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A Homotopy Theory for Diffeological Spaces

Enxin Wu
The University of Western Ontario

Supervisor
Dan Christensen
The University of Western Ontario

Graduate Program in Mathematics
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A Homotopy Theory for Diffeological Spaces

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by

Enxin Wu

Department of Mathematics

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Supervisor:	**Examiners:**
____________________	____________________
Dr. Dan Christensen	Dr. Rob Corless
	____________________
	Dr. Graham Denham
	____________________
	Dr. Dan Isaksen
	____________________
	Dr. Gordon Sinnamon

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Abstract

Smooth manifolds are central objects in mathematics. However, the category of smooth manifolds is not closed under many useful operations. Since the 1970's, mathematicians have been trying to generalize the concept of smooth manifolds. J. Souriau's notion of diffeological spaces is one of them. P. Iglesias-Zemmour and others developed this theory, and used it to simplify and unify several important concepts and constructions in mathematics and physics.

We further develop the diffeological space theory from several aspects: categorical, topological and differential geometrical. Our main concern is to build a suitable homotopy theory (also called a model category structure) on the category of diffeological spaces, which encodes the usual smooth homotopy theory of smooth manifolds and the diffeological bundle theory of Iglesias.

This is a huge task, and at the moment, we have not yet completely proved the existence theorem. However, in the process, we can see the beauty of the merging of differential geometry and homotopy theory. (More details are explained in the Introduction.) These results should be of some interest to people working in these fields.

Keywords: Diffeological spaces, D-topology, smooth homotopy groups, tangent spaces, diffeological bundles, irrational torus, simplicial sets, model category theory, homogeneous spaces.

To my parents.

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Finally, I want to dedicate this thesis to my parents for their unconditional love, encouragement and support throughout my life.

Enxin Wu.

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Contents

Introduction

Smooth manifolds are some of the most important objects in mathematics. They contain a lot of geometric information: tangent spaces, tangent bundles, differential forms, de Rham cohomology, etc. This information can be put to great use in proving results. However, the category $\mathfrak{Mfd}$ of smooth manifolds is not closed under many useful constructions, such as subspaces, quotients, function spaces, etc.

On the other hand, the category of compactly generated weak Hausdorff topological spaces is closed under these constructions, but the geometric information is missing.

Can we have the best of both worlds?

Since the 1970's, mathematicians have been trying to generalize the concept of smooth manifold. Based on the fact that the smooth structure of a smooth manifold M can be tested by the set $I(M)$ of all smooth maps from some test objects to the manifold, or by the set $O(M)$ of all smooth maps from the manifold to some test objects, and these two sets $I(M)$ and $O(M)$ have compatibility conditions, A. Stacey in [St] classified some of these approaches into three classes: the mapping-in approaches (which use $I(M)$ for defining the smooth structure, for example, the Chen spaces [Ch], and J. Souriau's diffeological spaces [So]), the mapping-out approaches (which use $O(M)$ for defining the smooth structure, for example, the Sikorski spaces [Si], and the Smith spaces [Sm]), and the balanced approach (which uses both $I(M)$ and $O(M)$ together with the compatibility condition for defining the smooth structure, for example, the Frolicher spaces [F]). For the relationships between these approaches, see [St].

We pick diffeological spaces as our main target for three reasons: (1) it is a relatively well-developed theory, see [DI, Do, He, HM, I1, I2, I3, La, MS, So], etc; (2) there are some amazing facts in diffeological bundle theory developed by P. Iglesias-Zemmour, see [I1, I2] and Section 1.7 of this thesis. First of all, a diffeological bundle is not defined to be a smooth map which is locally trivial under the D-topology (see Section 1.3), but is defined to be a smooth map such that the pullback along any plot is locally trivial (Definition 1.7.4). Then, given a Lie group G and a subgroup H (not necessarily closed), the projection $G \to G/H$ is a diffeological bundle, where G/H is the set of left or right cosets (Proposition 1.7.12). Instead of (continuous) homotopy groups, we use smooth homotopy groups (see Section 1.4), and we get a long exact sequence of smooth homotopy groups for every diffeological bundle (Theorem 1.7.13). As a consequence of this, the smooth fundamental group of the irrational torus (Example 1.1.6) is non-trivial, but the continuous fundamental group is trivial (Example 1.7.14); (3) if we take the mapping-out approaches, in order to keep the underlying topology, the irrational torus becomes trivial. Similarly, if we take the Frolicher space approach, in order to balance $I(M)$ and $O(M)$ and keep the underlying topology, the irrational torus becomes trivial as well.

Our main goal of the thesis is to develop a homotopy theory on the category $\mathfrak{Diff}$ of diffeological spaces, which encodes the usual homotopy theory of smooth manifolds and the diffeological bundle theory. By homotopy theory, we mean a model category structure on $\mathfrak{Diff}$. In 1967, D. Quillen introduced the concept of model categories in his famous book [Q] by axiomatizing the basic properties and relationship between the category $\mathfrak{Top}$ of topological spaces and its homotopy category $\mathrm{Ho}(\mathfrak{Top})$.

In the following 45 years, only slight changes have been made, and now model categories are a standard tool to do homotopy theory on any nice enough category. The advantage of this abstract tool is that it unifies some ideas from different branches of mathematics. For example, CW-approximations in $\mathfrak{Top}$ and projective resolutions of R-modules are cofibrant replacements in some model category structures on $\mathfrak{Top}$ and $\mathrm{Ch}^{\geq 0}(R)$, respectively.

We set up our basic definitions of weak equivalences, fibrations and cofibrations of $\mathfrak{Diff}$ in Section 2.4. We use this definition for the following reasons: (1) we do not use Theorem A.2.35 to check the existence of a model category structure on $\mathfrak{Diff}$ since it is hard to choose suitable sets I and J. Note that in the proof (see [Ho]) that the standard model category structure on $\mathfrak{Top}$ (Example A.2.53) satisfies the model category axioms (Definition A.2.4), we need to use different forms of S^n, for example, $[0,1]^n/\partial[0,1]^n$, etc. However, in $\mathfrak{Diff}$ they are not diffeomorphic; (2) we have the usual compact cosimplicial object $|\Delta^\bullet|'_{\tilde{D}}$ in $\mathfrak{Diff}$ with each $|\Delta^n|'_{\tilde{D}} = \{(x_0, x_1, \cdots, x_n) \in (\mathbb{R}^{\geq 0})^{n+1} \mid \sum_{i=0}^n x_i = 1\}$ equipped with the sub-diffeology of $\mathbb{R}^{n+1}$. We use the noncompact cosimplicial object $\mathbb{A}^\bullet$ instead in our setup since these $|\Delta^n|'_{\tilde{D}}$ are too complicated (Remark 1.6.11, 1.8.2 and Example 1.8.1 (2)); (3) A^1 homotopy theory also uses a noncompact cosimplicial object. There is a folk belief that to get the model structure, one can lift the standard model category structure from $\mathfrak{s}$Set directly, instead of passing to the projective model category structure on simplicial presheaves; (4) note that we also have an adjoint pair between $\mathfrak{Diff}$ and $\mathfrak{Top}$ (Theorem 1.3.4). We do not talk about co-lifting the standard model category structure on $\mathfrak{Top}$ to $\mathfrak{Diff}$ since otherwise the striking example of the irrational torus would disappear.

Unfortunately, we have not yet completed the proof that under these definitions $\mathfrak{Diff}$ is a model category. Our main tool for the proof is Kan's theorem (Theorem

A.2.36). We have proved condition (1) of Kan's theorem (Theorem 2.1.3), but we have some difficulty in proving condition (2).

Nevertheless, we are able to prove many new results about diffeological spaces and their homotopy theory. Our main results for this project are in Section 2.6 and 2.7 about some characterizations of fibrant and cofibrant objects in $\mathfrak{Diff}$ (see below for more details).

We also develop some basics of diffeological spaces. See below for more details.

Thesis Organization

Chapter 1:

We summarize and develop some basics of diffeological spaces in this chapter.

J. Souriau [So] introduced diffeological spaces in 1980. We review the basic properties of this category in Section 1.1: it contains $\mathfrak{Mfd}$ as a full subcategory (Theorem 1.1.3), and it is complete, cocomplete and cartesian closed (Theorem 1.1.9 and 1.1.13). We recall the example of the irrational torus (Example 1.1.6) and introduce another important example Λ^2 (Example 1.1.10).

E.J. Dubuc [Du] introduced the concept of concrete sites (Definition 1.2.5) and concrete (pre)sheaves (Definition 1.2.10) over a concrete site in 1977. J. Baez and A. Hoffnung [BH] gave a sheaf theoretical approach to diffeological spaces in 2009. They proved that $\mathfrak{Diff}$ is equivalent to the category of concrete sheaves over the diffeological site DS (Definition 1.2.7 and Theorem 1.2.12), and that the category of concrete sheaves over any subcanonical concrete site is a quasi-topos. We further explore this approach in Section 1.2. We show that the subcanonical condition can be dropped and most of the results in [BH] still hold (Theorem 1.2.22). We also give conditions which imply that a presheaf is concrete (Lemma 1.2.24, Proposition 1.2.25 and 1.2.28). Furthermore, we study the example of lifting the standard model category structure on $\mathfrak{s}$Set to a category of concrete (pre)sheaves over the simplicial category Δ (Example 1.2.15) as a test example. The results are: we could not lift the standard model structure to the concrete presheaf category (Proposition 1.2.35), but we can lift the standard model structure to the concrete sheaf category (Theorem 1.2.38).

P. Iglesias-Zemmour introduced a natural topology (called the D-topology) for every diffeological space in his thesis [I2]. We further develop the general topology aspect of diffeological space in Section 1.3. We have the following original results: (1) there is an adjoint pair between $\mathfrak{D}$iff and $\mathfrak{T}$op (Theorem 1.3.4); (2) using results from [KM], we observe that the D-topology is totally determined by all smooth curves, while the diffeology is not (Remark 1.3.9); (3) the image of the D-topology for all diffeological spaces is exactly the Δ-generated spaces (Definition 1.3.15 and Proposition 1.3.16); (4) We study which topological spaces are Δ-generated. M. Laubinger [La] gave a necessary condition (Proposition 1.3.3) which is not sufficient (Example 1.3.23), and we give a sufficient condition (Proposition 1.3.18) and an example showing this condition is not both necessary and sufficient (Example 1.3.20); (5) we compare the D-topology on a function space with the compact-open topology, and show that the D-topology is usually strictly finer than the compact-open topology (Example 1.3.25 and 1.3.26, and Lemma 1.3.27); (6) some properties of $\mathfrak{M}$fd could not be extended to $\mathfrak{D}$iff: the usual notion of a ringed space does not totally determine the diffeology (Example 1.3.29), and not every diffeological space has the smooth variety property (Definition 1.3.31 and Example 1.3.33). Also, we can associate two topologies for a sub-diffeological space, and in general they are different (Example 1.3.6). We give some conditions under which they coincide (Lemma 1.3.8, Example 1.3.14 and 1.3.14).

P. Iglesias-Zemmour introduced smooth homotopy theory in his thesis [I2]. We further develop this theory in Section 1.4. Our main contribution is that several alternative definitions of the smooth homotopy groups of a diffeological space match Iglesias' original definition (Definition 1.4.1, Theorem 1.4.3, 1.4.4, 1.4.8 and 1.4.9). Two more equivalent characterizations will be given in Section 2.7 (Remark 2.7.1 and Proposition 2.7.2).

J. Souriau [So] introduced the differential forms and de Rham cohomology for diffeological spaces. P. Iglesias-Zemmour, etc. [I1, MS] further developed them. We summarize the basic theory in Section 1.5. Note that de Rham cohomology is smooth homotopy invariant (Theorem 1.5.12). We calculate the de Rham cohomology of the irrational torus (Example 1.5.8) and Λ^2 (Example 1.5.11). In $\mathfrak{D}$iff the de Rham theorem does not hold (Remark 1.5.9).

In Section 1.6, we compare two approaches to defining the tangent space of a pointed diffeological space, one introduced by G. Hector (we call it the internal tangent space, see Section 1.6.1), and the other suggested by A. Kock (we call it the external tangent space, see Section 1.6.2). Note that we can define the internal tangent bundle in a natural way, but we have trouble to define external tangent bundle in general. We compare the two tangent space approaches with the germ approaches, and develop some calculational tools (see Section 1.6.3). Although these approaches match for smooth manifolds, in general they are different. The main examples are the two tangent spaces for the pointed irrational torus (Example 1.6.8) and for the closed half line pointed at the boundary (Example 1.6.10).

To have theoretical understanding of their previous calculations in [DI], P. Iglesias-Zemmour introduced the diffeological bundle theory in his thesis [I2]. We summarize this theory together with some basic diffeological group theory in Section 1.7: the definition of a diffeological bundle (Definition 1.7.4), an example from diffeological groups (Proposition 1.7.12), the long exact sequence of smooth homotopy groups for every diffeological bundle (Theorem 1.7.13) and the difference between continuous and smooth fundamental groups of the irrational torus (Example 1.7.14). These results serve as a motivation for the development of a model category structure on $\mathfrak{Diff}$ in the next chapter.

Finally we summarize the dimension theory of diffeological spaces in Section 1.8. Some new examples are worked out there (Lemma 1.8.3 and 1.8.6, Example 1.8.4, 1.8.5 and 1.8.7, and Proposition 1.8.8).

Chapter 2:

In this chapter, we are trying to develop a homotopy theory (that is, a model category structure) on $\mathfrak{Diff}$ which encodes the usual homotopy theory of smooth manifolds and the diffeological bundle theory of Iglesias (see Section 1.7). However, we haven't completed the proof of the existence of the model category structure from our definitions of weak equivalences, fibrations and cofibrations (Definition 2.4.2) on $\mathfrak{Diff}$. We have partial results as follows. With a few exceptions, the material in this chapter is original.

In Section 2.1, we prove by definition that $\mathfrak{Diff}$ is locally presentable (Theorem 2.1.3). More generally, a similar proof shows that the category of all concrete sheaves over a concrete site is locally presentable (Remark 2.1.4).

In Section 2.2, we recall an adjoint functor theorem (Theorem 2.2.1) together with three famous classical examples (Example 2.2.2, 2.2.3 and 2.2.4). We also set up the adjoint pair between $\mathfrak{s}\mathrm{Set}$ and $\mathfrak{Diff}$ using the noncompact cosimplicial object $\mathbb{A}^\bullet$ (Example 2.2.5).

In Section 2.3, we prove that there is a model category structure on the category of presheaves over the diffeological site DS (Corollary 2.3.3).

In Section 2.4, we introduce the diffeological spaces Λ^n, Λ^n_{sub}, $\partial\mathbb{A}^n$ and $\partial\mathbb{A}^n_{sub}$, and we define the weak equivalences, fibrations and cofibrations in $\mathfrak{Diff}$ (Definition

2.4.2).

In Section 2.5, we study some basic properties of the diffeological realization functor $\text{j?j}_{\tilde{D}}$ and the smooth singular functor $S_{\tilde{D}}$. The main results are Lemma 2.5.1 and Proposition 2.5.2, 2.5.3 and 2.5.5.

In Section 2.6, we give characterizations of cofibrant objects and fibrant objects in $\mathfrak{D}$iff. The main results are: (1) we have a partial factorization (Proposition 2.6.2); (2) S^1 is cofibrant (Theorem 2.6.9); (3) every homogeneous diffeological space is fibrant (Theorem 2.6.23), in particular, every diffeological group (Proposition 2.6.12) and every smooth manifold without boundary (Corollary 2.6.25) is fibrant; (4) every topological space with the continuous diffeology is fibrant (Example 2.6.26); (5) some functional spaces are fibrant (Proposition 2.6.27); (6) every diffeological bundle with fibrant fiber is a fibration (Lemma 2.6.20); (7) not every diffeological space is fibrant (Example 2.6.30, 2.6.31 and 2.6.32), and in particular, no smooth manifold with boundary is fibrant (Example 2.6.33); (8) not every diffeological space is cofibrant (Example 2.6.21).

In Section 2.7, we give two more equivalent definitions of the smooth homotopy groups of a pointed diffeological space (Remark 2.7.1 and Proposition 2.7.2), and we use these characterizations to prove that the smooth homotopy groups of a fibrant diffeological space is bijective to the simplicial homotopy groups of its smooth singular complex (Theorem 2.7.3).

Appendix A:

This appendix contains some basics of model category theory.

Section A.1 contains the basics of left and right Kan extensions. A good reference is [Mac].

Section A.2 summarizes the basics of model category theory: the definition of a model category, its homotopy category, Quillen pairs, Quillen equivalence, the small object argument, cofibrant generation, properness, simplicial model categories, and Reedy model categories. We also review the standard model category structures on $\mathfrak{Top}$ and on $\mathfrak{s}$Set, and the projective model category structure on $\mathrm{Ch}^{\geq 0}(R)$. Good references for these are [DS, GJ, Hi, Ho, Q].

Notations and Conventions

- We use $\mathbb{N}$ to denote all natural numbers $\{0, 1, 2, \cdots\}$, and we use $\mathbb{Z}^+$ to denote all positive integers $\{1, 2, 3, \cdots\}$.

- Any category in this thesis is assumed to be locally small, in the sense that given any two objects in this category, the class of morphisms from one object to another is actually a set.

- We understand limits and colimits to be small, that is, they are defined with respect to functors out of small categories.

- We use $\mathfrak{S}$et to denote the category of sets with set maps.

- We use $\text{Vect}_{\mathbb{R}}$ to denote the category of real vector spaces with $\mathbb{R}$-linear maps.

- We use $\mathfrak{M}$fd to denote the category of finite dimensional real smooth manifolds without boundary with smooth maps (that is, C^∞ maps). All smooth manifolds are assumed to be second countable.

- We use $\mathfrak{T}$op to denote the category of topological spaces with continuous maps.

- We use $\mathfrak{s}$Set to denote the category of simplicial sets with simplicial maps.

- Given a site C (Definition 1.2.3), we use $\mathfrak{P}\text{re}(\mathsf{C})$ (or $\mathfrak{S}\text{h}(\mathsf{C})$) to denote the category of presheaves (or sheaves) on C with natural transformations (Definition 1.2.4).

- Given a concrete site C (Definition 1.2.5), we use $\mathfrak{C}\text{Pre}(\mathsf{C})$ (or $\mathfrak{C}\text{Sh}(\mathsf{C})$) to denote the category of concrete presheaves (or concrete sheaves) on C with natural transformations (Definition 1.2.10).

- We use DS to denote the diffeological site (Example 1.2.7).

[2] We use $\mathfrak{D}$iff to denote the category of diffeological spaces with smooth maps (see the beginning of Section 1.1.1).

[2] Given a category C, and given two objects A and B of C, we write $\mathsf{C}(A, B)$ for the hom-set of all morphisms from A to B in C.

[2] By an adjoint pair $F : \mathsf{C} \rightleftharpoons \mathsf{D} : G$, we always mean that F is a left adjoint and G is a right adjoint.

Chapter 1

Basics of Diffeological Spaces

Smooth manifolds are ubiquitous objects in mathematics. They contain a lot of important geometric information: tangent spaces, tangent bundles, differential forms, de Rham cohomology, etc. However, the category $\mathfrak{Mfd}$ of smooth manifolds is not closed under many useful constructions, such as subsets, quotients, functional spaces, etc.

In 1980, J. Souriau introduced a larger category called the category $\mathfrak{Diff}$ of diffeological spaces [So]. This category is closed under these usual constructions, and on every diffeological space, we can still do differential geometry.

In Section 1.1, We summarize the basic properties of this category $\mathfrak{Diff}$: it is complete, cocomplete and cartesian closed, and it contains $\mathfrak{Mfd}$ as a full subcategory.

In Section 1.2, we develop the concrete sheaf theory following J. Baez and A. Hoffnung. In [BH], they proved that $\mathfrak{Diff}$ is equivalent to the category of concrete sheaves over the diffeological site DS, and that the category of concrete sheaves over any subcanonical concrete site is a quasi-topos. We show that the subcanonical condition can be dropped and most of the results in [BH] still hold (Theorem 1.2.22). We also give conditions which imply that a presheaf is concrete. Furthermore, we study the possibility of lifting the standard model category structure on $\mathfrak{s}$Set to a category of concrete (pre)sheaves over the simplicial category Δ.

In Section 1.3, we develop the general topology aspect (called the D-topology) of diffeological spaces after P. Donato [Do]. We discuss the following five themes in this section: (1) the D-topology induces an adjoint pair between $\mathfrak{D}$iff and $\mathfrak{T}$op; (2) we can associate two topologies for a sub-diffeological space, and in general they are different. We give some conditions under which they coincide; (3) we characterize which topological spaces can be realized as the D-topology of a diffeological space; (4) we discuss the D-topology on function spaces; (5) some properties of $\mathfrak{M}$fd fail for $\mathfrak{D}$iff. The main original results for this section are Theorem 1.3.4, Remark 1.3.9, Proposition 1.3.16 and 1.3.18, Lemma 1.3.27, Example 1.3.20, 1.3.25, 1.3.26, 1.3.29, and 1.3.33.

In Section 1.4, we develop the smooth homotopy theory of diffeological spaces after P. Iglesias-Zemmour [I1, I2]. Our main contribution is that several alternative definitions of the smooth homotopy groups of a diffeological space match Iglesias' original definition.

In Section 1.5, we summarize some basics of the differential forms and de Rham cohomology theory for diffeological spaces [So, I1].

In Section 1.6, we compare two approaches to defining the tangent space of a pointed diffeological space, one introduced by G. Hector (we call it the internal tangent space), and the other suggested by A. Kock (we call it the external tangent space). We also compare them with the germ approaches, and develop some calculational tools. Although these approaches match for smooth manifolds, in general they are different. The main examples are the two tangent spaces for the pointed irrational torus (Example 1.6.8) and for the closed half line pointed at the boundary (Example 1.6.10).

In Section 1.7, we summarize the diffeological bundle theory of P. Iglesias-

Zemmour [I1, I2], as a motivation for the possible model category structure in the next chapter.

In Section 1.8, we summarize the dimension theory for diffeological spaces. A few new examples are worked out there.

1.1 Diffeological spaces, a set-theoretic point of view

Definition 1.1.1 ([So]). A *diffeological space* is a set X together with a specified set $\mathcal{D}_X$ of maps $U \to X$ (called *plots*) for every open set U in $\mathbb{R}^n$ and for each $n \in \mathbb{N}$, such that for all open subsets $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$:
(1) (Covering) Every constant map $U \to X$ is a plot;
(2) (Smooth Compatibility) If $U \to X$ is a plot, and $V \to U$ is smooth, then the composition $V \to U \to X$ is also a plot;
(3) (Sheaf Condition) If $U = \cup_i U_i$ is an open cover and $U \to X$ is a set map such that each restriction $U_i \to X$ is a plot, then $U \to X$ is a plot.

Without confusion, we usually use the underlying set X to represent the diffeological space $(X, \mathcal{D}_X)$.

Definition 1.1.2 ([So]). Let X and Y be two diffeological spaces, and let $f : X \to Y$ be a set map. We call f *smooth* if for every plot $u : U \to X$ of X, the composition $f \circ u$ is a plot of Y.

The collection of all diffeological spaces with smooth maps forms a category, and we will denote it by $\mathfrak{D}$iff. Given two diffeological spaces X and Y, we write $\mathfrak{D}$iff(X, Y) for the set of all smooth maps from X to Y. An isomorphism in $\mathfrak{D}$iff will be called a *diffeomorphism*.

Theorem 1.1.3 ([So]). *There is a fully faithful functor $\mathfrak{M}fd \to \mathfrak{D}iff$.*

Proof. Every smooth manifold M is canonically a diffeological space with the same underlying set and plots all smooth maps $U \to M$ in the usual sense. We call this the *standard diffeology* on M. By using charts, it is easy to see that smooth maps in the usual sense between smooth manifolds coincide with smooth maps between them with the standard diffeology. □

From now on, without specification, every smooth manifold considered as a diffeological space is equipped with the standard diffeology.

Remark 1.1.4 ([I1])**.** Similar results holds for smooth manifolds with boundary. This is a surprising result since it means that smoothness on the boundary can be tested by smooth functions from all open subsets of $\mathbb{R}^n$ for all $n \in \mathbb{N}$. Actually this also follows from Kriegl-Michor's Theorem (Theorem 24.5 of [KM]).

Proposition 1.1.5 ([I1])**.** *Given a set X, let D be the set of all diffeologies on X ordered by inclusion. Then D is a complete lattice.*

Proof. This follows from the fact that D is closed under arbitrary (small) intersection. The largest element in D is called the *indiscrete diffeology* on X, which consists of all set maps $U \to X$, and the smallest element in D is called the *discrete diffeology* on X, which consists of all locally constant maps $U \to X$. □

The smallest diffeology $\mathsf{D}_X(A)$ on X containing a set of maps $A = \{U_i \to X\}_{i \in I}$ is called the diffeology *generated* by A. More precisely, $\mathsf{D}_X(A)$ consists of all maps $f : U \to X$ such that there exists an open cover $\{V_j\}$ of U together with the property that f restricted to each V_j factors through some element $U_i \to X$ in A via a smooth map $V_j \to U_i$. The standard diffeology on a smooth manifold is generated by a smooth atlas on the manifold. For every diffeological space X, D_X is generated by $\cup_{n \in \mathbb{N}} \mathfrak{D}\mathrm{iff}(\mathbb{R}^n, X)$. We call elements in this generating set *global plots*.

Generalizing the previous paragraph, let $A = \{f_j : X_j \to X\}_{j \in J}$ be a set of functions from some diffeological spaces to a fixed set X. Then there exists a smallest diffeology on X making all f_j smooth, and we call it the *final diffeology* defined by A. For a diffeological space X with an equivalence relation $\sim$, the final diffeology defined by $\{X \twoheadrightarrow X/\sim\}$ is called the *quotient diffeology*. Similarly, let $B = \{g_k : Y \to Y_k\}_{k\in K}$ be a set of functions from a fixed set Y to some diffeological spaces. Then there exists a largest diffeology on Y making all g_k smooth, and we call it the *initial diffeology* defined by B. For a diffeological space X and a subset A of X, the initial diffeology defined by $\{A \hookrightarrow X\}$ is called the *sub-diffeology*.

Example 1.1.6 ([DI, I1]). Let $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the usual 2-torus, and let $\mathbb{R}_\theta$ be the image of $\{y = \theta x\}$ under the natural map $\mathbb{R}^2 \to T^2$ with θ irrational. Note that T^2 is a Lie group, and $\mathbb{R}_\theta$ is a subgroup. The set $T^2/\mathbb{R}_\theta$ of left cosets with the quotient diffeology is called the *irrational torus of slope θ*. One can show that $T^2/\mathbb{R}_\theta$ is diffeomorphic to $\mathbb{R}/(\mathbb{Z} + \theta\mathbb{Z})$ (denoted by T^2_θ).

Here are some basic properties of irrational tori:

(1) Let $p : \mathbb{R} \to T^2_\theta$ be the quotient map. Then for any interval $J \subseteq \mathbb{R}$, $p(J) = T^2_\theta$. Therefore, as a diffeological space, T^2_θ is neither discrete nor indiscrete. However, for any $a \in T^2_\theta$, $T^2_\theta - \{a\}$ as a sub-diffeological space is discrete. As a topological space with the quotient topology, T^2_θ is indiscrete.

