Vibration Behaviour Of In-plane Loaded Thin Rectangular Plates With Initial Geometrical Imperfections

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LA THÈSE A ÉTÉ MICROFILMÉE TELLE QUE NOUS L'AVONS REÇUE
VIBRATION BEHAVIOUR OF IN-PLANE LOADED THIN RECTANGULAR PLATES WITH INITIAL GEOMETRICAL IMPERFECTIONS

by

Sinniah Ilanko

Faculty of Engineering Science

Submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy

Faculty of Graduate Studies
The University of Western Ontario
London, Ontario
November, 1985

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ABSTRACT

The effect of in-plane loading on the natural frequencies of simply supported thin rectangular plates with initial geometrical imperfection is investigated theoretically and experimentally. It is shown that the natural frequencies depend on applied in-plane load, initial geometrical imperfection and the in-plane boundary conditions.

In the theoretical analysis, the natural frequencies, out-of-plane static displacements and in-plane stress distribution are calculated using the Rayleigh-Ritz minimization technique. A concept of 'connection coefficients' has been used to reduce the computational work. In this concept, the relationship between the out-of-plane and in-plane displacement coefficients are first determined by solving the equations resulting from the minimization of the total potential energy with respect to the in-plane displacement coefficients. This relationship is then substituted into the equations obtained by minimizing the total potential energy with respect to the out-of-plane displacement coefficients.

In the experimental side of the work, tests were carried out on several thin (thickness ranging from 0.36 mm to 1.15 mm) mild steel plates (300 mm x 250 mm). Uniaxial
in-plane loading was applied through two 'V' grooved edge beams. The other two edges were supported between two rows of ball bearings placed in 'V' grooves, carefully adjusted to minimize the friction along these edges.

The agreement between the measured and calculated values of natural frequencies, out-of-plane central displacements and static strain distribution is very good. An interesting observation from the result is that a simple approximate linear relationship between a load-frequency parameter (involving the fundamental natural frequency and the state of in-plane stress) and the square of the central deflection is obtained. Further experimental and theoretical work in this field is strongly recommended.
ACKNOWLEDGEMENTS

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NOMENCLATURE

The following is a list of the main symbols used in this thesis. Other symbols are defined as they appear in the text.

\( a \) Dimension of a plate in \( x \)-direction

\( A \) Cross-sectional area of a beam

\( A_{1,j}, B_{1,j} \) Static displacement coefficients

\( \bar{A}_{1,j}, \bar{B}_{1,j} \) Dynamic displacement coefficients

\( b \) Dimension of a plate in \( y \)-direction

\( D \) Plate rigidity defined by \( D = \frac{Eh^3}{12(1-\nu^2)} \)

\( E \) Young's modulus

\( F \) Airy stress function for the static analysis

\( \bar{F} \) Airy stress function for the dynamic analysis

\( f_{ui}(x), f_{vi}(x) \) Shape functions (ith) in \( x \) direction for the static displacements in \( x,y \) directions

\( \bar{f}_{ui}(x), \bar{f}_{vi}(x) \) Shape functions (ith) in \( x \) direction for the dynamic displacements in \( x,y \) directions

\( g_{uj}(y), g_{vj}(y) \) Shape functions (jth) in \( y \) direction for the static displacements in \( x,y \) directions

\( \bar{g}_{uj}(y), \bar{g}_{vj}(y) \) Shape functions (jth) in \( y \) direction for the dynamic displacements in \( x,y \) directions

\( [G] \) Dynamic connection coefficients matrix

\( [H] \) Static connection coefficients matrix

\( h \) Thickness of a plate
H_{1,j} or H_{1}  Dynamic out-of-plane displacement coefficients
l  Integer
I  Second moment of area of a beam
I_{1}  Also used as an integer in Chapter 3
j, k  Integers
k_{1}, k_{2}, k_{e}  Stiffness factor
L  Length of a beam
L  Also used as an integer in Chapter 3
l, m  Integers
\bar{m}  Mass density
m, N, p  Integers
P  Static axial load
P'  Dynamic axial force
P_{E}  Euler load
P_{x}  In-plane load along x-direction
q, r, s  Integers
t  Time
\bar{T}  Kinetic energy
u, v, u_{s}, v_{s}  Static in-plane displacements along x-y directions
u_{d}, v_{d}, w  Dynamic displacements in x, y, z directions measured from the equilibrium position
\bar{U}, \bar{V}  Strain energy, potential energy due to static loading
\bar{U}, \bar{V}  Strain energy, potential energy due to the vibration
$x, y, z$ Cartesian co-ordinates

$z(x, y)$ Static out-of-plane deflection of a plate measured from the plane of the supports

$z_{o}(x, y)$ Initial out-of-plane imperfection

$z_{i, j}$ or $z$ Static out-of-plane displacement coefficients

$z_{o i, j}$ Fourier coefficients in the out-of-plane initial imperfection

$\varepsilon_{x} , \varepsilon_{y}, \varepsilon_{xy}$ In-plane strains due to static loading

$\varepsilon_{x} , \varepsilon_{y}, \varepsilon_{xy}$ or $\varepsilon_{x} , \varepsilon_{y}, \varepsilon_{xy}$ In-plane strains due to vibration

$\varepsilon_{x} , \varepsilon_{y}, \varepsilon_{xy}$ Central displacement of a plate

$c$ Load ratio

$c$ Frequency ratio

$c_{o}$ Deflection parameter

$c_{o}$ Non-dimensional initial imperfection at the centre

$\sigma_{x} , \sigma_{y}, \sigma_{xy}$ Static in-plane stresses

$\sigma_{x} , \sigma_{y}, \sigma_{xy}$ Dynamic in-plane stresses

$\nu$ Poisson's ratio

$\omega_{i, j}$ Natural frequency corresponding to $(i, j)$ mode

$\tilde{\omega}_{i, j}$ Theoretical natural frequency of an unstressed flat plate corresponding to $(i, j)$ mode
CHAPTER 1

INTRODUCTION

1.1 INTRODUCTORY REMARKS

A considerable amount of work has been done on the practically important problem of the vibration of rectangular plates subject to in-plane loads. A preponderance of this work has dealt with plates having a uniform in-plane stress distribution, although numerous studies have included the effect of in-plane stresses which vary over the area of the plate. Almost all of the studies to date have been of a theoretical nature and have been concerned with plates which vibrate about the perfectly flat state. In practice, plates cannot be perfectly flat and, so called, geometric imperfection (curvature) must exist. Any initial curvature in an unstressed plate can be significantly magnified when the plate is subjected to compressive in-plane loads. Experimental work conducted by a minority of researchers has shown this to be the case for very slight initial curvatures and the resulting effect upon the natural frequencies of the plates, when compressively loaded, is very significant, causing the behaviour of the plate to deviate drastically from that predicted using flat plate theory. Very recently, theoretical
studies have suggested similar behaviour but, to date, no adequate agreement has been achieved between theoretically predicted and experimentally obtained natural frequencies of rectangular plates having initial geometric imperfection and in-plane loaded in compression; nor have the experimental studies accommodated in-plane loads substantially beyond the first buckling load. Such a study is the subject of the present thesis.

1.2 PRESENT STATE OF STUDIES ON THE VIBRATION BEHAVIOUR OF IN-PLANE LOADED RECTANGULAR PLATES

The basic theory of vibration of unstressed plates was laid down by Lord Rayleigh [1,2], and the initial solutions were obtained by Timoshenko [3]. A comprehensive literature survey in a NASA special publication by Leissa [4] reveals that very little work had been done on the free vibration of in-plane loaded rectangular plates prior to the time it was published (1969). Since then, there have been a number of papers on this problem. Dickinson et al. [5-11] published results for natural frequencies of flat rectangular plates under various in-plane stress distributions and boundary conditions, using analytical and numerical methods. An exact procedure applicable to certain boundary conditions, using a complex stiffness matrix, has been developed by Wittrick and
Williams [12,13]. A finite strip method of analysis was proposed by Dawe and Morris [14] for the vibration of circularly curved plate assemblies subjected to membrane stresses. The effect of curvature magnification due to the stresses was not considered. Application of Galerkin's method to the vibration of in-plane stressed flat plates was illustrated in a publication by Laura and Romanelli [15]. Other references on this subject and other complicating effects on plate vibration can be found in two of the papers by Leissa [16,17].

As mentioned, a complicating factor in the vibration analyses of in-plane stressed plates is the presence of geometrical imperfections (deviation from flatness). Experimental results reported by Phillips and Jubb [18] show that the frequencies of essentially unstressed clamped curved plates increase with distortion. Their results indicate a linear relationship to exist between the square of the fundamental natural frequency and the square of the central distortion. This observation was compared with the theoretical results for spherically curved plates published by Reissner [19].

Experimental results for the natural frequencies of in-plane loaded plates appear to have been first reported by Lurie [20]. Measured values of natural frequencies
were found to be higher than those computed by using the classical theory for flat plates. The plot of the square of the frequency against load deviated from the theoretical straight line. The discrepancy was attributed to the presence of initial geometrical imperfections. This argument was based on some earlier theoretical work carried out by Massonnet [21] for imperfect circular plates. An experimental investigation on the vibration of box columns (assembly of four rectangular plates) under in-plane loading was reported by Jubb, Phillips and Becker [22]. In this case, the agreement between the observed and predicted (based on the flat plate theory) values of the natural frequencies were very good until close to the buckling load. The frequencies first reduced with in-plane compressive load, but took a sharp turn near the buckling load and began to increase. This was explained as being the influence of distortion which lead to an increase in stiffness resulting from the membrane action.

An experimental study conducted at the University of Manchester, U.K., on rectangular plates with simply supported and clamped (one edge) boundary conditions indicated a deviation in the natural frequencies at high loadings [23,24], as earlier reported by Lurie. The effect of non-uniformity in the in-plane stress distribution
was investigated using a finite difference approach. The strain distribution and displacement pattern were also measured along with the natural frequencies. It was found that the non-uniformity in the stress distribution contributed to the discrepancy between the measured and calculated values, but a substantial part of the discrepancy remained.

Hui and Leissa [25] published the results of a theoretical study on the vibration of in-plane stressed rectangular plates with initial geometrical imperfection. This appears to be the first theoretical publication on this problem. The Von Kármán's large deflection equations and the linear shell vibration equations were solved using Galerkin's method. Airy stress functions were used in the compatibility and equilibrium equations. These functions were chosen to satisfy the compatibility equation exactly for any one out-of-plane displacement (or vibration) mode. This method is directly applicable to certain in-plane boundary conditions and simply supported out-of-plane boundary conditions. It was assumed that the flexural vibration modes are decoupled and are similar to the static buckling modes. Consideration of several vibration modes and buckling modes can change the results for large values of displacements as will be shown in Chapter 5 of the present thesis.
To the author's knowledge, no experimental work had been reported on the vibration behaviour of plates subject to in-plane loadings substantially higher than the lowest buckling load. The experimental results published so far have not been quantitatively compared with appropriate theoretical results.

The first part of the title problem is the calculation of static displacement and stresses due to the applied in-plane load. The equilibrium approach has been the popular method for post buckling analysis. Surprisingly, the number of publications on experimental studies appear to be very limited. Yamaki [26,27,28] and Coan [29] reported some experimental and theoretical results for the displacement and stress distribution of plates under large in-plane loadings (up to about three times the lowest critical load). Maximum central displacements were about three times the plate thickness at maximum applied load. The agreement between the experimental and theoretical results was good.

An approximate solution for the post buckling problem of plates with edges elastically restrained (against rotation) was published by Bhattacharya [30]. The energy method has been employed for post buckling analysis using Airy stress functions [31]. This procedure
may be difficult to apply for practical boundary conditions (where in-plane displacements are partially restrained) such as in the case of an experimental study since the displacements at the boundaries have to be calculated by integrating the stress functions.

1.3 THE SCOPE OF PRESENT WORK

The object of this project is to investigate the influence of in-plane loading, on the natural frequencies of simply supported rectangular plates. The effect of membrane stiffness on the frequencies and the rate of growth of out-of-plane displacements is studied theoretically and experimentally. The membrane stiffness at loadings lower than the lowest critical load is caused by the presence of initial geometrical imperfections (curvature) which are amplified by the application of load. (From here onwards, in-plane static load will be referred to as 'load'.) It should be mentioned however, that at loads above the lowest buckling load, even a plate that was perfectly flat prior to the loading can develop membrane stiffness. This is due to the fact that at such loads, a stable equilibrium state associated with out-of-plane displacements is possible. Effect of non-uniformity in the in-plane stress distribution is also taken into account.
In the theoretical analysis, the Rayleigh-Ritz method is used to calculate the deflections, stress distribution and natural frequencies. The in-plane and out-of-plane displacements are expressed as the summation of a series of the products of 'shape functions' and corresponding 'displacement coefficients'. Total potential energy is expressed in terms of these displacement coefficients which can be determined by solving the equations resulting from the minimization of the potential energy.

As far as the author is aware, the application of the energy method for vibration or post-buckling problems of rectangular plates with undetermined in-plane displacement coefficients has not been reported in the literature. Despite the lack of information on the merits and demerits of this method, it was decided to use this approach because of a significant advantage in its applicability to the experimental problem. Using this approach, the shape function for the in-plane displacements do not have to satisfy the natural boundary conditions. This permits certain modifications in the mathematical modelling, such as allowing for the effect of the flexibility of the testing frame and the inertia of the loading head. As explained later, such modifications can be done very simply when using the energy method with undetermined displacement coefficients.
The task of solving the non-linear post buckling analysis with undetermined displacement coefficients in all three cartesian directions appeared to be very difficult initially. However, the introduction of a concept of connection coefficients significantly reduced the computational effort. This concept is explained in the next chapter, where a simpler, related, beam problem is used to illustrate its applicability. Of particular importance in the use of the energy method is the choice of appropriate shape functions as these significantly affect the accuracy of the results and the rapidity of convergence. This too is illustrated using the simpler beam problem.

The application of the Rayleigh-Ritz method to the post buckling and vibration analysis of initially curved plates is described in Chapter 3.

To establish some confidence in the analytical approach used, results were computed for stress free curved plates having certain standard boundary conditions and are compared with results from a finite element package program in Chapter 5.

An equilibrium approach, using Galerkin's method, was developed initially and used to investigate the influence of membrane stretching on the frequencies of
plates with in-plane free and shear diaphragm boundary conditions. It was recognized that although this approach was attractive for such plates, it could not be adapted readily to suit boundary conditions likely to be met in experimentation. Hence, the use of this approach was not pursued but, for completeness, the analysis is given in Appendix A, and some numerical results are presented in Chapter 5.

In the experimental work, the natural frequencies of several rectangular plates under uniaxial in-plane loading were measured. The maximum applied load was well above the lowest critical load in most cases and in one case more than four times as large. At each increment of loading, the fundamental natural frequency and central deflection of the plate were measured. At certain values of load, the deflections at several points on the plate were measured. In one case, the strain distribution in the plate was measured with electric resistance strain gauges. For most of the plates tested, the second natural frequency was also measured. Due to the difficulties in measuring the amplitude of the static deflection corresponding to the second mode, and the computational problems in introducing the anti-symmetrical displacement shapes (the computer program was written for a fully symmetrical
shape), the theoretical values of the second natural frequency were not calculated. However, the experimental results are reported in Appendix B.

The experimental procedure is explained in Chapter 4 and the results are compared with the theoretical values in Chapter 5.

One interesting outcome of this study is an approximate linear relationship between the square of the central deflection and a frequency-load parameter, defined by the sum of the square of the non-dimensional natural frequency and the ratio of the applied load to the critical load. The probability of existence of such an exact linear relationship appears to be mathematically remote. However, the theoretical and experimental results do indicate an approximately linear relationship for the plates tested.

For a simply supported curved beam, a linear relationship between the frequency-load parameter and the square of the central deflection has been established analytically [32]. This is also illustrated in a simpler way in Chapter 2. An attempt has been made in Appendix C to show why an approximately linear relationship for plates may exist. If this relationship is valid generally, it may prove to be very useful in the prediction
of the natural frequencies of curved plates (such as aircraft panels) when the shape of the plates and the stress level are known. Alternatively, if the frequencies and the shape are known, the level of stress could be estimated. It is hoped that future work in this area will lead to more definite conclusions.
CHAPTER 2

INTRODUCTORY ANALYSIS

2.1 INTRODUCTORY REMARKS

Before tackling the description of the analysis of the plate problem, it is considered desirable to treat the simpler, but related, problem of the vibration of a slightly curved beam, subject to axial load. This will serve to introduce the concepts used in the plate analysis and permit a direct comparison with 'exact' results. First, the 'exact' analysis using the equilibrium method is given, followed by the approximate Rayleigh-Ritz method, where the concept of 'connection coefficients' is introduced, and which permits a study of the effect of using different 'admissible' shape functions. Additionally, the effect of unequal, partial axial restraint, analogous to that encountered in the plate experiments, is illustrated, showing that, at least for the beam, the unequal restraint can be replaced by an equivalent equal restraint. This will be an approximation in the case of the plate problem but is a very useful simplification, without which the symmetry of the problem is destroyed.

2.2 VIBRATION ANALYSIS OF AN AXIALLY LOADED SIMPLY SUPPORTED CURVED BEAM USING AN (EXACT) EQUILIBRIUM METHOD

Consider the vibration of a simply supported beam with
an initial curvature in the form of a half sine wave subjected to an axial load $P$ as shown in Figure 2.2.1.

![Figure 2.2.1](image)

Let the initial shape of the beam $z_0$ be given by

$z_0(x) = z_0 \sin \left( \frac{\pi x}{L} \right)$.

It is well known [33] that the deflection under load $P$ will then be given by

$z(x) = z_0 \sin \left( \frac{\pi x}{L} \right)$,

where $z = z_0/(1-P/P_E)$, in which $P_E$ is the Euler load or the lowest buckling load of the beam.

Let $w(x)$ be the dynamic displacement at the time of maximum positive excursion, measured from the static equilibrium position $z(x)$. Assuming the motion of the beam to be simple harmonic, the beam vibration equations given in Appendix D are as follows:
\[ EI \frac{4w}{3x^4} + p \frac{2w}{3x^2} + p' \frac{2w}{3x^2} - \omega^2 w = 0 \]  

(2.2.1)

and

\[ p' = EA \left( \frac{3u}{\delta_3} + \frac{3w}{\delta_3} \cdot \frac{3z}{\delta_3} \right) \]  

(2.2.2)

where \( u(x) \) is the axial displacement.

Neglecting the axial inertia of the beam,

\[ \frac{3p'}{\delta x} = 0 \]  

(2.2.3)

This means \( w = H \sin \left( \frac{\pi X}{L} \right) \) is a non-trivial solution of equations (2.2.1) and (2.2.2) where \( H \) is an undetermined displacement coefficient.

From equation (2.2.2),

\[ \frac{3u}{\delta x} = \frac{p'}{EA} - \frac{3w}{\delta x} \cdot \frac{3z}{\delta x} \]

\[ = \frac{p'}{EA} - \frac{\pi ZH}{L^2} \cos^2 \left( \frac{\pi X}{L} \right) \]

\[ = \frac{p'}{EA} - \frac{\pi ZH}{2L^2} \left( \cos \left( \frac{2\pi X}{L} \right) + 1 \right) \]

\[ u = \frac{p'}{EA} \frac{X}{4L} \sin \left( \frac{2\pi X}{L} \right) - \frac{\pi ZH}{2L^2} x + c \]  

(2.2.4)

where \( c \) is an integration constant and depends on the axial end conditions. The solution will be obtained for some simple axial boundary conditions as described below.
Case 1. Both Ends Axially Restrained (After the application of the static load).

at $x = 0$, $u \neq 0$ gives $c = 0$.

at $x = L$, $u = 0$ gives $P'L = \frac{\pi^2}{2} \frac{ZH}{L^2}$.

or $\frac{P'}{EA} = \frac{\pi^2}{2} \frac{ZH}{L^2}$ \hspace{1cm} (2.2.5)

Substituting these in equation (2.2.4) gives,

$$ u = -\frac{\pi}{4L} ZH \sin\left(\frac{2\pi x}{L}\right) \hspace{1cm} (2.2.6) $$

Substituting equations (2.2.5) and (2.2.6) into equation (2.2.1) yields

$$ \left[ EI \left(\frac{\pi}{L}\right)^4 - P \left(\frac{\pi}{L}\right)^2 - m_0^2 \right] w + EA \frac{\pi^2}{2L^2} ZH \left(\frac{\pi}{L}\right)^2 z = 0. $$

Since $H \cdot z = 2 \cdot w$, this becomes

$$ \left( EI \left(\frac{\pi}{L}\right)^4 - P \left(\frac{\pi}{L}\right)^2 - m_0^2 + EA \left(\frac{\pi}{L}\right)^4 \frac{Z^2}{2} \right) w = 0. $$

Dividing by $\frac{\pi^4 EI}{L^4}$ gives

$$ \left(1 - \frac{P}{PE} - \frac{m_0^2}{m_0^2} + \frac{m_0^2}{2k^2}\right) w = 0, $$

where $PE = \frac{\pi^2}{2} \frac{EI}{L^2}$ (Euler Load),

$$ m_0 = \frac{\pi^2}{2} \sqrt{EI/m} \hspace{1cm} \text{(Fundamental natural frequency of a straight simply supported beam without axial load)}, $$

$$ k = \sqrt{I/A} \hspace{1cm} \text{(Radius of gyration about the axis of bending)}. $$
For non-trivial solution of \( \omega \),
\[
1 - \frac{P}{P_E} - \frac{\omega^2}{\Omega^2} + \frac{1}{2} \frac{Z^2}{k^2} = 0
\]
or
\[
\omega^2 = \Omega^2 (1 - \frac{P}{P_E} + \frac{1}{2} \frac{Z^2}{k^2}) \quad (2.2.7)
\]

This simple formula gives the natural frequency of an axially loaded simply supported curved beam. A more general form of this equation can be found in a publication by Plaut and Johnson [32]. If both ends are free to move axially, it can be shown that the curvature will have no influence in the natural frequency as explained in the following paragraphs.

Case 2. Both Ends Free to Move Axially

\[
\begin{array}{c}
\begin{array}{cc}
\text{P} & \text{P} \\
\hline
\end{array}
\end{array}
\]

Figure 2.2.2

In this case, \( P' = 0 \).

Equation (2.2.1) reduces to that of a straight beam and the fundamental natural frequency will be given by
\[
\omega^2 = \Omega^2 (1 - \frac{P}{P_E}) \quad (2.2.3)
\]

Substituting \( P' = 0 \) in equation (2.2.4) gives
\[
u = c - \frac{2Hx}{2L^2} - \frac{2H}{4L} \sin(\frac{2 \pi x}{L})
\]
The constant of integration $c$, in this case, is undetermined. This is understandable since a free rigid body motion along the axis is permissible. However, for comparing with an approximate solution, let us assume that the motion is symmetrical about the centre of the beam.

At $x = L/2$, $u = 0$ gives $c = \frac{\pi^2 ZH}{4L}$,

$$u = \frac{\pi^2 ZH}{2L^2} \left( \frac{L}{2} - x \right) - \frac{\pi}{4L} \cdot ZH \sin\left(\frac{2\pi x}{L}\right). \tag{2.2.9}$$

Another type of boundary condition that is likely to be found in practice is partially restrained motion of the ends. During the conduct of the experiments, on the plates, it was recognized that the top and bottom edges of the plate were actually partially restrained. Therefore, it is useful to study the vibration behaviour of a curved beam with partially restrained boundaries.

Case 3. Both Ends Partially Restrained Axially

\[ \begin{array}{c}
\frac{F_1}{P + P} \\
\frac{F_2}{P + P}
\end{array} \]

Figure 2.2.3

Let the axial displacements at $x = 0$ and $x = L$ be $\delta_1$ and $\delta_2$ respectively. If the axial stiffnesses at the ends are $K_1$, $K_2$, then
\[ \frac{\beta_1}{\kappa_1}, \quad \frac{\beta_2}{\kappa_2} = -\frac{P'}{\kappa_2} \]

At \( x = 0 \), \( u = \beta_1 = \frac{P'}{\kappa_1} \).

At \( x = L \), \( u = \beta_2 = \frac{-P'}{\kappa_2} = c + \frac{P'L}{EA} - \frac{\pi^2 ZH}{2L} \)
\[ = \frac{P'}{\kappa_1} + \frac{P'L}{EA} - \frac{\pi^2 ZH}{2L} \]

From this,
\[ p' = \frac{\pi^2 ZH}{2L} \left/ \left[ \frac{L}{EA} + \frac{1}{\kappa_1} + \frac{1}{\kappa_2} \right] \right. \]

Expressing the stiffnesses in a non-dimensional form,
\[ i.e. \quad k_1 = \frac{\kappa_1 L}{EA}, \quad k_2 = \frac{\kappa_2 L}{EA}, \]
\[ p' = \frac{\pi^2 EA ZH}{2L^2} \left/ \left( 1 + \frac{1}{k_1} + \frac{1}{k_2} \right) \right. \]
\[ = \frac{\pi^2 EA ZH k_1 k_2}{2L^2 (k_1 k_2 + k_1 + k_2)} \] (2.2.10)

Substituting this in equation (2.2.1) gives,
\[ \frac{\omega^2}{\alpha^2} = 1 - \frac{P}{\alpha P_E} + \frac{1}{2} \frac{Z^2}{\alpha^2} \frac{k_1 k_2}{k_1 k_2 + k_1 + k_2} \] (2.2.11)

**Equivalent Equal Springs**

Note that the same frequencies will be obtained if two springs of equal stiffness \( k_e \) are placed at the ends such that,
\[
\frac{k_e^2}{k_e^2 + 2k_e} = \frac{k_1 k_2}{(k_1 + k_2 + k_1 k_2)}.
\]

i.e. \( k_e^2 (k_1 + k_2) = k_1 k_2 (k_e^2 + 2k_e) \)

\( k_e^2 (k_1 + k_2) = k_1 k_2 - 2k_e \)

\[
k_e = \frac{2k_1 k_2}{k_1 + k_2}.
\] (2.2.12)

If \( k_1 \gg k_2 \),

\[
k_e = \frac{2k_2}{1 + k_2/k_1} \approx 2k_2.
\] (2.2.13)

This means for a beam with one end axially fully restrained, the frequency can be calculated by applying twice the value of the other end stiffness on each side. The two systems shown in Figure 2.2.4 will have the same fundamental natural frequency.

![Figure 2.2.4](image)

If an equivalent equal stiffness is to be used, equation (2.2.11) simplifies to

\[
\frac{\omega^2}{\Omega^2} = 1 - \frac{p}{P_E} + \frac{1}{2} \frac{z^2}{k} \left( \frac{k_e}{2 + k_e} \right).
\] (2.2.14)
Note that if $k_e = 0$, equation (2.2.14) reduces to equation (2.2.8) and if $k_e = \infty$ it reduces to equation (2.2.7).

2.3 APPROXIMATE ANALYSIS OF THE VIBRATION OF A CURVED BEAM USING THE RAYLEIGH-RITZ METHOD

![Figure 2.3.1](image)

The problem treated in section 2.2 is solved using the Rayleigh-Ritz approach in this section (2.3). A concept of 'connection coefficients' is introduced to reduce the computational effort in calculating the frequencies. The accuracy of the solution depends on the choice of shape functions for the axial displacement. This is demonstrated by comparing the results for two different shape functions with the 'exact' results obtained in the previous section.

The maximum total potential energy of the beam during vibration is given by

\[
\mathcal{U} = \frac{1}{2} \int_{x=0}^{L} \left[ EA \left( \frac{\partial u}{\partial x} + \frac{\partial w}{\partial x} \right)^2 + EI \left( \frac{\partial^2 w}{\partial x^2} \right)^2 \right] dx
- \frac{P (\frac{\partial^2 w}{\partial x})^2}{2} dx \quad (2.3.1)
\]

Maximum kinetic energy $\mathcal{T} = \frac{m u^2}{2} \int_{x=0}^{L} \omega^2 dx \quad (2.3.2)$

Let $w = H_1 \cdot \sin \left( \frac{\pi x}{L} \right) + H_2 \cdot \sin \left( \frac{3\pi x}{L} \right)$
and \[ u = B_1 f_1(x) + B_2 f_2(x), \]

where \( f_1(x), f_2(x) \) are the shape functions for axial displacement and \( B_1, B_2 \) are the undetermined axial displacement coefficients, then

\[
\frac{3u}{3x} = B_1 \left( \frac{3f_1}{3x} \right) + B_2 \left( \frac{3f_2}{3x} \right),
\]

\[
\frac{3w}{3x} = \frac{\pi^2}{L^2} \frac{2}{H_1 \cos \left( \frac{\pi x}{L} \right)} (H_1 \cos \left( \frac{\pi x}{L} \right) + 3H_2 \cos \left( \frac{3\pi x}{L} \right))
\]

\[
= \frac{\pi^2}{L^2} \left[ \frac{1}{2} (\cos \left( \frac{2\pi x}{L} \right) + 1) + \frac{3H_2}{2} (\cos \left( \frac{4\pi x}{L} \right) + \cos \left( \frac{2\pi x}{L} \right)) \right],
\]

\[
\frac{\partial^2 w}{\partial x^2} = -\frac{\pi^2}{L^2} \left[ H_1 \sin \left( \frac{\pi x}{L} \right) + 9H_2 \sin \left( \frac{3\pi x}{L} \right) \right],
\]

and \( \frac{\partial w}{\partial x} = \frac{\pi}{L} \left[ H_1 \cos \left( \frac{\pi x}{L} \right) + 3H_2 \cos \left( \frac{3\pi x}{L} \right) \right]. \)

Substitution into equation (2.3.1) gives

\[
\bar{U} = \frac{E A}{2} \int_{x=0}^{L} \left[ B_1 \left( \frac{3f_1}{3x} \right) + B_2 \left( \frac{3f_2}{3x} \right) + \frac{\pi^2}{2L^2} (H_1 \left( 1 + \cos \left( \frac{2\pi x}{L} \right) \right) + 3H_2 \left( \cos \left( \frac{4\pi x}{L} \right) + \cos \left( \frac{2\pi x}{L} \right) \right)) \right]^2 dx
\]

\[
+ \frac{EI}{2} \int_{x=0}^{L} \left( \frac{\pi}{L} \right)^4 \left( H_1 \sin \left( \frac{\pi x}{L} \right) + 9H_2 \sin \left( \frac{3\pi x}{L} \right) \right)^2 dx
\]

\[
- \frac{P}{2} \left( \frac{\pi}{L} \right)^2 \int_{x=0}^{L} \left( H_1 \cos \left( \frac{\pi x}{L} \right) + 3H_2 \cos \left( \frac{3\pi x}{L} \right) \right)^2 dx
\]

or

\[
\bar{U} = \frac{E A}{2} \int_{x=0}^{L} \left[ B_1 \left( \frac{3f_1}{3x} \right) + B_2 \left( \frac{3f_2}{3x} \right) + \frac{\pi^2}{2L^2} (H_1 \left( 1 + \cos \left( \frac{2\pi x}{L} \right) \right) + 3H_2 \left( \cos \left( \frac{4\pi x}{L} \right) + \cos \left( \frac{2\pi x}{L} \right) \right)) \right]^2 dx
\]

\[
+ \frac{EI}{2} \int_{x=0}^{L} \left( \frac{\pi}{L} \right)^4 \left( H_1 \sin \left( \frac{\pi x}{L} \right) + 9H_2 \sin \left( \frac{3\pi x}{L} \right) \right)^2 dx
\]

\[
- \frac{P}{2} \left( \frac{\pi}{L} \right)^2 \int_{x=0}^{L} \left( H_1 \cos \left( \frac{\pi x}{L} \right) + 3H_2 \cos \left( \frac{3\pi x}{L} \right) \right)^2 dx
\]

\[
[H_1 \left( 1 + \cos \left( \frac{2\pi x}{L} \right) \right) + 3H_2 \left( \cos \left( \frac{4\pi x}{L} \right) + \cos \left( \frac{2\pi x}{L} \right) \right)]
\]
\[ + \left( \frac{\pi}{L} \right)^4 \frac{2}{4} [H_1^2 (1+\cos^2(\frac{2\pi x}{L})+2 \cos(\frac{2\pi x}{L})) + 9H_2^2 (\cos^2(\frac{2\pi x}{L})) \\
+ \cos^2(\frac{4\pi x}{L}) + 2 \cos(\frac{2\pi x}{L}) \cos(\frac{4\pi x}{L}) + 6H_1 H_2 (\cos^2(\frac{2\pi x}{L})) \\
+ \cos(\frac{4\pi x}{L}) + \cos^2(\frac{2\pi x}{L}) + \cos(\frac{2\pi x}{L}) \cos(\frac{4\pi x}{L})) \} \] \]
\[ + \frac{EI}{2} \left( \frac{\pi}{L} \right)^4 \left( H_1^2 \left( \frac{L}{2} \right)^2 + 81H_2^2 \left( \frac{L}{2} \right)^2 \right) - \frac{P}{2} \left( \frac{\pi}{L} \right)^2 \left( H_1 \left( \frac{L}{2} \right)^2 + 9H_2 \left( \frac{L}{2} \right)^2 \right) \]

which reduces to

\[ \overline{U} = \frac{EA}{2} \int_{x=0}^{L} \left[ B_1 \left( \frac{\partial f_1}{\partial x} \right)^2 + B_2 \left( \frac{\partial f_2}{\partial x} \right)^2 \right] \left[ B_1 \left( \frac{\partial^2 f_1}{\partial x^2} \right)^2 + B_2 \left( \frac{\partial^2 f_2}{\partial x^2} \right)^2 \right] \left( \frac{\pi}{L} \right)^2 \]
\[ + \frac{EA}{2} \left( \frac{\pi}{L} \right)^4 \frac{2}{4} [H_1^2 (\frac{3L}{2})^2 + 9H_2^2 (\frac{2L}{2})^2 + 6H_1 H_2 (\frac{L}{2})^2] \]
\[ + \left( \frac{\pi}{L} \right)^4 \frac{EI}{2} \left( \frac{L}{2} \right)^2 (H_1^2 + 81H_2^2) - \left( \frac{\pi}{L} \right)^2 \frac{P}{2} \left( \frac{L}{2} \right)^2 (H_1^2 + 9H_2^2) \] \( \text{(2.3.3)} \)

Also, the kinetic energy expression becomes

\[ \overline{T} = \frac{mw^2}{2} \int_{x=0}^{L} [H_1 \sin(\frac{\pi x}{L}) + H_2 \sin(\frac{3\pi x}{L})]^2 dx \]
\[ = \frac{mw^2}{2} \left( \frac{L}{2} \right) [H_1^2 + H_2^2] \] \( \text{(2.3.4)} \)

Using the Rayleigh-Ritz method,

\[ \frac{\partial \overline{U}}{\partial B_1} - \frac{\partial \overline{T}}{\partial f_1} = 0 \] \( \text{(2.3.5)} \)
\[ \frac{\partial \overline{U}}{\partial B_2} - \frac{\partial \overline{T}}{\partial f_2} = 0 \] \( \text{(2.3.6)} \)
\[
\frac{\partial U}{\partial H_1} - \frac{\partial T}{\partial H_1} = 0 \tag{2.3.7} \\
\frac{U}{H_2} - \frac{T}{H_2} = 0 \tag{2.3.8}
\]

\(T\) contains the unknown frequency \(\omega\). These eigenvalue equations can be solved simultaneously to calculate the eigenvalue \(\omega\) and the eigenvector \(B_1, B_2, H_1\) and \(H_2\). In the case of a curved plate, the undetermined coefficients in all three cartesian coordinate directions must be calculated by solving the minimization equations. A considerable saving in the computational effort can be achieved by the introduction of a concept of 'connection coefficients'.

In this method, the relationship between the axial (in-plane in the case of a curved plate) and transverse (out-of-plane in the case of a plate) displacement coefficients are first determined by solving the equations resulting from the minimization of the total potential energy with respect to the axial (in-plane for plates) displacement coefficients for unit values of the transverse (out-of-plane for plates) displacement coefficients. This relationship, which is expressed as a matrix called the 'connection coefficient matrix', does not depend on the magnitude of the displacement coefficients and can be conveniently substituted into the set of equations resulting from the minimization of the total potential energy with respect to each transverse (out-of-plane for plates) displacement coefficient. These equations can then be solved simultaneously for the frequencies (eigenvalues).
and transverse displacement coefficients (eigenvectors).

A similar idea has been employed by Coan [29] and Yamaki [26] in post buckling analysis of rectangular plates where the coefficients for Airy stress functions are expressed as functions of out-of-plane displacement coefficients.

In the beam problem, equations (2.3.5) and (2.3.6) can be first solved for unit values of $H'_1$, $H'_2$ to calculate the connection coefficients. The results can be substituted in equations (2.3.7) and (2.3.8). These two equations can then be solved for $\omega$ and the transverse vibration mode $(H'_1, H'_2)$. In the current example, the problem of solving four eigenvalue equations is reduced to that of solving two eigenvalue equations. The reduction in the size of this problem may not appear to be significant to justify the lengthy calculation of connection coefficients. However, for a more complicated case (such as a plate) the use of connection coefficients can result in substantial saving of computational effort by reducing the number of eigenvalue equations for the vibration problem and the number of non-linear equations for the post buckling analysis. The method will be illustrated by applying it to solve equations (2.3.5) to (2.3.8).

The first step is to solve equations (2.3.5) and (2.3.6) in terms of $H'_1$ and $H'_2$ as follows.
\[ \frac{\partial \bar{f}}{\partial B_1} = 0. \]

Therefore,

\[ \frac{\partial \bar{U}}{\partial B_1} = E A \int \limits_{x=0}^{L} \left( B_1 \left( \frac{\partial f_1}{\partial x} \right)^2 + B_2 \frac{\partial f_1}{\partial x} \cdot \frac{\partial f_2}{\partial x} + \left( \frac{2}{L} \right)^2 \frac{\partial f_1}{\partial x} \right) \left( H_1 (1 + \cos \left( \frac{2\pi x}{L} \right)) + 3H_2 (\cos \left( \frac{2\pi x}{L} \right) + \cos \left( \frac{4\pi x}{L} \right)) \right) dx = 0. \]

This can be written as,

\[ S_{1,1}^* B_1 + S_{1,2}^* B_2 = X_{1,1}^* H_1 + X_{1,2}^* H_2 \quad (2.3.9) \]

where

\[ S_{1,1} = \int \limits_{x=0}^{L} \left( \frac{\partial f_1}{\partial x} \right)^2 dx, \quad (2.3.9a) \]

\[ S_{1,2} = \int \limits_{x=0}^{L} \frac{\partial f_1}{\partial x} \cdot \frac{\partial f_2}{\partial x} dx, \quad (2.3.9b) \]

\[ X_{1,1} = -\left( \frac{\pi}{L} \right)^2 \frac{2}{2} \int \limits_{x=0}^{L} \frac{\partial f_1}{\partial x} (1 + \cos \left( \frac{2\pi x}{L} \right)) dx, \quad (2.3.9c) \]

and

\[ X_{1,2} = -3 \left( \frac{\pi}{L} \right)^2 \frac{2}{2} \int \limits_{x=0}^{L} \frac{\partial f_1}{\partial x} \left( \cos \left( \frac{2\pi x}{L} \right) + \cos \left( \frac{4\pi x}{L} \right) \right) dx. \quad (2.3.9d) \]

Similarly, \( \frac{\partial \bar{U}}{\partial B_2} = 0 \) can be written as

\[ S_{2,1}^* B_1 + S_{2,2}^* B_2 = X_{2,1}^* H_1 + X_{2,2}^* H_2 \quad (2.3.10) \]

where

\[ S_{2,1} = S_{1,2} \quad (2.3.10a) \]
\[ S_{2,2} = \int_{x=0}^{L} (\frac{3f_2}{3x})^2 \, dx, \quad (2.3.10b) \]

\[ x_{2,1} = -\left( \frac{\pi}{L} \right)^2 \frac{2}{2} \int_{x=0}^{L} \frac{3f_2}{3x} (1 + \cos \left( \frac{2\pi x}{L} \right)) \, dx, \quad (2.3.10c) \]

and

\[ x_{2,2} = -3 \left( \frac{\pi}{L} \right)^2 \frac{2}{2} \int_{x=0}^{L} \frac{3f_2}{3x} \left[ \cos \left( \frac{2\pi x}{L} \right) \right. \]

\[ + \cos\left( \frac{4\pi x}{L} \right) \} \, dx. \quad (2.3.10d) \]

In matrix form equations (2.3.9) and (2.3.10) can be written as

\[
\begin{bmatrix}
S_{1,1} & S_{1,2} \\
S_{2,1} & S_{2,2}
\end{bmatrix}
\begin{bmatrix}
B_1 \\
B_2
\end{bmatrix}
= \begin{bmatrix}
x_{1,1} & x_{1,2} \\
x_{2,1} & x_{2,2}
\end{bmatrix}
\begin{bmatrix}
H_1 \\
H_2
\end{bmatrix}. \quad (2.3.11)
\]

The solution proceeds as follows:

First, the following two sets of equations are solved.

\[
\begin{bmatrix}
S_{1,1} & S_{1,2} \\
S_{2,1} & S_{2,2}
\end{bmatrix}
\begin{bmatrix}
B_{1,1}' \\
B_{2,1}'
\end{bmatrix}
= \begin{bmatrix}
x_{1,1}' \\
x_{2,1}'
\end{bmatrix}. \quad (2.3.12)
\]

\[
\begin{bmatrix}
S_{1,1} & S_{1,2} \\
S_{2,1} & S_{2,2}
\end{bmatrix}
\begin{bmatrix}
B_{1,2}' \\
B_{2,2}'
\end{bmatrix}
= \begin{bmatrix}
x_{1,2}' \\
x_{2,2}'
\end{bmatrix}. \quad (2.3.13)
\]

Hence,

\[
\begin{bmatrix}
B_1 \\
B_2
\end{bmatrix}
= \begin{bmatrix}
B_{1,1}' & B_{1,2}' \\
B_{2,1}' & B_{2,2}'
\end{bmatrix}
\begin{bmatrix}
H_1 \\
H_2
\end{bmatrix}. \quad (2.3.14)
\]
The coefficients $B_{i,j}$ are called 'connection coefficients' since they relate the axial displacement coefficients to the transverse displacement coefficients. Note that the calculation of connection coefficients does not require the calculation of the transverse displacement coefficients.

Having calculated the connection coefficients, the frequencies and modeshapes can be found as follows:

Substituting equations (2.3.3) and (2.3.4) in equation (2.3.7) gives

$$
\frac{EA}{2} \left[ \left( \frac{\pi}{L} \right)^4 \frac{Z^2}{4} \left( \frac{L}{2} \right) [6H_1 + 6H_2] + \int_{x=0}^{L} \left( 1 + \cos \left( \frac{2\pi x}{L} \right) \right) \frac{Z}{L} (B_1 \frac{\partial f_1}{\partial x}) \right. \\
+ B_2 \frac{\partial f_2}{\partial x} \right] dx + \frac{EI}{L} \left( \frac{\pi}{L} \right)^4 \left( \frac{L}{2} \right) 2H_1 - \left( \frac{\pi}{L} \right)^2 \frac{P}{4} \cdot 2H_1 \cdot L
$$

$$
- \frac{m\omega^2 L}{2} \cdot H_1 = 0.
$$

i.e. $H_1 \left[ \frac{6EA}{16L^3} \cdot \frac{4\pi Z^2}{L^3} + \frac{4\pi EI}{2L^3} - \frac{2P}{2L^3} + \frac{6\pi^4 EAZ^2}{16L^3} \cdot H_2 \\
- EA \{X_{1,1} B_1 + X_{2,1} B_2 \} - \frac{m\omega^2 L}{2} \cdot H_1 = 0.
$$

Substituting $B_1 = B_{1,1} H_1 + B_{1,2} H_2$ from equation (2.3.14) into this gives

$$
H_1 \left[ \frac{3EA}{8L^3} \cdot \frac{\pi^2 Z^2}{L^3} + \frac{4\pi EI}{2L^3} - \frac{\pi^2 P}{2L^3} \right] + \frac{3\pi^4 EAZ^2}{8L^3} \cdot H_2

- EA \{X_{1,1} B_{1,1} H_1 + X_{2,1} B_{2,1} H_1 \\
+ X_{1,1} B_{1,2} H_2 + X_{2,1} B_{2,2} H_2 \} - \frac{m\omega^2 L}{2} \cdot H_1 = 0.
$$
This can be written as

\[ \mathbf{S}_{1,1} H_1 + \mathbf{S}_{1,2} H_2 - \frac{\bar{m} \omega^2 L}{2} H_1 = 0 , \quad (2.3.15) \]

where

\[ \mathbf{S}_{1,1} = \frac{3EA_1^2 z^2}{8L^3} + \frac{\pi^4 EI}{2L^3} - \frac{\pi^2 P}{2L} - EA(X_{1,1} B_{1,1}^1 + X_{2,1} B_{2,1}^1) \]

\[ \text{(2.3.15a)} \]

and

\[ \mathbf{S}_{1,2} = \frac{3EA_2^2 z^2}{8L^3} - EA(X_{1,1} B_{1,2}^1 + X_{2,1} B_{2,2}^1) \]

\[ \text{(2.3.15b)} \]

Similarly, equation (2.3.8) can be transformed into

\[ \mathbf{S}_{2,1} H_1 + \mathbf{S}_{2,2} H_2 - \frac{\bar{m} \omega^2 L}{2} H_2 = 0 , \quad (2.3.16) \]

where

\[ \mathbf{S}_{2,1} = \frac{3\pi^4 EA_2^2 z^2}{8L^3} - EA(X_{1,2} B_{1,1}^1 + X_{2,2} B_{2,1}^1) \]

\[ \text{(2.3.16a)} \]

and

\[ \mathbf{S}_{2,2} = \frac{9\pi^4 EA_2^2 z^2}{4L^3} + \frac{81\pi^4 EI}{2L^3} - \frac{9\pi^2 P}{2L} - EA(X_{1,2} B_{1,2}^1 + X_{2,2} B_{2,2}^1) \]

\[ \text{(2.3.16b)} \]

The natural frequencies can be found by solving the eigenvalue equations

\[
\begin{bmatrix}
\mathbf{S}_{1,1} & \mathbf{S}_{1,2} \\
\mathbf{S}_{2,1} & \mathbf{S}_{2,2}
\end{bmatrix}
\begin{bmatrix}
H_1 \\
H_2
\end{bmatrix}
= \frac{\bar{m} \omega^2 L}{2}
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
H_1 \\
H_2
\end{bmatrix}
= 0 .
\]

\[ (2.3.17) \]
In this simple case this reduces to

\[(\bar{S}_{1,1} - \frac{m\omega^2 L}{2}) (\bar{S}_{2,2} - \frac{m\omega^2 L}{2}) - \bar{S}_{1,2} \bar{S}_{2,1} = 0\]. \hspace{1cm} (2.3.17a)

It is worth noting that the substitution for axial displacement coefficients should be done after the minimization. The Rayleigh-Ritz method requires that each displacement coefficient must be treated as independent during minimization.

