a house and this purpose cannot be altered by the
shuffle of cheques that took place on October 27, 1988.”

Singleton then appealed to the Federal Court of Appeal which allowed his appeal by a ma-
majority decision. In this case, the issue was whether the borrowed funds were used as a business
investment, or were used to finance a home purchase? Put another way, should the same day
transactions be viewed as a series of connected activities, or each a distinct transaction? The
Federal Court of Appeal concluded that the transactions should be treated independently in
order to reflect the reality of what occurred. Mr. Singleton had his own funds in his capital
account which he withdrew and used to purchase a house and he used the borrowed funds
to replenish his capital account. Therefore, the borrowed funds were used for the purpose of
refinancing his capital investment in the law firm.

The CRA appealed to the Supreme Court of Canada in hope of a friendlier decision. The
Supreme Court of Canada dismissed this appeal and ruled that Singleton made distinct separate
transactions. He was entitled to deduct the interest payments from his income as there was a
direct link between the borrowed money and its eligible use. Regardless of the sequence of the transactions, he used the borrowed funds for the purpose of refinancing his partnership capital account with debt. The Court stated: “It is an error to treat this as one transaction — the transactions must be viewed independently.” Viewing the transactions as one simultaneous transaction suggests that the direct use of the funds was to purchase the house. It ignores what Singleton actually did — he used the borrowed funds to replace the funds required for his capital account at the firm.

Taxpayers are entitled to structure their transactions in a manner that reduces taxes. The fact that the structures may be complex arrangements does not remove the right to do so. The Supreme Court of Canada ruled that Singleton could write off his mortgage interest as he structured his affairs to reduce his tax burden. The Supreme Court of Canada paved the way for interest expenses to be tax deductible if the borrowed money is used for investment purposes even if a home is pledged as collateral.

**Mechanism of Re-advanceable Mortgage**

American homeowners have long been able to use their home mortgage interest costs as an income tax deductible expense [1]. This is not the case for Canadians having a traditional mortgage. According to Canadian tax law the interest on loans used for investment purposes is tax deductible [3] while interest on mortgage loans is not. One way of transforming from non-tax deductible to tax deductible interest expenses is to borrow against home equity to make investments. This can be effectively achieved through a re-advanceable mortgage and this strategy is known in the industry as the Smith Manoeuvre, so-named after financial planner Fraser Smith, who initiated the strategy.

A re-advanceable mortgage is a traditional mortgage coupled with a home equity line of credit. The principal associated with the line of credit increases at the time of each mortgage payment and by an amount equivalent to the principal component of each payment. In other words, each time a payment is made, the mortgagor re-borrows an amount equal to the principal component of the payment through an increase in the available home equity line of credit. Over time, the principal associated with the home equity line of credit will increase to an amount equal to the original mortgage principal. Thus the homeowner does not pay down the total amount borrowed as the principal shifts from the traditional mortgage to the line of credit.

Home equity line of credit is often used to finance major expenses such as home repairs, medical bills or college education. Note that the line of credit used for personal expenses would not be tax deductible according to Canadian tax law. The homeowner can also use the line of credit to invest in income producing entities like stocks or rental property. Since interest paid on borrowing for investment purposes is tax-deductible, this effectively transforms non-tax deductible interest payments into tax deductible ones. This is the basis of the Smith Manoeuvre [24] and the homeowner benefits in at least two ways:

- interest payments on the line of credit are tax-deductible; and
- the investment portfolio has a higher average rate of return than the interest rate paid on the home equity line of credit.

The tax savings can be applied to increase the payments made on the mortgage, hence accelerating the benefit as the principal is being paid down faster. The marginal tax rate of the
2.1. Introduction

A homeowner affects the amount of tax savings. For example, if a homeowner paid $6,000 in interest payments for a year and he/she has marginal tax rate of 40%, then the tax refund will be $2,400. If the marginal tax rate is 50%, then the refund is $3,000. These funds can be applied to the original mortgage principal which creates further home equity and increases the amount borrowed on the line of credit.

One concern a mortgagor might have is how to make payments on both the mortgage and the home equity line of credit. The mortgagor is responsible for mortgage payments along with the interest payments on the home equity line of credit while implementing the Smith Manoeuvre. For example, if one gets a $200,000 home equity line of credit at 6%, that’s an extra $1000 per month on top of the existing monthly mortgage payment to support this strategy. Capitalizing the interest is a way in which a homeowner will not have to make the extra interest payments out of pocket while the primary mortgage exists. Capitalizing the interest simply means withdrawing the monthly interest due from the home equity line of credit account and redepositing the amount as a payment. In other words, capitalizing the interest means to fund the investment loan with the loan itself.

Figure 2.1: Promotional material for the Smith Manoeuvre (see [25]).
The Smith Manoeuvre has been promoted by personal financial planners as a way of significantly decreasing the time required to pay off a mortgage and the associated total interest cost. The promotional materials (e.g., Figure [2.1]) [25] for this strategy typically make assumptions about the interest rate paid on the borrowed money and the rate of return earned by the investment. What is missing from this promotional material, however, is the notion of risk associated with the investment holdings (typically a stock or mutual fund). Although the investment has a higher average rate of return than the interest paid on the home equity line of credit there is risk associated with unknown future values (e.g., portfolio volatility). What is also missing is the protective benefit provided by the investment portfolio that protects the homeowner from default in the event of job loss.

2.2 Mathematical Formulation of Re-advanceable Mortgage

2.2.1 Mortgage Repayment

Mortgages are typically paid off in incremental payments that gradually chip away at the principal of the loan. The mortgagor decides between monthly, bi-weekly or weekly payments. In this section we derive a formula for monthly payments. The same formula can be used for bi-weekly and weekly payment frequencies by simply multiplying with an appropriate factor.

A monthly mortgage payment, \( P \), depends on the original principal, \( L \), interest rate, \( R \), and the amortization term, \( m \), where \( m \) is measured in months. The interest rate \( R \) is an annual percentage nominal rate, therefore the monthly rate \( R \approx \frac{R}{12} \). Each month the outstanding principal \( L \) increases by a factor of \( 1 + r \) and with no repayment, after \( m \) months the value of the principal increases to \( L(1 + r)^m \).

Now we determine the monthly payment, \( P \), required to pay off the mortgage in \( m \) months. Assume that the value of each payment earns interest at the same monthly rate as the principal. Then, after \( m \) months the total value of the payments is \( P[(1 + r) + (1 + r)^2 + ... + (1 + r)^{m-1}] \) which equals

\[
P((1 + r)^m - 1) = \frac{P((1 + r)^m - 1)}{r}.
\]

Equating the total accumulated value of the payments to the accumulated value of the debt gives the monthly mortgage payment

\[
P = \frac{Lr(1 + r)^m}{(1 + r)^m - 1}.
\]

(2.1)

For example, consider a 25-year, 6\% mortgage with principal of $300,000. The monthly payment is:

\[
P = \frac{300000 \times 0.005 \times (1 + 0.005)^{300}}{((1 + 0.005)^{300} - 1)} = \$1932.91.
\]

During the amortization term, the interest and principal portions of the monthly payments decrease and increase, respectively.
2.2.2 Home Equity Line of Credit and Risky Investment Funding

Let \( B_t \) be the principal outstanding after \( t \) mortgage payments are made. Consider the time interval \([0, m]\), which has been discretised into \( m \) subintervals with monthly time steps. At time 0 we have \( B_0 = L \). At the end of the first month, the mortgagor makes the first payment with a portion going to principal repayment and the rest to interest. The remaining debt balance is the debt balance at the time 0 plus the accrued interest on \( B_0 \) less the payment amount \( P \), i.e.,
\[
B_1 = L + rL - P = L(1 + r) - P.
\]
Similarly, at the end of the second month the total remaining debt balance is
\[
B_2 = B_1 + rB_1 - P = (1 + r)B_1 - P = (1 + r)[L(1 + r) - P] - P = L(1 + r)^2 - P(1 + r) - P.
\]
Repeating the above process for \( t \) months we get
\[
B_t = L(1 + r)^t - P(1 + r)^{t-1} - P(1 + r)^{t-2} - \cdots - P(1 + r) - P
\]
where \( t \leq m \). Summing the geometric series gives
\[
B_t = L(1 + r)^t - P\frac{(1 + r)^t - 1}{r},
\]
which, using Equation 2.1, simplifies to
\[
B_t = \begin{cases} 
L \times \left[ \frac{1 - ((1 + r)^m - (1 + r)^t)}{(1 + r)^m - 1}\right], & \text{if } 0 \leq t \leq m, \\
0, & \text{if } t > m. 
\end{cases} \tag{2.2}
\]
After \( m \) years the outstanding principal balance is zero since the original mortgage is paid off.

As the mortgage is paid the principal of the home equity line of credit increases by the principal portion of the mortgage payment. Thus after \( t \) payments the line of credit balance, \( C_t \), is
\[
C_t = L - B_t
\]
\[
= L - \left[ 1 - \frac{[(1 + r)^m - (1 + r)^t]}{(1 + r)^m - 1}\right]
\]
\[
= \begin{cases} 
L \left[ \frac{(1 + r)^{t-1}}{(1 + r)^m - 1}\right], & \text{if } 0 \leq t \leq m, \\
L, & \text{if } t > m. 
\end{cases} \tag{2.3}
\]
Note that Equation 2.3 yields \( L \) at the end of the amortization term \( m \), illustrating that the original mortgage principal is not repaid. Figure 2.2 shows the growth of the ratio of home equity line of credit principal to the original principal of mortgage. After 25 years, the whole mortgage amount will be transformed to the home equity line of credit with tax-deductible interest when used for investment. At time 0, the ratio of home equity line of credit principal \( C_0 = 0 \) to the original mortgage principal \( L \) is 0. As time marches along this ratio grows and at time \( m \) it grows to 1.
The homeowner invests the line of credit borrowings to get a tax refund on the interest cost and to earn a potentially higher return than the interest paid on borrowings. Consider the time interval \((t_{j-1}, t_j]\) and \(t_j = j\Delta t, j = 0, 1, 2, ..., m\) where \(\Delta t\) is the time step. Assume that the annual interest rate on line of credit is \(i\) and the annual tax rate is \(c\). After the \(j^\text{th}\) payment the new money available from the increased line of credit is the difference between the new principal \(C_{t_j}\) and the previous principal \(C_{t_{j-1}}\). From this new money, we deduct the capitalized time-
\(t_j\) monthly interest cost, \(i \times C_{t_{j-1}} \times \Delta t\), and add the (monthly) tax refund, \(i \times c \times C_{t_{j-1}} \times \Delta t\). Let \(I_{t_j}\) be the investment capital at time \(t_j\). Therefore, after the \(j^\text{th}\) payment, the amount of money available for new investment is

\[
\Delta I_{t_j} = \Delta C_{t_j} - i(1-c)C_{t_{j-1}}\Delta t + P\mathbb{1}_{t_j>m}\Delta t
\]

where \(\Delta t = \frac{1}{12}, t_j = j\Delta t, j = 0, 1, 2, ..., m,\) and \(I_0 = 0\). This framework makes the simplifying assumption that the tax rebate from the interest payments is realized contemporaneously with the payment. Also, note that \(\mathbb{1}_{t>m}\) is an indicator function that is zero if \(t < m\) and is one for all \(t > m\). After the amortization term the homeowner keeps making the mortgage payments which contributes towards the new money available for investment. Therefore,

\[
\Delta I_{t_j} = \begin{cases} 
\Delta C_{t_j} - i(1-c)C_{t_{j-1}}\Delta t, & \text{if } 0 \leq t_j \leq m, \\
P\Delta t - i(1-c)L\Delta t, & \text{if } t_j > m.
\end{cases}
\]
2.2.3 Risky Investment Dynamics and Portfolio Value.

We assume that the homeowner invests the entire proceeds from the home equity line of credit into a single risky asset (e.g., stock or mutual fund) whose evolution is described by geometric Brownian motion (GBM). GBM is a continuous-time stochastic process and is widely used for modeling financial prices \cite{8, 11, 13}. Let \( S_t \) denote the time- \( t \) value of the risky asset which we call a stock. The evolution of the stock price is governed by the following stochastic differential equation,

\[
dS_t = \mu S_t dt + \sigma S_t dW_t, \tag{2.6}
\]

where \( W \) is a Wiener process, \( \mu \) is the drift and \( \sigma \) is the volatility. Note that all variables are defined on a standard filtered probability space, \((\Omega, \mathcal{F}, \mathbb{P})\), where \( \Omega \) is the set of all possible outcomes, \( \mathcal{F} \) is a \( \sigma \)-Algebra of subsets of \( \Omega \) and \( \mathbb{P} \) is the real-world probability measure. The filtration \( \{\mathcal{F}_t\}_{t \geq 0} \) satisfies the usual conditions\(^2\) and is generated by \( W \)\(^3\). Given the initial value \( S_0 \), the stochastic differential Equation 2.6 has solution

\[
S_t = S_0 e^{(\mu - \frac{\sigma^2}{2}) t + \sigma W_t}.
\]

Figure 2.3 is a plot of 15 sample paths of GBM with drift 8\% (\( \mu = 0.08 \)), volatility 20\% (\( \sigma = 0.2 \)), and \( S_0 = 1 \) over a time horizon of 75 years.

The homeowner invests all proceeds from the home equity line of credit into the stock. At each payment time, the number of units of stock bought is determined by dividing the money

\(^2\)In other words it is complete and the filtration is right-continuous.

\(^3\)In extended version of this basic model, the filtration is extended to include more information, such as job status, variable interest rate and housing price.
available for investment by the current stock price. At time \( t_j \) the new investment capital available in the home equity line of credit is given by Equation (2.5). The number of new units, \( \Delta N_{t_j} \), of stock bought at time \( t_j \) is given by

\[
\Delta N_{t_j} = \frac{\Delta t_{t_j}}{S_{t_j}}, \tag{2.7}
\]

and the time-\( t_j \) value of the risky investment portfolio is \( V_{t_j} = S_{t_j} \sum_{k=1}^{j} \Delta N_{t_k} \). The change in \( V_{t_j} \) over \( (t_{j-1}, t_j) \) is given by

\[
\Delta V_{t_j} = V_{t_j} - V_{t_{j-1}}
= S_{t_j} \sum_{k=1}^{j} \Delta N_{t_k} - S_{t_{j-1}} \sum_{k=1}^{j-1} \Delta N_{t_k}
= \Delta S_{t_j} \sum_{k=1}^{j} \Delta N_{t_k} + \Delta I_{t_j}, \tag{2.8}
\]

Using Equation (2.5) and the discretised form of Equation (2.6) Equation (2.8) becomes

\[
\Delta V_{t_j} = (\mu S_{t_j} \Delta t + \sigma S_{t_j} \Delta W_{t_j}) \sum_{k=1}^{j} \Delta N_{t_k} + \Delta C_{t_j} - i(1-c)C_{t_{j-1}} \Delta t + P_{\delta_{t_j}>m} \Delta t
= \begin{cases} 
(\mu V_{t_j} - \frac{\sigma(1-c)}{\Delta t} C_{t_{j-1}}) \Delta t + \sigma V_{t_j} \Delta W_{t_j} + \Delta C_{t_j}, & \text{if } 0 \leq t_j \leq m, \\
(\mu V_{t_j} - \frac{\sigma(1-c)}{\Delta t} L) \Delta t + \sigma V_{t_j} \Delta W_{t_j} + P \Delta t, & \text{if } t_j > m. 
\end{cases} \tag{2.9}
\]

Note that \( \frac{\sigma(1-c)}{\Delta t} \) is an annualized quantity since \( i \) and \( c \) are monthly and \( \Delta t = \frac{1}{12} \). To get the formula for the value of \( V_{t_j} \) take the summation of the above equation from \( k = 1 \) to \( k = j \).

\[
\sum_{k=1}^{j} \Delta V_{t_k} = \begin{cases} 
\sum_{k=1}^{j} \left( \mu V_{t_k} - \frac{\sigma(1-c)}{\Delta t} C_{t_{k-1}} \right) \Delta t + \sigma V_{t_k} \Delta W_{t_k} + \Delta C_{t_k}, & \text{if } 0 \leq t_j \leq m, \\
\sum_{k=1}^{j} \left( \mu V_{t_k} - \frac{\sigma(1-c)}{\Delta t} L \right) \Delta t + \sigma V_{t_k} \Delta W_{t_k} + P \Delta t, & \text{if } t_j > m. 
\end{cases} \tag{2.10}
\]

\[
\Rightarrow V_{t_j} = \begin{cases} 
\sum_{k=1}^{j} \left( \mu V_{t_k} - \frac{\sigma(1-c)}{\Delta t} C_{t_{k-1}} \right) \Delta t + \sigma \sum_{k=1}^{j} V_{t_k} \Delta W_{t_k} + \sum_{k=1}^{j} \Delta C_{t_k}, & \text{if } 0 \leq t_j \leq m, \\
\sum_{k=1}^{j} \left( \mu V_{t_k} - \frac{\sigma(1-c)}{\Delta t} L \right) \Delta t + \sigma \sum_{k=1}^{j} V_{t_k} \Delta W_{t_k} + \sum_{k=1}^{j} P \Delta t, & \text{if } t_j > m. 
\end{cases} \tag{2.11}
\]

### 2.2.4 Continuous-Time Formulation

In this section we derive a continuous-time model for the total value of the investment portfolio. We acknowledge that this is an approximation to actual mortgage payment and tax rebate frequencies, which are typically monthly and yearly, respectively. However, the approximation of a discrete-time with a continuous-time process is a well established tool in finance, allowing for the derivation of elegant results. We consider the same fixed-rate mortgage as in Section...
2.2. MATHEMATICAL FORMULATION OF RE-ADVANCEABLE MORTGAGE

2.2.1 with an original principal of \( L \) and amortization term of \( m \) years. Assume that \( r \) is the annualized continuously-compounded rate of interest on the mortgage and the original mortgage is paid continuously at rate \( P \) which is given by

\[
L = \int_0^m Pe^{-ru} \, du = \frac{P}{r} \left(1 - e^{-rm}\right),
\]

which implies that

\[
P = \frac{Le^{rm}}{e^{rm} - 1}.
\] (2.12)

The time-\( t \) value of the principal amount outstanding is given by

\[
B_t = e^{rt} \int_t^m Pe^{-ru} \, du = \frac{Pe^{rt}}{r} \left(e^{-rt} - e^{-rm}\right) = \begin{cases} \frac{t(e^{rm} - e^{rt})}{e^{rm} - 1}, & \text{if } 0 \leq t \leq m, \\ 0, & \text{if } t > m. \end{cases}
\] (2.13)

As the mortgage is paid the principal of the home equity line of credit increases by the principal portion of the mortgage payment. Thus the time-\( t \) line of credit balance, \( C_t \), is

\[
C_t = L - B_t = \begin{cases} Le^{rt - 1}, & \text{if } 0 \leq t \leq m, \\ L, & \text{if } t > m. \end{cases}
\] (2.14)

The rate of change of \( C_t \) is

\[
dC_t = -dB_t = \begin{cases} \frac{Lre^{rt}}{e^{rm} - 1} \, dt, & \text{if } 0 \leq t \leq m, \\ 0, & \text{if } t > m. \end{cases}
\] (2.15)

The interest on the line of credit and the tax rebate are paid continuously at the annualized rates of \( i \) and \( c \), respectively. Consider the time interval \((t - dt, t)\). At time \( t \), the increment in the line of credit balance is the difference between the new principal \( C_t \) and the previous principal \( C_{t-dt} \). From this new money, we deduct the capitalized interest cost, \( i \times C_{t-dt} \), and add the tax refund, \( i \times c \times C_{t-dt} \), over the time interval. If \( I_t \) is the time-\( t \) investment capital then the time-\( t \) amount of money available for new investment is

\[
I_t - I_{t-dt} = C_t - C_{t-dt} - i(1 - c)C_{t-dt} \, dt + Pdt \mathbb{1}_{t>m},
\]

where \( \mathbb{1}_{t>m} \) is an indicator function that is zero if \( t < m \) and is one for all \( t > m \). After the amortization term the homeowner keeps making periodic payments equivalent to mortgage payments in amount which contributes towards the new money available for investment.

As \( dt \to 0^+ \) we get

\[
dI_t = dC_t - i(1 - c)C_{t} \, dt + Pdt \mathbb{1}_{t>m} = \begin{cases} dC_t - i(1 - c)C_t \, dt, & \text{if } 0 \leq t \leq m, \\ Pdt - i(1 - c)Ldt, & \text{if } t > m. \end{cases}
\] (2.16)

Using Equations 2.12, 2.14 and 2.15 we get

\[
dI_t = \begin{cases} \frac{L}{e^{rm} - 1} \left[re^{rt} - i(1 - c)(e^{rt} - 1)\right] \, dt, & \text{if } 0 \leq t \leq m, \\ \frac{L}{e^{rm} - 1} \left[re^{rm} - i(1 - c)(e^{rm} - 1)\right] \, dt, & \text{if } t > m. \end{cases}
\] (2.17)
The time-\( t \) number of new units of stock bought with the time-\( t \) money available, \( dI_t \), is given by \( dN_t = \frac{dI_t}{S_t} \) where the evolution of \( S_t \) is given by Equation 2.6. Thus, the time-\( t \) value of the risky investment portfolio is

\[
V_t = S_t \int_{0}^{t} dN_u = S_t \int_{0}^{t} \frac{dI_u}{S_u}. \tag{2.18}
\]

Differentiating Equation 2.18 and using Equation 2.17 gives a stochastic differential equation for the dynamics of the investment portfolio value,

\[
dV_t = dS_t \int_{0}^{t} \frac{dI_u}{S_u} + S_t \frac{dI_t}{S_t} = (\mu S_t dt + \sigma S_t dW_t) \int_{0}^{t} dN_u + dI_t
\]

\[
= (\mu dt + \sigma dW_t)V_t + dC_t - i(1-c)C_t dt + Pd\Psi_{t>m}
\]

\[
= \begin{cases} 
\left( \mu V_t + L_{\frac{b-\alpha t}{e^{m-1}}} - i(1-c)L_{\frac{b-\alpha t}{e^{m-1}}} \right) dt + \sigma V_t dW_t, & \text{if } 0 \leq t \leq m, \\
\left( \mu V_t + L_{\frac{b-\alpha t}{e^{m-1}}} - i(1-c)L \right) dt + \sigma V_t dW_t, & \text{if } t > m.
\end{cases} \tag{2.19}
\]

Let \( A = \frac{L_{\frac{b-\alpha t}{e^{m-1}}}}{e^{m-1}} \) and \( B = \frac{i(1-c)L_{\frac{b-\alpha t}{e^{m-1}}}}{e^{m-1}} \). Define \( \beta(t) = (A - B)e^{\alpha t} + B \) and \( \alpha = (A - B)e^{\alpha t} + B \), hence Equation 2.19 becomes

\[
dV_t = \begin{cases} 
[\mu V_t + \beta(t)] dt + \sigma V_t dW_t, & \text{if } 0 \leq t \leq m, \\
[\mu V_t + \alpha] dt + \sigma V_t dW_t, & \text{if } t > m.
\end{cases} \tag{2.20}
\]

We know that the stochastic differential equation

\[
dV_t = \mu V_t dt + \sigma V_t dW_t \tag{2.21}
\]

has a general solution \( V_t = a\Psi_t \), where \( a \) is a constant and

\[
\Psi_t = e^{\left(\mu - \frac{1}{2} \sigma^2\right) t + \sigma W_t}. \tag{2.22}
\]

Define a stochastic process \( Y_t \) such that

\[
Y_t = \Psi_t^{-1} V_t, \tag{2.23}
\]

where

\[
\Psi_t^{-1} = e^{-(\mu - \frac{1}{2} \sigma^2) t - \sigma W_t}. \tag{2.24}
\]

\( \Psi_t^{-1} \) is a geometric Brownian motion with drift \( -\mu + \sigma^2 \) and volatility \( -\sigma \), therefore, it satisfies the following stochastic differential equation

\[
d\Psi_t^{-1} = (\mu - \frac{1}{2} \sigma^2)\Psi_t^{-1} dt - \sigma \Psi_t^{-1} dW_t. \tag{2.25}
\]

Using the stochastic version of integration by parts [21] we find the dynamics of \( Y_t \) as follows

\[
dY_t = d(\Psi_t^{-1} V_t) = V_t d\Psi_t^{-1} + \Psi_t^{-1} dV_t + dV_t d\Psi_t^{-1}. \tag{2.26}
\]
Using Equations 2.20 and 2.25, the above stochastic differential equation can be simplified as

\[ dY_t = \begin{cases} \beta(t)\Psi_t^{-1}dt, & \text{if } 0 \leq t \leq m, \\ \alpha\Psi_t^{-1}dt, & \text{if } t > m. \end{cases} \]  

(2.27)

Integrating we get

\[ Y_t = \begin{cases} Y_0 + \int_0^t \beta(s)\Psi_s^{-1}ds, & \text{if } 0 \leq t \leq m, \\ Y_m + \int_m^t \alpha\Psi_s^{-1}ds, & \text{if } t > m. \end{cases} \]  

(2.28)

which after rearranging for \( V_t \) and noting that \( V_0 = 0 \) and \( \Psi_0^{-1} = 1 \) gives

\[ Y_t = \begin{cases} \Psi_t \int_0^t \beta(s)\Psi_s^{-1}ds, & \text{if } 0 \leq t \leq m, \\ V_m\Psi_m^{-1}\Psi_t + \Psi_t\alpha\int_m^t \Psi_s^{-1}ds, & \text{if } t > m. \end{cases} \]  

(2.29)

Using Equations 2.22 and 2.24, the solution of the stochastic differential in Equation 2.20 is

\[ V_t = \begin{cases} e^{(µ-\frac{1}{2}σ^2)t+σW_t} \int_0^t \beta(s)e^{-(µ-\frac{1}{2}σ^2)s-σW_s}ds, & \text{if } 0 \leq t \leq m, \\ Ve^{(µ-\frac{1}{2}σ^2)(t+m)+σ(W_t+W_m)+αe^{(µ-\frac{1}{2}σ^2)s-σW_s}} \int_m^t e^{-(µ-\frac{1}{2}σ^2)s-σW_s}ds, & \text{if } t > m. \end{cases} \]  

(2.30)

Using the initial condition, \( V_0 = 0 \), the integral form of Equation 2.20 is

\[ V_t = \begin{cases} \int_0^t (µV_0 + \beta(s))ds + \sigma \int_0^t V_s dW_s, & \text{if } 0 \leq t \leq m, \\ V_m + \int_m^t (µV_0 + \alpha)ds + \sigma \int_m^t V_s dW_s, & \text{if } t > m. \end{cases} \]  

(2.31)

The mean of \( V_t \) is derived in Appendix A.1 and is given by

\[ \mathbb{E}[V_t] = \begin{cases} \frac{A-B}{r-µ} (e^{rt} - e^{rt-µ}), & \text{if } 0 \leq t \leq m, \\ \frac{B}{r} (e^{rt}-1) - \frac{A-B}{r}e^{(r-µ)tm} + \frac{B}{r} e^{(r-µ)t}, & \text{if } t > m. \end{cases} \]  

(2.32)

We find the limit as \( µ \) approaches \( r \) of Equation 2.32

\[ \lim_{µ \to r} \mathbb{E}[V_t] = \begin{cases} (A-B) \lim_{µ \to r} e^{e^{rt-µ}} + B \lim_{µ \to r} (e^{rt} - 1), & \text{if } 0 \leq t \leq m, \\ -\frac{A-B}{r} e^{(r-µ)m} + \frac{B}{r} e^{(r-µ)t} \lim_{µ \to r} (A-B)e^{(r-µ)m} + B - µA, & \text{if } t > m. \end{cases} \]  

(2.33)

The terms \( e^{e^{rt-µ}} \) and \( r(A-B)e^{(r-µ)m} + B - µA \) yield 0/0 intermediate forms upon taking the limit as \( µ \) approaches \( r \). We use L’Hôpital’s rule wherever we come across the 0/0 intermediate form in order to get the limit. Therefore,

\[ \lim_{µ \to r} e^{er^t-µ} = te^{rt}, \]  

(2.34)

\[ \lim_{µ \to r} \frac{r(A-B)e^{(r-µ)m} + B - µA}{r-µ} = rm(A-B) + A. \]  

(2.35)

Substituting the above two equations into Equation 2.33 gives

\[ \lim_{µ \to r} \mathbb{E}[V_t] = \begin{cases} (A-B)te^{rt} + \frac{B}{r} (e^{rt} - 1), & \text{if } 0 \leq t \leq m, \\ -\frac{A-B}{r} e^{(r-µ)m} + \frac{B}{r} e^{(r-µ)t} + \frac{A-B}{r} + \frac{A}{r} (A-B)m + \frac{A}{2} \right)e^{rt}, & \text{if } t > m. \end{cases} \]  

(2.36)
We equate $\mathbb{E}[V_t]$ given by Equation 2.32 to the above equation for the case when $\mu = r$ which makes the mean function $\mathbb{E}[V_t]$ continuous.

The second moment of the investment portfolio, $\mathbb{E}[V_t^2]$, is derived in Appendix A.2 and is given by

$$
\mathbb{E}[V_t^2] = \begin{cases} \frac{(A - B)^2}{(r - \mu)(2r - 2\mu - \sigma^2)} + \frac{(A - B)(rB - \mu A)}{\mu(r - \mu)(r - \mu - \sigma^2)} + \frac{B(A - B)(2\mu - r)}{\mu(r - \mu)(r - 2\mu - \sigma^2)}e^{2rt} - \frac{2\alpha^2}{\mu(2\mu + \sigma^2)^2} - \frac{2\alpha e^\mu}{\mu + \sigma^2}e^{2\mu t} + P_2 e^{(2\mu + \sigma^2)t}, & \text{if } 0 \leq t \leq m, \\ \frac{2B^2}{r(2r + \sigma^2)} + 2D_1 e^{(2\mu + \sigma^2)t}, & \text{if } t > m. \end{cases}
$$

(2.37)

where

$$
D_1 = \frac{(A - B)^2}{(r - \mu)(2r - 2\mu - \sigma^2)} - \frac{(A - B)(rB - \mu A)}{\mu(r - \mu)(r - \mu - \sigma^2)} - \frac{B(A - B)(2\mu - r)}{\mu(r - \mu)(r - 2\mu - \sigma^2)}
$$

(2.38)

$$
P_1 = \frac{r(A - B)}{\mu(r - \mu)} e^{(r-\mu)t} + \frac{rB - \mu A}{\mu(r - \mu)}, \text{ and}
$$

(2.39)

$$
P_2 = \left( \mathbb{E}[V_m^2] - \frac{2\alpha^2}{\mu(2\mu + \sigma^2)^2} + \frac{2\alpha e^\mu}{\mu + \sigma^2}e^{2\mu t} \right) e^{-(2\mu + \sigma^2)t}.
$$

(2.40)

As above, we compute the limit as $\mu$ approaches $r$ to get

$$
\lim_{\mu \to r} \mathbb{E}[V_t^2] = \begin{cases} \frac{2(A - B)^2}{(r - \mu)(2r - 2\mu - \sigma^2)} + \frac{(A - B)(rB - \mu A)}{\mu(r - \mu)(r - \mu - \sigma^2)} + \frac{B(A - B)(2\mu - r)}{\mu(r - \mu)(r - 2\mu - \sigma^2)}e^{2rt} + \frac{2B^2}{r(2r + \sigma^2)} + 2D_1 e^{(2\mu + \sigma^2)t} \lim_{\mu \to r} D_1, & \text{if } 0 \leq t \leq m \\ \frac{2\alpha^2}{r(2r + \sigma^2)} + \frac{2\alpha e^\mu}{\mu + \sigma^2}e^{2\mu t} \lim_{\mu \to r} P_1 + e^{2(\mu + \sigma^2)t} \lim_{\mu \to r} P_2, & \text{if } t > m. \end{cases}
$$

(2.41)

where

$$
\lim_{\mu \to r} D_1 = \frac{(A - B)(B\sigma^2 + r(A - B))}{rr^4 + (r + \sigma^2)B(2B - A) + rB(A - B)} - \frac{B^2}{r(2r + \sigma^2)}
$$

(2.42)

$$
\lim_{\mu \to r} P_1 = \frac{rm(A - B) + A}{r}, \text{ and}
$$

(2.43)

$$
\lim_{\mu \to r} P_2 = \left( \lim_{\mu \to r} \mathbb{E}[V_m^2] - \frac{2\alpha^2}{r(2r + \sigma^2)} + \frac{2\alpha (rm(A - B) + A)}{r(2r + \sigma^2)} e^{2\mu t} \right) e^{-(2r + \sigma^2)m}.
$$

(2.44)

The variance, $\mathbb{V}ar[V_t]$, can be computed by using the following formula

$$
\mathbb{V}ar[V_t] = \mathbb{E}[V_t^2] - (\mathbb{E}[V_t])^2.
$$

(2.45)

Consider the example of a 25-year mortgage, 6% fixed-rate and with the home equity line of credit interest rate of 6%. For the evolution of the stock price we take $\mu = 0.08$ and $\sigma = 0.20$. Over the time horizon of 75 years we simulate ten thousand investment portfolio sample paths to estimate its mean and variance. The left panel of Figure 2.4 shows the theoretical estimated mean, and the 95% confidence interval of the investment portfolio mean. The right panel is a plot of the theoretical variance, estimated variance and the 95% confidence interval of the investment portfolio variance. We can see that the theoretical mean given by Equation A.8 and the simulated mean of the SDE of $V_t$ given by Equation 2.19 are in good agreement.
2.2.5 Quantities of Interest

These are things which we would like to compute, either analytically or by estimating them through simulation.

- One quantity of interest is the mortgage payoff time, which we define as the first time the value of the risky investment portfolio is at least the value of the original mortgage amount. In mathematical form the payoff time \( \tau \) is defined as

\[
\tau = \inf_t \{ V_t \geq L \}.
\]  

(2.46)

By simulating sample price paths for the risky investment, we generate corresponding sample paths for the portfolio value and hence a simulated value for the mortgage payoff time. By investigating the distribution of the payoff time and comparing with the amortization term we can assess one aspect of the risk.

- Another quantity of interest is the total interest cost, \( I^*_m \), of the original mortgage which is defined as

\[
I^*_m = Pm - L.
\]  

(2.47)

The total cost of the re-advanceable mortgage will in general be different from \( I^*_m \) depending on the payoff time \( \tau \). If \( \tau \) is less than the amortization term then the total interest cost will be reduced as well and vice versa. If \( P \) is the mortgage payment rate over the time interval \([0, \tau]\) then the total value of payments is \( \int_0^\tau P \, dt = P \tau \). For \( \tau < m \), \( P \tau \) is the total cost of repaying \( C \) by time \( \tau \) and for \( \tau > m \), \( P \tau \) is the total cost of repaying the original principal \( L \) including payments required to make up for losses on the risky investment. Let \( I^*_\tau \) be the total cost of the re-advanceable mortgage, defined as

\[
I^*_\tau = \begin{cases} 
P \tau - C, & \text{if } \tau < m, \\
P \tau - L, & \text{if } \tau \geq m.
\end{cases}
\]  

(2.48)

Note that since \( P \) is fixed and for \( \tau < m \) we have \( C < L \), it is not possible to have a sample path which pays off mortgage in less than \( m \) years but have the total cost \( I^*_\tau \)
greater than the total interest cost $I_m$. Also note that in our model we did not include the
transaction cost on stock and commission to a broker for completing the transactions on
stock.

- In the event of job loss the re-advanceable mortgage scheme can protect the home owner
from default. A part of the investment portfolio can be sold to make mortgage payments
during the period of unemployment. This originates another quantity of interest - the
time it takes to accumulate $x$-years worth of payments in the investment portfolio. This
is another random variable defined by

$$
\tau_x = \inf\{V_t \geq x - \text{years worth of mortgage payments}\}.
$$

- It will be interesting to see how different marginal tax rates affect the re-advanceable
mortgage scheme.

Note that the hitting times defined in Equations 2.46 and 2.49 do not have easily-derived
distributions due to the time-dependent drift in the portfolio value evolution (Equation 2.9).

### 2.3 Example

Let us consider an example of a $300,000, 25-year closed 6\% fixed-rate mortgage and assume
that the mortgage term is same as the amortization term. Using Equation 2.1, the monthly
mortgage payment is $1932.91. And Equation 2.12 gives a constant (annual) payment rate of
$1930.83. The interest rate on the home equity line of credit is assumed to be 6\% and the
marginal income tax rate on the interest expenses is 31.15\%. Ten thousand sample price paths
of the stock are simulated with $\mu = 0.08, \sigma = 0.20, S_0 = 1$ and $\Delta t = \frac{1}{12}$. The distributions of
simulated mortgage payoff times and the corresponding interest cost are given in Figure 2.5.
The average payoff time is 21.22 years which is less than the amortization term of 25 years.
The 95\% confidence interval for the average payoff time is [21.11 21.33].

A homeowner can expect to pay off the mortgage before the end of 25 year-amortization
using the re-advanceable mortgage strategy. One crucial point to be noted is the right tail of the
payoff time distribution. The estimated standard deviation of this distribution is almost 5.48
years, implying a considerable risk that the homeowner will take longer than 25 years to pay
off the mortgage with an estimated probability of 0.22. According to left panel of Figure 2.5
the probability of taking it longer than 30 years, 35 years and 40 years to pay off the mortgage
are 0.07, 0.02 and 0.004, respectively.

Using Equation 4.10 the total interest cost of a traditional 25-year closed 6\% fixed-rate
mortgage is $1932.91 \times 300 - 300000 = $279873. The average interest cost of the interest cost
distribution given in the right panel of Figure 2.5 is almost $233001. A homeowner can ex-
pect to reduce the total interest cost by almost $46871 while implementing the re-advanceable
mortgage scheme. However, we cannot ignore the right tail of the interest cost distribution
which has the standard deviation of $70970.59.

