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Comparison of option pricing between ARMA-GARCH and GARCH-M models

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COMPARISON OF OPTION PRICING BETWEEN ARMA-GARCH AND GARCH-M MODELS
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by

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Abstract

Option pricing is a major area in financial modeling. Option pricing is sometimes based on normal GARCH models. Normal GARCH models fail to capture the skewness and the leptokurtosis in financial data. The variant GARCH-in-mean (GARCH-M) model is widely used in the option pricing literature. It adds a heteroskedasticity term to the mean equation, which is interpreted as a risk premium, and also incorporates a type of asymmetry.

Our goal is to compare option valuation between GARCH-M and ARMA-GARCH models with normal and non-normal, z-distributed innovations. The models are fitted to the historical return data, and risk neutral measures are based on the conditional Esscher transform and the extended Girsanov principle. We compare European Calls on the S&P 500 with the model predictions. The TGARCH is best for ARMA-GARCH/GARCH-M models. Neither normal nor z dominates the other, but overall z-TGARCH-M (z-innovations) seems to be best, ARMA-TGARCH is surprisingly good.

Keywords: option pricing, ARMA-GARCH, GARCH-in-Mean, z-distribution, Esscher transform, Extended Girsanov principle

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Chapter 1

Introduction

The theory of option pricing is an important topic in the financial literature. The seminal works of Black and Scholes and Merton were the starting point for European option pricing. Following the finding that these model prices systematically differ from market prices, the literature on option valuation has formulated a number of theoretical models designed to capture these empirical biases. Many empirical studies on asset price dynamics have demonstrated that characteristics such as time-varying volatility, volatility clustering, non-normality, and leverage effect etc. should be taken into account when modelling financial data. Therefore various models and techniques were developed in both discrete and continuous time to incorporate some or all of the above properties.

The significant contribution in the continuous-time financial literature includes the stochastic volatility models (Wiggings [Wig87], Hull and White [HW87], Scott [Sco87], Stein and Stein [SS91], Heston [Hes93]), jump-diffusion models (Bates [Bat96], Bakshi *et al.* [BCC97], and Scott [Sco97]) and models with jumps in both the asset price and volatility (Duffie *et al.* [DPS00] and Chernov *et al.* [CGGT03]). More recently, Carr *et al.*[CGMY03] investigated the option pricing performance of a time-changed Lévy process. Although the continuous time models hold advantages in constructing closed form solutions for European option prices, their Markovian structure is not consistent with the empirical findings. In practice they are difficult to implement and test.

The discrete time literature has been dominated by the class of autoregressive conditional heteroscedastic models (ARCH) introduced by Engle [Eng82] or its generalization (GARCH) as first defined by Borellslev [Bor86]. The main advantage of these models stands in the relative ease of estimation. Thus, in the last few years, much interest has given to GARCH option price. However, simple GARCH specifications can capture only some of the skewness and excess kurtosis found in financial data, and this has led to the developments of a large number of extensions, all trying to give a better description of the data.

The first important extension was to investigate return process for GARCH models with non-normal innovations. Excess kurtosis can be taken account for by the heavier-tailed distributions such as Students t (Bollerslev [Bor87]) or the GED distribution (Nelson [Nel91]), but these were unable to explain excess skewness. Asymmetry can be incorporated using leverage effects (Nelson [Nel91] and Glosten *et al.* [GJ93]), or by assuming skewed innovation densities such as normal inverse Gaussian distribution (Forsberg and Bollerslev [FB02]) and inverse Gaussian density (Christoffersen *et al.* [CHJ06]). Various any other parametric distributions are also implemented in a GARCH framework: Shifted Gamma (Siu *et al.* [ST04]), Generalized Error (Duan [Dua99]), $\alpha-$stable (Menn and Rachev [MR05]), Normal Inverse Gaussian (Stentoft [Ste06]), mixture of normals (Badescu *et al.* [BKL08]), Poisson-normal innovations (Duan *et al.* [DRS06]) and the z-distribution (Lanne and Saikkonen [LS05]).

A second important issue addressed in the GARCH option pricing literature is the impact of different volatility specifications. Replacing standard GARCH model with asymmetric ones such as exponential GARCH or threshold GARCH, is another extension. Hardle and Hafner [HH00] proposed a model relying on the Glosten *et al.* [GJ93] asymmetric volatility process, called the GJR model, while Christofferson and Jacobs [CJ04] argued that a simple leverage effect in the conditional variance process outperforms most of the extensions considered in the literature relative to option prices.

It is well known that in the GARCH setup markets are incomplete, so there exist contingent claims which cannot be replicated exactly by constructing a self-financing hedge. Therefore there is an infinite number of risk neutral measures under which one can price derivatives. Option pricing in GARCH models has been typically done using the local risk neutral valuation relationship (LRNVR) pioneered by Duan [Dua95], but his method depends very strongly on the Gaussian innovations' distribution. Since this method does not apply when relaxing the conditional normality assumption of the asset returns, researchers try to exploit other possible choices for the pricing kernels. Follmer and Schweizer [FS91] constructed minimal martingale measure (MMM), which is consistent with common criteria to find strategies that minimize cost process. Elliott and Madan [EM98] introduced an extended Girsanov principle (EGP) to construct a risk neutral measure which is supported by finding similar hedging strategies. A well-known tool in actuarial science is the Esscher transform. It was introduced in the option pricing literature by Gerber and Shiu [GS94]. Another martingale measure is a mean correcting martingale measure (MCMM) driven by geometric Lévy processes (Schoutens [Sch03]).

GARCH-type models used for option pricing are usually GARCH-M models; see for example Duan [Dua95] for normal innovation or Badescu [Bad07] for non-normal innovation. GARCH-M models include a risk premium as proposed by Engle, Lilien and Robins [ELR87]. The main motivation of this thesis is to test the necessity of the risk premium within the return process for option pricing. Thus, we compare the option pricing performance between the widely used GARCH-M and ARMA-GARCH models. If an ARMA-GARCH model predicts option prices as well as a GARCH-M model it raises questions about the interpretation of the risk premium parameter for the GARCH-M.

Another motivation for this comparison stands on the model's simplicity and understandability. The estimation theory of ARMA-GARCH models provided by QMLE and MLE method is consistent and asymptotically normal; see Francq and Zakoïan [FZ04]. However, the asymptotic normality of GARCH-M model has not yet been established. Given that, ARMA-GARCH model is more fully understood as compared with GARCH-M.

The goal of this thesis is to provide a general analysis of option valuation between GARCH-M and ARMA-GARCH models driven by normal and z-distributed innovations. The remainder of this paper is organized as follows. Chapter 2 introduces the necessary background for financial time series models. Chapter 3 is devoted to estimate the GARCH models by MLE method. z-distribution is introduced in estimation process. We specify the estimation procedures of various GARCH models with QMLE and MLE under assumption of z-distributed innovation. A numerical example of fitting S&P 500 Index is provide in the end of this chapter.

In the first part of Chapter 4, we introduce certain notation, definitions and preliminary results which are very useful in constructing martingale measures and describe two well-known martingale measures, the conditional Esscher transform and the extended Girsanov principle. In the second part, we apply these two risk neutral measures for GARCH models to derive their risk neutral dynamics.

In Chapter 5, we compute prices for European Call options using Monte Carlo simulation by the two martingale measures introduced before. Two sample sets of European Call option written on S&P 500 Index are used for test option pricing for GARCH-M and ARMA-GARCH models. Our numerical study shows that: 1) under both risk neutral measures, z-distributed TGARCH model outperform the normal GARCH model; 2) the pricing errors when using Esscher transform are smaller than EGP method; 3) TGARCH option pricing model based on the z-distribution outperform the normal TGARCH model for in-the-money and long maturity options, while the latter provides a better for short maturity and out-of-the-money options. 4) ARMA-GARCH models price option as nearly good, but slightly worse than GARCH-M.

The conclusion and related future research are presented in the Chapter 6.

Chapter 2

Financial Series and Models

In this chapter, we provide a brief introduction of the financial return process in Section 2.1. Section 2.2 presents the ARMA model that is used to model the conditional expectation of the return process. Section 2.3 is devoted to the GARCH models and its extensions of conditional volatility specification. The properties of the GARCH(1,1) model are deeply discussed in Section 2.3.1. All models used to fit financial data in this thesis are proposed in the end of the chapter.

2.1 Financial Time Series

The financial world is filled with uncertainty. Modeling financial series is a quite complex issue. The complexity stems not only from the variety of financial products in the market (i.e. stock, index, exchange rates, interest rate) but also from the existence of stylized facts. Most of these stylized facts (i.e. volatility clustering, fat-tailed distribution, and etc.) are put forward in a paper of Mandelbrot [Man63], which are common to a large amount of financial series. However, they are difficult to generate artificially by stochastic models.

To investigate the regularities and patterns, we use return instead of the asset price itself. In practical analysis, the return is conventionally defined as the logarithmic price changes, which is close to the relative price change.

Definition 2.1.1 *Denote a financial asset with price S_t at time t (t is an integer) and price S_{t-1} at $t-1$, the return is defined as:*

$$y_t = \ln \frac{S_t}{S_{t-1}}.$$

In contrast to the prices, return is scale-free, which facilitates comparisons between assets. Moreover, return series with more attractive statistical properties are easier than working with the price process directly. The properties are mainly concerned with financial series with daily changing.

Time series is regarded as a discrete stochastic process, e.g. $\{X_t, t \in Z\}$. With respect to the financial data used in this thesis, the continuously compounded return process, $\{y_t, t \in Z\}$, is a time series. Generally, this series can be decomposed into two elements:

$$
\begin{aligned}
y_t &= m_t + \epsilon_t \\
\epsilon_t &= \sigma_t \varepsilon_t,
\end{aligned}
$$

where m_t is a predictable process and ϵ_t is a nondeterministic process driven by a noise random variable ε_t. Here, $\{\varepsilon_t\}$ is iid with mean zero and unit variance. Consider the filtration associated with the model, $\mathcal{F}_t$ is a sequence of increasing σ-algebras of $\mathcal{F}$ representing all market information up to time t. Hence, m_t and σ_t^2 represent the conditional mean and variance of y_t:

$$
\begin{aligned}
m_t &= E[y_t|\mathcal{F}_{t-1}] \qquad (2.1)\\
\sigma_t^2 &= Var[y_t|\mathcal{F}_{t-1}]. \qquad (2.2)
\end{aligned}
$$

In the following sections, we will introduce several time series models which are widely used in financial time series analysis.

2.2 ARMA Model

In the statistical analysis of time series, the class of autoregressive-moving-average (ARMA) models is the most broadly utilized for the prediction of second-order stationary stochastic process. The ARMA model is a tool for understanding and analyzing the causal structure, or to obtain the predictions of the future values in this series. The model consists of two parts, one for autoregressive (AR) and the second for moving average (MA). The model is usually referred to as the ARMA(P, Q) process where P is the order of the autoregressive part and Q is the order of the moving average part.

Definition 2.2.1 ([FZ10]) *A second-order stationary process* $\{y_t\}$ *is called an ARMA(P, Q) process, if there exist real coefficients* $c, \phi_1, ..., \phi_P, \theta_1, ..., \theta_Q$*, where P and Q are integers, so*

$$
y_t - \sum_{i=1}^{P} \phi_i y_{t-i} = c + \epsilon_t + \sum_{j=1}^{Q} \theta_j \epsilon_{t-j}, \quad \forall t \in \mathbb{Z}, \qquad (2.3)
$$

where $\{\epsilon_t\}$ *is the white noise (0,* σ^2*).*

Denote B as the back-shift operator such as $B^k y_t = y_{t-k}$. Using B, rewrite (2.3) as $\phi(B)y_t = \theta(B)\epsilon_t$. The polynomials are described as

$$
\phi(B) = 1 - \phi_1 B - ... - \phi_p B^P \quad \textit{and} \quad \theta(B) = 1 + \theta_1 B + ... + \theta_Q B^Q .
$$

If $\phi(z) \equiv 1$ the process is a Moving Average (MA) process while if $\theta(z) \equiv 1$ it is an Autoregressive (AR) process. It is possible for us to obtain the transfer function from our operator notation. Let

$$
\psi(B) = \frac{\theta(B)}{\phi(B)} .
$$

Then,

$$
y_t = \psi(B)\epsilon_t , \quad \psi(B) = 1 + \psi_1 B + \psi_2 B^2 + ... ,
$$

where the coefficients ψ_k are obtained by the Taylor series expansion of $\frac{\theta(z)}{\phi(z)}$ about $z_0 = 0$. Similarly, denote

$$\pi(B) = \psi^{-1}(B) = \frac{\phi(B)}{\theta(B)} .$$

In this case,

$$\epsilon_t = \pi(B)y_t, \pi(B) = 1 + \pi_1 B + \pi_2 B^2 + ...$$

Proposition 2.2.2 *If an ARMA(P,Q) process $\{y_t\}$ can be written as $y_t = 1 + \sum_{i=1}^{\infty} \psi_i \varepsilon_{t-i}$ for all t, with $\sum_{i=1}^{\infty} |\psi_i| < \infty$, the process y_t is stationarity.*

Proposition 2.2.3 *If an ARMA(P,Q) process $\{y_t\}$ can be written as $\epsilon_t = 1 + \sum_{i=1}^{\infty} \pi_i y_{t-i}$ for all t, with $\sum_{i=1}^{\infty} |\pi_i| < \infty$, the process y_t is invertible.*

Proposition 2.2.4 *If an ARMA(P,Q) process defined by $\phi(B)y_t = \theta(B)\epsilon_t$ is stationarity, the roots of $\phi(B) = 0$ lie outside the unit circle.*

Proposition 2.2.5 *If an ARMA(P,Q) process defined by $\phi(B)y_t = \theta(B)\epsilon_t$ is invertible, the roots of $\theta(B) = 0$ lie outside the unit circle.*

For a special case ARMA(1,1) model, the conditions for stationary and invertibility is $|\theta_1| < 1$ and $|\phi_1| < 1$.

The advantage of the ARMA model is that it can successfully capture the movements of conditional mean. However, the assumption of constant variance indicates that the conditional variance is time-invariant and contains no past information. This measure of unconditional variance ignores the possible predictable pattens of volatility in exploring real financial market.

2.3 GARCH Model

Autoregressive Conditional Heteroskedasticity (ARCH) stochastic models were introduced by Engle [Eng82] in 1982. The ARCH model specifies the conditional variance as a linear function of past squared returns, which effectively explains the volatility clustering and heavy-tailed financial returns. Inspired by the idea of the ARCH model and the ARMA model, Bollerslev [Bor86] generalized GARCH model by adding the past conditional variance into the conditional variance term.

Definition 2.3.1 ([FZ10]) *The process $\{y_t\}$ called the GARCH (p, q) process is of the form:*

$$\begin{aligned} y_t &= c + \epsilon_t , \\ \epsilon_t &= \varepsilon_t \sigma_t , \qquad (2.4) \\ \sigma_t^2 &= \alpha_0 + \sum_{i=1}^{p} \alpha_i \epsilon_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2 , \qquad (2.5) \end{aligned}$$

where $\alpha_0 > 0$, $\alpha_i \geq 0$, $1 \leq i \leq p$, $\beta_j > 0$, $1 \leq j \leq q$ and c are constant. we also assume that $\{\varepsilon_t,\ t \in Z\}$ is a sequence of i.i.d. *(0,1) random variables.*

If $q = 0$, the above process can be reduced to an ARCH(p) process. Rewriting the equation (2.4) in terms of back-shift operator B, we can get

$$\sigma_t^2 = \alpha_0 + \alpha(B)\epsilon_t^2 + \beta(B)\sigma_t^2 \ , \tag{2.6}$$

where

$$\begin{aligned} \alpha(B) &= \alpha_1 B + \alpha_2 B^2 + ... + \alpha_p B^p \ , \\ \beta(B) &= \beta_1 B + \beta_2 B^2 + ... + \beta_p B^q \ . \end{aligned}$$

If the roots of the characteristic equation $1 - \beta_1 x - \beta_2 x^2 - ... - \beta_p x^q = 0$ $1 - (\alpha_1 + \beta_1)x - \ldots - (\alpha_m + \beta_m)x^m = 0$ lie outside the unit circle, the process $\{y_t\}$ is covariance stationary. Here, $m = \max(p, q)$, $\alpha_i = 0$ for $i > p$, and $\beta_j = 0$ for $j > q$. Then we can write (2.6) as

$$\begin{aligned} \sigma_t^2 &= \frac{\alpha_0}{1-\beta(1)} + \frac{\alpha(B)}{1-\beta(B)}\epsilon_t^2 \\ &= \alpha_0^* + \sum_{i=1}^{\infty} \delta_i \epsilon_{t-i}^2 \ , \end{aligned} \tag{2.7}$$

where $\alpha_0^* = \frac{\alpha_0}{1-\beta(1)}$ and δ_i are coefficients of B^i in the expansion of $\alpha(B)[1 - \beta(B)]^{-1}$. Note that the expression (2.7) tells us that the GARCH(p,q) process can be expressed as an ARCH process of infinite order with a fractional structure of the coefficients.

Define $\upsilon_t = \epsilon_t^2 - \sigma_t^2$. Through rearranging equation(2.4), we have

$$\epsilon_t^2 = \alpha_0 + \sum_{i=1}^{m} (\alpha_i + \beta_i)\epsilon_{t-i}^2 + \upsilon_t - \sum_{j=1}^{q} \beta_j \upsilon_{t-j} \ ,$$

where $m = \max(p, q)$, $\alpha_i = 0$ for $i > p$, and $\beta_j = 0$ for $j > q$. Thus a GARCH model can be represented in the form of an ARMA model in ϵ_t^2. Based on that, the GARCH model easily inherits many properties from the corresponding ARMA model. With this representation, many stylized facts : volatility clustering, fat tails and volatility mean reversion are successfully captured by the GARCH model.

2.3.1 Properties of the GARCH(1,1) Models

Although GARCH models with higher order than (1,1) allow for more complex autocorrelation structure, GARCH(1,1) is more commonly used because of its simplicity. In addition, empirical studies suggest that coefficients corresponding to higher lags is insignificant. Thus, the success of the simple GARCH(1,1) model to explain a variety of financial time series is doubtless. The univariate GARCH(1,1) model with additional assumption of normal innovation can be defined as

$$\begin{aligned} y_t &= m_t + \epsilon_t, \\ \epsilon_t &= \sigma_t \varepsilon_t \qquad \varepsilon_t \sim i.i.d.N(0, 1), \\ \sigma_t^2 &= \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2 \ , \end{aligned}$$

where y_t is nonnegative process if $\alpha_0, \alpha_1, \beta_1 > 0$.

Under the assumption of $N(0, 1)$, the conditional distribution of y_t is Gaussian. As notated in Eq. (2.1-2.2), the conditional mean is m_t and the conditional variance is σ_t^2. In the simplest case, m_t is assumed as a constant independent of time. Usually, m_t is a deterministic process given by the filtration $\mathcal{F}_{t-1}$, which can be defined by different models. This will be discussed in the next section. The conditional variance changes over time. The unconditional variance is constant and given by

$$\sigma^2 = Var(y_t) = \frac{\alpha_0}{1 - \alpha_1 - \beta_1} .$$

The necessary and sufficient requirement for existence of the unconditional variance is $\alpha_1 + \beta_1 < 1$. The volatility will settle down in the long run to its stationary value, which also suggests the mean-reversion characteristic of volatility.

All autocorrelations of squared returns in GARCH(1,1) model are positive with an exponential decay. If $\alpha_1 + \beta_1$ is close to one, the decay is slow. Thus, $\alpha_1 + \beta_1$ can be called as the "persistence" parameter of the GARCH(1,1) model. The closer the persistence parameter is to one, the longer time the periods of volatility clustering will last. In addition, the larger α_1 relative to β_1 will contributes to the higher immediate impact of lagged squared returns on volatility.

As referred in [FZ10], $\beta_1^2 + 2\alpha_1\beta_1 + 3\alpha_1^2 < 1$ is the necessary and sufficient condition for finite fourth moments. The kurtosis of y_t is given by

$$\kappa = 3 + \frac{6\alpha_1^2}{1 - \beta_1^2 - 2\alpha_1\beta_1 - 3\alpha_1^2} .$$

Since the second term on the right hand side is positive, the kurtosis is larger than three. Thus, the GARCH(1,1) model exhibits leptokurtosis compared with normal distribution. However, comparing with the sample kurtosis observed for most returns time series, the kurtosis implied by the GARCH model is typically smaller. So, several non-normal distributions are proposed. Bai *et al.* [BRT03] and Lanne and Saikkonen [LS03] found that the z-distribution can capture some stylized facts exhibited by financial data such as skewness and leptokurtosis. In the next Chapter, we will discuss the innovation based on z-distribution in detail.

2.3.2 Extensions of GARCH Model

In many cases, the basic GARCH model is reasonably good for analyzing financial time series and estimating conditional volatility. However, it is obvious that simple specification cannot capture all properties of the observed financial time series, which leads to lots of extensions. This section will introduces several extended GARCH models from two perspectives: the conditional mean and the conditional variance.

In the standard GARCH model, positive and negative shocks have the same effect on volatility since the model depends only on the squared previous shocks. It fails to capture the "leverage effect". Under the leverage effect, 'bad' news in financial market brings about more rapid volatility response than that corresponding 'good' news. Hence, Exponential GARCH or EGARCH [Nel91] and Threshold GARCH or the similar GJR-GARCH [GJ93] are proposed to capture this asymmetric responding mechanism.

In the EGARCH model, the innovation ϵ_t satisfies an equation of the form

$$\begin{aligned}\epsilon_t &= \sigma_t \varepsilon_t \,, \\ \varepsilon_t &\sim i.i.d.\,(0,1), \\ \sigma_t^2 &= e^{\alpha_0} \prod_{i=1}^{p} \exp\{\alpha_i g(\varepsilon_{t-i})\} \prod_{j=1}^{q} (\sigma_{t-j}^2)^{\beta_j} \,,\end{aligned}$$

where $g(\varepsilon_{t-i}) = \varpi_i \varepsilon_{t-i} + |\varepsilon_{t-i}|$, α_0, α_1, β_1 and ϖ are real numbers. When there is good news, the total effect of ε_{t-i} is $\alpha_i(1+\varpi_i)$. On the contrary, when there is bad news, the total effect of ε_{t-i} is $\alpha_i(\varpi_i - 1)$. The value of $\varpi_i - 1$ should be negative since a larger impact on volatility under bad news.