(2) Let α, β be two irrational numbers. Then for every smooth map $f : T^2_\alpha \to T^2_\beta$, there exists $\lambda, \mu \in \mathbb{R}$ such that λ and $\alpha\lambda \in \mathbb{Z} + \beta\mathbb{Z}$ and the affine map $F : \mathbb{R} \to \mathbb{R}$ sending x to $\lambda x + \mu$ makes the following diagram commutative:

$$\begin{tikzcd} \mathbb{R} \arrow[r, "F"] \arrow[d] & \mathbb{R} \arrow[d] \\ T^2_\alpha \arrow[r, "f"'] & T^2_\beta. \end{tikzcd}$$

Furthermore, there exists a non-constant smooth map $f : T^2_\alpha \to T^2_\beta$ if and only if $\alpha = \frac{a+b\beta}{c+d\beta}$ with $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2(\mathbb{Q})$ and $a, b, c, d \in \mathbb{Z}$ if and only if there exists a surjective smooth map $f : T^2_\alpha \to T^2_\beta$.

Also, there is a smooth bijection $f : T^2_\alpha \to T^2_\beta$ if and only if $\alpha = \frac{a+b\beta}{c+d\beta}$ with $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2(\mathbb{Z})$ if and only if there is a diffeomorphism $f : T^2_\alpha \to T^2_\beta$.

Remark 1.1.7. Let X be a subset of $\mathbb{R}^n$ with the sub-diffeology. Then by Boman's theorem (Corollary 3.14 of [KM]), $p : U \to X$ is a plot if and only if for every smooth function $f : \mathbb{R} \to U$, the composition $p \circ f : \mathbb{R} \to X$ is a plot. In other words, $\mathfrak{D}\mathrm{iff}(\mathbb{R}, X)$ determines the smooth structure of X. Therefore, all the techniques developed in [KM] apply in this case.

Example 1.1.8. In contrast, let $X = \mathbb{R}^2$ be equipped with the diffeology generated by all maps $\mathbb{R} \to \mathbb{R}^2$ which are smooth in the usual sense. Then the set map $id : \mathbb{R}^2 \to X$ is not a plot, although for every smooth map $f : \mathbb{R} \to \mathbb{R}^2$, $id \circ f = f : \mathbb{R} \to X$ is a plot.

Theorem 1.1.9 ([So])**.** *$\mathfrak{D}$iff is both complete and cocomplete.*

The forgetful functor $\mathfrak{D}\mathrm{iff} \to \mathfrak{S}\mathrm{et}$ preserves both limits and colimits since it has both left (discrete diffeology) and right (indiscrete diffeology) adjoints. The diffeology on the limit (colimit) is the initial (final) diffeology defined by the natural maps from the universal properties. More precisely, for the limits, let $F : J \to \mathfrak{D}\mathrm{iff}$ be a functor from a small category J. Then $U \to \lim F$ is a plot of $\lim F$ if and only if the composition $U \to \lim F \to F(j)$ is a plot of $F(j)$ for any $j \in \mathrm{Obj}(J)$. And for the colimit, let $F : J \to \mathfrak{D}\mathrm{iff}$ be a functor from a small category J. Then $U \to \operatorname{colim} F$ is a plot of $\operatorname{colim} F$ if and only if locally it factors through $F(j) \to \operatorname{colim} F$ for some $j \in \mathrm{Obj}(J)$, with the factorization a plot of $F(j)$.

Example 1.1.10. Let Λ^2 be the pushout of $\mathbb{R} \xleftarrow{i} \mathbb{R}^0 \xrightarrow{i} \mathbb{R}$ with $\mathrm{Im}(i) = \{0\}$. As a set, $\Lambda^2 = \{(x, y) \in \mathbb{R}^2 \mid x = 0 \text{ or } y = 0\}$. If we denote the sub-diffeological space of $\mathbb{R}^2$ with the same underlying set as Λ^2 by Λ^2_{sub}, then we have a smooth map $\Lambda^2 \to \Lambda^2_{sub}$ which is identity on the underlying sets. However, this map is not a diffeomorphism. This is because there exists a smooth function $f : \mathbb{R} \to \mathbb{R}$ such that $f(t) > 0$ for all $t > 0$, and $f(t) = 0$ for all $t \leq 0$, for example,

$$f(t) = \begin{cases} e^{-\frac{1}{t}} & \text{if } t > 0 \\ 0 & \text{if } t \leq 0. \end{cases}$$

Then the smooth map $\mathbb{R} \to \mathbb{R}^2$ defined by $t \mapsto (f(t), f(-t))$ induces a plot of Λ^2_{sub}, but not a plot of Λ^2 since no neighborhood of 0 has a smooth lifting to either of the axes of $\mathbb{R}^2$.

We may call the crossing point $(0, 0)$ of Λ^2 and Λ^2_{sub} a *border point* and a *stop sign*, respectively, since every smooth curve making a turn in Λ^2 must stop for an amount of time, while in Λ^2_{sub} it only needs to stop instantaneously.

Example 1.1.11. Let X be the pushout of

$$\mathbb{R}^n \times (-\infty, \epsilon) \hookleftarrow \mathbb{R}^n \times (-\delta, \epsilon) \hookrightarrow \mathbb{R}^n \times (-\delta, \infty)$$

for some $\epsilon, \delta > 0$. Then X is diffeomorphic to $\mathbb{R}^{n+1}$. This result will be used extensively for the gluing of smooth maps.

Example 1.1.12. In contrast to the previous example, the pushout of

$$\mathbb{R}^n \times \mathbb{R}^{\leq 0} \hookleftarrow \mathbb{R}^n \times \{0\} \hookrightarrow \mathbb{R}^n \times \mathbb{R}^{\geq 0}$$

is not diffeomorphic to $\mathbb{R}^{n+1}$, for any $n \in \mathbb{N}$. This can be proved by Lemma 1.8.6 and (2) of Example 1.8.1.

Theorem 1.1.13 ([So]). *$\mathfrak{D}$iff is cartesian closed.*

More precisely, given two diffeological spaces X and Y, the set of maps $\{U \to \mathfrak{D}\text{iff}(X, Y) \mid U \times X \to Y \text{ is smooth}\}$ forms a diffeology on $\mathfrak{D}\text{iff}(X, Y)$. We call it the *functional diffeology* on $\mathfrak{D}\text{iff}(X, Y)$, and without specification, we always give any hom-set the functional diffeology. Furthermore, for any diffeological space Y, $? \times Y : \mathfrak{D}\text{iff} \rightleftharpoons \mathfrak{D}\text{iff} : \mathfrak{D}\text{iff}(Y, ?)$ is an adjoint pair.

1.2 Diffeological spaces, a sheaf-theoretic point of view

In this section, we explore more generally to study 'something like diffeological spaces'. They are called concrete sheaves over a concrete site. There are three themes here: (1) the basic properties of these categories; (2) conditions under which a presheaf is concrete; (3) lifting of the standard model category structure on $\mathfrak{s}$Set to $\mathfrak{C}$Pre(Δ) and $\mathfrak{C}$Sh(Δ).

1.2.1 Concrete (pre)sheaves over a concrete site

There is a long history of the definition of a site, and there are many variations. The following four definitions are essentially from [Jo]:

Definition 1.2.1. A *family* is a collection of morphisms with a common codomain.

Definition 1.2.2. A *coverage* on a category C is a function assigning to each object U in C a collection $\mathfrak{J}(U)$ of families $\{f_i : U_i \to U\}_{i\in I}$, called the *covering families*, with the following properties:

(1) given a covering family $\{f_i : U_i \to U\}_{i\in I}$ and a morphism $g : V \to U$, there exists a covering family $\{h_j : V_j \to V\}_{j\in J}$ such that each morphism $g \circ h_j$ factors through some f_i;

(2) let $\{U_i \to U\}_{i\in I}$ be a covering family of U, and for each $i \in I$, let $\{U_{ij} \to U_i\}_{j\in J_i}$ be a covering family of U_i. Then $\{U_{ij} \to U_i \to U\}_{i\in I, j\in J_i}$ is also a covering family of U;

(3) $\{id : U \to U\}$ is a covering family of U.

Definition 1.2.3. A *site* is a small category together with a coverage.

Definition 1.2.4. Let C be a site.

(1) A *presheaf* on C is a functor $\mathsf{C}^{op} \to \mathfrak{Set}$.

(2) Let F be a presheaf on C, let U be an object of C, and let $\{f_i : U_i \to U\}_{i\in I}$ be a covering family of U. A function $x : I \to \bigcup_{i\in I} F(U_i)$ with $x(i) \in F(U_i)$ for each $i \in I$ is called *compatible* with respect to this covering family, if for any $i, j \in I$, any object V of C, and any morphisms $g : V \to U_i$ and $h : V \to U_j$ with $f_i \circ g = f_j \circ h$, we have $F(g)(x(i)) = F(h)(x(j))$.

(3) A *sheaf* F on C is a presheaf $F : \mathsf{C}^{op} \to \mathfrak{Set}$ satisfying the following condition: for any object U of C, any covering family $\{f_i : U_i \to U\}_{i\in I}$ of U, and any compatible $x : I \to \bigcup_{i\in I} F(U_i)$ with respect to this covering family, there exists a unique $y \in F(U)$ such that $F(f_i)(y) = x(i)$ for any $i \in I$.

The category of all presheaves (or sheaves) on C with natural transformations will be denoted by $\mathfrak{Pre}(\mathsf{C})$ (or $\mathfrak{Sh}(\mathsf{C})$).

Definition 1.2.5. A site C is called *concrete* if it has a terminal object 1, such that

(1) $\mathsf{C}(1, ?) : \mathsf{C} \to \mathfrak{Set}$ is faithful;

(2) for any object U in C and any covering family $\{U_j \to U\}_{j\in J}$ of U, the natural map $\coprod_{j\in J} \mathsf{C}(1, U_j) \to \mathsf{C}(1, U)$ is surjective.

Compared to the definition of a concrete site in [BH], we drop the subcanonical condition.

Example 1.2.6. Let X be a topological space. Then the poset $\mathsf{O}(X)$ with objects all open subsets of X ordered by inclusion together with the usual open covering is a site with terminal object X. It is concrete if and only if for any covering family $\{U_i \hookrightarrow X\}_{i\in I}$ of X, there exists $i \in I$ such that $U_i = X$.

Example 1.2.7 ([BH])**.** Let DS be the category with objects all open subsets of $\mathbb{R}^n$ with $n \in \mathbb{N}$, and morphisms all smooth maps between them. We say $\{U_i \hookrightarrow U\}_{i\in I}$ is a covering family of U in DS if each U_i is open in U and

$\bigcup_{i\in I} U_i = U$. Then DS forms a concrete site with terminal object $\mathbb{R}^0$, and the functor $\mathsf{DS}(\mathbb{R}^0, \star) : \mathsf{DS} \to \mathfrak{Set}$ sends every object U of DS to its underlying set U. We call DS the *diffeological site*.

Example 1.2.8 ([BH])**.** Let CS be the category with objects all convex subsets of $\mathbb{R}^n$ for $n \in \mathbb{N}$ with nonempty interior, morphisms all smooth functions, and coverings the relative open coverings. Then CS forms a concrete site.

Remark 1.2.9. Similarly, we can construct many such examples of concrete sites. For example, we can change the morphisms in DS or CS to be continuous maps, or C^k-maps, or analytic maps, etc, to get concrete sites.

Also we can take the category with objects all open subsets of $\mathbb{C}^n$ with $n \in \mathbb{N}$, morphisms all holomorphic maps, and coverings the usual open coverings to get a holomorphic concrete site, etc.

Definition 1.2.10. A *concrete presheaf* (or *sheaf*) X over a concrete site C is a presheaf (or sheaf) such that for any object U in C, the natural map $\alpha : X(U) \to \mathfrak{Set}(\mathsf{C}(1,U), X(1))$ defined by $\alpha(f)(u) = u^*(f)$ is injective. The category of all concrete presheaves (or sheaves) over C with presheaf maps will be denoted by $\mathfrak{C}\mathrm{Pre}(\mathsf{C})$ (or $\mathfrak{C}\mathrm{Sh}(\mathsf{C})$).

Remark 1.2.11. By replacing $X(U)$ by its image under α, we can make the following observation: giving a concrete presheaf over a concrete site C is the same as giving a set A (thought of as $X(1)$) together with a set $\mathfrak{P}_A \subseteq \bigcup_{U\in \mathrm{Obj}(\mathcal{C})} \mathfrak{Set}(\mathsf{C}(1,U), A)$, such that

(1) every set map $\mathsf{C}(1,1) \to A$ is in $\mathfrak{P}_A$, so every constant map $\mathsf{C}(1,U) \to A$ is in $\mathfrak{P}_A$;

(2) if $g : V \to U \in \mathrm{Mor}(\mathsf{C})$, and $f : \mathsf{C}(1,U) \to A \in \mathfrak{P}_A$, then $f \circ g_* \in \mathfrak{P}_A$, where $g_* : \mathsf{C}(1,V) \to \mathsf{C}(1,U)$ is induced by g.

A concrete sheaf is a concrete presheaf such that

(3) if $f : \mathsf{C}(1, U) \to A$ is a set map, and there exists a covering family $\{U_i \to U\}_{i\in I}$ of U such that the composition $\mathsf{C}(1, U_i) \longrightarrow \mathsf{C}(1, U) \xrightarrow{f} A$ is in $\mathfrak{P}_A$ for any $i \in I$, then $f \in \mathfrak{P}_A$.

Let A and B be two concrete presheaves (or sheaves) over the concrete site C in the above sense. Then a set map $f : A \to B$ is a morphism in $\mathfrak{C}\mathrm{Pre}(\mathsf{C})$ (or $\mathfrak{C}\mathrm{Sh}(\mathsf{C})$) if $f(\mathfrak{P}_A) \subseteq \mathfrak{P}_B$.

This shows that $\mathfrak{C}\mathrm{Sh}(\mathsf{C})$ is equivalent to a category of sets with certain types of plots.

Take $\mathsf{C} = \mathsf{DS}$ as an example. The category whose objects and morphisms are described in the above remark is exactly $\mathfrak{D}\mathrm{iff}$. There is an equivalence $\beta : \mathfrak{D}\mathrm{iff} \rightleftharpoons \mathfrak{C}\mathrm{Sh}(\mathsf{DS}) : \gamma$ given by:

for a diffeological space X, $\beta(X)(U) =$ the set of all plots $U \to X$, and

for a concrete sheaf Y over DS, $\gamma(Y) = Y(\mathbb{R}^0)$ and $\mathsf{D}_{\gamma(Y)} = \cup_{U\in \mathrm{Obj}(\mathcal{DS})} \alpha(Y(U))$.

This proves the following:

Theorem 1.2.12 ([BH])**.** *There is an equivalence of categories between $\mathfrak{D}$iff and $\mathfrak{C}$Sh(DS).*

Example 1.2.13. Fix $n \in \mathbb{Z}^+$. Then $\mathsf{DS} \to \mathrm{Vect}_{\mathbb{R}}^{op}$ given by $U \mapsto \Omega^n(U)$, the set of all smooth n-forms on U, is a sheaf, which is concrete if and only if $n = 0$.

Proposition 1.2.14. *Let C be a concrete site. Then the Yoneda functor $\mathsf{C} \to \mathfrak{P}re(\mathsf{C})$ factors through the inclusion functor $\mathfrak{C}Pre(\mathsf{C}) \to \mathfrak{P}re(\mathsf{C})$.*

Proof. We need to show that for any object U in C, $\mathsf{C}(\star, U)$ is concrete. This is just because by the definition of a concrete site, $\mathsf{C}(1, \star) : \mathsf{C} \to \mathfrak{S}\mathrm{et}$ is faithful. $\square$

Example 1.2.15. In general, the Yoneda functor does not factor through the inclusion functor $\mathfrak{C}\mathrm{Sh}(\mathsf{C}) \to \mathfrak{P}\mathrm{re}(\mathsf{C})$. For example, let C be the simplicial category

Δ (see Example A.2.44 for more details). Then $\underline{0}$ is the terminal object, and $\Delta(\underline{0}, \underline{n}) \cong \{0, 1, 2, \cdots, n\}$. If we define $\{f_j : \underline{n_j} \to \underline{n}$ injective $\in \Delta(\underline{n_j}, \underline{n})\}$ to be a covering family of $\underline{n}$ if $\bigcup_j f_j(\underline{n_j}) = \underline{n}$, then Δ becomes a concrete site. Clearly, $\Delta(\underline{1}, \underline{1})$ has three elements. But if we cover the domain with two singletons, and note that $\Delta(\underline{0}, \underline{1})$ has 2 elements, the sheaf condition would require that $\Delta(\underline{1}, \underline{1})$ have 4 elements. In other words, $\Delta(?, \underline{1})$ is a (concrete) presheaf, but not a sheaf.

On the other hand, $\mathsf{C}(?, 1)$ is a concrete sheaf for any concrete site C with terminal object 1, since it is constant.

Definition 1.2.16. We call a concrete site C *subcanonical* if the Yoneda functor factors through the inclusion functor $\mathfrak{C}\mathrm{Sh}(\mathsf{C}) \to \mathfrak{P}\mathrm{re}(\mathsf{C})$.

Example 1.2.17 ([BH])**.** Both DS and CS are subcanonical.

Proposition 1.2.18 ([BH])**.** *There is an adjoint pair $c : \mathfrak{P}re(\mathsf{C}) \rightleftharpoons \mathfrak{C}Pre(\mathsf{C}) : i$, with i the forgetful functor, and $c(X)(U) = X(U)/\sim$, where $f \sim g \in X(U)$ if and only if they have the same image under the natural map $\alpha : X(U) \to \mathfrak{S}et(\mathsf{C}(1, U), X(1))$.*

We call $c : \mathfrak{P}\mathrm{re}(\mathsf{C}) \to \mathfrak{C}\mathrm{Pre}(\mathsf{C})$ in the above proposition the *concretization functor*. Clearly, $c \circ i = id$, and $c(X)(1) = X(1)$ for any presheaf X.

Proposition 1.2.19. *The concretization functor preserves products.*

Proof. This means that for any set of presheaves $\{X_i\}_{i\in I}$ over a concrete site C, the natural presheaf map $c(\prod_{i\in I} X_i) \to \prod_{i\in I} c(X_i)$ is an isomorphism in $\mathfrak{C}\mathrm{Pre}(\mathsf{C})$, and it follows directly from the definition of the concretization functor in the above proposition. □

Proposition 1.2.20 ([BH])**.** *There is an adjoint functor $\mathfrak{F} : \mathfrak{C}Pre(\mathsf{C}) \rightleftharpoons \mathfrak{C}Sh(\mathsf{C}) : i$, where $\mathfrak{F}$ is the usual sheafification functor restricted to $\mathfrak{C}Pre(\mathsf{C})$.*

Clearly, $\mathfrak{F} \circ i = id$.

Theorem 1.2.21 ([BH]). *Let* C *be a subcanonical concrete site, then* $\mathfrak{C}Sh(\mathsf{C})$ *is complete, cocomplete, and (locally) cartesian closed.*

In fact, we can drop the subcanonical condition and get:

Theorem 1.2.22. *Let* C *be a concrete site, then both* $\mathfrak{C}Pre(\mathsf{C})$ *and* $\mathfrak{C}Sh(\mathsf{C})$ *are complete, cocomplete, and cartesian closed.*

Proof. For $\mathfrak{C}\mathrm{Pre}(\mathsf{C})$, limits are calculated sectionwise, and colimits are calculated sectionwise followed by concretization. Let X and Y be two concrete presheaves over C. We can define another concrete presheaf $\underline{\mathfrak{C}}\mathrm{Pre}(\mathsf{C})(X,Y)$ with $\underline{\mathfrak{C}}\mathrm{Pre}(\mathsf{C})(X,Y)(U) = \mathfrak{C}\mathrm{Pre}(\mathsf{C})(X \times \mathsf{C}(-,U),Y)$. Then we have the following adjoint pair $- \times X : \mathfrak{C}\mathrm{Pre}(\mathsf{C}) \rightleftharpoons \mathfrak{C}\mathrm{Pre}(\mathsf{C}) : \underline{\mathfrak{C}}\mathrm{Pre}(\mathsf{C})(X,-)$.

For $\mathfrak{C}\mathrm{Sh}(\mathsf{C})$, limits are calculated sectionwise, and colimits are calculated sectionwise followed by concretization and sheafification. Let X and Y be two concrete sheaves over C. Then $\underline{\mathfrak{C}}\mathrm{Pre}(\mathsf{C})(X,Y)$ (now we write it as $\underline{\mathfrak{C}}\mathrm{Sh}(\mathsf{C})(X,Y)$) is in fact a concrete sheaf, since if $\mathsf{C}(1,U) \to \underline{\mathfrak{C}}\mathrm{Sh}(\mathsf{C})(X,Y)$ is a set map such that there exists a covering $\{U_i\}_{i\in I}$ of U with the property that the composition $\mathsf{C}(1,U_i) \to \mathsf{C}(1,U) \to \underline{\mathfrak{C}}\mathrm{Sh}(\mathsf{C})(X,Y) \in \underline{\mathfrak{C}}\mathrm{Sh}(\mathsf{C})(X,Y)(U_i)$ for each $i \in I$, then for any $(f,g) : \mathsf{C}(1,V) \to \mathsf{C}(1,U) \times X(1) \in \mathfrak{P}_{\mathcal{C}(1,U)\times X(1)}$, there exists a covering family $\{V_j \to V\}_{j\in J}$ of V such that we have commutative diagrams of the form

$$\begin{tikzcd}
\mathsf{C}(1,V) \arrow[r, "(f,g)"] & \mathsf{C}(1,U) \times X(1) \\
\mathsf{C}(1,V_j) \arrow[u] \arrow[r] & \mathsf{C}(1,U_i) \times X(1) \arrow[u]
\end{tikzcd}$$

with the bottom horizontal map in $\mathfrak{P}_{\mathcal{C}(1,U_i)\times X(1)}$, by the definition of coverage. Since Y is a sheaf, the composition

$$\mathsf{C}(1,V) \xrightarrow{(f,g)} \mathsf{C}(1,U) \times X(1) \longrightarrow Y(1)$$

is in $\mathfrak{P}_{Y(1)}$. $\square$

Remark 1.2.23. The above proof shows that if X is a concrete presheaf and Y is a concrete sheaf over a concrete site C, then $\underline{\mathfrak{C}}\mathrm{Pre}(\mathsf{C})(X,Y)$ is in fact a concrete sheaf.

Let $F : J \to \mathfrak{C}\mathrm{Sh}(\mathsf{C})$ be a functor. Write $F(j) = \{A_j, \mathfrak{P}_{A_j}\}$ with A_j a set. Then $\mathrm{colim}(F)(1) = \mathrm{colim}_{j\in J} A_j$ as a set, and $\mathsf{C}(1,U) \to \mathrm{colim}(F)(1)$ is a plot if and only if there exists a covering family $\{U_i \to U\}_{i\in I}$ of U such that the composition $\mathsf{C}(1,U_i) \to \mathsf{C}(1,U) \to \mathrm{colim}(F)(1)$ factors through some $A_j \to \mathrm{colim}(F)(1)$ via a plot $\mathsf{C}(1,U_i) \to A_j$ for each $i \in I$.

1.2.2 When concreteness is automatic?

Now let's explore some conditions under which a presheaf over a concrete site is guaranteed to be concrete.

Lemma 1.2.24. *Every sub-presheaf A of a concrete presheaf X over a concrete site is again concrete.*

Proof. Let C be a concrete site with terminal object 1. We have the following commutative diagram in $\mathfrak{S}$et:

$$\begin{tikzcd}
A(U) \arrow[r] \arrow[d] & \mathfrak{S}et(\mathsf{C}(1,U), A(1)) \arrow[d] \\
X(U) \arrow[r] & \mathfrak{S}et(\mathsf{C}(1,U), X(1))
\end{tikzcd}$$

for any object U in C. Since A is a sub-presheaf of X, the left vertical map is an injection, and since X is a concrete presheaf, the bottom horizontal map is an injection. Hence the top horizontal map is also an injection. $\square$

Proposition 1.2.25. *Let C be a concrete site. Then any coproduct in $\mathfrak{P}re(\mathsf{C})$ of concrete presheaves is again concrete.*

Proof. This directly follows from the definition. $\square$

Remark 1.2.26. In general, the coequalizer in $\mathfrak{P}\mathrm{re}(\mathsf{C})$ of concrete presheaves may not be concrete.

For example let $\mathsf{C} = \mathsf{DS}$, let $X = \mathbb{R}^0 = \{x\}$ and $Y = \mathbb{R}^0 \coprod \mathbb{R}^0 = \{y_1, y_2\}$ be two objects in $\mathfrak{C}\mathrm{Pre}(\mathsf{DS})$, and define maps $f_i : X \to Y$ by $x \mapsto y_i$ for $i = 1, 2$. Write A for the coequalizer of $X \overset{f_1}{\underset{f_2}{\rightrightarrows}} Y$ in $\mathfrak{P}\mathrm{re}(\mathsf{DS})$. Then for any object $U = U_1 \coprod U_2$ in DS with both U_1, U_2 connected, we will have three elements for $A(U)$. Since $A(\mathbb{R}^0)$ is a one-point set, A is not concrete.

Definition 1.2.27. Let C be a concrete site with terminal object 1. A morphism $i : A \to B$ in $\mathfrak{C}\mathrm{Pre}(\mathsf{C})$ is called an *induction* if

(1) $i_1 : A(1) \to B(1)$ is injective;

(2) for any $f : \mathsf{C}(1, c) \to B(1) \in B(c)$ with $\mathrm{Im}(f) \subseteq A(1)$, there exists $g : \mathsf{C}(1, c) \to A(1) \in A(c)$ such that $f = i_1 \circ g$.

We call A a *subspace* of B when there exists an induction $A \to B$.

Proposition 1.2.28. *Let* $Y \xleftarrow{f} X \xrightarrow{g} Z$ *be a diagram in* $\mathfrak{C}\mathit{Pre}(\mathsf{C})$*, such that both* f_1 *and* g_1 *are injective, and one of* f *and* g *is an induction. Let*

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "f"'] & Z \arrow[d, "h"] \\ Y \arrow[r, "k"'] & W \end{tikzcd}$$

be a pushout diagram in $\mathfrak{P}\mathit{re}(\mathsf{C})$*. Then* W *is concrete.*

Proof. Note that both h_1 and k_1 are injective, and the rest is easy. □

Remark 1.2.29. In the above proposition, the condition that one of f and g is an induction is essential. See the proof of Proposition 1.2.35.

Let C be a concrete site, and let $f : X \to Y$ be a morphism in $\mathfrak{C}\mathrm{Pre}(\mathsf{C})$. Then the graph map $g = (id_X, f) : X \to X \times Y$ is an induction, since there is an isomorphism between X and $\mathrm{Im}(g)$ as a subspace of $X \times Y$.

1.2.3 Concrete (pre)sheaves over Δ

In this part, we take a simple concrete site Δ (see Example 1.2.15), and study the liftings of the standard model category structure on $\mathfrak{s}$Set to $\mathfrak{C}$Pre(Δ) and $\mathfrak{C}$Sh(Δ), respectively (see Definition A.2.37). They serve as testing examples of the idea of the liftings of the standard model category structure on $\mathfrak{s}$Set to $\mathfrak{C}$Pre(C) and $\mathfrak{C}$Sh(C), respectively, for arbitrary concrete site C. We will talk in more detail in Chapter 2 for $\mathsf{C} = \mathsf{DS}$, the diffeological site. No results in this part will be used in the rest of the thesis.