In the above example, the rate of interest on mortgage and the home equity line of credit
are the same. Generally, the interest rate on the home equity line of credit is higher than the
2.4 Income Tax Rate Effect

In this section we investigate how different income tax rates affect the efficiency of the re-advanceable mortgage scheme. In Canada personal income is taxed at both the Federal and Provincial levels. There are four Federal tax brackets [2] and the rates for 2008 - 2010 are given in Table 2.1.

For example someone earning $60,000 in taxable income in 2010 is in the 22% federal marginal tax bracket. For someone earning $100,000 the marginal tax bracket is 26%. The marginal tax rate is the rate paid on any additional dollars earned to the next tax bracket. One misconception about tax rates is that one’s entire income is taxed at the marginal tax rate, but this is not the case. For example a person with $60,000 in taxable income would pay $10,382 \times 0.00 + (40,970 - 10,383) \times 0.15 + (60,000 - 40,971) \times 0.22 = 8774.43$ in taxes. Table 2.2 gives the provincial tax rate brackets in Ontario for 2008 - 2010. The provincial income tax is added to the federal income tax. For example someone with $60,000 in taxable income has...
Table 2.1: Federal Marginal Tax Rates

<table>
<thead>
<tr>
<th>2010 Taxable Income</th>
<th>2009 Taxable Income</th>
<th>2008 Taxable Income</th>
<th>Tax Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0 - $10,382</td>
<td>$0 - $10,320</td>
<td>$0 - $9,600</td>
<td>0%</td>
</tr>
<tr>
<td>$10,383 - $40,970</td>
<td>$10,321 - $40,726</td>
<td>$9,601 - $37,885</td>
<td>15%</td>
</tr>
<tr>
<td>$40,971 - $81,941</td>
<td>$40,727 - $81,452</td>
<td>$37,886 - $75,769</td>
<td>22%</td>
</tr>
<tr>
<td>$81,942 - $127,021</td>
<td>$81,453 - $126,264</td>
<td>$75,770 - $123,184</td>
<td>26%</td>
</tr>
<tr>
<td>over $127,021</td>
<td>over $126,264</td>
<td>over $123,184</td>
<td>29%</td>
</tr>
</tbody>
</table>

The marginal tax rates given in 2.1 are on the total income. The total income is a combination of all the incomes coming from different sources. Basically, there is earned income, dividend income, and taxable gain income. While implementing the re-advanceable mortgage scheme we assume that the homeowner invests the entire proceeds from the home equity line of credit into a single risky asset, we call it a stock. The investment value may increase or decrease with time depending on the rise or fall in the stock price. If any part of the investment portfolio is sold then the homeowner either realizes a capital gain, or a capital loss. The gain or loss is the difference between how much is paid for the asset and how much it is sold for. For example, if there is a stock bought for $100 and sold for $200, then there is a capital gain of $100 and if it is sold for $50 then there is a capital loss of $50. The government of Canada has treated capital gains differently over time. Currently, half (50%) of all capital gains are considered taxable income. This means that 50% of the value of any taxable gain is added to any other income earned in a year, and then the tax rates are applied dependant on the marginal tax brackets that apply. On the other hand, a capital loss can be used to offset and reduce capital gains. If there are more losses than gains in a particular year the net loss may be carried back up to three tax years to reduce net capital gains reported previously. This may result in a refund of taxes already paid. Alternatively, a capital loss may be carried forward indefinitely to offset future capital gains. We conclude that the income from the investment whether in the form of capital gain or capital loss can change the total income of a homeowner and hence affects the tax rate bracket. In our study we ignore this effect.

Consider the same example of a 25-year, 6% fixed rate closed mortgage, a 6% rate on the line of credit and a stock with $\mu = 0.08$ and $\sigma = 0.20$. Figure 2.6 gives box and whisker plots of the simulated payoff time distribution for different marginal tax rates. The horizontal solid straight line represents the original mortgage term of 25-years. It is clear from the box plots that as the tax rate increases the average and standard deviation of payoff time decreases.
implying the strategy is most beneficial to high wage earners.

Figure 2.6: Box and whisker plots of the simulated payoff time distribution versus marginal tax rates ($\mu = 0.08$, $\sigma = 0.20$, and $r = 0.06$).

For the $20.05\%$ tax rate the average payoff time is 21.98 years compared with 20.90 years for the tax rate $40.16\%$. The chance of taking it longer than 25 years to pay off the mortgage decreases from 27.44% to 20.55%. This is expected since, as the marginal tax rate increases, the tax rebate on the interest expense increases, thus increasing the amount of new money available for investment hence growing the investment portfolio at a faster rate and decreasing the payoff time. Interestingly we see that the standard deviation decreases with the marginal tax rate, implying more risk for homeowners with lower incomes. Higher moments are also important, particularly as they reflect the weight in the right tail of the distribution (e.g., large payoff times). For the $20.05\%$ tax rate the skewness and kurtosis of the payoff time distribution are 0.79283 and 3.7084, respectively. The skewness and the kurtosis are decreased to 0.65805 and 3.3288, respectively, for the $40.16\%$ tax rate, showing the advantage of the strategy to high-wage earners.

2.5 Job Loss and Mortgage Default

If a mortgagor loses his/her job then it is quite possible that the homeowner defaults, which is a situation when a mortgagor is not able to make the mortgage payments. Job loss is one of the main factors in mortgage defaults [23]. Here we investigate whether the Smith Manoeuvre is effective in lowering the risk to the homeowner of defaulting on his/her mortgage due to loss of income. To describe job status we use a two-state Markov chain assuming that all jobs the homeowner may hold have the same income and are otherwise equivalent. Table 2.3 details the Markov chain used for the job status. A homeowner employed at time $t$ maintains this status at time $t + \Delta t$ with probability $p$. Similarly if unemployed at time $t$ the homeowner remains
unemployed at time $t + \Delta t$ with probability $q$. The change of job status from employed to unemployed and from unemployed to employed happens with probabilities $1 - p$ and $1 - q$, respectively. Let $Q$ be the transition matrix of the two-state Markov chain which is given by

$$Q = \begin{bmatrix} p & 1 - p \\ 1 - q & q \end{bmatrix}$$

Let $\pi$ be the equilibrium distribution of the Markov chain which can be evaluated by solving the system of equations given by

$$\pi Q = \pi \quad (2.50)$$

where $\pi = [\pi_{Employed}, \pi_{Unemployed}]$ is a row vector of probabilities and $\pi_{Employed} + \pi_{Unemployed} = 1$. Solving the system of equations we get

$$\pi_{Employed} = \frac{1 - q}{2 - p - q}, \quad (2.51)$$
$$\pi_{Unemployed} = \frac{1 - p}{2 - p - q}. \quad (2.52)$$

<table>
<thead>
<tr>
<th>Time $t$</th>
<th>$t + \Delta t$</th>
<th>$t + 2\Delta t$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Employed</strong></td>
<td>$p$</td>
<td>$1 - p$</td>
</tr>
<tr>
<td><strong>Unemployed</strong></td>
<td>$1 - q$</td>
<td>$q$</td>
</tr>
</tbody>
</table>

Table 2.3: The two-state Markov Chain model for job loss status.

Figure 2.7: Left panel is a simulated employment status sample path. Right panel is the corresponding investment portfolio value sample path.
2.5. **Job Loss and Mortgage Default**

We assume that job status and the risky stock evolve independently, though our model could allow for correlation. When unemployed, we assume that the homeowner funds the required mortgage payment by selling an equivalent dollar amount of the investment portfolio. Thus, we define default as the first time the value of the investment portfolio falls below the mortgage payment (and the job status is unemployed). Simulated sample paths of employment status are paired with investment portfolio value sample paths to determine either the simulated default time or mortgage payoff time.

![Figure 2.8: 100 sample paths of the investment portfolio value versus time.](image)

Continuing the same example of the 25-year fixed rate mortgage and with monthly time increments, we take \( p = 0.999 \) and \( q = 0.900 \). The left panel of Figure 2.7 shows an employment status sample path and the right panel gives the corresponding sample path of investment portfolio value. Along this sample path, we see that between the fourth and fifth years the homeowner lost a job for a short period of time and then was employed again. Default did not occur during that time as the investment portfolio value was greater than the monthly mortgage payment and it was partly sold to fund the payment. Between the time period seven years and nine years, the homeowner was unemployed for a longer period of time and did not default immediately after becoming unemployed. This is due to the fact that the investment portfolio value was large enough to fund the mortgage payments for some period of time while unemployed. Eventually the investment portfolio value became smaller than the monthly mortgage payment, resulting in default at around the 8-year time mark, 4 years after first becoming unemployed.

Figure 2.8 shows 100 sample paths of the investment portfolio values against time (assuming no job loss) for the first five years along with amounts corresponding to 1 month, 1 year and 2 years worth of mortgage payments. This shows that in a relatively short period of time, the investment portfolio value can accumulate to cover a whole year’s worth of mortgage payments, thus protecting the homeowner from default due to job loss. The average time it takes...
to get the investment portfolio value equal to 1 month, 1 year and 2 years worth of mortgage payments is around half a year, four years and seven years, respectively. Their respective confidence intervals for the average time are \([0.4981, 0.4989]\), \([3.93, 4.12]\) and \([6.74, 6.80]\). Selling part of the investment portfolio to make mortgage payments increases the expected payoff time, given default does not occur.

If no default occurs the payoff time distribution is similar to the case in which job loss is not considered. The left panel of Figure 2.9 gives a mortgage payoff time distribution while taking job loss factor into account. In our simulations we adjusted the tax rate in case of job loss depending on the annual income after incorporating the unemployment duration. The estimated conditional mortgage default time distribution is given on the right panel of Figure 2.9. The average time it takes a mortgagor to default is over 3 years. The standard deviation of default time distribution is 3.01 years. It is clear from the distribution that most of the defaults occur at an early stage. As discussed before this is due to the fact that the investment portfolio is not big enough in the early stages to protect the homeowner against default.

In the above example, the rate of interest on the mortgage and the home equity line of credit are the same. Generally, the interest rate on the home equity line of credit is higher than the mortgage interest rate. Figure 2.10 shows that holding the mortgage rate at 6% and changing the line of credit interest rate to 7% increases the average payoff time from 21.50 years to 22.38 years and standard deviation from 5.64 years to 6.32 years. The skewness increases from 0.8519 to 1.0173 and the kurtosis increases from 4.1019 to 4.7671. Holding the line of credit interest rate at 6% and changing the mortgage interest rate to 7% increases the average payoff time by only three months and the standard deviation decreases from 5.58 years to 5.22 years. The skewness decreases from 0.8519 to 0.7386 and the kurtosis decreases from 4.1019 to 3.9571. Thus, a change in interest rates does not affect the default time distribution that
2.6 Conclusion

The re-advanceable mortgage benefits the homeowner such that the average mortgage payoff time is less than the original mortgage term. Consequently, the average total interest cost is also less. However, there is a risk associated with this strategy; namely a reasonable chance that the payoff time will be much longer, particularly if the investment performs poorly. Different income tax rates affect the strategy’s efficiency. Higher income homeowners enjoy a payoff time distribution with both a lower average and a lower standard deviation than low-income homeowners. In the event of job loss, the investment portfolio protects the homeowner from default as part of the investment portfolio can be sold to fund mortgage payments, another benefit that is not typically advertised.

Consider a scenario where homeowner sells his/her house and buy a new house. In that case homeowner will have to pay off the mortgage on the old house and the line of credit balance. Buying a new house and selling old one which already has a secured line of credit does not guarantee a new line of credit against new house. This is a hidden risk of this strategy.
Chapter 3

Sensitivity Analysis

3.1 Introduction

The ability of the re-advanceable mortgage to improve a homeowner’s position depends on a number of parameters. Consider an $m$-year fixed rate mortgage with original principal $L$ and continuous mortgage interest rate $r$. The home equity line of credit (HELOC) and tax rebate are paid continuously with annualized continuous HELOC interest rate $i$ and continuous tax rebate rate $c$, respectively. We concluded in Chapter 1 that high tax rate bracket homeowners benefit more from the re-advanceable scheme as it reduces the expected mortgage payoff time and lowers payoff time variability. In this chapter we fix the tax rate to 31.15% and we see how changing the values of other parameters affect the re-advanceable mortgage scheme.

We make the same assumption as in Chapter 1 that the homeowner invests the entire proceeds from the home equity line of credit into a single risky asset. Let $S_t$ denote the time-$t$ value of the risky asset which we call a stock. The evolution of the stock price is governed by the following stochastic differential equation,

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

where $W$ is a Wiener process, $\mu$ is the drift and $\sigma$ is the volatility.

A list of parameters that affect the ability of the re-advanceable mortgage scheme includes the mortgage rate, $r$, the line of credit interest rate, $i$, drift, $\mu$ and the volatility, $\sigma$. We investigate the rate of change in investment portfolio mean and standard deviation with respect to these parameters.

3.2 Sensitivity of Investment Capital

The original mortgage is paid continuously at rate $P$ which is given by

$$P = \frac{Lre^{rm}}{e^{rm} - 1}.$$  

(3.2)
3.2. Sensitivity of Investment Capital

As the mortgage is paid the principal of the home equity line of credit increases by the principal portion of the mortgage payment. Thus, the time-\(t\) line of credit balance, \(C_t\), is

\[
C_t = \begin{cases} 
L \frac{e^{rt}-1}{e^{rm}-1}, & \text{if } 0 \leq t \leq m, \\
L, & \text{if } t > m.
\end{cases}
\] (3.3)

The rate of change of \(C_t\) is

\[
dC_t = \begin{cases} 
\frac{Lr e^{rt}}{e^{rm}-1} dt, & \text{if } 0 \leq t \leq m, \\
0, & \text{if } t > m.
\end{cases}
\] (3.4)

Consider the time interval \((t−dt, t]\). If \(I_t\) is the time-\(t\) investment capital then the time-\(t\) amount of money available for new investment is

\[
dI_t = dC_t - i(1-c)C_t dt + P dt \mathbb{1}_{t>m}
\] (3.5)

where \(\mathbb{1}_{t>m}\) is an indicator function taking the value one if \(t > m\) is true and zero otherwise.

Substituting Equations 3.3 and 3.4 in the above equation we get

\[
dI_t = \begin{cases} 
\frac{Lr}{e^{rm}-1} \left[ e^{rt} - i(1-c)(e^{rt}-1) \right] dt, & \text{if } 0 \leq t \leq m, \\
\frac{Lr}{e^{rm}-1} \left[ e^{re^m} - i(1-c)(e^{re^m}-1) \right] dt, & \text{if } t > m.
\end{cases}
\] (3.6)

Equation 3.6 only makes sense if the change in investment capital \(dI_t\) stays greater than zero otherwise additional funds are required to support the strategy. For different values of the line of credit rate \(i\) and the mortgage rate \(r\) we investigate the sign of \(dI_t\). The denominator in the above equation is always positive since \(r\) and \(m\) are greater than zero, therefore, \(e^{rm} > 1\). It is the numerator which is crucial in our investigation. Let \(f(r, i, t)\) be defined as

\[
f(r, i, t) = \begin{cases} 
e^{rt}(r - i(1-c)) + i(1-c), & \text{if } 0 \leq t \leq m, \\
e^{rm}(r - i(1-c)) + i(1-c), & \text{if } t > m.
\end{cases}
\] (3.7)

i.e., \(dI_t = \frac{Lr}{e^{rm}-1} f(r, i, t) dt\). Note that \(e^{rt} \geq 1\) for all values of \(r\) and \(t\), therefore, \(e^{rt} - 1\) is always greater than or equal to zero. The same is true for \(e^{rm} - 1\). The function \(f(r, i, t)\) is positive for all choices of \(r, i,\) and \(t\) if \(r - i(1-c)\) is positive. If \(r - i(1-c)\) is negative then \(f(r, i, t)\) is positive only for the values of \(r, i,\) and \(t\) for which the absolute value of \(e^{rt}(r - i(1-c))\) is less than \(i(1-c)\) for the case \(0 \leq t \leq m\) or the absolute value of \(e^{rm}(r - i(1-c))\) is less than \(i(1-c)\) for the case \(t > m\).

Figure 3.1 gives the sign plot of \(r - i(1-c)\) for different values of \(r\) and \(i\). The blue and red regions correspond to negative and positive values for \(r - i(1-c)\), respectively. We fix the tax rate \(c\) at 31.15%. The range of values taken for the mortgage rate \(r\) and the line of credit rate \(i\) is from 1\% to 20\% with a step size of 0.1\%. If we fix the tax rate at 31.15\% then it means that mortgage rate, \(r\), should be greater than the 68.85\% of the line of credit rate, \(i\), for \(r - i(1-c)\) to be positive. For example, if \(r\) is 6\% then \(i\) should be less than or equal to 8\%.

We are interested in looking at the values where it is negative so that we can determine pairs \((r, i)\) which give negative investment capital changes. From Figure 3.1 we conclude that the investment capital change will always be positive if the line of credit interest rate is less than the mortgage rate. There are also the cases when the line of credit interest rate is greater than
Figure 3.1: Sign plot of $r - i(1 - c)$ versus $(r, i)$. The tax rate $c$ is fixed at 31.15%.

the mortgage rate and the investment capital is positive. One such example is the pair $(r, i) = (7\%, 10\%)$ for which $r - i(1 - c)$ is positive, the reason being $r = 7\%$ is greater than the 68.85\% of $i = 10\%$.

Considering the 25-year fixed rate mortgage, Figure 3.2 is a level surface plot of Equation 3.6 and shows four level surfaces of $dI_t$ corresponding to $t = 5$, $t = 10$, $t = 15$ and $t = 30$, respectively. As proved above, $dI_t$ is positive when $r - i(1 - c)$ is positive. We investigate the pairs $(r, i)$ for which $dI_t$ is negative.

We have discussed before that $f(r, i, t)$ can be positive even if $r - i(1 - c)$ is negative provided it satisfies the following conditions

$$e^{rt}(r - i(1 - c)) > -i(1 - c),$$

$$\Rightarrow t < \frac{1}{r} \ln \left( \frac{i}{i(1 - c) - r} \right), 0 \leq t \leq m. \quad (3.8)$$

and

$$e^{rm}(r - i(1 - c)) > -i(1 - c),$$

$$\Rightarrow m < \frac{1}{r} \ln \left( \frac{i(1 - c)}{i(1 - c) - r} \right), t > m. \quad (3.9)$$

If we fix the values of $r$ and $i$ then the above condition 3.9 tells us the time period for which the $dI_t$ stays positive.

Treating $r$ and $i$ as constants and taking the derivative of Equation 3.7 with respect to time $t$ we get

$$\frac{df}{dt} = \begin{cases} re^{rt}(r - i(1 - c)), & \text{if} \quad 0 \leq t \leq m \\ 0, & \text{if} \quad t > m \end{cases}. \quad (3.10)$$
3.2. Sensitivity of Investment Capital

Figure 3.2: Level surface plot of investment capital change, \( dI_t \), with \( c = 31.15\% \). Top left, top right, bottom left and bottom right panels correspond to \( t = 5 \), \( t = 10 \), \( t = 15 \) and \( t = 30 \), respectively.

Note that \( \frac{dI_t}{dt} < 0 \) for \( 0 \leq t \leq m \) since we are investigating the case where \( r - i(1 - c) < 0 \). Therefore, \( f(r, i, t) \) is a decreasing function of time. Figure 3.3 gives one such scenario where \( r - i(1 - c) < 0 \) for \( r = 6\% \) and \( i = 10\% \). Plugging the values of \( r \) and \( i \) into Inequality 3.8 gives \( t < 34.19 \) years. It means that for these values of \( r \) and \( i \), \( f(r, i, t) \) is positive over the time interval \([0, 34.19]\). This pair also satisfies the second condition given by Inequality 3.9 that \( m < 34.19 \) since \( m \) is 25 years. It means that the investment capital change is positive everywhere for the pair \((r, i) = (6\%, 10\%)\). In Figure 3.3 we consider the time interval \([0, 45]\). It is clear from the figure that \( f \) is positive for all different values of time \( t \), in other words, \( dI_t \) is positive for all \( t \).

Tables 3.1 and 3.2 give the time intervals on which \( dI_t \) is positive for the choices of \( r \) and \( i \) such that \( r - i(1 - c) \) is negative. These time intervals are derived using Inequality 3.8 which is for the case when \( 0 \leq t < m \). If a time interval consists of all the values in the interval \([0, m]\) then we say that \( dI_t \) is positive everywhere and denote it with a ’+’ in the tables. If a time interval is a subset of \([0, m]\) then \( dI_t \) is not positive for all values of \( t \). For example, if \( r = 1\% \) and \( i = 8\% \) then the time interval over which \( dI_t \) is positive is \([0, 20.04]\). Since \([0, 20.04]\) is a subset of \([0, 25]\), therefore, \( dI_t \) is negative over the time interval \((20.04, 25]\). If we closely look at the pairs \((r, i)\) such that \( dI_t \) is negative for some time period, we find that many of those pairs are very unlikely to happen. For example, with a mortgage rate 1\%, the line of credit rate
Table 3.1: The time interval on which $dI_t$ is positive for the case $0 \leq t \leq m$.

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Table 3.2: Cont’d: The time interval on which $dI_t$ is positive for the case $0 \leq t \leq m$.

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Chapter 3. Sensitivity Analysis

must be 7% or higher in order for \( dI_t \) to be negative prior to the original mortgage term, \( m \). Considering all possible pairs we find that only if the difference, \( i - r \), is more than 5% will \( dI_t \) become negative for some \( t < m \). A spread this large is not likely to occur in practice. Having this much difference between the mortgage rate and the line of credit rate affects the efficiency of the Smith Manoeuvre. If the line of credit rate is very high compared to the mortgage rate then a lot of money from the principal portion of the mortgage payment is required to pay the interest cost on the line of credit. That leaves one with less new investment capital.

Note that \( dI_t \) does not depend on \( t \) for the case \( t > m \). Therefore, for all \( t > m \) the value of \( dI_t \) is either positive or negative, depending on the values of \( r \) and \( i \). Figure 3.4 gives the sign plot of \( dI_t \) for the case \( t > m \). It gives a graphical representation of Tables 3.1 and 3.2. We plot the sign of \( dI_t \) for the case \( t > m \) because if \( dI_t \) is positive for \( t > m \) then it will definitely be positive for all values \( t \) on the interval \( 0 \leq t \leq m \). If \( dI_t \) is negative for \( t \geq m \) then it will only be positive over a subset of \( 0 \leq t \leq m \). We get the same conclusion from Figure 3.4 as from the Tables 3.1 and 3.2 that if \( i - r \) is more than 5% then \( dI_t \) is negative. Figure 3.5 is a level surface plot of \( dI_t \) and shows four level surfaces corresponding to \( t = 5 \), \( t = 10 \), \( t = 15 \) and \( t = 30 \), respectively. It is a figure where we delete all possible pairs \((i, r)\) for which \( dI_t \) is negative.

We collected mortgage and home equity line of credit rates from the web sites of five different banks on February 23, 2012. Table 3.3 compares the HELOC rate with the 10-year fixed-rate closed mortgage of the five largest Canadian banks. We discussed above that the investment capital change, \( dI_t \), is negative when the spread between the mortgage rate and the HELOC rate is greater than or equal to 5%. The maximum spread we get from Table 3.3 is 3.25% (for RBC). In reality this spread will be less than this as the mortgage rate is the posted rate and it will be less than 6.75% after negotiation. Therefore, we conclude that it is unlikely
3.2. Sensitivity of Investment Capital

Figure 3.4: Sign plot of \( dI_t \) for the case when \( t \geq m \). The tax rate \( c \) is fixed at 31.15%.

to get a spread of 5% or greater between the mortgage rate and the HELOC rate\(^1\). Thus, we can safely say that \( dI_t \) will always be positive for realistic values of \( r \) and \( i \).

Table 3.3: The data of 10-year fixed-rate closed mortgage and the HELOC rate from five different banks, collected on February 23, 2012.

<table>
<thead>
<tr>
<th>Bank Name</th>
<th>Mortgage rate (( r ))</th>
<th>HELOC rate (( i ))</th>
<th>Product Name</th>
<th>( r - i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>CIBC</td>
<td>6.75%</td>
<td>4.00%</td>
<td>The CIBC Home Power Line of Credit</td>
<td>2.75%</td>
</tr>
<tr>
<td>TD Trust</td>
<td>6.75%</td>
<td>4.00%</td>
<td>Home Equity Line of Credit</td>
<td>2.75%</td>
</tr>
<tr>
<td>RBC</td>
<td>6.75%</td>
<td>3.50%</td>
<td>RBC Homeline Plan</td>
<td>3.25%</td>
</tr>
<tr>
<td>Scotia Bank</td>
<td>6.29%</td>
<td>4.00%</td>
<td>Scotia Total Equity Plan</td>
<td>2.29%</td>
</tr>
<tr>
<td>BMO</td>
<td>6.29%</td>
<td>3.10%</td>
<td>Homeowner Readiline</td>
<td>3.19%</td>
</tr>
</tbody>
</table>

\(^1\)Note that in the Table 3.3 we have long run mortgage rates (10-year fixed rates). The variable rates are going to be less than the fixed rates and hence, closer to the home equity line of credit rates. For example, variable rate of a mortgage with 5 years term is 3.00% and this makes the spread \( r - i \) as -1%. 
3.3 Sensitivity of Investment Portfolio Value

The homeowner invests the entire proceeds from the line of credit into a risky asset (a stock) and its time-t price is \( S_t \). The time-t value of the risky investment portfolio is

\[
V_t = S_t \int_0^t \frac{dI_u}{S_u}.
\]

(3.11)

The dynamics of the investment portfolio value are given by

\[
dV_t = \begin{cases} 
[\mu V_t + \beta(t)] \ dt + \sigma V_t dW_t, & \text{if } 0 \leq t \leq m, \\
[\mu V_t + \alpha] \ dt + \sigma V_t dW_t, & \text{if } t > m.
\end{cases}
\]

(3.12)

where the functions \( \beta(t) \) and \( \alpha \) are defined as \((A - B)e^{rt} + B\) and \((A - B)e^{rm} + B\), respectively, with \( A = \frac{Le - r}{e^{m-1}} \) and \( B = \frac{1 - c - cLe}{e^{m-1}} \). We perform a sensitivity analysis of the mean and standard deviation of the investment portfolio value for different values of parameters, \( t, \mu, i, r \) and \( \sigma \).

We derived the expressions for the mean and variance (and hence standard deviation) of the investment portfolio value in Chapter 1. The mean of the investment portfolio, \( \mathbb{E}[V_t] \), is

\[
\mathbb{E}[V_t] = \begin{cases} 
\frac{A-B}{r} (e^{rt} - e^{\mu t}) + \frac{B}{\mu} (e^{\mu t} - 1), & \text{if } 0 \leq t \leq m, \\
-\frac{A-B}{r} + \frac{e^{\mu t}}{\mu (r-\mu)} (r(A - B)e^{(r-\mu)m} + rB - \mu A), & \text{if } t > m.
\end{cases}
\]

(3.13)
3.3. Sensitivity of Investment Portfolio Value

We find the limit as $\mu$ approaches $r$ of the above equation. We use L'Hôpital's rule wherever we come across the $0/0$ indeterminant form in order to get the limit. The resulting limit is

$$
\lim_{\mu \to r} \mathbb{E}[V_t] = \begin{cases} 
(A - B)e^{rt} + \frac{B}{r}(e^{rt} - 1), & \text{if } 0 \leq t \leq m, \\
-\frac{(A - B)e^{\mu + rB}}{r} + (A - B)m + \frac{4}{r}e^{\mu}, & \text{if } t > m.
\end{cases} \tag{3.14}
$$

We equate $\mathbb{E}[V_t]$ given by Equation 3.13 to the above equation for the case when $\mu = r$ which makes the mean function $\mathbb{E}[V_t]$ continuous.

The second moment of the investment portfolio, $\mathbb{E}[V_t^2]$, is

$$
\mathbb{E}[V_t^2] = \begin{cases} 
\frac{2(A - B)^2}{(r - \mu)(2r - 2\mu - \sigma^2)} e^{2rt} + \frac{2(A - B)(rB - \mu A)}{2(r - \mu)(r - \mu - \sigma^2)} e^{(r - \mu)t} + \frac{2B(A - B)(2\mu - r)}{\mu(r - \mu)(r - 2\mu - \sigma^2)} e^{rt} - \\
\frac{2B(e^{2t} - 1)}{\mu(\mu - \sigma^2)} + 2D_1 e^{2(\mu + \sigma^2)t}, & \text{if } 0 \leq t \leq m, \\
\frac{2\alpha^2}{\mu + \sigma^2} e^{\mu} + P_2 e^{(2+\sigma^2)t}, & \text{if } t > m.
\end{cases} \tag{3.15}
$$

where

$$
D_1 = \frac{(A - B)^2}{(r - \mu)(2r - 2\mu - \sigma^2)} - \frac{(A - B)(rB - \mu A)}{\mu(r - \mu)(r - \mu - \sigma^2)} - \frac{B(A - B)(2\mu - r)}{\mu(r - \mu)(r - 2\mu - \sigma^2)} (rB - \mu A) + \frac{\mu(r - \mu)(\mu + \sigma^2)}{\mu(2\mu + \sigma^2)} - \frac{B^2}{\mu(r - \mu)(\mu + \sigma^2)}, \tag{3.16}
$$

$$
P_1 = \frac{r(A - B)}{\mu(r - \mu)} e^{(r - \mu)t} + \frac{rB - \mu A}{\mu(r - \mu)}, \tag{3.17}
$$

$$
P_2 = \left(\mathbb{E}[V_m^2] - \frac{2\alpha^2}{\mu(2\mu + \sigma^2)} + \frac{2\alpha P_1}{\mu + \sigma^2} e^{\mu t}\right) e^{-2(\mu + \sigma^2)m}. \tag{3.18}
$$

As above, we compute the limit as $\mu$ approaches $r$ to get

$$
\lim_{\mu \to r} \mathbb{E}[V_t^2] = \begin{cases} 
\frac{2(A - B)^2((2r - 2\mu - \sigma^2) - 2A(2\mu - r))}{r(\sigma^2 - r(2r - 2\mu - \sigma^2))} e^{2rt} + \\
\frac{2(B(A - B) - A(2A - B))}{r(\sigma^2 - r(2r + \sigma^2))} e^{rt} + \\
\frac{2B^2}{r(2r + \sigma^2)} + 2e^{(2r + \sigma^2)t} \lim_{\mu \to r} D_1, & \text{if } 0 \leq t \leq m, \\
\frac{2\alpha^2}{r(2r + \sigma^2)} - \frac{2\alpha e^{rt}}{r(2r + \sigma^2)} \lim_{\mu \to r} P_1 + e^{(2r + \sigma^2)t} \lim_{\mu \to r} P_2, & \text{if } t > m.
\end{cases} \tag{3.19}
$$

where

$$
\lim_{\mu \to r} D_1 = \frac{(A - B)(Br^2 + (rA - B))}{r\sigma^4} + \frac{(r + \sigma^2)B(2B - A) + rB(A - B)}{r(\sigma^2 - r(2r + \sigma^2))} = \frac{B^2}{r(2r + \sigma^2)} \tag{3.20}
$$

$$
\lim_{\mu \to r} P_1 = \frac{rm(A - B) + A}{r}, \tag{3.21}
$$

$$
\lim_{\mu \to r} P_2 = \left(\lim_{\mu \to r} \mathbb{E}[V_m^2] - \frac{2\alpha^2}{r(2r + \sigma^2)} + \frac{2\alpha(rm(A - B) + A)}{r(\sigma^2 - r(2r + \sigma^2))} e^{rt}\right) e^{-(2r + \sigma^2)m}. \tag{3.22}
$$

The variance, $\forall \text{Var}[V_t]$, can be computed by using the following formula

$$
\forall \text{Var}[V_t] = \mathbb{E}[V_t^2] - (\mathbb{E}[V_t])^2. \tag{3.23}
$$
We use the standard deviation, \( SD[V_t] = \sqrt{\text{var}[V_t]} \), as it is in the same units as \( \mathbb{E}[V_t] \) (and \( V_t \)).

We investigate how the set of parameters, \((t, r, i, \mu, \sigma)\), affect the growth of investment portfolio value, \( V_t \). The change in the mean value and standard deviation of \( V_t \) with respect to the parameters \((t, r, i, \mu, \sigma)\) gives us a better picture of the sensitivity of the distribution of \( V_t \) to these parameters. Derivatives with respect to each of the parameters are given in Appendix B.

The sensitivities of \( SD[V_t] \) can be computed using the chain rule. For example

\[
\frac{\partial SD[V_t]}{\partial r} = \frac{1}{2 \sqrt{\text{var}[V_t]}} \left( \frac{\partial \mathbb{E}[V_t^2]}{\partial r} - 2 \mathbb{E}[V_t] \frac{\partial \mathbb{E}[V_t]}{\partial r} \right). \tag{3.24}
\]

### 3.3.1 Sensitivity to mortgage rate, \( r \), the line of credit rate, \( i \) and time, \( t \)

Consider an example of a fixed-rate mortgage with original principal of $300,000 and amortization term of 25 years. Figure 3.6 makes a comparison of \( \mathbb{E}[V_t] \) and its derivatives with respect to the parameters, \( r, \mu, i \) and \( t \) at time 5 years with \( \mu = 0.08 \), and tax rate is 31.15%. Note that the function \( \mathbb{E}[V_t] \) does not depend on \( \sigma \) therefore its derivative with respect to \( \sigma \) is zero. In the figure top left, top right, middle left, middle right and bottom left panels correspond to \( \mathbb{E}[V_t] \), \( d\mathbb{E}[V_t]/dr \), \( d\mathbb{E}[V_t]/d\mu \), \( d\mathbb{E}[V_t]/di \) and \( d\mathbb{E}[V_t]/dt \), respectively. The range of values for the mortgage rate, \( r \) and the line of credit, \( i \) is from 1% to 20% with an increment size of 1%. However, as discussed in the previous section we exclude the pairs \((r, i)\) for which \( dI_t \) is negative. We see that as \( r \) increases \( \mathbb{E}[V_t] \) decreases which makes sense. When the mortgage rate, \( r \), gets higher the interest portion of the mortgage payment gets bigger than the principal portion. Hence, less money would be available for investment which effectively decreases the mean value of the investment portfolio. We get similar behavior when we compare \( \mathbb{E}[V_t] \) versus the line of credit rate, \( i \). When \( i \) is small the mean value is higher than the mean value when \( i \) is big. The rate of change of \( \mathbb{E}[V_t] \) with respect to \( r \) is an increasing function of \( r \) and \( i \). The derivative with respect to \( \mu \), \( d\mathbb{E}[V_t]/d\mu \) decreases with an increase in \( r \) and \( i \). It means that when the line of credit and mortgage rate increase the rate of change in \( \mathbb{E}[V_t] \) with respect to \( \mu \) is going to decrease. The derivative with respect to \( t \), \( d\mathbb{E}[V_t]/dt \), shows the same behavior as \( d\mathbb{E}[V_t]/d\mu \). The derivative with respect to \( i \), \( d\mathbb{E}[V_t]/di \) is an increasing function of \( r \) but it does not change with \( i \).

Figure 3.7 gives the plot of mean of the investment portfolio and its derivatives across time for different values of mortgage rate with tax rate 31.15%, \( i = 0.06 \), \( \mu=0.08 \) and \( \sigma=0.20 \). The top left, top right, middle left, middle right and bottom left panels correspond to \( \mathbb{E}[V_t] \), \( d\mathbb{E}[V_t]/dt \), \( d\mathbb{E}[V_t]/dr \), \( d\mathbb{E}[V_t]/di \) and \( d\mathbb{E}[V_t]/d\mu \), respectively. Different colored curves represent different mortgage rates. We investigate how the mean and its derivatives evolve with time. The mean grows with time. It is evident from the top left panel that the lower the mortgage rate the higher the \( \mathbb{E}[V_t] \) until almost 25 years. After 25 years we see that the curves flipping in a way that the higher the mortgage rate the higher the \( \mathbb{E}[V_t] \). The reason for this flip is that after the amortization term of 25 years, the homeowner keeps on making the mortgage payment until the value of the investment portfolio is at least the original principal. After 25 years the whole amount of mortgage payments is invested, therefore, if the mortgage rate is higher more money is available for new investment. The availability of more money for new investment accelerates the growth of investment portfolio, hence, the \( \mathbb{E}[V_t] \) increases for high mortgage rates after 30 years.
Figure 3.6: Level surface plot of mean, $\mathbb{E}[V_t]$, and its derivatives at time 5 years. The tax rate, $c = 31.15\%$, drift, $\mu = 0.08$ and volatility, $\sigma = 0.20$. Top left, top right, middle left, middle right and bottom left panels correspond to $\mathbb{E}[V_t]$, $d\mathbb{E}[V_t]/dr$, $d\mathbb{E}[V_t]/d\mu$, $d\mathbb{E}[V_t]/di$ and $d\mathbb{E}[V_t]/dt$, respectively.

The top right panel of the figure corresponds to the rate of change in $\mathbb{E}[V_t]$ with respect to time, $t$. It is an increasing function of time, however, there is a flip of high mortgage rate curves to low mortgage rate curves between the 20 and 25 years. After 20 years, the higher the mortgage rate the bigger the change in the value of $\mathbb{E}[V_t]$ with respect to $t$.

From the middle left panel of the figure we see that, $d\mathbb{E}[V_t]/dr$, the rate of change in $\mathbb{E}[V_t]$ with respect to the mortgage rate has concave up curves for different values of the mortgage rate. As time marches along the curves representing the different mortgage rates spread apart from each other. Between the starting time of 0 years and the amortization term of 25 years, $d\mathbb{E}[V_t]/dr$ is decreasing and negative. After 25 years it is increasing and positive. Between the years 0 and 25, the lower the mortgage rate, the lower the decrease in the value of $\mathbb{E}[V_t]$. After 25 years, the higher the mortgage rate, the higher the increase in the value of $\mathbb{E}[V_t]$. This is due to the fact that all the mortgage payments made after 25 years are contributed into the risky investment.

