1986

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Tom Nik Markovich

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**LA THÈSE A ÉTÉ
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A TALBERIAN THEOREM CONCERNING BOREL-TYPE
AND CESÀRO METHODS OF SUMMABILITY

by

Tom Nik Markovich

Department of Mathematics

Submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy

Faculty of Graduate Studies
The University of Western Ontario

London, Ontario

July, 1986

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ISBN 0-315-33024-4

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ABSTRACT

Let $r > 0$, $\alpha > 0$, $\alpha N_0 + \beta > 0$ where N_0 is a non-negative integer, and let $s_n = \sum_{k=0}^n a_k$. The Borel-type summability method (B, α, β) is defined as follows: $s_n \rightarrow l(B, \alpha, \beta)$ or $\sum_{n=0}^{\infty} a_n = l(B, \alpha, \beta)$

$$\text{if } ae^{-x} \sum_{n=N_0}^{\infty} s_n \frac{x^{\alpha n + \beta - 1}}{\Gamma(\alpha n + \beta)} \rightarrow l \text{ as } x \rightarrow \infty.$$

The special case $(B, 1, 1)$ is the ordinary Borel exponential method B .

The main result of this thesis is the following Tauberian theorem for summability (B, α, β) :

If $\{s_n\}$ is a real sequence that satisfies the slowly decreasing condition

$$(T_r): \lim_{\delta \rightarrow 0+} \liminf_{n \rightarrow \infty} \min_{n \leq m \leq n + \delta \sqrt{n}} \frac{s_m - s_n}{n^r} > 0,$$

and $s_n \rightarrow l(B, \alpha, \beta)$, then $s_n \rightarrow l(C_{2r})$.

Here C_{2r} denotes the Cesàro method of order $2r$.

In particular, if $a_n = O_L(n^{r-1/2})$, $n = 1, 2, \dots$, then (T_r) holds and thus the result extends a theorem due to Koebe.

It is shown, by a Vijayaraghavan type theorem, that (T_r) and $s_n \rightarrow l(B, \alpha, \beta)$ together imply that $s_n = O(n^r)$. Next a known method of summability V_α is considered. This is a special case of the Valiron method and it is shown that summability (B, α, β) is equivalent to summability V_α provided that $s_n = O(n^r)$. The desired Tauberian theorem is then deduced from established results.

ACKNOWLEDGEMENT

I would like to thank Prof. D. Borwein for his advice and academic guidance during my years at Western.

TABLE OF CONTENTS

	Page
CERTIFICATE OF EXAMINATION	11
ABSTRACT	111
ACKNOWLEDGEMENTS	1v
TABLE OF CONTENTS	v
CHAPTER 0: INTRODUCTION	1
CHAPTER 1: PRELIMINARY LEMMAS AND KNOWN RESULTS	13
CHAPTER 2: A VIJAYARAGHAVAN TYPE THEOREM	21
CHAPTER 3: AN EQUIVALENCE THEOREM FOR THE (B, α, β) AND V_α METHODS. 3	48
CHAPTER 4: A TAUBERIAN THEOREM FOR (B, α, β) SUMMABILITY	86
REFERENCES	89
VITA	92

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CHAPTER 0

INTRODUCTION

(1) Definitions and discussion.

Let $\{s_n\}$ be a sequence of complex numbers. If a given method of summability P assigns a "limit" l to the sequence $\{s_n\}$ we say that it is limitable P to l and write $s_n \rightarrow l(P)$. In the case of a series $\sum_{n=0}^{\infty} a_n$ of complex terms, if P assigns a "sum" l to $\sum_{n=0}^{\infty} a_n$ we say that it is summable P to l and write $\sum_{n=0}^{\infty} a_n = l(P)$.

The method P is said to be regular if $s_n \rightarrow l$ as $n \rightarrow \infty$ implies $s_n \rightarrow l(P)$. Necessary and sufficient conditions for regularity of a wide class of summability methods can be found in Hardy ([12], Chapter 3).

Assume that we now have two methods of summability P and Q . We say that Q includes P if $s_n \rightarrow l(P) \Rightarrow s_n \rightarrow l(Q)$, and that P and Q are equivalent if $s_n \rightarrow l(P) \Leftrightarrow s_n \rightarrow l(Q)$.

Suppose l, a_n ($n = 0, 1, 2, \dots$) are arbitrary complex numbers, and that $\alpha > 0, x > 0$. Furthermore, suppose $\beta \in \mathbb{R}$ and N_0 is a non-negative integer such that $\alpha N_0 + \beta > 0$. Let $s_n = \sum_{r=0}^n a_r$, $s_{-1} = 0$, $l_{N_0} = l - s_{N_0-1}$ and define Borel-type sums $s_{\alpha, \beta}$ and $S_{\alpha, \beta}$ by

$$s_{\alpha, \beta}(x) = \sum_{n=N_0}^{\infty} a_n \frac{x^{\alpha n + \beta - 1}}{\Gamma(\alpha n + \beta)} ; \quad S_{\alpha, \beta}(x) = \sum_{n=N_0}^{\infty} s_n \frac{x^{\alpha n + \beta - 1}}{\Gamma(\alpha n + \beta)}$$

Borel-type means are defined by

$$A_{\alpha, \beta}(x) = \int_0^x e^{-t} s_{\alpha, \beta}(t) dt ; \quad S_{\alpha, \beta}(x) = \alpha e^{-x} s_{\alpha, \beta}(x)$$

Borel-type methods are defined as follows:

(A) Summability:

$$(I) \quad s_n \rightarrow l(B', \alpha, \beta) \text{ if } A_{\alpha, \beta}(x) \rightarrow l_{N_0} \text{ as } x \rightarrow \infty.$$

$$(II) \quad s_n \rightarrow l(B, \alpha, \beta) \text{ if } S_{\alpha, \beta}(x) \rightarrow l \text{ as } x \rightarrow \infty.$$

The summability methods $(B, 1, 1)$ and $(B', 1, 1)$ are the ordinary Borel exponential and Borel integral methods (see Hardy [12]). The (B, α, β) and (B', α, β) methods were introduced by Borwein in [4]. A further discussion of these methods can also be found in [5], [6], and [7].

The Cesàro method is probably the most well-known summability method. For $p > -1$, the sequence $\{s_n\}$ is limitable in the Cesàro sense, written $s_n \rightarrow l(C_p)$, if

$$C_n^{(p)} = \frac{S_n^{(p)}}{A_n^{(p)}} \rightarrow l \text{ as } n \rightarrow \infty, \text{ where } A_n^{(p)} = \binom{n+p}{n} \text{ and}$$

$$S_n^{(p)} = \sum_{v=0}^n A_{n-v}^{(p-1)} s_v. \quad C_n^{(p)} \text{ is called the } p\text{-th Cesàro mean of } \{s_n\}.$$

A discussion of these methods can be found in Hardy [12].

The main result of this thesis is a theorem which states that $s_n \rightarrow l(C_{2r})$ if $s_n \rightarrow l(B, \alpha, \beta)$ and $\{s_n\}$ satisfies a condition (called a Tauberian condition) involving a fixed non-negative real number r .

In general neither summability (B, α, β) nor summability C_p includes

the other. We now give examples which show that neither of these inclusions is true in general. Consider the divergent series ([12], p. 213)

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} n^{a-1} \exp(A n^a) \quad \text{where } A > 0 \text{ and } 0 < a < 1/2. \quad \text{It is clear}$$

that $a_n = O(n^{-1/2})$. By the Borwein and Robinson Tauberian theorem [7] the series $\sum_{n=1}^{\infty} a_n$ is not summable (B, α, β) for any α and β . But by a

result due to Hardy and Littlewood ([12], Theorem 84 with $\beta = 1-a$) the series $\sum_{n=1}^{\infty} a_n$ is summable C_p for every $p > 0$. This example shows

that the (B, α, β) method does not include the C_p method. On the other hand, let $a > 1$ and consider the series

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} (-a)^n. \quad \text{It is known (see Hardy [12], p. 80) that}$$

$$\sum_{n=0}^{\infty} (-a)^n = \frac{1}{1+a} (B). \quad \text{Now for any non-negative integer } M > -\beta/\alpha, \text{ the}$$

$$\sum_{n=M}^{\infty} \frac{(-a)^n}{\Gamma(\alpha n + \beta)} x^n \text{ converges for all } x, \text{ and hence by a result due to}$$

$$\text{Borwein ([5], Result (I)) } \sum_{n=0}^{\infty} (-a)^n = \frac{1}{1+a} (B, \alpha, \beta) \text{ where } 1 > \alpha > 0$$

and β is real. However, since $a_n = (-a)^n$ is not $O(n^p)$ for any p it

follows from the limitation theorem for Cesàro summability (see Hardy

[12], Theorem 46) that $\sum_{n=0}^{\infty} (-a)^n$ is not summable C_p for any p . Thus, the C_p method does not include the (B, α, β) method.

Let r be any fixed non-negative number, and let $\{s_n\}$ be a sequence of real numbers. We consider the following (Tauberian) condition

$$(T_r): \quad \lim_{\delta \rightarrow 0+} \liminf_{n \rightarrow \infty} \min_{n \leq m \leq n + \delta \sqrt{n}} \frac{s_m - s_n}{n^r} > 0.$$

Clearly if $\{s_n\}$ is an increasing sequence then the condition (T_r) is satisfied. We will discuss the condition (T_r) at greater length in Chapter 2, and in particular we show there, that if

$s_n - s_{n-1} > -Kn^{r-1/2}$, $n = 1, 2, \dots$ (K is a constant) then (T_r) holds.

A condition of the form (T_r) is often called a slowly decreasing condition (see [12] or [19]).

The main result of this thesis is the following Tauberian theorem: If $s_n \rightarrow l(B, \alpha, \beta)$ and (T_r) holds, then $s_n \rightarrow l(C_{2r})$.

This is Theorem (4.2) of the thesis; see the end of this introduction for the method of numbering.

We shall also be concerned with the circle family of summability methods consisting of the Euler, Borel, Meyer-König, Taylor and Valiron methods.

(i) For $\alpha > 0$, the sequence $\{s_n\}$ is said to converge to l in the sense of the Valiron method of summability with parameter α if

$$\sqrt{\frac{\alpha}{2\pi n}} \sum_{i=0}^{\infty} s_i \exp\left\{-\frac{\alpha}{2} \frac{(1-n)^2}{n}\right\} \rightarrow l \text{ as } n \rightarrow \infty, \quad n = 1, 2, \dots, \text{ we then}$$

write $s_n \rightarrow l(V_\alpha)$.

The Borel exponential method B has already been mentioned but for completeness we shall define it here.

$$(ii) \quad s_n \rightarrow l(B) \text{ means } \sum_{i=0}^{\infty} s_i \frac{e^{-x} x^i}{i!} \rightarrow l \text{ as } x \rightarrow \infty.$$

(iii) The Euler method E_α , $\alpha \in (0, 1)$, for which

$$s_n \rightarrow l(E_\alpha) \text{ means } \sum_{i=0}^n \binom{n}{i} \alpha^i (1-\alpha)^{n-i} s_i \rightarrow l \text{ as } n \rightarrow \infty.$$

(iv) The Meyer-König method S_α , $\alpha \in (0, 1)$, for which

$$s_n \rightarrow l(S_\alpha) \text{ means } (1-\alpha)^{n+1} \sum_{i=0}^{\infty} \binom{n+1}{i} \alpha^i s_i \rightarrow l \text{ as } n \rightarrow \infty.$$

(v) The Taylor method T_α , $\alpha \in (0, 1)$, for which

$$s_n \rightarrow l(T_\alpha) \text{ means } (1-\alpha)^{n+1} \sum_{i=0}^{\infty} \binom{n+1}{i} \alpha^i s_{n+1} \rightarrow l \text{ as } n \rightarrow \infty.$$

For comparison purposes, Hardy ([12], Chapter 9), defines summability (e, c) , $c > 0$, for which $s_n \rightarrow l(e, c)$ means

$$\left(\frac{c}{\pi n}\right)^{1/2} \sum_{h=-n}^{\infty} s_{n+h} \exp\left(-c \frac{h^2}{n}\right) \rightarrow l \text{ as } n \rightarrow \infty.$$

Therefore the V_α method of summability is the $(e, \alpha/2)$ method of summability.

We now state some theorems and show how our main result is obtained. The first theorem which follows is Theorem (2.15) of this thesis, the second is Theorem (3.21), and the others are known.

Theorem 1: If $b(x) = ae^{-x} \sum_{n=N_0}^{\infty} s_n \frac{x^{an+\beta-1}}{\Gamma(an+\beta)} = O(x^r)$ as $x \rightarrow \infty$ and (T_r) , then $s_n = O(n^r)$.

Theorem 2: If $s_n = O(n^r)$ then $s_n \rightarrow l(B, \alpha, \beta) \Leftrightarrow s_n \rightarrow l(V_\alpha)$.

Theorem 3: (Faulhaber [10] or Bingham [2]) If $s_n = O(n^r)$ and $0 < \alpha < 1$, then $s_n \rightarrow l(S_\alpha) \Leftrightarrow s_n \rightarrow l(V_{1-\alpha})$.

Theorem 4: (Sitaraman [18]) If $s_n \rightarrow l(S_\alpha)$ and (T_r) holds, then $s_n \rightarrow l(C_{2r})$.

Theorem 5: (Borwein [5]) If $\alpha > \gamma > 0$, and for any non-negative integer $M > -\delta/\gamma$,

$\sum_{n=M}^{\infty} \frac{a_n}{\Gamma(\gamma n + \delta)} x^n$ is convergent for all x , then

$s_n \rightarrow l(B, \alpha, \beta) \Leftrightarrow s_n \rightarrow l(B, \gamma, \delta)$.

The proof of our main result is as follows:

$s_n \rightarrow \ell(B, \alpha, \beta)$ implies that $b(x) = O(x^r)$ as $x \rightarrow \infty$ and hence, by Theorem 1, that $s_n = O(n^r)$. Theorem 2 now shows that $s_n \rightarrow \ell(V_\alpha)$ while Theorem 5 shows that there is no loss in generality in making the restriction $0 < \alpha < 1$. It then follows, by Theorem 3, that $s_n \rightarrow \ell(S_{1-\alpha})$ and hence, by Theorem 4, that $s_n \rightarrow \ell(C_{2r})$.

The special case $\alpha - \beta = 1$, $r = 0$ of Theorem (4.2) with (T_0) replaced by $a_n = O(n^{-1/2})$ is the original O-Tauberian theorem for Borel summability due to Hardy and Littlewood [13]. The Borel summability case $\alpha - \beta = 1$, $r > 0$ of Theorem (4.2) has been proved by Rajagopal [17], and the corresponding theorem with summability (B, α, β) replaced by summability S_α is due to Sitaraman [18]. More recently Bingham [3] proved the theorem with summability (B, α, β) replaced by summability methods of random walk-type (see below for definitions). As for the general (B, α, β) method, the case $r > 0$ of Theorem (4.2) with (T_r) replaced by $a_n = o(n^{r-1/2})$ is due to Borwein [6], and the case $r = 0$ with (T_0) replaced by $a_n = O(n^{-1/2})$ is due to Borwein and Robinson [7]. The most general result to-date for the (B, α, β) method is due to Kwee [15] who proved the case of Theorem (4.2) with (T_r) replaced by $a_n = O(n^{r-1/2})$.

We now define summability methods of the random walk-type. Let P be a probability law with mean μ and variance σ^2 . Let $X, X_1, X_2, \dots$ be a sequence of independent and identically distributed random variables

with law P , $S_n = \sum_{k=1}^n X_k$, $S_0 = 0$ (the sequence of random variables $\{S_n\}$ is called a random walk generated by P). Bingham [2], [3], writes

$$s_n \rightarrow \ell(P) \text{ if } \sum_{k=0}^{\infty} s_k P(S_n = k) \rightarrow \ell \text{ as } n \rightarrow \infty.$$

This resulting summability method P is called a random walk method

Of the five types of circle methods, the Euler and Meyer-König methods are of random walk-type, the Borel and Taylor methods are 'nearly' so,

while the Valiron method V_α with $\alpha = \mu/\sigma^2$ is linked with the

P -methods (see [2]). The weights in these P -methods, $P(S_n = k)$, are the n -fold convolutions of the probability laws on the integers.

Specifically, to calculate $P(S_n = k)$ it follows by a result given in

Feller ([11], p. 215) that if $X_1, \dots, X_n$ are non-negative integer-valued independent random variables with generating functions $G_1, \dots, G_n$

$$\left(\text{i.e. } G_j(s) = \sum_{k=0}^{\infty} s^k P(X_j = k) \text{ on some interval } (-s_0, s_0) \right)$$

$$\text{then } S_n = \sum_{k=1}^n X_k \text{ has generating function } \prod_{j=1}^n G_j.$$

$$\left(\text{i.e. } \prod_{j=1}^n G_j(s) = \sum_{k=0}^{\infty} s^k P(S_n = k) \text{ on } (-s_0, s_0) \right)$$

Furthermore, if we assume that $X_1, \dots, X_n$ have a common distribution

so that $G_1 = \dots = G_n = G$, then

$$G^n(s) = \sum_{k=0}^{\infty} s^k P(S_n = k).$$

As an example, let P be the Bernoulli law with parameter $\alpha \in (0,1)$.

Then $P(X=1) = \alpha$, $P(X=0) = 1-\alpha$, and the generating function G is given

by $G(s) = 1 - \alpha + \alpha s$. Since $G^n(s) = \sum_{k=0}^n \binom{n}{k} \alpha^k (1-\alpha)^{n-k} s^k$ it

follows that $P(S_n=k) = \binom{n}{k} \alpha^k (1-\alpha)^{n-k}$. Thus with the Bernoulli law P

$s_n \rightarrow \ell(P)$ is $s_n \rightarrow \ell(E_\alpha)$.

In the case where P is the geometric law with parameter $\alpha \in (0,1)$

we have that $P(X=k) = (1-\alpha)\alpha^k$, $k = 0,1,\dots$; and hence that

$P(S_n=k) = \binom{n+k-1}{k} \alpha^k (1-\alpha)^n$ (see [11], p. 217). It follows that

$s_n \rightarrow \ell(P)$ is $(1-\alpha)^n \sum_{k=0}^{\infty} \binom{n+k-1}{k} \alpha^k s_k \rightarrow \ell$ as $n \rightarrow \infty$, and this is

equivalent to $s_n \rightarrow \ell(S_\alpha)$.

The (B, α, β) method of summability is in fact equivalent to a special case of a very general type of summability method, usually known as the J method or integral function method.

Let $J(z) = \sum_{n=0}^{\infty} j_n z^n$, where $j_n > 0$ for $n > 0$, and J is an entire function but not a polynomial. Then $s_n \rightarrow \ell(J)$ means $\sum_{n=0}^{\infty} s_n j_n x^n \sim J(x)$ as $x \rightarrow \infty$. Since $\sum_{n=N_0}^{\infty} \frac{x^{\alpha n + \beta - 1}}{\Gamma(\alpha n + \beta)} \sim \alpha^{-1} e^x$ as $x \rightarrow \infty$ (see Borwein [5], p. 130,

with $\delta = \beta - 1$) it follows that the (B, α, β) method is equivalent to the J method with

$$j_n = \begin{cases} \frac{1}{\Gamma(\alpha n + \beta)} & , n > N_0 \\ 0 & , 0 \leq n \leq N_0 \end{cases}$$

Then $x^{\beta-1} \sum_{n=0}^{\infty} J_n x^{\alpha n} \sim \alpha^{-1} e^x$ as $x \rightarrow \infty$. Letting $y = x^\alpha$ we obtain

$$y^{\frac{\beta-1}{\alpha}} \sum_{n=0}^{\infty} J_n y^n \sim \alpha^{-1} e^{y^{\frac{1}{\alpha}}} \text{ as } y \rightarrow \infty, \text{ i.e. } x^{\frac{\beta-1}{\alpha}} J(x) \sim \alpha^{-1} e^{x^{\frac{1}{\alpha}}} \text{ as } x \rightarrow \infty.$$

Hence:

$$\frac{1}{J(x)} \sum_{n=0}^{\infty} s_n J_n x^n \sim \alpha e^{-x^{\frac{1}{\alpha}}} x^{\frac{\beta-1}{\alpha}} \sum_{n=0}^{\infty} s_n J_n x^n = \alpha e^{-u} \sum_{n=N_0}^{\infty} s_n \frac{u^{\alpha n + \beta - 1}}{\Gamma(\alpha n + \beta)}$$

where $u = x^{1/\alpha}$. Since the J methods are regular (see Hardy [12], Theorem 33) it follows that summability (B, α, β) is also regular.

Borwein ([5], III and IV) has proved the following result:

Let h be an analytic function of $z = x+iy$ in the region $x > x_0$, and suppose that

(i) when $x > x_0$ and $|z|$ is large

$$h(z) = z^{\alpha z + \beta} e^{\gamma z} \left\{ C + O\left(\frac{1}{|z|}\right) \right\}, \text{ where } C > 0, \alpha > 0, \beta, \gamma \in \mathbb{R} \text{ and}$$

(ii) $h(x)$ is real for $x > x_0$,

$$\text{Let } J(z) = \sum_{n=N_0}^{\infty} \frac{z^n}{h(n)}.$$

Then $s_n \rightarrow t(J) \iff s_n \rightarrow t(B, \alpha, \beta + 1/2)$.

In particular, taking $h(z) = \{\Gamma(\alpha z + b)\}^c (z+p)^{qz+r}$ where $b, c, p, q, r \in \mathbb{R}$, $\alpha > 0$ and $\alpha c + q > 0$ so that

$$J(z) = \sum_{n=N_0}^{\infty} \frac{z^n}{\{\Gamma(\alpha n + b)\}^c (n+p)^{qn+r}}$$

$$s_n \rightarrow l(J) \Leftrightarrow s_n \rightarrow l(B, ac+q, bc+r-c/2+1/2).$$

It follows that Theorem (4.2) is in fact true for quite a wide class of summability methods. Theorem (4.2) remains true if $s_n \rightarrow l(B, \alpha, \beta)$ is replaced by $s_n \rightarrow l(B', \alpha, \beta)$. This is a consequence of the following known result, Borwein ([4] Theorem 2):

$$s_n \rightarrow l(B, \alpha, \beta+1) \Leftrightarrow s_n \rightarrow l(B', \alpha, \beta).$$

(B) Strong summability with index $p > 0$.

$$(i) \quad s_n \rightarrow l(B', \alpha, \beta)_p \quad \text{if} \quad \int_0^x e^t |A_{\alpha, \beta-1}(t) - l_{N_0}|^p dt = o(e^x) \quad \text{as} \quad x \rightarrow \infty.$$

$$(ii) \quad s_n \rightarrow l(B, \alpha, \beta)_p \quad \text{if} \quad \int_0^x e^t |S_{\alpha, \beta-1}(t) - l|^p dt = o(e^x) \quad \text{as} \quad x \rightarrow \infty.$$

These strong methods of summability were introduced by Borwein and Shawyer in [9]. Various theorems linking strong summability with summability (B, α, β) are known (see [8], and [9]). For example in ([8], Theorem 3) $s_n \rightarrow l[B, \alpha, \beta]_1 \Leftrightarrow s_n \rightarrow l(B, \alpha, \beta)$ and so strong summability $[B, \alpha, \beta]_1$ can replace the hypothesis of summability (B, α, β) in Theorem (4.2).

(2) Method of Numbering

The thesis contains this introduction and four chapters. Theorems, corollaries, lemmas, remarks, and equations to which we will later refer are numbered consecutively within each chapter and placed in the margin for ease of reference. Thus in chapter three,

our first result is listed as (3.1) ~~Lemma~~ and would be referred to as (3.1) or ~~lemma~~ (3.1).

We close this introductory chapter with the following note concerning notation.

Note: Whenever the symbol $a/b(\cdot)$ appears it always means $\frac{a}{b}(\cdot)$.

For example $1/2(j+1)$ means $\frac{1}{2}(j+1)$.

CHAPTER 1

PRELIMINARY LEMMAS AND KNOWN RESULTS

Our aim in this chapter is to state certain results which are needed in later chapters.

(1.1) **Lemma:** If $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has radius of convergence $R > 0$,

then for $|x| < r < R$, $f(x) = \sum_{n=0}^m a_n x^n + O(|x|^{m+1})$.

Proof: $\left| f(x) - \sum_{n=0}^m a_n x^n \right| \leq \sum_{n=m+1}^{\infty} |a_n| |x|^n = |x|^{m+1} \sum_{n=m+1}^{\infty} |a_n| |x|^{n-m-1}$

$(|x|^{m+1} \sum_{n=m+1}^{\infty} |a_n| r^{n-m-1}) = M_r |x|^{m+1}$, since the series converges.