This procedure is illustrated in the following examples. The importance of the choice of shape functions for the axial displacement is also demonstrated in these examples. The choice of shape functions for the in-plane displacements of the curved plate will be explained in Chapter 3.

**Example 1**

Consider the vibration of the curved beam for axially restrained end conditions with the following shapes:

\[w = H_1 \sin(\frac{\pi x}{L}) , \quad u = B_1 \sin(\frac{2\pi x}{L})\]

(These functions are the same as those found in the exact analysis.)

The equations in the preceding pages can be used with \(H_2 = 0, B_2 = 0, f_2 = 0\) and \(f_1 = \sin(\frac{2\pi x}{L})\).

\[S_{1,1} = \int_{x=0}^{L} \left(\frac{\partial^2 f_1}{\partial x^2}\right)^2 dx = \frac{2\pi^2}{L} , \quad S_{1,2} = S_{2,1} = S_{2,2} = 0\].
\[ X_{1,1} = -\left(\frac{\pi}{L}\right)^2 \frac{2}{2L} \int_{x=0}^{L} \frac{2}{L} \cos\left(\frac{2\pi x}{L}\right) \left[1 + \cos\left(\frac{2\pi x}{L}\right)\right] dx = -\frac{\pi^2 Z}{2L^2} \]

\[ X_{1,2} = -\frac{3\pi^2 Z}{2L^2} \int_{x=0}^{L} \left[\cos\left(\frac{2\pi x}{L}\right) + \cos\left(\frac{4\pi x}{L}\right)\right] \left(\frac{2\pi x}{L}\right) \cos\left(\frac{2\pi x}{L}\right) dx = -\frac{3\pi^3 Z}{2L^2} \]

\[ B_{1,1} = \frac{X_{1,1}}{S_{1,1}} = -\frac{2\pi}{4L} \]

\[ S_{1,1} = \frac{3EA}{2L^3} + \frac{\pi^4 EI}{2L^3} - \frac{\pi^2 P}{2L} + \pi^2 \left(\frac{3Z}{2L^2}\right) EA \]

\[ = \frac{\pi^4 EAZ^2}{4L^3} + \frac{\pi^4 EI}{2L^3} - \frac{\pi^2 P}{2L} \]

Hence,

\[ \frac{\bar{m} \omega^2 L}{2} = \frac{\pi^4}{2L^3} \frac{EA Z^2}{2} + EI - \frac{PL^2}{2 \pi} \]

i.e. \[ \frac{\bar{m} \omega^2 L}{EI \pi} = 1 + \frac{AZ^2}{2I} - \frac{P}{PE} \]

or \[ \frac{\omega^2}{\Omega^2} = (1 - \frac{P}{PE} + \frac{Z^2}{2k^2}) \]

This is the correct exact result.

**Example 2**

To illustrate the importance of the choice of shape functions, the problem in example 1 can be solved with the following alternative shape functions.
\[ w = H_1 \sin \left( \frac{\pi x}{L} \right) + H_2 \sin \left( \frac{3\pi x}{L} \right), \]

\[ u = B_1 f_1(x), \] where \( f_1(x) = \cos \left( \frac{\pi x}{L} \right) + \frac{2x}{L} - 1. \)

This new function \( f_1(x) \) also satisfies the forced boundary conditions that the axial displacements at the ends are zero, which is a requirement for the application of the Rayleigh-Ritz method.

\[ \frac{\partial f_1}{\partial x} = -\frac{\pi}{L} \sin \left( \frac{\pi x}{L} \right) + \frac{2}{L} x. \]

\[ S_{1,1} = \int_0^L \left( \frac{\partial f_1}{\partial x} \right)^2 dx = \int_0^L \left( \frac{4}{L^2} + \left( \frac{\pi}{L} \right)^2 \sin^2 \left( \frac{\pi x}{L} \right) \right. \]

\[ - \frac{4\pi}{L^2} \sin \left( \frac{\pi x}{L} \right) ) dx \]

\[ = \frac{(\pi^2-8)}{2L}, \]

\[ X_{1,1} = -\left( \frac{\pi}{L} \right)^2 \frac{2}{L} \int_0^L \left[ \frac{2}{L} - \frac{\pi}{L} \sin \left( \frac{\pi x}{L} \right) \right] [1 + \cos \left( \frac{2\pi x}{L} \right)] dx \]

\[ = \frac{\pi^2 Z}{3L^2}, \]

\[ X_{1,2} = \int_0^L \cos \left( \frac{2\pi x}{L} \right) + \cos \left( \frac{4\pi x}{L} \right) \left[ \frac{2}{L} - \frac{\pi}{L} \sin \left( \frac{\pi x}{L} \right) \right] \left( \frac{3\pi^2}{2L^2} \right) dx \]

\[ = \frac{6\pi^2 Z}{5L^2}, \]

\[ B_{1,1} = \frac{X_{1,1}}{S_{1,1}} = \frac{\pi^2 Z}{3L^2} \frac{2L}{(\pi^2-8)} \]

\[ = \frac{2\pi^2 Z}{3L(\pi^2-8)}. \]
\[ B_{1,2} = \frac{X_{1,2}}{S_{1,1}} = \frac{6\pi^3 Z}{5L^2} \left( \frac{2L}{\pi^2 - 8} \right) \]
\[ = \frac{12\pi^3 Z}{5L(\pi^2 - 8)} \]

\[ \bar{S}_{1,1} = \frac{3EA\pi^4 Z^2}{8L^3} + \frac{\pi^4 EI}{2L^3} - \frac{\pi^2 P}{2L} - \frac{EA}{3L} \frac{2\pi^2 Z}{\pi^2 - 8} \frac{\pi^2 Z}{3L^2} \]

\[ \bar{S}_{1,2} = \frac{3\pi^4 EAZ^2}{8L^3} - \frac{12\pi^3 Z}{5L(\pi^2 - 8)} \frac{\pi^2 Z}{3L^2} \]

\[ \bar{S}_{2,1} = \frac{3\pi^4 EAZ^2}{8L^3} - \frac{2\pi^2 Z}{3L(\pi^2 - 8)} \frac{6\pi^3 Z}{5L^2} \]

\[ \bar{S}_{2,2} = \frac{9\pi^4 EAZ^2}{4L^3} + 81 \frac{4EI}{2L^3} - \frac{9\pi^2 P}{14L} - \frac{EA}{5L(\pi^2 - 8)} \frac{6\pi^3}{5L^2} \]

Substituting these into equation (2.3.17a) and dividing by \((\pi^4 EI/2L^3)^2\) yields

\[
\left[ 1 - \frac{P}{P_E} + \frac{Z^2}{k^2} \left( \frac{3}{4} \right) \left( \frac{4}{9(\pi^2 - 8)} - \frac{\mu^2}{\pi^2} \right) \right] \left[ 81 - \frac{9P}{P_E} + \frac{Z^2}{k^2} \left( \frac{9}{25(\pi^2 - 8)} - \frac{\mu^2}{\pi^2} \right) \right] = 0.
\]

\[ \left( \frac{\pi^2}{k^2} \left( \frac{3}{4} \right) \frac{8}{5(\pi^2 - 8)} \right)^2 = 0. \quad (2.3.18) \]

Let \( \lambda^2 = \frac{\mu^2}{\pi^2} \), \( c = \frac{P}{P_E} \) and \( \phi^2 = \frac{Z^2}{k^2} \),

then

\[ (1-\lambda^2 - \phi^2) (81 - 9\phi - \lambda^2 + 1.419 \phi^2 - 1058 \phi^4 - 0.108 \phi^4 = 0. \quad (2.3.18a) \]

One term solution gives \( \lambda^2 = 1 - \phi + 0.5 \phi \).

The exact solution is \( \lambda^2 = 1 - \phi + 0.5 \phi \).

The two term solution is given by the following equation
\[
\lambda^4 - \lambda^2 (82 - 10\varphi + 1.931\varphi^2) + (1 - \varphi + 0.512\varphi^2)(81 - 9\varphi + 1.419\varphi^2) - 1.058^2\varphi^4 = 0,
\]
from which
\[
\lambda^2 = \frac{41 - 5\varphi + 0.9657\varphi^2 + \sqrt{(41 - 5\varphi + 0.9657\varphi^2)^2}}{-(1 - \varphi + 0.512\varphi^2)(81 - 9\varphi + 1.419\varphi^2) + 1.058^2\varphi^4}
\]

It is interesting to note the variation of \(\lambda^2\) with \(\varphi^2\).

Limit \(\frac{\Delta \lambda^2}{\Delta \varphi^2}\) as \(\varphi \to 0\)
\[
= .9657 + \frac{1}{2} \left(2 \times 0.9657^2\varphi^2 - 2 \times 1.4191 \times 0.512 - 2 \times 1.058^2\right)
\]
\[
\frac{\sqrt{0.9657^2 - 1.4191 \times 0.512 + 1.058^2}}{\sqrt{0.9657^2 - 1.4191 \times 0.512 + 1.058^2}} = .5
\]

This example illustrates an interesting point. The approximate solution for a curved beam vibration may exhibit a non-linear relationship between the square of the non-dimensional fundamental frequency and the square of the central deflection, although an exact linear relationship exists.

The application of the Rayleigh-Ritz method for the curved plate problem is explained in Chapter 3.
CHAPTER 3
THEORETICAL ANALYSIS OF THE PLATE PROBLEM

3.1 INTRODUCTORY REMARKS

This chapter consists of the theoretical derivations associated with the application of the Rayleigh-Ritz method to the post buckling and vibration analysis of simply supported curved rectangular plates. Application of the Rayleigh-Ritz method to the post buckling analysis is discussed in section 3.2. The derivation of the formulae that are necessary to calculate the static displacements and in-plane stress distribution is given in this section. Under applied in-plane loading, the plate can vibrate freely about its equilibrium state. Having calculated the deflected shape and stresses under the applied load, the natural frequencies can be calculated by using the Rayleigh-Ritz method. This procedure is explained in section 3.3. A brief discussion on the choice of shape functions can be found in section 3.4. Some notes on a computer program which was developed to obtain numerical values for the natural frequencies, displacements and stress distribution of practical plates using the analysis explained in this chapter are given in section 3.5.
3.2 APPLICATION OF THE RAYLEIGH-RITZ METHOD TO THE POST BUCKLING ANALYSIS OF SIMPLY SUPPORTED RECTANGULAR PLATES

This section deals with the application of the Rayleigh-Ritz method to calculate the static displacements and in-plane stresses due to an applied in-plane load.

Consider the equilibrium of a rectangular plate, subject to uniaxial load $P_x$, applied at two points on the edges $x=0$ and $x=a$, through two edge beams as shown in Figure 3.2.1.

![Figure 3.2.1](image)

For a plate simply supported along all four edges, the initial out-of-plane displacement (initial imperfection) can be expressed as a Fourier series.

$$z_0 = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} z_{0i,j} \sin\left(\frac{i \pi x}{a}\right) \sin\left(\frac{j \pi y}{b}\right)$$  \hspace{1cm} (3.2.1)

where for displacements symmetrical about the axes $x=a/2$, $y=b/2$, $a_1=(2i-1)\pi$.

The displacement $z$ at a load $P_x$ can also be given by a Fourier series,
\[ z = \sum_{i} \sum_{j} Z_{i,j} \sin \left( \frac{\alpha_{i} x}{a} \right) \sin \left( \frac{\alpha_{j} y}{b} \right). \]  

(3.2.2)

It is assumed that the plate is initially stress free. Let the in-plane displacements along \( x, y \) directions be \( u_{s}, v_{s} \) respectively.

\( u_{s} \) and \( v_{s} \) are expressed as the sum of a series of the products of in-plane displacement coefficients and the corresponding shape functions as given by the following equations:

\[ u_{s} = \sum_{i} \sum_{j} A_{i,j} f_{ui}(x) g_{uj}(y), \]  

(3.2.3)

\[ v_{s} = \sum_{i} \sum_{j} B_{i,j} f_{vi}(x) g_{vj}(y). \]  

(3.2.4)

The shape functions \( f_{ui}, g_{uj}, f_{vi} \) and \( g_{vj} \) must satisfy the geometric boundary conditions as will be explained in section 3.4.

The total potential energy of the plate and the edge beams consists of the following components:

(a) Strain Energy due to bending of the plate given by [31],

\[
\overline{U_{be}} = \frac{Eh^{3}}{24(1-v^{2})} \int_{x=0}^{a} \int_{y=0}^{b} \left[ \frac{\partial^{2}(z-z_{o})}{\partial x^{2}} + \frac{\partial^{2}(z-z_{o})}{\partial y^{2}} \right]^{2} dx \ dy - 2(1-v) \left[ \frac{\partial^{2}(z-z_{o})}{\partial x^{2}} \frac{\partial^{2}(z-z_{o})}{\partial y^{2}} - \left( \frac{\partial^{2}(z-z_{o})}{\partial x \partial y} \right)^{2} \right] dx \ dy. \]  

(3.2.5a)
(b) Strain Energy due to the stretching of the middle surface given by [31],

\[
\tilde{U}_{\text{str}} = \frac{Eh}{2(1-\nu^2)} \int_a^b \int_{x=0}^{x=b} \left[ \varepsilon_x^2 + \varepsilon_y^2 + 2\nu \varepsilon_x \varepsilon_y + \frac{(1-\nu)}{2} \gamma_{xy}^2 \right] dx \, dy,
\]

(3.2.5b)

where the middle surface strains are related to the derivatives of displacements as follows:

\[
\begin{align*}
\varepsilon_x &= \frac{\partial u_s}{\partial x} + \frac{1}{2} \left( \frac{\partial z}{\partial x} \right)^2 - \frac{1}{2} \left( \frac{\partial \sigma}{\partial x} \right)^2 \\
\varepsilon_y &= \frac{\partial v_s}{\partial y} + \frac{1}{2} \left( \frac{\partial z}{\partial y} \right)^2 - \frac{1}{2} \left( \frac{\partial \sigma}{\partial y} \right)^2 \\
\gamma_{xy} &= \frac{\partial u_s}{\partial y} + \frac{\partial v_s}{\partial x} + \frac{\partial z}{\partial x} \frac{\partial z}{\partial y} - \frac{1}{2} \left( \frac{\partial \sigma}{\partial x} \frac{\partial \sigma}{\partial y} \right)
\end{align*}
\]

(3.2.5c)

Substituting equations (3.2.5c) into equation (3.2.5b) gives

\[
\tilde{U}_{\text{str}} = \frac{Eh}{2(1-\nu^2)} \int_a^b \int_{x=0}^{x=b} \left[ \left( \frac{\partial u_s}{\partial x} + \frac{1}{2} \left( \frac{\partial z}{\partial x} \right)^2 - \frac{1}{2} \left( \frac{\partial \sigma}{\partial x} \right)^2 \right)^2 \\
+ \left( \frac{\partial v_s}{\partial y} + \frac{1}{2} \left( \frac{\partial z}{\partial y} \right)^2 - \frac{1}{2} \left( \frac{\partial \sigma}{\partial y} \right)^2 \right)^2 \\
+ 2\nu \left[ \frac{\partial u_s}{\partial x} + \frac{1}{2} \left( \frac{\partial z}{\partial x} \right)^2 - \frac{1}{2} \left( \frac{\partial \sigma}{\partial x} \right)^2 \right] \left[ \frac{\partial v_s}{\partial y} + \frac{1}{2} \left( \frac{\partial z}{\partial y} \right)^2 - \frac{1}{2} \left( \frac{\partial \sigma}{\partial y} \right)^2 \right] \\
+ \frac{(1-\nu)}{2} \left[ \left( \frac{\partial u_s}{\partial y} + \frac{\partial v_s}{\partial x} + \frac{\partial z}{\partial x} \frac{\partial z}{\partial y} - \frac{1}{2} \left( \frac{\partial \sigma}{\partial x} \frac{\partial \sigma}{\partial y} \right) \right)^2 \right] \right] dx \, dy.
\]

(3.2.5d)

(c) Change in the potential energy due to the movement of the load \( P_x \) is given by


\[ \overline{V}_{\text{LOAD}} = - P_x \left[ u_s \right]_{x=a, y=b} - u_s \left[ x=b, y=0 \right] \]  \hspace{1cm} (3.2.5e)

(d) Strain Energy due to the bending of the edge beam is given by

\[ \tilde{U}_{\text{beam}} = \frac{EI}{2} \int_0^b \left[ \left( \frac{\partial^4 u_s}{\partial y^2} \right)^2 \left|_{x=0} \right. \right. \left. \left. + \left( \frac{\partial^4 u_s}{\partial y^2} \right)^2 \left|_{x=a} \right. \right. \right] dy. \]

\hspace{1cm} (3.2.5f)

Total potential energy of the deflected plate and the edge beam is given by

\[ \overline{V}_T = \overline{U}_{\text{be}} + \overline{U}_{\text{str}} + \overline{V}_{\text{LOAD}} + \overline{U}_{\text{beam}}. \]  \hspace{1cm} (3.2.6)

Using the Rayleigh-Ritz method,

\[ \left\{ \frac{\partial \overline{V}_T}{\partial A_{i,j}} \right\} = 0, \]  \hspace{1cm} (3.2.7)

\[ \left\{ \frac{\partial \overline{V}_T}{\partial B_{i,j}} \right\} = 0, \]  \hspace{1cm} (3.2.8)

\[ \left\{ \frac{\partial \overline{V}_T}{\partial Z_{i,j}} \right\} = 0. \]  \hspace{1cm} (3.2.9)

**Calculation of Connection Coefficients**

Substituting equations (3.2.5a), (3.2.5d), (3.2.5e), (3.2.5f) and (3.2.6) into equation (3.2.7) and, noting that only functions of \( u_s \) will yield non-zero terms for \( \frac{\partial \overline{V}_T}{\partial A_{i,j}} \), leads to
\[
\frac{Eh}{2(1-\nu^2)} \frac{3}{z A_{i,j}} \int_a^b \int_{x=0}^{y=0} \left( \frac{3u_s}{\beta x} \right)^2 + \frac{3u_s}{\beta x} \left[ \frac{3z}{\beta x} - \frac{3z_o}{\beta x} \right]^2 \right. \\
+ 2\nu \frac{3u_s}{\beta x} \frac{3v_s}{\beta y} + \nu \frac{3u_s}{\beta x} \left[ \frac{3z_o}{\beta y} \right]^2 - \frac{3z_o}{\beta y} \right) + (1-\nu) \left( \frac{3u_s}{\beta y} \right)^2 \\
+ (1-\nu) \frac{3u_s}{\beta y} \frac{3v_s}{\beta y} + (1-\nu) \frac{3u_s}{\beta y} \left[ \frac{3z_o}{\beta y} - \frac{3z_o}{\beta y} \right] \right) dx \, dy \\
- \frac{p_x}{3A_{i,j}} \left[ \frac{3u_s}{3y} \right]_{x=0, y=bl}^{x=a, y=bl} \\
+ \frac{EI}{2} \left[ \frac{3u_s}{3y} \right]_{x=0}^{x=a} \right|_y=0 \\
\right|_y=a \\

This can be rearranged to give
\[
\frac{1}{2} \frac{3}{z A_{i,j}} \int_a^b \int_{x=0}^{y=0} \left( \frac{3u_s}{\beta x} \right)^2 + \frac{1-\nu}{2} \left( \frac{3u_s}{\beta y} \right)^2 \\
+ \frac{3u_s}{\beta x} \left[ \frac{3z}{\beta x} - \frac{3z_o}{\beta x} \right]^2 + \nu \left[ \frac{3z}{\beta y} - \frac{3z_o}{\beta y} \right] \right) \\
+ 2\nu \frac{3u_s}{\beta x} \frac{3v_s}{\beta y} + (1-\nu) \frac{3u_s}{\beta y} \frac{3v_s}{\beta y} \\
+ (1-\nu) \frac{3u_s}{\beta y} \left[ \frac{3z_o}{\beta y} - \frac{3z_o}{\beta y} \right] \right) dx \, dy \\
- \frac{p_x}{3A_{i,j}} \left[ \frac{3u_s}{3y} \right]_{x=0, y=bl}^{x=a, y=bl} \\
+ \frac{EI}{2} \left[ \frac{3u_s}{3y} \right]_{x=0}^{x=a} \right|_y=0 \\
+ \frac{1}{(Eh)} \frac{3u_s}{3y} \right|_{x=a} \\
\right|_y=0 \\
\right|_{x=a} \\
\right| = (3.2.10)
\]

The following treats the various parts of this expression.
separately. Using equation (3.2.3),

\[
\frac{3u_x}{2x} + \frac{(1-v)}{2} \left( \frac{3u_y}{2y} \right)^2 \sum_{ij} A_{ij} \left( \frac{\partial^2 u_i}{\partial x^2} + \frac{\partial^2 u_j}{\partial y^2} \right)
\]

\[
+ \frac{(1-v)}{2} \left( \sum_{ij} A_{ij} \left( \frac{\partial^2 u_i}{\partial x^2} + \frac{\partial^2 u_j}{\partial y^2} \right) \right)^2
\]

thus \( \frac{1}{2} \sum_{ij} A_{ij} \left( \frac{3u_x}{2x} + \frac{(1-v)}{2} \left( \frac{3u_y}{2y} \right)^2 \right) \)

\[
= \int_a^b \int_c^d \frac{\partial f_{ui}}{\partial x} \frac{\partial f_{uk}}{\partial x} \sum_{k,l} A_{k,l} \left( \frac{\partial u_k}{\partial x} \right) \left( \frac{\partial u_l}{\partial y} \right) dx dy
\]

\[
+ \frac{(1-v)}{2} \int_a^b \int_c^d \frac{\partial f_{ui}}{\partial x} \frac{\partial f_{uk}}{\partial x} \sum_{k,l} A_{k,l} \left( \frac{\partial u_k}{\partial x} \right) \left( \frac{\partial u_l}{\partial y} \right) dx dy
\]

\[
= \sum_{k,l} A_{k,l} \left( R_{i,j,k,l} \right),
\]

where \( R_{i,j,k,l} = \int_a^b \int_c^d \frac{\partial f_{ui}}{\partial x} \frac{\partial f_{uk}}{\partial x} \sum_{k,l} A_{k,l} \left( \frac{\partial u_k}{\partial x} \right) \left( \frac{\partial u_l}{\partial y} \right) dx dy
\]

\[
+ \frac{(1-v)}{2} \int_a^b \int_c^d \frac{\partial f_{ui}}{\partial x} \frac{\partial f_{uk}}{\partial x} \sum_{k,l} A_{k,l} \left( \frac{\partial u_k}{\partial x} \right) \left( \frac{\partial u_l}{\partial y} \right) dx dy.
\]

Using equations (3.2.1) to (3.2.3) it can be shown that

\[
\frac{1}{2} \sum_{ij} A_{ij} \left( \frac{3u_x}{2x} + \frac{(1-v)}{2} \left( \frac{3u_y}{2y} \right)^2 \right) + y \left( \frac{3z_x}{3x} - \frac{3z_y}{3y} \right)\left( \frac{3z_x}{3x} - \frac{3z_y}{3y} \right)
\]

\[
+ \frac{(1-v)}{2} \left( \frac{3z_x}{3x} - \frac{3z_y}{3y} \right) \left( \frac{3z_x}{3x} - \frac{3z_y}{3y} \right) dx dy
\]

\[
= \sum_{p,q,r,s} \left( Z_{p,q,r,s} \left( \frac{Z_{p,q,r,s}}{Z_{p,q,r,s}} - \frac{Z_{p,q,r,s}}{Z_{p,q,r,s}} \right) R_{i,j,p,q,r,s} \right)
\]

(3.2.12a)
where \( R_{i,j,p,q,r,s} \),

\[
\begin{align*}
&= \int_{x=0}^{a} \int_{y=0}^{b} f_{ui} \left( - \frac{a}{a} \cos\left( \frac{a}{a} \right) \cos\left( \frac{b}{b} \right) \sin\left( \frac{c}{c} \right) \sin\left( \frac{d}{d} \right) \right) \\
&\quad + \frac{a}{b} \sin\left( \frac{b}{b} \right) \sin\left( \frac{c}{c} \right) \cos\left( \frac{a}{a} \right) \cos\left( \frac{d}{d} \right) \\
&\quad + \frac{b}{b} \sin\left( \frac{b}{b} \right) \cos\left( \frac{a}{a} \right) \cos\left( \frac{d}{d} \right) \sin\left( \frac{c}{c} \right) \sin\left( \frac{d}{d} \right) \right) \, dx \, dy,
\end{align*}
\]

(3.2.12b)

in which, for symmetrical shapes,

\[ x_p = (2p - 1)\pi, \quad \beta_r = (2r - 1)\pi, \quad a_q = (2q - 1)\pi, \quad \gamma_s = (2s - 1)\pi. \]

Also,

\[
\begin{align*}
\Delta_{i,j} &= \frac{P_x (1 - \nu^2)}{E h} \int_{y=0}^{b} \left[ u_x \right]_{x=0, y=b} - u_x \right|_{x=a, y=b} \\
&= \left( - \frac{P_x (1 - \nu^2)}{E h} R_{3_i,j} \right)
\end{align*}
\]

(3.2.13a)

where \( R_{3_i,j} = [f_{ui}(0) - f_{ui}(a)] \) \( g_{uj}(b) \)

(3.2.13b)

and

\[
\begin{align*}
\frac{EI}{(Eh)} \int_{y=0}^{b} \left[ \frac{a^2}{a} \frac{\partial^2 u_{s}}{\partial y^2} \right] \left[ \frac{a^2}{a} \frac{\partial^2 u_{s}}{\partial x^2} \right] + \left[ \frac{a^2}{a} \frac{\partial u_{s}}{\partial y} \right] \right) \, dy \\
= K_{beam} \left( \sum_{k,l} A_{x,k} \left( R_{4_{i,j,k}} \right) \right)
\end{align*}
\]

(3.2.14a)
where

\[ R^4_{i,j,k,l} = \left[ \frac{\partial^2 f_{uk}}{\partial y^2} \right]_{x=0} + \left[ \frac{\partial^2 f_{uk}}{\partial y^2} \right]_{x=a} \int_{y=0} b^2 \frac{\partial^2 g_{ui}}{\partial y^2} \frac{\partial^2 g_{ul}}{\partial y^2} \, dy \]

\[ = \left[ f_{ui}(0) f_{uk}(0) + f_{ai}(a) f_{uk}(a) \right] \int_{y=0} b^2 \frac{\partial^2 g_{ui}}{\partial y^2} \frac{\partial^2 g_{ul}}{\partial y^2} \, dy \]

(3.2.14b)

and \( K_{\text{beam}} = (EI)_{\text{beam}} \left( \frac{1-v^2}{Eh} \right)_{\text{plate}} \) (3.2.14c)

Finally, using equations (3.2.3) and (3.2.4) it can be shown that,

\[ \frac{3}{2R_{i,j}} \int_{x=0}^{a} \int_{y=0}^{b} \left[ 2v \frac{\partial u_s}{\partial x} \frac{\partial v_s}{\partial y} + (1-v) \frac{\partial u_s}{\partial y} \frac{\partial v_s}{\partial x} \right] \, dx \, dy \]

\[ = \{ \Sigma B_{m,n} R^5_{i,j,m,n} \} \]

(3.2.15a)

where \( R^5_{i,j,m,n} = \int_{x=0}^{a} \int_{y=0}^{b} \left[ (v \frac{\partial f_{ui}}{\partial x} \frac{\partial g_{uj}}{\partial y} + \frac{\partial g_{vn}}{\partial y} \right] \, dx \, dy \]

\[ + \left( \frac{1-v}{2} \right) f_{ui} \frac{\partial f_{vm}}{\partial x} \frac{\partial g_{uj}}{\partial y} g_{vn} \, dx \, dy \]

(3.2.15b)

Substituting equations (3.2.11a), (3.2.12a), (3.2.13a), (3.2.14a) and (3.2.15a) into equation (3.2.10) gives,

\[ \{ \Sigma A_{k,l}(R^1_{i,j,k,l} + R^4_{i,j,k,l}) + \Sigma B_{m,n} R^5_{i,j,m,n} \]

\[ + \Sigma \Sigma \Sigma Z_{p,q,r,s} - Z_{op,q} Z_{or,s} R^1_{i,j,p,q,r,s} \]

\[ - P_x R^3_{i,j} \} = 0 \]

(3.2.16)

For different values of \( i \) and \( j \) this will result in different equations. The total number of equations will be
equal to the total number of in-plane shapes taken for \( u_s \).

This set of equations represents an approximation to the
equation of equilibrium in the \( x \)-direction. The same
operations in the \( y \)-direction will result in the following
type of equations:

\[
\sum_{i} \sum_{j,k,l} R^{i}_{j,k,l} A_{k,l} + \sum_{m,n} B_{m,n} S^{i}_{j,m,n}
+ \sum_{p,q,r,s} \left( Z_{p,q,r,s} - Z_{p,q,r,s} \right) S^{2}_{i,j,p,q,r,s} = 0,
\]

where

\[
S^{1}_{i,j,m,n} = \int_{x=0}^{a} \int_{y=0}^{b} \left[ f_{vi} f_{vm} \frac{\partial q_{vi}}{\partial y} \frac{\partial q_{vn}}{\partial y} \right] dx dy
+ \frac{(1-v)}{2} \left[ \frac{\partial f_{vi}}{\partial x} \frac{\partial f_{m}}{\partial x} q_{vj} q_{vn} \right] dx dy
\]

\[
S^{2}_{i,j,p,q,r,s} = \int_{x=0}^{a} \int_{y=0}^{b} \left[ \frac{\partial q_{vi}}{\partial y} \cos\left(\frac{\alpha x}{a}\right) \sin\left(\frac{\beta x}{a}\right) \right] dx dy
+ \int_{x=0}^{a} \int_{y=0}^{b} \frac{\partial f_{vi}}{\partial x} \frac{\partial f_{vi}}{\partial x} q_{vj} \left( \frac{\partial q_{vi}}{\partial x} \cos\left(\frac{\beta x}{a}\right) \cos\left(\frac{\alpha x}{a}\right) \sin\left(\frac{\beta x}{a}\right) \right) dx dy
+ \frac{\alpha p}{ab} \frac{\beta y}{b} \cos\left(\frac{\alpha x}{a}\right) \cos\left(\frac{\beta x}{a}\right) \cos\left(\frac{\alpha y}{a}\right) \cos\left(\frac{\beta y}{b}\right) \] dx dy.

(3.2.17)

(3.2.18a)

(3.2.18b)
Equations (3.2.16) and (3.2.17) can be used to calculate the relationship between the in-plane and out-of-plane displacement coefficients. Since these equations are linear in $A_{k,l}$ and $B_{m,n}$, the solution can be obtained as follows:

Let $A_{k,l}$ and $B_{k,l}$ be the solutions of,

\[
\sum_{k} \sum_{l} A_{k,l} (R_{i,j,k,l} + R_{i,j,k,l}^4) + \sum_{m} \sum_{n} B_{m,n} (R_{i,j,m,n}^5) = - (R_{i,j})
\]  
(3.2.19a)

and

\[
\sum_{k} \sum_{l} A_{k,l} (R_{i,j,k,l}^4) + \sum_{m} \sum_{n} B_{m,n} (S_{i,j,m,n}) = \{0\}.
\]  
(3.2.19b)

Also, let $A_{k,l,p,q,r,s}$ and $B_{m,n,p,q,r,s}$ be the solutions of,

\[
\sum_{k} \sum_{l} A_{k,l}^{''} (R_{i,j,k,l}^4 + R_{i,j,k,l}^4) + \sum_{m} \sum_{n} B_{m,n}^{''} (R_{i,j,m,n}^5) = - (R_{i,j}^3)
\]  
(3.2.19c)

and

\[
\sum_{k} \sum_{l} A_{k,l}^{''} (R_{i,j,k,l}^4) + \sum_{m} \sum_{n} B_{m,n}^{''} (S_{i,j,m,n}) = \{0\}.
\]  
(3.2.19d)

Then from linear algebra,

\[
\{A_{i,j}\} = \{P_{x} A_{i,j}^{'}, + \sum_{p} \sum_{q} \sum_{r} (Z_{p,q,r,s} - Z_{p,q,r,s}^{0}) A_{i,j}^{''} \}
\]  
(3.2.20a)

and

\[
\{B_{i,j}\} = \{P_{x} B_{i,j}^{'}, + \sum_{p} \sum_{q} \sum_{r} (Z_{p,q,r,s} - Z_{p,q,r,s}^{0}) B_{i,j}^{''} \}
\]  
(3.2.20b)

Equations (3.2.19a) - (3.2.19d) can be solved using Gaussian elimination. The resulting values of $A_{i,j,p,q,r,s}$ and
$B_{i,j,p,q,r,s}$ are the connection coefficients which relate the in-plane displacement coefficients to the various products of out-of-plane displacement coefficients. $A_{i,j}$ and $B_{i,j}$ give the in-plane displacement coefficients due to the displacement of the applied in-plane load and can be considered as a special set of connection coefficients resulting from the change in the position of the applied load.

At this stage it becomes necessary to introduce the following definitions and matrix notations.

**Definitions:**

- $N_{ux}$: The maximum number of shape functions for $f_u$
- $N_{uy}$: The maximum number of shape functions for $g_u$
- $N_{vx}$: The maximum number of shape functions for $f_v$
- $N_{vy}$: The maximum number of shape function for $g_v$
- $P_m$: The maximum number of out-of-plane displacement shapes in $x$ direction
- $q_m$: The maximum number of out-of-plane displacement shapes in $y$ direction
- $I_p$: A position indicator for $Z_{p,q}$ defined by $I_p = q + (p-1)q_m$
- $I_r$: A position indicator for $Z_{r,s}$ defined by $I_r = s + (r-1)q_m$
- $I_{pq}$: The total number of shapes in the out-of-plane displacement series given by $I_{pq} = p_m q_m$
\( L \) : A position indicator for the connection coefficients 
defined as 
\[ L = l + I_{p} \cdot (p - 1)I_{pq} - I_{p} (I_{pq} - 1)/2 \]

\( \hat{L} \) : Maximum value of \( L \); 
\[ \hat{L} = l + I_{pq}^2 - I_{pq} (I_{pq} - 1)/2 \]

(The use of \( L \) and \( \hat{L} \) will be explained later)

\( N_u \) : Total number of shapes in the series for \( u \) and is 
given by 
\[ N_u = N_{ux} \times N_{uy} \]

\( N_v \) : Total number of shapes in the series for \( v \) and is 
given by 
\[ N_v = N_{vx} \times N_{vy} \]

\( N_n \) : Total number of in-plane displacement coefficients 
and is given by 
\[ N_n = N_u + N_v \]

The left hand side of the set of equations (3.2.16) and (3.2.17) can be written in matrix form.

i.e. L.H.S. of equations (3.2.16) and (3.2.17) = \([SZ][C]\) 

(3.2.21)

where \( C(I) = A_{i,j} \) for \( I \leq N_u \), 
in which 
\[ I = j + (i-1)N_{uy} \] 

(3.2.21a)

\( C(I) = B_{i,j} \) for \( I > N_u \), 
in which, \( I = N_u + j + (i-1)N_{vy} \) 

(3.2.21b)

\( SZ(I,J) = (R_{i,j,k,l} + R_{4,i,j,k,l}) \) for \( I \leq N_u \) and \( J \leq N_u \) 
in which, \( I = j + (i-1)N_{uy} \) 
and \( J = k + (k-1)N_{uy} \) 

(3.2.21c)
\[ \text{SZ}(I,J) = R^2_{i,j,m,n} \quad \text{for } I \leq N_u \text{ and } J > N_u \]

in which \[ I = j + (i-1)N_{uy} \]

and \[ J = N_u + n + (m-1)N_{vy} \]  

(3.2.21d)

\[ \text{SZ}(I,J) = R^2_{i,j,k,l} \quad \text{for } I > N_u \text{ and } J < N_u \]

in which \[ J = k + (i-1)N_{uy} \]

and \[ I = N_u + j + (i-1)N_{vy} \]  

(3.2.21e)

\[ \text{SZ}(I,J) = S^1_{i,j,m,n} \quad \text{for } I > N_u \text{ and } J > N_u \]

in which \[ J = N_u + n + (m-1)N_{vy} \]

and \[ I = N_u + j + (i-1)N_{vy} \]  

(3.2.21f)

Equations (3.2.16) and (3.2.17) cannot be solved at this stage since the out-of-plane displacement coefficients are yet to be determined. However, equations (3.2.19a) to (3.2.19d) can be solved as follows:

Let the connection coefficient matrix be \([H]\), such that \(H(I,L)\) gives the displacement coefficient \(C(I)\) for a unit value of either the load \(P\) (if \(L = 1\)) or for a product of two out-of-plane displacement coefficients of unit magnitude, the identities of which are indicated by a connection index \(L\) as explained below.

\[ L = 1 + \frac{I_x}{I_{pq}} + (I_{pq} - 1)I_{pq} - I_p(I_p - 1)/2 \]

\( L = 1 \) corresponds to the contribution from displacement of the load.
i.e. \( H(I,l) = A_{i,j}^l \) for \( I \leq N_u \)

where \( I = j + (i-1)N_{uy} \) \hspace{1cm} (3.2.22a)

\( H(I,l) = B_{i,j}^l \) for \( I > N_u \)

where \( I = N_u + j + (i-1)N_{vy} \). \hspace{1cm} (3.2.22b)

If \( L > 1 \), \( L \) indicates the out-of-plane displacement coefficients that are considered. This can be explained through an example as follows:

**Example:**

Consider two shape functions in each direction \((x,y)\) for \( z \).

The displacement coefficients involved are,

\[
\begin{align*}
Z_{1,1}, \quad Z_{1,2}, \quad Z_{2,1}, \quad Z_{2,2} \\
p_m = q_m = 2, \\
l_{pq} = 4.
\end{align*}
\]

Using the definition of position indicators, the displacement coefficients can be written as

\[
Z_{1,1} = Z(1), \quad Z_{1,2} = Z(2), \quad Z_{2,1} = Z(3), \quad Z_{2,2} = Z(4).
\]

The following ten combinations of products are then possible:

\[
\begin{align*}
Z(1) \times Z(1) & \quad Z(1) \times Z(2) & \quad Z(1) \times Z(3) & \quad Z(1) \times Z(4) \\
Z(2) \times Z(2) & \quad Z(2) \times Z(3) & \quad Z(2) \times Z(4) \\
Z(3) \times Z(3) & \quad Z(3) \times Z(4) \\
Z(4) \times Z(4)
\end{align*}
\]
It can be checked by substitution that the positions of these products are given by the connection index \( L \) as follows:

\[
\begin{align*}
2 & \quad 3 & \quad 4 & \quad 5 \\
6 & \quad 7 & \quad 8 \\
9 & \quad 10 \\
11
\end{align*}
\]

For example, consider the position of \( Z(2) \) \( Z(4) \):

\[
I_p = 2, \quad I_r = 4 \quad \text{gives,} \quad L = 1 + \frac{1}{4} \cdot (2-1)4 - \frac{2(2-1)}{2} = 8.
\]

Therefore, the in-plane displacement coefficients due to the change in the product of two out-of-plane displacement coefficients (having unit value) is given by the matrix \([H]\)

where for \( L > 1, \)

\[
H(I, L) = A''_{i}, j, p, q, r, s \quad \text{for} \quad I \leq N_u
\]

in which \( I = j + (i-1)N_{uy} \).

\[
(3.2.22c)
\]

\[
H(I, L) = B''_{i}, j, p, q, r, s, \quad \text{for} \quad I > N_u
\]

in which \( I = N_u + j + (i-1)N_{uy} \).

\[
(3.2.22d)
\]

From equations (3.2.20a), (3.2.20b), (3.2.21a) and (3.2.21b) it follows that,

\[
C(I) = H(I, L) P_x + \sum_{L=2}^{\hat{L}} H(I, L) \times (Z_{p, q, s}^{Z_{r, q, s}} - Z_{0, p, q, s}^{Z_{0, r, q, s}})
\]

\[
(3.2.23)
\]

Furthermore, the R.H.S. of equations (3.2.19a) to (3.2.19d)
can be written as

\[ \text{R.H.S. of equations (3.2.19a) to (3.2.19d)} = [ZB], \]

\[ (3.2.24a) \]

where

\[ ZB(\ell,1) = -R_{S_{ij}} \text{ for } \ell \leq N_u, \text{ in which } \ell = j + (i-1)N_{u_y} \]

\[ ZB(\ell,1) = 0.0 \text{ for } \ell > N_u, \text{ in which } \ell = N_u + j + (i-1)N_{u_y} \]

\[ (3.2.24b) \]

and

\[ ZB(\ell,L) = -R_{S_{ijpqrs}} \text{ for } \ell \leq N_u, \text{ in which } \ell = j + (i-1)N_{u_y} \]

\[ ZB(\ell,L) = -S_{ijpqrs} \text{ for } \ell > N_u, \text{ in which } \ell = N_u + j + (i-1)N_{u_y} \]

\[ (3.2.24c) \]

Hence,

\[ [SZ][H] = [ZB]. \]

\[ (3.2.25) \]

The unknowns in [H] can be found by Gaussian elimination for each value of \( L \).

i.e. \([SZ][H(L1)] = [ZB(L1)] \) (for \( L = L1 \)) can be solved for any value of \( L1 \). The reduction of \([SZ]\) to a triangular matrix needs to be done only once. A special Gaussian elimination procedure was written for this purpose.

The in-plane displacement coefficients \( \{C\} \) can be found after calculating the out-of-plane displacement coefficients as described below.
Calculation of Out-of-plane Displacement Coefficients

The solution of equation (3.2.9) is obtained using a modified version of Newton-Raphson's method [34] as described below. The idea is demonstrated through a simple example in Appendix E.

\[
\frac{\partial \bar{V}_T}{\partial Z_{i,j}} = \{0\} \text{ is found when}
\]

\[
[\frac{d}{dZ_{r,s}} \left( \frac{\partial \bar{V}_T}{\partial Z_{i,j}} \right)] \{\Delta Z_{r,s} \} = -\left( \frac{\partial \bar{V}_T}{\partial Z_{i,j}} \right), \tag{3.2.26}
\]

as \(\{\Delta Z_{r,s}\} = \{0\}\) is satisfied.

After calculating \(\left\{ \frac{\partial \bar{V}_T}{\partial Z_{i,j}} \right\}\) in terms of \(A_{i,j}, B_{i,j}\) and \(Z_{i,j}\), the relationship between the in-plane and out-of-plane displacement coefficients can be substituted in equation (3.2.26). However, at this stage, the in-plane displacement coefficients should be treated as dependent variables, since the relationship between these and the out-of-plane coefficients is used. The following distinctions must be clearly made.

All displacement coefficients are treated as independent variables when applying the Rayleigh-Ritz method to form equations (3.2.7), (3.2.8) and (3.2.9). When solving equations (3.2.9), the relationship between the in-plane and out-of-plane displacement coefficients is used and therefore the in-plane displacement coefficients are treated as dependent variables, giving
\[
\frac{d}{dZ_{r,s}} \left( \frac{3\vec{V}_T}{3Z_{i,j}} \right) = \frac{3^2\vec{V}_T}{3Z_{r,s} 3Z_{i,j}} + \sum_k \left( \frac{3^2\vec{V}_T}{3Z_{r,s} 3Z_{i,j}} \right) \frac{dA_{k,\ell}}{dZ_{r,s}} \\
+ \sum_m \sum_n \left( \frac{3^2\vec{V}_T}{3Z_{r,s} 3Z_{i,j}} \right) \frac{dB_{m,n}}{dZ_{r,s}}.
\]

(3.2.27)

Hence the required equation

\[
\frac{3^2\vec{V}_T}{3Z_{r,s} 3Z_{i,j}} + \left( \frac{3^2\vec{V}_T}{3Z_{r,s} 3Z_{i,j}} \right) \frac{dA_{k,\ell}}{dZ_{r,s}} \\
+ \left( \frac{3^2\vec{V}_T}{3Z_{r,s} 3Z_{i,j}} \right) \left( \frac{dB_{m,n}}{dZ_{r,s}} \right) \Delta Z_{r,s} = - \left( \frac{3\vec{V}_T}{3Z_{i,j}} \right).
\]

(3.2.28)

Using equations (2.2.21a) and (2.2.21b) this can be written as

\[
\frac{3^2\vec{V}_T}{3Z_{r,s} 3Z_{i,j}} + \left( \frac{3^2\vec{V}_T}{3Z_{r,s} 3Z_{i,j}} \right) \left( \frac{dC(I)}{dZ_{r,s}} \right) \Delta Z_{r,s} = - \left( \frac{3\vec{V}_T}{3Z_{i,j}} \right).
\]

(3.2.28a)

This is the matrix iteration equation.