The derivative $d\mathbb{E}[V_t]/di$ is a decreasing function of time and it has negative values. It means that there is a decrease in the value of $\mathbb{E}[V_t]$ with respect to the line of credit rate, $i$. The lower the mortgage rate, the lower the rate of change, and this rate of change becomes more pronounced with time.
Figure 3.7: Plot of mean, $\mathbb{E}[V_t]$, and its derivatives with tax rate 31.15%, $i = 0.06$ and $\mu = 0.08$. Different colored curves represent different mortgage rates, $r$.

The derivative $d\mathbb{E}[V_t]/d\mu$ is an increasing function of time. If the mortgage rate is lower, the rate of change in $\mathbb{E}[V_t]$ with respect to $\mu$ is higher. In other words, the less interest homeowner pays on mortgage, the more the gain from investment portfolio, and this rate of change increases with time.

Figure 3.8 gives the plot of mean of the investment portfolio and its derivatives across time for different values of the line of credit rate with tax rate 31.15%, $r = 0.15$, $\mu = 0.08$ and $\sigma = 0.20$. We fix the mortgage rate at 15% because this way the difference $i - r$ is less than or equal to 5% which guarantees the positive change in the new money available for investment for all the values of line of credit rate in the interval [0%, 20%]. The top left, top right, middle left, middle right and bottom left panels correspond to $\mathbb{E}[V_t]$, $d\mathbb{E}[V_t]/dt$, $d\mathbb{E}[V_t]/dr$, $d\mathbb{E}[V_t]/di$ and $d\mathbb{E}[V_t]/d\mu$, respectively. Different colored curves represent different line of credit rates ($i$). It is evident from the figure that $\mathbb{E}[V_t]$, $d\mathbb{E}[V_t]/dt$, and $d\mathbb{E}[V_t]/d\mu$ are increasing functions of time and decreasing functions of the line of credit rate. As the line of credit rate increases less new money is available for investment, hence, the investment portfolio mean decreases. The derivative, $d\mathbb{E}[V_t]/dr$, appears concave (middle left panel). The function is decreasing before 25 years and increasing afterwards. The $d\mathbb{E}[V_t]/dr$ is an increasing function of line of credit rate. It suggests that the change in the values of $\mathbb{E}[V_t]$ with respect to the mortgage rate get bigger with time and this change increases with the line of credit rate.

Figure 3.9 makes a comparison of the standard deviation of investment portfolio and its derivatives with respect to the parameters, $r$, $\mu$, $i$, $t$ and $\sigma$ at time 5 years with $\mu = 0.08$, $\sigma = 0.2$ and tax rate is 31.15%. In the figure top left, top right, middle left, middle right, bottom left and bottom right panels correspond to $SD[V_t]$, $dSD[V_t]/dt$, $dSD[V_t]/dr$, $dSD[V_t]/d\mu$, $dSD[V_t]/di$, $dSD[V_t]/dt$ and $dSD[V_t]/d\sigma$, respectively. The standard deviation of the investment portfolio is high when the mortgage rate is low. If mortgage rate is low, more principal gets paid, hence,
the amount of new money available for investment grows faster, resulting in higher standard deviation. As we discussed earlier, $\mathbb{E}[V_t]$ increases with decrease in mortgage rate. Collectively the behaviors of $\mathbb{E}[V_t]$ and $SD[V_t]$ imply that high expected return from the investment portfolio comes with high standard deviation. The derivatives of standard deviation with respect to $\mu$ and $t$ decrease with increase in mortgage rate and line of credit rate which are the same patterns as the derivatives of mean with respect to $\mu$ and $t$. The derivatives of standard deviation with respect to $r$ and $i$ increase with increase in mortgage rate. As the mortgage rate increases the derivatives $dSD[V_t]/dr$ and $dSD[V_t]/di$ approach to zero. If mortgage rate is high then the principal is paid down slowly which slows down the growth of the investment portfolio and results in low rate of change in standard deviation with respect to $r$. Similarly, if the line of credit rate is high the new money available for investment grows slowly and hence the rate of change in standard deviation of the investment portfolio is small with respect to line of credit rate. The derivative of standard deviation with respect to $\sigma$ decreases with increase in the mortgage rate and the line of credit rate.

Figure 3.10 gives the plot of standard deviation of the investment portfolio and its derivatives across time for different values of mortgage rate with tax rate 31.15%, $i = 0.06$, $\mu = 0.08$ and $\sigma = 0.20$. The top left, top right, middle left, middle right and bottom left panels correspond to $SD[V_t]$, $SD[V_t]/dt$, $SD[V_t]/dr$, $SD[V_t]/di$, $SD[V_t]/d\mu$ and $SD[V_t]/d\sigma$, respectively. Different colored curves represent different mortgage rates. We investigate how the standard deviation and its derivatives evolve with time. With time, the standard deviation of the investment portfolio increases. The curves of the standard deviation for different mortgage rates spread apart as time marches along. With time the principal portion of the mortgage payment increases, hence, the amount of new money available for investment grows faster, resulting in higher standard deviation with time. The derivatives of the standard deviation with respect to
Figure 3.9: Level surface plot of standard deviation, $SD[V_t]$, and its derivatives at time 5 years. The tax rate, $c = 31.15\%$, drift, $\mu = 0.08$ and volatility, $\sigma = 0.20$. Top left, top right, middle left, middle right, bottom left and bottom right panels correspond to $SD[V_t]$, $dSD[V_t]/dr$, $dSD[V_t]/d\mu$, $dSD[V_t]/di$, $dSD[V_t]/d\sigma$ and $dSD[V_t]/dt$, respectively.

$t, \mu$ and $\sigma$ also show the same pattern that they are increasing functions of time and decreasing functions of mortgage rate. The derivatives of the standard deviation with respect to $r$ and $i$ are decreasing functions of time and increasing functions of mortgage rate.

Figure 3.11 gives the plot of standard deviation of the investment portfolio and its derivatives across time for different values of line of credit rate with tax rate 31.15\%, $r = 0.15$, $\mu = 0.08$ and $\sigma = 0.20$. The top left, top right, middle left, middle right and bottom left panels correspond to $SD[V_t]$, $SD[V_t]/dt$, $SD[V_t]/dr$, $SD[V_t]/di$, $SD[V_t]/d\mu$ and $SD[V_t]/d\sigma$, respectively. Different colored curves represent different line of credit rates. The curves of the standard deviation for different line of credit rates spread apart as time marches along. For instance, consider 1\% and 20\% line of credit rates. At any time $t$, the standard deviation, $SD[V_t]$, corresponding to 1\% line credit rate is greater than that for 20\% line of credit rate. The interest cost on the line of credit balance is deducted from the line of credit balance itself. Thus, higher the line of credit rate, the smaller the amount of new money available for investment. Therefore, investment portfolio standard deviation is lower when line of credit rate is higher. The derivatives of the standard deviation with respect to $t, \mu$ and $\sigma$ also show the same pattern that they are increasing functions of time and decreasing functions of line of credit rate. The derivatives of the standard deviation with respect to $r$ and $i$ are decreasing functions of time and increasing functions of line of credit rate.
3.3. Sensitivity of Investment Portfolio Value

Figure 3.10: Plot of standard deviation, $SD[V_t]$, and its derivatives with $i = 0.06$ and $\mu = 0.08$, for different values of $r$.

3.3.2 Sensitivity to drift, $\mu$ and volatility, $\sigma$

In this section we see how sensitive the mean and the standard deviation of the investment portfolio are to the different values of growth rate, $\mu$, and volatility, $\sigma$, of the risky investment. The range of values we take for $\mu$ and $\sigma$ is from 1% to 20% with an increment of 1%. Figure 3.12 makes a comparison of $E[V_t]$ and its derivatives with respect to the parameters, $r$, $\mu$, $i$, and $t$ at time 5 years with $r = 0.06$, $i = 0.06$ and tax rate is 31.15%. Along horizontal axis we have the growth rate, $\mu$. In the figure top left, top right, middle left, middle right and bottom left panels correspond to $E[V_t]$, $dE[V_t]/dr$, $dE[V_t]/d\mu$, $dE[V_t]/di$ and $dE[V_t]/dt$, respectively. Note that the mean of the investment portfolio does not depend on the volatility, therefore, the change in $\sigma$ does not affect mean, $E[V_t]$, and its derivatives. The mean, $E[V_t]$ increases with increase in $\mu$. It is because higher the growth rate of the stock the higher the mean of the investment portfolio. The derivatives of $E[V_t]$ with respect to $t$ and $\mu$ increase with $\mu$. The derivatives of $E[V_t]$ with respect to $r$ and $i$ decrease with $\mu$. The rates of decrease increase with $\mu$.

Figure 3.13 gives the plots of the mean and its derivatives across time for different values of $\mu$ with $r=6\% = i$ and tax rate of 31.15%. The top left, top right, middle left, middle right and bottom left panels correspond to $E[V_t]$, $dE[V_t]/dt$, $dE[V_t]/dr$, $dE[V_t]/di$ and $dE[V_t]/d\mu$, respectively. Different colored curves represent different growth rates, $\mu$. We investigate how the mean and its derivatives evolve with time for different values of $\mu$. It is evident from the figure that the mean $E[V_t]$ and its derivatives with respect to $t$ and $\mu$ increase with time. We have discussed this before that the investment portfolio value grows with time and with expected return on the stock. One crucial point to be noted here is that for small values of $\mu$ i.e., from 1% to 10%, the mean $E[V_t]$ does not grow that much. It is clear that the mean $E[V_t]$ grows quickly with time when $\mu$ is higher than 10%. The curves for $\mu$ greater than 10% spread apart when time marches along implying that the higher the $\mu$ the higher the values of $E[V_t]$. 
Figure 3.11: Plot of standard deviation, $SD[V_t]$, and its derivatives with $r = 0.15$ and $\mu = 0.08$, for different values of $i$.

This is one of the advantages of the re-advanceable mortgage strategy that the homeowner invests the entire proceeds of the line of credit into a stock with an expectation that it grows at a faster rate than the mortgage rate and line of credit rate. The derivatives of $E[V_t]$ with respect to $r$ and $i$ decrease with time.

Figure 3.14 makes a comparison of the standard deviation $SD[V_t]$ and its derivatives with respect to the parameters, $r$, $\mu$, $i$ and $t$ at time 5 years with $r = 0.06$, $i = 0.06$ and tax rate 31.15%. In the figure top left, top right, middle left, middle right, bottom left and bottom right panels correspond to $SD[V_t]$, $dSD[V_t]/dr$, $dSD[V_t]/d\mu$, $dSD[V_t]/di$, $dSD[V_t]/dt$ and $dSD[V_t]/d\sigma$, respectively. The standard deviation, $SD[V_t]$ increases with $\sigma$ and $\mu$. It is the highest when $\mu$ and $\sigma$ are the highest (i.e., 20%). Therefore, while considering the re-advanceable mortgage scheme one should keep in mind that investing in a high growth risky asset increases the expected return but it comes with the cost of high standard deviation of the investment portfolio. The derivatives of $SD[V_t]$ with respect to $r$ and $i$ decrease with $\mu$ and $\sigma$.

Figure 3.15 gives the plots of the standard deviation and its derivatives across time for different values of $\mu$ with tax rate 31.15%, $r=6\%=i$ and $\sigma = 0.2$. The top left, top right, middle left, middle right, bottom left and bottom right panels correspond to $SD[V_t]$, $dSD[V_t]/dt$, $dSD[V_t]/dr$, $dSD[V_t]/d\mu$, $dSD[V_t]/di$ and $dSD[V_t]/d\sigma$, respectively. Different colored curves represent different growth rates, $\mu$. We investigate how the standard deviation and its derivatives evolve with time for different values of $\mu$. The standard deviation and its derivatives with respect to $t$, $\mu$ and $\sigma$ increase with time and also with $\mu$. When $\mu$ is small the standard deviation is small but that corresponds with a low mean value of the investment portfolio. The derivatives of standard deviation with respect to $r$ and $i$ decrease with time and also with $\mu$.

Figure 3.16 gives the plots of the standard deviation and its derivatives across time for different values of $\sigma$ with tax rate 31.15%, $r=6\%=i$ and $\mu = 0.08$. The top left, top right, middle...
3.3. Sensitivity of Investment Portfolio Value

Figure 3.12: Plot of mean, $\mathbb{E}[V_t]$, and its derivatives with $r = 0.06$, $i = 0.06$ and $t = 5$ years.

Left, middle right, bottom left and bottom right panels correspond to $SD[V_t]$, $dSD[V_t]/dt$, $dSD[V_t]/dr$, $dSD[V_t]/di$, $dSD[V_t]/d\mu$ and $dSD[V_t]/d\sigma$, respectively. Different colored curves represent different $\sigma$ values. The $SD[V_t]$ and its derivatives with respect to $t$, $\mu$ and $\sigma$ increase $\sigma$. For small values of $\sigma$, the $SD[V_t]$ and its derivatives with respect to $t$, $\mu$ and $\sigma$ grow bigger with time. This was not the case with $\mathbb{E}[V_t]$, $d\mathbb{E}[V_t]/dt$ and $d\mathbb{E}[V_t]/d\mu$, for them for small $\mu$ the change was not big. This shows that even for small values of $\sigma$, the rate of change in standard deviation of the investment portfolio can be bigger than the rate of change in the mean of the investment portfolio.
Figure 3.13: Plot of mean, $\mathbb{E}[V_t]$, and its derivatives with $r = 0.06 = i$ and tax rate 31.15% for different values of $\mu$.

Figure 3.14: Level surface plot of variance, $SD[V_t]$, and its derivatives at time 5 years with tax rate, $c = 31.15\%$, $r = 0.06$ and $i = 0.06$. Top left, top right, middle left, middle right, bottom left and bottom right panels correspond to $SD[V_t]$, $dSD[V_t]/dr$, $dSD[V_t]/d\mu$, $dSD[V_t]/di$, $dSD[V_t]/d\sigma$ and $dSD[V_t]/dt$, respectively.
3.3. Sensitivity of Investment Portfolio Value

Figure 3.15: Plot of standard deviation, $SD[V_t]$, and its derivatives with $r = 0.06$, $i = 0.06$ and $\sigma = 0.20$ for different values of $\mu$.

Figure 3.16: Plot of standard deviation, $SD[V_t]$, and its derivatives with $r = 0.06$, $i = 0.06$ and $\mu = 0.08$ for different values of $\sigma$. 
3.3.3 A comprehensive set of figures

A comprehensive set of figures showing the sensitivities of $\mathbb{E}[V_t]$ and $SD[V_t]$ for various parameter settings is provided in Appendix C.1. Tables 3.4 and 3.5 summarize the list of figures according to values of $\mu$, $\sigma$, $r$ and $i$. The range of values taken for $r$ and $i$ are the same from 1% to 20% with an increment of 1%. Plots only show the curves for the pairs $(r, i)$ so that $dI_t > 0$.

Table 3.4: List of figures for mean, $\mathbb{E}[V_t]$ and its derivatives.

<table>
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<th>$\mu$</th>
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<th>Mortgage rate $r$</th>
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Table 3.5: List of figures for standard deviation, $SD[V_t]$ and its derivatives.

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3.4 Conclusion

In this chapter we studied the behavior of the investment capital to different values of the mortgage rate, \( r \) and the line of credit rate \( i \). The range of values for \( r \) and \( i \) is from 1\% to 20\% with an increment size of 1\%. Considering all possible pairs of \((r, i)\) we find that only if the difference, \( i - r \), exceeds 5\% will \( dI_t \) become negative for some \( t > 0 \). A spread this large is unlikely to occur in practice. We collected mortgage and the line of credit rates from five different banks on February 23, 2012. The maximum \( r - i \) spread we get is 3.25\%. Thus, we can safely say that \( dI_t \) will always be positive for realistic values of \( r \) and \( i \).

We investigate the sensitivity of the mean and standard deviation of the risky investment to different parameters such as time \( t \), mortgage rate, \( r \), the line of credit rate, \( i \), growth rate, \( \mu \), and volatility, \( \sigma \). The mean, \( \mathbb{E}[V_t] \), and the standard deviation, \( SD[V_t] \), grow with time. The derivatives of mean with respect to \( t \) and \( \mu \), and the derivatives of standard deviation with respect to \( t \), \( \mu \) and \( \sigma \) are increasing functions of time. The derivatives of mean and standard deviation with respect to \( i \) are decreasing functions of time. The derivative of mean with respect to \( r \) appears concave up. The function decreases between the time period 0 years and amortization term. After the amortization term it increases. The derivatives of standard deviations with respect to \( r \) and \( i \) are decreasing functions of time.

After the amortization term the whole amount of mortgage payment is invested, therefore, if mortgage rate is higher the more money is available for new investment. It is for this reason that the mean, \( \mathbb{E}[V_t] \) and its derivative with respect to \( t \), are decreasing functions of \( r \) before the amortization term and they are increasing functions of \( r \) after the amortization term. The derivatives of mean with respect to \( r \) and \( i \) are increasing functions of \( r \). The derivative of mean with respect to \( \mu \) is a decreasing function of \( r \). The standard deviation, \( SD[V_t] \), and its derivatives with respect to \( t \), \( \sigma \) and \( \mu \) are decreasing functions of \( r \). The derivatives of \( SD[V_t] \) with respect to \( r \) and \( i \) are increasing functions of \( r \).

As the line of credit rate, \( i \), increases less new money is available for investment, hence, \( \mathbb{E}[V_t] \) decreases. The derivatives of mean with respect to \( t \) and \( \mu \) are also decreasing functions of \( i \). The derivative of mean with respect to \( r \) is an increasing function of \( i \). The standard deviation, \( SD[V_t] \), and its derivatives with respect to \( t \), \( \sigma \) and \( \mu \) are decreasing functions of \( i \). The derivatives of \( SD[V_t] \) with respect to \( r \) and \( i \) are increasing functions of \( i \).

The \( \mathbb{E}[V_t] \) and its derivatives with respect to \( t \) and \( \mu \) are increasing functions of \( \mu \). The derivatives of \( \mathbb{E}[V_t] \) with respect to \( r \) and \( i \) are decreasing functions of \( \mu \). The growth of \( \mathbb{E}[V_t] \) is not big enough if \( \mu \) is small (between 1\% to 10\%). The higher the \( \mu \) the higher the values of \( \mathbb{E}[V_t] \). The \( SD[V_t] \) and its derivatives with respect to \( t \), \( \sigma \) and \( \mu \) increases with \( \mu \) and \( \sigma \). The derivatives of \( SD[V_t] \) with respect to \( r \) and \( i \) decrease with \( \mu \) and \( \sigma \).
Chapter 4

Incorporating Variable Housing Prices and Interest Rates

4.1 Introduction

In Chapter 1 we considered a fixed-rate mortgage and presented a mathematical model for the re-advanceable mortgage scheme. A homeowner has a mortgage of principal $L$ with amortization term $m$ years with fixed annual continuous rate $r$. The homeowner repays the mortgage continuously at the annual rate $P$ given by

$$P = \frac{Lre^rm}{e^rm - 1}. \quad (4.1)$$

The mortgage principal outstanding at time $t$ is calculated by

$$B_t = \begin{cases} 
\frac{t(e^rm - e^{rt})}{e^rm - 1}, & \text{if } 0 \leq t \leq m, \\
0, & \text{if } t > m. 
\end{cases} \quad (4.2)$$

As the mortgage is paid the principal of the home equity line of credit increases by the principal portion of the mortgage payment. Thus, the time-$t$ line of credit balance, $C_t$, is

$$C_t = L - B_t = \begin{cases} 
L \frac{e^{rt} - 1}{e^rm - 1}, & \text{if } 0 \leq t \leq m, \\
L, & \text{if } t > m. 
\end{cases} \quad (4.3)$$

The rate of change of $C_t$ is

$$dC_t = dB_t = \begin{cases} 
\frac{Lre^{rt}}{e^rm - 1} dt, & \text{if } 0 \leq t \leq m, \\
0, & \text{if } t > m. 
\end{cases} \quad (4.4)$$

The interest on the line of credit and the tax rebate are paid continuously at the annualized rates of $i$ and $c$, respectively. If $I_t$ is the time-$t$ investment capital then the time-$t$ amount of money available for new investment is

$$dl_t = dC_t - i(1 - c)C_t dt + PdtI_t > m = \begin{cases} 
\frac{L}{e^rm - 1} \left[ re^{rt} - i(1 - c)(e^{rt} - 1) \right] dt, & \text{if } 0 \leq t \leq m, \\
\frac{L}{e^rm - 1} \left[ re^{rm} - i(1 - c)(e^{rm} - 1) \right] dt, & \text{if } t > m. 
\end{cases} \quad (4.5)$$
where $I_A$ is an indicator function that is one if $A$ is true and is zero otherwise.

The homeowner invests the entire proceeds from the line of credit borrowing into a risky asset, a stock. The dynamics of the stock price, $S_t$, are given by geometric Brownian motion

$$dS_t = \mu S_t dt + \sigma_S S_t dW_t,$$

where $W$ is a Wiener process, $\mu$ is the drift and $\sigma_S$ is the volatility. The stochastic differential equation for the investment portfolio value, $V_t$, is given by

$$dV_t = (\mu V_t + \sigma_S V_t dW_t) dt + \sigma S V_t dW_t,$$

$$= \begin{cases} 
(\mu V_t + L \frac{e^{r_p t}}{r_p t} - i(1-c)L \frac{e^{r_p t}}{r_p t-1}) dt + \sigma S V_t dW_t, & \text{if } 0 \leq t \leq m, \\
(\mu V_t + L \frac{e^{r_p t}}{r_p t} - i(1-c)L) dt + \sigma S V_t dW_t, & \text{if } t > m. 
\end{cases}$$

(4.7)

A quantity of interest is the mortgage payoff time, $\tau$, which we defined as the first time the value of the risky investment portfolio is at least the value of the original mortgage amount. The payoff time is defined as

$$\tau = \inf \{ t \geq 0 | V_t \geq L \}.$$  

(4.8)

Another quantity of interest is the total interest cost, $I^*_m$, of the original mortgage which is defined as

$$I^*_m = Pm - L.$$  

(4.9)

The total cost of the re-advanceable mortgage can be different from $I^*_m$ depending on the payoff time $\tau$ and it is defined as

$$I^*_\tau = \begin{cases} 
P\tau - C\tau, & \text{if } \tau < m, \\
P\tau - L, & \text{if } \tau \geq m. 
\end{cases}$$

(4.10)

For $\tau < m$, $P\tau$ is the total cost of repaying $C\tau$ by time $\tau$.

The homeowner keeps on making mortgage payment after the amortization term $m$ years if $\tau > m$. This way there would be new money available for the investment as no more money can be re-borrowed in the line of credit once $C\tau$ is $L$. In other words, if $\tau > m$ then the homeowner needs to make up investment losses by making mortgage payments until $V_t \geq L$. When $\tau = m$, $I^*_\tau$ is exactly equal to $I^*_m$.

As mentioned earlier so far in this thesis a mathematical model for the re-advanceable mortgage investment strategy was derived assuming that the mortgage rate remains fixed throughout the mortgage term. For simplicity we assumed that the mortgage term is same as the amortization term. In reality this is not the case as the mortgage rate gets renewed on the expiry of a mortgage term (which is typically from six months to 10 years). In this chapter we incorporate the fact that the mortgage rate is not fixed by considering a variable rate mortgage. We still assume that the homeowner makes a fixed mortgage payment but the ratio between the principal and interest components of the payment fluctuates. When interest rates are low less of the mortgage payment goes to interest and more goes to principal. If rates are high more goes to interest and less to principal. This typically has the effect of introducing uncertainty in the payoff time for a traditional mortgage.

In this chapter we also incorporate the fact that housing prices change with time. If there is a rise in the house price then the home equity increases. This additional home equity allows the homeowner to increase the line of credit limit and invest more money in the stock.
4.2 Interest Rate Model

The re-advanceable mortgage scheme was discussed in detail in Chapter 1 for a fixed-rate mortgage. Here, we consider a variable rate mortgage which changes continuously throughout the mortgage life. First we consider a mean reverting interest rate model without any diffusion term. Subsequently, we incorporate diffusion into the mean reverting model by using the Cox-Ingersoll-Ross (CIR) interest rate model.

4.2.1 Mean Reverting Interest Rate Model

The mean reverting interest rate model without a diffusion term is given as

\[ dr_t = \kappa(\theta - r_t)dt, \]  

(4.11)

where \( \kappa \) and \( \theta \) are positive constants and denote respectively, the mean reversion speed and the mean reversion level of the short rate \( r_t \). Using the initial interest rate \( r_0 \) the solution of Equation 4.11 is given as

\[ r_t = \theta - (\theta - r_0)e^{-\kappa t}. \]  

(4.12)

Figure 4.1 is a plot of \( r_t \) given by Equation 4.12 for different values of \( r_0 \) and \( \kappa \). We fix the long run interest rate \( \theta \) at 6% and we consider two values of the initial interest rate, \( r_0 \), as 2% and 8%. The mean reversion speed \( \kappa \) takes values 0.1, 0.3 and 0.5. As time marches along the interest rate \( r_t \) moves towards its equilibrium level \( \theta \). The bigger the value of \( \kappa \) the faster \( r_t \) reverts to the long run interest rate \( \theta \). Note that if \( r_0 = \theta \), then the interest rate is constant.

Mathematical Formulation of Re-advanceable Mortgage

Consider a mortgage with principal amount \( L \) and amortization term of \( m \) years. Assume that the homeowner pays back the mortgage at continuous rate \( P \) and the mortgage rate is driven
by mean reverting rate model given by Equation 4.11. If we accumulate all the mortgage payments made over \( m \) years and multiply that by the discount factor \( e^{-\int_0^t r_s \, ds} \) then we get the present value of the payments which is \( L \) i.e., \( L = \int_0^m P e^{-\int_0^t r_s \, ds} \, dt \). As \( P \) is constant it can be taken out of the integral. Using this expression we calculate the mortgage payment rate as

\[
P = \frac{L}{\int_0^m e^{-\int_0^t r_s \, ds} \, dt}.
\]  

(4.13)

The mortgage principal outstanding balance, \( B_t \), is defined as

\[
B_t = \begin{cases} 
Le^{\int_0^t r_s \, ds} - P \int_0^t e^{\int_0^s r_u \, du} \, ds, & \text{if } 0 \leq t \leq m, \\
0, & \text{if } t > m.
\end{cases}
\]  

(4.14)

Using Equation 4.12 we can evaluate the following integrals

\[
\int_0^t r_u \, du = \theta t + \frac{r_0 - r_t}{\kappa},
\]  

(4.15)

\[
\int_s^t r_u \, du = \theta(t - s) + \frac{r_s - r_t}{\kappa}, \text{ and}
\]  

(4.16)

\[
\int_0^t e^{\int_0^s r_u \, du} \, ds = \int_0^t e^{\theta(t-s)+\frac{r_s-r_0}{\kappa}} \, ds.
\]  

(4.17)

Let

\[
u = r_s = \theta - (\theta - r_0)e^{-xs},
\]  

(4.18)

\[
du = \kappa(\theta - r_0)e^{-xs} \, ds = \kappa(\theta - r_s)ds = \kappa(\theta - u)ds.
\]  

(4.19)

Using the Equation 4.18 we get

\[
s = -\frac{1}{\kappa} \ln \left( \frac{\theta - u}{\theta - r_0} \right).
\]  

(4.20)

When \( s = 0 \) then \( u = r_0 \) and when \( s = t \) then \( u = r_t \). Use Equations 4.18, 4.19 and 4.20 to evaluate the integral given by Equation 4.17

\[
\int_0^t e^{\theta(t-s)+\frac{r_s-r_0}{\kappa}} \, ds = e^{\theta t - \frac{1}{\kappa}} \int_0^t e^{\frac{r_s-r_0}{\kappa} + \frac{\theta}{\kappa}} \frac{du}{\kappa(\theta - u)}
\]  

\[
= e^{\theta t - \frac{1}{\kappa}} \frac{1}{\kappa} \int_0^t e^{\ln \left( \frac{\theta - u}{\theta - r_0} \right) + \frac{\theta}{\kappa}} \frac{1}{\theta - u} \, du,
\]  

(4.21)

\[
= e^{\theta t - \frac{1}{\kappa}} \frac{1}{\kappa} \left( \frac{1}{\theta - r_0} \right)^{\frac{\theta}{\kappa}} \int_0^t (\theta - u)^{\frac{\theta}{\kappa}-1} e^{\frac{\theta}{\kappa}u} \, du.
\]  

(4.22)

Substitute \( f = \frac{\theta - u}{\kappa} \) and \( df = -\frac{1}{\kappa} \, du \). The limits of the integral will change accordingly. There-
4.2. INTEREST RATE MODEL

Therefore,
\[
\int_0^t e^{\theta(t-s)+\frac{t-r_0}{\kappa}} ds = e^{\theta \frac{t-u}{\kappa}} \left( \frac{1}{\theta - r_0} \right) ^{\frac{\theta}{\kappa}} \int_{\theta = 0}^{\theta = \frac{\theta}{\kappa}} (\kappa f)^{\frac{\theta}{\kappa}-1} e^{\frac{\theta}{\kappa} f} (-\kappa) df.
\]

\[
= e^{\theta \frac{t-u}{\kappa}} \left( \frac{1}{\theta - r_0} \right) ^{\frac{\theta}{\kappa}} \int_{\theta = 0}^{\theta = \frac{\theta}{\kappa}} (\kappa f)^{\frac{\theta}{\kappa}-1} e^{\frac{\theta}{\kappa} f} df,
\]

\[
= e^{\theta \frac{t-u}{\kappa}} \left( \frac{1}{\theta - r_0} \right) ^{\frac{\theta}{\kappa}} \left[ \int_{0}^{\theta = \frac{\theta}{\kappa}} (\kappa f)^{\frac{\theta}{\kappa}-1} e^{\frac{\theta}{\kappa} f} df - \int_{0}^{\theta = \frac{\theta}{\kappa}} (\kappa f)^{\frac{\theta}{\kappa}-1} e^{\frac{\theta}{\kappa} f} df \right],
\]

\[
e^{\theta \frac{t-u}{\kappa}} \left( \frac{1}{\theta - r_0} \right) ^{\frac{\theta}{\kappa}} \left[ \gamma \left( \frac{\theta}{\kappa}, \frac{\theta - r_0}{\kappa} \right) - \gamma \left( \frac{\theta}{\kappa}, \frac{\theta - r_t}{\kappa} \right) \right]. \quad (4.24)
\]

In the above equation \( \gamma(s, x) \) is the lower incomplete gamma function and it is defined as
\[
\gamma(s, x) = \int_0^x t^{s-1} e^{-t} dt. \quad (4.25)
\]

Note that for Equation 4.24 to make sense we must have \( r_0 < \theta \). For the case \( r_0 > \theta \) we need to substitute \( s = -\frac{1}{\kappa} \ln \left( \frac{u - \theta}{r_0 - \theta} \right) \) to get
\[
\int_0^t e^{\theta(t-s)+\frac{t-r_0}{\kappa}} ds = e^{\theta \frac{t-u}{\kappa}} \int_{r_0}^{t} e^{\frac{\theta}{\kappa} \ln \left( \frac{u - \theta}{r_0 - \theta} \right) + \frac{\theta}{\kappa} u} \frac{du}{\kappa(\theta - u)} \quad (4.26)
\]

\[
= e^{\theta \frac{t-u}{\kappa}} \left( \frac{1}{r_0 - \theta} \right) ^{\frac{\theta}{\kappa}} (-1) \int_{r_0}^{\theta} (u - \theta)^{\frac{\theta}{\kappa}-1} e^{\frac{\theta}{\kappa} u} du. \quad (4.27)
\]

Substitute \( f = \frac{u-\theta}{r_0-\theta} \) and \( df = \frac{1}{\kappa} du \). The limits of the integral will change accordingly. Therefore,
\[
\int_0^t e^{\theta(t-s)+\frac{t-r_0}{\kappa}} ds = e^{\theta \frac{t-u}{\kappa}} \left( \frac{1}{r_0 - \theta} \right) ^{\frac{\theta}{\kappa}} (-1) \int_{r_0}^{\theta} (\kappa f)^{\frac{\theta}{\kappa}-1} e^{\frac{\theta}{\kappa} f} \kappa df.
\]

\[
= e^{\theta \frac{t-u}{\kappa}} \left( \frac{1}{r_0 - \theta} \right) ^{\frac{\theta}{\kappa}} \int_{r_0}^{\theta} (\kappa f)^{\frac{\theta}{\kappa}-1} e^{\frac{\theta}{\kappa} f} df. \quad (4.28)
\]

Define \( J_1 = \int_0^u x^{\frac{\theta}{\kappa}-1} e^x dx \) where \( c = \frac{\theta - \theta}{\kappa}, d = \frac{\theta - \theta}{\kappa}, y = \frac{\theta}{\kappa} \) and \( x = f \). \( J_1 \) is an ill defined definite integral. To evaluate this integral we can either approximate it numerically or approximate the integrand by using a Taylor series expansion of \( e^x \) and then integrate term-by-term. We use a Taylor series expansion and it is defined as
\[
e^x = P_k(x) + R_k(x) = \sum_{n=0}^{k} \frac{g^{(n)}(0)}{n!} x^n + \frac{g^{(k+1)}(\xi x)}{(k+1)!} x^{k+1}. \quad (4.29)
\]

In the above equation, \( R_k(x) = \frac{g^{(k+1)}(\xi)}{(k+1)!} x^{k+1} \) is a Lagrange form of remainder for some \( 0 < \xi x < x \). \( g^{(n)}(0) \) is the \( n \)-th derivative of \( e^x \) at 0 which is 1. Therefore,
\[
e^x = \sum_{n=0}^{k} \frac{x^n}{n!} + \frac{e^\xi x}{(k+1)!} x^{k+1}. \quad (4.30)
\]
Using Equation [4.30] the integral $I_1$ becomes

$$J_1 = \int_c^d x^{y-1} \left( \sum_{n=0}^{k} \frac{x^n}{n!} + \frac{e^{\xi x}}{(k+1)!} x^{k+1} \right) \, dx,$$

$$= \int_c^d x^{y-1} \sum_{n=0}^{k} \frac{x^n}{n!} \, dx + \int_c^d \frac{e^{\xi x}}{(k+1)!} x^{k+1} \, dx,$$

$$= \sum_{n=0}^{k} \frac{1}{n!} \int_c^d x^{y+n-1} \, dx + \int_c^d \frac{e^{\xi x}}{(k+1)!} x^{k+1} \, dx,$$

$$= \sum_{n=0}^{k} \frac{1}{n!} \left( \frac{d^{y+n} - c^{y+n}}{y + n} \right) + \int_c^d \frac{e^{\xi x}}{(k+1)!} x^{k+1} \, dx. \quad (4.31)$$

Note that since $d \geq x \geq c$ and $e^x$ is an increasing function, we have $e^d > e^{\xi x} > e^0 = 1$. Using this inequality we have

$$0 < \int_c^d \frac{e^{\xi x}}{(k+1)!} x^{k+1} \, dx < \frac{e^d}{(k+1)!} \int_c^d x^{k+1} \, dx,$$

$$= \frac{e^d}{(k+1)!} \frac{d^{y+k+1} - c^{y+k+1}}{y + k + 1}. \quad (4.32)$$

Thus we can find the number of terms, $k$, in the Taylor polynomial so that $\int_c^d \frac{e^{\xi x}}{(k+1)!} x^{k+y} \, dx$ is arbitrarily small. The value of $k$ depends on $r_0$, $\kappa$, $\theta$, and $r_t$ through the definitions of $c$, $d$ and $y$. We can give a simple upper bound on $\int_c^d \frac{e^{\xi x}}{(k+1)!} x^{k+y} \, dx$ that does not depend on $r_t$ (and hence $t$).

In Equation [4.32] we established that

$$\int_c^d \frac{e^{\xi x}}{(k+1)!} x^{k+y} \, dx < \frac{e^d}{(k+1)!} \frac{d^{y+k+1} - c^{y+k+1}}{y + k + 1},$$

$$< \frac{e^d}{(k+1)!} \frac{d^{y+k+1}}{y + k + 1}. \quad (4.33)$$

For a given $\epsilon > 0$, choose $k$ such that

$$\frac{e^d}{(k+1)!} \frac{d^{y+k+1}}{y + k + 1} < \epsilon. \quad (4.34)$$

This will ensure that

$$\left| \int_c^d x^{y-1} e^x \, dx - \sum_{n=0}^{k} \frac{1}{n!} \int_c^d x^{y+n-1} \, dx \right| < \epsilon, \quad (4.35)$$

for all $t$.