The lemma follows. ///

(1.2) **Remark:** It is well known that for $|x| < 1$,

$$\log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n \quad \text{and} \quad (1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n.$$

Hence by Lemma (1.1)

$$\log(1+x) = \sum_{n=1}^m \frac{(-1)^{n-1}}{n} x^n + O(|x|^{m+1}) \quad \text{and}$$

$$(1+x)^k = \sum_{n=0}^m \binom{k}{n} x^n + O(|x|^{m+1}), \quad \text{for } |x| < r < 1.$$

(1.3) Lemma: Let ϕ be a function which decreases on $[n_0, n_1]$. Then

$$\sum_{n=n_0}^{n_1} \phi(n) \leq \int_{n_0}^{n_1} \phi(t) dt + \phi(n_0).$$

Proof: Since ϕ is decreasing on $[n_0, n_1]$ we have:

$$\int_{n-1}^n \phi(t) dt \geq \phi(n), \quad n = n_0 + 1, n_0 + 2, \dots, n_1.$$

$$\text{Thus, } \int_{n_0}^{n_1} \phi(t) dt = \sum_{n=n_0+1}^{n_1} \int_{n-1}^n \phi(t) dt \geq \sum_{n=n_0+1}^{n_1} \phi(n) = \sum_{n=n_0}^{n_1} \phi(n) - \phi(n_0).$$

The result follows. ///

(1.4) Lemma: Let $\phi(t) \geq 0$ on $[n_0, n_1]$, increasing on $[n_0, x_0]$ and decreasing on $[x_0, n_1]$. Then for $n_1 \geq x_0 + 1$,

$$\sum_{n=n_0}^{n_1} \phi(n) \leq \int_{n_0}^{n_1} \phi(t) dt + \phi(x_0).$$

Proof: Let s_0 denote the integer part of x_0 . If $n_0 < s_0 < x_0 < s_0 + 1 < n_1$ (the case $n_0 = s_0$ is immediate), then

$$\sum_{n=n_0}^{s_0-1} \phi(n) \leq \sum_{n=n_0}^{s_0-1} \int_n^{n+1} \phi(t) dt = \int_{n_0}^{s_0} \phi(t) dt,$$

$$\sum_{n=s_0+2}^{n_1} \phi(n) = \sum_{n=s_0+1}^{n_1-1} \phi(n+1) \leq \sum_{n=s_0+1}^{n_1-1} \int_n^{n+1} \phi(t) dt = \int_{s_0+1}^{n_1} \phi(t) dt$$

Thus,

$$(1.5) \quad \sum_{n=n_0}^{n_1} \phi(n) = \sum_{n=n_0}^{s_0-1} \phi(n) + \phi(s_0) + \phi(s_0+1) + \sum_{n=s_0+2}^{n_1} \phi(n)$$

$$= \int_{n_0}^{s_0-1} \phi(t) dt + \phi(s_0) + \phi(s_0+1) + \int_{s_0+2}^{n_1} \phi(t) dt$$

Now consider the two cases: (i) $\phi(s_0) \leq \phi(s_0+1)$ and (ii) $\phi(s_0) > \phi(s_0+1)$.

If (i) holds, then $\phi(s_0) \leq \phi(t)$ for all $t \in [s_0, s_0+1]$, so that

$$\phi(s_0) \leq \int_{s_0}^{s_0+1} \phi(t) dt \quad \text{and, since } x_0 \text{ is a maximum point } \phi(s_0+1) \leq \phi(x_0).$$

$$\text{Thus, } \phi(s_0) + \phi(s_0+1) \leq \int_{s_0}^{s_0+1} \phi(t) dt + \phi(x_0).$$

If (ii) holds, then $\phi(s_0+1) \leq \int_{s_0}^{s_0+1} \phi(t) dt$ and, since x_0 is a maximum point $\phi(s_0) \leq \phi(x_0)$.

$$\text{Thus, } \phi(s_0) + \phi(s_0+1) \leq \int_{s_0}^{s_0+1} \phi(t) dt + \phi(x_0).$$

So in case (i) or (ii), $\phi(s_0) + \phi(s_0+1) \leq \int_{s_0}^{s_0+1} \phi(t) dt + \phi(x_0)$.

The result now follows from (1.5). ///

Let us now fix some notation which will be used throughout this thesis. Set $h_n = n - x/\alpha$ and $k_n = -h_n$.

The next few lemmas will be used to obtain estimates (when $x \rightarrow \infty$)

for sums of the form:

$$\sum_{n=n_0}^{n_1} h_n^c \exp\left(-\frac{a}{x} h_n^2\right), \quad \sum_{n=n_0}^{n_1} k_n^c \exp\left(-\frac{a}{x} k_n^2\right), \quad \text{where } a > 0 \text{ and } c > 0.$$

(1.6) Lemma: Let f be a decreasing function on $[h_{n_0}, h_{n_1}]$. Then,

$$\sum_{n=n_0}^{n_1} f(h_n) \leq f(h_{n_0}) + \int_{h_{n_0}}^{h_{n_1}} f(y) dy.$$

The result may be obtained directly as in Lemma (1.3), however, we choose to deduce the result.

Proof: Define ϕ by $\phi(t) = f(t - x/a)$, so that $f(h_n) = \phi(n)$. Now $f(t)$ decreasing on $[h_{n_0}, h_{n_1}] = [n_0 - x/a, n_1 - x/a]$ is equivalent to $\phi(t)$ decreasing on $[n_0, n_1]$.

Thus, by Lemma (1.3),

$$\begin{aligned} \sum_{n=n_0}^{n_1} f(h_n) &= \sum_{n=n_0}^{n_1} \phi(n) \leq \int_{n_0}^{n_1} \phi(t) dt + \phi(n_0) \\ &= \int_{h_{n_0}}^{h_{n_1}} \phi(y + x/a) dy + \phi(n_0), \text{ where } t = y + x/a \\ &= \int_{h_{n_0}}^{h_{n_1}} f(y) dy + f(h_{n_0}). \quad /// \end{aligned}$$

(1.7) Lemma: Let f be a decreasing function on $[k_{n_1}, k_{n_0}]$. Then,

$$\sum_{n=n_0}^{n_1} f(k_n) \leq f(k_{n_1}) + \int_{k_{n_1}}^{k_{n_0}} f(y) dy.$$

Proof: Define g by $g(y) = f(-y)$. The function g satisfies all the

hypotheses of Lemma (1.6). Thus ,

$$\sum_{n=n_0}^{n_1} f(k_n) = \sum_{n=n_0}^{n_1} g(h_n) \leq g(h_{n_0}) + \int_{h_{n_0}}^{h_{n_1}} g(y) dy = f(k_{n_0}) + \int_{k_{n_1}}^{k_{n_0}} f(y) dy \\ \leq f(k_{n_1}) + \int_{k_{n_1}}^{k_{n_0}} f(y) dy . \quad ///$$

In the next lemma, x_0 is not necessarily an integer, however, we write $h_{x_0} = x_0 - x/\alpha$ simply to denote the point where $f(y)$ attains its maximum. Similarly, in Lemma(1.9), k_{x_0} denotes the point where $f(y)$ attains its maximum.

(1.8) Lemma: Let $f(y) > 0$ on $[h_{n_0}, h_{n_1}]$, increasing on $[h_{n_0}, h_{x_0}]$ and decreasing on $[h_{x_0}, h_{n_1}]$. Then

$$\sum_{n=n_0}^{n_1} f(h_n) \leq \int_{h_{n_0}}^{h_{n_1}} f(y) dy + f(h_{x_0}).$$

Proof: Define ϕ by $\phi(t) = f(t - x/\alpha)$, so that $f(h_n) = \phi(n)$. $f(t)$ increasing on $[h_{n_0}, h_{x_0}]$ is equivalent to $\phi(t)$ increasing on $[n_0, x_0]$ and $f(t)$ decreasing on $[h_{x_0}, h_{n_1}]$ is equivalent to $\phi(t)$ decreasing on $[x_0, n_1]$. Moreover, $f(t) > 0$ on $[h_{n_0}, h_{n_1}]$ is equivalent to $\phi(t) > 0$ on $[n_0, n_1]$.

Thus, $\phi(t)$ satisfies all the hypotheses of Lemma(1.4), and so

$$\sum_{n=n_0}^{n_1} f(h_n) = \sum_{n=n_0}^{n_1} \phi(n) \leq \int_{n_0}^{n_1} \phi(t) dt + \phi(x_0)$$

$$= \int_{h_{n_0}}^{h_{n_1}} \phi(y+x/a) dy + \phi(x_0), \text{ where } t = y + x/a.$$

$$= \int_{h_{n_0}}^{h_{n_1}} f(y) dy + f(h_{x_0}). \quad ///$$

(1.9) Lemma: Let $f(y) \geq 0$ on $[k_{n_1}, k_{n_0}]$, increasing on $[k_{n_1}, k_{x_0}]$ and decreasing on $[k_{x_0}, k_{n_0}]$. Then

$$\sum_{n=n_0}^{n_1} f(k_n) \leq \int_{k_{n_1}}^{k_{n_0}} f(y) dy + f(k_{x_0}).$$

Proof: This is, of course, just a relabelling of Lemma(1.8). ///

(1.10) Lemma: For $a > 0$, $\mu = 1, 2, \dots$

$$\left(\frac{a}{2\pi\mu}\right)^{1/2} \sum_{n=0}^{\infty} \exp\left\{-\frac{a}{2} \left(\frac{n-\mu}{\mu}\right)^2\right\} = 1 + O(1/\sqrt{\mu}).$$

Proof: Write $S(\mu) = \sum_{n=0}^{\infty} \exp\left\{-\frac{a}{2} \left(\frac{n-\mu}{\mu}\right)^2\right\} = \sum_{n=-\mu}^{\infty} \exp\left\{-\frac{a}{2} \frac{n^2}{\mu}\right\}$

$$= 2 \sum_{n=1}^{\infty} \exp\left\{-\frac{a}{2} \frac{n^2}{\mu}\right\} - \sum_{n=\mu+1}^{\infty} \exp\left\{-\frac{a}{2} \frac{n^2}{\mu}\right\} + 1$$

$$= 2 S_{\mu} - T_{\mu} + 1.$$

Since $\exp\left\{-\frac{\alpha}{2} \frac{n^2}{\mu}\right\} \leq \int_{n-1}^n \exp\left\{-\frac{\alpha}{2} \frac{t^2}{\mu}\right\} dt \leq \exp\left\{-\frac{\alpha}{2} \frac{(n-1)^2}{\mu}\right\}$, we have:

$$\sum_{n=1}^{\infty} \exp\left\{-\frac{\alpha}{2} \frac{n^2}{\mu}\right\} \leq \sum_{n=1}^{\infty} \int_{n-1}^n \exp\left\{-\frac{\alpha}{2} \frac{t^2}{\mu}\right\} dt \leq \sum_{n=1}^{\infty} \exp\left\{-\frac{\alpha}{2} \frac{(n-1)^2}{\mu}\right\}$$

$$\text{i.e. } S_{\mu} \leq \int_0^{\infty} \exp\left\{-\frac{\alpha}{2} \frac{t^2}{\mu}\right\} dt \leq 1 + S_{\mu}$$

Letting $\left\{\frac{\alpha}{2\mu}\right\}^{1/2} t = \sqrt{x}$ in the above integral we have:

$$S_{\mu} \leq \left\{\frac{\mu}{2\alpha}\right\}^{1/2} \int_0^{\infty} e^{-x} x^{-1/2} dx = \left\{\frac{\pi\mu}{2\alpha}\right\}^{1/2} \leq 1 + S_{\mu}$$

$$\text{Hence, } S_{\mu} = \left\{\frac{\pi\mu}{2\alpha}\right\}^{1/2} + O(1).$$

From the proof of Lemma(1.3) with $n_0 = \mu$ and $\phi(t) = \exp\left\{-\frac{\alpha}{2} \frac{t^2}{\mu}\right\}$ we obtain:

$$\sum_{n=\mu+1}^{n_1} \exp\left\{-\frac{\alpha}{2} \frac{n^2}{\mu}\right\} \leq \int_{\mu}^{n_1} \exp\left\{-\frac{\alpha}{2} \frac{t^2}{\mu}\right\} dt \leq \int_{\mu}^{\infty} \exp\left\{-\frac{\alpha}{2} \frac{t^2}{\mu}\right\} dt.$$

It follows that $0 \leq T_{\mu} \leq \int_{\mu}^{\infty} \exp\left\{-\frac{\alpha}{2} \frac{t^2}{\mu}\right\} dt$ and hence,

$$0 \leq T_{\mu} \leq \frac{1}{\alpha\mu/2} \int_0^{\infty} e^{-x} dx = o(1) \text{ as } \mu \rightarrow \infty.$$

Thus, $S(\mu) = \left\{\frac{2\pi\mu}{\alpha}\right\}^{1/2} + O(1)$, i.e. $\left\{\frac{\alpha}{2\pi\mu}\right\}^{1/2} S(\mu) = 1 + O(1/\sqrt{\mu})$, which is the required result. ///

(1.11) Theorem: The V_{α} method is regular.

Proof: We must show that $s_n \rightarrow l$, $n \rightarrow \infty$ implies $s_n \rightarrow l(V_{\alpha})$.

Given any $\epsilon > 0$, there is a positive integer N such that $|s_n - l| < \epsilon$

for all $n > M$.

Next,

$$|S(\mu)| = \left\{ \left\{ \frac{a}{2\pi\mu} \right\}^{1/2} \sum_{n=0}^{\infty} (s_n - \ell) \exp \left\{ -\frac{a}{2} \left(\frac{n-\mu}{\mu} \right)^2 \right\} \right\} \\
= \left\{ \left\{ \frac{a}{2\pi\mu} \right\}^{1/2} \left\{ \left(\sum_{n=0}^M + \sum_{n=M+1}^{\infty} \right) (s_n - \ell) \exp \left\{ -\frac{a}{2} \left(\frac{n-\mu}{\mu} \right)^2 \right\} \right\} \right\} \\
= S_1 + S_2.$$

$$\text{If } A = \max_{0 \leq n \leq M} |s_n - \ell|, \text{ then clearly } S_1 \leq A \left\{ \frac{a}{2\pi\mu} \right\}^{1/2} \sum_{n=0}^M \exp \left\{ -\frac{a}{2} \left(\frac{n-\mu}{\mu} \right)^2 \right\} \\
= o(1) \text{ as } \mu \rightarrow \infty.$$

While in S_2 where $n > M+1$, $|s_n - \ell| < \epsilon$ so that

$$S_2 < \epsilon \left\{ \frac{a}{2\pi\mu} \right\}^{1/2} \sum_{n=M+1}^{\infty} \exp \left\{ -\frac{a}{2} \left(\frac{n-\mu}{\mu} \right)^2 \right\} < \epsilon \left\{ \frac{a}{2\pi\mu} \right\}^{1/2} \sum_{n=0}^{\infty} \exp \left\{ -\frac{a}{2} \left(\frac{n-\mu}{\mu} \right)^2 \right\} \\
= \epsilon(1+o(1)) \text{ as } \mu \rightarrow \infty, \text{ by}$$

Lemma(1.10).

Thus, $|S(\mu)| \leq o(1) + \epsilon(1+o(1)) = \epsilon + o(1)$ as $\mu \rightarrow \infty$, and

$$0 \leq \liminf_{\mu \rightarrow \infty} |S(\mu)| \leq \limsup_{\mu \rightarrow \infty} |S(\mu)| \leq \limsup_{\mu \rightarrow \infty} (\epsilon + o(1)) = \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we have that $S(\mu) = o(1)$ as $\mu \rightarrow \infty$.

Therefore, by Lemma(1.10),

$$\left\{ \frac{a}{2\pi\mu} \right\}^{1/2} \sum_{n=0}^{\infty} s_n \exp \left\{ -\frac{a}{2} \left(\frac{n-\mu}{\mu} \right)^2 \right\} =$$

$$S(\mu) + \ell \left\{ \frac{a}{2\pi\mu} \right\}^{1/2} \sum_{n=0}^{\infty} \exp \left\{ -\frac{a}{2} \left(\frac{n-\mu}{\mu} \right)^2 \right\} \rightarrow \ell \text{ as } \mu \rightarrow \infty. \quad ///$$

CHAPTER 2

A VIJAYARAGHAVAN-TYPE THEOREM

In this chapter we show that the conditions (T_r) and

$$ae^{-x} \sum_{n=N_0}^{\infty} s_n \frac{x^{an+\beta-1}}{\Gamma(an+\beta)} = O(x^r) \text{ as } x \rightarrow \infty$$

together imply that $s_n = O(n^r)$ where $r > 0$. The method of proof is that of Vijayaraghavan [19]. In particular we follow Sitaraman ([18],

Lemma 5 and Theorem 1) who deals with the S_a method of summability.

Since $\frac{ae^{-x} s_n x^{an+\beta-1}}{\Gamma(an+\beta)} \rightarrow 0$ as $x \rightarrow \infty$, it is clear that the actual choice of N_0 is immaterial. We therefore will assume that N_0 satisfies the condition: $N_0 + \beta - r > 0$, where $r > 0$ is some fixed number.

We also let $c_n(x) := \frac{ae^{-x} x^{an+\beta-1}}{\Gamma(an+\beta)}$, $n \geq N_0$ and $b(x) := \sum_{n=N_0}^{\infty} s_n c_n(x)$.

Note that $c_n(x) > 0$ for $n \geq N_0$ and $x > 0$, and that $\sum_{n=N_0}^{\infty} c_n(x) \sim 1$ as $x \rightarrow \infty$ (see Borwein [5], p. 130 with $\delta = \beta - 1$).

The next lemma is well known and will be used extensively.

(2.1) **Lemma** (Borwein and Robinson [7]). Let $x > 0$, $h_n = n - x/a$,

$1/2 < \xi < 2/3$, and $0 < \eta < 2\xi - 1$. Then

$$e^{-x} \sum_{|h_n| > x^\xi} \frac{x^{an+\beta-1}}{\Gamma(an+\beta)} = O(e^{-x^\eta}), \text{ and}$$

$$\begin{aligned} \frac{ae^{-x} x^{an+\beta-1}}{\Gamma(an+\beta)} &= \frac{a}{\sqrt{2\pi x}} \exp\left(-\frac{a^2 h_n^2}{2x}\right) \left\{ 1 + O\left(\frac{|h_n| + 1}{x}\right) + O\left(\frac{|h_n|^3}{x^2}\right) \right\} \\ &= \frac{a}{\sqrt{2\pi x}} \exp\left(-\frac{a^2 h_n^2}{2x}\right) \left\{ 1 + O(x^{3\xi-2}) \right\}, \text{ if } |h_n| < x^\xi. \end{aligned}$$

(2.2) Lemma: $\frac{(an)^r}{\Gamma(an+\beta)} \sim \frac{1}{\Gamma(an+\beta-r)} \text{ as } n \rightarrow \infty.$

Proof: It follows from Stirling's formula that

$$\Gamma(an+\beta) = \sqrt{(2\pi)} e^{-an} (an)^{an+\beta-1/2} \left\{ 1 + O\left(\frac{1}{n}\right) \right\}.$$

(see [1], formula 3, 1.18, p. 47).

Thus $\frac{(an)^r \Gamma(an+\beta-r)}{\Gamma(an+\beta)} \rightarrow 1 \text{ as } n \rightarrow \infty$, and the lemma follows. ///

(2.3) Lemma: Let $1/2 < \xi < 2/3$. Then as $x \rightarrow \infty$

$$\sum_{N_0 \leq n < x/\alpha-x} n^r c_n(x) = o(1),$$

$$\sum_{n > x/\alpha+x} n^r c_n(x) = o(1).$$

Proof: It follows, from Lemma (2.2) and Lemma (2.1), that

$$\begin{aligned} \sum_{N_0 \leq n < x/\alpha-x} n^r c_n(x) &= O\left\{ x^r e^{-x} \sum_{N_0 \leq n < x/\alpha-x} \frac{x^{an+\beta-r-1}}{\Gamma(an+\beta-r)} \right\} \\ &= O\left\{ x^r e^{-x^\eta} \right\}, \quad 0 < \eta < 2\xi - 1 \\ &= o(1) \text{ as } x \rightarrow \infty. \end{aligned}$$

The second sum is dealt with similarly. ///

In the next lemma we will be using a slowly decreasing condition of the form :

$$\lim_{\delta \rightarrow 0+} \liminf_{u \rightarrow \infty} \min_{u \leq t < u+\delta\phi(u)} \frac{s(t) - s(u)}{u^r} > 0, \text{ where } \phi(u) > 0.$$

To see what this means in terms of $\epsilon > 0$ consider the following :

$$\text{let } m_\delta(u) = \min_{u \leq t < u+\delta\phi(u)} \frac{s(t) - s(u)}{u^r}, \text{ and } m_\delta = \liminf_{u \rightarrow \infty} m_\delta(u).$$

The slowly decreasing condition above says that

$m_\delta \rightarrow m > 0$ say, as $\delta \rightarrow 0^+$. Therefore for any $\epsilon > 0$, there exists a $\delta_0 = \delta_0(\epsilon) > 0$ such that $m_\delta - m > -\epsilon/2$ whenever $0 < \delta \leq \delta_0$.

In particular $m_{\delta_0} > -\epsilon/2$.

Since $m_{\delta_0} = \liminf_{u \rightarrow \infty} m_{\delta_0}(u)$, then there exists a $u_0 = u_0(\epsilon) > 0$ such that $m_{\delta_0}(u) > m_{\delta_0} - \epsilon/2$ when $u > u_0$, and hence $m_{\delta_0}(u) > -\epsilon$ when $u > u_0$.

Thus $\frac{s(t) - s(u)}{u^\Gamma} > -\epsilon$ when $u > u_0$, $u \leq t \leq u + \delta_0 \phi(u)$.

Summarizing, we see that the slowly decreasing condition means that

for any $\epsilon > 0$, there is a $\delta_0 > 0$ and a $u_0 > 0$ such that

$\frac{s(t) - s(u)}{u^\Gamma} > -\epsilon$ when $u > u_0$, and $u \leq t \leq u + \delta_0 \phi(u)$.

The first part of the next lemma is essentially due to Hardy [12].

As for the last part this is due to Rajagopal [17] who merely states the result. For the sake of completeness we shall give a proof.

(2.4) Lemma: ([12], Theorem 239 and [17], Lemma 1)

(i) If $\phi(u) > 0$ and differentiable for $u > 1$, $\phi(u) \rightarrow \infty$, $0 < \phi'(u) < C$, where C is independent of u ,

$\phi(u) = \int_1^u \frac{dt}{\phi(t)}$ and, if

(ii) $s(t) = s_n$, $n \leq t < n+1$, satisfies

$\lim_{\delta \rightarrow 0^+} \liminf_{u \rightarrow \infty} \min_{u \leq t \leq u + \delta \phi(u)} \frac{s(t) - s(u)}{u^\Gamma} > 0$,

then there are positive constants K and K' such that

$s(q) - s(p) > -Kq^\Gamma [\phi(q) - \phi(p)] - K'p^\Gamma$, for $q > p > 1$.

Furthermore, if

$\lim_{\delta \rightarrow 0^+} \liminf_{n \rightarrow \infty} \min_{n \leq t \leq n + \delta \phi(n)} \frac{s_n - s_n}{n^\Gamma} > 0$, then

$$s_q - s_p > -Kq^r [\phi(q) - \phi(p)] - K'p^r, \text{ for } q > p > 1.$$

In particular, taking $\phi(u) = 2\sqrt{u}$,

$$(T_r) : \lim_{\delta \rightarrow 0^+} \liminf_{n \rightarrow \infty} \inf_{n \leq m \leq n+\delta\sqrt{n}} \frac{s_m - s_n}{n^r} > 0$$

implies for $q > p > 1$,

$$\begin{aligned} s_q - s_p &> -Kq^r (q^{1/2} - p^{1/2}) - K'p^r \\ &> -K(q^{r+1/2} - p^{r+1/2}) - K'p^r. \end{aligned}$$

Proof: By condition (11), there exists a $\delta_0 > 0$, and a $u_0 > 0$ such that $\frac{s(t) - s(u)}{u^r} > -1$ when $u > u_0$, $u \leq t \leq u + \delta_0 \phi(u)$. If on the other hand $u < u_0$, $t \leq u + \delta_0 \phi(u_0)$, then $\frac{s(t) - s(u)}{u^r}$ has a lower bound depending only on u_0 .