Terms in the derivatives of \( \vec{V}_T \) can be evaluated successively until all values of \( \Delta Z_{r,s} \) become very small. After each iteration, the values of \( Z_{i,j} \) are corrected by adding \( \Delta Z_{r,s} \) calculated by solving equation (3.2.28a) using Gaussian elimination.
Using equation (3.2.6) and noting that $\bar{V}_{\text{LOAD}}$ and $\bar{U}_{\text{beam}}$ do not depend on $Z_{i,j}$, the following equation is obtained

$$\frac{\partial \bar{V}_T}{\partial Z_{i,j}} = \frac{\partial \bar{U}_{\text{be}}}{\partial Z_{i,j}} + \frac{\partial \bar{U}_{\text{str}}}{\partial Z_{i,j}}. \quad (3.2.29)$$

The term $\frac{\partial \bar{U}_{\text{be}}}{\partial Z_{i,j}} = \frac{Eh^3}{24(1-\nu^2)} \left[ \left( \frac{a}{a} \right)^2 + \left( \frac{b}{b} \right)^2 \right]^2 \frac{ab}{4}(Z_{i,j} - Z_{o,i,j}) \times 2$

$$= \frac{Eh^3}{24(1-\nu^2)} G_1. \quad (3.2.30)$$

where

$$G_1 = \frac{ab}{2}(Z_{i,j} - Z_{o,i,j}). \left[ \left( \frac{\alpha_i}{a} \right)^2 + \left( \frac{\alpha_j}{b} \right)^2 \right]^2, \quad (3.2.30a)$$

in which

$$\alpha_i = (2i-1)\pi \quad (3.2.30b)$$

$$\alpha_j = (2j-1)\pi \quad (3.2.30b)$$

It can be shown that,

$$\frac{\partial \bar{U}_{\text{str}}}{\partial Z_{i,j}} = \frac{Eh^3}{(1-\nu^2)} \sum_{m=2}^{10} G_m. \quad (3.2.31)$$

$G_2$ to $G_{10}$ can be obtained as follows:

$$G_2 = \frac{1}{2} \frac{1}{8} \int_{0}^{a} \int_{0}^{b} \left( \frac{\partial Z}{\partial x} \right)^4 + \left( \frac{\partial Z}{\partial y} \right)^4 \, dx \, dy
$$

$$= \frac{1}{8} \int_{0}^{a} \int_{0}^{b} \left[ 4 \left( \frac{\partial Z}{\partial x} \right)^3 \frac{\partial Z}{\partial x} \frac{\partial Z}{\partial y} \right] + 4 \left( \frac{\partial Z}{\partial y} \right)^3 \frac{\partial Z}{\partial y} \frac{\partial Z}{\partial Z_{i,j}} \, dx \, dy
$$

$$= \frac{1}{8} \sum_{r,s,p,q,k,l} \left( Z_{r,s,p,q,k,l} \right) \cdot (3.2.32)$$
where

\[ Q_2 = \frac{\beta}{a} \int_0^a \cos\left(\frac{r}{a}\right) \cos\left(\frac{x}{a}\right) \cos\left(\frac{y}{a}\right) \cos\left(\frac{\phi}{a}\right) dx \]

and

\[ b \int_0^a \sin\left(\frac{r}{b}\right) \sin\left(\frac{s}{b}\right) \sin\left(\frac{y}{b}\right) \sin\left(\frac{\phi}{b}\right) dy + \]

\[ y = 0 \]

\[ \frac{a}{\beta} \int_0^a \sin\left(\frac{r}{a}\right) \sin\left(\frac{s}{a}\right) \sin\left(\frac{y}{a}\right) \sin\left(\frac{\phi}{a}\right) dx \]

\[ x = 0 \]

\[ b \int_0^a \cos\left(\frac{r}{b}\right) \cos\left(\frac{s}{b}\right) \cos\left(\frac{y}{b}\right) \cos\left(\frac{\phi}{b}\right) dy \] \hspace{1cm} (3.2.32a)

in which

\[ a_i = (2i-1)\pi, \quad a_j = (2j+1)\pi \]

\[ \beta_k = (2r-1)\pi, \quad \beta_k = (2s-1)\pi \]

\[ \gamma_p = (2m-1)\pi, \quad \gamma_q = (2q-1)\pi \]

\[ \phi_k = (2k-1)\pi, \quad \phi_k = (2l-1)\pi \] \hspace{1cm} (3.2.32b)

(The definition of these angles \( a_i \) to \( \phi_k \) applies to all the following equations in section 3.2.)

\[ G_3 = \frac{1}{Z_{i,j}} \int_0^a \frac{1}{4} \int_0^a \frac{b}{x} \left(\frac{\partial^2}{\partial x^2}\right) \left(\frac{\partial^2}{\partial y^2}\right) dx \] \hspace{1cm} (3.2.33)

\[ \int_0^b \frac{1}{y} \]
where
\[
Q_3 = (a_1, b_2, c_1) \int_0^a \cos \left( \frac{a_1 x}{a} \right) \cos \left( \frac{b_2 x}{b} \right) \sin \left( \frac{c_1 x}{a} \right) \sin \left( \frac{c_1 x}{b} \right) dx +
\]
\[
\int_0^b \sin \left( \frac{a_1 y}{b} \right) \cos \left( \frac{b_2 y}{b} \right) \cos \left( \frac{c_1 y}{b} \right) \cos \left( \frac{c_1 y}{b} \right) dy.
\]

(3.2.33a)

\[
G_4 = \frac{1}{3} \int_{1, j} \left( \frac{1}{4} \right) \int_0^a \int_0^b \left[ \left( \frac{3 z}{3 x} \right)^2 + \left( \frac{3 z}{3 y} \right)^2 \right] dx \, dy.
\]

(3.2.34)

\[
G_5 = \frac{1}{3} \int_{1, j} \left( \frac{1}{4} \right) \int_0^a \int_0^b \left[ \left( \frac{3 z}{3 x} \right)^2 + \left( \frac{3 z}{3 y} \right)^2 \right] dx \, dy.
\]

(3.2.35)
\[
G_b = -\frac{(1-v)}{2} \frac{\partial}{\partial z_{1,j}} \int_a^b \int_{x=0}^{b} \frac{\partial z}{\partial x} \frac{\partial z}{\partial y} \frac{\partial z_0}{\partial x} \frac{\partial z_0}{\partial y} \, dx \, dy
\]

\[
= -\frac{(1-v)}{2} \int_a^b \left[ \int_{x=0}^{b} \left( \frac{\partial z}{\partial z_{1,j}} \right) \frac{\partial z}{\partial x} \frac{\partial z_0}{\partial y} \right. \frac{\partial z_0}{\partial y} \left. \right] \, dx \, dy
\]

\[
+ \left( \frac{\partial}{\partial z_{1,j}} \right) \frac{\partial z}{\partial x} \frac{\partial z_0}{\partial x} \frac{\partial z_0}{\partial y} \, dx \, dy
\]

\[
= -\frac{(1-v)}{2} \sum_{r,s} \sum_{p,q} \sum_{k,l} (Z_{r,s} Z_{op,q} Z_{k,l}) \times Q_4,
\]

(3.2.36)

where

\[
Q_4 = \left( \frac{\partial}{\partial a} \right) \frac{\partial}{\partial \phi} \int_a^b \int_{y=0}^{b} \cos\left( \frac{a}{\alpha} \right) \sin\left( \frac{b}{\alpha} \right) \cos\left( \frac{a}{\beta} \right) \sin\left( \frac{b}{\beta} \right) \phi \, dx \, dy
\]

\[
= \int_a^b \int_{y=0}^{b} \cos\left( \frac{a}{\alpha} \right) \sin\left( \frac{b}{\alpha} \right) \cos\left( \frac{a}{\beta} \right) \sin\left( \frac{b}{\beta} \right) \phi \, dx \, dy.
\]

(3.2.36a)

\[
G_7 = \frac{1}{2} \frac{\partial}{\partial z_{1,j}} \int_a^b \int_{y=0}^{b} \frac{\partial \tilde{u}_s}{\partial x} \left[ (\frac{\partial z}{\partial x})^2 + \nu (\frac{\partial z}{\partial y})^2 \right] \, dx \, dy
\]

\[
= \int_a^b \int_{y=0}^{b} \left[ \frac{\partial \tilde{u}_s}{\partial x} \frac{\partial z}{\partial x} \frac{\partial z_{1,j}}{\partial x} \right] \, dx \, dy
\]

\[
+ \nu \frac{\partial \tilde{u}_s}{\partial x} \frac{\partial z}{\partial y} \frac{\partial z_{1,j}}{\partial y} \, dx \, dy.
\]
\[ = \sum \sum Z_{rs}, s \left\{ \int_a^b \frac{\partial u_s}{\partial x} \left( \frac{1}{a^2} \frac{\partial}{\partial x} \right) \cos \left( \frac{x}{a} \right) \cos \left( \frac{y}{a} \right) \right|_{x=0, y=0} \right. \\
\left. + \int_a^b \int_b^b \alpha \frac{s y}{b^2} \sin \left( \frac{s y}{b} \right) dy \, dy \right\} \\
\int_a^b \int_b^b \alpha \frac{s x}{b^2} \sin \left( \frac{s x}{b} \right) dy \, dy \right\}.
\]

but, \( \frac{\partial u_s}{\partial x} = \sum_{I=1}^{N_u} \frac{\partial f_{uj}}{\partial x} \bar{q}_{ul} \),

where \( I = \lambda + (k-1) N_{uy} \).

therefore \( G_7 = \sum \sum \sum \sum Z_{rs}, s \times C(I) \times Q_5 \),

\[ (3.2.37) \]

where \( Q_5 = \left( \frac{1}{a^2} \right) \int_a^b \frac{\partial f_{uk}}{\partial x} \cos \left( \frac{x}{a} \right) \cos \left( \frac{y}{a} \right) dx \int_b^b q_{ul} \sin \left( \frac{s y}{b} \right) dy \)

\[ (3.2.37a) \]

\[ G_8 = \left( \frac{1-\nu}{2} \right) \frac{\partial}{\partial z}, j \right|_{x=0, y=0} \int_a^b \int_b^b \frac{\partial u_s}{\partial y} \frac{\partial z}{\partial x} \frac{\partial z}{\partial y} dx \, dy \]

\[ = \left( \frac{1-\nu}{2} \right) \int_a^b \int_b^b \left[ \frac{\partial u_s}{\partial y} \frac{\partial z}{\partial x} \frac{\partial z}{\partial y} \frac{\partial z}{\partial y} \frac{\partial z}{\partial y} \frac{\partial z}{\partial y} \frac{\partial z}{\partial y} \frac{\partial z}{\partial y} \frac{\partial z}{\partial y} \right] dx \, dy, \]

but, \( \frac{\partial u_s}{\partial y} = \sum_{I=1}^{N_u} C(I) \frac{\partial q_{ul}}{\partial y} \)
where \( I = l + (k-1) N_{uy} \)

due to

\[
G_6 = \left( \frac{1}{2} \right) \sum_{r} \sum_{s} N_{u} Z_{r,s} C(I) \times Q_6 \quad (3.2.38)
\]

where

\[
Q_6 = \frac{a \beta s}{ab} \int_{y=0}^{a} f_{uk} \cos(\frac{1}{a}x) \sin(\frac{r}{a}x) dx \left( \frac{b \beta q u k}{a} \right) \sin(\frac{\alpha y}{b})
\]

\[
+ \frac{\beta s y}{b} \int_{y=0}^{a} f_{uk} \sin(\frac{1}{a}x) \cos(\frac{r}{a}x) dx \left( \frac{x a \beta}{b} \right)
\]

\[
\left( \frac{b \beta q u k}{a} \right) \cos(\frac{1}{b}x) \sin(\frac{y}{b}) dy.
\]

\[
G_9 = \frac{1}{2} \sum_{i,j}^{N_n} \int_{x=0}^{a} \int_{y=0}^{b} \frac{\beta q v m}{b} \left( \frac{3}{2} \right) \left( \frac{\beta z}{b} \right)^2 + v \left( \frac{r z}{x} \right)^2 \right) dx dy
\]

\[
G_9 = \sum_{r} \sum_{s} N_{u} Z_{r,s} C(I) \times Q_6
\]

where

\[
Q_7 = \frac{a \beta s}{b} \int_{x=0}^{a} f_{vm} \sin(\frac{1}{a}x) \sin(\frac{r}{a}x) dx \left( \frac{b \beta q v m}{a} \right) \cos(\frac{\alpha y}{b})
\]

\[
+ \frac{1}{a} \frac{\beta r}{b} \int_{x=0}^{a} f_{vm} \cos(\frac{1}{a}x) \cos(\frac{r}{a}x) dx \left( \frac{x a \beta}{b} \right)
\]

\[
\left( \frac{b \beta q v m}{a} \right) \sin(\frac{1}{b}x) \sin(\frac{y}{b}) dy.
\]

in which \( I = N_{u} = n + (m-1) N_{vy} \)
\[
G_{10} = \left( \frac{1-v}{2} \right) \frac{\partial}{\partial x} \int_{x=0}^{a} \int_{y=0}^{b} \frac{3y}{3x} \frac{3y}{3y} \, dx \, dy
\]

By analogy with \( G_{8} \) it can be shown that

\[
G_{10} = \left( \frac{1-v}{2} \right) \sum_{r,s=I}^{N_{r}} \sum_{I=N_{u}+1}^{N_{n}} Z_{r,s} C(I) \times Q_{8}, \tag{3.2.40}
\]

where

\[
Q_{8} = \frac{a_{0}^{3} \alpha_{s}}{ab} \int_{x=0}^{a} \frac{\partial f_{vm}}{\partial x} \cos(\frac{\alpha_{x}}{a}) \sin(\frac{\beta_{x}}{a}) \, dx \times
\]

\[
\int_{y=0}^{b} q_{vn} \sin(\frac{\alpha_{y}}{b}) \cos(\frac{\beta_{y}}{b}) \, dy
\]

\[
+ \frac{\alpha_{r} \beta_{r}}{ab} \int_{x=0}^{a} \frac{\partial f_{vm}}{\partial x} \sin(\frac{\alpha_{x}}{a}) \cos(\frac{\beta_{x}}{a}) \, dx \times
\]

\[
\int_{y=0}^{b} q_{vn} \cos(\frac{\alpha_{y}}{b}) \sin(\frac{\beta_{y}}{b}) \, dy. \tag{3.2.40a}
\]

From equations (3.2.29), (3.2.30) and (3.2.31),

\[
\frac{\partial \mathcal{V}_{T}}{\partial Z_{i,j}} = \frac{Eh^{3}}{24(1-v^{2})} \mathcal{G}_{1} + \frac{Eh^{3}}{(1-v^{2})} \mathcal{G}_{m} \tag{3.2.41}
\]

For each value of \( i,j \), this gives a value for \( \frac{\partial \mathcal{V}_{T}}{\partial Z_{i,j}} \).

This can be expressed in matrix form as

\[
\{ R \}, \text{ where } R(J) = \frac{\partial \mathcal{V}_{T}}{\partial Z_{i,j}},
\]

in which

\[
J = j + (i-1)q_{m}
\]

- \( \{ R \} \) is the R.H.S. of equation (3.2.28a)
The terms in the L.H.S. of equation (3.2.28a) can be calculated as follows:

\[ T_1 = \frac{\partial^2 \psi}{\partial T_1, j} = \frac{Eh}{24(1-v^2)} \frac{\partial^3}{\partial r^3} G_{1, j} \cdot \left( \frac{\partial}{\partial r} \right)^{10} \frac{3G_m}{\partial Z_r, s} \]

(3.2.42)

The terms \(\partial G_m/\partial Z_r, s\) are given as follows.

\[ \frac{\partial G_1}{\partial Z_r, s} = \left[ \left( \frac{a}{b} \right)^2 + \left( \frac{b}{a} \right)^2 \right] \frac{ab}{4} \text{ if } i=r \text{ and } j=s \]

(3.2.42a)

\[ = \delta \text{ if } i \neq r \text{ or } j \neq s \]

\[ \frac{\partial G_2}{\partial Z_r, s} = \frac{1}{8} \int_0^a \int_0^b \left\{ 12 \left( \frac{\partial z}{\partial x} \right) \frac{\partial}{\partial r} \left( \frac{\partial z}{\partial y} \right) \partial Z_{i, j} \right\} dx \, dy \]

\[ + 12 \left( \frac{\partial z}{\partial y} \right) \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial x} \right) \partial Z_{i, j} \]

\[ = \frac{3}{2} \sum_{p \neq q \neq k \neq l} \left( z_p, z_q, z_k, z_l \right) \times Q_2 \]

(3.2.42b)

\[ \frac{\partial G_3}{\partial Z_r, s} = \frac{1}{2} \left\{ \int_0^a \int_0^b \left( \frac{\partial^2 z}{\partial x^2} \right) \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial x} \right) \partial Z_{i, j} \right\} dx \, dy \]

\[ + \int_0^a \int_0^b \left( \frac{\partial^2 z}{\partial x} \right) \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial x} \right) \partial Z_{i, j} \right\} dx \, dy \]

\[ + \int_0^a \int_0^b \left( \frac{\partial z}{\partial y} \right) \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial x} \right) \partial Z_{i, j} \right\} dx \, dy \]

\[ = \frac{1}{2} \sum_{p \neq q \neq k \neq l} \left( z_p, z_q, z_k, z_l \right) \times \left( Q_3 + Q_6 + Q_{10} \right) \]

(3.2.42c)
where
\[
Q_g = 2 \frac{1}{a^2 b^2} \int_{x=0}^{a} \int_{y=0}^{b} \left[ \alpha x + \beta y \right] \left[ \cos\left(\frac{x}{a}\right) \sin\left(\frac{y}{b}\right) \cos\left(\frac{y}{b}\right) \cos\left(\frac{y}{b}\right) \cos\left(\frac{y}{b}\right) \right] \, dx \, dy
\]

and
\[
Q_{10} = 2 \frac{1}{a^2 b^2} \int_{x=0}^{a} \int_{y=0}^{b} \left[ \alpha x + \beta y \right] \left[ \cos\left(\frac{x}{a}\right) \sin\left(\frac{y}{b}\right) \cos\left(\frac{y}{b}\right) \cos\left(\frac{y}{b}\right) \cos\left(\frac{y}{b}\right) \right] \, dx \, dy
\]

\[\frac{\partial G_4}{\partial r, s} = - \frac{1}{2} \sum_{p, q, k, l} (z_{op}, q, z_{ok}, l) \cdot Q_2 \quad (3.2.42d)\]

\[\frac{\partial G_5}{\partial r, s} = - \frac{\nu}{2} \sum_{p, q, k, l} (z_{op}, q, z_{ok}, l) \cdot Q_3 \quad (3.2.42e)\]

\[\frac{\partial G_6}{\partial r, s} = - \frac{(1-\nu)}{2} \sum_{p, q, k, l} (z_{op}, q, z_{ok}, l) \cdot Q_4 \quad (3.2.42f)\]

\[\frac{\partial G_7}{\partial r, s} = N_u \sum_{I=0}^{C(I)} \cdot Q_5 \quad (3.2.42g)\]

\[\frac{\partial G_8}{\partial r, s} = N_u \sum_{I=0}^{C(I)} \cdot Q_6 \quad (3.2.42h)\]

\[\frac{\partial G_9}{\partial r, s} = N_n \sum_{I=N_u+1}^{C(I)} \cdot Q_7 \quad (3.2.42i)\]
\[ \frac{\partial G_{10}}{\partial z_{r,s}} = \sum_{I=N_u+1}^{N_i} C(I) \times Q_8 \times \frac{(1-\nu)}{2} \]  \hspace{1cm} (3.2.42j)

Now the terms in the L.H.S. of equation (3.2.28a) resulting from

\[ \left\{ \frac{\partial^2 V_T}{\partial C(I) \partial z_{i,j}} \right\}^T \left\{ \frac{dC(I)}{dz_{r,s}} \right\} \]

can be evaluated as follows:

By definition, \( C(I) = H(I,L) \times P_x + \sum_{L} H(I,L) (Z_p,q \times Z_{r,s} - Z_{op,q} Z_{or,s}) \) and

\[ \frac{dC(I)}{dz_{r,s}} = \frac{L_2}{L_1} \times H(I,L) \times Z_{p,q} \times F_2, \]  \hspace{1cm} (3.2.43)

where

\[ L_1 = 1 + I_r \]

\[ L_2 = 1 + I_r + (I_pq-1)I_{pq}/2 \]  \hspace{1cm} (3.2.43a)

\[ F_2 = 1.0 \text{ if } p \neq r \text{ or } q \neq s \]

\[ F_2 = 2.0 \text{ if } p=r \text{ and } q=s \]

\[ \frac{\partial^2 V_T}{\partial C(I) \partial z_{i,j}} = \frac{Eh}{(1-\nu)^2} \frac{\partial}{\partial C(I)} \left[ G_7 + G_8 + G_9 + G_{10} \right] \]

\[ = \frac{Eh}{(1-\nu)^2} \sum_{r,s} Z_{r,s} \left( Q_5 + Q_6 \frac{(1-\nu)}{2} + \Omega_7 + \Omega_8 \frac{(1-\nu)}{2} \right). \]  \hspace{1cm} (3.2.44)

From equations (3.2.43), (3.2.43a) and (3.2.44),
\[ \frac{3^2 \varphi_T}{3C(I)} \frac{d^2 c(I)}{d z^2_{r,s}} = \frac{E h}{(1 - \nu^2)} Z_{r,s} Z_{p,q} \times \]
\[ F_2 \left[ Q_5 + Q_6 \frac{(1 - \nu)}{2} + Q_7 + Q_8 \frac{(1 - \nu)}{2} \right] \times \frac{L_2}{L_1} H(I,L) = T_2 . \quad (3.2.45) \]

Hence, the L.H.S. of equation (3.2.28a) can be written in matrix form as,

\[ [S_{\lambda}] \{ \triangle Z \} = \text{L.H.S. of equation (3.2.28a)} , \quad (3.2.46) \]

where \( S_{\lambda}(I,J) = T_1 + T_2 . \) \quad (3.2.46a)

From equations (3.2.28a) and (3.2.41),

\[ [S_{\lambda}] \{ \triangle Z \} = - \{ R \} . \quad (3.2.47) \]

This set of equations can be solved using Gaussian elimination until \( \{ \triangle Z \} \) becomes sufficiently small. Before each iteration, the coefficients of the matrix \( [S_{\lambda}] \) and the vector \( \{ R \} \) must be calculated using the latest value of \( Z_{i,j} . \)

The strains and stresses at a particular point \( \bar{x}, \bar{y} \) are found by using the following formulae:

From equations (3.2.1), (3.2.2), (3.2.3), (3.2.4) and (3.2.5c),

\[ \varepsilon_x = \sum_i \sum_j A_{i,j} \frac{\partial f}{\partial x_i} (\bar{x}) q_{u,j}(\bar{y}) + \frac{1}{2} \sum_{p} \sum_{q} \sum_{r} \sum_{s} (Z_{p,q} Z_{r,s} - Z_{p,q} Z_{r,s}) \]
\[ \times \left( \frac{\alpha r}{a^2} \cos \left( \frac{B r}{a} \right) \cos \left( -\frac{r a}{a} \right) \sin \left( -\frac{q y}{b} \right) \right) \]
\[ \sin \left( -\frac{s y}{b} \right) , \quad (3.2.48) \]
\[ \varepsilon_y = \sum_i \sum_j B_{ij} f_{vi} (x) \frac{\partial q_{uj}}{\partial y} + \sum_p \sum_{q,s} \sum_{r,s} (z_{op,q} z_{or,s}) \alpha_p \alpha_q \alpha_r \alpha_s \beta_p \beta_q \beta_r \beta_s \frac{a x}{b} \sin(\frac{p x}{a}) \sin(\frac{q x}{a}) \cos(\frac{r x}{b}) \cos(\frac{s x}{b}), \]

(3.2.49)

and \[ \gamma_{xy} = \sum_i \sum_j A_{ij} f_{ui} (x) \frac{\partial q_{uj}}{\partial y} + \sum_i \sum_j B_{ji} \frac{\partial f_{vi}}{\partial x} q_{vj} (y) \]

(3.2.50)

where

\[ \begin{align*}
\alpha_p &= (2p-1)\pi, \quad \beta_p = (2r-1)\pi, \\
\alpha_q &= (2q-1)\pi, \quad \beta_q = (2s-1)\pi.
\end{align*} \]

(3.2.51)

The in-plane stresses at point \((x, y)\) are calculated using the following stress-strain relationships and equations (3.2.48), (3.2.49), and (3.2.50).

\[ \sigma_x = \frac{E}{(1-\nu^2)} (\varepsilon_x + \nu \varepsilon_y), \]

\[ \sigma_y = \frac{E}{(1-\nu^2)} (\varepsilon_y + \nu \varepsilon_x), \]

(3.2.52)

and \[ \tau_{xy} = \frac{E}{2(1+\nu)} \gamma_{xy}. \]
3.3 APPLICATION OF THE RAYLEIGH-RITZ METHOD TO THE FREE VIBRATION ANALYSIS OF SIMPLY SUPPORTED RECTANGULAR CURVED PLATES SUBJECT TO IN-PLANE STRESSES

Consider the vibration of the rectangular plate treated in section 3.2. Figure 3.3.1 shows a section of the plate at the time of maximum positive excursion.

![Diagram showing plate vibration](image)

Figure 3.3.1

Assuming the motion to be simple harmonic, the dynamic displacement \((w')\) of the plate from the equilibrium configuration \((z)\) is given by

\[
w' = \sum_{i} \sum_{j} H_{i,j} \sin\left(\frac{\alpha_{i}x}{a}\right) \sin\left(\frac{\alpha_{j}y}{b}\right) \sin(\omega t) \tag{3.3.1}\]

where, for symmetrical modes of vibration,

\[
\alpha_{i} = (2i-1)\pi
\]

At the time of maximum positive excursion,

\[
w' = w = \sum_{i} \sum_{j} H_{i,j} \sin\left(\frac{\alpha_{i}x}{a}\right) \sin\left(\frac{\alpha_{j}y}{b}\right) \tag{3.3.1a}
\]

\(i, j = 1, 2, 3\ldots\)
Let the maximum dynamic in-plane displacement in \( x, y \) directions be \( u_d', v_d' \),

\[ u_d = \sum_{i,j} \bar{A}_{ij} f_{ui}(x) g_{uj}(y), \quad (3.3.2) \]

\[ v_d = \sum_{i,j} \bar{B}_{ij} f_{vi}(x) g_{vj}(y) \quad (3.3.3) \]

The total potential energy of the plate and the supporting frame consists of the following:

(a) Strain energy due to dynamic bending of the plate given by

\[
\hat{U}_{be} = \frac{Eh^3}{24(1-\nu^2)} \int_{x=0}^{a} \int_{y=0}^{b} \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)^2 \]

\[
- 2(1-\nu) \left( \frac{\partial w}{\partial x} \frac{\partial^2 w}{\partial x \partial y} - \left( \frac{\partial w}{\partial x} \frac{\partial^2 w}{\partial y^2} \right) \right) dx \, dy , \quad (3.3.4)\]

(b) Strain energy due to the dynamic stretching of the middle surface given by

\[
\hat{U}_{str} = \frac{Eh}{2(1-\nu^2)} \int_{x=0}^{a} \int_{y=0}^{b} \left( \dot{w}^2 - \dot{v}^2 \right) \dot{x} \dot{y} + \frac{(1-\nu)}{2} \dot{xy}^2 \right) dx \, dy , \quad (3.3.5a)\]

where the middle surface dynamic strains are related to the derivatives of the displacements as follows:

\[
\varepsilon_x = \frac{\partial u_d}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial z}{\partial x} + \frac{1}{2} \frac{\partial^2 w}{\partial x^2} ,
\]

\[
\varepsilon_y = \frac{\partial v_d}{\partial y} + \frac{\partial w}{\partial y} \frac{\partial z}{\partial y} + \frac{1}{2} \frac{\partial^2 w}{\partial y^2} ,
\]

\[
\gamma_{xy} = \frac{\partial u_d}{\partial y} + \frac{\partial v_d}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial z}{\partial y} + \frac{\partial z}{\partial x} \frac{\partial w}{\partial y} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} .
\]

\[
\]

\[
\]

\[
\]
Substituting equations (3.2.5b) into equation (3.3.5a) and neglecting the non-linear terms, as explained in Appendix F,

\[
\hat{V}_{str} = \frac{Eh}{2(1-\nu^2)} \int_{x=0}^{a} \int_{y=0}^{b} \left[ \frac{\partial u_d}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial z}{\partial x} \right]^2 + \left[ \frac{\partial v_d}{\partial y} + \frac{\partial w}{\partial y} \frac{\partial z}{\partial y} \right]^2
+ 2\nu \left[ \frac{\partial u_d}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial z}{\partial x} \right] \left[ \frac{\partial v_d}{\partial y} + \frac{\partial w}{\partial y} \frac{\partial z}{\partial y} \right]
+ \left( \frac{1-\nu}{2} \right) \left[ \frac{\partial u_d}{\partial y} + \frac{\partial v_d}{\partial x} \right]^2
+ \frac{\partial w}{\partial x} \frac{\partial z}{\partial y} + \frac{\partial w}{\partial y} \frac{\partial z}{\partial x} \right] \, dx \, dy. \tag{3.3.5c}
\]

(c) Potential energy due to the change in the position of the static in-plane stress distribution is given by

\[
\hat{V}_{str} = \int_{x=0}^{a} \int_{y=0}^{b} h \left( \sigma_{x} \overline{\epsilon}_{x} + \sigma_{y} \overline{\epsilon}_{y} + \tau_{xy} \overline{\gamma}_{xy} \right) \, dx \, dy,
\]

where \(\sigma_x, \sigma_y, \tau_{xy}\) are the static in-plane stresses. In terms of strains,

\[
\hat{V}_{str} = \int_{x=0}^{a} \int_{y=0}^{b} \frac{Eh}{2(1-\nu^2)} \left[ (\varepsilon_x + \nu \varepsilon_y) \overline{\epsilon}_x + (\varepsilon_y + \nu \varepsilon_x) \overline{\epsilon}_y \right]
+ G h \gamma_{xy} \overline{\gamma}_{xy} \, dx \, dy.
\]

Taking only the linear terms, as explained in Appendix F,

\[
\hat{V}_{str} = \frac{Eh}{(1-\nu^2)} \int_{x=0}^{a} \int_{y=0}^{b} \left[ \frac{\partial u_d}{\partial x} + \frac{1}{2} \left( \frac{\partial z}{\partial x} \right)^2 - \frac{1}{2} \left( \frac{\partial z}{\partial x} \right)^2 \right]
+ \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 + \nu \left( \frac{\partial w}{\partial y} \right)^2
\]
\[
+ \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 + \nu \left( \frac{\partial w}{\partial x} \right)^2
+ \left( 1-\nu \right) \left[ \frac{\partial u_d}{\partial y} + \frac{\partial v_d}{\partial x} \right]^2
+ \frac{\partial w}{\partial y} \frac{\partial z}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial z}{\partial y} \right] \, dx \, dy.
\]
\[- \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} \{ \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \} \, dx \, dy. \quad (3.3.6)\]

(d) Energy stored in working against partially restraining supports is given by

\[ \hat{U}_{\text{bound}} = \frac{1}{2} \int_{y=0}^{b} k_x (u_d^2 \bigg|_{x=0} + u_d^2 \bigg|_{x=a}) \, dy \]

\[ + \frac{1}{2} \int_{x=0}^{a} k_y (v_d^2 \bigg|_{y=0} + v_d^2 \bigg|_{y=b}) \, dx \quad (3.3.7) \]

where \( k_x, k_y \) are the boundary support stiffnesses in \( x, y \) directions respectively. (The calculation of the boundary stiffness for the test apparatus is explained in Appendix H.)

For a free boundary \( k=0 \) and for a fully restrained boundary \( k=\infty \).

Total potential energy due to vibration is given by

\[ \hat{V}_T = \hat{U}_{\text{be}} + \hat{U}_{\text{str}} + \hat{V}_{\text{str}} + \hat{U}_{\text{bound}} \quad (3.3.8) \]

Neglecting the in-plane inertia, maximum kinetic energy

\[ \hat{T} = \frac{1}{2} \omega^2 \int_{x=0}^{a} \int_{y=0}^{b} \bar{m} \omega^2 \, dx \, dy, \quad (3.3.9) \]

where \( \omega \) is the frequency of vibration and \( \bar{m} \) is the mass density of the plate (mass/unit area).

Using the Rayleigh-Ritz method,

\[ \frac{\partial \hat{V}_T}{\partial A_{l,j}} = 0, \quad (3.3.10) \]
\[
\frac{3V_{T}}{3B_{i,j}} = 0 \tag{3.3.11}
\]
and
\[
\frac{3\hat{V}}{3H_{i,j}} - \frac{3\hat{T}}{3H_{i,j}} = 0 \tag{3.3.12}
\]

**Calculation of the Dynamic Connection Coefficients**

Substituting equations (3.3.4), (3.3.5c), (3.3.6), (3.3.7) and (3.3.8) into equation (3.3.10) and noting that only terms associated with \(u_d\) will yield non-zero values in equation (3.3.10) gives

\[
\frac{3}{3A_{i,j}} \int_{x=0}^{a} \int_{y=0}^{b} \frac{Eh}{2(1-\nu^2)} \left( \frac{\partial u_d}{\partial x} \right)^2 + 2 \frac{\partial u_d}{\partial x} \frac{\partial w}{\partial x} \frac{\partial z}{\partial x} + \nu \frac{\partial w}{\partial y} \frac{\partial z}{\partial y} \right) \\
+ \frac{3}{2} \left( \frac{\partial u_d}{\partial x} \right)^2 + 2 \nu \frac{\partial u_d}{\partial x} \frac{\partial v_d}{\partial y} + (1-\nu) \frac{\partial u_d}{\partial y} \frac{\partial v_d}{\partial x} + \frac{\partial w}{\partial y} \frac{\partial z}{\partial x} \\
+ \frac{\partial w}{\partial y} \frac{\partial z}{\partial x} \right) dx \ dy + \frac{1}{2} \int_{x=0}^{a} \int_{y=0}^{b} \left( u_d \right)_{x=0}^{x=a} + \frac{u_d^2}{2} \right) dy = 0.
\tag{3.3.13}
\]

This can be rearranged to give

\[
\frac{3}{3A_{i,j}} \left[ \frac{1}{2} \int_{x=0}^{a} \int_{y=0}^{b} \left( \frac{\partial u_d}{\partial x} \right)^2 \right] + (1-\nu) \left( \frac{\partial u_d}{\partial y} \right)^2 + 2 \nu \frac{\partial u_d}{\partial x} \frac{\partial v_d}{\partial y} \\
+ (1-\nu) \frac{\partial u_d}{\partial y} \frac{\partial v_d}{\partial x} \right) dx \ dy + \int_{y=0}^{b} \frac{1}{2} \left( u_d \right)_{x=0}^{x=a} + \frac{u_d^2}{2} \right) dy \right) \\
= - \frac{3}{3A_{i,j}} \left[ \frac{1}{2} \int_{x=0}^{a} \int_{y=0}^{b} \left( \frac{\partial u_d}{\partial x} \right)^2 \right] + \frac{3}{2} \frac{\partial u_d}{\partial x} \frac{\partial w}{\partial x} \frac{\partial z}{\partial x} + \nu \frac{\partial w}{\partial y} \frac{\partial z}{\partial y} \\
+ (1-\nu) \frac{\partial u_d}{\partial y} \frac{\partial w}{\partial y} \frac{\partial z}{\partial x} + \frac{\partial w}{\partial y} \frac{\partial z}{\partial x} \right) dx \ dy \tag{3.3.13a}
\]
Let \[ \frac{\partial}{\partial a_{i,j}} \left[ \int_{x=0}^{b} \left( \frac{3u_{d}}{2} \right)^2 + (1-\nu) \left( \frac{3u_{d}}{2} \right)^2 \right] dx \, dy, \]

\[ = \sum_{k} \sum_{l} \bar{A}_{k,l} \cdot SU_{1,i,j,k,l}, \]

where \[ SU_{1,i,j,k,l} = \left\{ \begin{array}{l}
\frac{a}{\partial x} \cdot \frac{\partial u_{i}}{\partial x} \cdot \frac{\partial u_{k}}{\partial x} \cdot \left( \int_{y=0}^{b} \bar{u}_{ij} \cdot \bar{u}_{kl} \, dy \right) \\
+ (1-\nu) \left( \int_{x=0}^{a} \bar{F}_{ui} \cdot \overline{F}_{uk} \, dx \right) \left( \int_{y=0}^{b} \overline{\bar{g}}_{uj} \cdot \overline{\bar{g}}_{ul} \, dy \right), \end{array} \right. \] (3.3.13b)

\[ \frac{\partial}{\partial a_{i,j}} \left[ \int_{x=0}^{b} \left( \frac{3u_{d}}{2} \right)^2 + \frac{3v_{d}}{2} \frac{\partial u_{d}}{\partial x} + \frac{3v_{d}}{2} \frac{\partial v_{d}}{\partial y} \right] dx \, dy, \]

\[ = \sum_{m} \sum_{n} \bar{B}_{m,n} \cdot SV_{2,i,j,m,n}, \]

where \[ SV_{2,i,j,m,n} = \left\{ \begin{array}{l}
\left( \int_{x=0}^{a} \frac{\partial F_{ui}}{\partial x} \cdot F_{vm} \, dx \right) \left( \int_{y=0}^{b} \overline{\bar{g}}_{uj} \cdot \overline{\bar{g}}_{vn} \, dy \right) \\
+ (1-\nu) \left( \int_{x=0}^{a} \bar{F}_{ui} \cdot \frac{\partial F_{vm}}{\partial x} \, dx \right) \left( \int_{y=0}^{b} \overline{\bar{g}}_{uj} \cdot \overline{\bar{g}}_{vn} \, dy \right), \end{array} \right. \] (3.3.13c)

\[ \frac{\partial}{\partial a_{i,j}} \left[ \int_{y=0}^{b} \frac{k_{x}(1-\nu^2)}{2Eh} \left( u_{d}^2 \right|_{x=0}^{x=a} + u_{d}^2 \right) \, dy = \sum_{k} \sum_{l} \bar{A}_{k,l} \cdot SU_{3,i,j,k,l}, \]

where \[ SU_{3,i,j,k,l} = (\overline{F}_{ui}(o) \cdot \overline{F}_{uk}(o) + \overline{F}_{ui}(a) \cdot \overline{F}_{uk}(a)) \]

\[ \times \int_{y=0}^{b} \frac{k_{x}(1-\nu^2)}{Eh} \overline{\bar{g}}_{uj} \cdot \overline{\bar{g}}_{ul} \, dy, \] (3.3.13d)

and
\[ \frac{\partial}{\partial A_{i,j}} \int_a^b \int_{\frac{b}{a}} \frac{\partial u_d}{\partial x} (\alpha_1 \alpha_2 + \beta_1 \beta_2) \frac{\partial u_p}{\partial x} \frac{\partial u_q}{\partial y} \frac{\partial w}{\partial y} \frac{\partial w}{\partial y} \]

\[ = \sum_p^r \sum_q^s H_{p,q} Z_{r,s} \left( a \left[ \frac{\partial f_{ui}}{\partial x} \cos \left( \frac{a}{a} \right) \cos \left( \frac{a}{a} \right) \right] x=0 \right) \]

\[ + \left( \frac{a \beta s}{b^2} \right) \left[ \int_a^b \frac{\partial f_{ui}}{\partial x} \sin \left( \frac{a}{a} \right) \sin \left( \frac{a}{a} \right) dx \right] x=0 \]

\[ \int_{\frac{b}{b}} \frac{\partial g_{uj}}{\partial x} \cos \left( \frac{a}{a} \right) \cos \left( \frac{a}{a} \right) dy \]

where \( \alpha_p = (2p-1)\pi, \alpha_q = (2q-1)\pi, \beta_r = (2r-1)\pi, \beta_s = (2s-1)\pi \).

This leads to

\[ \int_a^b \int_{\frac{b}{a}} \frac{\partial u_d}{\partial x} (\alpha_1 \alpha_2 + \beta_1 \beta_2) \frac{\partial u_p}{\partial x} \frac{\partial u_q}{\partial y} \frac{\partial w}{\partial y} \frac{\partial w}{\partial y} \]

\[ = \sum_p^r \sum_q^s Z_{D_{I,j},p,q} \]  

(3.3.13e)

where \( Z_{D_{I,j},p,q} = \sum_p^r Z_{r,s} \left( a \left[ \frac{\partial f_{ui}}{\partial x} \cos \left( \frac{a}{a} \right) \cos \left( \frac{a}{a} \right) \right] x=0 \right) \)

\[ + \frac{a \beta s}{b^2} \int_a^b \frac{\partial f_{ui}}{\partial x} \sin \left( \frac{a}{a} \right) \sin \left( \frac{a}{a} \right) dx \]

\[ + \frac{a \beta s}{b^2} \int_{\frac{b}{b}} \frac{\partial g_{uj}}{\partial x} \cos \left( \frac{a}{a} \right) \cos \left( \frac{a}{a} \right) dy \]

\[ \int_{\frac{b}{b}} \frac{\partial g_{uj}}{\partial x} \cos \left( \frac{a}{a} \right) \cos \left( \frac{a}{a} \right) dy \]

(3.3.13f)

Similarly,
\[
\frac{3}{\partial A_{i,j}} \left[ \int_{x=0}^{a} \int_{y=0}^{b} \left( \frac{1-\gamma}{2} \right) \left( \frac{3u_{d}}{3y} \frac{3w_{d}}{3y} \frac{3z_{d}}{3y} \frac{3w}{3x} \frac{3z}{3x} \frac{3}{3x} \right) dx \ dy \right] \right]

= \sum_{p} \sum_{q} H_{p,q} \cdot ZD_{i,j,p,q} \cdot 2ZD_{i,j,p,q} \quad (3.3.13g)

where

\[
ZD_{i,j,p,q} = \left( \frac{1-\gamma}{2} \right) \sum_{r} \sum_{s} Z_{r,s} \left[ \left( \frac{\alpha \beta}{ab} \right) \int_{x=0}^{a} \Sigma_{u_{1}} \cos \left( \frac{\alpha x}{a} \right) \sin \left( \frac{\beta x}{a} \right) dx \right.

\left. \int_{y=0}^{b} \frac{3\gamma_{u_{1}}}{3y} \cdot \sin \left( \frac{\alpha y}{b} \right) \cos \left( \frac{\beta y}{b} \right) dy + \left( \frac{\alpha \beta}{ab} \right) \int_{x=0}^{a} \Sigma_{v_{1}} \cos \left( \frac{\alpha y}{a} \right) \cos \left( \frac{\beta y}{b} \right) \sin \left( \frac{\beta y}{b} \right) dy \right]. \quad (3.3.13h)

Substituting equations (3.3.13b) to (3.3.13h) into equation (3.3.13a) yields

\[
\sum_{k} \sum_{l} (SU_{i,j,k,l} + SU_{i,j,k,l}) \overline{A}_{k,l} + \sum_{m} \sum_{n} \overline{B}_{m,n} \cdot SV_{i,j,m,n} = \sum_{p} \sum_{q} H_{p,q} (ZD_{i,j,p,q} + ZD_{i,j,p,q}) \quad (3.3.14)

This is the result of minimizing \( V_{T} \) with respect to \( \overline{A}_{i,j} \).

The following equation can be obtained by minimizing \( V_{T} \) with respect to \( \overline{B}_{m,n} \):

\[
\sum_{k} \sum_{l} (SU_{i,j,k,l} + SU_{i,j,k,l}) \overline{A}_{k,l} + \sum_{m} \sum_{n} \overline{B}_{m,n} \cdot SV_{i,j,m,n} = \sum_{p} \sum_{q} H_{p,q} (ZD_{i,j,p,q} + ZD_{i,j,p,q}) \quad (3.3.15)
\]
where,

\[
SU_{i,j,k,l} = \left\{ \begin{array}{l}
\left[ \int_{x=0}^{a} \frac{\partial F_{vi}}{\partial x} \frac{\partial F_{uk}}{\partial x} \ dx \right] \int_{y=0}^{b} \frac{\partial g_{vi}}{\partial y} \cdot g_{uj} \ dy
\\+
\left[ \frac{1-c}{2} \right] \int_{x=0}^{a} \frac{\partial F_{vi}}{\partial x} \ F_{uk} \ dx \int_{y=0}^{b} \frac{\partial g_{vi}}{\partial y} \ dy, \end{array} \right. \tag{3.3.15a}
\]

\[
SV_{i,j,m,n} = \left\{ \begin{array}{l}
\left[ \int_{x=0}^{a} \frac{\partial F_{vi}}{\partial x} \ F_{vm} \ dx \right] \left[ \int_{y=0}^{b} \frac{\partial g_{vi}}{\partial y} \cdot \frac{\partial g_{vn}}{\partial y} \ dy \right]
\\+
\left[ \frac{1-c}{2} \right] \int_{x=0}^{a} \frac{\partial F_{vi}}{\partial x} \ F_{vm} \ dx \int_{y=0}^{b} \frac{\partial g_{vi}}{\partial y} \ dy, \end{array} \right. \tag{3.3.15b}
\]

\[
SV_{3i,j,m,n} = (\overline{g}_{vj}(o) \cdot \overline{g}_{vn}(o) + \overline{g}_{vj}(b) \cdot \overline{g}_{vn}(b))
\\\int_{x=0}^{a} \frac{k}{Eh} \ F_{vi} \cdot F_{vm} \ dx, \tag{3.3.15c}
\]

\[
ZD_{3i,j,p,q} = \sum_{r,s} Z_{rs} \left[ \left( \frac{\alpha_{rs}}{b} \right)^{2} \left( \int_{x=0}^{a} \ F_{vi} \cdot \sin \left( \frac{\alpha_{rs} x}{b} \right) \ dx \right) \right]
\\\left( \int_{y=0}^{b} \frac{\partial g_{vi}}{\partial y} \cdot \cos \left( \frac{\alpha_{rs} y}{b} \right) \cdot \cos \left( \frac{\beta_{rs} y}{b} \right) \ dy \right)
\\+
\left( \frac{\sqrt{c}}{a} \right) \left[ \int_{x=0}^{a} \ F_{vi} \cdot \cos \left( \frac{\alpha_{rs} x}{b} \right) \cos \left( \frac{\beta_{rs} x}{b} \right) \ dx \right]
\\\left( \int_{y=0}^{b} \frac{\partial g_{vi}}{\partial y} \cdot \sin \left( \frac{\alpha_{rs} y}{b} \right) \sin \left( \frac{\beta_{rs} y}{b} \right) \ dy \right), \tag{3.3.15d}
\]

and

\[
ZD_{4i,j,p,q} = \left[ \frac{1-c}{2} \right] \sum_{r,s} Z_{rs} \left[ \left( \frac{\alpha_{rs}}{ab} \right)^{2} \left( \int_{x=0}^{a} \frac{\partial F_{vi}}{\partial x} \cdot \cos \left( \frac{\alpha_{rs} x}{b} \right) \sin \left( \frac{\beta_{rs} x}{b} \right) \ dx \right) \right]
\\\left( \int_{y=0}^{b} \ F_{vj} \cdot \sin \left( \frac{\alpha_{rs} y}{b} \right) \cos \left( \frac{\beta_{rs} y}{b} \right) \ dy \right) + \left( \frac{\sqrt{c}}{ab} \right) \left[ \int_{x=0}^{a} \frac{\partial F_{vi}}{\partial x} \right]
\\\left( \int_{y=0}^{b} \ F_{vj} \cdot \cos \left( \frac{\alpha_{rs} y}{b} \right) \sin \left( \frac{\beta_{rs} y}{b} \right) \ dy \right), \tag{3.3.15e}
\]

Equations (3.3.14) and (3.3.15) will result in NN equations where NN is the total number of in-plane displacement coefficients. These equations can be written in matrix form as follows:

\[
[SX][C] = [ZD][H]
\]  (3.3.16)

where
\[I = j+(i-l) \times N_{uy}\] for \(I \leq N_u\),
\[I = N_u+j+(i-l) \times N_{vy}\] for \(I > N_u\),
\[J = l+(k-l) \times N_{uy}\] for \(J \leq N_u\),
\[J = N_u+n+(m-l) \times N_{vy}\] for \(J > N_u\),

and
\[SX(I,J) = (SU_{1,i,j,k,l}^1 + SU_{3,i,j,k,l})\] for \(I \leq N_u\) and \(J \leq N_u\),
\[SX(I,J) = SV_{2,i,j,m,n}\] for \(I \leq N_u\) and \(J > N_u\),
\[SX(I,J) = SU_{2,i,j,k,l}\] for \(I > N_u\) and \(J \leq N_u\),
\[SX(I,J) = (SV_{1,i,j,m,n}^1 + SV_{3,i,j,m,n})\] for \(I > N_u\) and \(J > N_u\),

\[\bar{C}(I) = \bar{A}_{i,j}\] for \(I \leq N_u\),
\[\bar{C}(I) = \bar{B}_{i,j}\] for \(I > N_u\).