Using the approximation of the integral $\int_c^d x^{y-1} e^x \, dx$ which is given by the first term in Equation [4.31] Equation [4.28] becomes

$$\int_0^\gamma e^{\theta (t-s) + \frac{\gamma r_t}{x}} \, ds = e^{\theta (t-s) + \frac{\gamma r_t}{x}} \left( \frac{1}{r_0 - \theta} \right)^{\frac{\gamma}{r_0 - \theta}} \left( \sum_{n=0}^{k} \frac{1}{n!} \left( \frac{d^{y+n} - c^{y+n}}{y + n} \right) \right). \quad (4.36)$$
where \( c = \frac{r_0 - \theta}{k}, d = \frac{r_0 - \theta}{k}, y = \frac{\theta}{k} \) and \( x = f \). Define \( J_2 = \int_0^t e^{\theta(t-s) + \frac{r_0 - \theta}{k}} ds \) and combine the previously derived two evaluations of \( J_2 \) for two cases: \( r_0 < \theta \) and \( r_0 > \theta \). This way we get a piece-wise function for \( J_2 \),

\[
J_2 = \begin{cases} 
  e^{\theta \frac{t - \theta}{k}} \left( \frac{1}{\theta - r_0} \right)^{\frac{\theta}{k}} \left( \gamma \left( \frac{\theta}{k}, \frac{\theta - r_0}{k} \right) - \gamma \left( \frac{\theta}{k}, \frac{\theta - r_0}{k} \right) \right), & \text{if } r_0 < \theta, \\
  e^{\theta \frac{t - \theta}{k}} \left( \frac{1}{r_0 - \theta} \right)^{\frac{\theta}{k}} \left( \gamma \left( \frac{\theta}{k}, \frac{\theta - r_0}{k} \right) - \gamma \left( \frac{\theta}{k}, \frac{\theta - r_0}{k} \right) \right), & \text{if } r_0 > \theta.
\end{cases}
\]  

(4.37)

Note that if \( r_0 = \theta \) the constant interest rate scenario analyzed earlier applies. Substituting Equations 4.15 and 4.37 into Equation 4.14 we get

\[
B_t = \begin{cases} 
  Le^{\theta \frac{t - \theta}{k}} - PJ_2, & \text{if } 0 \leq t \leq m, \\
  0, & \text{if } t > m.
\end{cases}
\]  

(4.38)

where \( P \) and \( J_2 \) are defined by Equations 4.13 and 4.37 respectively. Note that the mortgage payment, \( P \), involves the integral \( \int_0^m e^{-\frac{\theta}{k} r_0} ds \). The evaluation of this integral follows the same series of steps as we did for the integral, \( J_2 \). Therefore,

\[
J_3 = \int_0^m e^{-\frac{\theta}{k} r_0} ds dt,
\]

\[
= \begin{cases} 
  e^{-\frac{\theta}{k} r_0} \left( \frac{1}{\theta - r_0} \right)^{\frac{\theta}{k}} \left( \gamma \left( \frac{\theta}{k}, \frac{\theta - r_0}{k} \right) - \gamma \left( \frac{\theta}{k}, \frac{\theta - r_0}{k} \right) \right), & \text{if } r_0 < \theta, \\
  e^{-\frac{\theta}{k} r_0} \left( \frac{1}{r_0 - \theta} \right)^{\frac{\theta}{k}} \left( \gamma \left( \frac{\theta}{k}, \frac{\theta - r_0}{k} \right) - \gamma \left( \frac{\theta}{k}, \frac{\theta - r_0}{k} \right) \right), & \text{if } r_0 > \theta.
\end{cases}
\]  

(4.39)

Substituting Equation 4.39 into Equation 4.13 gives

\[
P = \begin{cases} 
  -\frac{\theta}{k} \left( \frac{1}{\theta - r_0} \right)^{\frac{\theta}{k}} \gamma \left( \frac{\theta}{k}, \frac{\theta - r_0}{k} \right), & \text{if } r_0 < \theta, \\
  \frac{\theta}{k} \left( \frac{1}{r_0 - \theta} \right)^{\frac{\theta}{k}} \gamma \left( \frac{\theta}{k}, \frac{\theta - r_0}{k} \right), & \text{if } r_0 > \theta.
\end{cases}
\]  

(4.40)

Note that this payment rate makes the amortization term exactly \( m \) years. This is possible since, for the mean reverting model, all future time-\( t \) interest rates \( r_i \) are known at time 0.

Let \( \Gamma_m^* \) be the total interest cost of the original mortgage if the amortization term is exactly \( m \) years for the given dynamics. It can be evaluated by subtracting the original principal, \( L \) from the total of all the mortgage payments.

\[
\Gamma_m^* = Pm - L
\]  

(4.41)

where \( P \) is given by Equation 4.40.

The dynamics of \( B_t \) can be found using the total derivative

\[
dB_t = \frac{\partial B_t}{\partial t} dt + \frac{\partial B_t}{\partial r_i} dr_i.
\]  

(4.42)

\[
\frac{\partial B_t}{\partial r_i} = L \theta e^{\theta \frac{t - \theta}{k}} - P \frac{\partial J_2}{\partial t}
\]  

(4.43)
We know that \( I_2 = \int_0^1 e^{\theta(t-s) + \frac{r(t-s)}{\kappa}} ds \). Let \( f(s, t) = e^{\theta(t-s) + \frac{r(t-s)}{\kappa}} \), therefore

\[
J_2 = \int_0^1 f(s, t) \, ds  
\]  \hspace{1cm} (4.44)

We use Leibniz’s rule in order to get the partial derivative of \( J_2 \) with respect to \( t \). Leibniz’s rule says that if there is a function \( f(s, t) \) such that its derivative with respect to \( t \), \( f_s(t, s) \) exists and it is continuous then

\[
\frac{d}{dt} \left( \int_{a(t)}^{b(t)} f(s, t) \, ds \right) = f(b(t), t)b'(t) - f(a(t), t)a'(t) + \int_{a(t)}^{b(t)} f_s(s, t) \, ds.  
\]  \hspace{1cm} (4.45)

Using the above rule and the derivative \( f_s(s, t) = \theta f(s, t) \) we get

\[
\frac{\partial J_2}{\partial t} = 1 + \theta \int_0^t f(s, t) \, ds = 1 + \theta J_2.  
\]  \hspace{1cm} (4.46)

Substituting the above result in Equation (4.43) gives

\[
\frac{\partial B_t}{\partial t} = L\theta e^{\theta t + \frac{r(t)}{\kappa}} - P(1 + \theta J_2).  
\]  \hspace{1cm} (4.47)

Now

\[
\frac{\partial B_t}{\partial r_t} = -\frac{L_r}{\kappa} e^{\theta t + \frac{r(t)}{\kappa}} - \frac{P}{\kappa} \frac{\partial J_2}{\partial r_t}  
\]  \hspace{1cm} (4.48)

where \( \frac{\partial J_2}{\partial r_t} = \int_0^t \frac{\partial f(s, t)}{\partial r_t} \, ds = -\frac{1}{\kappa} J_2 \). Therefore,

\[
\frac{\partial B_t}{\partial r_t} = -\frac{L_r}{\kappa} e^{\theta t + \frac{r(t)}{\kappa}} + \frac{P}{\kappa} J_2.  
\]  \hspace{1cm} (4.49)

Substituting Equations (4.47) and (4.49) into Equation (4.42) gives

\[
dB_t = \left( L\theta e^{\theta t + \frac{r(t)}{\kappa}} - P(1 + \theta J_2) \right) dt + \left( -\frac{L_r}{\kappa} e^{\theta t + \frac{r(t)}{\kappa}} + \frac{P}{\kappa} J_2 \right) dr_t,  
\]

\[
= \left( L\theta e^{\theta t + \frac{r(t)}{\kappa}} - P(1 + \theta J_2) \right) dt + \left( -\frac{L_r}{\kappa} e^{\theta t + \frac{r(t)}{\kappa}} + \frac{P}{\kappa} J_2 \right) \kappa(\theta - r_t) dt,  
\]

\[
= \left( Lr_t e^{\theta t + \frac{r(t)}{\kappa}} - P(1 + r_t J_2) \right) dt.  
\]  \hspace{1cm} (4.50)

\[
\Rightarrow dB_t = \begin{cases} 
Lr_t e^{\theta t + \frac{r(t)}{\kappa}} - P(1 + r_t J_2) & \text{if } 0 \leq t \leq m, \\
0 & \text{if } t > m. 
\end{cases}  
\]  \hspace{1cm} (4.51)

We know that time-\( t \) value of line of credit balance, \( C_t \), is evaluated as

\[
C_t = L - B_t = \begin{cases} 
L \left( 1 - e^{\theta t + \frac{r(t)}{\kappa}} \right) + PJ_2 & \text{if } 0 \leq t \leq m, \\
L & \text{if } t > m. 
\end{cases}  
\]  \hspace{1cm} (4.52)
and the dynamics of $C_t$ are

$$
dC_t = \begin{cases} 
-Lr_t e^{\theta t + \frac{r - r_t}{2}} - P(1 + r_t J_2) \ dt, & \text{if } 0 \leq t \leq m, \\
0, & \text{if } t > m. 
\end{cases}
$$

(4.53)

Note that after the amortization term $m$ the homeowner will keep on making the mortgage payment which contributes towards the line of credit balance. Let $i$ and $c$ be the continuous rate of interest on line of credit balance and tax rebate on the interest cost, respectively. The new money, $dI_t$, available for investment can be found by using the relation $dI_t = dC_t - i(1 - c)C_t dt + PdW_{t \geq m}$. Therefore, $dI_t$ is

$$
dI_t = \begin{cases} 
(P(1 + r_t J_2) - Lr_t e^{\theta t + \frac{r - r_t}{2}} - i(1 - c)\left(L\left(1 - e^{\theta t + \frac{r - r_t}{2}}\right) + PJ_2\right)) dt, & \text{if } 0 \leq t \leq m, \\
Pdt - i(1 - c)L dt, & \text{if } t > m. 
\end{cases}
$$

(4.54)

The homeowner invests entire the proceeds from the line of credit into a single risky asset, a stock, whose dynamics are given by Equation 4.6. The investment portfolio value, $V_t$, evolves according to the following stochastic differential equation

$$
dV_t = (\mu dt + \sigma S dW_t) V_t + dI_t.
$$

(4.55)

$$
dV_t = \begin{cases} 
\left[\mu V_t + P(1 + r_t J_2) - Lr_t e^{\theta t + \frac{r - r_t}{2}} - i(1 - c)\left(L\left(1 - e^{\theta t + \frac{r - r_t}{2}}\right) + PJ_2\right)\right] dt + \\
\sigma S V_t dW_t, & \text{if } 0 \leq t \leq m, \\
[\mu V_t + P - i(1 - c)L] dt + \sigma S V_t dW_t, & \text{if } t > m. 
\end{cases}
$$

(4.56)

The total cost $I^*_\tau$ of the re-advanceable mortgage is given as

$$
I^*_\tau = \begin{cases} 
P \tau - C \tau, & \text{if } \tau < m, \\
P \tau - L, & \text{if } \tau \geq m. 
\end{cases}
$$

(4.57)

where $\tau$ is the first time when the investment portfolio value is at least the original principal $L$ and $P$ is given by Equation 4.40. Note that $I^*_m = Pm - L$ is different from $I^*_\tau$ in the sense that former is the total interest cost of the original mortgage and the latter is the total cost of the re-advanceable scheme. $I^*_\tau$ includes the cost to make up investment losses when $\tau > m$. Note that $I^*_\tau = I^*_m$ exactly when $\tau = m$.

Consider a scenario in which the mortgage interest rate evolves according to the mean reverting model (with no diffusion) but the mortgage payment rate is given by Equation 4.1. the mortgage payment rate of the fixed-rate mortgage. In this scenario depending on whether the value of $r_0$ is less or greater than $\theta$, the original mortgage will be paid off before or after the amortization term. Let the payoff time be $\tau_m$ which is defined as the first time when $B_t = 0$,

$$
\tau_m = \inf \{ t \geq 0 \mid B_t = 0 \},
$$

(4.58)

$$
= \inf \left\{ t \geq 0 \mid L e^{\int_0^t r_u du} - P \int_0^t e^{\int_0^u r_v dv} = 0 \right\}
$$

(4.59)
Note that \( \tau_m \) is deterministic. Under this scenario the total interest cost of the original mortgage is
\[
I'_{\tau_m} = P\tau_m - L; \quad (4.60)
\]
The total cost of the re-advanceable mortgage with the mortgage payoff time \( \tau \) is
\[
I'_\tau = \begin{cases} 
P\tau - C_\tau, & \text{if } \tau < \tau_m, \\
P\tau - L, & \text{if } \tau \geq \tau_m. 
\end{cases} \quad (4.61)
\]
where \( P \) is given by Equation 4.1. \( I'_\tau \) includes the cost to make up the investment losses when \( \tau > m \). Note that since \( P \) is fixed and for \( \tau < m \) we have \( C_\tau < L \), it is not possible to have a sample path which pays off mortgage in less than \( m \) years but have the total cost \( I'_\tau \) greater than the total interest cost \( I'_m \).

Example

Consider a mortgage with amortization period of 25 years and principal amount of $300,000. We compare the three scenarios,

I fixed-rate mortgage,

II variable-rate mortgage where the payment rate amortizes the mortgage in 25 years and

III variable-rate mortgage where the payment rate is the same as for the fixed-rate mortgage, giving amortization term different from 25 years.

For an example, we set the long run interest rate \( \theta \) at 6% and the mean reversion speed at 0.2. We use three different values of the initial interest rate, \( r_0 \), i.e., 7%, 6% and 5%. Note that for the fixed-rate mortgage the mortgage rate is same as the long run interest rate \( \theta \).

<table>
<thead>
<tr>
<th>Scenario</th>
<th>Mortgage Payment Rate</th>
<th>Total Interest Cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>$1903.8</td>
<td>$279,250</td>
</tr>
<tr>
<td>II ( r_0 = 5% )</td>
<td>$1880.9</td>
<td>$264,260</td>
</tr>
<tr>
<td>II ( r_0 = 7% )</td>
<td>$1982.0</td>
<td>$294,590</td>
</tr>
<tr>
<td>III ( r_0 = 5% )</td>
<td>$1903.8</td>
<td>$248,090</td>
</tr>
<tr>
<td>III ( r_0 = 7% )</td>
<td>$1903.8</td>
<td>$318,820</td>
</tr>
</tbody>
</table>

Table 4.1: Mortgage payment rate and total interest cost \( (I'_m) \) of the three scenarios with \( \kappa = 0.3 \) and \( \theta = 0.06 \)

For a fixed-rate mortgage at the annual rate of 6% (i.e., when \( r_0 = \theta \)), the monthly mortgage payment rate is $1930.8 which is calculated using Equation 4.1. For the mean-reverting mortgage rate the monthly payment rates are $1982.0 and $1880.9, respectively for \( r_0 = 7\% \) and \( r_0 = 5\% \), they are calculated from Equation 4.40. Table 4.1 gives the mortgage payment rates and total interest cost \( (I'_m) \) for the three cases. Comparing all three cases we see that the monthly
Figure 4.2: The left and right panels are the outstanding principal balance, $B_t$, and the instantaneous interest cost, respectively, of the mortgage with $\theta = 0.06$, $\kappa = 0.3$ and three different values of $r_0 = 0.05, 0.06$ and 0.07.

<table>
<thead>
<tr>
<th>Scenario</th>
<th>Average</th>
<th>95% CI</th>
<th>std</th>
<th>$\Pr(\tau &gt; 25)$</th>
<th>$\Pr(\tau &gt; 30)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Scenario I</td>
<td>21.29</td>
<td>[20.94 21.63]</td>
<td>5.46</td>
<td>0.213</td>
<td>0.063</td>
</tr>
<tr>
<td>Scenario II</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$r_0 = 5%$</td>
<td>21.04</td>
<td>[20.68 21.40]</td>
<td>5.65</td>
<td>0.211</td>
<td>0.067</td>
</tr>
<tr>
<td>$r_0 = 7%$</td>
<td>21.63</td>
<td>[21.30 21.97]</td>
<td>5.25</td>
<td>0.221</td>
<td>0.063</td>
</tr>
<tr>
<td>Scenario III</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$r_0 = 5%$</td>
<td>20.03</td>
<td>[19.70 20.37]</td>
<td>5.36</td>
<td>0.159</td>
<td>0.045</td>
</tr>
<tr>
<td>$r_0 = 7%$</td>
<td>23.00</td>
<td>[22.61 23.35]</td>
<td>5.53</td>
<td>0.324</td>
<td>0.098</td>
</tr>
</tbody>
</table>

Table 4.2: Summary statistics of simulated payoff time distribution with $\kappa = 0.2$, $\theta = 0.06$, $\mu = 0.08$, $\sigma_S = 0.2$ and the tax rate is 31.15%}

payment rate is highest when $r_0 = 7\%$ and it is the lowest when $r_0 = 5\%$. This because of the fact that at the beginning of the mortgage, the portion of the monthly mortgage payment which contributes towards the interest cost is bigger as compared to the one which goes to principal. It implies that if $r_0 > \theta$ then the homeowner needs to increase the monthly payment (as compared to fixed-rate mortgage) in order to cover the interest cost within the amortization term. On the other hand, if $r_0 < \theta$ then homeowner enjoys the benefit of lower monthly mortgage payments (as compared to fixed-rate mortgage) because now the interest cost is less. Figure 4.2 gives the plot of outstanding principal balance, $B_t$, versus time in the left panel and the plot of instantaneous monthly interest cost versus time in the right panel for the original mortgage with fixed-rate (annual rate of 6%) and with mean-reverting mortgage rate for two values of $r_0$; 7% and 5%. The instantaneous interest cost is the amount of each mortgage payment that goes to pay the interest portion. This is clear from the figure that the outstanding principal balance
Figure 4.3: The left and right panels are the outstanding principal balance, $B_t$, and the instantaneous (monthly) interest cost, respectively, of the mortgage with $\theta = 0.06$, $\kappa = 0.3$ and three different values of $r_0 = 0.05$, 0.06 and 0.07. $P$ is the same for the two scenarios.

<table>
<thead>
<tr>
<th>Scenario</th>
<th>Average</th>
<th>95% CI</th>
<th>std</th>
<th>$\Pr(I^<em>_r &gt; I^</em>_m)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Scenario I</td>
<td>280560</td>
<td>[276030 285080]</td>
<td>71546</td>
<td>0.306</td>
</tr>
<tr>
<td>Scenario II</td>
<td>263930</td>
<td>[254400 268470]</td>
<td>71766</td>
<td>0.293</td>
</tr>
<tr>
<td></td>
<td>$r_0 = 5%$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$r_0 = 7%$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Scenario III</td>
<td>249310</td>
<td>[244900 253730]</td>
<td>69845</td>
<td>0.226</td>
</tr>
<tr>
<td></td>
<td>$r_0 = 5%$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$r_0 = 7%$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 4.3: Summary statistics of simulated interest cost distribution with $\kappa = 0.3$, $\theta = 0.06$, $\mu = 0.08$, $\sigma_S = 0.2$ and the tax rate is 31.15%.

The curve is higher for the case $r_0 = 7\%$ than the curve for the case $r_0 = 6\%$ (fixed-rate mortgage) and the outstanding principal balance is lower for the case $r_0 = 5\%$ than the curve for the case $r_0 = 6\%$ (fixed-rate mortgage). It is also obvious from the figure that the total interest cost is highest for the case when $r_0 = 7\%$ and lowest when $r_0 = 5\%$. For the mean reverting mortgage rate scenario the total interest cost is $264,260$ when $r_0 = 5\%$ and it is $294,590$ when $r_0 = 7\%$. For the fixed-rate mortgage scenario the total interest cost is $279,250$.

Consider the scenario where the monthly payment is evaluated using Equation 4.1 for both the fixed-rate mortgage and the mean-reverting mortgage rate. In other words replace $P$ in Equation 4.38 by the one given in Equation 4.1 which makes the monthly mortgage payment rate the same for the fixed-rate and the mean reverting rate mortgages. We investigate three different scenarios. Let us call the fixed-rate mortgage as Scenario I, the mean reverting interest rate as Scenario II and the mean reverting interest rate with the mortgage payment rate
4.2. Interest Rate Model

Figure 4.4: Right panel is mortgage payoff time and left panel is interest cost distributions of 1000 sample paths for the mean reverting-rate mortgage. The parameters are $\kappa = 0.3$, $\theta = 0.06$, $r_0 = 0.05$, $\mu = 0.08$, $\sigma_S = 0.2$. The tax rate is 31.15% and the line of credit rate is $0.0025 + r_t$.

same as the one for the fixed-rate mortgage (Equation 4.1) as Scenario III. The left panel of Figure 4.3 plots the outstanding balance, $B_t$, versus time of Scenario III for different values of $r_0$ by keeping the $\theta$ and $\kappa$ fixed at 6% and 0.3, respectively. The right panel of the Figure 4.3 is the plot of instantaneous monthly interest cost of original mortgage for $r_0 = 5\%$, $6\%$ and $7\%$. For $r_0 = 7\%$, it takes almost 27 years (longer than 25 years) for the homeowner to pay off the entire mortgage. Whereas, for $r_0 = 5\%$ it takes the homeowner about 23 years (less than 25 years) to pay off the mortgage. Similar behavior is shown in the instantaneous monthly interest cost curves for the different values of $r_0$. Also, the total interest cost increases from $294,260$ to $318,820$ when $r_0 = 7\%$ and it decreases from $264,260$ to $248,090$ when $r_0 = 5\%$ when comparing Scenarios II and III. As mentioned earlier the fixed-rate mortgage (Scenario I) has total interest cost of $279,250$. If the current rate, $r_0$, is less than the long run rate ($\theta$) then the Scenario III can save the homeowner thousands of dollars in interest cost. In the case when the current rate, $r_0$, is greater than the long term rate ($\theta$) then the Scenario I produce best results.

We apply the re-advanceable mortgage investment strategy to the three different scenarios and simulate 1000 sample paths of the risky asset, each corresponding to a simulated mortgage payoff time and total cost. For mean reverting interest rate of Scenario II histograms of the distribution of mortgage payoff time and total cost are given in Figure 4.4. The left panel corresponds to a histogram of the simulated mortgage payoff times and the right panel corresponds to a histogram of the simulated total costs with $\kappa = 0.3$, $\theta = 0.06$, $r_0 = 0.05$, $\mu = 0.08$, $\sigma_S = 0.2$. The tax rate is 31.15% and the line of credit rate is $0.0025 + r_t$. In Chapter 1 we have discussed how the line of credit rate affects the re-advanceable mortgage scheme when it is less than, greater than or equal to the mortgage rate. We do not repeat that discussion here. Instead of taking the line of credit exactly same as the mortgage rate we introduce a spread of 0.25%
between them. We add the spread to \( r_t \) and take it as the line of credit rate, this makes the line of credit rate a mean reverting process as well. The average time it takes to repay the mortgage is almost 21 years with the standard deviation of 5.65 years. The chances that it can take longer than 25 years and 35 years to payoff the entire mortgage are 0.211 and .067, respectively (see Table 4.2). The average cost of the strategy is \$263,930 and the probability that the cost of the strategy is greater than the interest cost of the mortgage is 0.293 (see Table 4.3).

Figure 4.5 is a box and whisker plot of the simulated payoff time distribution and the simulated interest cost distribution versus the Scenario I, Scenario II and Scenario III for \( r_0 = 0.05 \) and \( r_0 = 0.07 \). The left panel corresponds to the payoff time distribution and right panel corresponds to interest cost distribution with \( r=0.06 = \theta, \kappa = 0.3, \mu = 0.08, \) and \( \sigma_S = 0.2 \). The tax rate is 31.15% and the line of credit rate is 0.0025 \( + r_t \). In the figure \( \text{S I}, \text{S II} \) and \( \text{S III} \) represent Scenarios I, II and III, respectively. We plot Scenarios II and III for both the cases when \( r_0 < \theta \) and \( r_0 > \theta \). The average mortgage payoff time (stopping time such that \( B_t = 0 \)) for Scenario III is 23.65 years when \( r_0 = 5\% \) and it is 26.71 years when \( r_0 = 7\% \). Figure 4.5 shows that the investment strategy is more efficient if \( r_0 < \theta \).

Tables 4.2 and 4.3 give summary statistics of the mortgage payoff time and interest cost distributions for the three different scenarios. We see that the lowest average payoff time is of the Scenario III when \( r_0 = 5\% \). For this scenario the probabilities that it can take longer than 25 years and 30 years are 0.159 and 0.045 which are the lowest as compared to other two scenarios. Similar behavior is investigated about the interest cost distribution. The lowest average total cost of the strategy is \$249,310 which is of Scenario III with \( r_0 = 5\% \).
4.2.2 Stochastic Interest Rate Model

In this section we add a diffusion term to the mean reverting interest rate whose dynamics are given by Equation 4.11. This introduces a stochastic mortgage interest rate. If the diffusion term is taken as \( \sigma_r \sqrt{r_t} dW_t \), then we end up with the Cox-Ingersoll-Ross (CIR) \([7]\) model for the short interest rate \( r_t \). This model is a standard short rate model and has been used for mortgage rates in \([15]\), \([5]\) and \([14]\). The stochastic differential equation for the short rate is

\[
dr_t = \kappa(\theta - r_t)dt + \sigma_r \sqrt{r_t}dW_t,
\]

where \( \kappa, \theta \) and \( \sigma_r \) are positive constants and denote respectively, the mean reversion speed, the mean reversion level and the volatility of the short rate \( r_t \). Note that under the conditions \( 2\kappa\theta > \sigma^2 \) and \( r_0 > 0 \) the process is always positive. The term \( W_t \) is a standard Brownian motion. The transition density of the process is related to the non-central chi-squared density. The transition density is the probability density function of a future value of the process given the current value. Specifically, fix any \( t > 0 \) and define

\[
a_t = \frac{4\kappa}{\sigma_r^2(1 - e^{-\kappa t})},
\]

\[
v = \frac{4\kappa\theta}{\sigma_r^2},
\]

\[
\lambda_t = a_t r_0 e^{-\kappa t},
\]

and

\[
Y_t = a_t r_t.
\]

\( Y_t \sim \chi^2_{(\nu)}(\lambda_t) \) is a non-central chi-squared random variable with \( \nu \) degrees of freedom and non-centrality parameter \( \lambda_t \). This gives the transition density of \( r_t \) given \( r_0 \). The probability density function of \( Y_t \) is given as

\[
g(y) = e^{-y/2} \sum_{k=0}^{\infty} \frac{y^{\nu/2+k-1} e^{-y/2}(\lambda/2)^k}{k! 2^{\nu/2+k} \Gamma(\nu/2 + k)},
\]

where \( \Gamma(\cdot) \) denotes the gamma function defined as

\[
\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt
\]

If \( n \) is a positive integer then gamma function is \( \Gamma(n) = (n - 1)! \).

If we simulate the process \( r_t \) directly from the transition density function, it is computationally expensive and hence we use an approximation. We transform the stochastic differential equation for the short rate to a unit diffusion process. The transformation that gives a unit or constant diffusion term is called the Lamberti transform and is widely used in the study and simulation of stochastic differential equations. Typically the Euler approximation of the transformed process is a better approximation than performing Euler on the original process. Define the transformed process \( X_t \) by

\[
X_t = \int_{0}^{r_t} \frac{1}{\sigma_r \sqrt{u}} du = \frac{2}{\sigma_r} \sqrt{r_t}.
\]
To find the dynamics of \( X_t \) we use Itô’s Lemma

\[
dX_t = \frac{\partial X_t}{\partial t} dt + \frac{\partial X_t}{\partial r} dr_t + \frac{1}{2} \frac{\partial^2 X_t}{\partial r^2} dr_t dr_t,
\]

or

\[
= \left[ \frac{2\kappa \theta}{\sigma_r^2} - \frac{1}{2} \right] \frac{1}{X_t} - \frac{\kappa X_t}{2} \right] dt + dW_t^r.
\]

Equation \( 4.71 \) gives the dynamics of a unit diffusion process and the one-step Euler discretization of the process over the interval \([0, t]\) is \( X_t = X_0 + \left( \frac{2\kappa \theta}{\sigma_r^2} - \frac{1}{2} \right) \frac{1}{X_0} - \frac{\kappa X_0}{2} t + \sqrt{t} Z_t \), giving

\[
X_t|X_0 \sim N \left( X_0 + \left( \frac{2\kappa \theta}{\sigma_r^2} - \frac{1}{2} \right) \frac{1}{X_0} - \frac{\kappa X_0}{2} t, t \right)
\]

or

\[
f(X_t|X_0) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(X_t - X_0)^2}{2t}},
\]

where \( X_0 = \frac{2\sqrt{m}}{\sigma_r} > 0 \) and \( N(a, b) \) is a normal distribution with mean \( a \) and variance \( b \). Transforming the unit diffusion process Equation \( 4.71 \) back to the original process \( r_t \) gives the following transition density function

\[
f(r_t|r_0) = f(-X_t|X_0) + f(X_t|X_0)
\]

\[
= \frac{1}{\sigma_r \sqrt{t}} \left( f \left( -\frac{2\sqrt{r_t}}{\sigma_r} \right) + f \left( \frac{2\sqrt{r_t}}{\sigma_r} \right) \right).
\]

where \( f(\cdot) \) is a probability density function of the normal distribution defined by Equation \( 4.73 \). Replace \( X_t \) by \( X_t = \frac{2\sqrt{m}}{\sigma_r} \) and \( X_0 \) by \( X_0 = \frac{2\sqrt{m}}{\sigma_r} \) in Equation \( 4.73 \) to get the required density function.

In comparison, the one step Euler discretization of the original process Equation \( 4.62 \) over the interval \([0, t]\) is given by \( r_t = r_0 + \kappa(\theta - r_0)t + \sigma_r \sqrt{t} Z_t \), giving transition density

\[
r_t|r_0 \sim N \left( r_0 + \kappa(\theta - r_0)t, \sigma_r^2 t \right),
\]

or

\[
f(r_t|r_0) = \frac{1}{\sqrt{2\pi \sigma_r^2 t}} e^{-\frac{(r_t - r_0 - \kappa(\theta - r_0)t)^2}{2\sigma_r^2 t}}.
\]

Equations \( 4.67 \), \( 4.75 \) and \( 4.77 \) represent, respectively, the exact transition density, transition density of Euler approximation of the transformed process and the transition density of the Euler approximation of the original process. Figure \( 4.6 \) shows the plot of these three transition densities at times 0.25, 0.5 and 1. The values of the parameters are \( r_0 = 0.06, \kappa = 0.2, \theta = 0.05 \) and \( \sigma_r = 0.1 \). As the time increment increases we observe that the Euler discretization of the transformed process gets closer to the exact transition density than the Euler discretization of the original process, with this improvement increasing with time increment. It is because of
4.2. Interest Rate Model

the fact that the Euler discretization of the original process draws samples from normal distribution and the Euler discretization of the transformed process draws samples from chi-squared distribution.

We simulate 1000 sample paths using the three different methods described above. We observed that the computational time taken by the exact pdf is much longer than the two Euler discretization methods. Of the two Euler discretizations we found that the transformed process is faster and also more accurate than the original process. Therefore, in our study we compute the sample paths of the process \( r_t \) using the Euler discretization of the transformed process.

**Mathematical Formulation of Re-advanceable Mortgage**

Consider a variable rate mortgage with mortgage rate \( r_t \) and the mortgage principal \( L \). Let the amortization term be \( m \) years. We assume that the mortgage term is same as the amortization term. The homeowner invests the line of credit balance into a single stock. The stock price dynamics are described by geometric Brownian motion with drift \( \mu \) and volatility \( \sigma_S \). The dynamics of the mortgage rate, \( r_t \), and stock price, \( S_t \), are respectively given by the following two stochastic differential equations

\[
\begin{align*}
    dr_t &= \kappa(\theta - r_t)dt + \sigma_r \sqrt{r_t}dW^1_t, \\
    dS_t &= \mu S_t dt + \sigma_S S_t dW^2_t,
\end{align*}
\]

where \( W^1_t \) and \( W^2_t \) are standard Brownian motions with instantaneous correlation \( \rho \). Let \( \Sigma \) be a covariance matrix such that

\[
\Sigma = \begin{bmatrix}
    dt & \text{cov}(dW^1_t, dW^2_t) \\
    \text{cov}(dW^2_t, dW^1_t) & dt
\end{bmatrix}.
\]
If $\rho$ is the instantaneous correlation between $dW_1^t$ and $dW_2^t$ then the covariance can be defined as $\text{cov}(dW_1^t, dW_2^t) = d\rho$. Therefore, the matrix becomes

$$
\Sigma = dt \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}.
$$

Two correlated normal random variables can be generated by first generating two independent standard normal variables, $Z_1$ and $Z_2$, and then defining $W_1$ and $W_2$ as

$$
W_1 = Z_1,
$$

$$
W_2 = \rho Z_1 + \sqrt{1-\rho^2} Z_2.
$$

Homeowners repay the mortgage by making mortgage payments continuously at the annual payment rate of $P$. We fix $P$ such that

$$
L = \mathbb{E} \left[ \int_0^m P e^{-\int_0^t r_s ds} dt \right],
$$

(4.80)

that is, on average the expected present value of the total payment over $m$ years equals the original mortgage amount. By interchanging the integral with the expectation we get

$$
L = P \int_0^m \mathbb{E} \left[ e^{-\int_0^t r_s ds} \right] dt,
$$

(4.81)

and rearranging gives

$$
P = \frac{L}{\int_0^m \mathbb{E} \left[ e^{-\int_0^t r_s ds} \right] dt}.
$$

(4.82)

Note that $\mathbb{E} \left[ e^{-\int_0^t r_s ds} \right]$ is the Laplace transform of $\int_0^t r_s ds$ evaluated at -1. For the CIR model the (bivariate) Laplace transformation of $(r_t, \int_0^t r_s ds)$ is well known. As given in [20] that if $r_t^{r_0}$ is a solution to the SDE 4.78 starting at $r_0$ then for any non-negative $\zeta$ and $\xi$, we have

$$
\mathbb{E} \left( e^{-\xi r_t^{r_0}} e^{-\zeta \int_0^t r_s ds} \right) = e^{-\lambda \phi_{\xi,\zeta}(t)} e^{-r_0 \psi_{\xi,\zeta}(t)}
$$

(4.83)

where the functions $\phi_{\xi,\zeta}$ and $\psi_{\xi,\zeta}$ are given by

$$
\phi_{\xi,\zeta}(t) = -\frac{2}{\sigma_r^2} \log \left( \frac{2\sigma e^{\frac{\zeta}{2\sigma_r^2}}}{\sigma_r^2 (e^{\sigma t} - 1) + \sigma - \kappa + e^{\sigma t} (\sigma + \kappa)} \right)
$$

(4.84)

and

$$
\psi_{\xi,\zeta}(t) = \frac{\zeta (\sigma + \kappa + e^{\sigma t} (\sigma - \kappa)) + 2\xi (e^{\sigma t} - 1)}{\sigma_r^2 \xi (e^{\sigma t} - 1) + \sigma - \kappa + e^{\sigma t} (\sigma + \kappa)}
$$

(4.85)

with

$$
\sigma = \sqrt{k^2 + 2\sigma_r^2}.
$$

(4.86)
To get $E\left[e^{-\int_0^t r_u \, ds}\right]$ plug $\zeta = 0$ and $\xi = 1$ in Equations 4.83 - 4.86 which gives

$$E\left[e^{-\int_0^t r_u \, ds}\right] = e^{-k\eta\varphi_{0,1}(t)}e^{-r_0\psi_{0,1}(t)}$$

(4.87)

where the functions $\varphi_{0,1}$ and $\psi_{0,1}$ are given by

$$\varphi_{0,1}(t) = -\frac{2}{\sigma^2} \log \left( \frac{2\sigma e^{-\frac{k}{2}}} {\sigma - \kappa + e^{\sigma t}(\sigma + \kappa)} \right)$$

(4.88)

and

$$\psi_{0,1}(t) = \frac{2(e^{\sigma t} - 1)} {\sigma - \kappa + e^{\sigma t}(\sigma + \kappa)}$$

(4.89)

with

$$\sigma = \sqrt{\kappa^2 + 2\sigma_0^2}.$$ 

(4.90)

Thus the Equation 4.87 can be used in Equation 4.82 to compute $P$.

Let $B_t$ denote the time-$t$ amount of outstanding mortgage principal with $B_0 = L$. The formula for $B_t$ is

$$B_t = \begin{cases} 
L e^{\int_0^t r_u \, du} - P \int_0^t e^{\int_0^s r_u \, du} \, ds, & \text{if } 0 \leq t \leq \tau_m, \\
0, & \text{if } t > \tau_m.
\end{cases}$$

(4.91)

Note that the mortgage payoff time $\tau_m$ is a random variable which is a stopping time defined by

$$\tau_m = \inf \{ t \geq 0 \mid B_t = 0 \},$$

(4.92)

$$= \inf \left\{ t \geq 0 \mid L e^{\int_0^t r_u \, du} - P \int_0^t e^{\int_0^s r_u \, du} \, ds = 0 \right\}.$$ 

(4.93)

$B_t$ involves the integrals $e^{\int_0^t r_u \, du}$ and $\int_0^t e^{\int_0^s r_u \, du} \, ds$ where $r_u$ is a stochastic process. We use a Riemann sum to approximate the integrals and define them as

$$e^{\int_0^t r_u \, du} \approx e^{\sum_{h=1}^n r_h \Delta t},$$

(4.94)

$$\int_0^t e^{\int_0^s r_u \, du} \, ds \approx \sum_{h=1}^n e^{\sum_{j=h}^n r_{j} \Delta t} \Delta t,$$ 

(4.95)

where $\Delta t$ is time increment and $n$ is a total number of subintervals of $[0,t]$. For example, $\Delta t = 1$ month if the homeowner makes monthly mortgage payments. To estimate the stopping time, $\tau_m$ we stop when the discretized approximation $L e^{\sum_{h=1}^n r_h \Delta t} - P \sum_{h=1}^n e^{\sum_{j=h}^n r_{j} \Delta t} \Delta t$ changes sign from positive to negative for the first time. Therefore, we estimate $\tau_m$ with $\hat{\tau}_m$ given by

$$\hat{\tau}_m = \inf \left\{ t \geq 0 \mid L e^{\sum_{h=1}^n r_h \Delta t} - P \sum_{h=1}^n e^{\sum_{j=h}^n r_{j} \Delta t} \Delta t \leq 0 \right\}.$$ 

(4.96)
Chapter 4. Incorporating Variable Housing Prices and Interest Rates

The total interest cost of the original mortgage with the stochastic interest is given by the following equation

\[ I^*_{\tau_m} = P\tau_m - L = \frac{L}{\int_0^{\tau_m} \mathbb{E} \left[ e^{-\int_0^s r_t ds} \right] dt} \tau_m - L, \]  

(4.97)

which is estimated by \( I^*_{\hat{\tau}_m} = P\hat{\tau}_m - L. \)

There are two ways to calculate the outstanding balance, \( B_t \), depending on the formula for \( P \). From now on, we call the case of obtaining \( B_t \) by using \( P \) given in Equation 4.82 as Scenario IV and the case of obtaining \( B_t \) by using \( P \) as given in Equation 4.1 as Scenario V. Consider a mortgage with principal amount of $300,000 and amortization term of 25 years. We simulate 1000 sample paths of \( B_{25} \) for Scenarios IV and V at time 25 years with \( \kappa = 0.3, \ \theta = 0.06, \ r_0 = 0.05 \) and \( \sigma_r = 0.02 \). Figure 4.7 has a histogram of \( B_{25} \) of Scenario IV on the left panel and a histogram of \( B_{25} \) of Scenario V on the right panel. From the figure we see that \( B_{25} \) can be positive which means that there is a chance that it can take longer than 25 years to pay off the entire mortgage. The probability that \( B_t \) is greater than zero at time 25 years is 0.413 for Scenario IV and it is 0.13 for Scenario V. It means that Scenario V has a less chance of a positive outstanding balance at the end of the amortization term of 25 years.