Thus there are numbers γ and δ_0 such that $\frac{s(t) - s(u)}{u^r} > -\gamma$ for all t and u such that $u \leq t \leq u + \delta_0 \phi(u)$.

Now write $p_0 = p$, $p_1 = p_0 + \delta_0 \phi(p_0)$, ..., $p_{k+1} = p_k + \delta_0 \phi(p_k)$, ... (so that $1 \leq p = p_0 < p_1 < p_2 < \dots$ and $p_1 \rightarrow \infty$ as $1 \rightarrow \infty$, since

$$p_1 - p_0 = \sum_{k=0}^{1-1} (p_{k+1} - p_k) = \delta_0 \sum_{k=0}^{1-1} \phi(p_k) > \delta_0 \sum_{k=0}^{1-1} \phi(p_0) = \delta_0 \phi(p_0) 1 \}$$

and suppose that $p_1 \leq q < p_{1+1}$.

For each $j = 0, 1, \dots, 1-1$, $0 < p_{j+1} - p_j = \delta_0 \phi(p_j)$, and

$$0 < q - p_1 < p_{1+1} - p_1 = \delta_0 \phi(p_1).$$

Therefore,

$$\begin{aligned} s(q) - s(p) &= \sum_{j=0}^{1-1} (s(p_{j+1}) - s(p_j)) + s(q) - s(p_1) \\ &> \sum_{j=0}^{1-1} (-\gamma p_j^r) + (-\gamma p_1^r) = -\gamma p^r - \gamma \sum_{j=1}^1 p_j^r > -\gamma p^r - \gamma \sum_{j=1}^1 q^r. \end{aligned}$$

$$= -\gamma p^r - \gamma q^r 1.$$

By the mean value theorem,

$\phi(p_{j+1}) - \phi(p_j) = (p_{j+1} - p_j)\phi'(\xi)$ for some $\xi \in (p_j, p_{j+1})$. Hence

$$\begin{aligned}\phi(p_{j+1}) &= \phi(p_j) + (p_{j+1} - p_j)\phi'(\xi) = \phi(p_j) + \delta_0 \phi(p_j)\phi'(\xi) \\ &< \phi(p_j)(1 + \delta_0 C), \text{ by hypothesis (1).}\end{aligned}$$

$$\text{i.e. } \frac{1}{\phi(p_j)} < \frac{1 + \delta_0 C}{\phi(p_{j+1})}.$$

$$\text{Thus, } 1/\delta_0 = \sum_{j=0}^{1-1} \delta_0 \frac{\phi(p_j)}{\phi(p_j)} = \sum_{j=0}^{1-1} \frac{p_{j+1} - p_j}{\phi(p_j)} < (1 + \delta_0 C) \sum_{j=0}^{1-1} \frac{p_{j+1} - p_j}{\phi(p_{j+1})}.$$

$$< (1 + \delta_0 C) \int_{p_0}^{p_1} \frac{du}{\phi(u)} < (1 + \delta_0 C) \int_p^q \frac{du}{\phi(u)} = (1 + \delta_0 C) [\phi(q) - \phi(p)]$$

Hence, $s(q) - s(p) > -\gamma q^r 1 - \gamma p^r > -\gamma \left(\frac{1}{\delta_0} + C \right) q^r [\phi(q) - \phi(p)] - \gamma p^r$.

Setting $K = \gamma \left(\frac{1}{\delta_0} + C \right)$, and $K' = \gamma$, which are both positive, we see that the proof of the first part of the lemma is complete.

As for the latter part of the lemma, the hypothesis means that given any $\epsilon > 0$ there is a $\delta_0 > 0$ and an $n_0 > 0$ such that

$$\frac{s_m - s_n}{n^r} > -\epsilon \text{ when } n \geq n_0, \quad n \leq m \leq n + \delta_0 \phi(n).$$

Next, given t and u , $t > u$, let $m \leq t < m+1$, $n \leq u < n+1$, so that

$$s(t) = s_m, \text{ and } s(u) = s_n.$$

Choose $u_0 \geq n_0 + 1$ such that $\frac{1}{\phi(u_0 - 1)} < \frac{\delta_0}{2}$ and then a positive $\delta' < \delta_0$ such that $\delta' \left(1 + \frac{C\delta_0}{2} \right) < \frac{\delta_0}{2}$.

For $t > u \geq u_0$ and $\frac{t-u}{\phi(u)} < \delta'$ we have that $[t] \geq [u] \geq [u_0] \geq n_0$

$$\text{i.e. } m \geq n \geq n_0, \text{ and } \frac{m-n}{\phi(n)} < \frac{t-(u-1)}{\phi(u-1)} = \frac{t-u}{\phi(u-1)} + \frac{1}{\phi(u-1)}$$

$$< \frac{\delta' \phi(u)}{\phi(u-1)} + \frac{1}{\phi(u-1)} < \delta' \left(1 + \frac{C}{\phi(u-1)} \right) + \frac{1}{\phi(u-1)}, \text{ since by the mean}$$

value theorem and hypothesis (1), $0 < \phi(u) - \phi(u-1) = \phi'(\xi) < C$ for some

$\xi \in (u-1, u)$, and so

$$0 < \frac{\phi(u)}{\phi(u-1)} - 1 < \frac{C}{\phi(u-1)}.$$

$$\text{Hence } \frac{s_m - s_n}{n^r} < \delta' \left(1 + \frac{C}{\phi(u-1)} \right) + \frac{1}{\phi(u-1)} < \delta' \left(1 + \frac{C}{\phi(u_0-1)} \right) + \frac{1}{\phi(u_0-1)}$$

$$< \delta' \left(1 + \frac{C\delta_0}{2} \right) + \frac{\delta_0}{2} < \delta_0.$$

Thus we have $m > n > n_0$, $\frac{s_m - s_n}{\phi(n)} < \delta_0$, and it follows that

$$\frac{s_m - s_n}{n^r} > -\epsilon.$$

If we have $s(t) > s(u)$, then obviously $\frac{s(t) - s(u)}{u^r} > -\epsilon$; on the other

$$\text{hand if } s(t) < s(u), \text{ then } \frac{s(u) - s(t)}{u^r} = \frac{s_n - s_m \left(\frac{n}{u} \right)^r}{n^r} < \frac{s_n - s_m}{n^r} < \epsilon.$$

$$\text{Thus, } \frac{s(t) - s(u)}{u^r} > -\epsilon, \text{ when } t > u > u_0, \frac{t - u}{\phi(u)} < \delta'.$$

Finally, following the proof of the first part of this lemma we have:

$$s(q) - s(p) > -Kq^r [\phi(q) - \phi(p)] - K'p^r, \text{ for } q > p > 1, \text{ and where } K, K'$$

are positive constants. Specializing to integers p and q , so that

$s(p) = s_p$ and $s(q) = s_q$ we have as required:

$$s_q - s_p > -Kq^r [\phi(q) - \phi(p)] - K'p^r, \text{ for } q > p > 1.$$

When $\phi(u) = 2\sqrt{u}$, then $\phi(u) = \sqrt{u} - 1$, and so (T_r) implies for $q > p > 1$,

$$s_q - s_p > -Kq^r (q^{1/2} - p^{1/2}) - K'p^r, \text{ and hence}$$

$$\begin{aligned} s_q - s_p &> -Kq^{r+1/2} + Kq^r p^{1/2} - K'p^r > -Kq^{r+1/2} + Kp^{r+1/2} - K'p^r \\ &= -K(q^{r+1/2} - p^{r+1/2}) - K'p^r, \end{aligned}$$

where K, K' are positive constants. ///

(2.5) Remark: In particular if $a_n = s_n - s_{n-1} = O_L(n^{r-1/2})$, $r > 0$ (i.e. $a_n > -Kn^{r-1/2}$ for some $K > 0$, and $n = 1, 2, \dots$) then (T_r) holds.

To see this consider the following:

$$\begin{aligned} \frac{s_m - s_n}{n^r} &= \frac{1}{n^r} \sum_{j=n+1}^m a_j > -\frac{K}{n^r} \sum_{j=n+1}^m j^{r-1/2} > -\frac{K(m-n)}{\sqrt{(n+1)}} \left(\frac{m}{n}\right)^r \\ &> -\frac{K\delta\sqrt{n}}{\sqrt{(n+1)}} \left(1 + \frac{\delta}{\sqrt{n}}\right)^r, \text{ for } n \leq m \leq n + \delta\sqrt{n} \\ &> -K\delta \left(1 + \frac{\delta}{\sqrt{n}}\right)^r. \end{aligned}$$

$$\text{and } \min_{n \leq m \leq n + \delta\sqrt{n}} \frac{s_m - s_n}{n^r} > -K\delta \left(1 + \frac{\delta}{\sqrt{n}}\right)^r.$$

Thus,

$$\lim_{\delta \rightarrow 0+} \liminf_{n \rightarrow \infty} \min_{n \leq m \leq n + \delta\sqrt{n}} \frac{s_m - s_n}{n^r} > \lim_{\delta \rightarrow 0+} \liminf_{n \rightarrow \infty} \inf \left\{ -K\delta \left(1 + \frac{\delta}{\sqrt{n}}\right)^r \right\} = 0.$$

(2.6) Lemma: Let M and N be any positive integers such that

$M > x/\alpha + t\sqrt{(x/\alpha)}$, $N_0 < N < x/\alpha - t\sqrt{(x/\alpha)}$. Then for any $r > 0$, and $x, t \rightarrow \infty$

we have :

$$(2.7) \quad \sum_{n=N_0}^N n^r c_n(x) = o(x^r),$$

$$(2.8) \quad \sum_{n=M}^{\infty} n^r c_n(x) = o(x^r),$$

$$(2.9) \quad \sum_{n=N}^M n^r c_n(x) = (x/\alpha)^r,$$

$$(2.10) \quad \sum_{n=M}^{\infty} (n^{r+1/2} - M^{r+1/2}) c_n(x) = o(x^r).$$

{ For example definition (2.9) means, as the others similarly do, the following: for every $\epsilon > 0$ there exists an X_0 such that for every $x > X_0$, $t > X_0$, and for every M and N , satisfying $M > x/a + t\sqrt{(x/a)}$, $N_0 < N < x/a - t\sqrt{(x/a)}$ we have

$$\left| \frac{1}{x} \sum_{n=N}^M n^r c_n(x) - \frac{1}{a^r} \right| < \epsilon.$$

Proof: Regarding (2.7) we first show that $\sum_{n=N_0}^N c_n(x) = o(1)$ as $x, t \rightarrow \infty$, and from this we deduce the final result.

$$\begin{aligned} \sum_{n=N_0}^N c_n(x) &= \sum_{N_0 < n < x/a - t\sqrt{(x/a)}} c_n(x) \\ &= \left(\sum_{N_0 < n < x/a - x^\xi} + \sum_{x/a - x^\xi < n < x/a - t\sqrt{(x/a)}} \right) c_n(x) = S_1 + S_2, \end{aligned}$$

where $1/2 < \xi < 2/3$.

By Lemma (2.3) with $r=0$, $S_1 = o(1)$ as $x \rightarrow \infty$.

(Since $3\xi - 2 < 0$, Lemma (2.1) implies that

$c_n(x) = O\left\{x^{-1/2} \exp\left(-\frac{a^2 k_n^2}{2x}\right)\right\}$ for $|k_n| = |x/a - n| < x^\xi$. In particular this estimate is valid for n such that $x^\xi > x/a - n > t\sqrt{(x/a)}$.

Thus,

$$\begin{aligned} S_2 &= O\left\{x^{-1/2} \sum_{x/a - x^\xi < n < x/a - t\sqrt{(x/a)}} \exp\left(-\frac{a^2 k_n^2}{2x}\right)\right\} \\ &= O\left\{x^{-1/2} \sum_{n=n_0}^{n_1} \exp\left(-\frac{a^2 k_n^2}{2x}\right)\right\}, \text{ where } n_0 \text{ is the smallest} \end{aligned}$$

integer $\lceil x/a - x^\xi \rceil$, and n_1 is the largest integer $\lceil x/a - t\sqrt{x/a} \rceil$.

Let $f(y) = \exp\{-\frac{a^2 y^2}{2x}\}$. Then $f(y)$ decreases for all $y > 0$. For x and t non-negative and by our choice of n_1 the inequality:

$0 < t\sqrt{x/a} < x/a - n_1 = k_{n_1}$ is valid. Applying Lemma (1.7) we have:

$$\begin{aligned} \sum_{n=n_0}^{n_1} \exp\left(-\frac{a^2 k_n^2}{2x}\right) &\leq \exp\left(-\frac{a^2 k_{n_1}^2}{2x}\right) + \int_{k_{n_1}}^{k_{n_0}} \exp\left(-\frac{a^2 y^2}{2x}\right) dy \\ &\leq \exp\left\{-\frac{a^2}{2x} \left(t\sqrt{x/a}\right)^2\right\} + \int_{t\sqrt{x/a}}^{x^\xi} \exp\left(-\frac{a^2 y^2}{2x}\right) dy, \end{aligned}$$

since $x^\xi > x/a - n_0 = k_{n_0}$ and $t\sqrt{x/a} < x/a - n_1 = k_{n_1}$.

$$\begin{aligned} \text{It follows that } \sum_{n=n_0}^{n_1} \exp\left(-\frac{a^2 k_n^2}{2x}\right) &\leq \exp\left(-\frac{at^2}{2}\right) + \int_{t\sqrt{x/a}}^{x^\xi} \exp\left(-\frac{a^2 y^2}{2x}\right) dy \\ &= \exp\left(-\frac{at^2}{2}\right) + \frac{\sqrt{2x}}{a} \int_{\sqrt{(a/2)t}}^{x^\xi} \exp(-u^2) du. \end{aligned}$$

$$\text{Thus, } S_2 = O\left\{x^{-1/2} \exp\left(-\frac{at^2}{2}\right)\right\} + O\left\{\int_{\sqrt{(a/2)t}}^{x^\xi} \exp(-u^2) du\right\} = o(1) \text{ as } x, t \rightarrow \infty.$$

Finally, $\sum_{n=N_0}^N c_n(x) \leq S_1 + S_2 = o(1) + o(1) = o(1)$ as $x, t \rightarrow \infty$, and

$$\text{hence } \sum_{n=N_0}^N n^r c_n(x) \leq N^r \sum_{n=N_0}^N c_n(x) \leq (x/a)^r o(1) = o(x^r) \text{ as } x, t \rightarrow \infty.$$

For (2.8) we must show that $S := x^{-r} \sum_{n=N}^{\infty} n^r c_n(x) = o(1)$ as $x, t \rightarrow \infty$.

$$S = x^{-r} \left(\sum_{N \leq n \leq x/a + x^\xi} + \sum_{n > x/a + x^\xi} \right) n^r c_n(x) = S_1 + S_2, \quad 1/2 < \xi < 2/3.$$

By Lemma (2.3), $S_2 = o(1)$ as $x \rightarrow \infty$.

Next,

$$S_1 = O \left\{ x^{-r} e^{-x} \sum_{x/a+t\sqrt{(x/a)} < n \leq x/a+x^\xi} n^r \frac{x^{an+\beta-1}}{\Gamma(an+\beta)} \right\}, \text{ since } M > x/a+t\sqrt{(x/a)}$$

$$= O \left\{ e^{-x} \sum_{x/a+t\sqrt{(x/a)} < n \leq x/a+x^\xi} \frac{x^{an+\beta-r-1}}{\Gamma(an+\beta-r)} \right\}, \text{ by Lemma (2.2).}$$

Since $3\xi-2 < 0$, Lemma (2.1) implies that

$$\frac{e^{-x} x^{an+\beta-r-1}}{\Gamma(an+\beta-r)} = O \left\{ x^{-1/2} \exp \left(-\frac{a^2 h_n^2}{2x} \right) \right\} \text{ for } |h_n| = |n-x/a| \leq x^\xi. \text{ In}$$

particular this estimate is valid for n such that $t\sqrt{(x/a)} < n-x/a \leq x^\xi$.

$$\text{Therefore, } S_1 = O \left\{ x^{-1/2} \sum_{x/a+t\sqrt{(x/a)} < n \leq x/a+x^\xi} \exp \left(-\frac{a^2 h_n^2}{2x} \right) \right\}$$

$$= O \left\{ x^{-1/2} \sum_{n=n_0}^{n_1} \exp \left(-\frac{a^2 h_n^2}{2x} \right) \right\}, \text{ where } n_0 \text{ is the smallest}$$

integer $> x/a+t\sqrt{(x/a)}$, and n_1 is the largest integer $\leq x/a+x^\xi$.

Applying Lemma (1.6) with $f(y) = \exp \left(-\frac{a^2 y^2}{2x} \right)$ we find, as in the proof of (2.7), that $S_1 = o(1)$ as $x, t \rightarrow \infty$.

Thus, $0 \leq S \leq S_1 + S_2 = o(1) + o(1) = o(1)$ as $x, t \rightarrow \infty$.

(2.9) for $r=0$, obviously follows from (2.7) and (2.8) with $r=0$, and from the fact that $\sum_{n=N_0}^{\infty} c_n(x) \sim 1$ as $x \rightarrow \infty$. Before we prove the result for $r>0$ we make the following comments which will aid us in the proof.

First, (2.7) may be written in the following equivalent form :

for any sequences $\{N_1\}, \{t_1\}, \{x_1\}$ such that $t_1 \rightarrow \infty$, $x_1 \rightarrow \infty$,
and $N_0 < N_1(x_1/\alpha - t_1\sqrt{(x_1/\alpha)})$ then $\sum_{n=N_0}^{N_1} n^r c_n(x_1) = o(x_1^r)$ as $1 \rightarrow \infty$.

Secondly, (2.8) may be written in the following equivalent form :

for any sequences $\{M_1\}, \{t_1\}, \{x_1\}$ such that $t_1 \rightarrow \infty$, $x_1 \rightarrow \infty$,
and $M_1 > x_1/\alpha + t_1\sqrt{(x_1/\alpha)}$ then $\sum_{n=M_1}^{\infty} n^r c_n(x_1) = o(x_1^r)$ as $1 \rightarrow \infty$.

Lastly, (2.9) may be written in the following equivalent form :

for any sequences $\{M_1\}, \{N_1\}, \{t_1\}, \{x_1\}$ such that $t_1 \rightarrow \infty$,
 $x_1 \rightarrow \infty$, and $M_1 > x_1/\alpha + t_1\sqrt{(x_1/\alpha)}$, $N_0 < N_1(x_1/\alpha - t_1\sqrt{(x_1/\alpha)})$

then $\sum_{n=N_1}^{M_1} n^r c_n(x_1) \sim (x_1/\alpha)^r$ as $1 \rightarrow \infty$.

Now to prove (2.9) let us choose arbitrary sequences

$\{M_1\}, \{N_1\}, \{t_1\}, \{x_1\}$, $t_1 \rightarrow \infty$, $x_1 \rightarrow \infty$, $M_1 > x_1/\alpha + t_1\sqrt{(x_1/\alpha)}$ and
 $N_0 < N_1(x_1/\alpha - t_1\sqrt{(x_1/\alpha)})$.

Having established (2.7) and (2.8) we will use their equivalent forms
and prove that $\sum_{n=N_1}^{M_1} n^r c_n(x_1) \sim (x_1/\alpha)^r$ as $1 \rightarrow \infty$.

Now having already chosen $\{t_1\}, \{x_1\}$, $t_1 \rightarrow \infty$, $x_1 \rightarrow \infty$, choose
another sequence $\{w_1\}$ such that $0 < w_1 < t_1$, $w_1 \rightarrow \infty$ and $w_1/\sqrt{(x_1/\alpha)} \rightarrow 0$,
as $1 \rightarrow \infty$.

This is possible by letting $w_1 = \min\{(x_1/\alpha)^{1/4}, t_1\}$.

Having already chosen $\{M_1\}$ and $\{N_1\}$, choose sequences of
positive integers $\{M'_1\}$ and $\{N'_1\}$ such that

$$M'_1 - 1(x_1/\alpha + w_1\sqrt{(x_1/\alpha)}) < M'_1 < M_1$$

and $N_1 \leq N'_1 \leq x_1/\alpha - w_1\sqrt{(x_1/\alpha)} \leq N'_1 + 1$.

Then,

$$(2.11) \quad \sum_{n=N_1}^{M_1} n^r c_n(x_1) = \left\{ \sum_{n=N_1}^{N'_1-1} + \sum_{n=N'_1}^{M'_1} + \sum_{n=M'_1+1}^{M_1} \right\} n^r c_n(x_1)$$

(The first series on the right side of (2.11) is defined to be zero when $N'_1 = N_1$, as is the last series when $M'_1 = M_1$.)

Since $x_1 \rightarrow \infty$, and $w_1/\sqrt{(x_1/\alpha)} \rightarrow 0$ as $1 \rightarrow \infty$, we have :

$$(N'_1)^r \sim (x_1/\alpha)^r \text{ and } (M'_1)^r \sim (x_1/\alpha)^r \text{ as } 1 \rightarrow \infty.$$

Hence from the inequality :

$$(N'_1)^r \sum_{n=N'_1}^{M'_1} c_n(x_1) \leq \sum_{n=N'_1}^{M'_1} n^r c_n(x_1) \leq (M'_1)^r \sum_{n=N'_1}^{M'_1} c_n(x_1)$$

$$\text{it follows that } \sum_{n=N'_1}^{M'_1} n^r c_n(x_1) \sim (x_1/\alpha)^r \sum_{n=N'_1}^{M'_1} c_n(x_1)$$

$$= (x_1/\alpha)^r \left\{ \sum_{n=N_0}^{\infty} - \sum_{n=N_0}^{N'_1-1} - \sum_{n=M'_1+1}^{\infty} \right\} c_n(x_1) \text{ as } 1 \rightarrow \infty.$$

Since $w_1 \rightarrow \infty$, $x_1 \rightarrow \infty$ as $1 \rightarrow \infty$, and N'_1, M'_1 are respectively less than $x_1/\alpha - w_1\sqrt{(x_1/\alpha)}$ and greater than $x_1/\alpha + w_1\sqrt{(x_1/\alpha)}$, the results (2.7) and (2.8) in their equivalent forms hold, as observed at the outset of this proof.

Therefore,

$$0 < \sum_{n=N_0}^{N'_1-1} c_n(x_1) < \sum_{n=N_0}^{N'_1} c_n(x_1) = o(1) \text{ as } 1 \rightarrow \infty, \text{ by (2.7) with } r=0,$$

$$0 < \sum_{n=M'_1+1}^{\infty} c_n(x_1) < \sum_{n=M'_1}^{\infty} c_n(x_1) = o(1) \text{ as } 1 \rightarrow \infty, \text{ by (2.8) with } r=0,$$

$$\text{and moreover } \sum_{n=N_0}^{\infty} c_n(x_1) \sim 1 \text{ as } 1 \rightarrow \infty, \text{ since } \sum_{n=N_0}^{\infty} c_n(x) \sim 1$$

Hence,

$$(2.12) \quad \sum_{n=N'_1}^{M'_1} n^r c_n(x_1) \sim (x_1/\alpha)^r \left\{ \sum_{n=N_0}^{\infty} - \sum_{n=N_0}^{N'_1-1} - \sum_{n=M'_1+1}^{\infty} \right\} c_n(x_1) \sim (x_1/\alpha)^r,$$

as $1 \rightarrow \infty$.

$$(2.13) \quad 0 < \sum_{n=N_1}^{N'_1-1} n^r c_n(x_1) < \sum_{n=N_0}^{N'_1} n^r c_n(x_1) = o(x_1^r) \text{ as } 1 \rightarrow \infty, \text{ by (2.7)}$$

$$(2.14) \quad 0 < \sum_{n=M'_1+1}^{M'_1} n^r c_n(x_1) < \sum_{n=M'_1}^{\infty} n^r c_n(x_1) = o(x_1^r) \text{ as } 1 \rightarrow \infty, \text{ by (2.8)}$$

Substituting (2.13), (2.12), and (2.14) in (2.11) gives

$$\sum_{n=N_1}^{M_1} n^r c_n(x_1) \sim (x_1/\alpha)^r \text{ as } 1 \rightarrow \infty.$$

The final result now follows from the equivalence mentioned at the beginning of this proof.