For the background of this part, see Example A.2.44 for the definition of the simplicial category Δ, and see Example A.2.54 for the standard model category structure on the category $\mathfrak{s}$Set of simplicial sets.

Since every simplicial set is a set-valued presheaf over Δ, we call a set-valued concrete presheaf over Δ a *concrete simplicial set*. The category of all concrete simplicial sets will be denoted by $\mathfrak{cs}$Set, and by Proposition 1.2.18, we have the following adjoint pair $c : \mathfrak{s}\mathrm{Set} \rightleftharpoons \mathfrak{cs}\mathrm{Set} : i$, with $c(X)_n = X_n/\sim$, and $a \sim b$ if and only if $s^*(a) = s^*(b)$ in X_0 for any $s \in \Delta(\underline{0}, \underline{n})$, or in other words, the ordered vertices of a and b match.

Lemma 1.2.30. *Let X be a simplicial set such that its non-degenerate elements have the following properties:*

(1) none of them has repeated vertices;

(2) any two of them have different ordered vertices.

Then X is concrete.

Proof. To show that the natural map $X_n \to \mathfrak{Set}(\underline{n}, X_0)$ is injective for all n, we write X_n as disjoint union of degenerate and non-degenerate elements, and check that the image of any two different degenerate elements, any degenerate element and any non-degenerate element, and any two non-degenerate elements, are all different.

The assumption guarantees the last two cases. For the first case, suppose $a = s_{i_1}(\cdots s_{i_m}(x))$ and $b = s_{j_1}(\cdots s_{j_l}(y))$ are two degenerate elements in X_n with the same ordered vertices, with both x and y non-degenerate, and $i_1 > \cdots > i_m$ and $j_1 > \cdots > j_l$. By the assumption, it is easy to see that $a = b$. Therefore, X is concrete. $\square$

Example 1.2.31. The simplicial sets Δ^n, $\partial\Delta^n$, Λ^n_k are all concrete.

Here are some more examples considered by other people earlier. They fit nicely into the language of concrete simplicial sets.

Definition 1.2.32 ([Ja2])**.** A simplicial set is called a *polyhedral complex* if it is a subcomplex of $B(P)$ for some poset P, where the classifying space functor B is introduced in Example 2.2.3.

Example 1.2.33. Let A be a poset. Then its classifying space $B(A)$ is a concrete simplicial set, and so is every polyhedral complex.

In particular, let X be a simplicial set, and let NX be the set of all non-degenerate simplices in X ordered by the face relation. Then $B : \mathfrak{s}\mathrm{Set} \to \mathfrak{s}\mathrm{Set}$ defined by $X \mapsto BX = B(NX)$ is a functor, since if $f : X \to Y$ is a morphism in $\mathfrak{s}\mathrm{Set}$, then $f_* : NX \to NY$ defined by $f(x) = tf_*(x)$ with $f_*(x) \in NY$ and t an iterated degeneracy is a functor. We call BX the *classifying space of the simplicial set* X, and it is always concrete.

Let P be a poset. Then there is a last vertex map $\gamma : BB(P) \to B(P)$ defined by $NB(P) \to P$ with $(x : \underline{n} \to P) \mapsto x(n)$. Note that, we have $B(\underline{n}) = \Delta^n$ and $B\Delta^n = Sd(\Delta^n) = sd(\underline{n})$, where the functors Sd and sd are also introduced in Example 2.2.4. The last vertex map induces $\gamma : B\Delta^n \to \Delta^n$, hence $\gamma : Sd(X) \to X$ and $Sd(X) \to BX$ for any simplicial set X. Moreover, [Ja2] shows that $c(Sd(X)) = BX$ for every simplicial set X, and $Sd(X) \to BX$ is an isomorphism for every polyhedral complex X.

Remark 1.2.34. In general, the concretization functor does not preserve equalizers. For example, let Y be the simplicial set generated by

$$\begin{tikzcd} x \arrow[r, bend left, "a"] \arrow[r, bend right, "b"'] & y \end{tikzcd}$$

Let Z be the equalizer of $\Delta^1 \underset{b}{\overset{a}{\rightrightarrows}} Y$ in $\mathfrak{s}$Set. Then $Z = \partial\Delta^1$, while the equalizer of $c(\Delta^1) \underset{c(b)}{\overset{c(a)}{\rightrightarrows}} c(Y)$ is $\Delta^1 \neq c(Z) = Z$.

Proposition 1.2.35. *We can not lift the standard model category structure of $\mathfrak{s}$Set to $\mathfrak{cs}$Set from the adjoint pair $c : \mathfrak{s}Set \rightleftharpoons \mathfrak{cs}Set : i$ (see Definition A.2.37).*

Proof. Actually, condition (2) of Kan's theorem fails. For example, the pushout of

$$\begin{tikzcd} \Lambda^2_1 \arrow[r, hook] \arrow[d] & \partial\Delta^2 \\ \Delta^2 & \end{tikzcd}$$

in $\mathfrak{cs}$Set is Δ^2, which is not weakly equivalent to $\partial\Delta^2$. □

Remark 1.2.36. By the definition of the coverage on Δ (see Example 1.2.15), every object in $\mathfrak{C}$Sh(Δ) is a Kan complex. Therefore, none of Δ^n, $\partial\Delta^n$, Λ^n_k are objects in $\mathfrak{C}$Sh(Δ) except Δ^0, Λ^1_0 and Λ^1_1.

Proposition 1.2.37. *Let X be a concrete simplicial set. Then its sheafification $\mathfrak{F}(X) = \mathfrak{S}et(?, X_0)$.*

Proof. It is easy to see that $\mathfrak{S}$et$(?, X_0)$ is a concrete sheaf over Δ. Clearly $\mathfrak{F}(X)$ μ $\mathfrak{S}$et$(?, X_0)$, and $\mathfrak{S}$et$(?, X_0)$ μ $\mathfrak{F}(X)$ can be proved by using the covering family of $\underline{n}$ by $n+1$ copies of $\underline{0}$'s, and the fact that all constant maps $\underline{n}$! X_0 are in X_n since X is concrete. □

Hence $\mathfrak{F}(\Delta^1) = \mathfrak{F}(\partial\Delta^1) = A$ with $A_n = \mathfrak{Set}(\{0, 1, \cdots, n\}, \{0, 1\})$. In fact, the geometric realization of A is S^∞. Therefore, it is not always true that if X is a concrete simplicial set then $X \to i \circ \mathfrak{F}(X)$ is a weak equivalence of simplicial sets.

The above proposition says that we have an equivalence of categories $\mathfrak{CSh}(\Delta) \simeq \mathfrak{Set}$. In fact from the adjoint pair $\mathfrak{F} \dashv c : \mathfrak{s}\mathrm{Set} \rightleftharpoons \mathfrak{C}\mathrm{Sh}(\Delta) : i$, we can figure out the weak equivalences, fibrations and cofibrations on $\mathfrak{Set}$ as follows. Since $\mathfrak{F}(\Lambda^n_k) \to \mathfrak{F}(\Delta^n)$ and $\mathfrak{F}(\partial\Delta^n) \to \mathfrak{F}(\Delta^n)$ are all identities for $n \geq 1$, $X \to Y$ in $\mathfrak{Set}$ is a fibration (or a trivial fibration) if and only if it has the right lifting property with respect to $d_0, d_1 : \{0\} \to \{0, 1\}$ (or $\emptyset \to \{0\}$). In other words, $X \to Y$ is a trivial fibration if and only if it is surjective, and it is a fibration if and only if it is surjective with $X \neq \emptyset$ or $X = \emptyset$ and Y is arbitrary. Then $A \to B$ is a cofibration, if and only if it is injective, and it is a trivial cofibration if and only if it is injective when $A \neq \emptyset$ or it is $\emptyset \to \emptyset$. Every map from a non-empty set is a weak equivalence, since $\mathfrak{Set}(?, X)$ is contractible for any set X via the homotopy $\mathfrak{Set}(?, X) \times \Delta^1 \to \mathfrak{Set}(?, X)$ defined by $(x_0, \cdots, x_n; g) \mapsto (x_0, \cdots, x_i, x, \cdots, x)$, where $g^{-1}(0) = \{0, 1, \cdots, i\}$. It is easy to see that this is a proper cofibrantly generated model structure on $\mathfrak{Set} \simeq \mathfrak{C}\mathrm{Sh}(\Delta)$. In other words, we have proved:

Theorem 1.2.38. *We can lift the standard model category structure on $\mathfrak{s}$Set to $\mathfrak{C}$Sh(Δ) from the adjoint pair $\mathfrak{F} \dashv c : \mathfrak{s}Set \rightleftharpoons \mathfrak{C}Sh(\Delta) : i$.*

1.3 The D-topology

A diffeological space is a set with some extra structure. We can associate to every diffeological space the following interesting topology:

Definition 1.3.1 ([Do]). Given a diffeological space X, the final topology induced by all its plots, where each domain is equipped with the standard topology, is called the *D-topology* on X.

More precisely, if (X, D) is a diffeological space, then a subset A of X is open (called *D-open*) in the D-topology of X if and only if $\phi^{-1}(A)$ is open for each $\phi \in \mathsf{D}$. If D is generated by a subset D', then A is D-open if and only if $\phi^{-1}(A)$ is open for each $\phi \in \mathsf{D}'$.

Example 1.3.2 ([Do]). (1) The D-topology on any smooth manifold with the standard diffeology coincides with the usual topology on the smooth manifold.

(2) The D-topology on a discrete diffeological space is discrete, and the D-topology on an indiscrete diffeological space is indiscrete.

Proposition 1.3.3 ([La]). *For any diffeological space, the D-topology is locally path-connected.*

Proof. This is because every path component of a D-open set is D-open. □

Theorem 1.3.4. *There is an adjoint pair $D : \mathfrak{Diff} \rightleftharpoons \mathfrak{Top} : T$ given by $D(X) =$ the set X with the D-topology, $T(Y)(U) = \{f : U \to Y \text{ continuous}\}$, and both $D(f)$ and $T(g)$ the same set maps for the underlying sets. Furthermore, we have $T \circ D \circ T = T$ and $D \circ T \circ D = D$.*

Proof. This is routine. □

Clearly, both functors D and T are faithful. But neither of them are full. Neither of them reflects isomorphisms, since for example, the irrational torus T^2_θ and the same set with the indiscrete diffeology have the same D-topology, and $\mathbb{Q}$ with the usual topology or with the discrete topology have the same image in $\mathfrak{Diff}$. And neither of them is essentially surjective, since the D-topology is always locally path-connected, and there is no topological space X such that $T(X) = T^2_\theta$, for the identity underlying set map $D(T(X))$! X is continuous. The functor T sends (in)discrete topological spaces to (in)discrete diffeological spaces, and it preserves arbitrary coproducts. However, T does not preserve coequalizers in general. For example, $\mathbb{R}^0 \overset{i_0}{\underset{i_1}{\rightrightarrows}} [0,1] \longrightarrow S^1$, with $i_s(\mathbb{R}^0) = s$ for $s = 0, 1$, is a coequalier in $\mathfrak{Top}$, while $T(\mathbb{R}^0) \overset{T(i_0)}{\underset{T(i_1)}{\rightrightarrows}} T([0,1]) \longrightarrow T(S^1)$ is not a coequalizer in $\mathfrak{Diff}$.

Here is one application of the D-topology:

Example 1.3.5. Let X and Y be diffeological spaces such that X is indiscrete and every point in $D(Y)$ is either open or closed. Then as diffeological spaces, $\mathfrak{Diff}(X,Y) \cong Y$.

1.3.1 Two D-topologies related to a subspace

Let X be a diffeological space, and let Y be a quotient set of X. Then we can give Y two topologies:

(1) the D-topology of the quotient diffeology on Y;

(2) the quotient topology of the D-topology on X.

Since $D : \mathfrak{Diff}$! $\mathfrak{Top}$ is a left adjoint, these two topologies are the same. [I1] has direct proof for this result.

Similarly, let X be a diffeological space, and let A be a subset of X. Then we can give A two topologies:

(1) $\tau_1 =$ the D-topology of the sub-diffeology on A;

(2) $\tau_2 =$ the sub-topology of the D-topology on X.

However, these two topologies are not the same in general. We can only conclude that $\tau_2 \subseteq \tau_1$.

Example 1.3.6. Let $X = \mathbb{R}$ have the standard diffeology, and let $A = \mathbb{Q}$. Then τ_1 is the discrete topology, which is strictly finer than the sub-topology τ_2.

Remark 1.3.7. Let $\mathbb{R}$ have the standard diffeology, and let $\mathbb{Q}$ have the sub-topology of $\mathbb{R}$. Then $\mathfrak{Diff}(T(\mathbb{Q}), \mathbb{R}) = \mathfrak{Set}(\mathbb{Q}, \mathbb{R})$.

We are interested in the conditions under which $\tau_1 = \tau_2$.

Lemma 1.3.8. *Let A be a locally convex subset of $\mathbb{R}^n$. Then $\tau_1 = \tau_2$.*

Proof. This is essentially (3) of Lemma 24.6 in [KM]. □

Remark 1.3.9. By the same proof of the above lemma, we can show that given any diffeological space X, $\mathfrak{Diff}(\mathbb{R}, X)$ totally determines the D-topology on X, although it does not determine the diffeology of X in general, see Example 1.1.8.

Example 1.3.10. For every open subset A of $\mathbb{R}^n$, $\tau_1 = \tau_2$.

More generally, we have the following:

Lemma 1.3.11. *If A is a D-open subset of a diffeological space X, then $\tau_1 = \tau_2$.*

Proof. Let B be τ_1-open, and let $p : U \to X$ be an arbitrary plot of X. Since A is D-open in X, $p^{-1}(A)$ is an open subset of U. Hence $p^{-1}(A)$ is an object in DS, and the composition of $p^{-1}(A) \hookrightarrow U \to X$ is also a plot for X, which factors through the inclusion map $A \hookrightarrow X$. Since $B \in \tau_1$, $(p|_{p^{-1}(A)})^{-1}(B)$ is open in $p^{-1}(A)$, which implies that $p^{-1}(B)$ is open in U. □

Lemma 1.3.12. *Let X be a diffeological space and let A be a subset of X. If there exists a D-open neighborhood C of A in X together with a smooth retraction $r : C \to A$ (here both C and A are equipped with the sub-diffeologies from X), then $\tau_1(A) = \tau_2(A)$.*

Proof. Let $B \in \tau_1(A)$. Then $r^{-1}(B) \in \tau_1(C) = \tau_2(C)$ is D-open in X. Therefore, $B = A \cap r^{-1}(B) \in \tau_2(A)$. □

Example 1.3.13. Given a smooth manifold M of dimension $n > 0$, by the strong Whitney Embedding Theorem, there is a smooth embedding $M \hookrightarrow \mathbb{R}^{2n}$. If we view M as a subset of $\mathbb{R}^{2n}$, then $\tau_1 = \tau_2$ since there is a tubular neighborhood U of M in $\mathbb{R}^{2n}$ together with a smooth retraction $U \to M$.

Here are some other examples:

Example 1.3.14. The D-topologies on $\Lambda^n, \Lambda^n_{sub}, \partial\mathbb{A}^n, \partial\mathbb{A}^n_{sub}$ coincide with the sub-topologies of $\mathbb{R}^n$, where these diffeological spaces are defined at the beginning of Section 2.4.

1.3.2 Relationship with Δ-generated topological spaces

Definition 1.3.15. A topological space X is called Δ-generated if the following condition holds: $A \subseteq X$ is open if and only if $f^{-1}(A)$ is open in $|\Delta^n|$, the standard n-simplex in $\mathfrak{T}$op, for any continuous map $f : |\Delta^n| \to X$ and any $n \in \mathbb{N}$.

More on Δ-generated topological spaces can be found in [Du1].

Proposition 1.3.16. *The objects in the image of the functor D are exactly the Δ-generated topological spaces.*

Proof. More generally, for any set S of topological spaces, we can define S-generated topological spaces in the same way. Clearly any topological space in S is S-generated, since the identity map works. Hence $\{S\text{-generated topological spaces}\} = \{T\text{-generated topological spaces}\}$ if and only if every element in T is S-generated and every element in S is T-generated.

Now let $S = \{\mathbb{R}^n \mid n \in \mathbb{N}\}$, and let $T = \{|\Delta^n| \mid n \in \mathbb{N}\}$. Then a topological space X is S-generated if and only if $D \circ T(X) = X$. $D \circ T \circ D = D$ implies that the image of the functor D is exactly the S-generated topological spaces. To see that $|\Delta^n|$ is S-generated, we only need to notice that the inclusion $|\Delta^n| \hookrightarrow \mathbb{R}^n$ has a continuous retract, and to see that $\mathbb{R}^n$ is T-generated, we only need to notice that for every small open ball B in $\mathbb{R}^n$, there is a continuous map $|\Delta^n| \to \mathbb{R}^n$ which sends an open ball in the interior of $|\Delta^n|$ homeomorphically to B. □

Remark 1.3.17. [SYH] has results similar to Theorem 1.3.4 and Proposition 1.3.16.

Proposition 1.3.18. *Every locally path-connected first countable topological space is Δ-generated.*

Proof. Let (X, τ) be a locally path-connected first countable topological space. Then for any $x \in X$, there exists a neighborhood basis $\{A_i\}_{i=1}^{\infty}$ of x, such that

(1) each A_i is path-connected;

(2) $A_{i+1} \subseteq A_i$.

This is because, for any neighborhood basis $\{B_i\}_{i=1}^{\infty}$ of x, we can define A_1 to be the path-component of B_1 containing x, and A_i to be the path-component of $A_{i-1} \cap B_i$ containing x for $i \geq 2$.

Now let τ' be the final topology of X for all continuous maps $|\Delta^n| \to (X, \tau)$ for all $n \in \mathbb{N}$. Clearly $\tau \subseteq \tau'$. Suppose A is in τ' but not in τ. This means that there exists $x \in A$ such that for any $U \in \tau$ which is a neighborhood of x, there exists $x_U \in U - A$. Let $\{A_i\}_{i=1}^{\infty}$ be a neighbourhood basis for x with the above two properties. Let's write $x_n \in A_n - A$ accordingly. Define $f : [0, 1] \to X$ by letting

$f|_{[\frac{1}{i+1},\frac{1}{i}]}$ be a continuous path connecting x_{i+1} and x_i in A_i, and $f(0) = x$. Easy to see that f is continuous for (X, τ), but $f^{-1}(A)$ is not open in $[0, 1]$. □

Example 1.3.19. The infinite earring $\bigcup_{i\in\mathbb{Z}^+}\{(x, y) \in \mathbb{R}^2 \mid (x - \frac{1}{i})^2 + y^2 = \frac{1}{i^2}\}$ with the sub-topology of $\mathbb{R}^2$ is Δ-generated, since it is locally path-connected and first countable.

However, the converse of the above proposition is not true:

Example 1.3.20. Let X be a set with the complement-finite topology. We write $card(X)$ for its cardinality. Then

(1) X is Δ-generated if $card(X) < card(\mathbb{N})$ or $card(X) \geq card(\mathbb{R})$;

(2) X is not Δ-generated if $card(X) = card(\mathbb{N})$.

Note that X is not first countable when $card(X) \geq card(\mathbb{R})$. This provides a counterexample to the converse of the above proposition.

Proof. (1.1) If X is a finite set, then the complement finite topology is the discrete topology. Hence X is Δ-generated.

(1.2) Assume $card(X) \geq card(\mathbb{R})$, and let B be a non-closed subset of X, that is, $B \neq X$ and $card(B) \geq card(\mathbb{N})$. We must construct a continuous map $f : \mathbb{R} \to X$ such that $f^{-1}(B)$ is not closed in $\mathbb{R}$. Note that in this case, every injection $\mathbb{R} \to X$ is continuous.

Take an injection $\tilde{f} : \{\frac{1}{n}\}_{n\in\mathbb{Z}^+} \to B$. We can extend this to an injection $f : \mathbb{R} \to X$ with $f(0) \in X - B$. This map is what we are looking for.

(2) If $card(X) = card(\mathbb{N})$, then every continuous map $\mathbb{R} \to X$ is constant. Otherwise, since every point in X is closed, $\mathbb{R}$ is a disjoint union of at least 2 and at most countably many non-empty closed subsets. But this is impossible both for finite disjoint union since $\mathbb{R}$ is connected, and for countable disjoint union because of the existence of the Cantor set. □

Remark 1.3.21. Assume the continuum hypothesis. Then the above example can be simply written as: a set X with complement finite topology is Δ-generated if and only if X is not an infinite countable set.

Example 1.3.22. Let κ be a field. Then $Spec(\kappa[x])$ with the Zariski topology is Δ-generated if $card(\kappa) < card(\mathbb{N})$ or $card(\kappa) \geq card(\mathbb{R})$. However, $Spec(\mathbb{Z})$ and $Spec(\mathbb{Z}[x])$ with the Zariski topology are not Δ-generated.

Actually, not every locally path-connected topological space is Δ-generated. The following example is given by J. Brazas:

Example 1.3.23. As a set, let X be the union of the closed unit intervals indexed by $I = \{(a,b) \mid a,b \text{ are ordinals with } a < b \leq \omega_1\}$, where ω_1 is the first uncountable ordinal. We write elements in X as $x_{a,b}$ with $x \in [0,1]$ and $(a,b) \in I$. Let Y be the quotient set $X/\sim$, where the only non-trivial relations are $0_{a,b} \sim 1_{c,a}$ for any $(a,b),(c,a) \in I$. Since we will only work with Y, by abuse of notation, we denote the elements in Y in the same way as those in X. The topology on Y is given by the following subbasis:

(1) open intervals $(x_{a,b}, y_{a,b})$ for any $0 \leq x < y \leq 1$ and $(a,b) \in I$;

(2) $(\bigcup_{b \leq c < d \leq \omega_1}[0_{c,d}, 1_{c,d}]) \cup (\bigcup_{a < b \leq e \leq \omega_1}(x_{a,e}, 1_{a,e}])$ for any $(b,\omega_1) \in I$ and any $x \in [0,1)$.

It is clear that Y is locally path-connected (but not first countable). However, Y is not Δ-generated. Here is the proof. Let $A = \bigcup_{a<\omega_1}(0_{a,\omega_1}, 1_{a,\omega_1}]$. Then A is not open in Y. For any continuous map $f : |\Delta^n| \to X$, we claim that $f^{-1}(A)$ is open in $|\Delta^n|$. Otherwise, there exists $u \in f^{-1}(A)$ such that no open neighborhood of u is contained in $f^{-1}(A)$. Then $f(u) = \omega_1$, and we can choose a sequence (u_i) converging to u, and each u_i is not in $f^{-1}(A)$. Since ω_1 is the first uncountable ordinal, $f(u_i)$ is not convergent to $f(u) = \omega_1$, which conflicts the continuity of f.

Remark 1.3.24. Except local path-connectedness, almost no property in general topology holds for an arbitrary Δ-generated space. Example 74 of [SS] shows that not every Δ-generated space is locally compact or paracompact.

1.3.3 The D-topology on infinite dimensional spaces

For any two diffeological spaces X and Y, we have a canonical continuous map $D(X \times Y) \to D(X) \times D(Y)$. If both $D(X)$ and $D(Y)$ are first countable, then by a proof similar to the proof of Proposition 1.3.18, we can show that the canonical map is a homeomorphism.

Example 1.3.25. Let M and N be two smooth manifolds. Then the D-topology on $\mathfrak{Diff}(M,N)$ is finer than the compact-open topology. This is because the compact-open topology has a subbasis $A(K,W) = \{f \in \mathfrak{Diff}(M,N) \mid f|_K \subseteq W\}$, where K is a compact subset of M and W is an open subset of N. Let $\phi : U \to \mathfrak{Diff}(M,N)$ be any plot of $\mathfrak{Diff}(M,N)$. Since the corresponding map $\bar{\phi} : U \times M \to N$ is smooth, $\bar{\phi}^{-1}(W)$ is open in $U \times M$. Then for any $u \in \phi^{-1}(A(K,W))$, $\{u\} \times K$ is in the open set $\bar{\phi}^{-1}(W)$. By the compactness of K and the definition of the product topology, $V \times K \subseteq \bar{\phi}^{-1}(W)$ for some open neighborhood V of u in U, which implies that $\phi^{-1}(A(K,W))$ is open in U. Thus $A(K,W)$ is open in the D-topology.

Actually, the D-topology is almost always strictly finer than the compact-open topology.

Example 1.3.26. Consider $\mathfrak{Diff}(\mathbb{R},\mathbb{R})$. Let's define $A(K_0, \cdots, K_n; W_0, \cdots, W_n) = \{f \in \mathfrak{Diff}(\mathbb{R},\mathbb{R}) \mid f^{(i)}(K_i) \subseteq W_i \text{ for all } i \in \{0, 1, \cdots, n\}\}$, where each K_i is a compact subset of $\mathbb{R}$ and each W_i is an open subset of $\mathbb{R}$. The same proof as above shows that $A(K_0, \cdots, K_n; W_0, \cdots, W_n)$ is D-open in $\mathfrak{Diff}(\mathbb{R},\mathbb{R})$. Now for example, let $U = A([-1,1],[-1,1];(-1,1),(-1,1))$. Clearly the zero function $\hat{0} \in U$. We claim that there is no open neighborhood of $\hat{0}$ in the compact-open topology of $\mathfrak{D}\mathrm{iff}(\mathbb{R},\mathbb{R})$ contained in U. Otherwise, we may assume $\hat{0} \in A(K,(-\epsilon,\epsilon)) \subseteq U$ for some $\epsilon > 0$, since if $\hat{0} \in A(K_1,W_1) \cap \cdots \cap A(K_m,W_m)$, then $0 \in W_i$ for each i and $\hat{0} \in A(K_1 \cup \cdots \cup K_m, W_1 \cap \cdots \cap W_m) \subseteq A(K_1,W_1) \cap \cdots \cap A(K_m,W_m)$. Then clearly $f : \mathbb{R} \to \mathbb{R}$ defined by $f(y) = \frac{\epsilon}{2}\sin(\frac{2y}{\epsilon})$ is in $A(K,(-\epsilon,\epsilon))$, but not in U. This example is due to G. Sinnamon.

Now we define a topology on $\mathfrak{D}\mathrm{iff}(\mathbb{R},\mathbb{R})$. Clearly $\{A_n(\epsilon)\}_{n\in\mathbb{Z}^+,\epsilon>0}$ with $A_n(\epsilon) = \{f \in \mathfrak{D}\mathrm{iff}(\mathbb{R},\mathbb{R}) \mid f^{(j)}([-n,n]) \subseteq (-\epsilon,\epsilon) \text{ for } j \in \{0,1,\cdots,n\}\}$ forms a neighborhood basis for $\hat{0} \in \mathfrak{D}\mathrm{iff}(\mathbb{R},\mathbb{R})$. Since $\mathfrak{D}\mathrm{iff}(\mathbb{R},\mathbb{R})$ is a diffeological abelian group (defined in Section 1.7) under pointwise addition with identity $\hat{0}$, we can translate the neighborhood basis of $\hat{0}$ to get a topology on $\mathfrak{D}\mathrm{iff}(\mathbb{R},\mathbb{R})$. Let's call this the *extended compact-open topology* of $\mathfrak{D}\mathrm{iff}(\mathbb{R},\mathbb{R})$. G. Sinnamon showed the following:

Lemma 1.3.27. *The D-topology on* $\mathfrak{D}\mathit{iff}(\mathbb{R},\mathbb{R})$ *coincides with the extended compact-open topology.*

Proof. From the previous example, we already know that the D-topology is finer than the extended compact-open topology on $\mathfrak{D}\mathrm{iff}(\mathbb{R},\mathbb{R})$.