The GJR-GARCH model is a variant of Threshold GARCH. The conditional variance of the GJR-GARCH(p, q) process takes the following form

$$\begin{aligned}\epsilon_t &= \sigma_t \varepsilon_t \,, \\ \varepsilon_t &\sim i.i.d.\,(0,1)\,, \\ \sigma_t^2 &= \alpha_0 + \sum_{i=1}^{p} \epsilon_{t-i}^2 \left(\alpha_i + \gamma_i I\left(\varepsilon_{t-i} < 0\right)\right) + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2 \,.\end{aligned}$$

We remark that the effect of ϵ_{t-i}^2 on the conditional variance is α_i if the shock is non-negative, and $\alpha_i + \gamma_i$ if the shock is negative. We use GJR-GARCH(1,1) model in our thesis to fit the financial data, which is record as TGARCH(1,1) for convenience. Here, we assume $F_\varepsilon()$ as the cumulative distribution function (CDF) of the driving noise ε_t. The uncondition variance is given by

$$\sigma^2 = Var(y_t) = \frac{\alpha_0}{1 - (\alpha_1 + \gamma F_\varepsilon(0^-) + \beta_1)}.$$

If $\alpha_1 + \gamma F_\varepsilon(0^-) + \beta_1 < 1$, the volatility itself is mean reverting. Under the assumption of $\varepsilon_t \sim i.i.d.N(0,1)$, a necessary and sufficient condition for existence of a strictly stationary TGARCH(1,1) when $\alpha_1 + \frac{1}{2}\gamma + \beta_1 < 1$, $\alpha_1 + \gamma \geq 0$, $\alpha_1, \beta_1 \geq 0$ and $\alpha_0 > 0$.

Another important extension comes from the dynamic frame of the condition mean. The ARMA-GARCH model combines an ARMA model for modeling the dynamic conditional mean and a GARCH model for modeling the dynamic conditional volatility. The conditional mean of an ARMA(P, Q)-GARCH(p, q) is of the form

$$m_t = c + \sum_{i=1}^{P} \phi_i y_{t-i} + \sum_{j=1}^{Q} \theta_j \epsilon_{t-j} \,. \tag{2.8}$$

In finance, the return of a financial asset may depend on its volatility. For example, we might expect the higher conditional variability causes higher returns, which is because the market demands a higher risk premium for higher risk. To model such a phenomenon, GARCH-in-mean (GARCH-M) was introduced by Engle, Lilien and Robins [ELR87], which takes the conditional volatility as a part of the expected returns. The GARCH-M model extends the conditional mean as follows

$$m_t = c + \lambda f(\sigma_t) \,. \tag{2.9}$$

where λ is a constant and f can be any arbitrary function of volatility σ_t, i.e. $f(\sigma_t) = \sigma_t$, $f(\sigma_t) = \sigma_t^2$, or $f(\sigma_t) = \ln \sigma_t$. For the GARCH-M model used in this thesis, the $f()$ function is specified as

$$f(\sigma_t) = \sigma_t.$$

The formulation of the GARCH-M model in Eq (2.9) implies that there are serial correlations in the return series y_t. These serial correlations are introduced by those in the volatility process $\{\sigma_t^2\}$.

We give a general formulation to summarize the GARCH models we will use in the following parts.

$$\begin{cases} y_t = m_t + \epsilon_t, \\ \epsilon_t = \sigma_t \varepsilon_t\,, \qquad \varepsilon_t \sim i.i.d.(0,1), \\ \sigma_t^2 = \alpha_0 + \alpha_1 \sigma_{t-1}^2 \omega(\varepsilon_{t-1}) + \beta_1 \sigma_{t-1}^2. \end{cases} \tag{2.10}$$

ARMA-GARCH	$m_t = c + \phi_1 y_{t-1} + \theta_1 \epsilon_{t-1}$	$\omega(\varepsilon_{t-1}) = \varepsilon_{t-1}^2$
ARMA-TGARCH	$m_t = c + \phi_1 y_{t-1} + \theta_1 \epsilon_{t-1}$	$\omega(\varepsilon_{t-1}) = \varepsilon_t^2(1 + \frac{\gamma}{\alpha_1} I(\varepsilon_{t-1} < 0))$
GARCH-M	$m_t = c + \lambda\sigma_t$	$\omega(\varepsilon_{t-1}) = \varepsilon_{t-1}^2$
TGARCH-M	$m_t = c + \lambda\sigma_t$	$\omega(\varepsilon_{t-1}) = \varepsilon_t^2(1 + \frac{\gamma}{\alpha_1} I(\varepsilon_{t-1} < 0))$

The normal innovation distribution cannot completely capture the skewness and leptokurtosis of the financial time series. The innovation distributions are extended to other non-Gaussian forms, such as Gamma, generalized error (GED), z or general hyperbolic distributions .

Chapter 3

Estimation of the GARCH Models

In this Chapter, our aim is to fit the GARCH models we discussed in Chapter 2. The chapter is organized as follows. Section 3.1 is devoted to a brief introduction of maximum likelihood estimation (MLE) and its extension quasi-maximum likelihood (QML). Section 3.2 specifies the estimation procedures of ARMA-GARCH/TGARCH and GARCH/TGARCH-M models with QMLE and MLE method under the assumption of z-distributed innovations. Section 3.3 analyzes the estimation performance for each models.

3.1 Estimation Methods

In Chapter2 we have discussed the GARCH models that are widely used to simulate financial time series. After selecting a reasonable model, we need to estimate the parameters to fit the models. There are many statistical methods can be applied in the estimation process. The simplest estimation method is ordinary least squares (OLS). Although this estimation procedure has the advantage of numerical simplicity, OLS is not useful for estimating GARCH models because OLS is not very efficient for this model. Maximum likelihood estimation (MLE) and its extension, quasi-maximum likelihood (QML) method, which are more efficient and outperform the OLS, will be applied in our thesis to fit the GARCH models.

3.1.1 Maximum Likelihood Estimation

The method of maximum likelihood is well-known in statistics. Earlier literature on inference from ARCH/GARCH models is based on MLE with a conditional Gaussian assumption on the innovation. Considering heavy-tailed and asymmetric innovation distributions documented by plenty of empirical evidence, Student's t or generalized Gaussian likelihood has been introduced, see e.g. Engle and Bollerslev [EB86], Bollerslev [Bor87], Hsieh [Hsi89] and Nelson [Nel91].

We briefly introduce the principle for the method of maximum likelihood. Recall the return process y_t, $y_t = m_t + \sigma_t \varepsilon_t$ with the appropriate initial conditions. Here, the innovations $\varepsilon = (\varepsilon_1, \varepsilon_2, ..., \varepsilon_n)$ are supposed to be independent and identically distributed with an unknown probability density function $f()$. It is surmised that the function $f()$ belongs to a certain family of distributions $\{f(|\varphi), \varphi \in \Phi\}$ (where φ is a vector of parameters from Φ). The value φ_0 is unknown and is called to as the true value of the parameter. The object is to find an estimator which would be as close to the true value φ_0 as possible.

To use the method of MLE, we one first specifies the joint density function for all innovations. Due to *i.i.d.*, the joint density function is

$$f_n(\varepsilon_1, \varepsilon_2, ..., \varepsilon_n|\varphi) = f(\varepsilon_1|\varphi) \times f(\varepsilon_2|\varphi) \times \cdots \times f(\varepsilon_n|\varphi).$$

Looking this function from another perspective, we consider the observed values ε to be fixed parameters, while φ to be the function's independent variables. This function will be called the likelihood:

$$L_n(\varphi) = L(\varphi; \varepsilon_1, \varepsilon_2, ..., \varepsilon_n) = f_n(\varepsilon_1, \varepsilon_2, ..., \varepsilon_n|\varphi) = \prod_{t=1}^{n} f(\varepsilon_t|\varphi) \tag{3.1}$$

The method of maximum likelihood estimates φ_0 by finding a value of φ that maximizes $L_n(\varphi)$. A maximum-likelihood estimator (MLE) of φ is defined as any solution $\hat{\varphi}$ of

$$\hat{\varphi} = \arg\max_{\varphi \in \Phi} L_n(\varphi)$$

In practice it is often more convenient to work with the logarithm of the likelihood function. The above MLE is equivalent to find $\hat{\varphi}$:

$$\hat{\varphi} = \arg\max_{\varphi \in \Phi} \ln L_n(\varphi)$$

The maximum likelihood estimator is efficient, and it achieves Cramér-Rao lower bound when the sample size tends to infinity. However, this method may lead to inconsistent estimates if the distribution of the innovation is misspecified. Alternatively, the Gaussian MLE, regarded as a quasi-maximum likelihood estimator (QMLE) may be consistent and asymptotically normal see Elie and Jeantheau [EJ95], provided that the innovation has a finite fourth moment, even if it is far from Gaussian, see Hall and Yao [HY03].

3.1.2 Quasi-Maximum Likelihood

The method of the maximum likelihood is an important estimation tool. It gives an estimator for a given model. There is a related question about finding a good model, generally known as a goodness of fit. GARCH-type models seem to be an appropriate family of models for some real financial data. Some model diagnostics are needed. QMLE is commonly used for financial models. We include a discussion here for completeness. Some of the notation given here is used later.

Quasi-likelihood was introduced by Robert Wedderburn [Wed74] to describe a function which has similar properties to the log-likelihood function but not corresponding to any actual probability distribution.

Francq and Zakoian [FZ04] has proposed that the Quasi-maximum likelihood (QML) method is particularly relevant for GARCH model. They proved the strong consistency and asymptotic normality of the quasi-maximum likelihood estimator of the parameters of pure GARCH processes, and of autoregressive moving-average models with a noise sequence driven by a GARCH model. Here, we use a simple GARCH(1,1) model to illustrate the the method of quasi-likelihood.

Recalling the pure GARCH(1,1) process, the observations $\epsilon_1, \epsilon_2, ..., \epsilon_n$ follow the formation:

$$\begin{cases} \epsilon_t = \sigma_t \varepsilon_t \\ \sigma_t^2 = \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2. \quad \forall t \in \mathbf{Z}, \end{cases} \tag{3.2}$$

where $\{\varepsilon_t\}$ is a sequence of *i.i.d.* variables of variance one and mean zero ($\alpha_0 > 0$, $\alpha_1 \geq 0$, $\beta_1 \geq 0$ and $\alpha_1 + \beta_1 < 1$). The vector of the parameters

$$\psi = (\alpha_0,\ \alpha_1,\ \beta_1)'$$

belong to a parameter space of the form

$$\Psi \subset (0, +\infty) \times [0, +\infty)^2 \ .$$

The true value of the parameter is unknown, and is denoted by

$$\psi_0 = (\alpha_{00},\ \alpha_{01},\ \beta_{01})' \ .$$

To write the likelihood of the model a distribution must be specified for the *i.i.d.* variables ε_t. Initial conditions about σ_0 and ϵ_0 are needed. However, we do not make any assumption on the distribution for QML. Here, we work with the Gaussian quasi-likelihood function, which coincides with the likelihood when the ε_t are standard normally distributed. Following Franqc and Zakoïan, we use $\tilde{\sigma}_t^2$ to correspond to Eq. (3.2). These are now observable objects, so QML can be defined. Given initial values ϵ_0 and $\tilde{\sigma_0}$, the conditional Gaussian quasi-likelihood is given by

$$L_n(\varphi) = L_n(\psi;\ \epsilon_1, \epsilon_2, ..., \epsilon_n) = \prod_{t=1}^{n} \frac{1}{\sqrt{2\pi\tilde{\sigma_t}^2}} \exp\left(-\frac{\epsilon_t^2}{2\tilde{\sigma_t}^2}\right) \ ,$$

where the $\tilde{\sigma_t}^2$ are recursively defined by $\tilde{\sigma_t}^2 = \tilde{\sigma_t}^2(\psi) = \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \beta_1 \tilde{\sigma}_{t-1}^2$. For a given value of parameters, under the second-order stationarity assumption, the unconditional variance is a reasonable choice for the unknown initial values: $\epsilon_0^2 = \tilde{\sigma_0}^2 = \frac{\alpha_0}{1-\alpha_1-\beta_1}$. In practice, the choice of initial values is important.

A QMLE of ψ is defined as the solution $\hat{\psi}$ of

$$\hat{\psi}_n = \arg\max_{\psi \in \Psi} L_n(\psi).$$

Taking the logarithm, it is seen that maximizing the likelihood is equivalent to minimizing with respect to ψ. Thus, a QMLE is equivalent to a measurable solution of the equation

$$\hat{\psi}_n = \arg\min_{\psi \in \Psi} I_n(\psi) \ . \tag{3.3}$$

Here, $I_n(\psi)$ is defined as

$$I_n(\psi) = n^{-1} \sum_{t=1}^{n} \tilde{\ell}_t, \quad \tilde{\ell}_t = \tilde{\ell}_t(\psi) = \frac{\epsilon_t^2}{\tilde{\sigma_t}^2} + \log \tilde{\sigma_t}^2 \ . \tag{3.4}$$

The method of Gaussian quasi-likelihood gains in robustness while it lose in efficiency. Theoretically, the divergence of Gaussian likelihood from the true innovation density may considerably increase the variance of the estimates, which thereby fail to reach the Cramér-Rao lower bound by a wide margin, reflecting the cost of not knowing the true innovation distribution. The empirical reason of Gaussian QMLE's efficiency loss is that financial data generally have stylized facts. Thus, there is some attention on inference using non-Gaussian QMLE. However, in general a non-Gaussian QMLE does not yield consist estimation when the true error distribution deviates from the likelihood. Therefore, a non-Gaussian QMLE method which is robust against error misidentification, more efficient than Gaussian QMLE, require more works in choosing an appropriate innovation distribution.

Quasi-likelihood method is a possible choice to estimate data following GARCH process. In the Gaussian case, QMLE is the same as MLE. In the non-Gaussian cases, MLE is more efficient than QMLE. Therefore in this thesis we only use maximum likelihood estimation.

3.2 Estimating ARMA-GARCH and GARCH-M Models

In this thesis, two kinds of return processes: ARMA-GARCH and GARCH-M are used in option pricing. Considering the gain/loss asymmetry in financial time series, except for simple GARCH model, threshold GARCH (TGARCH) model is applied for conditional variance specification as well. Moreover, since the normal innovation distribution cannot completely capture the skewness and leptokurtosis of the financial time series, another innovation assumption on z-distribution is also involved in our model estimation process.

Thus, there are a total of eight GARCH models used for fitting the financial data, which are recorded as follows.

GARCH-M	the GARCH(1,1)-in-mean model with normal innovations.
TGARCH-M	the threshold GARCH(1,1)-in-mean model with normal innovations.
ARMA-GARCH	the ARMA(1,1)-GARCH(1,1)-in-mean model with normal innovations.
ARMA-TGARCH	the ARMA(1,1)-TGARCH(1,1) model with normal innovations.
z-GARCH-M	the GARCH(1,1)-in-mean model with z-distributed innovations
z-TGARCH-M	the GARCH(1,1)-in-mean model with z-distributed innovations
z-ARMA-GARCH	the ARMA(1,1)-GARCH(1,1) model with z-distributed innovations
z-ARMA-TGARCH	the ARMA(1,1)-GARCH(1,1) model with z-distributed innovations

Under the normal distribution, the estimation method by MLE is equivalent to Gaussian QMLE. Firstly, we denote the vector of parameters as $\varphi = (\vartheta', \psi')'$, where $\vartheta = (c, \phi_1, \theta_1)'$ is for ARMA-GARCH and $\vartheta = (c, \lambda)'$ is for GARCH-M. If the conditional variance follows TGARCH model, ψ is defined by $\psi = (\alpha_0, \alpha_1, \gamma, \beta_1)$. We can calculate the value $\tilde{\epsilon}_t(\vartheta)$ and $\tilde{\sigma_t}^2(\psi)$ for $t = 1, 2, ...n$ depending on the observations and model specification. The specified equations used in calculating $\tilde{\epsilon}_t(\vartheta)$ and $\tilde{\sigma_t}^2(\psi)$ are listed in Table 3.1.

Since the normal distribution cannot completely capture the skewness and leptokurtosis of the financial time series, the z-distribution is introduced for financial time series data by Lanne and Saikkonen [LS05]). They found that it can capture some of the stylized facts exhibited by financial data. According to Barndorff-Nielsen *et al.* [BNKS82], the density function of a

Table 3.1: $\tilde{\epsilon}_t(\vartheta)$ and $\tilde{\sigma_t}^2(\psi)$ in GARCH models

ARMA-GARCH	$\tilde{\epsilon}_t = \tilde{\epsilon}_t(\vartheta) = y_t - c - \phi_1 y_{t-1} - \theta_1 \tilde{\epsilon}_{t-1}$
	$\tilde{\sigma_t}^2 = \tilde{\sigma_t}^2(\psi) = \alpha_0 + \alpha_1 \tilde{\epsilon}_{t-1}^2 + \beta_1 \tilde{\sigma}_{t-1}^2$
ARMA-TGARCH	$\tilde{\epsilon}_t = \tilde{\epsilon}_t(\vartheta) = y_t - c - \phi_1 y_{t-1} - \theta_1 \tilde{\epsilon}_{t-1}$
	$\tilde{\sigma_t}^2 = \tilde{\sigma_t}^2(\psi) = \alpha_0 + \tilde{\epsilon}_{t-1}^2 (\alpha_1 + \gamma I(\tilde{\epsilon}_{t-1} < 0)) + \beta_1 \tilde{\sigma}_{t-1}^2$
GARCH-M	$\tilde{\epsilon}_t = \tilde{\epsilon}_t(\vartheta) = y_t - c - \lambda \tilde{\sigma}_t$
	$\tilde{\sigma_t}^2 = \tilde{\sigma_t}^2(\psi) = \alpha_0 + \alpha_1 \tilde{\epsilon}_{t-1}^2 + \beta_1 \tilde{\sigma}_{t-1}^2$
TGARCH-M	$\tilde{\epsilon}_t = \tilde{\epsilon}_t(\vartheta) = y_t - c - \lambda \tilde{\sigma}_t$
	$\tilde{\sigma_t}^2 = \tilde{\sigma_t}^2(\psi) = \alpha_0 + \tilde{\epsilon}_{t-1}^2 (\alpha_1 + \gamma I(\tilde{\epsilon}_{t-1} < 0)) + \beta_1 \tilde{\sigma}_{t-1}^2$

z-distributed random variable X, $X \sim \mathrm{z}(\alpha, \beta, \delta, \mu)$, is given by:

$$f(x, \alpha, \beta, \delta, \mu) = \frac{1}{\delta B(\alpha, \beta)} \cdot \frac{(\exp[(x-\mu)/\delta])^{\alpha}}{(1 + \exp[(x-\mu)/\delta])^{\alpha+\beta}}$$

where $x, \mu \in \mathbf{R}$, $\alpha, \beta, \delta > 0$, and $B(\alpha, \beta) = \Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha+\beta)$ is the beta function, where $\Gamma(\cdot)$ is gamma function. Here, μ and δ represent the location and scale parameters respectively. When $\alpha = \beta$, the distribution is symmetric, while $\alpha > \beta$ ($\alpha < \beta$) correspond to a skew density to the right (left). Various special cases can be obtained from the z-distribution.

As stated in Barndorff-Nielsen *et al.* [BNKS82] for any μ, $-\frac{\alpha}{\delta} < \mu < \frac{\beta}{\delta}$, the moment generating function, expected value, and variance of the driving noises are given by:

$$\begin{aligned} M_{\varepsilon_t}(u) &= \frac{B(\alpha + \delta u, \beta - \delta u)}{B(\alpha, \beta)} \cdot e^{\mu u} \\ E[\varepsilon_t] &= \mu + \delta \left(\frac{\partial \ln \Gamma(u)}{\partial u}|_{u=\alpha} - \frac{\partial \ln \Gamma(u)}{\partial u}|_{u=\beta} \right) \\ Var[\varepsilon_t] &= \delta^2 \left(\frac{\partial^2 \ln \Gamma(u)}{\partial u^2}|_{u=\alpha} + \frac{\partial^2 \ln \Gamma(u)}{\partial u^2}|_{u=\beta} \right) . \end{aligned}$$

Here, denote $\varpi(\alpha, \beta)$ and $\iota(\alpha, \beta)$ as:

$$\begin{aligned} \varpi(\alpha, \beta) &= \frac{\partial \ln \Gamma(u)}{\partial u}|_{u=\alpha} - \frac{\partial \ln \Gamma(u)}{\partial u}|_{u=\beta} \\ \iota(\alpha, \beta) &= \frac{\partial^2 \ln \Gamma(u)}{\partial u^2}|_{u=\alpha} + \frac{\partial^2 \ln \Gamma(u)}{\partial u^2}|_{u=\beta} . \end{aligned}$$

To ensure the innovation process ε_t has a z-distribution with mean zero and variance one, we can set $\tilde{\delta} = 1/\sqrt{\iota(\alpha, \beta)}$ and $\tilde{\mu} = -\varpi(\alpha, \beta)/\sqrt{\iota(\alpha, \beta)}$. We write $\varepsilon_t \sim \mathrm{z}(\alpha, \beta, \tilde{\delta}, \tilde{\mu})$. Note that the pair of inequalities $-\frac{\alpha}{\delta} < \mu < \frac{\beta}{\delta}$ automatically hold.

Depending on the parameters and the observations, we can calculate $\tilde{\varepsilon}_t = \tilde{\epsilon}_t / \tilde{\sigma}_t$ for $t = 1, ..., n$ from Table 3.1. Plugging them in density function of z-distribution, we can derive the likelihood function by Eq. (3.1) and estimate the parameters by maximizing $L_n(\varphi)$ or $\ln L_n(\varphi)$.

Both MLE and QMLE method can be used to estimate model parameters for the GARCH-M and ARMA-GARCH models. The estimation theory of ARMA-GARCH models provided by QMLE and MLE method has been proved to be consistent and asymptotically normal; see

Francq and Zakoïan [FZ04]. However the asymptotic normality of the estimators GARCH-M model has not yet been established. In other words, the estimators of ARMA-GARCH coefficients obtained from MLE/QMLE converges to the true value of the parameters in probability is known to be true.

$$\hat{\varphi}_n \xrightarrow[n\to\infty]{a.s.} \varphi_0$$

For GARCH-M parameters, while it is believed to be so, we can not make sure our estimators are asymptotically normal.

3.3 Model Fitting Analysis

For estimation purposes, we consider S&P500 daily index closing prices. The S&P 500, is a stock market index based on the market capitalizations of 500 leading companies traded publically on the U.S. stock market, as determined by Standard & Poor's. It is one of the most commonly followed equity indices and many consider it the best representation of the market as well as the U.S. economy.