For (2.10), we must show that

$$S := \sum_{n=M'}^{\infty} (n^{r+1/2} - M^{r+1/2}) c_n(x) = o(x^r) \text{ as } x, t \rightarrow \infty.$$

By the mean value theorem it follows that

$n^{r+1/2} - M^{r+1/2} + (2r+1)n^r(n^{1/2} - M^{1/2})$ and hence

$O(S) \leq (2r+1) \sum_{n=M}^{\infty} (n^{1/2} - M^{1/2}) n^r c_n(x)$. Therefore it suffices to prove the following

Claim: $S_1 = x^{-r} \sum_{n=M}^{\infty} (n^{1/2} - M^{1/2}) n^r c_n(x) = o(1)$ as $x, t \rightarrow \infty$.

Since $M > x/a$ it follows that $n^{1/2} + M^{1/2} > 2M^{1/2} > 2\sqrt{x/a}$ and
 $(n - M) = (n^{1/2} - M^{1/2})(n^{1/2} + M^{1/2}) > 2\sqrt{x/a}(n^{1/2} - M^{1/2})$.

Thus,

$$S_1 \leq \frac{a^{1/2} x^{-r-1/2}}{2} \sum_{n=M}^{\infty} (n-M) n^r c_n(x) \\ = \frac{a^{1/2} x^{-r-1/2}}{2} \left\{ \sum_{M \leq n \leq x/a+x} + \sum_{n > x/a+x} \right\} (n-M) n^r c_n(x)$$

$= S_{1,1} + S_{1,2}$, where $1/2 < \xi < 2/3$.

Now,

$$S_{1,1} = O \left\{ x^{-r-1/2} e^{-x} \sum_{M \leq n \leq x/a+x} (n-M) n^r \frac{x^{an+\beta-1}}{\Gamma(an+\beta)} \right\} \\ = O \left\{ x^{-1/2} e^{-x} \sum_{x/a+t\sqrt{x/a} < n \leq x/a+x} (n-x/a) n^r \frac{x^{an+\beta-r-1}}{\Gamma(an+\beta-r)} \right\}, \text{ since}$$

$M > x/a + t\sqrt{x/a} > x/a$, and $n-M < n-x/a$.

By Lemma (2.2),

$$S_{1,1} = O \left\{ x^{-1/2} e^{-x} \sum_{x/a+t\sqrt{x/a} < n \leq x/a+x} (n-x/a) \frac{x^{an+\beta-r-1}}{\Gamma(an+\beta-r)} \right\}$$

As $3\xi - 2 < 0$, Lemma (2.1) implies that

$$\frac{e^{-x} x^{an+\beta-r-1}}{\Gamma(an+\beta-r)} = O \left\{ x^{-1/2} \exp \left(-\frac{a^2 h_n^2}{2x} \right) \right\} \text{ for } |h_n| = |n-x/a| \leq x^\xi.$$

It follows that for n such that $t\sqrt{x/a} < n - x/a \leq x^\xi$

$$e^{-x} \frac{x^{an+\beta-r-1}}{\Gamma(an+\beta-r)} h_n = O\left\{x^{-1/2} h_n \exp\left(-\frac{a^2 h_n^2}{2x}\right)\right\}.$$

$$\begin{aligned} \text{Therefore, } S_{1,1} &= O\left\{x^{-1} \sum_{x/a+t\sqrt{x/a} < n \leq x/a+x^\xi} h_n \exp\left(-\frac{a^2 h_n^2}{2x}\right)\right\} \\ &= O\left\{x^{-1} \sum_{n=n_0}^{n_1} h_n \exp\left(-\frac{a^2 h_n^2}{2x}\right)\right\}, \text{ where } n_0 \text{ is the} \end{aligned}$$

smallest integer $> x/a+t\sqrt{x/a}$, and n_1 is the largest integer $\leq x/a+x^\xi$.

Let $f(y) = y \exp\left\{-\frac{a^2 y^2}{2x}\right\}$. Then $f(y)$ decreases for all $y \geq \sqrt{x/a}$. For $t > 1/\sqrt{a}$, and by our choice of n_0 the inequality $\sqrt{x/a} < t\sqrt{x/a} < n_0 - x/a = h_{n_0}$ is valid. By Lemma (1.6),

$$\begin{aligned} \sum_{n=n_0}^{n_1} h_n \exp\left(-\frac{a^2 h_n^2}{2x}\right) &\leq h_{n_0} \exp\left(-\frac{a^2 h_{n_0}^2}{2x}\right) + \int_{h_{n_0}}^{h_{n_1}} y \exp\left(-\frac{a^2 y^2}{2x}\right) dy \\ &\leq t\sqrt{x/a} \exp\left\{-\frac{a^2}{2x} \left\{t\sqrt{x/a}\right\}^2\right\} + \int_{t\sqrt{x/a}}^{x^\xi} y \exp\left(-\frac{a^2 y^2}{2x}\right) dy, \end{aligned}$$

since $x^\xi \geq n_1 - x/a = h_{n_1}$ and $t\sqrt{x/a} < n_0 - x/a = h_{n_0}$.

Further,

$$\sum_{n=n_0}^{n_1} h_n \exp\left(-\frac{a^2 h_n^2}{2x}\right) \leq t\sqrt{x/a} \exp\left\{-\frac{at^2}{2}\right\} + \int_{t\sqrt{x/a}}^{x^\xi} y \exp\left(-\frac{a^2 y^2}{2x}\right) dy$$

Therefore,

$$S_{1,1} = O\left\{x^{-1/2} t \exp\left(-\frac{at^2}{2}\right)\right\} + O\left\{x^{-1} \int_{t\sqrt{x/a}}^{x^\xi} y \exp\left(-\frac{a^2 y^2}{2x}\right) dy\right\}$$

$$= o(1) + O\left\{\int_{\sqrt{(a/2)t}}^{\infty} u \exp(-u^2) du\right\} = o(1) \text{ as } x, t \rightarrow \infty.$$

Finally,

$$\begin{aligned} S_{1,2} &= O\left\{x^{1/2} e^{-x} \sum_{n > x/a+x} \xi^{n^{r+1}} \frac{x^{an+\beta-r-2}}{\Gamma(an+\beta)}\right\} \\ &= O\left\{x^{1/2} e^{-x} \sum_{n > x/a+x} \xi \frac{x^{an+\beta-r-2}}{\Gamma(an+\beta-r-1)}\right\}, \text{ by Lemma (2.2)} \\ &= O\left\{x^{1/2} e^{-x^\eta}\right\}, \quad 0 < \eta < 2\xi-1, \text{ by Lemma (2.1)} \\ &= o(1) \text{ as } x \rightarrow \infty. \end{aligned}$$

Thus, $O(S_1)(S_{1,1} + S_{1,2}) = o(1) + o(1) = o(1)$ as $x, t \rightarrow \infty$.

This proves the claim, and as explained at the outset the proof of (2.10) is now complete. ///

(2.15) Theorem: If $b(x) = O(x^r)$ as $x \rightarrow \infty$ and (T_r) holds, then $s_n = O(n^r)$.

Proof: Following Sitaraman ([18], proof of Theorem 1) write,

$$\sigma_n = \frac{s_n}{n^r} \quad (n \geq 1), \quad \sigma_1(n) = \max_{v \leq n} \sigma_v, \quad \sigma_2(n) = \max_{v \leq n} (-\sigma_v).$$

There are two logical possibilities:

Case (A): $\sigma_1(n) > \sigma_2(n)$ for infinitely many values of n .

or Case (B): $\sigma_1(n) < \sigma_2(n)$ for all n sufficiently large.

We prove that the assumptions case (A), $\{\sigma_n\}$ unbounded, and (T_r) lead to a contradiction of the hypothesis $b(x) = O(x^r)$ as $x \rightarrow \infty$, and also that case (B), $\{\sigma_n\}$ unbounded, and (T_r) contradict the hypothesis $b(x) = O(x^r)$ as $x \rightarrow \infty$.

Suppose first that case (A) holds and that $\{\sigma_n\}$ is unbounded.

We first show that $\sigma_1(n) \rightarrow \infty$ as $n \rightarrow \infty$.

If $\{\sigma_1(n)\}$ is bounded above, then there exists an L such that $\sigma_1(n) \leq L$ for all n . This in turn implies that $\sigma_n \leq L$ for all n . On the other hand, because of case (A), we have $\sigma_2(n) \leq L$ for infinitely many values of n . This implies that $\sigma_m \leq L$ for any m , since there exists an n such that $\sigma_m \leq \sigma_2(n) \leq L$ for any $m \leq n$. Hence altogether we have shown that $\sigma_n \leq L$ for all n , which is a contradiction. Thus $\{\sigma_1(n)\}$ is unbounded above, moreover it is increasing in n , hence $\sigma_1(n) \rightarrow \infty$ as $n \rightarrow \infty$.

Now write,

$$(2.16) \quad b(x) = \left(\sum_{n=N_0}^{N-1} + \sum_{n=N}^{M-1} + \sum_{n=M}^{\infty} \right) c_n(x) \frac{x^n}{n!} = T_1(x) + T_2(x) + T_3(x),$$

where first N and then M are chosen as follows. Corresponding to any positive $H > H_0 = \sigma_1(N_0)$ there are $N = N(H) > N_0$ for which

$$(2.17) \quad \sigma_N = \sigma_1(N) > 2H, \quad \sigma_1(N) > \sigma_2(N)$$

We choose the least $N = N(H)$ satisfying (2.17).

(To see that there are $N = N(H) > N_0$ such that (2.17) is satisfied consider the following: the fact that $\sigma_1(n) \rightarrow \infty$ as $n \rightarrow \infty$, and the hypothesis of case (A) imply that for any positive $H > H_0$ there exists an N' such that $\sigma_1(N') > 2H$ and $\sigma_1(N') > \sigma_2(N')$. Clearly $\sigma_1(N') = \sigma_N$ for some $N \leq N'$, and hence $\sigma_1(N') > \sigma_1(N) > \sigma_N = \sigma_1(N')$. Thus $\sigma_N = \sigma_1(N) = \sigma_1(N')$ and it follows that $\sigma_N = \sigma_1(N) > 2H$. Furthermore, $\sigma_2(N') > \sigma_2(N)$ implies that $\sigma_1(N) = \sigma_1(N') > \sigma_2(N') > \sigma_2(N)$. Lastly $N > N_0$, since $2H > H_0 = \sigma_1(N_0)$ implies that $\sigma_1(N) > 2H > \sigma_1(N_0)$.)

Next choose the least $M = M(H) > N$ for which

$$(2.18) \quad \sigma_M \leq \frac{1}{2} \sigma_N = \frac{1}{2} \sigma_1(N)$$

{There are such M when H is large, because of the following arguments.

First we prove the

Claim: There exists an $H_1 > H_0$ such that $\sigma_n < H_1$ for infinitely many values of n . In order to obtain a contradiction suppose the claim is false. Therefore, for all $H_1 > H_0$, $\sigma_n > H_1$ for all n sufficiently large, i.e. $\sigma_n \rightarrow \infty$ as $n \rightarrow \infty$. Now since there can only be finitely many values of n such that $\sigma_n < 0$, and since the choice of N_0 is immaterial and satisfies $\alpha N_0 + \beta > 0$ it can be increased if necessary so that $\sigma_n > 0$ for $n > N_0$, and $\alpha N_0 + \beta - r > 0$.

By the proof of Lemma (2.2), $\frac{(\alpha n)^r \Gamma(\alpha n + \beta - r)}{\Gamma(\alpha n + \beta)} \sim 1$ as $n \rightarrow \infty$. There is

therefore a positive constant L such that $\frac{(\alpha n)^r \Gamma(\alpha n + \beta - r)}{\Gamma(\alpha n + \beta)} > L$, $n > N_0$.

Hence $\frac{n^r \sigma_n}{\Gamma(\alpha n + \beta)} > C \frac{\sigma_n}{\Gamma(\alpha n + \beta - r)}$, $n > N_0$, where $C = \frac{L}{\alpha^r}$ is a positive constant.

$$\text{Thus, } b(x) = \sum_{n=N_0}^{\infty} c_n(x) s_n = a e^{-x} \sum_{n=N_0}^{\infty} \sigma_n n^r \frac{x^{\alpha n + \beta - 1}}{\Gamma(\alpha n + \beta)}$$

$$> C a e^{-x} \sum_{n=N_0}^{\infty} \sigma_n \frac{x^{\alpha n + \beta - 1}}{\Gamma(\alpha n + \beta - r)}$$

Hence,

$$(2.19) \quad \frac{b(x)}{x^r} > C a e^{-x} \sum_{n=N_0}^{\infty} \sigma_n \frac{x^{\alpha n + \beta - r - 1}}{\Gamma(\alpha n + \beta - r)}$$

Now since the (B, α, β) method is regular in the usual sense, and a positive method, it is a totally regular method (see Hardy [12],

Theorem 9). Therefore $\sigma_n \rightarrow \infty$ as $n \rightarrow \infty$, implies that $\sigma_n \rightarrow \infty (B, \alpha, \beta - r)$,

and hence from (2.19) $\frac{b(x)}{x^r} \rightarrow \infty$, which is a contradiction with

the hypothesis $b(x) = O(x^r)$ as $x \rightarrow \infty$.

Having proven the claim, we have for all $H > H_1$, $\sigma_n < H$ for infinitely

many values of n . By (2.17) we have $\sigma_n \leq H < \frac{1}{2} \sigma_N = \frac{1}{2} \sigma_1(N)$ for infinitely many values of n . Thus when H is large there are $M=M(H) > N$ satisfying $\sigma_M \leq \frac{1}{2} \sigma_N = \frac{1}{2} \sigma_1(N)$.

Since (T_r) holds, Lemma (2.4) implies for $M, N > 1$,

$$s_M - s_N \geq -KM^r(M^{1/2} - N^{1/2}) - K'N^r, \text{ where } K \text{ and } K' \text{ are some positive constants.}$$

The choice of M and N in (2.18) and (2.17) are such that

$$s_M - s_N = \sigma_M M^r - \sigma_N N^r \leq \frac{1}{2} \sigma_N M^r - \sigma_N N^r = \sigma_1(N) \left(\frac{1}{2} M^r - N^r \right).$$

$$\text{Therefore, } \sigma_1(N) \left(\frac{1}{2} M^r - N^r \right) \geq s_M - s_N \geq -KM^r(M^{1/2} - N^{1/2}) - K'N^r.$$

Since $M > N$ it follows that,

$$K(M^{1/2} - N^{1/2}) \geq \sigma_1(N) \left\{ \left(\frac{M}{N} \right)^r - \frac{1}{2} \right\} - K'.$$

Now for a fixed $\delta \in (0, 1/2)$ we have either

$$\left(\frac{M}{N} \right)^r > \frac{1}{2} + \delta \text{ or } \left(\frac{M}{N} \right)^r \geq \left(\frac{1}{2} + \delta \right)^{-1} > 1.$$

In the first case,

$$K(M^{1/2} - N^{1/2}) \geq \sigma_1(N) \left\{ \left(\frac{M}{N} \right)^r - \frac{1}{2} \right\} - K' \geq \delta \sigma_1(N) - K',$$

while in the second case,

$$M^{1/2} - N^{1/2} = N^{1/2} \left\{ \left(\frac{M}{N} \right)^{1/2} - 1 \right\} \geq N^{1/2} \left\{ \left(\frac{1}{2} + \delta \right)^{-1/2r-1} \right\}.$$

$$\text{It follows that } t := t(H) = \frac{1}{2} (M^{1/2} - N^{1/2}) \rightarrow \infty \text{ as } N \rightarrow \infty$$

(or $H \rightarrow \infty$).

(Note: by (2.17), $\sigma_1(N) = \sigma_1(N(H)) \rightarrow \infty$ as $H \rightarrow \infty$, and hence $N=N(H) \rightarrow \infty$ as $H \rightarrow \infty$.)

Setting,

$$(2.20) \quad x := x(H) = \frac{\alpha}{4} (M^{1/2} + N^{1/2})^2$$

we see that $x \rightarrow \infty$ as $H \rightarrow \infty$, since $M > N \rightarrow \infty$ as $H \rightarrow \infty$.

From (2.20) we have:

$$2t = M^{1/2} - N^{1/2}$$

$$2\sqrt{x/a} = M^{1/2} + N^{1/2}$$

$$\text{and so } M^{1/2} = \sqrt{x/a} + t, \quad N^{1/2} = \sqrt{x/a} - t.$$

Therefore,

$$(2.21) \quad \begin{cases} M = (\sqrt{x/a} + t)^2 > (\sqrt{x/a} + t)\sqrt{x/a} = x/a + t\sqrt{x/a} \\ N < (\sqrt{x/a} - t)^2 < (\sqrt{x/a} - t)\sqrt{x/a} = x/a - t\sqrt{x/a}, \end{cases}$$

where x and $t \rightarrow \infty$ as $H \rightarrow \infty$.

In the analysis of case (A), which follows, we suppose that N, M , and x are chosen as in (2.17), (2.18), and (2.20), that $t = 1/2 (M^{1/2} - N^{1/2})$, and hence that (2.21) is satisfied. Clearly the choice of M and N satisfy the hypotheses of Lemma (2.6) and so as $x, t \rightarrow \infty$ (or $H \rightarrow \infty$) we have the properties (2.7) through (2.10).

Now,

$$(2.22) \quad T_1(x) := \sum_{n=N_0}^{N-1} c_n(x) s_n > -\sigma_2(N) \sum_{n=N_0}^{N-1} n^r c_n(x), \text{ since } \sigma_2(N) = \max_{n \leq N} \left(-\frac{s_n}{n^r} \right) \\ > -\sigma_1(N) \sum_{n=N_0}^{N-1} n^r c_n(x), \text{ since } \sigma_1(N) > \sigma_2(N) \\ > -\sigma_1(N) \sum_{n=N_0}^N n^r c_n(x), \text{ since } \sigma_1(N) > 0 \\ = -\sigma_1(N) o(1), \text{ as } x, t \rightarrow \infty \text{ (or } H \rightarrow \infty \text{),}$$

by (2.7) of Lemma (2.6).

Next, M is the least integer $> N$ which satisfies (2.18), so for $n = N+1, N+2, \dots, M-1$ we have $\sigma_n > \frac{1}{2} \sigma_N = \frac{1}{2} \sigma_1(N)$, and trivially $\sigma_N = \sigma_1(N) > \frac{1}{2} \sigma_1(N)$. So for $n = N, N+1, \dots, M-1$, $s_n > \frac{1}{2} \sigma_1(N) n^r$. Thus,

$$(2.23) \quad T_2(x) := \sum_{n=N}^{H-1} c_n(x) s_n > \frac{1}{2} \sigma_1(N) \sum_{n=N}^{H-1} n^r c_n(x) - \frac{1}{2} \sigma_1(N) (x/\alpha)^r \text{ as}$$

$x, t \rightarrow \infty$ (or $H \rightarrow \infty$), by (2.9) of Lemma (2.6).

Now by Lemma (2.4),

$$s_n - s_{n-1} > -K(n^{r+1/2} - (n-1)^{r+1/2}) - K'(n-1)^r, \quad n \geq M, \text{ for some positive constants } K, K'.$$

Therefore,

$$s_n > s_{n-1} - K(n^{r+1/2} - n^{r+1/2}) - O(n^{r-1/2}) - K'n^r.$$

Now since M is the least integer $> N$ which satisfies (2.18) we have by

(2.17),

$$\sigma_{M-1} > \frac{1}{2} \sigma_M > H \text{ i.e. } s_{M-1} > H(M-1)^r > 0.$$

$$\text{Hence } s_n > H(M-1)^r - K(n^{r+1/2} - M^{r+1/2}) - O(M^{r-1/2}) - K'M^r,$$

$$\text{i.e. } s_n > -K(n^{r+1/2} - M^{r+1/2}) - O(M^r), \quad n \geq M.$$

Thus,

$$(2.24) \quad T_3(x) := \sum_{n=M}^{\infty} c_n(x) s_n > \sum_{n=M}^{\infty} (-K(n^{r+1/2} - M^{r+1/2}) - O(M^r)) c_n(x)$$

$$> -K \sum_{n=M}^{\infty} (n^{r+1/2} - M^{r+1/2}) c_n(x) - O(1) \sum_{n=M}^{\infty} n^r c_n(x) > -o(x^r) - O(1)o(x^r)$$

$$= -o(x^r) \text{ as } x, t \rightarrow \infty \text{ (or } H \rightarrow \infty), \text{ by (2.10) and (2.8) of Lemma (2.6).}$$

Consequently by (2.16), (2.22), (2.23), and (2.24),

$$\frac{b(x)}{x^r} = \frac{T_1(x)}{x^r} + \frac{T_2(x)}{x^r} + \frac{T_3(x)}{x^r} > \sigma_1(N) \left(\frac{1}{2} \alpha^{-r} - o(1) \right) - o(1) \rightarrow \infty \text{ as } H \rightarrow \infty, \text{ since } \sigma_1(N) \rightarrow \infty, \text{ as } N \text{ (or } H) \rightarrow \infty. \text{ This shows that } \frac{b(x)}{x^r} \text{ is unbounded above and therefore we have a contradiction with the hypothesis } b(x) = O(x^r) \text{ as } x \rightarrow \infty.$$

Next we suppose that case (B) holds (i.e. there exists an M_0 such that $\sigma_2(n) > \sigma_1(n)$, $n > M_0$) and that $\{\sigma_n\}$ is unbounded. This supposition is ruled out by an argument similar to the preceding one.

Specifically we follow Rajagopal ([17], case (11) of Theorem 1).

We first show that $\sigma_2(n) \rightarrow \infty$ as $n \rightarrow \infty$:

if $\{\sigma_2(n)\}$ is bounded above, then there exists an L such that $\sigma_2(n) \leq L$ for all n . This implies that $-\sigma_n \leq L$ for all n . On the other hand, because of case (B), we have that $\sigma_1(n) < \sigma_2(n) \leq L$ for $n > M_0$. It follows that $\sigma_n \leq L$ for all n . Thus $|\sigma_n| \leq L$ for all n , which is a contradiction. Hence $\{\sigma_2(n)\}$ is unbounded above, moreover it is increasing in n , therefore $\sigma_2(n) \rightarrow \infty$ as $n \rightarrow \infty$.

Now write,

$$(2.25) \quad b(x) = \left(\sum_{n=N_0}^N + \sum_{n=N+1}^M + \sum_{n=M+1}^{\infty} \right) c_n(x) s_n = T_1(x) + T_2(x) + T_3(x),$$

where first M and then N are chosen as follows. Corresponding to any positive $H > H_0 = \sigma_2(M_0)$, choose the least $M = M(H)$ such that

$$(2.26) \quad \sigma_2(n) > \sigma_1(n), \quad n > M, \quad \sigma_M = -\sigma_2(M) < -2H.$$

(To see that there are $M = M(H)$ such that (2.26) is satisfied consider the following: given any positive $H > H_0$ there exists an M' such that $\sigma_2(M') > 2H$, since $\sigma_2(n) \rightarrow \infty$. Next, $\sigma_2(M') = -\sigma_M$ for some $M \leq M'$, and hence $\sigma_2(M') > \sigma_2(M) > -\sigma_M = \sigma_2(M')$. Thus $-\sigma_M = \sigma_2(M) = \sigma_2(M')$ and it follows that $-\sigma_M = \sigma_2(M) > 2H$. Furthermore, $2H > H_0 = \sigma_2(M_0)$ implies that $\sigma_2(M) > \sigma_2(M_0)$, hence $M > M_0$ and $\sigma_2(n) > \sigma_1(n)$, $n > M$.)

Next, choose the largest $N = N(H)$, $N \in (N_0, M)$ for which

$$(2.27) \quad \sigma_N > \frac{1}{2} \sigma_M = -\frac{1}{2} \sigma_2(M).$$

(There are such N when H is large because of the following arguments.

First, we prove the

Claim: There exists an $H_1 > H_0$ such that $\sigma_n > -H_1$ for infinitely many values of n .

For the sake of a contradiction we suppose the claim is false.

Therefore, for all $H_1 > H_0$, $\sigma_n < -H_1$ for all n sufficiently large, i.e.