Figure 4.8 makes a comparison of the stopping time \( \tau_m \) for the two scenarios; Scenario IV and Scenario V. The left panel is a histogram of \( \tau_m \) for Scenario IV and right panel is a histogram of \( \tau_m \) for Scenario V with \( \kappa = 0.3, \ r_0 = 0.05, \ \theta = 0.06 \) and \( \sigma_r = 0.02 \). The mean stopping time of Scenario IV is almost 25 years as expected since we define \( P \) in a way that on average the mortgage is paid off in 25 years. The average stopping time of Scenario V is 23.43
4.2. Interest Rate Model

Figure 4.8: Left panel is a histogram of stopping time, $\tau_m$ of Scenario IV and right panel is a histogram of stopping time, $\tau_m$ of Scenario V with $\kappa = 0.3$, $r_0 = 0.05$, $\theta = 0.06$ and $\sigma_r = 0.02$.

years. The estimated probability that the stopping time is greater than 25 years is 0.393 and 0.120 for Scenarios IV and V, respectively.

The line of credit balance, $C_t$ and the new money available for investment, $dI_t$ are defined in the same manner as in the case of fixed-rate mortgage. The time-$t$ value of $C_t$ and $dC_t$ are given by

$$C_t = L - B_t = \begin{cases} L - Le^{\int_0^t r_u du} + P \int_0^t e^{\int_0^s r_u du} ds & \text{if } 0 \leq t \leq \tau_m \\ L & \text{if } t > \tau_m \end{cases}.$$  (4.98)

To estimate the time-$t$ instantaneous changes in $B_t$ and $C_t$ we use discretised approximations as

$$\Delta B_t = \begin{cases} B_t - B_{t-\Delta t} & \text{if } 0 \leq t \leq \tau_m \\ 0 & \text{if } t > \tau_m \end{cases}.$$  (4.99)

and

$$\Delta C_t = -\Delta B_t = \begin{cases} B_{t-\Delta t} - B_t & \text{if } 0 \leq t \leq \tau_m \\ 0 & \text{if } t > \tau_m \end{cases}.$$  (4.100)

Let $I_t$ be time-$t$ amount of money available for new investment which is defined as

$$dI_t = dC_t - i(1 - c)C_t dt + Pd\nu_{t > \tau_m}$$  (4.101)

The time-$t$ number of new units of stock bought with the time-$t$ amount of new money available, $dI_t$, is given by $dN_t = dI_t / dS_t$. Thus, the time-$t$ values of the risky investment portfolio is

$$V_t = S_t \int_0^t dN_u = S_t \int_0^t \frac{dI_u}{S_u}.$$  (4.102)
To derive a stochastic differential equation describing the dynamics of $V_t$, requires the derivatives of the integrals $\int_0^t r_u \, du$ and $\int_0^t e^{\int_0^s r_u \, du} \, ds$ with respect to $t$ and $r_t$. This is not required to compute these quantities within our simulation since we use their discretised approximations.

The total cost of the re-advanceable mortgage is

$$I^*_\tau = \begin{cases} P\tau - C_\tau, & \text{if } \tau < \tau_m, \\ P\tau - L, & \text{if } \tau \geq \tau_m. \end{cases} \quad (4.103)$$

where $\tau$ is the time when the investment portfolio value hits the mortgage principal $L$.

**Example**

Consider an example of a mortgage with principal amount of $300,000 and amortization term of 25 years. We assume that the mortgage term is same as the mortgage amortization term. Mortgage payments are made monthly. We simulate one thousand stock price and CIR process sample paths pairing them up so that each stock price sample path is paired with an interest rate sample path. Using these pairs we compute the corresponding sample paths of the investment portfolio value for Scenarios IV and V. Using different sets of parameter values we investigate their effect on the re-advanceable mortgage strategy. We consider a wide range of parameter values which are in line with those reported in the literature ([18], [17], [32], [9], [19], [15], [12], [5] and [14]).

![Figure 4.9](image.png)

Figure 4.9: The left and right panels are histograms of the simulated mortgage payoff times and total costs, respectively, for Scenarios V with $\kappa = 0.3$, $\theta = 0.06$, $\sigma_r = 0.02$, $r_0 = 0.05$, $\mu = 0.08$, $\sigma_S = 0.2$ and $\rho = 0.1$. The tax rate is 0.3115 and the line of credit rate is $0.0025 + r_t$.

Figure 4.9 plots the histograms of the simulated mortgage payoff time and interest cost distributions. The left panel is the mortgage payoff time distribution and right panel is the interest cost distribution with $\kappa = 0.3$, $\theta = 0.06$, $\sigma_r = 0.02$, $r_0 = 0.05$, $\mu = 0.08$ and $\sigma_S =$
4.2. Interest Rate Model

0.2. The instantaneous correlation coefficient between the Brownian motions driving the stock price and interest rate movements is \( \rho = 0.1 \). The tax rate is 0.3115 and the line of credit rate is \( 0.0025 + r_t \). The mortgage payment rate is $1884. The average mortgage payoff time of the re-advanceable scheme is 20.82 years and the sample standard deviation of the mortgage payoff time distribution is 5.74 years. The estimated probability that it can take longer than 25 years and 30 years to pay off the entire mortgage is 0.21 and 0.063, respectively. We defined earlier two stopping times \( \tau_m \) and \( \tau \). The former is the stopping time when the original mortgage outstanding balance is zero, that is the time it takes to pay off the original mortgage. The latter is the stopping when the investment portfolio value is at least the original principal that is the time to pay off the re-advanceable mortgage. The estimated probability that \( \tau \) is greater than \( \tau_m \) is 0.217. Using Equation 4.97 we calculate the total interest cost, \( I^*_{\tau_m} \) of the original mortgage for each \( \tau_m \). The average total interest cost of the original mortgage is $260,290. Using Equation 4.103 we calculate the total cost, \( I_\tau \), of the re-advanceable mortgage corresponding to each \( \tau \). The average total cost of the re-advanceable mortgage is $259,850. The estimated probability that \( I_\tau \) is greater than \( I^*_{\tau_m} \) is 0.316.

![Figure 4.10](image_url)

Figure 4.10: The left and right panels are box and whisker plots of the simulated payoff times and total costs, respectively for Scenarios I to V. \( (r_0 = 0.05, \theta = 0.06, \kappa = 0.3, \sigma_r = 0.02, \mu = 0.08, \sigma_S = 0.2 \) and \( \rho = 0.1 \). The tax rate is 0.3115 and the line of credit rate is \( 0.0025 + r_t \).)

Figure 4.10 gives a comparison of the simulated mortgage payoff time and total cost distributions across all five scenarios. The left and right panels are box and whisker plots of the simulated mortgage payoff time and simulated total cost distributions, respectively, with \( r_0 = 0.05, \theta = 0.06, \kappa = 0.3, \sigma_r = 0.02, \mu = 0.08, \sigma_S = 0.2 \) and \( \rho = 0.1 \). The tax rate is 0.3115 and the line of credit rate is \( 0.0025 + r_t \). Scenarios III and V have similar results and they produce better results as compared to other three scenarios. Scenario III has the average payoff time of 20.04 years with standard deviation of 5.36 years. It has the average total cost of $249,320 with standard deviation of $69845. Scenario V has the average payoff time of 19.91 years with stan-
Chapter 4. Incorporating Variable Housing Prices and Interest Rates

Standard deviation of 5.53 years. It has the average total cost of $246,580 with standard deviation of $769,74. Scenario V has less average payoff time and average total cost than Scenario III, however this comes with increased risk. Both of the scenarios have the same monthly mortgage payment rate of $1930.8. Scenario III represents a mortgage which has mean reverting interest rate with zero diffusion term. Scenario V represents a mortgage which has mean reverting interest rate with nonzero diffusion term (CIR process). Because of the diffusion term, Scenario V has lower payoff time and total cost on average but that increases the standard deviation of the distributions.

Figure 4.11: The left and right panels are box and whisker plots of the simulated payoff time and simulated total cost distributions, respectively, for Scenarios IV and V and with \( r_0 = 0.05 \) and \( r_0 = 0.07 \). (\( \theta = 0.06, \kappa = 0.3, \sigma_r = 0.02, \mu = 0.08, \sigma_S = 0.2 \) and \( \rho = 0.1 \). The tax rate is 0.3115 and the line of credit rate is 0.0025 + \( r_t \).)

Figure 4.11 is a box and whisker plot of the simulated mortgage payoff time distribution versus Scenarios IV and V (left panel) and the simulated interest cost distribution versus Scenarios IV and V (right panel). Parameter values used are \( \kappa = 0.3, \theta = 0.06, \sigma_r = 0.02, \mu = 0.08 \) and \( \sigma_S = 0.2 \). The tax rate is 0.3115 and the line of credit rate is 0.0025 + \( r_t \). The distributions are compared for two different values of \( r_0 \) one above (7%) and one below (5%) the long term rate \( \theta = 6\% \). It is clear from the figure that the average payoff time and total cost for both Scenarios IV and V increase when we increase \( r_0 \) from 5\% to 7\%. Scenario V with \( r_0 = 5\% \) has the lowest average mortgage payoff time of 19.90 years and also the lowest average cost of the re-advanceable mortgage which is $246,580. The standard deviations of the mortgage payoff time and total cost distributions of Scenario V decrease when we decrease \( r_0 \) from 7\% to 5\%. Therefore, under Scenario V with \( r_0 = 5\% \), homeowner enjoys the benefit of paying off in less time and also saving thousands of dollars in interest cost on average. Total average interest cost increases more for Scenario V than for Scenario IV when \( r_0 \) changes from 5\% to 7\%. 
4.2. Interest Rate Model

Figure 4.12: The left and right panels are box and whisker plots of the simulated payoff times and total costs, respectively, for Scenarios IV and V with $\sigma_r = 0.02, 0.04$ and $0.08$. $r_0 = 0.05$, $\theta = 0.06$, $\kappa = 0.3$, $\mu = 0.08$, and $\sigma_S = 0.2$. The tax rate is 0.3115 and the line of credit rate is $0.0025 + r_t$.

We investigate the effect of interest rate volatility using Scenarios IV and V. We choose three different values of $\sigma_r$ as $0.02, 0.04$ and $0.08$ and compare the two scenarios. The mortgage payment rates and the total interest costs of the two scenarios with different values of $\sigma_r$ are given in Table 4.4. In the table the total costs for Scenarios IV and Scenario V are the average total costs represented by $\mathbb{E}[I_{\tau_m}]$. We start at $r_0=0.05$ (below $\theta$), therefore, when $\sigma_r$ increases the payment rate $P$ decreases. If the payment rate is less then one can expect to pay off the original mortgage in time greater than 25 years. That is why the total cost of Scenario IV is increasing with $\sigma_r$ because there is more risk and the mortgage payment rate has decreased. The left and right panels of Figure 4.12 are box and whisker plots of the simulated mortgage payoff times and total costs, respectively, for Scenarios IV and V with $\kappa = 0.3$, $\theta = 0.06$, $\sigma_r = 0.02$, $\mu = 0.08$ and $\sigma_S = 0.2$. The tax rate is 0.3115 and the line of credit rate is $0.0025 + r_t$. Table 4.5 and 4.6 give some statistics of the simulated mortgage payoff time and total cost distributions. We see a pattern from the tables that with an increase in the value of $\sigma_r$ there is an increase in the average payoff time and average total cost of the re-advanceable scheme. For Scenarios IV there is a difference of almost two years in the average payoff time and a difference of over $40,000$ in the average total cost when $\sigma_r$ increases from $0.02$ to $0.08$. For Scenario V when $\sigma_r$ increases from $0.02$ to $0.08$ the average payoff time increases by almost a year and the average total cost increases by over $20,000$. The change in Scenario IV is more than that in Scenario V because of the payment rate which is fixed and higher for Scenario V. Another crucial point is the tails of payoff time and total cost distributions. It is clear from the figure that there is more risk when $\sigma_r$ is big. We take the time horizon of 75 years over which we simulate the payoff time and total cost distributions. If within 75 years the investment
Chapter 4. Incorporating Variable Housing Prices and Interest Rates

<table>
<thead>
<tr>
<th></th>
<th>( P )</th>
<th>( \mathbb{E}[I^*_\tau_m] )</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Scenario IV</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \sigma_r = 0.02 )</td>
<td>1885.1</td>
<td>262170</td>
</tr>
<tr>
<td>( \sigma_r = 0.04 )</td>
<td>1879.9</td>
<td>273120</td>
</tr>
<tr>
<td>( \sigma_r = 0.08 )</td>
<td>1864.3</td>
<td>272040</td>
</tr>
<tr>
<td><strong>Scenario V</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \sigma_r = 0.02 )</td>
<td>1930.8</td>
<td>244860</td>
</tr>
<tr>
<td>( \sigma_r = 0.04 )</td>
<td>1930.8</td>
<td>254210</td>
</tr>
<tr>
<td>( \sigma_r = 0.08 )</td>
<td>1930.8</td>
<td>248050</td>
</tr>
</tbody>
</table>

Table 4.4: Mortgage payment rate and average total interest cost of the original mortgage for Scenarios IV and V with \( r_0 = 0.05, \kappa = 0.3, \theta = 0.06, \sigma_r = 0.02, 0.04, 0.08, \mu = 0.08, \sigma_S = 0.2 \) and \( \rho = 0.1 \). Tax rate is 0.3115 and the line of credit rate is \( r_t + 0.25\% \).

<table>
<thead>
<tr>
<th></th>
<th>Average</th>
<th>95% CI</th>
<th>std</th>
<th>( \Pr(\tau &lt; 15) )</th>
<th>( \Pr(\tau &gt; 25) )</th>
<th>( \Pr(\tau &gt; 30) )</th>
<th>( \Pr(\tau &gt; \tau_m) )</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Scenario IV</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \sigma_r = 0.02 )</td>
<td>20.82</td>
<td>[20.46 21.18]</td>
<td>5.74</td>
<td>0.137</td>
<td>0.210</td>
<td>0.063</td>
<td>0.216</td>
</tr>
<tr>
<td>( \sigma_r = 0.04 )</td>
<td>21.44</td>
<td>[20.99 21.88]</td>
<td>7.75</td>
<td>0.158</td>
<td>0.257</td>
<td>0.098</td>
<td>0.218</td>
</tr>
<tr>
<td>( \sigma_r = 0.08 )</td>
<td>23.05</td>
<td>[22.38 25.72]</td>
<td>7.67</td>
<td>0.192</td>
<td>0.322</td>
<td>0.185</td>
<td>0.260</td>
</tr>
<tr>
<td><strong>Scenario V</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \sigma_r = 0.02 )</td>
<td>19.90</td>
<td>[19.55 20.25]</td>
<td>5.54</td>
<td>0.168</td>
<td>0.163</td>
<td>0.047</td>
<td>0.223</td>
</tr>
<tr>
<td>( \sigma_r = 0.04 )</td>
<td>20.31</td>
<td>[19.91 20.71]</td>
<td>7.75</td>
<td>0.188</td>
<td>0.197</td>
<td>0.063</td>
<td>0.225</td>
</tr>
<tr>
<td>( \sigma_r = 0.08 )</td>
<td>21.19</td>
<td>[20.64 21.76]</td>
<td>7.42</td>
<td>0.237</td>
<td>0.263</td>
<td>0.129</td>
<td>0.255</td>
</tr>
</tbody>
</table>

Table 4.5: Summary statistics of simulated mortgage payoff time distribution for Scenarios IV and V with \( r_0 = 0.05, \kappa = 0.3, \theta = 0.06, \sigma_r = 0.02, 0.04, 0.08, \mu = 0.08, \sigma_S = 0.2 \) and \( \rho = 0.1 \). Tax rate is 0.3115 and the line of credit rate is \( r_t + 0.25\% \).

Portfolio value never reaches the level \( L \) it means that along that sample path the homeowner takes more than 75 years to pay off the mortgage. As this happens infrequently, we ignore this censoring and set the payoff time equal to 75 years. We can see that the probability that the homeowner pays off the entire mortgage in 75 years is small (1/1000) but it is nonzero. Also note the estimated probabilities \( \Pr(\tau > \tau_m) \) and \( \Pr(I^*_\tau > I^*_\tau_m) \) increase when there is more risk (with \( \sigma_r \)). For instance, for Scenario IV the probability that \( \tau \) is greater than \( \tau_m \) is 26% and the probability that \( I^*_\tau \) is greater than \( I^*_\tau_m \) is 31.9% when \( \sigma_r \) is 0.08 compared with 21.6% and 26.2% when \( \sigma_r = 0.02 \). These probabilities are significant. In our model we are not considering the fact that there is going to be taxes on income earned on the investment portfolio (e.g. capital gains). If we include tax paid on the investment portfolio in our model then we can expect an increase in the probabilities \( \Pr(\tau > \tau_m) \) and \( \Pr(I^*_\tau > I^*_\tau_m) \) as the investment portfolio value must exceed the original mortgage amount, net of taxes paid. However, there is a trade off between risk and reward. From Table 4.5 we see that the probability that it can take less than 15 years to pay off the entire mortgage increases with \( \sigma_r \) for both the Scenarios IV and V.

We investigate the effect of \( \rho \), the instantaneous correlation between the Brownian motions driving the stock price and interest rate processes on the re-advanceable mortgage scheme for Scenarios IV and V. Figure 4.13 makes a comparison of the simulated mortgage payoff time and total cost distributions for different values of \( \rho \). We choose five different values for \( \rho \).
### 4.2. Interest Rate Model

#### Table 4.6: Summary statistics of simulated interest cost distribution for Scenarios IV and V

<table>
<thead>
<tr>
<th>Scenario IV</th>
<th>( \sigma_r = 0.02 )</th>
<th>259840</th>
<th>[254910 264780]</th>
<th>78032</th>
<th>0.262</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \sigma_r = 0.04 )</td>
<td>272790</td>
<td>[265370 280210]</td>
<td>117330</td>
<td>0.268</td>
</tr>
<tr>
<td></td>
<td>( \sigma_r = 0.08 )</td>
<td>300560</td>
<td>[287170 313950]</td>
<td>211750</td>
<td>0.319</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Scenario V</th>
<th>( \sigma_r = 0.02 )</th>
<th>246540</th>
<th>[241670 251410]</th>
<th>76996</th>
<th>0.288</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \sigma_r = 0.04 )</td>
<td>285470</td>
<td>[248720 262220]</td>
<td>106730</td>
<td>0.291</td>
</tr>
<tr>
<td></td>
<td>( \sigma_r = 0.08 )</td>
<td>300560</td>
<td>[287170 313950]</td>
<td>211750</td>
<td>0.319</td>
</tr>
</tbody>
</table>

Table 4.6: Summary statistics of simulated interest cost distribution for Scenarios IV and V with \( r_0 = 0.05, \kappa = 0.3, \theta = 0.06, \sigma_r = 0.02, 0.04, 0.08, \mu = 0.08, \sigma_S = 0.2 \) and \( \rho=0.1 \). Tax rate is 0.3115 and the line of credit rate is \( r_r + 0.25\% \).

#### Table 4.7: Summary statistics of simulated payoff time distribution for Scenarios IV and V

<table>
<thead>
<tr>
<th>Scenario IV</th>
<th>( \rho = -1 )</th>
<th>21.60</th>
<th>[21.18 22.03]</th>
<th>6.74</th>
<th>0.272</th>
<th>0.113</th>
<th>0.246</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \rho = -0.5 )</td>
<td>21.39</td>
<td>[20.98 21.81]</td>
<td>6.53</td>
<td>0.264</td>
<td>0.104</td>
<td>0.246</td>
</tr>
<tr>
<td></td>
<td>( \rho = 0 )</td>
<td>20.94</td>
<td>[20.57 21.31]</td>
<td>5.88</td>
<td>0.220</td>
<td>0.069</td>
<td>0.218</td>
</tr>
<tr>
<td></td>
<td>( \rho = 0.5 )</td>
<td>20.74</td>
<td>[20.39 21.08]</td>
<td>5.44</td>
<td>0.189</td>
<td>0.060</td>
<td>0.216</td>
</tr>
<tr>
<td></td>
<td>( \rho = 1 )</td>
<td>20.57</td>
<td>[20.25 20.89]</td>
<td>5.02</td>
<td>0.176</td>
<td>0.039</td>
<td>0.224</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Scenario V</th>
<th>( \rho = -1 )</th>
<th>20.56</th>
<th>[20.15 20.97]</th>
<th>6.49</th>
<th>0.227</th>
<th>0.083</th>
<th>0.259</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \rho = -0.5 )</td>
<td>20.32</td>
<td>[19.93 20.71]</td>
<td>6.19</td>
<td>0.210</td>
<td>0.078</td>
<td>0.257</td>
</tr>
<tr>
<td></td>
<td>( \rho = 0 )</td>
<td>19.97</td>
<td>[19.61 20.32]</td>
<td>5.64</td>
<td>0.172</td>
<td>0.043</td>
<td>0.222</td>
</tr>
<tr>
<td></td>
<td>( \rho = 0.5 )</td>
<td>19.73</td>
<td>[19.41 20.06]</td>
<td>5.11</td>
<td>0.146</td>
<td>0.040</td>
<td>0.214</td>
</tr>
<tr>
<td></td>
<td>( \rho = 1 )</td>
<td>19.55</td>
<td>[19.24 19.85]</td>
<td>4.80</td>
<td>0.127</td>
<td>0.024</td>
<td>0.221</td>
</tr>
</tbody>
</table>

Table 4.7: Summary statistics of simulated payoff time distribution for Scenarios IV and V with \( r_0 = 0.05, \kappa = 0.3, \theta = 0.06, \sigma_r = 0.02, \mu = 0.08, \sigma_S = 0.2 \) and \( \rho=-1, -0.5, 0, 0.5, 1 \). Tax rate is 0.3115 and the line of credit rate is \( r_r + 0.25\% \).

#### Table 4.8: Summary statistics of simulated total cost distribution for Scenarios IV and V

<table>
<thead>
<tr>
<th>Scenario IV</th>
<th>( \rho = -1.0 )</th>
<th>271980</th>
<th>[265760 278190]</th>
<th>98275</th>
<th>0.297</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \rho = -0.5 )</td>
<td>268190</td>
<td>[262360 274010]</td>
<td>92066</td>
<td>0.305</td>
</tr>
<tr>
<td></td>
<td>( \rho = 0 )</td>
<td>261430</td>
<td>[256320 266530]</td>
<td>80700</td>
<td>0.263</td>
</tr>
<tr>
<td></td>
<td>( \rho = 0.5 )</td>
<td>258210</td>
<td>[253650 262780]</td>
<td>72213</td>
<td>0.266</td>
</tr>
<tr>
<td></td>
<td>( \rho = 1.0 )</td>
<td>253830</td>
<td>[249900 257760]</td>
<td>62119</td>
<td>0.271</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Scenario V</th>
<th>( \rho = -1.0 )</th>
<th>257920</th>
<th>[251800 264030]</th>
<th>96707</th>
<th>0.311</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \rho = -0.5 )</td>
<td>253000</td>
<td>[247360 258640]</td>
<td>89207</td>
<td>0.311</td>
</tr>
<tr>
<td></td>
<td>( \rho = 0 )</td>
<td>247310</td>
<td>[242280 252330]</td>
<td>79455</td>
<td>0.287</td>
</tr>
<tr>
<td></td>
<td>( \rho = 0.5 )</td>
<td>243250</td>
<td>[238920 247580]</td>
<td>68431</td>
<td>0.278</td>
</tr>
<tr>
<td></td>
<td>( \rho = 1.0 )</td>
<td>239380</td>
<td>[235510 243260]</td>
<td>61331</td>
<td>0.284</td>
</tr>
</tbody>
</table>

Table 4.8: Summary statistics of simulated total cost distribution for Scenarios IV and V with \( r_0 = 0.05, \kappa = 0.3, \theta = 0.06, \sigma_r = 0.02, \mu = 0.08, \sigma_S = 0.2 \) and \( \rho=-1, -0.5, 0, 0.5, 1 \).
which are -1, -0.5, 0, 0.5 and 1. The top and bottom panels are box and whisker plots of the simulated mortgage payoff time and total cost distributions, respectively, for Scenarios IV and V. The values of other parameters are, \( r_0 = 0.05, \theta = 0.06, \kappa = 0.3, \sigma_r = 0.02, \mu = 0.08, \) and \( \sigma_S = 0.2 \). Tables 4.7 and 4.8 provide some summary statistics of the simulated payoff time and total cost distributions. First of all consider that case when the Brownian motions driving \( S_t \) and \( r_t \) are uncorrelated i.e., \( \rho = 0 \). The average payoff time for Scenario IV is 20.94 years which is more than that for Scenario V which has the average payoff time of 19.97 years. Similarly, the average total cost for Scenario IV is more than that of Scenario V. Now lets move to the cases when \( \rho \) is nonzero. We can see from the tables that when the Brownian motions are negatively correlated then the average payoff time and total cost increase for both of the scenarios. However, when the Brownian motions are positively correlated then the average payoff time and total cost decrease. A similar pattern holds for the standard deviations of the payoff time and total cost distributions. Thus if the mortgage rate is strongly negatively correlated with the risky investment, the homeowner is subject to both a higher expected cost and more risk than the case of strong positive correlation. Personal financial advisors should, therefore, advise clients considering this wealth building strategy to invest in an asset that is positively correlated with mortgage rates.

4.3 Housing Price Model

For mortgage valuation, it is typically assumed that housing price, \( H_t \), follows a geometric Brownian motion process \([18, 17, 32, 9, 27, 31, 16]\). The dynamics of \( H_t \) are given by the
following stochastic differential equation

$$dH_t = \phi H_t dt + \sigma_H H_t dW^3_t,$$  \hspace{1cm} (4.104)

where $\phi$ and $\sigma_H$ are the drift and the volatility, respectively. The process $W^3_t$ is a standard Brownian motion. Given the initial value $H_0$, the above stochastic differential equation has solution

$$H_t = H_0 e^{(\phi - \frac{\sigma^2}{2}) t + \sigma_H W^3_t}.$$  

Consider a mortgage of principal $L$ and the amortization term $m$ years. We assume that the homeowner makes a down payment of $20\%$ of $H_0$, the purchase price of the house which implies that $H_0 = \frac{5}{4}L$. In this section we include the housing price model in our re-advanceable mortgage scheme with three different scenarios of the mortgage interest rate. In Chapter 1 we have discussed the mortgage with fixed rate in detail. In the previous section we covered the two scenarios of mortgage rates which are the mean reverting model without the diffusion term and the CIR process.

Thus the total debt outstanding for a homeowner was fixed at $L$ at all times while implementing the re-advanceable mortgage scheme. Whenever there is a mortgage payment, the homeowner re-borrows the principal portion of the payment in the line of credit this makes the total borrowing equal to $L$. When we incorporate the housing price model, the total borrowing will change according to the movements in housing price and this will effectively increase the line of credit borrowing limit. We assume that the homeowner can re-borrow $80\%$ of the increase in the housing price in the line of credit and there is no new borrowing if house price falls (and no existing borrowing paid back). Therefore, the line of credit balance at time $t$ consists of two parts, one is the principal repaid on the original mortgage and the other is $80\%$ of the increase in the house price. At any time the total debt is equal to the sum of the original mortgage principal and $80\%$ of the increase in the housing price at time $t$. In the rest of the chapter we discuss with the help of an example the effect of housing price incorporation on the re-advanceable mortgage scheme.

### 4.3.1 Fixed-rate Mortgage and Housing Price Model

For a fixed-rate mortgage the mortgage payment rate and the time-$t$ amount of outstanding balance, $B_t$, are given by Equations \[4.1\] and \[4.2\] respectively. The line of credit balance can be obtained by subtracting the outstanding principal balance, $B_t$, from the original principal, $L$. The definition of the line of credit balance, $C_t$, will be modified to incorporate the housing price model. The line of credit balance will increase with an increase in housing price and this will effectively increase the line of credit borrowing limit. We assume that the homeowner has increased home equity which can be used to secure further borrowing. Let $M_t$ be the time-$t$ running maximum housing price defined as

$$M_t = \max_{0 \leq s \leq t} H_s.$$  \hspace{1cm} (4.105)

Therefore, the time-$t$ value of $C_t$ is

$$C_t = \begin{cases} 
L - \frac{L(e^m - e^t)}{e^m - 1} + 0.8(M_t - H_0), & \text{if } 0 \leq t \leq m \\
L + 0.8(M_t - H_0), & \text{if } t > m.
\end{cases}$$  \hspace{1cm} (4.106)
The term $0.8(M_t - H_0)$ in the above equation represents the 80% of the increase in the house price. Differentiating the above equation gives the dynamics of $C_t$.

$$dC_t = -dB_t + 0.8dM_t = \begin{cases} \frac{L}{e^{r(t)}} dt + 0.8dM_t, & \text{if } 0 \leq t \leq m, \\ 0.8dM_t, & \text{if } t > m. \end{cases} \quad (4.107)$$

Figure 4.14 illustrates the modified definition of the line of credit balance, $C_t$. The left panel is a plot of a housing price sample path and its running maximum with $\phi = 0.02$, $\sigma_H = 0.20$ and $H_0 = $400,000. The solid thick line in the left panel is the corresponding running maximum house price sample path. The right panel plots two corresponding line of credit balance sample paths. The solid thick line is the sample path with the housing price model. While the dotted line corresponds to the sample path without the housing price model. We see that when there is an increase in the running maximum house price the line of credit balance increases by the 80% of that amount less $H_0$. Note that $C_t$ does not decline when the housing price declines.

Figure 4.14: The left panel is a sample path of $H_t$ and the corresponding $M_t$. The right panel is the corresponding line of credit balance sample path with (solid line) and without housing price model (dotted line).

The time-$t$ amount of money, $dI_t$, available for new investment can be calculated by using Equations (4.106) and (4.107)

$$dI_t = \begin{cases} \frac{L}{e^{r(t)}} dt + dM_t - i(1-c) \left[ L + \frac{tL(e^{r(t)} - e^{r(m-1)})}{e^{r(m-1)}} - 0.8(M_t - H_0) \right] dt, & \text{if } 0 \leq t \leq m \\ \frac{L}{e^{r(m-1)}} dt + dM_t - i(1-c) \left[ L + 0.8(M_t - H_0) \right] dt, & \text{if } t > m \end{cases} \quad (4.108)$$

The dynamics of the time-$t$ value of the investment portfolio, $V_t$, can be calculated by substituting Equation (4.108) into the following equation

$$dV_t = \mu V_t dt + \sigma_S V_t dW^2_t + dI_t, \quad (4.109)$$
We redefine the mortgage payoff time since the total borrowing limit has changed. The mortgage payoff time is defined as the first time the risky investment portfolio is at least \( L + 0.8(M_t - H_0) \). That is,

\[
\tau = \inf \{ t \geq 0 \mid V_t \geq L + 0.8(M_t - H_0) \}.
\]  

(4.111)

Example

Consider a mortgage with principal amount $300,000, amortization term of 25 years and the homeowner makes monthly mortgage payments. We investigate how the re-advanceable scheme with a fixed-rate mortgage changes with the incorporation of the housing price model. Let us call the fixed rate mortgage as Scenario I and the fixed-rate mortgage with housing price model as Scenario I-h. In this example we simulate 1000 sample paths of the mortgage payoff time and total cost of the two scenarios. We compare the two scenarios based on different sets of parameter values of \( \phi, \sigma_H \) and \( \rho_{SH} \). Here \( \rho_{SH} \) is the instantaneous correlation between the Brownian motions driving the stock and housing price models. The range of values taken for \( \phi \) and \( \sigma_H \) are in accordance with \([18, 17, 32, 9, 19, 27, 31, 28, 26, 6]\) and \([10]\).

Figure 4.15 is a plot of the histograms of the simulated payoff time and total cost of Scenario I-h with \( r = 0.06, \mu = 0.08, \sigma_S = 0.2, \phi = 0.02, \sigma_H = 0.05 \) and tax rate is 0.3115. The line of credit interest rate is \( r + 0.25\% \). The instantaneous correlation coefficient between the Brownian motions driving the stock price and housing price movements is \( \rho_{SH} = 0.1 \). The time horizon over which we simulated the distributions is 75 years. We take the payoff time, \( \tau \) as 75 years in the case when the investment portfolio values does not reach the level \( L + 0.8(M_t - H_0) \) within 75 years. On average the mortgage payoff time of the re-advanceable scheme with housing price model is 19.75 years with the standard deviation of 8.65 years. The probability that it can take longer than 25 and 30 years to payoff the mortgage are 0.186 and 0.091, respectively. The average interest cost of the re-advanceable scheme with housing price model is \$275,760 and the probability that the total cost of the re-advanceable scheme can be greater than the interest cost of the mortgage is 0.245.

Figure 4.16 gives box and whisker plots of the simulated payoff times and total costs in the left and right panels, respectively, with \( r=0.06, i=0.0625, \mu = 0.08, \sigma_S = 0.2, \phi = 0.02, \sigma_H = 0.05, \) and \( \rho_{SH} = 0.1 \). The tax rate is 0.3115 and the line of credit rate is 0.0025 + \( r \). We compare Scenarios I and I-h in this figure. We see that the expected mortgage payoff time and the expected total cost are reduced when the housing price model is incorporated in the re-advanceable scheme. However the differences between the average payoff times and average total cost of two scenarios are not that big. The difference between the average payoff times of Scenarios I and I-h is over 1 year. The difference between the average costs of the two scenarios is \$4800. These small differences come at the cost of substantial increased standard deviations in the payoff time and total cost distributions. The differences between the standard
deviations of payoff time and total cost distributions of two scenarios are over 3 years and $68,000, respectively. The right tails of the payoff time and total cost distributions for Scenario I-h are much longer as compared to those for Scenario I. The incorporation of the housing price model in the re-advanceable mortgage can result in an increased likelihood that the homeowner is not able to pay off the entire loan in 75 years. This is because the total borrowed amount has increased from $L$ to $L + 0.8(M_t - H_0)$ at any time $t$. This implies that the homeowner expects to pay off the mortgage little bit earlier with savings of a few thousands of dollars in interest cost while implementing the re-advanceable mortgage with the house price model. This comes at the cost of significantly increased risk.

<table>
<thead>
<tr>
<th>Scenario</th>
<th>Average</th>
<th>95% CI</th>
<th>std</th>
<th>$Pr(\tau &gt; 25)$</th>
<th>$Pr(\tau &gt; 30)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Scenario I</td>
<td>21.29</td>
<td>[20.04 21.63]</td>
<td>5.46</td>
<td>0.213</td>
<td>0.063</td>
</tr>
<tr>
<td>Scenario I-h</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\phi = 0.005$</td>
<td>20.43</td>
<td>[19.96 20.89]</td>
<td>7.40</td>
<td>0.210</td>
<td>0.084</td>
</tr>
<tr>
<td>$\phi = 0.01$</td>
<td>20.26</td>
<td>[19.76 20.75]</td>
<td>7.80</td>
<td>0.204</td>
<td>0.087</td>
</tr>
<tr>
<td>$\phi = 0.05$</td>
<td>17.34</td>
<td>[16.79 17.89]</td>
<td>8.70</td>
<td>0.137</td>
<td>0.076</td>
</tr>
<tr>
<td>$\phi = 0.10$</td>
<td>13.39</td>
<td>[12.94 13.84]</td>
<td>7.05</td>
<td>0.059</td>
<td>0.032</td>
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</table>

Table 4.9: Summary statistics of simulated mortgage payoff time distribution for Scenarios I and I-h with $r = 0.06$, $i = 0.0625$, $\mu = 0.08$, $\sigma_S = 0.2$, $\sigma_H = 0.05$, $\rho_{SH} = 0.1$ and tax rate is 0.3115.

Figure 4.17 compares the simulated mortgage payoff times (left panel) and total costs
4.3. Housing Price Model

Figure 4.16: The right and left panels are the box and whisker plots of the simulated payoff times and total costs, respectively, for the fixed-rate mortgage and the fixed-rate mortgage with the housing price effect. ($r=0.06, i=0.0625, \mu = 0.08, \sigma_S = 0.2, \phi = 0.02, \sigma_H = 0.05, \rho_{SH} = 0.1$ and tax rate is 0.3115.)

(right panel) for different values of the average growth rate, $\phi$, of housing prices. We choose four different values of $\phi$ i.e., 0.005, 0.01, 0.05 and 0.1. Other parameter values are $r = 0.06$, $i=0.0625$, $\mu = 0.08$, $\sigma_S = 0.2$, $\sigma_H = 0.05$ and $\rho_{SH} = 0.1$. Tables 4.9 and 4.10 provide some summary statistics of the simulated mortgage payoff times and total cost distributions, respectively. When $\phi=0.005$, we see that there is not a lot of difference between the average payoff time and total cost of the two scenarios. But even with a small value of $\phi$ there is an introduction of greater risk in the payoff time and total cost distributions. We see that as $\phi$ increases the average payoff time and average total cost decreases. Keeping the fluctuation rate, $\sigma_H$, in the housing prices fixed means that the housing prices are going to be higher for higher $\phi$. In return that increases the money available for investment and that decreases the average payoff time and average total cost of the re-advanceable strategy. Notice the case, $\phi = 0.05$, this is when the risk (standard deviation) in the simulated payoff and total cost distributions is the highest as compared to the other three values of $\phi$.