Here, index data of S&P500 from January 02, 1988 to January 06, 2004, a total of 4040 observations from Bloomberg database, are used. We want to compare the results with Han's M.Sc project, so we use the same data. These fitted models are applied in option valuation in Section 5.3.2. Figure 3.1 plots the daily closing prices and returns, which used to fit the models. The initial condition employed in MLE is : $\varepsilon_0 = 0$, $y_0 = 0$ and $\sigma_0 = S(y_n)$ (standard deviation of historical observations).

The models' parameter estimates and their standard errors are reported in Table 3.2 and 3.3. To compare the goodness of fit between models, we employ three likelihood based criteria: the maximum log-likelihood (LLF) obtained using MLE, the Akaike information criterion (AIC), and the Bayesian information criterion (BIC). By careful analysis of the parameters and σ_0 in corresponding models, it is obvious that the parameters used in conditional variances (GARCH) specification are similar for both ARMA-GARCH and GARCH-M models. In addition, those criteria for goodness of fit also present this consistency between two kinds of models. Then, another rule we find is that LLF is larger for TGARCH models and the models with z-distributed innovations, and AIC and BIC are smaller for the same cases. Given that, we believe TGARCH and the assumption of z-distributed innovations will contribute to a better model specification.

Figure 3.2 compares the log-densities of observed standardized residuals with their theoretical distribution, standard normal distribution. In Figure 3.2a, the 'dash-dash' line presents the log-density of the standardized residuals of GARCH-M model and 'dash-dot' line presents the log-density of the standardized residuals of ARMA-GARCH model. These two lines appear to overlap, which shows GARCH-M and ARMA-GARCH model have the similar fitting performance. However, it is obvious to find that both models with the assumption on normal innovations fit poorly to their theoretical density (solid line). The standardized residuals show apparent heavy-tailed and left-skewness. The consistency of fitting performance between TGARCH-M and ARMA-TGARCH can also be found in Figure 3.2b. Similarly, standardized residuals generated from the TGARCH models are heavy-tailed and skewed to the left side. But

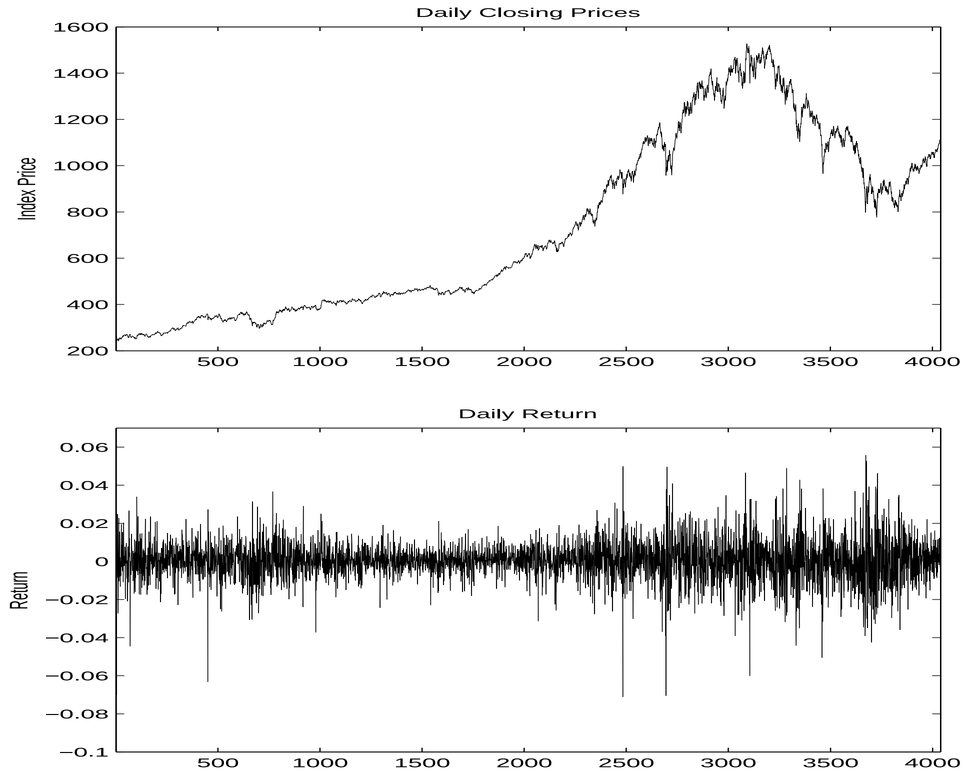


Figure 3.1: Daily closing prices and returns of S&P 500 Index from January 02, 1988 to January 06, 2004

we find that TGARCH models actually contribute some improvement in model fitting. Comparing with the density of the residual of the left tail in Figure 3.2a, that exhibited on Figure 3.2b is pretty thinner. Generally, we can make a conclusion similar to the previous literature, normal GARCH model fails to capture the skewness and leptokurtosis in financial data.

Figure 3.3 plots the log-densities of observed residuals vs. their theoretical distribution for different GARCH models. Similarly, there is a high degree of consistency of residuals' distribution between GARCH/TGARCH-M and ARMA-GARCH/TGARCH models. Compared with Figure 3.2, the log-density curves of standardized residuals estimated by z-GARCH models show a great improvement in fitting their theoretical distribution, even though they do not completely capture the behavior of their theoretical density in the tails. Given that, we conclude that GARCH models driven by z-distributed innovations perform better in fitting financial data. Based on that, we expect z-GARCH models have a much better performance in option pricing. The details will be discussed in Chapter 5.

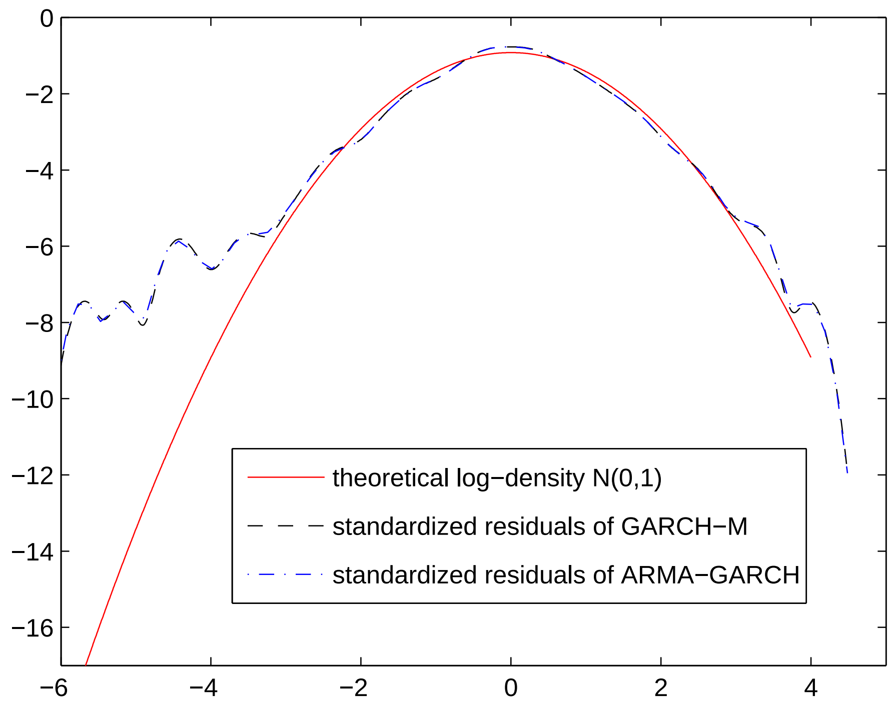
Alternatively, we could use appropriate QQ plots to compare the standardized residuals to the normal or z distributions. However, we would then need to construct a QQ plot for the z-distribution.

Table 3.2: GARCH and TGARCH parameters estimated by MLE with normal innovations using daily closing prices of S&P 500 from January 02,1988 to January 06, 2004

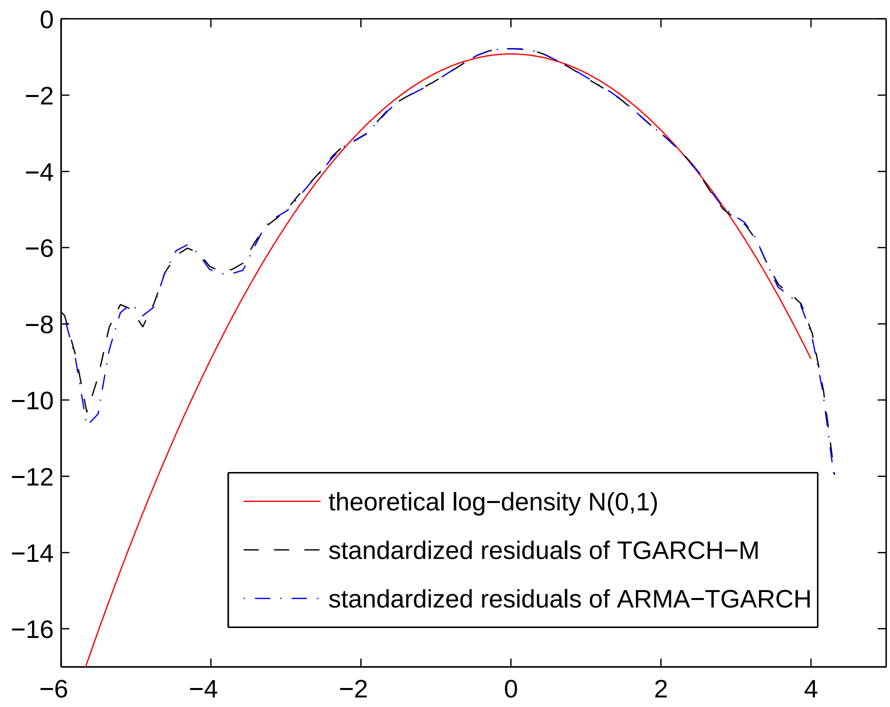
Parameters	GARCH-M	TGARCH-M	ARMA-GARCH	ARMA-TGARCH
c	1.1377×10^{-4}	1.1377×10^{-4}	9.87×10^{-4}	3.40×10^{-4}
	-	-	(2.63×10^{-4})	(1.36×10^{-4})
ϕ_1	-	-	-0.924	0.0131
	-	-	(0.117)	(8.8×10^{-3})
θ_1	-	-	0.916	7.21×10^{-3}
	-	-	(0.122)	(8.8×10^{-3})
λ	0.0609	0.0441	-	-
	(0.0013)	(0.0188)	-	-
α_0	4.87×10^{-7}	1.096×10^{-6}	4.52×10^{-7}	9.53×10^{-7}
	(1.5×10^{-7})	(2.6×10^{-7})	(8.8×10^{-8})	(1.21×10^{-7})
α_1	0.0413	0.0059	0.0402	5.15×10^{-3}
	(0.0058)	0.0043	(2.99×10^{-3})	(5.23×10^{-3})
β_1	0.954	0.9424	0.956	0.9461
	(0.0065)	0.0143	(3.3×10^{-3})	(4.4×10^{-3})
γ	-	0.0783	-	0.0768
	-	(0.015)	-	(7.6×10^{-3})
σ_0	0.0072	0.0061	0.0072	0.0062
LLF	1.3160×10^4	1.3192×10^4	1.3160×10^4	1.3192×10^4
AIC	-2.6312×10^4	-2.6373×10^4	-2.6308×10^4	-2.6370×10^4
BIC	-2.6286×10^4	-2.6342×10^4	-2.6270×10^4	-2.6325×10^4

Table 3.3: GARCH and TGARCH parameters estimated by MLE, which under the assumption on z-distributed noise, using daily closing prices of S&P 500 from January 02, 1988 to January 06, 2004
Note: in this table, the subscript in α_z or β_z denote the z-distribution case.

Parameters	GARCH-M	TGARCH-M	ARMA-GARCH	ARMA-TGARCH
c	1.1377×10^{-4}	1.1377×10^{-4}	9.34×10^{-5}	6.973×10^{-4}
	-	-	(3.3×10^{-5})	(2.42×10^{-4})
ϕ_1	-	-	0.8071	-0.9337
	-	-	(0.0568)	(0.0292)
θ_1	-	-	-0.8494	0.9259
	-	-	(0.0512)	(0.0319)
λ	0.0565	0.0419	-	-
	(0.0158)	(0.016)	-	-
α_0	3.04×10^{-7}	7.83×10^{-6}	2.63×10^{-7}	6.30×10^{-7}
	(1.3×10^{-7})	(2×10^{-7})	(1.2×10^{-7})	(1.88×10^{-7})
α_1	0.0405	0.0075	0.0389	0.0072
	(0.0065)	(0.006)	(0.0064)	(0.0056)
β_1	0.9568	0.9443	0.9589	0.9488
	(0.0069)	(0.009)	(0.0066)	(0.008)
γ	-	0.0787	-	0.0745
	-	(0.015)	-	(0.0138)
α_z	0.698	0.7654	0.6181	0.7625
	(0.096)	(0.112)	(0.0977)	(0.111)
β_z	0.7901	0.901	0.7273	0.9053
	(0.119)	(0.147)	(0.126)	(0.147)
σ_0	0.0071	0.0059	0.0071	0.0059
LLF	1.3299×10^4	1.3322×10^4	1.3306×10^4	1.3323×10^4
AIC	-2.6585×10^4	-2.6630×10^4	-2.6596×10^4	-2.6545×10^4
BIC	-2.6547×10^4	-2.6586×10^4	-2.6572×10^4	-2.6572×10^4

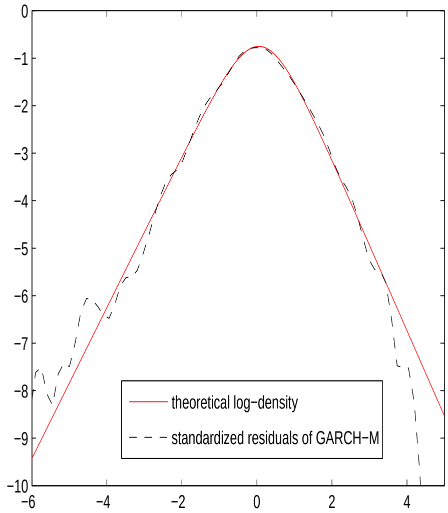


(a) GARCH models

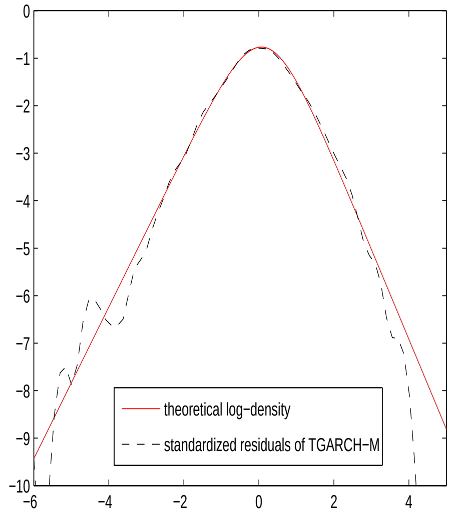


(b) TGARCH models

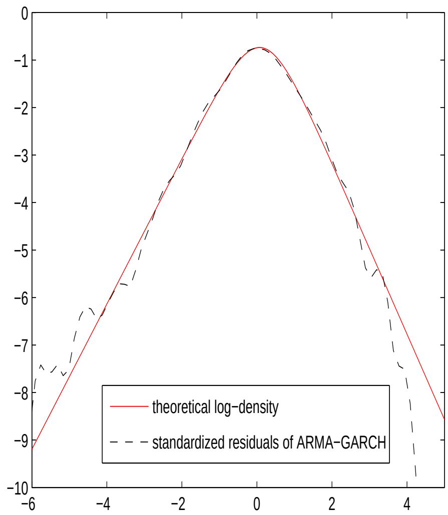
Figure 3.2: Log-density of observed residuals versus their theoretical density for Normal distribution



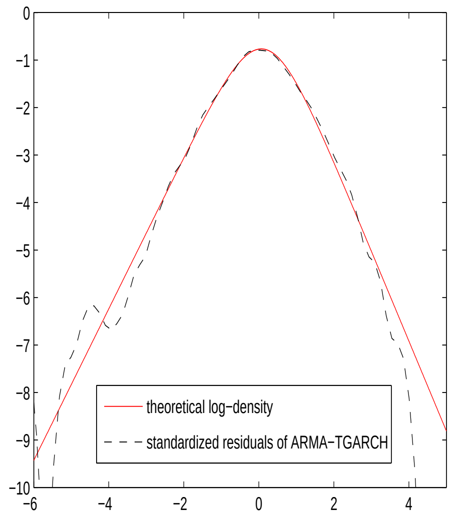
(a) GARCH-M



(b) TGARCH-M



(c) ARMA-GARCH



(d) ARMA-TGARCH

Figure 3.3: Log-density of observed residuals versus their theoretical density for z-distribution

Chapter 4

Risk Neutral Measures Under GARCH Models

All the definitions given in this Chapter are taken from Badescu [Bad07] who gives the appropriate references and attribution. These are given here in order to make this thesis self contained. When a definition or theorem does not have an explicit reference, it can be found in Badescu.

In this Chapter, we introduce two candidates of risk neutral measures which one can utilize for option pricing and normal and non-normal applications for discrete time GARCH models. In Section 4.1 we introduce certain notation, definitions and preliminary results which are very useful in constructing martingale measures. In Section 4.2 we describe two well known risk measures defined for discrete time markets, conditional Esscher transform and the extended Girsanov principle. The application of these two risk neutral measure for GARCH models to derive their risk neutralized dynamics will be introduced in Section 4.3.

4.1 Definitions and Notations

In discrete time financial models, trading dates are considered to form a discrete time index $\{t|t = 0, 1, ..., T\}$ where $T < \infty$ is the finite expiration time. The time index t refers to trading day and T is the expiration time in trading days. Corresponding to this time scale, the interest rate r is daily interest rate. Denote $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ as a complete filtered probability space, where P is the historical probability measure and $\mathcal{F}_t$ is a sequence of increasing σ-fields of $\mathcal{F}$ containing all information up to time t. We can assume $\mathcal{F}_0 = \{0, \Omega\}$ and $\mathcal{F}_T = \mathcal{F}$.

In order to price options written on a single stock, the financial market can be simplified as one consisting of only a reference asset or bond S_t^0 and a risky security S_t adapted to the filtration $\mathcal{F}$. The dynamics of the bond S_t^0 is $S_t^0 = e^{-\sum_{k=1}^{t} r_k}$, where r_t is a $\mathcal{F}_t$-predictable process is the interest rate over the period $[t-1, t]$. The discounted stock price is thus $\tilde{S}_t = e^{-\sum_{k=1}^{t} r_k} S_t$. In our numerical experiments, we consider a constant continuously compounded risk-free rate r for the whole period so that $S_t^0 = e^{-rt}$ and $\tilde{S}_t = e^{-rt} S_t$.

An arbitrage strategy aims at exploiting price differentials that exist as a result of market inefficiencies. To avoid arbitrage opportunities, we assume that the price process admits an equivalent martingale measure (EMM).

Definition 4.1.1 (EMM [Bad07]) *A probability measure Q is an equivalent martingale measure w.r.t. to P if the following relations have been satisfied:*

- $Q \approx P \quad (i.e. \quad \forall B \in \mathcal{F}, \quad Q(B) = 0 \Leftrightarrow P(B) = 0)$
- *the discounted price process* $\tilde{S}_t$ *is a martingale under Q w.r.t. to* $\mathcal{F}_t$, *that is* $E^Q[\tilde{S}_t|\mathcal{F}_{t-1}] = \tilde{S}_{t-1}$

Let $y_t = \ln \frac{S_t}{S_{t-1}}$ be the continuously compounded (log) return process, the above martingale condition of discounted stock price can be replaced by :

$$E^Q[e^{y_t}|\mathcal{F}_{t-1}] = e^r.$$

Let $\mathcal{M}^e(P) = \{Q|Q \text{ is an EMM w.r.t. } P\}$, where, $\mathcal{M}^e(P)$ is the set of all martingale measures equivalent to P.

Options are part of a larger class of financial derivatives. In our study we only focus on evaluating European options.

Example 4.1.2 *European Options*

- *Call option*
  An option which conveys the right to buy an asset at the expiration time T for a fixed predefined strike price X is called a call.The payoff function is :

$$h^{Call}(S_T) = \begin{cases} S_T - X & \text{if } S_T > X \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

- *Put option*
  An option which conveys the right to sell an asset at the expiration time T for a fixed predefined strike price X is called a put. The payoff function is :

$$h^{put}(S_T) = \begin{cases} X - S_T & \text{if } S_T < X \\ 0 & \text{otherwise.} \end{cases}$$

We denote Π_t^{Call} and Π_t^{Put} as the prices at time t of the Call and Put contracts.

The First Fundamental Theorem of Asset Pricing [FS04], it states that a market model is arbitrage-free if, and only if, there exists at least one risk neutral probability measure that is equivalent to the original probability measure P. The second Fundamental Theorem of Assert Pricing [FS04] states that the model is complete if and only if the risk neutral measure is unique. The Black-Scholes option pricing model which follows a Geometric Brownian Motion (GBM), admits a complete and the unique risk neutral measure given by the Girsanov theorem. With a few exceptions such as the binomial model, discrete time series models for the stock process are always incomplete market models. Thus it is not possible to build a self-financing portfolio to perfectly replicate any contingent claim and there exist an infinite number of risk neutral martingale measures as the following theorem.

Definition 4.1.3 (Badescu [Bad07]) *Suppose that the set of all equivalent martingale measures $M^e(P)$ is nonempty. Then the family of arbitrage-free prices at time t of a derivative security with payoff $h(S_T)$ is non-empty and is given by:*

$$\Pi_t^Q(h(S_T)) = \left\{E^Q\left[e^{-r(T-t)}h(S_T)|\mathcal{F}_t\right] \mid Q \in M^e(P),\ E^Q\left[h(S_T)|\mathcal{F}_t\right] < \infty\right\}$$

In the next section we will discuss several most important risk-neutral measures used in the discrete time framework. Here, we state a lemma first, which will be utilized in constructing risk-neutral measures throughout this and subsequent chapter.