Also:
\[H(L) = H_{p,q}\] where \(L = q+(p-l) \times q_m\),
\[ZD(I,L) = ZD_{1,i,j,p,q} + ZD_{2,i,j,p,q}\] if \(I \leq N_u\),
and \[ZD(I,L) = ZD_{3,i,j,p,q} + ZD_{4,i,j,p,q}\] if \(I > N_u\).

Equation (3.3.16) is linear in \(\bar{C}\) and therefore \(\bar{C}\) can be
calculated in the following manner:

If \( \bar{G}(I,L) = \bar{C}(I) \) when \( H(L) = 1.0 \) for \( L = L_1 \),
\[
H(L) = 0.0 \quad \text{for} \quad L \neq L_1,
\]

then, \( \bar{C}(I) = \sum L L_1 \bar{G}(I,L) \cdot H(L) \), from linear algebra. \( \bar{G}(I,L) \)
\( L \)
\( \bar{L}_1 \)
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\[
\frac{\partial U^{tr}}{\partial H_{i,j}} = \frac{E_h}{(1-\nu^2)} \sum_{r=1}^{8} \hat{x}_r , \tag{3.3.22}
\]

where \( \hat{x}_r \) can be found as follows:

Let \( r = (2r-1)*, \) \( p = (2p-1)*, \) \( k = (2k-1)*, \) \( s = (2s-1)*, \)

\( q = (2q-1)*, \) \( \phi = (2\phi-1)*. \)

\[
\hat{x}_1 = \frac{3}{\partial H_{i,j}} \frac{1}{2} \int_{x=0}^{a} \int_{y=0}^{b} \left[ \left( \frac{3w}{\partial x} \frac{3z}{\partial y} \right)^2 + \left( \frac{3w}{\partial y} \frac{3z}{\partial x} \right)^2 \right] dx \, dy
\]

\[
\hat{x}_2 = \frac{3}{\partial H_{i,j}} \frac{1}{2} \int_{x=0}^{a} \int_{y=0}^{b} \frac{3w}{\partial x} \frac{3z}{\partial y} \left( \frac{3w}{\partial x} \frac{3z}{\partial y} \right) dx \, dy
\]

\[
= \sum_{r,s} \sum_{p,q,k,l} \left( \frac{a_i b_r g_{pq,kl}}{a^4} \right) \cdot TX1 \cdot TY2
\]

\[
+ \frac{a_i b_r g_{pq,kl}}{a^4} \cdot TX2 \cdot TY1) Z_{p,q} Z_{k,l} \tag{3.3.23a}
\]

where

\[
TX1 = \int_{x=0}^{a} \int_{y=0}^{b} \left( \frac{a_i}{a} \right) \left( \frac{\beta r X}{a} \right) \left( \frac{\gamma_{pq} X}{a} \right) \left( \frac{\phi_{kl} X}{a} \right) \ dx \, dy
\]

\[
TX2 = \int_{x=0}^{a} \left( \frac{a_i}{a} \right) \left( \frac{\beta r X}{a} \right) \left( \frac{\gamma_{pq} X}{a} \right) \left( \frac{\phi_{kl} X}{a} \right) \ dx
\]

\[
TY1 = \int_{y=0}^{b} \left( \frac{a_i}{a} \right) \left( \frac{\beta s Y}{b} \right) \left( \frac{\gamma_{pq} Y}{b} \right) \left( \frac{\phi_{kl} Y}{b} \right) \ dy
\]

\[
TY2 = \int_{y=0}^{b} \left( \frac{a_i}{a} \right) \left( \frac{\beta s Y}{b} \right) \left( \frac{\gamma_{pq} Y}{b} \right) \left( \frac{\phi_{kl} Y}{b} \right) \ dy
\]
\[ \begin{align*}
\alpha \beta s y \phi k \frac{a}{b^2} & \cdot TX3 + \left( \frac{\alpha \beta s y}{a b^2} \right) \cdot TX4 \cdot TY3, \quad (3.3.23b)
\end{align*} \]

where

\[ \begin{align*}
TX3 &= \int_a^b \cos \left( \frac{a}{x} \right) \sin \left( \frac{b}{x} \right) \cos \left( \frac{\gamma x}{a} \right) \sin \left( \frac{\phi x}{a} \right) dx, \\
TX4 &= \int_a^b \sin \left( \frac{\alpha x}{a} \right) \cos \left( \frac{\beta x}{a} \right) \sin \left( \frac{\gamma x}{a} \right) \cos \left( \frac{\phi x}{a} \right) dx, \\
TY4 &= \int_b^y \sin \left( \frac{\alpha y}{b} \right) \cos \left( \frac{\beta y}{b} \right) \sin \left( \frac{\gamma y}{b} \right) \cos \left( \frac{\phi y}{b} \right) dy,
\end{align*} \]

and

\[ \begin{align*}
\hat{x}_3 &= \frac{3}{2} \frac{(1 - \nu)}{\beta H_i} \int_a^b \int_0^y \left[ \frac{\partial w}{\partial x} \frac{\partial z}{\partial y} \right] dx dy \\
&= \frac{1 - \nu}{2} \int_a^b \int_0^y \left[ \frac{\partial w}{\partial x} \frac{\partial z}{\partial y} \right] dx dy \\
&= \frac{1 - \nu}{2} \sum_{r, s} H_{r,s} \sum_{p, q, k} \frac{\alpha \beta \gamma \phi}{a b^2} \cdot (TX5) \cdot (TY6) \\
&\quad + \frac{\alpha \beta \gamma \phi}{a b^2} \cdot (TX6) \cdot (TY5), \quad (3.3.23c)
\end{align*} \]

where

\[ \begin{align*}
TX5 &= \int_a^b \cos \left( \frac{a}{x} \right) \cos \left( \frac{b}{x} \right) \sin \left( \frac{\gamma x}{a} \right) \sin \left( \frac{\phi x}{a} \right) dx, \\
TX6 &= \int_a^b \sin \left( \frac{a}{x} \right) \sin \left( \frac{b}{x} \right) \cos \left( \frac{\gamma x}{a} \right) \cos \left( \frac{\phi x}{a} \right) dx.
\end{align*} \]
\[
TY5 = \int_{y=0}^{b} \cos\left(\frac{a_y y}{b}\right) \cos\left(\frac{b_y y}{a}\right) \sin\left(\frac{a_y y}{b}\right) \sin\left(\frac{b_y y}{a}\right) dy,
\]

and
\[
TY6 = \int_{y=0}^{b} \sin\left(\frac{a_y y}{b}\right) \sin\left(\frac{b_y y}{a}\right) \cos\left(\frac{a_y y}{b}\right) \cos\left(\frac{b_y y}{a}\right) dy.
\]

\[\hat{x}_4 = \frac{\partial}{\partial H_{i,j}} \left( \frac{1-v}{2} \right) \int_{x=0}^{a} \int_{y=0}^{b} \frac{\partial w}{\partial x} \cdot \frac{\partial z}{\partial y} \cdot \frac{\partial z}{\partial x} \cdot \frac{\partial w}{\partial y} \cdot \frac{\partial}{\partial H_{i,j}} \left( \frac{\partial w}{\partial y} \right) dx dy
\]

\[= \left( \frac{1-v}{2} \right) \int_{x=0}^{a} \int_{y=0}^{b} \frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial y} \cdot \frac{\partial w}{\partial y} \cdot \frac{\partial}{\partial H_{i,j}} \left( \frac{\partial w}{\partial y} \right) + \frac{\partial w}{\partial x} \cdot \frac{\partial}{\partial H_{i,j}} \left( \frac{\partial w}{\partial y} \right) dx dy
\]

\[= \left( \frac{1-v}{2} \right) \sum_{r,s} H_{r,s} \sum_{p,q} Z_{p,q} \left( \frac{a_y b_y}{2} \right) \cdot TX4 \cdot TX3 \cdot \left( \frac{a_y b_y}{2} \right)
\]

\[+ \frac{a_y b_y}{2} \cdot TX4 \cdot TX3 \cdot Z_{p,q} \cdot Z_{k,l} \quad (3.3.23d)
\]

\[\hat{x}_5 = \frac{\partial}{\partial H_{i,j}} \left( \frac{1-v}{2} \right) \int_{x=0}^{a} \int_{y=0}^{b} \left( \frac{\partial w}{\partial x} \cdot \frac{\partial z}{\partial x} + \frac{\partial w}{\partial x} \cdot \frac{\partial z}{\partial y} \cdot \frac{\partial}{\partial H_{i,j}} \left( \frac{\partial w}{\partial y} \right) + \frac{\partial w}{\partial x} \cdot \frac{\partial z}{\partial y} \cdot \frac{\partial}{\partial H_{i,j}} \left( \frac{\partial w}{\partial y} \right) \right) dx dy
\]

\[= \left( \frac{1-v}{2} \right) \sum_{r,s} H_{r,s} \sum_{p,q} Z_{p,q} \left( \frac{a_y b_y}{2} \right) \cdot TX5 \cdot TX3 \cdot \left( \frac{a_y b_y}{2} \right)
\]

\[\quad \left( \frac{a_y b_y}{2} \right) \cdot TX4 \cdot TX3 \cdot Z_{p,q} \cdot Z_{k,l} \quad (3.3.23e)
\]

\[\hat{x}_6 = \frac{\partial}{\partial H_{i,j}} \left( \frac{1-v}{2} \right) \int_{x=0}^{a} \int_{y=0}^{b} \frac{\partial w}{\partial y} \cdot \frac{\partial z}{\partial x} + \frac{\partial w}{\partial y} \cdot \frac{\partial z}{\partial y} \cdot \frac{\partial}{\partial H_{i,j}} \left( \frac{\partial w}{\partial y} \right) dx dy
\]

\[= \left( \frac{1-v}{2} \right) \sum_{r,s} H_{r,s} \sum_{p,q} Z_{p,q} \left( \frac{a_y b_y}{2} \right) \cdot TX5 \cdot TX3 \cdot \left( \frac{a_y b_y}{2} \right)
\]

\[\quad \left( \frac{a_y b_y}{2} \right) \cdot TX4 \cdot TX3 \cdot Z_{p,q} \cdot Z_{k,l} \]
\[
\begin{align*}
\hat{X}_g &= \frac{\partial}{\partial h_{ij}} \left( \frac{1 - \gamma}{2} \right) \int_{x=0}^{a} \int_{y=0}^{b} \left( \frac{\partial v}{\partial y} \frac{\partial w}{\partial y} \frac{\partial z}{\partial y} + \frac{\partial w}{\partial y} \frac{\partial z}{\partial x} \frac{\partial z}{\partial x} \right) dx \cdot dy \\
&= \left( \frac{1 - \gamma}{2} \right) \sum_{m,n} \sum_{p,q} \left[ \frac{a_i x}{a} \right] \frac{v_{mn}}{a} \cdot \frac{w_{pq}}{a} \cdot \frac{z_{pq}}{a} \\
&\quad + \left( \frac{a_i y}{ab} \right) \left( \int_{x=0}^{a} \frac{v_{mn}}{a} \cdot \frac{w_{pq}}{a} \cdot \frac{z_{pq}}{a} \cdot \frac{z_{pq}}{a} dx \right) \\
&\quad + \left( \frac{a_i x}{a} \right) \left( \int_{y=0}^{b} \frac{v_{mn}}{a} \cdot \frac{w_{pq}}{a} \cdot \frac{z_{pq}}{a} \cdot \frac{z_{pq}}{a} dy \right) \\
&\quad + \left( \frac{a_i y}{b} \right) \left( \int_{x=0}^{a} \frac{v_{mn}}{a} \cdot \frac{w_{pq}}{a} \cdot \frac{z_{pq}}{a} \cdot \frac{z_{pq}}{a} dx \right) \\
&\quad + \left( \frac{a_i x}{b} \right) \left( \int_{y=0}^{b} \frac{v_{mn}}{a} \cdot \frac{w_{pq}}{a} \cdot \frac{z_{pq}}{a} \cdot \frac{z_{pq}}{a} dy \right).
\end{align*}
\]
\[
\frac{\partial V_{\text{str}}}{\partial H_{i,j}} = \frac{E_h}{(1-\nu^2)} \sum_{r=1}^{7} \hat{V}_r,
\]

where \( \hat{V}_r \) can be found as follows:

\[
\hat{V}_1 = \frac{3}{\partial H_{i,j}} \int_{x=0}^{a} \int_{y=0}^{b} \frac{1}{2} \frac{\partial u_s}{\partial x} \left[ \left( \frac{\partial w}{\partial x} \right)^2 + \frac{\partial w}{\partial y} \right] dx \, dy
\]

\[
= \sum_{r,s} \int_{x=0}^{a} \int_{y=0}^{b} \frac{\partial u_s}{\partial x} \cdot \cos(\frac{r \cdot x}{a}) \cos(\frac{r \cdot x}{a}) \sin(\frac{r \cdot x}{b}) \sin(\frac{r \cdot x}{b}) \, dx \, dy
\]

\[
= \sum_{r,s} \int_{x=0}^{a} \int_{y=0}^{b} \frac{\partial u_s}{\partial x} \cdot \frac{\partial w}{\partial x} \cdot \cos(\frac{r \cdot x}{a}) \cos(\frac{r \cdot x}{b}) \sin(\frac{r \cdot x}{b}) \sin(\frac{r \cdot x}{b}) \, dx \, dy
\]

\[
\hat{V}_2 = \frac{1}{\partial H_{i,j}} \int_{x=0}^{a} \int_{y=0}^{b} \frac{1}{2} \frac{\partial u_s}{\partial x} \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \, dx \, dy
\]

\[
= \frac{1}{2} \int_{x=0}^{a} \int_{y=0}^{b} \frac{\partial u_s}{\partial y} \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} + \frac{\partial u_s}{\partial y} \frac{\partial w}{\partial y} \frac{\partial w}{\partial y} \, dx \, dy
\]

\[
= \frac{1}{2} \sum_{r,s} \sum_{k,l} \frac{\partial V_{\text{str}}}{\partial H_{i,j}} \left[ \left( \frac{1}{ab} \right) \int_{x=0}^{a} \frac{\partial u_s}{\partial y} \cos(\frac{r \cdot x}{a}) \, dx \right]
\]
\[
\hat{y}_j = \frac{\partial}{\partial H_{ij,j}} \left\{ \int_a^b \int_{\frac{3x}{\partial y}}^{3y} \frac{3y}{\partial y} \frac{1}{2} \left( \frac{3w}{\partial y} \right)^2 + \nu \left( \frac{3w}{\partial y} \right)^2 \right\} dx \, dy \\
= \frac{\partial}{\partial H_{ij,j}} \left\{ \int_a^b \int_{\frac{3x}{\partial y}}^{3y} \frac{3y}{\partial y} \frac{1}{2} \left( \frac{3w}{\partial y} \right)^2 + \nu \left( \frac{3w}{\partial y} \right)^2 \right\} dx \, dy \\
= \frac{\partial}{\partial H_{ij,j}} \left\{ \int_a^b \int_{\frac{3x}{\partial y}}^{3y} \frac{3y}{\partial y} \frac{1}{2} \left( \frac{3w}{\partial y} \right)^2 + \nu \left( \frac{3w}{\partial y} \right)^2 \right\} dx \, dy \\
= \frac{\partial}{\partial H_{ij,j}} \left\{ \int_a^b \int_{\frac{3x}{\partial y}}^{3y} \frac{3y}{\partial y} \frac{1}{2} \left( \frac{3w}{\partial y} \right)^2 + \nu \left( \frac{3w}{\partial y} \right)^2 \right\} dx \, dy \\
(3.3.25b)
\]

\[
\hat{y}_4 = \frac{\partial}{\partial H_{ij,j}} \left\{ \int_a^b \int_{\frac{3x}{\partial y}}^{3y} \frac{3y}{\partial y} \frac{1}{2} \left( \frac{3w}{\partial y} \right)^2 + \nu \left( \frac{3w}{\partial y} \right)^2 \right\} dx \, dy \\
= \frac{\partial}{\partial H_{ij,j}} \left\{ \int_a^b \int_{\frac{3x}{\partial y}}^{3y} \frac{3y}{\partial y} \frac{1}{2} \left( \frac{3w}{\partial y} \right)^2 + \nu \left( \frac{3w}{\partial y} \right)^2 \right\} dx \, dy \\
= \frac{\partial}{\partial H_{ij,j}} \left\{ \int_a^b \int_{\frac{3x}{\partial y}}^{3y} \frac{3y}{\partial y} \frac{1}{2} \left( \frac{3w}{\partial y} \right)^2 + \nu \left( \frac{3w}{\partial y} \right)^2 \right\} dx \, dy \\
= \frac{\partial}{\partial H_{ij,j}} \left\{ \int_a^b \int_{\frac{3x}{\partial y}}^{3y} \frac{3y}{\partial y} \frac{1}{2} \left( \frac{3w}{\partial y} \right)^2 + \nu \left( \frac{3w}{\partial y} \right)^2 \right\} dx \, dy \\
(3.3.25c)
\]

\[
\hat{y}_5 = \frac{\partial}{\partial H_{ij,j}} \left\{ \int_a^b \int_{\frac{3x}{\partial y}}^{3y} \frac{3y}{\partial y} \frac{1}{2} \left( \frac{3w}{\partial y} \right)^2 + \nu \left( \frac{3w}{\partial y} \right)^2 \right\} dx \, dy \\
= \frac{\partial}{\partial H_{ij,j}} \left\{ \int_a^b \int_{\frac{3x}{\partial y}}^{3y} \frac{3y}{\partial y} \frac{1}{2} \left( \frac{3w}{\partial y} \right)^2 + \nu \left( \frac{3w}{\partial y} \right)^2 \right\} dx \, dy \\
= \frac{\partial}{\partial H_{ij,j}} \left\{ \int_a^b \int_{\frac{3x}{\partial y}}^{3y} \frac{3y}{\partial y} \frac{1}{2} \left( \frac{3w}{\partial y} \right)^2 + \nu \left( \frac{3w}{\partial y} \right)^2 \right\} dx \, dy \\
(3.3.25d)
\]
\[
\hat{\gamma}_6 = \frac{3}{\delta H_{i,j}} \int_a^b \int_x^y \left[ \frac{1}{2} \left( \frac{3z}{3y} \right)^2 - \frac{1}{2} \left( \frac{3z}{3y} \right)^2 + \frac{1}{2} \left( \frac{3w}{3y} \right)^2 + \frac{1}{2} \left( \frac{3w}{3x} \right)^2 \right] dx dy
\]
\[
\hat{\gamma}_7 = \frac{3}{\delta H_{i,j}} \left( 1 - \nu \right) \int_a^b \int_x^y \left[ \frac{1}{2} \frac{3w}{3x} \frac{3w}{3y} \frac{3z}{3x} \frac{3z}{3y} + \frac{1}{2} \frac{3w}{3x} \frac{3w}{3y} \frac{3z}{3x} \frac{3z}{3y} \right] dx dy
\]

From equations (3.3.19), (3.3.20), (3.3.21), (3.3.22) and (3.3.24),

\[
\frac{\partial \hat{V}_T}{\partial H_{i,j}} = \frac{Eh^3}{24(1-\nu^2)} \left[ \frac{a}{4} \left( \frac{a}{4} \right)^2 + \frac{b}{4} \left( \frac{b}{4} \right)^2 \right] H_{i,j}
\]

\[
+ \sum_{rs} \sum_{r,s} C_1 \sum_{rs} \sum_{r,s} \sum_{r,s} C_3
\]

(3.3.26)

where \(C_1\), \(C_2\) and \(C_3\) can be found from equations (3.3.22) to (3.3.25g). Using the connection coefficients, equation (3.3.26) can be transformed into the following form:

\[
\frac{\partial \hat{V}_T}{\partial H_{i,j}} = C_4 \cdot H_{i,j} + \sum_{rs} \sum_{r,s} \sum_{r,s} C_5
\]

(3.3.26a)
where the bending stiffness \( C_4 = \frac{Eh^3}{48(1-\nu^2)} \left( \frac{\alpha_1}{a} \right)^2 + \frac{\alpha_1}{b} \right)^2 \cdot ab \).

\( C_5 \) can be found by substituting equations (3.3.17) and (3.3.18) into equation (3.3.26).

From equation (3.3.9),

\[
\frac{\partial^2}{\partial H_{i,j}} = -\frac{\omega}{H_{i,j} \left( \frac{ab}{4} \right)} \tag{3.3.27}
\]

Substituting equations (3.3.26) and (3.3.27) in equation (3.3.12) gives

\[
C_4 \cdot H_{i,j} + \sum \sum C_5 H_{r,s} - \frac{\omega}{H_{i,j} \left( \frac{ab}{4} \right)} = 0.
\]

This can be expressed in matrix form as

\[
[SK][H] - \omega^2 [MASS][H] = 0 \tag{3.3.28}
\]

where \([SK]\) is a dynamic stiffness matrix,
and \([MASS]\) is a diagonal mass matrix.

This is a standard eigenvalue problem. Natural frequencies \(\omega\) can be found by solving equation (3.3.28) in an iterative way.

3.4 SHAPE FUNCTIONS FOR IN-PLANE DISPLACEMENTS

The choice of shape functions depends on the in-plane boundary conditions. A study on the vibration of a curved beam indicates that the following shape functions are suitable for use in the Rayleigh-Ritz analysis for the symmetrical vibration modes.
(i) For normally fully restrained boundary conditions, 
\[ f_{ui}(x) = \sin\left(\frac{2i \pi x}{a}\right) \] and 
\[ q_{vi}(y) = \sin\left(\frac{2i \pi y}{b}\right) \] are satisfactory 
since these will give zero values at the restrained edges 
and the centrelines of the plate (axes of symmetry), as 
shown in Figure 3.4.1.

![Figure 3.4.1](image)

(ii) For normally free or partially restrained boundaries, 
in addition to the above functions, the following functions 
are also used to allow for the displacement at the edges:

\[ f_{uo}(x) = \left(\frac{x}{a} - \frac{1}{2}\right) \] \[ q_{vo}(y) = \left(\frac{y}{b} - \frac{1}{2}\right) \] 

The shape of these functions is shown in Figure 3.4.2.

![Figure 3.4.2](image)

(iii) For tangentially fully restrained boundary conditions 
the following shapes (shown in Figure 3.4.3) are 
satisfactory:
\[ f_{vi}(x) = \sin\left(\frac{(2i-1)\pi x}{a}\right) \]
\[ g_{ui}(y) = \sin\left(\frac{(2i-1)\pi y}{b}\right) \]

Figure 3.4.3

(iv) For tangentially free or partially restrained boundaries, the following functions are used, in addition to the above functions, to allow for the edge displacements (see Figure 3.4.4):
\[ f_{vo}(x) = 1.0 \quad g_{vo}(y) = 1.0 \]

Figure 3.4.4

The same shape functions may be used in the static displacement calculation and vibration analysis if the boundary conditions remain unchanged. For a partially restrained edge, the shape functions are the same as that for an in-plane free edge. The effect of restraining stiffness is taken into account by adding the extra energy spent in
working against the restraining boundary forces as in equation 3.3.7. Numerical results indicate that if the restraining stiffness is very high, the displacement coefficients which contribute to the work of restraint at the boundary approach zero. This results in shapes that approach the shapes for the fully restrained boundaries.

3.5 GENERAL OUTLINES OF A COMPUTER PROGRAM TO SOLVE THE RAYLEIGH-RITZ MINIMIZATION EQUATION FOR THE POST BUCKLING AND VIBRATION ANALYSIS

A Fortran program has been developed to solve the equations derived in sections 3.2 and 3.3. Some features and limitations of this program are outlined in the following paragraphs.

The static displacements and in-plane stress (and strain) distribution due to the applied load are calculated in the first part of the program. Using these calculated values, the natural frequencies and mode shapes are calculated in the second part.

Computation of Static Displacements

The in-plane displacement shape functions are set up using the subroutine SETUP. The program and all the integral subroutines are capable of generating and analytically integrating the products of, any combination of the following shapes:
\[ f_1 = \text{Constant} \]
\[ f_2 = \sin \left( \frac{x}{a} \right) \]
\[ f_3 = \cos \left( \frac{x}{a} \right) \]

It is possible to approximately derive the function

\[ f_4 = \left( \frac{x}{a} - \frac{1}{2} \right) = a_1 \sin \left( \frac{2x}{a} \right) - a_2 \cos \left( \frac{2x}{a} \right) \]

where \( a_1 = \frac{1}{2} \cos \left( \frac{x}{2} \right) \), \( a_2 = \frac{1}{2} \sin \left( \frac{x}{2} \right) \)

in which \( \epsilon = 0 \).

The computation of the coefficients of [SZ] and [ZB] in the equation (3.2.25) requires several subroutines of the integral of products such as,

\[ \int_{x=0}^{a} f(x) \cos \left( \frac{\pi x}{a} \right) \cos \left( \frac{\pi x}{a} \right) \, dx \]

\[ \int_{x=0}^{a} f'(x) \cos \left( \frac{\pi x}{a} \right) \cos \left( \frac{\pi x}{a} \right) \, dx \]

\[ \int_{x=0}^{a} \cos \left( \frac{\pi x}{a} \right) \cos \left( \frac{\pi x}{a} \right) \cos \left( \frac{\pi x}{a} \right) \cos \left( \frac{\pi x}{a} \right) \, dx \]

The computation of these integrals is carried out in the subroutines analytically based on the following five relationships.

\[ (1) \quad \int_{0}^{a} \cos \left( \frac{\pi x}{a} \right) = \frac{a}{\pi} \sin \left( \frac{\pi x}{a} \right) \quad \text{if} \quad \epsilon \neq 0 \]
\[ = a \quad \text{if} \quad \epsilon = 0 \]

\[ (2) \quad \int_{0}^{a} \sin \left( \frac{\pi x}{a} \right) = \frac{a}{\pi} \left[ 1 - \cos \left( \frac{\pi a}{a} \right) \right] \]
\[
\sin\left(\frac{\lambda x}{a}\right) \sin\left(\frac{5 \lambda x}{a}\right) = \frac{1}{2} \cos\left[\frac{(1-\beta) \lambda x}{a}\right] - \frac{1}{2} \cos\left[\frac{(1+\beta) \lambda x}{a}\right]
\]

\[
\cos\left(\frac{\lambda x}{a}\right) \cos\left(\frac{3 \lambda x}{a}\right) = \frac{1}{2} \cos\left[\frac{(1-\beta) \lambda x}{a}\right] + \frac{1}{2} \cos\left[\frac{(1+\beta) \lambda x}{a}\right]
\]

\[
\cos\left(\frac{\lambda x}{a}\right) \sin\left(\frac{3 \lambda x}{a}\right) = \frac{1}{2} \sin\left[\frac{(1+\beta) \lambda x}{a}\right] - \frac{1}{2} \sin\left[\frac{(1-\beta) \lambda x}{a}\right]
\]

Repeated application of the last three relationships gives the integrals of multiple products of trigonometric functions. This is done in the program using simple subroutines.

The out-of-plane and in-plane displacement functions used in the program are correct for symmetric out-of-plane displacements. To include anti-symmetrical terms, the setup of out-of-plane shape angles \((\gamma_x, \beta_x, \text{etc.})\) that are currently set to take only odd multiples of \(\gamma\) must be altered. The setup of in-plane shape functions must also be corrected accordingly. The integral in equation (3.3.13d) involves the calculation of the stiffness of the supporting frame. This integration has been done numerically using a computer program STIFCAL which is attached in Appendix G. Analytical derivations associated with this, based on the slope deflection analysis of the frame is attached in Appendix H.

The effect of the mass of the loading head is introduced as a spring stiffness. The justification for this
is explained in Appendix I.

The listing of the program and a typical output are attached in Appendix J for completeness.
CHAPTER 4

EXPERIMENTAL PROCEDURES

4.1 INTRODUCTION TO THE EXPERIMENTS

The object of the experiments conducted was to measure the first few natural frequencies and the out-of-plane deflection profiles of thin rectangular plates under various in-plane loadings and prescribed boundary conditions.

Providing the boundary conditions that can be accurately and conveniently modelled in the theoretical analysis was a difficult task. In the theoretical analysis, out-of-plane simply supported boundaries can be treated more conveniently than any other type of boundaries. For this reason, it was decided to design the experimental apparatus to provide simply supported boundaries along all four edges.

A Denison loading machine was used to apply the in-plane loading for most of the plates tested as shown in Figure 4.1.1. In one case, however, 'weights' were used to apply the load as shown in Figure 4.1.2 because, for the test plate used in that experiment, the buckling load was too small for the efficient use of the Denison machine. Tests were carried out at loads that were higher than the lowest buckling load in most cases and in one case (a 0.86 mm thick plate) the plate was loaded up to more than four times the lowest
Figure 4.1.1 Testing Rig in the Loading Machine
buckling load.

Non contacting, electro-magnetic transducers were used to excite the plates in vibration and to pick up the response which was then transmitted to an oscilloscope for visual observation as explained in section 4.3. A capacitance displacement transducer was used to measure the out-of-plane static displacements. In one case, the static strain distribution was measured. Details of the methods of measurement are given in section 4.3.

4.2 DESIGN OF THE TESTING RIG

The design of the testing rig was governed by the following requirements:

(1) to hold the plate in a suitable position with respect to the loading machine and to transfer the load smoothly to the plate;
(2) to provide the necessary boundary conditions at the edges of the plate;
(3) to allow the attachment of a displacement measuring device.

The rig (Figure 4.2.1) was made of four (76.2 x 31.75 x 6.35 mm) channel sections welded together to form a rectangular frame. A detachable circular loading head with two stout circular bars which could slide through two collars
Figure 4.2.1 - The Testing Rig
mounted on top of the channel was provided for transferring the load. The rods rested on top of a 'V' groove support, as shown in Figure 4.2.2. The bottom support was also a 'V' groove which was firmly attached to the channel base. The top and bottom edges of most of the test plates were machined to form knife edges which allowed rotation to take place. This arrangement closely satisfied the requirements for simply supported boundaries.

The top and bottom supports were very rigid. This setup was expected to constrain the normal in-plane displacements to be constant during loading. However, the flexibility of the supports was considered in the theory in an approximate manner. (The results obtained were very close to the results for a plate with absolutely constant edge displacements.)

To isolate the machine vibration, three layers of rubber were used between the loading machine and the loading head. One piece of rubber was 12.7 mm thick and the other two were somewhat thinner. The rubber, while transmitting the static force supplied by the loading machine, essentially eliminated any contribution from the machine to the in-plane constraint normal to the top edge of the plate during vibration. Initially, it had been intended that the top edge should have been considered as essentially in-plane clamped both experimentally and theoretically. However, the effect of the isolation was to cause the in-plane boundary condition along
Figure 4.2.2 - Supporting Arrangement at the Top Edge
the top edge to become one of constant in-plane motion normal to the edge. The tangential restraint was preserved, however. The vibration of the mass of the loading head provided some restraint which was calculated in an approximate manner, as explained in Appendix I and is included in the analysis.

The supporting arrangement for the sides consisted of two rows of ball bearings on 'V' grooves holding the plate on each side of both vertical edges as shown in Figure 4.2.3. By carefully adjusting the side screws, the contact force between the plate and the ball bearings could be minimized so that the plate could move freely in its plane and could rotate without significant restraint. However, the frictional restraint was not completely avoidable as the side screws had to be sufficiently tightened to straighten the plate edges and to hold the plate in the correct position. This was evident from the load-deflection graphs in Chapter 5, where a hysteresis can be observed. Under static loading, additional contact forces might have been induced as the ball bearings restrained the bending of the plate. It is believed that the frictional resistance generated by this increase in the contact force, was likely to be substantially smaller than the forces that were required to prevent static in-plane displacements normal to the plate edge. Therefore, the ball bearings did not prevent the slippage of the plate during loading. Tangential in-plane displacements could easily take place, since the balls could roll along the 'V' grooves.
Figure 4.2.3 Side Support
Some grease was applied to the ball bearings to minimize the friction. This arrangement was expected to minimize the loss of applied load through friction at the sides.

In-plane boundary conditions for the vibration were different. Since the amplitude of vibration was very small compared to the amplitude of the static displacement, the forces that were necessary to prevent slippage of the plate at the ball bearings were also small. The contact forces which were induced during the static loading, were likely to have provided sufficient frictional restraint against normal slippage. This however, does not mean that the ball bearings provided a fully normally restrained boundary, since the supporting frame has some flexibility. An equivalent boundary stiffness calculation is explained in Appendix H, to model this partial restraint. In this calculation, the inertia of the frame has been neglected. To simplify the analysis, the 'V' groove supports are considered to run over the full length of the frame (in calculating the second moment of area of the section), although in the experiment the 'V' grooves terminated just below the top support. The effect of making these simplifications is expected to be negligible.

Some experiments were carried out with another type of side support, where the ball bearings were placed on two channel grooves instead of 'V' grooves as shown in Figure 4.2.4. In this arrangement, rolling of the ball bearings
Figure 4.2.4 Channel Groove
across the groove was allowed by placing these between two strips of rubber. It was expected that this would reduce the resistance to in-plane displacement normal to the edges. However, it was found that the natural frequencies were significantly higher than the predicted values even at zero loading in some cases. The discrepancy was found to be random in nature and seemed to change each time the side supports were reset. It is thought that a misalignment of the ball bearings may have resulted in non-straight supports which would effectively apply some rotational restraint. Use of firm rubber strips, precisely cut to fill the gap between the ball bearings and the sides of the channel groove, may help to overcome this problem.

4.3 METHODS OF MEASUREMENTS

Natural Frequencies

A power signal generator (Brüel & Kjaer signal generator, type 1024) and an electro magnetic transducer were used to excite the plates. Another magnetic transducer picked up the response signal, that was observed using an oscilloscope (Phillips PM3232). Both transducers were mounted on an adjustable stand with a clamping arrangement (Retort stand and clamp). These probes were placed near different points on the plate to observe various modes of vibration. Figure 4.1.2 shows this setup. In some of the experiments, the output signal from the transducer was sent to the oscilloscope through
a frequency filter (KROHN-HITE, Model 3500 filter). The frequencies of the input signal were measured using a Hewlett Packard 3734 electronic counter.

**Deflection Profile**

A capacitance probe was connected to a digital display unit (Hitec Proximic 3101-SP), which indicated the distance between the probe and the plate in units of 0.0254 mm (1/10,000 of an inch) directly. The deflection was measured at each point of intersection of the grid lines marked on the plates, and at a number of points as close to the edges as possible. Figure 4.3.1 shows the points on the plates, where the deflections were measured. Figure 4.3.2 illustrates this experimental setup.

The capacitance probe was mounted on an aluminum block holder. This holder could move along a horizontal bar which could be slid vertically on two parallel circular bars. These bars were attached to the channel frame by adjustable screws. The screws were adjusted to set the orientation of the vertical bars so that they were parallel. A machined angle block was used as the reference surface for the initial setting up.

**Strain Distribution**

One plate specimen was fitted with 27 strain gauges on
Figure 4.3.1 Grid lines for Deflection Measurements
each side at corresponding points to measure the strains. 6.35 mm (1/4 inch) strain gauges with a gauge factor of 2.1 were used. The gauges were connected to a balancing unit with digital display through six multi-channel switching units. Strain readings were taken at various loads. The average value of strains on both sides of the plate at a point gives the in-plane strain. The difference between the two gives twice the value of the maximum bending strain at the surface. The detailed results of the strain measurements are given in Appendix B. The measured in-plane strain variation at each point is compared with the theoretical values in Chapter 5. Figure 4.3.3 shows the plate with strain gauges. The locations of these gauges are shown in Figure 4.3.4.

Loading

The load was measured using the dial on the Denison machine (except for the 0.56 mm plate which was loaded using weights). The accuracy of this scale was first verified using a load cell (for up to 800 lbs).

4.4 PLATE SPECIMENS

Six mild steel rectangular plates with thickness ranging from 0.56 mm to 1.15 mm were tested. The properties of the mild steel were taken as follows:
Figure 4.3.3 Plate with Strain Gauges
Figure 4.3.4 Locations of Strain Gauges
Young's modulus  $E = 207 \text{ MPa}$

Poisson's ratio  $\nu = 0.3$

Density  $= 7.738 \text{ kg/m}^3$

The density was taken from the measurements of weight and area. The Young's modulus and Poisson's ratio were assumed to be the normal values which are given in standard specifications.

All plates had overall dimensions of $0.3 \text{ m} \times 0.256 \text{ m}$. The distance between the centreline of the vertical rows of ball bearings was $0.25 \text{ m}$. The extra $6 \text{ mm}$ in the width of the plates were allowed for placing in the rig.

The thicknesses of the plates are listed in the table below with identification numbers which will be used from hereonwards.

<table>
<thead>
<tr>
<th>Plate Identification Number</th>
<th>Thickness (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 (fitted with strain gauges)</td>
<td>1.0</td>
</tr>
<tr>
<td>2</td>
<td>1.0</td>
</tr>
<tr>
<td>3</td>
<td>1.0</td>
</tr>
<tr>
<td>4</td>
<td>0.86</td>
</tr>
<tr>
<td>5</td>
<td>1.15</td>
</tr>
<tr>
<td>6</td>
<td>0.56</td>
</tr>
</tbody>
</table>

Plate 1 was fitted with strain gauges after the completion of the frequency measurement, so that the mass of the wires would not influence the natural frequencies.
CHAPTER 5
RESULTS AND DISCUSSION

5.1 THEORETICAL RESULTS
Comparison With Existing Results

Before comparing the experimental and theoretical results, it is necessary to study the accuracy of the theoretical results. Unfortunately, the theoretical results which exist in the literature can not be compared directly with the experimental results, since their applizability is limited to certain standard boundary conditions. To establish confidence in the theoretical approach used in this thesis, results obtained for some simple cases are compared with existing theoretical results or results generated using a package finite element program.

The static displacement values computed by using the Rayleigh-Ritz analysis will be compared with the results published by Yamaki [26] for simply supported out-of-plane boundary conditions and the following in-plane boundary conditions:

(i) Loaded edges free to slide tangentially (no shear) and having constant normal displacement.

(ii) Sides free in both directions (normally and tangentially).
Yamaki's results were obtained for a square plate with Poisson's ratio ($\nu$) of 1/3. Results from the Rayleigh-Ritz analysis for the same plate with zero initial imperfection are compared with Yamaki's results in Table 5.1.1. In the table, the actual deflection of the plate is given by

$$z(x, y) = Z_{1,1} \frac{\pi x}{a} \sin \left( \frac{n_x}{a} \right) \sin \left( \frac{n_y}{b} \right) + Z_{3,1} \frac{3\pi x}{a} \sin \left( \frac{3n_x}{a} \right) \sin \left( \frac{3n_y}{b} \right) + Z_{1,3} \frac{\pi x}{a} \sin \left( \frac{n_x}{a} \right) \sin \left( \frac{3n_y}{b} \right) + Z_{3,3} \frac{3\pi x}{a} \sin \left( \frac{3n_x}{a} \right) \sin \left( \frac{3n_y}{b} \right).$$

**TABLE 5.1.1 Comparison of Theoretical Values of Static Displacements**

<table>
<thead>
<tr>
<th>LOAD RATIO</th>
<th>CENTRAL* DEFLECTION</th>
<th>DEFORMATION COEFFICIENTS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rayleigh-Ritz</td>
<td>1.456</td>
<td>3.850</td>
</tr>
<tr>
<td>Yamaki's</td>
<td>1.456</td>
<td>3.905</td>
</tr>
</tbody>
</table>

The agreement between the two results is good. The discrepancy in the central deflection is only about 1.4%. The Rayleigh-Ritz solution was obtained with four symmetrical out-of-plane displacement coefficients and twenty-five in-plane displacement coefficients. The Rayleigh-Ritz method usually gives a lower bound solution for the displacement. This was verified in a convergence study for the case of an experimental plate discussed later in this chapter.

The natural frequencies calculated by using the Rayleigh-Ritz method were to be compared with results from a finite element package program [35] which was applicable to analyze
unstressed shells. A preliminary analysis using this program illustrated the significance of in-plane boundary conditions. The results for the fundamental natural frequencies of plate 1, under simply supported out-of-plane boundary conditions and various in-plane boundary conditions without any in-plane stress are given in Table 5.1.2. It can be observed that the frequency increases with restraining the boundaries.

In the finite element program, making use of symmetry, a quarter of the plate was divided into fifty triangular elements. The static shape of the plate was taken as

\[ z = 0.8 \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi y}{b}\right) \text{ mm.} \]

<table>
<thead>
<tr>
<th>In-plane Boundary Conditions</th>
<th>Fundamental Natural Frequency (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>All edges in-plane free (normally and tangentially).</td>
<td>73.12</td>
</tr>
<tr>
<td>All edges tangentially free, normally constrained to move with constant displacement.</td>
<td>79.84</td>
</tr>
<tr>
<td>All edges normally free, tangentially restrained (Shear Diaphragm).</td>
<td>80.37</td>
</tr>
<tr>
<td>All edges normally restrained, tangentially free.</td>
<td>109.19</td>
</tr>
<tr>
<td>All edges normally and tangentially restrained.</td>
<td>109.30</td>
</tr>
<tr>
<td>Long edges free, short edges normally and tangentially restrained.</td>
<td>83.69</td>
</tr>
<tr>
<td>Long edges free, short edges normally restrained.</td>
<td>82.18</td>
</tr>
<tr>
<td>Flat Plate</td>
<td>66.71</td>
</tr>
</tbody>
</table>

The analytically calculated value of the fundamental frequency of the flat plate was 66.65 Hz. This agrees well with the finite element result of 66.71 Hz.
The curvature has increased the frequency for all in-plane boundary conditions. From Table 5.1.2, it is clear that any restraint at the boundary increases the frequency of a curved plate. It can also be observed that for a plate with normally restrained edges, tangential restraining does not change the frequency significantly.

The frequencies for various magnitudes of curvature for a plate with short edges normally restrained and long edges in-plane free are tabulated along with the amplitude of static deflection in Table 5.1.3. The deflection parameter \( \mu \) is given by \( \mu = \frac{\text{deflection at the centre}}{\text{plate thickness h}} \).

The shape of the plate is \( z = \mu \cdot h \cdot \sin\left(\frac{nX}{a}\right) \cdot \sin\left(\frac{nY}{b}\right) \).

**TABLE 5.1.3 Variation of Frequency With Curvature Using Finite Element Package Program**

<table>
<thead>
<tr>
<th>( \mu )</th>
<th>0.2</th>
<th>0.475</th>
<th>0.588</th>
<th>0.80</th>
<th>0.95</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \omega_{1,1} ) (Hz)</td>
<td>67.79</td>
<td>72.57</td>
<td>75.49</td>
<td>82.18</td>
<td>87.67</td>
</tr>
</tbody>
</table>

It is useful at this stage to introduce a frequency parameter \( \lambda^2 \) which is defined by \( \lambda^2 = \left(\frac{\omega_{1,1}}{\Omega_{1,1}}\right)^2 \), where \( \omega_{1,1} \) and \( \Omega_{1,1} \) are the theoretical values of the fundamental natural frequencies of a curved plate and the corresponding flat plate respectively.

\( \lambda^2 \) is plotted against \( \mu^2 \) (which will be called the deflection parameter from hereonwards) in Figure 5.1.1. It
Figure 5.1.1 Finite Elements Result for Plate 1
is clear that the frequency parameter varies approximately linearly with the deflection parameter.

\[ \lambda^2 = 1 + k \mu^2, \]

where \( k \) is the gradient of the straight line in Figure 5.1.1. The corresponding numerical values are given in Table 5.1.3a.