We make a comparison of the re-advanceable scheme of fixed-rate mortgage with that of the fixed-rate mortgage for different values of $\sigma_H$. We choose four different values of $\sigma_H$ which are 0.05, 0.1, 0.2 and 0.3. Tables 4.11 and 4.12 provide some summary statistics of the simulated mortgage payoff times and total cost distributions, respectively. Figure 4.18 has box and whisker plots of the simulated payoff times (left panel) and total costs (right panel) for Scenarios I and Scenario I-h for $\sigma_H = 0.05, 0.1, 0.2$ and 0.3. Other parameters are $r=0.06$, $i=0.0625$, $\mu = 0.08$, $\sigma_S = 0.2$, $\phi = 0.02$, $\rho_{SH} = 0.1$ and tax rate is 0.3115. It is evident from the figure the simulated payoff time and total cost averages decrease but do not change that much with an increase in the values of $\sigma_H$. The average payoff time is 19.75 years and average total...
Chapter 4. Incorporating Variable Housing Prices and Interest Rates

Table 4.10: Summary statistics of simulated total cost distribution for Scenarios I and I-h with $r = 0.06$, $i=0.0625$, $\mu = 0.08$, $\sigma_S = 0.2$, $\sigma_H = 0.05$, $\rho_{SH} = 0.1$ and tax rate is 31.15%.

<table>
<thead>
<tr>
<th>Scenario</th>
<th>Average</th>
<th>95% CI</th>
<th>std</th>
<th>$\Pr(I_t &gt; I_m)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Scenario I</td>
<td>280560</td>
<td>[276030 285080]</td>
<td>71546</td>
<td>0.306</td>
</tr>
<tr>
<td>Scenario I-h</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\phi = 0.005$</td>
<td>278190</td>
<td>[271180 285190]</td>
<td>110700</td>
<td>0.280</td>
</tr>
<tr>
<td>$\phi = 0.01$</td>
<td>277920</td>
<td>[270270 285570]</td>
<td>120950</td>
<td>0.273</td>
</tr>
<tr>
<td>$\phi = 0.05$</td>
<td>249390</td>
<td>[240980 257810]</td>
<td>133010</td>
<td>0.175</td>
</tr>
<tr>
<td>$\phi = 0.10$</td>
<td>199750</td>
<td>[193510 205990]</td>
<td>98680</td>
<td>0.080</td>
</tr>
</tbody>
</table>

Table 4.11: Summary statistics of simulated mortgage payoff time distribution for Scenarios I and I-h with $r = 0.06$, $i=0.0625$, $\mu = 0.08$, $\sigma_S = 0.2$, $\phi = 0.02$, $\sigma_H = 0.05$ and tax rate is 31.15%.

<table>
<thead>
<tr>
<th>Scenario</th>
<th>Average</th>
<th>95% CI</th>
<th>std</th>
<th>$\Pr(\tau &gt; 25)$</th>
<th>$\Pr(\tau &gt; 30)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Scenario I</td>
<td>21.29</td>
<td>[20.04 21.63]</td>
<td>5.46</td>
<td>0.213</td>
<td>0.063</td>
</tr>
<tr>
<td>Scenario I-h</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sigma_H = 0.05$</td>
<td>19.75</td>
<td>[19.20 20.29]</td>
<td>8.65</td>
<td>0.186</td>
<td>0.091</td>
</tr>
<tr>
<td>$\sigma_H = 0.1$</td>
<td>19.46</td>
<td>[18.82 20.10]</td>
<td>10.08</td>
<td>0.190</td>
<td>0.092</td>
</tr>
<tr>
<td>$\sigma_H = 0.2$</td>
<td>18.86</td>
<td>[18.08 19.63]</td>
<td>12.23</td>
<td>0.192</td>
<td>0.106</td>
</tr>
<tr>
<td>$\sigma_H = 0.3$</td>
<td>18.46</td>
<td>[17.62 19.31]</td>
<td>13.34</td>
<td>0.196</td>
<td>0.114</td>
</tr>
</tbody>
</table>

The correlation coefficient, $\rho_{SH}$, between the Brownian motions driving the stock price, $S_t$, and house price, $H_t$, is another parameter whose effect we investigate. We take $\rho_{SH} = -1.0$, -0.5, 0, 0.5 and 1.0. Tables 4.13 and 4.14 provide some summary statistics of the simulated mortgage payoff time and total cost distributions, respectively. Figure 4.19 has box and whisker plots of the simulated payoff times (left panel) and total costs (right panel) for Scenarios I and I-h. The values of other parameters are $r=0.06$, $i=0.0625$, $\mu = 0.08$, $\sigma_S = 0.2$, $\phi = 0.02$, $\sigma_H = 0.05$ and tax rate is 31.15%. Consider the case when the Brownian motions driving the $S_t$ and $H_t$ are uncorrelated, i.e., $\rho_{SH} = 0$, the average payoff time is 19.69 years and the average total cost of the re-advanceable scheme is $275,480. Also note that in this case the standard deviations of the payoff time and total cost are the highest. Once non-zero correlation between the Brownian motions is introduced, the average payoff time and average total cost go up and down depending on the positive or negative correlation. If the Brownian motions are strongly negatively correlated ($\rho_{SH}=-1$), average payoff time and total cost drop down to 18.59 years and $261,540$, respectively. Similarly, if the Brownian motions are strongly positively correlated ($\rho_{SH}=1$), the average payoff time and total cost rise up to 20.19 years and $277,000$, respectively. The estimated probabilities that the mortgage payoff time is greater than 25 and
4.3. Housing Price Model

Figure 4.17: The left and right panels are box and whisker plots of the simulated payoff times and total costs, respectively, for Scenarios I and I-h with $\phi = 0.005, 0.01, 0.05$ and $0.1$. ($r=0.06, i=0.0625, \mu = 0.08, \sigma_S = 0.2, \sigma_H = 0.05, \rho_{SH} = 0.1$ and tax rate is 0.3115)

30 years increase as $\rho_{SH}$ increases from -1 to 1. Similarly, the estimated probability that the total cost of the re-advanceable scheme is greater than the interest cost of the original mortgage increases as $\rho_{SH}$ increases from -1 to 1. Thus, if house prices are strongly positively correlated with the risky investment, the homeowner is cautioned against using increased home equity to finance additional borrowing.

4.3.2 Mean Reverting Interest Rate and Housing Price Model

Consider a variable rate mortgage whose interest rate evolves according to Equation 4.11. When we incorporate the housing price effect, the line of credit balance, $C_t$, given by Equation 4.112 is modified as

$$C_t = \begin{cases} L\left(1 - e^{rt - \frac{\mu H_0}{\gamma}}\right) + PJ_2 + 0.8(M_t - H_0), & \text{if } 0 \leq t \leq m, \\ L + 0.8(M_t - H_0), & \text{if } t > m. \end{cases}$$ (4.112)

where $J_2$ and $P$ are defined by Equations 4.37 and 4.40 respectively. Note that these equations fix the payoff time of the mortgage to $m$ years. The dynamics of $C_t$ are given as

$$dC_t = \begin{cases} (P(1 + rJ_2) - Lr_t e^{rt - \frac{\mu H_0}{\gamma}})dt + 0.8dM_t, & \text{if } 0 \leq t \leq m, \\ 0.8dM_t, & \text{if } t > m. \end{cases}$$ (4.113)

The new money, $dI_t$, available for investment can be found by using the relation $dI_t =$
Table 4.12: Summary statistics of simulated total cost distribution for Scenarios I and I-h with $r = 0.06, i=0.0625, \mu = 0.08, \sigma_S = 0.2, \phi = 0.02, \rho_{SH} = 0.1$ and tax rate is 0.3115.

<table>
<thead>
<tr>
<th>Scenario</th>
<th>Average</th>
<th>95% CI</th>
<th>std</th>
<th>$\Pr(I_t &gt; I_m)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Scenario I</td>
<td>280560</td>
<td>[276030 285080]</td>
<td>71546</td>
<td>0.306</td>
</tr>
<tr>
<td>Scenario I-h</td>
<td>$\sigma_H = 0.05$</td>
<td>275760</td>
<td>[266910 284610]</td>
<td>139940</td>
</tr>
<tr>
<td>Scenario I-h</td>
<td>$\sigma_H = 0.1$</td>
<td>275670</td>
<td>[264870 286480]</td>
<td>170860</td>
</tr>
<tr>
<td>Scenario I-h</td>
<td>$\sigma_H = 0.2$</td>
<td>274620</td>
<td>[260920 288310]</td>
<td>216550</td>
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<tr>
<td>Scenario I-h</td>
<td>$\sigma_H = 0.3$</td>
<td>272990</td>
<td>[257960 288020]</td>
<td>237630</td>
</tr>
</tbody>
</table>

The investment portfolio value, $V_t$, evolves according to the following stochastic differential equation

$$dV_t = (\mu dt + \sigma_S dW_t^2)V_t + dI_t.$$  \hspace{1cm} (4.115)

$$dV_t = \begin{cases} 
\mu V_t + P (1 - c)L - Lr_i e^{r(t-m)} - i(1-c)\left[L\left(1 - e^{r(t-m)}\right) + PJ_2\right]
+ 0.8(M_t - H_0) dt,
\text{ if } 0 \leq t \leq m, \\
P dt - i(1-c)L dt + 0.8dM_t,
\text{ if } t > m.
\end{cases}$$

$$dV_t = \begin{cases} 
\left[\mu V_t + P (1 - c)L - Lr_i e^{r(t-m)} - i(1-c)\left[L\left(1 - e^{r(t-m)}\right) + PJ_2 + 0.8(M_t - H_0)\right]\right] dt + \sigma_S V_t dW_t^2 + 0.8dM_t,
\text{ if } 0 \leq t \leq m, \\
\left[\mu V_t + P - i(1-c)L\right] dt + \sigma_S V_t dW_t^2 + 0.8dM_t,
\text{ if } t > m.
\end{cases}$$

**Example**

Consider an example of a mortgage whose interest rate evolves according to mean reverting model without diffusion. The original principal of the mortgage is $300,000 with an amortization term of 25 years. The mortgage payments are made monthly. We incorporate the
4.3. Housing Price Model

Figure 4.18: The left and right panels are box and whisker plots of the simulated payoff times and total costs, respectively, for Scenarios I and I-h with $\sigma_H = 0.05, 0.1, 0.2$ and $0.3$. ($r=0.06$, $i=0.0625$, $\mu = 0.08$, $\sigma_S = 0.2$, $\phi = 0.02$, $\rho_{SH} = 0.1$ and tax rate is 0.3115)

housing price model in both of the Scenarios II and III, and denote them Scenarios II-h and III-h, respectively. We simulate 1000 sample paths of the mortgage payoff time and the cost of the re-advanceable scheme for the two scenarios. Figure 4.20 gives the histograms of the payoff time (left panel) and the interest cost (right panel) for Scenario II-h with $r_0=0.05$, $\kappa = 0.3$, $\theta = 0.06$, $\mu = 0.08$, $\sigma_S = 0.2$, $\phi = 0.02$, $\sigma_H = 0.05$, and $\rho_{SH} = 0.1$. The line of credit rate is $r_t + 0.25\%$ and the tax rate is 0.3115. The average payoff time is 18.99 years with the standard deviation of 9.19 years. The probabilities that it can take longer than 25 and 30 years to payoff the entire mortgage are 0.185 and 0.094, respectively. The average cost of the re-advanceable scheme of Scenario II-h is $260,890$ with the probability of 0.239 that the cost of the re-advanceable scheme is greater than the interest cost of the mortgage.

Figure 4.21 is a box and whisker plot of the simulated payoff time distribution of Scenarios II, III, II-h and III-h (left panel). The right panel is the simulated interest cost distribution of the corresponding scenarios. The set of parameters taken is $r_0 = 0.05$, $\theta = 0.06$, $\kappa = 0.3$, $\sigma_r = 0.02$, $\mu = 0.08$, $\sigma_S = 0.2$, $\phi = 0.01$, $\sigma_H = 0.05$ and $\rho_{SH} = 0.1$. The tax rate is 0.3115 and the line of credit rate is 0.0025 + $r_t$. There is not much difference in the average payoff time and total cost with the incorporation of housing price model across the scenarios. There is a small decrease in the values of average payoff times and total costs which comes with increased standard deviations for Scenarios II-h and III-h. For instance, for Scenario II the average payoff time is 21.04 years which reduces to 19.57 years when we incorporate housing price model but the standard deviation increases from 5.65 years to 9.19 years. Summary statistics for the simulated payoff time and total cost distributions are given in Tables 4.15 and 4.16.
Table 4.14: Summary statistics of simulated total cost distribution for Scenarios I and I-h with $r = 0.06$, $i = 0.0625$, $\mu = 0.08$, $\sigma_S = 0.2$, $\phi = 0.02$ and $\sigma_H = 0.05$.

<table>
<thead>
<tr>
<th>Scenario</th>
<th>Average</th>
<th>95% CI</th>
<th>std</th>
<th>$\text{Pr}(I_r &gt; I_m)$</th>
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</thead>
<tbody>
<tr>
<td>Scenario I</td>
<td>280560</td>
<td>[276030 285080]</td>
<td>71546</td>
<td>0.306</td>
</tr>
<tr>
<td>Scenario I-h</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\rho_{SH} = -1.0$</td>
<td>261540</td>
<td>[254200 268880]</td>
<td>116080</td>
<td>0.169</td>
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<tr>
<td>$\rho_{SH} = -0.5$</td>
<td>272100</td>
<td>[263280 280930]</td>
<td>139500</td>
<td>0.210</td>
</tr>
<tr>
<td>$\rho_{SH} = 0.0$</td>
<td>275480</td>
<td>[266410 284550]</td>
<td>143380</td>
<td>0.238</td>
</tr>
<tr>
<td>$\rho_{SH} = 0.5$</td>
<td>276420</td>
<td>[267990 284850]</td>
<td>133300</td>
<td>0.267</td>
</tr>
<tr>
<td>$\rho_{SH} = 1.0$</td>
<td>277000</td>
<td>[270000 284000]</td>
<td>110670</td>
<td>0.309</td>
</tr>
</tbody>
</table>

Table 4.15: Summary statistics of simulated mortgage payoff time distribution for Scenarios I and I-h with $r = 0.06$, $\mu = 0.08$, $\sigma_S = 0.2$, $\phi = 0.02$, $\sigma_H = 0.05$ and $\rho_{SH} = 0.1$. The tax rate is 0.3115 and the line of credit rate is 0.0025 + $r_t$.

<table>
<thead>
<tr>
<th>Scenario</th>
<th>Average</th>
<th>95% CI</th>
<th>std</th>
<th>$\text{Pr}(\tau &gt; 25)$</th>
<th>$\text{Pr}(\tau &gt; 30)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Scenario II</td>
<td>21.04</td>
<td>[20.68 21.40]</td>
<td>5.65</td>
<td>0.211</td>
<td>0.067</td>
</tr>
<tr>
<td>Scenario III</td>
<td>20.03</td>
<td>[19.70 20.37]</td>
<td>5.36</td>
<td>0.159</td>
<td>0.045</td>
</tr>
<tr>
<td>Scenario II-h</td>
<td>19.57</td>
<td>[18.99 20.15]</td>
<td>9.19</td>
<td>0.185</td>
<td>0.094</td>
</tr>
<tr>
<td>Scenario III-h</td>
<td>20.03</td>
<td>[18.14 19.21]</td>
<td>8.45</td>
<td>0.158</td>
<td>0.075</td>
</tr>
</tbody>
</table>

4.3.3 Stochastic Interest Rate and Housing Price Model

Consider a mortgage with a variable rate $r_t$ which is driven by the CIR process. The housing price, $H_t$, evolves according to geometric Brownian motion. The three factor model is given as

\[
\begin{align*}
\frac{dr_t}{r_t} &= \kappa(\theta - r_t)dt + \sigma_r \sqrt{r_t}dW^1_t, \\
\frac{dS_t}{S_t} &= \mu S_t dt + \sigma_S S_t dW^2_t, \\
\frac{dH_t}{H_t} &= \phi H_t dt + \sigma_H H_t dW^3_t.
\end{align*}
\]

where $W^1_t$, $W^2_t$ and $W^3_t$ are correlated standard Brownian motions. Let $\Sigma$ be a covariance matrix such that

\[
\Sigma = \begin{bmatrix}
dt & \text{cov}(dW^1_t, dW^2_t) & \text{cov}(dW^1_t, dW^3_t) \\
\text{cov}(dW^2_t, dW^1_t) & dt & \text{cov}(dW^2_t, dW^3_t) \\
\text{cov}(dW^3_t, dW^1_t) & \text{cov}(dW^3_t, dW^2_t) & dt
\end{bmatrix}
\]

If $\rho_{i,j}$ is the instantaneous correlation between $dW^i_t$ and $dW^j_t$ then the covariance can be defined as $\text{cov}(dW^i_t, dW^j_t) = dt \rho_{i,j}$ for $i \neq j$. Therefore the covariance matrix becomes

\[
\Sigma = dt \begin{bmatrix}
1 & \rho_{1,2} & \rho_{1,3} \\
\rho_{1,2} & 1 & \rho_{2,3} \\
\rho_{1,3} & \rho_{2,3} & 1
\end{bmatrix}
\]

A standard Brownian motion is normally distributed with mean 0 and variance 1. Now, we have a vector of three standard Brownian motions and the joint distribution of their increments
4.3. Housing Price Model

Figure 4.19: The left and right panels are box and whisker plots of the simulated payoff times and total costs for Scenarios I and Scenario I-h with $\rho_{SH} = -1.0, -0.5, 0.0, 0.5$ and $1.0$. ($r=0.06$, $i=0.0625$, $\mu = 0.08$, $\sigma_S = 0.2$, $\phi = 0.02$, $\sigma_H = 0.05$ and tax rate is 0.3115)

<table>
<thead>
<tr>
<th>Scenario</th>
<th>Average</th>
<th>95% CI</th>
<th>std</th>
<th>$\Pr(I^<em>_i &gt; I^</em>_m)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Scenario II</td>
<td>26370</td>
<td>[259100 268240]</td>
<td>72250</td>
<td>0.292</td>
</tr>
<tr>
<td>Scenario III</td>
<td>249320</td>
<td>[244900 253730]</td>
<td>69845</td>
<td>0.302</td>
</tr>
<tr>
<td>Scenario II-h</td>
<td>260890</td>
<td>[251540 270240]</td>
<td>147850</td>
<td>0.239</td>
</tr>
<tr>
<td>Scenario III-h</td>
<td>246130</td>
<td>[237430 254840]</td>
<td>137640</td>
<td>0.253</td>
</tr>
</tbody>
</table>

Table 4.16: Summary statistics of simulated total cost distribution for Scenarios I and I-h with $r = 0.06$, $\mu = 0.08$, $\sigma_S = 0.2$, $\phi = 0.02$, $\sigma_H = 0.05$ and $\rho_{SH} = 0.1$. The tax rate is 0.3115 and the line of credit rate is $0.0025 + r_i$.

is given as

$$dW = \begin{bmatrix} dW^1_i \\ dW^2_i \\ dW^3_i \end{bmatrix} \sim MVN(\vec{0}, \Sigma)$$

where $MVN(\vec{\mu}, \Sigma)$ is the multivariate normal distribution with mean vector $\vec{\mu}$ and covariance matrix $\Sigma$.

Example

As before, consider a mortgage with original principal amount of $300,000 and an amortization period of 25 years. The mortgage payments are made monthly. We simulate 1000 sample paths of the investment strategy for all the five scenarios with the house price model. Figure 4.22 gives histograms of the simulated payoff time (left panel) and total cost (right panel) for Scenario IV-h with $\kappa = 0.3$, $\theta = 0.06$, $\sigma_r = 0.02$, $r_0 = 0.05$, $\mu = 0.08$, $\sigma_S = 0.2$, $\phi = 0.02$ and $\sigma_H$
Figure 4.20: The left and right panels are histograms of the simulated payoff time and total cost distributions, respectively, for Scenario II-h with \( r_0 = 0.05 \), \( \kappa = 0.3 \), \( \theta = 0.06 \), \( \mu = 0.08 \), \( \sigma_S = 0.2 \), \( \phi = 0.02 \), \( \sigma_H = 0.05 \), \( \rho_{SH} = 0.1 \) and tax rate is 0.3115. The line of credit rate is \( r_t + 0.25\% \).

<table>
<thead>
<tr>
<th>Scenario</th>
<th>Average</th>
<th>95% CI</th>
<th>std</th>
<th>( \Pr(\tau &gt; 25) )</th>
<th>( \Pr(\tau &gt; 30) )</th>
<th>( \Pr(\tau &gt; \tau_m) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>I-h</td>
<td>19.57</td>
<td>[19.00 20.13]</td>
<td>8.91</td>
<td>0.175</td>
<td>0.090</td>
<td></td>
</tr>
<tr>
<td>II-h</td>
<td>19.50</td>
<td>[18.92 20.08]</td>
<td>9.16</td>
<td>0.181</td>
<td>0.093</td>
<td></td>
</tr>
<tr>
<td>III-h</td>
<td>18.61</td>
<td>[18.07 19.15]</td>
<td>8.49</td>
<td>0.152</td>
<td>0.074</td>
<td></td>
</tr>
<tr>
<td>IV-h</td>
<td>19.19</td>
<td>[18.62 19.75]</td>
<td>9.00</td>
<td>0.174</td>
<td>0.086</td>
<td>0.178</td>
</tr>
<tr>
<td>V-h</td>
<td>18.47</td>
<td>[17.93 19.00]</td>
<td>8.48</td>
<td>0.153</td>
<td>0.075</td>
<td>0.189</td>
</tr>
</tbody>
</table>

Table 4.17: Summary statistics of simulated mortgage payoff time distributions of all five scenarios with \( \kappa = 0.3 \), \( \theta = 0.06 \), \( r_0 = 0.05 \), \( \sigma_r = 0.02 \), \( \mu = 0.08 \), \( \sigma_S = 0.2 \), \( \phi = 0.01 \) and \( \sigma_H = 0.05 \) and \( \rho_{1,2} = \rho_{1,3} = \rho_{2,3} = 0 \). Tax rate is 0.3115 and the line of credit rate is \( r_t + 0.25\% \).

= 0.05. Tax rate is 0.3115 and the line of credit rate is \( r_t + 0.25\% \). For now we are considering the case when \( \rho_{1,2} = \rho_{1,3} = \rho_{2,3} = 0 \). Later in this section we consider different values of the instantaneous correlation factors to see how they affect the re-advanceable scheme. The average payoff time of the re-advanceable scheme is 19.19 years with the standard deviation of 9 years. The probability that it can take longer than 25 and 30 years to pay off the mortgage is 0.174 and 0.086, respectively. The probability that it can take longer than the stopping time \( \tau_m \) to pay off the entire mortgage is 0.178. The average total cost of the re-advanceable scheme is $254,670 with the probability of 0.228 that the total cost of the re-advanceable scheme is greater than the interest cost of the original mortgage.

Figure 4.23 compares all five scenarios of mortgage rates with the housing price model. The left and right panels are box and whisker plots of the simulated mortgage payoff times and total cost, respectively, for Scenarios I-h to V-h. Here the Brownian motions driving the stock price, \( S_t \), house price, \( H_t \) and the interest rate, \( r_t \) are not correlated. The parameter values are \( \kappa=0.3 \), \( \theta = 0.06 \), \( \sigma_r = 0.02 \), \( r_0 = 0.05 \), \( \mu = 0.08 \), \( \sigma_S = 0.2 \), \( \phi = 0.02 \), and \( \sigma_H = 0.05 \).
Figure 4.21: The left and right panels are box and whisker plots of the simulated payoff times and total costs, respectively, for Scenarios II, III, II-h and III-h. with \( r_0 = 0.05 \), \( \theta = 0.06 \), \( \kappa = 0.3 \), \( \sigma_r = 0.02 \), \( \mu = 0.08 \), \( \sigma_S = 0.2 \), \( \phi = 0.02 \), \( \sigma_H = 0.05 \) and \( \rho_{SH} = 0.1 \). The tax rate is 0.3115 and the line of credit rate is 0.0025 + \( r_t \).

<table>
<thead>
<tr>
<th>Scenario</th>
<th>Average</th>
<th>95% CI</th>
<th>std</th>
<th>( \Pr(I_t &gt; I_m) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>I-h</td>
<td>274820</td>
<td>[265500 284140]</td>
<td>147350</td>
<td>0.232</td>
</tr>
<tr>
<td>II-h</td>
<td>260160</td>
<td>[250820 269490]</td>
<td>147620</td>
<td>0.234</td>
</tr>
<tr>
<td>III-h</td>
<td>245640</td>
<td>[236790 254490]</td>
<td>139930</td>
<td>0.244</td>
</tr>
<tr>
<td>IV-h</td>
<td>254670</td>
<td>[245430 263900]</td>
<td>146020</td>
<td>0.206</td>
</tr>
<tr>
<td>V-h</td>
<td>242610</td>
<td>[233740 251470]</td>
<td>140150</td>
<td>0.228</td>
</tr>
</tbody>
</table>

Table 4.18: Summary statistics of simulated total cost distributions of all five scenarios with \( \kappa = 0.3 \), \( \theta = 0.06 \), \( r_0 = 0.05 \), \( \sigma_r = 0.02 \), \( \mu = 0.08 \), \( \sigma_S = 0.2 \), \( \phi = 0.02 \), \( \sigma_H = 0.05 \) and \( \rho_{1,2} = \rho_{1,3} = \rho_{2,3} = 0 \). Tax rate is 0.3115 and the line of credit rate is \( r_t + 0.25\% \).

Tables 4.17 and 4.18 provide some summary statistics of the mortgage payoff time and total cost distributions. All scenarios produce similar results in terms of the average payoff time and average total cost. Out of all five scenarios, Scenario V-h has the least on average payoff time of 18.47 years and least on average total cost of $242,610. The payoff time and total cost distributions for Scenarios III-h and V-h are very similar to each other.

Now we introduce a non zero correlation between the Brownian motions driving, \( S_t \), \( r_t \) and \( H_t \). First, consider a case when all the three correlation coefficients are negative such that \( \rho_{1,2} = \rho_{1,3} = \rho_{2,3} = -0.7 \). Figure 4.24 gives box and whisker plots of the simulated mortgage payoff time and total cost in the left and right panels, respectively, for the five scenarios. The other parameter values are the same as for the zero-correlation case. Tables 4.19 and 4.20 provide the summary statistics of the mortgage payoff time and total cost distributions, respectively. The average payoff times and total costs of all five scenarios decrease as compared to case when there the correlation coefficients were zeros. The standard deviations of the distributions...
Figure 4.22: The left and right panels are histograms of the simulated payoff times and total cost, respectively, for Scenarios IV-h with $\theta = 0.06$, $\mu = 0.08$, $\sigma_S = 0.2$, $\phi = 0.02$, $\sigma_H = 0.05$, and $\rho_{1,2} = \rho_{1,3} = \rho_{2,3} = 0.0$. Tax rate is 0.3115 and the line of credit rate is $r_t + 0.25\%$.

Table 4.19: Summary statistics of simulated mortgage payoff time distributions of all five scenarios with $\kappa = 0.3$, $\theta = 0.06$, $r_0 = 0.05$, $\sigma_r = 0.02$, $\mu = 0.08$, $\sigma_S = 0.2$, $\phi = 0.02$ and $\sigma_H = 0.05$ and $\rho_{1,2} = \rho_{1,3} = \rho_{2,3} = -0.7$. Tax rate is 0.3115 and the line of credit rate is $r_t + 0.25\%$.

<table>
<thead>
<tr>
<th>Scenario</th>
<th>Average</th>
<th>95% CI</th>
<th>std</th>
<th>Pr($\tau &gt; 25$)</th>
<th>Pr($\tau &gt; 30$)</th>
<th>Pr($\tau &gt; \tau_m$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>I-h</td>
<td>19.20</td>
<td>[18.67 19.75]</td>
<td>8.60</td>
<td>0.162</td>
<td>0.080</td>
<td></td>
</tr>
<tr>
<td>II-h</td>
<td>19.12</td>
<td>[18.56 19.68]</td>
<td>8.86</td>
<td>0.164</td>
<td>0.083</td>
<td></td>
</tr>
<tr>
<td>III-h</td>
<td>18.27</td>
<td>[17.76 18.78]</td>
<td>8.08</td>
<td>0.140</td>
<td>0.065</td>
<td></td>
</tr>
<tr>
<td>IV-h</td>
<td>18.74</td>
<td>[18.17 19.30]</td>
<td>8.91</td>
<td>0.161</td>
<td>0.079</td>
<td>0.137</td>
</tr>
<tr>
<td>V-h</td>
<td>17.98</td>
<td>[17.44 18.52]</td>
<td>8.49</td>
<td>0.138</td>
<td>0.064</td>
<td>0.157</td>
</tr>
</tbody>
</table>

also decrease with the introduction of nonzero correlation. The effect of the correlation seems to be modest.

Now, consider a case when all three correlation coefficients are positive such that $\rho_{1,2} = \rho_{1,3} = \rho_{2,3} = 0.7$. Figure [4.25] gives box and whisker plots of the simulated mortgage payoff time and total cost on the left and right panels, respectively for all five scenarios. The other parameter values remain the same. Tables [4.21] and [4.22] provide the summary statistics of the mortgage payoff time and total cost distributions, respectively. For Scenario I-h the average payoff times are 19.57, 19.20 and 19.91 years for $\rho_{2,3} = 0$, -0.7 and 0.7, respectively. We can see that when $S_t$ and $H_t$ are negatively correlated then the average payoff time decreases but when they are positively correlated the average payoff time increases. If stock and house prices move in opposite directions i.e., one increases and the other one decreases that balances the investment portfolio growth and the line of credit balance and hence decreases the average mortgage payoff time. However, if both stock price and house price go in the same direction that is both increase or decrease then that can increase the average payoff time. The similar
### 4.3. Housing Price Model

<table>
<thead>
<tr>
<th>Scenario</th>
<th>Average</th>
<th>95% CI</th>
<th>std</th>
<th>$\text{Pr}(I^<em>_r &gt; I^</em>_m)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Scenario I-h</td>
<td>270790</td>
<td>[261820 279760]</td>
<td>141860</td>
<td>0.205</td>
</tr>
<tr>
<td>Scenario II-h</td>
<td>255790</td>
<td>[246740 264840]</td>
<td>143070</td>
<td>0.207</td>
</tr>
<tr>
<td>Scenario III-h</td>
<td>241950</td>
<td>[233640 250270]</td>
<td>131450</td>
<td>0.208</td>
</tr>
<tr>
<td>Scenario IV-h</td>
<td>250360</td>
<td>[241110 259600]</td>
<td>146240</td>
<td>0.175</td>
</tr>
<tr>
<td>Scenario V-h</td>
<td>237990</td>
<td>[228920 247060]</td>
<td>143420</td>
<td>0.186</td>
</tr>
</tbody>
</table>

Table 4.20: Summary statistics of simulated total cost distributions of all five scenarios with $\kappa = 0.3$, $\theta = 0.06$, $r_0 = 0.05$, $\sigma_r = 0.02$, $\mu = 0.08$, $\sigma_S = 0.2$, $\phi = 0.02$, $\sigma_H = 0.05$ and $\rho_{1,2} = \rho_{1,3} = \rho_{2,3} = -0.7$. Tax rate is 0.3115 and the line of credit rate is $r_t + 0.25\%$.

<table>
<thead>
<tr>
<th>Scenario</th>
<th>Average</th>
<th>95% CI</th>
<th>std</th>
<th>$\text{Pr}(\tau &gt; 25)$</th>
<th>$\text{Pr}(\tau &gt; 30)$</th>
<th>$\text{Pr}(\tau &gt; \tau_m)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Scenario I-h</td>
<td>19.91</td>
<td>[19.36 20.46]</td>
<td>8.68</td>
<td>0.213</td>
<td>0.101</td>
<td></td>
</tr>
<tr>
<td>Scenario II-h</td>
<td>19.87</td>
<td>[19.30 20.44]</td>
<td>8.99</td>
<td>0.221</td>
<td>0.107</td>
<td></td>
</tr>
<tr>
<td>Scenario III-h</td>
<td>18.92</td>
<td>[18.41 19.43]</td>
<td>8.13</td>
<td>0.171</td>
<td>0.081</td>
<td></td>
</tr>
<tr>
<td>Scenario IV-h</td>
<td>19.15</td>
<td>[18.66 19.64]</td>
<td>7.75</td>
<td>0.169</td>
<td>0.082</td>
<td>0.207</td>
</tr>
<tr>
<td>Scenario V-h</td>
<td>18.25</td>
<td>[17.79 18.71]</td>
<td>7.23</td>
<td>0.138</td>
<td>0.064</td>
<td>0.195</td>
</tr>
</tbody>
</table>

Table 4.21: Summary statistics of simulated mortgage payoff time distributions of all five scenarios with $\kappa = 0.3$, $\theta = 0.06$, $r_0 = 0.05$, $\sigma_r = 0.02$, $\mu = 0.08$, $\sigma_S = 0.2$, $\phi = 0.02$ and $\sigma_H = 0.05$ and $\rho_{1,2} = \rho_{1,3} = \rho_{2,3} = 0.7$. Tax rate is 0.3115 and the line of credit rate is $r_t + 0.25\%$.

<table>
<thead>
<tr>
<th>Scenario</th>
<th>Average</th>
<th>95% CI</th>
<th>std</th>
<th>$\text{Pr}(I^<em>_r &gt; I^</em>_m)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Scenario I-h</td>
<td>276710</td>
<td>[268170 285260]</td>
<td>135140</td>
<td>0.273</td>
</tr>
<tr>
<td>Scenario II-h</td>
<td>262600</td>
<td>[253920 271290]</td>
<td>137310</td>
<td>0.276</td>
</tr>
<tr>
<td>Scenario III-h</td>
<td>246650</td>
<td>[238750 254540]</td>
<td>124790</td>
<td>0.291</td>
</tr>
<tr>
<td>Scenario IV-h</td>
<td>248820</td>
<td>[241770 255870]</td>
<td>11480</td>
<td>0.231</td>
</tr>
<tr>
<td>Scenario V-h</td>
<td>235090</td>
<td>[228430 241760]</td>
<td>105430</td>
<td>0.245</td>
</tr>
</tbody>
</table>

Table 4.22: Summary statistics of simulated total cost distributions of all five scenarios with $\kappa = 0.3$, $\theta = 0.06$, $r_0 = 0.05$, $\sigma_r = 0.02$, $\mu = 0.08$, $\sigma_S = 0.2$, $\phi = 0.02$, $\sigma_H = 0.05$ and $\rho_{1,2} = \rho_{1,3} = \rho_{2,3} = 0.7$. Tax rate is 0.3115 and the line of credit rate is $r_t + 0.25\%$. 
Figure 4.23: Left panel is a box and whisker plot of the simulated payoff time distribution of all five scenarios. Right panel is a box and whisker plot of the simulated interest cost distribution of all the five scenarios with $\kappa=0.3$, $\theta=0.06$, $\sigma_r = 0.02$, $r_0 = 0.05$, $\mu = 0.08$, $\sigma_S = 0.2$, $\phi = 0.02$, $\sigma_H = 0.05$, $\rho_{1,2} = \rho_{1,3} = \rho_{2,3} = 0.0$. Tax rate is 0.3115 and the line of credit rate is $r_t + 0.25\%$.

<table>
<thead>
<tr>
<th>Case</th>
<th>$\rho_{1,2}$</th>
<th>$\rho_{1,3}$</th>
<th>$\rho_{2,3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case 1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Case 2</td>
<td>-0.5</td>
<td>-0.5</td>
<td>-0.5</td>
</tr>
<tr>
<td>Case 3</td>
<td>0.7</td>
<td>0.7</td>
<td>0.7</td>
</tr>
<tr>
<td>Case 4</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>Case 5</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>Case 6</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
</tr>
</tbody>
</table>

Table 4.23: Cases for different correlation coefficients values.

pattern is observed for the average total cost. The other four scenarios also give the same pattern.

So far we have considered the cases where all the three correlation coefficients have the same values. Next, we consider three cases where one of three correlation coefficients is positive and other two are negative. We keep all the other parameters same as in the previous examples. Table 4.23 shows six different cases we considered depending on the values of $\rho_{1,2}$, $\rho_{1,3}$ and $\rho_{2,3}$. The box and whisker plots of simulated payoff times and total costs for Cases 4-6 are given in Appendix D. Tables containing the summary statistics of the mortgage payoff time and total cost distributions are also given in the same appendix. Figure 4.26 gives plots of average payoff time (top left), standard deviation of payoff time (top right), average total cost (bottom left) and standard deviation of total cost (bottom right) for Scenarios I-h to V-h versus Cases 1-6. For Scenarios I-h to III-h the average payoff time and total cost decrease (increase) when there is negative (positive) correlation between the Brownian motions driving stock price and housing price ($\rho_{2,3}$). The standard deviations of the payoff time and total cost distributions
4.3. Housing Price Model

Figure 4.24: The left and right panels are box and whisker plots of the simulated payoff times and total costs, respectively, for Scenarios I-h to V-h with $\kappa=0.3$, $\theta=0.06$, $\sigma_r=0.02$, $r_0=0.05$, $\mu=0.08$, $\sigma_S=0.2$, $\phi=0.02$, $\sigma_H=0.05$ and $\rho_{1,2}=\rho_{1,3}=\rho_{2,3}=-0.7$. Tax rate is 0.3115 and the line of credit rate is $r_t+0.25\%$.

are the highest when $\rho_{2,3}=0$. The standard deviations are lower for negative correlation as compared to positive correlation. For Scenarios IV-h and V-h the average payoff time and total cost are the highest for Case 5 ($\rho_{1,2}=-1$, $\rho_{1,3}=-1$ and $\rho_{2,3}=1$) and the lowest for Case 6 ($\rho_{1,2}=1$, $\rho_{1,3}=-1$ and $\rho_{2,3}=-1$). The standard deviations of the payoff time and total cost distributions are the highest for Case 4 ($\rho_{1,2}=-1$, $\rho_{1,3}=1$ and $\rho_{2,3}=-1$) and the lowest for Case 6. It means that strong positive correlation between stock price and interest rate, and strong negative correlation between stock price and house price, and between interest rate and house price, increases the reward and decreases the risk.
Figure 4.25: The left and right panels are box and whisker plots of the simulated payoff times and total costs, respectively, for Scenarios I-h to V-h with $\kappa=0.3$, $\theta=0.06$, $\sigma_r = 0.02$, $r_0 = 0.05$, $\mu = 0.08$, $\sigma_S = 0.2$, $\phi = 0.02$, $\sigma_H = 0.05$ and $\rho_{1,2} = \rho_{1,3} = \rho_{2,3} = 0.7$. Tax rate is 0.3115 and the line of credit rate is $r_t + 0.25\%$.