Lemma 4.1.4 (Badescu [Bad07]) *Let P and Q be equivalent measures defined on on the measurable space $(\Omega, \mathcal{F})$. Then there exists an almost surely positive r.v. Z_t such that $E^P[Z_t|\mathcal{G}_t] = 1$ and $Q(A) = E^P[I_A Z_t|\mathcal{G}_t]$ for all $A \in \mathcal{G}_t$ ($\mathcal{G}_t$ is a finite sub-σ-algebra of $\mathcal{F}$). Here, Z_T is called the Radon-Nikodym derivative on the filtration $\mathcal{G}_T$. Then for any $0 \le t \le T$ we have: P1: the conditional Radon-Nikodym derivative of Q w.r.t. P on $\mathcal{G}_T$ is given by:*

$$Z_t := \frac{dQ}{dP}\Big|\mathcal{G}_t = E^P\Big[\frac{dQ}{dP}\Big|\mathcal{G}_t\Big].$$

P2: for any $\mathcal{G}_t(s \ge t)$ and Q-integrable measurable function g,we have:

$$E^Q[g|\mathcal{G}_t] = \frac{E^P[Z_s g|\mathcal{G}_t]}{Z_t}.$$

4.2 Risk Neutral Measures

During the recent decades, some of the widely used risk neutral measures are identified for general discrete time models. The stochastic discount factor (SDF) approach which is a common approach, whose relation with the risk neutral valuation relationship (RNVR) principle is justified by an equilibrium argument. The minimal martingale measure (MMM) constructed by Follmer and Schweizer [FS91] was also studied in the financial literature. Another two well known tools are the conditional Esscher transform, which was first applied to option pricing by Gerber and Shiu [GS94], and the extended Girsanov principle (EGP) introduced by Elliot and Madan [EM98]. Badescu [Bad07] investigated some relationships between the Esscher transform, SDF and MMM and utility maximization. He proposed that the Esscher transform and EGP are two good candidate risk neutral measures, and the Esscher transform performs best for option pricing. Hence, we restrict our attention to these two measures in our thesis.

4.2.1 Conditional Esscher Transform

The Esscher transform is a powerful tool used in actuarial science. Gerber and Shiu [GS94] show that the Esscher transform can be applied to price derivative securities if the log return process has stationary and independent increments (Lévy process). The increments do not have to have a normal distribution. The idea is to choose the Esscher parameter which makes the discounted return process of each underlying asset become a martingale under the Esscher transformed probability measure. The independent increments assumption is inappropriate for time series models. A conditional version of the Esscher transform was proposed by Buhlmann *et al.* [BDES96] for a more general discrete model. The conditional version is related to a utility maximization problem for some specific form of the utility function.

Suppose that the conditional moment generating function of the returns y_t w.r.t. $\mathcal{F}_{t-1}$ exists for all t, $0 \leq t \leq T$:

$$M^P_{y_t|\mathcal{F}_{t-1}}(c) = E^p[e^{cy_t}|\mathcal{F}_{t-1}] < \infty, \quad c \in D \subseteq \mathbf{R}.$$

Definition 4.2.1 (Conditional Esscher transform [GS94]) *Denote δ_t as a predictable process w.r.t. a σ-algebra $\mathcal{G}_t$. The probability measure $\hat{P}$ is called the conditional Esscher transformed measure of P if conditional moment generating functions exist:*

$$\frac{d\hat{P}}{dP}\bigg|_{\mathcal{G}_t} = \prod_{k=1}^{t} \frac{e^{\delta_k X_k}}{M_{X_k|\mathcal{G}_{k-1}}(\delta_k)}, \tag{4.2}$$

where, X_t represents the stochastic process and δ_t is denoted as the Esscher parameter with respect to the filtration $\mathcal{G}_t$.

Based on the Gerber and Shiu [GS94] formulation, we can characterize the Esscher risk-neutralized measure by the following theorem.

Theorem 4.2.2 (Badescu [Bad07]) *Let the process Z_t defined by:*

$$Z_t = \prod_{k=1}^{t} \frac{e^{\delta^*_k y_k}}{M^P_{y_k|\mathcal{F}_{k-1}}(\delta^*_k)},$$

where $Z_0 = 1$ and δ^*_k is a predictable process and δ^*_k is the unique solution of the equation:

$$M^P_{y_k|\mathcal{F}_{k-1}}(1 + \delta_k) = e^r M^P_{y_k|\mathcal{F}_{k-1}}(\delta_k), \tag{4.3}$$

for all $k \in \overline{1...T}$. Let the measure Q^{ess} be defined by:

$$\frac{\mathrm{d}Q^{ess}}{\mathrm{d}P} = Z_T,$$

Then Q^{ess} is called the conditional Esscher transform of P generated by the process y_t and the Esscher parameter δ_t.

The conditional Esscher transform price of a contingent claim with payoff function $h(S_T)$ is

$$\Pi_t^{Q^{ess}}(h(S_T)) = E^{Q^{ess}}\left[e^{-r(T-t)} h(S_T)|\mathcal{G}_t\right].$$

Badescu [Bad07] discusses various properties of the Esscher pricing method including relations to SDF and minimal entropy optimal risk neutral measures. In particular there is a pricing kernel that corresponds to the Esscher pricing.

4.2.2 Extended Girsanov Principle

The extended Girsanov Principle was introduced by Elliot and Madan [EM98] and gives another tool in choosing a probability measures within the infinite class of equivalent martingale measures under the discrete time framework.

The construction of the EGP transform is similar to the minimal martingale measure approach, which depends on the multiplicative Doob decomposition of the discounted stock price.

$$\widetilde{S}_t = \widetilde{S}_0 A_t M_t \ ,$$

where M_t is an $\mathcal{G}_t$ martingale and A_t is a predictable process with respect to $\mathcal{G}_t$. The process A_t is given by the unique representation:

$$A_t = \prod_{k=1}^{t} E^P\Big[\frac{\widetilde{S}_k}{\widetilde{S}_{k-1}}|\mathcal{G}_{k-1}\Big]$$

and where M_t is defined by:

$$M_t = \frac{\widetilde{S}_t}{\widetilde{S}_0 A_t} \ . \tag{4.4}$$

According to Eq. (4.4), we can easily show that M_t is a P-martingale:

$$E^P[M_t|\mathcal{G}_{t-1}] = E^P\Big[\frac{\widetilde{S}_t}{\widetilde{S}_{t-1}A_t}\Big|\mathcal{G}_{t-1}\Big] = \frac{\widetilde{S}_{t-1}E^P[\frac{\widetilde{S}_k}{\widetilde{S}_{k-1}}|\mathcal{G}_{k-1}]}{\tilde{S}_0 A_{t-1}} = M_{t-1} \ .$$

The dynamics of the discounted stock price process under P has the following representation

$$\widetilde{S}_t = \widetilde{S}_{t-1} e^{\nu_t} W_t \ . \tag{4.5}$$

where $W_t = M_t/M_{t-1}$ is a $\mathcal{G}_t$ martingale under P with the unit mean and ν_t represents the one period discounted excess returns meeting the following relation:

$$\nu_t = -r + \ln E^P[e^{y_t}|\mathcal{G}_{t-1}] \ .$$

Definition 4.2.3 (Extended Girsanov Principle [Bad07]) *A probability Q with respect to $\mathcal{G}$ is said to satisfy the Extended Girsanov Principle (EGP) if the conditional law of the discounted stock price under the new measure is equal to the conditional law where their martingale component from the multiplicative Doob decomposition prior to the change of measure:*

$$\mathcal{L}^Q\left(\frac{\widetilde{S}_t}{\widetilde{S}_{t-1}}\Big|\mathcal{G}_{t-1}\right) = \mathcal{L}^P(W_t|\mathcal{G}_{t-1}) \tag{4.6}$$

Thus, the discounted stock price $\widetilde{S}_t$ is a martingale under Q. The form of the Radon-Nikodym process of the risk neutral measure that satisfies the EGP in the following theorem.

Theorem 4.2.4 (Elliot and Madan [EM98]) *Let the process Z_T defined by:*

$$Z_T = \prod_{t=1}^{T} \frac{g_t^P\left(\frac{\widetilde{S}_t}{\widetilde{S}_{t-1}}\right) e^{v_t}}{g_t^P\left(e^{-v_t}\frac{\widetilde{S}_t}{\widetilde{S}_{t-1}}\right)} \tag{4.7}$$

where $g_t^P(w_t)$ is the conditional pdf of W_t given $\mathcal{G}_{t-1}$. Q^{egp} is given by:

$$\frac{dQ^{egp}}{dP} = Z_T \ . \tag{4.8}$$

Then we notice that Q^{egp} is the unique equivalent probability measure that satisfies Eq. (4.6).

One advantage of the EGP is that it does not require any distributional assumption about the returns. Thus this principle can be applied to investigate pricing and hedging for various types of discrete time models.

4.3 Applications for GARCH Models

For the family of discrete time GARCH models, most research investigating pricing performance assume the return process is conditionally normal distributed. Thus we discuss the the option pricing in GARCH models with normal innovation. However, a GARCH model with innovations which are normally distributed cannot capture the skewness and leptokurtosis of the financial data. Given that, we introduce another non-Gaussian distribution, z-distribution and derive the derive their risk neutralized dynamics under both risk neutral measures.

4.3.1 GARCH with Normal Innovation

We recall the following GARCH specification for the returns y_t we introduced in Section 2.3 with independent and identically distributed driving noise:

$$y_t = m_t + \epsilon_t \tag{4.9}$$

$$\epsilon_t = \sigma_t \varepsilon_t \ , \qquad \varepsilon_t \sim i.i.d.\ \mathbf{N}(0,1) \tag{4.10}$$

$$\sigma_t^2 = \alpha_0 + \alpha_1 \sigma_{t-1}^2 \omega(\varepsilon_{t-1}) + \beta_1 \sigma_{t-1}^2 \tag{4.11}$$

where the conditional mean return m_t is assumed to be a predictable process, which are set as Eq. (2.8) or (2.9) in this thesis. And, the function ω indicates the volatility specification. When $\omega(\varepsilon_{t-1}) = \varepsilon_{t-1}^2$, the return process follow the standard GARCH model. While $\omega(\varepsilon_{t-1}) = \varepsilon_{t-1}^2\left(1 + \frac{\lambda}{\alpha_1} I(\varepsilon_{t-1} < 0)\right)$, the return process corresponds to the TGARCH model. Denote the conditional generating function of y_t w.r.t. $\mathcal{F}_{t-1}$ under measure P as

$$M^P_{y_t|\mathcal{F}_{t-1}}(u) = E^P[e^{uy_t}|\mathcal{F}_{t-1}].$$

For normal GARCH models, a local risk neutral valuation relationship (LRNVR) developed by Duan [Dua95] is widely used. Under the risk neutral measure Q^{lrnvr} given in local risk neutral valuation relationship (LRNVR), the return dynamics are given by:

$$y_t = r - \frac{1}{2}\sigma_t^2 + \sigma_t\eta_t \tag{4.12}$$
$$\eta_t \sim i.i.d.\ N(0,1) \tag{4.13}$$
$$\sigma_t^2 = \alpha_0 + \alpha_1\sigma_{t-1}^2\omega\left(\frac{r - \sigma_{t-1}^2/2 - m_{t-1}}{\sigma_{t-1}} + \eta_{t-1}\right) + \beta_1\sigma_{t-1}^2\ . \tag{4.14}$$

The form of m_t only makes a difference in the variance specification.

The return dynamics under the Esscher transform and EGP for normal GARCH models are derived in Badescu [Bad07]. Comparing with the specific return dynamics under measure Q, we care more about the tractable form of the Radon-Nikodym derivative. Based on it, we are able to perform option pricing by simulating stock paths under the physical measure P.

Under the Esscher transform measure Q^{ess}, the Radon-Nikodym derivative is given by:

$$\frac{\mathrm{d}Q^{ess}}{\mathrm{d}P} = \prod_{t=1}^{T} \frac{e^{\delta_t^* y_t}}{M^P_{y_t|\mathcal{F}_{t-1}}(\delta_t^*)}\ . \tag{4.15}$$

Since $y_t \sim N(m_t,\ \sigma_t^2)$ under measure P, $M^P_{y_t|\mathcal{F}_{t-1}}(c) = \exp\left(cm_t + c^2\sigma_t^2/2\right)$. $M^{Q^{ess}}_{y_t}$ w.r.t. $\mathcal{F}_{t-1}$ is given by $E^{Q^{ess}}[y_t|\mathcal{F}_{t-1}] = \exp\left(c(m_t + \delta_t^*\sigma_t^2) + c^2\frac{\sigma_t^2}{2}\right)$. The Esscher parameter can be solved from Eq. (4.3). The solution is

$$\delta_t^* = \frac{1}{\sigma_t^2}\left(r_t - m_t - \frac{\sigma_t^2}{2}\right)\ .$$

Under the extended Girsanov principle measure, the Radon-Nikodym derivative is given by:

$$\frac{\mathrm{d}Q^{egp}}{\mathrm{d}P} = \prod_{t=1}^{t} \frac{f_t^P\left(y_t - r + \ln M_{y_t|\mathcal{F}_{t-1}}(1)\right)}{f_t^P(y_t)}\ . \tag{4.16}$$

Where, $\ln M_{y_k|\mathcal{F}_{t-1}}(1) = m_t + \sigma_t^2/2$ and $f_t^P()$ is the conditional pdf of y_t given $\mathcal{F}_{t-1}$, $f_t^P(u) = \frac{1}{\sqrt{2\pi}\sigma_t} e^{\frac{-(u-m_t)^2}{2\sigma_t^2}}$.

Badescu [Bad07] shows that in case of GARCH with normal innovation, Duan's LRNVR is consistent with the Esscher transform and the measure given by EGP methods as well. However Duan's method is specific to normal innovations and does not naturally extend to non-normal GARCH.

4.3.2 GARCH with *Z*-distributed Innovation

We recall the density function of z-distribution mentioned in Section 3.2, the innovation ε_t, $\varepsilon_t \sim \mathrm{z}(\alpha, \beta, \tilde{\delta}, \tilde{\mu})$, is given by:

$$f(\varepsilon_t, \alpha, \beta, \tilde{\delta}, \tilde{\mu}) = \frac{1}{\tilde{\delta}B(\alpha,\beta)} \cdot \frac{\left(\exp[(\varepsilon_t - \tilde{\mu})/\tilde{\delta}]\right)^{\alpha}}{\left(1 + \exp[(\varepsilon_t - \tilde{\mu})/\tilde{\delta}]\right)^{\alpha+\beta}}$$

where ε_t, $\tilde{\mu} \in \mathbf{R}$, α, β, $\tilde{\delta} > 0$, and $B(\alpha,\beta) = \Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha+\beta)$ is the beta function. Here, $\tilde{\delta} = 1/\sqrt{\iota(\alpha,\beta)}$, $\tilde{\mu} = -\varpi(\alpha,\beta)/\sqrt{\iota(\alpha,\beta)}$ to ensure ε_t is mean zero and variance one and $\Gamma(\cdot)$ represents gamma function.

The return process y_t, $y_t = m_t + \sigma_t\varepsilon_t$. Supposing $y_t = g(\varepsilon_t)$, $g : \varepsilon_t \mapsto g(\varepsilon_t)$ is 1 to 1, invertible and differentiable. $y_t = g(\varepsilon_t)$ has inverse $\varepsilon_t = h(y_t) = \frac{y_t - m_t}{\sigma_t}$, we also write $h = g^{-1}$.

$$\begin{aligned} f_y(y_t) &= f_\varepsilon(h(y_t))\left|\frac{\mathrm{d}h(y_t)}{\mathrm{d}y_t}\right| \\ &= \frac{1}{\tilde{\delta}\sigma_t B(\alpha,\beta)} \cdot \frac{\left(\exp[(y_t - m_t - \tilde{\mu}\sigma_t)/\tilde{\delta}\sigma_t]\right)^{\alpha}}{\left(1 + \exp[(y_t - m_t - \tilde{\mu}\sigma_t)/\tilde{\delta}\sigma_t]\right)^{\alpha+\beta}} \end{aligned}$$

Thus, the returns y_t are z-distributed as well:

$$y_t|_{\mathcal{F}_{t-1}} \sim \mathrm{z}(\alpha, \beta, \tilde{\delta}\sigma_t, m_t + \tilde{\mu}\sigma_t). \tag{4.17}$$

The following proposition describes the dynamics of y_t obtained using the conditional Esscher transform and the Extended Girsanov Principle approach, respectively.

Proposition 4.3.1 *Let $y := y_{t\{t\in T\}}$ denote the asset return process satisfy Eq. (2.10) under P, and assume the driving innovation $\varepsilon_t \sim \mathrm{z}(\alpha,\beta,\tilde{\delta},\tilde{\mu})$. Then, under the Esscher transform measure Q^{ess}, the return dynamics are given by:*

$$y_t = m_t + \sigma_t(\tilde{\mu} + \tilde{\delta}\varpi(\alpha_t^*,\beta_t^*)) + \sigma_t\tilde{\delta}\sqrt{\iota(\alpha_t^*,\beta_t^*)}\eta_t , \tag{4.18}$$

$$\eta_t|_{\mathcal{F}_{t-1}} \sim \mathrm{z}(\alpha_t^*,\beta_t^*, \frac{1}{\sqrt{\iota(\alpha_t^*,\beta_t^*)}}, -\frac{\varpi(\alpha_t^*,\beta_t^*)}{\sqrt{\iota(\alpha_t^*,\beta_t^*)}}) , \tag{4.19}$$

where α_t^* and β_t^* have the following representations:

$$\alpha_t^* = \alpha + \sigma_t\tilde{\delta}\theta_t^* , \tag{4.20}$$

$$\beta_t^* = \beta - \sigma_t\tilde{\delta}\theta_t^* , \tag{4.21}$$

and θ_t^* is the unique solution of the equation:

$$\ln\frac{B(\alpha + \sigma_t\tilde{\delta}(1+\theta_t), \beta - \sigma_t\tilde{\delta}(1+\theta_t))}{B(\alpha + \sigma_t\tilde{\delta}\theta_t, \beta - \sigma_t\tilde{\delta}\theta_t)} = r - m_t - \tilde{\mu}\sigma_t . \tag{4.22}$$

Proof. Using the density function defined under P, we calculate the MGF of the return process y_t under P:

$$M_{y_t}^P(u) = \frac{B(\alpha + \sigma_t\tilde{\delta}u, \beta - \sigma_t\tilde{\delta}u)}{B(\alpha,\beta)} \cdot e^{(m_t + \tilde{\mu}\sigma_t)u} \tag{4.23}$$

Thus, the conditional MGF of y_t under Q^{ess} satisfies

$$\begin{aligned}
\ln\left(M_{y_t}^{Q^{ess}}(c)\right) &= \ln\left(M_{y_t}^{P}(c+\theta_t^*)\right) - \left(M_{y_t}^{P}(\theta_t^*)\right) \\
&= \ln\left(\frac{B\left(\alpha+\sigma_t\tilde{\delta}(c+\theta_t^*),\beta-\sigma_t\tilde{\delta}(c+\theta_t^*)\right)}{B(\alpha,\beta)}\right) \\
&- \ln\left(\frac{B(\alpha+\sigma_t\tilde{\delta}\theta_t^*,\beta-\sigma_t\tilde{\delta}\theta_t^*)}{B(\alpha,\beta)}\right) + (m_t+\tilde{\mu}\sigma_t)(c+\theta_t^*) - (m_t+\tilde{\mu}\sigma_t)\theta_t^* \\
&= \ln\frac{B\left(\alpha+\sigma_t\tilde{\delta}(c+\theta_t^*),\beta-\sigma_t\tilde{\delta}(c+\theta_t^*)\right)}{B(\alpha+\sigma_t\tilde{\delta}\theta_t^*,\beta-\sigma_t\tilde{\delta}\theta_t^*)} + (m_t+\tilde{\mu}\sigma_t)c\ .
\end{aligned}$$

Corresponding to the Eq. (4.23), y_t, under measure Q^{ess}, is verified as another z-distributed random variable.

$$y_t|_{\mathcal{F}_{t-1}} \sim \mathrm{z}(\alpha^*,\ \beta^*,\ \tilde{\delta}\sigma_t,\ m_t+\tilde{\mu}\sigma_t).$$

where

$$\begin{aligned}
\alpha_t^* &= \alpha+\sigma_t\tilde{\delta}\theta_t^* \\
\beta_t^* &= \beta-\sigma_t\tilde{\delta}\theta_t^*\ ,
\end{aligned}$$

Here, θ_t^* is the Esscher parameter, which is solved from the martingale equation $M^P_{y_t|\mathcal{F}_{t-1}}(1+\theta_t) = e^r M^P_{y_t|\mathcal{F}_{t-1}}(\theta_t)$. Expanding the moment generating functions, we have:

$$\ln\frac{B\left(\alpha+\sigma_t\tilde{\delta}(1+\theta_t),\beta-\sigma_t\tilde{\delta}(1+\theta_t)\right)}{B(\alpha+\sigma_t\tilde{\delta}\theta_t,\beta-\sigma_t\tilde{\delta}\theta_t)} = r - m_t - \tilde{\mu}\sigma_t.$$

The conditional mean and variance are rewritten as:

$$\begin{aligned}
E[y_t|\mathcal{F}_{t-1}] &= m_t + \sigma_t\left(\tilde{\mu}+\tilde{\delta}\varpi(\alpha_t^*,\ \beta_t^*)\right) \\
Var[y_t|\mathcal{F}_{t-1}] &= \tilde{\delta}^2\sigma_t^2\iota(\alpha_t^*,\ \beta_t^*)\ .
\end{aligned}$$

Thus, the return process under Q^{ess} have the form listed on Eq. (4.18).