**TABLE 5.1.3a** Variation of \( \lambda^2 \) vs \( \mu^2 \) Using Finite Element Program for In-Plane Free-Short Edges and Normally Restrained Long Edges

<table>
<thead>
<tr>
<th>( \mu^2 )</th>
<th>0.040</th>
<th>0.226</th>
<th>0.346</th>
<th>0.640</th>
<th>0.903</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda^2 )</td>
<td>1.03</td>
<td>1.19</td>
<td>1.28</td>
<td>1.52</td>
<td>1.73</td>
</tr>
</tbody>
</table>

As mentioned earlier, the use of Galerkin's method was explored for calculating the frequencies to be compared with the experimental results. However, due to difficulties encountered in modelling the experimental boundary conditions in the stress formulation, this approach was abandoned in favour of the Rayleigh-Ritz method. Nevertheless, certain results are included here for completeness.

The finite element results for the in-plane boundary conditions listed below are compared with results obtained using Galerkin's method (described in Appendix A) in Table 5.1.4, for \( \mu = 0.8 \).
The in-plane boundary conditions:

- case (i) - All edges in-plane free.
- case (ii) - All edges normally free, tangentially restrained.

<table>
<thead>
<tr>
<th>In-Plane Boundary Condition</th>
<th>Case (i)</th>
<th>Case (ii)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Finite Element Result</td>
<td>73.12 Hz</td>
<td>80.37 Hz</td>
</tr>
<tr>
<td>Galerkin's Method Result</td>
<td>73.16 Hz</td>
<td>81.67 Hz</td>
</tr>
</tbody>
</table>

Table 5.1.4 Comparison of Fundamental Natural Frequencies Obtained by Using Galerkin's Method and Finite Element Program

In both cases, the agreement between the results is reasonably good. Galerkin's method was used with one out-of-plane displacement term only. A multi-term result may improve the agreement. Results for higher values of curvature were not calculated since the single term solution was not expected to give good results at high curvatures.

The results from the Rayleigh-Ritz method with undetermined displacement coefficients are compared with the corresponding finite element results for a stress free curved plate in Table 5.1.5. The Rayleigh-Ritz results were obtained using seventeen in-plane displacement shapes with one or four fully symmetric out-of-plane displacement shapes. The following in-plane boundary conditions were treated:

- case (iii) - All edges fully restrained in-plane,
- case (iv) - Short edges fully restrained, long edges in-plane free.
TABLE 5.1.5 Comparison of Fundamental Natural Frequencies Obtained By Using the Rayleigh-Ritz Method and Finite Element Method

<table>
<thead>
<tr>
<th>In-Plane Boundary Condition</th>
<th>Case (iii) 0.8</th>
<th>Case (iv) 0.8</th>
<th>Case (iii) 5.0</th>
<th>Case (iv) 5.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Finite Element Result</td>
<td>109.3</td>
<td>83.69</td>
<td>439.73</td>
<td>288.02</td>
</tr>
<tr>
<td>Result Using the Rayleigh-Ritz Method</td>
<td>one out-of-plane term 110.01</td>
<td>83.81</td>
<td>510.16</td>
<td>324.45</td>
</tr>
<tr>
<td></td>
<td>four out-of-plane terms 109.88</td>
<td>83.62</td>
<td>434.03</td>
<td>287.10</td>
</tr>
</tbody>
</table>

The agreement between the results from both methods is good. The Rayleigh-Ritz method gives an upper bound for the frequencies. Non-conforming elements were used in the finite element method, thus it is not certain whether the frequencies determined are upper bounds or not.

This illustrates that the Rayleigh-Ritz method with undetermined displacement coefficients can be used satisfactorily for the calculation of the natural frequencies of curved plates.

A Note On the In-Plane Boundary Conditions At The Top And Bottom (Loaded) Edges of the Test Plates

In using the Rayleigh-Ritz method for the analysis of the experimental plates, the following simplification was made to reduce the computational effort, while taking into account the practical boundary conditions.

For static deflection calculations, the bending of the
edge beams has been taken into account in an approximate manner. The beams at the top and bottom have different second moments of area, but the analysis is performed assuming symmetry about both axes through the centre of the plate. Therefore, a weighted average value of the stiffness is used. The bottom support is a part of the channel frame, but it has been taken as a simply supported beam. The load is applied at the top edge on two points as shown in Figure 5.1.2(a). The reactions at the bottom are at the edges of

![Diagram](image)

**Figure 5.1.2**

the beam which rests on two crossbars welded on the channel to keep the apparatus stable. In the analysis, the points of application of the load are taken as the midpoints between
the bottom and top loading points. The approximate model is shown in Figure 5.1.2(b).

All these simplifications, however, are not likely to cause any significant error in the calculations, since the flexural rigidities of the edge beams are very high. This is illustrated in the following example. The results obtained using these simplifications (case (i)) are compared with those for a constant normal edge displacement (case (ii)) in Table 5.1.6, where it is seen that slightly lower deflections occur for the experimental condition than for the case of infinitely rigid supports. These results are given for plate 4, which had an initial imperfection $\varepsilon_0$ of 0.47. The load ratio in Table 5.1.6 is defined as the in-plane load divided by the lowest buckling load (1967N in this case).

**TABLE 5.1.6 Calculated Deflection Ratios For Plate 4**

<table>
<thead>
<tr>
<th>Load Ratio $P/P_c$</th>
<th>0.755</th>
<th>1.209</th>
<th>1.663</th>
<th>2.117</th>
<th>3.021</th>
<th>4.149</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u$ for case (i)</td>
<td>1.100</td>
<td>1.686</td>
<td>2.244</td>
<td>2.752</td>
<td>3.652</td>
<td>4.602</td>
</tr>
<tr>
<td>$u$ for case (ii)</td>
<td>1.109</td>
<td>1.708</td>
<td>2.270</td>
<td>2.781</td>
<td>3.676</td>
<td>4.620</td>
</tr>
<tr>
<td>Deviation (%)</td>
<td>0.8</td>
<td>1.3</td>
<td>1.2</td>
<td>1.1</td>
<td>0.7</td>
<td>0.4</td>
</tr>
</tbody>
</table>
5.2 COMPARISON OF EXPERIMENTAL AND THEORETICAL RESULTS

The experimental and theoretical results for plate 1 are compared graphically in Figure 5.2.1. The theoretical results were calculated using four fully symmetric out-of-plane displacement coefficients corresponding to the (1,1), (1,3), (3,1) and (3,3) modes, where the mode numbers \((m,n)\) represent the number of half sine waves in \(x, y\) directions respectively. For in-plane displacement in \(x\) direction 3,4 terms were taken in \(x, y\) directions respectively. For in-plane displacement in \(y\) direction 3 terms in each direction were taken. (In the preliminary analysis, only three terms in each direction were taken for in-plane displacement in each direction. A fourth term for the constant displacement shape was included later to improve the accuracy of the solution.)

The variation of the measured and calculated values of the central deflection with load is illustrated in Figure 5.2.1a. The deflection parameter \(\mu^2\) is defined as the square of the ratio of the central deflection to the thickness of the plate. The load ratio is the ratio of the in-plane load to the lowest critical load of the plate. The deflection measurement during unloading was generally higher than that during loading. This hysteresis is thought to be due to the friction at the ball bearings. The slippage at the ball bearings may have been prevented initially, until
Figure 5.2.1 Results for Plate 1
sufficient in-plane forces developed to overcome the friction. The initial frictional forces that can be induced, depend on the tightening of the ball bearings. In later tests, grease was applied to the ball bearings which resulted in significant reduction in the hysteresis.

The theoretical values were calculated for initial imperfection amplitudes ($u_0$) of 0.15 and 0.25. Southwell's method [33] was used to estimate the magnitude of the initial imperfection from the experimental data. The results from the data points for loading and unloading indicated initial imperfection magnitudes of 0.19 and 0.25 times the thickness respectively. (The data for the Southwell plots are given in Appendix B.) The magnitude of initial central deflection was also calculated by subtracting the displacement reading at the centre from the average of the displacement readings near the four corners of the plate prior to loading. This was found to be about 0.12 times the plate thickness. The discrepancy between this value and that indicated by Southwell's plot (0.19) may be due to the following factors:

(a) The bending of the plate may have resulted in some displacements near the corners where it was assumed to be zero.

(b) Although care was taken to set the side supports so that they were parallel, in most of the tests a small skew was present. This skew was calculated by taking the difference between the sums of the displacement
readings near the corners on each diagonal. In most cases this skew was found to be less than 0.06 mm.

(c) Imperfections in the displacement measuring apparatus such as the bending of the guide frame on which the capacitance probe was mounted.

(d) In the theoretical analysis, all the initial imperfections are assumed to be of the form 
\[ z_o = z_o^0 \cdot h \cdot \sin(\frac{x}{a}) \cdot \sin(\frac{y}{b}). \]
(It can also be expressed as a Fourier series. The first coefficients of this series were computed using the deflection measurements at all the grid points. These agreed very well with the magnitude of initial imperfection at the centre as shown in Table B.27.) The presence of other shapes of imperfection can influence the measurement of initial imperfection amplitude as well as the estimation of \( z_o \) using Southwell's method. This is, however, not likely to change the results significantly at high loadings, since the effect of the actual magnitude of initial imperfection is not very significant at high loadings.

(e) Measurement of small values of displacement is less accurate than larger values. Southwell's method makes use of measured values of larger displacements, and therefore is considered to give a better estimation of the initial imperfection.

For the above listed reasons, the initial imperfection was estimated using Southwell's plot method whenever possible. For plate 4 and plate 6 however, this was not possible, because these plates had large initial imperfections.
Southwell's method is not applicable for plates with large initial imperfections since the membrane stretching affects the linearity of Southwell's plot. (It is not possible to draw a straight line through the data points.)

Figure 5.2.1b, shows the variation of the square of the non-dimensional natural frequency defined as the frequency parameter \( \lambda^2 \), where \( \lambda^2 = \frac{\omega_{1,1}^2}{\tilde{\omega}_{1,1}} \), in which \( \omega_{1,1} \) is the measured fundamental natural frequency and \( \tilde{\omega}_{1,1} \) is the theoretical value of the fundamental natural frequency of the corresponding flat plate. The effect of friction can be seen on this plot also. The measured frequencies were higher than the calculated values generally. This is primarily due to the discrepancy between the calculated and measured values of the deflections. The calculated values of the deflections are smaller, thus causing smaller membrane stretching effect and hence lower frequencies.

Figure 5.2.1c shows the variation of the load-frequency parameter with deflection parameter. The load-frequency parameter is given by the summation of the load ratio and the frequency parameter \( (P/P_c + \lambda^2) \). It is interesting to notice that the experimental and theoretical results lie approximately on a straight line. Another interesting point is that the theoretical lines for \( \nu_0 = 0.15 \) and \( \nu_0 = 0.25 \) almost coincide with each other. The hysteresis in the experimental results has almost disappeared. This indicates that the
discrepancy between the experimental and the theoretical results for the frequency is mainly due to the discrepancy in the deflections.

The results for plates 2 and 3 which had the same overall dimensions are shown in Figures 5.2.2 and 5.2.3 respectively. The measured imperfection in these cases varied between 0.14 and 0.25 times the plate thickness as indicated by Southwell's plot.

Tests were carried out on a thinner plate (plate 4 having 0.86 mm thickness) up to about three times the lowest critical load. The results are shown in Figure 5.2.4. Further tests with the same specimen were carried out for loads of up to 4.15 times the lowest critical load. The results are illustrated in Figure 5.2.5. For this plate, the initial imperfection was calculated using the displacement readings at the centre and at the corners, because it was not possible to draw a straight line in Southwell's format. The measured imperfection ratio was 0.47.

In Figures 5.2.4c and 5.2.5c, the end points of the theoretical results are connected to show the deviation of the results from a straight line. The slope of the theoretical curve increases with deflection. This increase may be attributed to the contribution from the vibration of the loading head. Since this plate is thinner than the previous ones, the restraint provided by the loading head, which
Figure 5.2.2 Results for Plate 2
Figure 5.2.3 Results for Plate 3
Figure 5.2.4 Results for Plate 4 (Test 1)
Figure 5.2.5 Further Results for Plate 4 (Test 2)
increases with the frequency, is higher than those for plates 1, 2 and 3. For plate 4, the agreement between the experimental and theoretical results is excellent. A further improvement in the agreement was observed when nine out-of-plane displacement coefficients and thirty-two in-plane displacement coefficients were included in the analysis as shown in Table 5.2.1.

<table>
<thead>
<tr>
<th></th>
<th>( u )</th>
<th>( \omega_{1,1} ) (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Theoretical - 4 term</td>
<td>4.60</td>
<td>211.16</td>
</tr>
<tr>
<td>Theoretical - 9 term</td>
<td>4.79</td>
<td>215.49</td>
</tr>
<tr>
<td>Experimental</td>
<td>4.89</td>
<td>220.0</td>
</tr>
</tbody>
</table>

The deviation in the central deflection reduced from 6\% to 2\% and the deviation in the frequency reduced from 4\% to 2\% when the number of out-of-plane displacement coefficients was increased from four to nine.

Test results for plate 5 are shown in Figures 5.2.6 to 5.2.8. The thickness of the plate was 1.15 mm. After the first two tests were carried out, it was found that there was some gap between the plate and the ball bearings at the bottom on one side. (This was found as rattling was observed when the amplitude of excitation was increased which resulted in some disturbance on the oscilloscope.
Figure 5.2.6 Results for Plate 5 (Test 1)
Figure 5.2.7 Further Results for Plate 5 (Test 2)
Figure 5.2.8 Further Results for Plate 5 (Test 3)
screen.) The bottom screws on the side support units were readjusted and the test was repeated. The final test results are shown in Figure 5.2.8. The upward curving of the load-frequency parameter vs deflection parameter plot is not noticeable for plate 5. This is because the effect of the inertia of the loading head is small. The agreement between the experimental and theoretical results is generally very good. The sharp change in the frequency parameter vs load ratio plot at the buckling load is because the plate is almost flat and a rapid change in displacement takes place near the buckling load.

The results for plate 6 (with an initial imperfection ratio \( \mu_0 = 2.7 \)) are shown in Figure 5.2.9. The experimental and theoretical values for the deflection and the frequency do not agree, but the trend in the variation of these parameters appears to be similar. The large discrepancies for this plate are thought to be due to the problems in the out-of-plane boundary conditions. The measured frequencies were lower than the calculated values. This may be due to a lack of fit at the top and bottom supports where the plate may have vibrated freely (flapping of the edges). The plate edges were straight (in-plane) when they were cut. When placing it in the rig, the sides which had some out-of-plane curvature, were straightened by 'clamping' between the side supports. This may have resulted in non-straight top and bottom edges. This problem may have occurred in other
Figure 5.2.9 Results for Plate 6
plates too, but to a smaller degree, since the smaller imperfections would have caused smaller deviation from straightness at the top and bottom edges.

It is interesting to observe the results for another set of in-plane boundary conditions. The natural frequencies of plate 4 were calculated for in-plane normally constrained (constant motion) loaded edges and in-plane fully restrained sides. The variation of load-frequency parameter with the deflection parameter is shown in Figure 5.2.10 (dotted line) along with the theoretical results for the experimental boundary conditions (continuous line) and the points corresponding to the experimental results.

For plate 1 the strain values at twenty-seven points on each side of the plate were measured at various loads. These are compared with the corresponding theoretical values in Figures 5.2.11. The location and orientation of the gauges are given in Appendix B in tabular form and in Figure 4.3.4. The agreement in the overall pattern of the load-strain relationship is reasonably good for most of the gauge points. The discrepancy at some points indicates that there may have been some initial lack of fit at the top and bottom supports. At gauge locations 1 and 3, there was no change in the strain until about 0.3 times the buckling load was applied. This indicates that there was a small gap between the plate and the support near these points.
Figure 5.2.10  Results for Plate 1 Compared With the Results For a Plate With Standard Boundary Conditions
(Results for Gauge 1)

(Results for Gauge 2)

(Results for Gauge 3)

(Results for Gauge 4)

$\bigcirc$ - Experimental (loading), $\blacksquare$ - Experimental (unloading)

--- Theoretical

Figure 5.2.11 Load-Strain Relationship for Plate 1
Figure 5.2.11 - continued
Figure 5.2.11 - Continued
Figure 5.2.11 - Continued
Figure 5.2.11 - Continued
Figure 5.2.11 - Continued
Figure 5.2.11 - Continued
At location 2 however, more strain was recorded experimentally than calculated. This indicates that more load was transferred to the plate near this point initially. From these observations it appears that the top edge of the plate was initially curved (in-plane) as shown in Figure 5.2.12.

Figure 5.2.12

The agreement between the measured and calculated values of the strains in the direction of loading at the horizontal centreline of the plate is very good.

The discrepancy between the experimental and theoretical values of the deflections and in-plane strain distribution may be attributed to the following factors:

(1) The edges of the plate not being straight in their planes causing initial lack of fit at the loading edges.
(2) Presence of some restraint against rotation of the plate at the edges.
(3) Friction at the ball bearings when the load is not sufficiently large to produce in-plane forces which
can overcome the frictional resistance.

(4) Influence of anti-symmetric type of initial geometrical imperfections.

(5) Shape of the initial imperfection being different from the assumed shape.

(6) Initial setting errors causing skewness of the support.

(7) Presence of initial residual stresses since the plates were not stress relieved prior to testing.

(8) Induced initial stresses due to flattening of edges on set up.

(9) Measurement errors and errors due to simplifying assumptions made in modelling as explained in the previous sections. These are expected to be small as explained in Appendix K.

Since the agreement between the theoretical and experimental results are generally good (except for plate 6), it can be said that the above listed factors have not significantly influenced the results of the experiments on plates with small initial imperfections (less than 1/2 the plate thickness). With the exception of plate 6, the calculated and measured values of the fundamental natural frequencies agreed reasonably well for all the plates tested. The discrepancies are likely to be primarily due to the discrepancies in the calculated and measured values of the out-of-plane deflections and also due to the discrepancies in the calculated and measured values of in-plane stress.
distribution. Therefore, all the factors mentioned in the comparison of displacement results may have contributed to the discrepancies in the natural frequencies. This is clearly seen from the variation of load-frequency parameter with the deflection parameter, in which the theoretical and experimental results agree remarkably well.

If the approximately linear relationship between the load-frequency parameter and the deflection parameter which was exhibited for the plates tested can be established for slightly curved plates in general, the results may lead to some significant applications such as non-destructive testing of curved plates. If, for instance, the frequency and deflected shapes at various loads can be measured, the actual stress level in a curved plate may be estimated. This may be useful in the aircraft industry where thin curved panels are often used. It may lead to ways of optimizing the shape of curved panels.
CHAPTER 6
CONCLUSIONS AND RECOMMENDATIONS
FOR FUTURE WORK

The following conclusions can be reached from the discussion in the previous chapter:

6.1 CONCLUDING REMARKS

1) The Rayleigh-Ritz method using undetermined in-plane and out-of-plane displacement coefficients has been successfully applied to calculate the static displacements, in-plane stress distributions and fundamental natural frequencies of simply supported rectangular plates subjected to static in-plane loadings varying from zero to well above the lowest buckling load.

2) Experiments have been conducted on several thin mild steel rectangular plates subject to uniaxial, in-plane loading, in which the static deflections, natural frequencies and, in one case, the static in-plane strain distribution were measured.

3) The calculated and measured values of the central deflections, strains and the fundamental natural frequencies agree very well for most of the plates tested.
4) It has been shown experimentally, as well as theoretically, that:
   a) The presence of initial imperfection influences the natural frequencies of rectangular plates. This effect increases with applied in-plane load due to the growth of deflection and change in stress distribution.
   b) The natural frequencies of curved plates depend on the in-plane boundary conditions; restraining in-plane displacement, generally increases the fundamental natural frequencies.
   c) The fundamental natural frequencies of curved plates are higher than those of the flat plates.

5) For the plates tested, there exists an approximate linear relationship between the square of the central deflection and a load-frequency parameter which can be defined as the summation of the square of the non-dimensional natural frequency and the in-plane load ratio.

6) This thesis represents the first successful attempt at comparing experimental and theoretical frequencies for geometrically imperfect rectangular plates subject to in-plane loads which are significantly larger than the lowest critical load.
6.2 RECOMMENDATIONS FOR FUTURE WORK

The work presented in this thesis can be extended in the following areas:

1) The theory for the vibration and postbuckling analysis using the Rayleigh-Ritz method with undetermined displacement coefficients may be extended for other out-of-plane boundary conditions.

2) Experimental work should be carried out for different in-plane and out-of-plane boundary conditions. Tests on plates with different aspect ratios should be carried out.

3) Theoretical and experimental investigation should be extended to include the higher modes of vibration and to include anti-symmetrical terms in the analysis.

4) A statistical study on practical plates may be carried out to verify the validity and limitation of the approximate linear relationship between the deflection parameter and the load-frequency parameter.
APPENDIX A

APPLICATION OF GALERKIN'S METHOD USING AIRY STRESS FUNCTIONS TO CALCULATE THE NATURAL FREQUENCIES AND DEFLECTIONS OF A SIMPLY SUPPORTED RECTANGULAR PLATE UNDER STATIC IN-PLANE LOADING

Von Kármán's non-linear large deflection equations and linear shell vibration equations in terms of the out-of-plane displacements and Airy stress functions have been solved by Hui and Leissa [25] for simply-supported curved rectangular plates with the following boundary conditions using Galerkin's method: All edges free to move tangentially (shear free), but constrained to move with a constant displacement in the direction normal to the edges. The solution was based on the assumption that the out-of-plane buckling modes (and vibration modes) are decoupled. The stress functions were obtained by solving the compatibility equation exactly, for a particular buckling (or vibration) mode. In this appendix, a method to obtain the single term solution for other simple in-plane boundary conditions, in which the Airy stress functions are taken as the summation of a series of the products of beam functions and undetermined coefficients is described; the undetermined coefficients are found by solving the compatibility equations approximately, using Galerkin's method.
Calculation of the Static Displacements

The compatibility equation is [25],

\[ \nabla^4 F = 2C_1 \left[ \left( \frac{\partial^2 z}{\partial x \partial y} \right)^2 - \frac{\partial^2 z}{\partial x^2} \frac{\partial^2 z}{\partial y^2} \right] + \frac{\partial^2 z}{\partial x^2} \frac{\partial^2 z}{\partial y^2} + \frac{\partial^2 z}{\partial x^2} \frac{\partial^2 z}{\partial y^2} \] (A.1)

where

Initial Deflection, \( z_0 = Z_0 \sin \left( \frac{k \pi x}{a} \right) \sin \left( \frac{\pi y}{b} \right) \) (A.1a)

Deflection under in-plane load, \( z = Z \sin \left( \frac{k \pi x}{a} \right) \sin \left( \frac{\pi y}{b} \right) \) (A.1b)

and

\[ C_1 = \frac{E \cdot h}{2} \] (A.1c)

Airy stress function \( F \) can be expressed as a series of products of beam functions. This follows from an analogy between the Airy stress function and the out-of-plane displacement of a plate which is explained in a Thesis by Bassily [36].

\[ i.e. \quad F = \sum_{p \neq m} \frac{P_m \psi_m}{\alpha_{p,q} \phi_p \psi_q} \] (A.2)

for \( p, q = 1, 2 \ldots \)

where \( \phi_p \) and \( \psi_q \) are the beam functions which satisfy the necessary boundary conditions for an analogous plate bending problem,

\( \alpha_{p,q} \) are the undetermined weighting coefficients,
\( p_m, q_m \) are the maximum number of functions in \( x,y \) directions.

Equation (A.2) can be written in matrix form as,

\[
F = \{\alpha\}^T \{\phi \cdot \psi\} \tag{A.3}
\]

where

\[
\alpha(J) = \alpha_{p,q}
\]

in which \( J = q + (p-1) \cdot q_m \)

Let \( \frac{\pi k^2 z^2}{ab} = R \),

then,

\[
\left( \frac{\partial^2 z}{\partial x^2} \right)^2 - \left( \frac{\partial^2 z}{\partial x \partial y} \right)^2 = R \cdot (z^2 - z_o^2) \cos^2\left( \frac{k \pi x}{a} \right) \cos^2\left( \frac{k \pi y}{b} \right)
\]

\[
\frac{\partial^2 z}{\partial x^2} \cdot \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial x^2} \cdot \frac{\partial^2 z}{\partial y^2} = -R \cdot (z^2 - z_o^2) \sin^2\left( \frac{k \pi x}{a} \right) \sin^2\left( \frac{k \pi y}{b} \right)
\]

Adding these two equations gives the R.H.S. of the compatibility equation as,

R.H.S. of equation (A.1)' = \( 2R \cdot C_1 (z^2 - z_o^2) \left[ \cos^2\left( \frac{2k \pi x}{a} \right) \cdot \sin^2\left( \frac{k \pi y}{b} \right) \right] \)

\[
- \sin^2\left( \frac{k \pi x}{a} \right) \cdot \sin^2\left( \frac{k \pi y}{b} \right) \right)
\]

\[
= R \cdot C_1 (z^2 - z_o^2) \left[ \cos\left( \frac{2k \pi x}{a} \right) + \cos\left( \frac{2k \pi y}{b} \right) \right]
\]

Now Galerkin's method can be applied to the compatibility equation. Let the weighting function be \( \phi \cdot \psi \).

This gives,
\[
\int_{x=0}^{a} \int_{y=0}^{b} \left( \phi_p \cdot \psi_q \right) \delta_{p,q} \cdot \nabla^4 (\phi_p \cdot \psi_q) \, dx \, dy
\]

This can be written in matrix form as

\[ [SK] \{a\} = (Z^2 - z_o^2) \{A\} \quad (A.5) \]

where,

\[ SK(I,J) = \int_{x=0}^{a} \int_{y=0}^{b} (\phi_p \cdot \psi_q) \nabla^4 (\phi_p \cdot \psi_q) \, dx \, dy \quad (A.5a) \]

and

\[ A(I) = \int_{x=0}^{a} \int_{y=0}^{b} C_1 \cdot R [\cos \left( \frac{2k \pi x}{a} \right) + \cos \left( \frac{2 \pi y}{b} \right)] (\phi_p \cdot \psi_q) \, dx \, dy \quad (A.5b) \]

in which,

\[ I = s + (r-1) \cdot q_m \]

and \[ J = q + (p-1) \cdot q_m \]

\[ \nabla^4 (\phi_p \cdot \psi_q) = \phi_p^{IV} \cdot \psi_q + 2 \phi_p^{II} \cdot \psi_q^{II} + \phi_p \cdot \psi_q^{IV} \]

Using this, equation (A.5a) can be written as,

\[ SK(I,J) = T_1 S_3 + 2 T_2 S_2 + T_3 S_3 \quad (A.6) \]

where
\[ T_1 = \int_{x=0}^{a} \phi_r \phi_p^IV \, dx; \quad S_1 = \int_{y=0}^{b} \psi_s \psi_q^IV \, dy; \]

\[ T_2 = \int_{x=0}^{a} \phi_r \phi_p^IV \, dx; \quad S_2 = \int_{y=0}^{b} \psi_s \psi_q^IV \, dy; \]

(A.6a)

\[ T_3 = \int_{x=0}^{a} \phi_r \phi_p^IV \, dx; \quad S_3 = \int_{y=0}^{b} \psi_s \psi_q^IV \, dy. \]

Equation (A.5b) can be written as

\[ A(I) = R \cdot C_1 (P_1 \cdot P_4 + P_2 \cdot P_3), \]  

where

\[ P_1 = \int_{x=0}^{a} \phi_r \cdot \cos \left( \frac{2\pi x}{a} \right) \, dx, \]

\[ P_2 = \int_{y=0}^{b} \psi_s \cdot \cos \left( \frac{2\pi y}{b} \right) \, dy, \]

(A.7a)

\[ P_3 = \int_{x=0}^{a} \phi_r \, dx, \]

\[ P_4 = \int_{y=0}^{b} \psi_s \, dy. \]

In equation (A.5), \( \{a\} \) and \( Z \) are unknowns. The solution of \( Z \) is a two step procedure. First, the solution of

\[ [S_K]\{a'\} = \{A\}, \]

(A.8)

is found.

\[ \{a\} = (Z^2 - Z_0^2)\{a'\} \]

(A.9)
Having calculated \( \{a'\} \), equation (A.9) can be substituted in the equilibrium equation to evaluate \( Z \) in an iterative procedure.

The equation of static equilibrium is [25],

\[
\nu^4(z - z_o) + 2C_1(\sigma_x \frac{\partial^2 z}{\partial x^2} + \sigma_y \frac{\partial^2 z}{\partial y^2}) = 2C_1(\frac{\partial^2 F}{\partial x^2} \cdot \frac{\partial^2 z}{\partial x^2})
+ \frac{\partial^2 F}{\partial x^2} \cdot \frac{\partial^2 z}{\partial y^2} - 2 \frac{\partial^2 F}{\partial x \partial y} \frac{\partial^2 z}{\partial x \partial y}
\]

(A.10)

Differentiating equations (A.1a) and (A.1b) appropriately and substituting in equation (A.10) gives

L.H.S. of equation (A.10) = \[ \pi^4 Q_1 (Z - Z_o) - 2C_1 \pi^2 Q_2 Z \].

\[
\sin\left(\frac{k \pi x}{a}\right) \sin\left(\frac{\pi y}{b}\right)
\]

(A.11)

where,

\[
Q_1 = \left(\frac{k^2}{a^2} + \frac{l^2}{b^2}\right) 2
\]

(A.11a)

and

\[
Q_2 = \frac{k^2 \sigma}{a^2} + \frac{l^2 \sigma}{b^2}
\]

(A.11b)

Using equations (A.3) and (A.9), it can be shown that,

R.H.S. of equation (A.10) = \[-2C_1 \pi^2 Z (Z - Z_o) \]

\[
[T \sin\left(\frac{k \pi x}{a}\right) \sin\left(\frac{\pi y}{b}\right) + U \cos\left(\frac{k \pi x}{a}\right) \cos\left(\frac{\pi y}{b}\right)]
\]

(A.12)

where,
\[ T = \frac{k^2}{a^2} (\alpha')^T \{(\phi' \cdot \psi')\} + \frac{l^2}{b^2} (\alpha')^T \{(\phi' \cdot \psi')\} \quad (A.12a) \]

and \[ U = -\frac{2k^2}{ab} (\alpha')^T \{(\phi' \cdot \psi')\}. \quad (A.12b) \]

Using \[ \sin\left(\frac{k\pi x}{a}\right) \cdot \sin\left(\frac{l\pi y}{b}\right) \] as the weighting function in Galerkin's method gives,

\[
\int_0^a \int_0^b \left[ \nabla^4(z - z_0) + 2C_1 \left( \frac{\partial^2 z}{\partial x^2} + \frac{\partial z}{\partial y} \frac{\partial^2 z}{\partial y^2} \right) \right] \sin\left(\frac{k\pi x}{a}\right) \sin\left(\frac{l\pi y}{b}\right) dx \, dy = 2C_1(z^2 - z_0^2) \cdot z \cdot \pi^2 \cdot Q_3, \quad (A.13)
\]

where

\[ Q_3 = \int_0^a \int_0^b \left[ T \sin^2\left(\frac{k\pi x}{a}\right) \sin^2\left(\frac{l\pi y}{b}\right) + U \cos\left(\frac{k\pi x}{a}\right) \cos\left(\frac{l\pi y}{b}\right) \sin\left(\frac{k\pi x}{a}\right) \sin\left(\frac{l\pi y}{b}\right) \right] dx \, dy. \quad (A.13a) \]

And \[ \int_0^a \sin^2\left(\frac{k\pi x}{a}\right) dx = \frac{a}{2}, \quad (x=0) \]

\[ \int_0^b \sin^2\left(\frac{l\pi y}{b}\right) dy = \frac{b}{2}. \quad (y=0) \]

Substituting these integrals and equation (A.11) into equation (A.12) gives,

\[ \pi^4 \cdot Q_1 \cdot (z - z_0) - 2C_1 \cdot \pi^2 \cdot z - 2C_1(z^2 - z_0^2) \cdot z \cdot \pi^2 \cdot Q_3 \cdot (4/ab) = 0 \quad (A.14) \]

This equation can be solved using an iterative procedure to compute \( z \).
Calculation of Natural Frequencies

Consider the vibration in \((m,n)\) mode.

Let the displacement \(w\), during vibration be given by,

\[ w = H \cdot \sin\left(\frac{mx}{a}\right) \sin\left(\frac{ny}{b}\right) \sin(\omega t) \]

For the following analysis, \(H\) can be taken as unity, since for free vibration analysis the natural frequency does not depend on the amplitude (for small amplitude vibrations).

For simplicity, the vibration at the time of maximum excursion \((\sin(\omega t) = 1.0)\) will be considered.

i.e. \[ w = \sin\left(\frac{mx}{a}\right) \sin\left(\frac{ny}{b}\right) \]. \hfill (A.15)

The compatibility equation is [25],

\[ \nabla^4 \bar{\mathbf{F}} = 2C_1 \left( \frac{\partial^2 \mathbf{z}}{\partial x \partial y} \cdot \frac{\partial^2 \mathbf{w}}{\partial x \partial y} - \frac{\partial^2 \mathbf{z}}{\partial x^2} \cdot \frac{\partial^2 \mathbf{w}}{\partial y^2} - \frac{\partial^2 \mathbf{z}}{\partial y^2} \cdot \frac{\partial^2 \mathbf{w}}{\partial x^2} \right) \hfill (A.16) \]

The dynamic Airy stress function can be expressed as the summation of a series of the products of beam functions.

i.e. \[ \bar{\mathbf{F}} = \sum_{p q} \sum_p \beta_p \phi_p \psi_q \hfill (A.17) \]

\[ \quad = \{\beta\}^T \{\phi \psi\} \hfill (A.17a) \]

By following the same procedure as in the static deflection calculation, it can be shown that the Galerkin's approximation to equation \((A.16)\) is given by,
\[ [SK](S) = Z[B] \] (A.18)

where,

\[ S(I) = \beta_{p,q} \] (A.18a)

\[ B(I) = \frac{2C_1 \pi^4}{a^2 b^2} \left[ 2k \cdot m \cdot n \cdot T_4 \cdot S_4 - (k^2 m^2 + n^2) T_5 S_5 \right] \] (A.18b)

in which,

\[ T_4 = \int_{x=0}^{a} \cos\left(\frac{k \pi x}{a}\right) \cdot \cos\left(\frac{m \pi x}{a}\right) \cdot \phi_r \, dx \]

\[ S_4 = \int_{y=0}^{b} \cos\left(\frac{k \pi y}{b}\right) \cdot \cos\left(\frac{n \pi y}{b}\right) \cdot \psi_s \, dy \] (A.18c)

\[ T_5 = \int_{x=0}^{a} \sin\left(\frac{k \pi x}{a}\right) \cdot \sin\left(\frac{m \pi x}{a}\right) \cdot \phi_r \, dx \]

\[ S_5 = \int_{y=0}^{b} \sin\left(\frac{k \pi y}{b}\right) \cdot \sin\left(\frac{n \pi y}{b}\right) \cdot \psi_s \, dy \]

and \[ I = s + (r-1) q_m \] (A.18d)

[SK] is defined in equation (A.5a).

The equation of motion is [25],

\[ y t w - \lambda^2 w = 2C_1 \left[ \frac{\partial^2 F}{\partial x^2} \cdot \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} \cdot \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 F}{\partial x \partial y} \cdot \frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 F}{\partial y \partial x} \cdot \frac{\partial^2 w}{\partial y \partial x} \right] \]

\[ + \frac{\partial^2 z}{\partial x^2} \cdot \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 z}{\partial y^2} \cdot \frac{\partial^2 F}{\partial x^2} - 2 \frac{\partial^2 z}{\partial x \partial y} \cdot \frac{\partial^2 F}{\partial x \partial y} \] (A.19)
where,
\[
\lambda^2 = \frac{\rho \omega^2}{D}
\]  
(A.19a)
\[
\frac{\partial^2 w}{\partial x^2} = -\frac{m^2 \pi^2}{a^2} w \quad \frac{\partial^2 w}{\partial y^2} = -\frac{n^2 \pi^2}{b^2} w
\]
and
\[
\nabla^4 w = \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2 w.
\]

Substituting these in equation (A.19) gives,
\[
\lambda^2 w = \left[\pi^4 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2 - 2C_1 \pi^2 \left(\frac{m^2}{a^2} \frac{\partial^2 \sigma}{\partial x^2} + \frac{n^2}{b^2} \frac{\partial^2 \sigma}{\partial y^2}\right)\right] w
\]
\[
+ 2C_1 \pi^2 \left(\frac{m^2}{a^2} \frac{\partial^2 F}{\partial y^2} + \frac{n^2}{b^2} \frac{\partial^2 F}{\partial x^2}\right) w
\]
\[
+ 4C_1 \frac{\partial^2 F}{\partial x \partial y} \frac{\partial^2 w}{\partial x \partial y}
\]
\[
+ 2C_1 \left(\frac{\partial^2 \sigma}{\partial x^2} \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 \sigma}{\partial y^2} \frac{\partial^2 F}{\partial x^2} \right) - 2 \frac{\partial^2 \sigma}{\partial x \partial y} \frac{\partial^2 F}{\partial x \partial y}
\]
\]
(A.20)

Using Galerkin's method (taking \(\sin\left(\frac{m \pi x}{a}\right) \cdot \sin\left(\frac{n \pi y}{b}\right)\) as the weighting function) gives,
\[
\lambda^2 \left(\frac{ab}{4}\right) = \left[\pi^4 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2 - 2C_1 \pi^2 \left(\frac{m^2 \sigma}{a^2} + \frac{n^2 \sigma}{b^2}\right)\right] \left(\frac{ab}{4}\right)
\]
\[
+ R_1 + R_2
\]
(A.21)

where,
\[ R_1 = 2C_1 \int_0^a \int_0^b \left( \pi^2 \left( \frac{m^2}{a^2} \frac{\partial^2 F}{\partial y^2} + \frac{n^2}{b^2} \frac{\partial^2 F}{\partial x^2} \right) \cdot w^2 \right) \]
\[ + 2 \frac{\partial^2 F}{\partial x \partial y} \cdot \frac{\partial^2 w}{\partial x \partial y} \cdot w \, dx \, dy \quad (A.21a) \]

and

\[ R_2 = -2C_1 \int_0^a \int_0^b \left( \frac{\partial^2 z}{\partial x^2} \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 z}{\partial y^2} \frac{\partial^2 F}{\partial x^2} \right) \]
\[ - 2 \frac{\partial^2 F}{\partial x \partial y} \cdot \frac{\partial^2 F}{\partial y \partial x} \cdot w \, dx \, dy \quad (A.21b) \]

Dividing equation (A.21) by \((\frac{ab}{4})\) gives,

\[ \lambda^2 = \pi^4 \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right) - 2C_1 \pi^2 \left( \frac{m^2}{a^2} \frac{\partial}{\partial x} + \frac{n^2}{b^2} \frac{\partial}{\partial y} \right) + \frac{4(R_1 + R_2)}{ab} \quad (A.22) \]

The natural frequency corresponding to the \((m,n)\) mode is given by,

\[ \omega_{m,n} = \sqrt{\lambda^2 Eh^3 / (12(1-\nu^2))} \quad (A.23) \]

\(R_1\) and \(R_2\) can be calculated as follows:

Substituting equation (A.3) in equation (A.21a) leads to,

\[ R_1 = 2C_1 \pi^2 (z^2 - z_0^2) \int_0^a \int_0^b \left[ \left( \frac{m^2}{a^2} \{a'\}^T \{\phi'\} \right) + \frac{n^2}{b^2} \{a'\}^T \{\phi'\} \right] \]
\[ \sin \left( \frac{mn \pi x}{a} \right) \sin \left( \frac{mn \pi y}{b} \right) + 2 \frac{mn}{ab} \{a'\}^T \{\phi'\} \right) \cos \left( \frac{mn \pi x}{a} \right) \cos \left( \frac{mn \pi y}{b} \right) \, dx \, dy \]

\[ R_2 = \frac{4(2C_1 \pi^2 - 2) \int_0^a \int_0^b \left[ \left\{ \frac{m^2}{a^2} \{a'\}^T \{\phi'\} \right. \right. \right. \]
\[ \left. \left. \left. + \frac{n^2}{b^2} \{a'\}^T \{\phi'\} \right] \left( \frac{\partial^2 F}{\partial x \partial y} \cdot \frac{\partial^2 F}{\partial y \partial x} \cdot w \right. \right. \]
\[ \left. \left. \cdot dx \cdot dy \right. \right. \right. \]

\[ (A.24) \]
Substituting equation (A.17a) in equation (A.21b) leads to,

\[ R_2 = 2C_1 \pi^2 z \int_{x=0}^{\frac{k_2}{a}} \int_{y=0}^{\frac{j_2}{b}} \left[ \left( \frac{k}{a} \right)^2 \{ \beta \}^T \{ \phi \} \{ \psi \} \right] \times \]

\[ \sin \left( \frac{k \pi x}{a} \right) \sin \left( \frac{j \pi y}{b} \right) \]

\[ + \frac{2k \beta}{ab} \{ \beta \}^T \{ \phi' \} \cos \left( \frac{k \pi x}{a} \right) \cos \left( \frac{j \pi y}{a} \right) \] \[dx \, dy \quad (A.25) \]

Choice of Beam Functions:

By analogy with plate bending problem, it can be shown [36] that in-plane shear diaphragm boundary conditions can be represented by simply supported out-of-plane boundary conditions, and in-plane free boundary conditions can be represented by out-of-plane clamped boundary condition. For example, \( \phi(x) = \sin \left( \frac{i \pi x}{a} \right) \) can be used for shear diaphragm boundary conditions at \( x=0 \) and \( x=a \).
APPENDIX B

NUMERICAL RESULTS

The numerical values for the theoretical and experimental results are given in this section. Tables B.1 to B.11 give the measured values of central displacements and the natural frequencies for the test plates. Two examples of the parameters required for the Southwell plot are given in tables B.12 and B.13. Calculation of the change in deflection due to the applied load (Δ) is simply done by subtracting the central deflection reading with the dead load (weight of the supporting beam and the loading head is 33 lbs) from the central deflection reading for a given load. Table B.12 and B.13 are derived from table B.1 and table B.3 respectively.

For example the deflection due to a load of 600 lbs. (actual load 633 lbs.) = 0.0542 - 0.0812 = -0.027 inches.

Theoretical results for the out-of-plane displacement coefficients (Z_{1,1}, Z_{1,3}, Z_{3,1}, Z_{3,3}), the central deflection parameter (μ) and the fundamental natural frequency (ω) are tabulated in tables B.14 to B.19. Table B.14 and table B.15 are for plate 1 with different initial imperfection (μ_0) as indicated. Table B.17 gives the theoretical results for plate 4 with the following in-plane boundary conditions: top and bottom edges normally constrained and sides fully restrained.

The deflection readings at various points on the plates (see Figure 4.3.1 for the identification of these points.) are given in table B.20 to table B.23.
The location of strain gauges are shown in table B.24, where the co-ordinate system used is defined in the figure. Table B.25 gives the strain readings on each side at all of the gauge points. The reduced values of these strains are compared with the theoretical results in table B.26.
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<thead>
<tr>
<th>LOAD (lbs)</th>
<th>$\Delta_e$ 10^{-3}$ inch</th>
<th>$\omega_{hi}$ (Hz)</th>
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<tr>
<td>893.0</td>
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<td>96.7</td>
</tr>
</tbody>
</table>

(b) Unloading

(c) Loading

Initial Displacement Readings at the corners: 86.9, 85.3, 86.0, 85.8

$\Delta_e = 81.2$ mill inches

Amplitude of initial imperfection = \( (86.9 + 85.3 + 86.0 + 85.8) / 4 - 31.2 \)

= 31.2 mill inches

Table 8.1 Experimental Results for Plate 1 - Test 1
(Fundamental Frequencies)
<table>
<thead>
<tr>
<th>LOAD (lbs)</th>
<th>$\Delta c$ (10^{-4} inches)</th>
<th>$\omega_{12}$ (Hz)</th>
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<tbody>
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<td>33.0</td>
<td>9.2</td>
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<td>123.5</td>
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<tr>
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<td>111.5</td>
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</tr>
<tr>
<td>893.0</td>
<td>18.7</td>
<td>158.4</td>
</tr>
</tbody>
</table>

(b) Unloading

Initial Displacement readings at the corners: 36.9, 85.3 \text{ mill inches}
36.0, 85.4

$\Delta c = 80 \cdot 2 \text{ mill inches}$

Amplitude of initial imperfection: \((36.9 - 85.3 - 36.0 - 85.4) / 4 - 80.2\)
\approx 5.7 \text{ mill inches}

Table 82 Experimental Results for Plate 1, Test 2
(Natural Frequency - 2nd mode)
<table>
<thead>
<tr>
<th>LOAD (lbs)</th>
<th>Δc (10⁻³ inches)</th>
<th>ω₁L (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>33.0</td>
<td>90.6</td>
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<tr>
<td>23.0</td>
<td>77.7</td>
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<td>43.0</td>
<td>70.3</td>
<td>224.6</td>
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<td>48.0</td>
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<td>222.1</td>
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<td>62.3</td>
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<td>237.5</td>
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<td>8.93.0</td>
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</tr>
</tbody>
</table>

### Table 83 Experimental Results for Plate 1 - 3rd Frequency (Test 3)

**Loading**

Initial Displacement Readings:
- 86.9, 85.3 (mill. inches)
- 86.0, 85.4

Δc = 80.6 mill. inches

Amplitude of initial imperfection = (86.9 + 85.3 + 86.0 + 85.4) / 4 - 85.4 = 5.3 mill. inches

Flapping of the top edge observed when pressed with a finger at the top. The frequency increased from 273.7 to 277.4 Hz at 33 lbs. After the application of load, the frequency was not affected by touching at its top edge. Flapping ceased.
<table>
<thead>
<tr>
<th>LOAD (lbs.)</th>
<th>$\Delta e \cdot 10^3$ inch</th>
<th>$\omega_1$</th>
<th>$\omega_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>330</td>
<td>9.49</td>
<td>630-640</td>
<td>144.0</td>
</tr>
<tr>
<td>1330</td>
<td>92.7</td>
<td>570-610</td>
<td>124.0-1300</td>
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<tr>
<td>2330</td>
<td>85.1</td>
<td>540-550</td>
<td>115.0-1500</td>
</tr>
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<td>3330</td>
<td>83.5</td>
<td>51.0</td>
<td>120.0</td>
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<td>3860</td>
<td>79.4</td>
<td>50.0</td>
<td>117.0</td>
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<td>74.3</td>
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<td>129.0</td>
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(a) Loading

<table>
<thead>
<tr>
<th>LOAD (lbs.)</th>
<th>$\Delta e \cdot 10^3$ inch</th>
<th>$\omega_1$</th>
<th>$\omega_2$</th>
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(b) Unloading

Table 84: Experimental Results for Plate 2
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<th>ω₁</th>
<th>ω₂</th>
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Table 35: Experimental Results for Plate 3

(a) Loading

(b) Unloading
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<th>LOAD (lbs)</th>
<th>$\Delta_c$ (in.)</th>
<th>$\omega_{11}$ (Hz)</th>
<th>$\omega_{12}$ (Hz)</th>
<th>LOAD (lbs)</th>
<th>$\Delta_c$ (in.)</th>
<th>$\omega_{11}$ (Hz)</th>
<th>$\omega_{12}$ (Hz)</th>
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<td></td>
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</table>

(a) Loading
(b) Unloading

Capacitance probe was reset

Initial Displacement Readings at the corners: 965, 950, 922, 959 (in. in.)