$\sigma_n \rightarrow -\infty$. Since there can only be finitely many values of n

such that $\sigma_n > 0$, and since the actual choice of N_0 is immaterial and

satisfies $\alpha N_0 + \beta - r > 0$, it may be increased if necessary so that $\sigma_n < 0$,

$n > N_0$, and $\alpha N_0 + \beta - r > 0$. Now exactly as in case (A) we have:

$$\frac{n^r \sigma_n}{\Gamma(\alpha + \beta)} \leq C \frac{\sigma_n}{\Gamma(\alpha + \beta - r)}, \quad n > N_0, \text{ where } C \text{ is a positive constant.}$$

$$\begin{aligned} \text{Thus } b(x) &= \sum_{n=N_0}^{\infty} c_n(x) s_n = a e^{-x} \sum_{n=N_0}^{\infty} \sigma_n n^r \frac{x^{\alpha + \beta - 1}}{\Gamma(\alpha + \beta)} \\ &\leq C a e^{-x} \sum_{n=N_0}^{\infty} \sigma_n \frac{x^{\alpha + \beta - 1}}{\Gamma(\alpha + \beta - r)}, \text{ and hence} \end{aligned}$$

$$(2.28) \quad \frac{b(x)}{x^r} \leq C a e^{-x} \sum_{n=N_0}^{\infty} \sigma_n \frac{x^{\alpha + \beta - r - 1}}{\Gamma(\alpha + \beta - r)}.$$

Since $\sigma_n \rightarrow -\infty$ as $n \rightarrow \infty$, total regularity implies that

$\sigma_n \rightarrow -\infty(B, \alpha, \beta - r)$. Hence from (2.28) it follows that $\frac{b(x)}{x^r} \rightarrow -\infty$

as $x \rightarrow \infty$, which is a contradiction with the hypothesis

$b(x) = O(x^r)$ as $x \rightarrow \infty$.

Having proven the claim we have that $\sigma_n > -H_1$ for infinitely many

values of $n > N_0$. Since $\sigma_2(H) = \sigma_2(H(H)) > 2H$ we have that $\sigma_2(H) \rightarrow \infty$

as $H \rightarrow \infty$, and this implies $M = M(H) \rightarrow \infty$ as $H \rightarrow \infty$. Thus there exists

an $H_2 > H_1$ such that for all $H > H_2$, $M = M(H) > N_0$ and (N_0, M) contains at

least one of the n 's for which $\sigma_n > -H_1 > -H > \frac{1}{2} \sigma_M = -\frac{1}{2} \sigma_2(M)$.

In other words there are N when H is large for which

$$\sigma_N > \frac{1}{2} \sigma_H = -\frac{1}{2} \sigma_2(H).$$

By the choice of H and N in (2.26) and (2.27),

$$s_H - s_N = \sigma_H H^r - \sigma_N N^r < \sigma_H H^r - \frac{1}{2} \sigma_H N^r = -\sigma_2(H) \left(H^r - \frac{1}{2} N^r \right).$$

Further, by Lemma (2.4),

$$s_H - s_N > -KH^r(H^{1/2} - N^{1/2}) - K'N^r, \text{ for some positive constants } K \text{ and } K'.$$

It follows that

$$-\sigma_2(H) \left(H^r - \frac{1}{2} N^r \right) > -KH^r(H^{1/2} - N^{1/2}) - K'N^r, \text{ and hence}$$

$$K(H^{1/2} - N^{1/2}) > \sigma_2(H) \left(1 - \frac{1}{2} \left(\frac{N}{H} \right)^r \right) - K' \left(\frac{N}{H} \right)^r > \frac{1}{2} \sigma_2(H) - K' \rightarrow \infty$$

as $H \rightarrow \infty$, since in this case $\sigma_2(H) \rightarrow \infty$ as $H \rightarrow \infty$.

That is $t = \frac{1}{2} (H^{1/2} - N^{1/2}) \rightarrow \infty$ as $H \rightarrow \infty$.

Letting, as in case (A),

$$(2.20) \quad x = \frac{\alpha}{4} (H^{1/2} + N^{1/2})^2,$$

we see that $x > (\alpha/4)H$, and hence $x \rightarrow \infty$ as $H \rightarrow \infty$.

Also, as in case (A), we obtain (2.21) from the definitions of x and t .

We now suppose that N, H , and x are chosen as in (2.26), (2.27), and (2.20), that $t = 1/2 (H^{1/2} - N^{1/2})$ and hence that (2.21) is satisfied.

Clearly the choice of H and N satisfy the hypothesis of Lemma (2.6), and so as $x, t \rightarrow \infty$ (or $H \rightarrow \infty$) the properties (2.7)-(2.10) are all valid.

With the choice of H in (2.27) and Lemma (2.4) we have for $n > H$, $s_n + H^r \sigma_2(H) = s_n - H^r \sigma_H = s_n - s_H > -K(n^{r+1/2} - H^{r+1/2}) - K'H^r$ for some positive constants K and K' .

It follows that,

$$\sigma_n = n^{-r} s_n > -H^r n^{-r} \sigma_2(H) - Kn^{-r} (n^{r+1/2} - H^{r+1/2}) - K'H^r n^{-r}$$

$$\begin{aligned}
&= -M^r n^{-r} (\sigma_2(M) + K') - K(n^{1/2} - n^{-r} M^{r+1/2}) \\
&> -(\sigma_2(M) + K') - K(n^{1/2} - n^{-r} M^{r+1/2}), \quad n \geq M' \\
&= -t_n \quad \text{say.}
\end{aligned}$$

i.e. $-\sigma_n < t_n$ for $n \geq M$.

Next, $-\sigma_M = \sigma_2(M) = \max_{n \leq M} (-\sigma_n)$, i.e. $-\sigma_M \geq -\sigma_n$ for $n=1, 2, \dots, M$.

Thus we have the following inequalities:

$$(2.29) \quad -\sigma_n < t_n \quad \text{for } n \geq M.$$

$$(2.30) \quad -\sigma_n \leq -\sigma_M < t_M \quad \text{for } n \leq M.$$

Since $\{t_n\}$ is increasing in n we have for $n \geq M$, $t_n \geq t_M$, and by (2.29):

$$(2.31) \quad -\sigma_n < t_M \leq t_n \quad \text{for } n \geq M.$$

Also for $n \geq M$, $t_n \geq t_M$, and by (2.30):

$$(2.32) \quad -\sigma_n < t_M \leq t_n \quad \text{for } n \geq M.$$

By (2.32) and (2.31), $-\sigma_n < t_n$ for $n \leq n_0$ and $n \geq M$. It follows from this that

$$(2.33) \quad \sigma_2(n) < t_n \quad \text{for } n \geq M, \text{ where } \sigma_2(n) = \max_{m \leq n} (-\sigma_m).$$

Now

$$T_3(x) = \sum_{n=M+1}^{\infty} \sigma_n c_n(x) = \sum_{n=M+1}^{\infty} n^r \sigma_n c_n(x) \leq \sum_{n=M+1}^{\infty} \sigma_1(n) n^r c_n(x),$$

since $\sigma_1(n) \geq \sigma_n$.

By (2.26) and (2.33),

$$T_3(x) < \sum_{n=M+1}^{\infty} \sigma_2(n) n^r c_n(x) < \sum_{n=M+1}^{\infty} t_n n^r c_n(x).$$

By the definition of t_n , and (2.8) and (2.10) of Lemma (2.6),

$$(2.34) \quad T_3(x) < \sum_{n=M+1}^{\infty} (\sigma_2(M) + K' + K(n^{1/2} - n^{-r} M^{r+1/2})) n^r c_n(x)$$

$$= (\sigma_2(M) + K') \sum_{n=M+1}^{\infty} n^r c_n(x) + K \sum_{n=M+1}^{\infty} (n^{r+1/2} - M^{r+1/2}) c_n(x)$$

$$= (\sigma_2(M) + K') o(x^r) + K o(x^r) = \sigma_2(M) o(x^r) + o(x^r)$$

as $x, t \rightarrow \infty$ (or $H \rightarrow \infty$).

Next,

$$T_1(x) = \sum_{n=N_0}^N c_n(x) s_n = \sum_{n=N_0}^N \sigma_n n^r c_n(x) < \sigma_1(M) \sum_{n=N_0}^N n^r c_n(x) \\ < \sigma_1(M) \sum_{n=N_0}^N n^r c_n(x), \text{ since } M > N \text{ and } \{\sigma_1(n)\} \text{ is increasing.}$$

It follows from (2.26), and (2.7) of Lemma (2.6) that,

$$(2.35) \quad T_1(x) < \sigma_2(M) \sum_{n=N_0}^N n^r c_n(x) = \sigma_2(M) o(x^r) \\ \text{as } x, t \rightarrow \infty \text{ (or } H \rightarrow \infty).$$

Next, N is the largest integer in (N_0, M) which satisfies (2.27):

hence for $n=N+1, N+2, \dots, M-1$, $\sigma_n < -\frac{1}{2} \sigma_2(M)$, and trivially

$$\sigma_M = -\sigma_2(M) < -\frac{1}{2} \sigma_2(M). \text{ So for } n=N+1, N+2, \dots, M, s_n < -\frac{1}{2} \sigma_2(M) n^r.$$

Thus, by (2.9) of Lemma (2.6),

$$(2.36) \quad T_2(x) = \sum_{n=N+1}^M c_n(x) s_n < -\frac{1}{2} \sigma_2(M) \sum_{n=N+1}^M n^r c_n(x) \\ \sim -\frac{1}{2} \sigma_2(M) (x/\alpha)^r \text{ as } x, t \rightarrow \infty \text{ (or } H \rightarrow \infty).$$

Therefore from (2.25) and using (2.34), (2.35), (2.36) we have:

$$\frac{b(x)}{x^r} = \frac{T_1(x)}{x^r} + \frac{T_2(x)}{x^r} + \frac{T_3(x)}{x^r} < -\sigma_2(M) \left(\frac{1}{2} \alpha^{-r} o(1) - o(1) \right) + o(1) \rightarrow -\infty \\ \text{as } H \rightarrow \infty, \text{ since } \sigma_2(M) \rightarrow \infty \text{ as } H \rightarrow \infty.$$

This shows that $\frac{b(x)}{x^r}$ is unbounded below and therefore contradicts the hypothesis $b(x) = O(x^r)$ as $x \rightarrow \infty$. Ruling out the supposition that case (B) holds and $\{\sigma_n\}$ is unbounded we see that the proof is complete. ///

(2.37) Corollary: Given $s_n \in \mathbb{R}$ such that $s_n - s_{n-1} = O_L(n^{r-1/2})$, $r > 0$, and $b(x) = O(x^r)$ as $x \rightarrow \infty$, then $s_n = O(n^r)$.

Proof: This follows from Remark (2.5) and Théorème (2.15). ///

CHAPTER 3

AN EQUIVALENCE THEOREM FOR THE (B, α, β) AND V_α METHODS

Borwein and Robinson ([7], Theorem 2) have proved the following result: If $s_n = o(\sqrt{n})$ then $s_n \rightarrow l(B, \alpha, \beta) \iff s_n \rightarrow l(V_\alpha)$.

It will be shown in this chapter that the result is still true when the condition $s_n = o(\sqrt{n})$ is replaced by $s_n = O(n^r)$, $r > 0$. In order to counter the effect of this widening of the order condition, a much closer approximation to the function $\frac{ae^{-x} x^{\alpha r + \beta - 1}}{\Gamma(\alpha r + \beta)}$ is required than is given in Lemma (2.1) and this is the point of the functions g_k and R_k introduced in Lemma (3.8). The proof of Lemma (3.8) is similar to one given by Faulhaber ([10], proof of Hilfssatz 5). The proof of our equivalence theorem, between the (B, α, β) method and the V_α method, is along the lines of ([10], Satz II).

(3.1) Lemma: (see Hyslop ([14], Lemma 1))

If $p, r > 0$, $1/2 < \xi < 1$, $h_n = n - x/\alpha$, then as $x \rightarrow \infty$,

$$(3.2) \quad \sum_{n > x/\alpha + x^\xi} n^r h_n^p \exp\left(-\frac{\alpha^2 h_n^2}{2x}\right) = o(1),$$

$$(3.3) \quad \sum_{0 \leq n < x/\alpha - x^\xi} n^r |h_n|^p \exp\left(-\frac{\alpha^2 h_n^2}{2x}\right) = o(1),$$

$$(3.4) \quad \sum_{n=0}^{\infty} n^r |h_n|^p \exp\left(-\frac{\alpha^2 h_n^2}{2x}\right) = O(x^{r+1/2(p+1)}).$$

Proof: For (3.2), $S(x) := \sum_{n > x/\alpha + x^\xi} n^r h_n^p \exp\left(-\frac{\alpha^2 h_n^2}{2x}\right)$

$$= \left\{ \sum_{x/a+x^\xi < n < 2x/a} + \sum_{n \geq 2x/a} \right\} n^r h_n^p \exp\left(-\frac{a^2 h_n^2}{2x}\right)$$

$$= (2x/a)^r \sum_{x/a+x^\xi < n < 2x/a} h_n^p \exp\left(-\frac{a^2 h_n^2}{2x}\right) + 2^r \sum_{n \geq 2x/a} h_n^{p+r} \exp\left(-\frac{a^2 h_n^2}{2x}\right),$$

since for $n \geq 2x/a$, $n \leq 2(n-x/a) = 2h_n$.

Therefore

$$\begin{aligned} S(x) &= O\left\{x^r \sum_{x/a+x^\xi < n < 2x/a} h_n^p \exp\left(-\frac{a^2 h_n^2}{2x}\right)\right\} + O\left\{\sum_{n \geq 2x/a} h_n^{p+r} \exp\left(-\frac{a^2 h_n^2}{2x}\right)\right\} \\ &= O\left\{x^r S_1(x)\right\} + O\left\{S_2(x)\right\}. \end{aligned}$$

Now, $S_1(x) = \sum_{n=n_0}^{n_1} h_n^p \exp\left(-\frac{a^2 h_n^2}{2x}\right)$, where n_0 is the smallest

integer $> x/a+x^\xi$, and n_1 is the largest integer $< 2x/a$.

Let $f(y) = y^p \exp\left(-\frac{a^2 y^2}{2x}\right)$. Then $f(y)$ decreases for $y > \sqrt{(px)/a}$. For x sufficiently large and by our choice of n_0 the inequality

$\sqrt{(px)/a} < x^\xi < n_0 - x/a = h_{n_0}$ is valid. Applying Lemma (1.6) we have:

$$\begin{aligned} S_1(x) &\leq h_{n_0}^p \exp\left(-\frac{a^2 h_{n_0}^2}{2x}\right) + \int_{h_{n_0}}^{h_{n_1}} y^p \exp\left(-\frac{a^2 y^2}{2x}\right) dy \\ &\leq x^{\xi p} \exp\left(-\frac{a^2 x^{2\xi}}{2x}\right) + \int_{x^\xi}^{x/a} y^p \exp\left(-\frac{a^2 y^2}{2x}\right) dy, \text{ since } h_{n_1} = n_1 - x/a < x/a \end{aligned}$$

and $h_{n_0} = n_0 - x/a > x^\xi$.

Since the integrand decreases on the interval $[x^\xi, x/a]$, where x is

large and $1/2 < \xi < 1$, it follows that

$$\begin{aligned} x^r S_1(x) &= O\left\{x^{\xi p+r} \exp\left(-\frac{\alpha^2 x^{2\xi-1}}{2}\right)\right\} + O\left\{x^r (x/\alpha - x^\xi) x^{\xi p} \exp\left(-\frac{\alpha^2 x^{2\xi-1}}{2}\right)\right\} \\ &= o(1) \text{ as } x \rightarrow \infty, \text{ since } 2\xi-1 > 0. \end{aligned}$$

For the sum $S_2(x)$, let n_0 be the smallest integer $> 2x/\alpha$ and

$$f(y) = y^{p+r} \exp\left(-\frac{\alpha^2 y^2}{2x}\right). \text{ The function } f(y) \text{ decreases for } y > \sqrt{(p+r)x/\alpha}.$$

For x sufficiently large and by our choice of n_0 the inequality

$$\sqrt{(p+r)x/\alpha} \leq x/\alpha \leq n_0 - x/\alpha = h_{n_0} \text{ is valid. In particular } f(y)$$

decreases on the interval $[h_{n_0}, h_m]$, where $m > n_0$.

By Lemma (1.6),

$$\begin{aligned} t_m &:= \sum_{n=n_0}^m h_n^{p+r} \exp\left(-\frac{\alpha^2 h_n^2}{2x}\right) \leq h_{n_0}^{p+r} \exp\left(-\frac{\alpha^2 h_{n_0}^2}{2x}\right) + \int_{h_{n_0}}^{h_m} y^{p+r} \exp\left(-\frac{\alpha^2 y^2}{2x}\right) dy \\ &\leq (x/\alpha)^{p+r} \exp\left(-\frac{\alpha^2 (x/\alpha)^2}{2x}\right) + \int_{x/\alpha}^{\infty} y^{p+r} \exp\left(-\frac{\alpha^2 y^2}{2x}\right) dy, \end{aligned}$$

since $h_{n_0} = n_0 - x/\alpha > x/\alpha$.

Since $\{t_m\}$ is increasing and bounded above we have:

$$\begin{aligned} S_2(x) &= \sum_{n=n_0}^{\infty} h_n^{p+r} \exp\left(-\frac{\alpha^2 h_n^2}{2x}\right) \leq (x/\alpha)^{p+r} \exp(-x/2) + \int_{x/\alpha}^{\infty} y^{p+r} \exp\left(-\frac{\alpha^2 y^2}{2x}\right) dy \\ &= O\left(x^{p+r} \exp(-x/2)\right) + x^{1/2(p+r+1)} \int_{\sqrt{x/\alpha}}^{\infty} u^{p+r} \exp\left(-\frac{\alpha^2 u^2}{2}\right) du, \end{aligned}$$

$$\text{where } u^2 = \frac{y^2}{x}.$$

To find an estimate for the integral, let $\lambda = 2(p+r+2)$ and

$$f(u) = u^{-\lambda} \exp\left(\frac{\alpha^2 u^2}{2}\right) \text{ for } u \in (\sqrt{x/\alpha}, \infty). \text{ } f(u) \text{ increases for } u > \lambda/\alpha \text{ so it}$$

certainly increases for $u > \sqrt{x}/\alpha$, where x is large. That is $f(u) > f(\sqrt{x}/\alpha)$ for $u \in (\sqrt{x}/\alpha, \infty)$ i.e. $\exp(-\frac{\alpha^2 u^2}{2}) < (\alpha u)^{-\lambda} x^{\lambda/2} \exp(-x/2)$ for $u \in (\sqrt{x}/\alpha, \infty)$.

Thus,

$$\begin{aligned} \int_{\sqrt{x}/\alpha}^{\infty} u^{p+r} \exp\left(-\frac{\alpha^2 u^2}{2}\right) du &= O\left\{x^{\lambda/2} \exp(-x/2) \int_{\sqrt{x}/\alpha}^{\infty} u^{p+r-\lambda} du\right\} \\ &= O\left\{x^{1/2(p+r+1)} \exp(-x/2)\right\}, \end{aligned}$$

since $\lambda = 2(p+r+2)$.

It follows that,

$$\begin{aligned} S_2(x) &= \sum_{n=n_0}^{\infty} h_n^{p+r} \exp\left(-\frac{\alpha^2 h_n^2}{2x}\right) = O\left(x^{p+r} \exp(-x/2)\right) + O\left(x^{p+r+1} \exp(-x/2)\right) \\ &= o(1) \text{ as } x \rightarrow \infty. \end{aligned}$$

Finally, $S(x) = O\left(x^r S_1(x)\right) + O\left(S_2(x)\right) = o(1) + o(1) = o(1)$ as $x \rightarrow \infty$.

To prove (3.3) it suffices to show that

$$S(x) := \sum_{0 \leq n \leq x/\alpha - x^\xi} n^r k_n^p \exp\left(-\frac{\alpha^2 k_n^2}{2x}\right) = o(1) \text{ as } x \rightarrow \infty, \text{ where } k_n = x/\alpha - n.$$

$$\begin{aligned} \text{Now, } S(x) &= O\left\{\left(x/\alpha - x^\xi\right)^r \sum_{0 \leq n \leq x/\alpha - x^\xi} k_n^p \exp\left(-\frac{\alpha^2 k_n^2}{2x}\right)\right\} \\ &= O\left\{\left(x/\alpha - x^\xi\right)^r \sum_{n=0}^{n_1} k_n^p \exp\left(-\frac{\alpha^2 k_n^2}{2x}\right)\right\}, \text{ where } n_1 \text{ is the largest} \end{aligned}$$

integer $< x/\alpha - x^\xi$.

The function $f(y) = y^p \exp\left(-\frac{\alpha^2 y^2}{2x}\right)$ decreases for $y > \sqrt{(px)}/\alpha$. For x sufficiently large and $\xi > 1/2$, the inequality $\sqrt{(px)}/\alpha < x/\alpha - n_1 = k_{n_1}$

is valid.

By Lemma (1.7) with $n_0=0$,

$$\begin{aligned} \sum_{n=0}^{n_1} k_n^p \exp\left(-\frac{a^2 k_n^2}{2x}\right) &\leq k_{n_1}^p \exp\left(-\frac{a^2 k_{n_1}^2}{2x}\right) + \int_{k_{n_1}}^{k_0} y^p \exp\left(-\frac{a^2 y^2}{2x}\right) dy \\ &\leq x^{\xi p} \exp\left(-\frac{a^2 x^{2\xi-1}}{2}\right) + \int_{x^{\xi}}^{x/\alpha} y^p \exp\left(-\frac{a^2 y^2}{2x}\right) dy, \end{aligned}$$

since $k_0 = x/\alpha$ and $k_{n_1} = x/\alpha - n_1 > x^{\xi}$.

Since the integrand decreases on the interval $[x^{\xi}, x/\alpha]$, where x is large and $1/2 < \xi < 1$, it follows that

$$\begin{aligned} S(x) &= O\left\{\left(x/\alpha - x^{\xi}\right)^r x^{\xi p} \exp\left(-\frac{a^2 x^{2\xi-1}}{2}\right)\right\} + O\left\{\left(x/\alpha - x^{\xi}\right)^{r+1} x^{\xi p} \exp\left(-\frac{a^2 x^{2\xi-1}}{2}\right)\right\} \\ &= o(1) \text{ as } x \rightarrow \infty, \text{ since } 2\xi - 1 > 0. \end{aligned}$$

$$\begin{aligned} \text{To prove (3.4) write, } S(x) &:= \sum_{n=0}^{\infty} n^r |h_n|^p \exp\left(-\frac{a^2 h_n^2}{2x}\right) \\ &= \left\{ \sum_{0 \leq n < 2x/\alpha} + \sum_{n \geq 2x/\alpha} \right\} n^r |h_n|^p \exp\left(-\frac{a^2 h_n^2}{2x}\right) \\ &= S_1(x) + S_2(x). \end{aligned}$$

Recalling the proof of (3.2) we see that $S_2(x) = o(1)$ as $x \rightarrow \infty$.

Next, setting $k_n = -h_n$ we have:

$$S_1(x) \leq (2x/\alpha)^r \sum_{0 \leq n < 2x/\alpha} |h_n|^p \exp\left(-\frac{a^2 h_n^2}{2x}\right)$$

$$\begin{aligned} & \leq (2x/a)^r \left\{ \sum_{0 \leq n \leq x/a} k_n^p \exp\left(-\frac{a^2 k_n^2}{2x}\right) + \sum_{x/a < n < 2x/a} h_n^p \exp\left(-\frac{a^2 h_n^2}{2x}\right) \right\} \\ & = (2x/a)^r (S_{1,1}(x) + S_{1,2}(x)). \end{aligned}$$

For the sum $S_{1,1}(x)$, let n_1 be the largest integer $\leq x/a$, so that

$$S_{1,1}(x) = \sum_{n=0}^{n_1} k_n^p \exp\left(-\frac{a^2 k_n^2}{2x}\right).$$

Let $f(y) = y^p \exp\left(-\frac{a^2 y^2}{2x}\right)$. Then $f(y)$ increases for $0 \leq y \leq \sqrt{(px)}/a$,

decreases for $y > \sqrt{(px)}/a$, and attains its maximum at $y = \sqrt{(px)}/a$. By the

choice of n_1 , $k_{n_1} = x/a - n_1 > 0$ so in particular $f(y)$ increases on

$[k_{n_1}, k_{x_0}]$, and decreases on $[k_{x_0}, k_0]$, where $k_{x_0} = x/a - x_0 = \sqrt{(px)}/a$ and

$k_0 = x/a$.

