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Conducting Wall Solutions For The Mhd Channel Flow Equations

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CONDUCTING WALL SOLUTIONS

FOR THE

MHD CHANNEL FLOW EQUATIONS

by

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ABSTRACT

Approximations are obtained for MHD channel flow configurations where the electrodynamic (Lorentz) forces are large compared with viscous forces, and the boundary walls are highly conducting. The systematic boundary layer method of matched asymptotic expansions has been employed. The major results obtained herein are:

(a) A demonstration, by means of approximate b[illegible] layer solutions, that a circular tube with con[illegible] walls requires the solution of three boundary layers whereas the tube with insulating walls requires only two boundary layers, at least to first order.

(b) The mass flow, to second order in a circular tube with walls of arbitrary conductivity.

(c) An explicit solution for the parabolic or obscure layer for a circular tube with arbitrary wall conductivity and an estimate of the flow deficit in this layer.

(d) An approximation for the rectangular duct corner region in the important case when the walls parallel to the applied field are highly conducting, and the perpendicular walls insulating, is obtained. An

observation that a variable comprising the sum of the magnetic field and fluid velocity displays boundary layers along only one of the two insulating walls, is used to construct the approximate solution.

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TABLE OF CONTENTS

LIST OF FIGURES

1.0 INTRODUCTION

Due to Faraday's law of induction, the motion of an electrically conducting fluid across a magnetic field causes electric currents to flow. The magnetic fields associated with these currents modify the magnetic field which created them and interaction of the electric currents with the magnetic field causes Lorentz forces which modify the fluid flow. Magnetohydrodynamics (MHD) is the study of this interaction between hydrodynamic and electrodynamic effects. MHD flow through a tube or channel is a branch of magnetohydrodynamics which has interesting practical applications.

MHD channel flow effects have been applied to flow meters for conducting fluids (blood for example) where the induced potential gradient provides a direct measure of flow rate without contamination of the fluid (Shercliff (1)). A recent flow meter application, an MHD angular rate sensor reported by Klass (2), consists of a mercury torus angular accelerometer spun at 12,000 rpm about an axis in the plane of the torus. A rate orthogonal to the spin axis results in a periodic flow. Measurement of the amplitude and phase of the induced potentials provides a measure of the angular rate about the two axes orthogonal to the spin axis.

MHD pumps, in which the Lorentz force replaces mechanical impellers, have been employed for moving corrosive conducting fluids such as liquid sodium coolants in nuclear reactors. A spacecraft application of the MHD pump which is perhaps worthy of consideration, would consist conceptually of a torus of mercury about the periphery of the spacecraft and performing triple duty as a reaction wheel for attitude stabilization during three axis stabilized phases, as a heat exchanger to transfer heat from internal dissipating components to external radiators, and as a passive (or active) nutation damper during spin stabilized phases. As well as performing three functions using a single component with a potential weight saving, the lack of a mechanical wearout mechanism could result in a high reliability for long life spacecraft.

The major area for application of MHD channel flows is in the development of MHD power generation technology which has promise of a fifty percent increase in fuel efficiency over conventional thermal generation methods (3). The working fluid in this case is normally a high temperature gas seeded with conducting material and forced to expand through a duct with a strong transverse magnetic field applied. The theoretical study of these high temperature gaseous MHD flows is formidable indeed and even the simplest steady incompressible flows have presented significant mathematical difficulties. It is expected, however, that some insight into the more complex flow configurations may be gained through an understanding of the simpler flows.

In general, engineering feasibility studies and detail design activities are greatly facilitated when simple expressions are available relating major parameters to one another. In the application of MHD channel flows, readily available approximations are highly desirable for quantities such as flow rate, power consumed or generated, or wall potentials. Such approximations are possible to obtain using boundary layer methods in the frequent practical situation where the electrodynamic (Lorentz) forces are large compared with viscous forces, which is the situation considered herein.

An illustration of the duct configuration is shown in Figure 1.1. The object is to obtain the fluid velocity and induced magnetic field profiles throughout the duct. The governing equations are a set of deceptively simple appearing coupled linear elliptic equations. The interesting solutions occur with large magnetic fields and highly conducting fluids in which case Lorentz forces are large compared with viscous forces. Singular perturbation techniques in the form of "Matched Asymptotic Expansions" have been successfully used for most of the known approximate solutions. This approximation technique is physically motivated as a result of thin boundary layers which form along the walls.

Insulating wall approximations are now reasonably well understood and the significant parameters may be calculated with relative ease. These flows are characterized by a large value for the Hartmann number M defined as $B_o L (\sigma/\rho\nu)^{1/2}$, B_o, L, σ, ρ and ν being the applied magnetic field, scale length, fluid conductivity, density and kinematic viscosity. This non-dimensional number is a measure of the ratio between Lorentz forces and viscous forces. In the simplest case of flow between plane parallel walls, Hartmann (4) obtained solutions which are characterized by two distinct flow regions, a core in which the dependent variables are relatively slowly varying and thin boundary layers along the walls where the fluid velocity and induced magnetic field vary rapidly. A similar division of the flow also occurs with closed ducts and in this case up to three different types of boundary layers occur with the type of layer dependent on the angle between the applied field and the boundary wall.

The division of the flow referred to above is a result of the magnetic forces dominating the viscous forces over most of the duct. It is only near boundaries or within regions of large shear that the viscous forces are comparable to the magnetic forces. A variety of such shear layers may occur and may be classified according to the direction of the applied magnetic field within the layer and the electrical or geometric nature of the walls intersected by the magnetic field. A review of several configurations is given by Hunt and Shercliff (5).

One of the more interesting two dimensional situations with a uniform transverse field occurs when the field lines intersect highly conducting walls with insulating walls parallel to the field. In this case a high speed boundary layer exists along the insulating wall and reverse flow occurs (Temperley and Todd [16]). Additional configurations are reported by Shercliff (20) with conducting walls parallel to the field but shorted externally, and insulating walls intersecting the field lines. The flow characteristics in this case are dependent on the shape of the insulating walls with respect to the midline. Velocities change by a factor M as this geometry changes. In some instances, high speed backflow occurs in the conducting wall boundary layer and regions of fast and slow core flow exists.

Non-uniform (transverse) magnetic fields are considered by Todd (21) using orthogonal curvilinear coordinates based on the applied field itself, rather than the geometric coordinates. The results for a uniform field are simply special cases of this more general formulation. An aspect of interest is the fact that free shear layers (or wakes) lie along the magnetic field lines whether these are curved or not. In addition, under some field non-uniformities, the core velocity may, with sufficiently large field strength, exhibit reverse flow in portions of the duct. Such a situation is also considered by Ranger (22). A tapered field (plane cusp geometry) has been studied by Regirer (23). In this case the Lorentz force accelerates the fluid stream.

Three dimensional flows with obstacles in the channel or with diverging channels have also been treated using boundary layer techniques. In these cases, the interaction parameter N relating magnetic to inertial forces is large as well as the Hartmann number. The result of greatest interest for the two dimensional flows considered in this thesis are that the entry length for an insulating pipe is $O(M^{1/2})$ as opposed to the entry length of $O(1)$ for a perfectly conducting pipe (Walker and Ludford [24]). Thus, fully developed flow may be difficult to obtain in practice with an insulated pipe.

Though we are concerned here with the matched asymptotic methods of solution, approximations have been obtained using variational principles in the Ritz or Galerkin manner. An example treated by Lu (25) gave poor results for large Hartmann number. However, Wenger (26) has derived a general variational principle for ducts with arbitrary wall conductivities. By applying this principle to a square channel using relatively simple trial functions to approximate the fluid velocity and electric potential, an approximate mean flow was obtained which was in good agreement with previous exact solutions for insulating walls, perfectly conducting walls and for thin walls of arbitrary conductivity.

Extremum principles in the form of inequalities for the mass flow were obtained by Smith (27) with application to the insulating wall rectangular duct. The mass flow so calculated, using the plane parallel Hartmann solutions to construct the trial function, resulted in a mass flow correct to first order. Later (Smith [28]) results correct to second order were obtained for both the rectangular and circular ducts. An extension of the principles to include conducting walls (Smith [29]) resulted in a mass flow for the circular tube which depends on three parameters, the Hartmann number, the ratio of fluid to wall conductivities, and the wall thickness. This differs from the results obtained for thin conducting walls using matched asymptotic expansions in which the wall thickness does not appear as a separate parameter. Smith's result, of order M^{-2}, is not entirely in agreement with the results of the present thesis if the walls are of small conductivity. A further extension and application of the extremum principles to a square duct with conducting walls (Smith [30]) results in a mass flow in agreement with those of Temperley and Todd (16) where boundary layer methods were used. Sloan (31) has presented extremum principles which are related to those of Smith.

The variational principles referred to above can apparently predict mass flow rates with some accuracy and are therefore of interest. However, it is unlikely that the approximating functions can provide the detailed flow characteristics which may be obtained through the use of matched asymptotic expansions.

The mathematical problem, particularly where insulating walls are concerned, is a special case of a more general theory of linear elliptic singular perturbations. Eckhaus (6) has presented a survey of results in this area, including some justification for the asymptotic boundary layer methods.

The more important configurations appropriate to MHD power generation are much more difficult than the insulating wall situation due to the complications introduced by portions of the wall being highly conducting. The major objective of this thesis has been to consider the effects of wall conductivity and obtain approximations for the flow and magnetic field profiles with highly conducting walls involved. To accomplish this the systematic boundary layer method of matched asymptotic expansions has been utilized. A number of new results are obtained, in particular:

(a) It is shown that three different boundary layers are required for a circular duct with conducting walls, whereas only two boundary layers are required with insulating walls, at least to first order.

(b) Solutions for the circular tube with arbitrary wall conductivity are obtained to second order from which the flow integral to second order is computed.

(c) Through the use of "Optimal" coordinates, a solution for the "obscure" or parabolic layer in a circular tube with arbitrary wall conductivity is obtained. The resulting flow deficit for this layer is shown to be of higher order than second.

(d) An approximate solution to the so called "corner" problem for rectangular ducts with perfectly conducting walls parallel to the applied field and insulating wall perpendicular to the field, is obtained.

In the following sections, the MHD channel flow equations and conducting wall boundary conditions are briefly developed followed by a discussion of the boundary layer approximations. Section 3.0 is devoted to conducting wall solutions for a circular pipe and section 4.0 deals with the rectangular channel. A general discussion of the results is given in section 5.0.

2.0 THE CHANNEL FLOW EQUATIONS

2.1 Derivation of the Differential Equations

The development of the MHD equation for a homogeneous, incompressible, electrically conducting fluid is treated in most MHD texts (for example (7)) and will not be repeated in detail here. The MHD approximations involved are:

(a) The fluid velocity $\underline{u}$ is small compared with the speed of light in which case the electric field $\underline{E}'$, magnetic field $\underline{B}'$ and current density $\underline{J}'$ in a moving fluid reference frame are:

$$\underline{E}' = \underline{E} + \underline{u} \times \underline{B}$$

$$\underline{B}' = \underline{B}$$

$$\underline{J}' = \underline{J}$$

where the unprimed quantities refer to a fixed laboratory reference.

(b) Displacement currents are negligible so that:

$$\text{curl } \underline{H} = \underline{J}$$

(c) The fluid is isotropic so that:

$$\underline{B} = \mu \underline{H}$$

$$\underline{D} = \epsilon \underline{E}$$

where μ and ϵ are the magnetic permeability and dielectric constants.

(d) The effect of convection of charge density is negligible so that ohm's law is:

$$\underline{J} = \sigma(\underline{E} + \underline{u} \times \underline{B})$$

where σ is the fluid conductivity.

(e) The electrostatic force due to volume charge densities is negligible so that the electromagnetic body force is simply:

$$\underline{f} = \underline{J} \times \underline{B}$$

Using (a) through (d), the electromagnetic induction equation becomes:

$$\frac{\partial \underline{H}}{\partial t} = \nabla \times \underline{u} \times \underline{H} + \eta \nabla^2 \underline{H} \qquad (2.1)$$

where $\eta = 1/\mu\sigma$ is the magnetic diffusivity.

Using (e), and incompressibility, Navier Stokes equation becomes:

$$\rho \frac{D\underline{u}}{Dt} = \mu (\nabla \times \underline{H}) \times \underline{H} - \nabla p + \rho \nu \nabla^2 \underline{u} \qquad (2.2)$$

where	p	=	pressure
	ρ	=	fluid density
	ν	=	kinematic viscosity

Finally, from Maxwell's equations and the incompressibility of the fluid:

$$\nabla \cdot \underline{B} = 0 \qquad (2.3)$$

$$\nabla \cdot \underline{u} = 0 \qquad (2.4)$$

These equations (1) through (4) are to be solved in the interior of the duct under the following conditions:

(a) The motion is steady

(b) $\nabla p = \frac{\partial p}{\partial x} \underline{x} + \frac{\partial p}{\partial y} \underline{y} - P_o \underline{z}$

where P_o is constant and $\underline{z}$ is the axial direction of the duct. $\underline{x}$, $\underline{y}$, $\underline{z}$ are unit vectors.

(c) $\underline{u}$ is entirely in the $\underline{z}$ direction and is independent of z, that is:

$$\underline{u} = u_z(x, y)\ \underline{z}.$$

(d) $\underline{H} = H_o\ \underline{x} + H_z\ \underline{z}$

where H_o is the large applied field and H_z the induced field.

With these conditions, equations (2.1) to (2.4) reduce exactly to:

$$\nabla^2_{x,y} H_z + \frac{H_o}{\eta}\frac{\partial u_z}{\partial x} = 0$$

$$\nabla^2_{x,y} u_z + \frac{\mu H_o}{\rho \nu}\frac{\partial H_z}{\partial x} = -\frac{P_o}{\rho \nu}$$

$$P = -\frac{1}{2}\mu H_z^2 + \text{constant}$$

where $\nabla^2_{x,y} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

The first two of these equations may be written in dimensionless form as:

$$\nabla^2 H + M \frac{\partial u}{\partial x'} = 0 \qquad (2.5)$$

$$\nabla^2 u + M \frac{\partial H}{\partial x'} = -P \qquad (2.6)$$

where $H = \sqrt{\frac{\mu\eta}{\rho\nu}}\,\frac{H_z}{u_o}$, $P = \frac{a^2 P_o}{\rho\nu u_o}$,

$u = u_z/u_o$, $x' = x/a$, $y' = y/a$

$M = H_o a \sqrt{\frac{\mu}{\rho\nu\eta}}$ (Hartmann number)

u_o = typical velocity

a = typical length (tube radius or rectangular channel half width)

$$\nabla^2 = \frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2}$$

Primes will not be displayed in subsequent equations and the tube radius or rectangular channel half width is taken as unity.

The only hydrodynamic boundary condition required is the "no slip" condition along the walls, that is:

$$u = 0 \tag{2.7}$$

The electromagnetic boundary condition for a thin walled duct may be obtained from continuity considerations. Electromagnetic quantities within the wall will be denoted by the subscript "i" and the non-dimensional wall thickness by t ($t \ll 1$). At the fluid-wall interface the tangential electric field must be continuous (assuming no interface resistance), that is:

$$\underline{E}\Big|_T = \underline{E}_i\Big|_T$$

But, $\underline{E}\Big|_T = \frac{\underline{J}}{\sigma}\Big|_T + \mu(\underline{u} \times \underline{H})\Big|_T$

$$= \frac{\underline{J}}{\sigma}\Big|_T$$

$$= \frac{1}{\sigma}\,(\nabla \times \underline{H})\Big|_T$$

Since $\underline{u} = 0$ at the interface and:

$$\underline{E}_i\Big|_T = \frac{\underline{J}_i}{\sigma_i}\Big|_T = \frac{1}{\sigma_i}\,(\nabla \times \underline{H}_i)\Big|_T$$

the tangential continuity of $\underline{E}$ implies:

$$\frac{1}{\sigma}(\nabla \times \underline{H})\Big|_T = \frac{1}{\sigma_i}(\nabla \times \underline{H}_i)\Big|_T$$

that is: $\frac{1}{\sigma}\frac{\partial H_z}{\partial n} = \frac{1}{\sigma_i}\frac{\partial H_{iz}}{\partial n}$

where n is the normal direction into the wall.

The tangential magnetic field must also be continuous at the interface, that is:

$$H_z = H_{iz}$$

and since $t \ll 1$,

$$H_{iz}\Big|_{n=t} \approx H_{iz}\Big|_{n=0} + t\frac{\partial H_{iz}}{\partial n}\Big|_{n=0}$$

Taking H_{iz} zero at the external wall boundary results in:

$$H_{iz}\Big|_{n=0} + t\frac{\partial H_i}{\partial n}\Big|_{n=0} = 0$$

Thus, $$H_z + t\frac{\sigma_i}{\sigma}\frac{\partial H_z}{\partial n} = 0$$

at the interface.

In dimensionless variables this electromagnetic boundary condition becomes:

$$H + c\frac{\partial H}{\partial n} = 0 \qquad (2.8)$$

where

$$c = t\frac{\sigma_w}{\sigma}$$

If $\sigma_w = 0$ equation (2.8) reduces to the simpler condition for non-conducting walls:

$$H = 0$$

Implicit in the above approximation is an assumption that the derivatives of H_{w} in the tangential direction are negligible with respect to derivatives in the normal direction, since H_{w} must satisfy Laplace's equation. Thus, in regions where the scale length tangential to the wall is of 0(1), it is necessary only that $t \ll 1$. However, in regions where H and H_{w} may change rapidly in a tangential direction, the appropriate scale length along the wall could be small and in some instances may be of $0(M^{-1})$. In this case, t must be very small with $t << O(M^{-1})$ in order for the thin wall approximation to remain valid. This super thin wall condition is not a major limitation, at least for the calculation of quantities such as flow rate, since the regions with a small tangential scale length generally provide a negligible contribution to the flow integral. In addition, the thin wall approximation reduces to the correct limits in the insulating and perfectly conducting cases. A more accurate representation of the flow characteristics with arbitrary wall conductivity would require the inclusion of a finite wall thickness in the analysis as was identified for example by Todd-Temperley (16) and Butler (32).

Symmetry conditions can be obtained by inspection of equations (2.5) and (2.6) and the boundary conditions (2.7) and (2.8). H is odd in x and even in y whereas u is even in both x and y. That is:

$$
\begin{aligned}
H(-x,y) &= -H(x,y) \\
H(x,-y) &= H(x,y) \\
u(-x,y) &= u(x,y) \\
u(x,-y) &= u(x,y)
\end{aligned}
\qquad (2.9)
$$

For convenience, a new set of dependent variables may be introduced which decouple the differential equations (2.5) and (2.6). Unfortunately, this leads to coupled boundary conditions, though some simplification of the problem results. The new variables are:

$$F = u + H + \frac{P}{M} x \tag{2.10}$$

$$G = u - H - \frac{P}{M} x \tag{2.11}$$

The differential equations then become:

$$\nabla^2 F + M \frac{\partial F}{\partial x} = 0 \tag{2.12}$$

$$\nabla^2 G - M \frac{\partial G}{\partial x} = 0 \tag{2.13}$$

With reference to Figures 2.1 (a) and 2.1 (b) the corresponding boundary conditions are:

(a) <u>Circular Duct, at r = 1</u>

$$F - G + C\left(\frac{\partial F}{\partial r} - \frac{\partial G}{\partial r}\right) = 2\frac{P}{M}(1 + C) \cos\theta$$

$$F + G = 0$$

(b) Rectangular Duct

$$F - G \pm C_A\left(\frac{\partial F}{\partial x} - \frac{\partial G}{\partial x}\right) = \pm 2\frac{P}{M}(1 + C_A) \quad @ \; x = \pm 1$$

$$F - G \pm C_B\left(\frac{\partial F}{\partial y} - \frac{\partial G}{\partial y}\right) = 2\frac{P}{M}x \quad @ \; y = \pm L$$

$$F + G = 0 \quad @ \; x = \pm 1, \quad y = \pm L$$

It will be noted that a common boundary between the walls A and B does not appear in the thin wall approximation.

2.2 Discussion of the Boundary Layer Approximation

Exact or formal solutions have been obtained for the above equations in some special cases. These solutions are often in the form of slowly convergent infinite series. As a result, most features of the flow may be difficult to obtain without calculating a large number of terms. For the practical cases of large Hartmann numbers, greater insight into the nature of the flow can be obtained through asymptotic solutions. Though asymptotic forms may be obtained from an exact convergent series, as, for example, Gold (8) obtained for a special case, asymptotic solutions may also be obtained by the more direct application of boundary layer techniques, that is matched asymptotic expansion.

With reference to equation (2.12) for F, a regular asymptotic expansion for large M such as:

$$F(x, y) \sim \sum_{i=0}^{n} \varepsilon_i(M) F_i(x, y)$$

where each of the $F_i(x, y)$ is of order unity and,

$$\lim_{M \to \infty} \frac{\varepsilon_{i+1}(M)}{\varepsilon_i(M)} = 0$$

may be substituted directly in (2.12) with the result that for the first term:

$$\frac{\partial F_o(x, y)}{\partial x} = 0$$

It is apparent that all highest order derivatives are lost in such a procedure rendering the solution incapable of satisfying the boundary conditions. This non-uniformity could be considered, according to Van Dyke (9), a result of the parameter M being the ratio of two disparate lengths. That is:

$$M = a/d_h$$

where a = tube radius for example, which is fixed and d_h = Hartmann depth which becomes small as either the applied field H_o or conductivity σ increases since:

$$d_h = \frac{1}{H_o}\sqrt{\frac{\eta\nu\rho}{\mu}} = \frac{1}{\mu H_o}\sqrt{\frac{\nu\rho}{\sigma}}$$

The Hartmann depth is the MHD analogue of the viscous depth. The appropriate Hartmann depth is in fact the depth normal to the tube wall, based on the component of field strength perpendicular to the wall. Thus, in the case of a circular tube, the parameter of greatest concern is M cos θ rather than M itself. This would suggest that when creating additional asymptotic expansions, from which uniformly valid approximations are to be obtained, there will be at least three distinct types of asymptotic equations:

(a) The regular expansion for $M \to \infty$ but appropriate only away from the boundary influence, that is, in the core.

(b) A boundary layer expansion for $M \cos\theta \to \infty$ to provide the link between the boundary and the core.

(c) One or more expansion in the boundary region where $\cos\theta \ll 1$.

The nature of the various boundary layer equations is most readily seen for the rectangular duct. In this case so called "local" boundary variables could be defined as:

$$X = \frac{1 - x}{\Delta_1(M)}, \qquad Y = \frac{L - y}{\Delta_2(M)}$$

where $\Delta_1(M)$ and $\Delta_2(M)$ are stretching coefficients to be determined. In these new variables, equations (2.12) and (2.13) become:

$$\frac{1}{\Delta_1^{\,2}} \frac{\partial^2 G}{\partial x^2} + \frac{1}{\Delta_2^{\,2}} \frac{\partial^2 G}{\partial y^2} + \frac{M}{\Delta_1} \frac{\partial G}{\partial x} = 0$$

$$\frac{1}{\Delta_1^{\,2}} \frac{\partial^2 F}{\partial x^2} + \frac{1}{\Delta_2^{\,2}} \frac{\partial^2 F}{\partial y^2} - \frac{M}{\Delta_1} \frac{\partial F}{\partial x} = 0$$

As identified by Eckhaus (6), there are four "significant" degenerations corresponding to the following four flow regions:

(a) The regular or core approximation with $\Delta_1 = \Delta_2 = 1$ in which case the regular expansion identified above is obtained.

(b) The ordinary boundary layers along the wall x = 1 (also called the Hartmann layer for the rectangular tube and Shercliff layer for the circular tube), where $\Delta_2 = 1, \Delta_1 = M^{-1}$. In this degeneration, the equations for the first term become:

$$\frac{\partial^2 G}{\partial x^2} + \frac{\partial G}{\partial x} = 0$$

$$\frac{\partial^2 F}{\partial x^2} - \frac{\partial F}{\partial x} = 0$$

These layers derive their name from the fact that the equations reduce to ordinary differential equations. It will be noted that the exponentially unbounded term in F must be discarded, the remaining term then being independent of x. Only G retains a rapidly varying term. Thus, the core F solution extends to the boundary x = 1. Near the opposite wall x = -1, the situation is reversed with F retaining a rapidly varying term and the core G solution extending to the wall.

(c) The parabolic layer occurs along the wall y = L, parallel to the applied magnetic field (also termed the secondary layer) with $\Delta_1 = 1$ and $\Delta_2 = M^{-1/2}$. In this degeneration the equations are parabolic with the first term being:

$$\frac{\partial^2 G}{\partial Y^2} + \frac{\partial G}{\partial X} = 0$$

$$\frac{\partial^2 F}{\partial Y^2} - \frac{\partial F}{\partial X} = 0$$

Solutions for F are obtained from boundary conditions along X = 0 and Y = 0, that is, along the walls x = 1 and y = L. Thus, as for the core, the first order F parabolic layer will satisfy the boundary conditions on x = 1, though this is no longer true in higher approximations since derivatives of the first order solution in general become singular at the origin. The G parabolic layer solution will, to first order, satisfy the boundary conditions at x = -1.