Let's make some notation. Fix a smooth function $h : \mathbb{R} \to \mathbb{R}$ such that $supp(h) \subseteq [1,4]$ and $h(2) = 1$. Now let's define $h_n(x) = h(4^n x)$ and let $M_n = \max_{x\in\mathbb{R},j\in\{0,1,\cdots,n\}}\{|h_n^{(j)}(x)|\}$.

Assume that W is D-open but not open in the extended compact-open topology. Without loss of generality, we may assume W is a D-open neighborhood of $\hat{0}$ in $\mathfrak{D}\mathrm{iff}(\mathbb{R},\mathbb{R})$. Then there exists $f_n \in A_n(\frac{1}{16^n M_n}) - W$ for each $n \in \mathbb{Z}^+$.

Now we claim that $\alpha : \mathbb{R} \to \mathfrak{D}\mathrm{iff}(\mathbb{R},\mathbb{R})$ defined by $\alpha(x) = \sum_{i=1}^{\infty} h_n(x) f_n$ is a well-defined smooth map, or in other words, that $\tilde{\alpha} : \mathbb{R}^2 \to \mathbb{R}$ defined by $\tilde{\alpha}(x,y) = \alpha(x)(y) = \sum_{i=1}^{\infty} h_n(x) f_n(y)$ is a well-defined smooth map. Note that, except for the y-axis, any point has an open neighborhood such that $\tilde{\alpha}$ is a finite sum of smooth functions. Also on the left half open plane, all partial derivatives of $\tilde{\alpha}(x, y)$ exist and are equal to 0. We are left to show that $\frac{\partial^{j+k}}{\partial x^j \partial y^k}\tilde{\alpha}(0, y) = 0$ for all $j, k \in \mathbb{N}$ and all $y \in \mathbb{R}$. We can prove this by induction on j, for each fixed k and y. Clearly $\tilde{\alpha}(0, y) = 0$. Assume that we have proved the equality for some j, then

$$\lim_{x \to 0^+} \frac{1}{x}\left(\frac{\partial^{j+k}}{\partial x^j \partial y^k}\tilde{\alpha}(x, y) - \frac{\partial^{j+k}}{\partial x^j \partial y^k}\tilde{\alpha}(0, y)\right) = 0$$

since we have

$$\left|\frac{\partial^{j+k}}{\partial x^j \partial y^k}\tilde{\alpha}(x, y)\right| = |h_n^j(x) f_n^k(y)| \leq M_n \times \frac{1}{16^n M_n} = \frac{1}{16^n} \leq x^2$$

if $n \geq \max\{j, k\}$, $x \in [\frac{1}{4^n}, \frac{1}{4^{n-1}}]$ and $y \in [-n, n]$. The rest of the claim is easy.

Note that $\alpha(0) = \hat{0}$ and $\alpha(\frac{2}{4^n}) = f_n$. Hence $\alpha^{-1}(W)$ is not open, which contradicts the assumption that W is D-open. □

By the same method, similar results holds for $\mathfrak{Diff}(\mathbb{R}^l, \mathbb{R}^m)$, with $A_n(\epsilon)$ changed to $\{f \in \mathfrak{Diff}(\mathbb{R}^l, \mathbb{R}^m) \mid D^\alpha f(\bar{B}_n^l) \subseteq B_\epsilon^m$ for all $0 \leq |\alpha| \leq n\}$, where $\bar{B}_n^l$ is the closed ball in $\mathbb{R}^l$ with center the origin and radius n, and B_ϵ^m is the open ball in $\mathbb{R}^m$ with center the origin and radius ϵ.

Here is another example:

Example 1.3.28. Let C be the poset $\mathbb{N}$ with the usual ordering. Define a functor $F : \mathsf{C} \to \mathfrak{Diff}$ by $(i \leq j) \mapsto (\mathbb{R}^i \to \mathbb{R}^j)$ with $(x_1, \cdots, x_i) \mapsto (x_1, \cdots, x_i, 0, \cdots, 0)$. Let $X = \mathrm{colim}(F)$. Then as a set, $X = \{x \in \mathbb{R}^{\mathbb{N}} \mid$ there exists $N \in \mathbb{N}$ such that $x_i = 0$ for all $i > N\}$. This set has a natural metric $d : X \times X \to \mathbb{R}$ given by $d(x, y) = \sum_{i \in \mathbb{N}}(x_i - y_i)^2$. There are natural smooth maps $f_n : \mathbb{R}^n \to X$ given by $(x_1, \cdots, x_n) \to (x_1, \cdots, x_n, 0, \cdots)$ for all $n \in \mathbb{N}$. However, the D-topology (which is the weak topology, that is, $U \subseteq X$ is D-open if and only if $f_n^{-1}(U)$ is open in $\mathbb{R}^n$ for any $n \in \mathbb{N}$) on X is strictly finer than the topology induced by the metric

d. For example, $A = \{x \in \mathbb{R}^{\mathbb{N}} \mid |\sum_{i\in\mathbb{N}} x_i| < 1\} \cap X$ is clearly D-open with $0 \in A$. But there exists no $\epsilon > 0$ such that $B(0,\epsilon) = \{x \in X \mid d(x,0) < \epsilon\} \subseteq A$. Also the D-topology on X is strictly finer than the topology induced by the box topology on $\mathbb{R}^{\mathbb{N}}$. For example, $B = (0,1) \cup (0,\frac{1}{2})^2 \cup (0,\frac{1}{3})^3 \cup \cdots$ is only open in the D-topology.

1.3.4 Ringed space and the smooth variety property

In this part, we talk about some properties for smooth manifolds, but not for all diffeological spaces.

Let X be a diffeological space. Then $\mathfrak{D}\mathrm{iff}(X,\mathbb{R})$ is a commutative $\mathbb{R}$-algebra with identity under pointwise addition, pointwise multiplication, and pointwise scalar multiplication.

Let $x_0 \in X$. Then $B = \{f \in \mathfrak{D}\mathrm{iff}(X,\mathbb{R}) \mid f(x_0) = 0\}$ is a maximal ideal of $\mathfrak{D}\mathrm{iff}(X,\mathbb{R})$, and as $\mathbb{R}$-algebras, $\mathfrak{D}\mathrm{iff}(X,\mathbb{R})/B \cong \mathbb{R}$.

Given a diffeological space X, we can define a ringed space $(D(X), \mathcal{O}_X)$, where $\mathcal{O}_X(U) = \mathfrak{D}\mathrm{iff}(U,\mathbb{R})$, with U equipped with the sub-diffeology of X. This gives a functor F from $\mathfrak{D}\mathrm{iff}$ to the category of ringed spaces. It is a classical result (see [DLORV]) that the restriction of this functor to $\mathfrak{M}\mathrm{fd}$ reflects isomorphisms, that is, if $f : M \to N$ is a smooth map between two smooth manifolds such that $F(f)$ is an isomorphism between the corresponding ringed spaces, then f is a diffeomorphism. However, the whole functor does not have this property:

Example 1.3.29. Let X_1 be $\mathbb{R}^2$ with the standard diffeology, and let X_2 be $\mathbb{R}^2$ with diffeology generated by all the usual smooth maps $\mathbb{R} \to \mathbb{R}^2$. Then $D(X_1) = D(X_2)$ by Remark 1.3.9. For any open subset U of $\mathbb{R}^2$, regard $U_i = U$ as a sub-diffeological space of X_i, $i = 1, 2$. By Boman's theorem (Corollary 3.14 of [KM]), $\mathcal{O}_{X_1}(U_1) = \mathfrak{D}\mathrm{iff}(U_1,\mathbb{R}) = \mathfrak{D}\mathrm{iff}(U_2,\mathbb{R}) = \mathcal{O}_{X_2}(U_2)$.

Now we talk about the smooth variety property. It is motivated by the following theorem (see [KP]):

Theorem 1.3.30. *For any closed subset A of $\mathbb{R}^n$, there exists a smooth map $f : \mathbb{R}^n \to \mathbb{R}$ such that $A = f^{-1}(0)$.*

Definition 1.3.31. We say that a diffeological space X has the *smooth variety property* if for every D-closed subset A of X, there exists a smooth map $f : X \to \mathbb{R}$ such that $A = f^{-1}(0)$.

Here is an easy corollary of Theorem 1.3.30:

Corollary 1.3.32. *Let X be a diffeological space together with a smooth map $f : X \to \mathbb{R}^n$ for some $n \in \mathbb{N}$ such that the D-topology on X is the initial topology of f. Then X has the smooth variety property.*

For example, Λ^n, Λ^n_{sub}, $\partial\mathbb{A}^n$, $\partial\mathbb{A}^n_{sub}$ (defined at the beginning of Section 2.4), and every smooth manifold has the smooth variety property.

However, not every diffeological space has the smooth variety property. The following example is due to J. Watts.

Example 1.3.33. Let X be the quotient diffeological space of $\mathbb{R}$ modulo the open interval $(-1, 1)$. Then the D-topology on X is the same as the quotient topology. Hence the point $[0] \in X$ is D-open. Note that every smooth map $X \to \mathbb{R}$ is a smooth map $\mathbb{R} \to \mathbb{R}$ which sends the open interval $(-1, 1)$ to a single point. So there is no smooth map $f : X \to \mathbb{R}$ such that $f^{-1}(0) = X - \{[0]\}$.

Also note that in this example the initial topology on X with respect to $\mathfrak{D}\mathrm{iff}(X, \mathbb{R})$ is strictly finer than the D-topology on X.

1.4 Naive smooth homotopy theory

Smooth homotopy groups for pointed diffeological spaces were first introduced by P. Iglesias-Zemmour in his Ph.D. thesis [I2]. We introduce them in a different way (although the idea was already hidden in [I1, I2]), and prove that they are equivalent to the original definition. The advantage of this approach is that the stationarity condition in the original definition is not essential for the underlying set.

Definition 1.4.1. Let (X, x) be a pointed diffeological space. The n^{th} *smooth homotopy group* $\pi_n^D(X, x)$ of X at x is defined to be $\{f \mid f \in \mathfrak{D}\mathrm{iff}(\mathbb{R}^n, X) \mid f|_{\partial I^n} = x\}/\sim$, with $f \sim g$ if and only if there exists $F \in \mathfrak{D}\mathrm{iff}(\mathbb{R}^{n+1}, X)$ such that $F(a, 0) = f(a)$, $F(a, 1) = g(a)$ and $F|_{\partial I^n \times \mathbb{R}} = x$. Here $I^n = [0, 1]^n$, the standard closed unit cube in $\mathbb{R}^n$.

$\sim$ is an equivalence relation. Reflexivity and symmetry are clear. For transitivity, let F_1 be a smooth pointed homotopy between f and g, and let F_2 be a smooth pointed homotopy between g and h. Then

$$F(a,t) = \begin{cases} F_1(a, \psi(2t)) & \text{if } t \leq \frac{1}{2} \\ F_2(a, \psi(2t - 1)) & \text{if } t > \frac{1}{2} \end{cases}$$

is a smooth pointed homotopy between f and h, where $\psi : \mathbb{R} \to \mathbb{R}$ is a cut-off function, that is, a smooth function such that there exists some $\epsilon > 0$ with $\psi((-\infty, \epsilon)) = 0$, $\psi((1 - \epsilon, \infty)) = 1$ and $\mathrm{Im}(\psi) = [0, 1]$.

Remark 1.4.2. For any $[f] \in \pi_n^D(X, x)$, we can find $g \in \mathfrak{D}\mathrm{iff}(\mathbb{R}^n, X)$ such that $g \sim f$ and $g|_{\mathbb{R}^n - \epsilon I^n} = x$ for some $\epsilon > 0$, where $\epsilon I^n = [\epsilon, 1 - \epsilon]^n$. Here is the reason: take a cut-off function $\phi : \mathbb{R} \to \mathbb{R}$, and define $g = f \circ \phi^n$. Clearly, $g|_{\mathbb{R}^n - \epsilon I^n} = x$, and $F(a_1, \cdots, a_n, t) = f((1 - \phi(t))a_1 + \phi(t)\phi(a_1), \cdots, (1 - \phi(t))a_n + \phi(t)\phi(a_n))$ is a smooth pointed homotopy between f and g.

The group structure on $\pi_n^D(X,x)$ for $n \in \mathbb{Z}^+$ is defined in the same way as that on the usual (continuous) homotopy groups in algebraic topology, by using the representatives with the above stationarity property.

Here is a quick review of P. Iglesias-Zemmour's definition of smooth homotopy groups $\tilde{\pi}_n^D(X,x)$ of a pointed diffeological space (X,x):

Let X be a pointed diffeological space. We define an equivalence relation on X by $x \sim' x'$ if and only if there is $f \in \mathfrak{D}\mathrm{iff}(\mathbb{R},X)$ such that there exists $\epsilon > 0$ with $f|_{(-\infty,\epsilon)} = x$ and $f|_{(1-\epsilon,\infty)} = x'$. We call such f a *stationary path* from x to x', and we define $\tilde{\pi}_0^D(X) = X/\sim'$.

Given a pointed diffeological space (X,x), define $\mathfrak{S}\mathrm{tLoops}^1(X,x)$ to be $\{f \in \mathfrak{D}\mathrm{iff}(\mathbb{R},X) \mid f$ is a stationary path from x to itself$\}$. The elements in $\mathfrak{S}\mathrm{tLoops}^1(X,x)$ will be called *stationary loops* of (X,x). Note that $\mathfrak{S}\mathrm{tLoops}^1(X,x)$ is a diffeological space with the sub-diffeology of the functional diffeology on $\mathfrak{D}\mathrm{iff}(\mathbb{R},X)$. There is a canonical basepoint $x_1 =$ the constant map in $\mathfrak{S}\mathrm{tLoops}^1(X,x)$. Now recursively define $\mathfrak{S}\mathrm{tLoops}^{n+1}(X,x) = \mathfrak{S}\mathrm{tLoops}^1(\mathfrak{S}\mathrm{tLoops}^n(X,x),x_n)$, where x_n is the canonical basepoint on $\mathfrak{S}\mathrm{tLoops}^n(X,x)$, that is, the constant map $f : \mathbb{R}^n \to X$ sending everything to x. The n^{th} smooth homotopy group of X at x is defined to be $\tilde{\pi}_n^D(X,x) = \tilde{\pi}_0^D(\mathfrak{S}\mathrm{tLoops}^n(X,x))$. The group structure on $\tilde{\pi}_n^D(X,x)$ for $n \in \mathbb{Z}^+$ is defined as usual.

When unraveling $\mathfrak{S}\mathrm{tLoops}^n(X,x)$ as smooth maps $\mathbb{R}^n \to X$ satisfying some stationarity conditions using the cartesian closedness of $\mathfrak{D}\mathrm{iff}$, these stationarity conditions are quite complicated since they vary for different variables. However, we have:

Theorem 1.4.3. *The two definitions of smooth homotopy groups for a pointed diffeological space match.*

Proof. Let (X, x) be a pointed diffeological space. By cartesian closedness of $\mathfrak{D}\mathrm{iff}$, we have the inclusion map $\mathfrak{S}\mathrm{tLoops}^n(X, x) \to \mathfrak{D}\mathrm{iff}(\mathbb{R}^n, X)$. Every stationary path connecting f and g in $\mathfrak{S}\mathrm{tLoops}^n(X, x)$ is a smooth pointed homotopy between f and g, which implies that we have a canonical well-defined map $i : \tilde{\pi}_n^D(X, x) \to \pi_n^D(X, x)$, and it is clearly a group homomorphism for $n \in \mathbb{Z}^+$. We have shown that i is surjective in Remark 1.4.2.

i is injective, since every smooth pointed homotopy in $\mathfrak{D}\mathrm{iff}$ can be made into a stationary path in $\mathfrak{S}\mathrm{tLoops}^n(X, x)$ by precomposing with a cut-off function for all the variables except the time variable. $\square$

The proof of the above theorem also indicates:

Theorem 1.4.4. *Let (X, x) be a pointed diffeological space. Then $\pi_n^D(X, x)$ can also be characterized as $\{f \in \mathfrak{D}\mathit{iff}(\mathbb{R}^n, X) \mid f|_{\mathbb{R}^n - \epsilon I^n} = x \text{ for some } \epsilon > 0\}/\sim_1$, where $f \sim_1 g$ if there exists $F \in \mathfrak{D}\mathit{iff}(\mathbb{R}^{n+1}, X)$ and $\delta > 0$ such that $F(a, 0) = f(a)$, $F(a, 1) = g(a)$, and $F|_{(\mathbb{R}^n - \delta I^n) \times \mathbb{R}} = x$.*

Given any pointed diffeological space (X, x), there is a canonical well-defined map $j_n : \pi_n^D(X, x) \to \pi_n(D(X), x)$ by restriction to I^n, which is a group homomorphism when $n \geq 1$.

Proposition 1.4.5 ([I1])**.** *Let X be a diffeological space. Then $j_0 : \pi_0^D(X) \to \pi_0(D(X))$ is a bijection, that is, $\pi_0^D(X)$ coincides with the usual (continuous) path components of X under the D-topology.*

The classical smooth approximation theorem shows:

Proposition 1.4.6. *Let (X, x) be a pointed smooth manifold with the standard diffeology. Then $j_n : \pi_n^D(X, x) \to \pi_n(D(X), x)$ is an isomorphism for any $n \in \mathbb{N}$.*

By a similar method to the proof of Theorem 1.4.3, it is easy to see that:

Proposition 1.4.7. *Let (X, x) be a pointed topological space. Then the canonical map $\pi_n^D(T(X), x) \to \pi_n(X, x)$ is an isomorphism for any $n \in \mathbb{N}$.*

However, in general, $\pi_n^D(X, x)$ and $\pi_n(D(X), x)$ might differ; see Example 1.7.14.

We can also define smooth homotopy groups for a pointed diffeological space (X, x) as $[(S^n, N), (X, x)]$, where $N = (0, \cdots, 0, 1)$ is the north pole of S^n, and $[(S^n, N), (X, x)] = \{f \in \mathfrak{D}\mathrm{iff}(S^n, X) \mid f(N) = x\}/\sim''$, with $f \sim'' g$ if there exists $F \in \mathfrak{D}\mathrm{iff}(S^n \times \mathbb{R}, X)$ such that $F(a, 0) = f(a)$, $F(a, 1) = g(a)$ and $F(N, t) = x$ for any $a \in S^n$ and any $t \in \mathbb{R}$.

For any $[g] \in [(S^n, N), (X, x)]$, there exists $g' \in \mathfrak{D}\mathrm{iff}(S^n, X)$ with $g'(x_0, \cdots, x_n) = x$ if $x_n > 1 - \epsilon$ for some $\epsilon > 0$, such that $g \sim'' g'$. In fact, g' can be constructed as the composition $g \circ h$ with $h \in \mathfrak{D}\mathrm{iff}(S^n, S^n)$ such that $h(x_0, \cdots, x_n) = N$ if $x_n > 1 - \epsilon$. We use the following charts for S^n: $\{(U_i, \phi_i)\}_{i=1,2}$ where $U_1 = \{(x_0, \cdots, x_n) \in S^n \mid x_n > -\epsilon\}$, $U_2 = \{(x_0, \cdots, x_n) \in S^n \mid x_n < \epsilon\}$, and ϕ_1 and ϕ_2 are stereographic projections with respect to the south pole $S = (0, \cdots, 0, -1)$ and the north pole N, respectively. h is defined by $h|_{U_2} = id$, and $\phi_1 \circ h|_{U_1} \circ \phi_1^{-1}$ by $(y_1, \cdots, y_n) \mapsto (y_1 \tau(r), \cdots, y_n \tau(r))$ for some cut-off function $\tau : \mathbb{R} \to \mathbb{R}$ and $r = \sqrt{\sum_{i=1}^n y_i^2}$. Clearly, h and id_{S^n} are smoothly homotopic.

Theorem 1.4.8. *For any pointed diffeological space (X, x), we have $\pi_n^D(X, x) = [(S^n, N), (X, x)]$.*

Proof. By the characterization of $\pi_n^D(X, x)$ in Theorem 1.4.4, we can define $j : \pi_n^D(X, x) \to [(S^n, N), (X, x)]$ by $[f] \mapsto [g]$ with

$$g(x_0, \cdots, x_n) = \begin{cases} f \circ \phi(x_0, \cdots, x_n), & \text{if } x_n \in U \\ x, & \text{otherwise,} \end{cases}$$

where $U = S^n - \{N\}$, and ϕ is the stereographic projection with respect to the south pole S. This is clearly well-defined.

j is surjective, since for any $[g] \in [(S^n, N), (X, x)]$ with $g(x_0, \cdots, x_n) = x$ if $x_n > 1 - \epsilon$ for some $\epsilon > 0$, we can define $f : \mathbb{R}^n \to X$ as the composition $\mathbb{R}^n \xrightarrow{s} \mathbb{R}^n \xrightarrow{\phi^{-1}} U \hookrightarrow S^n \xrightarrow{g} X$, where s is a re-scaling so that $[f] \in \pi_n^D(X, x)$. It is clear that $j([f]) = [g]$.

j is injective, since any smooth homotopy in $[(S^n, N), (X, x)]$ can be made stationary by composing with $h \in \mathfrak{D}\mathrm{iff}(S^n, S^n)$ defined above. □

In other words, we have also proved the following:

Theorem 1.4.9. *$\pi_n^D(X, x)$ can be written as $\{f \in \mathfrak{D}iff(S^n, X) \mid f(x_0, \cdots, x_n) = x$ if $x_n > 1 - \epsilon$ for some $\epsilon > 0\}/\sim_2$, with $f \sim_2 g$ if there exists $F \in \mathfrak{D}iff(S^n \times \mathbb{R}, X)$ such that $F(a, 0) = f(a)$, $F(a, 1) = g(a)$ and $F((x_0, \cdots, x_n), t) = N$ if $x_n > 1 - \delta$ for some $\delta > 0$, for any $a \in S^n$ and any $t \in \mathbb{R}$.*

If $n \geq 1$, then we can define $[(S^n, N), (X, x)] \times [(S^n, N), (X, x)] \to [(S^n, N), (X, x)]$ by $([f], [g]) \mapsto [h]$, with f, g as described in the above theorem, and $h : S^n \to X$ defined by

$$h(x_0, \cdots, x_n) = \begin{cases} f \circ \sigma_1(x_0, \cdots, x_n), & \text{if } x_{n-1} < 0 \\ g \circ \sigma_2(x_0, \cdots, x_n), & \text{if } x_{n-1} > 0 \\ x, & \text{otherwise,} \end{cases}$$

where σ_1, σ_2 are some diffeomorphisms between $\{(x_0, \cdots, x_n) \in S^n \mid x_{n-1} < 0\}$, $\{(x_0, \cdots, x_n) \in S^n \mid x_{n-1} > 0\}$ and $S^n - \{N\}$, respectively. We can pick suitable σ_1, σ_2 so that this coincides with the group structure of $\pi_n^D(X, x)$, and the map j defined in the proof of Theorem 1.4.8 is a group homomorphism (for $n \geq 1$).

From Iglesias' definition, $\pi_n^D(X, x) = \mathfrak{S}\mathrm{tLoops}^n(X, x)/\sim'$, we can give $\pi_n^D(X, x)$ the quotient diffeology from $\mathfrak{S}\mathrm{tLoops}^n(X, x)$. As [I1] shows, this is the discrete diffeology.

Definition 1.4.10 ([I1, I2])**.** Let X and Y be two diffeological spaces. We say two smooth maps $f, g : X \to Y$ are *smoothly homotopic* if $[f] = [g] \in \pi_0^D(\mathfrak{D}\mathrm{iff}(X, Y))$.

Lemma 1.4.11 ([I1, I2])**.** *The equivalence relation of smooth homotopy is compatible with both left and right compositions.*

Proposition 1.4.12 ([I1, I2])**.** $\pi_n^D : \mathfrak{D}\mathit{iff}_* \to \mathfrak{S}\mathit{et}$ *is a functor, which factors through* $\mathfrak{G}\mathit{rp}$ *when* $n \geq 1$*, and factors through* $\mathfrak{A}\mathit{b}$ *when* $n \geq 2$*.*

Clearly, j_n is a natural transformation $\pi_n^D \to \pi_n \circ D$.

As usual, we have:

Proposition 1.4.13. *If* X *is a diffeological group (which is introduced in Section 1.7), then* $\pi_0^D(X)$ *is a group.*

Proof. This is formal. □

Proposition 1.4.14. *Let* $\{(X_j, x_j)\}_{j \in J}$ *be a set of pointed diffeological spaces. Then the canonical map* $\pi_n^D(\prod_{j\in J} X_j, (x_j)_{j\in J}) \to \prod_{j\in J} \pi_n^D(X_j, x_j)$ *is an isomorphism, for any* $n \in \mathbb{N}$*.*

Proof. This is formal. □

Given a diffeological space X, its fundamental smooth groupoid $\pi_1^D(X)$ is defined in [I1] to be the category with objects points of X, and morphisms $\pi_1^D(X)(x, x') = \pi_0^D(\mathfrak{P}\mathrm{aths}(X; x, x'))$, where $\mathfrak{P}\mathrm{aths}(X; x, x') = \{f \in \mathfrak{D}\mathrm{iff}(\mathbb{R}, X) \mid f(0) = x$ and $f(1) = x'\}$ with the sub-diffeology of $\mathfrak{D}\mathrm{iff}(\mathbb{R}, X)$. By using cut-off functions as above, $\pi_1^D(X)(x, x')$ can also be written as $\pi_0^D(\mathfrak{S}\mathrm{tPaths}(X; x, x'))$, where $\mathfrak{S}\mathrm{tPaths}(X; x, x')$ is the set of all stationary paths in X from x to x', with the sub-diffeology of $\mathfrak{D}\mathrm{iff}(\mathbb{R}, X)$. The composition in $\pi_1^D(X)$ is the usual composition of paths using stationary paths.

For the rest of this section, let's discuss relative smooth homotopy groups and the long exact sequence of a diffeological pair (X, A), that is, X is a diffeological space and A is a non-empty sub-diffeological space of X:

Definition 1.4.15 ([I1, I2]). Let (X, A) be a diffeological pair, and let $a \in A$. Write $\mathfrak{P}\mathrm{aths}(X, A, a) = \{f \in \mathfrak{D}\mathrm{iff}(\mathbb{R}, X) \mid f(0) \in A, f(1) = a\}$ with the sub-diffeology of $\mathfrak{D}\mathrm{iff}(\mathbb{R}, X)$. Define $\pi_n^D(X, A, a) = \pi_0^D(\mathfrak{P}\mathrm{aths}(\mathfrak{S}\mathrm{tLoops}^{n-1}(X, a), \mathfrak{S}\mathrm{tLoops}^{n-1}(A, a), a_{n-1}))$ for any $n \in \mathbb{Z}^+$, where a_{n-1} is the constant map $\mathbb{R}^{n-1} \to X$ sending everything to a, and as a convention $\mathfrak{S}\mathrm{tLoops}^0(X, a) = X$.