By Lemma (1.9) with $n_0 = 0$,

$$\begin{aligned} S_{1,1}(x) & \leq k_{x_0}^p \exp\left(-\frac{a^2 k_{x_0}^2}{2x}\right) + \int_{k_{n_1}}^{k_0} y^p \exp\left(-\frac{a^2 y^2}{2x}\right) dy \\ & \leq \left(\frac{\sqrt{(px)}}{a}\right)^p \exp\left(-\frac{a^2 px}{2a^2 x}\right) + \int_0^{x/a} y^p \exp\left(-\frac{a^2 y^2}{2x}\right) dy \\ & = O(x^{p/2}) + \int_0^{x/a} y^p \exp\left(-\frac{a^2 y^2}{2x}\right) dy, \text{ since } k_{n_1} > 0. \end{aligned}$$

Next,

$$S_{1,2}(x) = \sum_{n=n_0}^{n_1} h_n^p \exp\left(-\frac{a^2 h_n^2}{2x}\right), \text{ where } n_0 \text{ is the smallest integer } > x/a, \text{ and } n_1 \text{ is the largest integer } < 2x/a.$$

Defining f as above, and letting $h_{x_0} = x_0 - x/a = \sqrt{(px)}/a$ we find by

applying Lemma (1.8) that,

$$S_{1,2}(x) \leq h_{x_0}^p \exp\left(-\frac{a^2}{2x} h_{x_0}^2\right) + \int_{h_{n_0}}^{h_{n_1}} y^p \exp\left(-\frac{a^2 y^2}{2x}\right) dy$$

$$\leq O(x^{p/2}) + \int_0^{x/\alpha} y^p \exp\left(-\frac{a^2 y^2}{2x}\right) dy, \text{ since } h_{n_1} = n_1 - x/\alpha < x/\alpha \text{ and } h_{n_0} = n_0 - x/\alpha > 0.$$

$$\text{Therefore, } S_1(x) \leq (2x/\alpha)^r (S_{1,1}(x) + S_{1,2}(x))$$

$$= O\left\{x^r \int_0^{x/\alpha} y^p \exp\left(-\frac{a^2 y^2}{2x}\right) dy\right\} + O(x^{r+p/2})$$

$$= O\left\{x^{r+1/2(p+1)} \int_0^1 u^{1/2(p-1)} \exp(-u) du\right\} + O(x^{r+p/2})$$

$$= O\left\{x^{r+1/2(p+1)}\right\}.$$

$$\begin{aligned} \text{Finally, } S(x) = S_1(x) + S_2(x) &= o(1) + O\left\{x^{r+1/2(p+1)}\right\} \text{ as } x \rightarrow \infty \\ &= O\left\{x^{r+1/2(p+1)}\right\}. \quad /// \end{aligned}$$

Definition ([2] or [3]): A sequence $\{s_n\}$ is said to be of finite order if $s_n = O(n^c)$ for some finite c .

(3.5) Lemma ([14], Lemma.3 or [10], Hilfssatz 3): Suppose $\{s_n\}$ is of finite order. Then for $\alpha > 0$ and $q > 0$,

$$\sum_{n=0}^{\infty} s_n \exp\left(-\frac{\alpha(n-x)^2}{2x}\right) = o(x^{1/2+q}) \text{ as } x \rightarrow \infty$$

implies for each integer $j \geq 0$, and for each $\eta > 0$

$$\sum_{n=0}^{\infty} s_n (n-x)^j \exp\left(-\frac{\alpha(n-x)^2}{2x}\right) = o\left(x^{1/2(j+1)+q+\eta}\right) \text{ as } x \rightarrow \infty.$$

Recalling that $h_n = n - x/\alpha$, it is obvious that the following result follows from the previous one.

(3.6) **Lemma:** Suppose $\{s_n\}$ is of finite order. Then for $\alpha > 0$ and $q > 0$

$$\sum_{n=0}^{\infty} s_n \exp\left(-\frac{\alpha^2 h_n^2}{2x}\right) = o\left(x^{1/2+q}\right) \text{ as } x \rightarrow \infty$$

implies for each integer $j \geq 0$, and each $\eta > 0$

$$\sum_{n=0}^{\infty} s_n h_n^j \exp\left(-\frac{\alpha^2 h_n^2}{2x}\right) = o\left(x^{1/2(j+1)+q+\eta}\right) \text{ as } x \rightarrow \infty.$$

(3.7) **Lemma** ([14], Theorem 2 or [10], Hilfssatz 4):

Suppose $\{s_n\}$ is of finite order. Then

$$\sum_{k=0}^{\infty} s_k \exp\left(-\frac{\alpha(n-k)^2}{2n}\right) = o(n^{1/2}) \text{ as } n \rightarrow \infty, n=1,2,\dots$$

implies

$$\sum_{k=0}^{\infty} s_k \exp\left(-\frac{\alpha(x-k)^2}{2x}\right) = o(x^{1/2}) \text{ as } x \rightarrow \infty.$$

The next lemma is obtained by following the arguments given by

Faulhaber ([10], Hilfssatz 5). For another similar result see Hvslop ([14], Lemma 5).

(3.8) Lemma: If $0 < H < 1$, $(1-H)x/\alpha \leq n \leq (1+H)x/\alpha$, and k is any integer ≥ 2 , then as $x \rightarrow \infty$

$$c_n(x) = \alpha/\sqrt{2\pi x} \exp\left(-\frac{\alpha^2 h_n^2}{2x}\right) + g_k + R_k$$

where $h_n = n - x/\alpha$, $R_k = O\left(\frac{|h_n|^{k+1} + 1}{x^k}\right)$, $g_k = \sum_{i=1}^k \sum_{j=0}^{i+1} b_{i,j} \frac{h_n^j}{x^i}$, and

$b_{i,j}$ ($1 \leq i \leq k$, $0 \leq j \leq i+1$) are constants depending on α and β . In particular $b_{1,2} = b_{k,k+1} = 0$.

Proof: We have the following form of Stirling's formula:

$$(3.9) \quad \log \Gamma(z+\beta) = (z+\beta-1/2) \log z - z + 1/2 \log(2\pi) +$$

$$\sum_{r=1}^k \frac{(-1)^{r+1} B_{r+1}(\beta)}{r(r+1)z^r} + O(z^{-k-1}), \quad k=1, 2, \dots, \quad |\arg z| < \pi.$$

(See [1], p. 48, equation 12.)

Here $B_{r+1}(\beta)$ ($1 \leq r \leq k$) are the Bernoulli polynomials. For example $B_2(\beta) = \beta^2 - \beta + 1/6$, $B_3(\beta) = \beta^3 - 3/2 \beta^2 - 1/2 \beta$, ...

Setting $z = \alpha n$, and $h_n = n - x/\alpha$ in (3.9) we have:

(3.10)

$$\begin{aligned} \log \Gamma(\alpha n + \beta) &= (\alpha h_n + x + \beta - 1/2) \log x + (\alpha h_n + x + \beta - 1/2) \log \left(\frac{\alpha h_n}{x} + 1 \right) \\ &\quad - \alpha h_n - x + 1/2 \log(2\pi) + \sum_{r=1}^k \frac{(-1)^{r+1} B_{r+1}(\beta)}{r(r+1)x^r} \left(\frac{\alpha h_n}{x} + 1 \right)^{-r} \end{aligned}$$

$$+ O\left\{x^{-k-1} \left(\frac{ah_n}{x} + 1\right)^{-k-1}\right\}, \quad \text{for } k=1, 2, \dots$$

Since $(1-H)x/\alpha \leq n \leq (1+H)x/\alpha$, $0 < H < 1$, then $1-H \leq \frac{ah_n}{x} + 1 \leq 1+H$ and $|\frac{ah_n}{x}| \leq H < 1$. It follows that

$$(3.11) \quad x^{-k-1} \left(\frac{ah_n}{x} + 1\right)^{-k-1} = O(x^{-k-1}) = O(x^{-k}).$$

Moreover,

$$(3.12) \quad \left(\frac{ah_n}{x} + 1\right)^{-r} = \sum_{j=0}^{k-r} \binom{-r}{j} \left(\frac{ah_n}{x}\right)^j + O\left\{\left(\frac{|h_n|}{x}\right)^{k-r+1}\right\}$$

and

$$(3.13) \quad \log\left(\frac{ah_n}{x} + 1\right) = \sum_{j=1}^k \frac{(-1)^{j-1}}{j} \left(\frac{ah_n}{x}\right)^j + O\left\{\left(\frac{|h_n|}{x}\right)^{k+1}\right\}$$

by Remark (1.2) with $\frac{ah_n}{x}$ in place of x .

From (3.12) we have:

$$\begin{aligned} (3.14) \quad & \sum_{r=1}^k \frac{(-1)^{r+1} B_{r+1}(\beta)}{r(r+1)x^r} \left(\frac{ah_n}{x} + 1\right)^{-r} \\ &= \sum_{r=1}^k \sum_{j=0}^{k-r} \frac{(-1)^{r+1} B_{r+1}(\beta)}{r(r+1)} \binom{-r}{j} \frac{(ah_n)^j}{x^{r+j}} \\ &+ \sum_{r=1}^k \frac{(-1)^{r+1} B_{r+1}(\beta)}{r(r+1)x^r} O\left\{\left(\frac{|h_n|}{x}\right)^{k-r+1}\right\} \\ &= \sum_{r=1}^k \sum_{j=0}^{k-r} d_{r,j} \frac{h_n^j}{x^{r+j}} + \sum_{r=1}^k \frac{1}{x^r} O\left\{\left(\frac{|h_n|}{x}\right)^{k-r+1}\right\} = t_k + u_k, \end{aligned}$$

where $d_{r,j}$ is some constant which depends on α and β .

Since $|h_n|^{k-r+1} \leq |h_n|^{k+1} + 1$ for $1 \leq r \leq k$, it follows that

$$u_k = O\left\{\frac{|h_n|^{k+1} + 1}{x^{k+1}}\right\} = O\left\{\frac{|h_n|^{k+1} + 1}{x^k}\right\} \quad \text{as } x \rightarrow \infty.$$

As for the sum t_k , inverting the summation we see that

$$t_k = \sum_{j=0}^{k-1} \sum_{r=1}^{k-j} d_{r,j} \frac{h_n^j}{x^{r+j}}.$$

$$\text{Letting } r+j=1, \quad t_k = \sum_{j=0}^{k-1} \sum_{1=1+j}^k d_{1-j,j} \frac{h_n^j}{x^1}.$$

Inverting once again we have:

$$(3.15) \quad t_k = \sum_{i=1}^k \sum_{j=0}^{i-1} d_{1-j,j} \frac{h_n^j}{x^1}.$$

Setting $R_k = O\left\{\frac{|h_n|^{k+1} + 1}{x^k}\right\}$, it follows from (3.14) that

$$(3.16) \quad \sum_{r=1}^k \frac{(-1)^{r+1} B_{r+1}(\beta) \left(\frac{ah_n}{x} + 1\right)^{-r}}{r(r+1)x^r} = t_k + R_k \quad \text{as } x \rightarrow \infty.$$

Substituting (3.11), (3.13), and (3.16) into (3.10) we have:

$$\log \Gamma(an+\beta) = (ah_n+x+\beta-1/2) \log x - ah_n - x + 1/2 \log(2\pi)$$

$$+ (ah_n+x+\beta-1/2) \left\{ \sum_{j=1}^k \frac{(-1)^{j-1}}{j} \left(\frac{ah_n}{x}\right)^j + O\left\{\left(\frac{|h_n|}{x}\right)^{k+1}\right\} \right\}$$

$$+ t_k + R_k, \quad k \rightarrow \infty, \quad k=1, 2, \dots$$

Consequently, for $x \rightarrow \infty$, $k = 1, 2, \dots$

(3.17)

$$\begin{aligned}
 \log c_n(x) &= \log \alpha - x + (\alpha n + \beta - 1) \log x - \log \Gamma(\alpha n + \beta) \\
 &= \log \alpha - x + (\alpha n + \beta - 1) \log x - (\alpha n + \beta - 1/2) \log x \\
 &\quad - (\alpha h_n + x + \beta - 1/2) \left\{ \sum_{j=1}^k \frac{(-1)^{j-1}}{j} \left(\frac{\alpha h_n}{x} \right)^j + O \left\{ \left(\frac{|h_n|}{x} \right)^{k+1} \right\} \right\} \\
 &\quad + \alpha h_n + x - 1/2 \log(2\pi) - t_k + R_k \\
 &= \log(\alpha / \sqrt{2\pi x}) + \alpha h_n + (\alpha h_n + x + \beta - 1/2) \sum_{j=1}^k \frac{(-1)^{j-1}}{j} \left(\frac{\alpha h_n}{x} \right)^j \\
 &\quad - t_k + O \left\{ h_n \left(\frac{|h_n|}{x} \right)^{k+1} \right\} + O \left\{ x \left(\frac{|h_n|}{x} \right)^{k+1} \right\} + O \left\{ \left(\frac{|h_n|}{x} \right)^{k+1} \right\} + R_k.
 \end{aligned}$$

Next, $\left(\frac{|h_n|}{x} \right)^{k+1} = O \left\{ \frac{|h_n|^{k+1}}{x^k} \right\}$ as $x \rightarrow \infty$, and since $\frac{|h_n|}{x} < \frac{H}{\alpha}$ it

follows that $h_n \left(\frac{|h_n|}{x} \right)^{k+1} = O \left\{ \frac{|h_n|^{k+1}}{x^k} \right\}$.

It is clear that all of the O -terms as well as the term

$\frac{(-1)^k}{k} \frac{(\alpha h_n)^{k+1}}{x^k}$ on the right side of (3.17) may be incorporated in R_k .

It follows from (3.17), that for $k=2, 3, \dots$ and $x \rightarrow \infty$,

$$\begin{aligned}
 (3.18) \quad \log c_n(x) &= \log(\alpha / \sqrt{2\pi x}) - \frac{\alpha^2 h_n^2}{2x} + \sum_{j=3}^k (-1)^j \left(\frac{1}{j} - \frac{1}{j-1} \right) \frac{(\alpha h_n)^j}{x^{j-1}} \\
 &\quad + (\beta - 1/2) \sum_{j=1}^k \frac{(-1)^j}{j} \left(\frac{\alpha h_n}{x} \right)^j - t_k + R_k.
 \end{aligned}$$

where the first summation on the right side is zero for $k=2$.

(Note: from (3.17) with $k=1$ we see, after the term $-\frac{(ah_n)^2}{x}$ and the 0-terms have been absorbed in R_1 , that

$$\log c_n(x) = \log\{\alpha/\sqrt{(2\pi x)}\} - (\beta-1/2)\frac{ah_n}{x} - t_1 + R_1$$

Recalling the definition of t_k , given by (3.15), it is clear that the entire sum:

$$\sum_{j=3}^k (-1)^j \left(\frac{1}{j} - \frac{1}{j-1} \right) \frac{(ah_n)^j}{x^{j-1}} + (\beta-1/2) \sum_{j=1}^k \frac{(-1)^j}{j} \cdot \left(\frac{ah_n}{x} \right)^j - t_k$$

may be written as

$$g_k = \sum_{i=1}^k \sum_{j=0}^{i+1} b_{i,j} \frac{h_n^j}{x^i}, \text{ where in particular } b_{1,2} = b_{k,k+1} = 0.$$

It follows from (3.18) that for $k=2,3,\dots$ and $x \rightarrow \infty$,

$$\log c_n(x) = \log\{\alpha/\sqrt{(2\pi x)}\} - \frac{a^2 h_n^2}{2x} + g_k + R_k.$$

The lemma follows. ///.

(3.19) Remark: Lemma (3.8) is also valid for all n such that

$|h_n| = |n - x/\alpha| < x^\xi$, $1/2 < \xi < 2/3$, because the inequality:

$(1-H)x/\alpha < x/\alpha - x^\xi < n < x/\alpha + x^\xi < (1+H)x/\alpha$ is certainly true for x sufficiently large.

(3.20) Remark: In the proof of the theorem which follows we will be interested in the form of g_k^μ , where μ is a positive integer. Since

$g_k = \sum_{i=1}^k \sum_{j=0}^{i+1} b_{i,j} \frac{h_n^j}{x^i}$, where $b_{1,2} = 0$, we can expand g_k^μ as a

sum of a finite number of terms of the form $\bar{b}_{i,j} \frac{h_n^j}{x^i}$. Our purpose is to determine the values of i and j appearing in this expansion. First, observe that in the expansion of g_k , the lowest power of x^{-1} is 1 and the highest is k . Hence in the expansion of g_k^μ , the lowest power of x^{-1} is μ and the highest is $k\mu$. That is i satisfies $\mu \leq i \leq k\mu$ in the expansion of g_k^μ . Next, to determine the values of j in the expansion of g_k^μ , write g_k^μ μ times as

$$\sum_{i_1=1}^k \sum_{j_1=0}^{i_1+1} b_{i_1,j_1} \frac{h_n^{j_1}}{x^{i_1}} \cdots \sum_{i_\mu=1}^k \sum_{j_\mu=0}^{i_\mu+1} b_{i_\mu,j_\mu} \frac{h_n^{j_\mu}}{x^{i_\mu}}.$$

When we consider g_k^μ we multiply terms of the form $b_{i_1,j_1} \frac{h_n^{j_1}}{x^{i_1}} \cdots$

$b_{i_\mu,j_\mu} \frac{h_n^{j_\mu}}{x^{i_\mu}}$ to obtain a term of the form $\bar{b}_{i,j} \frac{h_n^j}{x^i}$ where

$$i = \sum_{r=1}^{\mu} i_r, \quad j = \sum_{r=1}^{\mu} j_r, \quad \text{and} \quad \bar{b}_{i,j} = \prod_{r=1}^{\mu} b_{i_r,j_r}.$$

Since $0 \leq j_r \leq i_r + 1$, $r=1, 2, \dots, \mu$, we have that $0 \leq j \leq \sum_{r=1}^{\mu} (i_r + 1) = i + \mu$.

i.e. j satisfies $0 \leq j \leq i + \mu$ in the expansion of g_k^μ . Hence

$$g_k^\mu = \sum_{i=\mu}^{k\mu} \sum_{j=0}^{i+\mu} \bar{b}_{i,j} \frac{h_n^j}{x^i}.$$

However, $\bar{b}_{i,j} = 0$ for $2\mu > j > \mu$. To see this we write, as above,

$$\bar{b}_{\mu,j} = \prod_{r=1}^{\mu} b_{1_r,j_r} \text{ where } \mu = \sum_{r=1}^{\mu} 1_r, 2\mu \geq j = \sum_{r=1}^{\mu} j_r > \mu, \text{ and } 1_r \geq 1,$$

$0 \leq j_r \leq 1_r + 1$ for $r=1,2,\dots,\mu$. It is immediate that $1_r=1$, $r=1,2,\dots,\mu$ and hence $0 \leq j_r \leq 2$, $r=1,2,\dots,\mu$. From the condition $\sum_{r=1}^{\mu} j_r > \mu$ it now follows that at least one $j_r=2$. Thus $b_{1_r,j_r} = b_{1,2} = 0$ and $\bar{b}_{\mu,j} = 0$, for $j > \mu$. Hence

$$g_k^{\mu} = \sum_{j=0}^{\mu} \bar{b}_{\mu,j} \frac{h_n^j}{x^j} + \sum_{1=\mu+1}^{k\mu} \sum_{j=0}^{1+\mu} \bar{b}_{1,j} \frac{h_n^j}{x^j}. \text{ In other words } g_k^{\mu} \text{ is a}$$

finite sum of terms of the form $\bar{b}_{1,j} \frac{h_n^j}{x^j}$ with values 1 and j satisfying $0 \leq j \leq \mu$ for $1=\mu$, and $0 \leq j \leq 1+\mu$ for $1 \geq \mu+1$.

(3.21) Theorem: Suppose $s_n = O(n^r)$, $r \geq 0$, then

$$s_n \rightarrow \ell(B, \alpha, \beta) \iff s_n \rightarrow \ell(V_{\alpha}).$$

Proof: Let $1/2 < \xi < 2/3$, $\bar{b}(x) = \sum_{|h_n| \leq x^{\xi}} c_n(x) s_n$, and

$$\bar{t}(x) = \alpha / \sqrt{2\pi x} \sum_{|h_n| \leq x^{\xi}} \exp\left(-\frac{\alpha^2 h_n^2}{2x}\right) s_n \text{ throughout the proof.}$$

We first show: If $s_n = O(n^r)$, $r \geq 0$, and $s_n \rightarrow \ell(V_{\alpha})$ then, $s_n \rightarrow \ell(B, \alpha, \beta)$.

By the regularity of both methods it is enough to prove the result for $\ell = 0$. Moreover, by Lemma (2.3) it suffices to show that $\bar{b}(x) = o(1)$ as $x \rightarrow \infty$.

Recall, from Remark (3.19), that for n such that $|h_n| \leq x^{\xi}$, any

integer $k \geq 2$, and x sufficiently large

$$(3.22) \quad c_n(x) = \alpha/\sqrt{2\pi x} \exp\left(-\frac{\alpha^2 h_n^2}{2x} + g_k + R_k\right),$$

where g_k and R_k are defined in Lemma (3.8).

Choosing an integer $k > 2r+1$ (so in particular $k \geq 2$) we see that for x sufficiently large,

$$\begin{aligned} b(x) - \bar{c}(x) &= \sum_{|h_n| \leq x} \xi s_n \left\{ c_n(x) - \alpha/\sqrt{2\pi x} \exp\left(-\frac{\alpha^2 h_n^2}{2x}\right) \right\} \\ &= \alpha/\sqrt{2\pi x} \sum_{|h_n| \leq x} \xi s_n \exp\left(-\frac{\alpha^2 h_n^2}{2x}\right) (\exp(g_k + R_k) - 1) \\ &= \alpha/\sqrt{2\pi x} \sum_{|h_n| \leq x} \xi s_n \exp\left(-\frac{\alpha^2 h_n^2}{2x}\right) \sum_{\mu=1}^{\infty} \frac{(g_k + R_k)^\mu}{\mu!} \\ &= A_1(x) + A_2(x) + A_3(x), \end{aligned}$$

where

$$(3.23) \quad A_1(x) = \alpha/\sqrt{2\pi x} \sum_{|h_n| \leq x} \xi s_n \exp\left(-\frac{\alpha^2 h_n^2}{2x}\right) \sum_{\mu=1}^{2s} \frac{g_k^\mu}{\mu!},$$

$$(3.24) \quad A_2(x) = \alpha/\sqrt{2\pi x} \sum_{|h_n| \leq x} \xi s_n \exp\left(-\frac{\alpha^2 h_n^2}{2x}\right) \sum_{\mu=1}^{2s} \frac{(g_k + R_k)^\mu}{\mu!} - \frac{g_k^\mu}{\mu!},$$

$$(3.25) \quad A_3(x) = \alpha/\sqrt{2\pi x} \sum_{|h_n| \leq x} \xi s_n \exp\left(-\frac{\alpha^2 h_n^2}{2x}\right) \sum_{\mu=2s+1}^{\infty} \frac{(g_k + R_k)^\mu}{\mu!},$$

and the positive integer $s > r-1/2$.

We proceed to show that $A_1(x) = o(1)$, $A_2(x) = o(1)$, $A_3(x) = o(1)$.

as $x \rightarrow \infty$.

To see that $A_1(x) = o(1)$, consider for $1 \leq \mu \leq 2s$

$$v(x) := \alpha / \sqrt{2\pi x} \sum_{|h_n| \leq x^\xi} s_n \exp\left(-\frac{\alpha^2 h_n^2}{2x}\right) \frac{g_k^\mu}{\mu!},$$

where g_k^μ is a finite sum of the form: $\sum_{1 \leq j} \bar{b}_{1,j} \frac{h_n^j}{x^{1/2}}$ and $0 \leq j \leq \mu$ for

$1 = \mu$, $0 \leq j \leq 1 + \mu$ for $1 \leq \mu < 1$ (see Remark (3.20)).

If we can show that $v_{1,j} := \bar{b}_{1,j} x^{-1/2} \sum_{|h_n| \leq x^\xi} s_n \exp\left(-\frac{\alpha^2 h_n^2}{2x}\right) \frac{h_n^j}{x^{1/2}}$

is $o(1)$ as $x \rightarrow \infty$ for those values of i and j mentioned above, then

it follows that $v(x) = o(1)$ as $x \rightarrow \infty$, and hence also that

$A_1(x) = o(1)$ as $x \rightarrow \infty$.

To this end, the hypothesis $s_n \rightarrow 0(V_\alpha)$ implies that

$$\sum_{k=0}^{\infty} s_k \exp\left\{-\frac{\alpha(n-k)^2}{2n}\right\} = o(n^{1/2}) \text{ as } n \rightarrow \infty, \quad n=1,2,\dots$$

As $\{s_n\}$ is of finite order, Lemma (3.7) implies that

$$\sum_{n=0}^{\infty} s_n \exp\left\{-\frac{\alpha(n-x)^2}{2x}\right\} = o(x^{1/2}) \text{ as } x \rightarrow \infty.$$

Therefore, by Lemma (3.5) with $q = 0$,

$$(3.26) \quad \sum_{n=0}^{\infty} s_n (n-x)^j \exp\left\{-\frac{\alpha(n-x)^2}{2x}\right\} = o(x^{1/2(j+1)+\eta}) \text{ as } x \rightarrow \infty,$$

for each integer $j \geq 0$, and each $\eta > 0$.