(d) The elliptic boundary layer, also termed the "corner", is obtained with $\Delta_1 = \Delta_2 = M^{-1}$ in which case the equation becomes:

$$\frac{\partial^2 G}{\partial X^2} + \frac{\partial^2 G}{\partial Y^2} + \frac{\partial G}{\partial X} = 0$$

$$\frac{\partial^2 F}{\partial X^2} + \frac{\partial^2 F}{\partial Y^2} - \frac{\partial F}{\partial X} = 0$$

It will be noted that no terms of the original equation have been lost in this degeneration (at least for the rectangular duct) with the only simplification being that a quarter-infinite region is now involved rather than a bounded plane. In addition, an exact solution for this corner layer will contain solutions to all orders which can be matched to parabolic layer solutions of all orders. As shown by Cook et. al. (10) for a situation related to the insulating wall case, the corner solution contains the parabolic layer solution to all orders. With these remarks, it is not surprising that additional approximations may be required for the solution of this corner problem, as is the case studied in this thesis where the wall $y = L$ is conducting.

The various regions of application of the asymptotic forms is illustrated in Figures 2.2(a) and 2.2(b).

3.0 THE CIRCULAR TUBE

As for the rectangular duct, asymptotic solutions for a circular tube involve a core region and three boundary layers, the ordinary, parabolic and elliptic. With an insulating tube, the core and ordinary layer solutions were obtained by Shercliff (11), (12). Roberts (13) was able to obtain a parabolic layer solution resulting in a uniform first order approximation throughout the tube. No elliptic layer was necessary. With conducting walls, this is no longer the case, as will be shown below. The asymptotic method of Matched Asymptotic Expansions (Van Dyke, (9)) will be followed in detail to obtain the solutions.

3.1 The Core

The regular perturbation expansions in the interior $r < 1$ is obtained by assuming F and G may be expanded in the following asymptotic sequences:

$$F_c(x, y) = \varepsilon_1^C(M)\ F_1(x, y) + \varepsilon_2^C(M)\ F_2(x, y) + \ldots$$

$$G_c(x, y) = \delta_1^C(M) G_1(x, y) + \delta_2^C(M) G_2(x, y) + \ldots$$

where the $F_k(x, y)$ and $G_k(x, y)$ are all of order unity and:

$$\lim_{M \to \infty} \frac{\varepsilon_{k+1}^C(M)}{\varepsilon_k^C(M)} = \lim_{M \to \infty} \frac{\delta_{k+1}^C(M)}{\delta_k^C(M)} \to 0 \qquad (3.1)$$

Substitution into equation (2.12) and (2.13) results in:

$$\varepsilon_1^C \nabla^2 F_1 + \varepsilon_2^C \nabla^2 F_2 + \ldots + M\left(\varepsilon_1^C \frac{\partial F_1}{\partial x} + \varepsilon_2^C \frac{\partial F_2}{\partial x} + \ldots\right) = 0$$

$$\delta_1^C \nabla^2 G_1 + \delta_2^C \nabla^2 G_2 + \ldots - M\left(\delta_1^C \frac{\partial G_1}{\partial x} + \delta_2^C \frac{\partial G_2}{\partial x} + \ldots\right) = 0$$

Dividing through by $M\varepsilon_1^C$ and $M\delta_1^C$ and taking the limit, using the property (3.1) results in:

$$\frac{\partial F_1}{\partial x} = 0 = \frac{\partial G_1}{\partial x}$$

Thus, F_1 and G_1 depend only on y. The leading terms in the regular expansion for F and G are therefore:

$$F_c^{(1)} \sim \varepsilon_1^C(M) F_1(y)$$

$$G_c^{(1)} \sim \delta_1^C(M)G_1(y)$$

Since 2H = F - G is odd in x by equations (2.9), $\varepsilon_1^C(M)F_1(y) - \delta_1^C(M)G_1(y)$ must also be odd in x, and since this quantity is independent of x it follows that:

$$F_c^{(1)} = G_c^{(1)} = \varepsilon_1^C(M)F_1(y) = \delta_1^C(M)G_1(y) \qquad (3.2)$$

The leading terms in the regular expansion for the velocity and magnetic field are:

$$u_c^{(1)} \sim \varepsilon_1^C(M)F_1(y) = \varepsilon_1^C(M)F_1(r\sin\theta)$$

$$H_o^{(1)} \sim -\frac{P}{M}x = -\frac{P}{M}r\cos\theta$$

These solutions cannot satisfy all the boundary conditions at r = 1. They must match appropriate "inner" expansions valid near r = 1. The unknowns $\varepsilon_1^C(M)$, $\delta_1^C(M)$, $F_1(y)$ are obtained from the matching conditions.

3.2 The Ordinary (Shercliff) Boundary Layer

In polar coordinates, the equation for F (equation 2.12) becomes:

$$\frac{\partial^2 F}{\partial r^2} + \frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2} + M(\cos\theta \frac{\partial F}{\partial r} - \frac{\sin\theta}{r}\frac{\partial F}{\partial \theta}) = 0$$

Following the usual procedure for matched asymptotic expansion, a stretched variable X_s or local coordinate near the wall is introduced;

$$X_s = \frac{1 - r}{\Delta(M)}$$

where $\Delta(M)$ is a stretching factor to be determined. The resulting equation for F is:

$$\frac{1}{\Delta^2}\frac{\partial^2 F}{\partial X_s^2} - \frac{1}{\Delta(1 - \Delta X_s)}\frac{\partial F}{\partial X_s} + \frac{1}{(1 - \Delta X_s)^2}\frac{\partial^2 F}{\partial \theta^2} -$$

$$M(\frac{\cos\theta}{\Delta}\frac{\partial F}{\partial X_s} + \frac{\sin\theta}{1 - \Delta X_s}\frac{\partial F}{\partial \theta}) = 0 \qquad (3.3)$$

An asymptotic expansion is now assumed with:

$$F_s(X_s, \theta) \sim \varepsilon_{1s}(M) F_{1s}(X_s, \theta) + \varepsilon_{2s}(M) F_{2s}(X_s, \theta) + \ldots$$

where, again, the $F_{ks}(X_s, \theta)$ are of order unity and a limit condition like (3.1) exists for the new order factors $\varepsilon_{ks}(M)$. Substitution into equation (3.3) followed with multiplication by $\Delta^2/\varepsilon_{1s}(M)$ results in:

$$\frac{\partial^2 F_{1s}}{\partial X_s^2} + \frac{\varepsilon_{2s}}{\varepsilon_{1s}} \frac{\partial^2 F_{2s}}{\partial X_s^2} + \ldots.$$

$$- \frac{\Delta}{(1 - \Delta X_s)} \left(\frac{\partial F_{1s}}{\partial X_s} + \frac{\varepsilon_{2s}}{\varepsilon_{1s}} \frac{\partial F_{2s}}{\partial X_s} + \ldots. \right)$$

$$+ \frac{\Delta^2}{(1 - \Delta X_s)^2} \left(\frac{\partial^2 F_{1s}}{\partial \theta^2} + \frac{\varepsilon_{2s}}{\varepsilon_{1s}} \frac{\partial^2 F_{2s}}{\partial \theta^2} + \ldots. \right)$$

$$- M \Big[\Delta \cos\theta \left(\frac{\partial F_{1s}}{\partial X_s} + \frac{\varepsilon_{2s}}{\varepsilon_{1s}} \frac{\partial F_{2s}}{\partial X_s} + \ldots. \right)$$

$$+ \frac{\Delta^2 \sin\theta}{(1 - \Delta X_s)} \left(\frac{\partial F_{1s}}{\partial \theta} + \frac{\varepsilon_{2s}}{\varepsilon_{1s}} \frac{\partial F_{2s}}{\partial \theta} + \ldots \right) \Big]$$

$$= 0$$

Taking the limit as $M \rightarrow \infty$, using the ordering property of the order factors $\varepsilon_{ks}(M)$, results in:

$$\frac{\partial^2 F_{1s}}{\partial X_s^2} - \lim_{M \rightarrow \infty} (M\Delta) \cos\theta \frac{\partial F_{1s}}{\partial X_s} = 0$$

By the principle of Minimum Degeneracy (Van Dyke, (9)) the stretching factor Δ must be $\Delta = M^{-1}$. Implicit in these steps is the assumption that:

$$\lim_{M \to \infty} \Delta \frac{\sin\theta}{\cos\theta} = 0$$

which is valid only if θ is sufficiently removed from $\pi/2$.

Using the same procedure for G_1 the two boundary layer equations become:

$$\frac{\partial^2 F_{1s}}{\partial X_s^2} - \cos\theta \frac{\partial F_{1s}}{\partial X_s} = 0$$

$$\frac{\partial^2 G_{1s}}{\partial X_s^2} + \cos\theta \frac{\partial G_{1s}}{\partial X_s} = 0$$

The general solutions which may be written by inspection are:

$$F_{1s} = A_1(\theta) + A_2(\theta) e^{X_s \cos\theta}$$

$$G_{1s} = B_1(\theta) + B_2(\theta) e^{-X_s \cos\theta} \qquad (3.4)$$

If attention is restricted to the region $\cos\theta > 0$ then $A_2(\theta) = 0$ in order that F_{1s} remain bounded. The boundary condition (equation 2.14 and 2.15), when written with the present inner variable, becomes:

$$\varepsilon_{1s}F_{1s} + \varepsilon_{2s}F_{2s} + \ldots - \delta_{1s}G_{1s} - \delta_{2s}G_{2s} \ldots$$

$$- MC[\varepsilon_{1s}\frac{\partial F_{1s}}{\partial X_s} + \varepsilon_{2s}\frac{\partial F_{1s}}{\partial X_s} + \ldots$$

$$- \delta_{1s}\frac{\partial G_{1s}}{\partial X_s} - \delta_{2s}\frac{\partial G_{2s}}{\partial X_s} \ldots]$$

$$= 2\frac{P}{M}(1 + C)\cos\theta$$

$$\varepsilon_{1s}F_{1s} + \varepsilon_{2s}F_{2s} + \ldots + \delta_{1s}G_{1s} + \delta_{2s}G_{2s} + \ldots = 0$$

It is convenient to introduce the parameter $\gamma = MC = C/\Delta$ and assume for the limiting process, that γ is of order unity. It is also convenient to introduce the "effective" pressure gradient $P' = P(1 + C)$. Dividing each of the boundary equations by ε_{1s} and taking the limit as $M \rightarrow \infty$ results in:

$$F_{1s} - \lim_{M\rightarrow\infty}\left(\frac{\delta_{1s}}{\varepsilon_{1s}}\right)G_{1s} - \gamma\left[\frac{\partial F_{1s}}{\partial X_s} - \lim_{M\rightarrow\infty}\left(\frac{\delta_{1s}}{\varepsilon_{1s}}\right)\frac{\partial G_{1s}}{\partial X_s}\right]$$

$$= \lim_{M\rightarrow\infty}\left(\frac{1}{\varepsilon_{1s}M}\right)2P'\cos\theta$$

$$F_{1s} + \lim_{M\rightarrow\infty}\left(\frac{\delta_{1s}}{\varepsilon_{1s}}\right)G_{1s} = 0$$

The principle of minimum degeneracy requires that $\varepsilon_{rs} = \delta_{1s} = M^{-1}$. Thus, at $X_s = 0$ the boundary conditions are:

$$F_{1s} - G_{1s} - \gamma\left(\frac{\partial F_{1s}}{\partial X_s} - \frac{\partial G_{1s}}{\partial X_s}\right) = 2P'\cos\theta$$

$$F_{1s} + G_{1s} = 0 \qquad (3.5)$$

Retention of the parameters $\gamma = Mc$ and $P' = P(1+c)$ in the analysis appears to be in contradiction with the asymptotic method of sizing relative terms. Thus it might appear that at least three different orders should be considered for the parameter c, that is:

$$0 \lesssim c \lesssim O(M^{-1})$$

$$O(M^{-1}) \lesssim c \lesssim O(1)$$

$$O(1) \lesssim c \lesssim \infty$$

since the parameter C occurs in the magnetic boundary condition in the form

$$h - Mc\frac{\partial h}{\partial X_s} = \frac{P}{M}(1+c)\cos\theta$$

With $\gamma = Mc$ it would appear that the results are limited to the case $c \sim O(M^{-1})$ or smaller. However, this is not the case which may be shown as follows.

With $C \sim O(1)$ or greater we may anticipate the sequence

$$h \sim \frac{1}{M^2} h_1^* + \frac{1}{M^3} h_2^* + \cdots$$

in which case the boundary conditions are:

$$\frac{\partial h_1^*}{\partial x_s} = - P(1 + 1/c) \cos\theta$$

$$\frac{\partial h_2^*}{\partial x_s} = \frac{1}{c} h_1^*$$

$$\vdots$$

This formulation includes the case $c \to \infty$ but is singular in the limit $c \to 0$ and is appropriate only for $c \sim O(1)$ or greater.

The alternate formulation is obtained by setting $\gamma = Mc$ and $P' = P(1+c)$ with the sequence

$$h \sim \frac{1}{M} h_1 + \frac{1}{M^2} h_2^* + \cdots$$

In this second case the boundary conditions are:

$$h_1 - \gamma \frac{\partial h_1}{\partial x_3} = P' \cos\theta$$

$$h_2 - \gamma \frac{\partial h_2}{\partial x_3} = 0$$

$$\vdots$$

This second formulation includes the case $c = 0$. It also includes the limit $c = \infty$, that is:

$$\frac{\partial h_1}{\partial x_3} = -\frac{P}{M} \cos\theta$$

$$\frac{\partial h_2}{\partial x_3} = 0$$

$$\vdots$$

The order of the result is raised with a net effect identical to that for the first formulation.

To show that the two formulations are equivalent for the case $c \sim O(1)$ we consider the one term, two term, etc., solutions. Thus, to first order,

$$h \sim \frac{1}{M^2} h_1^*$$

so that

$$h - Mc\frac{\partial h}{\partial x_s} \sim \frac{1}{M^2} h_1^* - \frac{c}{M}\frac{\partial h_1^*}{\partial x_s}$$

$$= \frac{P}{M}(1+c)\cos\theta + \frac{1}{M^2} h_1^*$$

$$\sim \frac{P}{M}(1+c)\cos\theta + O(M^{-2})$$

alternatively, with

$$h \sim \frac{1}{M} h_1$$

$$h - Mc\frac{\partial h}{\partial x_s} \sim \frac{1}{M} h_1 - c\frac{\partial h_1}{\partial x_s}$$

$$= \frac{1}{M}\left(h_1 - Mc\frac{\partial h_1}{\partial x_s}\right)$$

$$= \frac{P}{M}(1+c)\cos\theta$$

To first order, the one term solutions obtained by either formulation satisfy the same boundary condition.

To second order,

$$h \sim \frac{1}{M^2} h_1^* + \frac{1}{M^3} h_2^*$$

$$h - Mc \frac{\partial h}{\partial x_s} \sim \frac{1}{M^2} h_1^* + \frac{1}{M^3} h_2^* - \frac{c}{M} \frac{\partial h_1^*}{\partial x_s} - \frac{c}{M^2} \frac{\partial h_1^*}{\partial x_s}$$

$$= \frac{p}{M}(1+c)\cos\theta + O(M^{-3})$$

Alternatively.

$$h \sim \frac{1}{M} h_1 + \frac{1}{M^2} h_2$$

and

$$h - Mc \frac{\partial h}{\partial x_s} \sim \frac{1}{M} h_1 + \frac{1}{M^2} h_2 - c \frac{\partial h_1}{\partial x_s} - \frac{c}{M} \frac{\partial h_2}{\partial x_s}$$

$$= \frac{1}{M}\left[h_1 - Mc \frac{\partial h_1}{\partial x_s} + \frac{1}{M}\left(h_2 - Mc \frac{\partial h_2}{\partial x_s} \right) \right]$$

$$= \frac{p}{M}(1+c)\cos\theta$$

and to second order, the boundary condition satisfied by the two formulations is the same. It is now apparent that the second formulation is valid for any value of the conductivity parameter C.

Applying the boundary conditions (3.5) to the solution (3.4) results in:

$$F_{1s} = \left(P' + \frac{\gamma}{2} B_2\right) \cos\theta$$

$$G_{1s} = -\left(P' + \frac{\gamma}{2} B_2\right) \cos\theta - B_2\left(1 - e^{-X_s \cos\theta}\right)$$

The leading terms in the ordinary layer solutions are then:

$$F_s^{(1)} \sim \frac{1}{M}\left(P' + \frac{\gamma}{2} B_2\right) \cos\theta \qquad (3.6)$$

$$G_s^{(1)} \sim -\frac{1}{M}\left[\left(P' + \frac{\gamma}{2} B_2\right) \cos\theta + B_2\left(1 - e^{-X_s \cos\theta}\right)\right]$$

The unknown function $B_2(\theta)$ will be obtained by matching with the core solutions. Writing equation (3.6) in the outer core coordinates (r,θ) and expanding for large M gives:

$$F_s^{(1)} \sim \frac{1}{M}\left(P' + \frac{\gamma}{2} B_2\right) \cos\theta$$

$$G_s^{(1)} \sim -\frac{1}{M}\left[\left(P' + \frac{\gamma}{2} B_2\right) \cos\theta + B_2\right]$$

The core solutions, equation 3.2, expressed in "inner" coordinates (X_s, θ) are:

$$F_c^{(1)} = G_c^{(1)} \sim \varepsilon_1^C(M) F_1 \left\{(1 - \frac{X_s}{M}) \sin\theta\right\}$$

$$\sim \varepsilon_1^C(M) F_1 (\sin\theta) \qquad \text{as } M \rightarrow \infty$$

In order that these limiting solutions match, it is necessary to set $\varepsilon_1^C(M) = M^{-1}$ and:

$$F_1(\sin\theta) = (P' + \frac{\gamma}{2} B_2) \cos\theta$$

$$= - [(P' + \frac{\gamma}{2} B_2) \cos\theta + B_2]$$

Therefore, $$B_2(\theta) = - \frac{2P' \cos\theta}{1 + \gamma \cos\theta}$$

Replacing γ and P' by MC and $P(1 + C)$ respectively, the outer and inner solutions are:

$$F_c^{(1)} = G_c^{(1)} \sim \frac{P(1 + C)\sqrt{1 - y^2}}{M[1 + MC\sqrt{1 - y^2}]}$$

$$F_s^{(1)} \sim \frac{P(1 + C) \cos\theta}{M[1 + MC \cos\theta]}$$

$$G_s^{(1)} \sim \frac{P(1 + C) \cos\theta}{M[1 + MC \cos\theta]} (1 - 2 e^{-X_s \cos\theta})$$

For $\cos\theta > 0$, the velocity and magnetic field are easily derived:

$$u_c^{(1)} \sim \frac{P(1 + C)\sqrt{1 - y^2}}{M[1 + MC\sqrt{1 - y^2}]} \tag{3.7}$$

$$u_s^{(1)} \sim \frac{P(1 + C)\cos\theta}{M[1 + MC\cos\theta]}[1 - e^{-M(1 - r)\cos\theta}] \tag{3.8}$$

$$H_c^{(1)} \sim -\frac{P}{M} r\cos\theta \tag{3.9}$$

$$H_s^{(1)} \sim -\frac{P}{M}\cos\theta\left[r - \frac{(1 + C)\ e^{-M(1 - r)\cos\theta}}{1 + MC\cos\theta}\right] \tag{3.10}$$

Solutions for $\cos\theta < 0$ may be obtained directly from the symmetry conditions (equation 2.9). The method used here can be readily extended to obtain higher order approximations (see for example Van Dyke, (9).

3.3 Discussion of the Results

The results given above are valid for any value of the wall conductivity parameter C. Figures 3.1 and 3.2 show the effects of wall conductivity on the velocity and Figures 3.3 and 3.4 show the effects on the magnetic field.

As indicated previously, the treatment above cannot be uniformly valid for all values of θ. This is clearly evident from equation (3.8), the boundary layer velocity. At $\theta = \pi/2$, this velocity is zero, independent of r, a result which cannot be correct. Thus, one or more additional boundary layers are required in the vicinity of $\theta = \pi/2$.

The solution $H_s^{(1)}$ (equation 3.10) contains $H_c^{(1)}$ and provides directly a composite solution. A composite solution for u may be easily constructed in the form:

$$u \sim \frac{P(1+c)\sqrt{1-r^2\sin^2\theta}}{M\left[1+Mc\sqrt{1-r^2\sin^2\theta}\right]}\left(1-e^{-M(1-r)\cos\theta}\right)$$

which reduces to $u_c^{(1)}$ or $u_s^{(1)}$ when the appropriate variables are substituted and limits taken.

3.4 The Parabolic Boundary Layer, Moderate Wall Conductivity

Following the preceding discussion, an additional solution valid near r = 1 and $\theta = \pi/2$ is necessary. This boundary layer should match both the core and the ordinary layer solutions.

An appropriate set of coordinates $R, \emptyset$ centered about $x = 0$, $y = 1$ is illustrated in Figure 3.5.

These coordinates are defined by:

$x = R \cos \emptyset$

$y = 1 - R \sin \emptyset$

The boundary defined by $x^2 + y^2 = 1$ is given by $R = 2 \sin \emptyset$ in the $R, \emptyset$ coordinates. Introducing the variable $s = \sin \emptyset$, the equations 2.12 and 2.13 are:

$$\frac{\partial^2 F}{\partial R^2} + \frac{1}{R}\frac{\partial F}{\partial R} + \frac{1}{R^2}\left[(1-s^2)\frac{\partial^2 F}{\partial s^2} - s\frac{\partial F}{\partial s}\right]$$

$$- M\sqrt{1-s^2}\left[\frac{\partial F}{\partial R} - \frac{s}{R}\frac{\partial F}{\partial s}\right] = 0$$

$$\frac{\partial^2 G}{\partial R^2} + \frac{1}{R}\frac{\partial G}{\partial R} + \frac{1}{R^2}\left[(1-s^2)\frac{\partial^2 G}{\partial s^2} - s\frac{\partial G}{\partial s}\right]$$

$$+ M\sqrt{1-s^2}\left[\frac{\partial F}{\partial R} - \frac{s}{R}\frac{\partial F}{\partial s}\right] = 0$$

The boundary conditions at $R = 2s$ become:

$$F - G + c\left[(R-s)\frac{\partial}{\partial R}(F-G) + \frac{(s^2-1)}{R}\frac{\partial}{\partial s}(F-G)\right]$$

$$= 2\frac{P}{M}(1+c)R\sqrt{1-s^2}$$

$$F + G = 0$$

As before, asymptotic sequences are assumed:

$$F_p \sim \varepsilon_{1p}(M) F_{1p} + \varepsilon_{2p}(M) F_{2p} + \dots$$

$$G_p \sim \delta_{1p}(M) G_{1p} + \delta_{2p}(M) G_{2p} + \dots$$

and now both independent variables are stretched:

$$X_p = \frac{R}{\Delta_1(M)}$$

$$Y_p = \frac{s}{\Delta_2(M)}$$

with ε_{kp}, δ_{kp}, Δ_1 and Δ_2 to be determined by the limit process.

The equations for the leading terms are, after some manipulation:

$$\frac{1}{X_p^2} \frac{\partial^2 F_{1p}}{\partial Y_p^2} - \lim_{M \to \infty} (M \Delta_1 \Delta_2^2) \left[\frac{\partial F_{1p}}{\partial X_p} - \frac{Y_p}{X_p} \frac{\partial F_{1p}}{\partial Y_p}\right] = 0$$

$$\frac{1}{X_p^2} \frac{\partial^2 G_{1p}}{\partial Y_p^2} + \lim_{M \to \infty} (M \Delta_1 \Delta_2^2) \left[\frac{\partial G_{1p}}{\partial X_p} - \frac{Y_p}{X_p} \frac{\partial G_{1p}}{\partial Y_p}\right] = 0$$

and the principle of minimum degeneracy dictates that:

$$\Delta_1 \Delta_2^2 = M^{-1}$$

The boundary conditions to first order are:

$$F_{1p} - G_{1p} - \lim_{M \to \infty} \left(\frac{C}{\Delta_1 \Delta_2}\right) \frac{1}{X_p} \frac{\partial}{\partial Y_p} (F_{1p} - G_{1p})$$

$$= \lim_{M \to \infty} \left(\frac{\Delta_1}{M \varepsilon_{1p}}\right) 2P' X_p$$

A modified conductivity parameter γ_p may be introduced with $\gamma_p = C/\Delta_1 \Delta_2$ and γ_p assumed finite in the limit process, a procedure similar to that employed with the ordinary layer. The factor $\Delta_1 X_p$ appearing on the right hand side of the boundary condition can equally well be replaced by $2\Delta_2 Y_p$ from the equation for the boundary, in which case the boundary condition would be:

$$F_{1p} - G_{1p} - \frac{\gamma_p}{X_p} \frac{\partial}{\partial Y_p} (F_{1p} - G_{1p}) = \lim_{M \to \infty} \left(\frac{\Delta_2}{M \varepsilon_{1p}}\right) 4P' Y_p$$

Minimum degeneracy now dictates that:

$$\frac{\Delta_1}{M \varepsilon_{1p}} = \frac{\Delta_2}{M \varepsilon_{1p}} = 1$$

or $\Delta_1 = \Delta_2 = M^{-1/3}$ and $\varepsilon_{1p} = M^{-4/3}$

The final set of parabolic equations for the leading terms is:

$$\frac{\partial^2 F_{1p}}{\partial Y_p^2} - X_p^2 \frac{\partial F_{1p}}{\partial X_p} + X_p Y_p \frac{\partial F_{1p}}{\partial Y_p} = 0 \qquad (3.11)$$

$$\frac{\partial^2 G_{1p}}{\partial Y_p^2} + X_p^2 \frac{\partial G_{1p}}{\partial X_p} - X_p Y_p \frac{\partial G_{1p}}{\partial Y_p} = 0$$

$$F_p \sim M^{-4/3} F_{1p}(X_p, Y_p) + \cdots \qquad (3.12)$$

$$G_p \sim M^{-4/3} G_{1p}(X_p, Y_p) + \cdots$$

and at $X_p = 2\,Y_p$

$$X_p\,(F_{1p} - G_{1p}) - \gamma_p\left(\frac{\partial F_{1p}}{\partial Y_p} - \frac{\partial G_{1p}}{\partial Y_p}\right) = 2\,P'\,X_p^2 \qquad (3.13)$$

$$F_{1p} + G_{1p} = 0$$

The assumption that γ_p remains finite (rather than γ) is equivalent to examining the case with moderate or large wall conductivity. To assume γ remains finite during the limit process would be in keeping with the ordinary layer analysis but this results in solutions valid only for small conductivity, a case to be treated in section 3.5 below.