Lemma 1.4.16 ([I1, I2]). *Let (X, A) be a diffeological pair, and let $a \in A$. Then $\pi_n^D(X, A, a) = \pi_{n-1}^D(\mathfrak{P}aths(X, A, a), a_1)$ with $a_1 \in \mathfrak{D}iff(\mathbb{R}, X)$ sending everything to a. In particular, $\pi_n^D(X, A, a)$ is a pointed space if $n = 1$, it is a group if $n = 2$, and it is an abelian group if $n \geq 3$.*

Theorem 1.4.17 ([I1, I2]). *For any diffeological pair (X, A) and any $a \in A$, there is a long exact sequence*

$$\cdots \to \pi_n^D(A, a) \xrightarrow{i_*} \pi_n^D(X, a) \xrightarrow{j_*} \pi_n^D(X, A, a) \xrightarrow{k_*} \pi_{n-1}^D(A, a) \to \cdots \to \pi_0^D(X),$$

where i_* is induced from the inclusion $i : A \to X$, j_* is induced from the inclusion $j : \mathfrak{S}tLoops(X, a) \to \mathfrak{P}aths(X, A, a)$, and k_* is induced from $k : \mathfrak{P}aths(X, A, a) \to A$ by sending f to $f(0)$.

1.5 Differential forms

Differential forms and de Rham cohomology for a diffeological space were introduced in [So] and explored carefully in [I1] as follows:

Given a diffeological space X, we have the full subcategory DS/X of $\mathfrak{Diff}/X$, with objects all plots $U \to X$, and morphisms commutative triangles

$$\begin{tikzcd} U \arrow[r] \arrow[d] & X \\ V \arrow[ur] & \end{tikzcd}$$

with the vertical map a morphism in DS. We call DS/X the *category of plots* of X. The set $\Omega^n(X)$ of all *smooth n-forms* on X is defined to be the limit of the following composition of functors $\mathsf{DS}/X \to \mathsf{DS} \to \mathrm{Vect}_{\mathbb{R}}^{op}$, with the first functor the forgetful functor and the second functor given by $(f : U \to V) \mapsto (f^* : \Omega^n(V) \to \Omega^n(U))^{op}$. In other words, a smooth n-form on X is a family of smooth n-forms on the domains of plots of X, compatible with smooth maps between all plots. We denote a smooth n-form on X by $\alpha = \{\alpha_p\}$. There is a wedge product on $\Omega^*(X)$ defined by plotwise wedge product.

The exterior derivative $d : \Omega^n(U) \to \Omega^{n+1}(U)$ induces an exterior derivative $d : \Omega^n(X) \to \Omega^{n+1}(X)$ defined by $d(\{\alpha_p\}) = \{d\alpha_p\}$. Since $d^2 = 0$, we can define the *de Rham complex* of X to be the following cochain complex

$$0 \longrightarrow \Omega^0(X) \xrightarrow{d} \Omega^1(X) \xrightarrow{d} \Omega^2(X) \xrightarrow{d} \cdots$$

$\Omega^*(X)$ together with the wedge product and the exterior derivative forms a differential graded-commutative $\mathbb{R}$-algebra.

The *de Rham cohomology* of X is defined to be the cohomology of the de Rham complex $H^*_{de}(X) = \ker(d)/\mathrm{Im}(d)$ as usual.

A smooth map $g : X \to Y$ between two diffeological spaces induces a functor $g_* : \mathsf{DS}/X \to \mathsf{DS}/Y$. So we have $g^* : \Omega^*(Y) \to \Omega^*(X)$ (called *pullback of smooth* *forms*), which turns out to be a morphism between differential graded-commutative $\mathbb{R}$-algebras. Hence, we have a graded-commutative $\mathbb{R}$-algebra map $g^* : H^*_{de}(Y) \to H^*_{de}(X)$. In other words, we have functors $\Omega^* : \mathfrak{D}\mathrm{iff} \to \mathrm{dgcAlg}^{op}_{\mathbb{R}}$ and $H^*_{de} : \mathfrak{D}\mathrm{iff} \to \mathrm{gcAlg}^{op}_{\mathbb{R}}$.

These are natural generalizations of the set of all smooth n-forms, the de Rham complex and the de Rham cohomology of a smooth manifold.

Proposition 1.5.1 ([I1]). *Let X be a diffeological space. Then $\Omega^0(X) = \mathfrak{D}\mathit{iff}(X, \mathbb{R})$.*

Proof. This also follows easily from Theorem 2.1.3. □

Differential forms are local, in the sense that:

Proposition 1.5.2 ([I1]). *Two smooth forms α and β of a diffeological space X coincide if and only if there is a D-open covering $\{U_i\}_{i\in I}$ of X such that for each U_i with the sub-diffeology, $\alpha|_{U_i} = \beta|_{U_i}$ for any $i \in I$.*

Proposition 1.5.3 ([I1]). *Two smooth k-forms α and β of a diffeological space X coincide if and only if $\alpha_p = \beta_p$ for every plot $p : U \to X$ with $\dim(U) = k$.*

Definition 1.5.4 ([I1, I2]). A morphism $f : X \to Y$ in $\mathfrak{D}\mathrm{iff}$ is called a *subduction* if the diffeology on Y coincides with the final diffeology defined by f.

Theorem 1.5.5 ([I1]). *Let $\pi : X \to Y$ be a subduction in $\mathfrak{D}\mathit{iff}$. Then $\pi^* : \Omega^k(Y) \to \Omega^k(X)$ is injective, with image $\{\alpha \in \Omega^k(X) \mid$ for any plots $p_1, p_2 : U \to X$ with $\pi \circ p_1 = \pi \circ p_2$, $\alpha_{p_1} = \alpha_{p_2}\}$.*

Corollary 1.5.6 ([I1]). *If X is a diffeological space with dimension n $(< \infty)$ (which is defined in Section 1.8), then $\Omega^m(X) = 0$ for any $m > n$.*

Remark 1.5.7. Given a diffeological space X, we can define its *de Rham dimension* $\dim_{\mathbb{R}}(X) = \sup\{n \in \mathbb{N} \mid H^n_{de}(X) \neq 0\}$. Then the above Corollary says $\dim_{\mathbb{R}}(X) \leq \dim(X)$.

Example 1.5.8 ([I1])**.** Let $\mathbb{R} \to \mathbb{R}/(\mathbb{Z}+\theta\mathbb{Z}) \cong T^2_\theta$ be the natural projection to the irrational torus of slope θ. By the above theorem, we can calculate that $\Omega^0(T^2_\theta) \cong$ all constant functions $\mathbb{R} \to \mathbb{R}$, $\Omega^1(T^2_\theta) \cong$ all 1-forms $f dx$ on $\mathbb{R}$ with $f : \mathbb{R} \to \mathbb{R}$ some constant function, and the differential d is zero. Hence, $H^0_{de}(T^2_\theta) \cong H^1_{de}(T^2_\theta) \cong \mathbb{R}$.

Remark 1.5.9. (1) Note that the usual cohomology group $H^1(D(T^2_\theta);\mathbb{R}) = 0$, which implies that de Rham theorem does not always hold in $\mathfrak{D}$iff.

(2) If we define the Euler characteristic $\chi(X)$ of a diffeological space X to be the alternating sum of the dimensions of its de Rham cohomology groups, then $\chi(T^2_\theta) = 0$.

Proposition 1.5.10 ([I1])**.** *Let (X, D_X) be a diffeological space, and let J be a generating set of D_X (see the beginning of Section 1.8). Then $\{(\alpha_p)_{p\in J} \mid \alpha_p \in \Omega^k(dom(p))\}$ is from a smooth k-form of X if and only if for any $p : U \to X$, $q : V \to X$ from J, and any morphisms $f : W \to U$ and $g : W \to V$ in DS such that $p \circ f = q \circ g$, we have $f^*(\alpha_p) = g^*(\alpha_q)$.*

Example 1.5.11. Let $X = \Lambda^2$ (see Example 1.1.10). Then it is easy to calculate that $\Omega^0(X) = \mathfrak{D}\mathrm{iff}(X,\mathbb{R}) = \{(f,g) \in (\mathfrak{D}\mathrm{iff}(\mathbb{R},\mathbb{R}))^2 \mid f(0) = g(0)\}$, $\Omega^1(X) = \Omega^1(\mathbb{R}) \oplus \Omega^1(\mathbb{R})$, and $\Omega^i(X) = 0$ for all $i \geq 2$, with $d^0(f,g) = (df, dg)$. Hence, $H^0_{de}(X) = \mathbb{R}$, and $H^j_{de}(X) = 0$ for all $j \geq 1$.

Theorem 1.5.12 ([I1])**.** *Let $f, g : X \to Y$ be smoothly homotopic maps between diffeological spaces. Then $f^* = g^* : H^*_{de}(Y) \to H^*_{de}(X)$.*

Proof. This follows directly from the fact that for any diffeological space Y, there exists a chain homotopy $K : \Omega^p(Y) \to \Omega^{p-1}(\mathfrak{D}\mathrm{iff}(\mathbb{R}, Y))$ for any $p \in \mathbb{Z}^+$, such that $K \circ d + d \circ K = t^* - s^*$, where $s, t : \mathfrak{D}\mathrm{iff}(\mathbb{R}, Y) \to Y$ are defined by $s(\gamma) = \gamma(0)$ and $t(\gamma) = \gamma(1)$. $\square$

1.6 Tangent spaces

There are two ways to talk about tangent spaces for a pointed diffeological space. One was introduced by G. Hector and uses plots, and we call it the internal tangent space. The other uses smooth functions, and we call it the external tangent space. We compare these tangent spaces through some examples, and find that in general they are different.

I'd like to thank A. Kock for mentioning the external tangent space approach, and its possible difference from the internal one.

1.6.1 Internal tangent spaces

The internal tangent space of a pointed diffeological space is defined using the plots. It was first introduced in [He] as follows:

Given a pointed diffeological space (X, x), we define a subcategory $(\mathsf{DS}/X)_x$ of DS/X with objects $f : U \to X$ such that $0 \in U$ and $f(0) = x$, and morphisms commutative triangles

$$\begin{tikzcd} U \arrow[r] \arrow[d] & X \\ V \arrow[ur] & \end{tikzcd}$$

with the vertical map also satisfying $0 \mapsto 0$. We call $(\mathsf{DS}/X)_x$ the *category of plots of X centered at x*. The *internal tangent space* $T_x(X)$ of X at x is defined to be the colimit of the following composition of functors $(\mathsf{DS}/X)_x \to \mathsf{DS} \to \mathrm{Vect}_{\mathbb{R}}$, with the first functor the forgetful functor and the second functor given by $(f : U \to V) \mapsto (f_0 : T_0(U) \to T_0(V))$.

The *internal tangent bundle* TX of X is defined to be the set $\coprod_{x \in X} T_x(X)$ with diffeology generated by all maps $Tf : TU \to TX$, where $f : U \to X$ is a plot of X, TU has the standard diffeology, and for any $u \in U$, $Tf_u : T_u(U) \to T_{f(u)}(X)$ is defined to be the composition $T_u(U) \longrightarrow T_0(U - u) \longrightarrow T_{f(u)}X$, with $U - u$ the translation of U by u. Therefore, the natural map $TX \to X$ is smooth. This gives us a functor $T : \mathfrak{D}\mathrm{iff} \to \mathfrak{D}\mathrm{iff}$ together with a natural transformation $T \to 1$.

These are natural generalizations of tangent spaces at a point and tangent bundles for smooth manifolds.

Example 1.6.1 ([He, HM])**.** Let (X, x) be a pointed topological space. Then $T_x(T(X)) = 0$, where $T(X)$ has the continuous diffeology on X (see Theorem 1.3.4).

1.6.2 External tangent spaces

The external tangent space of a pointed diffeological space (X, x) is defined using the functional space $\mathfrak{D}\mathrm{iff}(X, \mathbb{R})$. Recall that $\mathfrak{D}\mathrm{iff}(X, \mathbb{R})$ is an $\mathbb{R}$-algebra under pointwise addition, pointwise multiplication and pointwise scalar multiplication.

Definition 1.6.2. Let (X, x) be a pointed diffeological space. An *external tangent vector* on X at x is an $\mathbb{R}$-linear map $F : \mathfrak{D}\mathrm{iff}(X, \mathbb{R}) \to \mathbb{R}$ such that the *Leibniz rule* holds: $F(fg) = F(f)g(x) + f(x)F(g)$.

The Leibniz rule implies that $F(f) = 0$ for every constant function f on X.

In the definition, we do not require an external tangent vector F to be smooth. However, if $\mathfrak{D}\mathrm{iff}(X, \mathbb{R}) = 0$, then every external tangent vector is smooth.

Definition 1.6.3. Let (X, x) be a pointed diffeological space. The *external tangent space* $\hat{T}_xX$ is defined to be the set of all external tangent vectors of X at x.

Clearly $\hat{T}_xX$ is an $\mathbb{R}$-vector space under pointwise addition and scalar multiplication.

Let $f : (X, x) \to (Y, y)$ be a pointed smooth map between two pointed diffeological spaces. Then it induces a canonical $\mathbb{R}$-linear map $f_* : \hat{T}_x(X) \to \hat{T}_y(Y)$

by $f_*(F)(g) = F(g \circ f)$ for $F \in \hat{T}_x(X)$ and $g \in \mathfrak{D}\mathrm{iff}(Y, \mathbb{R})$. This gives a functor $\hat{T} : \mathfrak{D}\mathrm{iff}_* \to \mathrm{Vect}_{\mathbb{R}}$.

As a set, we can define the external tangent bundle $\hat{T}(X)$ of a diffeological space X to be $\coprod_{x \in X} \hat{T}_x(X)$, and there is a canonical set map Pr: $\hat{T}(X) \to X$ sending F to x if $F \in \hat{T}_x(X)$. However, it seems difficult to give a suitable diffeology on $\hat{T}(X)$ to extend the concept of tangent bundles of smooth manifolds, since Pr in general is not locally trivial (see Definition 1.7.2).

1.6.3 Comparisons and computations

In general, it is not easy to calculate the internal and external tangent spaces for a pointed diffeological space. We are going to develop some useful calculational tools: local generating sets and germs.

Let (X, x) be a pointed diffeological space. We call a set A_x of objects in $(\mathsf{DS}/X)_x$ a *local generating set* of X at x, if for any object $V \to X$ in $(\mathsf{DS}/X)_x$, there exist an open neighborhood W of 0 in V, an object $U \to X$ in A_x, and a smooth map $W \to U$ with $0 \mapsto 0$, such that the following diagram is commutative:

$$\begin{tikzcd} W \arrow[r, hook] \arrow[d] & V \arrow[d] \\ U \arrow[r] & X. \end{tikzcd}$$

It is easy to see that $\cup_{x \in X} \mathsf{A}_x$ is a generating set for D_X if each A_x is a local generating set of X at x.

Lemma 1.6.4. *Let (X, x) be a pointed diffeological space, and let $\hat{\mathsf{A}}_x$ be the full subcategory of $(\mathsf{DS}/X)_x$ consisting of elements in a local generating set A_x of X at x. If we write F for the composition of functors $\hat{\mathsf{A}}_x \hookrightarrow (\mathsf{DS}/X)_x \to \mathsf{DS} \to$ Vect$_{\mathbb{R}}$, then there is a canonical surjective map* $\mathrm{colim}(F) \to T_x(X)$ *in Vect$_{\mathbb{R}}$.*

Proof. There is a canonical map $\mathrm{colim}(F) \to T_x(X)$ by the universal property of colimit, and surjectivity follows from the definition of a local generating set. $\square$

Let (X, x) be a pointed diffeological space. We say that two morphisms $f, g : (U \to X) \to (V \to X)$ in $(\mathsf{DS}/X)_x$ are *equivalent* (denoted by $f \sim g$) if there exists an open neighborhood W of $0 \in U$ such that $f|_W = g|_W : W \to V$. This is clearly an equivalence relation on $(\mathsf{DS}/X)_x((U \to X), (V \to X))$ which is compatible with compositions. We define the *germ category* $\mathsf{G}(X, x)$ of (X, x) to be the corresponding quotient category.

There is a functor $\mathsf{G}(X, x) \to \mathrm{Vect}_{\mathbb{R}}$ defined by $([f] : (U \to X) \to (V \to X)) \mapsto f_* : T_0(U) \to T_0(V)$. Clearly the colimit of this functor is $T_x(X)$. And it is easy to see that there is an epimorphism $\oplus T_0(\mathbb{R}) \to T_x(X)$, where the direct sum is over $[p]$ for all $p : \mathbb{R} \to X \in (\mathsf{DS}/X)_x$.

Let A_x be a local generating set of X at x, and let $\tilde{\mathsf{A}}_x$ be the full subcategory of $\mathsf{G}(X, x)$ consisting of objects $[f]$ with f an element in A_x. If $\tilde{\mathsf{A}}_x$ is final, then $\mathrm{colim}(\tilde{\mathsf{A}}_x \hookrightarrow \mathsf{G}(X, x) \to \mathrm{Vect}_{\mathbb{R}}) \cong T_x(X)$ [Mac]. And always, we have an epimorphism from this colimit to $T_x(X)$.

Example 1.6.5. Let (X, x) be a pointed smooth manifold, and let A_x be the set of all charts $U \to X$ with $0 \in U$ and $0 \mapsto x$. Then A_x is a local generating set of X at x. It is easy to see that $\tilde{\mathsf{A}}_x$ is final. Because of the existence of smooth partitions of unity on smooth manifolds, it is easy to see that every external tangent vector is smooth. And it is a classical result that $T_x(X) \cong \mathbb{R}^n \cong \hat{T}_x(X)$, with $n = \dim(X)$.

Example 1.6.6. Let $X = \Lambda^2$, and let $x = (0, 0) \in X$. Then $\mathsf{A}_x = \{ i_0 : \mathbb{R}^0 \to X, i_1 : \mathbb{R} \to X, i_2 : \mathbb{R} \to X \}$ is a local generating set of X at x. It is easy to see that $\tilde{\mathsf{A}}_x$ is final, and the non-identity morphisms in $\tilde{\mathsf{A}}_x$ are

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "i_1"] \\ \mathbb{R}^0 \arrow[ur, "0"] \arrow[r, "i_0"] \arrow[dr, "0"'] & X \\ & \mathbb{R}. \arrow[u, "i_2"'] \end{tikzcd}$$

Therefore, $T_x(X) = \mathbb{R}^2$. Also it is easy to see that $T_y(X) = \mathbb{R}$ for any $x \neq y \in X$. These results are also in [HM].

We can also show that $\hat{T}_x(X) \cong \hat{T}_0(\mathbb{R}) \oplus \hat{T}_0(\mathbb{R}) = \mathbb{R}^2$. Here is the proof. For any $f \in \mathfrak{D}\mathrm{iff}(\mathbb{R}, \mathbb{R})$, we can extend it to $\hat{f}_i \in \mathfrak{D}\mathrm{iff}(X, \mathbb{R})$ by $\hat{f}_i(x_1, x_2) = f(x_i)$ for $i = 1, 2$. For any $F \in \hat{T}_x(X)$, we can define $F_i \in \hat{T}_0(\mathbb{R})$ by $F_i(f) = F(\hat{f}_i)$. For $F_1, F_2 \in \hat{T}_0(\mathbb{R})$ and $f \in \mathfrak{D}\mathrm{iff}(X, \mathbb{R})$, we can define $F \in \hat{T}_x(X)$ by $F(f) = F_1(f \circ i_1) + F_2(f \circ i_2)$. The rest is easy. And it is easy to see that $\hat{T}_y(X) = \mathbb{R}$ for any $x \neq y \in X$.

Remark 1.6.7. This example shows that $TX \to X$ is not always a diffeological bundle (defined in the next section).

Example 1.6.8. Let X be the irrational torus of slope θ, and let $x \in X$.

Since the D-topology on X is indiscrete, the only smooth maps $X \to \mathbb{R}$ are the constant maps, which implies $\hat{T}_x X = 0$.

On the other hand, since $\mathsf{A}_x = \{\mathbb{R} \to X\}$ is a local generating set of X at $[0]$, and $\mathbb{Z} + \theta\mathbb{Z}$ is totally disconnected in $\mathbb{R}$, $\hat{\mathsf{A}}_x$ is a trivial category. Hence, there is a canonical surjective map $\mathbb{R} = T_0(\mathbb{R}) \to T_x(X)$. Moreover, by the same trick, we can show that $\tilde{\mathsf{A}}_x$ is final. Therefore, $T_x(X) = \mathbb{R}$.

In other words, $T_x(X)$ and $\hat{T}_x(X)$ are not always isomorphic.

Let (X, x) be a pointed diffeological space. We say $f \sim_x g \in \mathfrak{D}\mathrm{iff}(X, \mathbb{R})$ if there exists a D-open neighborhood U of x in X such that $f|_U = g|_U$. Clearly, this is an equivalence relation on $\mathfrak{Diff}(X,\mathbb{R})$, and $\mathfrak{Diff}(X,\mathbb{R})/\sim_x$ is again an $\mathbb{R}$-algebra. The map $\mathfrak{Diff}(X,\mathbb{R})/\sim_x \to \mathbb{R}$ given by $[f] \mapsto f(x)$ is a well-defined $\mathbb{R}$-algebra homomorphism, and its kernel $F_x(X)$ is called the *external germ* of X at x. We define $T'_x(X) = \mathrm{Vect}_{\mathbb{R}}(F_x(X)/F_x^2(X),\mathbb{R})$ as an $\mathbb{R}$-vector space.

We say that X *has bump functions* at x if for every D-open neighborhood U of x in X, there exists $f \in \mathfrak{Diff}(X,\mathbb{R})$ such that $f(x) = 1$ and $supp(f) \subseteq U$. For example, every smooth manifold (or more generally, any diffeological space X which admits a smooth injective map $f : X \to \mathbb{R}^n$ for some $n \in \mathbb{N}$ such that the D-topology on X is the initial topology of f) and every diffeological space with (in)discrete D-topology (in particular, any irrational torus) has bump functions at any point. If X has bump functions at x, then the Leibniz rule implies a well-defined $\mathbb{R}$-linear map $\hat{T}_x(X) \to T'_x(X)$, which turns out to be an isomorphism by the same proof as Lemma 1.16 in [Wa].

Here are some applications of this approach:

Example 1.6.9. Let (X,x) be a pointed discrete diffeological space. Then $T_x(X) = 0$, since $\tilde{\psi}_x = \{x : \mathbb{R}^0 \to X\}$ is final; and $\hat{T}_x(X) = 0$, since $F_x(X) = 0$.

Similarly, let (X,x) be a pointed indiscrete diffeological space. Then $\hat{T}_x(X) = 0$ since $\mathfrak{Diff}(X,\mathbb{R})$ only contains constant functions; and $T_x(X) = 0$ since for any $p : \mathbb{R} \to X \in (\mathsf{DS}/X)_0$, there exists $q : \mathbb{R} \to X \in (\mathsf{DS}/X)_0$ such that $q \circ f = p$, where $f : \mathbb{R} \to \mathbb{R}$ sends x to x^3.

Example 1.6.10. Let $X = [0,\infty)$ be the sub-diffeological space of $\mathbb{R}$. Then X has bump functions at 0, which implies $\hat{T}_0(X)$ is isomorphic to $T'_0(X)$. [KM] says that $\mathfrak{Diff}(X,\mathbb{R}) = \{f \in \mathfrak{Diff}((0,\infty),\mathbb{R}) \mid f^{(n)}(0+) \text{ exists for all } n \in \mathbb{N}\} = \{f|_X \mid f \in \mathfrak{Diff}(\mathbb{R},\mathbb{R})\}$. Hence, the Taylor formula with the Lagrange form of the remainder for any representative f of $[f] \in F_0(X)$ is given by $f(x) = \sum_{i=1}^n \frac{f^{(i)}(0)}{i!}x^i + \frac{f^{(n+1)}(y)}{(n+1)!}x^{n+1}$ for some y between 0 and x, for any $n \in \mathbb{Z}^+$. In other words, $f(x) - f'(0)x = xg(x)$

for some smooth function $g : X \to \mathbb{R}$ such that $g(0) = 0$. So any element in $T_0'(X)$ acting on $f(x)$ is the same as acting on $f'(0)x$. Therefore, $\hat{T}_0(X) = T_0'(X) = \mathbb{R}$.

Now let's calculate $T_0(X)$. We claim that the square function $\{f : \mathbb{R} \to X \mid f(x) = x^2\}$ is a local generating set for all curves in X at 0. In other words, every $p \in \mathfrak{D}\mathrm{iff}(\mathbb{R}, X)$ with $p(0) = 0$ locally factors through f at 0. Note that $p \in \mathfrak{D}\mathrm{iff}(\mathbb{R}, X)$ with $p(0) = 0$ means that $p : \mathbb{R} \to \mathbb{R}$ is a smooth function such that $p(x) \geq 0$ for all x, and $p(0) = 0$. Suppose $p^{(n)}(0) \neq 0$ for some $n \in \mathbb{N}$. Then the first such n must be an even integer, say $2m$, with a positive coefficient, by using the Taylor formula for p. That is, $p(x) = x^{2m} g(x)$ for some smooth function $g : \mathbb{R} \to \mathbb{R}$ with $g(0) > 0$. Therefore, locally at 0, $x \mapsto x^m \sqrt{g(x)}$ is a well-defined smooth function whose square is f. The rest of the proof for this claim is due to G. Sinnamon. Now suppose $p \in \mathfrak{D}\mathrm{iff}(\mathbb{R}, \mathbb{R})$ is such that $p(x) \geq 0$ for all x, and $p^{(n)}(0) = 0$ for all $n \in \mathbb{N}$. Let $A = \bigcap_{k=0}^{\infty} (p^{(k)})^{-1}(0)$. Then A is closed in $\mathbb{R}$, and by the mean value theorem, A contains all the limit points of $p^{-1}(0)$. Therefore, we can define a smooth map $g : \mathbb{R} - A \to \mathbb{R}$ by suitable modification (choose appropriate sign) of the function discussed above, so that $p|_{\mathbb{R}-A} = g^2$. Now we define $h : \mathbb{R} \to \mathbb{R}$ by

$$h(x) = \begin{cases} 0, & \text{if } x \in A \\ g(x), & \text{otherwise.} \end{cases}$$

Then clearly h is continuous and at each point $x \in \mathbb{R}$, $f(x) = h^2(x)$. To show that h is smooth, we will show by induction on n that $h^{(n)}(a) = 0$ for all $a \in \partial A$. $n = 0$ is the same as h continuous. Suppose we have $h^{(k)}(a) = 0$ for all $k \leq n - 1$. Then

$$h^{(n)}(a) = \lim_{x \to a} \frac{h^{(n-1)}(x) - h^{(n-1)}(a)}{x - a} = \lim_{x \to a} \frac{h(x)}{(n-1)!(x-a)^n},$$

and

$$\lim_{x \to a} \left| \frac{h(x)}{(x-a)^n} \right| = \lim_{x \to a} \sqrt{\frac{f(x)}{(x-a)^{2n}}} = \lim_{x \to a} \sqrt{\frac{f^{(2n)}(x)}{(2n)!}} = 0.$$

Note that we have a commutative diagram

$$\begin{tikzcd} \mathbb{R} \arrow[r, "-id_{\mathbb{R}}"] \arrow[dr, "f"'] & \mathbb{R} \arrow[d, "f"] \\ & X, \end{tikzcd}$$

which implies $T_0(X) = 0$.