Amplitude of initial imperfection = $\Delta_c$ - Average initial reading at the corners = \[
\frac{[965 + 950 + 922 + 959]}{4} = 679 \text{ in.}
\]

Table 86: Experimental Results for Plate 4 (Test 1)
<table>
<thead>
<tr>
<th>Load (lbs)</th>
<th>Δc $10^3$ inch</th>
<th>$\omega_{1,1}$</th>
<th>$\omega_{1,2}$</th>
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<tbody>
<tr>
<td>33.0</td>
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<td></td>
</tr>
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<td>77.0</td>
<td></td>
</tr>
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<td>82.0</td>
<td></td>
</tr>
<tr>
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<td>63.3</td>
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</tr>
<tr>
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<td>45.1</td>
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</tr>
<tr>
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<td></td>
</tr>
<tr>
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</table>

TABLE B7 Experimental Results for Pad 4 (Test 7)
<table>
<thead>
<tr>
<th>Load (lbs)</th>
<th>$\Delta_c$</th>
<th>$\omega_{1}$</th>
<th>$\omega_{2}$</th>
</tr>
</thead>
<tbody>
<tr>
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<td>73.6</td>
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</tr>
<tr>
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<td>72.0</td>
<td>161.0</td>
</tr>
<tr>
<td>2350</td>
<td>83.5</td>
<td>69.0</td>
<td>157.5</td>
</tr>
<tr>
<td>3330</td>
<td>84.0</td>
<td>65.5</td>
<td>150.5</td>
</tr>
<tr>
<td>4330</td>
<td>83.9</td>
<td>62.0</td>
<td>143.0</td>
</tr>
<tr>
<td>5330</td>
<td>84.8</td>
<td>58.0</td>
<td>137.0</td>
</tr>
<tr>
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<td>86.6</td>
<td>54.0</td>
<td>129.0</td>
</tr>
<tr>
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<td>88.0</td>
<td>49.0</td>
<td>121.0</td>
</tr>
<tr>
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<td>92.5</td>
<td>47.0</td>
<td>116.0</td>
</tr>
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<td>100.0</td>
<td>49.0</td>
<td>112.0</td>
</tr>
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<td>112.6</td>
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<td>108.0</td>
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<tr>
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<td>114.5</td>
<td>62.0</td>
<td>107.0</td>
</tr>
<tr>
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<td>72.0</td>
<td>107.0</td>
</tr>
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<td>12330</td>
<td>84.8</td>
<td>69.5</td>
<td>139.5</td>
</tr>
<tr>
<td>13330</td>
<td>92.2</td>
<td>74.0</td>
<td>149.0</td>
</tr>
</tbody>
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<table>
<thead>
<tr>
<th>Load (lbs)</th>
<th>$\Delta_c$</th>
<th>$\omega_{1}$</th>
<th>$\omega_{2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>14330</td>
<td>99.5</td>
<td>77.5</td>
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<tr>
<td>15330</td>
<td>106.7</td>
<td>83.0</td>
<td>171.0</td>
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<tr>
<td>17330</td>
<td>125.1</td>
<td>100.5</td>
<td>172.0</td>
</tr>
</tbody>
</table>

Initial Displacement Readings at the corners: 100, 880, 73.4, 72.0 (mill inches)

$\Delta_c = 81.9$ (mill inches)

Amplitude of initial imperfection = $\Delta_c = (800 + 80.0 + 73.4 + 72.0) / 4$

= 5.5 (mill inches)

Table 8-8 Experimental Results for Plate 5 (Test 1)
<table>
<thead>
<tr>
<th>Load (lbs)</th>
<th>( \Delta_c \times 10^3 \text{inch} )</th>
<th>( \omega_{1} )</th>
<th>( \omega_{1.2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>33.0</td>
<td>35.7</td>
<td>74.5</td>
<td>166.5</td>
</tr>
<tr>
<td>27.0</td>
<td>36.3</td>
<td>69.0</td>
<td>157.0</td>
</tr>
<tr>
<td>24.0</td>
<td>37.1</td>
<td>60.0</td>
<td>142.0</td>
</tr>
<tr>
<td>13.0</td>
<td>40.3</td>
<td>53.5</td>
<td>123.0</td>
</tr>
<tr>
<td>8.32</td>
<td>49.2</td>
<td>45.5</td>
<td>117.0</td>
</tr>
<tr>
<td>10.38</td>
<td>68.2</td>
<td>62.0</td>
<td>126.0</td>
</tr>
<tr>
<td>12.32</td>
<td>82.4</td>
<td>70.0</td>
<td>143.5</td>
</tr>
<tr>
<td>14.33</td>
<td>96.9</td>
<td>78.0</td>
<td>161.0</td>
</tr>
<tr>
<td>16.33</td>
<td>110.4</td>
<td>88.0</td>
<td>179.0</td>
</tr>
<tr>
<td>18.34</td>
<td>123.7</td>
<td>102.0</td>
<td>192.5</td>
</tr>
<tr>
<td>18.49</td>
<td>123.2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>20.32</td>
<td>107.1</td>
<td>114.0</td>
<td>206.0</td>
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<tr>
<td>22.33</td>
<td>117.3</td>
<td>128.5</td>
<td>219.0</td>
</tr>
</tbody>
</table>

(a) Loading

1 Not clear mode

(b) Unloading

II Load increased

Initial Displacement Readings at the corners: 32.0, 35.2, \( \frac{1}{2} \) milli-inches

\( \Delta_c = 35.7 \) milli-inches

Amplitude of initial imperfection = 35.7 - (32.0 + 35.2 + 25.1 + 24.7) = 3.95 milli-inches

Table 89 Experimental Results for Plate 5 (Test 2)
### Table 8.10 Experimental Results for Plate 5 (Test 3)

<table>
<thead>
<tr>
<th>LOAD (lbs)</th>
<th>( \Delta L )</th>
<th>( \omega_{1} )</th>
<th>( \omega_{2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>83.0</td>
<td>32.2</td>
<td>7.0</td>
<td>165.5</td>
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<tr>
<td>243.3</td>
<td>33.2</td>
<td>6.9</td>
<td>156.0</td>
</tr>
<tr>
<td>433.0</td>
<td>33.7</td>
<td>6.0</td>
<td>142.5</td>
</tr>
<tr>
<td>633.0</td>
<td>33.0</td>
<td>5.5</td>
<td>126.8</td>
</tr>
<tr>
<td>833.0</td>
<td>33.5</td>
<td>4.0</td>
<td>110.0</td>
</tr>
<tr>
<td>870.0</td>
<td>32.2</td>
<td>3.7</td>
<td>105.5</td>
</tr>
<tr>
<td>910.0</td>
<td>34.9</td>
<td>3.0</td>
<td>102.0</td>
</tr>
<tr>
<td>950.0</td>
<td>35.9</td>
<td>3.5</td>
<td>98.5</td>
</tr>
<tr>
<td>993.0</td>
<td>37.7</td>
<td>3.0</td>
<td>98.5</td>
</tr>
<tr>
<td>1033.0</td>
<td>44.4</td>
<td>3.4</td>
<td>97.5</td>
</tr>
<tr>
<td>1133.0</td>
<td>63.9</td>
<td>3.3</td>
<td>110.0</td>
</tr>
<tr>
<td>1233.0</td>
<td>72.4</td>
<td>3.4</td>
<td>122.5</td>
</tr>
<tr>
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<td>91.1</td>
<td>7.6</td>
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<tr>
<td>1833.0</td>
<td>117.4</td>
<td>103.0</td>
<td>192.5</td>
</tr>
<tr>
<td>2033.0</td>
<td>129.0</td>
<td>116.0</td>
<td>206.0</td>
</tr>
</tbody>
</table>

#### Initial Displacement Readings at the Corners

\[ \begin{align*}
\Delta_L &= 32.2 \text{ milli inches} \\
(24.7, 23.7) & \text{ milli inches}
\end{align*} \]

\[ \Delta_L = 32.2 \text{ milli inches} \]

Amplitude of imperfection: 

\[ 32.2 - \left( \frac{(24.7 + 23.7)}{2} \right) = 3.6 \text{ milli inches} \]

Initial Displacement Readings at the Corners: 

\[ \begin{align*}
32.8, 33.2, 24.7, 23.7
\end{align*} \] milli inches.

Table 8.10 Experimental Results for Plate 5 (Test 3)
<table>
<thead>
<tr>
<th>Load (kips)</th>
<th>$\Delta_e$ $10^{-3}$ inch</th>
<th>$\omega_{1}$</th>
<th>$\omega_{2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>82.7</td>
<td>65.5</td>
<td>104.5</td>
</tr>
<tr>
<td>15</td>
<td>85.2</td>
<td>71.0</td>
<td>107.5</td>
</tr>
<tr>
<td>25</td>
<td>89.0</td>
<td>76.0</td>
<td>118.0</td>
</tr>
<tr>
<td>35</td>
<td>93.6</td>
<td>77.0</td>
<td>118.5</td>
</tr>
<tr>
<td>45</td>
<td>98.1</td>
<td>79.0</td>
<td>119.5</td>
</tr>
<tr>
<td>55</td>
<td>104.7</td>
<td>83.0</td>
<td>135.5</td>
</tr>
<tr>
<td>60</td>
<td>104.2</td>
<td>87.5</td>
<td>138.0</td>
</tr>
<tr>
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<td>105.6</td>
<td>88.5</td>
<td>138.0</td>
</tr>
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<td>70</td>
<td>107.5</td>
<td>92.5</td>
<td>139.5</td>
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<td>75</td>
<td>107.0</td>
<td>94.5</td>
<td>140.5</td>
</tr>
<tr>
<td>81</td>
<td>111.3</td>
<td>97.5</td>
<td>142.5</td>
</tr>
<tr>
<td>85</td>
<td>112.3</td>
<td>98.5</td>
<td>143.5</td>
</tr>
</tbody>
</table>

(G) Loading

<table>
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<tr>
<th>Load (kips)</th>
<th>$\Delta_e$ $10^{-3}$ inch</th>
<th>$\omega_{1}$</th>
<th>$\omega_{2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>81</td>
<td>111.5</td>
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<td>143.0</td>
</tr>
<tr>
<td>75</td>
<td>109.5</td>
<td>96.5</td>
<td>140.5</td>
</tr>
<tr>
<td>70</td>
<td>107.8</td>
<td>91.5</td>
<td>140.5</td>
</tr>
<tr>
<td>65</td>
<td>106.4</td>
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</tr>
<tr>
<td>60</td>
<td>104.9</td>
<td>86.5</td>
<td>140.5</td>
</tr>
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<td>102.7</td>
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<td>89.0</td>
<td>71.0</td>
<td>109.0</td>
</tr>
<tr>
<td>5</td>
<td>86.1</td>
<td>69.0</td>
<td>104.5</td>
</tr>
</tbody>
</table>

(b) Unloading

Initial Displacement Readings at the corners: 28.6, 122.7 $\text{mill inches}$
28.8, 237.3 $\text{mill inches}$

$\Delta_e = 82.7 \text{ mill inches}$

Amplitude of initial imperfection: $82.7 - \left(28.6 + 122.7 + 28.8 + 237.3\right) / 4 = 57.38 \text{ mill inches}$

Table 36: Experimental Results for Plate G (Test 1)
<table>
<thead>
<tr>
<th>LOAD (P) (lbs)</th>
<th>( \Delta ) (milli inches)</th>
<th>( \Delta / P \times 10^3 )</th>
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<tr>
<td>0</td>
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<td>0/0</td>
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<td>100</td>
<td>1.0</td>
<td>10.0</td>
</tr>
<tr>
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<td>4.9</td>
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<td>20.3</td>
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<td>45.0</td>
</tr>
<tr>
<td>622</td>
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<td>55.5</td>
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<td>60.5</td>
</tr>
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<table>
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<th>( \Delta )</th>
<th>( \Delta / P \times 10^3 )</th>
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<tbody>
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<td>71.0</td>
</tr>
<tr>
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<td>42.0</td>
<td>65.6</td>
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Table B.12 Southwell's plot Data for Plate 1 - Test 1
<table>
<thead>
<tr>
<th>LOAD (P) (lbs)</th>
<th>$\Delta$ (milli inch)</th>
<th>$\Delta/P \times 10^3$</th>
<th>P</th>
<th>$\Delta$</th>
<th>$\Delta/P \times 10^3$</th>
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<td>760</td>
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<td>75.8</td>
</tr>
<tr>
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<td>73.6</td>
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<tr>
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<td>39.6</td>
<td>66.0</td>
</tr>
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<td>550</td>
<td>31.4</td>
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<td>36.6</td>
<td>500</td>
<td>23.1</td>
<td>46.2</td>
</tr>
<tr>
<td>550</td>
<td>24.8</td>
<td>45.1</td>
<td>400</td>
<td>12.2</td>
<td>30.5</td>
</tr>
<tr>
<td>620</td>
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Table B12: Southwell's plot data for Plate 1 - Test 3
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<th>$Z_{u1}$</th>
<th>$Z_{u1}$</th>
<th>$\mu$</th>
<th>$\omega_{ul}$ (Hz)</th>
</tr>
</thead>
<tbody>
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<td>0.0</td>
<td>0.0</td>
<td>0.1576</td>
<td>65.49</td>
</tr>
<tr>
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<td>-0.0005</td>
<td>-0.0002</td>
<td>0.2273</td>
<td>55.37</td>
</tr>
<tr>
<td>333</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.2830</td>
<td>50.16</td>
</tr>
<tr>
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<td>-0.0014</td>
<td>-0.0005</td>
<td>0.3901</td>
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<td>0.002</td>
<td>0.001</td>
<td>0.463</td>
<td>44.35</td>
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<tr>
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<td>0.002</td>
<td>0.001</td>
<td>0.557</td>
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<td>0.0006</td>
<td>0.0011</td>
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<td>0.003</td>
<td>0.001</td>
<td>0.932</td>
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<td>0.014</td>
<td>0.000</td>
<td>1.197</td>
<td>62.21</td>
</tr>
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Table 84: Theoretical Results for 1 mm plate with

$y_o = 0.15 h \sin \left( \frac{\pi x}{a} \right) \sin \left( \frac{\pi y}{b} \right)$
<table>
<thead>
<tr>
<th>LOAD (lbs)</th>
<th>Z_{41}</th>
<th>Z_{13}</th>
<th>Z_{31}</th>
<th>Z_{33}</th>
<th>$\mu$</th>
<th>$\omega_{n1}$ (Hz)</th>
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<tr>
<td>33</td>
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<td>0.682</td>
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<td>0.0317</td>
<td>2.5365</td>
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Table 8.15 Theoretical Results for 1 mm. plate with

$\theta_0 = 0.25 \sin(\eta_y) \sin(\eta_x)$
<table>
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<tr>
<th>LOAD (lbs)</th>
<th>$Z_{0,1}$</th>
<th>$Z_{1,3}$</th>
<th>$Z_{3,1}$</th>
<th>$Z_{3,3}$</th>
<th>$\mu$</th>
<th>$\omega_{n1}$ (Hz)</th>
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<tbody>
<tr>
<td>133</td>
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<td>0.0000</td>
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<td>0.0000</td>
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<td>63.20</td>
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<td>0.0220</td>
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<td>1.3900</td>
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</tr>
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<td>0.482</td>
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Table B.6: Theoretical Results for Plate 4 (with $\mu = 0.47$)
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<th>LOAD (lbs.)</th>
<th>Z_{1,1}</th>
<th>Z_{1,3}</th>
<th>Z_{3,1}</th>
<th>Z_{3,3}</th>
<th>\mu</th>
<th>\omega_{11} (Hz)</th>
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<td>-</td>
<td>-</td>
<td>-</td>
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</tr>
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Table B17: Theoretical Results for Plate 4 with Standard Boundary Conditions.
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<th>$\omega_{v1}$ (Hz)</th>
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<td>-0.0009</td>
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Table B-18: Theoretical Results for Plate 5 ($\mu = 0.05$)
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<th>$Z_{u3}$</th>
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<th>$\omega_{u1}$ (Hz)</th>
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Table B.19 Theoretical Results for Plate 6 with $\nu = 0.3$.
### Deflection Readings at Load = 33.18 lbs

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<td>92.3</td>
<td>92.9</td>
<td>92.1</td>
<td>92.4</td>
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### Deflection Readings at Load = 63.3 lbs

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### Deflection Readings at Load = 73.3 lbs

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<td>76.9</td>
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Table B.20: Deflection Readings for Plate 1.
### Deflection Readings at a Load of 791 lbs

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*Table 320—Continued*
### Deflection Readings at a Load of 833 lbs

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### Deflection Readings at a Load of 993 lbs

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Table B.20 - Continued
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## Deflection Readings at a Load of 393 lbs

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Table 8.20-Continued
### Deflection Readings at a Load of 533 lbs

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### Deflection Readings at a Load of 33 lbs

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Table B-20 - Continued
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Table B.21 Deflection Readings for Plate 4
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### Deflection Readings at a Load of 2245 lbs

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* Probe on contact with the pad

Table 822 Deflection Readings for Plate 5
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### Table B.23 Deflection Readings at a Load of 55 kgf

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Table 8.24 - LOCATION AND ORIENTATION OF THE STRAIN GAUGES
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Table 8.25 - Continued.
## Table B26: Comparison of Theoretical and Experimental Values of Strains

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<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
</tbody>
</table>

Table 826 - Continued
The first Fourier coefficients of the initial imperfection were calculated from the deflection readings at various points by using the following procedure:

\[
Z_{0,1,1} = \frac{4}{ab} \int_a^b \int_{y=0}^{b} z_0(x,y) \sin(\pi x/a) \sin(\pi y/b) \, dx \, dy
\]

The integration was carried out numerically using the measured values of \(z_0(x,y)\) at the 49 grid points applying Simpson's rule.

The calculated values of \(Z_{0,1,1}\) are compared with the magnitudes of the initial imperfection at the centre in the following table. The deflection values for plates 2 and 3 were not measured at all the grid points.

<table>
<thead>
<tr>
<th>Plate Number</th>
<th>Magnitude of Initial Imperfection at the Centre (1/1000 inches)</th>
<th>First Fourier Coefficient (1/1000 inches)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4.70</td>
<td>4.76</td>
</tr>
<tr>
<td>4 (Test 1)</td>
<td>17.22</td>
<td>17.75</td>
</tr>
<tr>
<td>5 (Test 1)</td>
<td>5.50</td>
<td>5.08</td>
</tr>
<tr>
<td>6</td>
<td>57.38</td>
<td>57.05</td>
</tr>
</tbody>
</table>
APPENDIX C

APPROXIMATE ANALYSIS OF THE POST BUCKLING AND VIBRATION BEHAVIOUR OF A SIMPLY SUPPORTED RECTANGULAR PLATE

The static displacements and natural frequencies of an imperfect plate subjected to uni-axial in-plane loading can be calculated in an approximate manner using the equilibrium approach. First, the deflections (static displacements) will be calculated. Having calculated the deflections and the static in-plane stresses, the same approach can be used for the calculation of natural frequencies.

Calculations of Deflections:

Consider the equilibrium of a simply supported rectangular plate subjected to uni-axial in-plane loading as shown in Figure C.1.

![Figure C.1](image)

Let the initial geometrical imperfection (distortion) of the plate be given by
\[ z_0(x,y) = Z_0 \sin \left( \frac{i \pi x}{a} \right) \sin \left( \frac{j \pi y}{b} \right) \]  \hspace{1cm} (C.1)

The deflection at a load \( \bar{N}_y \) (average in-plane load in \( y \)-direction per unit length) be expressed as
\[ z(x,y) = \sum_{i=1}^{L} \sum_{j=1}^{L} z_{ij} \sin \left( \frac{i \pi x}{a} \right) \sin \left( \frac{j \pi y}{b} \right) \]  \hspace{1cm} (C.2)

Consider the following in-plane boundary conditions:
(i) At sides \( x=0 \) and \( x=a \), the normal and tangential in-plane forces are zero (i.e. in-plane free).
(ii) At sides \( y=0 \) and \( y=b \), the normal displacements are constant (i.e. the load is applied via a rigid beam) and the tangential in-plane forces are zero (i.e. shear free).

The governing differential equation of equilibrium is
\[ D V^4 (z-z_0) + N_x \frac{\partial^2 z}{\partial x^2} + N_y \frac{\partial^2 z}{\partial y^2} - 2 N_{xy} \frac{\partial^2 z}{\partial x \partial y} = 0, \]  \hspace{1cm} (C.3)

where \( N_x, N_y \), and \( N_{xy} \) are the intensity of in-plane forces acting on the plate at a general point \( x, y \), and \( D \) is the flexural rigidity of the plate.

The analysis can be simplified with the assumption that the in-plane shear force is negligible at all points on the plate
\[ i.e. N_{xy} = 0. \]  \hspace{1cm} (C.4)

For equilibrium in the \( x \)-direction, \[ \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0. \]  \hspace{1cm} (C.5)

From equation (C.4) and (C.5), \[ \frac{\partial N_x}{\partial x} = 0. \]
But $N_x = 0$ at $x = 0$ and at $x = a$.

Therefore, $N_x = 0$ \hspace{1cm} (C.5a)

at all points on the plate. Similarly equation of equilibrium in the $y$-direction gives,

$\frac{\partial N_y}{\partial y} = 0$

This means, $N_{xy}$ is a function of $x$ only. An expression for $N_y$ may be formulated as follows:

Consider the displacement of a vertical strip of width $dx$ at a distance $x$ from the origin, as shown in Figure C.2.

![Diagram showing a vertical strip](image)

Figure C.2

If the plate was allowed to move freely at the top and bottom, the change in the curved length of the strip due to the loading is given by

$$\Delta l = \int_0^b \left( \frac{\partial z}{\partial y} \right)^2 + \left( \frac{\partial z}{\partial y} \right)^2 \, dy = \frac{1}{2} \int_0^b \left( \frac{\partial z}{\partial y} \right)^2 - \left( \frac{\partial z}{\partial y} \right)^2 \, dy$$

Taking only the first term in the series for $z$ from equation (C.2),
\[ \Delta \beta = \frac{\pi^2}{2b^2} \left( z_o^2 - z_0^2 \right) \sin^2 \left( \frac{\pi x}{a} \right) \int_y^b \cos \left( \frac{\pi y}{b} \right) dy \]

\[ = \frac{\pi^2}{4b} \left( z_o^2 - z_0^2 \right) \sin^2 \left( \frac{\pi x}{a} \right) \]  \hspace{1cm} (C.6)

In order to maintain a constant displacement at the top and bottom, this strip must be stretched back by a distance of \( \Delta \beta \). The intensity of restraining stretching force required is given by:

\[ R = -Eh \left( \frac{\Delta \beta}{b} \right) = -\frac{\pi^2 Eh}{4b^2} \left( z_o^2 - z_0^2 \right) \sin^2 \left( \frac{\pi x}{a} \right) \]  \hspace{1cm} (C.7)

The resultant force from \( R \) is given by:

\[ \bar{R} = \int_{x=0} R \, dx = -\frac{\pi^2 EhA}{8b^2} \left( z_o^2 - z_0^2 \right) \]  \hspace{1cm} (C.8)

The constant displacement condition at the top and bottom edges will not be violated if a constant uniform force intensity of \(-\bar{R}/a\) is applied to maintain the equilibrium. Final distribution of net restraining force is then given by:

\[ N_y = \bar{N}_y + R - \frac{\bar{R}}{a} \]

\[ = \bar{N}_y - \frac{\pi^2 Eh}{4b^2} \left( z_o^2 - z_0^2 \right) \sin^2 \left( \frac{\pi x}{a} \right) + \frac{\pi^2 Eh}{8b^2} \left( z_o^2 - z_0^2 \right) \]

\[ = \bar{N}_y + \frac{\pi^2 Eh}{8b^2} \left( z_o^2 - z_0^2 \right) \cos \left( \frac{2\pi x}{a} \right) \]  \hspace{1cm} (C.9)

Substituting equations (C.4), (C.5a) and (C.9) into equation (C.3) yields the following equation:
\[ D^{-4} \left( \frac{1}{a^2} + \frac{1}{b^2} \right)^2 (z_1, l - z_o) + \left[ N_y + \frac{2\pi E h}{8b^2} (z_1, l - z_o) \cos \left( \frac{2\pi x}{a} \right) \right] \left( \frac{-\pi^2 z_1, l}{b^2} \right) \cdot \sin \left( \frac{\pi x}{a} \right) \sin \left( \frac{\pi y}{b} \right) = 0 \] (C.10)

Since the parameters within the square brackets are not functions of \( y \),

\[ \left[ D^{-4} \left( \frac{1}{a^2} + \frac{1}{b^2} \right)^2 (z_1, l - z_o) \right. \]

\[ + \left[ N_y + \frac{2\pi E h}{8b^2} (z_1, l - z_o) \cos \left( \frac{2\pi x}{a} \right) \right] \left( \frac{-\pi^2 z_1, l}{b^2} \right) x \]

\[ \sin \left( \frac{\pi x}{a} \right) = 0 \]

Galerkin's method with the weighting function \( \sin \left( \frac{\pi x}{a} \right) \) gives

\[ \left[ D^{-4} \left( \frac{1}{a^2} + \frac{1}{b^2} \right)^2 (z_1, l - z_o) \right. \]

\[ + \left[ N_y + \frac{2\pi E h}{8b^2} (z_1, l - z_o) \cos \left( \frac{2\pi x}{a} \right) \right] \frac{-\pi^2 z_1, l}{b^2} \left. \right] x = 0 \]

\[ \int_0^a \sin^2 \left( \frac{\pi x}{a} \right) dx = \left. \frac{a^2}{2} \right|_{x=0} \] (C.11a)

but,

\[ \int_0^a \cos \left( \frac{2\pi x}{a} \right) \sin^2 \left( \frac{\pi x}{a} \right) dx = -\frac{a^2}{4} \] (C.11b)

Substituting equations (C.11a) and (C.11b) in equation (C.11) gives:

\[ \left[ D^{-4} \left( \frac{1}{a^2} + \frac{1}{b^2} \right)^2 (z_1, l - z_o) \right. \]

\[ + \left[ N_y + \frac{2\pi E h}{8b^2} (z_1, l - z_o) \cos \left( \frac{2\pi x}{a} \right) \right] \frac{-\pi^2 z_1, l}{b^2} \left. \right] \frac{a}{2} \]

\[ \int_0^a \cos \left( \frac{2\pi x}{a} \right) \sin^2 \left( \frac{\pi x}{a} \right) dx = 0 \] (C.12)

Let \( u = \frac{z_1, l}{h} \) and \( u_o = \frac{z_o}{h} \).
Dividing equation (C.12) by \( D \pi^4 \left( \frac{1}{a^2} + \frac{1}{b^2} \right)^2 \frac{a^2 b^2}{2} \)

\[
(u - u_0) - \frac{\tilde{N}_Y}{\tilde{N}_Y} + 1.5 \frac{(1-u_0^2)}{(1+\gamma^2)^2} u (u^2 - u_0^2) = 0
\]  
(C.13)

where \( \gamma \) is the aspect ratio given by \( \gamma = b/a \), \hspace{1cm} (C.13a)

and \( \tilde{N}_Y \) is the critical force intensity given by

\[
\tilde{N}_Y = \pi^2 D b^2 \left( \frac{1}{a^2} + \frac{1}{b^2} \right)^2
\]  
(C.13b)

Let \( \rho \) be defined as a load ratio such that,

\[
\rho = \frac{\tilde{N}_Y}{\tilde{N}_Y}
\]  
(C.13c)

Equation (C.13) can be written as

\[
(u - u_0) - \rho u + 1.5 \frac{(1-u_0^2)}{(1+\gamma^2)^2} u (u^2 - u_0^2) = 0
\]

or

\[
[1 - \frac{u_0}{u} - \rho + C(u^2 - u_0^2)]u = 0
\]  
(C.14)

in which \( C = 1.5 \frac{(1-u_0^2)}{(1+\gamma^2)^2} \) \hspace{1cm} (C.14a)

for very small values of \( u_0 \) (i.e. if \( u_0 \ll u \)),

\[
[1 - \rho + C u^2]u = 0
\]  
(C.15)

This will be satisfied if \( u = 0 \) \hspace{1cm} (C.15a),

or

\[
u^2 = \left( \frac{1-u_0}{C} \right)
\]  
(C.15b).

Equations (C.15a) and (C.15b) represents the solution for
the deflection of an initially flat plate. Figure C.3
illustrates the solution graphically

![Graph](image)

Figure C.3

For \( \rho < 1.0 \), \( \mu = 0 \) since the alternative solution \( \mu^2 = \frac{1-\nu}{C} \)
gives an imaginary value for \( \mu \).

**Vibration Analysis**

Neglecting the dynamic in-plane shear stresses in the plate, the plate vibration equation can be shown to be:

\[
DV^4w + N_y \frac{\partial^2 w}{\partial y^2} + S_y \frac{\partial^2 z}{\partial x^2} - \ddot{m} \omega^2 w = 0 \quad (C.16)
\]

where \( S_y \) is the dynamic in-plane force intensity in the \( y \)-direction

and \( w \) is the dynamic out-of-plane displacement given by

\[
w(x, y, t) = H(t) \sin \left( \frac{\pi x}{a} \right) \sin \left( \frac{\pi y}{b} \right), \quad (C.16a)
\]

in which for simple harmonic motion,

\[
H(t) = \ddot{H} \sin(\omega t) \quad (C.16b)
\]

For dynamic analysis, the sides \( x = 0 \) and \( x = a \) can be taken as in-plane stress free (normal and shear). The top and bottom edges may be taken as being shear free (free to slide tangentially) and having constant normal displacements. Two
simple cases are treated.

Case 1. **Resultant Dynamic Forces at the Top and Bottom Edges are Zero**

For this case $S_y$ can be calculated as follows:

Following the same procedure as in the case of deflection calculations, during vibration the change in length of a strip of plate at distance $x$ from the origin is given by

$$\Delta l = \frac{1}{2} \int_{y=0}^{b} \left[ \left( \frac{\partial (z+w)}{\partial y} \right)^2 - \left( \frac{\partial z}{\partial y} \right)^2 \right] dy$$

$$\approx \int_{y=0}^{b} \frac{\partial z}{\partial y} \frac{\partial w}{\partial y} dy \quad \text{for small amplitude vibrations}$$

i.e. \( \Delta l = \frac{\pi^2 b^2 \sin^2 \left( \frac{\pi x}{a} \right)}{b^2} \)\( \sin^2 \left( \frac{\pi x}{a} \right) \)

$$= \frac{\pi^2 b^2 \sin^2 \left( \frac{\pi x}{a} \right)}{2b}$$

The corresponding intensity of stretching force is

$$S_y = Eh \frac{\Delta l}{b} = \frac{2Eh^2}{2b^2} \sin^2 \left( \frac{\pi x}{a} \right)$$

The resultant force $S$ is

$$S_y = \int_{x=0}^{b} S_y dx = \pi^2 \frac{Eh^2}{4b^2} \sin^2 \left( \frac{\pi x}{a} \right)$$

For equilibrium, the net force intensity $S_y = S_y - \frac{\pi}{a}$
\[ -\pi^2 \frac{Eh^2}{4b^2} \mu H \cos\left(\frac{2\pi x}{a}\right) \]  \hspace{1cm} (C.20)

Substituting equations (C.9) and (C.20) in equation (C.16) gives:

\[ \sum_{\pm} \left[ \left( \frac{D\pi^4}{a^2} + \frac{1}{b^2} \right) - \frac{\pi^2}{y b^2} \right] - \frac{4}{8b^2} \left( \frac{Eh^3}{4b^4} \right) (\mu^2 - \mu_o^2) \]

\[ + \pi^4 \frac{Eh^3}{4b^4} (\mu^2) \cos\left(\frac{2\pi x}{a}\right) - \frac{\pi^2}{y b^2} \right] \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi y}{b}\right) = 0 \]

For non-trivial solution of \( H \),

\[ \{ \left[ \frac{D\pi^4}{a^2} \frac{1}{b^2} + \frac{1}{b^2} \right] - \frac{\pi^2}{y b^2} \} - \frac{4}{8b^2} \left( \frac{Eh^3}{4b^4} \right) (\mu^2 - \mu_o^2) \]

\[ + \pi^4 \frac{Eh^3}{4b^4} (\mu^2) \cos\left(\frac{2\pi x}{a}\right) - \frac{\pi^2}{y b^2} \right] \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi y}{b}\right) = 0 \]

Following the same procedure as in the deflection analysis, this can be transformed into:

\[ (1 - \rho + C(\mu^2 - \mu_o^2 + 2\mu^2) - \frac{\omega^2}{\Omega^2}) = 0 \]  \hspace{1cm} (C.21)

where \( \Omega^2 = D \frac{\pi^4}{m} \left( \frac{1}{a^2} + \frac{1}{b^2} \right) \)  \hspace{1cm} (C.21a)

\( \Omega \) can be recognized as the fundamental natural frequency of the unstressed plate.

Let \( \lambda^2 = \frac{\omega^2}{\Omega^2} \)  \hspace{1cm} (C.22)

Then, \( 1 - \rho + C(\mu^2 - \mu_o^2) + 2C\mu^2 - \lambda^2 = 0 \)

or \( \lambda^2 + \rho = 1 + 2C\mu^2 + C(\mu^2 - \mu_o^2) \)  \hspace{1cm} (C.23)
In equation (C.23), $2C_{\mu}^2$ represents the effect of dynamic stretching of the plate (membrane stiffness on the frequency, and $C(\mu^2 - \mu_0^2)$ represents the effect of non-uniform static stress distribution on the frequency.

For $\mu_0 \ll \mu$ (almost flat plate), the effect of membrane stretching is twice the effect of non-uniform stress distribution. Equation (C.23) is illustrated graphically in Figure C.4.

![Graph](image)

Figure C.4

Case 2: **Top and Bottom Edges Fully Restrained Against Normal In-plane Movement**

In this case, the analysis is similar to that for the previous case, except that no stress redistribution is required to maintain zero overall edge force.

$$i.e. \quad S_y = S = \pi^2 \frac{E h^2}{2 b^2} \mu H \sin^2 \left( \frac{\pi x}{a} \right)$$

(C.24)

The effect of $S_y$ in the dynamic equilibrium equation can be determined by calculating the integral:
\[ \int_{y=0}^{a} y \left( \frac{2 \xi}{a} \right) \sin \left( \frac{\pi \xi}{a} \right) d\xi = 3 \frac{\pi^4}{16b^4} Eh^3 \mu^2 H \]

This leads to

\[ 1 - \rho + C(\mu^2 - \mu_o^2) + 6C\mu^2 - \lambda^2 = 0 \]

or \[ \lambda^2 + \rho = 1 + 6C\mu^2 + C(\mu^2 - \mu_o^2) \]

\[(c.25)\]

The effect of dynamic membrane stretching is about six times as large as that of non-uniform stress distribution for an almost perfect plate (\( \mu_o \ll \mu \)).

From these relationships it can be observed that for a perfectly flat plate the following frequency load diagram is obtained.

![Diagram](image)

Figure C.5

The slope of the \( \lambda^2 \) vs \( \rho \) plot in the post buckling region is given by

\[ \theta = \tan^{-1} \left( \frac{1}{3C} \right) \] for case 1

and \[ \theta = \tan^{-1} \left( \frac{1}{7C} \right) \] for case 2.
From the above derivations it is clear that if the shear stress distribution is neglected and the static and dynamic out-of-plane displacements are assumed to take the shape of buckling and vibration of the corresponding flat plate, there exists a linear relationship between \((\lambda^2 + \nu)\) and \(\mu^2\). Such a relationship can also be derived for a spherically curved simply supported rectangular plate subjected to a constant uni-axial static in-plane stress distribution by adding the stress effect in the analysis published by Reissner [19].
APPENDIX D

DERIVATION OF THE EQUATION OF MOTION FOR A VIBRATING CURVED BEAM SUBJECTED TO STATIC AXIAL LOAD

The equation of motion for a curved beam subjected to static axial load can be derived using Newton's 2nd law of motion.

Consider the motion of the curved beam shown in Figure D.1, vibrating in its plane of curvature.

![Diagram of curved beam with applied forces and displacements](image)

**Figure D.1**

Let $z(x)$ be the transverse static displacement due to the applied compressive force $P$. Let $u(x,t)$ and $w(x,t)$ be the axial and transverse dynamic displacements (from the static equilibrium position) respectively. Let $P'$ be the dynamic axial stretching force (tensile) induced on the beam during the vibration.
First consider the change in the transverse forces acting on a small element of length $\Delta x$.

The forces acting on the beam when it passes the static equilibrium position, and when it reaches the maximum positive excursion position are shown in Figure D.2 and Figure D.3 respectively.

For Figure D.2,

$$M_{s2} - M_{s1} = F_{s2} \cdot \Delta x + P \cdot \Delta z$$

As $\Delta x \rightarrow 0$,

$$= F_{s2} \cdot \Delta x + P \cdot \frac{\partial z}{\partial x} \cdot \Delta x$$

$$= (F_{s2} + P \frac{\partial z}{\partial x}) \Delta x$$

or

$$\frac{\Delta M_s}{\Delta x} = F_{s2} + P \frac{\partial z}{\partial x}$$
as \( \Delta x \to 0 \),
\[
\frac{\partial M_s}{\partial x} = F_{s2} + P \frac{\partial z}{\partial x}
\]
\[
F_{s2} = \frac{\partial M_s}{\partial x} - P \frac{\partial z}{\partial x}
\]
\[
F_{s2} - F_{s1} = \Delta F_s = \frac{\partial F_{s2}}{\partial x} \Delta x
\]
\[
= \left( \frac{\partial^2 M_s}{\partial x^2} - P \frac{\partial^2 z}{\partial x^2} \right) \Delta x
\]

But, using the Euler-Bernoulli's beam bending formula
\[
\frac{\partial^2 M_s}{\partial x^2} = -EI \frac{\partial^4 z}{\partial x^4}
\]

The net transverse force is,
\[
\Delta F_s = \left( -EI \frac{\partial^4 z}{\partial x^4} - P \frac{\partial^2 z}{\partial x^2} \right) \Delta x
\]

Similarly, by considering the equilibrium of forces shown in Figure D.3, it can be shown that
\[
\Delta F_s + \Delta F_D = \left( -EI \frac{\partial^4 (z+w)}{\partial x^4} - P \frac{\partial^2 (z+w)}{\partial x^2} + P' \frac{\partial^2 (z+w)}{\partial x^2} \right) \Delta x
\]

where \( P' \) is the axial force induced during vibration and can be calculated by considering the axial equilibrium.

The change in the transverse force during vibration is then given by
\[
\Delta F_D = \left( -EI \frac{\partial^4 w}{\partial x^4} - P \frac{\partial^2 w}{\partial x^2} + P' \frac{\partial^2 (z+w)}{\partial x^2} \right) \Delta x
\]

Using Newton's 2nd law of motion,
where, $\bar{m}$ is the mass per unit length.

This gives

$$-EI \frac{\partial^4 w}{\partial x^4} - p \frac{\partial^2 w}{\partial x^2} + p' \frac{\partial^2 w}{\partial x^2} + p' \frac{\partial^2 w}{\partial x^2} - \bar{m} \frac{\partial^2 w}{\partial t^2} = 0 \quad (D.1)$$

$p'$ is amplitude dependent and the product $p' \frac{\partial^2 w}{\partial x^2}$ can be neglected for small amplitude vibrations. For simple harmonic motion, $\frac{\partial^2 w}{\partial t^2} = -\omega^2 w$. Hence, equation (D.1) reduces to:

$$EI \frac{\partial^4 w}{\partial x^4} + p \frac{\partial^2 w}{\partial x^2} - p' \frac{\partial^2 w}{\partial x^2} - \bar{m} \omega^2 w = 0 \quad (D.2)$$

Neglecting the axial inertia $p' = EA \epsilon_x'$

where $\epsilon_x'$ is the dynamic axial strain given by

$$\epsilon_x' = \frac{\partial u}{\partial x} + \frac{1}{2} \left[ \left( \frac{\partial (z + w)}{\partial x} \right)^2 - \left( \frac{\partial z}{\partial x} \right)^2 \right]$$

$$= \frac{\partial u}{\partial x} + \frac{\partial w}{\partial x} \cdot \frac{\partial z}{\partial x} + \frac{1}{2} \frac{\partial^2 w}{\partial x^2}$$

$$= \frac{\partial u}{\partial x} + \frac{\partial w}{\partial x} \cdot \frac{\partial z}{\partial x} \quad \text{(for small amplitude vibrations)}$$

or $p' = EA \left( \frac{\partial u}{\partial x} + \frac{\partial w}{\partial x} \cdot \frac{\partial z}{\partial x} \right) \quad (D.3)$

The terms in equation (D.2) can be interpreted as follows:

Dynamic transverse force resisting the bending: $EI \frac{\partial^4 w}{\partial x^4}$

Dynamic transverse force resulting from the change in position of the applied axial force: $p \frac{\partial^2 w}{\partial x^2}$
Dynamic transverse resisting force due to the 'stretching' action of the beam: \( p' \frac{\partial^2 z}{\partial x^2} \)

The force resulting from the change in position of the stretching force \( p' \frac{\partial^2 w}{\partial x^2} \) can be neglected for small amplitude vibrations.

Negative inertia force: \(-mw^2\)

All these forces maintain the dynamic equilibrium of the beam.
APPENDIX E

APPLICATION OF NEWTON-RAPHSON'S METHOD WITH A CONDITIONAL EQUATION

Consider the equations

\[ A - uZ = 0 \quad \text{(E.1)} \]

and

\[ u = Z^2 \quad \text{(E.2)} \]

where \( A \) is a constant.

Solving for \( Z \) using these two equations is equivalent to solving

\[ A - Z^3 = 0 \quad \text{(E.3)} \]

Equation (E.3) can be solved using Newton-Raphson's method as follows:

Let \( f(Z) = A - Z^3 \).

\[ \frac{df}{dZ} = -3Z^2 \]

\[ \Delta f = \frac{df}{dZ} \Delta Z = -3Z^2 \Delta Z \quad \text{(E.4)} \]

If equations (E.1) and (E.2) are treated separately, i.e.

\[ f = A - uZ ; u = Z^2 \]

\[ \Delta f = \frac{df}{dZ} \Delta Z + \frac{df}{du} \Delta u = \frac{df}{dZ} \Delta Z + \frac{df}{du} \frac{du}{dz} \Delta Z \]

\[ \Delta f = \frac{df}{dZ} \Delta Z + \frac{df}{du} \frac{du}{dz} \Delta Z \]

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\[
\frac{\partial^2 f}{\partial z^2} = \left( \frac{\partial f}{\partial u} \right) \frac{\partial^2 u}{\partial z^2} + 2 \frac{\partial f}{\partial z} \frac{\partial^2 u}{\partial z^2} \Delta z = 0
\]

This is the same as equation (E.4), and will lead to the iterative equation,

\[
f_n = -3z_n^2 (z_{n+1} - f_n)
\]  \hspace{1cm} (E.5)

Hence, when using Newton-Raphson's method for a set of equations with several unknowns, if the relationship between some of the unknowns is known, then it is necessary to solve only some of the equations in terms of the independent unknowns - the relationship between which is not known. However, the effect of other unknowns is included by taking the total differential terms which contain the relationship between the independent and dependent unknowns.

In applying the Rayleigh-Ritz principle, first, all the displacement coefficients are treated as independent parameters when substituting in equations (3.2.7), (3.2.8) and (3.2.9). After this, the relationship between the in-plane and out-of-plane coefficients is established. Hereafter, the in-plane displacement coefficients are treated as dependent variables in solving the equation (3.2.9).
APPENDIX F

LINEARIZATION OF THE STRAIN ENERGY EXPRESSIONS FOR A VIBRATING CURVED BEAM UNDER AXIAL LOADING

Consider the free vibration of the beam shown in Figure F.1.

Definitions
The static and dynamic displacements along x axis are given by \( u_s \) and \( u_d \) respectively.
The initial shape of the beam is given by \( z_0 \).
The transverse deflection of the beam under load \( P \) is \( z \).
The transverse dynamic displacement from the equilibrium configuration is \( w \).
\( z \) and \( w \) can be taken as a series with unknown coefficients, i.e.
\[
\begin{align*}
z &= \sum_{i=1,2} Z_i \phi_i \tag{F.1} \\
w &= \sum_{i=1,2} H_i \psi_i \tag{F.2}
\end{align*}
\]

where, \( \phi_i \) and \( \psi_i \) are transverse shape functions that can represent the shape of the beam subject to boundary conditions.