The equations considered by Roberts (13) may be obtained by expressing equation (3.11) in the coordinates $\xi = X_p(2Y_p - X_p)/4$ and $\eta = X_p$. For insulating walls, Roberts was able to find an exact solution involving infinite integrals of Airy functions. An attempt to apply Roberts' procedures to the more general boundary conditions considered here, failed. No other methods for obtaining solutions in closed form could be found. It is possible however, to obtain approximate solutions for small X_p by introducing the coordinates:

$$T = X_p^{1/2} \qquad Q = X_p^{1/2}(2Y_p - X_p)/4$$

The equations (3.11) become:

$$\frac{\partial^2 F_{1p}}{\partial Q^2} + 2(Q + T^3)\frac{\partial F_{1p}}{\partial Q} - 2T\frac{\partial F_{1p}}{\partial T} = 0$$
$$\frac{\partial^2 G_{1p}}{\partial Q^2} - 2(Q + T^3)\frac{\partial G_{1p}}{\partial Q} + 2T\frac{\partial G_{1p}}{\partial T} = 0 \qquad (3.14)$$

and at Q = 0, equation (3.13) is:

$$T(F_{1p} - G_{1p}) - \frac{\gamma_p}{2}\left(\frac{\partial F_{1p}}{\partial Q} - \frac{\partial G_{1p}}{\partial Q}\right) = 2P'T^3$$

$$F_{1p} + G_{1p} = 0 \qquad (3.15)$$

Approximate solutions valid for T small may be obtained by assuming an expansion of the form:

$$F_{1p}(Q, T) = V_o(Q) + TV_1(Q) + T^2V_2(Q) + \ldots$$

$$G_{1p}(Q, T) = W_o(Q) + TW_1(Q) + T^2W_2(Q) + \ldots$$

Substituting these expansions in equation (3.14) and equating coefficients of like powers of T results in:

$$V_n'' + 2QV_n' - 2nV_n = 0$$

$$n = 0, 1, 2, 3$$

$$W_n'' - 2QW_n' + 2nW_n = 0$$

$$V_n'' + 2QV_n' - 2nV_n = -2V_{n-3}'$$

$$n = 4, 5, 6, \ldots$$

$$W_n'' - 2QW_n' + 2nW_n = 2W_{n-3}'$$

Solutions for the homogeneous parts are the Hermite polynomials and integrated error functions:

$$V_n(Q) = H_n^*(Q); \qquad i^n\text{erfc}(Q)$$

$$W_n(Q) = H_n(Q); \qquad i^n\text{erfc}(iQ)$$

where $\qquad H_k^*(Q) = (-i)^k H_k(iQ)$

The integrated error functions of imaginary argument are unbounded for large Q and must be rejected. Construction of the solutions consists of determining the arbitrary constants from the boundary conditions and matching with the ordinary layer. This procedure is straightforward but tedious and will not be reproduced here, other than to note that only the linear terms in cos θ (cubic terms in T) will match correctly for the G solution. This is not surprising since during the matching it has been assumed that $\cos\theta \gtrsim (2 - 2\sin\theta)^{1/2}$, a relation which is valid only to linear terms. Writing γ_p as $CM^{2/3}$, the solutions are:

$$F_p^{(1)} \sim \frac{1}{M^2} P(1 + 1/C)$$

$$G_p^{(1)} \sim \frac{1}{M^2} P(1 + 1/C)[-1 - \frac{1}{3}T^3 H_3(Q)$$

$$+ T^6 (\frac{1}{120} H_6(Q) - \frac{1}{2}H_2(Q)) - \frac{2}{CM^{2/3}} TH_1(Q)$$

$$+ \frac{T^4}{CM^{2/3}} (\frac{1}{12} H_4(Q) - H_o(Q)) + \dots]$$

The velocity and magnetic field are:

$$u_p^{(1)} \sim \frac{1}{M^2} \frac{P}{2}(1 + 1/C) [- \frac{1}{3}T^3 H_3(Q)$$

$$+ \frac{T^6}{2}[\frac{1}{60} H_6(Q) - H_2(Q)] - \frac{2}{CM^{2/3}} TH_1(Q)$$

$$+ \frac{T^4}{CM^{2/3}}(\frac{1}{12} H_4(Q) - H_o(Q)) + \dots]$$

$$H_p^{(1)} \sim - \frac{P}{M}r \cos\theta + \frac{1}{M^2} \frac{P}{2}(1 + 1/C)[1 + \frac{1}{3} T^3 H_3(Q)$$

$$- \frac{T^2}{2}(\frac{1}{60} H_6(Q) - H_2(Q)) + \frac{2}{CM^{2/3}} TH_1(Q)$$

$$- \frac{T^4}{CM^{2/3}}(\frac{1}{12} H_4(Q) - H_o(Q)) + \dots]$$

From the symmetry conditions, equations 2.9, the magnetic field must be zero along the line $\theta = \pi/2$, and in particular, at the point $r = 1$, $\theta = \pi/2$. This point corresponds to $Q = T = 0$ and it is apparent that the above solution results in a non-zero field at this point. This dilemma indicates that, at least for moderately and highly conducting walls, the parabolic layer approximation cannot be applied in the immediate vicinity of $r = 1$, $\theta = \pi/2$. This result is perhaps surprising since Roberts was able to construct a uniformly valid approximation for the pipe with insulating walls using only a parabolic layer solution. To confirm this, the following solutions for small wall conductivity will be obtained by the same approximate methods used in the preceding analysis.

3.5 The Parabolic Boundary Layer, Small Wall Conductivity

With walls of small conductivity, the equation 3.11 remains valid. However, the boundary conditions and ε_{1p}, δ_{1p}, must be redetermined. Introducing $C = \gamma/M$ in the boundary condition and making the assumption that $\gamma \sim O(1)$ as $M \to \infty$, we obtain, as before, $\varepsilon_{1p} = \varepsilon_{1p} = M^{-4/3}$ but the boundary conditions at $X_p = 2Y_p$ are:

$$F_{1p} = PX_p = 2PY_p$$

$$G_{1p} = -PX_p = -2PY_p$$

It will be noted that γ is eliminated from the boundary conditions for the leading terms. Introducing the variables Q, T and making the substitutions:

$$v = T^{-2/3}F_{1p}(Q, T)$$

$$w = T^{-2/3}G_{1p}(Q, T)$$

into equation 3.11 results in:

$$\frac{\partial^2 v}{\partial Q^2} + 2(Q + T)\frac{\partial v}{\partial Q} - 6T\frac{\partial v}{\partial T} - 4v = 0$$

$$\frac{\partial^2 w}{\partial Q^2} - 2(Q + T)\frac{\partial w}{\partial Q} + 6T\frac{\partial w}{\partial T} + 4w = 0$$

and at $Q = 0$, the boundary conditions are:

$$v = -w = P$$

Expanding w and v in powers of T, results in equations with repeated error integrals and Hermite polynomials, as before. Discarding the unbounded solutions, introducing the boundary condition and matching to first order in T with the ordinary layer solution results in:

$$F_{1p}(Q, T) = PT^{2/3}[\frac{1}{2} H_2^*(Q) + T (\frac{1}{2} H_1^*(Q) - \frac{1}{120} H_5^*(Q))$$

$$+ T^2(\frac{1}{8} H_o^*(Q) - \frac{1}{48} H_4^*(Q) + \frac{1}{13440} H_8^*(Q))$$

$$+ O(T^3)]$$

$$G_{1p}(Q,T) = PT^{2/3}[\frac{1}{2}H_2(Q) + T(\frac{1}{2}H_1(Q) + \frac{1}{40}H_5(Q))$$

$$+ T^2(\frac{1}{8}H_o(Q) + \frac{5}{80}H_4(Q) - \frac{7}{13440}H_8(Q))$$

$$+ O(T^3)]$$

It is apparent that these solutions will satisfy the necessary condition at the point $Q = T = 0$, that is, the magnetic field derived from these solutions will be zero. Thus, valid solutions for small wall conductivity have been obtained using a method virtually identical to that employed for highly conducting walls where the solution was not uniformly valid at the point $Q = T = 0$.

The only conclusion that can be reached from this is that with highly conducting walls, an additional boundary layer is necessary near the point $Q = T = 0$.

3.6 The Elliptic Boundary Layer Equations

As shown in the preceding sections, an asymptotic expansion in the parabolic boundary layer leads to uniformly valid solutions only with insulating walls, or walls of small conductivity. For moderate or large wall conductivities, yet another boundary layer must be obtained near $r = 1$ and $\Theta = \pi/2$. To develop the equations for this layer, it is necessary to consider only the equation for F since the development for G is identical.

Introducing new independent variables $R = 1 - r$, $S = \cos\Theta$, equation (2.12) becomes:

$$\frac{\partial^2 F}{\partial R^2} - \frac{1}{(1 - R)}\frac{\partial F}{\partial R} + \frac{1}{(1 - R)^2}\left[(1 - S^2)\frac{\partial^2 F}{\partial S^2} - S\frac{\partial F}{\partial S}\right]$$

$$-M\left[S\frac{\partial F}{\partial R} - \frac{(1 - S^2)}{(1 - R)}\frac{\partial F}{\partial S}\right] = 0$$

With the stretched local variables X and Y defined by:

$$X = \frac{R}{\Delta_1(M)}, \qquad Y = \frac{S}{\Delta_2(M)}$$

and an asymptotic sequence introduced:

$$F_e \sim \mathcal{E}_{1e}(M) F_{1e}(x,y) + \mathcal{E}_{2e}(M) F_{2e}(x,y) + \ldots$$

This equation becomes:

$$\frac{1}{\Delta_1^2}\left(\mathcal{E}_{1e}\frac{\partial^2 F_{1e}}{\partial X^2} + \mathcal{E}_{2e}\frac{\partial^2 F_{2e}}{\partial X^2} + \ldots\right)$$

$$- \frac{1}{(1 - \Delta_1 X)} \frac{1}{\Delta_1}\left(\mathcal{E}_{1e}\frac{\partial F_{1e}}{\partial X} + \mathcal{E}_{2e}\frac{\partial F_{2e}}{\partial X} + \ldots\right)$$

$$+ \frac{1}{(1 - \Delta_1 X)^2}\left[\frac{(1 - \Delta_2^2 Y^2)}{\Delta_2^2}\left(\mathcal{E}_{1e}\frac{\partial^2 F_{1e}}{\partial Y^2} + \mathcal{E}_{2e}\frac{\partial^2 F_{1e}}{\partial Y^2} + \ldots\right)\right.$$

$$\left. - Y\left(\mathcal{E}_{1e}\frac{\partial F_{1e}}{\partial Y} + \mathcal{E}_{2e}\frac{\partial F_{2e}}{\partial Y} + \ldots\right)\right]$$

$$-M\left[\frac{\Delta_2}{\Delta_1} Y\left(\mathcal{E}_{1e}\frac{\partial F_{1e}}{\partial X} + \mathcal{E}_{2e}\frac{\partial F_{2e}}{\partial X} + \ldots\right)\right.$$

$$\left. - \frac{(1 - \Delta_2^2 Y^2)}{(1 - \Delta_1 X)\Delta_2}\left(\mathcal{E}_{1e}\frac{\partial F_{1e}}{\partial Y} + \mathcal{E}_{2e}\frac{\partial F_{2e}}{\partial Y} + \ldots\right)\right]$$

$$= 0$$

Multiplying through by $\Delta_1^2/\mathcal{E}_{1e}$ and taking the limit as $M \to \infty$ results in:

$$\frac{\partial^2 F_{1e}}{\partial X^2} + \lim_{M\to\infty}\left(\frac{\Delta_1^2}{\Delta_2^2}\right)\frac{\partial^2 F_{1e}}{\partial Y^2} - \lim_{M\to\infty}(M\Delta_1\Delta_2)Y\frac{\partial F_{1e}}{\partial X}$$

$$+ \lim_{M\to\infty}\left(M\frac{\Delta_1^2}{\Delta_2}\right)\frac{\partial F_{1e}}{\partial Y} = 0$$

We now note that if $\Delta_2 = 1$ and $\Delta_1 = M^{-1}$, only the first and third terms remain as the ordinary boundary layer equation. If $\Delta_1 = M^{-2/3}$ and $\Delta_2 = M^{-1/3}$, the parabolic layer equation is obtained in the form:

$$\frac{\partial^2 F_{1e}}{\partial X^2} - Y\frac{\partial F_{1e}}{\partial X} + \frac{\partial F_{1e}}{\partial Y} = 0$$

This is the equation used by Roberts, and as indicated in Section 3.4 above, the parabolic layer equations used in that section, could, by an appropriate change of variable, be written in this form.

An additional limit may be obtained by setting $\Delta_1 = \Delta_2 = M^{-1}$ in which case the elliptic boundary layer equation results:

$$\frac{\partial^2 F_{1e}}{\partial X^2} + \frac{\partial^2 F_{1e}}{\partial Y^2} + \frac{\partial F_{1e}}{\partial Y} = 0 \qquad (3.16)$$

An identical development for G results in:

$$\frac{\partial^2 G_{1e}}{\partial X^2} + \frac{\partial^2 G_{1e}}{\partial Y^2} - \frac{\partial G_{1e}}{\partial Y} = 0 \qquad (3.17)$$

These equations are very similar to the original equations, though they are now defined over a semi-infinite plane rather than a finite circular region. To complete the set, the ordering factors for F and G are $\varepsilon_{1e}(M) = \delta_{1e}(M) = M^{-3}$ as obtained from perfectly conducting wall boundary conditions, which are, at X = 0:

$$\frac{\partial F_{1e}}{\partial X} - \frac{\partial G_{1e}}{\partial X} = -2PY \qquad (3.18)$$

$$F_{1e} + G_{1e} = 0 \qquad (3.19)$$

An integral equation formulation for this problem may be obtained by noting that if $f = \exp(-Y/2)F_{1e}$ and $g = \exp(Y/2)G_{1e}$, the equations reduce to:

$$\frac{\partial^2 f}{\partial X^2} + \frac{\partial^2 f}{\partial Y^2} - \frac{1}{4}f = 0$$

$$\frac{\partial^2 g}{\partial X^2} + \frac{\partial^2 g}{\partial Y^2} - \frac{1}{4}g = 0$$

for which a Green's function may readily be obtained and the integral equation found from the boundary conditions. Intrinsic difficulties arise from the unbounded nature of condition 3.18 and since the resulting solutions will be of order M^{-3}, contributions from this elliptic region to the major parameters of interest will be very small, of order M^{-5}. Second order terms for the core and ordinary layers are of more practical interest.

3.7 Second Order Solutions

The results of sections 3.1 and 3.2 (the core and ordinary layer) may be extended to include second order terms. Thus, the second order core is written as:

$$F_c \sim M^{-1} F_{1c}(y) + \varepsilon_{2c} F_{2c} + \cdots$$

$$G_c \sim M^{-1} F_{1c}(y) + \delta_{2c} G_{2c} + \cdots$$

with $F_{1c}(y) = \dfrac{P'(1-y^2)^{1/2}}{1+\gamma(1-y^2)^{1/2}}$

in which case the governing equations are:

$$\frac{1}{M}\frac{\partial^2 F_{1c}}{\partial y^2} + \varepsilon_{2c}\nabla^2 F_{2c} + M\varepsilon_{2c}\frac{\partial F_{2c}}{\partial x} + \cdots = 0$$

$$\frac{1}{M}\frac{\partial^2 F_{1c}}{\partial y^2} + \delta_{2c}\nabla^2 G_{2c} - M\delta_{2c}\frac{\partial G_{2c}}{\partial x} + \cdots = 0$$

The principle of minimum degeneracy dictates that $\varepsilon_{2c} = \delta_{2c} = M^{-2}$ and

$$\frac{\partial F_{2c}}{\partial x} = -\frac{\partial^2 F_{1c}}{\partial y^2}$$

$$\frac{\partial G_{2c}}{\partial x} = \frac{\partial^2 F_{1c}}{\partial y^2}$$

Solutions are easily obtained in the form

$$F_{2c} = \frac{P'}{M^2}\left[f_{2c}(y) + \frac{x(1-y^2)^{1/2}}{\left[1+\gamma(1-y^2)^{1/2}\right]^2}\left\{ 1 + \frac{2\gamma y^2 (1-y^2)^{1/2}}{1+\gamma(1-y^2)^{1/2}} \right\}\right]$$

$$G_{2c} = \frac{P'}{M^2}\left[f_{2c}(y) - \frac{x(1-y^2)^{1/2}}{\left[1+\gamma(1-y^2)^{1/2}\right]^2}\left\{ 1 + \frac{2\gamma y^2 (1-y^2)^{1/2}}{1+\gamma(1-y^2)^{1/2}} \right\}\right]$$

The unknown function $f_{2c}(y)$ must be obtained through matching with the second order ordinary layer. Thus, defining

$$F_s \sim M^{-1} F_{1s} + \varepsilon_{2s} F_{2s} + \cdots$$

$$G_s \sim M^{-1} G_{1s} + \delta_{2s} G_{2s} + \cdots$$

with

$$F_{1s} = \frac{P' \cos\theta}{1+\gamma\cos\theta}$$

$$G_{1s} = \frac{P' \cos\theta}{1+\gamma\cos\theta}\left(1 - 2e^{-x_s\cos\theta}\right)$$

one obtains: $\varepsilon_{2s} = \delta_{2s} = M^{-2}$

$$\frac{\partial^2 F_{2s}}{\partial X_s^2} - \cos\theta \frac{\partial F_{2s}}{\partial X_s} = - \frac{P' \sin^2\theta}{(1+\gamma\cos\theta)^2}$$

$$\frac{\partial^2 G_{2s}}{\partial X_s^2} + \cos\theta \frac{\partial G_{2s}}{\partial X_s} = \frac{2 P' \cos^2\theta}{1+\gamma(\cos\theta)} e^{-X_s\cos\theta}$$

$$+ \frac{P' \sin^2\theta}{(1+\gamma\cos\theta)^2} \gamma\left(1 - 2 e^{-X_s\cos\theta}\right)$$

$$+ \frac{2 P' X_s \cos\theta \sin^2\theta}{1+\gamma\cos\theta} e^{-X_s\cos\theta}$$

with boundary conditions at $X_s = 0$

$$F_{2s} - G_{2s} - \gamma\left(\frac{\partial F_{2s}}{\partial X_s} - \frac{\partial G_{2s}}{\partial X_s}\right) = 0$$

$$F_{2s} + G_{2s} = 0$$

Bounded solutions for the differential equations are

$$F_{2s} = A_1(\theta) + \frac{P' X_s \sin\theta \tan\theta}{(1+\gamma\cos\theta)^2}$$

$$G_{2s} = B_1(\theta) + B_2(\theta) e^{-X_s\cos\theta}$$

$$+ \frac{P' \sin\theta \tan\theta}{(1+\gamma\cos\theta)^2} X_s \left(1 + 2 e^{-X_s\cos\theta}\right)$$

$$- \frac{2 P' \cos\theta}{1+\gamma\cos\theta} X_s e^{-X_s\cos\theta}$$

$$- \frac{2 P' \sin\theta \tan\theta}{1+\gamma\cos\theta}\left(X_s + \frac{1}{2} X_s^2 \cos\theta\right) e^{-X_s\cos\theta}$$

Application of the boundary conditions is straight forward as is matching with the core. An interesting feature of the matching is the occurrence of higher order contributions when the first term core is expanded in ordinary layer variables, that is:

$$F_c \sim \frac{p'}{M}\left[\frac{\cos\theta}{1+\gamma\cos\theta} + \frac{1}{M} X_s \frac{\sin\theta\tan\theta}{(1+\gamma\cos\theta)^2} + \cdots\right]$$

$$+ \frac{p'}{M^2}\left[f_{2c}(\sin\theta) + \frac{1}{\cos^2\theta(1+\gamma\cos\theta)^2}\left\{1 + \frac{2\gamma\sin^2\theta\cos\theta}{1+\gamma\cos\theta}\right\}\right] + \cdots$$

This "mixing" of orders is slightly undesirable and will be rectified through the use of "optimal" coordinates in section 3.9 below. Roberts (13) encountered a similar mixing of orders in his insulating wall parabolic layer solutions.

The core and ordinary layer solutions may now be written as:

$$F_c \sim \frac{p'}{M}\frac{(1-y^2)^{1/2}}{1+\gamma(1-y^2)^{1/2}}$$

$$-\frac{p'}{M^2}\left[\frac{(1-y^2)^{-3/2}}{[1+\gamma(1-y^2)^{1/2}]^2}\left\{\frac{(1-y^2)^{1/2}}{1+\gamma(1-y^2)^{1/2}} - x\right\}\left\{1 + \frac{2\gamma y^2(1-y^2)^{1/2}}{1+\gamma(1-y^2)^{1/2}}\right\} - \frac{\gamma[\gamma+(1-y^2)^{1/2}]}{[1+\gamma(1-y^2)^{1/2}]^3}\right]$$

$$G_c \sim \frac{P'}{M}\frac{(1-y^2)^{1/2}}{1+\gamma(1-y^2)^{1/2}}$$

$$-\frac{P'}{M^2}\left[\frac{(1-y^2)^{-3/2}}{[1+\gamma(1-y^2)^{1/2}]^2}\left\{\frac{(1-y^2)^{1/2}}{1+\gamma(1-y^2)^{1/2}}+x\right\}\left\{1+\frac{2\gamma y^2(1-y^2)^{1/2}}{1+\gamma(1-y^2)^{1/2}}\right\}-\frac{\gamma\left[\gamma+(1-y^2)^{1/2}\right]}{[1+\gamma(1-y^2)^{1/2}]^3}\right]$$

$$F_s \sim \frac{P'}{M}\frac{\cos\theta}{1+\gamma\cos\theta}$$

$$+\frac{P'}{M^2}\left[\frac{x_s\sin\theta\tan\theta}{(1+\gamma\cos\theta)^2}+\frac{\gamma(\gamma+\cos\theta)}{(1+\gamma\cos\theta)^3}\right.$$

$$\left.+\frac{\gamma}{\cos\theta(1+\gamma\cos\theta)^3}\left\{1+\frac{2\gamma\sin^2\theta\cos\theta}{1+\gamma\cos\theta}\right\}\right]$$

$$G_s \sim \frac{P'}{M}\frac{\cos\theta}{1+\gamma\cos\theta}\left(1-2e^{-x_s\cos\theta}\right)$$

$$+\frac{P'}{M^2}\left[\frac{x_s\sin\theta\tan\theta}{(1+\gamma\cos\theta)^2}+\frac{\gamma(\gamma+\cos\theta)}{(1+\gamma\cos\theta)^3}\left(1-2e^{-x_s\cos\theta}\right)\right.$$

$$-\frac{\gamma}{\cos\theta(1+\gamma\cos\theta)^3}\left\{1+\frac{2\gamma\sin^2\theta\cos\theta}{1+\gamma\cos\theta}\right\}$$

$$-\frac{2}{\cos^2\theta(1+\gamma\cos\theta)^3}\left\{1+\frac{2\gamma\sin^2\theta\cos\theta}{1+\gamma\cos\theta}\right\}\left(1-e^{-x_s\cos\theta}\right)$$

$$-\frac{2(\gamma+\cos\theta)}{(1+\gamma\cos\theta)^2}x_s e^{-x_s\cos\theta}$$

$$\left.-\frac{\sin^2\theta}{(1+\gamma\cos\theta)}x_s^2 e^{-x_s\cos\theta}\right]$$

The above solutions are valid for any γ including the limit $\gamma \to \infty$ in which case the solutions are:

$$F_c \sim \frac{P}{M^2} + \frac{P}{M^3}\frac{1}{(1-y^2)^{3/2}}$$

$$G_c \sim \frac{P}{M^2} + \frac{P}{M^3}\frac{1}{(1-y^2)^{3/2}}$$

$$F_s \sim \frac{P}{M^2} + \frac{P}{M^3}\frac{1}{\cos^3\theta}$$

$$G_s \sim \frac{P}{M^2}\left(1-2e^{-X_s\cos\theta}\right)$$

$$+\frac{P}{M^3}\left[\frac{1}{\cos^3\theta}\left(1-2e^{-X_s\cos\theta}\right)-\frac{2X_s}{\cos^2\theta}e^{-X_s\cos\theta}\right.$$

$$\left.-\sin\theta\tan\theta\, X_s^2 e^{-X_s\cos\theta}\right]$$

Verification of this limiting case is easily obtained by direct substitution in the appropriate core and ordinary layer equations and perfectly conducting boundary conditions.