Replacing x with x/a in (3.26) we obtain.

$$(3.27) \quad \sum_{n=0}^{\infty} s_n h_n^j \exp\left(-\frac{\alpha^2 h_n^2}{2x}\right) = o\left(x^{1/2(j+1)+\eta}\right) \text{ as } x \rightarrow \infty,$$

for each integer $j \geq 0$, and each $\eta > 0$.

Since $s_n = O(n^r)$, (3.2) and (3.3) of Lemma (3.1), and (3.27) imply that,

$$(3.28) \quad \sum_{|h_n| \leq x} s_n h_n^j \exp\left(-\frac{\alpha^2 h_n^2}{2x}\right) = o\left(x^{1/2(j+1)+\eta}\right) \text{ as } x \rightarrow \infty,$$

for each integer $j \geq 0$, and each $\eta > 0$.

Consequently, by (3.28)

$$v_{1,j} = \bar{b}_{1,j} x^{-1/2-j} \sum_{|h_n| \leq x} s_n h_n^j \exp\left(-\frac{\alpha^2 h_n^2}{2x}\right) = o\left(x^{-1/2-j+1/2(j+1)+\eta}\right)$$

as $x \rightarrow \infty$, for each $\eta > 0$.

Now, for $1 \leq \mu$, $0 \leq j \leq \mu$

$$v_{1,j} = o\left(x^{-1+1/2(j+\eta)}\right) = o\left(x^{-\mu+1/2(j+\eta)}\right) = o\left(x^{-\mu+1/2(\mu+\eta)}\right)$$

$$= o\left(x^{-1/2(\mu+\eta)}\right) = o\left(x^{-1/2+\eta}\right), \mu \geq 1$$

$= o(1)$ as $x \rightarrow \infty$, since $\eta > 0$ can be chosen arbitrarily small.

(e.g. $\eta = 1/4$)

Next, for $1 \leq j \leq \mu$, $0 \leq j \leq 1 + \mu$.

$$v_{1,j} = o\left(x^{-1/2-1+1/2(j+1)+\eta}\right) = o(x^{-1/2+\eta}), \text{ since } 1 \leq j \leq j-1+1, \text{ that is } -1+1/2(j+1) \leq 0.$$

Choosing $\eta = 1/4$ we have: $v_{1,j} = o(1)$ as $x \rightarrow \infty$.

Thus $v_{1,j} = o(1)$ as $x \rightarrow \infty$, for all the appropriate values of i and j .

It follows that $A_1(x) = o(1)$ as $x \rightarrow \infty$.

In showing that $A_2(x) = o(1)$ as $x \rightarrow \infty$, observe that this will be done independently of the hypothesis $s_n \rightarrow 0(V_\alpha)$; we only use $s_n = O(n^\gamma)$.

Since $s_n = O(n^\gamma)$, it follows from (3.24) that,

$$|A_2(x)| \leq L\alpha/\sqrt{(2\pi x)} \sum_{|h_n| \leq x^\xi} n^\gamma \exp\left(-\frac{\alpha^2 h_n^2}{2x}\right) \sum_{\mu=1}^{2s} \left| \frac{(g_k + R_k)^\mu - g_k^\mu}{\mu!} \right|,$$

where L is some positive constant.

Consider for $1 \leq \mu \leq 2s$,

$$u(x) := \frac{L\alpha}{\mu! \sqrt{(2\pi x)}} \sum_{|h_n| \leq x^\xi} n^\gamma \exp\left(-\frac{\alpha^2 h_n^2}{2x}\right) \left| (g_k + R_k)^\mu - g_k^\mu \right|.$$

If we can show that $u(x) = o(1)$ as $x \rightarrow \infty$, then it follows that

$$A_2(x) = o(1) \text{ as } x \rightarrow \infty.$$

Now for $k \geq 2$, $1/2 < \xi < 2/3$, $|h_n| \leq x^\xi$, $x \rightarrow \infty$, and by Lemma (3.8),

$$R_k = O\left(\frac{|h_n|^{k+1} + 1}{x^k}\right) = O(1) \text{ and } g_k = O(1).$$

Hence for $j=1,2,\dots,\mu$

$$|R_k|^j |g_k|^{\mu-j} = |R_k|^{j-1} |R_k| |g_k|^{\mu-j} = O\left\{\frac{|h_n|^{k+1} + 1}{x^k}\right\}.$$

It now follows from $|(g_k + R_k)^\mu - g_k^\mu| \leq \sum_{j=1}^{\mu} \binom{\mu}{j} |R_k|^j |g_k|^{\mu-j}$ that

$$(g_k + R_k)^\mu - g_k^\mu = O\left\{\frac{|h_n|^{k+1} + 1}{x^k}\right\}.$$

Thus,

$$\begin{aligned} u(x) &= O\left\{x^{-1/2-k} \sum_{|h_n| \leq x^{\frac{1}{2}}} n^r (1 + |h_n|^{k+1}) \exp\left(-\frac{a^2 h_n^2}{2x}\right)\right\} \\ &= O\left\{x^{-1/2-k} \sum_{n=0}^{\infty} n^r \exp\left(-\frac{a^2 h_n^2}{2x}\right)\right\} + O\left\{x^{-1/2-k} \sum_{n=0}^{\infty} n^r |h_n|^{k+1} \exp\left(-\frac{a^2 h_n^2}{2x}\right)\right\}. \end{aligned}$$

By Lemma (3.1),

$$\begin{aligned} u(x) &= O\left\{x^{-1/2-k} x^{r+1/2}\right\} + O\left\{x^{-1/2-k} x^{r+1/2(k+1+1)}\right\} \text{ as } x \rightarrow \infty \\ &= o(1) \text{ as } x \rightarrow \infty \text{ since } k > 2r+1. \end{aligned}$$

It follows that $A_2(x) = o(1)$ as $x \rightarrow \infty$.

In showing that $A_3(x) = o(1)$ as $x \rightarrow \infty$, observe (as in the proof of $A_2(x) = o(1)$ as $x \rightarrow \infty$) that this will be done by only using the hypothesis $g_n = O(n^r)$.

First, for $1/2 < k < 2/3$, $|h_n| \leq x^{\frac{1}{2}}$, $x \rightarrow \infty$, and by Lemma (3.8),

$$\begin{aligned} g_k + R_k &= g_2 + R_2 = \sum_{i=1}^2 \sum_{j=0}^{1+i} b_{i,j} \frac{h_n^j}{x^i} + O\left\{\frac{|h_n|^3 + 1}{x^2}\right\} \\ &= O\left(\frac{1}{x}\right) + O\left(\frac{|h_n|}{x}\right) + O\left(\frac{1}{x^2}\right) + O\left(\frac{|h_n|}{x^2}\right) + O\left(\frac{|h_n|^2}{x^2}\right) \end{aligned}$$

$$\begin{aligned}
& + O\left\{\frac{|h_n|^3}{x^2}\right\} + O\left(\frac{1}{x}\right), \text{ since } b_{1,2} = 0 \\
& = O\left\{\frac{|h_n| + 1}{x}\right\} + O\left\{\frac{|h_n|^3}{x^2}\right\},
\end{aligned}$$

since $\frac{1}{x^2} = O\left(\frac{1}{x}\right)$, $\frac{|h_n|}{x^2} = O\left(\frac{|h_n|}{x}\right)$, and $\frac{|h_n|}{x} = O(x^{\xi-1}) = O(1)$

so that $\frac{|h_n|^2}{x^2} = O\left(\frac{|h_n|}{x}\right)$.

In particular, since $\frac{|h_n|}{x} = O(x^{\xi-1}) = o(1)$, and $\frac{|h_n|^3}{x^2} = O(x^{3\xi-2}) = o(1)$ as $x \rightarrow \infty$, it follows that $g_k + R_k = o(1)$ as $x \rightarrow \infty$. Consequently, for x sufficiently large, $|h_n| \leq x^\xi$, $1/2 < \xi < 2/3$, we obtain $|g_k + R_k| \leq C < 1$.

From this we have:

$$\sum_{\mu=2s+1}^{\infty} \frac{|g_k + R_k|^\mu}{\mu!} = O(|g_k + R_k|^{2s+1}).$$

By (3.25), the hypothesis $s_n = O(n^\Gamma)$, and the last estimate

$$\begin{aligned}
(3.31) \quad A_3(x) &= O\left\{x^{-1/2} \sum_{|h_n| \leq x^\xi} n^r \exp\left(-\frac{a^2 h_n^2}{2x}\right) \sum_{\mu=2s+1}^{\infty} \frac{|g_k + R_k|^\mu}{\mu!}\right\} \\
&= O\left\{x^{-1/2} \sum_{|h_n| \leq x^\xi} n^r \exp\left(-\frac{a^2 h_n^2}{2x}\right) |g_k + R_k|^{2s+1}\right\}
\end{aligned}$$

Next, it is immediate that for constants $a, b, c > 0$,

$(a+b+c)^k \leq 3^k(a^k + b^k + c^k)$. So from

$$g_k + R_k = O\left\{\frac{|h_n| + 1}{x}\right\} + O\left\{\frac{|h_n|^3}{x^2}\right\} \text{ we obtain:}$$

$$|g_k + R_k|^{2s+1} \leq \frac{M}{x^{2s+1}} \left\{ 1 + |h_n| + \frac{|h_n|^3}{x} \right\}^{2s+1} \\ \leq \frac{3^{2s+1} M}{x^{2s+1}} \left\{ 1 + |h_n|^{2s+1} + \frac{|h_n|^{6s+3}}{x^{2s+1}} \right\}$$

where M is some constant.

Substituting the last inequality for $|g_k + R_k|^{2s+1}$ into (3.31) gives:

$$(3.32) \quad \Lambda_3(x) = O\left\{x^{-2s-3/2} \sum_{|h_n| \leq x^\xi} n^r \exp\left(-\frac{\alpha^2 h_n^2}{2x}\right) \left(1 + |h_n|^{2s+1} + \frac{|h_n|^{6s+3}}{x^{2s+1}}\right)\right\} \\ = O\left\{x^{-2s-3/2} \sum_{|h_n| \leq x^\xi} n^r \exp\left(-\frac{\alpha^2 h_n^2}{2x}\right)\right\} \\ + O\left\{x^{-2s-3/2} \sum_{|h_n| \leq x^\xi} n^r |h_n|^{2s+1} \exp\left(-\frac{\alpha^2 h_n^2}{2x}\right)\right\} \\ + O\left\{x^{-4s-5/2} \sum_{|h_n| \leq x^\xi} n^r |h_n|^{6s+3} \exp\left(-\frac{\alpha^2 h_n^2}{2x}\right)\right\} \\ = O(\Lambda_{3,1}(x)) + O(\Lambda_{3,2}(x)) + O(\Lambda_{3,3}(x))$$

Since we chose $s > r - 1/2$ it follows that

$$(3.33) \quad \Lambda_{3,1}(x) = O\left(x^{-2s-3/2} x^{r+1/2}\right) = o(1) \quad \text{as } x \rightarrow \infty,$$

by Lemma (3.1) with $p=0$.

$$(3.34) \quad \Lambda_{3,2}(x) = O\left(x^{-2s-3/2} x^{r+1/2(2s+1+1)}\right) = o(1) \quad \text{as } x \rightarrow \infty,$$

by Lemma (3.1) with $p=2s+1$.

2

$$(3.35) \quad A_{3,3}(x) = O\left(x^{-4s-5/2} x^{r+1/2(6s+3+1)}\right) = o(1) \text{ as } x \rightarrow \infty,$$

by Lemma (3.1) with $p=6s+3$.

Substituting (3.33), (3.34), (3.35) in (3.32) we have, as required,

$$A_3(x) = o(1) \text{ as } x \rightarrow \infty.$$

Therefore, we have shown that

$$(3.36) \quad \bar{b}(x) - \bar{t}(x) = A_1(x) + A_2(x) + A_3(x) = o(1) \text{ as } x \rightarrow \infty.$$

Next, since $s_n \rightarrow 0(V_\alpha)$, it follows by Lemma (3.7) that

$$(3.37) \quad \left(\frac{\alpha}{2\pi x}\right)^{1/2} \sum_{n=0}^{\infty} s_n \exp\left\{-\frac{\alpha(n-x)^2}{2x}\right\} = o(1) \text{ as } x \rightarrow \infty.$$

Replacing x by x/α in (3.37) we have:

$$(3.38) \quad \alpha/\sqrt{(2\pi x)} \sum_{n=0}^{\infty} s_n \exp\left(-\frac{\alpha^2 h_n^2}{2x}\right) = o(1) \text{ as } x \rightarrow \infty.$$

Recalling that $s_n = O(n^r)$, (3.38), and (3.2), (3.3) of Lemma(3.1) with $p=0$, imply that

$$(3.39) \quad \bar{t}(x) = o(1) \text{ as } x \rightarrow \infty.$$

Therefore, from (3.36) and (3.39) it follows that $\bar{b}(x) = o(1) x \rightarrow \infty$.

This completes the first part of the theorem.

Now show: if $s_n = O(n^r)$, $r > 0$, and $s_n \rightarrow \ell(B, \alpha, \beta)$, then $s_n \rightarrow \ell(V_\alpha)$.

Again we assume $\ell=0$. By Remark (3.19), Lemma (2.3), and our hypothesis,

$$\alpha/\sqrt{2\pi x} \sum_{|h_n| \leq x} s_n \exp\left(-\frac{\alpha^2 h_n^2}{2x} + g_k + R_k\right) = o(1) \text{ as } x \rightarrow \infty$$

that is

$$\begin{aligned} (3.40) \quad & \alpha/\sqrt{2\pi x} \sum_{|h_n| \leq x} s_n \exp\left(-\frac{\alpha^2 h_n^2}{2x}\right) \\ & + \alpha/\sqrt{2\pi x} \sum_{|h_n| \leq x} s_n \exp\left(-\frac{\alpha^2 h_n^2}{2x}\right) (\exp(g_k + R_k) - 1) \\ & = \bar{t}(x) + \alpha/\sqrt{2\pi x} \sum_{|h_n| \leq x} s_n \exp\left(-\frac{\alpha^2 h_n^2}{2x}\right) \sum_{\mu=1}^{2s} \frac{g_k^\mu}{\mu!} \\ & + \alpha/\sqrt{2\pi x} \sum_{|h_n| \leq x} s_n \exp\left(-\frac{\alpha^2 h_n^2}{2x}\right) \sum_{\mu=1}^{2s} \frac{(g_k + R_k)^\mu - g_k^\mu}{\mu!} \\ & + \alpha/\sqrt{2\pi x} \sum_{|h_n| \leq x} s_n \exp\left(-\frac{\alpha^2 h_n^2}{2x}\right) \sum_{\mu=2s+1}^{\infty} \frac{(g_k + R_k)^\mu}{\mu!} \\ & = \bar{t}(x) + A_1(x) + A_2(x) + A_3(x) = o(1) \text{ as } x \rightarrow \infty, \end{aligned}$$

where $A_1(x)$, $A_2(x)$, and $A_3(x)$ are defined by (3.23), (3.24) and (3.25) and $k > 2r+1$ and the positive integer $s > r - 1/2$.

When we dealt with $A_2(x)$ and $A_3(x)$ previously it was clear that both were $o(1)$ only because of the condition $s_n = O(n^r)$, which we have here. Thus, from (3.40) we obtain:

$$\bar{t}(x) + A_1(x) = o(1) \text{ as } x \rightarrow \infty.$$

Next, by the hypothesis $s_n = O(n^r)$, and (3.4) of Lemma (3.1) with $p=0$,

$$(3.41) \quad \bar{t}(x) = O\left\{x^{-1/2} \sum_{|h_n| \leq x} n^r \exp\left(-\frac{\alpha^2 h_n^2}{2x}\right)\right\}$$

$$= O\left\{x^{-1/2} \sum_{n=0}^{\infty} n^r \exp\left(-\frac{\alpha^2 h_n^2}{2x}\right)\right\} = O(x^r).$$

Let $\gamma := \inf\{\bar{\gamma} : \bar{t}(x) = O(x^{\bar{\gamma}})\}$. It follows from (3.41) that $\gamma \leq r$. If $\gamma < 0$, then $\bar{t}(x) = o(1)$ as $x \rightarrow \infty$, which is what we eventually want to show. Therefore let us assume that $0 \leq \gamma \leq r$, and prove that $\bar{t}(x) = o(1)$ as $x \rightarrow \infty$.

From the definition of γ we have: $\bar{t}(x) = o(x^{\gamma+\epsilon})$ as $x \rightarrow \infty$, for any $\epsilon > 0$, and that $\bar{t}(x) = o(x^{\gamma-\delta})$ as $x \rightarrow \infty$ for no $\delta > 0$.

Consider,

$$A_1(x) = \alpha/\sqrt{2\pi x} \sum_{|h_n| \leq x} \xi s_n \exp\left(-\frac{\alpha^2 h_n^2}{2x}\right) \sum_{\mu=1}^{2s} \frac{g_k^\mu}{\mu!}, \text{ and for } 1 \leq \mu \leq 2s \text{ let}$$

$$p(x) = \alpha/\sqrt{2\pi x} \sum_{|h_n| \leq x} \xi s_n \exp\left(-\frac{\alpha^2 h_n^2}{2x}\right) \frac{g_k^\mu}{\mu!}, \text{ where } g_k^\mu \text{ is a finite sum of}$$

the form $\sum_{1,j} \bar{b}_{1,j} \frac{h_n^j}{x^1}$, $0 \leq j \leq \mu$ for $1 \leq \mu$, $0 \leq j \leq 1+\mu$ for $1 \geq \mu+1$. (see Remark

(3.20)).

Next, let

$$p_{1,j} = \bar{b}_{1,j} x^{-1/2} \sum_{|h_n| \leq x} \xi s_n \exp\left(-\frac{\alpha^2 h_n^2}{2x}\right) \frac{h_n^j}{x^1} \text{ for those values of } 1$$

and j mentioned above.

Since $\bar{t}(x) = o(x^{\gamma+\epsilon})$ as $x \rightarrow \infty$, for any $\epsilon > 0$, it follows that

$$\sum_{|h_n| \leq x} \xi s_n \exp\left(-\frac{\alpha^2 h_n^2}{2x}\right) = o(x^{1/2+\gamma+\epsilon}) \text{ as } x \rightarrow \infty, \text{ for any } \epsilon > 0.$$

As $s_n = O(n^\gamma)$, it follows by (3.2), (3.3) of Lemma (3.1) with $p=0$ that

$$\sum_{n=0}^{\infty} s_n \exp\left(-\frac{\alpha^2 h_n^2}{2x}\right) = o(x^{1/2+\gamma+\epsilon}) \text{ as } x \rightarrow \infty, \text{ for any } \epsilon > 0.$$

Since $\{s_n\}$ is of finite order and $\gamma+\epsilon > 0$, it follows by Lemma (3.6) with $q=\gamma+\epsilon$ that

$$(3.42) \quad \sum_{n=0}^{\infty} s_n h_n^j \exp\left(-\frac{\alpha^2 h_n^2}{2x}\right) = o(x^{1/2(j+1)+\gamma+\epsilon+\eta})$$

as $x \rightarrow \infty$, for each integer $j \geq 0$ and each $\eta > 0$.

Since $s_n = O(n^\gamma)$, (3.2) and (3.3) of Lemma (3.1), and (3.42) imply that

$$\sum_{|h_n| \leq x^\xi} s_n h_n^j \exp\left(-\frac{\alpha^2 h_n^2}{2x}\right) = o(x^{1/2(j+1)+\gamma+\epsilon+\eta})$$

as $x \rightarrow \infty$, for each integer $j \geq 0$ and each $\eta > 0$.

Consequently, $p_{1,j} = o(x^{-1/2-1+1/2(j+1)+\gamma+\epsilon+\eta})$ as $x \rightarrow \infty$,

where $\epsilon > 0$ and $\eta > 0$ are both arbitrary.

Now for $1 \leq \mu$, $0 \leq j \leq \mu$,

$$p_{1,j} = o(x^{-1/2-\mu+1/2(j+1)+\gamma+\epsilon+\eta})$$

$$= o(x^{-\mu+1/2+\gamma+\epsilon+\eta}), \quad 0 \leq j \leq \mu$$

$$= o(x^{-1/2+\gamma+\epsilon+\eta}), \quad \mu \geq 1.$$

Choosing $\epsilon > 0$ and $\eta > 0$ small enough we have: $p_{1,j} = o(x^{\gamma-\delta_0})$ for a

$\delta_0 > 0$. (e.g. for $\epsilon=\eta=1/8$, then $\delta_0=1/4$)

In the second case: $1 \leq \mu+1$, $0 \leq j \leq 1+\mu$,

$$\begin{aligned} p_{1,j} &= o(x^{-1/2-1+1/2(j+1)+\gamma+\epsilon+\eta}) \\ &= o(x^{-1/2-1/2+1+1/2(j-1+1)+\gamma+\epsilon+\eta}) \\ &= o(x^{-1/2-1/2(\mu+1)+1/2(\mu+1)+\gamma+\epsilon+\eta}), \quad 1 \leq \mu+1, \quad j \leq 1+\mu \\ &= o(x^{-1/2+\gamma+\epsilon+\eta}), \end{aligned}$$

$$= o(x^{\gamma-\delta_0}) \text{ with the same } \delta_0 > 0 \text{ as above.}$$

Thus, for the appropriate values of i and j , $p_{i,j} = o(x^{\gamma-\delta_0})$ for a $\delta_0 > 0$. As $p(x)$ is a finite sum of terms of the form $p_{i,j}$ it follows that $p(x) = o(x^{\gamma-\delta_0})$ for a $\delta_0 > 0$. A similar reason shows that $A_1(x) = o(x^{\gamma-\delta_0})$ for a $\delta_0 > 0$. It follows from $\bar{t}(x) + A_1(x) = o(1)$, and $A_1(x) = o(x^{\gamma-\delta_0})$ that $\bar{t}(x) = o(x^{\gamma-\delta_0}) + o(1)$ for a $\delta_0 > 0$. Now, if $\gamma - \delta_0 > 0$, then $\bar{t}(x) = o(x^{\gamma-\delta_0})$ for a $\delta_0 > 0$, which contradicts the minimality of γ . Thus, $\gamma - \delta_0 \leq 0$ and this implies that $\bar{t}(x) = o(1)$ as $x \rightarrow \infty$.

Finally, since $s_n = O(n^\gamma)$, (3.2), (3.3) of Lemma (3.1) and $\bar{t}(x) = o(1)$ as $x \rightarrow \infty$, imply that

$$\alpha/\sqrt{2\pi x} \sum_{n=0}^{\infty} s_n \exp\left(-\frac{\alpha^2 h_n^2}{2x}\right) = o(1) \text{ as } x \rightarrow \infty.$$

Replacing x with αx we have:

$$\left(\frac{\alpha}{2\pi x}\right)^{1/2} \sum_{n=0}^{\infty} s_n \exp\left(-\frac{\alpha(n-x)^2}{2x}\right) = o(1) \text{ as } x \rightarrow \infty, \text{ and of course,}$$

$$\left(\frac{\alpha}{2\pi m}\right)^{1/2} \sum_{n=0}^{\infty} s_n \exp\left(-\frac{\alpha(n-m)^2}{2m}\right) = o(1) \text{ as } m \rightarrow \infty, m=1,2,\dots$$

$$\text{i.e. } s_n \rightarrow 0(V_\alpha). ///$$

The next theorem is an equiconvergence result applicable to $\{s_n\}$ for which neither convergence statement $s_n \rightarrow l(V_\alpha)$, $s_n \rightarrow l(B, \alpha, \beta)$ need hold. Theorem (3.21) is a more general theorem than (3.43) but less general in that the equiconvergence property is lost.

(3.43) Theorem (see Borwein and Robinson [7], Theorem 2): If

$s_n = O(n^r)$, $r < 1/2$, then

$$\sum_{n=N_0}^{\infty} s_n \left\{ c_n(x) - \alpha/\sqrt{2\pi x} \exp\left(-\frac{\alpha^2 h_n^2}{2x}\right) \right\} = o(1) \text{ as } x \rightarrow \infty$$

In particular $s_n \rightarrow l(B, \alpha, \beta) \Leftrightarrow s_n \rightarrow l(V_\alpha)$.

Proof: Let $a_n(x) := c_n(x) - \alpha/\sqrt{2\pi x} \exp\left(-\frac{\alpha^2 h_n^2}{2x}\right)$ for $n > N_0$, and $1/2 < \xi < 2/3$.