Assuming that the beam is initially stress free, the static strain at a load \( P \) is given by,
\[
\varepsilon = \frac{3u}{\delta x} + \frac{1}{2} \left( \frac{\partial z}{\partial x} \right)^2 - \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \tag{F.3}
\]

The strain at the time of maximum positive excursion is given by,
\[
\varepsilon = \varepsilon + \varepsilon' \tag{F.4}
\]

where
\[
\varepsilon' = \frac{3u}{\delta x} + \frac{1}{2} \left( \frac{\partial z}{\partial x} \right)^2 + \frac{\partial w}{\partial x} \cdot \frac{\partial z}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2
\]

\[
= \frac{\partial u}{\partial x} + \frac{\partial w}{\partial x} \cdot \frac{\partial z}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \tag{F.5}
\]

The total potential energy of the beam at the time of maximum positive excursion is given by,
\[ \hat{\nu} = \frac{1}{2} \int_{x=0}^{L} EI \left( \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 \phi}{\partial x^2} \right)^2 dx + \frac{1}{2} \int_{x=0}^{L} EA (\varepsilon + \varepsilon')^2 dx + \frac{1}{2} \left. P(u_s + u_d) \right|_{x=0} \\
- \left. P(u_s + u_d) \right|_{x=L} \\
= \frac{1}{2} \int_{x=0}^{L} EI \left( \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 \phi}{\partial x^2} \right)^2 dx + \frac{1}{2} \int_{x=0}^{L} EA \left( \frac{\partial u_s}{\partial x} + \frac{\partial u_d}{\partial x} \right)^2 dx \\
+ \frac{1}{2} \left( \frac{\partial z}{\partial x} \right)^2 - \frac{1}{2} \left( \frac{\partial \phi}{\partial x} \right)^2 + \frac{\partial w}{\partial x} \cdot \frac{\partial z}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right] \right|_{x=L} \\
- \left. P(u_s + u_d) \right|_{x=0} \quad \text{(F.6)} \\
\]

This can be expressed as,
\[ \hat{\nu} = \hat{U}_1 + \hat{U}_2 + \hat{U}_3 + \hat{U}_4 \quad \text{(F.7)} \]

where
\[ \hat{U}_1 \] consists of static displacement terms only and is given by,
\[ \hat{U}_1 = \frac{1}{2} \int_{x=0}^{L} EI \left( \frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 \phi}{\partial x^2} \right)^2 dx + \frac{1}{2} \int_{x=0}^{L} EA \left( \frac{\partial u_s}{\partial x} + \frac{\partial u_d}{\partial x} \right)^2 dx \\
+ \frac{1}{2} \left( \frac{\partial z}{\partial x} \right)^2 - \frac{1}{2} \left( \frac{\partial \phi}{\partial x} \right)^2 \right|_{x=L} - \left. P(u_s) \right|_{x=0} \quad \text{(F.7a)} \\
\]

\[ \hat{U}_2 \] consists of first order dynamic terms, and is given by,
\[ \hat{U}_2 = \int_{x=0}^{L} \left( EI \frac{\partial^2 z}{\partial x^2} - \frac{\partial z}{\partial x} \frac{\partial^2 w}{\partial x^2} \right) dx + \int_{x=0}^{L} EA \left( \frac{\partial^3 u_s}{\partial x^3} \right) dx + \frac{1}{2} \left( \frac{\partial z}{\partial x} \right)^2 - \frac{1}{2} \left( \frac{\partial z_0}{\partial x} \right)^2 \left( \frac{\partial u_d}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial z}{\partial x} \right) dx \]

\[ - P u_d \bigg|_{x=0} \]  

(F.7b)

\[ \hat{U}_3 \] is given by

\[ \hat{U}_3 = \frac{1}{2} \int_{x=0}^{L} \left( \frac{1}{2} \frac{\partial w}{\partial x} \right)^2 dx + \frac{1}{2} \int_{x=0}^{L} EA \left( \frac{\partial u_d}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial z}{\partial x} \right)^2 + \frac{\partial u_s}{\partial x} \frac{\partial u_s}{\partial x} \left( \frac{\partial z}{\partial x} \right)^2 \right) dx \]  

(F.7c)

\[ \hat{U}_4 \] is given by

\[ \hat{U}_4 = \frac{1}{2} \int_{x=0}^{L} EA \left[ \frac{1}{4} \frac{\partial w}{\partial x} \right]^4 + \frac{\partial u_d}{\partial x} \left( \frac{\partial w}{\partial x} \right)^2 + \frac{\partial w}{\partial x} \left( \frac{\partial z}{\partial x} \right)^3 \right] dx \]  

(F.7d)

When applying Rayleigh-Ritz method to the vibration analysis,

(i) \( \hat{U}_1 \) may be omitted since all terms in \( \hat{U}_1 \) are independent of dynamic displacements.

(ii) \( \hat{U}_4 \) consists of terms that are negligible compared to all other values (\( \hat{U}_2 \) and \( \hat{U}_3 \)) for small amplitude vibrations.

(iii) It can be shown that although the individual terms in \( \hat{U}_2 \) are larger than those in \( \hat{U}_3 \), \( \hat{U}_2 \) must vanish by using the principle of virtual work as follows.
Let $M$ be the bending moment induced at a section of the beam due to the applied load $P$

i.e. \[ M = EI \left( \frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 Q}{\partial x^2} \right) \]

\[ P = EAE = EA \left[ \frac{\partial u_s}{\partial x} + \frac{1}{2} \left( \frac{\partial z}{\partial x} \right)^2 - \frac{1}{2} \left( \frac{\partial Q}{\partial x} \right)^2 \right] \]

Substituting equations (F.8a) and (F.8b) in equation (F.7b) gives

\[ \hat{U}_2 = \int_{x=0}^{L} \left[ M \frac{\partial^2 w}{\partial x^2} + P \left( \frac{\partial u_d}{\partial x} + \frac{\partial w}{\partial x} \cdot \frac{\partial z}{\partial x} \right) \right] dx - P [u_d]^L_0 \]

For small amplitude vibrations, the dynamic axial strain \( \epsilon' \) is given by,

\[ \epsilon' = \frac{\partial u_d}{\partial x} + \frac{\partial w}{\partial x} \cdot \frac{\partial z}{\partial x} \]

Therefore

\[ \hat{U}_2 = \int_{x=0}^{L} \left( M \frac{\partial^2 w}{\partial x^2} + P \epsilon' \right) dx - P \delta L \]

where the elongation of the beam

\[ \delta L = u_d(L) - u_d(0) \]

Since the dynamic displacement $w$ is a geometrically compatible shape for the beam, it can be considered as a virtual displacement shape that induces a virtual strain $\epsilon'$ in the beam. $P \delta L$ can be considered as the external virtual work due to the virtual displacement of the end.
forces \( (P) \), \( \int_{x=0}^{L} \left( M \frac{\partial^2 \phi}{\partial x^2} + P \phi \right) dx \) can be considered as the virtual internal work done on the beam. \( \phi_2 \) then gives the net virtual work, which must be zero since \( M \) and \( P \) are the forces in equilibrium. This can be explicitly proven as follows.

Applying the Rayleigh-Ritz method for the static displacement analysis gives,

\[
\frac{\partial \phi}{\partial \xi} = 0
\]

i.e.

\[
\frac{\partial}{\partial \xi} \left( \frac{1}{2} \int_{x=0}^{L} \left( EI \left( \frac{\partial^2 \phi}{\partial x^2} \right) \right) dx + \frac{1}{2} \int_{x=0}^{L} EA \left( \frac{\partial^2 \phi}{\partial x^2} \right) dx \right.
\]

\[
+ \frac{1}{2} \left( \frac{\partial \phi}{\partial x} \right)^2 - \frac{1}{4} \left( \frac{\partial \phi}{\partial x} \right)^2 \left( \frac{1}{2} \int_{x=0}^{L} \left( \frac{\partial \phi}{\partial x} \right)^2 dx - P \phi \right) \]

\[
= \int_{x=0}^{L} EI \left( \frac{\partial^2 \phi}{\partial x^2} \right) \left( \frac{\partial^2 \phi}{\partial x^2} \right) dx
\]

\[
+ \int_{x=0}^{L} EA \left( \frac{\partial \phi}{\partial x} \right) \left( \frac{\partial \phi}{\partial x} \right) dx + \int_{x=0}^{L} \frac{\partial u}{\partial x} \left( \frac{\partial \phi}{\partial x} \right) dx = 0
\]

\[
= \int_{x=0}^{L} EI \left( \frac{\partial^2 \phi}{\partial x^2} \right) \left( \frac{\partial^2 \phi}{\partial x^2} \right) dx + \int_{x=0}^{L} EA \left( \frac{\partial \phi}{\partial x} \right) \left( \frac{\partial \phi}{\partial x} \right) dx
\]

\[
+ \frac{1}{2} \left( \frac{\partial \phi}{\partial x} \right)^2 - \frac{1}{2} \left( \frac{\partial \phi}{\partial x} \right)^2 \left( \frac{1}{2} \int_{x=0}^{L} \left( \frac{\partial \phi}{\partial x} \right)^2 dx \right) \]

\[
= \int_{x=0}^{L} \left( M \frac{\partial^2 \phi}{\partial x^2} + P \phi \right) dx
\]

Since \( \phi \) is a valid shape function to be used in a virtual
work concept, equation (F.11) can be considered as a statement of virtual work. If we replace \( \phi_i \) by \( \psi_i \), which is also a compatible displacement shape,

\[
\int_{x=0}^{L} \left( M \frac{\partial^2 \psi_i}{\partial x^2} + P \frac{\partial \psi_i}{\partial x} \frac{\partial z}{\partial x} \right) dx = 0.
\]

Multiplying by \( H_i \) and summing gives

\[
\sum_{i=1,2} \int_{x=0}^{L} \left( M H_i \frac{\partial^2 \psi_i}{\partial x^2} + P H_i \frac{\partial \psi_i}{\partial x} \frac{\partial z}{\partial x} \right) dx = 0.
\]

i.e.

\[
\int_{x=0}^{L} \sum_{i=1,2} \left( M H_i \frac{\partial^2 \psi_i}{\partial x^2} + P H_i \frac{\partial \psi_i}{\partial x} \frac{\partial z}{\partial x} \right) dx = 0.
\]

\[
\int_{x=0}^{L} \left( M \frac{\partial^2 \psi}{\partial x^2} + P \frac{\partial \psi}{\partial x} \frac{\partial z}{\partial x} \right) dx = 0. \tag{F.12}
\]

Equation (F.9) can be transformed as

\[
\hat{U}_2 = \int_{x=0}^{L} \left( M \frac{\partial^2 \psi}{\partial x^2} + P \frac{\partial \psi}{\partial x} \frac{\partial z}{\partial x} \right) dx + \int_{x=0}^{L} P \frac{\partial u}{\partial x} dx - P[u_d]_0^L - P[u_d]_0^L
\]

\[
= \int_{x=0}^{L} \left( M \frac{\partial^2 \psi}{\partial x^2} + P \frac{\partial \psi}{\partial x} \frac{\partial z}{\partial x} \right) dx + P[u_d]_0^L - P[u_d]_0^L
\]

\[
= 0 \quad \text{(using equation (F.12))} \tag{F.13}
\]

Hence, in the Rayleigh-Ritz analysis, it is only necessary to consider the terms associated with the second order dynamic displacement products for small amplitude vibrations. This results in linear minimization equations.
APPENDIX G

LISTING OF THE COMPUTER PROGRAM TO CALCULATE
THE EFFECT OF THE FLEXIBILITY OF THE FRAME.

This program is used to calculate the terms associated
with each deflection coefficient in the integral
\[ \int_{x_0}^{x_1} k_y \left[ v^2_{d, y=0} + v^2_{d, y=b} \right] dx, \]
where \( k_y \) is given by \( k_y = 1/(f_b_1 + f_b_2 - f_a_x) \) as explained in
Appendix H.

In the program \( f_{b_1} \) is denoted as \( S_1 \). \( S_2 \) and \( S_3 \) are the
flexibilities due to the bending of the web of the channel
and the axial straining respectively. The overall stiffness
\( ST \) (\( k_y \) in Appendix H) is given by \( ST = 1/(S_1 + S_2 + S_3) \).

The frame is subdivided into \( N \) elements and the integration
is done numerically using trapezoidal method. JM and KM
represent the mode numbers associated with the deflection
coefficients. AX is the length of the plate and XL is the
length of the vertical channels. XB is the length of channel
near the top and bottom supports where the flexibility due
to the bending of the web of the channel can be ignored.
SK is the required integral. For a given IM, JM the
calculated value of SK can be used in the main program given
in Appendix J, for \( ST IM2(IM, JM) \).

The program listing is given below:
PROGRAM ILAG0 (INPUT, OUTPUT, STIF, FLEX, TAPE5=STIF, TAPE6=FLEX)
READ(5,*) N, NM, RH, AX, XN, X0
PI = 4.*ATAN(1.0)
N = N-1
DO 3 J = 1, JM
  DO 1 X = 1, XN
    SUM = 0.0
    X = X/AX
    DO 10 I = 1, N-1
      TF = 1.0 + (52.1*I*(I.EQ.N))/FAC
      H = X*T**2*(X1-Y)*X62.306
      1 = 277.00*X1*X13+Y*(0.229055-Z)
      1 = IF(X.LT.XB)*Z7*0.0
      ST = 1.0/ST
      1 = IF(X.GE.XB)*ST**1.0/24.0
      ST = (2.*ST**12.0)**1/16.9
      ST = (X1**1+Y**1)**1/8.9
      ST = 1.0/ST
      15 SUM = SUM + FAC*ST*ZIM(X2**J)*SIN(X2**J)
      SK = SUM/(2.0*N)
   10 CONTINUE
   CONTINUE
   CONTINUE
   CONTINUE
END
STOP
END
APPENDIX H

CALCULATIONS FOR THE FLEXIBILITY OF THE TEST RIG

The dynamic displacement of the ball bearings for 'no slippage' condition will be resisted by the testing rig. The rig can not provide a completely rigid support. The flexibility of the rig results from the bending of the channel frame, the bending of the web of the channel and the axial straining of the top and bottom channel beams. These effects can be estimated as follows:

(1) Flexibility of the Support Due to Bending of the Frame

![Figure H.1](image)

Consider the rectangular frame shown in Figure H.1. When a force $F$ is applied at a distance $aL$ from the base of the rig symmetrically on both sides, the displacement $\Delta$ at these points can be shown to be given by,
\[ \Delta = \frac{FL^3}{EI} \alpha^2 (1-\alpha)^2 \left( (1+n^1) c_1^2 + (\alpha-(1-\alpha))^2 c_2^2 (\eta^1 + \frac{1}{3}) \right) \]

\[ - \left( c_1 + \frac{(\alpha-(1-\alpha))^2}{3} c_2 \right) + \frac{1}{3} \]  

(H.1)

where, \( n^1 = n \cdot \frac{I_1}{I_2} \)

\[ c_1 = \frac{k_2}{2(k_1+k_2)} \], \[ c_2 = \frac{k_2}{2(k_2+3k_1)} \]

in which \( k_1 = I_1/L \), \( k_2 = I_2/(nL) \) and \( I_1, I_2 \) are second moments of area about the neutral axes of the frame.

The flexibility due to bending of the frame is then given by,

\[ f_{b_1} = \frac{\Delta}{F} \]  

(H.2)

(2) **Flexibility Due to the Bending of the Web of the Channel**

![Figure H.2](image_url)
The effect of the bending of the web is approximately estimated using an effective span of bending \( w \) where \( w \) is the sum of all the clear space between the various edge beams attached to the channel. The table in Appendix B has been used to calculate the flexibility of the web \( f_{b_1} \) in three zones in the channel, two of which are near the joints and one in the middle, where the web can be taken as a one way spanning plate.

(3) **Flexibility Due to the Axial Straining of the Channel**

At the top and bottom edge of the channel, the only flexibility is due to the axial straining of the top and bottom channels. In order to improve the accuracy of the numerical integration of the effect of the overall flexibility, the axial straining is also considered as follows:

![Diagram](image)

*Figure H.3*
The end forces $R_1$ and $R_2$ are given by

$$R_1 = F(1-\alpha), \quad R_2 = \alpha F$$

$$\Delta_1 = \frac{R_1}{K_{ax}} = \frac{F(1-\alpha)}{K_{ax}}$$

$$\Delta_2 = \frac{\alpha F}{K_{ax}}$$

where $K_{ax}$ is the axial stiffness of the horizontal channel and other attached beams, and is given by

$$K_{ax} = \frac{EA_h}{nL}$$

in which $A_h$ is the cross-sectional area of the horizontal channel and other attached beams. $\Delta = \Delta_1 + \alpha(\Delta_2 - \Delta_1)$. The axial flexibility

$$f_{ax} = \frac{\Delta}{F} = \frac{nL}{EA_h} \left( \alpha^2 + (1-\alpha)^2 \right) \quad (H.3)$$

The effect of all these flexibilities has been included in the evaluation of the frequencies for comparison with the experimental results. That is, the overall boundary stiffness $k_y = \frac{1}{f_{b_1} + f_{b_2} + f_{ax}}$ has been included in the minimization equation. The integral $\int_{x=0}^{a} k_y \left(V^2_{yd} \bigg|_{y=0} + V^2_{yd} \bigg|_{y=b}\right) dx$ has been calculated numerically and added in the main program appropriately in place of equation (3.3.15c).
APPENDIX I

EFFECT OF END MASSES ON THE VIBRATION OF A CURVED BEAM

Consider the vibration of a curved simply supported beam with two masses \( \ddot{m} \) attached at the ends as shown in Figure I.1.

\[
F_e = \ddot{m} \frac{\partial^2 u}{\partial t^2} = -\ddot{m}\omega^2 u_{x=0}
\]

This has the same effect as that of a spring at the end with a stiffness \( k = \ddot{m}\omega^2 \) (see Figure I.2).

\[
F_e = -\ddot{m}\omega^2 u_{x=0} = F_e = k u_{x=0}
\]

\( k = \ddot{m}\omega^2 \)
Hence the end mass can be replaced by an equivalent spring with stiffness \( k = \bar{m} \omega^2 \).

This idea is used to calculate the effect of the mass of the loading head, which is isolated from the loading machine by three layers of rubber. The stiffnesses of some sample pieces of rubber that are similar to the ones that were used in the vibration experiment were found to be small compared to the effect of the mass of the loading head. Since the stiffness depends on the frequency, an estimate for \( \omega \) was used in the calculation of stiffness. It was found that the inaccuracy in the stiffness calculation did not change the natural frequency noticeably in the range of interest.
APPENDIX J

LISTING OF THE COMPUTER PROGRAM FOR
THE POSTBUCKLING AND VIBRATION ANALYSIS

This program can be used to calculate the natural frequencies, displacements, in-plane stresses and strains for symmetrical modes of vibration of symmetrically curved plates. The modes and initial shapes must be symmetrical about both centrelines of the plate.

Input parameters that are required in this program are listed below in the order they appear on the program:

IFREQ - An integer flag number to indicate whether the natural frequency calculations are required or not. Any number other than zero will give frequency calculations. If only static deflections are required zero should be used.

IPR1 - A flag number to indicate whether the connection coefficients are to be printed or not. If these values are not required zero may be used. Any other input will result in printing all of the dynamic connection coefficients.

IPR2 - A flag number to indicate whether the static in-plane stress, strain results are required or not. If these are not required the input should be zero.

E - Young's modulus of the plate material.

PO - Poisson's ratio of the plate material.

Q - Mass density of the plate.
XX - A test parameter used in a preliminary analysis.
This should be set to 1.0.
AX,AY - Dimensions of the plate in x,y directions.
T - Plate thickness.
NST,MST - Number of out-of-plane buckling mode shapes in x,y directions.
N,M - Number of out-of-plane vibration mode shapes in x,y directions.
IUSM - Number of mode shapes for $u_s$ in x direction.
IUXM - Number of shapes for $u_d$ in x direction.
IUYM - Number of shapes for $u_s,u_d$ in y direction.
IVXM - Number of shapes for $v_s,v_d$ in x direction.
IVYM - Number of shapes for $v_s,v_d$ in y direction.
K,L - Integers used in a preliminary analysis. Not used in this program.
NLQ - Number of load cases to be treated.
DZI - A step parameter for Z used in a preliminary analysis. Not used in this program.
Z0 - An initial imperfection parameter used in a preliminary analysis.
NSTF,MSTF - Same as nNST,MST.
NCOMP - A flag number to indicate whether the stress, strain calculations are required or not. If these calculations are not required zero may be used. Any other value will result in the calculation of stress and strain.
ITMAX - Maximum number of iterations in the static analysis.
ZZ)1(1) TO ZZ0(4) - Initial imperfection coefficients $Z_{01,1}, Z_{03,1}, Z_{01,3}, Z_{03,3}$
ZREAD - Initial trial value for $Z_{1,1}$.

FIN1, FIN2, CHEC1 to CHEC6 - Test parameters used in a preliminary analysis.

STIFM - Stiffness due to the vibration of the loading head but should be redefined for each frequency calculation as can be seen later.

STIFM3 - Stiffness parameter to allow for the flexibility of the top and bottom supports. A very high value for this will result in the solution for a plate with rigid supports having absolutely constant movement.

STIFM2(I,J) - Stiffness factor to be calculated using the program in Appendix G.

POINT - Distance between the vertical centreline of the side support and the point of application of load as shown in the approximate model in Figure 5.1.2.

NDP - Number of gauge points.

For each gauge,

NGAGE(I) - Type of strain required: 1 - along x direction,
2 - along y direction, 3 - at 45° to the x-axis, 4 - at -45° to the x-axis.

XCO(I) - x co-ordinate of the gauge.

YCO(I) - y co-ordinate of the gauge.

PX(I) - In-plane load.

For each loading case,

Z2C(1) to Z2C(4) - Initial trial values for $Z_{1,1}$, $Z_{1,3}$, $Z_{3,1}$, $Z_{3,3}$.

IPRI - Defined earlier.

STIFM - Defined earlier.
The important output parameters are as follows:

\( Z_{2C(1)} \) to \( Z_{2C(4)} \) - Out-of-plane displacement coefficients 
\( Z_{1,1} \) to \( Z_{3,3} \).

\textbf{STRAIN} - Strain at a point, the type of which is defined by \( \text{NGAGE}(I) \).

\textbf{ALFR2}(I) - Natural frequency of the curved plate.
PROGRAM LA01 INPUT,OUTPUT,HSYS,ZEPS,TAPE5=HSYS,TAPE6=ZEPS
DIMENSION H(21,11),Z(21,11),S(21,22),IC(2,2),IZM(21,22)
DIMENSION ID(21),IZZ(21),IZZ(21)
DIMENSION C(21),CC(4),CAN(4,4),DELC(4),ZIZO(4),IZC(4),AVXZ(3)
DIMENSION CVXZ(3),CVXZ(3)
DIMENSION BYX1(11),BYX0(11),BYX1(11),BYX1(11),BYX1(11)
DIMENSION U(11,11),V(11,11),W(11,11),X(11,11),Y(11,11)
DIMENSION RUX(11),RUX(11),RUX(11)
DIMENSION FXZ(11,11),FYS(11,11),FYS(11,11)
DIMENSION EPX(11,11),EPY(11,11),EPX(11,11),EPY(11,11)
DIMENSION FXY(11,11)
DIMENSION ALCUX(3),ALGUX(3),ALGUX(3),ALGUX(3),ALGUX(3),D(4),S(4)
DIMENSION ALFRZ(4),ALFRZ(4),ALFRZ(4),ALFRZ(4),ALFRZ(4),DM(4)
DIMENSION XJK(4,4),XJK(4,4),XJK(4,4),XJK(4,4),XJK(4,4),XJK(4,4)
DIMENSION PX(11),SCH(10)
DIMENSION C(21,4),ZB(21,4),SB(21,4),SB(21,4)
DIMENSION BUX(3),BUZ(3),CUZ(3),CUZ(3),BUZ(3),BUZ(3),AUXZ(4)
DIMENSION CVZ(3),CVZ(3),CVZ(3),ALFXZ(3),ALFXZ(3),ALFXZ(3),ALFXZ(3)
DIMENSION STIFMZ(3,3)
DIMENSION SQ(4),FQ(4)
DIMENSION SNX(11,11),SNT(11,11),SNX(11,11)
DIMENSION SRX(11,11),SRX(11,11),SRX(11,11)
DIMENSION EXST1(32),EXST2(32),EXST3(32),RAGE(32),EXST4(32)
DIMENSION SXO(32),SXO(32),SVO(32),SVO(32),SVO(32)
DIMENSION SXV(32),SXV(32),SXV(32),XCO(32),YCO(32)
READ(5,*)IPRM,IPRL,IPR3
READ(5,*)E,PQ,AX,XX
READ(5,*)AX,AY,T
READ(5,*)NST,NSM,H,M,IMSM,IXX,IXY,IVXM,K,IVY,IVYM,K,L,NL0
READ(5,*)IXM1,IZM2,IZM1
READ(5,*)STIF,STIF4
READ(5,*)STIF,STIF4
DO 7777 MVX=1,IVXM
READ(5,*)KSTIF,HSTF,HSTP,ISTMP,IZO(1),IZO(2)
FORMAT(3,2X,13,F3.0,3X,2X,14,7.7)
WRITE(6,*)E,PQ,RO,AX,AY,T
WRITE(6,*)NST,NSM,H,M,IMSM,IXX,IXY,IVXM,K,IVY,IVYM,K,L,NL0
IF(IZM1)WRITE(6,BSZ1,20)
FORMAT(6,8D13.7)
7 FORMAT(12,2X,13)
8 FORMAT(2,2X,14.7)
READ(5,*)FIN,FIM1
WRITE(6,*)FIN,FIM1
READ(5,*)CHRC1,CHRC2,CHRC3,CHRC4,CHRC5,CHRC6
WRITE(6,*)CHRC1,CHRC2,CHRC3,CHRC4,CHRC5,CHRC6
PO2=PO+PO
READ(5,*)POINT
READ(5,*)NCP
DO 7778 ID=1,NDP
READ(5,*)NAGE(ID),XCO(ID),YCO(ID)
READ(5,*)ID1,EX1,EX1
READ(5,*)ID2,EX2,EX2
READ(5,*)ID3,EX3,EX3
7778 CONTINUE
DO 5 ID=1,NL0
SCH(ID)=0.0
READ(5,*)MX(ID)
2C(ID)=20
66    ZC(2, 1) = 0.0
67    ZC(1, 2) = 0.0
68    ZC(2, 2) = 0.0
69    ER5 = (1.0 - PO2)
70    CP = 1.0
71    CALL NATSET(U, 11, 11)
72    CALL NATSET(SNY, 11, 11)
73    CALL NATSET(SRX, 11, 11)
74    CALL NATSET(SNY, 11, 11)
75    CALL NATSET(SRX, 11, 11)
76    CALL NATSET(SRX, 11, 11)
77    CALL NATSET(SRX, 11, 11)
78    CALL NATSET(SRX, 11, 11)
79    CALL NATSET(SRX, 11, 11)
80    CALL NATSET(SRX, 11, 11)
81    CALL NATSET(SRX, 11, 11)
82    CALL NATSET(SRX, 11, 11)
83    CALL NATSET(SRX, 11, 11)
84    CALL NATSET(SRX, 11, 11)
85    CALL NATSET(SRX, 11, 11)
86    CALL NATSET(SRX, 11, 11)
87    CALL NATSET(SRX, 11, 11)
88    CALL NATSET(SRX, 11, 11)
89    CALL NATSET(SRX, 11, 11)
90    CALL NATSET(SRX, 11, 11)
91    PI = (1.0 - PO2) / 2.0
92    D12 = (1.0 - PO2) / 2.0
93    D12 = (1.0 - PO2) / 2.0
94    PI = (1.0 - PO2) / 2.0
95    PI1 = PI / PI
96    PI2 = PI1 * PI
97    PI3 = PI2 * PI
98    IVM = IVM * PI
99    IVM = IVM * PI
100   VM = VM * PI
101   VM = VM * PI
102   VM = VM * PI
103   VM = VM * PI
104   VM = VM * PI
105   VM = VM * PI
106   VM = VM * PI
107   VM = VM * PI
108   VM = VM * PI
109   VM = VM * PI
110   VM = VM * PI
111   VM = VM * PI
112   VM = VM * PI
113   VM = VM * PI
114   VM = VM * PI
115   VM = VM * PI
116   VM = VM * PI
117   VM = VM * PI
118   VM = VM * PI
119   VM = VM * PI
120   10 CONTINUE
121   DO 10 IG = 1, IVM
122     IG = 1
123     IG = 1
124     IG = 1
125     IG = 1
126     IG = 1
127     IG = 1
128     IG = 1
129     IG = 1
130     IG = 1
131     IG = 1
132     IG = 1
ALUY3=ALGYU(IUX)
BUY3=BUY5(IUT)
AUX3=AUX5(IUY)
CUY3=CUY5(IUY)
LU=(IUX+1)*IUX+IUY
II=1
DO 25 IN=1,NSTF
BE=(INT-1)*PI
DO 25 IN=1,NSTF
BE=(INT-1)*PI
DO 25 IN=1,NSTF
GAX=(IX+2-1)*PI
T2=DSSN30(AK,BEX,GAX,0,AUX3,BUX3,CUX3,ALUX3)
T3=DSSN30(AK,GAX,BEX,0,AUX3,BUX3,CUX3,ALUX3)
T6=CSCSR0(AK,BEX,GAX,1,AUX3,BUX3,CUX3,ALUX3)
T7=SNSR30(AK,BEX,GAX,1,AUX3,BUX3,CUX3,ALUX3)
154
IF (IX.EQ.1) ING=IN
DO 25 IL=ING,NSTF
GAT=(IL+2-1)*PI
U1=CSCSR0(AK,BEX,GAX,0,AUX3,BUX3,CUX3,ALUX3)
U4=SNSR30(AK,BEX,GAX,0,AUX3,BUX3,CUX3,ALUX3)
U6=DSSN30(AK,GAX,BEX,1,AUX3,BUX3,CUX3,ALUX3)
U7=DSSN30(AK,BEX,GAX,1,AUX3,BUX3,CUX3,ALUX3)
CMU=0.0
IF (I2M.EQ.1) AND (IM.EQ.IL) CMU=-0.5
UV1=VALUE(AK,0.0,ALUX3,AUX3,BUX3,CUX3,0)
UV2=VALUE(AK,AX,ALUX3,AUX3,BUX3,CUX3,0)
S1=TS*U4*BE*XGAX/(AX*AX)
S2=PO*TS*U1*XGAY*AY/AY
S3=DI*UT*TS*BE*XGAY/(AX*AY)
S4=DI*UT*T3*XGAX/(AX*AY)

UUU1=VALUE(AY,POINT,ALUY3,AUX3,BUX3,CUX3,0)
IF (II.EQ.1) ID(IU,1)=UUU1/(UUU1-UV2)/(2.568*DI)
II=II+1
Z2=ID(IU,1)
ZD(IU,II)=CMU*(S1+S2+S3+S4)
IF (II.EQ.2) IDD(IU,1)=ID(IU,2)

25 CONTINUE

DO 15 JUX=1,1,IUSM
ALUX4=ALGYU(JUX)
BUY4=BUY5(JUX)
CUX4=CUX5(JUX)
TU1=PRAM(AK,AUX3,BUX3,CUX3,ALUX3,AUX4,BUX4,CUX4,ALUX4,1,1)
TU2=PRAM(AK,AUX3,BUX3,CUX3,ALUX3,AUX4,BUX4,CUX4,ALUX4,0,0)
JUX=1

10 CONTINUE
AUX4=AUX5(JUX)
BUY4=BUY5(JUX)
CUX4=CUX5(JUX)
ALDY4=ALGUY(JUY)
JU=(JUX-1) * IUYM+JUY
SU=PRIN(AY,AUX3,BUY3,CUY3,ALDY3,AY4,BUY4,CUY4,ALDY4,1,1)
SU2=PRIN(AY,AUX3,BUY3,CUY3,ALDY3,AY4,BUY4,CUY4,ALDY4,0,0)
JY=YU+1
SU3=PRIN(AY,AUX3,BUY3,CUY3,ALDY3,AY4,BUY4,CUY4,ALDY4,2,1)
SU4=PRIN(AY,AUX3,BUY3,CUY3,ALDY3,AY4,BUY4,CUY4,ALDY4,1,0)
S2(IU,JV)=SU1+SU2+SU3+SU4/(T+T)
S1(IU,JV)=S2(IU,JV)
IF(JUY.LE.IUYM) GOTO 30
30 CONTINUE
DO 40 JUX=1,IVYM
ALVX4=ALGUV(JUX)
AVX4=AVX5(JUX)
BVX4=BVX5(JUX)
CVX4=CVX5(JUX)
TU3=PRIM(AX,AUX2,BUX2,CUX2,ALUX3,AVX4,BVX4,CVX4,ALVX4,0,1)
TV4=PRIM(A2,AUX2,BUX2,CUX2,ALUX3,AVX4,BVX4,CVX4,ALVX4,1,0)
40 CONTINUE
DO 40 JUX=1,IVYM
ALVX4=ALGUV(JUX)
BVY4=BVY5(JUX)
CVY4=CVY5(JUX)
JUX+JUX=JUX+1
S3(IU,JV)=SU1+SU2+SU4/((T-T))
S2(IU,JV)=S3(IU,JV)
IF(JUY.LE.IUYM) GOTO 45
45 CONTINUE
DO 50 IVX=1,IVYM
ALVX3=ALGUV(IVX)
AVX3=AVX5(IVX)
BVX3=BVX5(IVX)
CVX3=CVX5(IVX)
50 CONTINUE
DO 50 IVY=1,IVYM
ALVY3=ALGUV(IVY)
BVY3=BVY5(IVY)
CVY3=CVY5(IVY)
IVY=IVY+1
50 CONTINUE
II=1
DO 55 IM=1,NSTF
BEX=IM+2-1*PI
55 CONTINUE
DO 55 IM=1,NSTF
BET=IM+2-1*PI
55 CONTINUE
DO 55 IM=1,NSTF
GAZ=(IM+2-1)*PI
55 CONTINUE
TI=CSCSNO(AX,BEX,GAX,0,AVX3,BVX3,CVX3,ALVX3)
T4=MSSHRO(AX,BEX,GAX,0,AVX3,BVX3,CVX3,ALVX3)
T8=OBSPRO(AX,BEX,GAX,1,AVX3,BVX3,CVX3,ALVX3)
T9=OBSPR0(AX,GAX,Avx,1,Avx3,Bvx3,Cvx3,Alvx3)
INO=1
IF (IX EQ IN) INC IN
DO 55 IL=IN,XXTF
GAY=(IL**2-1)**P1
U2=OSSNRO (AY, BEY, GAY, S, AVY3, BVY3, CVY3, ALVY3)
U3=OSSNRO (AY, GAY, BEY, S, AVY3, BVY3, CVY3, ALVY3)
U4=CSSNRO (AY, BEY, GAY, S, AVY3, BVY3, CVY3, ALVY3)
U5=CSSNRO (AY, BEY, GAY, S, AVY3, BVY3, CVY3, ALVY3)
CMU=1.0
IF ((IN EQ IX).AND. (IN EQ IL)) CMU=-0.5
S1=U8*T4*BEY*GAY/(AY*AY)
S2=PO*U9*T1*BEY*GAY/(AX*AX)
S3=D1*T9*U3*GAY*BEY/(AX*AY)
S4=D1*T9*U2*GAY*BEY/(AX*AY)
IF ((III, EQ, 1) .GT. (IV, IV) ) .LT. 0.0.
ZIV (IV, IV) = 0.0.
III=III+1
ZIV (IV, III) =CMU*(S1+S2+S3+S4)
ZIV (IV, IV) =ZIV (IV, IV) +
III=III+1
CONTINUE

DO 65 JUV=1, IUVX
ALUX4=ALUX (JUX)
BUXX=BUXX (JUX)
CUXX=CUXX (JUX)
TV3=PRIN (AX, AVX3, BVX3, CVX3, ALVX3, AUVX4, BUXX, CUXX, ALUX4, 0, 1)
TV4=PRIN (AX, AVX4, BVX4, CVX4, ALVX4, AUVX4, BUXX, CUXX, ALUX4, 0, 1)
JUV=JUV+1
CONTINUE
ALUX4=ALUX (JUX)
BUXX=BUXX (JUX)
CUXX=CUXX (JUX)
JUV=JUV+1
CONTINUE
SV3=PRIN (AY, AVY3, BVY3, CVY3, ALVY3, AVY4, BUXX, CUXX, ALUX4, 0, 1)
SV4=PRIN (AY, AVY4, BVY4, CVY4, ALVY4, AVY4, BUXX, CUXX, ALUX4, 0, 1)
JUV=JUV+1
SV4 = (PO + TV3 + SV4 * D1 + TV4 * SV3)/(T*T7)
ZIV (IV, JUV) =ZIV (IV, JUV) +
JUV=JUV+1
CONTINUE

ALVY4=ALVY (JUV)
BUYY=BUYY (JUV)
AVY4=AVY4 (JUV)
CUYY=CUYY (JUV)
JUV=JUV+1
CONTINUE
SVY=PRIN (AY, AVY3, BVY3, CVY3, ALVY3, AVY4, BUYY, CUYY, ALVY4, 0, 1)
SVY=PRIN (AY, AVY4, BVY4, CVY4, ALVY4, AVY4, BUYY, CUYY, ALVY4, 0, 1)
JUV=JUV+1
CONTINUE
SVY=PRIN (AY, AVY3, BVY3, CVY3, ALVY3, AVY4, BUYY, CUYY, ALVY4, 0, 1)
JUV=JUV+1
CONTINUE
SVY=PRIN (AY, AVY3, BVY3, CVY3, ALVY3, AVY4, BUYY, CUYY, ALVY4, 0, 1)
CONTINUE
SVY=PRIN (AY, AVY3, BVY3, CVY3, ALVY3, AVY4, BUYY, CUYY, ALVY4, 0, 1)
CONTINUE
UQ1=VALUE (AY, 0.2, ALVY3, AVY3, BVY3, CVY3, 0)
WRITE (6,72)
92 FORMAT ('/) /2X,"CONNECTION STIFFNESS MATRIX"/,/)
DO 74 IQ=1,NNST
WRITE (6,73) (S2(IQ,JQ),JQ=1,NNST)
73 FORMAT (2X,E14.7)
74 CONTINUE
CALL GAUSEL(NNST,IM,SZ,ZD,H)
DO 4015 I=1,NNST
SUMO=0.0
SUM1=0.0
DO 4010 J=1,NNST
SUMO=SUMO+SZ(I,J)*NJ(J,2)
4010 SUM0=SUM0+SZ(J,I)*NJ(I,2)
IZE(I)=IZE(I)-SUMO
IZE(J)=IZE(J)-SUMO
WRITE (6,4013) (IZE(I),JZD(I),IM01,TH(I,J),S2D(I),VZ(I,2))
4013 FORMAT (2X,13,6(2X,E14.7))
4015 CONTINUE
WRITE (6,75)
75 FORMAT (//,2X,"IN PLANE MODE INFLUENCE COEFFICIENTS"/,/)
DO 82 IQ=1,NNST
WRITE (6,80) IQ,ATD(IQ),AM(IQ),Q(II),Q(IQ,II)
80 FORMAT (2X,E14.7)
82 CONTINUE
DO 3995 ILO=2,NLO
READ (5,12) ZIC(1),ZIC(2),ZIC(3),ZIC(4),IPRM,STIFM
DO 7779 IM=1,NDP
EXY(IN)=0.0
EXST(IN)=0.0
EXST2(IN)=0.0
EXST3(IN)=0.0
7779 CONTINUE
WRITE (6,8024) ZIC(1),ZIC(2),ZIC(3),ZIC(4),Q(KLO)
8024 FORMAT (2X,E14.7)
DO 8025 IG=1,NNST
C(IG)=H(IG,1)*Q(ILQ)
8025 CONTINUE
DO 8030 IH=1,NNSTF
H(IH)=H(IH,1)
8030 CONTINUE
DO 8040 IF=1,NSTF
C(IF)=H(IF,1)*Q(IF)
8040 CONTINUE
DO 3110 IMAX=1,ITMAX
DO 3100 IN=1,NSTF
A[(2-1)*PI]
3100 CONTINUE
DO 3120 IN=1,NSTF
ALR[(2-1)*PI]
3120 CONTINUE
DO 388 DFAC=4.0
388 CONTINUE
3110 CONTINUE
DO 390 UMM=0.0
390 CONTINUE
391 UMM=0.0
392 UMM=0.0
393 UMM=0.0
394 UMM=0.0
395 CC(1CO)=0.0
396
397  RR4=0.0
398  RS6=0.0
399  DO 360 ZK=1.0
400  DO 360 ZK=1.0
401  DO 3600 1=1,NSTP
402  BEX=1.0*RS6
403  RCL=BEX*AY/(AX*AY)
404  DO 3600 IS=1,NSTP
405  BEX=1.0*RS6
406  J=(IR-IS)*NSTP+1
407  DSCF=1.0
408  JCO1=J
409  RC2=RS6*ALY/(AY*AY)
410  RC3=0.5*(1.0-PO)*BEX*ALY/(AX*AY)
411  RC4=0.5*(1.0-PO)*BEX*ALY/(AX*AY)
412  RC5=0.0
413  RS6=0.0
414  ZUM=0.0
415  ZUM=0.0
416  DO 3665 JUX=1.0
417  ALUX3=ALUX(JUX)
418  BUX3=BUX3(JUX)
419  CUX3=CUX3(JUX)
420  JUX=1
421  CONTINUE
422  ALUX3=ALUX(JUX)
423  BUX3=BUX3(JUX)
424  CUX3=CUX3(JUX)
425  JUX=1
426  CONTINUE
427  CUX5=CUX5(JUX)
428  JUX5=1
429  CONTINUE
430  JUX=1
431  CONTINUE
432  \#III=1
433  DO 1063 IP=1,NSTF
434  DO 1063 IQ=1,NSTF
435  IF(IP=IQ) 1060=PO
436  DO 1063 IP=1,NSTF
437  CONTINUE
438  IP=IQ*(IP-1)*NSTF
439  DO 1063 IP=IQ,NSTF
440  IQ=1
441  CONTINUE
442  IF[(IK.EQ.IPL)] IP=IQ
443  DO 1063 IP=IQ,NSTF
444  CONTINUE
445  IP=IQ*(IP-1)*NSTF
446  CONTINUE
447  \#III=1
448  DIFC=1.0
449  DIFC=DIFC*(JUX,JUX)
450  RRR1=(CUX1*PO*CUX1+CUX1+CUX1)*TTT*ZIC(JCO1)
451  CAM=(IK1,IP)<CAM=(IK1,IP)+RRR1*ZIC(IP)
452  CAM=(IK1,IK1)<CAM=(IK1,IK1)+RRR1*ZIC(IP)
453  3063 CONTINUE
454  JUX=JUX+1
455  IF(JUX.LE.IUXM) GOTO 1071
456  3065 CONTINUE
457  CONTINUE
458  CAM=(IK1,IK1)<RRR1=CAM=(IK1,IK1)
459  ZUM=ZUM+RRR1*ZIC(JCO1)
460  DO 3668 JUX=1
461  DO 3668 JUX=1
462  \#III=1
GX=KCO(ID)
SX0(ID)=VALUE(AX,GX,ALUX3,AUX3,BUX3,CUX3,0)
SX1(ID)=VALUE(AX,GX,ALUX3,AUX3,BUX3,CUX3,1)
3297 CONTINUE
IUY+=1
9501 CONTINUE
ALUY3=ALGY(IUY)
AUY3=AUY5(IUY)
BUY3=BUY5(IUY)
CUX3=CUX5(IUY)
ID=(IUX-1)*IUYM+IUY
DO 3300 IT=1,11
3301 Y=(IT-1)*AY/10.0
SUY0(IY)=VALUE(AY,Y,ALUY3,AUX3,BUY3,CUY3,0)
SUY1(IY)=VALUE(AY,Y,ALUY3,AUX3,BUY3,CUY3,1)
3300 CONTINUE
DO 3302 ID=1,NDP
3303 T=UCO(ID)
3304 SUY0(ID)=VALUE(AY,Y,ALUY3,AUX3,BUY3,CUY3,0)
3305 SUY1(ID)=VALUE(AY,Y,ALUY3,AUX3,BUY3,CUY3,1)
3302 CONTINUE
DO 3303 IX=1,11
3304 DO 3303 IY=1,11
3305 SNX(IX,IY)=SNX(IX,IY)-C(IU)*RUX1(IX)*BUY0(IY)
3306 SNX(IX,IY)=SNX(IX,IY)-C(IU)*RUX0(IX)*BUY1(IY)
3307 CONTINUE
DO 3304 ID=1,NDP
3308 EXST1(ID)=EXST1(ID)-C(IU)*SUX1(ID)*BUY0(ID)
3309 EXY(ID)=EXY(ID)-C(IU)*SUX0(ID)*BUY1(ID)
3308 CONTINUE
IUY=IUY+1
IF(IUY,100,J0M) GOTO 3301
3305 CONTINUE
DO 3322 IVX=1,IVXM
ALVX3=ALGVX(IVX)
AVX3=AVX5(IVX)
BVX3=BVX5(IVX)
CVX3=CVX5(IVX)
3323 DO 3323 IX=1,11
3324 GX=(IX-1)*AX/10.0
3325 RVX0(IX)=VALUE(AX,GX,ALVX3,AVX3,BVX3,CVX3,0)
3326 RVX1(IX)=VALUE(AX,GX,ALVX3,AVX3,BVX3,CVX3,1)
3323 CONTINUE
DO 3312 ID=1,NDP
GX=KCO(ID)
SVX0(ID)=VALUE(AX,GX,ALVX3,AVX3,BVX3,CVX3,0)
SVX1(ID)=VALUE(AX,GX,ALVX3,AVX3,BVX3,CVX3,1)
3312 CONTINUE
DO 3322 IVY=1,IVYM
IVY=10M+IVY-1*IVYM+IVY
ALVY3=ALGVY(IVY)
BVY3=BVY5(IVY)
CVY3=CVY5(IVY)
3323 DO 3323 IX=1,11
3324 Y=(IX-1)*AY/10.0
3325 RVY0(IX)=VALUE(AY,Y,ALVY3,AVY3,BVY3,CVY3,0)
3326 RVY1(IX)=VALUE(AY,Y,ALVY3,AVY3,BVY3,CVY3,1)
3323 CONTINUE
DO 3316 ID=1,NDP
Y=UCO(ID)
SVY0(ID)=VALUE(AY,Y,ALVY3,AVY3,BVY3,CVY3,0)
SVY1(ID)=VALUE(AY,Y,ALVY3,AVY3,BVY3,CVY3,1)
DO 3320 IX=1,11
DO 3320 IY=1,11
SNXY(IY,IX)*SNXY(IY,IX)+C(IY)*RVX0(IY)*RVY0(IY)
DO 3320 IY=1,11
SNXY(IY,IX)*SNXY(IY,IX)+C(IY)*RVX1(IY)*RVY0(IY)
DO 3320 IX=1,11
DO 3322 ID=1,NDF
EXST1(ID)=EXST1(ID)+C(IY)*RVX0(ID)*SVY1(ID)
EXY(ID)=EXY(ID)+C(IY)*SVX1(ID)*SVY0(ID)
DO 3322 CONTINUE
DO 3340 IX=1,11
GX=(IX-1)/10.0
DO 3340 IY=1,11
Y=(IY-1)/10.0
DO 3340 IX=1,11
ALX=PI*(IX-1)
RX=GX+ALX
DO 3340 IY=1,11
ALY=PI*(INY-1)
IM=IN-(IN-1)*NSTF
RTW=TX+ALY
DO 3340 IX=1,11
BEK=IN-1+PI
SXX=BEX+GX
DO 3340 IY=1,11
BEY=(IY-1)*PI
SYY=BEY+Y
IR=IS+1*(IR-1)*NSTF
IZFU=(IZC(IMN)+IZC(IRS)-IZO(IMN)*IZO(IRS))*TT
VAL1=ALX*BEX*COS(RX)+COS(SXX)*SIN(RY)+SIN(SYY)/(AX*AY)
VAL2=ALY*BEY*COS(RW)+COS(SXY)*SIN(RX)+SIN(SYW)/(AY*AY)
VAL3=ALEY*BEY*COS(RW)+COS(SXY)*SIN(RX)+SIN(SYW)/(AX*AY)
VAL4=ALY*BEY*COS(RW)+COS(SXY)*SIN(RX)+SIN(SYW)/(AX*AY)
SNXX(IY,IX)=SNXX(IY,IX)+0.5*IZFU*VAL2
SNXY(IY,IX)=SNXY(IY,IX)+0.5*IZFU*VAL2
DO 3340 IX=1,11
DO 3340 IY=1,11
DO 3342 ID=1,NDF
GX=KCO(ID)
TX=KCO(ID)
DO 3342 IX=1,11
ALX=PI*(IX-1)*NSTF
RX=ALX*GX
DO 3342 IY=1,11
ALY=PI*(INY-1)*NSTF
RTW=TX+ALY
DO 3342 IX=1,11
BEK=IN-1+PI
SXX=BEX+GX
DO 3342 IY=1,11
BEY=(IY-1)*PI
SYY=BEY+Y
IR=IS+1*(IR-1)*NSTF
IZFU=(IZC(IMN)+IZC(IRS)-IZO(IMN)*IZO(IRS))*TT
VAL1=ALX*BEX*COS(RX)+COS(SXX)*SIN(RY)+SIN(SYY)/(AX*AY)
VAL2=ALY*BEY*COS(RW)+COS(SXY)*SIN(RX)+SIN(SYW)/(AY*AY)
VAL3=ALEY*BEY*COS(RW)+COS(SXY)*SIN(RX)+SIN(SYW)/(AX*AY)
VAL4=ALY*BEY*COS(RW)+COS(SXY)*SIN(RX)+SIN(SYW)/(AX*AY)
SNXX(IY,IX)=SNXX(IY,IX)+0.5*IZFU*VAL2
SNXY(IY,IX)=SNXY(IY,IX)+0.5*IZFU*VAL2
DO 3342 CONTINUE
DO 3345 IX=1,11
DO 3345 IY=1,11
SRX(IX, IY) = (SNX(IX, IY) + PO*SNX(IX, IY)) * E / (1.0 - PO + PO)
SRY(IX, IY) = (SNY(IX, IY) + PO*SNX(IX, IY)) * E / (1.0 - PO + PO)
SRX(IX, IY) = SNX(IX, IY) * E / (2.0 * (1.0 + PO))
IF(IPR. EQ. 0) GOTO 1192
CALL MATPR(SNX, 11, 11)
CALL MATPR(SNY, 11, 11)
CALL MATPR(SRX, 11, 11)
CALL MATPR(SRY, 11, 11)
CALL MATPR(SRX, 11, 11)
1192 CONTINUE
IF(IPREM.EQ.0)GOTO 371
IUN=IUXM*IUYM
IVM=IVXM*IYVM
NN=N*M
NN=IUN+IVM
AUX1=0.0
AVX1=0.0
AUX2=0.0
AVX2=0.0
AUX1=0.0
AVX1=0.0
AUX2=0.0
AVX2=0.0
CALL SETUP(IUXM, BUAY, CX, ALFX, 0.0, 0.0, 1.0, 0.0, 0.0, 0.0, 1.0, 0.0, 1.2)
CALL SETUP(IYVM, BAYU, CY, ALFY, 0.0, 0.0, 0.0, 0.0, 1.0, 0.0, 0.0, 0.0, 0.0, 2.2)
CALL SETUP(IUXM, BUAY, CX, ALFX, 0.0, 0.0, 1.0, 0.0, 0.0, 0.0, 0.0, 0.0, 1.2)
CALL SETUP(IYVM, BAYU, CY, ALFY, 0.0, 0.0, 0.0, 0.0, 1.0, 0.0, 0.0, 0.0, 0.0, 2.2)
DO 155 I=1, IYVM
AUX(I)=0.0
155 CONTINUE
AUX(I)=1.0
DO 1175 IUX+1, IUXM
AUX(I)=AU1(IUX)
BUX=BUAY(IUX)
CUX=CUX(IUX)
IUX+1
1175 CONTINUE
ALFY1=ALFY(IY)
AUX1=AUX(IY)
BAYU=BUAY(IY)
CUXY=CUX(IY)
IY=IY+1
DO 170 IM=1, M
BEX=(IM*2-1)*PI
DO 170 IM=1, M
BEX=(IM*2-1)*PI
II=(IM-1)*M+1
S1=0.0
S2=0.0
S3=0.0
S4=0.0
DO 160 IM=1, MSTF
GAX=(IM*2-1)*PI
160 CONTINUE
T2=DSSNR0(AX, BEX, GAX, 0, AUX1, BUAY, CUXY, ALFY1)
T3=DSSNR0(AX, GAX, BEX, 0, AUX1, BUAY, CUXY, ALFY1)
T4=CCSNO0(AX, BEX, GAX, 1, AUX1, BUAY, CUXY, ALFY1)
T5
T7=SNSMRO(AX,BEX,GAX,1,AUX1,BUX1,CUX1,ALUX1)
GAY=IEX=I*1)*PI
J2=(IX=1)*L+IL