3.8 The Flow Integral

From the definition of F and G, the fluid velocity is simply:

$$\mathcal{U} = \frac{1}{2}(F + G)$$

so that in the core

$$\mathcal{U}_c \sim \frac{P_0'}{M}\frac{(1-y^2)^{1/2}}{1+\gamma(1-y^2)^{1/2}} - \frac{P'}{M^2}\left[\frac{(1-y^2)^{-1}}{[1+\gamma(1-y^2)^{1/2}]^3}\left\{1 + \frac{2\gamma y^2(1-y^2)^{1/2}}{1+\gamma(1-y^2)^{1/2}}\right\} - \frac{\gamma[\gamma+(1-y^2)^{1/2}]}{[1+\gamma(1-y^2)^{1/2}]^3}\right]$$

and in the ordinary layer

$$\mathcal{U}_s \sim \frac{P'}{M}\frac{\cos\theta}{1+\gamma\cos\theta}\left(1-e^{-X_s\cos\theta}\right)$$

$$+ \frac{P'}{M^2}\left[\frac{X_s\sin\theta\tan\theta}{(1+\gamma\cos\theta)^2} + \frac{\gamma(\gamma+\cos\theta)}{(1+\gamma\cos\theta)^3}\left(1-e^{-X_s\cos\theta}\right)\right.$$

$$- \frac{1}{\cos^2\theta(1+\gamma\cos\theta)^3}\left\{1 + \frac{2\gamma\sin^2\theta\cos\theta}{1+\gamma\cos\theta}\right\}\left(1-e^{-X_s\cos\theta}\right)$$

$$\left. + \frac{(\gamma+\cos\theta)}{(1+\gamma\cos\theta)^2}X_s e^{-X_s\cos\theta} - \frac{\sin^2\theta}{(1+\gamma\cos\theta)}\frac{1}{2}X_s^2 e^{-X_s\cos\theta}\right]$$

A composite solution u is easily formed from these, in which case, in the boundary layer,

$$u_c - u = \frac{P'}{M}\frac{\cos\theta}{1+\gamma\cos\theta}e^{-x_s\cos\theta}$$

$$+\frac{P'}{M^2}\left[\frac{\gamma(\gamma+\cos\theta)}{(1+\gamma\cos\theta)^3} - \frac{1}{\cos^3\theta(1+\gamma\cos\theta)^3}\left\{1+\frac{2\gamma\sin^2\theta\cos\theta}{1+\gamma\cos\theta}\right\}\right.$$

$$\left. + \frac{(\gamma+\cos\theta)}{(1+\gamma\cos\theta)^2}x_s + \frac{\sin^2\theta}{1+\gamma\cos\theta}\frac{1}{2}x_s^2\right]e^{-x_s\cos\theta}$$

The flow integral may now be defined as:

$$Q = Q_c - Q_d$$

where

$$Q_c = 4\int_0^1\int_0^{(1-y^2)^{1/2}} u_c(x,y)\,dx\,dy$$

is the flow integral due to the core and

$$Q_d = \frac{4}{M}\int_0^{\pi/2}\int_0^{\infty}(u_c - u)\left(1-\frac{x_s}{M}\right)dx_s\,d\theta$$

is the flow deficit due to the boundary layer. Defining $\eta = (1-y^2)^{1/2}$, Qc and Qd to second order are:

$$Q_c = \frac{4P'}{M}\int_0^1\int_0^{\eta}\frac{\eta^2\,dx\,d\eta}{(1+\gamma\eta)(1-\eta^2)^{1/2}}$$

$$-\frac{4P'}{M^2}\int_0^1\int_0^{\eta}\left[\frac{1}{(1+\gamma\eta)^3}\frac{1}{\eta^2}\left(1+\frac{2\gamma\eta(1-\eta^2)}{1+\gamma\eta}\right) - \frac{\gamma(\gamma+\eta)}{(1+\gamma\eta)^3}\right]\frac{\eta\,dx\,d\eta}{(1-\eta^2)^{1/2}}$$

$$= \frac{4P'}{M}\int_0^1 \frac{\eta^3\, d\eta}{(1+\gamma\eta)(1-\eta^2)^{1/2}}$$

$$- \frac{4P'}{M^2}\int_0^1 \left[\frac{1}{(1+\gamma\eta)^2} + \frac{2\gamma\eta(1-\eta^2)}{(1+\gamma\eta)^4} - \frac{\gamma(\gamma+\eta)\eta^2}{(1+\gamma\eta)^3}\right]\frac{d\eta}{(1-\eta^2)^{1/2}}$$

$$Q_d \sim \frac{4P'}{M^2}\int_0^{\pi/2} \frac{d\theta}{1+\gamma\cos\theta}$$

Evaluation of the above integrals is straightforward, though algebraically tedious, with the following results:

(a) Insulating Walls ($\gamma=0$)

$$Q = \frac{4P}{M}\left(\frac{2}{3} - \frac{\pi}{M}\right)$$

(b) The Case $0<\gamma<1$

$$Q = \frac{4P'}{M}\left\{\frac{3}{\gamma^3}\left[\left(1+\frac{\gamma^2}{2}\right)\frac{\pi}{4} - \frac{\gamma}{2} - \frac{1}{\sqrt{1-\gamma^2}}\left(\tan^{-1}\sqrt{\frac{1+\gamma}{1-\gamma}} - \tan^{-1}\frac{\gamma}{\sqrt{1-\gamma^2}}\right)\right]\right.$$

$$- \frac{1}{M}\left[1-2\gamma+\frac{3}{\gamma} - \frac{1}{\gamma^2}\left(1+\frac{\pi}{2}\right) + \frac{2}{3}\left(\frac{2-\gamma}{\gamma^2-1}\right) - \frac{1}{6}\frac{(2\gamma-1)}{(\gamma^2-1)(\gamma+1)}\right.$$

$$+ 2\left(\frac{1}{\gamma^2} - \frac{\gamma^2}{\gamma^2-1} + \frac{1}{2}\frac{1}{(\gamma^2-1)^2}\right)\left(\frac{1}{2} - \gamma + \frac{1}{\sqrt{1-\gamma^2}}\left(\tan^{-1}\sqrt{\frac{1+\gamma}{1-\gamma}} - \tan^{-1}\frac{\gamma}{\sqrt{1-\gamma^2}}\right)\right)$$

$$\left.\left. - \frac{2}{\sqrt{1-\gamma^2}}\tan^{-1}\sqrt{\frac{1-\gamma}{1+\gamma}}\right]\right\}$$

(c) The Limit $\gamma=1$ ($C=M^{-1}$)

$$Q = \frac{4P'}{M}\left(\frac{3}{4}\pi - 2 - \frac{1}{M}\left\{\frac{34}{15} - \frac{\pi}{2}\right\}\right)$$

(d) The Case $1 < \gamma < \infty$

$$Q = \frac{4P'}{M}\left\{\frac{2}{\gamma^3}\left[\left(1+\frac{\gamma^2}{2}\right)\frac{\pi}{4} - \frac{\gamma}{2} - \frac{1}{2\sqrt{1-\gamma^2}}\log\frac{1+\gamma+\sqrt{\gamma^2-1}}{1+\gamma-\sqrt{\gamma^2-1}}\right]\right.$$

$$-\frac{1}{M}\left[1-2\gamma+\frac{3}{8}-\frac{1}{\gamma^2}\left(1+\frac{\pi}{2}\right)+\frac{2}{3}\frac{(2-\gamma)}{(\gamma^2-1)}-\frac{1}{6}\frac{(2\gamma-1)}{(\gamma^2-1)(\gamma+1)}\right.$$

$$+2\left(\frac{1}{\gamma^2}-\frac{\gamma^2}{\gamma^2-1}+\frac{1}{2}\frac{1}{(\gamma^2-1)^2}\right)\left(\frac{1}{2}-\gamma+\frac{1}{2\sqrt{\gamma^2-1}}\log\frac{1+\gamma+\sqrt{\gamma^2-1}}{1+\gamma-\sqrt{\gamma^2-1}}\right)$$

$$\left.\left.+\frac{1}{\sqrt{\gamma^2-1}}\log\frac{1+\gamma+\sqrt{\gamma^2-1}}{1+\gamma-\sqrt{\gamma^2-1}}\right]\right\}$$

(e) Perfectly Conducting Walls ($\gamma = \infty$)

$$Q = \frac{4P}{M^2}\left(\frac{\pi}{4} - \frac{7}{3}\frac{1}{M}\right)$$

It will be shown in the following sections that contributions from the parabolic layer are of a high order and therefore may be neglected in this second order approximation.

3.9 Optimal Coordinates

The circular tube problem may be cast in a slightly different light by means of a set of optimal coordinates as discussed by Van Dyke (9). The form of the core solution suggests the following set for the first quadrant:

$$\eta = (1-y^2)^{1/2} \qquad\qquad x = \eta - \xi_1$$

$$\xi_1 = (1-y^2)^{1/2} - x \qquad\qquad y = (1-\eta^2)^{1/2}$$

In these new coordinates the equations to be solved are:

$$\frac{\partial^2 F}{\partial \xi_1^2} + (1-\eta^2)\left(\frac{\partial^2 F}{\partial \eta^2} + 2\frac{\partial^2 F}{\partial \xi_1 \partial \eta}\right) - \frac{1}{\eta}\left(\frac{\partial F}{\partial \eta} + \frac{\partial F}{\partial \xi_1}\right) - M\eta^2 \frac{\partial F}{\partial \xi_1} = 0$$

$$\frac{\partial^2 G}{\partial \xi_1^2} + (1-\eta^2)\left(\frac{\partial^2 G}{\partial \eta^2} + 2\frac{\partial^2 G}{\partial \xi_1 \partial \eta}\right) - \frac{1}{\eta}\left(\frac{\partial G}{\partial \eta} + \frac{\partial G}{\partial \xi_1}\right) + M\eta^2 \frac{\partial G}{\partial \xi_1} = 0$$

and the boundary conditions at $\xi_1 = 0$ are

$$F + G = 0$$

$$F - G - \frac{c}{\eta}\left\{(1-\eta^2)\left(\frac{\partial F}{\partial \eta} - \frac{\partial G}{\partial \eta}\right) + \left(\frac{\partial F}{\partial \xi_1} - \frac{\partial G}{\partial \xi_1}\right)\right\} = 2\frac{p'}{M}\eta$$

A similar set of coordinates may be defined for the second quadrant with

$$\eta = (1-y^2)^{1/2} \qquad x = \xi_2 - \eta$$

$$\xi_2 = (1-y^2)^{1/2} + x \qquad y = (1-\eta^2)^{1/2}$$

and $\xi_2 = 2\eta - \xi_1$

The symmetry relations may be expressed in the form

$$F(\xi_1, \eta) = G(\xi_2, \eta)$$

$$G(\xi_1, \eta) = F(\xi_2, \eta) = F(2\eta - \xi_1, \eta)$$

The solutions obtained previously for the core will be obtained briefly in these new coordinates. Thus defining:

$$F_c \sim \varepsilon_{c_1} F_{c_1} + \varepsilon_{c_2} F_{c_2} + \cdots$$

$$G_c \sim \delta_{c_1} G_{c_1} + \delta_{c_2} G_{c_2} + \cdots$$

$$\therefore \frac{\partial F_{c_1}}{\partial \xi_1} = \frac{\partial G_{c_1}}{\partial \xi_1} = 0,$$

$$\therefore F_{c_1} = F_{c_1}(\eta), \quad G_{c_1} = G_{c_1}(\eta)$$

But by symmetry $\varepsilon_{c_1} F_{c_1}(\eta) = \delta_{c_1} G_{c_1}(\eta)$

$$\therefore \quad \varepsilon_{c1} = \delta_{c1} \quad , \quad F_{c1}(\eta) = G_{c1}(\eta)$$

For the next term:

$$M \varepsilon_{c2} \eta^2 \frac{\partial F_{c2}}{\partial \xi_1} = \varepsilon_{c1} \left\{ (1-\eta^2) \frac{\partial^2 F_{c1}}{\partial \eta^2} - \frac{1}{\eta} \frac{\partial F_{c1}}{\partial \eta} \right\},$$

$$M \delta_{c2} \eta^2 \frac{\partial G_{c2}}{\partial \xi_1} = \varepsilon_{c1} \left\{ (1-\eta^2) \frac{\partial^2 F_{c1}}{\partial \eta^2} - \frac{1}{\eta} \frac{\partial F_{c1}}{\partial \eta} \right\}$$

$$\therefore \quad \varepsilon_{c2} = \delta_{c2} = M^{-1} \varepsilon_{c1}$$

$$F_{c2} = \left\{ (1-\eta^2) \frac{\partial^2 F_{c1}}{\partial \eta^2} - \frac{1}{\eta} \frac{\partial F_{c1}}{\partial \eta} \right\} \frac{\xi_1}{\eta^2} + A_{c2}(\eta)$$

G_{c2} is obtained directly from the symmetry condition and the second order core may be written as:

$$F_c \sim \varepsilon_{c1} \left[F_{c1} + \frac{1}{M} \left\{ \left((1-\eta^2) \frac{\partial^2 F_{c1}}{\partial \eta^2} - \frac{1}{\eta} \frac{\partial F_{c1}}{\partial \eta} \right) \frac{\xi_1}{\eta^2} + A_{c2}(\eta) \right\} + \cdots \right]$$

$$G_c \sim \varepsilon_{c1} \left[F_{c1} + \frac{1}{M} \left\{ \left((1-\eta^2) \frac{\partial^2 F_{c1}}{\partial \eta^2} - \frac{1}{\eta} \frac{\partial F_{c1}}{\partial \eta} \right) \frac{(2\eta - \xi_1)}{\eta^2} + A_{c2}(\eta) \right\} + \cdots \right]$$

The unknown factor $\varepsilon_{c1}(M)$ and functions $F_{c1}(\eta)$ and $A_{c2}(\eta)$ remain to be determined through matching with a boundary layer. The origin of the singularity as $\eta \to 0$ in the second term is now quite readily apparent. This singularity will propagate with increasing severity in higher order terms and it is only with an expansion for small η that the singularity can be resolved.

With the stretched boundary layer variable $X_s = \xi_1/\Delta$ introduced into the equations, minimum degeneracy dictates that $\Delta = M^{-1}$ in which case:

$$\frac{\partial^2 F}{\partial X_s^2} - \eta^2 \frac{\partial F}{\partial X_s} = \frac{1}{M}\left(\frac{1}{\eta}\frac{\partial F}{\partial X_s} - 2(1-\eta^2)\frac{\partial^2 F}{\partial X_s \partial \eta}\right)$$

$$+ \frac{1}{M^2}\left(\frac{\partial F}{\partial \eta} - (1-\eta^2)\frac{\partial^2 F}{\partial \eta^2}\right)$$

$$\frac{\partial^2 G}{\partial X_s^2} + \eta^2 \frac{\partial G}{\partial X_s} = \frac{1}{M}\left(\frac{1}{\eta}\frac{\partial G}{\partial X_s} - 2(1-\eta^2)\frac{\partial^2 G}{\partial X_s \partial \eta}\right)$$

$$+ \frac{1}{M^2}\left(\frac{\partial G}{\partial \eta} - (1-\eta^2)\frac{\partial^2 G}{\partial \eta^2}\right)$$

and at $X_s = 0$

$$F + G = 0$$

$$F - G - \frac{\gamma}{\eta}\left(\frac{\partial F}{\partial X_s} - \frac{\partial G}{\partial X_s}\right) = 2\frac{P'}{M}\eta + \frac{\gamma}{M}\frac{(1-\eta^2)}{\eta}\left(\frac{\partial F}{\partial \eta} - \frac{\partial G}{\partial \eta}\right)$$

with $\gamma = Mc$ as before.

F and G may now be expanded and solutions to second order obtained. During the matching procedure a mixing of the orders no longer occurs.

What is more, the first order core now occurs directly as a limiting form of the first order boundary layer, a result which is to be expected of optimal coordinates. Without repeating the details, the final results are:

$$F_c \sim \frac{\rho'}{M}\frac{\eta}{1+\tau\eta} - \frac{\rho'}{M^2}\left[\left\{\frac{\xi_1}{\eta^3(1+\tau\eta)^2} - \frac{\tau}{\eta(1+\tau\eta)^3}\right\}\left\{1+\frac{2\tau\eta(1-\eta^2)}{1+\tau\eta}\right\} - \frac{\tau(\tau+\eta)}{(1+\tau\eta)^3}\right]$$

$$G_c \sim \frac{\rho'}{M}\frac{\eta}{1+\tau\eta} - \frac{\rho'}{M^2}\left[\left\{\frac{2\eta-\xi_1}{\eta^3(1+\tau\eta)^2} - \frac{\tau}{\eta(1+\tau\eta)^3}\right\}\left\{1+\frac{2\tau\eta(1-\eta^2)}{1+\tau\eta}\right\} - \frac{\tau(\tau+\eta)}{(1+\tau\eta)^3}\right]$$

$$F_s \sim \frac{\rho'}{M}\frac{\eta}{1+\tau\eta} + \frac{\rho'}{M^2}\left[\frac{\tau}{\eta(1+\tau\eta)^3}\left\{1+\frac{2\tau\eta(1-\eta^2)}{1+\tau\eta}\right\} + \frac{\tau(\tau+\eta)}{(1+\tau\eta)^3}\right]$$

$$G_s \sim \frac{\rho'}{M}\frac{\eta}{1+\tau\eta}\left(1-2e^{-\eta^2X_s}\right)$$

$$+\frac{\rho'}{M^2}\left[\left(\frac{\tau}{\eta(1+\tau\eta)^3}\left\{1+\frac{2\tau\eta(1-\eta^2)}{1+\tau\eta}\right\} + \frac{\tau(\tau+\eta)}{(1+\tau\eta)^3}\right)\left(1-2e^{-\eta^2X_s}\right)\right.$$

$$-\frac{2}{\eta^2(1+\tau\eta)^2}\left\{1+\frac{2\tau\eta(1-\eta^2)}{(1+\tau\eta)}\right\}\left(1-e^{-\eta^2X_s}\right)$$

$$\left.-\frac{2}{1+\tau\eta}\left\{\left(1-\frac{2(1-\eta^2)}{1+\tau\eta}\right)X_s + 2(1-\eta^2)X_s^2\right\}e^{-\eta^2X_s}\right]$$

When written in terms of x and y, the above core solutions reproduce exactly the results obtained previously. The boundary layer solutions differ slightly from those obtained previously, particularly in the exponent $\eta^2 X_s$.

Perhaps the most interesting aspect of these optimal coordinates is that the parabolic layer is considerably simplified. Thus, with $Y_p = \eta/\Delta$ substituted in the equations one finds that Δ must be $M^{-1/4}$ to satisfy the minimum degeneracy principle. The resulting equations are:

$$\frac{\partial^2 F}{\partial Y_p^2} - \frac{1}{Y_p}\frac{\partial F}{\partial Y_p} - \frac{\partial F}{\partial \xi_1} = M^{-1/4}\left(\frac{1}{Y_p}\frac{\partial F}{\partial \xi_1} - 2\frac{\partial^2 F}{\partial \xi_1 \partial Y_p}\right)$$

$$- M^{-1/2}\left(\frac{\partial^2 F}{\partial \xi_1^2} - Y_p^2 \frac{\partial^2 F}{\partial Y_p^2}\right)$$

$$+ 2M^{-3/4} Y_p^2 \frac{\partial^2 F}{\partial \xi_1 \partial Y_p}$$

$$\frac{\partial^2 G}{\partial Y_p^2} - \frac{1}{Y_p}\frac{\partial G}{\partial Y_p} + \frac{\partial G}{\partial \xi_1} = M^{-1/4}\left(\frac{1}{Y_p}\frac{\partial G}{\partial \xi_1} - 2\frac{\partial^2 G}{\partial \xi_1 \partial Y_p}\right)$$

$$- M^{-1/2}\left(\frac{\partial^2 G}{\partial \xi_1^2} - Y_p^2 \frac{\partial^2 G}{\partial Y_p^2}\right)$$

$$+ 2M^{-3/4} Y_p^2 \frac{\partial^2 G}{\partial \xi_1 \partial Y_p}$$

By introducing an asymptotic sequence:

$$F_p \sim \varepsilon_{p1} F_{p1} + \varepsilon_{p2} F_{p2} + \cdots$$

$$G_p \sim \delta_{p1} G_{p1} + \delta_{p2} G_{p2} + \cdots$$

the equations for the first term are obtained as:

$$\frac{\partial^2 F_{p1}}{\partial Y_p^2} - \frac{1}{Y_p}\frac{\partial F_{p1}}{\partial Y_p} - \frac{\partial F_{p1}}{\partial \xi_1} = 0$$

$$\frac{\partial^2 G}{\partial Y_p^2} - \frac{1}{Y_p}\frac{\partial G_{p1}}{\partial Y_p} + \frac{\partial G_{p1}}{\partial \xi_1} = 0$$

It will be noted that the highest derivative in ξ_1 has been lost in this degeneration and therefore the boundary conditions cannot be imposed. This layer must be considered as intermediate between the core and small η, with an additional boundary layer near the wall $\xi_1 = 0$ required. If a new variable $v = Y_p^2/2$ is defined, the simple diffusion equation results, that is:

$$\frac{\partial^2 F_{p1}}{\partial v^2} - \frac{\partial F_{p1}}{\partial \xi_1} = 0$$

A solution may be written in the form

$$F_{p1} = \frac{1}{2\sqrt{\pi \xi_1}} \int_0^\infty A_{p1}(t)\, e^{-\frac{(v-t)^2}{4\xi_1}}\, dt$$

and G_{p1} obtained from the symmetry condition, that is,

$$G_{p1} = \frac{1}{2\sqrt{\pi \xi_2}} \int_0^\infty A_{p1}(t)\, e^{-\frac{(v-t)^2}{4\xi_2}}\, dt$$

To obtain the unknown function $A_{p1}(t)$ and the factor $\mathcal{E}_{p1}$ it is sufficient to match with the first term core. Thus, expressing F_{p1} in core variables.

$$F_{p1} = \frac{1}{2\sqrt{\pi \xi_1}} \int_0^\infty A_{p1}(t)\, e^{-\frac{\left(\frac{1}{2}M^{1/2}\eta^2 - t\right)^2}{4\xi_1}}\, dt$$

$$\sim \frac{1}{\sqrt{\pi}} A_{p1}\left(\tfrac{1}{2}M^{1/2}\eta^2\right) \int_{-\frac{M^{1/2}\eta^2}{4\xi_1^{1/2}}}^{\infty} e^{-u^2} du + \cdots$$

$$\sim A_{p1}\left(\tfrac{1}{2}M^{1/2}\eta^2\right)$$

$$= A_{p1}(v)$$

therefore,

$$F_p \sim \mathcal{E}_{p1} A_{p1}(v)$$

The first term core written in parabolic coordinates is

$$F_c \sim \frac{P'}{M^{5/4}} \cdot \frac{Y_p}{1 + Y_p' Y_p}$$

where $\gamma_p = M^{-1/4}\gamma = cM^{3/4}$ has been introduced so as not to lose the arbitrary nature of the wall conductivity. Thus,

$$\varepsilon_{p_1} = M^{-5/4}$$

$$A_{p_1} = \frac{P' Y_p}{1+\gamma_p Y_p} = \frac{P'\sqrt{2v}}{1+\gamma_p\sqrt{2v}}$$

The first term parabolic solutions may now be written as:

$$F_p \sim \frac{P'}{M^{5/4}}\,\frac{1}{2\sqrt{\pi\xi_1}}\int_0^\infty \frac{\sqrt{2t}}{1+\gamma_p\sqrt{2t}}\,e^{-\frac{(v-t)^2}{4\xi_1}}\,dt + \cdots$$

$$G_p \sim \frac{P'}{M^{5/4}}\,\frac{1}{2\sqrt{\pi\xi_2}}\int_0^\infty \frac{\sqrt{2t}}{1+\gamma_p\sqrt{2t}}\,e^{-\frac{(v-t)^2}{4\xi_2}}\,dt + \cdots$$

The additional boundary layer at the wall $\xi_1 = 0$ is easily obtained by introducing the new stretched variable $X_p = \xi_1/\Delta$ with $Y_p = M^{1/4}\eta$. Minimum degeneracy requires that $\Delta = M^{-1/2}$ and

$$\frac{\partial^2 F}{\partial X_p^2} - Y_p^2\frac{\partial F}{\partial X_p} = M^{-1/4}\left(\frac{1}{Y_p}\frac{\partial F}{\partial X_p} - 2\frac{\partial^2 F}{\partial X_p\partial Y_p}\right)$$

$$+ M^{-1/2}\left(\frac{1}{Y_p}\frac{\partial F}{\partial Y_p} - \frac{\partial^2 F}{\partial Y_p^2}\right)$$

$$+ 2M^{-3/4}Y_p^2\frac{\partial^2 F}{\partial X_p\partial Y_p} + M^{-1}Y_p^2\frac{\partial^2 F}{\partial Y_p^2}$$

$$\frac{\partial^2 G}{\partial X_p^2} + Y_p^2 \frac{\partial G}{\partial X_p} = M^{-1/4}\left(\frac{1}{Y_p}\frac{\partial G}{\partial X_p} - 2\frac{\partial^2 G}{\partial X_p \partial Y_p}\right)$$

$$+ M^{-1/2}\left(\frac{1}{Y_p}\frac{\partial G}{\partial Y_p} - \frac{\partial^2 G}{\partial Y_p^2}\right)$$

$$+ 2M^{-3/4} Y_p^2 \frac{\partial^2 G}{\partial X_p \partial Y_p} + M^{-1} Y_p^2 \frac{\partial^2 G}{\partial Y_p^2}$$

and at $X_p = 0$

$$F + G = 0$$

$$F - G - \frac{\gamma_p}{Y_p}\left(\frac{\partial F}{\partial X_p} - \frac{\partial G}{\partial X_p}\right) = 2\frac{P' Y_p}{M^{5/4}} + M^{-1/4}\frac{\gamma_p}{Y_p}\left(\frac{\partial F}{\partial Y_p} - \frac{\partial G}{\partial Y_p}\right)$$

$$- M^{-3/4}\gamma_p Y_p^2\left(\frac{\partial F}{\partial Y_p} - \frac{\partial G}{\partial Y_p}\right)$$

Now defining

$$F_{ip} \sim \varepsilon_{ip_1} F_{ip_1} + \varepsilon_{ip_2} F_{ip_2} + \cdots$$

$$G_{ip} \sim \delta_{ip_1} G_{ip_1} + \delta_{ip_2} G_{ip_2} + \cdots$$

the first order equations provide

$$\varepsilon_{ip_1} = \delta_{ip_1} = M^{-3/4}$$

$$\frac{\partial^2 F_{ip1}}{\partial X_p^2} - Y_p^2 \frac{\partial F_{ip1}}{\partial X_p} = 0$$

$$\frac{\partial G_{ip1}}{\partial X_p^2} + Y_p^2 \frac{\partial G_{ip1}}{\partial X_p} = 0$$

and at $X_p = 0$

$$F_{ip1} + G_{ip1} = 0$$

$$F_{ip1} - G_{ip1} - \frac{\gamma_p}{Y_p}\left(\frac{\partial F_{ip1}}{\partial X_p} - \frac{\partial G_{ip1}}{\partial X_p}\right) = 2\rho' Y_p$$

When considering the match with the ordinary layer, solutions may be written by inspection as

$$F_{ip} \sim \frac{\rho'}{M^{5/4}} \frac{Y_p}{1+\gamma_p Y_p} + \cdots$$

$$G_{ip} \sim \frac{\rho'}{M^{5/4}} \frac{Y_p}{1+\gamma_p Y_p}\left(1 - 2e^{-Y_p^2 X_p}\right)$$

which are nothing more than the ordinary layer solutions written in the X_p, Y_p variables.