Remark 1.6.11. Here is another observation made by G. Sinnamon: not every smooth map $\mathbb{R}^n \to [0, \infty)$ with $0 \mapsto 0$ locally factors through the square function. One example is $n = 2$, $p : \mathbb{R}^2 \to \mathbb{R}$ given in polar coordinates as $(r, \theta) \mapsto e^{-\frac{1}{r}}(1 - \cos(3\theta))$. It is not too hard to check that p is smooth (in xy-coordinates). $p^{-1}(0)$ consists of three rays $\theta = 0, \frac{2}{3}\pi, \frac{4}{3}\pi$, which cut $\mathbb{R}^2$ into three connected pieces. If there is a smooth (in xy-coordinates) function $g : \mathbb{R}^2 \to \mathbb{R}$ such that $f = g^2$ in a small neighborhood of 0, then g has fixed sign at each connected piece, and g has to change sign when passing through any ray, which makes a contradiction.

It is easy to check that $\frac{\partial^2 p}{\partial x^2}(0)$, $\frac{\partial^2 p}{\partial x \partial y}(0)$ and $\frac{\partial^2 p}{\partial y^2}(0)$ are all 0. Hence, the square function does not factor locally at 0 through p as well.

Remark 1.6.12. With a similar method, we can see that if X is a locally convex subset of $\mathbb{R}^n$ with non-empty interior, together with the sub-diffeology, then $\hat{T}_x(X) = T'_x(X) = \mathbb{R}^n$. This is based on the following facts from analysis: (1) every smooth map $f : X \to \mathbb{R}$ is a restriction of a smooth map from some open neighborhood of X in $\mathbb{R}^n$ to $\mathbb{R}$ (see [KM]); (2) let $f : X \to \mathbb{R}$ be a smooth function with $f(a) = 0$ for some fixed $a \in X$. Then $f(x) = \sum_{i=1}^n x_i \frac{\partial f}{\partial x_i}(a) + \sum_{i=1}^n x_i g_i(x)$ for some smooth functions $g_i : X \to \mathbb{R}$ with $g_i(a) = 0$.

Remark 1.6.13. It is much easier to see that $T_{[0]}(X_n) = 0$ by Lemma 1.6.4, where X_n is introduced in (2) of Example 1.8.1.

1.7 Diffeological bundles and diffeological groups

Definition 1.7.1 ([I1, I2])**.** Let F be a diffeological space. A smooth map $f : X \to Y$ between two diffeological spaces is called *trivial of fiber type F*, if there exists a diffeomorphism $h : X \to F \times Y$, where $F \times Y$ is equipped with the product diffeology, such that the following diagram is commutative:

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[dr, "f"'] & F \times Y \arrow[d, "pr_2"] \\ & Y. \end{tikzcd}$$

Definition 1.7.2 ([I1, I2])**.** Let F be a diffeological space. A morphism $f : A \to U$ in $\mathfrak{D}$iff with U an open subset of $\mathbb{R}^n$ is called *locally trivial of fiber type F*, if there exists an open covering $\{U_i\}_{i \in I}$ of U such that $f|_{f^{-1}(U_i)} : f^{-1}(U_i) \to U_i$ is trivial of fiber type F for each $i \in I$.

Proposition 1.7.3 ([I1, I2])**.** *Let $f : X \to Y$ be a smooth surjective map between two diffeological spaces. The following two statements are equivalent:*
(1) There exists a diffeological space F such that the pullback of f along any plot of Y is locally trivial of fiber type F;
(2) There exists a diffeological space F such that the pullback of f along any global plot of Y (that is, plots of the form $\mathbb{R}^n \to Y$) is trivial of fiber type F.

Definition 1.7.4 ([I1, I2])**.** A smooth surjective map $f : X \to Y$ between two diffeological spaces is called a *diffeological bundle* if one of the conditions in the above theorem holds.

Remark 1.7.5. Every smooth fiber bundle over a smooth manifold is a diffeological bundle.

Definition 1.7.6 ([So])**.** A *diffeological group G* is a group object in $\mathfrak{D}$iff, that is, it is both a diffeological space and a group, such that the multiplication and inverse maps are smooth.

Example 1.7.7 ([So])**.** Any Lie group with the standard diffeology is a diffeological group.

Example 1.7.8. Let G be a topological group. Then $T(G)$ is a diffeological group.

Remark 1.7.9 ([I1])**.** Let G be a diffeological group, and let H be any subgroup of G. Then H with the sub-diffeology of G is automatically a diffeological group. If H is a normal subgroup of G, then G/H with the quotient diffeology is also a diffeological group.

Example 1.7.10 ([So])**.** Let X be a diffeological space. Write $\mathfrak{D}\mathrm{iff}(X)$ for the set of all diffeomorphisms from X to itself. Then $\{\phi : U \to \mathfrak{D}\mathrm{iff}(X) \mid U \times X \to X$ defined by $(u, x) \mapsto \phi(u)(x)$ and $U \times X \to X$ defined by $(u, x) \mapsto (\phi(u))^{-1}(x)$ are both smooth$\}$ is a diffeology on $\mathfrak{D}\mathrm{iff}(X)$, which makes $\mathfrak{D}\mathrm{iff}(X)$ a diffeological group.

Example 1.7.11 ([So])**.** Let M be a smooth manifold, viewed as a diffeological space with the standard diffeology. Then the set $\mathfrak{D}\mathrm{iff}(M)$ with the sub-diffeology of the functional diffeology on $\mathfrak{D}\mathrm{iff}(M, M)$ is already a diffeological group, since the implicit function theorem implies that this diffeology is the same as the diffeology defined in the above example.

Proposition 1.7.12 ([I1, I2])**.** *Let G be a diffeological group, and let H be a subgroup of G. Then $G \to G/H$ is a diffeological bundle of fiber type H (equipped with the sub-diffeology of G), where G/H is the set of left (or right) cosets of H in G, with the quotient diffeology.*

Theorem 1.7.13 ([I1, I2])**.** *Let $f : X \to Y$ be a diffeological bundle of fiber type $F = f^{-1}(y)$ (equipped with the sub-diffeology of X) for some $y \in Y$. Pick $x \in F$.*

Then f induces an isomorphism $\pi_j^D(X,F,x) \to \pi_{j-1}^D(Y,y)$ of pointed sets for $j=1$ and of groups for $j \geq 2$. Therefore, we have the following long exact sequence:

$$\cdots \longrightarrow \pi_n^D(F,x) \xrightarrow{i_*} \pi_n^D(X,x) \longrightarrow \pi_n^D(Y,y) \longrightarrow \pi_{n-1}^D(F,x) \longrightarrow \cdots \longrightarrow \pi_0^D(M) \longrightarrow 0.$$

Example 1.7.14 ([I1, I2])**.** Let $T^2_\theta \cong T^2/\mathbb{R}_\theta$ be the irrational torus of slope θ. Then we have a diffeological bundle $T^2 \to T^2/\mathbb{R}_\theta$ with fiber $\mathbb{R}_\theta$. By Theorem 1.7.13, $\pi_1^D(T^2_\theta) \cong \pi_1^D(T^2/\mathbb{R}_\theta) \cong \mathbb{Z} \oplus \mathbb{Z}$. But as a topological space with the D-topology, $\pi_1(T^2_\theta) \cong 0$.

1.8 Dimension theory

The dimension of a diffeological space is defined in [I1, I3] and [La] as follows:

Given a diffeological space $(X, \mathcal{D}_X)$, define a map $d : \mathcal{D}_X \to \mathbb{N}$ by $d(\phi) = n$ for $\phi : U \to X$ with U open in $\mathbb{R}^n$. Let A be a *generating set* for the diffeology $\mathcal{D}_X$ (that is, the smallest diffeology on X containing A is exactly $\mathcal{D}_X$), and denote $d_A = \sup\{d(\phi) \mid \phi \in A\}$. The *dimension* of $(X, \mathcal{D}_X)$ is defined to be $\dim(X) = \inf\{d_A \mid A \in J\}$, with J the set of all generating sets for the diffeology $\mathcal{D}_X$ on X. This is a generalization of the definition of dimension for smooth manifolds. It is easy to see that dimension is invariant under any diffeomorphism.

Therefore, if $\dim(X) = n < \infty$, then there exists a generating set A for $\mathcal{D}_X$ which consists of some plots of the form $\mathbb{R}^n \to X$.

Here are some (in)equalities about the dimensions of diffeological spaces, with the first three from [I1]:

(1) $\dim(X) = 0$ if and only if X is a discrete diffeological space;

(2) $\dim(X/\sim) \leq \dim(X)$;

(3) $\max\{\dim(X), \dim(Y)\} \leq \dim(X \times Y) \leq \dim(X) + \dim(Y)$;

(4) $\dim(\coprod_{i\in I} X_i) = \max_{i\in I}\{\dim(X_i)\}$.

Here is another characterization of dimension of a diffeological space from [I1, I3]:

By the notation we introduced in Section 1.6.3, we can define $\dim_x(X) = \inf\{d_A \mid A$ is a local generating set of X at $x\}$. In fact, $\dim(X) = \sup_{x\in X}\{\dim_x(X)\}$, and if $f : X \to Y$ is a diffeomorphism, then $\dim_x(X) = \dim_{f(x)}(Y)$ for any $x \in X$.

Example 1.8.1 ([I1, I3]). (1) The dimension of the irrational torus T^2_θ is 1.

(2) We can equip $[0, \infty)$ with the sub-diffeology of $\mathbb{R}$ (denoted by X_∞), or the quotient diffeology of $\mathbb{R}^n/O(n, \mathbb{R})$ (denoted by X_n) under the identification $[x] \to$

$\|x\|^2$. Then $\dim_0(X_n) = n$, $\dim_0(X_\infty) = \infty$, and $\dim_x(X_n) = \dim_x(X_\infty) = 1$ for any $0 \neq x \in X$.

(3) The unit interval $X = [0, 1]$ with the sub-diffeology of $\mathbb{R}$ has dimension ∞, since $\dim_0(X) = \dim_1(X) = \infty$ and $\dim_x(X) = 1$ if $x \in (0, 1)$.

Remark 1.8.2. The identity set maps (under the identifications in (2) of the previous example) $f : X_n \to X_\infty$ and $i_n : X_n \to X_{n+1}$ are smooth. Moreover, by (a) or (c) of Theorem 1.2 in [BBCP],

$$f_* : \operatorname{colim}(\, X_1 \xrightarrow{i_1} X_2 \longrightarrow \cdots \longrightarrow X_n \xrightarrow{i_n} X_{n+1} \longrightarrow \cdots) \to X_\infty$$

is not a diffeomorphism. It is easy to check that the dimension of this colimit is ∞.

Lemma 1.8.3. *Let X be a diffeological space, and let Y be a D-open subset of X with the sub-diffeology. Then* $\dim(Y) \leq \dim(X)$.

Proof. Let $A = \{f_i : U_i \to X\}_{i \in I}$ be a generating set for $\mathcal{D}_X$ with $\dim(X) = \sup_{i \in I}\{\dim(U_i)\}$. Since Y is a D-open subset of X, $f_i^{-1}(Y)$ is open in U_i for each $i \in I$. Each f_i induces a plot $f_i^{-1}(Y) \to Y$ of Y, and it is easy to check that the set of all such plots forms a generating set for $\mathcal{D}_Y$. □

Example 1.8.4. Let X be a diffeological space with diffeology generated by $\{U_i \to X\}_{i \in I}$. Clearly, $\dim(X) \leq \sup\{\dim(U_i) \mid i \in I\}$. The equality does not always hold, even if I only contains one element. For example, let $X = \{0, 1\}$ be equipped with the diffeology generated by the map $f : \mathbb{R}^2 \to X$ with $f^{-1}(0) = \mathbb{R} \times \mathbb{R}^{\geq 0}$. Then $\dim(X) = 1$, since the composition $\mathbb{R}^2 \xrightarrow{g} \mathbb{R} \xrightarrow{h} \mathbb{R}^2 \xrightarrow{f} X$ is f, where $g(x, y) = y$ and $h(x) = (0, x)$.

Example 1.8.5. Let $X = \{0, 1\}$ be equipped with the diffeology generated by the map $f : \mathbb{R}^2 \to X$ with $f^{-1}(0) = \mathbb{Q}^2$. Then $\dim(X) = 2$. Here is the reason. Clearly $1 \leq \dim(X) \leq 2$. Suppose $\dim(X) = 1$, that is, there exists an open covering

$\{V_i\}_{i\in I}$ of $\mathbb{R}$ such that f restricted to each V_i is equal to $V_i \xrightarrow{g_i} \mathbb{R} \xrightarrow{h_i} X$ with g_i a smooth map, $g_i(\mathbb{Q}^2 \cap V_i) \cap g_i((\mathbb{R}^2 - \mathbb{Q}^2) \cap V_i) = \emptyset$ and h_i a plot of X. Then $\mathbb{Q}^2$ must contain a regular point a of g_i, that is, $(\frac{\partial g_i}{\partial x}(a), \frac{\partial g_i}{\partial y}(a)) \neq (0,0)$, by its density in $\mathbb{R}^2$. Hence there exists a neighborhood U_i of a in V_i such that $g_i(a)$ is a regular value of $g_i|_{U_i} : U_i \to \mathbb{R}$. Therefore, $g_i|_{U_i}^{-1}(g_i(a))$ is a dimension 1 submanifold of U_i which is contained in $\mathbb{Q}^2$. This is impossible. This example also shows that $\{0,1\}$ with the indiscrete diffeology has dimension ≥ 2. In fact, the dimension of $\{0,1\}$ with the indiscrete diffeology is infinity, and the following proof is due to R. Shafikov. Assume the dimension is $n < \infty$. Then $f : \mathbb{R}^{n+1} \to \{0,1\}$ with $f^{-1}(0) = \mathbb{Q}^{n+1}$ must locally factor through some set map $\mathbb{R}^n \to \{0,1\}$ via a smooth map. Let $g : U \to \mathbb{R}^n$ be such a smooth map with U an open subset of $\mathbb{R}^{n+1}$. Let $b \in U$ be a point with the highest rank. Then there exists a neighborhood $U' \subseteq U$ of b such that $g|_{U'}$ has constant rank. Pick a point $a \in U' \cap \mathbb{Q}^{n+1}$. By the constant rank theorem, $g_{U'}^{-1}(g(a))$ is a submanifold of U' with dimension $n+1-\operatorname{rank}(a)$, and the same argument as above shows that there is a contradiction.

Lemma 1.8.6. *Let $f : A \to X$ and $g : X \to A$ be morphisms in $\mathfrak{D}iff$ such that $g \circ f = id_A$. Then A is diffeomorphic to a sub-diffeological space of X and* $\dim(A) \leq \dim(X)$.

Proof. $g \circ f = id_A$ implies that f is injective. Let $p : U \to X$ be a plot such that $Im(p) \subseteq Im(f)$. Then there is a set map $h : U \to A$ with $p = f \circ h$. So $h = g \circ f \circ h = g \circ p : U \to A$ is a plot. In other words, A is diffeomorphic to $Im(f)$ with the sub-diffeology of X.

Assume that $S = \{p_i : U_i \to X\}_{i\in I}$ is a generating set of the diffeology $\mathcal{D}_X$. Let $q : V \to A$ be a plot. Then there is a covering $\{V_j\}_{j\in J}$ of V such that for each $j \in J$, there exists $p_{i(j)} : U_{i(j)} \to X \in S$ and a smooth map $V_j \to U_{i(j)}$ making the following diagram commutative

$$\begin{tikzcd} V_j \arrow[r, hook, "i"] \arrow[drr] & V \arrow[r, "q"] & A \arrow[r, "f"] & X \\ & & U_{i(j)} \arrow[ur, "p_{i(j)}"'] & \end{tikzcd}$$

Note that $q = g \circ f \circ q$ and $q \circ i = g \circ f \circ q \circ i$, which implies that $\{g \circ p_i : U_i \to A\}$ is a generating set of the diffeology $\mathcal{D}_A$. □

As a direct corollary of this lemma and Example 1.8.5, any set X with the indiscrete diffeology has dimension ∞ if $card(X) > 1$. And as a direct corollary of this lemma and Example 1.8.1, there are no smooth retracts $\mathbb{R} \to [0,1]$ or $\mathbb{R} \to X_\infty$.

Example 1.8.7. $\dim(\Lambda^n) = \dim(\partial\mathbb{A}^n) = n - 1$, where Λ^n and $\partial\mathbb{A}^n$ are defined at the beginning of Section 2.4. Therefore, it is not always true that $\dim_x(X) = \dim(T_x(X))$, for example, $X = \Lambda^2$ and $x = (0,0)$.

Proposition 1.8.8. *Let $f : X \to Y$ be a diffeological bundle with fiber F. Then $\dim(X) \leq \dim(F) + \dim(Y)$.*

Proof. Assume $\dim(F) = k$ and $\dim(Y) = l$ with $k, l < \infty$. Then for any plot $p : \mathbb{R}^n \to X$, there exists an open covering $\{U_i\}_{i \in I}$ of $\mathbb{R}^n$, smooth functions $\{h_i : U_i \to \mathbb{R}^l\}_{i \in I}$ and plots $\{g_i : \mathbb{R}^l \to Y\}_{i \in I}$ such that $f \circ p|_{U_i} = g_i \circ h_i$ for each $i \in I$. Hence, we have the following commutative diagrams for each $i \in I$

$$\begin{tikzcd} U \arrow[r, hook] \arrow[dr, dotted, "(h_i, w_i)"] \arrow[ddr, bend right, "h_i"'] & \mathbb{R}^n \arrow[dr, bend left, "p"] & \\ & \mathbb{R}^l \times F \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ & \mathbb{R}^l \arrow[r, "g_i"'] & Y, \end{tikzcd}$$

where the bottom square is the pullback square, and the dotted arrow exists by the universal property of pullback. Therefore, there exists an open covering $\{V_{ij}\}_{j \in J}$ of

U_i, smooth functions $s_{ij} : V_{ij} \to \mathbb{R}^k$ and plots $t_{ij} : \mathbb{R}^k \to F$ such that $w_i|_{V_{ij}} = t_{ij} \circ s_{ij}$. The inequality follows. $\square$

Chapter 2

A Homotopy Theory for Diffeological Spaces

Our main goal in this chapter is to construct a homotopy theory (that is, a model category structure) on $\mathfrak{Diff}$, which extends the usual homotopy theory of smooth manifolds, and recovers the diffeological bundle theory.

With the exception of the adjoint functor theorem and the first three examples of Section 2.2, this chapter is original. Here is the structure of the chapter:

In Section 2.1, we prove that $\mathfrak{Diff}$ is locally presentable (Theorem 2.1.3). More generally, the category of all concrete sheaves over a concrete site is locally presentable. We also reach a characterization of Δ-generated topological spaces as colimits of second countable locally path-connected topological spaces.

In Section 2.2, we recall an adjoint functor theorem together with three famous classical examples. We also set up the adjoint pair between $\mathfrak{s}\mathrm{Set}$ and $\mathfrak{Diff}$ using a noncompact cosimplicial object.

In Section 2.3, we prove that there is a model category structure on $\mathfrak{Pre}(\mathsf{DS})$.

In Section 2.4, we define the weak equivalences, fibrations and cofibrations in

$\mathfrak{D}$iff.

In Section 2.5, we study some basic properties of the diffeological realization functor and the smooth singular functor.

In Section 2.6, we give characterizations of cofibrant objects and fibrant objects in $\mathfrak{D}$iff. The main results are: (1) we have a partial factorization (Proposition 2.6.2); (2) S^1 is cofibrant (Theorem 2.6.9); (3) every homogeneous diffeological space is fibrant (Theorem 2.6.23), in particular, every diffeological group and every smooth manifold without boundary is fibrant; (4) every topological space with the continuous diffeology is fibrant (Example 2.6.26); (5) some functional spaces are fibrant (Proposition 2.6.27); (6) every diffeological bundle with fibrant fiber is a fibration (Lemma 2.6.20); (7) not every diffeological space is fibrant (Examples 2.6.30, 2.6.31 and 2.6.32), and in particular, no smooth manifold with boundary is fibrant (Example 2.6.33); (8) not every diffeological space is cofibrant (Example 2.6.21).

In Section 2.7, we give two more equivalent definitions of the smooth homotopy groups of a pointed diffeological space, and we use these characterizations to prove that the smooth homotopy groups of a fibrant diffeological space is bijective to the simplicial homotopy groups of its smooth singular complex (Theorem 2.7.3).

2.1 $\mathfrak{Diff}$ is locally presentable

Locally presentable categories are studied carefully in [AR]. The following definition is taken from [Lu]:

Definition 2.1.1. A category C is *locally presentable* if it satisfies the following conditions:

(1) C is cocomplete;

(2) There exists a set S of objects of C such that every object in C is a colimit of a functor $F : J \to \mathsf{C}$ taking values in S;

(3) Every object in S is small (see Definition A.2.30).

Remark 2.1.2. Clearly, if C is a locally presentable category, then any set I of morphisms in C permits the small object argument (see Definition A.2.31).

Theorem 2.1.3. *$\mathfrak{Diff}$ is locally presentable.*

Proof. (1) We already know that $\mathfrak{Diff}$ is cocomplete (see Theorem 1.1.9).

(2) Let S be the set of all open subsets of $\mathbb{R}^n$ for all $n \in \mathbb{N}$. We claim that every diffeological space X is the colimit of the composition $F : \mathsf{DS}/X \to \mathsf{DS} \to \mathfrak{Diff}$, where the category DS is introduced in Section 1.5, the first functor here is the forgetful functor, and the second functor is the inclusion. For any diffeological space Y and any cocone $F \to Y$, there is a unique set map $f : X \to Y$ defined pointwise as follows. Since X is a diffeological space, for any $x \in X$, $x : \mathbb{R}^0 \to X$ is an object in DS/X. f is defined to be the unique set map making all the following diagrams commutative:

$$\begin{tikzcd} \mathbb{R}^0 \arrow[d, "x"'] \arrow[dr] & \\ X \arrow[r, "f"'] & Y, \end{tikzcd}$$

where $\mathbb{R}^0 \to Y$ is from the cocone $F \to Y$. f is smooth, since for any plot $U \to X$ of X, the following diagram is commutative, which can be checked pointwise as above

$$\begin{tikzcd} U \arrow[d] \arrow[dr] & \\ X \arrow[r, "f"'] & Y, \end{tikzcd}$$

where $U \to Y$ is also from the cocone $F \to Y$.

(3) Let U be an object in DS. We claim that U is κ-small, where κ is some cardinal larger than $2^{\aleph_0}$. That is, for every κ-filtered ordinal J (see Definition A.2.29) and every functor $F : J \to \mathfrak{Diff}$, the natural set map $\operatorname{colim}_{j\in J} \mathfrak{Diff}(U, F(j)) \to \mathfrak{Diff}(U, \operatorname{colim}_{j\in J} F(j))$ is a bijection.

This map is surjective for the following reason. For any $f \in \mathfrak{Diff}(U, \operatorname{colim}_{j\in J} F(j))$, by the definition of colimit in $\mathfrak{Diff}$, for any $u \in U$, there exists a neighborhood V_u of u in U such that there exists some $j_u \in J$ and a smooth map $V_u \to F(j_u)$, making the following diagram commutative:

$$\begin{tikzcd} & F(j_u) \arrow[dr] & \\ V_u \arrow[ur] \arrow[r, hook] & U \arrow[r, "f"'] & \operatorname{colim}_{j\in J} F(j). \end{tikzcd}$$

For any $u, u' \in U$ and any $a \in V_u \cap V_{u'}$, the image of a under the two compositions $V_u \cap V_{u'} \to V_u \to F(j_u) \to \operatorname{colim}_{j\in J} F(j)$ and $V_u \cap V_{u'} \to V_{u'} \to F(j_{u'}) \to \operatorname{colim}_{j\in J} F(j)$ are the same. Since J is filtered, by the construction of colimits in $\mathfrak{Set}$, there exists an upper bound $j_{u,u',a}$ of j_u and $j_{u'}$ such that the image of a under the two compositions $V_u \cap V_{u'} \to V_u \to F(j_u) \to F(j_{u,u',a})$ and $V_u \cap V_{u'} \to V_{u'} \to F(j_{u'}) \to F(j_{u,u',a})$ are the same. $J_0 = \{ j_{u,u',a} \mid u, u' \in U, a \in V_u \cap V_{u'} \}$ is a subset of J with cardinality $\leq 2^{\aleph_0} < \kappa$. Hence by the definition of κ-filteredness of J, there exists an upper bound j for J_0 in J. By construction, we have a smooth map $U \to F(j)$

making the following diagram commutative

$$\begin{tikzcd} F(j) \arrow[r] & \operatorname{colim}_{j\in J} F(j) \\ U, \arrow[u] \arrow[ur] & \end{tikzcd}$$

which implies the surjectivity.

This map is injective for the following reason. Suppose $f, g \in \operatorname{colim}_{j\in J} \mathfrak{D}\mathrm{iff}(U, F(j))$ have the same image in $\mathfrak{D}\mathrm{iff}(U, \operatorname{colim}_{j\in J} F(j))$. Since J is filtered, this just means that there exists $j \in J$ and two smooth maps $f, g : U \to F(j)$, such that their composition with $F(j) \to \operatorname{colim}_{j\in J} F(j)$ agree. For any $u \in U$, there exists $j_u \geq j$ in J such that the image of $f(u)$ and $g(u)$ agree under $F(j) \to F(j_u)$. $J_0 = \{j_u \mid u \in U\}$ is a subset of J with cardinality $\leq 2^{\aleph_0} < \kappa$. Therefore, there exists an upper bound j'' of J_0 in J, and $f, g : U \to F(j)$ composed with $F(j) \to F(j'')$ agree, which implies the injectivity. $\square$

Remark 2.1.4. More generally, a similar proof shows that the category of all concrete sheaves over any concrete site C (note that we define a site to be a small category to start with, see Definition 1.2.3) is locally presentable. By the adjoint $\mathfrak{F} : \mathfrak{C}\mathrm{Pre}(\mathsf{C}) \rightleftharpoons \mathfrak{C}\mathrm{Sh}(\mathsf{C}) : i$, $\mathfrak{C}\mathrm{Sh}(\mathsf{C})(\mathfrak{F}(\mathsf{C}(\star, c)), X) \cong \mathfrak{C}\mathrm{Pre}(\mathsf{C})(\mathsf{C}(\star, c), X) = X(c)$. We only need to change two things: (1) fix a concrete sheaf X over C. The set S is changed to $\{\mathfrak{F}(\mathsf{C}(\star, c)) \mid c \in \mathrm{Obj}(C)\}$. The composition of functors $\mathsf{DS}/X \to \mathsf{DS} \to \mathfrak{D}\mathrm{iff}$ is changed to the forgetful functor $\tilde{S} \to \mathfrak{C}\mathrm{Sh}(\mathsf{C})$ with the index category $\tilde{S}$ having objects all morphisms in $\mathfrak{C}\mathrm{Sh}(\mathsf{C})$ of the form $\mathfrak{F}(\mathsf{C}(\star, c)) \to X$ and morphisms all commutative diagrams

$$\begin{tikzcd} \mathfrak{F}(\mathsf{C}(\star, c)) \arrow[r] \arrow[d] & \mathfrak{F}(\mathsf{C}(\star, d)) \arrow[dl] \\ X, & \end{tikzcd}$$

with c, d any two objects of C and the horizontal map induced by a morphism

$c \to d$ in C, and the forgetful functor sends the above commutative triangle to the horizontal map $\mathfrak{F}(\mathsf{C}(?, c)) \to \mathfrak{F}(\mathsf{C}(?, d))$; (2) $2^{\aleph_0}$ is changed to an infinite cardinal which is larger than the cardinality of the underlying set of any object in the site C. S. Isaacson had essentially the same proof at the same time. Moreover, P. Johnstone already has the same result in [Jo].