Write,

$$\begin{aligned} (3.44) \quad V(x) &:= \left| \sum_{n=N_0}^{\infty} s_n a_n(x) \right| \\ &= \left| \left\{ \sum_{N_0 < n < x/\alpha - x} + \sum_{|h_n| < x^\xi} + \sum_{n > x/\alpha + x} \right\} s_n a_n(x) \right| \\ &= (V_1(x) + V_2(x) + V_3(x)) \end{aligned}$$

where,

$$V_1(x) = \left| \sum_{N_0 \leq n < x/\alpha - x} s_n a_n(x) \right|, \quad V_2(x) = \left| \sum_{|h_n| \leq x} s_n a_n(x) \right|, \text{ and}$$

$$V_3(x) = \left| \sum_{n > x/\alpha + x} s_n a_n(x) \right|.$$

$$(3.45) \quad V_1(x) = O \left\{ \sum_{N_0 \leq n < x/\alpha - x} n^r c_n(x) \right\} + O \left\{ x^{-1/2} \sum_{N_0 \leq n < x/\alpha - x} n^r \exp \left(-\frac{\alpha^2 h_n^2}{2x} \right) \right\}$$

$= o(1)$ as $x \rightarrow \infty$, by Lemma (2.3), and (3.3) of

Lemma (3.1) with $p=0$.

$$(3.46) \quad V_3(x) = O \left\{ \sum_{n > x/\alpha + x} n^r c_n(x) \right\} + O \left\{ x^{-1/2} \sum_{n > x/\alpha + x} n^r \exp \left(-\frac{\alpha^2 h_n^2}{2x} \right) \right\}$$

$= o(1)$ as $x \rightarrow \infty$, by Lemma (2.3), and (3.2) of

Lemma (3.1) with $p=0$.

For the sum $V_2(x)$ we use Lemma (2.1), so that

$$(3.47) \quad V_2(x) = \frac{\alpha}{\sqrt{2\pi x}} \sum_{|h_n| \leq x} s_n \exp \left(-\frac{\alpha^2 h_n^2}{2x} \right) \left\{ O \left(\frac{|h_n|+1}{x} \right) + O \left(\frac{|h_n|^3}{x^2} \right) \right\}$$

$$= O \left\{ x^{-1/2} \sum_{|h_n| \leq x} n^r \exp \left(-\frac{\alpha^2 h_n^2}{2x} \right) \left(\frac{|h_n|+1}{x} + \frac{|h_n|^3}{x^2} \right) \right\}$$

$$= O \left\{ x^{-3/2} \sum_{|h_n| \leq x} n^r |h_n| \exp \left(-\frac{\alpha^2 h_n^2}{2x} \right) \right\}$$

$$\begin{aligned}
& + O \left\{ x^{-3/2} \sum_{|h_n| \leq x} \xi^{n^r} \exp \left(-\frac{\alpha^2 h_n^2}{2x} \right) \right\} \\
& + O \left\{ x^{-5/2} \sum_{|h_n| \leq x} \xi^{n^r} |h_n|^3 \exp \left(-\frac{\alpha^2 h_n^2}{2x} \right) \right\} \\
& = O(V_{2,1}(x)) + O(V_{2,2}(x)) + O(V_{2,3}(x))
\end{aligned}$$

Since $r < 1/2$,

$$V_{2,1}(x) = O \left(x^{-3/2} x^{r+1/2(1+1)} \right) = O(x^{r-1/2}) = o(1) \text{ as } x \rightarrow \infty,$$

by Lemma (3.1) with $p=1$.

$$V_{2,2}(x) = O \left(x^{-3/2} x^{r+1/2} \right) = O(x^{r-1}) = o(1) \text{ as } x \rightarrow \infty,$$

by Lemma (3.1) with $p=0$.

$$V_{2,3}(x) = O \left(x^{-5/2} x^{r+1/2(3+1)} \right) = O(x^{r-1/2}) = o(1) \text{ as } x \rightarrow \infty,$$

by Lemma (3.1) with $p=3$.

It follows from (3.47) that

$$(3.48) \quad V_2(x) = o(1) \text{ as } x \rightarrow \infty.$$

Substituting (3.45), (3.48), and (3.46) in (3.44) yields the required result.

For the final part, assume first, that $s_n \rightarrow \ell(B, \alpha, \beta)$,

$$1. \quad \sum_{n=N_0}^{\infty} s_n c_n(x) \rightarrow \ell \text{ as } x \rightarrow \infty. \text{ Then as } x \rightarrow \infty,$$

$$\alpha/J(2\pi x) \sum_{n=N_0}^{\infty} s_n \exp \left(-\frac{\alpha^2 h_n^2}{2x} \right) = \sum_{n=N_0}^{\infty} s_n c_n(x) - \sum_{n=N_0}^{\infty} s_n a_n(x) \rightarrow \ell.$$

Replacing x with αx we have:

$$\left(\frac{\alpha}{2\pi x}\right)^{1/2} \sum_{n=N_0}^{\infty} s_n \exp\left\{-\frac{\alpha(n-x)^2}{2x}\right\} \rightarrow \ell \text{ as } x \rightarrow \infty.$$

Since the finite sum $\left(\frac{\alpha}{2\pi x}\right)^{1/2} \sum_{n=0}^{N_0-1} s_n \exp\left\{-\frac{\alpha(n-x)^2}{2x}\right\} = o(1)$

we have: $\left(\frac{\alpha}{2\pi x}\right)^{1/2} \sum_{n=0}^{\infty} s_n \exp\left\{-\frac{\alpha(n-x)^2}{2x}\right\} \rightarrow \ell \text{ as } x \rightarrow \infty$, and of

course, $\left(\frac{\alpha}{2\pi m}\right)^{1/2} \sum_{n=0}^{\infty} s_n \exp\left\{-\frac{\alpha(n-m)^2}{2m}\right\} \rightarrow \ell \text{ as } m \rightarrow \infty, m=1,2,\dots$

i.e. $s_n \rightarrow \ell(V_\alpha)$.

Conversely, we may assume by regularity that $\ell = 0$, so that $s_n \rightarrow 0(V_\alpha)$ is our hypothesis and we want to show that $s_n \rightarrow 0(B, \alpha, \beta)$.

Since $\left(\frac{\alpha}{2\pi n}\right)^{1/2} \sum_{k=0}^{\infty} s_k \exp\left\{-\frac{\alpha(n-k)^2}{2n}\right\} = o(1) \text{ as } n \rightarrow \infty, n=1,2,\dots$

and $\{s_n\}$ is of finite order, Lemma (3.7) implies that

$$\left(\frac{\alpha}{2\pi x}\right)^{1/2} \sum_{n=0}^{\infty} s_n \exp\left\{-\frac{\alpha(n-x)^2}{2x}\right\} = o(1) \text{ as } x \rightarrow \infty.$$

Hence, $\alpha/\sqrt{(2\pi x)} \sum_{n=0}^{\infty} s_n \exp\left(-\frac{\alpha^2 h_n^2}{2x}\right) = o(1)$, which implies that

$$\alpha/\sqrt{(2\pi x)} \sum_{n=N_0}^{\infty} s_n \exp\left(-\frac{\alpha^2 h_n^2}{2x}\right) = o(1) \text{ as } x \rightarrow \infty.$$

Thus, as $x \rightarrow \infty$

$$\sum_{n=N_0}^{\infty} s_n \tilde{e}_n(x) = \sum_{n=N_0}^{\infty} s_n a_n(x) + \alpha/\sqrt{(2\pi x)} \sum_{n=N_0}^{\infty} s_n \exp\left(-\frac{\alpha^2 h_n^2}{2x}\right) = o(1).$$

i.e. $s_n \rightarrow O(B, \alpha, \beta)$.

The next four results are due to Faulhaber [10] and Bingham [2] and are stated as lemmas. These will be used with Theorem (3.21) to obtain equivalences between the (B, α, β) method and those of Euler, Meyer-König, and Taylor, under the assumption $\{s_n\}$ is of finite order. Later we obtain inclusions under the weaker assumption (T_r) .

(3.49) Lemma([2], corollary 3): For $\{s_n\}$ of finite order,

$$s_n \rightarrow L(B) \iff s_n \rightarrow L(V_1).$$

This is, of course, a special case of Theorem (3.21) with $\alpha = \beta = 1$.

(3.50) Lemma([2], corollary 4): For $0 < \alpha < 1$, and $\{s_n\}$ of finite order,

$$s_n \rightarrow L(E_\alpha) \iff s_n \rightarrow L(V_{\frac{1}{1-\alpha}}).$$

(3.51) Lemma([2], corollary 5): For $0 < \alpha < 1$, and $\{s_n\}$ of finite order,

$$s_n \rightarrow L(S_\alpha) \iff s_n \rightarrow L(V_{1-\alpha}).$$

(3.52) Lemma([2], corollary 6): For $0 < \alpha < 1$, and $\{s_n\}$ of finite order,

$$s_n \rightarrow L(T_\alpha) \iff s_n \rightarrow L(V_{\frac{1}{\alpha}-1}).$$

Lemma (3.53) is also due to Bingham [2] but has also been proved earlier by Borwein and Robinson([7], Theorem 3) under the condition $s_n = o(\sqrt{n})$.

(3.53) Lemma ([2], Theorem 3): If $0 < \alpha < \alpha' \leq 1$ and $\{s_n\}$ is of finite order, then $s_n \rightarrow l(V_{\alpha'}) \Leftrightarrow s_n \rightarrow l(V_{\alpha})$.

Note: The proof of Lemma (3.53) is obtained by using some known results linking the circle methods of convergence. But, we can get the same inclusion under the same hypotheses by using Theorem (3.21) and a result due to Borwein (see Result (I) in Remark (3.58)) as follows:

$$s_n \rightarrow l(V_{\alpha'}) \Leftrightarrow s_n \rightarrow l(B, \alpha', \beta) \Leftrightarrow s_n \rightarrow l(B, \alpha, \beta) \Leftrightarrow s_n \rightarrow l(V_{\alpha}).$$

(3.54) Corollary: For $0 < \alpha < 1$, and $\{s_n\}$ of finite order,

$$s_n \rightarrow l(E_{\alpha}) \Leftrightarrow s_n \rightarrow l(B, \frac{1}{1-\alpha}, \beta).$$

Proof: By Lemma (3.50) and Theorem (3.21) we have:

$$s_n \rightarrow l(E_{\alpha}) \Leftrightarrow s_n \rightarrow l(V_{\frac{1}{1-\alpha}}) \Leftrightarrow s_n \rightarrow l(B, \frac{1}{1-\alpha}, \beta). \quad ///$$

(3.55) Corollary: For $0 < \alpha < 1$, and $\{s_n\}$ of finite order,

$$s_n \rightarrow l(S_{\alpha}) \Leftrightarrow s_n \rightarrow l(B, 1-\alpha, \beta).$$

Proof: By Lemma (3.51) and Theorem (3.21) we obtain:

$$s_n \rightarrow l(S_{\alpha}) \Leftrightarrow s_n \rightarrow l(V_{1-\alpha}) \Leftrightarrow s_n \rightarrow l(B, 1-\alpha, \beta). \quad ///$$

(3.56) Corollary: For $0 < \alpha < 1$, and $\{s_n\}$ of finite order,

$$s_n \rightarrow l(T_{\alpha}) \Leftrightarrow s_n \rightarrow l(B, \frac{1}{\alpha} - 1, \beta).$$

Proof: By Lemma (3.52) and Theorem (3.21) we have:

$$s_n \rightarrow \ell(T_\alpha) \Leftrightarrow s_n \rightarrow \ell(V_{\frac{1}{\alpha}-1}) \Leftrightarrow s_n \rightarrow \ell(B, \frac{1}{\alpha}-1, \beta). \quad ///$$

(3.57) Corollary: For $0 < \alpha < \alpha' < \infty$, and $\{s_n\}$ of finite order,

$$s_n \rightarrow \ell(B, \alpha', \beta') \Leftrightarrow s_n \rightarrow \ell(B, \alpha, \beta).$$

Proof: By applying Theorem (3.21) twice, and Lemma (3.53) we have: $s_n \rightarrow \ell(B, \alpha', \beta') \Leftrightarrow s_n \rightarrow \ell(V_{\alpha'}) \Leftrightarrow s_n \rightarrow \ell(V_\alpha) \Leftrightarrow s_n \rightarrow \ell(B, \alpha, \beta). \quad ///$

(3.58) Remark: Borwein([5], Result (I)) has proved the following result:

(I) If $\alpha' > \alpha > 0$ and, for any non-negative integer $M > -\beta/\alpha$,

$\sum_{n=M}^{\infty} \frac{a_n}{\Gamma(\alpha n + \beta)} x^n$ is convergent for all $x > 0$, then

$$\sum_{n=0}^{\infty} a_n = \ell(B, \alpha', \beta') + \sum_{n=0}^M a_n = \ell(B, \alpha, \beta).$$

(I) is stronger than Corollary (3.57) in that for $\{s_n\}$ of finite order (i.e. $s_n = O(n^r)$ for some r), it then follows that

$\sum_{n=M}^{\infty} \frac{a_n}{\Gamma(\alpha n + \beta)} x^n$ is convergent for all $x > 0$, where $a_n = s_n - s_{n-1}$.

$a_{-1} = 0$. This can be shown as follows: Since $s_n = O(n^r)$, then also $a_n = O(n^r)$, i.e. $|a_n| \leq Kn^r$ for some $K > 0$. Therefore,

$$\frac{|a_n|}{\Gamma(\alpha n + \beta)} x^n \leq \frac{Kn^r}{\Gamma(\alpha n + \beta)} x^n, \quad n > M. \quad \text{Using the version of Stirling's}$$

formula given in the proof of Lemma (2.2), we see that

$\frac{\Gamma(\alpha n + \beta)}{\Gamma(\alpha(n+1) + \beta)} = o(1)$ as $n \rightarrow \infty$. It follows by the ratio test that

$\sum_{n=N}^{\infty} \frac{n^r}{\Gamma(\alpha n + \beta)} x^n$ is convergent for all x . The conclusion follows.

From what we have done so far we may weaken the hypothesis in the previous corollaries, but when this is done some of the equivalences are lost.

(3.59) Corollary: Suppose that $\alpha > \alpha' > 0$, and (T_r) holds. Then $s_n \rightarrow l(B, \alpha, \beta)$ implies $s_n \rightarrow l(B, \alpha', \beta')$.

Proof: Recalling the notation introduced in Chapter 2 we have:

$$b(x) = \sum_{n=N_0}^{\infty} s_n c_n(x) \rightarrow l \text{ as } x \rightarrow \infty. \text{ Hence } b(x) = O(x^r) \text{ as } x \rightarrow \infty.$$

$r > 0$. Next, (T_r) and $b(x) = O(x^r)$ as $x \rightarrow \infty$ imply that $s_n = O(n^r)$, by Theorem (2.15). By Theorem (3.21), $s_n \rightarrow l(B, \alpha, \beta)$ and $s_n = O(n^r)$ imply $s_n \rightarrow l(V_\alpha)$. Since $\{s_n\}$ is of finite order, and $\alpha > \alpha' > 0$ we have by Lemma (3.53), $s_n \rightarrow l(V_{\alpha'})$. Finally, by Theorem (3.21) $s_n \rightarrow l(B, \alpha', \beta')$. ///

Note the hypotheses of Corollary (3.59) imply $s_n = O(n^r)$. This in turn implies that the series

$$\sum_{n=N}^{\infty} \frac{s_n}{\Gamma(\alpha'n + \beta')} x^n \text{ is convergent for all } x, \text{ where } \alpha'H + \beta' > 0$$

(see Remark (3.58)). Thus, Corollary (3.59) also follows from

Borwein's Result (I).

(3.60) Corollary: Suppose that $\alpha > 1$, and that (T_r) holds. Then

$$s_n \rightarrow \ell(B, \alpha, \beta) \text{ implies } s_n \rightarrow \ell(E_{1-\frac{1}{\alpha}}).$$

Proof: As in the last proof, our hypotheses imply that $s_n = O(n^r)$. By Theorem (3.21), $s_n \rightarrow \ell(V_\alpha)$. Since $\{s_n\}$ is of finite order we have by Lemma (3.50), $s_n \rightarrow \ell(E_{1-\frac{1}{\alpha}})$. Note for $\alpha > 1$,

$0 < 1 - \frac{1}{\alpha} < 1$, which is required in the definition of the $E_{1-\frac{1}{\alpha}}$. ///

(3.61) Corollary: Suppose that $0 < \alpha < 1$, and (T_r) holds. Then

$$s_n \rightarrow \ell(B, \alpha, \beta) \text{ implies } s_n \rightarrow \ell(S_{1-\alpha}).$$

Proof: As in the previous proof our hypotheses imply that $s_n = O(n^r)$, and hence $s_n \rightarrow \ell(V_\alpha)$. Since $\{s_n\}$ is of finite order, Lemma (3.51) implies that $s_n \rightarrow \ell(S_{1-\alpha})$. Note for $0 < \alpha < 1$, $0 < 1 - \alpha < 1$ which is required in the definition of the $S_{1-\alpha}$ method. ///

(3.62) Corollary: Suppose that (T_r) holds and that $s_n \rightarrow \ell(B, \alpha, \beta)$.

$$\text{Then } s_n \rightarrow \ell(T_{1/(1+\alpha)}).$$

Proof: The hypotheses imply that $\{s_n\}$ is of finite order, and hence that $s_n \rightarrow \ell(V_\alpha)$. By Lemma (3.52), $s_n \rightarrow \ell(T_{1/(1+\alpha)})$. Note $\alpha > 0$ is assumed in the definition of the (B, α, β) method and this

implies that that $0 < \frac{1}{1+\alpha} < 1$ which is required in the definition of the $T_{1/(1+\alpha)}$ method. ///

(3.63) Corollary: Suppose that $0 < \alpha < 1$, and (T_r) holds. Then $s_n \rightarrow l(S_{1-\alpha})$ implies $s_n \rightarrow l(B, \alpha, \beta)$.

Proof: Recalling that $s_n \rightarrow l(S_{1-\alpha})$ means $S_{1-\alpha}(m) = \alpha^{m+1} \sum_{i=0}^m \binom{m+1}{i} (1-\alpha)^i s_i \rightarrow l$ as $m \rightarrow \infty$, $m = 0, 1, 2, \dots$ we have in particular that $S_{1-\alpha}(m) = O(m^r)$ as $m \rightarrow \infty$, $r > 0$.

By a result due to Sitaraman ([18], Theorem 1), (T_r) and $S_{1-\alpha}(m) = O(m^r)$ as $m \rightarrow \infty$, imply that $s_n = O(n^r)$. As $\{s_n\}$ is of finite order, the hypothesis $s_n \rightarrow l(S_{1-\alpha})$ yields $s_n \rightarrow l(V_\alpha)$, by Lemma (3.51). The result now follows from Theorem (3.21): ///

Combining Corollaries (3.61) and (3.63) we have:

(3.64) Corollary: Suppose that $0 < \alpha < 1$, and (T_r) holds. Then $s_n \rightarrow l(S_{1-\alpha}) \Leftrightarrow s_n \rightarrow l(B, \alpha, \beta)$.

(3.65) Corollary: Suppose that (T_r) holds and that $s_n \rightarrow l(B)$. Then $s_n \rightarrow l(S_\alpha)$.

Proof: It follows from a result due to Rajagopal ([17], Theorem 1) or by Theorem (2.15) with $\alpha = \beta = 1$, that $s_n = O(n^r)$ i.e. $\{s_n\}$ is of finite order. By Lemma (3.49) or Theorem (3.21), $s_n \rightarrow l(V_1)$. Since

$1 > 1 - \alpha > 0$, Lemma (3.53) implies that $s_n \rightarrow \ell(V_{1-\alpha})$. By Lemma (3.51), $s_n \rightarrow \ell(S_\alpha)$. ///

(3.66) Corollary: Suppose that (T_0) holds and that $s_n \rightarrow \ell(S_\alpha)$. Then $s_n \rightarrow \ell(B)$.

Proof: The hypothesis $s_n \rightarrow \ell(S_\alpha)$ implies that $S_\alpha(m) = (1-\alpha)^{m+1} \sum_{i=0}^m \binom{m+1}{i} \alpha^i s_i = O(1)$, and hence $s_n = O(1)$, by Sitaraman ([18], Theorem 1 with $r = 0$). By a result due to Meyer-König ([16], Satz 25), $s_n \rightarrow \ell(S_\alpha)$ and $s_n = O(1)$ imply that $s_n \rightarrow \ell(B)$. ///

Combining the last two results we have:

(3.67) Corollary: Suppose that (T_0) holds. Then $s_n \rightarrow \ell(S_\alpha) \iff s_n \rightarrow \ell(B)$.

CHAPTER 4

A TAUBERIAN THEOREM FOR (B, α, β) SUMMABILITY

In this chapter we prove our main result, Theorem (4.2). This result will be deduced from a corresponding theorem for S_α summability due to Sitaraman.

(4.1) Lemma (Sitaraman [18], Theorem 2): If (T_r) holds and $s_n \rightarrow l(S_\alpha)$, then $s_n \rightarrow l(C_{2r})$.

Note the corresponding Tauberian theorem for Borel summability is due to Rajagopal ([17], Theorem 2), and can be deduced by applying Corollary (3.65) and Sitaraman's Theorem.

(4.2) Theorem: If (T_r) holds and $s_n \rightarrow l(B, \alpha, \beta)$, then $s_n \rightarrow l(C_{2r})$.

Proof: It is enough to prove the theorem for $1 > \alpha > 0$. That is, it suffices to show: (T_r) and $s_n \rightarrow l(B, \alpha', \beta)$ imply $s_n \rightarrow l(C_{2r})$, where $1 > \alpha' > 0$. For suppose $\alpha > 1 > \alpha' > 0$. By Corollary (3.59) the hypotheses will imply $s_n \rightarrow l(B, \alpha', \beta)$. Thus the result for all $\alpha > 0$ follows from the corresponding result with $1 > \alpha > 0$.

Restricting to $1 > \alpha > 0$, Corollary (3.61) implies $s_n \rightarrow l(S_{1-\alpha})$. The conclusion, $s_n \rightarrow l(C_{2r})$, now follows from Lemma (4.1). ///

(4.3) Remark: A slightly different proof of Theorem (4.2) is as

follows: By Theorem (2.15) we have $s_n = O(n^r)$, and it follows by Theorem (3.21) that $s_n \rightarrow \ell(V_\alpha)$. Since $\{s_n\}$ is of finite order, an application of Lemma (3.53) shows that $s_n \rightarrow \ell(V_\beta)$, $\beta = \frac{\alpha}{\alpha+1}$, and hence by Lemma (3.51), $s_n \rightarrow \ell(S_{1/(\alpha+1)})$. As $0 < \frac{1}{\alpha+1} < 1$, for any $\alpha > 0$, we may apply Lemma (4.1) once again to get the conclusion, $s_n \rightarrow \ell(C_{2r})$.

(4.4) Corollary: If $a_n = s_n - s_{n-1} = O_L(n^{r-1/2})$, $r > 0$, and

$$\sum_{n=0}^{\infty} a_n = \ell(B, \alpha, \beta), \text{ then } \sum_{n=0}^{\infty} a_n = \ell(C_{2r}).$$

Proof: By Remark (2.5), $a_n = O_L(n^{r-1/2})$ implies (T_r) , and

$$\sum_{n=0}^{\infty} a_n = \ell(B, \alpha, \beta) \text{ is the same as } s_n \rightarrow \ell(B, \alpha, \beta). \text{ So by Theorem (4.2),}$$

$$s_n \rightarrow \ell(C_{2r}), \text{ i.e. } \sum_{n=0}^{\infty} a_n = \ell(C_{2r}). \quad ///$$

Kwee ([15], Theorem 2) has given the corollary with 0 instead of O_L .

Recalling that the circle methods consist of the B , V_β , E_α , S_α , and T_α methods ($0 < \alpha < 1$, $\beta > 0$), the O-version of Corollary (4.4), Theorem (3.21), and Corollaries (3.54), (3.55), (3.56) imply that:

(4.5) Corollary: If C is any circle method, $a_n = O(n^{r-1/2})$, $r > 0$,

$$\text{and } \sum_{n=0}^{\infty} a_n = \ell(C), \text{ then } \sum_{n=0}^{\infty} a_n = \ell(C_{2r}).$$

Note when C is the B method or the S_α method, the corollary is a special case of the results due to Ragagopal ([17], Theorem 2) and Sitaraman ([18], Theorem 2).

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