By expanding the parabolic layer solutions in X_p, Y_p variables, matching can be verified, that is:

$$F_p \sim \frac{\rho'}{M^{5/4}} \frac{M^{1/4}}{2\sqrt{\pi X_p}} \int_0^\infty \frac{\sqrt{2t}}{1+\gamma_p\sqrt{2t}} e^{-\frac{M^{1/2}(\frac{1}{2}Y_p^2-t)^2}{4X_p}} dt$$

$$= \frac{\rho'}{M^{5/4}} \frac{1}{\sqrt{\pi}} \int_{-\frac{M^{1/4}Y_p^2}{4X_p^{1/2}}}^{\infty} \frac{\sqrt{Y_p^2+4M^{-1/4}X_p^{1/2}u}}{1+\gamma_p\sqrt{Y_p^2+4M^{-1/4}X_p^{1/2}u}} e^{-u^2} du$$

$$\sim \frac{\rho'}{M^{5/4}} \frac{Y_p}{1+\gamma_p Y_p} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2} du + \cdots$$

$$= \frac{\rho'}{M^{5/4}} \frac{Y_p}{1+\gamma_p Y_p}$$

Similarly,

$$G_p \sim \frac{\rho'}{M^{6/4}} \frac{M^{1/8}}{2\sqrt{\pi(2Y_p - M^{-1/4}X_p)}} \int_0^\infty \frac{\sqrt{2t}}{1+\gamma_p\sqrt{2t}} e^{-\frac{M^{1/4}(\frac{1}{2}Y_p^2-t)^2}{4(2Y_p-M^{-1/4}X_p)}} dt$$

$$\sim \frac{\rho'}{M^{6/4}} \frac{Y_p}{1+\gamma_p Y_p} + \cdots$$

and

$$G_{ip} \sim \frac{\rho'}{M^{5/4}} \frac{Y_p}{1+\gamma_p Y_p} \left(1 - e^{-M^{1/2} Y_p^2 \xi_1}\right)$$

$$\sim \frac{\rho'}{M^{5/4}} \frac{Y_p}{1+\gamma_p Y_p} + \cdots$$

The parabolic layer therefore matches with the ordinary layer.

3.10 Parabolic Layer Flow Deficit

The parabolic layer velocity is

$$u_p = \frac{1}{2}\left(F_p + G_p\right)$$

and the flow integral is

$$Q_p = 4\int_0^1\int_0^{(1-y^2)^{1/2}} u_p \, dx\, dy = 4\int_0^1\int_0^{\eta} u_p \frac{\eta}{(1-\eta^2)^{1/2}} \, d\xi_1\, d\eta$$

so that

$$Q_p = 2\int_0^1 \frac{\eta\, d\eta}{(1-\eta^2)^{1/2}}\int_0^{\eta}\left[F(\xi_1,\eta) + G(\xi_1,\eta)\right] d\xi_1$$

From the symmetry conditions one may write:

$$\int_0^{\eta} G(\xi_1, \eta)\, d\xi_1 = \int_{\eta}^{2\eta} F(\xi_2, \eta)\, d\xi_2$$

The flow integral may now be written as

$$Q_p = 2\int_0^1 \frac{\eta\, d\eta}{(1-\eta^2)^{1/2}}\int_0^{2\eta} F(\xi,\eta)\, d\xi$$

that is:

$$Q_p = \frac{P'}{M^{3/4}}\frac{1}{\sqrt{\pi}}\int_0^1 \frac{\eta\, d\eta}{(1-\eta^2)^{1/2}}\int_0^{2\eta}\frac{d\xi}{\xi^{1/2}}\int_0^{\infty}\frac{(2t)^{1/2}}{1+\gamma_p(2t)^{1/2}}\, e^{-\frac{\left(\frac{1}{2}M^{1/2}\eta^2 - t\right)^2}{4\xi}}\, dt$$

Letting $t = \frac{1}{2}M^{1/2}s^2$ and $\gamma = M^{1/4}\gamma_p$; this may be re-written in the form:

$$Q_p = \frac{\rho'}{M^{1/2}} \frac{1}{\sqrt{\pi}} \int_0^1 \frac{\eta \, d\eta}{(1-\eta^2)^{1/2}} \int_0^{2\eta} \frac{d\xi}{\xi^{1/2}} \int_0^{\infty} \frac{s^2}{1+\gamma s} e^{-\frac{M}{16}\frac{(\eta^2-s^2)^2}{\xi}} ds$$

A parabolic layer flow deficit may be obtained by removing from this the core flow, which may be expressed asymptotically as

$$Q_c \sim \frac{\rho'}{M^{1/2}\sqrt{\pi}} \int_0^1 \frac{\eta \, d\eta}{(1-\eta^2)^{1/2}} \int_0^{2\eta} \frac{d\xi}{\xi^{1/2}} \int_0^{\infty} \frac{\eta s}{1+\gamma\eta} e^{-\frac{M}{16}\frac{(\eta^2-s^2)^2}{\xi}} ds$$

Thus, the flow deficit is:

$$Q_d \sim \frac{\rho'}{M^{1/2}\sqrt{\pi}} \int_0^1 \frac{\eta \, d\eta}{(1-\eta^2)^{1/2}} \int_0^{2\eta} \frac{d\xi}{\xi^{1/2}} \int_0^{\infty} \left(\frac{s}{1+\gamma s} - \frac{\eta}{1+\gamma\eta}\right) s \, e^{-\frac{M}{16}\frac{(\eta^2-s^2)^2}{\xi}} ds$$

which may be written as:

$$Q_d \sim \frac{\rho'}{M^{1/2}} \frac{1}{\sqrt{\pi}} \int_0^1 \frac{\eta \, d\eta}{(1-\eta^2)^{1/2}(1+\gamma\eta)} \int_0^{2\eta} \frac{d\xi}{\xi^{1/2}} \int_0^{\infty} \frac{(s^2-\eta^2)\, s}{(1+\gamma s)(\eta+s)} e^{-\frac{M}{16\xi}(\eta^2-s^2)^2} ds$$

Introducing the variable u with $s^2 = \eta^2(1+u)$ results in:

$$Q_d \sim \frac{\rho'}{M^{1/2}} \frac{1}{2\sqrt{\pi}} \int_0^1 \frac{\eta^4 \, d\eta}{(1-\eta^2)^{1/2}(1+\gamma\eta)} \int_0^{2\eta^4} \frac{d\xi}{\xi^{1/2}} \int_{-1}^{\infty} \frac{e^{-\frac{M\eta^4}{16\xi}u^2} \, u \, du}{[1+\gamma\eta(1+u)^{1/2}][1+(1+u)^{1/2}]}$$

Since the major contribution from the inner integral will occur near u = 0, one may write approximately:

$$Q_d \sim \frac{P'}{M^{1/2}} \frac{1}{4\sqrt{\pi}} \int_0^1 \frac{\eta^4 d\eta}{(1-\eta^2)^{1/2}(1+\gamma\eta)^2} \int_0^{2\eta} \frac{d\xi}{\xi^{1/2}} \int_{-1}^{\infty} e^{-\frac{M\eta^4}{16\xi}u^2} u\,du$$

$$= \frac{P'}{M^{3/2}} \frac{2}{\sqrt{\pi}} \int_0^1 \frac{d\eta}{(1-\eta^2)^{1/2}(1+\gamma\eta)^2} \int_0^{2\eta} \xi^{1/2} e^{-\frac{M\eta^4}{16\xi}} d\xi$$

Replacing ξ by ηw results in

$$Q_d \sim \frac{P'}{M^{3/2}} \frac{2}{\sqrt{\pi}} \int_0^1 \frac{\eta^{3/2} d\eta}{(1-\eta^2)^{1/2}(1+\gamma\eta)^2} \int_0^2 w^{1/2} e^{-\frac{M\eta^3}{16w}} dw$$

$$= \frac{P'}{M^{3/2}} \frac{2}{\sqrt{\pi}} \int_0^2 w^{1/2} dw \int_0^1 \frac{\eta^{3/2} e^{-\frac{M\eta^3}{16w}}}{(1-\eta^2)^{1/2}(1+\gamma\eta)^2} d\eta$$

With $z = \frac{M\eta^3}{16w}$, Qd may be written as

$$Q_d \sim \frac{P'}{M^{7/3}} \frac{2}{\sqrt{\pi}} \frac{(16)^{5/6}}{3} \int_0^2 w^{4/3} dw \int_0^{\frac{M}{16w}} \frac{z^{-1/6} e^{-z} dz}{\left[1-M^{-2/3}(16wz)^{2/3}\right]^{1/2}\left[1+\gamma M^{-1/3}(16wz)^{1/3}\right]^2}$$

For the insulating wall case,

$$Q_d \sim \frac{P}{M^{7/3}} \frac{2}{\sqrt{\pi}} \frac{(16)^{5/6}}{3} \int_0^2 w^{4/3} dw \int_0^{\infty} z^{-1/6} e^{-z} dz + O(M^{-3})$$

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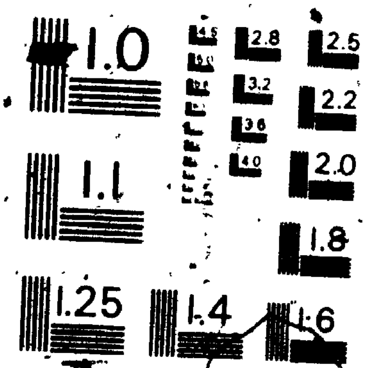

MICROCOPY RESOLUTION TEST CHART
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$$\sim \frac{P}{(M/2)^{7/3}} \frac{2^{1/3}}{\sqrt{\pi}} \frac{16}{7} \Gamma(5/6)$$

$$= 1.83 \frac{P}{(M/2)^{7/3}}$$

The coefficient 1.83 is very close to the value 1.732 obtained by Roberts (13) using a numerical integration. This verifies that for insulating walls, the parabolic flow deficit is of higher order than second.

For very large wall conductivity, that is $\gamma \gg M$, we may write:

$$Q_d \sim \frac{P}{M^{8/3}} \frac{1}{\gamma} \frac{2}{\sqrt{\pi}} \frac{(16)^{1/6}}{3} \int_0^2 w^{2/3} dw \int_0^{\frac{M}{16w}} z^{-5/6} e^{-z} dz$$

$$\sim \frac{P}{(M/2)^{8/3}} \frac{2^{2/3}}{\sqrt{\pi}} \frac{\Gamma(1/6)}{5} \frac{1}{\gamma}$$

This is again of higher order than second and vanishes for perfectly conducting walls.

It is apparent therefore, that to second order, the flow deficit due to the parabolic layer may be neglected.

4.0 THE RECTANGULAR DUCT

Insulating wall solutions for the core and all three boundary layers are known for the rectangular duct. The core, ordinary and parabolic layers were found by Shercliff (14) with the core and ordinary layer obtained from a large M expension of an exact Fourier series expansion. Generalization of the exact solution to arbitrarily oriented ducts have also been considered (for example by Eraslan, (15)). Cook, Ludford and Walker (10) have further shown that an elliptic layer solution may actually contain the whole structure of the parabolic layer.

Various permutations of wall conductivities have also been considered and a general method of treatment has been explored by Temperley and Todd (16). In the most important case, with insulating walls perpendicular to the applied field and perfectly conducting walls parallel to the field, solutions for the core, ordinary layer and parabolic layer have been obtained by Hunt and Stewartson (17). The parabolic layer was also obtained by Chiang and Lundgren (18). In both of these publications, the parabolic layer was obtained through solutions of an integral equation.

Due to the difficulties involved and the small contribution to the integrated flow, the elliptic or corner layer is often ignored, particularly with a conducting wall parallel to the field. An approximation of the solution for this layer will be obtained in Section 4.2 below.

4.1 The Core, Ordinary and Parabolic Boundary Layers

Solutions for the core and ordinary layers are easily obtained, either directly from the asymptotic equations or by expansion of the Hartmann solutions for large M. These are:

a) Core

$$F_o = G_o \sim \frac{P}{M} \left(\frac{1 + C_A}{1 + MC_A}\right)$$

where C_A is the conductivity parameter for the walls AA (see Figure 2.1 (b)) perpendicular to the field.

(b) Ordinary Layer

$$F_H \sim \frac{P}{M} \left(\frac{1 + C_A}{1 + MC_A}\right)$$

$$G_H \sim \frac{P}{M} \left(\frac{1 + C_A}{1 + MC_A}\right)(1 \quad 2e^{-M(1-x)})$$

The leading term parabolic solution obtained by Chiang and Lundgren (18) when written in the variables used here are:

$$F_p \sim \frac{P}{M} \left[1 - \frac{2\sqrt{2\pi}}{\Gamma^2(1/4)} \int_x^1 (1-\beta^2)^{-3/4} \operatorname{erfc}\frac{\eta}{2\sqrt{\beta - x}}\, d\beta\right]$$

$$G_p \sim \frac{P}{M} \left[1 - \frac{2\sqrt{2\pi}}{\Gamma^2(1/4)} \int_{-1}^x (1-\beta^2)^{-3/4} \operatorname{erfc}\frac{\eta}{2\sqrt{x - \beta}}\, d\beta\right]$$

where $\eta = M^{1/2}(L - y)$ and the walls parallel to the field are perfect conductors, and those perpendicular to the field are insulating.

As anticipated previously, in Section 2.2, the solution for F_p satisfies the boundary conditions at $x = 1$ as well as $\eta = 0$ (conversely, G_p satisfies the conditions at $x = -1$), a fact which will be used to advantage for the elliptic layer approximate solution.

For the construction of the elliptic layer, it is necessary to obtain the expansion of the parabolic solutions into the corner. Thus, changing variables to:

$$X = M(1 - x), \quad Y = M^{1/2}\eta = M(L - y)$$

the solution F_p is:

$$F_p \sim \frac{P}{M}[1 - \frac{2\sqrt{2\pi}}{\Gamma^2(1/4)} \int_{1-M^{-1}X}^{1} (1 - \beta^2)^{-3/4}$$

$$\times \operatorname{erfc} \frac{M^{-1/2}X}{2\sqrt{\beta - (1 - M^{-1}X)}} d\beta]$$

$$\sim \frac{P}{M}[1 - \frac{2^{1/4}\sqrt{2\pi}}{M^{1/4}\Gamma^2(1/4)} \int_0^X u^{-3/4}(1 - \frac{u}{2M})^{-3/4}$$

$$\times \operatorname{erfc} \frac{Y}{2\sqrt{X - u}} du]$$

where $u = M(1 - \beta)$

Expanding the integrand gives:

$$F_p \sim \frac{P}{M}[1 - \frac{2^{1/4}\sqrt{2\pi}}{M^{1/4}\Gamma^2(1/4)} \int_0^X u^{-3/4} \operatorname{erfc} \frac{Y}{2\sqrt{X - u}} du$$

$$+ O(M^{-5/4})]$$

which, after integration by parts is:

$$F_p \sim \frac{P}{M}[1 - \frac{2^{3/4} \cdot 8}{M^{1/4}\Gamma^2(1/4)} \int_0^X u^{1/4} e^{-\frac{Y^2}{4(X-u)}}$$

$$\times \frac{Y}{4(X-u)^{3/4}} du]$$

which becomes, on letting $t = (1 - \frac{u}{X})^{-1}$

$$F_p \sim \frac{P}{M}[1 - \frac{2}{M^{1/4}} \frac{2^{3/4}}{\Gamma^2(1/4)} \frac{Y}{X^{1/4}} \int_1^\infty e^{-\frac{Y^2}{4X}t}$$

$$\times (t-1)^{1/4} t^{-3/4} dt]$$

This may be expressed as a confluent hypergeometric function $U(a, b, z)$, (19) that is:

$$F_p \sim \frac{P}{M}[1 - \frac{2}{M^{1/4}} \frac{2^{3/4}}{\Gamma^2(1/4)} \frac{Y}{X^{1/4}} e^{-\frac{Y^2}{4X}}$$

$$\times \Gamma(5/4) U(5/4, 3/2, \frac{Y^2}{4X})]$$

Along the wall Y = 0, this reduces to:

$$F_p(X, 0) \sim \frac{P}{M}[1 - \frac{2^{1/4}}{M^{1/4}} \frac{2}{\Gamma^2(1/4)} 4X^{1/4}] \qquad (4.1)$$

For matching purposes, the asymptotic solution is rewritten in terms of η and $\xi = 1 - x$ as:

$$F_{p1}(\xi,\eta) \sim \frac{P}{M}[1 - \frac{2^{-1/4}}{\Gamma^2(1/4)} \frac{\eta}{\xi^{1/4}} e^{\frac{\eta^2}{4\xi}} U(5/4,3/2,\frac{\eta^2}{4\xi})] \tag{4.2}$$

where $F_p = \frac{1}{M} F_{p1}$

Following a somewhat similar expansion procedure, the expansion for G_p is:

$$G_{p1}(\xi,\eta) \sim -\frac{P}{M}[1 - \frac{2^{-1/4}}{\Gamma^2(1/4)} \frac{\eta}{\xi^{1/4}} U(1/4,1/2,\frac{\eta^2}{4\xi})] \tag{4.3}$$

where $G_p = \frac{1}{M} G_{p1}$

4.2 The Elliptic Boundary Layer

In a preceding section (Section 2.2) the elliptic layer equations were obtained and it was noted that all terms of the original equations were retained in the process. Similarly, the leading term boundary condition contains all the original boundary data. Thus, if an asymptotic sequence were defined for the elliptic layer, all terms other than the leading term will be obtained as solutions to a homogeneous equation with homogeneous boundary conditions. Since the solution must be at least bounded at infinity, and the equations are elliptic, it is reasonable to suggest that the solution must be constant with value zero. This of course implies that the leading term contains solutions to all orders, which is much more than was originally sought, that is, first order approximate solutions only.

In the following, an integral equation will be derived for perfectly conducting walls parallel to the field and insulating walls perpendicular to the field.

The equation for F and G, written in the elliptic corner variables are:

$$F = \frac{1}{M} F_1$$

$$G = \frac{1}{M} F_2$$

$$\frac{\partial^2 F_1}{\partial X^2} + \frac{\partial^2 F_1}{\partial Y^2} - \frac{\partial F_1}{\partial X} = 0$$

$$\frac{\partial^2 G_1}{\partial X^2} + \frac{\partial^2 G_1}{\partial Y^2} + \frac{\partial G_1}{\partial X} = 0$$

with $X = M(l - x)$, $Y = M(L - y)$, and the boundary conditions are:

$$F_1 = P, \quad G_1 = -P \quad \text{at } X = 0 \tag{4.4}$$

$$\left.\begin{aligned} &\frac{\partial F_1}{\partial Y} - \frac{\partial G_1}{\partial Y} = 0 \quad (4.5) \\ &F_1 + G_1 = 0 \quad (4.6) \end{aligned}\right\} \text{at } Y = 0$$

By making the substitutions:

$$F_1 = e^{X/2} f, \qquad G_1 = e^{-X/2} g$$

the equation may be written as:

$$\frac{\partial^2 f}{\partial X^2} + \frac{\partial^2 f}{\partial Y^2} - \frac{1}{4} f = 0$$

$$\frac{\partial^2 g}{\partial X^2} + \frac{\partial^2 g}{\partial Y^2} - \frac{1}{4} g = 0$$

A Green's function which is zero on both boundaries is:

$$\frac{1}{2\pi}\Big[K_o\left(\tfrac{1}{2}\sqrt{(X_o - X)^2 + (Y_o - Y)^2}\right)$$

$$- K_o\left(\tfrac{1}{2}\sqrt{(X_o - X)^2 + (Y_o + Y)^2}\right) - K_o\left(\tfrac{1}{2}\sqrt{(X_o + X)^2 + (Y_o - Y)^2}\right)$$

$$+ K_o\left(\tfrac{1}{2}\sqrt{(X_o + X)^2 + (Y_o + Y)^2}\right) \Big]$$

where X_o, Y_o is the source point and K_o the modified Bessel function of the second kind. The solution, expressed in terms of F_1 and G_1 may now be formally written as:

$$F_1(X, Y) = \frac{PX}{\pi} e^{X/2} \int_0^Y \frac{K_1\left(\tfrac{1}{2}\sqrt{X^2 + u^2}\right) du}{(X^2 + u^2)^{1/2}}$$

$$- \frac{1}{\pi} e^{X/2} \frac{\partial}{\partial Y} \int_0^\infty F_1(X_o, 0) e^{-X_o/2}$$

$$\times \left[K_o(r_1) - K_o(r_2)\right] dX_o$$

with $r_1 = \frac{1}{2}\sqrt{(X_o - X)^2 + Y^2}$

$r_2 = \frac{1}{2}\sqrt{(X_o + X)^2 + Y^2}$

$$G_1(X, Y) = -\frac{Px}{\pi} e^{-X/2} \int_0^Y \frac{K_1(\frac{1}{2}\sqrt{X^2 + u^2})\, du}{(X^2 + u^2)^{1/2}}$$

$$+ \frac{1}{\pi} e^{-X/2} \frac{\partial}{\partial Y} \int_0^\infty F_1(X_o, 0) e^{X_o/2}$$

$$\times \left[K'_o(r_1) - K_o(r_2)\right] dx_o$$

where the boundary conditions (4.4) and (4.6) have been used. The boundary condition (4.5) will now produce an integral equation when it is noted that:

$$\frac{\partial^2}{\partial Y^2} K_o(r_{1,2}) = \frac{1}{4} K_o(r_{1,2}) - \frac{\partial^2}{\partial X^2} K_o(r_{1,2})$$

except at $r_1 = 0$ so that if we differentiate $F_1(X, Y)$ for example, that is:

$$\frac{\partial F_1}{\partial Y} = \frac{PX}{\pi} e^{X/2} \frac{K_1(\frac{1}{2}\sqrt{X^2 + Y^2})}{(X^2 + Y^2)^{1/2}}$$

$$- \frac{1}{\pi} e^{X/2} \frac{\partial^2}{\partial Y^2} \int_0^\infty F_1(X_o, 0)\; e^{-X_o/2}$$

$$\times \left[K_o(r_1) - K_o(r_2)\right] dx_o$$

We obtain, after interchanging differentiation with respect to X with differentiation with respect to X_o and integrating by parts:

$$\frac{\partial F_1}{\partial Y}(X,Y) = \frac{1}{\pi} e^{X/2} \int_0^\infty \left[\frac{\partial^2 F_1(X_o, 0)}{\partial X_o^2} - \frac{\partial F_1(X_o, 0)}{\partial X_o}\right]$$

$$\cdot\, e^{-X_o/2} [K_o(r_1) - K_o(r_2)]\, dX_o$$

Similarly:

$$\frac{\partial G_1}{\partial Y}(X,Y) = -\frac{1}{\pi} e^{-X/2} \int_0^\infty \left[\frac{\partial^2 F_1(X_o, 0)}{\partial X_o^2} + \frac{\partial F_1(X_o, 0)}{\partial X_o}\right]$$

$$\cdot\, e^{X_o/2} [K_o(r_1) - K_o(r_2)]\, dX_o$$

If we now take the difference, set $Y = 0$ and use boundary condition (4.5), the result is:

$$\int_0^\infty \left[\frac{\partial^2 F_1(X_o, 0)}{\partial X_o^2} \cosh\frac{(X_o - X)}{2} - \frac{\partial F_1(X_o, 0)}{\partial X_o} \sinh\frac{(X_o - X)}{2}\right]$$

$$\cdot\, \left[K_o\frac{|X_o - X|}{2} - K_o\frac{(X_o + X)}{2}\right] dX_o = 0 \qquad (4.7)$$

A method for solving this equation in closed form is not readily apparent, consequently iterative methods have been sought. One such procedure employs the parabolic solutions to obtain a first approximation. As noted previously, the parabolic F solution, unlike G, actually extends to the insulating wall. Thus, it would be anticipated that a reasonably accurate first approximation may be obtained directly by using the parabolic F solution to decouple the boundary conditions on the conducting wall. This procedure has proved fruitful as will be shown in the following section.