Proposition 4.3.2 *Let $y := y_{t\{t\in T\}}$ denote the asset return process satisfy Eq. (2.10) under P, and assume the driving innovation $\varepsilon_t \sim \mathrm{z}(\alpha,\beta,\tilde{\delta},\tilde{\mu})$. Then, under the EGP measure Q^{egp}, the return dynamics are given by:*

$$y_t = r - \ln\frac{B(\alpha+\sigma_t\tilde{\delta},\ \beta-\sigma_t\tilde{\delta})}{B(\alpha,\beta)} + \sigma_t\tilde{\delta}\varpi(\alpha,\beta) + \sigma_t\eta_t \tag{4.24}$$

$$\eta_t|_{\mathcal{F}_{t-1}} \sim \mathrm{z}(\alpha,\beta,\tilde{\delta},\tilde{\mu}), \tag{4.25}$$

Proof. The risk neutral measure given by the extended Girsanov Principle is

$$\begin{aligned}
Z_t = \frac{\mathrm{d}Q^{egp}}{\mathrm{d}P} &= \prod_{k=1}^{t}\frac{f_{zk}^P\left(y_k - r + \ln M_{y_k|\mathcal{F}_{k-1}}(1)\right)}{f_{zk}^P(y_k)} \\
&= \frac{f_{zt}^P\left(y_t - r + \ln M_{y_t|\mathcal{F}_{t-1}}(1)\right)}{f_{zt}^P(y_t)}\cdot Z_{t-1}\ .
\end{aligned}$$

Here, $f_{zt}^{P}(y_t)$ is the conditional pdf of y_t given by $\mathcal{F}_{t-1}$. Setting $A_t = -r + \ln M_{y_t|\mathcal{F}_{t-1}}(1)$, we compute the conditional moment generating function of y_t w.r.t. $\mathcal{F}_{t-1}$ under Q^{egp}.

$$\begin{aligned}
M_{y_t|\mathcal{F}_{t-1}}^{Q^{egp}}(c) &= E^{Q^{egp}}[\exp(cy_t)|\mathcal{F}_{t-1}] \\
&= E^{P}[\exp(cy_t)\frac{\mathrm{d}Q^{egp}}{\mathrm{d}P}|\mathcal{F}_{t-1}] \\
&= E^{P}[\exp(cy_t)\frac{f_{zt}^{P}(y_t+A_t)}{f_{zt}^{P}(y_t)}\cdot Z_{t-1}|\mathcal{F}_{t-1}] \\
&= E^{P}[\exp(cy_t)\frac{f_{zt}^{P}(y_t+A_t)}{f_{zt}^{P}(y_t)}|\mathcal{F}_{t-1}]\cdot E^{P}[Z_{t-1}|\mathcal{F}_{t-1}] \\
&= \int_{-\infty}^{\infty}\exp(cy_t)f_{zt}^{P}(y_t+A_t)dy_t \\
&= M_{y_t|\mathcal{F}_{t-1}}^{P}(c)\exp(-A_t c)\ .
\end{aligned}$$

Due to $y_t \sim \mathrm{z}(\alpha,\beta,\tilde{\delta}\sigma_t, m_t+\tilde{\mu}\sigma_t)$, We can get that:

$$M_{y_t|\mathcal{F}_{t-1}}^{Q^{egp}}(c) = \frac{B(\alpha+\sigma_t\tilde{\delta}c,\ \beta-\sigma_t\tilde{\delta}c)}{B(\alpha,\beta)}\cdot e^{(m_t+\tilde{\mu}\sigma_t-A_t)c}$$

which implies that y_t follows a z-distribution under Q^{egp}. Expanding $A_t = -r+\ln B(\alpha+\sigma_t\tilde{\delta},\beta-\sigma_t\tilde{\delta})-\ln B(\alpha,\beta)+m_t+\tilde{\mu}\sigma_t$, the return process y_t has the following from under Q^{egp}:

$$y_t|_{\mathcal{F}_{t-1}} \sim \mathrm{z}\left(\alpha,\beta,\tilde{\delta}\sigma_t,\ r-\ln\frac{B(\alpha+\sigma_t\tilde{\delta},\ \beta-\sigma_t\tilde{\delta})}{B(\alpha,\beta)}\right).$$

Thus, the representations of the conditional mean and variance are given by

$$\begin{aligned}
E^{Q^{egp}}[y_t|\mathcal{F}_{t-1}] &= r-\ln B(\alpha+\sigma_t\tilde{\delta},\ \beta-\sigma_t\tilde{\delta})+\ln B(\alpha,\beta)+\tilde{\delta}\sigma_t\varpi(\alpha,\beta)\ , \\
Var^{Q^{egp}}[y_t|\mathcal{F}_{t-1}] &= \tilde{\delta}\sigma_t^2\iota(\alpha,\beta)=\sigma_t^2\ .
\end{aligned}$$

Thus, the return process under Q^{epg} have the form listed on Eq. (4.24).

The above propositions show that the two changes of measures used for z-GARCH model lead to two different risk neutral specifications. A nice feature is that both methods agree that, after the change of measure, the conditional returns distribution stays in the z-distribution family. In the case of EGP, the conditional returns variance is unchanged after the change of measure. However, we showed that applying an Esscher transform to this setup leads to a different risk-neutral conditional variance. Thus we can conclude that the Esscher transform and EGP for z-GARCH models are no longer consistent with each other and may give rise to different derivative prices.

Chapter 5

Numerical Experiments

In this section we compare the pricing performance of the ARMA-GARCH/TGARCH and that of GARCH/TGARCH-M model under both risk neutral transformations presented above. Through our numerical study, we intend to answer/explain the following question: 1) which is a better risk neutral measure, conditional Esscher transform or extended Girsanov principle; 2) do GARCH models with z-distributed noise for the option pricing outperform those with normal ones ; 3) determine if ARMA-GARCH models have the similar pricing accuracy as GARCH-M models.

This Chapter is organized as follows. Section 5.1 exhibits a Monte Carlo option pricing algorithm for European Call options. Section 5.2 is devoted to data description for option pricing. Section 5.3 presents the experiment analysis for two option data sets.

5.1 Simulation Steps

In Section 3 we have introduced ARMA-GARCH/GARCH-in-Mean models in forecasting the trend of stock price. We also discussed two risk neutral measures: the Esscher transformation and the extend Girsanov principle, widely used in evaluating option prices in Section 4. Thus, the following goal is to compute option prices within the class of GARCH models. In the financial literature, two approaches are proposed. First we recall that the price at time t of an European Call option for a given equivalent measure $Q \in \mathcal{M}^e(P)$, the set of risk neutral measures, with strike X, maturity T and payoff function $h^{Call}(S_T) = (S_T - X, 0)^+$ is given by $e^{-r(T-t)}E^Q[h(S_T)|\mathcal{F}_t]$. In general, in the GARCH framework there is no closed form for option price, so the Monte Carlo techniques are used.

The return process could simulated under Q-measure which can then be used as the framework Monte-Carlo method. One may also use a simulation based on the P-measure. Note that

$$E^P[h(S_T)\frac{dQ}{dP}] = E^Q[h(S_T)] \ .$$

Simulation under the P-measure allows one to approximate the expected payoff with respect to the Q-measure.

$$\frac{1}{M}\sum_{m=1}^{M} h\left(S_T(m)\right)\frac{dQ}{dP}(m) \xrightarrow[M\to\infty]{P\ probability} E^P[h(S_T)\frac{dQ}{dP}] = E^Q[h(S_T)]$$

where $S_T(m)$ is the m-th stock path simulated under the physical measure P and $\frac{dQ}{dP}(m)$ is the m-th path of the Radon-Nikodym derivative computed on this path. In the thesis of Badescu [Bad07], a simulation study of option pricing has been made under both methods to compare the the speed of convergence of the estimator. When the returns follow a conditionally normal, we found that simulation under the physical measure P is more efficient that one using the risk neutralized dynamics of the returns.

In the following, we propose a Monte Carlo option pricing algorithm for the European option price when the return process satisfies Eqs.(2.10). The model parameter estimates are obtained by the maximum-likelihood estimation (MLE) under an assumed density of the innovation.

Algorithm 1

Step 1. Estimate $\hat{\varphi}$ by MLE using all historical data, where n is the number of observations. The fitting algorithm also gives $\hat{\sigma}_n^2$ and ε_n which are needed as the initial conditions of the GARCH process going forward.

Step 2. For each $m = \overline{1...M}$, generate randomly $\underline{\varepsilon}^*(m) = (\varepsilon^*_{t+1}(m), ..., \varepsilon^*_T(m))$ based on the innovation distribution. Using MLE method, the innovations are assumed as standard normal or z-distribution.

Step 3. Simulate recursively the variance and return process using the general function Eq.(2.10) based on the estimated historical information $\mathcal{F}_t$ and $\underline{\varepsilon}^*(m)$:

$$\begin{aligned}
\sigma^2_{s+1}(\hat{\varphi}_n,\ m) &= \hat{\alpha}_0 + \hat{\alpha}_1\hat{\sigma}^2_s(\hat{\varphi}_n)\omega(\varepsilon^*_s) + \hat{\beta}_1\hat{\sigma}^2_s(\hat{\varphi}_n),\\
y_{s+1}(\hat{\varphi}_n,\ m) &= m_{s+1}(\hat{\varphi}_n) + \sigma_{s+1}(\hat{\varphi}_n)\varepsilon^*_{s+1},
\end{aligned}$$

where $t \le s \le T$. The m-th path of the simulated stock price is given by:

$$S_T(\ \hat{\varphi}_n, m) = S_0 \exp\left(\sum_{t=1}^{T} y_t(\hat{\varphi}_n,\ m)\right).$$

Step 4. Evaluate the m-th path of Radon-Nikodym derivatives $\frac{dQ^{ess}(m)}{dP}$ and $\frac{dQ^{egp}(m)}{dP}$ respectively described in Section 4.2.1 and 4.2.2. For notational convenience, we drop $\hat{\varphi}_n$.

Step 5. The values of an European option at time 0 are given by:

$$C_0^{ess}(\hat{\varphi}_n) = \frac{e^{-rT}}{M}\sum_{m=1}^{M} h(S_T(m))\frac{dQ^{ess}(m)}{dP}\ , \tag{5.1}$$

$$C_0^{egp}(\hat{\varphi}_n) = \frac{e^{-rT}}{M}\sum_{m=1}^{M} h(S_T(m))\frac{dQ^{egp}(m)}{dP}. \tag{5.2}$$

Here, the European payoff function has been defined in Eq. (4.1), $h^{Call}(S_T)$ or $h^{Put}(S_T)$.

In simulating the volatility process at Step 3 we need to specify an initial value σ_0 for the conditional variance which is given in Step 1. A reliable choice is to use the last estimate of the conditional volatility. The pricing process using the Esscher transform is much more time consuming than that using extended Girsanov principle since for the former we need to solve the Esscher parameters for each $t \in \overline{0...T}$.

In Step 5, we compute Monte Carlo prices based on $M = 50,000$ simulated stock paths. Here, the value of $M = 50,000$ is taken directly from Badescu thesis [Bad07]. To check the accuracy of the price estimates, we could use the confidence interval for the expected payoff function. For example, 95% confidence interval is

$$\bar{h} \pm \frac{1.96 s_h}{\sqrt{M}}$$

where h denotes the generic payoff for a given path, $\bar{h}$ and s_h the sample mean and sample standard deviation of the M simulated payoffs. We can then choose M to achieve a desired accuracy, say .001.

5.2 Option Data Description

In Section 3.3 we fit our models to some data from the S&P 500 index. Here, we illustrate two data sets that consist of real options data used for testing our models. Data set 1 is for 1-day ahead pricing, while data set 2 is for out-of-sample pricing.

Option Data Set 1

The first data set we used is the European Call option taken from Schoutens [Sch03]. This data consists of 54 European Call options on this index at the close of the market on April 18, 2002. The closing price on that day was $S_0 = \$1124.47$, the annual risk free rate is $r = 1.9\%$ and the dividend yield is $d = 1.2\%$. When there is a dividend, the interest rate r in the discount factor is replaced by $r - d$. Note the effective risk neutral rate $r_{\text{year}} = r - d = 1.9 - 1.2 = 0.7\%$; see Badescu for further details [Bad07]. The strike price ranges from \$975 to \$1325 and we consider options with maturities $T = 22, 46, 109, 173$, and 234 days. The average option price is \$56.94. The model parameters are estimated using daily closing price of S&P 500 from January 04, 1988 to April 17, 2002, for a total of 3606 observation. The data set is the same as the data set 1 in Section 6 from Badescu and Kulperger [BK08]. Although the information this data set provides may not be sufficient for option valuation, our object is to compare and test risk neutral measures and give an overall pricing errors for each model.

Option Data Set 2

Another option data set is retrieved from STRICKNET INC. This data set is used in Han Zhang's M.Sc project [Zha12]. The data set is sampled every Wednesday at closing prices from January 07, 2004 to December 29, 2004. We use the average of the bid-ask quotes as the option observed prices. The data set consists of 1582 European call options. These options are actively traded with daily trading volume more than 200 in addition of at least 500 open interest or bids for the option.

Table 5.1: S&P 500 call option prices from Schoutens (2003)

	Day To Maturity				
Strike	22	46	109	173	234
975			161.6	173.3	
995			144.8	157	
1025			120.1	133.1	146.5
1050		84.5	100.7	114.8	
1075		64.3	82.5	97.6	
1090	43.1				
1100	35.6		65.5	81.2	
1110		39.5			
1120	22.9	33.5			
1125	20.2	30.7	51	66.9	81.7
1130		28			
1135		25.6	45.5		
1140	13.3	23.2		58.9	
1150		19.1	38.1	53.9	68.3
1160		15.3			
1170		12.1			
1175		10.9	27.7	42.5	56.6
1200			19.6	33	46.1
1225			13.2	24.9	36.9
1250				18.3	29.3
1275				13.2	22.5
1300					17.2
1325					12.8

We divide the option data into several categories based on maturity and moneyness. The days to maturity is defined as the number of trading days up to the expiration time of the option, and the moneyness, denoted as M_o, is defined as the ratio of the strike price over the underlying stock price. i.e. $M_o = K/S_0$. A call option is said to be out-of-the-money if the moneyness of the call option is greater than 1 ($M_o > 1$), and is said to be in-the-money if its moneyness less than 1 ($M_o < 1$). In order to examine closely the accuracy of option pricing results on different level of moneyness, we divide our option data into nine intervals based on the values of M_o. The option data has also been classified into four groups by the day to maturity (DTM). According to our classification, an option is short-term maturity if the option has less than 40 trading days to expire, a medium-term maturity if the number of days to maturity is between 40 and 80 days, a long-term maturity for the days to maturity between 80 and 180 days, or a very-long-term maturity if the option has more than 180 days to expire.

Table 5.2: Number of Call option contracts (S&P 500 Index, January 07, 2004 to December 29, 2004)

M_o	DTM < 40	40 ≤ DTM < 80	80 ≤ DTM < 180	DTM ≥ 180	All
[0.8, 0.9)	8	1	3	2	14
[0.9, 0.95)	31	4	4	4	43
[0.95, 0.975)	39	6	2	2	49
[0.975, 0.99)	88	19	8	9	124
[0.99, 1.01)	259	73	40	21	393
[1.01, 1.025)	201	37	18	6	262
[1.025, 1.05)	232	55	34	4	325
[1.05, 1.1)	154	75	46	21	296
[1.1, 1.2)	21	13	29	13	76
All	1033	283	184	82	1582

Table 5.3: Average price of Call option contracts (S&P 500 Index, January 07, 2004 to December 29, 2004)

M_o	DTM < 40	40 ≤ DTM < 80	80 ≤ DTM < 180	DTM ≥ 180	All
[0.8, 0.9)	165.00	177.80	173.93	152.65	166.06
[0.9, 0.95)	82.65	86.55	103.45	119.53	88.38
[0.95, 0.975)	46.54	52.75	67.40	92.85	50.04
[0.975, 0.99)	25.08	40.99	61.25	75.06	33.48
[0.99, 1.01)	13.47	30.67	46.82	65.37	22.83
[1.01, 1.025)	5.59	19.60	37.60	55.50	10.91
[1.025, 1.05)	2.72	11.69	28.00	47.33	7.43
[1.05, 1.1)	0.71	4.41	15.47	27.96	5.87
[1.1, 1.2)	0.19	0.90	5.37	12.32	4.36
ALL	12.84	19.68	32.22	52.28	18.36

The performance of all models is measured by three indicators which are described below: (i) the dollar root mean square error (RMSE), (ii) the average relative pricing error (ARPE) and (iii) the average absolute error (APE).

$$RMSE(\$) = \sqrt{\sum_{j=1}^{NO} \frac{(C_j^{market} - C_j^{model})^2}{NO}} \ , \tag{5.3}$$

$$ARPE(\%) = \frac{1}{NO} \sum_{j=1}^{NO} \frac{|C_j^{market} - C_j^{model}|}{C_j^{market}} \times 100 \ , \tag{5.4}$$

$$APE(\%) = \frac{1}{NO \cdot \overline{C}^{market}} \sum_{j=1}^{NO} |C_j^{market} - C_j^{model}| \times 100 \ , \tag{5.5}$$

where NO represents the total number of options, C_j^{market} and C_j^{model} respectively represent the real traded price and predicted price estimated by our models of underlying option j and $\overline{C}^{market}$ is the average option price.

5.3 Empirical Analysis

5.3.1 Option Data Set 1

In this section, we study the option prices for eight models fitted in 3. These yield twelve risk neutral measures, since in normal case, the Esscher transform and the extended Girsanov principle give the same measures.

— **GARCH-M** – the GARCH(1,1)-in-mean model with normal innovations.
— **TGARCH-M** – the TGARCH(1,1)-in-mean model with normal innovations.
— **ARMA-GARCH** – the ARMA(1,1)-GARCH(1,1)-in-mean model with normal innovations.
— **ARMA-TGARCH** – the ARMA(1,1)-TGARCH(1,1) model with normal innovations.
— **ESS-GARCH-M** – the GARCH(1,1)-in-mean model with z-distributed innovations under Esscher transform Q^{ess}.
— **ESS-TGARCH-M** – the TGARCH(1,1)-in-mean model with z-distributed innovations under Esscher transform Q^{ess}.
— **ESS-ARMA-GARCH** – the ARMA(1,1)-GARCH(1,1) model with z-distributed innovations under Esscher transform Q^{ess}.
— **ESS-ARMA-TGARCH** – the ARMA(1,1)-TGARCH(1,1) model with z-distributed innovations under Esscher transform Q^{ess}.
— **EGP-GARCH-M** – the GARCH(1,1)-in-mean model with z-distributed innovations under the measure Q^{egp} obtained from the extended Girsanov principle.
— **EGP-TGARCH-M** – the TGARCH(1,1)-in-mean model with z-distributed innovations under the measure Q^{egp} obtained from the extended Girsanov principle.
— **EGP-ARMA-GARCH** – the ARMA(1,1)-GARCH(1,1) model with z-distributed innovations under the measure Q^{egp} obtained from the extended Girsanov principle.
— **EGP-ARMA-TGARCH** – the ARMA(1,1)-TGARCH(1,1) model with z-distributed innovations under the measure Q^{egp} obtained from the extended Girsanov principle.

Table 5.5 and 5.6 list GARCH models parameter estimates and their criteria. The following Table 5.4 summarizes the overall pricing errors of the various models considered. Upon viewing the results, we remark that the TGARCH model outperforms the GARCH model for conditional normal distribution, which was indicated in Badescu and Kulperger [BK08]. For those models estimated with z-distributed innovations, the same feature also holds on corresponding option pricing algorithms using either Esscher transform or extended Girsanov principle.

There are two main objects for us to study the option prices based on this small data set. One is to investigate the difference in prices when using normal or z-distributed innovations. The result indicated that the choice of risk neutral measure may play a key role in the judgement. For the standard GARCH model family (GARCH-M and ARMA-GARCH), although the option prices using the Esscher transform does not bring about an obvious reduction in terms of RMSE, another indicator, ARPE is reduced by over 15%. However, GARCH models slightly outperform EGP-GARCH.

Table 5.4: Overall pricing errors for European Call options on April 18, 2002

Model	RMSE	ARPE(%)	APE(%)
ARMA-GARCH	3.9796	9.8497	5.8492
ARMA-TGARCH	4.0203	7.0526	5.5015
GARCH-M	3.7274	9.5042	5.4763
TGARCH-M	3.6525	6.0863	4.9552
ESS-ARMA-GARCH	3.2374	7.2098	4.6386
ESS-ARMA-TGARCH	1.7757	3.9643	2.4649
ESS-GARCH-M	3.7721	7.6723	5.2631
ESS-TGARCH-M	1.6129	4.2553	2.3602
EGP-GARCH-M	4.3826	7.5794	5.9114
EGP-TGARCH-M	2.6177	5.2802	3.5959
EGP-ARMA-GARCH	3.9109	7.1065	5.2638
EGP-ARMA-TGARCH	2.8453	5.4911	3.8452

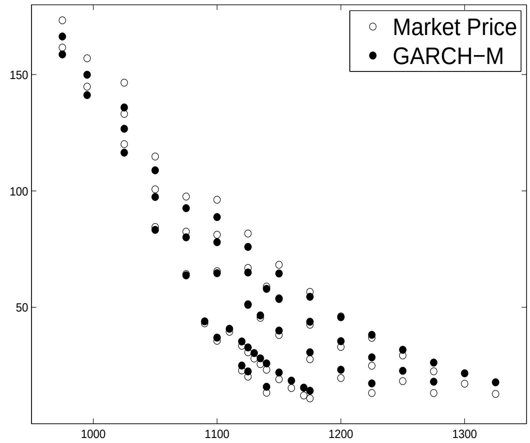
As to the threshold GARCH model family, a group of asymmetric models are constructed by including a leverage effect in the variance equation. Christofferson and Jacobs [CJ04] found that TGARCH performs the best in terms of pricing European options when the driving noise is normally distributed, which was demonstrated in [BK08]. Similarly, this conclusion also holds on our result. The z-TGARCH models (their conditional variance follow TGARCH model, innovations are z-distributed) under both risk neutral measures, whose source of asymmetry from both returns innovation distribution and the volatility equation, perform better than TGARCH with normal innovations for all the three indicators. We notice that ESS-TGARCH-M is the best option pricing model from the threshold processes. The RMSE is reduced by \$2.039 when using this model instead of normal TGARCH for an average option price of 56.94. See also Figure 5.1g and 5.1h and Figure 5.2g and 5.2h.

To test the accuracy of the indicators for each model, more analysis is required. An easy way to do this is to replicate the M paths of the simulation R times. Therefore, R independent replicates of the evaluation criteria are obtained and hence a 95% confidence interval of the indicators will be obtained for each models. If these confidence intervals do not overlap, then M is sufficiently large. Since this is computationally intensive, parallel computing techniques will be useful. We are not exploring this further in the thesis.

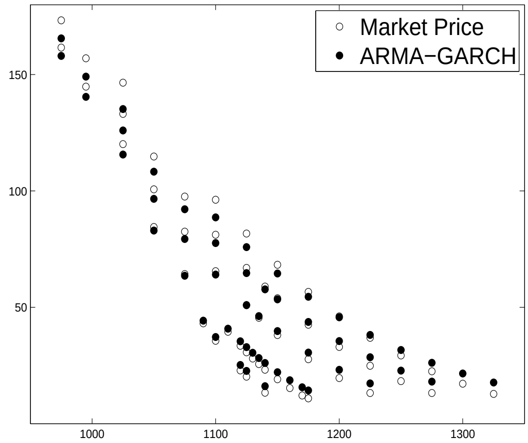
Figure 5.1 plots market prices and predicted prices for each model according to their corresponding strike prices and Figure 5.2 plots the pricing errors. From checking these plots we can conclude the following. For short maturity options (T = 22, 46 days), the option prices are slightly overpriced for all GARCH models. The model option prices severely underprice as maturity increase, especially for deep in-the-money options. The leverage effect in normal TGARCH models help in reducing parts of pricing errors for in-the-money options. The Esscher transform combining with z-distribution not only minimize the pricing error, but also makes the prices of in-the-money medium maturity options from underpriced into a slight overpricing. From Figure 5.2g and 5.2h, we see that the ESS-TGARCH-M/-ARMA-TGARCH model is able to is able to replicate the market behavior for short and medium maturity options independent of moneyness. EGP-TGARCH is the good model to price short maturity options (T = 22, 46 days); they also price relatively well deep out-of-the-money options compared to

GARCH or TGARCH. On the other hand, it performs poorly for pricing long maturity options especially for deep in-the-money options, which are severely underpriced as maturity increases.