Remark 2.1.5. $\mathfrak{Top}$ is not locally presentable, since not every topological space is small. For example, the Sierpinski space and the indiscrete space on two points are not small (see [Ho]).

Remark 2.1.6. Let X be a diffeological space, and let $\hat{\mathsf{A}}$ be a full subcategory of DS/X whose objects forms a generating set of X and contains $\mathfrak{Diff}(\mathbb{R}^0, X)$. Then by the same proof as the above theorem, we can see that the colimit of the composition of functors $\hat{\mathsf{A}} \hookrightarrow \mathsf{DS}/X \to \mathsf{DS} \to \mathfrak{Diff}$ is X.

Corollary 2.1.7. *A topological space is Δ-generated if and only if it is a colimit of second countable locally path-connected topological spaces.*

Proof. ($\Leftarrow$) This follows from Proposition 1.3.18 and the facts that $D : \mathfrak{Diff} \to \mathfrak{Top}$ is a left adjoint, and a topological space X is Δ-generated if and only if $D(T(X)) = X$.

($\Rightarrow$) By the previous remark and the above theorem, $T(X)$ is the colimit of the composition (denoted by F) of functors $\hat{\mathsf{A}} \hookrightarrow \mathsf{DS}/T(X) \to \mathsf{DS} \to \mathfrak{Diff}$, with $\mathrm{Obj}(\hat{\mathsf{A}}) = \cup_{n=0}^{\infty} \mathfrak{Top}(\mathbb{R}^n, X)$. Then $X = D(T(X)) = D(\operatorname{colim} F) = \operatorname{colim}(D \circ F)$. $\square$

2.2 An adjoint functor theorem

Here is a general theorem from [MM]:

Theorem 2.2.1. *Given a small category* C*, a cocomplete category* D*, and a functor* $F : \mathsf{C} \to \mathsf{D}$*, there is an adjoint pair* $L : \mathfrak{P}re(\mathsf{C}) \rightleftharpoons \mathsf{D} : R$ *with* $R(d)(c) = \mathsf{D}(F(c), d)$ *and* $L(X) = \operatorname{colim}_{\mathcal{C}(\star,c)\to X} F(c)$*, where* c *is any object in* C *and* d *is any object in* D*.*

If we take C to be the simplicial category Δ (see Example A.2.44), then the above theorem says that, given a *cosimplicial object* in a cocomplete category D (that is, a functor $\Delta \to \mathsf{D}$), we get an adjoint pair $\mathfrak{s}\mathrm{Set} \rightleftharpoons \mathsf{D}$.

Here are three standard examples from [MM] and [GJ]. Note that the second and the third examples are not used anywhere else in this chapter.

Example 2.2.2. If we take $F : \Delta \to \mathfrak{T}\mathrm{op}$, sending $\underline{n}$ to $|\Delta^n| = \{(x_0, x_1, \cdots, x_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^n x_i = 1, x_i \geq 0$ for all $i \in \{0, 1, \cdots, n\}\}$ with the sub-topology of $\mathbb{R}^{n+1}$, then we get the usual adjoint pair $|\star| : \mathfrak{s}\mathrm{Set} \rightleftharpoons \mathfrak{T}\mathrm{op} : s$.

Example 2.2.3. If we take $F : \Delta \to \mathfrak{C}\mathrm{at}$, sending $\underline{n}$ to the poset $\{0 < 1 < \cdots < n\}$, where $\mathfrak{C}\mathrm{at}$ is the category of all small categories with functors, then we get an adjoint pair $L : \mathfrak{s}\mathrm{Set} \rightleftharpoons \mathfrak{C}\mathrm{at} : B$. The right adjoint B is called the *classifying space functor* with $B(\mathsf{C}) = \mathfrak{C}\mathrm{at}(F(\underline{n}), \mathsf{C})$.

Hence the functor B commutes with products, which implies that B sends a natural transformation to a naive simplicial homotopy. Therefore, B sends every adjoint pair (in particular, an equivalence between two categories) to a naive simplicial homotopy equivalence. Also B sends any small category with initial or terminal object to a contractible simplicial set.

Example 2.2.4. If we define $sd : \Delta \to \mathfrak{s}\mathrm{Set}$ by $sd(\underline{n}) = B(P'\underline{n})$, where $P'\underline{n}$ is the power set of the set $\{0, 1, \cdots, n\}$ without the empty set, ordered by inclusion, and

$B : \mathfrak{Cat} \to \mathfrak{s}\mathrm{Set}$ is the classifying space functor, then we get an adjoint pair $Sd : \mathfrak{s}\mathrm{Set} \rightleftharpoons \mathfrak{s}\mathrm{Set} : Ex$. More precisely, for any simplicial sets X, Y, $Sd(X) = \mathrm{colim}_{\Delta^n \to X}\, sd(\underline{n})$ and $(Ex(Y))_n = \mathfrak{s}\mathrm{Set}(sd(\underline{n}), Y)$.

Here is the most important example for the rest of the chapter:

Example 2.2.5. We write $\mathbb{A}^n = \{(x_0, x_1, \cdots, x_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^n x_i = 1\}$ with the sub-diffeology. It is diffeomorphic to $\mathbb{R}^n$, by forgetting the first coordinate, for example. As for the standard cosimplicial object in $\mathfrak{Top}$, $\mathbb{A}^\bullet$ is a cosimplicial object in $\mathfrak{Diff}$. Hence, we get an adjoint pair $|\star|_{\tilde{D}} : \mathfrak{s}\mathrm{Set} \rightleftharpoons \mathfrak{Diff} : S_{\tilde{D}}$. We call $|\star|_{\tilde{D}}$ the *(diffeological) realization functor* and $S_{\tilde{D}}$ the *(smooth) singular functor*.

More precisely, $S_{\tilde{D}}(X)_n = \mathfrak{Diff}(\mathbb{A}^n, X) = \mathfrak{Diff}(\mathbb{R}^n, X)$ and $|A|_{\tilde{D}} = \mathrm{colim}_{\Delta^n \to A}\, \mathbb{A}^n = \mathrm{colim}_{\Delta^n \to A}\, \mathbb{R}^n =$ coequalizer of $\coprod_{\phi : \underline{n} \to \underline{m} \in \mathrm{Mor}(\Delta)} \mathbb{R}^n \times A_m \underset{\phi_*}{\overset{\phi^*}{\rightrightarrows}} \coprod_{n \in \mathbb{N}} \mathbb{R}^n \times A_n$, with $\phi^* = 1 \times \phi^* : \mathbb{R}^n \times A_m \to \mathbb{R}^n \times A_n$, and $\phi_* = \phi_* \times 1 : \mathbb{R}^n \times A^m \to \mathbb{R}^m \times A^m$. In other words, on $\coprod_{n \in \mathbb{N}} \mathbb{R}^n \times A_n$, we can define an equivalence relation generated by $\mathbb{R}^n \times A_n \ni (a, x) \sim (a', x') \in \mathbb{R}^m \times A_m$ if there is a morphism $f : \underline{n} \to \underline{m}$ in Δ such that $f_*(a) = a'$ and $f^*(x') = x$, where $f_* : \mathbb{R}^n \to \mathbb{R}^m$ and $f^* : A_m \to A_n$ are induced from f. And $|A|_{\tilde{D}} = \coprod_{n \in \mathbb{N}} \mathbb{R}^n \times A_n / \sim$ has the quotient diffeology.

2.3 A model category structure on $\mathfrak{P}\mathrm{re}(\mathsf{DS})$

Let C be a site, and let $\tilde{F} : \Delta \to \mathsf{C}$ be a cosimplicial object in C. Composing with the Yoneda embedding $\mathsf{C} \to \mathfrak{P}\mathrm{re}(\mathsf{C})$, we get a cosimplicial object $F : \Delta \to \mathfrak{P}\mathrm{re}(\mathsf{C})$ in $\mathfrak{P}\mathrm{re}(\mathsf{C})$. By the adjoint functor theorem in the previous section, we get an adjoint pair: $L : \mathfrak{s}\mathrm{Set} \rightleftharpoons \mathfrak{P}\mathrm{re}(\mathsf{C}) : R$.

Theorem 2.3.1. *We can lift the model category structure of $\mathfrak{s}$Set to $\mathfrak{P}re(\mathsf{C})$ through the above adjoint pair $L : \mathfrak{s}Set \rightleftharpoons \mathfrak{P}re(\mathsf{C}) : R$ if $R(L(\Lambda^n_k)) \to R(L(\Delta^n))$ is a trivial cofibration in $\mathfrak{s}$Set for any $n \in \mathbb{Z}^+$ and $0 \leq k \leq n$.*

Proof. We are going to use Kan's theorem (Theorem A.2.36) to prove this:

[AR] says $\mathfrak{P}\mathrm{re}(\mathsf{C})$ is locally presentable. Also, $\mathfrak{P}\mathrm{re}(\mathsf{C})$ is both complete and cocomplete. So condition (1) of Kan's theorem holds.

Condition (2) of Kan's theorem holds, since

(i) By assumption, R sends LJ to trivial cofibrations in $\mathfrak{s}$Set.

(ii) the colimits in any set-valued presheaf category are taken as sectionwise colimits in $\mathfrak{S}$et, which implies that the functor R commutes with colimits, in particular, with pushouts and transfinite compositions. Because any pushout along a trivial cofibration in $\mathfrak{s}$Set is again a trivial cofibration, and any transfinite composition of weak equivalences in $\mathfrak{s}$Set is again a weak equivalence, R sends relative LJ-cell complexes to weak equivalences in $\mathfrak{s}$Set. □

Let $F = \mathbb{A}^\bullet : \Delta \to \mathfrak{D}$iff be the cosimplicial object defined in Example 2.2.5. Since $\mathbb{A}^\bullet \cong \mathbb{R}^\bullet$ as cosimplicial objects, we can also view F as a cosimplicial object in DS. As in the above setting, we get an adjoint pair $|?|_D : \mathfrak{s}\mathrm{Set} \rightleftharpoons \mathfrak{P}\mathrm{re}(\mathsf{DS}) : S_D$.

We call a presheaf X over DS *contractible* if there is a presheaf map $H : X \times$

$|\Delta^1|_D \to X$ such that the following diagram is commutative:

$$\begin{tikzcd}
X \times |\Delta^0|_D \arrow[d, "1\times|d_0|_D"'] \arrow[dr, "c"] & \\
X \times |\Delta^1|_D \arrow[r, "H"] & X \\
X \times |\Delta^0|_D, \arrow[u, "1\times|d_1|_D"] \arrow[ur, "p"'] &
\end{tikzcd}$$

where c is the constant map and p is the projection onto the first coordinate.

Lemma 2.3.2. *Let $f : X \to Y$ be a presheaf map between contractible presheaves over* DS. *Then $S_D(f) : S_D(X) \to S_D(Y)$ is a weak equivalence in $\mathfrak{s}$Set.*

Proof. By the two out of three property of weak equivalences (see Definition A.2.4 and Example A.2.54), it is enough to prove that $S_D(X) \to \Delta^0$ is a weak equivalence. And it is enough to prove that $S_D(X)$ is contractible in $\mathfrak{s}$Set.

Since S_D is a right adjoint, we have the following commutative diagram

$$\begin{tikzcd}
S_D(X) \times \Delta^0 \arrow[r, "1\times\eta_{\Delta^0}"] \arrow[d, "1\times d_0"'] & S_D(X) \times S_D(|\Delta^0|_D) \arrow[d, "1\times S_D|d_0|_D"'] \arrow[dr, "S_D c"] & \\
S_D(X) \times \Delta^1 \arrow[r, "1\times\eta_{\Delta^1}"] & S_D(X) \times S_D(|\Delta^1|_D) \arrow[r, "S_D H"] & S_D(X) \\
S_D(X) \times \Delta^0 \arrow[r, "1\times\eta_{\Delta^0}"] \arrow[u, "1\times d_1"] & S_D(X) \times S_D(|\Delta^0|_D), \arrow[u, "1\times S_D|d_1|_D"] \arrow[ur, "p"'] &
\end{tikzcd}$$

where η is the unit of the adjunction $|-|_D : \mathfrak{s}\mathrm{Set} \rightleftharpoons \mathfrak{P}\mathrm{re}(\mathsf{DS}) : S_D$. Hence $S_D(X)$ is contractible. □

Corollary 2.3.3. *We can lift the model category structure of $\mathfrak{s}$Set to $\mathfrak{P}$re(DS) through the adjoint pair $|-|_D : \mathfrak{s}Set \rightleftharpoons \mathfrak{P}re(\mathsf{DS}) : S_D$.*

Proof. $S_D(|\Lambda^n_k|_D) \to S_D(|\Delta^n|_D)$ is a trivial cofibration in $\mathfrak{s}$Set. It is a cofibration since it is clearly injective sectionwise. It is a weak equivalence by the above lemma since both $|\Lambda^n_k|_D$ and $|\Delta^n|_D$ are contractible in $\mathfrak{P}$re(DS). □

Remark 2.3.4. $\mathsf{j}\Lambda^n_k\mathsf{j}_D$ is concrete, $\mathfrak{F}(\mathsf{j}\Lambda^n_k\mathsf{j}_D) = \mathsf{j}\Lambda^n_k\mathsf{j}_{\tilde{D}}$, and there is a canonical presheaf map $\mathsf{j}\Lambda^n_k\mathsf{j}_D\ !\ \ \mathfrak{F}(\mathsf{j}\Lambda^n_k\mathsf{j}_D)$ which is the identity for the underlying sets.

2.4 Definitions of weak equivalences and (co)fibrations in $\mathfrak{D}$iff

Recall that in Example 2.2.5, we have the adjoint pair $|?|_{\tilde{D}} : \mathfrak{s}\mathrm{Set} \rightleftharpoons \mathfrak{D}\mathrm{iff} : S_{\tilde{D}}$. Let's make some conventions here:

Since Λ^n_k is the coequalizer of $\coprod_{0\leq i<j\leq n, i,j\neq k} \Delta^{n-2} \rightrightarrows \coprod_{0\leq i\leq n, i\neq k} \Delta^{n-1}$ in $\mathfrak{s}$Set, $|\Lambda^n_k|_{\tilde{D}}$ is the coequalizer of $\coprod_{0\leq i<j\leq n, i,j\neq k} \mathbb{A}^{n-2} \rightrightarrows \coprod_{0\leq i\leq n, i\neq k} \mathbb{A}^{n-1}$, for the (diffeological) realization is a left adjoint. It is easy to see that all $|\Lambda^n_k|_{\tilde{D}}$'s are diffeomorphic to $\Lambda^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i = 0 \text{ for some } 1 \leq i \leq n\}$ with the coequalizer diffeology from $\coprod_{i=1}^{\binom{n}{2}} \mathbb{R}^{n-2} \rightrightarrows \coprod_{i=1}^{n} \mathbb{R}^{n-1}$ if $n \geq 2$, and to $\Lambda^1 = \mathbb{R}^0$ if $n = 1$.

Since $\partial\Delta^n$ is the coequalizer of $\coprod_{0\leq i<j\leq n} \Delta^{n-2} \rightrightarrows \coprod_{0\leq i\leq n} \Delta^{n-1}$ in $\mathfrak{s}$Set, $|\partial\Delta^n|_{\tilde{D}}$ is the coequalizer of $\coprod_{0\leq i<j\leq n} \mathbb{A}^{n-2} \rightrightarrows \coprod_{0\leq i\leq n} \mathbb{A}^{n-1}$, for the (diffeological) realization is a left adjoint. It is easy to see that it is diffeomorphic to $\partial\mathbb{A}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i = 0 \text{ for some } i \in \{1, 2, \ldots, n\} \text{ or } \sum_{i=1}^n x_i = 1\} = \Lambda^n \cup \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid \sum_{i=1}^n x_i = 1\}$ with the coequalizer diffeology if $n \geq 2$, and to $\partial\mathbb{A}$, the coproduct of two copies of $\mathbb{R}^0$, if $n = 1$.

Remark 2.4.1. As explained in Example 1.1.10, Λ^n and $\partial\mathbb{A}^n$ are not sub-diffeological spaces of $\mathbb{R}^n$ for any $n \geq 2$. If we denote the sub-diffeological spaces of $\mathbb{R}^n$ with the same underlying sets as Λ^n and $\partial\mathbb{A}^n$ by Λ^n_{sub} and $\partial\mathbb{A}^n_{sub}$, respectively, then we have smooth maps $\Lambda^n \to \Lambda^n_{sub}$ and $\partial\mathbb{A}^n \to \partial\mathbb{A}^n_{sub}$ which are both identities on the underlying sets.

Definition 2.4.2. We define a morphism $X \to Y$ in $\mathfrak{D}$iff to be a *weak equivalence* (or *fibration*) if $S_{\tilde{D}}X \to S_{\tilde{D}}Y$ is a weak equivalence (or fibration) in $\mathfrak{s}$Set. We define a morphism $X \to Y$ in $\mathfrak{D}$iff to be a *cofibration* if it has the left lifting property with respect to all trivial fibrations.

The following proposition is only used in property (2) of the next section.

Proposition 2.4.3. *Let* $\mathbb{R}^\bullet \in \mathsf{DS}^\Delta$ *be defined as above. Then* $\mathsf{DS}^\Delta(\mathbb{R}^\bullet, \mathbb{R}^\bullet) \cong \{g \in \mathfrak{D}\mathit{iff}(\mathbb{R}, \mathbb{R}) \mid g(0) = 0, g(1) = 1\}$.

Proof. Let $f \in \mathsf{DS}^\Delta(\mathbb{R}^\bullet, \mathbb{R}^\bullet)$. We write $f_n : \mathbb{R}^n \to \mathbb{R}^n$ in coordinates by sending $(x_1, \cdots, x_n)$ to $(f_{n1}(x_1, \cdots, x_n), \cdots, f_{nn}(x_1, \cdots, x_n))$. Clearly $id = f_0 : \mathbb{R}^0 \to \mathbb{R}^0$, and $f_1 \circ d^i = d^i \circ f_0$ for $i = 0, 1$ implies that $f_1(0) = 0$ and $f_1(1) = 1$.

In fact f is determined by f_1 for the following reason. Assume that f_{n-1} is determined by f_1 for some $n \geq 2$. Then $s^i \circ f_n = f_{n-1} \circ s^i$ for $i \in \{0, 1\}$ means that f_n is determined by f_{n-1}, hence determined by f_1. Moreover, we can calculate that $f_{ni}(x_1, \cdots, x_n) = f_1(x_i + \cdots + x_n) - f_1(x_{i+1} + \cdots + x_n)$ for any $i \in \{1, 2, \cdots, n\}$.

Moreover, for any $g \in \mathfrak{D}\mathrm{iff}(\mathbb{R}, \mathbb{R})$ with $g(0) = 0$ and $g(1) = 1$, we can define $f : \mathbb{R}^\bullet \to \mathbb{R}^\bullet$ by $f_{ni}(x_1, \cdots, x_n) = g(x_i + \cdots + x_n) - g(x_{i+1} + \cdots + x_n)$. By direct calculation, we can see that $f \in \mathsf{DS}^\Delta(\mathbb{R}^\bullet, \mathbb{R}^\bullet)$. □

In fact, $\mathsf{DS}^\Delta(\mathbb{R}^\bullet, \mathbb{R}^\bullet)$ is a (non-commutative) monoid under composition, and for any diffeological space X, $\mathsf{DS}^\Delta(\mathbb{R}^\bullet, \mathbb{R}^\bullet)$ naturally acts on the simplicial set $S_{\tilde{D}}X$.

2.5 Properties of the functors $|\star|_{\tilde{D}}$ and $S_{\tilde{D}}$

Now let's explore some properties of the two functors $|\star|_{\tilde{D}} : \mathfrak{s}\mathrm{Set} \to \mathfrak{D}\mathrm{iff}$ and $S_{\tilde{D}} : \mathfrak{D}\mathrm{iff} \to \mathfrak{s}\mathrm{Set}$:

(1) $S_{\tilde{D}}$ is faithful, since if $f, g : X \to Y$ in $\mathfrak{D}\mathrm{iff}$ induces $Sf = Sg : SX \to SY$ in $\mathfrak{s}\mathrm{Set}$, then $f = (Sf)_0 = (Sg)_0 = g : X = (SX)_0 \to (SY)_0 = Y$.

(2) Proposition 2.4.3 implies that the functor $S_{\tilde{D}}$ is not full. For example, let X be the set $\{0, 1\}$ with the indiscrete diffeology. Then any non-identity map $f \in \mathcal{DS}^{\Delta}(\mathbb{R}^\bullet, \mathbb{R}^\bullet)$ induces a non-identity simplicial map $S_{\tilde{D}}X \to S_{\tilde{D}}X$, but $(S_{\tilde{D}}X)_0 \to (S_{\tilde{D}}X)_0$ is the identity $X \to X$.

(3) $S_{\tilde{D}}$ is not essentially surjective, that is, there exists a simplicial set X such that there is no diffeological space Y with $X \cong S_{\tilde{D}}Y$. For example, let A be the set $\mathbb{R}$ with the diffeology $\mathcal{D}_A$ generated by $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x^2$. Let $X = sk_1(S_{\tilde{D}}A)$, that is, the sub-simplicial set of $S_{\tilde{D}}A$ generated by $(S_{\tilde{D}}A)_0$ and $(S_{\tilde{D}}A)_1$. If $X \cong S_{\tilde{D}}Y$ for some diffeological space $(Y, \mathcal{D}_Y)$, then as a set $Y \cong \mathbb{R}$, and $\mathcal{D}_A \subseteq \mathcal{D}_Y$ since $f \in \mathcal{D}_Y$. However $(S_{\tilde{D}}Y)_2 = X_2 \subseteq (S_{\tilde{D}}A)_2$. The inclusion is proper since $\mathbb{R}^2 \to \mathbb{R}$ given by $(x, y) \mapsto x^2y^2$ factors through f, while it does not locally factor through f by s^0_* or s^1_*.

Note that if X is a non-discrete diffeological space with $card(X) < \infty$, then there exists a plot $U \to X$ which is not locally constant. Let $n = \dim(U)$. Then $(S_{\tilde{D}}(X))_n$ is an infinite set. Hence $(S_{\tilde{D}}(X))_m$ is an infinite set for any $m \in \mathbb{Z}^+$. This implies that if A is a simplicial set which is not a coproduct of finitely many Δ^0's, and there exists some $n \in \mathbb{Z}^+$ such that A_n is a finite set, then there is no diffeological space X with the property that $A \cong S_{\tilde{D}}(X)$.

(4) $S_{\tilde{D}}$ reflects isomorphisms. Assume that $f : X \to Y$ in $\mathfrak{D}\mathrm{iff}$ induces an isomorphism $S_{\tilde{D}}f : S_{\tilde{D}}X \to S_{\tilde{D}}Y$ in $\mathfrak{s}\mathrm{Set}$. Write $g : S_{\tilde{D}}Y \to S_{\tilde{D}}X$ for the inverse of $S_{\tilde{D}}f$. Then $f \circ g_0 = id_Y$ and $g_0 \circ f = id_X$ as set maps. In fact, $g_0 : Y \to X$ is smooth.

Let $l : U \to Y$ be a plot of Y, and let $U = \cup_{i \in I} U_i$ with each U_i diffeomorphic to $\mathbb{R}^n$ via f_i, where $n = \dim(U)$. Write h for the composition $\mathbb{R}^n \xrightarrow{(f_i)^{-1}} U_i \hookrightarrow U \xrightarrow{l} Y$. We have the following commutative diagram

$$\begin{tikzcd}
\mathbb{R}^n \arrow[r, "g_n(h)"] \arrow[dr, "h"'] & X \arrow[d, "f"] \\
& Y \arrow[d, "g_0"] \\
& X.
\end{tikzcd}$$

Note the composition of the two vertical maps is identity, which implies $g_0 \circ h$ is a plot of X. Therefore, by the sheaf condition, $g_0 \circ l$ is a plot of X, which implies the smoothness of g_0.

(5) We have the following lemma connecting naive smooth homotopy and simplicial homotopy:

Lemma 2.5.1. *$S_{\tilde{D}} : \mathfrak{D}iff \to \mathfrak{s}Set$ sends smoothly homotopic maps to simplicially homotopic maps. Hence smoothly homotopic maps induce the same map on simplicial homotopy groups.*

Proof. $f, g : X \to Y$ being smoothly homotopic (see Definition 1.4.10) means that there exists a stationary path in $\mathfrak{D}\mathrm{iff}(X, Y)$ connecting f and g. By cartesian closedness of $\mathfrak{D}\mathrm{iff}$, we have the following commutative diagram in $\mathfrak{D}\mathrm{iff}$:

$$\begin{tikzcd}
X \times \{0\} \arrow[d] \arrow[dr, "f"] & \\
X \times \mathbb{R} \arrow[r] & Y \\
X \times \{1\}. \arrow[u] \arrow[ur, "g"'] &
\end{tikzcd}$$

Since $S_{\tilde{D}}$ is a right adjoint, we have the following commutative diagram in $\mathfrak{s}$Set

$$\begin{tikzcd}
 & S_{\tilde{D}}X \times \Delta^0 \arrow[dl] \arrow[d] \arrow[dr, "S_{\tilde{D}}f"] & \\
S_{\tilde{D}}X \times \Delta^1 \arrow[r, "1\times\nu"] & S_{\tilde{D}}X \times S_{\tilde{D}}\mathbb{R} \arrow[r] & S_{\tilde{D}}Y \\
 & S_{\tilde{D}}X \times \Delta^0, \arrow[ul] \arrow[u] \arrow[ur, "S_{\tilde{D}}g"'] &
\end{tikzcd}$$

where ν corresponds to the identity map $\mathbb{R} \to \mathbb{R}$ by the Yoneda lemma. $\square$

We call a diffeological space X *smoothly contractible*, if the identity map $X \to X$ is smoothly homotopic to a constant map $X \to X$.

Therefore, if a diffeological space X is smoothly contractible, then $X \to \mathbb{R}^0$ is a weak equivalence. In particular, the map $\Lambda^n \to \Lambda^n_{sub}$ introduced in Remark 2.4.1 is a weak equivalence.