4.3 Approximate Solution for the Elliptic Layer

An approximation for the elliptic layer or corner region may be obtained through use of the parabolic layer solution for F which we have seen satisfies the boundary conditions, even within the corner region. Thus, we set:

$$F_1(X_o, 0) = P[1 - \frac{2^{1/4}}{M^{1/4}} \frac{\sqrt{2\pi}}{\Gamma^2_{(1/4)}} 4X_o^{1/4}]$$

which is obtained from the expansion (equation 4.1) of the F_p solution on the wall Y = 0. We may now write:

$$F_1(X,Y) = \frac{PX}{\pi} e^{X/2} \int_0^Y \frac{K_1(\frac{1}{2}\sqrt{X^2 + u^2})}{(X^2 + u^2)^{1/2}} du$$

$$+ \frac{PY}{2\pi} e^{X/2} \int_0^\infty e^{-X_o/2} \left\{ \frac{K_1(\frac{1}{2}\sqrt{(X_o - X)^2 + Y^2})}{[(X_o - X)^2 + Y^2]^{1/2}} - \frac{K_1(\frac{1}{2}\sqrt{(X_o + X)^2 + Y^2})}{[(X_o + X)^2 + Y^2]^{1/2}} \right\} dX_o$$

$$- \frac{P}{M^{1/4}} \frac{2^{1/4}}{\Gamma^2_{(1/4)}} \frac{4Y}{\sqrt{2\pi}} e^{X/2} \int^\infty X_o^{1/4} e^{-X_o/2}$$

$$\cdot \left\{ \frac{K_1(\frac{1}{2}\sqrt{(X_o - X)^2 + Y^2})}{[(X_o - X)^2 + Y^2]^{1/2}} - \frac{K_1(\frac{1}{2}\sqrt{(X_o + X)^2 + Y^2})}{[(X_o + X)^2 + Y^2]^{1/2}} \right\} dX_o$$

$$G_1(X,Y) = -\frac{PX}{\pi}e^{-X/2}\int_0^{Y}\frac{K_1\left(\frac{1}{2}\sqrt{X^2+u^2}\right)}{(X^2+u^2)^{1/2}}\,du$$

$$-\frac{PY}{\pi}e^{-X/2}\int_0^{\infty} e^{X_o/2}$$

$$\left\{\frac{K_1\left(\frac{1}{2}\sqrt{(X_o-X)^2+Y^2}\right)}{[(X_o-X)^2+Y^2]^{1/2}} - \frac{K_1\left(\frac{1}{2}\sqrt{(X_o+X)^2+Y^2}\right)}{[(X_o+X)^2+Y^2]^{1/2}}\right\}dX_o$$

$$+\frac{P}{M^{1/4}}\frac{2^{1/4}}{\Gamma^2(1/4)}\frac{4Y}{\sqrt{2\pi}}e^{-X/2}\int_0^{\infty} X_o^{1/4}e^{X_o/2}$$

$$\left\{\frac{K_1\left(\frac{1}{2}\sqrt{(X_o-X)^2+Y^2}\right)}{[(X_o-X)^2+Y^2]^{1/2}} - \frac{K_1\left(\frac{1}{2}\sqrt{(X_o-X)^2+Y^2}\right)}{[(X_o-X)^2+Y^2]^{1/2}}\right\}dX_o$$

By construction, these solutions satisfy the differential equations, the boundary conditions at $X = 0$ and the no slip (equation 4.6) condition at $Y = 0$. It remains to verify that a match occurs with the parabolic layer and that the derivative boundary condition (equation 4.5) is satisfied with sufficient accuracy.

Verification of the matching is lengthy and is presented in the appendix.

Since the solutions are approximate, the derivative boundary condition will not be satisfied exactly. However, the match with the parabolic layer ensures that, at least within the overlap region, the condition will be satisfied to sufficient order. It should be noted that the magnetic field is an electric current stream function and the normal derivative condition simply states that tangential currents should be zero along a wall of infinite conductivity. Of greater importance physically is the requirement that the total current into the wall be zero, which is satisfied exactly with the present solution since, by construction, the field on the walls $x = \pm 1$ is zero.

The fluid velocity and magnetic field may now be written directly as:

$$u_e \sim \frac{1}{2M}[F_1(X,Y) + G_1(X,Y)]$$

$$\sim \frac{PX}{M\pi}\sinh\left(\frac{X}{2}\right)\int_0^Y \frac{K_1\left(\frac{1}{2}\sqrt{X^2+u^2}\right)}{(X^2+u^2)^{1/2}}\,du$$

$$+ \frac{PY}{2\pi M}\int_0^\infty \sinh\left(\frac{X-X_0}{2}\right)\left\{\frac{K_1\left(\frac{1}{2}\sqrt{(X_0-X)^2+Y^2}\right)}{[(X_0-X)^2+Y^2]^{1/2}} - \frac{K_1\left(\frac{1}{2}\sqrt{(X_0+X)^2+Y^2}\right)}{[(X_0+X)^2+Y^2]^{1/2}}\right\}dX_0$$

$$- \frac{P}{M^{5/4}} \frac{2^{1/4}}{\Gamma^2(1/4)} \frac{4Y}{\sqrt{2\pi}} \int_0^\infty X_0^{1/4} \sinh(\frac{X-X_0}{2})$$

$$\left\{\frac{K_1(\frac{1}{2}\sqrt{(X_0 - X)^2 + Y^2})}{[(X_0 - X)^2 + Y^2]^{1/2}} - \frac{K_1(\frac{1}{2}\sqrt{(X_0 + X)^2 + Y^2})}{[(X_0 + X)^2 + Y^2]^{1/2}}\right\} dX_0$$

$$H_e \sim - \frac{P}{M}(1 - \frac{X}{M}) + \frac{1}{2M} [F_1(X,Y) - G_1(X,Y)]$$

$$\sim - \frac{P}{M} (1 - \frac{X}{M})$$

$$+ \frac{P}{M} \frac{X}{\pi} \cosh(\frac{X}{2}) \int_0^Y \frac{K_1(\frac{1}{2}\sqrt{X^2 + u^2})}{(X^2 + u^2)^{1/2}} du$$

$$+ \frac{PY}{2\pi M} \int_0^\infty \cosh(\frac{X - X_0}{2}) \left\{\frac{K_1(\frac{1}{2}\sqrt{(X_0 - X)^2 + Y^2})}{[(X_0 - X)^2 + Y^2]^{1/2}}\right.$$

$$\left. - \frac{K_1(\frac{1}{2} (X_0 + X)^2 + Y^2)}{[(X_0 + X)^2 + Y^2]^{1/2}}\right\} dX_0$$

$$- \frac{P}{M^{5/4}} \frac{2^{1/4}}{\Gamma^2(1/4)} \frac{4Y}{\sqrt{2\pi}} \int_0^\infty X_0^{1/4} \cosh(\frac{X - X_0}{2})$$

$$\left\{\frac{K_1(\frac{1}{2}\sqrt{(X_0 - X)^2 + Y^2})}{[(X_0 - X)^2 + Y^2]^{1/2}} - \frac{K_1(\frac{1}{2}\sqrt{(X_0 + X)^2 + Y^2})}{[(X_0 + X)^2 + Y^2]^{1/2}}\right\} dX_0$$

The corresponding u and H solutions in the parabolic layer are:

$$u_p \sim \frac{P}{M}[1 - \frac{\sqrt{2\pi}}{\Gamma^2(1/4)} \int_{-1}^1 (1 - \beta^2)^{-3/4} \operatorname{erfc} \frac{\eta}{2|\beta - x|^{1/2}} d\beta]$$

$$H_p \sim - \frac{P}{M} [x + \frac{\sqrt{2\pi}}{\Gamma^2(1/4)} \int_{-1}^1 (1 - \beta^2)^{-3/4} \operatorname{signum}(\beta - x) \operatorname{erfc} \frac{\eta}{2|\beta - x|^{1/2}} d\beta]$$

These solutions, along with the core and ordinary layer, provide a uniformly valid first approximation.

5.0 CONCLUSIONS

For the circular tube, conducting wall solutions for the core and ordinary boundary layers may be easily obtained using the systematic method of matched asymptotic expansions. Even though the resulting solutions are not uniformly valid, in that additional singular regions exist near the point where the wall is parallel to the applied field, it has been possible to obtain the flow integral to second order.

In order to improve the accuracy of the boundary layer approximations in the singular region noted, a second boundary layer, characterized by a parabolic differential equation, was solved approximately with the result that a singular region still existed when the walls were conducting. An approximate solution with the walls of small or zero wall conductivity does not display this singular behaviour. Thus, a fundamental difference apparently exists between small and large wall conductivities and an additional boundary region characterized by an elliptic differential equation was identified. The error incurred by simply ignoring this additional boundary layer is of a very small order and the labour required to obtain explicit solutions is probably difficult to justify. Through the use of a set of "optimal" coordinates it was found possible to obtain an explicit solution for the parabolic region, valid for arbitrary values of the wall conductivity. To second order, it was shown that the flow deficit due to this region may be ignored.

With the rectangular duct, three boundary layer regions also exist, but in this case the contributions from elliptic layers, in the corners of the duct, are of larger order than for the circular tube elliptic region. Consequently, there is more justification for obtaining solutions. The differential equations in this case are the same as those for the complete problem itself with the asymptotic method providing very little in the way of simplification. These equations have been solved approximately for the case of practical importance where the walls parallel to the applied field are perfectly conducting and those perpendicular to the field are insulating. With the sum and difference of the fluid velocity and magnetic field as dependent variables, boundary layers occur only near one of the insulating walls, with the sum and difference variables having boundary layers on opposite walls. This observation is used to decouple boundary conditions in the corner region and, therefore, to obtain the approximate solution.

APPENDIX

Verification of the Matching

To verify the matching between $F_1(X,Y)$ and $G_1(X,Y)$ of section 4.3 with the parabolic solutions shown in section 4.1, $F_1(X,Y)$ will be considered first.

Making the subsitutions:

$$u = M^{1/2}v$$

in the first term and

$$v = M^{-1}(X_0 - X)$$

or

$$v = M^{-1}(X_0 + X)$$

as appropriate in the remaining terms, and substituting

$$X = M\xi$$

$$Y = M^{1/2},$$

one obtains, for $F_1(X,Y)$:

$$F_1(\xi,\eta) = \frac{PM^{1/2}\xi}{\pi} e^{\frac{M\xi}{2}} \int_0^{\eta} \frac{K_1\left(\frac{M}{2}\sqrt{\xi^2 + v^2/M}\right)}{(\xi^2 + v^2/M)^{1/2}} dv \quad (I_1)$$

$$+ \frac{PM^{1/2}\eta}{2\pi} \int_0^{\xi} e^{\frac{M}{2}v} \frac{K_1\left(\frac{M}{2}\sqrt{v^2 + \eta^2/M}\right)}{(v^2 + \eta^2/M)^{1/2}} dv \quad (I_2)$$

$$+ \frac{PM^{1/2}\eta}{2\pi} \int_0^{\infty} e^{-\frac{M}{2}v} \frac{K_1\left(\frac{M}{2}\sqrt{v^2 + \eta^2/M}\right)}{(v^2 + \eta^2/M)^{1/2}} dv \quad (I_3)$$

$$- \frac{PM^{1/2}\eta}{2\pi} e^{M\xi} \int_{\xi}^{\infty} e^{-\frac{M}{2}v} \frac{K_1\left(\frac{M}{2}\sqrt{v^2 + \eta^2/M}\right)}{(v^2 + \eta^2/M)^{1/2}} dv \quad (I_4)$$

$$- \frac{PM^{1/2}\eta}{\sqrt{2}\pi} \frac{2^{1/4}}{\Gamma^2(1/4)} 4 \int_0^{\xi} (\xi - v)^{1/4} e^{\frac{M}{2}v}$$

$$\times \frac{K_1\left(\frac{M}{2}\sqrt{v^2 + \eta^2/M}\right)}{(v^2 + \eta^2/M)^{1/2}} dv \quad (I_5)$$

$$- \frac{PM^{1/2}\eta}{\sqrt{2}\pi} \frac{2^{1/4}}{\Gamma^2(1/4)} 4 \int_0^{\infty} (\xi + v)^{1/4} e^{-\frac{M}{2}v}$$

$$\times \frac{K_1\left(\frac{M}{2}\sqrt{v^2 + \eta^2/M}\right)}{(v^2 + \eta^2/M)^{1/2}} dv \quad (I_6)$$

$$+ \frac{PM^{1/2}\eta}{\sqrt{2\pi}} \frac{2^{1/4}}{\Gamma^2(1/4)} 4e^{M\xi} \int_{\xi}^{\infty} (v - \xi)^{1/4} e^{-\frac{M}{2}v}$$

$$\times \frac{K_1\left(\frac{M}{2}\sqrt{v^2 + \eta^2/M}\right)}{(v^2 + \eta^2/M)^{1/2}}\, dv \qquad (I_7)$$

Each term is identified by I_1, I_2, etc. Asymptotic expansions will be obtained term by term. In each case the asymptotic expansion for the Bessel function will be used, since even where v = 0, the argument of the Bessel functions will be of order $M^{1/2}$. Two new variables, defined as follows, will be used frequently:

$$Z_1 = (v^2 + \eta^2/M)^{1/2} - v$$

$$Z_2 = (v^2 + \eta^2/M)^{1/2} + v$$

The following properties of these variables will be required:

$$v^2 + \eta^2/M = \left[\frac{Z_1^{\,2} + \eta^2/M}{2Z_1}\right]^2 = \left[\frac{Z_2^{\,2} + \eta^2/M}{2Z_2}\right]^2$$

$$\frac{dv}{dZ_1} = -\frac{(Z_1^{\,2} + \eta^2/M)}{2Z_1^{\,2}}$$

$$\frac{dv}{dZ_2} = \frac{(Z_2^{\ 2} + \eta^2/M)}{2Z_2^{\ 2}}$$

$$v = \frac{1}{2Z_1}\ (\eta^2/M - Z_1^{\ 2}) = \frac{1}{2Z_2}\ (Z_2^{\ 2} - \eta^2/M)$$

$$\xi = \frac{1}{2Z_1(\xi)}\ (\eta^2/M - Z_1^{\ 2}(\xi)) = \frac{1}{2Z_2(\xi)}\ (Z_2^{\ 2}(\xi) - \eta^2/M)$$

$$Z_1(\xi) = (\xi^2 + \eta^2/M) - \xi^2$$

$$Z_2(\xi) = (\xi^2 + \eta^2/M) + \xi^2$$

$$Z_1(0) = Z_2(0) = \frac{\eta}{\sqrt{M}}$$

$$Z_1(\infty) = 0$$

$$Z_2(\infty) = \infty$$

In the expansions which follow, most of the algebraic manipulations have been deleted.

Expansion for I_1

$$I_1 \sim \frac{P\xi}{\sqrt{\pi}} e^{\frac{M}{2}\xi} \int_0^{\eta} \frac{\exp[-\frac{M}{2}(\xi^2 + v^2/M)^{1/2}]}{(\xi^2 + v^2/M)^{3/4}}$$

$$[1 + \frac{3}{4M}(\xi^2 + v^2/M)^{-1/2} + 0(M^{-2})]\,dv$$

$$\sim \frac{P}{\sqrt{\pi}} \frac{e^{\frac{M}{2}\xi}}{\sqrt{\xi}} \int_0^{\eta} \frac{\exp[-\frac{M}{2}\xi[1 + \frac{v^2}{2M\xi^2} - \frac{1}{8}\frac{v^4}{\xi^4 M^2} + 0(M^{-3})]}{(1 + \frac{v^2}{M\xi^2})^{3/4}}$$

$$\times [1 + \frac{3}{4M\xi} + 0(M^{-2})]\,dv$$

$$\sim \frac{P}{\sqrt{\pi}} \frac{1}{\sqrt{\xi}} \int_0^{\eta} \exp(-\frac{v^2}{4\xi})[1 + \frac{v^4}{16M\xi^3} + 0(M^{-2})]$$

$$\times [1 - \frac{3v^2}{4M\xi^2} + 0(M^{-2})][1 + \frac{3}{4M\xi} + 0(M^{-2})]\,dv$$

$$\sim P\frac{2}{\sqrt{\pi}} \int_0^{\frac{\eta}{2\sqrt{\xi}}} e^{-t^2}[1 + \frac{1}{M}(\frac{t^4}{\xi} - \frac{3t^2}{\xi} + \frac{3}{4\xi}) + 0(M^{-2})]\,dt$$

where $t = \frac{v}{2\sqrt{\xi}}$

$$I_1 \sim P\,\mathrm{erf}(\frac{\eta}{2\sqrt{\xi}}) + 0(M^{-1})$$

Expansion of I_2

$$I_2 \sim \frac{P\eta}{2\sqrt{\pi}} \int_0^{\xi} \frac{\exp \frac{M}{2}[v - (v^2 + \eta^2/M)^{1/2}]}{(v^2 + \eta^2/M)^{3/4}}$$

$$\times [1 + \frac{3}{4M}(v^2 + \frac{\eta^2}{M})^{-1/2} + 0(M^{-2})]dv$$

Using $Z_1(v)$, this becomes:

$$I_2 \sim \frac{P\eta}{\sqrt{2\pi}} \int_{Z_1(\xi)}^{Z_1(0)} \frac{\exp(-\frac{M}{2}Z_1)}{(Z_1{}^2 + \eta^2/M)^{1/2}} [1 + \frac{3}{4M}(\frac{2Z_1}{Z_1{}^2 + \eta^2/M})$$

$$+ 0(M^{-2})] \frac{dZ_1}{Z_1{}^{1/2}}$$

$$\sim P\frac{2}{\sqrt{\pi}} \int_{\sqrt{\frac{M}{2}Z_1(\xi)}}^{\sqrt{\frac{M}{2}Z_1(0)}} \exp(-t^2)(1 + \frac{4t^4}{M\eta^2})^{-1/2}$$

$$\times [1 + \frac{3}{M\eta^2}t^2 + 0(M^{-2})]dt$$

where $t^2 = \frac{M}{2}Z_1$

$$I_2 \sim P \operatorname{erfc}\sqrt{\frac{M}{2}Z_1(\xi)} - P \operatorname{erfc}\sqrt{\frac{M}{2}Z_1(0)} + 0(M^{-1})$$

$$\therefore \quad I_2 \sim P \operatorname{erfc}(\frac{\eta}{2\sqrt{\xi}}) + 0(M^{-1})$$

since, $Z_1(\xi) = (\xi^2 + \eta^2/M)^{1/2} - \xi \sim \frac{\eta^2}{2M\xi} + O(M^{-2})$

Expansion of I_3

$$I_3 \sim \frac{P\eta}{2\sqrt{\pi}} \int_0^\infty \frac{\exp -\frac{M}{2}[v + (v^2 + \eta^2/M)^{1/2}]}{(v^2 + \eta^2/M)^{3/4}} \, dv$$

The higher order terms for the Bessel function have been omitted here and will not be included in subsequent expansions. Rewriting this in terms of the variable $Z_2(v)$ results in:

$$I_3 \sim \frac{P\eta}{\sqrt{2\pi}} \int_{\eta/\sqrt{M}}^\infty \frac{\exp(-\frac{M}{2}Z_2)}{(Z_2^{\,2} + \eta^2/M)^{1/2}} \frac{dZ_2}{Z_2^{\,1/2}}$$

$$\sim P\frac{2}{\sqrt{\pi}} \int_{\sqrt{\frac{M^{1/2}\eta}{2}}}^\infty \exp(-t^2)\left(1 + \frac{4t^4}{M\eta^2}\right)^{-1/2} dt$$

$$\sim P \operatorname{erfc}\sqrt{\frac{M^{1/2}\eta}{2}}$$

$\sim$ exponentially small

Expansion of I_4

$$I_4 \sim \frac{P\eta}{2\sqrt{\pi}} \exp(M\xi) \int_{\xi}^{\infty} \frac{\exp\left\{-\frac{M}{2}[v + (v^2 + \eta^2/M)^{1/2}]\right\}}{(v^2 + \eta^2/M)^{3/4}} \, dv$$

$$\sim \frac{P\eta}{\sqrt{2\pi}} \exp(M\xi) \int_{Z_2(\xi)}^{\infty} \frac{\exp(-\frac{M}{2} Z_2)}{(Z_2{}^2 + \eta^2/M)^{1/2}} \frac{dZ_2}{Z_2{}^{1/2}}$$

$$\sim P \exp(M\xi) \frac{2}{\sqrt{\pi}} \int_{\sqrt{\frac{M}{2}Z_2(\xi)}}^{\infty} \exp(-t^2)(1 + \frac{4t^4}{M\eta^2})^{-1/2} dt$$

$$\sim P \exp(M\xi) \operatorname{erfc}\sqrt{\frac{M}{2} Z_2(\xi)}$$

$$\sim \frac{P}{\sqrt{M\pi\xi}} \exp(-\frac{\eta^2}{4\xi})$$

Expansion of I_5

$$I_5 \sim P\frac{2^{3/4}}{\Gamma^2(1/4)}\, 2\eta \int_0^{\xi} (\xi - v)^{1/4}$$

$$\times \frac{\exp \frac{M}{2}[v - (v^2 + \eta^2/M)^{1/2}]}{(v^2 + \eta^2/M)^{3/4}}\, dv$$

$$\sim \frac{4P\eta^{1/2}M^{1/4}}{\Gamma^2(1/4)} \int_{Z_1(\xi)}^{Z_1(0)} \left(\frac{Z_1}{Z_1(\xi)} - 1\right)^{1/4}$$

$$\times \left(1 + \frac{MZ_1(\xi)Z_1}{\eta^2}\right)^{1/4} \frac{\exp(-\frac{M}{2} Z_1)}{\left(1 + \frac{MZ_1^2}{\eta^2}\right)^{1/2}}\, \frac{dZ_1}{Z_1^{3/4}}$$

$$\sim \frac{4P\eta^{1/2}M^{1/4}Z_1^{1/4}(\xi)}{\Gamma^2(1/4)} \int_1^{\frac{Z_1(0)}{Z_1(\xi)}} (t - 1)^{1/4} t^{-3/4}$$

$$\times \exp(-\frac{M}{2}Z_1(\xi) t)\, \frac{\left(1 + \frac{MZ_1^2(\xi)t}{\eta^2}\right)^{1/4}}{\left(1 + \frac{MZ_1^2(\xi)t^2}{\eta^2}\right)^{1/2}}\, dt$$

where $t = Z_1/Z_1(\xi)$ in this case.