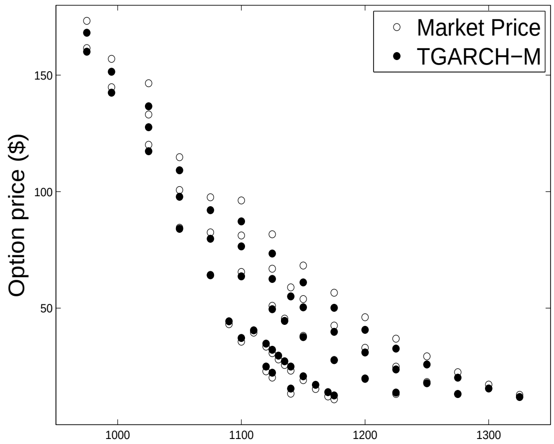
Another goal was to study the pricing performance between GARCH and ARMA-GARCH models. The results from Table 5.4 shows GARCH-M model outperforms ARMA-GARCH ones. The one exception is ARMA-GARCH with z-distributed innovations, with present smaller price errors for medium and long maturity options, especially for in-the-money ones. The comparison of these two models will be continue in the next section.



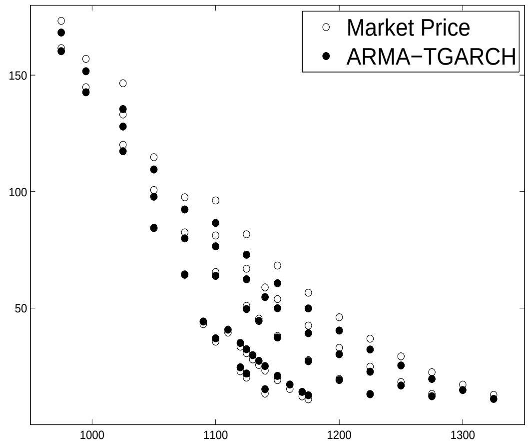
(a) GARCH-M vs. Market price



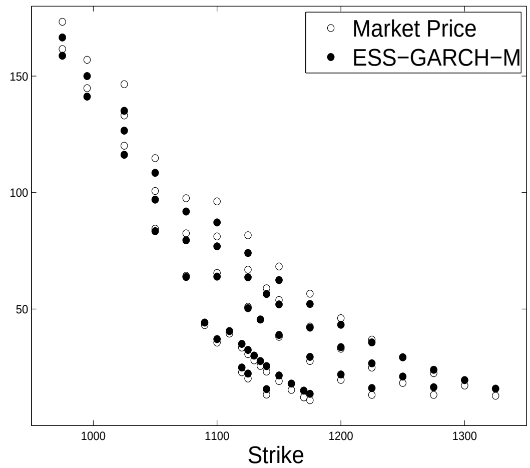
(b) ARMA-GARCH vs. Market price



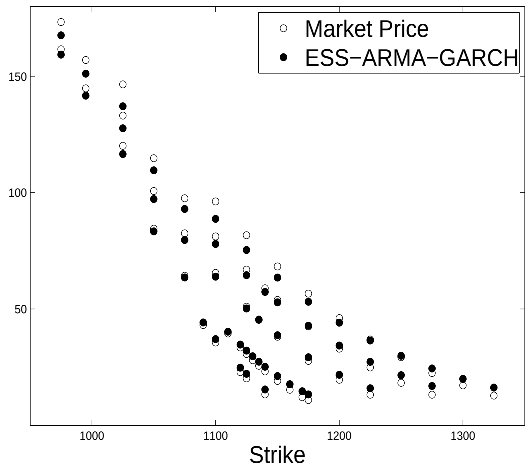
(c) TGARCH-M vs. Market price



(d) ARMA-TGARCH vs. Market price

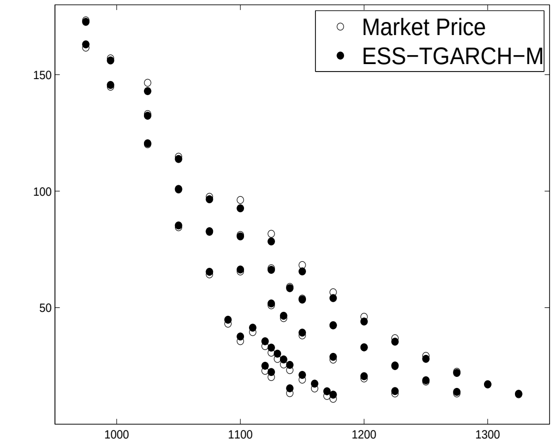


(e) ESS-GARCH-M vs. Market price

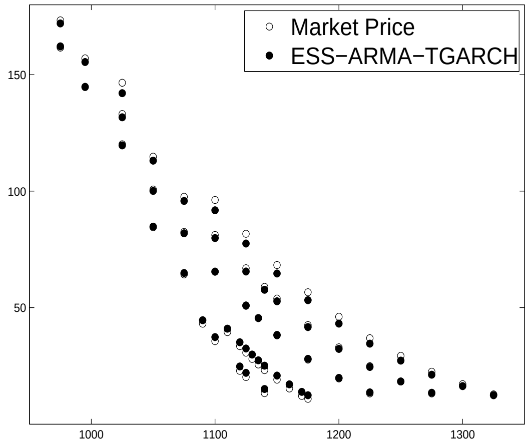


(f) ESS-ARMA-GARCH vs. Market price

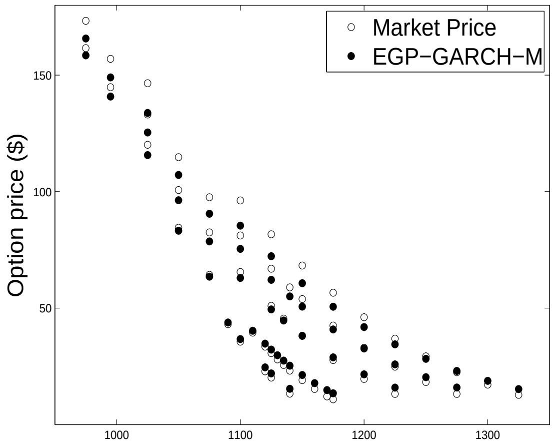
Figure 5.1: European Call option price evaluated on April 18, 2002 for different models with the maturities $T = 22, 46, 109, 173,$ and 234 days; the closing stock price on that day was $S_0 = \$1124.47$.



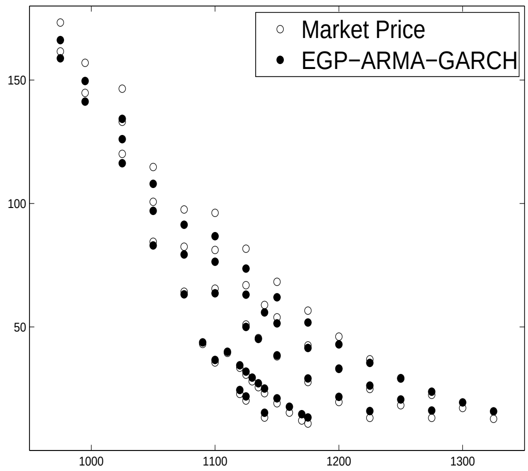
(g) ESS-TGARCH-M vs. Market price



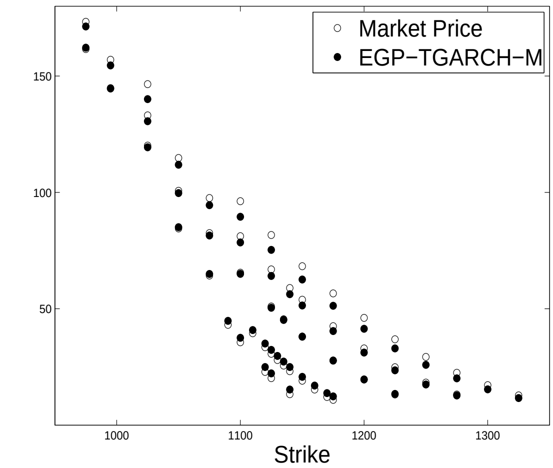
(h) ESS-ARMA-TGARCH vs. Market price



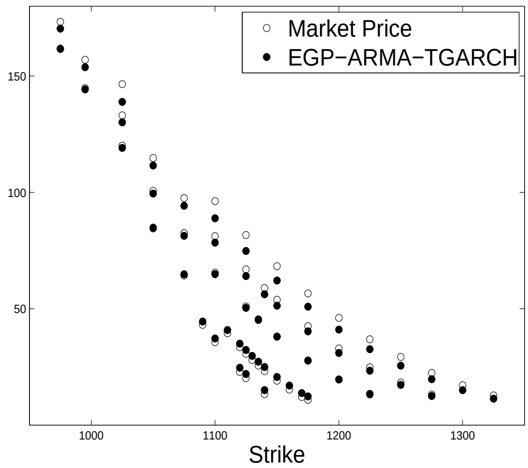
(i) EGP-GARCH-M vs. Market price



(j) EGP-ARMA-GARCH vs. Market price

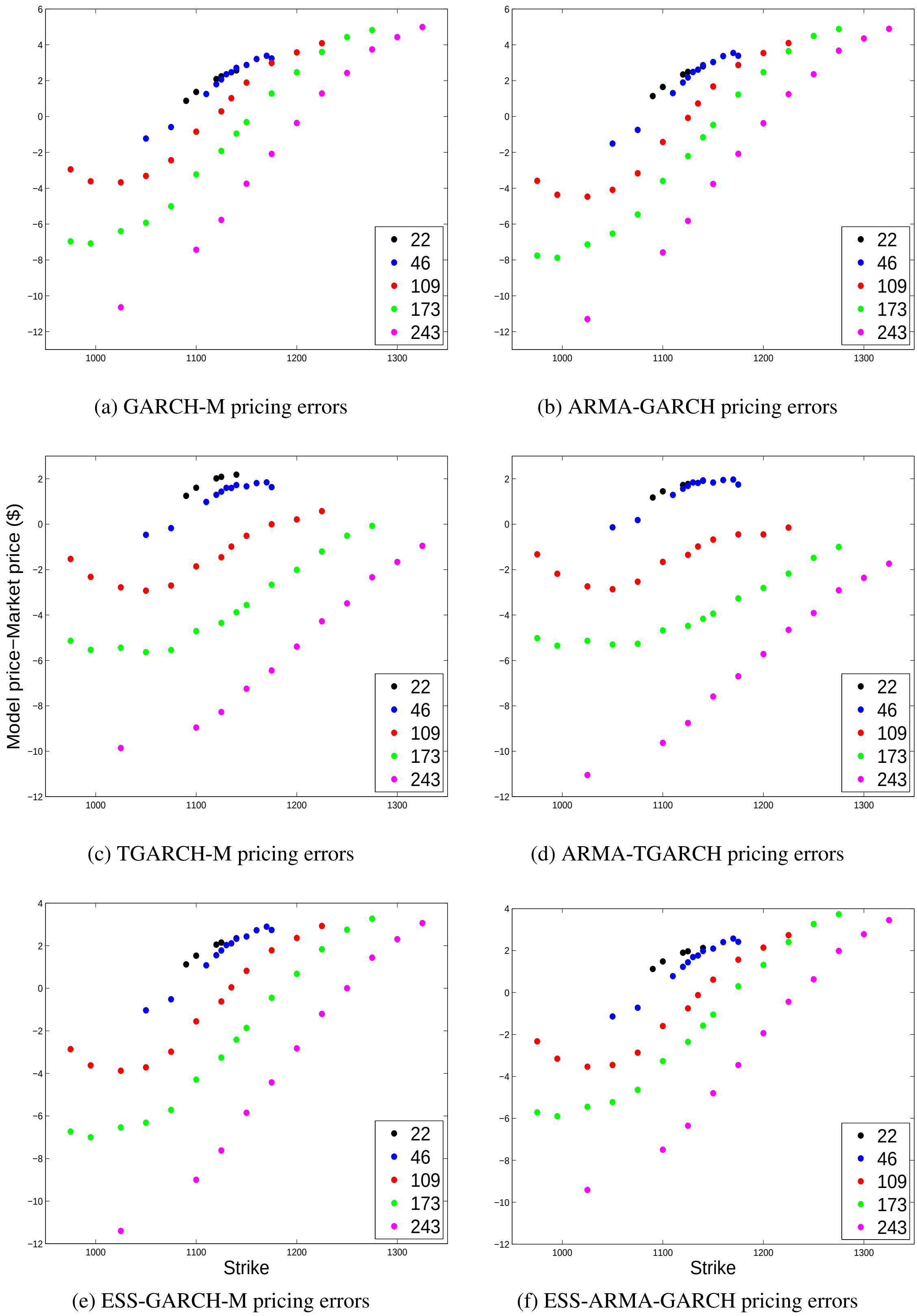


(k) EGP-TGARCH-M vs. Market price



(l) EGP-ARMA-TGARCH vs. Market price

Figure 5.1: *Continued Table*: European Call option price evaluated on April 18, 2002 for different models with the maturities $T = 22, 46, 109, 173,$ and 234 days; the closing stock price on that day was $S_0 = \$1124.47$.



(a) GARCH-M pricing errors

(b) ARMA-GARCH pricing errors

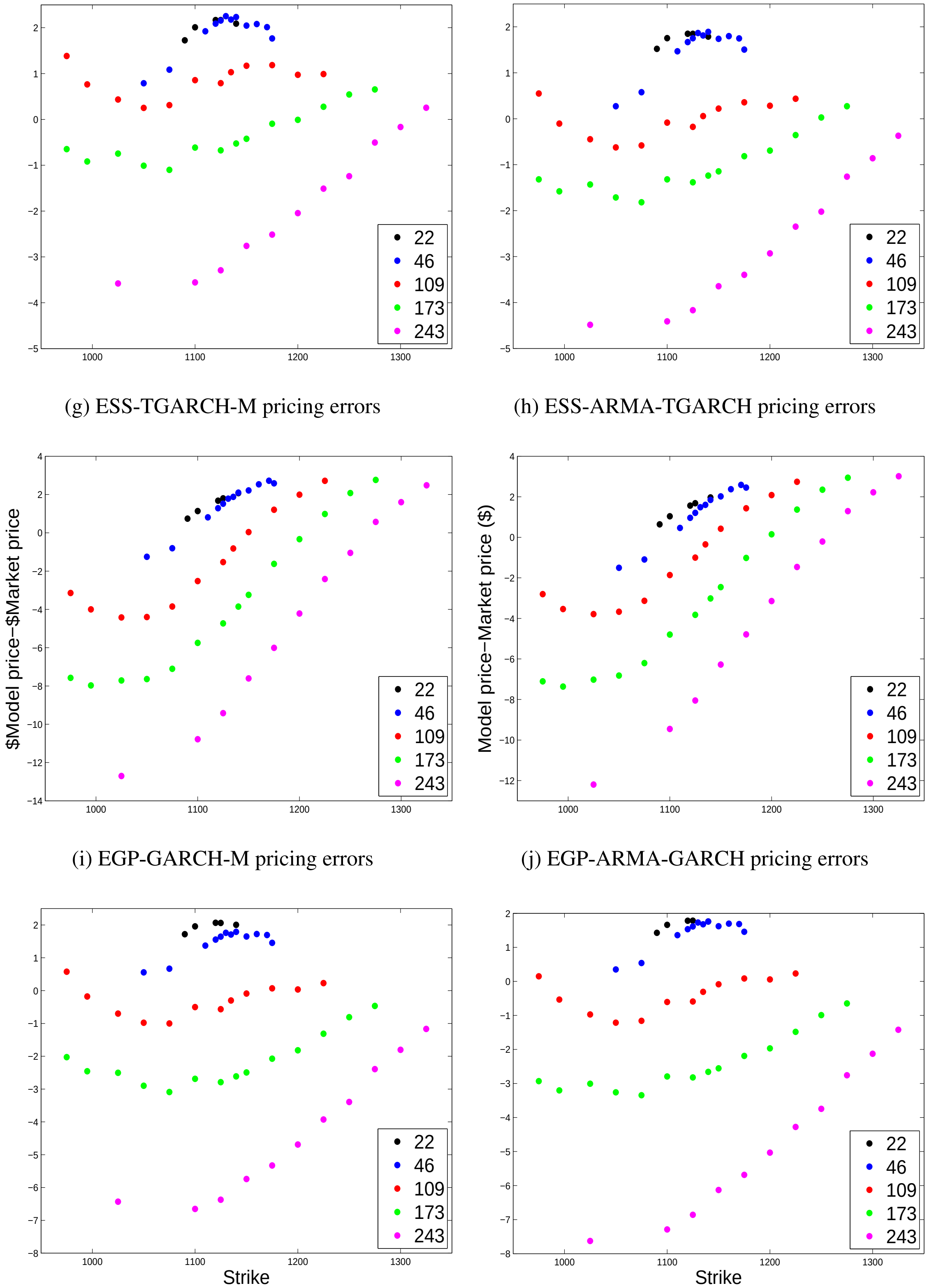
(c) TGARCH-M pricing errors

(d) ARMA-TGARCH pricing errors

(e) ESS-GARCH-M pricing errors

(f) ESS-ARMA-GARCH pricing errors

Figure 5.2: Model pricing errors for European Call options on April 18, 2002.



(g) ESS-TGARCH-M pricing errors

(h) ESS-ARMA-TGARCH pricing errors

(i) EGP-GARCH-M pricing errors

(j) EGP-ARMA-GARCH pricing errors

(k) EGP-TGARCH-M pricing errors

(l) EGP-ARMA-TGARCH pricing errors

Figure 5.2: *Continued Table*: Model pricing errors for European Call options on April 18, 2002.

Table 5.5: GARCH and TGARCH parameters estimated by MLE with normal distribution using daily closing prices of S&P 500 from January 02, 1988 to April 17, 2002

Parameters	GARCH-M	TGARCH-M	ARMA-GARCH	ARMA-TGARCH
c	2.778×10^{-5}	2.778×10^{-5}	5.0666×10^{-4}	2.9373×10^{-4}
	-	-	(1.439×10^{-4})	(1.20×10^{-4})
ϕ_1	-	-	-0.0288	0.1716
	-	-	(8.7×10^{-3})	(0.085)
θ_1	-	-	0.0543	−0.1348
	-	-	(8.7×10^{-3})	(0.086)
λ	0.0603	0.0457	-	-
	(0.0012)	(0.0175)	-	-
α_0	4.28×10^{-7}	1.0301×10^{-6}	4.14×10^{-7}	9.44×10^{-7}
	(1.5×10^{-7})	(2.8×10^{-7})	(8.68×10^{-8})	(2.51×10^{-7})
α_1	0.0396	0.0165	0.0392	0.0056
	(0.0065)	(0.0066)	(2.83×10^{-3})	(7.7×10^{-3})
β_1	0.9567	0.9403	0.9573	0.9467
	(0.0072)	0.0107	(3.15×10^{-3})	(9.5×10^{-3})
γ	-	0.0625	-	0.0735
	-	(0.0133)	-	(0.0144)
σ_0	0.0102	0.0109	0.0103	0.0111
LLF	1.1888×10^{4}	1.1906×10^{4}	1.1888×10^{4}	1.1908×10^{4}
AIC	-2.3767×10^{4}	-2.3803×10^{4}	-2.3756×10^{4}	-2.3803×10^{4}
BIC	-2.3742×10^{4}	-2.3772×10^{4}	-2.3719×10^{4}	-2.3759×10^{4}

Table 5.6: GARCH and TGARCH parameters estimated by MLE, which under the assumption on z-distributed noise, using daily closing prices of S&P 500 from January 02, 1988 to April 17, 2002
Note: in this table, the subscript in α_z or β_z denote the z-distribution case.

Parameters	GARCH-M	TGARCH-M	ARMA-GARCH	ARMA-TGARCH
c	2.778×10^{-5}	2.778×10^{-5}	1.428×10^{-4}	1.718×10^{-4}
	-	-	(4.92×10^{-5})	(1.136×10^{-4})
ϕ_1	-	-	0.6851	0.1716
	-	-	(0.078)	(0.085)
θ_1	-	-	-0.7324	−0.1348
	-	-	(0.079)	(0.086)
λ	0.0603	0.0457	-	-
	(0.0012)	(0.0175)	-	-
α_0	4.28×10^{-7}	1.0301×10^{-6}	2.55×10^{-7}	9.44×10^{-7}
	(1.5×10^{-7})	(2.8×10^{-7})	(1.22×10^{-7})	(2.51×10^{-7})
α_1	0.0396	0.0165	0.0351	0.0056
	(0.0065)	(0.0066)	(0.0071)	(7.7×10^{-3})
β_1	0.9567	0.9403	0.9624	0.9467
	(0.0072)	0.0107	(0.0074)	(9.5×10^{-3})
γ	-	0.0625	-	0.0735
	-	(0.0133)	-	(0.0144)
α_z	0.698	0.7654	0.5412	0.7625
	(0.096)	(0.112)	(0.0844)	(0.111)
β_z	0.7901	0.901	0.6438	0.9053
	(0.119)	(0.147)	(0.111)	(0.147)
σ_0	0.0071	0.0059	0.0102	0.011
LLF	1.1888×10^4	1.1906×10^4	1.2036×10^4	1.205×10^4
AIC	-2.3767×10^4	-2.3803×10^4	-2.4055×10^4	-2.4082×10^4
BIC	-2.3742×10^4	-2.3772×10^4	-2.4006×10^4	-2.4026×10^4

5.3.2 Option Data Set 2

In this section, our object is to test the out-of-sample performance of our pricing methodology. The difference between this data set from the previous one is that the option are not traded at the same date. There are 1582 options, which are traded every Wednesday in the following 52 weeks after January 6th, 2004. The model parameters for the return process of S&P 500 Index from January 02, 1988 to January 06, 2004 has been estimated for each model in Section 3.3.