U1=CSSMRO(AY,REY,GAY,0,AUY1,BUY1,CUY1,ALUY1)
U4=CSSMRO(AY,REY,GAY,0,AUY1,BUY1,CUY1,ALUY1)
U6=CSSMRO(AY,GAY,REY,1,AUY1,BUY1,CUY1,ALUY1)
U7=CSSMRO(AY,REY,GAY,1,AUY1,BUY1,CUY1,ALUY1)

S1=S1-ZIC(J2)*T6+U4*BEY*GAX/AX*AY)
S2=S2-ZIC(J2)*PO+T*U1*GAY*BEY/(AY*AY)
S3=S3-ZIC(J2)*D1+U6*TT*BEY/GAY/(AX*AY)
S4=S4-ZIC(J2)*D1*U7+TT*BEY*GAX/(AX*AY)

160 CONTINUE
170 CONTINUE

DO 173 JUX=1,LIUXM
ALUX2=ALFUX(JUX)
BUX2=BUX(JUX)
CUX2=CUX(JUX)

UV2=VALUE(AY,0.0,ALUX1,AUX1,BUX1,CUX1,0)
UY4=VALUE(AY,0.0,ALUX2,AUX2,BUX2,CUX2,0)
TU1=PRIN(AY,AUX1,BUX1,CUX1,ALUX1,AUX2,BUX2,CUX2)
TU2=PRIN(AY,AUX1,BUX1,CUX1,ALUX1,AUX2,BUX2,CUX2)
JUX=1

CONTINUE

ALUX2=ALFUX(JUX)
BUY2=BUY(JUX)
AUX2=AUX(JUX)
CUX2=CUX(JUX)
JU=(JUX-1)+IUYM+JUX

SU1=PRIN(AY,AUY1,BUY1,CUY1,ALUY1,AUY2,BUY2,CUY2,ALUY2,1,1)
SU2=PRIN(AY,AUY1,BUY1,CUY1,ALUY1,AUY2,BUY2,CUY2,ALUY2,0,0)

UV2=VALUE(AY,POINT,ALUY1,AUY1,BUY1,CUY1,0)
UY2=VALUE(AY,POINT,ALUY2,AUY2,BUY2,CUY2,0)

TU2=PRIN(AY,BUY1,CUY1,ALUY1,AUY2,BUY2,CUY2,ALUY2,2,2)

TU2=TPU+VALUE(AY,0.0,ALUX1,AUX1,BUX1,CUX1,0)

S2=S2+SU1+SU2+SU3+SU1+IUSM+UV)+UY2+U1+TPU

CONTINUE

DO 175 JUX=1,LIUXM
ALUX2=ALFUX(JUX)
BUY2=BUX(JUX)
CUX2=CUX(JUX)

TU3=PRIN(AY,AUX1,BUX1,CUX1,ALUX1,AUX2,BUX2,CUX2,ALUX2,0,1)
TU4=PRIN(AY,AUX1,BUX1,CUX1,ALUX1,AUX2,BUX2,CUX2,ALUX2,1,0)

CONTINUE

DO 175 JUX=1,LIUXM
ALUX2=ALFUX(JUX)
BUY2=BUX(JUX)
CUX2=CUX(JUX)

CONTINUE
JV=IUM+IVM+(IVX-1)*IVM
SU3=PRIN(AY,AYUL,BUTL,CYUL1,ALUY1,AYVT2,BVV2,CVY2,ALVT2,0.1)
SU4=PRIN(AY,AYUL,BUTL,CYUL1,ALUY1,AYVT2,BVV2,CVY2,ALVT2,1.0)
175 SX1(U,JV)=PO+TU4*SU3+D1*TU3*SU4
IVT=IVT+1
IF(IVT.LE.IUVM) GOTO 156
175 CONTINUE
867 DO 185 IVX=1,IVM
868 ALVX1=ALVX(IVX)
870 BVX1=BVX(IVX)
871 CVX1=CVX(IVX)
872
873 DO 185 IVY=1,IVYM
874 ALVY1=ALVY(IVY)
875 BVY1=BVY(IVY)
876 CVY1=CVY(IVY)
877 IV=IV+1+IV(IVX-1)*IVM
878
879 DO 180 IN=1,N
880 BEX+(IN*2-1)*PI
881
882 DO 180 IN=1,N
883 BET+(IN*2-1)*PI
884 IN=(IN-1)*N+IN
885
886 S1=0.0
887 S2=0.0
888 S3=0.0
889 S4=0.0
890
891 DO 179 IX=1,MSTT
892 GAX+(IX*2-1)*PI
893 TI=CSCSM(AX,BEX,GAX,0,AVX1,BVX1,CVX1,ALVX1)
894 T4=SNNSRO(AX,BEX,GAX,0,AVX1,BVX1,CVX1,ALVX1)
895 T5=SNNSRO(AX,BEX,GAX,1,AVX1,BVX1,CVX1,ALVX1)
896 T9=SNNSRO(AX,GAX,BEX,1,AVX1,BVX1,CVX1,ALVX1)
897
898
899 DO 179 IX=1,MSTT
900 GAY+(IX*2-1)*PI
901 J4=(IX-1)*L+IL
902
903 U9=SNNSRO(AX,BEY,GAY,0,AVY1,BVY1,CVY1,ALVY1)
904 U10=SNNSRO(AX,BEY,GAY,0,AVY1,BVY1,CVY1,ALVY1)
905 U9=SNNSRO(AX,BEY,GAY,1,AVY1,BVY1,CVY1,ALVY1)
906 U9=SNNSRO(AX,BEY,GAY,1,AVY1,BVY1,CVY1,ALVY1)
907
908 S1=S1-ZIC(JJ)+US*T4*BEY*GAY/A(AY)AY)
909 S2=S2-ZIC(JJ)+PO*US*TI*BEY*GAX/A(AX)AX)
910 S3=S3-ZIC(JJ)+DI*T4*US*BEY*GAY/A(AY)AY)
911 S4=S4-ZIC(JJ)+DI*T4*US*BEY*GAY/A(AY)AY)
912 179 CONTINUE
913 IB=IV,JJ)=S1+S2+S3+S4
915 180 CONTINUE
917 DO 183 JUX=1,IUXM
919 ALUX2=ALFUX(JUX)
920 BUX2=BUX(JUX)
921 CUX2=CUX(JUX)
922 TV4=PRIN(AX,AVX1,BVX1,CVX1,ALVX1,AUX2,BUX2,CUX2,ALUX2,0.1)
924 TV4=PRIN(AX,AVX1,BVX1,CVX1,ALVX1,AUX2,BUX2,CUX2,ALUX2,1.0)
DO 210 IN=1,N
   M = (IN-1) * N + IN
   IF (M .LE. JX) XM(JX) = MO * AX * AY / 4.0
   IF (M .GT. JX) XM(JX) = 0.0
   IF (M .GT. JY) XM(JY) = 0.0
   IF (M .GE. JZ) XM(JZ) = 0.0
   RR1 = 0.0
   RR2 = 0.0
   RS1 = 0.0
   RS2 = 0.0
   RS3 = 0.0
   RM = 0.0
   RST = 0.0
   RSV = 0.0
   RR3 = 0.0
   DO 210 IR=1,NSTF
      GAX = (IX * 2 - 1) * PI
      RC1 = GAX * ALX / (AX * AX)
   210    CONTINUE
   DO 220 IUX = 1, IUXM
      ALUX = ALFU(IUX)
      DO 220 IUX = 1, IUXM
      BUX = BU(X(IUX))
      CUX = CUX(IUX)
      IUX = 1
      CONTINUE
   220    CONTINUE
   DO 220 IYU = 1, IYUM
      ALUY = ALFUY(IYU)
      AUX1 = AUXY(IYU)
      CUY1 = CUY(IYU)
      IU = (IYU - 1) * IYUM + IYU
      CCM = CCMIU(IU, J) * CZCSRO(AX, GAY, ALY, 0, AUX1, BU1, CUX1, ALUX1)
      CCM = CCMIU(IU, J) * CSNSRO(AX, GAY, ALY, 0, AUX1, BU1, CUX1, ALUX1)
      CCM = CCMIU(IU, J) * CSNSRO(AX, GAY, ALY, 0, AUX1, BU1, CUX1, ALUX1)
      CCM = CCMIU(IU, J) * CSNSRO(AX, GAY, ALY, 0, AUX1, BU1, CUX1, ALUX1)
      R2R = RR2 * PO * R2C * CCM
      CCH = CCHIU(IU, J) * CSHSNRO(AX, GAY, ALY, 0, AUX1, BU1, CUX1, ALUX1)
      CCM = CCMIU(IU, J) * CSNSRO(AX, GAY, ALY, 0, AUX1, BU1, CUX1, ALUX1)
      CCM = CCMIU(IU, J) * CSNSRO(AX, GAY, ALY, 0, AUX1, BU1, CUX1, ALUX1)
      RR3 = RR3 * RC2 * CCM
      IF (IYU .LT. IYUM) GOTO 199
      199
      CONTINUE
      200
      DO 205 IVX = 1, IVXM
      ALVX = ALFVX(IVX)
      BVX1 = Bvx(IVX)
      CVX1 = CVX(IVX)
      DO 205 IVY = 1, IVYM
      CONTINUE
255

1057    ALVY1=ALFVY(IVY)
1058    BVY1=BVY(IVY)
1059    CVY1=CVVY(IVY)
1060    IV=IVM+(IVV-1)*IVY
1061    CCL=((IV,J)*ZJC(JJ)*SNSRO(AX,GAX,ALX,0,AUX3,BVX1,CVX1,ALVX1)
1062    CCL=CCCL*CSSRO(AX,GAX,ALX,0,AUX3,BVX1,CVX1,ALVX1)
1063    ACCC=CCCP*SNSRO(AY,GAY,ALY,1,AVY1,BVY1,CVY1,ALVY1)
1064    RS1*RS2+PO=MCC*CPC
1065    RS2+RS2=MCC+CC
1067    CCL=((IV,J)*ZJC(JJ)*SNSRO(AX,GAX,ALX,1,AUX3,BVX1,CVX1,ALVX1)
1068    CCL=CCL*CCL*CSSRO(AY,ALY,0,AUX3,BVY1,CVY1,ALVY1)
1069    CCL=((IV,J)*ZJC(JJ)*SNSRO(AX,GAX,1,AUX3,BVX1,CVX1,ALVX1)
1070    CCL=CCL*CSSRO(AY,ALY,0,AUX3,BVY1,CVY1,ALVY1)
1071    RS3*RS3=CSSC*RC3=CSSC*RC

205 CONTINUE

1073    ARAT=AX*AY*AY
1074    DO 212 IP=1,1001
1076    PHIX=(IP-1)*PI
1077    DO 210 IQ=1,1001
1078    PHIY=(IQ-1)*PI
1079    IPO=(IP-1)*IQ
1080    RM1=GAX*BE+ALX*PHIX/(AX**4.0)
1081    RM1=RM1+CCCTI(AX,GAY,ALX,PHIX,ALX)
1082    RM1=RM1+SSSIN(AY,GAY,PHY,BEY,ALY)
1083    RM2=GAY*PHIX*ALX*BEY/(AY**4.0)
1084    RM2=RM2+SSSIN(AX,GAY,PHY,BEY,ALX)
1085    RM2=RM2+CCCTI(AY,GAY,PHY,BEY,ALX)
1086    RM3=GAX*PHIX*ALY*BEY(AY,ARAT
1087    RM3=RM3+CCCTI(AX,GAY,ALY,PHIX,AL)
1088    RM3=RM3+SSSIN(AY,GAY,PHIY,AY,ALY)
1089    RM4=GAX*PHIX*ALY*BEY(AY,ARAT
1090    RM4=RM4+CCCTI(AX,GAY,ALY,PHIX,AL)
1091    RM4=RM4+SSSIN(AY,GAY,PHIY,AY,ALY)
1092    RM5=GAY*PHIX*ALX*BEY(AX,ARAT
1093    RM5=RM5+CCCTI(AX,GAY,ALX,PHIX,AL)
1094    RM5=RM5+SSSIN(AY,GAY,PHIY,AY,ALY)
1095    RM6=GAY*PHIX*ALX*BEY(AX,ARAT
1096    RM6=RM6+CCCTI(AX,GAY,ALX,PHIX,AL)
1097    RM6=RM6+SSSIN(AX,GAY,PHIY,AY,ALY)
1098    RM=RM+ZJC(JJ)*ZJC(IP)*RM1+RM2+RM3+RM4+RM5+RM6
1099    FQ1=0.5*FIM*ZJC(JJ)*ZJC(IP)*ZJC(JJ)*ZJC(IP).
1100    RM=RMS*RM*FQ1*(RM1+RM2+RM3+RM4+RM5+RM6)

210 CONTINUE

1102    DO 212 KUX=1,1001
1103    ALUX3=ALGXK(KUX)
1104    BUX3=BUX(KUX)
1105    CUX3=CUX(KUX)
1106    KUX=1

211 CONTINUE

1109    ALUX3=ALGUY(KUX)
1110    AUY3=AUY(KUX)
1111    BUX=BU3(KUX)
1112    CUX=CU3(KUX)
1113    KUX=KUX-1
1114    TCK=CSSRO(AX,ALX,BEX,1,AUX3,BUX3,CUX3,ALUX3)
1115    TCK=TCAX*SWORO(AX,ALY,BEY,0,AUY3,BUX3,CUY3,ALUY3)
1116    TCK=TCAX*ALX*BEY/(AX,AX)
1117    TCK=CSSRO(AY,ALY,BEY,0,AUY3,BUX3,CUY3,ALUY3)
1118    TCK=TCAX*ALY*BEY0/(AY,AY)
1119    TCK=CSSRO(AX,ALX,BEX,0,AUX3,BUX3,CUX3,ALUX3)
1120    TCK=TCAX*ALY*BEY0/(AY,AY)
1121    TCK=CSSRO(AY,ALY,BEY,0,AUY3,BUX3,CUY3,ALUY3)
1123 TCN=TCM+0.5*(1.0-PO)*ALX*BEY/(AX*AY)
1124 TCN=DSSNR0/AX,BEX,ALX,0,AVX3,BUX3,CUX3,ALUX3)
1125 TCN=TCM*110*0.5*(1.0-PO)*ALX*BEY/(AX*AY)
1126 TCN=TCM*110*0.5*(1.0-PO)*ALX*BEY/(AX*AY)
1127 TC*N=TCM+TCN+TCM/T+T
1128 RST=RCYC*(RY)*TCM/R
1129 KYT=KYT+1
1130 IF(KYT.LE.10VH) GOTO 211
1131 211 CONTINUE
1132 DO 215 KVX=1,1VXM
1133 ALVX3=ALGVX(KVX)
1134 AVX3=AVXS(KVX)
1135 BVX3=BVXS(KVX)
1136 CVX3=CVXS(KVX)
1137 DO 215 KVY=1,1VYM
1138 ALVY3=ALGVY(KVY)
1139 BVY3=BVYS(KVY)
1140 CVY3=CVYS(KVY)
1141 2143 KV=KUNS+(KVX-1)*IVYM+KVY
1142 TCL=DSSNR0/AX,BEX,ALX,0,AVX3,BUX3,CUX3,ALUX3)
1143 TCL=TCL*CSCSR00/AY,BEX,ALY,1,AVY3,BVY3,CVY3,ALVY3)
1144 TCL=TCL*BEY*/ALY/(AX*AY)
1145 TCP=CSCSR00/AX,BEX,ALX,0,AVX3,BUX3,CUX3,ALUX3)
1146 TCP=TCP*PO*BEY*/ALX*/AX*AX
1147 TCP=DSSNR00/AX,BEX,ALX,0,AVX3,BUX3,CUX3,ALUX3)
1148 TCP=TCP*PO*BEY*/ALX*/AX*AX
1149 TCP=DSSNR00/AX,BEX,ALX,0,AVX3,BUX3,CUX3,ALUX3)
1150 TCP=TCP*PO*BEY*/ALX*/AX*AX
1151 TCP=DSSNR00/AX,BEX,ALX,0,AVX3,BUX3,CUX3,ALUX3)
1152 TCP=DSSNR00/AX,BEX,ALX,0,AVX3,BUX3,CUX3,ALUX3)
1153 TCP=DSSNR00/AX,BEX,ALX,0,AVX3,BUX3,CUX3,ALUX3)
1154 TCP=DSSNR00/AX,BEX,ALX,0,AVX3,BUX3,CUX3,ALUX3)
1155 TCP=DSSNR00/AX,BEX,ALX,0,AVX3,BUX3,CUX3,ALUX3)
1156 TCP=DSSNR00/AX,BEX,ALX,0,AVX3,BUX3,CUX3,ALUX3)
1157 RWM=RWM*CHEC2
1158 RSV=RSV*CHEC2
1159 S1= (RNM+RWM+RST*CHEC1+RSSV+RR1+RR2+RS1+RS2+RS3)
1160 IF (I EQ 2) * (ALX*AX)**2.0, + (ALY*AY)**2.0.**2.0
1161 IF (1. EQ.2) * (1.0/4.0)*12.0**5/SS1)
1162 IF (1. EQ.2) * (1.0/4.0)*12.0**5/SS1)
1163 IF (1. EQ.2) * (1.0/4.0)*12.0**5/SS1)
1164 220 CONTINUE
1165 220 CONTINUE
1166 WRITE (6,8) XX(1,1),XX(2,2)
1167 WRITE (6,8) XX(1,2),XX(2,2)
1168 WRITE (6,8) XX(1,1),XX(2,2)
1169 WRITE (6,8) XX(1,2),XX(2,2)
1170 CALL MCG (NK,NM,NN,XX,XM,ALF,ALF2,XX,XK,1,XERR)
1171 225 CONTINUE
1172 DO 225 J=1,NH
1173 ALFR1(J)=XLR(J)*CP*CF/XET(J)
1174 225 CONTINUE
1175 K1=NM-1
1176 DO 225 KF=1,K1
1177 K1=KF+1
1178 DO 225 I=K1,NM
1179 IF ((ABS(ALFR1(KF)) .LT. ABS(ALFR1(I))) GO TO 235
1180 A1=ALFR1(KF)
1181 ALFR1(KF)=ALFR1(I)
1182 ALFR1(I)=A1
1183 DO 225 J=1,NM
1184 AI=XX(J,1)
1185 XH(J,1)=XH(J,1)
1186 XH(1,KF)=AI
1187 230 CONTINUE
1188
235 CONTINUE
236 DO 240 J=1,NM
237 ALFR2(J)=ABS(ALFR1(J))
238 ALFR2(J)=SQRT(ALFR2(J)/(PI+PI))
239 SQ(J)=SQRT(SQ(J))
240 CONTINUE
241 DO 275 I=1,NM
242 WRITE(6,245)
243 WRITE(6,245) FORMAT(1X)
244 AI=KH(I,1)
245 KF=1
246 DO 250 J=1,NM
247 IF(ABS(AI).GT.ABS(KH(J,1))) GO TO 250
248 KF=0
249 AI=KH(J,1)
250 CONTINUE
251 IX=MOD(KF,N)
252 JM=(KF-IX)/N+1
253 IF(IX.EQ.0)IX=N
254 IF(JM.EQ.N)JM=KF/N
255 IF(I.EQ.1) GO TO 265
256 WRITE(6,260)
257 260 FORMAT(2X,'FREQ.
SQUARED',7X,'FREQY.',7X,'FREQ2./FLAT',5X,'FREQ/F
CLAT',7X,'CHANGE',9X,'I',9X,'2X','J')
258 265 CONTINUE
259 ALFR3(I)=FR(KP)/PQ
260 DS(I)=100.0*((SQ(KP)*SQ(KP)-ALFR1(I)))/(ALFR1(I)-ALFR3(I))
261 SQ(I)=SQ(KP)/(PI+PI)
262 DP(I)=100.0*((ALFR1(I)/ALFR3(I))-1.0)
263 ALFR4(I)=SQRT(ALFR3(I))/(PI+PI)
264 WRITE(6,270)ALFR1(I),ALFR2(I),ALFR3(I),ALFR4(I),DP(I),IQ,JM
265 270 FORMAT(1X,5(E14.7,2X),12,2X,12)
266 275 CONTINUE
267 WRITE(6,280)
268 280 FORMAT(2X,'1 TERM SOLN.',4X,'MULTI.TERM SOLN.',3X,'PCNTGE.ERROR',/,
C/,/)
269 C/
270 DO 290 I=1,NM
271 WRITE(6,285)SQ(I),ALFR2(I),DS(I)
272 285 FORMAT(2X,3(E14.7,3X))
273 290 CONTINUE
274 371 CONTINUE
275 WRITE(6,7774)P(ILO)
276 7774 FORMAT(2X,'LOAD= ',E14.7,/)
CCSIN = (SNSNCS(A,PS,SH,E3) + SNSNCS(A,PS,SH,E4))/2.0
RETURN
END

REAL FUNCTION CCCCIN(A,AL,BE,PS,SH)
REAL A,AL,BE,PS,SH,E1,E4
E1 = AL + BE
E4 = AL + BE
CCCCIN = (CSSCS3(A,PS,SH,E1) + CSSCS3(A,PS,SH,E4))/2.0
RETURN
END

REAL FUNCTION SSSSSN(A,AL,BE,PS,SH)
REAL A,AL,BE,PS,SH,E3,E4
E3 = AL + BE
E4 = AL + BE
SSSSSN = (SNSNCS(A,PS,SH,E3) - SNSNCS(A,PS,SH,E4))/2.0
RETURN
END

REAL FUNCTION CSCSSN(A,AL1,AL2,AL3)
REAL A,AL1,AL2,AL3,E5,E6
E5 = AL1 + AL2
E6 = AL1 + AL2
CSCSSN = (CSSN(A,E5,AL3) + CSSN(A,E6,AL1))/2.0
RETURN
END

REAL FUNCTION SNSSSN(A,AL1,AL2,AL3)
REAL A,AL1,AL2,AL3,E5,E6
E5 = AL1 + AL2
E5 = AL1 + AL2
SNSSSN = (CSSN(A,E6,AL3) - CSSN(A,E5,AL1))/2.0
RETURN
END

REAL FUNCTION SNSSCS(A,AL1,AL2,AL3)
REAL A,AL1,AL2,AL3,E5,E6
E5 = AL1 + AL2
E6 = AL1 + AL2
SNSSCS = (CSCS(A,E6,AL3) - CSCS(A,E5,AL1))/2.0
RETURN
END

REAL FUNCTION CSCSCS(A,AL1,AL2,AL3)
REAL A,AL1,AL2,AL3,E5,E6
E5 = AL1 + AL2
E6 = AL1 + AL2
CSCSCS = (CSCS(A,E5,AL3) + CSCS(A,E6,AL1))/2.0
RETURN
END

REAL FUNCTION CSSRCS(A,E2,BE1,AL,B1,C1,CC1)
REAL A,B2,BE1,AL,B1,C1,A1,B2,B1,CC1
A1 = A
B2 = B1
CC1 = C1
IF (N, BE1) GOTO 530
A1 = 0.0
530 IF (N.NE.2) GOTO 535
AA1 = 0.0
BB1 = B1 * BE1 / (A * A)
Cl = B1 * BE1 / (A * A)
535 R1 = AA1 * CSCL (A, BE2, BE3)
R1 = R1 + BB1 * CSCS (A, BE1, BE2, BE3)
R1 = R1 + CC1 * CSSNN (A, BE2, BE3, BE1)
CSSNR = R1
RETURN
END

REAL FUNCTION DSSNR (A, BE2, BE3, N, A1, B1, Cl, BE1)
REAL A, BE1, BE2, BE3, A1, B1, Cl, R1, AA1, BB1, CC1
AA1 = A1
BB1 = B1
CC1 = Cl
IF (N.NE.1) GOTO 540
AA1 = 0.0
BB1 = B1 * BE1 / (A * A)
CC1 = B1 * BE1 / (A * A)
540 IF (N.NE.2) GOTO 545
AA1 = 0.0
BB1 = B1 * BE1 / (A * A)
CC1 = B1 * BE1 / (A * A)
545 R1 = AA1 * CSSN (A, BE2, BE3)
R1 = R1 + BB1 * CSCS (A, BE1, BE2, BE3) + CC1 * CSSN (A, BE1, BE3, BE2)
DSSNR = R1
RETURN
END

REAL FUNCTION CSSNN (A, BE2, BE3, N, A1, B1, Cl, BE1)
REAL A, BE1, BE2, BE3, A1, B1, Cl, R1, AA1, BB1, CC1
AA1 = A1
BB1 = B1
CC1 = Cl
IF (N.NE.1) GOTO 550
AA1 = 0.0
BB1 = B1 * BE1 / (A * A)
CC1 = B1 * BE1 / (A * A)
550 IF (N.NE.2) GOTO 555
AA1 = 0.0
BB1 = B1 * BE1 / (A * A)
CC1 = B1 * BE1 / (A * A)
555 R1 = AA1 * CSSN (A, BE2, BE3)
R1 = R1 + BB1 * CSSNN (A, BE2, BE3, BE1) + CC1 * CSSNN (A, BE1, BE3, BE2)
CSSNN = R1
RETURN
END

REAL FUNCTION CSCS (A, B, C)
REAL A, B, C, F, FF
F = B * SIN (B) * COS (C) - C * COS (B) * SIN (C)
R2 = ABS (B) - ABS (C)
EPS = 0.00000001
IF (ABS (B, LE, EPS)) AND (ABS (C, LE, EPS)) GOTO 601
IF (R2 .GT. EPS) FF = A * F / (B * B - C * C)
IF (R2 .LT. EPS) FF = (ABS (B) * EPS) AND (ABS (C) .LE. EPS) FF = A
601 IF (ABS (B) .LE. EPS) AND (ABS (C) .LE. EPS) FF = A
CSCS = FF
RETURN
END

REAL FUNCTION CSSN(A,B,C)
REAL A,B,C,F,FF
EPS=0.00000001
F=*SIN(B)*SIN(C)-C*COS(B)*COS(C)-C
R1=ABS(ABS(B)-ABS(C))
IF((ABS(B).LE.EPS).AND.(ABS(C).LE.EPS))GOTO 602
IF(R1.GE.EPS)FF=F/(B*B-C*C)
IF(R1.LT.EPS)FF=F/(B*B-C*C)
602 IF((ABS(B).LE.EPS).AND.(ABS(C).LE.EPS))FF=0.0
CSSN=FF
RETURN
END

REAL FUNCTION SNSN(A,B,C)
REAL A,B,C,F,FF
F=C*SIN(B)*COS(C)-B*COS(B)*SIN(C)
R1=ABS(ABS(B)-ABS(C))
IF((R1.GE.EPS)FF=F/(B*B-C*C)
IF((ABS(B).LE.EPS).AND.(ABS(C).LE.EPS))GOTO 603
IF(R1.LT.EPS)FF=F/(A/2.0)
603 IF((ABS(B).LE.EPS).AND.(ABS(C).LE.EPS))FF=0.0
SNSN=FF
RETURN
END

REAL FUNCTION PIn(A,B1,B1,Cl,DE1,A2,B2,C2,DE2,II,II)
REAL A,B1,B1,Cl,A2,B2,C2,DE1,A2,B2,C2
AA=A1
BB=B1
CC=Cl
AA=A2
BB=B2
CC=C2
IF(II.NE.1)GOTO 610
AA=0.0
BB=C1*DE1/A
CC=DE1/A
610 IF(II.NE.2)GOTO 615
AA=0.0
BB=B1*DE1+DE1/(A*A)
CC=DE1/(A*A)
615 IF(II.NE.1)GOTO 620
AA=0.0
BB=C2*DE2/A
CC=DE2/A
620 IF(II.NE.2)GOTO 622
AA=0.0
BB=B2*DE2+DE2/(A*A)
CC=DE2/(A*A)
622 CONTINUE
EPS=0.00000001
IF(Abs(DE1).GT.EPS)UG=SIN(DE1)/DE1
IF(ABS(DE1).LE.EPS)XG=1.0
1454 IF(ABS(DE2).GT.EPS)UG=SIGN(DE2)/DE2
1455 IF(ABS(DE2).LE.EPS)UG=1.0
1456 IF(ABS(DE1).GT.EPS)VG=(1.0-COS(DE1))/DE1
1457 IF(ABS(DE1).LE.EPS)VG=0.0
1458 IF(ABS(DE2).GT.EPS)VE=(1.0-COS(DE2))/DE2
1459 IF(ABS(DE2).LE.EPS)VE=0.0
1460 R1=AA**2/2+(AA**2BB**3UG**2)-AAL**2BB**2UG
1461 R2=(AA**2CC**2/2+AA**2CC**2CH-AAL**2CC**2/2)
1462 R3=BB**2CC**2/2CDE2/DE1+CC**2CDE2/ADE2.
1463 R3=R3**2**2CC**2CDE2/DE1+CC**2CDE2/ADE2
1464 PRINT=1+R2+R3
1465 RETURN
1466 END
1467 SUBROUTINE GAUSX(N,S,X)
1468 DIMENSION X(N:N),S(N:N)
1470 DO 3700 K=1,N
1471 K2=K+1
1472 IF(K2.GT.N)GOTO 3700
1473 DO 3685 I=K2,N
1474 IF(S(K,K).EQ.0.0)R=0.0
1475 IF(S(K,K).EQ.0.0)WRITE(6,6680)
1476 IF(S(K,K).EQ.0.0)WRITE(6,4680)
1477 IF(S(K,K).EQ.0.0)WRITE(6,6680)
1478 DO 3680 J=K,N
1479 3680 S(I,J)=S(I,J)-R*S(K,J)
1480 DO 3680 J=K,N
1481 3685 CONTINUE
1482 CONTINUE
1483 IF(S(N,N).EQ.0.0)X(N)=0.0
1484 IF(S(N,N).EQ.0.0)WRITE(6,4680)
1485 IF(S(N,N).EQ.0.0)WRITE(6,4680)
1486 IF(S(N,N).EQ.0.0)WRITE(6,4680)
1487 NH=1
1488 DO 3900 K=1,NH
1489 K=N-K
1490 SM=SM+K
1491 DO 3800 JJ=K,1,1
1492 3800 SM=SM+S(K,J)*X(J)
1493 IF(AABX(S(K,K)).LT.(1.10.**LB.1))GOTO 3850
1495 X(K)=SM/S(K,K)
1496 GOTO 3875
1497 3850 X(K)=0.0
1498 WRITE(6,3860)K
1499 3860 FORMAT(2X,"EMPT ROW NO.",I3)
1500 3875 CONTINUE
1501 3900 CONTINUE
1502 RETURN
1503 END
1504 SUBROUTINE GAUSEI(N,M,S,B,X)
1505 REAL SM,R
1506 DIMENSION X(N:N),B(N:N),S(N:N)
1508 DO 700 K=1,N
1509 K=K+1
1510 IF(K2.GT.N)GOTO 700
1511 DO 685 I=K2,N
1512 IF(S(K,K).EQ.0.0)R=0.0
1513 IF(S(K,K).EQ.0.0)WRITE(6,6680)
1514 6680 FORMAT(2X,"WARNING "S(*,I2," *)=0.0")
1515 IF(S(K,K).EQ.0.0)WRITE(6,6680)
1516 DO 680 J=K,N
1517 680 S(I,J)=S(I,J)-R*S(J,J)
1518 685
DO 582 L=1,M
582 B(I,L)=B(I,L)-R*B(K,L)
583 CONTINUE
584 DO 905 L=1,K
585 IF(S(N,N).NE.0.0)X(N,L)=B(N,L)/S(N,N)
586 IF(S(N,N).EQ.0.0)X(N,L)=0.0
587 WRITE(6,5681)
588 5681 FORMAT(2X,"S(N,N)=0.0")
589 M=M+1
590 DO 900 KK=1,NH
591 K=K+K
592 S(K)=B(K,L)
593 DO 800 JJ=1,KK
594 J=J+1-J
595 800 SM=SM+S(K,J)*Y(J,L)
596 IF(S(K,K).EQ.0.0)GOTO 850
597 X(K,L)=SM/S(K,K)
598 GOTO 875
599 850 X(K,L)=0.0
600 WRITE(6,860)K
601 860 FORMAT(2X,"EMPTY ROW NO.",I2)
602 875 CONTINUE
603 900 CONTINUE
604 905 CONTINUE
605 RETURN
606 END

SUBROUTINE SETUP(N,B,ALP,21,22,23,24,IND)
REAL 21,22,23,24
DIMENSION B(N),ALP(N)
PI=4.0*ATAN(1.0)
I=1
IF(I.LT.0.01)GOTO 444
IF(IND.EQ.2)GOTO 442
ALP(1)=0.001
B(1)=-SIN(0.0005)
C(1)=COS(0.0005)
ALP(2)=PI/2.
B(2)=0.0
C(2)=1.0
ALP(3)=PI/4.
B(3)=0.0
C(3)=1.0
GOTO 448
442 IF(IND.EQ.1)GOTO 444
ALP(2)=PI
B(2)=0.0
C(2)=1.0
B(1)=0.0
C(1)=0.0
ALP(1)=0.0
ALP(3)=PI/3.
C(3)=1.0
B(3)=0.0
ALP(4)=PI/5.
C(4)=1.0
B(4)=0.0
GOTO 448
444 CONTINUE
K=1
CONTINUE
B(1)=11
C(1)=22
CONTINUE
1585 IF (IND.EQ.1) ALP(1) = PI*(2*K-1)
1586 IF (IND.EQ.2) ALP(1) = PI*2*K
1587 IF (IND.EQ.3) ALP(1) = PI*K
1588 K = K + 1
1589 I = I + 1
1590 IF (I .LE. N) GOTO 445
1591 IF (K .LE. 0.5) GOTO 446
1592 ALP(N) = 0.0001
1593 S(N) = SIN (0.00005)
1594 C(N) = COS (0.00005)
1595 CONTINUE
1596 RETURN
1597 END
1598 REAL FUNCTION VALUE (A, X, AL, A1, A2, A3, N)
1599 REAL A, X, AL, A1, A2, A3, N
1600 R = AL / A
1601 IF (N .EQ. 0) VALUE = A1 + A2*COS (R) + A3*SIN (R)
1602 IF (N .EQ. 1) VALUE = A2*(AL/A)*SIN (R) + A3*(AL/A)*COS (R)
1603 IF (N .EQ. 2) VALUE = AL*A3*(A2*COS (R) + A3*SIN (R)) / (A*A)
1604 RETURN
1605 END
1606 SUBROUTINE NAMTR(S, N, M)
1607 DIMENSION S(N, M)
1608 DO 1000 I = 1, N
1609 WRITE (6, 999) (S(I, J), J = 1, N)
1610 999 FORMAT (2X, 4(E14.7, 2X))
1611 1000 CONTINUE
1612 WRITE (6, 1001)
1613 1001 FORMAT (' ')
1614 RETURN
1615 END
1616 SUBROUTINE NAMSET(S, N, M)
1617 DIMENSION S(N, M)
1618 DO 1002 I = 1, N
1619 DO 1002 J = 1, M
1620 S(I, J) = 0
1621 1002 CONTINUE
1622 RETURN
1623 END
1624 REAL FUNCTION DEFF (AX, AY, ZF, ZO, TEM1, TEM2)
1625 TFM = 9.0 / (AX**4.0) + (9.0 / (AY**4.0)) + (2.0 / (AX*AY*AY*AY))
1626 PI = 4.0 / ATAN (1.0)
1627 PI4 = PI**4.0
1628 RN1 = 2.0 / (ZF*ZF - ZO*ZO)
1629 RN2 = 2.0 / (ZF*ZF*ZF)
1630 RN3 = PI4 * RN1 + (1.0 / (AX*AX)) + (1.0 / (AY*AY))**2.0
1631 RN3 = RN1 + RN2 + RN3 + RN4
1632 RETURN
1633 END
1634 INTEGER FUNCTION ICONX(N1, N2, NN)
1635 IF (N1.GT.N2) X = N2
1636 IF (N1 .GT. N2) J = N1
1637 IF (N2 .GT. N1) J = N2
1638 I = (K-1)*NN + J
1639 ICONX = I - (K*K - K)/2 + 1
1640 RETURN
1641 END
APPENDIX K

ERRORS DUE TO MEASUREMENTS AND SIMPLIFYING ASSUMPTIONS

The accuracy of the calculated values of the frequencies depend on the accuracy of the input parameters. The maximum possible error in the fundamental frequency (calculated) of an unstressed flat plate due to the measurement errors is given by [23],

\[ \frac{\delta \Omega}{\Omega} = \frac{2}{(1/a^2 + 1/b^2)} \left( \frac{1}{a^2} \cdot \frac{\delta a}{a} + \frac{1}{b^2} \cdot \frac{\delta b}{b} \right) \]

\[ \pm \frac{1}{2} \left( \frac{\delta E}{E} \right) \pm \frac{3}{2} \left( \frac{\delta h}{h} \right) \pm \frac{1}{2} \left( \frac{\delta m}{m} \right) \pm \frac{\nu^2}{(1-\nu^2)} \cdot \frac{\delta \nu}{\nu} \]

The maximum possible error in the fundamental frequency of a stressed plate can be estimated as follows:

\[ \omega \approx \Omega \sqrt{1 - \frac{P}{P_c} + k \omega^2} \]

\[ \frac{\delta \omega}{\omega} = \frac{\delta \Omega}{\Omega} \pm \frac{1}{2} \left( \frac{\delta h}{h} \right)^2 \left( \frac{P}{P_c} \right) \pm k \omega^2 \left( \frac{\delta \omega}{\omega} \right) \cdot \left( \frac{\delta \nu}{\nu} \right) \]

For plate 4 at a load ratio \( (P/P_c) \) of 4.15, the above equations give \( \frac{\delta \omega}{\omega} = 0.00495 \) (4.95%). This is based on the assumption that all possible errors occur in such a way that the error in the frequency accumulates, and that the errors in the input parameters are as listed below:

\[ \delta E/E = .01 \]
\[ \delta a/a = .004 \] (1 mm for \( a = 250 \) mm)
\[ \delta b/b = .003 \] (1 mm for \( b = 300 \) mm)
\[ \frac{\Delta h}{h} = 0.003 \quad (10^{-4} \text{ inch for } h = 0.86 \text{ mm}) \]

\[ \frac{\Delta m}{m} = 0.01 \]

\[ \frac{\Delta v}{v} = 0.02 \]

\[ \frac{\Delta P}{P} = 0.0022 \quad (4 \text{ lbs at 1829 lbs}) \]

\[ \frac{\Delta u}{u} = 0.013 \quad (2 \times 10^{-3} \text{ inch at } 155 \times 10^{-3} \text{ inch}) \]

Errors caused by the simplifying assumptions, such as in the case of neglecting damping and neglecting the in-plane inertia, are expected to be very small. For example, it was assumed that the vibration takes place in a vacuum. The experiments were conducted in air. However, including the aerodynamic damping in the analysis (say for a damping ratio of 0.1%), does not change the resonance frequency significantly (order of 0.002%).
REFERENCES


35. PAFEC 75 Package Program, Pafec Ltd., Strelley, Nottingham, U.K.

END
08 04 87
FIN