But, $Z_1(\xi) = (\xi^2 + \eta^2/M)^{1/2} - \xi$

$$= \frac{\eta^2}{2M\xi}\left[\frac{2}{1 + (1 + \frac{\eta^2}{M\xi})^{1/2}}\right]$$

$$= \frac{\eta^2}{2M\xi}\left[1 + \frac{1 - (1 + \frac{\eta^2}{M\xi})^{1/2}}{1 + (1 + \frac{\eta^2}{M\xi})^{1/2}}\right]$$

$$I_5 \sim \frac{4P\eta^{1/2}}{\Gamma^2(1/4)} M^{1/4} Z_1^{1/4}(\xi) \int_1^{\frac{Z_1(0)}{Z_1(\xi)}} (t-1)^{1/4} t^{-3/4}$$

$$\exp(-\frac{\eta^2}{4\xi}t) \quad \exp\left[\frac{\eta^2 t}{4\xi} \frac{(1 + \frac{\eta^2}{M\xi})^{1/2} - 1}{(1 + \frac{\eta^2}{M\xi})^{1/2} + 1}\right]$$

$$\frac{(1 + \frac{MZ_1^2(\xi)t}{\eta^2})^{1/4}}{(1 + \frac{MZ_1^2(\xi)t^2}{\eta^2})^{1/4}} dt$$

$$I_5 \sim P\frac{2^{-1/4}}{\Gamma^2(1/4)} \frac{4\eta}{\xi^{1/4}}[1 + 0(M^{-1})] \int_1^{\infty} (t-1)^{1/4} t^{-3/4}$$

$$\exp(-\frac{\eta^2}{4\xi}t)[1 + 0(M^{-1})]\, dt$$

+ exponentially small term

$$I_5 \sim P\frac{2^{-1/4}}{\Gamma(1/4)}\frac{\eta}{\xi^{1/4}} \exp\left(-\frac{\eta^2}{4\xi}\right) U\left(5/4,3/2,\frac{\eta^2}{4\xi}\right)$$

Expansion of I_6

$$I_6 \sim P\frac{2^{3/4}}{\Gamma^2(1/4)} 2\eta \int_0^\infty (\xi + v)^{1/4} \frac{\exp\left\{-\frac{M}{2}[v + (v^2 + \eta^2/M)^{1/2}]\right\}}{(v^2 + \eta^2/M)^{3/4}} dv$$

$$\sim \frac{4P}{\Gamma^2(1/4)} M^{1/2} Z_2^{1/4}(\xi) \int_{\frac{\eta}{\sqrt{M}}}^\infty \left(1 + \frac{Z_2}{Z_2(\xi)}\right)^{1/4}$$

$$\times \left(1 - \frac{\eta^2}{MZ_2(\xi) Z_2}\right)^{1/4} \frac{\exp(-\frac{M}{2}Z_2)}{\left(1 + \frac{MZ_2}{\eta^2}\right)^{1/2}} \frac{dZ_2}{Z_2^{1/2}}$$

$$\sim \frac{8\sqrt{2}\, P}{\Gamma^2(1/4)} Z_2^{1/4}(\xi) \int_{\sqrt{\frac{M^{1/2}\eta}{2}}}^\infty \exp(-t^2)\left(1 - \frac{\eta^2}{2Z_2(\xi) t^2}\right)^{1/4}$$

$$\times \frac{\left(1 + \frac{2t^2}{MZ_2(\xi)}\right)^{1/4}}{\left(1 + \frac{4t^4}{M\eta^2}\right)^{1/2}} dt$$

where $t^2 = \frac{M}{2} Z_2$

$$I_6 \sim \frac{8\ 2^{3/4} P}{\Gamma^2_{(1/4)}} [\xi^{1/4} + O(M^{-1})] \int_{\sqrt{\frac{M^{1/2}\eta}{2}}}^{\infty} \exp(-t^2)$$

$$\times [1 + O(M^{-1/2})]\ dt$$

$$\sim \frac{4\sqrt{\pi}\ 2^{3/4} P}{\Gamma^2_{(1/4)}}\ \xi^{1/4} \operatorname{erfc}\sqrt{\frac{M^{1/2}\eta}{2}}$$

$\sim$ exponentially small.

Expansion of I_7

$$I_7 \sim \frac{P\ 2^{3/4}}{\Gamma^2_{(1/4)}}\ 2\eta \exp(M\xi) \int_{\xi}^{\infty} (v - \xi)^{1/4}$$

$$\times \frac{\exp -\frac{M}{2}[v + (v^2 + \eta^2/M)^{1/2}]}{(v^2 + \eta^2/M)^{3/4}}\ dv$$

Defining $u = \frac{M}{2}[Z_2(v) - Z_2(\xi)]$

$$I_7 \sim \frac{P}{M^{5/4}} \frac{2^{1/4}}{\Gamma^2_{(1/4)}} \frac{8\eta}{Z_2^{3/2}(\xi)} \exp[M(\xi - \frac{Z_2(\xi)}{2})]$$

$$\int_0^{\infty} u^{1/4}\ e^{-u}\ \frac{[1 + \frac{\eta^2}{2MZ_2^2(\xi)}(1 + \frac{2u}{MZ_2(\xi)})^{-1}]}{[(1 + \frac{2u}{MZ_2(\xi)})^2 + \frac{\eta^2}{MZ_2^2(\xi)}]^{1/2}}$$

$$\times \frac{du}{[1 + \frac{2u}{MZ_2(\xi)}]^{1/2}}$$

$$\sim \frac{P}{M^{5/4}} \frac{2^{-1/4}}{\Gamma^2(1/4)} \frac{4\eta}{\xi^{3/2}} \exp(-\frac{\eta^2}{4\xi})[1 + O(M^{-1})]$$

$$\int_0^\infty u^{1/4} e^{-u}[1 + O(M^{-1})]\, du$$

$$\sim \frac{P}{M^{5/4}} \frac{2^{-1/4}}{\Gamma(1/4)} \frac{\eta}{\xi^{3/2}} \exp(-\frac{\eta^2}{4\xi})$$

We may now form the sum, to obtain:

$$F_1(\xi,\eta) \sim P \operatorname{erf}(\frac{\eta}{2\sqrt{\xi}}) + P \operatorname{erfc}(\frac{\eta}{2\sqrt{\xi}})$$

$$- P\frac{2^{-1/4}}{\Gamma^2(1/4)} \frac{\eta}{\xi^{1/4}} \exp(-\frac{\eta^2}{4\xi}) U(5/4, 3/2, \frac{\eta^2}{4\xi})$$

$$+ O(M^{-1/2})$$

Thus,

$$F_1(\xi,\eta) \sim P[1 - \frac{2^{-1/4}}{\Gamma^2(1/4)} \frac{\eta}{\xi^{1/4}} e^{-\frac{\eta^2}{4\xi}} U(5/4, 3/2, \frac{\eta^2}{4\xi})]$$

which is the same as $F_{p1}(\xi,\eta)$ from equation (4.2).

To verify the matching for $G_1(X, Y)$, substitutions similar to those made for $F_1(X, Y)$ are introduced resulting in the following set of integrals to expand:

$$G_1(\xi,\eta) = -\frac{PM^{1/2}\xi}{\pi} e^{-\frac{M}{2}\xi} \int_0^{\eta} \frac{K_1\left(\frac{M}{2}\sqrt{\xi^2 + v^2/M}\right)}{(\xi^2 + v^2/M)^{1/2}}\, dv \qquad (J_1)$$

$$- \frac{PM^{1/2}\eta}{2\pi} \int_0^{\xi} e^{-\frac{M}{2}v} \frac{K_1\left(\frac{M}{2}\sqrt{v^2 + \eta^2/M}\right)}{(v^2 + \eta^2/M)^{1/2}}\, dv \qquad (J_2)$$

$$- \frac{PM^{1/2}\eta}{2\pi} \int_0^{\infty} e^{-\frac{M}{2}v} \frac{K_1\left(\frac{M}{2}\sqrt{v^2 + \eta^2/M}\right)}{(v^2 + \eta^2/M)^{1/2}}\, dv \qquad (J_3)$$

$$+ \frac{PM^{1/2}\eta}{2\pi} e^{-M\xi} \int_{\xi}^{\infty} e^{\frac{M}{2}v} \frac{K_1\left(\frac{M}{2}\sqrt{v^2 + \eta^2/M}\right)}{(v^2 + \eta^2/M)^{1/2}}\, dv \qquad (J_4)$$

$$+ \frac{PM^{1/2}\eta}{2\pi} \frac{2^{1/4}\sqrt{2\pi}}{\Gamma^2(1/4)} 4 \int_0^{\xi} (\xi - v)^{1/4} e^{-\frac{M}{2}v}$$

$$\times \frac{K_1\left(\frac{M}{2}\sqrt{v^2 + \eta^2/M}\right)}{(v^2 + \eta^2/M)^{1/2}}\, dv \qquad (J_5)$$

$$+ \frac{PM^{1/2}\eta}{2\pi} \frac{2^{1/4}\sqrt{2\pi}\, 4}{\Gamma^2(1/4)} \int_0^{\infty} (v + \xi)^{1/4} e^{\frac{M}{2}v}$$

$$\times \frac{K_1\left(\frac{M}{2}\sqrt{v^2 + \eta^2/M}\right)}{(v^2 + \eta^2/M)^{1/2}}\, dv \qquad (J_6)$$

$$- \frac{PM^{1/2}\eta}{2\pi} \frac{2^{1/4}\sqrt{2\pi}}{\Gamma^2(1/4)} 4e^{-M\xi} \int_{\xi}^{\infty} (v-\xi)^{1/4} e^{\frac{M}{2}v}$$

$$\times \frac{K_1\left(\frac{M}{2}\sqrt{v^2+\eta^2/M}\right)}{(v^2+\eta^2/M)^{1/2}} dv \qquad (J_7)$$

Expansion of J_1

This term is identical to I_1, except for the exponential factor, with an identical expansion procedure. The result is:

$$J_1 \sim Pe^{-M\xi} \operatorname{erf}\left(\frac{\eta}{2\sqrt{\xi}}\right)$$

$\sim$ exponentially small.

Expansion of J_2

$$J_2 \sim \frac{P\eta}{2\sqrt{\pi}} \int_0^{\xi} \frac{\exp -\frac{M}{2}[v + (v^2+\eta^2/M)^{1/2}]}{(v^2+\eta^2/M)^{3/4}} dv$$

$$\sim \frac{P\eta}{\sqrt{2\pi}} \int_{Z_2(0)}^{Z_2(\xi)} \frac{\exp(-\frac{M}{2} Z_2)}{(Z_2{}^2+\eta^2/M)^{1/2}} \frac{dZ_2}{Z_2{}^{1/2}}$$

$$\sim P\frac{2}{\sqrt{\pi}} \int_{\sqrt{\frac{M}{2}Z_2(0)}}^{\sqrt{\frac{M}{2}Z_2(\xi)}} \exp(-t^2) \cdot \frac{dt}{(1+\frac{4t^4}{M\eta^2})^{1/2}}$$

where $t^2 = \frac{M}{2}Z_2$

$$J_2 \sim P\ \mathrm{erfc}\sqrt{\frac{M}{2}Z_2(0)} - P\ \mathrm{erfc}\sqrt{\frac{M}{2}Z_2(\xi)}$$

$$\sim P\ \mathrm{erfc}\ \sqrt{\frac{M^{1/2}\eta}{2}}$$

$$\sim \text{exponentially small.}$$

<u>Expansion of J_3</u>

$$J_3 \sim \frac{P\eta}{2\sqrt{\pi}} \int_0^{\infty} \frac{\exp \frac{M}{2}[v - (v^2 + \eta^2/M)^{1/2}]}{(v^2 + \eta^2/M)^{3/4}}\, dv$$

$$\sim \frac{P\eta}{\sqrt{2\pi}} \int_0^{\frac{\eta}{\sqrt{M}}} \frac{\exp(-\frac{M}{2}Z_1)}{(Z_1^{\,2} + \eta^2/M)^{1/2}} \frac{dZ_1}{Z_1^{\,1/2}}$$

since $Z_1(\infty) = 0$. Let $t^2 = \frac{M}{2}Z_1$ and

$$J_3 \sim P\frac{2}{\sqrt{\pi}} \int_0^{\sqrt{\frac{M^{1/2}\eta}{2}}} \exp(-t^2) \frac{dt}{(1 + \frac{4t^4}{M\eta^2})^{1/2}}$$

$$\sim P + 0(M^{-1})$$

Expansion of J_4

$$J_4 \sim \frac{P\eta}{2\sqrt{\pi}} \exp(-M\xi) \int_{\xi}^{\infty} \frac{\exp \frac{M}{2}[v - (v^2 + \eta^2/M)^{1/2}]}{(v^2 + \eta^2/M)^{3/4}} dv$$

$$\sim \frac{P\eta}{\sqrt{2\pi}} \exp(-M\xi) \int_0^{Z_1(\xi)} \frac{\exp(-\frac{M}{2}Z_1)}{(Z_1^{\,2} + \eta^2/M)^{1/2}} \cdot \frac{dZ_1}{Z_1^{\,1/2}}$$

$$\sim P \exp(-M\xi)\frac{2}{\sqrt{\pi}} \int_0^{\frac{M}{2}Z_1(\xi)} \exp(-t^2) \frac{dt}{(1 + \frac{4t^4}{M\eta^2})^{1/2}}$$

$$\sim P \exp(-M\xi)[\mathrm{erf}\sqrt{\tfrac{M}{2}Z_1(\xi)} + O(M^{-1})]$$

$$\sim P \exp(-M\xi) \; \mathrm{erf}(\frac{\eta}{2\xi^{1/2}})$$

$\sim$ exponentially small.

Expansion of J_5

$$J_5 \sim P\frac{\eta 2^{-1/4}}{\Gamma^2(1/4)} 4 \int_0^{\xi} (\xi - v)^{1/4}$$

$$\cdot \frac{\exp -\frac{M}{2}[v + (v^2 + \eta^2/M)^{1/2}]}{(v^2 + \eta^2/M)^{3/4}} dv$$

$$\sim \frac{4P\eta}{\Gamma^2(1/4)} \int_{Z_2(0)}^{Z_2(\xi)} [Z_2(\xi) - Z_2]^{1/4} [1 + \frac{\eta^2}{M} \frac{1}{Z_2(\xi) Z_2}]^{1/4}$$

$$\times \frac{\exp(-\frac{M}{2}Z_2)}{(Z_2{}^2 + \eta^2/M)^{1/2}} \frac{dZ_2}{Z_2{}^{1/4}}$$

$$\sim \frac{4PM^{1/2}}{\Gamma^2(1/4)} Z_2{}^{1/4}(\xi) \int_{\frac{\eta}{\sqrt{M}}}^{Z_2(\xi)} [1 - \frac{Z_2}{Z_2(\xi)}]^{1/4}$$

$$\times [1 + \frac{\eta^2}{M} \frac{1}{Z_2(\xi) Z_2}]^{1/4} \frac{\exp(-\frac{M}{2}Z_2)}{[1 + \frac{MZ_2{}^2}{\eta^2}]^{1/2}} \frac{dZ_2}{Z_2{}^{1/2}}$$

$$\sim \frac{8\sqrt{2}\, P}{\Gamma^2(1/4)} Z_2{}^{1/4}(\xi) \int_{\sqrt{\frac{M^{1/2}\eta}{2}}}^{\sqrt{\frac{M}{2}Z_2(\xi)}} [1 - \frac{2t^2}{MZ_2(\xi)}]^{1/4}$$

$$\times [1 + \frac{\eta^2}{2Z_2(\xi) t^2}]^{1/4} \frac{\exp(-t^2)}{[1 + \frac{4t^4}{M\eta^2}]^{1/2}} dt$$

where $t^2 = \frac{M}{2}Z_2$. Now expanding the second factor $[1 + \eta^2/(2Z_2(\xi)t^2)]^{1/4}$ about the lower limit of integration plus the usual elementary expansion of the other factors results in:

$$J_5 \sim \frac{8\sqrt{2}\,P}{\Gamma^2_{(1/4)}}\, 2^{1/4}\, \xi^{1/4}[1 + O(M^{-1})] \int_{\sqrt{\frac{M^{1/2}\eta}{2}}}^{\sqrt{\frac{M}{2}Z_2(\xi)}}$$

$$\times \exp(-t^2)[1 + O(M^{-1/2})]\, dt$$

$$\sim \frac{4\sqrt{\pi}\, 2^{3/4}}{\Gamma^2_{(1/4)}}\, \xi^{1/4}\left[\operatorname{erfc}\sqrt{\frac{M^{1/2}\eta}{2}} - \operatorname{erfc}\sqrt{\frac{MZ_2(\xi)}{2}}\right]$$

$\sim$ exponentially small

<u>Expansion of J_6</u>

$$J_6 \sim P\eta \frac{2^{-1/4}}{\Gamma^2_{(1/4)}}\, 4 \int_0^\infty (v + \xi)^{1/4}$$

$$\times \frac{\exp \frac{M}{2}[v - (v^2 + \eta^2/M)^{1/2}]}{(v^2 + \eta^2/M)^{3/4}}\, dv$$

$$\sim \frac{4P\eta}{\Gamma^2_{(1/4)}} \int_0^{Z_1(\xi)} [Z_1 + Z_1(\xi)]^{1/4}\left[\frac{\eta^2}{MZ_1(\xi)Z_1} - 1\right]^{1/4}$$

$$\times \frac{\exp(-\frac{M}{2}Z_1)}{(Z_1^{\,2} + \eta^2/M)^{1/2}}\, \frac{dZ_1}{Z_1^{\,1/2}}$$

$$\sim \frac{4P}{\Gamma^2(1/4)} \eta^{1/2} M^{1/4} \int_0^{Z_1(0)} \left[1 + \frac{Z_1}{Z_1(\xi)}\right]^{1/4}$$

$$\times \left[1 - \frac{M Z_1(\xi) Z_1}{\eta^2}\right]^{1/4} \frac{\exp(-\frac{M}{2} Z_1)}{(1 + \frac{M Z_1^2}{\eta^2})^{1/2}} \frac{dZ_1}{Z_1^{3/4}}$$

$$\sim \frac{4P \eta^{1/2}}{\Gamma^2(1/4)} M^{1/4} Z_1^{1/4}(\xi) \int_0^{\frac{Z_1(0)}{Z_1(\xi)}} (1 + t)^{1/4} t^{-3/4}$$

$$\times \exp(-\frac{M}{2} Z_1(\xi) t) \frac{\left[1 - \frac{M Z_1^2(\xi) t}{\eta^2}\right]^{1/4}}{\left[1 + \frac{M Z_1^2(\xi) t^2}{\eta^2}\right]^{1/2}} dt$$

where $t = Z_1/Z_1(\xi)$. Expanding the factors in the same way as was done for I_5 results in:

$$J_6 \sim P \frac{2^{-1/4}}{\Gamma^2(1/4)} 4 \frac{\eta}{\xi^{1/4}} [1 + O(M^{-1})] \int_0^{\infty} (1 + t)^{1/4} t^{-3/4}$$

$$\times \exp(-\frac{\eta^2}{4\xi} t) [1 + O(M^{-1})] \, dt$$

$$+ \text{ exponentially small term}$$

$$\sim P \frac{2^{-1/4}}{\Gamma(1/4)} 4 \frac{\eta}{\xi^{1/4}} U(1/4, 3/2, \frac{\eta^2}{4\xi})$$

Expansion of J_7

$$J_7 \sim \frac{P\eta}{\Gamma^2(1/4)} 2^{-1/4} 4\exp(-M\xi) \int_{\xi}^{\infty} (v - \xi)^{1/4}$$

$$\times \frac{\exp \frac{M}{2}[v - (v^2 + \eta^2/M)^{1/2}]}{(v^2 + \eta^2/M)^{3/4}} dv$$

$$\sim \frac{4P\eta}{\Gamma^2(1/4)} \exp(-M\xi) \int_0^{Z_1(\xi)} [Z_1(\xi) - Z_1]^{1/4}$$

$$\times [\frac{\eta^2}{MZ_1(\xi)Z_1} + 1]^{1/4} \frac{\exp(-\frac{M}{2}Z_1)}{(Z_1{}^2 + \eta^2/M)^{1/2}} \frac{dZ_1}{Z_1{}^{1/2}}$$

$$\sim \frac{4P\eta^{1/2}}{\Gamma^2(1/4)} M^{1/4} Z_1{}^{1/4}(\xi) \exp(-M\xi) \int_0^1 (1 - t)^{1/4} t^{-3/4}$$

$$\times \exp(-\frac{M}{2}Z_1(\xi)t) \frac{[1 + \frac{MZ_1{}^2(\xi)t}{\eta^2}]^{1/4}}{[1 + \frac{MZ_1{}^2(\xi)t^2}{\eta^2}]^{1/2}} dt$$

with $t = Z_1/Z_1(\xi)$. Expanding as before:

$$J_7 \sim \frac{P\, 2^{-1/4}}{\Gamma^2(1/4)} 4\frac{\eta}{\xi^{1/4}} \exp(-M\xi) \int_0^1 (1 - t)^{1/4}$$

$$\times t^{-3/4} \exp(-\frac{\eta^2}{4\xi t}) dt$$

$\sim$ exponentially small.

The sum of these terms is now:

$$G_1(\xi,\eta) \sim -P + P\,\frac{2^{-1/4}}{\Gamma(1/4)}\,4\,\frac{\eta}{\xi^{1/4}}\,U(1/4,3/2,\frac{\eta^2}{4\xi}) + O(M^{-1})$$

$$-P[1 - \frac{2^{-1/4}}{\Gamma(1/4)}\,4\,\frac{\eta}{\xi^{1/4}}\,U(1/4,3/2,\frac{\eta^2}{4\xi})]$$

which matches $G_{P1}(\xi,\eta)$ from equation (4.3).

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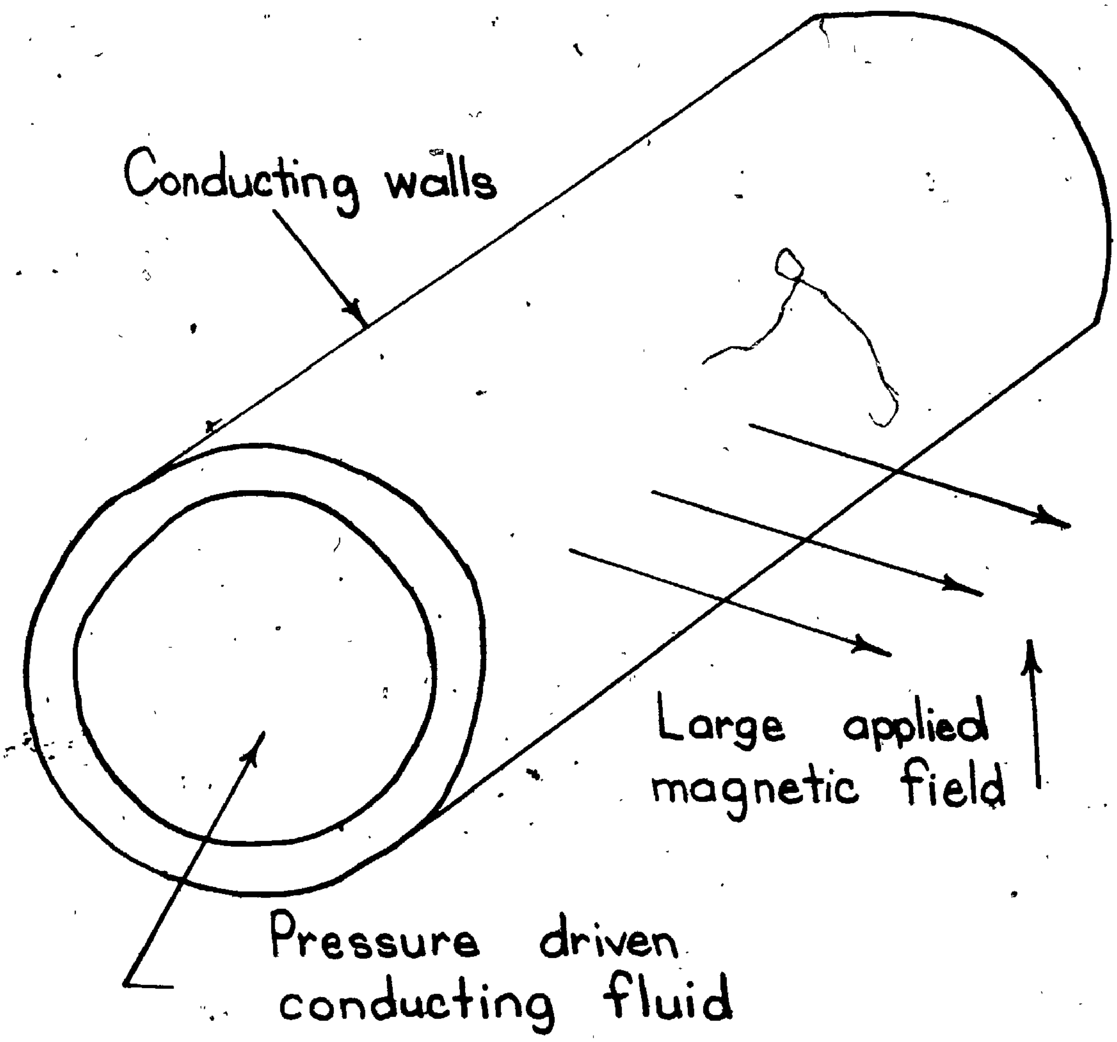