As we know, the choice of the starting value for the conditional variance is crucial for computing option prices within a GARCH framework. Thus, to ensure that our simulations on different dates are starting with the most accurate/updated initial volatility, we use an updating scheme by constructing a series of volatilities σ_t by using observed returns from 2004. A detailed example is described below to illustrate this method. For example, we intend to calculate the value of an option that was traded on Jan 14, 2004, and would expire in 25 trading days. Our GARCH model is fitted by observed historical data from Jan 02,1988 to Jan 06, 2004. Thus, we can get the estimated conditional volatility $\sigma_0(\hat{\varphi})$ on January 06, 2004 from our model fitting. We can calculate the observed noise $\varepsilon_t(\hat{\varphi})$ and hence the conditional volatility $\sigma_t(\hat{\varphi})$ at each date between Jan 07 to Jan 14 using model specification with observed real data ($S\&P$ 500 Index on Jan 07 to Jan 14) rather than refitting the model. Taking the estimated conditional volatility on Jan 14 as the initial volatility, the option price can be calculated by Monte Carlo techniques using simulated stock price and Radon-Nikodym derivatives.

Since from the previous section we noticed that extended Girsanov principle (EGP) is not appropriate risk neutral measure comparing with the Esscher transform. Due to its poor performance illustrated by previous example, we discarded the models using EGP measure in this section. Thus, all models end with prefix ‘z-’ are specified as the models using the conditional Esscher transform.

The overall pricing performance is reported in Table 5.7. Similarly, the replication of M path simulation could be conducted to test the accuracy of the indicators for each model. In this table, we find that the second indicator ARPE, it is the ratio of the absolute difference between model price and market price to the market option price. A small market price as denominator contributes a large ARPE. The market prices of 576 options are below \$5 in this data set, while all market price are above \$10 in the pervious example. That is the main reason to explain the high APE here. Similarly, lower average market price of \$18.36 cause the third indicator APE is slight higher than that in Table 5.4, where average option price is \$56.94. With understanding the effect of low market price, we can interpret the results listed on Table A.5-A.8 more easily.

Table 5.7: Overall pricing errors for European Call options in data set 2

Model	RMSE($)	ARPE(%)	APE(%)
ARMA-GARCH	3.1842	23.0401	10.3098
ARMA-TGARCH	2.1841	21.8342	7.566
GARCH-M	3.1384	23.0234	10.1846
TGARCH-M	2.142	21.4463	7.4492
z-ARMA-GARCH	3.4568	22.0189	11.2563
z-ARMA-TGARCH	2.2325	22.1975	8.0226
z-GARCH-M	3.7732	22.9206	12.2332
z-TGARCH-M	2.1005	22.083	7.6545

In Table 5.7, we notice that the advantage of TGARCH-M(ARMA-TGARCH)-Z model in option pricing is not as significant as that in Section 5.3.1. The normal GARCH models even outperform that with z-distributed innovations. These overall results seem to be inconsistent with the conclusion we obtain in Section 5.3.1. In order to answer/explain this discrepancy, we make Tables A.5-A.8 with addition information of moneyness and maturity for each model. Another two Table 5.8 and 5.9 as follows, summarize the price errors regarding moneyness and maturity respectively for all models, which greatly improve the readability.

Upon viewing the results in Table 5.8 and 5.9, we remark that the normal TGARCH models (including ARMA-TGARCH and TGARCH-M) outperform normal GARCH model in terms of three indicators for all moneyness and maturity. Comparing with GARCH/TGARCH-M and ARMA-GARCH/TGARCH models for normal innovations, GARCH-M model family with a risk premium, has a better pricing performance than ARMA-GARCH models for all situations.

Using z-distribution innovation instead of normal ones, the TGARCH models price better than the GARCH ones as well. The leverage effect in TGARCH model bring about more price error reduction for in-the-money options. Although z-TGARCH-M model outperforms z-ARMA-TGARCH model, z-GARCH-M model performs worse than z-ARMA-GARCH model.

In Table 5.7, we find that the assumption of z-distributed innovations in model estimation does not make the same price performance enhancement as Table 5.4 presents. However, according to the result listed on Table 5.8 and 5.9, we notice that TGARCH models with z-distributed innovations have lower pricing error of all indicators for in-the-money options. In addition, for the long maturity options, z-TGARCH models perform best. For example, it reduces \$1.086 in terms of RMSE for options with over 180 days to maturity. Thus we can draw a conclusion that z-TGARCH models are especially good for in-the-money and long maturity options. Otherwise, for out-of-the-money and short maturity options, the simple normal TGARCH models are much better for pricing.

When we look closely at this data set, we notice that it consists of more out-of-the-money options (959 contracts) than in-the-money options (230 contracts). Also, there are roughly 1,033 short-term maturity options (DTM < 40) and only 266 option contracts with days to maturity greater than 80 days. Thus, the imbalance of the data set is in favor of the overall performance of the normal TGARCH models. Therefore, we can assume that, compared with the normal TGARCH, it would suggest that the z-TGARCH would perform better under a balanced option data set.

A good model of an underlying asset and corresponding good choice of a risk neutral measure should make a good option price prediction. Thus these price predictions should then produce a similar Black-Scholes (BS) implied volatility as the observed market prices. Figure 5.3 shows the BS implied volatilities of the market option prices and the prices simulated from normal and Esscher z prices for option data set 2. Market and model prices are plunged into BS price formula and solved the implied volatility. The plot at a given moneyness presents the average implied volatility over all the expiration time at this moneyness. From this plot, we can clearly see that the implied volatility of z-TGARCH is closer to the real market implied volatility than that of the normal TGARCH model for in-the-money option. While the implied volatility of normal TGARCH is closer to the real market implied volatility than that of the z-TGARCH models for out-of-the-money option. This interpretation is consistent with the one we previously made. Also, the graphs show that the implied volatility of ARMA-TGARCH pricing models are worse than that of TGARCH-M models with both normal or z-distributed noises for in-the-money options, but both of their performance are similar for out-of-the-money options. However, z-ARMA-GARCH model makes better option prices than z-GARCH-M, especially for deep in-the-money options.

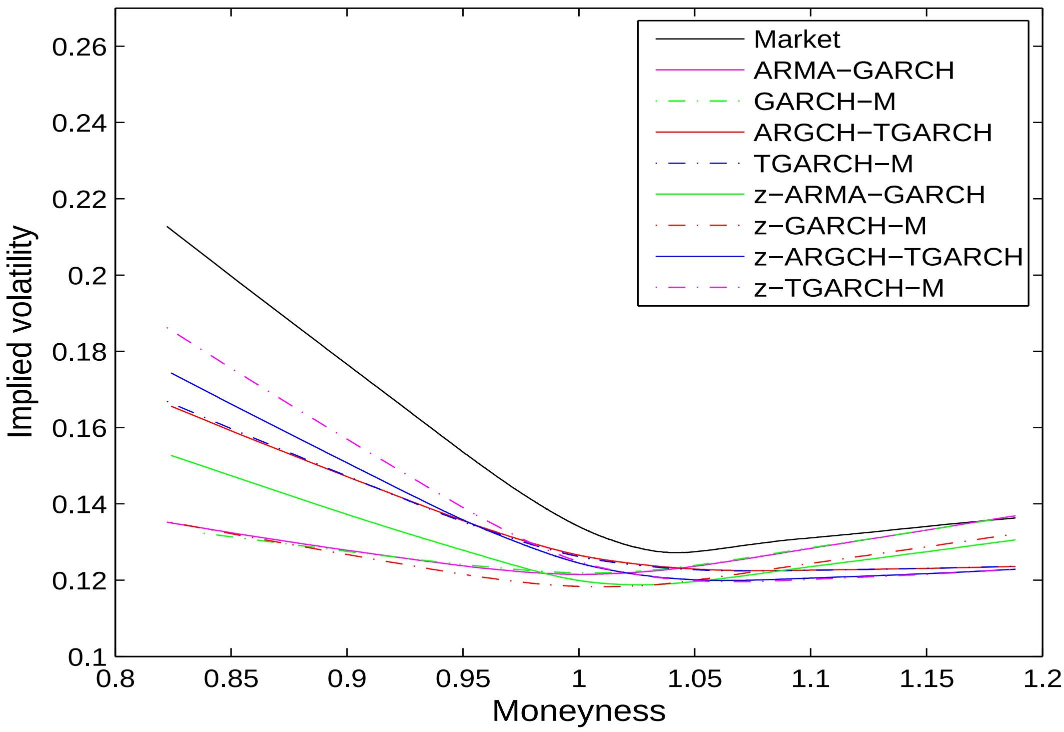


Figure 5.3: Implied volatility smiles based on MLE estimates using returns from January 07, 2004 to December 29, 2004

Table 5.8: Overall pricing errors regarding to moneyness(Call option contracts of S&P 500 Index, January 07, 2004 to December 29, 2004)
In this table, the models are presented in abbreviated form.
G: GARCH TG: TGARCH z-G: z-GARCH z-TG: z-TGARCH AG: ARMA-GARCH ATG: ARMA-TGARCH z-AG: z-ARMA-GARCH z-ATG: z-ARMA-TGARCH

RMSE($)								
M_o	G	TG	z-G	z-TG	AG	ATG	z-AG	z-ATG
[0.8, 0.9)	4.9094	2.9464	5.0767	2.0021	4.8543	3.0269	4.7470	2.4700
[0.9, 0.95)	3.9749	2.4919	4.3679	2.0036	4.0976	2.3685	3.1557	2.0660
[0.95, 0.975)	3.5138	2.5093	3.9779	2.4275	3.5809	2.5855	3.5178	2.5947
[0.975, 0.99)	4.3630	2.9405	5.0774	2.7322	4.5090	3.0137	4.8327	2.8887
[0.99, 1.01)	4.1922	2.7767	5.0291	2.7063	4.2289	2.8342	4.5836	2.8910
[1.01, 1.025)	2.7063	1.8770	3.2968	1.9418	2.7291	1.9433	2.9742	2.0359
[1.025, 1.05)	2.3637	1.7214	2.9772	1.8286	2.4018	1.7390	2.8013	1.9356
[1.05, 1.1)	1.6885	1.3210	2.3240	1.3333	1.7222	1.3510	2.1960	1.4187
[1.1, 1.2)	1.2249	0.8628	1.1630	0.8690	1.1818	0.8419	1.1824	0.9390
ALL	3.1384	2.1420	3.7732	2.1005	3.1842	2.1841	3.4568	2.2325

ARPE(%)								
M_o	G	TG	z-G	z-TG	AG	ATG	z-AG	z-ATG
[0.8, 0.9)	2.1123	1.2932	2.2054	0.8536	2.1117	1.3102	1.9400	1.1018
[0.9, 0.95)	3.0468	2.1313	3.2550	1.8285	3.1037	2.1273	2.5134	1.9137
[0.95, 0.975)	5.2509	4.0705	5.8927	3.9127	5.3861	4.0810	4.9758	4.1200
[0.975, 0.99)	7.9380	6.0569	9.0811	6.0578	8.2102	6.1248	8.5656	6.1717
[0.99, 1.01)	12.2793	10.3960	13.8661	10.5123	12.2557	10.5260	13.1677	10.7532
[1.01, 1.025)	23.6455	21.6830	23.5318	20.8782	23.2272	22.5266	21.9733	21.4828
[1.025, 1.05)	30.1273	29.3479	29.0362	30.1510	30.3939	29.7469	28.4539	29.8962
[1.05, 1.1)	36.3714	35.7498	35.8056	38.5150	36.4284	36.3031	34.3562	38.3861
[1.1, 1.2)	45.2984	39.2270	39.8010	40.8007	45.2850	39.7415	40.0513	41.0325
ALL	23.0234	21.4463	22.9206	22.0830	23.0401	21.8342	22.0189	22.1975

APE(%)								
M_o	G	TG	z-G	z-TG	AG	ATG	z-AG	z-ATG
[0.8, 0.9)	2.0225	1.2411	2.1117	0.8182	2.0418	1.2567	1.9295	1.0428
[0.9, 0.95)	3.2836	2.2135	3.5349	1.8678	3.3521	2.2054	2.6563	1.9575
[0.95, 0.975)	5.4837	4.0489	6.1929	3.8754	5.6147	4.1095	5.2307	4.1083
[0.975, 0.99)	9.2490	6.5416	10.7899	6.2741	9.5749	6.6282	10.0650	6.5343
[0.99, 1.01)	12.2521	8.5468	14.9853	8.7255	12.3438	8.6275	13.7246	9.1070
[1.01, 1.025)	14.6470	11.1770	17.8131	12.1361	14.6653	11.5285	16.3617	12.6550
[1.025, 1.05)	16.8803	13.5093	20.6974	15.2289	17.0851	13.7788	19.8476	15.8236
[1.05, 1.1)	16.0133	12.9732	20.9247	13.7029	16.2863	13.2871	19.5418	14.5424
[1.1, 1.2)	19.2318	12.5470	15.8201	12.6346	18.6612	12.4016	17.8530	13.7605
ALL	10.1846	7.4492	12.2332	7.6545	10.3098	7.5660	11.2563	8.0226

Table 5.9: Overall pricing errors regarding to maturity (Call option contracts of S&P 500 Index, January 07, 2004 to December 29, 2004)

RMSE($)					
Model	DTM				
	[1, 40)	[40, 80)	[80, 180)	[180, 250)	All
GARCH-M	1.5005	3.3023	5.4682	7.5450	3.1384
TGARCH-M	1.2125	2.3071	3.5638	4.8096	2.1420
z-GARCH-M	1.6671	3.8606	6.6725	9.3976	3.7732
z-TGARCH-M	1.3308	2.5610	3.4239	3.7237	2.1005
ARMA-GARCH	1.5200	3.3213	5.5566	7.6907	3.1842
ARMA-TGARCH	1.2405	2.3517	3.6256	4.9053	2.1841
z-ARMA-GARCH	1.5800	3.7259	6.2761	7.9245	3.4568
z-ARMA-TGARCH	1.3597	2.6924	3.6664	4.2049	2.2325
ARPE(%)					
Model	DTM				
	[1, 40)	[40, 80)	[80, 180)	[180, 250)	All
GARCH-M	26.2152	18.1212	17.1841	12.8364	23.0234
TGARCH-M	26.6335	13.2940	10.6344	8.4972	21.4463
z-GARCH-M-Z	25.1868	19.1150	19.1655	15.9334	22.9206
z-TGARCH-M-Z	26.8577	16.0464	11.2595	7.0532	22.0830
ARMA-GARCH	26.2082	18.1957	17.2249	12.8983	23.0401
ARMA-TGARCH	27.0169	13.8811	10.8571	8.6247	21.8342
z-ARMA-GARCH-Z	24.3286	18.5645	18.1214	13.5901	22.0189
z-ARMA-TGARCH-Z	26.7404	16.3540	12.1439	7.6936	22.1975
APE(%)					
Model	DTM				
	[1, 40)	[40, 80)	[80, 180)	[180, 250)	All
GARCH-M	7.3486	12.4290	13.4035	11.5896	10.1846
TGARCH-M	6.4417	8.6724	8.6783	7.2768	7.4492
z-GARCH-M-Z	8.1103	15.0978	16.7313	15.0440	12.2332
z-TGARCH-M-Z	6.9875	10.0748	8.2195	5.7921	7.6545
ARMA-GARCH	7.4153	12.5474	13.5905	11.8188	10.3098
ARMA-TGARCH	6.5403	8.8028	8.7764	7.4578	7.5660
z-ARMA-GARCH-Z	7.7306	14.5896	15.5168	11.9393	11.2563
z-ARMA-TGARCH-Z	7.1064	10.5548	8.9211	6.3245	8.0226

Chapter 6

Conclusion and Future research

6.1 Conclusion

This thesis studies in detail option pricing issues when the return process is modeled by discrete time GARCH-in-mean or ARMA-GARCH models.

Chapter 2 is devoted to a brief introduction of these two kinds of models and their extensions. Then, we estimate the GARCH models for $S\&P$ 500 index using maximum likelihood estimation (MLE) method. The fitting process involves two different assumptions on unknown innovation distribution, (i) the standard normal distribution and (ii) the z-distribution. Han [Zha12] showed that the z-distribution can capture some of the stylized facts exhibited by financial data. The parameter estimates and likelihood-based goodness of fit criteria indicate that both ARMA-GARCH and GARCH-M have similar fitting performance, especially for the conditional variance part. The figures of log-densities of observed residuals vs. their theoretical distribution for eight GARCH models in our study reveal more model fitting issues in detail. First, for the normal distribution, the observed standardized residuals present excess kurtosis and extreme skewness in left tail. The TGARCH model with a leverage effect can offset a bit of asymmetry of observed residuals. Generally, the fit for the normal distribution is still poor when compared with its theoretical density. Then, upon replacing the normal with a z-distribution, the plots of the residual log-density shows an improvement in fitting over the normal case. On one side, the observed residuals do not show obvious leptokurtosis to its theoretical density. On the other side, the matched range between residual log-density and its theoretical density has expanded, even though it does not capture the behavior of its theoretical density well in both tails. Generally speaking, we could conclude that GARCH models with z-distributed innovations present better performance in fitting the return process.

Since most discrete time markets are incomplete, various choices of risk neutral measures for derivative pricing are discussed in pervious financial literature. Following the work of Badescu [Bad07], in Chapter 4 we restrict our attention on the conditional Esscher transform developed by Gerber and Shiu [GS94] and the extended Girsanov principle introduced by Elliot and Madan [EM98]. We show that the conditional Esscher transform and the extended Girsanov principle lead to the same result as the well known local risk neutral valuation relationship proposed by Duan [Dua95]. For GARCH models with z-distribution innovations, we give risk neutralized dynamics using the conditional Esscher transform and the extended

Girsanov principle. We find that for both measures the conditional returns still follow two distinct z-distribution. However, we notice that the conditional variance function changes under the Esscher transform while under the extend Girsanov principle it remains the same as the historical measures P. Thus, when the normality assumption is violated, using different risk neutral measures leads to different option prices. Based on this remark, the choice of the risk neutral measure is crucial in option pricing.

In Chapter 5, we applied our GARCH models discussed above in an option pricing experiment. Two sample data sets, for which we have observed option prices, are used in our numerical test. We compute option prices using the Monte-Carlo technique by simulating under P the stock and volatility paths and the Radon-Nikodym derivatives. For the purpose of option valuation we consider these two risk neutral measure: the conditional Esscher transform and the extended Girsanov principle.

Our findings suggest that normal leverage TGARCH models outperform the simple GARCH ones, which is consistent with the model fitting results presented by the log-density plots in Section 3.3. Then, we numerically showed that both the Esscher transform and extended Girsanov principle for z-distributed leverage TGARCH models outperform the usual normal leverage ones. Another appealing conclusion is that the Esscher method is the best choice of martingale measure since it is able to price very well for both short and long maturity options compared with the extended Girsanov principle measure. We also notice that z-TGARCH models are very good at option pricing for in-the-money and long maturity options. However, for those out-of-the-money and short maturity options, normal TGARCH is much better in pricing.

One of important object in this thesis is to compare the pricing performance between GARCH-in-mean and ARMA-GARCH models. Our numerical test in both sample data shows that, under normal innovation assumption, GARCH/TGARCH-M model family with a risk premium, price slightly better than ARMA-GARCH/-TGARCH models. When it comes to z-distributed innovation, z-TGARCH-M model still outperforms z-ARMA-TGARCH model. However, z-ARMA-GARCH model makes better option prices than z-GARCH-M, especially for deep in-the-money option. Generally, ARMA-GARCH model is good in option pricing. Due to asymptotic normality of its estimators, the ARMA-GARCH model is the more easily understood in theory.

6.2 Future Research

Some of the interesting results presented in this thesis can be further exploited.

One could consider different choices for the innovation distribution. In Section 3.3, we find that the standardized residuals generated by z-TGARCH models cannot catch the behavior of theoretical density in left-tail. Thus, another heavy-tailed distribution innovation which could contribute to a better fitting performance will probably improve the option pricing.

We have used some variations and extensions of the GARCH process. As we know, TGARCH model with a leverage effect, fits the real financial series better. Thus we believe that an effective method to enhance this asymmetric effect of TGARCH model would make a positive effect.

Finally, the results from two sample data sets of this thesis is not totally consistent. Only call options written on $S\&P$ 500 Index are used for test. In order to validate the universality of our conclusion, more numerical experiments involving options in different time periods, written on distinct underlying assets are required.

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Appendix A

Tables of Pricing Errors for Example 2

These tables are further tables of numerical experiment on option data set 2 in Chapter 5. They contain more detail information about pricing errors in terms of the three indicators regarding different moneyness and maturity for each GARCH model used in this thesis.