Figure 4.26: The top left, top right, bottom left and bottom right panels are plots of the simulated average payoff times, standard deviation of payoff times, average total cost and standard deviation of total costs, respectively, for Scenarios I-h to V-h versus Cases 1-6 with $\kappa=0.3$, $\theta=0.06$, $\sigma_r = 0.02$, $r_0 = 0.05$, $\mu = 0.08$, $\sigma_S = 0.2$, $\phi = 0.02$, $\sigma_H = 0.05$. Tax rate is 0.3115 and the line of credit rate is $r_t + 0.25\%$. 
4.4 Conclusion

As compared to the fixed-rate mortgage (Scenario I), the mean reverting rate mortgage (Scenario II) reduces the average payoff time average total cost if the process starts below the long run rate. The average payoff time and average total cost increase if the process starts above the long run rate. However, an interesting result is that the payoff time and total cost standard deviations are less if mean reverting process starts above the long run rate. Thus, there is a trade off between the risk and reward. For Scenario III the average payoff time and the average total cost are the least as compared to Scenarios I and II if the mean reverting rate process starts below the long run rate and are the highest if the process starts above the long run rate.

A mortgage with stochastic interest rate (Scenarios IV and V) decreases the average payoff time and average total cost of the re-advanceable mortgage scheme. Comparing all five Scenarios I-V we find that Scenarios III and V produce best results as compared to the other three scenarios. Scenario V has less average payoff time and total cost than Scenario III, however, this comes with increased risk. We find that the average payoff time and average total cost increase with $\sigma_r$. However, increase in $\sigma_r$ affects the left tail of the payoff time distribution in a positive way. We see that the probability that it can take less than 15 years to pay off the mortgage increases with $\sigma_r$. The instantaneous correlation between the Brownian motions driving the stock price and CIR process (mortgage rate) affects the performance of re-advanceable mortgage scheme. If the mortgage rate is strongly negatively correlated with the risky investment, the homeowner is subject to both a higher expected total cost and more risk than the case of strong positive correlation.

When we incorporate the housing price model in the re-advanceable scheme of fixed-rate mortgage the homeowner can expect to pay off the mortgage slightly earlier with expected savings of a few thousands of dollars in interest cost as compared to the re-advanceable scheme without the housing price model. These small differences in the expected mortgage payoff time and the expected total cost come at the cost of increased risk. The incorporation of the housing price model in the re-advanceable mortgage scheme increases the total borrowed amount and hence it can result in an increased likelihood that the homeowner is not able to pay off the entire loan in 75 years. The expected payoff time and the expected total cost decrease with increase in the drift and volatility of the housing price model but again this comes at the cost of increased standard deviations in the payoff time and total cost distributions. We also find that if house prices are strongly positively correlated with the risky investment, the homeowner is cautioned against using increased home equity to finance additional borrowing.

For the variable-rate mortgage the effect of the incorporation of housing price model in the re-advanceable mortgage scheme is the same that it reduces the average payoff time and average total cost little bit at the cost of increased risk. The effect of correlations between stock price, house price and mortgage rate seems to be modest.
Chapter 5

Summary

The re-advanceable mortgage benefits the homeowner by reducing the average mortgage payoff time. However, there is a considerable variation in the payoff times with a significant probability of a payoff time exceeding the original mortgage amortization term. The efficiency of the re-advanceable mortgage depends on many factors including the investment portfolio performance. If investment performs poorly the risk associated with the re-advanceable strategy increases. Different income tax rates also affect the strategy’s efficiency. Higher income homeowners enjoy a payoff time distribution with both a lower average and a lower standard deviation than low-income homeowners. In the event of job loss, the investment portfolio protects the homeowner from default as part of the investment portfolio can be sold to fund mortgage payments, another benefit that is not typically advertised. This work is important to lenders, financial planners and homeowners to more fully understand the benefits and risk associated with this strategy.

A number of parameters such as time, mortgage rate, the line of credit rate, growth rate and volatility of risky asset affect the performance of investment capital and portfolio. We find that, if the spread between the line of credit rate and mortgage rate is large enough, the change in investment capital can become negative. However, a spread of this magnitude is not likely to occur in practice. The mean and the standard deviation of investment portfolio value grow with time. The increase in mortgage rate and the line of credit rate have inverse effects on the mean and the standard deviation. The mean and standard deviation increase with the growth rate of the risky asset. Unsurprisingly, the standard deviation increases with the volatility of the risky asset.

We considered the variable rate mortgage where the mortgage rate either evolves according to a deterministic mean reverting model or CIR process. As compared to the fixed-rate mortgage, the mean reverting rate mortgage reduces the average payoff time average total cost if the process starts below the long run rate. The average payoff time and average total cost increase if the process starts above the long run rate. However, an interesting result is that the payoff time and total cost standard deviations are less if mean reverting process starts above the long run rate. Thus, there is a trade off between the risk and reward. The mean reverting rate and stochastic rate mortgages produce similar results. They further improve the results if the mortgage payment rate used is that computed as for the fixed-rate mortgage. A stochastic rate mortgage has a lower average payoff time and total cost than mean reverting rate mortgage, however, this comes with increased risk. We find that the average payoff time and average total...
cost increase with volatility in the interest rate model. However, an increase in the volatility of
the interest rate model affects the left tail of the payoff time distribution in a positive way - the
probability that it can take less than the 15 years (considering that the amortization term in 25
years) to pay off the mortgage increases. The instantaneous correlation between the Brownian
motions driving the stock price and CIR processes (mortgage rate) affects the performance of
re-advanceable mortgage scheme. If the mortgage rate is strongly negatively correlated with
the risky investment, the homeowner is subject to both a higher expected total cost and more
risk than the case of strong positive correlation.

When we incorporate the housing price model in the re-advanceable scheme of the fixed-rate and variable-rate mortgages the homeowner can expect to pay off the mortgage slightly earlier with expected savings of a few thousand dollars in interest cost as compared to the re-advanceable scheme without the housing price model. These small differences in the expected mortgage payoff time and the expected total cost come at the cost of increased risk. The incorporation of the housing price model in the re-advanceable mortgage scheme increases the total borrowed amount and hence it results in an increased likelihood that the homeowner is not able to pay off the entire loan in a very long period of time e.g., 75 years. The expected payoff time and the expected total cost decrease with the drift and volatility of the housing price model but again this comes at the cost of increased standard deviations in the payoff time and total cost distributions. We also find that if house prices are strongly positively correlated with the risky investment, the homeowner is cautioned against using increased home equity to finance additional borrowing. The effect of correlations between stock price, house price and mortgage rate seems to be modest.

We considered a re-advanceable mortgage where the homeowner invests the entire proceeds of the line of credit into a single risky asset. We make an assumption that the homeowner continuously withdraw money from the line of credit account and continuously invest it in a stock. To invest in a stock the commission of a stock broker was not included in our model. We also did not include the transaction cost on stock in our model. Buying a number of units of stock each month can cost homeowner a large transaction costs on stock. These two factors of cost can increase the total cost of the re-advanceable mortgages scheme versus the total interest cost of a traditional mortgage. In order to reduce total transaction cost on stock the homeowner can let the line of credit balance pile up and then invest periodically (a period longer than a month). Our model can be extended to incorporate an optimal control problem which allows homeowner to optimize the waiting period to invest money out of the line of credit balance.

The single risky asset is assumed to follow a geometric Brownian motion. A future extension of this work can be to include a different model for the risky asset, for example one with jumps. A modified version of this study can include a scenario of diversifying the investment portfolio that is to invest in more than one risky asset. While implementing the re-advanceable scheme we ignore income taxes on the investment capital gains. Our model can be modified to include this factor and see how it affects the mortgage payoff time and total cost distributions. For the variable rate mortgage we fixed the mortgage payment rate. Further study can be done by recalculating the mortgage payment rate whenever the interest rate gets so high that it is unable to cover the interest portion of the mortgage payment.

The number of householders choosing the re-advanceable mortgage over the traditional mortgage can effect the stock market. Our model can be extended to study the aggregate effect of re-advanceable mortgage scheme on stock market.
Bibliography


Appendix A

Moments of Investment Portfolio Value

Here we derived the first two moments of the investment portfolio value.

A.1 First Moment

Take the expectation of Equation 2.31 to get the expected value of \( V_t \) and assuming we can change the integral and expectation order

\[
\mathbb{E}[V_t] = \begin{cases} 
\int_0^t (\mu \mathbb{E}[V_s] + \beta(s)) ds + \sigma \mathbb{E}\left[ \int_0^t V_t dW_s \right], & \text{if } 0 \leq t \leq m, \\
V_m + \int_m^t (\mu \mathbb{E}[V_s] + \alpha) ds + \sigma \mathbb{E}\left[ \int_m^t V_s dW_s \right], & \text{if } t > m.
\end{cases}
\]  

(A.1)

Note that \( \int_0^t V_s dW_s \) and \( \int_m^t V_s dW_s \) are stochastic integrals and their expectations are zero [4]. Let \( \mathbb{E}[V_t] = m_1(t) \) with initial conditions \( m_1(0) = \mathbb{E}[V_0] = V_0 = 0 \) and \( m_1(m) = \mathbb{E}[V_m] = V_m \). Then

\[
m_1(t) = \begin{cases} 
\int_0^t (\mu m_1(s) + \beta(s)) ds, & \text{if } 0 \leq t \leq m, \\
V_m + \int_m^t (\mu m_1(s) + \alpha) ds, & \text{if } t > m.
\end{cases}
\]  

(A.2)

Differentiating the above equation with respect to \( t \) and rearranging the terms we have

\[
m_1'(t) - \mu m_1(t) = \begin{cases} 
\beta(t), & \text{if } 0 \leq t \leq m, \\
\alpha, & \text{if } t > m.
\end{cases}
\]  

(A.3)

The above equation is an ordinary differential equation that can be solved using an integrating factor of \( e^{-\mu t} \). Multiplying Equation A.3 by \( e^{-\mu t} \) and integrating with respect to \( t \) gives

\[
\int d(e^{-\mu t}m_1(t)) = \begin{cases} 
\int \beta(t)e^{-\mu t} dt, & \text{if } 0 \leq t \leq m, \\
\alpha \int e^{-\mu t} dt, & \text{if } t > m.
\end{cases}
\]  

(A.4)

Simplify Equation A.4 to get

\[
e^{-\mu t}m_1(t) = \begin{cases} 
\frac{A-B}{\mu}e^{(t-\mu)m} - \frac{B}{\mu}e^{-\mu t} + D_1, & \text{if } 0 \leq t \leq m, \\
-\frac{A-B}{\mu}e^{m-\mu t} + \frac{B}{\mu}e^{-\mu t} + P_1, & \text{if } 0 \leq t \leq m.
\end{cases}
\]  

(A.5)
where \( D_1 \) and \( P_1 \) are the constants of integration and their values are

\[
D_1 = \frac{A - B}{r - \mu} + \frac{B}{\mu}, \quad (A.6)
\]

\[
P_1 = \frac{r(A - B)}{\mu(r - \mu)}e^{(r - \mu)t} + \frac{rB - \mu A}{\mu(r - \mu)}. \quad (A.7)
\]

Substituting the values of \( D_1 \) and \( P_1 \) in Equation (A.5) and rearranging the terms we have

\[
\mathbb{E}[V_t] = \begin{cases} 
\frac{A-B}{r-\mu} (e^{rt} - e^{\mu t}) + \frac{B}{\mu} (e^{\mu t} - 1), & \text{if } 0 \leq t \leq m \\
\frac{e^{\mu t}}{r(A-B)e^{\mu t} + rB - \mu A}, & \text{if } t > m.
\end{cases} \quad (A.8)
\]

### A.2 Second Moment

To get the second moment of \( V_t \) we first find the dynamics of \( V_t^2 \). Using Itô’s lemma, we get

\[
d(V_t^2) = \begin{cases} 
(2\mu + \sigma^2)V_t^2 + 2\beta(t)V_t \, dt + 2\sigma V_t^2 \, dW_t, & \text{if } 0 \leq t \leq m \\
(2\mu + \sigma^2)V_t^2 + 2\alpha V_t \, dt + 2\sigma V_t^2 \, dW_t, & \text{if } t > m.
\end{cases} \quad (A.9)
\]

In integral form we have

\[
V_t^2 = \begin{cases} 
V_0^2 + \int_0^t [(2\mu + \sigma^2)V_s^2 + 2\beta(s)V_s] \, ds + 2\sigma \int_0^t V_s^2 \, dW_s, & \text{if } 0 \leq t \leq m \\
V_m^2 + \int_m^t [(2\mu + \sigma^2)V_s^2 + 2\alpha V_s] \, ds + 2\sigma \int_m^t V_s^2 \, dW_s, & \text{if } t > m.
\end{cases} \quad (A.10)
\]

Take expectation of both sides of the above equation. Use the fact that the expectation of stochastic integral is zero. Also note that \( \mathbb{E}[V_0^2] = 0 \). Interchanging the order of expected value and integral, we get

\[
\mathbb{E}[V_t^2] = \begin{cases} 
(2\mu + \sigma^2)\int_0^t \mathbb{E}[V_s^2] \, ds + 2\int_0^t \mathbb{E}[V_s]\beta(s) \, ds, & \text{if } 0 \leq t \leq m \\
\mathbb{E}[V_m^2] + (2\mu + \sigma^2)\int_m^t \mathbb{E}[V_s^2] \, ds + 2\int_m^t \mathbb{E}[V_s]\beta(s) \, ds, & \text{if } t > m.
\end{cases} \quad (A.11)
\]

Let \( \mathbb{E}[V_t^2] = m_2(t) \) with initial condition \( m_2(0) = V_0^2 = 0 \). Equation (A.11) becomes

\[
m_2(t) = \begin{cases} 
(2\mu + \sigma^2)\int_0^t m_2(s) \, ds + 2\int_0^t m_1(s)\beta(s) \, ds, & \text{if } 0 \leq t \leq m \\
m_2(m) + (2\mu + \sigma^2)\int_m^t m_2(s) \, ds + 2\int_m^t m_1(s)\beta(s) \, ds, & \text{if } t > m.
\end{cases} \quad (A.12)
\]

Differentiating both sides of the above equation with respect to \( t \) gives

\[
m_2'(t) = \begin{cases} 
(2\mu + \sigma^2)m_2(t) + 2m_1(t)\beta(t), & \text{if } 0 \leq t \leq m \\
(2\mu + \sigma^2)m_2(t) + 2\alpha m_1(t), & \text{if } t > m.
\end{cases} \quad (A.13)
\]

\[
\Rightarrow m_2'(t) - (2\mu + \sigma^2)m_2(t) = \begin{cases} 
2m_1(t)\beta(t), & \text{if } 0 \leq t \leq m \\
2\alpha m_1(t), & \text{if } t > m.
\end{cases} \quad (A.14)
\]

The above equation is an ordinary differential equation in \( m_2(t) \). The integrating factor of it is \( e^{-(2\mu+\sigma^2)t} \). Multiplying Equation (A.14) by the integrating factor and integrating with respect to \( t \) gives
\[ m_2(t) = 2e^{-2\frac{\alpha^2}{\mu^2}} \int_0^t \left[ m_1(t) \beta(t) e^{-2\frac{\alpha^2}{\mu^2}t^2} \right] dt, \quad \text{if} \quad 0 \leq t \leq m, \quad (A.15) \]
\[ m_2(t) = \int_0^t \left[ m_1(t) e^{-2\frac{\alpha^2}{\mu^2}t^2} \right] dt, \quad \text{if} \quad t > m. \]

Using the formula for \( m_1(t) \) from Equation A.8 and replacing \( \beta(t) \) by \( (A - B)e^{rt} + B \) the integrands of A.15 becomes
\[ m_1(t)\beta(t)e^{-2\frac{\alpha^2}{\mu^2}t^2} = \frac{(A - B)^2}{r - \mu} e^{2(r-2\mu-\sigma^2)t} + (A - B) \left[ \frac{B}{r - \mu} \frac{A - B}{\mu} \right] e^{(r-\mu-\sigma^2)t} \]
\[ + \frac{B(A - B)(2\mu - r)}{\mu(r - \mu)} e^{(r-2\mu-\sigma^2)t} + B \left[ \frac{B}{\mu} - \frac{A - B}{r - \mu} \right] e^{-\mu+\sigma^2 \mu} \]
\[ \frac{B^2}{\mu} e^{-2\frac{\alpha^2}{\mu^2}t^2}, \quad (A.16) \]

and
\[ m_1(t)\alpha(t)e^{-2\frac{\alpha^2}{\mu^2}t^2} = -\frac{\alpha^2}{\mu} e^{-2\frac{\alpha^2}{\mu^2}t^2} + P_1 e^{-\mu+\sigma^2 \mu}. \quad (A.17) \]

Substituting the above integrands in Equation A.15 and integrating gives
\[ \mathbb{E}[V_t^2] = \begin{cases} \frac{2(A-B)^2}{(r-\mu)(2r-2\mu-\sigma^2)} e^{2rt} + 2 \frac{(A-B)(2\mu - r)}{\mu(r-\mu)} e^{(r-2\mu-\sigma^2)t}, & \text{if} \quad 0 \leq t \leq m, \\ \frac{2B^2}{\mu^2} + 2D_3 e^{2\mu+\sigma^2 \mu}, & \text{if} \quad t > m \end{cases} \quad (A.18) \]

and
\[ D_3 = -\frac{(A - B)^2}{(r - \mu)(2r - 2\mu - \sigma^2)} - \frac{(A - B)}{(r - \mu - \sigma^2)} \left[ \frac{B}{\mu} - \frac{A - B}{r - \mu} \right] - \frac{B(A - B)(2\mu - r)}{\mu(r - \mu)(r - 2\mu + \sigma^2)} \]
\[ + \frac{B}{\mu + \sigma^2} \left[ \frac{B}{\mu} - \frac{A - B}{r - \mu} \right] - \frac{B^2}{\mu(2\mu + \sigma^2)}, \quad (A.19) \]
\[ P_1 = \frac{r(A - B)}{\mu(r - \mu)} e^{(r-\mu)t} + \frac{rB - \mu A}{\mu(r - \mu)}, \quad (A.20) \]
\[ P_3 = \left( \mathbb{E}[V^2_m] - \frac{2\alpha^2}{\mu(2\mu + \sigma^2)} + \frac{2\alpha P_2}{\mu + \sigma^2} e^{\mu m} \right) e^{-\mu+\sigma^2 \mu}. \quad (A.21) \]
Appendix B

Derivatives of Mean and Second Moment of Investment Portfolio Value.

In this appendix we give the derivatives of mean and variance of the investment portfolio value with respect to the set of parameters \((t, r, i, \mu, \sigma)\).

B.1 Mean and Its Derivatives

The mean of the investment portfolio value is

\[
\mathbb{E}[V_t] = \begin{cases} 
\frac{A-B}{r} (e^{rt} - e^{\mu t}) + \frac{B}{r} (e^{rt} - 1), & \text{if } 0 \leq t \leq m, \\
-\frac{(A-B)e^{\mu t} + B}{r} + \frac{e^{rt}}{\mu(r-\mu)} \left( r(A-B)e^{(r-\mu)t} + rB - \mu A \right), & \text{if } t > m.
\end{cases}
\] (B.1)

The limit as \(\mu\) approaches \(r\) of \(\mathbb{E}[V_t]\) exists and is

\[
\lim_{\mu \to r} \mathbb{E}[V_t] = \begin{cases} 
(A-B)te^{rt} + \frac{B}{r} (e^{rt} - 1), & \text{if } 0 \leq t \leq m, \\
-Ae^{rt} e^{\mu t} + \frac{r}{r-\mu} \left( e^{rt} - 1 \right) e^{rt}, & \text{if } t > m.
\end{cases}
\] (B.2)

The derivative with respect to time \(t\) is

\[
\frac{d\mathbb{E}[V_t]}{dt} = \begin{cases} 
\frac{A-B}{r} (re^{rt} - \mu e^{\mu t}) + Be^{rt}, & \text{if } 0 \leq t \leq m, \\
\frac{e^{rt}}{\mu(r-\mu)} (r(A-B)e^{(r-\mu)t} + rB - \mu A), & \text{if } t > m.
\end{cases}
\] (B.3)

The limit as \(\mu\) approaches \(r\) of \(\frac{d\mathbb{E}[V_t]}{dt}\) exists and is

\[
\lim_{\mu \to r} \frac{d\mathbb{E}[V_t]}{dt} = \begin{cases} 
[A + (A-B)tr] e^{rt}, & \text{if } 0 \leq t \leq m, \\
[A + (A-B)mr] e^{rt}, & \text{if } t > m.
\end{cases}
\] (B.4)

The derivative of \(\mathbb{E}[V_t]\) with respect to the mortgage rate, \(r\), is

\[
\frac{d\mathbb{E}[V_t]}{dr} = \begin{cases} 
\frac{e^{rt} - e^{\mu t}}{A} \left( \frac{A-B}{r} (e^{rt} - e^{\mu t}) - \frac{A-B}{r-\mu} + \frac{(A-B)e^{rt}}{r-\mu} - \frac{Bme^{\mu (e^{rt} - 1)}}{\mu (e^{rt}-1)} \right), & \text{if } 0 \leq t \leq m, \\
- (rB - \mu A) \left( \frac{\mu}{r(r-\mu)} + \frac{me^{rt} - m}{e^{rt} - 1} \right), & \text{if } t > m
\end{cases}
\] (B.5)
The limit as $\mu$ approaches $r$ of $\frac{d\mathbb{E}[V]}{dr}$ exists and is
\[
\lim_{\mu \to r} \frac{d\mathbb{E}[V]}{dr} = \begin{cases} 
\text{te}^{r(\mu - 1)} + \frac{(A-B)e^{r\mu}}{2r} - Bme^{\mu}e^{r(\mu - 1)}, & \text{if } 0 \leq t \leq m, \\
\frac{Ae^{r\mu}(e^{r(1 - e^{\mu})} - e^{r\mu})}{e^{r(1 - e^{\mu})}} - \frac{2m(2A - B) - 2A(\frac{1}{r} - e^{r/2})}{e^{r/2}}, & \text{if } t > m.
\end{cases}
\] (B.6)

The derivative of $\mathbb{E}[V]$ with respect to the line of credit rate, $i$, is
\[
d\mathbb{E}[V] \frac{d}{di} = \begin{cases} 
\frac{B}{r} \left( \frac{e^{r-t} - e^{rt}}{r-rt} + \frac{e^{rt}}{\mu - rt} \right), & \text{if } 0 \leq t \leq m, \\
\frac{B}{r} \left( e^{rt} - rme^{r(t-m)} \right), & \text{if } t > m.
\end{cases}
\] (B.7)

The limit as $\mu$ approaches $r$ of $\frac{d\mathbb{E}[V]}{d\mu}$ exists and is
\[
\lim_{\mu \to r} \frac{d\mathbb{E}[V]}{d\mu} = \begin{cases} 
\frac{B}{r} \left( e^{rt}(1 - rt) - 1 \right), & \text{if } 0 \leq t \leq m, \\
\frac{B}{r} \left( e^{rt} - rme^{r(1 - t)} - 1 \right), & \text{if } t > m.
\end{cases}
\] (B.8)

The derivative of $\mathbb{E}[V]$ with respect to the drift, $\mu$, is
\[
d\mathbb{E}[V] \frac{d}{d\mu} = \begin{cases} 
\frac{A-B}{(r-\mu)^2} \left( [e^{t+1} - e^{rt}] + \frac{B}{r} \left( e^{rt}(\mu t - 1) + 1 \right) \right) + \frac{B}{r} \left( e^{rt} - r(t - \mu)e^{rt} - 1 \right), & \text{if } 0 \leq t \leq m, \\
\frac{A-B}{(r-\mu)^2} + \frac{B}{r} \left( e^{rt} - r(t - \mu)e^{rt} - 1 \right), & \text{if } t > m.
\end{cases}
\] (B.9)

The derivative of $\mathbb{E}[V]$ with respect to the volatility, $\sigma$, is
\[
\frac{d\mathbb{E}[V]}{d\sigma} = 0.
\] (B.11)

## B.2 Second Moment and Its Derivatives

The second moment of the investment portfolio, $\mathbb{E}[V^2]$, is
\[
\mathbb{E}[V^2] = \begin{cases} 
\frac{2(2B-r)^2}{(r-\mu)(2r-2\mu-\sigma^2)} e^{2rt} - \frac{2(A-B)(rB-\mu A)e^{2(\mu+\sigma)^2}}{\mu(r-\mu)(2r-2\mu-\sigma^2)} e^{2rt} - \frac{2B(r-\mu\mu)e^{2rt}}{\mu(r-\mu)(2r-2\mu-\sigma^2)}, & \text{if } 0 \leq t \leq m \\
\frac{2A}{\mu^2} e^{2(\mu+\sigma)^2} + P_2 e^{2(\mu+\sigma)^2} e^{2rt}, & \text{if } t > m.
\end{cases}
\] (B.12)

and
\[
D_1 = -\frac{(A-B)^2}{(r-\mu)(2r-2\mu-\sigma^2)} - \frac{(A-B)(rB-\mu A)}{\mu(r-\mu)(2r-2\mu-\sigma^2)} - \frac{B(rB-\mu A)}{\mu(r-\mu)(2r-2\mu-\sigma^2)} e^{2rt} + \frac{B(rB-\mu A)}{\mu(r-\mu)(2r-2\mu-\sigma^2)} e^{2rt} - \frac{B^2(rB-\mu A)}{\mu(r-\mu)(2r-2\mu-\sigma^2)},
\] (B.13)

\[
P_1 = \frac{r(A-B)}{\mu(r-\mu)} e^{r(t-\mu)m} + \frac{r(rB-\mu A)}{\mu(r-\mu)},
\] (B.14)

\[
P_2 = \frac{(\mathbb{E}[V^2_1] - \frac{2}\mu(2\mu+\sigma^2) + \frac{2\alpha P_1}{\mu+\sigma^2} e^{2\mu+\sigma}) e^{-(2\mu+\sigma)^2} m}{2\alpha P_1}\end{array}
\] (B.15)
The limit as $\mu$ approaches $r$ of $\mathbb{E}[V_t^2]$ exists and is

$$
\lim_{\mu \to r} \mathbb{E}[V_t^2] = \begin{cases} 
\frac{2(A-B)^2(\sigma^2-2\sigma^2 r-2A(B-A)r^2)e^{2rt}}{(2B(A-B)-ABrB(A-B)(r^2+\sigma^2)-2B(A-B))e^{rt}} & \text{if } 0 \leq t \leq m \\
\frac{2B^2}{r(2r+\sigma^2)^2} + 2e^{(2r+\sigma^2)t} \lim_{\mu \to r} D_1, & \text{if } t > m.
\end{cases}
$$

(B.16)

where

$$
\lim_{\mu \to r} D_1 = \frac{(A-B)(B\sigma^2 + r(A-B))}{r(2r+\sigma^2)^2} + \frac{(r+\sigma^2)B(2B-A) + rB(A-B)}{r(2r+\sigma^2)^2} - \frac{B^2}{r(2r+\sigma^2)}.
$$

(B.17)

$$
\lim_{\mu \to r} P_1 = \frac{rm(A-B) + A}{r},
$$

(B.18)

$$
\lim_{\mu \to r} P_2 = \left(\lim_{\mu \to r} \mathbb{E}[V_t^2] - \frac{2\alpha^2}{r(2r+\sigma^2)} \right) + \frac{2\alpha(rm(A-B) + A)}{r(2r+\sigma^2)} e^{(2r+\sigma^2)t}.
$$

(B.19)

The derivative of $\mathbb{E}[V_t^2]$ with respect to the time, $t$, is

$$
\frac{d\mathbb{E}[V_t^2]}{dt} = \begin{cases} 
\frac{4r(A-B)^2}{2\mu(Br-Br\mu+2A(B-A))e^{rt}} + \frac{2(r+\sigma^2)(B(2r-\mu)(t-\mu+\sigma^2))e^{rt}}{2\mu(Br-Br\mu+2A(B-A))e^{rt}} & \text{if } 0 \leq t \leq m \\
\frac{2B^2}{r(2r+\sigma^2)^2} + 2e^{(2r+\sigma^2)t} + 2\alpha \frac{P_1}{\mu + \sigma^2} & \text{if } t > m.
\end{cases}
$$

(B.20)

The limit as $\mu$ approaches $r$ of $\frac{d\mathbb{E}[V_t^2]}{dt}$ exists and is

$$
\lim_{\mu \to r} \frac{d\mathbb{E}[V_t^2]}{dt} = \begin{cases} 
\frac{-(A-B)^2(4r^2+4r-2\sigma^2)-4A(A-B)^2}{2\mu(Br-Br\mu+2A(B-A))e^{rt}} + \frac{4r(A-B)^2}{2\mu(Br-Br\mu+2A(B-A))e^{rt}} & \text{if } 0 \leq t \leq m \\
\frac{2B^2}{r(2r+\sigma^2)^2} + 2e^{(2r+\sigma^2)t} + 2\alpha \frac{P_1}{\mu + \sigma^2} & \text{if } t > m.
\end{cases}
$$

(B.21)

The derivative of $\mathbb{E}[V_t^2]$ with respect to the mortgage rate, $r$, is

$$
\frac{d\mathbb{E}[V_t^2]}{dr} = \begin{cases} 
\frac{-2B^2(2r-\mu)(r-\mu+\sigma^2)e^{rt}}{2r(2r+\sigma^2)^2} + \frac{4r(A-B)^2}{2r(2r+\sigma^2)^2} & \text{if } 0 \leq t \leq m \\
2rB(A-B)(r-\mu) + 2B(A-B)e^{rt} & \text{if } t > m
\end{cases}
$$

(B.22)
where

\[
\frac{dD_1}{dr} = \frac{(A - B)^2(4r - 4\mu - \sigma^2)}{(r - \mu)^2(2r - 2\mu - \sigma^2)^2} + \frac{(A - B)(rB - \mu A)(2r - 2\mu - \sigma^2)}{\mu(r - \mu)^2(r - \mu - \sigma^2)^2} - \frac{B(A - B)}{\mu(r - \mu)(r - \mu - \sigma^2)^2} + \frac{B(2A - B)}{r^2}(r - \mu)(2r - 3\mu - \sigma^2)^2 + \frac{B^2}{\mu(r - \mu)(r + \sigma^2)^2} - 
\]

\[
\frac{B(rB - \mu A)}{\mu(r + \sigma^2)(r - \mu)^2} - \frac{(r - \mu)(2r - 2\mu - \sigma^2)^2}{\mu(r - \mu)(r - \mu - \sigma^2)^2} - \frac{1}{\mu(r - \mu)(r - \mu - \sigma^2)^2} \left[ \frac{(A - B)me^m}{e^m - 1} \right] (rB - \mu A) - 
\]

\[
\frac{2(2\mu - r)}{\mu(r - \mu)(2r - 2\mu - \sigma^2)^2} \left[ -mB(A - B)e^m \right] e^m + B \left( \frac{A}{r} - \frac{(A - B)me^m}{e^m - 1} \right) - 
\]

\[
\frac{1}{\mu(r - \mu)(2\mu + \sigma^2)^2} \left[ mB(rB - \mu A)e^m \right] e^m + B \left( \frac{rmBe^m}{e^m - 1} + \mu \left( \frac{A}{r} - \frac{mAe^m}{e^m - 1} \right) \right) + 
\]

\[
\frac{2mB^2e^m}{\mu(2\mu + \sigma^2)^2(e^m - 1)} 
\]

\[
\frac{dP_1}{dr} = \frac{e^{(r-\mu)m}}{\mu(r - \mu)^2} \left[ (A - B) \left( rm(r - \mu) - \mu - \frac{rm(r - \mu)e^m}{e^m - 1} \right) + A(r - \mu) \right] 
\]

\[
\frac{1}{\mu(r - \mu)^2} \left[ \mu(A - B) - (r - \mu) \left( \frac{rmBe^m}{e^m - 1} + \mu \left( \frac{A}{r} - \frac{Ame^m}{e^m - 1} \right) \right) \right] 
\]

\[
\frac{dP_2}{dr} = \left[ \frac{dE[V^2]}{dr} + \frac{Ae^m}{\mu + \sigma^2} \frac{dP_1}{dr} \right] e^{-(2\mu + \sigma^2)m} 
\]

The limit as \( \mu \) approaches to \( r \) of \( \frac{dE[V^2]}{dr} \) exists and is

\[
\lim_{\mu \to r} \frac{dE[V^2]}{dr} = \left\{ \begin{array}{ll}
\frac{-(A - B)[tr^4(2B + 3tr(A - B)) + 2\sigma^4(B + 3tr(A - B)) + 6\sigma^4(A - B)]}{r^2(2\sigma^2)^3} e^{2\sigma^2} + \\
\frac{B[A - B][tr^4(2\sigma^2)^3 + 2\sigma^4(2 - 2r - 2\sigma^2)^3]}{r^2(2\sigma^2)^3} e^{2\sigma^2} + \\
\frac{2\sigma^2}{r^2} \left[ 4(A - B)(tr\sigma^2 + r - \sigma^2) \left( \frac{A}{r} - \frac{(A - B)me^m}{e^m - 1} \right) + \\
2\sigma^2 \left( 2A - B \left( \frac{A}{r} - \frac{Ame^m}{e^m - 1} + \frac{ABme^m}{e^m - 1} \right) \right) \right] + \\
\frac{2\sigma^2}{e^m - 1} \left[ -1 - \frac{2mB(A - B)e^m}{e^m - 1} + \frac{ABme^m}{e^m - 1} \right] e^{2\sigma^2} + \\
\frac{Ae^m}{\mu + \sigma^2} \frac{dD_1}{dr} + 2e^{2r + \sigma^2 \mu} \lim_{\mu \to r} \frac{dD_1}{dr}, \quad \text{if} \quad 0 \leq t \leq m, \\
\frac{Ae^m}{\mu + \sigma^2} \frac{dP_1}{dr} + e^{2r + \sigma^2 \mu} \lim_{\mu \to r} \frac{dP_1}{dr}, \quad \text{if} \quad t > m. 
\end{array} \right.
\]
where

\[
\lim_{\mu \to r} \frac{dD_1}{dr} = \frac{(A - B)(\sigma^4 B + 3r(A - B))}{r\sigma^6} - \frac{B(A - B)\sigma^2}{r(r + \sigma^2)^3} + \frac{1}{r\sigma^4} \left[ 2(r - \sigma^2)(A - B) \left( \frac{A}{r} - \frac{(A - B)me^m}{e^{rm} - 1} \right) + \sigma^2 \left( 2(A - B) \left( \frac{A}{r} - \frac{Ame^m}{e^{rm} - 1} \right) + ABme^m \right) \right] - \frac{1}{r(r + \sigma^2)^2} \left[ - \frac{2mB(A - B)e^{rm}}{e^{rm} - 1} + \frac{AB}{r} \right] (r + 2\sigma^2) - (B.33)
\]

\[
\lim_{\mu \to r} \frac{dP_1}{dr} = \frac{m(A - B)}{2r} \left( rm + 2 - \frac{2rme^m}{e^{rm} - 1} \right) + A \frac{r^2}{2} \left( rm + 1 - \frac{rme^m}{e^{rm} - 1} \right), \quad (B.28)
\]

\[
\lim_{\mu \to r} \frac{dP_2}{dr} = \left[ \lim_{\mu \to r} \frac{d\mathbb{E}[V^2_m]}{dr} + Ae^m \left( \frac{1}{r} - \frac{m}{e^{rm} - 1} \right) \left( \frac{2e^{rm}}{r + \sigma^2} \lim_{\mu \to r} P_1 - \frac{4\alpha}{r(2r + \sigma^2)} \right) + \frac{2\alpha e^{rm}}{r + \sigma^2} \lim_{\mu \to r} \frac{dP_1}{dr} \right] e^{-(2r + \sigma^2)m} \quad (B.30)
\]