MHD Duct Flow Illustration

Fig. 1·1

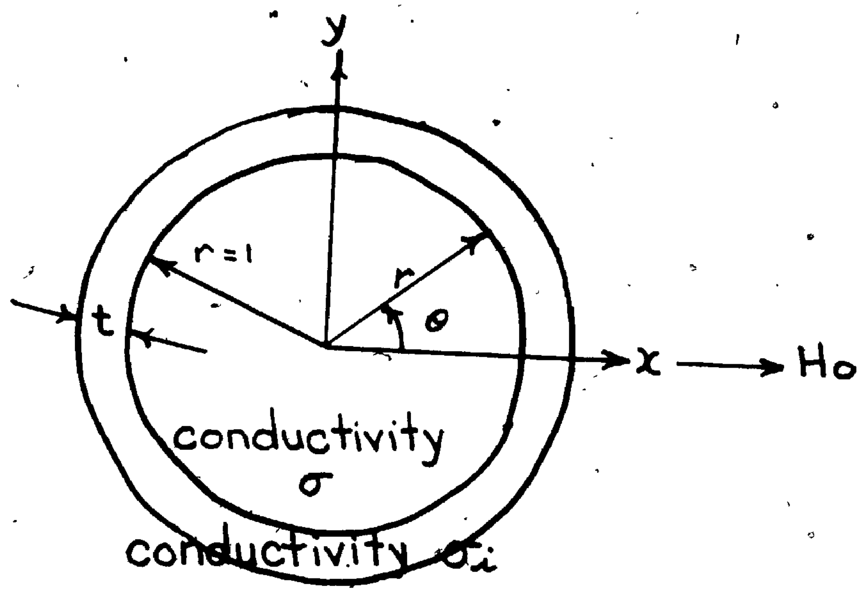


Fig. 2·1(a) Circular Duct Geometry

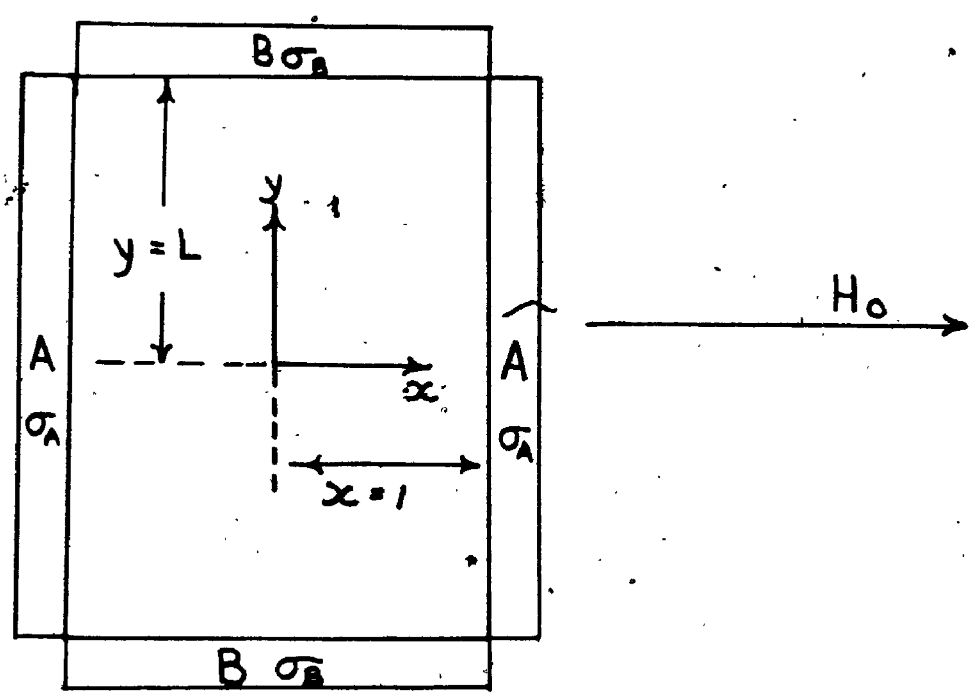


Fig 2·1(b) Rectangular Duct Geometry

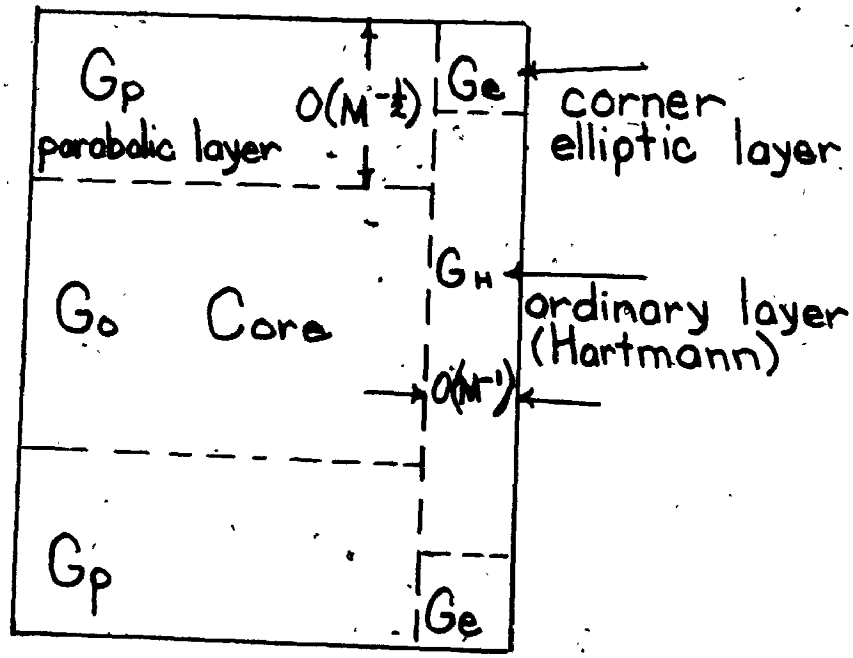


Fig. 2·2(a) Asymptotic Region For G.

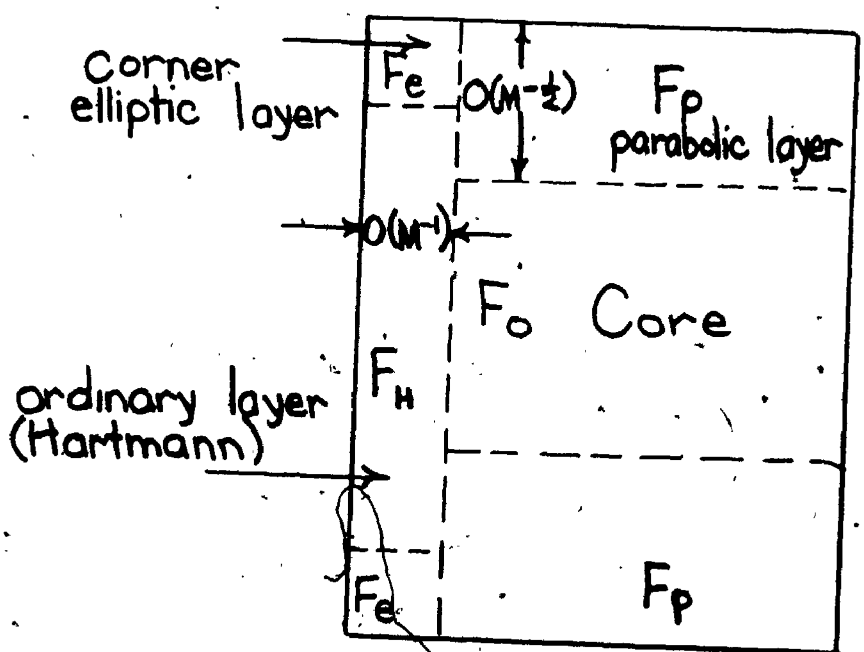


Fig. 2·2(b) Asymptotic Region For F.

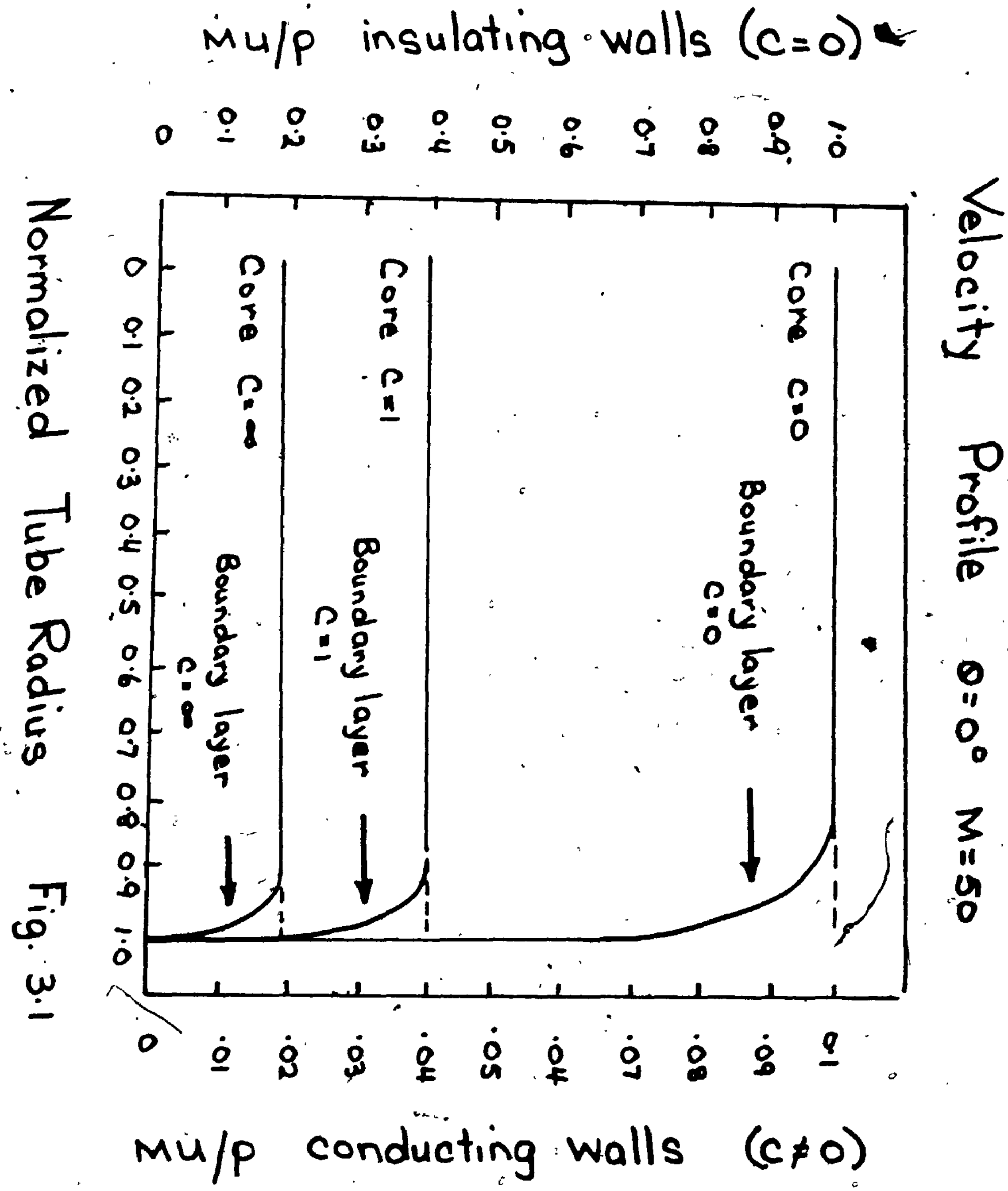
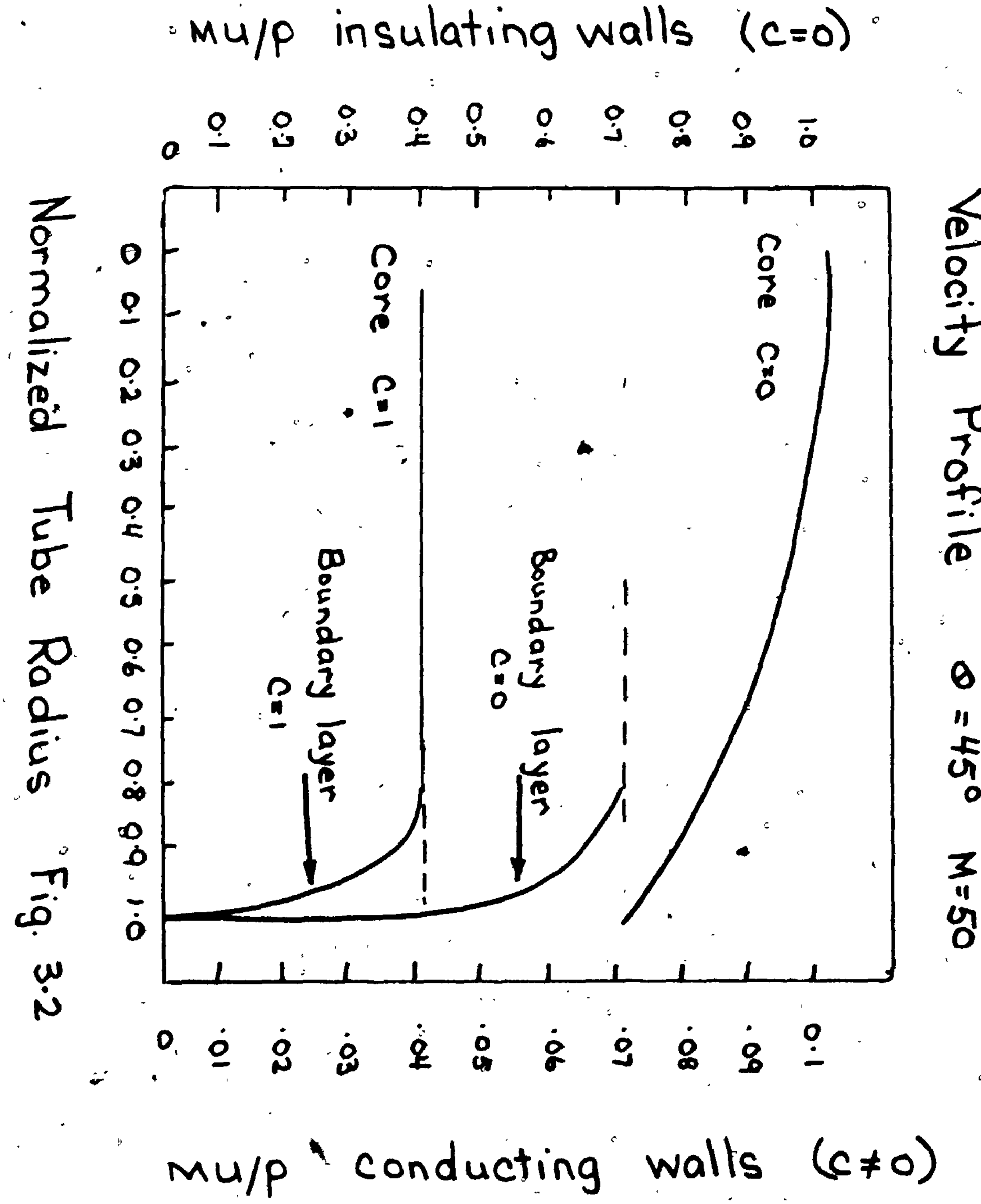
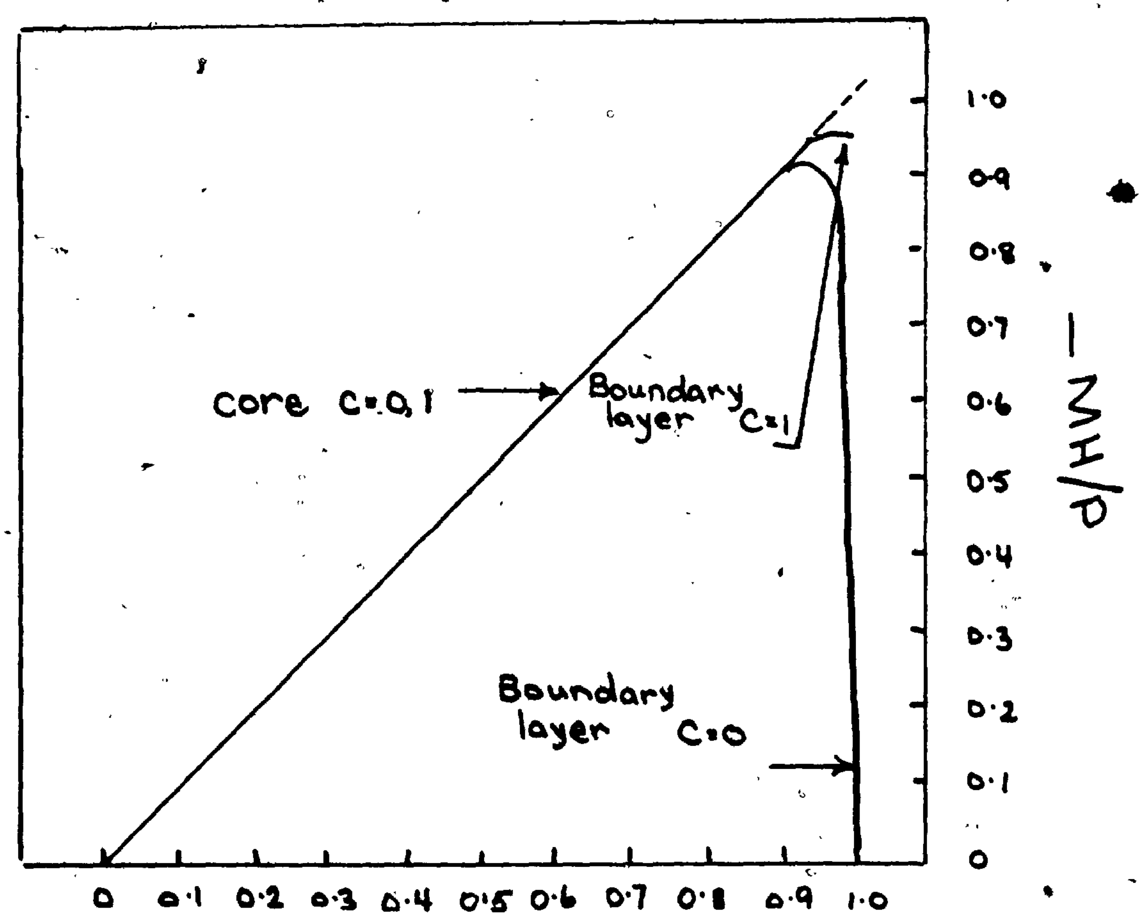

Mu/p insulating walls (c=0)
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0
Velocity Profile θ=0° M=50
Core C=0
Boundary layer C=0
Core C=1
Boundary layer C=1
Core C=∞
Boundary layer C=∞
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0
Normalized Tube Radius
0 .01 .02 .03 .04 .05 .06 .07 .08 .09 .1
Mu/p conducting walls (c≠0)

Fig. 3.1

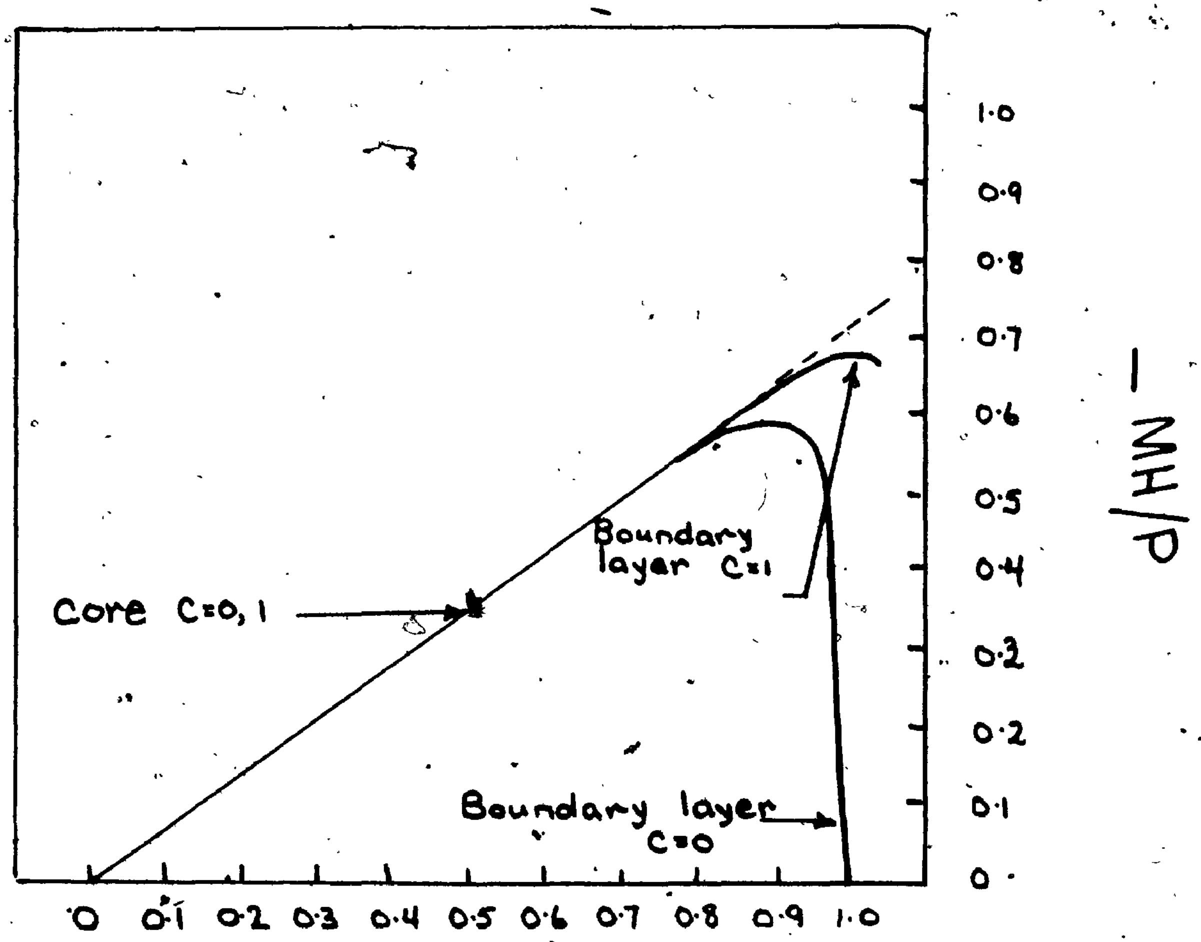

Mu/p insulating walls (c=0)
Velocity Profile θ = 45° M=50
Normalized Tube Radius Fig. 3.2
Core C=0
Core C=1
Boundary layer C=0
Boundary layer C=1
Mu/p conducting walls (c≠0)

Magnetic Field $\theta = 0^\circ$ M=50



Normalized Tube Radius Fig. 3·3

Magnetic Field $\Phi = 45°$ M=50



Normalized Tube Radius Fig. 3.4

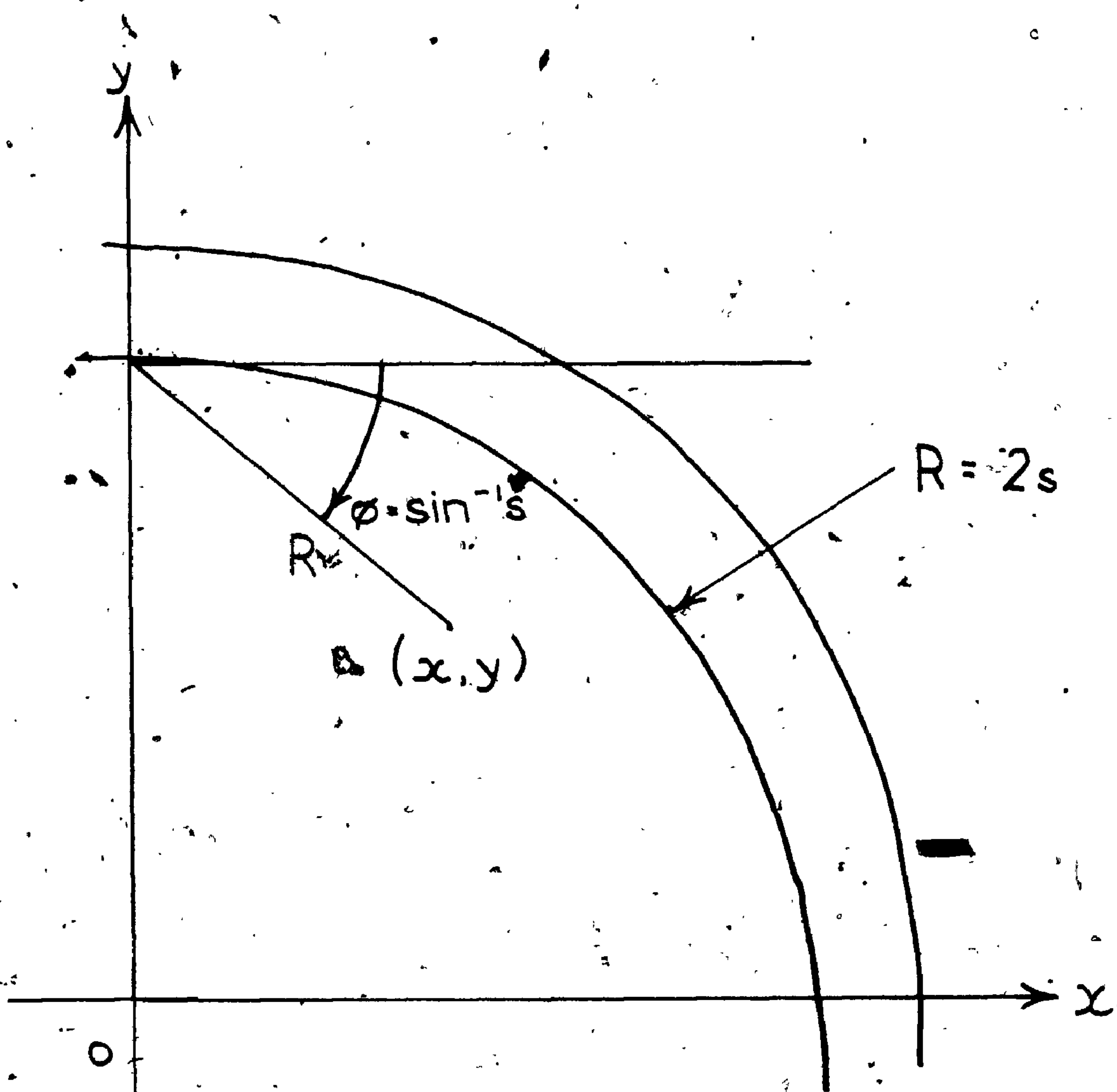


Illustration of the co-ordinates R and φ

Fig. 3.5