Table A.1: Pricing errors for GARCH-M (Call option contracts of S&P 500 Index, January 07, 2004 to December 29, 2004)

RMSE($)

	DTM				
M_o	[1, 40)	[40, 80)	[80, 180)	[180, 250)	All
[0.8, 0.9)	1.2575	2.7723	7.9171	8.0331	4.9094
[0.9, 0.95)	1.7397	4.6883	7.6892	8.0801	3.9749
[0.95, 0.975)	2.5512	4.8184	7.7207	6.8060	3.5138
[0.975, 0.99)	2.1451	5.1140	7.8288	10.3725	4.3630
[0.99, 1.01)	1.9534	4.6336	7.3573	10.2023	4.1922
[1.01, 1.025)	1.3360	3.4254	6.4494	7.9292	2.7063
[1.025, 1.05)	0.8891	2.1049	5.8805	7.2981	2.3637
[1.05, 1.1)	0.4346	1.1846	2.6982	4.2242	1.6885
[1.1, 1.2)	0.2024	0.5490	1.2199	2.2550	1.2249
All	1.5005	3.3023	5.4682	7.5450	3.1384

ARPE(%)

	DTM				
M_o	[1, 40)	[40, 80)	[80, 180)	[180, 250)	All
[0.8, 0.9)	0.4756	1.5592	4.6326	5.1552	2.1123
[0.9, 0.95)	1.7769	5.3845	7.1370	6.4605	3.0468
[0.95, 0.975)	4.2908	8.9132	11.5393	6.6984	5.2509
[0.975, 0.99)	6.2780	11.5672	12.5909	12.3722	7.9380
[0.99, 1.01)	11.6164	13.1337	14.4039	13.4388	12.2793
[1.01, 1.025)	26.7467	12.9074	15.1796	11.3707	23.6455
[1.025, 1.05)	36.6042	12.7874	16.1970	11.3017	30.1273
[1.05, 1.1)	50.5411	26.9831	14.7921	13.2594	36.3714
[1.1, 1.2)	78.1626	51.4287	31.5552	16.7377	45.2984
ALL	26.2152	18.1212	17.1841	12.8364	23.0234

APE(%)

	DTM				
M_o	[1, 40)	[40, 80)	[80, 180)	[180, 250)	All
[0.8, 0.9)	0.3849	1.5592	4.5106	5.1203	2.0225
[0.9, 0.95)	1.7872	5.3142	7.2119	6.4325	3.2836
[0.95, 0.975)	4.3275	8.8724	11.3892	6.7216	5.4837
[0.975, 0.99)	6.7078	11.6569	12.4242	12.4721	9.2490
[0.99, 1.01)	9.7486	13.3604	14.5074	13.7311	12.2521
[1.01, 1.025)	15.6816	13.6236	15.3660	11.9261	14.6470
[1.025, 1.05)	21.7488	12.8067	17.3097	12.3206	16.8803
[1.05, 1.1)	37.2183	20.0095	13.1690	13.2689	16.0133
[1.1, 1.2)	72.7374	48.4110	19.8722	15.1602	19.2318
ALL	7.3486	12.4290	13.4035	11.5896	10.1846

Table A.2: Pricing errors for TGARCH-M (Call option contracts of S&P 500 Index, January 07, 2004 to December 29, 2004)

RMSE($)					
	DTM				
M_o	[1, 40)	[40, 80)	[80, 180)	[180, 250)	All
$[0.8, 0.9)$	1.0605	2.0079	5.3279	3.4167	2.9464
$[0.9, 0.95)$	1.5765	2.2119	4.9249	4.2831	2.4919
$[0.95, 0.975)$	2.1301	3.4319	4.9198	2.5004	2.5093
$[0.975, 0.99)$	1.6350	3.8683	4.3614	6.6703	2.9405
$[0.99, 1.01)$	1.5208	3.0716	4.7253	6.3593	2.7767
$[1.01, 1.025)$	1.1063	2.3967	4.0646	5.2771	1.8770
$[1.025, 1.05)$	0.7909	1.6563	4.0650	5.1294	1.7214
$[1.05, 1.1)$	0.4336	0.9962	1.9748	3.3364	1.3210
$[1.1, 1.2)$	0.2397	0.2914	0.8787	1.5658	0.8628
ALL	1.2125	2.3071	3.5638	4.8096	2.1420
ARPE(%)					
	DTM				
M_o	[1, 40)	[40, 80)	[80, 180)	[180, 250)	All
$[0.8, 0.9)$	0.4237	1.1293	3.0965	2.1481	1.2932
$[0.9, 0.95)$	1.6581	2.5498	4.4145	3.0961	2.1313
$[0.95, 0.975)$	3.6699	6.2063	7.2601	2.2852	4.0705
$[0.975, 0.99)$	5.2241	8.7916	6.6199	7.9266	6.0569
$[0.99, 1.01)$	11.3761	8.2822	9.0959	8.1326	10.3960
$[1.01, 1.025)$	25.4248	9.5661	9.5479	7.4581	21.6830
$[1.025, 1.05)$	36.6358	11.1998	11.4506	8.3129	29.3479
$[1.05, 1.1)$	54.6205	19.9271	9.9951	10.2895	35.7498
$[1.1, 1.2)$	89.8524	36.7333	16.4656	10.7163	39.2270
ALL	26.6335	13.2940	10.6344	8.4972	21.4463
APE(%)					
	DTM				
M_o	[1, 40)	[40, 80)	[80, 180)	[180, 250)	All
$[0.8, 0.9)$	0.3451	1.1293	3.0265	2.1287	1.2411
$[0.9, 0.95)$	1.6521	2.5200	4.4879	3.0316	2.2135
$[0.95, 0.975)$	3.6261	6.2029	7.0979	2.2965	4.0489
$[0.975, 0.99)$	5.3252	8.7234	6.5357	8.0046	6.5416
$[0.99, 1.01)$	8.3924	8.4387	9.1456	8.2985	8.5468
$[1.01, 1.025)$	14.1259	9.7239	9.5129	7.7808	11.1770
$[1.025, 1.05)$	20.0748	10.6124	12.0935	8.5701	13.5093
$[1.05, 1.1)$	38.8559	16.3658	9.7462	10.1639	12.9732
$[1.1, 1.2)$	86.2316	24.0612	11.8632	10.5455	12.5470
ALL	6.4417	8.6724	8.6783	7.2768	7.4492

Table A.3: Pricing errors for ARMA-GARCH (Call option contracts of S&P 500 Index, January 07, 2004 to December 29, 2004)

RMSE($)

	DTM				
M_o	[1, 40)	[40, 80)	[80, 180)	[180, 250)	All
[0.8, 0.9)	1.2022	3.1587	7.6742	8.1142	4.8543
[0.9, 0.95)	1.7957	4.3740	8.1741	8.3399	4.0976
[0.95, 0.975)	2.5771	4.9082	7.8612	7.1124	3.5809
[0.975, 0.99)	2.2076	5.1821	8.0972	10.8393	4.5090
[0.99, 1.01)	1.9780	4.6562	7.4242	10.2990	4.2289
[1.01, 1.025)	1.3321	3.4276	6.5819	7.9609	2.7291
[1.025, 1.05)	0.8976	2.1381	5.9988	7.2971	2.4018
[1.05, 1.1)	0.4375	1.2001	2.7084	4.3806	1.7222
[1.1, 1.2)	0.2064	0.5381	1.2265	2.1098	1.1818
ALL	1.5200	3.3213	5.5566	7.6907	3.1842

ARPE(%)

	DTM				
M_o	[1, 40)	[40, 80)	[80, 180)	[180, 250)	All
[0.8, 0.9)	0.4858	1.7765	4.4684	5.2484	2.1117
[0.9, 0.95)	1.8251	4.9953	7.5079	6.7174	3.1037
[0.95, 0.975)	4.4126	9.0222	11.7450	7.1009	5.3861
[0.975, 0.99)	6.4905	11.8116	13.0844	13.0897	8.2102
[0.99, 1.01)	11.5459	13.1534	14.4850	13.6429	12.2557
[1.01, 1.025)	26.1400	13.1540	15.4158	11.2022	23.2272
[1.025, 1.05)	36.8332	13.2881	16.4198	10.9011	30.3939
[1.05, 1.1)	50.7227	26.5909	15.0274	13.6153	36.4284
[1.1, 1.2)	79.4396	52.0792	30.7366	15.7724	45.2850
ALL	26.2082	18.1957	17.2249	12.8983	23.0401

APE(%)

	DTM				
M_o	[1, 40)	[40, 80)	[80, 180)	[180, 250)	All
[0.8, 0.9)	0.4079	1.7765	4.4065	5.2187	2.0418
[0.9, 0.95)	1.8322	4.9352	7.6137	6.6630	3.3521
[0.95, 0.975)	4.4251	9.0090	11.5842	7.1232	5.6147
[0.975, 0.99)	6.9189	11.8494	12.9464	13.1842	9.5749
[0.99, 1.01)	9.8426	13.3741	14.6102	13.9289	12.3438
[1.01, 1.025)	15.4431	13.8683	15.6420	11.7933	14.6653
[1.025, 1.05)	21.7156	13.2969	17.5735	12.0537	17.0851
[1.05, 1.1)	37.6700	20.1426	13.3358	13.7163	16.2863
[1.1, 1.2)	74.4278	47.6722	19.6513	14.2061	18.6612
ALL	7.4153	12.5474	13.5905	11.8188	10.3098

Table A.4: Pricing errors for ARMA-TGARCH (Call option contracts of S&P 500 Index, January 07, 2004 to December 29, 2004)

RMSE($)

	DTM				
M_o	[1, 40)	[40, 80)	[80, 180)	[180, 250)	All
[0.8, 0.9)	1.0960	0.9779	5.6512	3.3088	3.0269
[0.9, 0.95)	1.6021	2.3704	4.6837	3.5858	2.3685
[0.95, 0.975)	2.1399	3.4772	5.4545	2.9082	2.5855
[0.975, 0.99)	1.6484	4.0286	4.5099	6.7989	3.0137
[0.99, 1.01)	1.5624	3.0766	4.8323	6.5450	2.8342
[1.01, 1.025)	1.1418	2.4781	4.1490	5.6314	1.9433
[1.025, 1.05)	0.8220	1.7187	4.0945	4.8382	1.7390
[1.05, 1.1)	0.4450	1.0254	1.9750	3.4606	1.3510
[1.1, 1.2)	0.2405	0.3087	0.8217	1.5650	0.8419
ALL	1.2405	2.3517	3.6256	4.9053	2.1841

ARPE(%)

	DTM				
M_o	[1, 40)	[40, 80)	[80, 180)	[180, 250)	All
[0.8, 0.9)	0.5008	0.5500	3.1981	2.0959	1.3102
[0.9, 0.95)	1.6858	2.5780	4.3719	2.8535	2.1273
[0.95, 0.975)	3.5895	6.3493	8.0270	2.9135	4.0810
[0.975, 0.99)	5.2563	8.8858	6.9240	8.0779	6.1248
[0.99, 1.01)	11.5473	8.2919	9.1480	8.3205	10.5260
[1.01, 1.025)	26.4107	9.9252	9.9359	7.8913	22.5266
[1.025, 1.05)	37.0729	11.6595	11.5481	8.2267	29.7469
[1.05, 1.1)	55.1185	20.9512	10.0482	10.6623	36.3031
[1.1, 1.2)	88.6285	40.4153	17.2261	10.3229	39.7415
ALL	27.0169	13.8811	10.8571	8.6247	21.8342

APE(%)

	DTM				
M_o	[1, 40)	[40, 80)	[80, 180)	[180, 250)	All
[0.8, 0.9)	0.4217	0.5500	3.1291	2.0786	1.2567
[0.9, 0.95)	1.6870	2.5790	4.3992	2.8141	2.2054
[0.95, 0.975)	3.5651	6.3398	7.8382	2.9224	4.1095
[0.975, 0.99)	5.3309	8.8831	6.7566	8.1733	6.6282
[0.99, 1.01)	8.4922	8.4360	9.2094	8.4901	8.6275
[1.01, 1.025)	14.4503	10.0631	9.8711	8.2378	11.5285
[1.025, 1.05)	20.7102	10.8787	12.1967	8.4660	13.7788
[1.05, 1.1)	39.8058	16.9409	9.6744	10.6798	13.2871
[1.1, 1.2)	85.6602	27.4838	11.5247	10.3388	12.4016
ALL	6.5403	8.8028	8.7764	7.4578	7.5660

Table A.5: Pricing errors for z-GARCH-M (Call option contracts of S&P 500 Index, January 07, 2004 to December 29, 2004)

RMSE($)					
	DTM				
M_o	[1, 40)	[40, 80)	[80, 180)	[180, 250)	All
$[0.8, 0.9)$	1.2733	2.9181	7.7296	8.9470	5.0767
$[0.9, 0.95)$	1.7620	4.6667	8.2538	9.5460	4.3679
$[0.95, 0.975)$	2.7051	5.6428	9.1717	8.0840	3.9779
$[0.975, 0.99)$	2.3447	5.7826	9.3526	12.3731	5.0774
$[0.99, 1.01)$	2.2278	5.4231	8.9720	12.5118	5.0291
$[1.01, 1.025)$	1.5075	4.1087	7.9999	10.1180	3.2968
$[1.025, 1.05)$	0.9758	2.6148	7.4176	10.1629	2.9772
$[1.05, 1.1)$	0.4292	1.3986	3.5557	6.3322	2.3240
$[1.1, 1.2)$	0.2140	0.3574	1.2905	1.9977	1.1630
ALL	1.6671	3.8606	6.6725	9.3976	3.7732

ARPE(%)					
	DTM				
M_o	[1, 40)	[40, 80)	[80, 180)	[180, 250)	All
$[0.8, 0.9)$	0.5019	1.6412	4.5257	5.8216	2.2054
$[0.9, 0.95)$	1.8209	5.3363	7.7605	7.7822	3.2550
$[0.95, 0.975)$	4.6785	10.3383	13.7013	8.4232	5.8927
$[0.975, 0.99)$	6.9505	13.3422	15.2015	15.4781	9.0811
$[0.99, 1.01)$	12.3111	16.1032	18.0072	17.3795	13.8661
$[1.01, 1.025)$	25.2252	17.4743	19.5552	16.0870	23.5318
$[1.025, 1.05)$	33.3068	16.5510	21.7429	15.0083	29.0362
$[1.05, 1.1)$	48.6474	24.6514	17.9811	20.5130	35.8056
$[1.1, 1.2)$	80.2685	37.6756	23.9359	11.9474	39.8010
ALL	25.1868	19.1150	19.1655	15.9334	22.9206

APE(%)					
	DTM				
M_o	[1, 40)	[40, 80)	[80, 180)	[180, 250)	All
$[0.8, 0.9)$	0.4120	1.6412	4.4169	5.7949	2.1117
$[0.9, 0.95)$	1.8311	5.2688	7.7978	7.7202	3.5349
$[0.95, 0.975)$	4.7004	10.2767	13.6032	8.4399	6.1929
$[0.975, 0.99)$	7.4629	13.4202	15.0362	15.5466	10.7899
$[0.99, 1.01)$	11.4622	16.2502	18.0844	17.6493	14.9853
$[1.01, 1.025)$	16.8837	18.0402	19.7126	16.5913	17.8131
$[1.025, 1.05)$	22.2798	17.2313	22.7209	17.0196	20.6974
$[1.05, 1.1)$	36.5630	22.2486	18.0252	20.7872	20.9247
$[1.1, 1.2)$	76.5828	30.1392	18.0221	11.1322	15.8201
ALL	8.1103	15.0978	16.7313	15.0440	12.2332

Table A.6: Pricing errors for z-TGARCH-M (Call option contracts of S&P 500 Index, January 07, 2004 to December 29, 2004)

RMSE($)

	DTM				
M_o	[1, 40)	[40, 80)	[80, 180)	[180, 250)	All
[0.8, 0.9)	0.9057	0.2736	3.6327	2.2239	2.0021
[0.9, 0.95)	1.4878	1.8279	3.9280	2.6884	2.0036
[0.95, 0.975)	2.1177	3.4098	4.2285	2.0405	2.4275
[0.975, 0.99)	1.7016	4.0308	3.7233	5.2836	2.7322
[0.99, 1.01)	1.7347	3.3220	4.4935	4.8089	2.7063
[1.01, 1.025)	1.2638	2.9133	3.8576	3.7626	1.9418
[1.025, 1.05)	0.9022	2.0826	4.1623	4.1924	1.8286
[1.05, 1.1)	0.4690	1.1790	2.1725	2.8535	1.3333
[1.1, 1.2)	0.2565	0.3748	0.9728	1.4342	0.8690
ALL	1.3308	2.5610	3.4239	3.7237	2.1005

ARPE(%)

	DTM				
M_o	[1, 40)	[40, 80)	[80, 180)	[180, 250)	All
[0.8, 0.9)	0.3731	0.1539	2.0237	1.3704	0.8536
[0.9, 0.95)	1.5298	1.8220	3.7311	2.2473	1.8285
[0.95, 0.975)	3.5459	6.3451	5.6327	2.0490	3.9127
[0.975, 0.99)	5.4326	9.0294	5.3672	6.5122	6.0578
[0.99, 1.01)	11.4842	9.4684	8.4261	6.1284	10.5123
[1.01, 1.025)	23.9254	12.4501	9.2314	5.7110	20.8782
[1.025, 1.05)	36.8133	14.5954	12.4740	7.8842	30.1510
[1.05, 1.1)	57.5045	24.3945	11.4365	9.0042	38.5150
[1.1, 1.2)	90.3567	41.5300	18.7290	9.2563	40.8007
ALL	26.8577	16.0464	11.2595	7.0532	22.0830

APE(%)

	DTM				
M_o	[1, 40)	[40, 80)	[80, 180)	[180, 250)	All
[0.8, 0.9)	0.3090	0.1539	2.0030	1.3814	0.8182
[0.9, 0.95)	1.5177	1.8282	3.7043	2.1833	1.8678
[0.95, 0.975)	3.5201	6.3530	5.3667	2.0423	3.8754
[0.975, 0.99)	5.5141	8.8958	5.1308	6.5635	6.2741
[0.99, 1.01)	9.4586	9.4833	8.2933	6.2158	8.7255
[1.01, 1.025)	15.7286	12.4719	8.9239	5.8219	12.1361
[1.025, 1.05)	22.5525	14.1953	12.5707	7.6784	15.2289
[1.05, 1.1)	41.6962	20.3281	10.3755	8.8010	13.7029
[1.1, 1.2)	89.1718	32.1803	12.8442	9.1105	12.6346
ALL	6.9875	10.0748	8.2195	5.7921	7.6545

Table A.7: Pricing errors for z-ARMA-GARCH (Call option contracts of S&P 500 Index, January 07, 2004 to December 29, 2004)

RMSE($)					
	DTM				
M_o	[1, 40)	[40, 80)	[80, 180)	[180, 250)	All
[0.8, 0.9)	1.0283	5.5016	9.0168	4.0519	4.7470
[0.9, 0.95)	1.6296	3.9990	6.0394	5.8314	3.1557
[0.95, 0.975)	2.3764	4.9892	7.9898	7.3866	3.5178
[0.975, 0.99)	2.2580	5.7316	8.6561	11.6608	4.8327
[0.99, 1.01)	2.1156	5.1435	8.4430	10.4990	4.5836
[1.01, 1.025)	1.4547	3.9975	7.1314	8.0165	2.9742
[1.025, 1.05)	0.9454	2.6682	7.1767	7.0770	2.8013
[1.05, 1.1)	0.4131	1.3889	3.5093	5.7323	2.1960
[1.1, 1.2)	0.2067	0.4136	1.2631	2.0915	1.1824
ALL	1.5800	3.7259	6.2761	7.9245	3.4568
ARPE(%)					
	DTM				
M_o	[1, 40)	[40, 80)	[80, 180)	[180, 250)	All
[0.8, 0.9)	0.4417	3.0943	5.2375	2.4096	1.9400
[0.9, 0.95)	1.6660	4.5668	5.0061	4.5349	2.5134
[0.95, 0.975)	3.8913	9.0960	11.9485	6.7891	4.9758
[0.975, 0.99)	6.5866	13.1776	13.7952	13.5307	8.5656
[0.99, 1.01)	12.0216	15.2233	16.4982	13.8148	13.1677
[1.01, 1.025)	23.6926	16.8369	17.0709	10.7557	21.9733
[1.025, 1.05)	32.6078	17.2105	20.4275	10.3472	28.4539
[1.05, 1.1)	47.0349	23.1990	17.8926	17.2894	34.3562
[1.1, 1.2)	79.0819	38.9752	23.4323	15.1511	40.0513
ALL	24.3286	18.5645	18.1214	13.5901	22.0189
APE(%)					
	DTM				
M_o	[1, 40)	[40, 80)	[80, 180)	[180, 250)	All
[0.8, 0.9)	0.3883	3.0943	5.1355	2.4353	1.9295
[0.9, 0.95)	1.6586	4.4951	5.2208	4.4525	2.6563
[0.95, 0.975)	3.9062	9.0862	11.8181	6.8218	5.2307
[0.975, 0.99)	7.0499	13.2401	13.5892	13.6981	10.0650
[0.99, 1.01)	11.0677	15.2899	16.5824	14.0259	13.7246
[1.01, 1.025)	16.6921	17.3638	17.2014	11.3591	16.3617
[1.025, 1.05)	21.7182	17.7638	21.6204	11.7732	19.8476
[1.05, 1.1)	34.3684	22.1099	17.7906	17.4625	19.5418
[1.1, 1.2)	70.8091	33.5796	19.0982	14.1859	17.8530
ALL	7.7306	14.5896	15.5168	11.9393	11.2563

Table A.8: Pricing errors for z-ARMA-TGARCH (Call option contracts of S&P 500 Index, January 07, 2004 to December 29, 2004)

RMSE($)

	DTM				
M_o	[1, 40)	[40, 80)	[80, 180)	[180, 250)	All
$[0.8, 0.9)$	1.2216	1.8333	4.4691	2.2578	2.4700
$[0.9, 0.95)$	1.5630	2.0302	3.9978	2.6170	2.0660
$[0.95, 0.975)$	2.1739	3.5881	5.5407	1.8617	2.5947
$[0.975, 0.99)$	1.7276	4.2974	3.6481	5.9140	2.8887
$[0.99, 1.01)$	1.7662	3.5108	4.7350	5.6908	2.8910
$[1.01, 1.025)$	1.2967	3.0062	4.3388	3.5297	2.0359
$[1.025, 1.05)$	0.9130	2.1626	4.4431	4.8944	1.9356
$[1.05, 1.1)$	0.4745	1.2120	2.2767	3.1811	1.4187
$[1.1, 1.2)$	0.2486	0.3859	1.1207	1.4506	0.9390
ALL	1.3597	2.6924	3.6664	4.2049	2.2325

ARPE(%)

	DTM				
M_o	[1, 40)	[40, 80)	[80, 180)	[180, 250)	All
$[0.8, 0.9)$	0.4921	1.0311	2.5096	1.4641	1.1018
$[0.9, 0.95)$	1.6146	2.0991	3.8079	2.1516	1.9137
$[0.95, 0.975)$	3.6547	6.5588	8.1664	1.8305	4.1200
$[0.975, 0.99)$	5.4168	9.5385	5.4010	7.1303	6.1717
$[0.99, 1.01)$	11.6003	9.8715	8.8278	7.0380	10.7532
$[1.01, 1.025)$	24.5490	12.8232	10.4520	5.2593	21.4828
$[1.025, 1.05)$	36.1945	15.1134	13.2965	8.9610	29.8962
$[1.05, 1.1)$	56.8514	24.2899	12.5552	9.8994	38.3861
$[1.1, 1.2)$	88.5052	42.3168	20.0454	9.8790	41.0325
ALL	26.7404	16.3540	12.1439	7.6936	22.1975

APE(%)

	DTM				
M_o	[1, 40)	[40, 80)	[80, 180)	[180, 250)	All
$[0.8, 0.9)$	0.4114	1.0311	2.4022	1.4565	1.0428
$[0.9, 0.95)$	1.6163	2.1016	3.8041	2.0835	1.9575
$[0.95, 0.975)$	3.6273	6.5583	7.9796	1.8236	4.1083
$[0.975, 0.99)$	5.5933	9.4594	5.1951	7.2074	6.5343
$[0.99, 1.01)$	9.5138	9.9522	8.7648	7.1619	9.1070
$[1.01, 1.025)$	16.1040	12.9248	10.2006	5.4284	12.6550
$[1.025, 1.05)$	22.7993	14.7483	13.3155	8.8203	15.8236
$[1.05, 1.1)$	42.2447	20.8840	11.2881	9.7659	14.5424
$[1.1, 1.2)$	86.4436	32.2568	15.5499	8.8738	13.7605
ALL	7.1064	10.5548	8.9211	6.3245	8.0226

Curriculum Vitae

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