The derivative of \( \mathbb{E}[V^2_m] \) with respect to the line of credit rate, \( i \), is

\[
\frac{d\mathbb{E}[V^2_m]}{di} = \begin{cases} 
- \frac{4(A - B)B}{i(r - \mu)(2r - 2\mu - \sigma^2)} e^{2rt} + \frac{2B((r + \mu)A - 2rB)}{i\mu(r - \mu)(r - \mu - \sigma^2)} e^{(r + \mu)t} + \frac{2B(A - 2B)(2r - \mu)}{i\mu(r - \mu)(2r - 2\mu - \sigma^2)} e^{t} - \frac{2B(2rB - A\mu)}{i\mu(2r + \sigma^2)} e^{t}, & \text{if } 0 \leq t \leq m, \\
\frac{2B(1 - e^{-(r - \mu)m})}{i\mu(2r + \sigma^2)} e^{t} - 2\alpha e^{tm} \frac{dP_1}{i\mu(2r + \sigma^2)} + e^{2(r + \sigma^2)m} \frac{dP_2}{di}, & \text{if } t > m.
\end{cases} \quad (B.31)
\]

where

\[
\frac{dD_2}{di} = \frac{2(A - B)B}{i(r - \mu)(2r - 2\mu - \sigma^2)} - \frac{B((r + \mu)A - 2rB)}{i\mu(r - \mu)(r - \mu - \sigma^2)} - \frac{B(A - 2B)(2r - \mu)}{i\mu(r - \mu)(2r - 2\mu - \sigma^2)} + \frac{B(2rB - A\mu)}{i\mu(2r + \sigma^2)}, \quad (B.32)
\]

\[
\frac{dP_1}{di} = \frac{rB(1 - e^{-(r - \mu)m})}{i\mu(r - \mu)}, \quad (B.33)
\]

\[
\frac{dP_2}{di} = \left( \frac{d\mathbb{E}[V^2_m]}{di} - \frac{4AeB(1 - e^{rm})}{i\mu(2r + \sigma^2)} + \frac{2B(1 - e^{rm})P_1}{i\mu(2r + \sigma^2)} e^{um} + \frac{2\alpha e^{um} dP_1}{i\mu + \sigma^2} \right) e^{-(2r + \sigma^2)m}. \quad (B.34)
\]
The limit as $\mu$ approaches to $r$ of $\frac{d\mathbb{E}[V_t^2]}{dt}$ exists and is

$$
\lim_{\mu \to r} \frac{d\mathbb{E}[V_t^2]}{dt} = \begin{cases} 
\frac{4(A-B)(r\sigma^2 + r - \sigma^2)}{ir\sigma^4} + \frac{2(2r+3\sigma^2)}{ir(r+\sigma^2)^2} e^{2rt} + \frac{2(r+\sigma^2)(AB-4B^2-rB(A-2B))-2rB(A-2B)}{ir(r+\sigma^2)^2} e^{rt} + & \text{if } 0 \leq t \leq m, \\
\frac{4B(1-e^{\mu_m})}{ir(2r+\sigma^2)} \lim_{\mu \to r} \mathbb{E}[V_t^2] - \frac{2\alpha e^{\mu_t}}{r+\sigma^2} \lim_{\mu \to r} dP_1 + e^{(2r+\sigma^2)\mu} \lim_{\mu \to r} \frac{dP_2}{dt}, & \text{if } t > m.
\end{cases}
$$

(B.35)

where

$$
\lim_{\mu \to r} \frac{dD_1}{dt} = \frac{2B(A-B)(\sigma^2 - r) - AB\sigma^2}{ir\sigma^4} - \frac{(r + \sigma^2)(AB - 4B^2) - rB(A - 2B)}{ir(r + \sigma^2)^2}
$$

(B.36)

$$
\lim_{\mu \to r} \frac{dP_2}{dt} = \frac{2B}{ir(2r + \sigma^2)},
$$

(B.37)

$$
\lim_{\mu \to r} \frac{dP_3}{dt} = \left( \lim_{\mu \to r} \frac{d\mathbb{E}[V_t^2]}{dt} - \frac{4\alpha B(1-e^{\mu_m})}{ir(2r + \sigma^2)} + \frac{2B(1-e^{\mu_m})e^{\mu_m}}{i(r + \sigma^2)} \right) \lim_{\mu \to r} \frac{dP_1}{dt} e^{(2r + \sigma^2)\mu}.
$$

(B.38)

The derivative of $\mathbb{E}[V_t^2]$ with respect to the drift, $\mu$, is

$$
\frac{d\mathbb{E}[V_t^2]}{d\mu} = \begin{cases} 
\frac{2(A-B)^2(4r-4\mu-\sigma^2)}{\mu^2(2r-2\mu-\sigma^2)^2} e^{2rt} - \frac{2(A-B)(rB-\mu)(r^2 + 3\mu^2 + 7\mu + 3\mu\sigma^2 + 2r(2\mu - \sigma^2))}{\mu^2(r-\mu)(2\mu^2 - 2\mu\sigma^2)} e^{(r+\mu)\mu} + & \text{if } 0 \leq t \leq m, \\
\frac{2B(r-B-\mu)(2\mu^2 - 2\mu\sigma^2)}{\mu^2(\mu-\mu)(2\mu^2 - 2\mu\sigma^2)^2} e^{2\mu t} - \frac{2B(c(B-\mu)(2\mu^2 - 2\mu\sigma^2))}{\mu^2(r-\mu)(2\mu^2 - 2\mu\sigma^2)} e^{(r+\mu)\mu} - & \\
\frac{2B^2(4r+\sigma^2)}{\mu^2(2\mu^2 + \sigma^2)^2} e^{2\mu t} + 4tD_1 e^{2\mu t)} + \frac{2B(2\mu^2 + \sigma^2)}{\mu^2(2\mu^2 + \sigma^2)^2} e^{(r+\mu)\mu} \frac{dD_1}{d\mu}, & \text{if } t > m.
\end{cases}
$$

(B.39)
and

\[
\frac{dD_1}{d\mu} = \frac{-(A - B)^2(4r - 4\mu - \sigma^2)}{(r - \mu)^2(2r - 2\mu - \sigma^2)^2} + \frac{(A - B)(rB - \mu A)(r^2 + 3\mu^2 - 4r\mu + \sigma^2(2\mu - r))}{\mu^2(r - \mu)^2(r - \mu - \sigma^2)^2} + \frac{\mu^2(r - \mu)(2\mu - r)}{2B(A - B)} + \frac{\mu(r - \mu)(r - \mu - \sigma^2)}{\mu(r - \mu)(r - 2\mu - \sigma^2)^2} + \frac{B(A - B)(2\mu - r)(r^2 - 6r\mu + 6\mu^2 + \sigma^2(2\mu - r))}{\mu^2(r - \mu)^2(r - 2\mu - \sigma^2)^2} - \frac{B(rB - \mu A)(2r\mu - 3\mu^2 + \sigma^2(r - 2\mu))}{\mu^2(r - \mu)^2(\mu + \sigma^2)^2} - \frac{AB}{\mu(r - \mu)(\mu + \sigma^2) + B^2(4\mu + \sigma^2) - \mu^2(2\mu + \sigma^2)^2}. \tag{B.40}
\]

\[
\frac{dP_1}{d\mu} = \frac{r(A - B)(m\mu^2 - m\mu r - 2\mu)}{\mu^2(r - \mu)^2}e^{(r - \mu)m} - \frac{Br^2 - 2B\mu + \mu^2 A}{\mu^2(r - \mu)^2}, \tag{B.41}
\]

\[
\frac{dP_2}{d\mu} = -2mP_2 + \left( \frac{dE[V^2_m]}{d\mu} + \frac{2\alpha^2(4\mu + \sigma^2)}{\mu^2(2\mu + \sigma^2)^2} + \frac{2\alpha P_1(m\mu + m\sigma^2 - 1)}{(\mu + \sigma^2)^2}e^{\mu m} + \frac{2\alpha e^{\mu m}dP_1}{\mu + \sigma^2 d\mu} \right)e^{-(2\mu + \sigma^2)m}. \tag{B.42}
\]

The limit as \( \mu \) approaches to \( r \) of \( \frac{dE[V^2_m]}{d\mu} \) exists and is

\[
\lim_{\mu \to r} \frac{dE[V^2_m]}{d\mu} = \begin{cases} 
\frac{(A-B)e^{2\mu} \left[ \mu^3(2B - 2rA + \mu + rA - B) - \frac{2B^2(4\mu + \sigma^2)}{r^2(2\mu + \sigma^2)^2} \left( r + \sigma^2 \right)^2 \right] + \frac{2B^2(4\mu + \sigma^2)}{r^2(2\mu + \sigma^2)^2}}{r^2(2\mu + \sigma^2)^2} + \frac{2B^2(4\mu + \sigma^2)}{r^2(2\mu + \sigma^2)^2} + 4te^{(2r + \sigma^2)t} \lim_{\mu \to r} D_1 + 2e^{(2r + \sigma^2)t} \lim_{\mu \to r} \frac{dD_1}{d\mu}, & \text{if } 0 \leq t \leq m, \\
\frac{2\alpha(2r + \sigma^2)}{r^2(2\mu + \sigma^2)^2} - \frac{2\alpha e^{(2r + \sigma^2)t}}{(2\mu + \sigma^2)^2} \lim_{\mu \to r} P_1 - \frac{2\alpha e^{(2r + \sigma^2)t}}{r^2(2\mu + \sigma^2)^2} \lim_{\mu \to r} \frac{dP_1}{d\mu} + 2te^{(2\mu + \sigma^2)t} \lim_{\mu \to r} \frac{dP_2}{d\mu}, & \text{if } t > m. 
\end{cases} \tag{B.43}
\]
where
\[
\lim_{\mu \to r} \frac{dD_1}{d\mu} = -\frac{(A - B)}{r^2 \sigma^5} (B \sigma^4 + r B \sigma^2 + 3r^2 (A - B)) - \frac{1}{r^2(r + \sigma^2)^3} \left[ (r + \sigma^2)^2 (AB - 2B^2) - (r + \sigma^2) (8rAB - 11rB^2 + 2\sigma^2AB - 4\sigma^2B^2) + B(A - B)(7r^2 + 4r\sigma^2) \right] + \frac{B^2 (4r + \sigma^2)}{r^2(2r + \sigma^2)^2},
\]
(B.44)
\[
\lim_{\mu \to r} \frac{dP_1}{d\mu} = -\frac{rm(A - B)(2 + rm) + 2A}{2r^2},
\]
(B.45)
\[
\lim_{\mu \to r} \frac{dP_2}{d\mu} = -2m \lim_{\mu \to r} P_2 + \left( \lim_{\mu \to r} \frac{d\mathbb{E}[V^2_m]}{d\mu} \right) + \frac{2\alpha^2 (4r + \sigma^2)}{r^2(2r + \sigma^2)^2} + \frac{2\alpha e^\mu m}{(r + \sigma^2)^2} \lim_{\mu \to r} P_1 + \frac{2\alpha e^\mu m}{(r + \sigma^2)^2} \lim_{\mu \to r} \frac{dP_1}{d\mu} e^{-(2r + \sigma^2)m}.
\]
(B.46)
The derivative of \( \mathbb{E}[V^2_t] \) with respect to the volatility, \( \sigma \), is
\[
\frac{d\mathbb{E}[V^2_t]}{d\sigma} = \left\{ \begin{array}{ll}
\frac{4\sigma (A - B)^2}{(r - \mu)(2r - 2\mu - \sigma^2)^2} + \frac{4\sigma (A - B)(rB - \mu A)}{r(\mu - \mu)(r - \mu - \sigma^2)^2} + \frac{4\sigma B (A - B) (2\mu - r)}{r(\mu - \mu)(r - 2\mu - \sigma^2)^2} \frac{e^{rt}}{e^\mu} + \\
\frac{2\sigma B^2}{(r - \mu)(\mu + \sigma^2)^2} + \frac{4\sigma P_1}{(\mu + \sigma^2)^2} + \frac{2\sigma P_2}{(\mu + \sigma^2)^2} & \text{if } 0 \leq t \leq m \\\n\end{array} \right.
\]
(B.47)
and
\[
\frac{dD_1}{d\sigma} = -\frac{2\sigma (A - B)^2}{(r - \mu)(2r - 2\mu - \sigma^2)^2} - \frac{2\sigma^2 (A - B)(rB - \mu A)}{r(r - \mu)(r - \mu - \sigma^2)^2} - \frac{2\sigma B (A - B)(2\mu - r)}{r(r - \mu)(r - 2\mu - \sigma^2)^2}
\]
\[
- \frac{2\sigma B (rB - \mu A)}{r(\mu - \mu)(r - \mu - \sigma^2)^2} + \frac{2\sigma B^2}{(r - \mu)(\mu + \sigma^2)^2},
\]
(B.48)
\[
\frac{dP_2}{d\sigma} = -2\alpha m P_2 + \left( \frac{d\mathbb{E}[V^2_m]}{d\sigma} \right) + \frac{4\sigma e^\mu}{(\mu + \sigma^2)^2} \lim_{\mu \to r} P_1 + \frac{4\sigma e^\mu}{(\mu + \sigma^2)^2} \lim_{\mu \to r} \frac{dP_1}{d\sigma} e^{-(2r + \sigma^2)m}.
\]
(B.49)
The limit as \( \mu \) approaches to \( r \) of \( \frac{d\mathbb{E}[V^2_t]}{d\sigma} \) exists and is
\[
\lim_{\mu \to r} \frac{d\mathbb{E}[V^2_t]}{d\sigma} = \left\{ \begin{array}{ll}
\frac{4(A - B)(\sigma^2 + B + r(A - B) + 2r(A - B))}{r(\sigma^2 + B + r(A - B) + 2r(A - B))} e^{2rt} + \frac{4\sigma r B(r + \sigma^2)(2B - A + r(A - B) + 8r r B(A - B))}{r(2r + \sigma^2)^3} \frac{e^{rt}}{e^\mu} - \\
\frac{4\sigma^2 B^2}{r(2r + \sigma^2)^3} + \frac{4\sigma e^\mu}{(r + \sigma^2)^2} \lim_{\mu \to r} D_1 + 2e^{(2r + \sigma^2)y} \lim_{\mu \to r} \frac{dD_1}{d\sigma}, & \text{if } 0 \leq t \leq m \\\n\frac{4\sigma^2 B^2}{r(2r + \sigma^2)^3} + \frac{4\sigma e^\mu}{(r + \sigma^2)^2} \lim_{\mu \to r} P_1 + 2\sigma e^{(2r + \sigma^2)y} \lim_{\mu \to r} \frac{dP_1}{d\sigma}, & \text{if } t > m
\end{array} \right.
\]
(B.50)
where
\[
\lim_{\mu \to r} \frac{dD_1}{d\sigma} = -\frac{2B(A - B)\sigma^2 - 4r(A - B)^2}{r\sigma^5} - \frac{2\sigma B (r + \sigma^2)(2B - A) + 4\sigma r B(A - B)}{r(r + \sigma^2)^3} + \frac{2\sigma B^2}{r(2r + \sigma^2)^2},
\]
(B.51)
\[
\lim_{\mu \to r} \frac{dP_2}{d\sigma} = -2\sigma m P_2 + \left( \frac{d\mathbb{E}[V^2_m]}{d\sigma} \right) + \frac{4\sigma e^\mu}{(r + \sigma^2)^2} \lim_{\mu \to r} P_1 + \frac{4\sigma e^\mu m}{(r + \sigma^2)^2} \lim_{\mu \to r} \frac{dP_1}{d\sigma} e^{-(2r + \sigma^2)m}.
\]
(B.52)
Appendix C

A List of Figures.

C.1 A List of Figures

Plots of mean, $\mathbb{E}[V_t]$, and its derivatives at different values of parameter, $r$, $i$ and $\mu$.

Plots of standard deviation, $SD[V_t]$, and its derivatives different parameter values of $r$, $i$, $\mu$ and $\sigma$. 
Figure C.1: Plot of mean, $\mathbb{E}[V_t]$, and its derivatives with $i = 0.03$ and $\mu = 0.05$. Different colored curves represent different mortgage rates.

Figure C.2: Plot of mean, $\mathbb{E}[V_t]$, and its derivatives with $i = 0.05$ and $\mu = 0.05$. Different colored curves represent different mortgage rates.
Figure C.3: Plot of mean, $\mathbb{E}[V_t]$, and its derivatives with $i = 0.08$ and $\mu = 0.05$. Different colored curves represent different mortgage rates.

Figure C.4: Plot of mean, $\mathbb{E}[V_t]$, and its derivatives with $r = 0.03$ and $\mu = 0.05$. Different colored curves represent different line of credit rates.
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Figure C.5: Plot of mean, $\mathbb{E}[V_t]$, and its derivatives with $r = 0.05$ and $\mu = 0.05$. Different colored curves represent different line of credit rates.

Figure C.6: Plot of mean, $\mathbb{E}[V_t]$, and its derivatives with $r = 0.08$ and $\mu = 0.05$. Different colored curves represent different line of credit rates.
Figure C.7: Plot of mean, $\mathbb{E}[V_t]$, and its derivatives with $i = 0.03$ and $\mu = 0.10$. Different colored curves represent different mortgage rates.

Figure C.8: Plot of mean, $\mathbb{E}[V_t]$, and its derivatives with $i = 0.05$ and $\mu = 0.10$. Different colored curves represent different mortgage rates.
Figure C.9: Plot of mean, $\mathbb{E}[V_t]$, and its derivatives with $i = 0.08$ and $\mu = 0.10$. Different colored curves represent different mortgage rates.

Figure C.10: Plot of mean, $\mathbb{E}[V_t]$, and its derivatives with $r = 0.03$ and $\mu = 0.10$. Different colored curves represent different line of credit rates.
Figure C.11: Plot of mean, $\mathbb{E}[V_t]$, and its derivatives with $r = 0.05$ and $\mu = 0.10$. Different colored curves represent different line of credit rates.

Figure C.12: Plot of mean, $\mathbb{E}[V_t]$, and its derivatives with $r = 0.08$ and $\mu = 0.10$. Different colored curves represent different line of credit rates.
Figure C.13: Plot of mean, $E[V_t]$, and its derivatives with $i = 0.03$ and $\mu = 0.15$. Different colored curves represent different mortgage rates.

Figure C.14: Plot of mean, $E[V_t]$, and its derivatives with $i = 0.05$ and $\mu = 0.15$. Different colored curves represent different mortgage rates.
Figure C.15: Plot of mean, $\mathbb{E}[V_t]$, and its derivatives with $i = 0.08$ and $\mu = 0.15$. Different colored curves represent different mortgage rates.

Figure C.16: Plot of mean, $\mathbb{E}[V_t]$, and its derivatives with $r = 0.03$ and $\mu = 0.15$. Different colored curves represent different line of credit rates.
Figure C.17: Plot of mean, $\mathbb{E}[V_t]$, and its derivatives with $r = 0.05$ and $\mu = 0.15$. Different colored curves represent different line of credit rates.

Figure C.18: Plot of mean, $\mathbb{E}[V_t]$, and its derivatives with $r = 0.08$ and $\mu = 0.15$. Different colored curves represent different line of credit rates.
Figure C.19: Plot of standard deviation, $SD[V_t]$, and its derivatives with $i = 0.03, \mu = 0.05$ and $\sigma = 0.10$. Different colored curves represent different mortgage rates.

Figure C.20: Plot of standard deviation, $SD[V_t]$, and its derivatives with $i = 0.05, \mu = 0.05$ and $\sigma = 0.10$. Different colored curves represent different mortgage rates.
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Figure C.21: Plot of standard deviation, $SD[V_t]$, and its derivatives with $i = 0.08$, $\mu = 0.05$ and $\sigma = 0.10$. Different colored curves represent different mortgage rates.

Figure C.22: Plot of standard deviation, $SD[V_t]$, and its derivatives with $r = 0.03$, $\mu = 0.05$ and $\sigma = 0.10$. Different colored curves represent different line of credit rates.
Figure C.23: Plot of standard deviation, $SD[V_t]$, and its derivatives with $r = 0.05$, $\mu = 0.05$ and $\sigma = 0.10$. Different colored curves represent different line of credit rates.

Figure C.24: Plot of standard deviation, $SD[V_t]$, and its derivatives with $r = 0.08$, $\mu = 0.05$ and $\sigma = 0.10$. Different colored curves represent different mortgage rates.
Figure C.25: Plot of standard deviation, $SD[V_t]$, and its derivatives with $i = 0.03$, $\mu = 0.05$ and $\sigma = 0.20$. Different colored curves represent different mortgage rates.

Figure C.26: Plot of standard deviation, $SD[V_t]$, and its derivatives with $i = 0.05$, $\mu = 0.05$ and $\sigma = 0.20$. Different colored curves represent different mortgage rates.
Figure C.27: Plot of standard deviation, $SD[V_t]$, and its derivatives with $i = 0.08$, $\mu = 0.05$ and $\sigma = 0.20$. Different colored curves represent different mortgage rates.

Figure C.28: Plot of standard deviation, $SD[V_t]$, and its derivatives with $r = 0.03$, $\mu = 0.05$ and $\sigma = 0.20$. Different colored curves represent different line of credit rates.
Figure C.29: Plot of standard deviation, $SD[V_t]$, and its derivatives with $r = 0.05$, $\mu = 0.05$ and $\sigma = 0.20$. Different colored curves represent different line of credit rates.

Figure C.30: Plot of standard deviation, $SD[V_t]$, and its derivatives with $r = 0.08$, $\mu = 0.05$ and $\sigma = 0.20$. Different colored curves represent different mortgage rates.
Figure C.31: Plot of standard deviation, $SD[V_t]$, and its derivatives with $i = 0.03$, $\mu = 0.05$ and $\sigma = 0.30$. Different colored curves represent different mortgage rates.

Figure C.32: Plot of standard deviation, $SD[V_t]$, and its derivatives with $i = 0.05$, $\mu = 0.05$ and $\sigma = 0.30$. Different colored curves represent different mortgage rates.
Figure C.33: Plot of standard deviation, $SD[V_t]$, and its derivatives with $i = 0.08$, $\mu = 0.05$ and $\sigma = 0.30$. Different colored curves represent different mortgage rates.

Figure C.34: Plot of standard deviation, $SD[V_t]$, and its derivatives with $r = 0.03$, $\mu = 0.05$ and $\sigma = 0.30$. Different colored curves represent different line of credit rates.
Figure C.35: Plot of standard deviation, $SD[V_t]$, and its derivatives with $r = 0.05$, $\mu = 0.05$ and $\sigma = 0.30$. Different colored curves represent different line of credit rates.

Figure C.36: Plot of standard deviation, $SD[V_t]$, and its derivatives with $r = 0.08$, $\mu = 0.05$ and $\sigma = 0.30$. Different colored curves represent different mortgage rates.
Figure C.37: Plot of standard deviation, $SD[V_t]$, and its derivatives with $i = 0.03$, $\mu = 0.10$ and $\sigma = 0.10$. Different colored curves represent different mortgage rates.
Figure C.38: Plot of standard deviation, $SD[V_t]$, and its derivatives with $i = 0.05$, $\mu = 0.10$ and $\sigma = 0.10$. Different colored curves represent different mortgage rates.

Figure C.39: Plot of standard deviation, $SD[V_t]$, and its derivatives with $i = 0.08$, $\mu = 0.10$ and $\sigma = 0.10$. Different colored curves represent different mortgage rates.
Figure C.40: Plot of standard deviation, $S D[V_t]$, and its derivatives with $r = 0.03$, $\mu = 0.10$ and $\sigma = 0.10$. Different colored curves represent different line of credit rates.

Figure C.41: Plot of standard deviation, $S D[V_t]$, and its derivatives with $r = 0.05$, $\mu = 0.10$ and $\sigma = 0.10$. Different colored curves represent different line of credit rates.
Figure C.42: Plot of standard deviation, $SD[V_t]$, and its derivatives with $r = 0.08$, $\mu = 0.10$ and $\sigma = 0.10$. Different colored curves represent different mortgage rates.

Figure C.43: Plot of standard deviation, $SD[V_t]$, and its derivatives with $i = 0.03$, $\mu = 0.10$ and $\sigma = 0.20$. Different colored curves represent different mortgage rates.
Figure C.44: Plot of standard deviation, $SD[V_t]$, and its derivatives with $i = 0.05$, $\mu = 0.10$ and $\sigma = 0.20$. Different colored curves represent different mortgage rates.

Figure C.45: Plot of standard deviation, $SD[V_t]$, and its derivatives with $i = 0.08$, $\mu = 0.10$ and $\sigma = 0.20$. Different colored curves represent different mortgage rates.
Figure C.46: Plot of standard deviation, $SD[V_t]$, and its derivatives with $r = 0.03$, $\mu = 0.10$ and $\sigma = 0.20$. Different colored curves represent different line of credit rates.

Figure C.47: Plot of standard deviation, $SD[V_t]$, and its derivatives with $r = 0.05$, $\mu = 0.10$ and $\sigma = 0.20$. Different colored curves represent different line of credit rates.
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Figure C.48: Plot of standard deviation, $S D[V_t]$, and its derivatives with $r = 0.08$, $\mu = 0.10$ and $\sigma = 0.20$. Different colored curves represent different mortgage rates.

Figure C.49: Plot of standard deviation, $S D[V_t]$, and its derivatives with $i = 0.03$, $\mu = 0.10$ and $\sigma = 0.30$. Different colored curves represent different mortgage rates.
Figure C.50: Plot of standard deviation, $SD[V_t]$, and its derivatives with $i = 0.05$, $\mu = 0.10$ and $\sigma = 0.30$. Different colored curves represent different mortgage rates.

Figure C.51: Plot of standard deviation, $SD[V_t]$, and its derivatives with $i = 0.08$, $\mu = 0.10$ and $\sigma = 0.30$. Different colored curves represent different mortgage rates.
Figure C.52: Plot of standard deviation, $SD[V_t]$, and its derivatives with $r = 0.03$, $\mu = 0.10$ and $\sigma = 0.30$. Different colored curves represent different line of credit rates.

Figure C.53: Plot of standard deviation, $SD[V_t]$, and its derivatives with $r = 0.05$, $\mu = 0.10$ and $\sigma = 0.30$. Different colored curves represent different line of credit rates.
Figure C.54: Plot of standard deviation, $SD[V_t]$, and its derivatives with $r = 0.08$, $\mu = 0.10$ and $\sigma = 0.30$. Different colored curves represent different mortgage rates.

Figure C.55: Plot of standard deviation, $SD[V_t]$, and its derivatives with $i = 0.03$, $\mu = 0.15$ and $\sigma = 0.10$. Different colored curves represent different mortgage rates.
Figure C.56: Plot of standard deviation, $SD[V_t]$, and its derivatives with $i = 0.05$, $\mu = 0.15$ and $\sigma = 0.10$. Different colored curves represent different mortgage rates.

Figure C.57: Plot of standard deviation, $SD[V_t]$, and its derivatives with $i = 0.08$, $\mu = 0.15$ and $\sigma = 0.10$. Different colored curves represent different mortgage rates.
Figure C.58: Plot of standard deviation, $SD[V_t]$, and its derivatives with $r = 0.03$, $\mu = 0.15$ and $\sigma = 0.10$. Different colored curves represent different line of credit rates.

Figure C.59: Plot of standard deviation, $SD[V_t]$, and its derivatives with $r = 0.05$, $\mu = 0.15$ and $\sigma = 0.10$. Different colored curves represent different line of credit rates.
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Figure C.60: Plot of standard deviation, $SD[V_t]$, and its derivatives with $r = 0.08$, $\mu = 0.15$ and $\sigma = 0.10$. Different colored curves represent different mortgage rates.

Figure C.61: Plot of standard deviation, $SD[V_t]$, and its derivatives with $i = 0.03$, $\mu = 0.15$ and $\sigma = 0.20$. Different colored curves represent different mortgage rates.
Figure C.62: Plot of standard deviation, $SD[V_t]$, and its derivatives with $i = 0.05$, $\mu = 0.15$ and $\sigma = 0.20$. Different colored curves represent different mortgage rates.

Figure C.63: Plot of standard deviation, $SD[V_t]$, and its derivatives with $i = 0.08$, $\mu = 0.15$ and $\sigma = 0.20$. Different colored curves represent different mortgage rates.
Figure C.64: Plot of standard deviation, $SD[V_t]$, and its derivatives with $r = 0.03$, $\mu = 0.15$ and $\sigma = 0.20$. Different colored curves represent different line of credit rates.

Figure C.65: Plot of standard deviation, $SD[V_t]$, and its derivatives with $r = 0.05$, $\mu = 0.15$ and $\sigma = 0.20$. Different colored curves represent different line of credit rates.
Figure C.66: Plot of standard deviation, $SD[V_t]$, and its derivatives with $r = 0.08$, $\mu = 0.15$ and $\sigma = 0.20$. Different colored curves represent different mortgage rates.

Figure C.67: Plot of standard deviation, $SD[V_t]$, and its derivatives with $i = 0.03$, $\mu = 0.15$ and $\sigma = 0.30$. Different colored curves represent different mortgage rates.
Figure C.68: Plot of standard deviation, $SD[V_t]$, and its derivatives with $i = 0.05$, $\mu = 0.15$ and $\sigma = 0.30$. Different colored curves represent different mortgage rates.

Figure C.69: Plot of standard deviation, $SD[V_t]$, and its derivatives with $i = 0.08$, $\mu = 0.15$ and $\sigma = 0.30$. Different colored curves represent different mortgage rates.
Figure C.70: Plot of standard deviation, $SD[V_t]$, and its derivatives with $r = 0.03$, $\mu = 0.15$ and $\sigma = 0.30$. Different colored curves represent different line of credit rates.

Figure C.71: Plot of standard deviation, $SD[V_t]$, and its derivatives with $r = 0.05$, $\mu = 0.15$ and $\sigma = 0.30$. Different colored curves represent different line of credit rates.
Figure C.72: Plot of standard deviation, $SD[V_t]$, and its derivatives with $r = 0.08$, $\mu = 0.15$ and $\sigma = 0.30$. Different colored curves represent different mortgage rates.
Appendix D

Different Values of Correlation Coefficients

We consider difference values of correlation coefficients, $\rho_{1,2}$, $\rho_{1,3}$ and $\rho_{2,3}$. The other parameter values are fixed at $\kappa=0.3$, $\theta=0.06$, $\sigma_r = 0.02$, $r_0 = 0.05$, $\mu = 0.08$, $\sigma_S = 0.2$, $\phi = 0.02$, and $\sigma_H = 0.05$. Tax rate is 31.15% and the line of credit rate is $r_t + 0.25\%$.

Figure D.1 gives box and whisker plots of the simulated mortgage payoff time and total cost in the left and right panels, respectively, for $\rho_{1,2} = -1$, $\rho_{1,3} = 1$ and $\rho_{2,3} = -1$. Tables D.1 and D.2 provide the summary statistics of the mortgage payoff time and total cost distributions, respectively.

![Box and whisker plots](image)

Figure D.1: The left and right panels are box and whisker plots of the simulated mortgage payoff time and total cost, respectively, for Scenarios I-h to V-h with $\kappa=0.3$, $\theta=0.06$, $\sigma_r = 0.02$, $r_0 = 0.05$, $\mu = 0.08$, $\sigma_S = 0.2$, $\phi = 0.02$, $\sigma_H = 0.05$, $\rho_{1,2} = -1$, $\rho_{1,3} = 1$ and $\rho_{2,3} = -1$. Tax rate is 31.15% and the line of credit rate is $r_t + 0.25\%$.

Figure D.2 gives box and whisker plots of the simulated mortgage payoff time and total cost in the left and right panels, respectively, with $\rho_{1,2} = \rho_{1,3} = -1$ and $\rho_{2,3} = 1$. Tables D.3 and D.4 provide the summary statistics of the mortgage payoff time and total cost distributions,
Table D.1: Summary statistics of simulated mortgage payoff time distributions of all five scenarios with \( \kappa = 0.3, \theta = 0.06, r_0 = 0.05, \sigma_r = 0.02, \mu = 0.08, \sigma_S = 0.2, \phi = 0.01 \) and \( \sigma_H = 0.05, \rho_{1,2} = -1, \rho_{1,3} = 1 \) and \( \rho_{2,3} = -1 \). Tax rate is 31.15% and the line of credit rate is \( r_t + 0.25\% \).

<table>
<thead>
<tr>
<th>Scenario</th>
<th>Average</th>
<th>95% CI</th>
<th>std</th>
<th>\Pr(\tau &gt; 25)</th>
<th>\Pr(\tau &gt; 30)</th>
<th>\Pr(\tau &gt; \tau_m)</th>
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<tbody>
<tr>
<td>I-h</td>
<td>18.34</td>
<td>[17.87 18.81]</td>
<td>7.44</td>
<td>0.125</td>
<td>0.063</td>
<td></td>
</tr>
<tr>
<td>II-h</td>
<td>18.22</td>
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<tr>
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<td>9.77</td>
<td>0.129</td>
<td>0.072</td>
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Table D.2: Summary statistics of simulated total cost distributions of all five scenarios with \( \kappa = 0.3, \theta = 0.06, r_0 = 0.05, \sigma_r = 0.02, \mu = 0.08, \sigma_S = 0.2, \phi = 0.02, \sigma_H = 0.05, \rho_{1,2} = -1, \rho_{1,3} = 1 \) and \( \rho_{2,3} = -1 \). Tax rate is 31.15% and the line of credit rate is \( r_t + 0.25\% \).

<table>
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<th>Scenario</th>
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<th>95% CI</th>
<th>std</th>
<th>\Pr(I^r &gt; I^m)</th>
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<td>176890</td>
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Table D.3: Summary statistics of simulated mortgage payoff time distributions of all five scenarios with \( \kappa = 0.3, \theta = 0.06, r_0 = 0.05, \sigma_r = 0.02, \mu = 0.08, \sigma_S = 0.2, \phi = 0.01 \) and \( \sigma_H = 0.05, \rho_{1,2} = -1, \rho_{1,3} = -1 \) and \( \rho_{2,3} = 1 \). Tax rate is 31.15% and the line of credit rate is \( r_t + 0.25\% \).

<table>
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<th>Scenario</th>
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<th>\Pr(\tau &gt; 25)</th>
<th>\Pr(\tau &gt; 30)</th>
<th>\Pr(\tau &gt; \tau_m)</th>
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<tr>
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<td>0.113</td>
<td>0.218</td>
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</table>

Figure D.3 gives box and whisker plots of the simulated mortgage payoff time and total cost in the left and right panels, respectively, with \( \rho_{1,2} = 1 \) and \( \rho_{1,3} = \rho_{2,3} = -1 \). Tables D.5 and D.6 provide the summary statistics of the mortgage payoff time and total cost distributions, respectively.
Figure D.2: The left and right panels are box and whisker plots of the simulated payoff times and total costs, respectively, for Scenarios I-h to V-h with $\kappa=0.3$, $\theta=0.06$, $\sigma_r=0.02$, $r_0=0.05$, $\mu=0.08$, $\sigma_S=0.2$, $\phi=0.02$, $\sigma_H=0.05$, $\rho_{1,2}=-1$, $\rho_{1,3}=-1$ and $\rho_{2,3}=1$. Tax rate is 31.15% and the line of credit rate is $r_t + 0.25\%$.

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<th>std</th>
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Table D.4: Summary statistics of simulated total cost distributions of all five scenarios with $\kappa=0.3$, $\theta=0.06$, $r_0=0.05$, $\sigma_r=0.02$, $\mu=0.08$, $\sigma_S=0.2$, $\phi=0.02$, $\sigma_H=0.05$, $\rho_{1,2}=-1$, $\rho_{1,3}=-1$ and $\rho_{2,3}=1$. Tax rate is 31.15% and the line of credit rate is $r_t + 0.25\%$.

<table>
<thead>
<tr>
<th>Scenario</th>
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<th>std</th>
<th>$\text{Pr}(\tau &gt; 25)$</th>
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<td>0.125</td>
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<tr>
<td>II-h</td>
<td>18.22</td>
<td>[17.74 18.70]</td>
<td>7.63</td>
<td>0.127</td>
<td>0.065</td>
<td></td>
</tr>
<tr>
<td>III-h</td>
<td>17.58</td>
<td>[17.11 18.05]</td>
<td>7.44</td>
<td>0.108</td>
<td>0.056</td>
<td></td>
</tr>
<tr>
<td>IV-h</td>
<td>17.69</td>
<td>[17.23 18.15]</td>
<td>7.32</td>
<td>0.093</td>
<td>0.045</td>
<td>0.115</td>
</tr>
<tr>
<td>V-h</td>
<td>17.01</td>
<td>[16.58 17.44]</td>
<td>6.86</td>
<td>0.073</td>
<td>0.035</td>
<td>0.114</td>
</tr>
</tbody>
</table>

Table D.5: Summary statistics of simulated mortgage payoff time distributions of all five scenarios with $\kappa=0.3$, $\theta=0.06$, $r_0=0.05$, $\sigma_r=0.02$, $\mu=0.08$, $\sigma_S=0.2$, $\phi=0.01$ and $\sigma_H=0.05$, $\rho_{1,2}=1$, $\rho_{1,3}=-1$ and $\rho_{2,3}=-1$. Tax rate is 31.15% and the line of credit rate is $r_t + 0.25\%$. 
Figure D.3: The left and right panels are box and whisker plots of the simulated payoff times and total costs, respectively, for Scenarios I-h to V-h with $\kappa = 0.3$, $\theta = 0.06$, $\sigma_r = 0.02$, $r_0 = 0.05$, $\mu = 0.08$, $\sigma_S = 0.2$, $\phi = 0.02$, $\sigma_H = 0.05$, $\rho_{1,2} = 1$, $\rho_{1,3} = -1$ and $\rho_{2,3} = -1$. Tax rate is 31.15% and the line of credit rate is $r_t + 0.25\%$.

<table>
<thead>
<tr>
<th>Scenario</th>
<th>Average</th>
<th>95% CI</th>
<th>std</th>
<th>$\Pr(I_t^* &gt; I_m^*)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>I-h</td>
<td>259160</td>
<td>[251800 266520]</td>
<td>116380</td>
<td>0.161</td>
</tr>
<tr>
<td>II-h</td>
<td>244110</td>
<td>[236730 251480]</td>
<td>116590</td>
<td>0.160</td>
</tr>
<tr>
<td>III-h</td>
<td>233530</td>
<td>[225990 241060]</td>
<td>119090</td>
<td>0.170</td>
</tr>
<tr>
<td>IV-h</td>
<td>232670</td>
<td>[225330 240010]</td>
<td>116060</td>
<td>0.133</td>
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<tr>
<td>V-h</td>
<td>220870</td>
<td>[213820 227920]</td>
<td>115100</td>
<td>0.145</td>
</tr>
</tbody>
</table>

Table D.6: Summary statistics of simulated total cost distributions of all five scenarios with $\kappa = 0.3$, $\theta = 0.06$, $r_0 = 0.05$, $\sigma_r = 0.02$, $\mu = 0.08$, $\sigma_S = 0.2$, $\phi = 0.02$, $\sigma_H = 0.05$, $\rho_{1,2} = 1$, $\rho_{1,3} = -1$ and $\rho_{2,3} = -1$. Tax rate is 31.15% and the line of credit rate is $r_t + 0.25\%$. 
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