(6) Here is a very unpleasant result:

Proposition 2.5.2. *The functor* $|\ast|_{\tilde{D}} : \mathfrak{s}Set \to \mathfrak{Diff}$ *does not commute with finite products.*

Proof. For simplicial sets X and Y, we have a natural map $|X \times Y|_{\tilde{D}} \to |X|_{\tilde{D}} \times |Y|_{\tilde{D}}$ induced from the projections. However, it is not always a diffeomorphism. For example, it is easy to see that $|\Delta^1 \times \Delta^1|_{\tilde{D}}$ is the pushout of

$$\begin{tikzcd}
|\Delta^1|_{\tilde{D}} \arrow[r, "|d^0_*|"] \arrow[d, "|d^2_*|"'] & |\Delta^2|_{\tilde{D}}, \\
|\Delta^2|_{\tilde{D}} &
\end{tikzcd}$$

hence not diffeomorphic to $\mathbb{R}^2 = |\Delta^1|_{\tilde{D}} \times |\Delta^1|_{\tilde{D}}$. (In fact, $|\Delta^1 \times \Delta^1|_{\tilde{D}} \cong \Lambda^2 \times \mathbb{R}$.) $\square$

(7) For any simplicial sets X and Y, the natural smooth map $f : |X \times Y|_{\tilde{D}} \to |X|_{\tilde{D}} \times |Y|_{\tilde{D}}$ induced from the projections is surjective. (The proof of Proposition 2.5.2 shows that this map is not injective in general.)

Before giving the proof, let's understand $|A|_{\tilde D}$ better. By using the coequalizer notation for $|A|_{\tilde D}$, we call a point (x,b) in $\tilde A = \coprod_{n\in\mathbb{N}} \mathbb{A}^n \times A_n$ *nondegenerate* if b is nondegenerate in A and x is *interior*, that is, not of the form $d^i(y)$. We can define $\alpha : \tilde A \to \tilde A$ by $(x,b) \mapsto (y, d_{i_1} \circ \cdots \circ d_{i_t}(b))$ if $x = d^{i_t} \circ \cdots \circ d^{i_1}(y)$ with y interior and $0 \leq i_1 < \cdots < i_t \leq n$, and define $\beta : \tilde A \to \tilde A$ by $(x,b) \mapsto (s^{j_1} \circ \cdots \circ s^{j_l}(x), c)$ if $b = s_{j_l} \circ \cdots \circ s_{j_1}(c)$ with c nondegenerate in A and $0 \leq j_1 < \cdots < j_l < n$. Then $\alpha \circ \beta$ sends each point in $\tilde A$ to a nondegenerate point. If $(x,b) \sim (x',b')$, then $\alpha \circ \beta(x,b) = \alpha \circ \beta(x',b')$. In other words, as a set $|A|_{\tilde D}$ is exactly the set of all nondegenerate points in $\tilde A$.

Here is the proof of the claim. For any nondegenerate representative $(x,a) \in |X|_{\tilde D}$ and $(y,b) \in |Y|_{\tilde D}$, we can construct $[(z, c \times d)] \in |X \times Y|_{\tilde D}$ as follows such that $(x,a) \sim (z,c)$ and $(y,b) \sim (z,d)$. If $x = (x_0, \cdots, x_n)$ and $y = (y_0, \cdots, y_m)$, then define $z = (x_0, \cdots, x_{n-1}, z_n, y_1, \cdots, y_m)$ with $z_n = x_n - \sum_{j=1}^m y_j = y_0 - \sum_{i=0}^{n-1} x_i$, and define $c = s_{n-1} \circ \cdots \circ s_0(a)$ and $d = s_{n+m-1} \circ \cdots \circ s_n(b)$. This defines a set map $g : |X|_{\tilde D} \times |Y|_{\tilde D} \to |X \times Y|_{\tilde D}$ such that $f \circ g = 1$.

(8) For every simplicial set A, the unit $\eta : A \to S_{\tilde D}|A|_{\tilde D}$ is a cofibration in $\mathfrak{s}$Set. For every diffeological space X, the counit $\epsilon : |S_{\tilde D}(X)|_{\tilde D} \to X$ is a subduction (see Definition 1.5.4).

Here is the proof. Let $a, b \in A_n$ such that $\eta_n(a) = \eta_n(b) : \mathbb{A}^n \to |A|_{\tilde D}$. This means that $[(x,a)] = \eta_n(a)(x) = \eta_n(b)(x) = [(x,b)]$ for any $x \in \mathbb{A}^n$. We can pick $x = (\frac{1}{n+1}, \cdots, \frac{1}{n+1})$. Then $a = s_{j_k} \circ \cdots \circ s_{j_1}(a')$ with a' nondegenerate in A and $0 \leq j_1 < \cdots < j_k < n$ implies $\alpha \circ \beta(x,a) = (s^{j_1} \circ \cdots \circ s^{j_k}(x), a')$. Hence, $a = b$.

Since $\epsilon([(x,a)]) = a(x)$, by concreteness, we know that ϵ is surjective. Since each open set U of $\mathbb{R}^n$ is locally diffeomorphic to $\mathbb{R}^n$, it is clear that ϵ is a subduction.

(9) Let's compare the three adjoint pairs $|?|_{\tilde D} : \mathfrak{s}\mathrm{Set} \rightleftharpoons \mathfrak{Diff}: S_{\tilde D}$, $D : \mathfrak{Diff} \rightleftharpoons \mathfrak{Top}: T$ and $|?| : \mathfrak{s}\mathrm{Set} \rightleftharpoons \mathfrak{Top}: s$.

Proposition 2.5.3. *Given any topological space A, there is a weak equivalence between $S_{\tilde{D}}(T(A))$ and sA in $\mathfrak{s}Set$.*

Proof. For any topological space A, $S_{\tilde{D}}(T(A)) = \mathfrak{D}\mathrm{iff}(\mathbb{A}^\bullet, T(A)) = \mathfrak{T}\mathrm{op}(D(\mathbb{A}^\bullet), A)$, and $sA = \mathfrak{T}\mathrm{op}(|\Delta^\bullet|, A)$.

Consider the Reedy model structure on $\mathfrak{T}\mathrm{op}^\Delta$. See the appendix for more details.

Note that every topological space is fibrant in the standard model category structure of $\mathfrak{T}\mathrm{op}$, both $D(\mathbb{A}^\bullet)$ and $|\Delta^\bullet|$ are cosimplicial resolutions of a point in $\mathfrak{T}\mathrm{op}$, and the natural inclusion map $i : |\Delta^\bullet| \to D(\mathbb{A}^\bullet)$ is a Reedy weak equivalence in $\mathfrak{T}\mathrm{op}^\Delta$ (since for any $n \in \mathbb{N}$, both $|\Delta^n|$ and $D(\mathbb{A}^n)$ are contractible). Therefore, $i^* : S_{\tilde{D}}(T(A)) \to sA$ is a weak equivalence of fibrant simplicial sets. □

Remark 2.5.4. I'd like to thank D. Dugger for the idea of the proof of this proposition.

Also we have the following:

Proposition 2.5.5. *Given any simplicial set X, there is a weak equivalence between $D(|X|_{\tilde{D}})$ and $|X|$ in $\mathfrak{T}op$.*

Proof. This follows from the fact that $i : |\Delta^\bullet| \to D(\mathbb{A}^\bullet)$ has a retract in $\mathfrak{T}\mathrm{op}^\Delta$, and $* \times I : \mathfrak{T}\mathrm{op} \to \mathfrak{T}\mathrm{op}$ is a left adjoint. □

2.6 Examples of cofibrations and fibrations in $\mathfrak{Diff}$

By the adjunction $|\star|_{\tilde{D}} : \mathfrak{s}\mathrm{Set} \rightleftharpoons \mathfrak{Diff} : S_{\tilde{D}}$ and Definition 2.4.2, $X \to Y$ is a fibration in $\mathfrak{Diff}$ if and only if it has the right lifting property with respect to $\Lambda^n \to \mathbb{R}^n$ for all $n \in \mathbb{Z}^+$, and $X \to Y$ is a trivial fibration in $\mathfrak{Diff}$ if and only if it has the right lifting property with respect to $\partial\mathbb{A}^n \to \mathbb{R}^n$ for all $n \in \mathbb{N}$. In particular, taking $n = 0$, we see that all trivial fibrations are surjective.

Also, if a smooth map $f : A \to B$ is a (diffeological) realization of a trivial cofibration in $\mathfrak{s}$Set, and $g : X \to Y$ is a fibration in $\mathfrak{Diff}$, then any commutative solid diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & X \arrow[d, "g"] \\ B \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

in $\mathfrak{Diff}$ has a smooth lift.

Proposition 2.6.1. $|\star|_{\tilde{D}} : \mathfrak{s}Set \to \mathfrak{Diff}$ *preserves cofibrations. The class of cofibrations in* $\mathfrak{Diff}$ *is closed under isomorphisms, pushouts, smooth retracts and (transfinite) compositions.*

Proof. This is formal. □

Proposition 2.6.2. *Every smooth map f in $\mathfrak{Diff}$ has a functorial factorization as $f = \alpha(f) \circ \beta(f)$ with $\alpha(f)$ a trivial fibration and $\beta(f)$ a cofibration.*

Proof. Apply the small object argument (Theorem A.2.32) to the set $I = \{\partial\mathbb{A}^n \to \mathbb{R}^n \mid n \in \mathbb{N}\}$. □

Corollary 2.6.3. *We have a functorial cofibrant replacement for every diffeological space.*

Example 2.6.4. $\Lambda^n \to \mathbb{R}^n$ for any $n \in \mathbb{Z}^+$ and $\partial\mathbb{A}^m \to \mathbb{R}^m$ for any $m \in \mathbb{N}$ are all cofibrations.

Example 2.6.5. $\mathbb{R}^n$ is cofibrant for any $n \in \mathbb{N}$, since $\mathbb{R}^n = |\Delta^n|_{\tilde{D}}$.

Example 2.6.6. Λ^2 is cofibrant, since it is the pushout of

$$\begin{tikzcd} \mathbb{R}^0 \arrow[r] \arrow[d] & \mathbb{R} \\ \mathbb{R}. & \end{tikzcd}$$

Example 2.6.7. More generally, all $\Lambda^n = |\Lambda^n_k|_{\tilde{D}}$ and $\partial\mathbb{A}^n = |\partial\Delta^n|_{\tilde{D}}$ are cofibrant. This can also be seen by building them as pushouts along the cofibrations in the previous examples, and along the way, we obtain other interesting cofibrations and cofibrant objects. For example, $\bigvee_{i=1}^n \mathbb{R}$ are all cofibrant.

Example 2.6.8. The pushout of

$$\begin{tikzcd} \partial\mathbb{A} \arrow[r] \arrow[d] & \mathbb{R}^0 \\ \mathbb{R} & \end{tikzcd}$$

will be denoted by $\hat{S}^1$, and it is cofibrant.

Clearly $\hat{S}^1$ is not diffeomorphic to S^1, because $\hat{S}^1$ has tails. But even the sub-diffeological space of $\hat{S}^1$ with the tails removed is not diffeomorphic to S^1, because of the point where the gluing occurs.

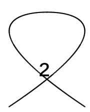


Figure 2.1: $\hat{S}^1$

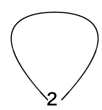
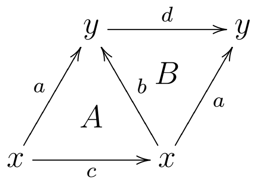


Figure 2.2: $\hat{S}^1$ with tails removed

We have the following amazing result:

Theorem 2.6.9. *S^1 is cofibrant.*

Proof. Let X be the simplicial set whose non-degenerate simplices are:



Then $|X|_{\tilde{D}}$ is cofibrant, and it is diffeomorphic to the union of the xy-plane and yz-plane in $\mathbb{R}^3$ modulo the following relation: $(\sqrt{3}y, y-\frac{1}{2}, 0) \sim (0, y, \frac{\sqrt{3}}{2}-\sqrt{3}y)$. We are going to prove that S^1 is a smooth retract of $|X|_{\tilde{D}}$. In other words, we are going to define smooth maps $f : S^1 \to |X|_{\tilde{D}}$ and $g : |X|_{\tilde{D}} \to S^1$ such that $g \circ f = id_{S^1}$. In order that $\mathrm{Im}(f)$ (with the sub-diffeology of $|X|_{\tilde{D}}$) and S^1 have the same smooth structure, $\mathrm{Im}(f)$ has to tangentially approach the lines along which the gluing occurs. These lines are the y-axis, the line $y = \frac{x}{\sqrt{3}} - \frac{1}{2}$ in the xy-plane, and the line $z = -\sqrt{3}y + \frac{\sqrt{3}}{2}$ in the yz-plane.

By modifying the smooth function $\mathbb{R} \to \mathbb{R}^2$ (see the graph below) with $t \mapsto (t, (1-\phi(2t))\sqrt{3}t + \phi(2t)(\frac{\sqrt{3}}{2} - \sqrt{3}t))$ for some cut-off function ϕ, we can define:

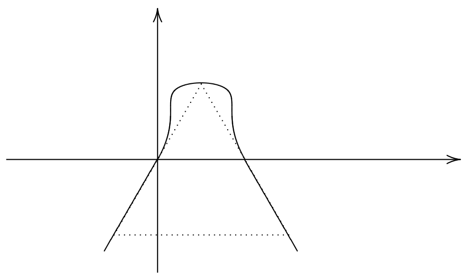


Figure 2.3: The smooth function

$$f(\theta) = \begin{cases} (\frac{\sqrt{3}}{4}(\frac{2\theta}{\pi} - 1)\phi(\frac{\theta}{\pi}), -\frac{\theta}{\pi} + \frac{3}{4}(\frac{2\theta}{\pi} - 1)\phi(\frac{\theta}{\pi}), 0) & \text{if } \theta \in [0, \pi] \\ (0, -\frac{\theta}{\pi} + \frac{3}{4}(\frac{2\theta}{\pi} + 1)\phi(-\frac{\theta}{\pi}), -\frac{\sqrt{3}}{4}(\frac{2\theta}{\pi} + 1)\phi(-\frac{\theta}{\pi})) & \text{if } \theta \in [-\pi, 0]. \end{cases}$$

and

$$g(x, y, z) = \begin{cases} e^{\pi i(\sqrt{3}x - y)}, & \text{if } z = 0 \\ e^{-\pi i(y + \sqrt{3}z)} & \text{if } x = 0. \end{cases}$$

□

Example 2.6.10. Every discrete diffeological space X is both cofibrant (since $X = |\coprod_{x\in X} \Delta^0|_{\tilde{D}}$) and fibrant (since every smooth map $\Lambda^n \to X$ must be constant).

Every indiscrete diffeological space X is fibrant, since every diagram

$$\begin{tikzcd}
\Lambda^n \arrow[r] \arrow[d] & X \\
\mathbb{R}^n &
\end{tikzcd}$$

has a lift in $\mathfrak{D}$iff. Moreover, X is smoothly contractible.

Example 2.6.11. Let $(X_1, \mathcal{D}_1)$, $(X_2, \mathcal{D}_2)$ be two diffeological spaces with the same underlying set X, such that $\mathcal{D}_1 \subseteq \mathcal{D}_2$. Then the identity set map $X_1 \to X_2$ is a fibration if and only if $\mathcal{D}_1 = \mathcal{D}_2$. Indeed, suppose $id : X_1 \to X_2$ is a fibration and let $f : \mathbb{R}^n \to X_2$ be a plot. Then

$$\begin{tikzcd}
\mathbb{R}^0 \arrow[rr, "id^{-1}(f(0))"] \arrow[d, "0"'] & & X_1 \arrow[d, "id"] \\
\Lambda^n \arrow[r] \arrow[urr, dotted, "g"] & \mathbb{R}^n \arrow[r, "f"'] & X_2
\end{tikzcd}$$

has a lift to X_1 in $\mathcal{D}$iff, since $0 : \mathbb{R}^0 \to \Lambda^n$ is a trivial cofibration (it is a cofibration since it is the diffeological realization of $\Delta^0 \to \Lambda^n_k$ for any k, and it is a weak equivalence since Λ^n is smoothly contractible). Hence

$$\begin{tikzcd}
\Lambda^n \arrow[r, "g"] \arrow[d] & X_1 \arrow[d, "id"] \\
\mathbb{R}^n \arrow[r, "f"'] \arrow[ur, dotted] & X_2
\end{tikzcd}$$

has a lift $\mathbb{R}^n \to X_1$ in $\mathcal{D}$iff making the whole diagram commutative, which implies that f is a plot in X_1.

Proposition 2.6.12. *Every diffeological group is fibrant.*

Proof. The right adjoint of an adjoint pair between two categories with finite products always sends group objects to group objects. The group objects in $\mathfrak{D}$iff and in $\mathfrak{s}$Set are precisely diffeological groups and simplicial groups, respectively, and Moore's lemma ([GJ, Lemma I.3.4]) says that every simplicial group is fibrant in $\mathfrak{s}$Set. □

Example 2.6.13. (1) Every Lie group viewed as a diffeological space with the standard diffeology is fibrant.

(2) Every irrational torus is fibrant.

Example 2.6.14. Here is a more concrete way to see that every diffeological abelian group A is fibrant. Given a solid diagram

$$\begin{tikzcd} \Lambda^n \arrow[r, "F"] \arrow[d] & A \\ \mathbb{R}^n \arrow[ur, dotted, "\tilde{F}"'] & \end{tikzcd}$$

in $\mathfrak{D}$iff, define the extension $\tilde{F}$ directly as follows. For any $0 \leq k < n$ and $1 \leq i_1 < \cdots < i_k \leq n$, write $P_{i_1,\cdots,i_k} : \mathbb{R}^n \to \Lambda^n$ for the orthogonal projection onto the coordinate plane $x_{i_1} \cdots x_{i_k}$. When $k = 0$, this is the constant map $\mathbb{R}^n \to \Lambda^n$ sending everything to 0. All of these projections are clearly smooth. Then the smooth map $\tilde{F} = \sum_{k=0}^{n-1} \sum_{1 \leq i_1 < \cdots < i_k \leq n} (-1)^{n-k+1} F \circ P_{i_1,\cdots,i_k}$ is an extension of F.

Example 2.6.15. Let A be a diffeological group. Then $\mathfrak{D}$iff(X, A) is also a diffeological group for any diffeological space X, hence fibrant.

Lemma 2.6.16. *Fibrant diffeological spaces are closed under coproducts in $\mathfrak{D}$iff, and if X is fibrant, then so is each path component.*

Proof. This is because both $D(\Lambda^n)$ and $D(\mathbb{R}^n)$ are connected. □

Proposition 2.6.17 (Right Proper)**.** *Let*

$$\begin{tikzcd} W \arrow[r, "h"] \arrow[d] & X \arrow[d, "f"] \\ Z \arrow[r, "g"'] & Y \end{tikzcd}$$

be a pullback diagram in $\mathfrak{D}$iff with f a fibration and g a weak equivalence. Then h is also a weak equivalence.

Proof. This follows from the right properness of the standard model category structure on $\mathfrak{s}$Set. □

Proposition 2.6.18. *The class of fibrations in $\mathfrak{D}$iff is closed under isomorphisms, pullbacks, smooth retracts and finite compositions.*

Proof. This is formal. □

Corollary 2.6.19. *Let $f : X \to Y$ be a fibration in $\mathfrak{D}$iff, then any fiber of f is fibrant, that is, for any $y \in Y$, $f^{-1}(y)$ with the sub-diffeology of X is fibrant.*

Lemma 2.6.20. *Any diffeological bundle with fibrant fiber is a fibration.*

Proof. Suppose $f : X \to Y$ is a diffeological bundle with fiber F. Given any commutative diagram in $\mathfrak{D}$iff

$$\begin{tikzcd} \Lambda^n \arrow[r, "b"] \arrow[d, "a"'] & X \arrow[d, "f"] \\ \mathbb{R}^n \arrow[r, "c"'] & Y, \end{tikzcd}$$

we have the following pullback diagram in $\mathfrak{D}$iff

$$\begin{tikzcd} \mathbb{R}^n \times F \arrow[r, "d"] \arrow[d, "\pi_1"'] & X \arrow[d, "f"] \\ \mathbb{R}^n \arrow[r, "c"'] & Y. \end{tikzcd}$$

Therefore, we have the following commutative diagram:

$$\begin{tikzcd}
\Lambda^n \arrow[dr, dotted, "{(a,e)}"] \arrow[drr, bend left, "b"] \arrow[ddr, bend right, "a"'] & & \\
& \mathbb{R}^n \times F \arrow[r, "d"] \arrow[d, "\pi_1"'] & X \arrow[d, "f"] \\
& \mathbb{R}^n \arrow[r, "c"'] & Y.
\end{tikzcd}$$

Let $g : \mathbb{R}^n \to F$ be any smooth map and consider the smooth section $(1, g) : \mathbb{R}^n \to \mathbb{R}^n \times F$. Then $f \circ d \circ (1, g) \circ \pi_1 = c \circ \pi_1 \circ (1, g) \circ \pi_1 = c \circ \pi_1$, and by the surjectivity of π_1, we have the following commutative triangle

$$\begin{tikzcd}
& X \arrow[d, "f"] \\
\mathbb{R}^n \arrow[ur, "d\circ(1,g)"] \arrow[r, "c"'] & Y.
\end{tikzcd}$$

We also want the triangle

$$\begin{tikzcd}
\Lambda^n \arrow[r, "b"] \arrow[d, "a"'] & X \\
\mathbb{R}^n, \arrow[ur, "d\circ(1,g)"'] &
\end{tikzcd}$$

to commute, which requires us to pick the smooth map g nicely. Since F is fibrant, we choose g to be a lifting of

$$\begin{tikzcd}
\Lambda^n \arrow[r, "e"] \arrow[d, "a"'] & F \\
\mathbb{R}^n \arrow[ur, dotted, "g"'] &
\end{tikzcd}$$

composed with the inclusion $F \hookrightarrow X$. Then for any $x \in \Lambda^n$, we have $d \circ (1, g) \circ a(x) = d(a(x), g \circ a(x)) = d(a(x), e(x)) = (d \circ (a, e))(x) = b(x)$. □

Example 2.6.21. Not every diffeological space is cofibrant. For example, the irrational torus T^2_θ (see Example 1.1.6) is not cofibrant, since the diffeomorphism $T^2_\theta \to T^2/\mathbb{R}_\theta$ by sending $[x]$ to $[(1, x)]$ has no smooth lift to T^2 by the connectedness of $\mathbb{R}$, and note that the quotient map $T^2 \to T^2/\mathbb{R}_\theta$ is a trivial fibration.

Definition 2.6.22. Let G be a diffeological group and let H be a subgroup of G. Then the set G/H of left cosets, with the quotient diffeology, is called a *homogeneous* diffeological space.

Note that we do not require H to be a closed subgroup of G. We can define a homogeneous diffeological space to be the set of right cosets, and all the corresponding results still hold.

Theorem 2.6.23. *Every homogeneous diffeological space is fibrant.*

Proof. We will give two proofs for this.

Here is the first proof:

By Proposition 7.5 (i) on page 26 of [May], we know that if $f : X \to Y$ is a fibration in $\mathfrak{D}\mathrm{iff}$, X is fibrant and $S_{\tilde{D}}f : S_{\tilde{D}}X \to S_{\tilde{D}}Y$ is onto, then Y is fibrant. Here $\mathfrak{D}\mathrm{iff}(\mathbb{R}^n, X) = (S_{\tilde{D}}X)_n \to (S_{\tilde{D}}Y)_n = \mathfrak{D}\mathrm{iff}(\mathbb{R}^n, Y)$ being onto just means that any global plot of Y has a smooth lift:

$$\begin{tikzcd} & X \arrow[d, "f"] \\ \mathbb{R}^n \arrow[r] \arrow[ur, dotted] & Y. \end{tikzcd}$$

In particular, if $X \to Y$ is a diffeological bundle with fibrant fiber and X is fibrant, then so is Y. Now $G \to G/H$ is a diffeological bundle with both G and H fibrant. Hence the result follows.

Here is the second proof:

Given $b : \Lambda^n \to G/H$, let $a : \mathbb{R}^0 \to G$ have $a(0) \in p^{-1}(b(0, \cdots, 0))$, where

$p : G \to G/H$ is the quotient map. Then we have the following smooth liftings:

$$\begin{tikzcd}
\mathbb{R}^0 \arrow[r, "a"] \arrow[d] & G \arrow[d, "p"] \\
\Lambda^n \arrow[r, "b"] \arrow[d] \arrow[ur, dotted, "\alpha"] & G/H \\
\mathbb{R}^n . \arrow[uur, dotted, "\beta"] \arrow[ur, dotted, "\gamma"'] &
\end{tikzcd}$$

The lifting α exists because $\mathbb{R}^0 \to \Lambda^n$ is a trivial cofibration in $\mathfrak{D}$iff and p is a fibration by Lemma 2.6.20. The lifting β exists because G is fibrant. And $\gamma = p \circ \beta$ is easily seen to be the required lifting. □

Remark 2.6.24. This theorem was suggested by P. Iglesias-Zemmour, and it generalizes the two classes of fibrant diffeological spaces: diffeological groups and smooth manifolds without boundary (see below).

The second proof of the theorem shows that if a smooth map $X \to Y$ is a fibration in $\mathfrak{D}$iff and a surjective set map, with X fibrant, then Y is also fibrant.

Corollary 2.6.25. *Every smooth manifold without boundary is fibrant.*

Proof. We will also give two proofs of this corollary.

Here is the first proof:

Use Theorem 2.6.23 and the fact that $\mathfrak{D}\mathrm{iff}(M)/stab(M, x) \cong M$ [Do], where M is an arbitrary connected smooth manifold and $x \in M$. $\mathfrak{D}\mathrm{iff}(M)$ is introduced in Example 1.7.11 and $stab(M, x) = \{f \in \mathfrak{D}\mathrm{iff}(M) \mid f(x) = x\}$ is a subgroup of $\mathfrak{D}\mathrm{iff}(M)$. Then use Lemma 2.6.16.

Here is a sketch of the second proof:

Let $\{(U_\alpha, f_\alpha)\}$ be an atlas of M, that is, each U_α is open in M, and each f_α :

$U_\alpha \to \mathbb{R}^m$ is a diffeomorphism, where m is the dimension of M. Given any diagram

$$\begin{tikzcd} \Lambda^n \arrow[r, "f"] \arrow[d] & M \\ \mathbb{R}^n, & \end{tikzcd}$$

take one U_α such that it contains $f((0, 0, \cdots, 0))$. Since U_α is diffeomorphic to $\mathbb{R}^m$, and $\mathbb{R}^m$ is fibrant, there is a smooth lift from a small neighborhood (we can think of it as a small open cube) of $(0, \cdots, 0)$ in $\mathbb{R}^n$ to U_α, hence to M. We need to further extend it to the whole $\mathbb{R}^n$. This can be done because $\mathbb{R}^m$ has the following weak right lifting property:

Fix any $\epsilon, \tau > 0$ with $\tau < \epsilon$. Let $A = \{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid x_n > -\epsilon\} \cup$ all the coordinate hyperplanes in $\mathbb{R}^n$. For any smooth map $g : A \to \mathbb{R}^m$, where the smoothness means that the restriction of g to the first part of A and to each coordinate hyperplane is smooth in the usual sense, we have a smooth map $\tilde{g} : \mathbb{R}^n \to \mathbb{R}^m$ such that $\tilde{g}$ restricted to $B = \{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid x_n > -\tau\} \cup$ all the coordinate hyperplanes in $\mathbb{R}^n$ is g.

This is because we also have a smooth extension h respecting all coordinate hyperplanes in $\mathbb{R}^n$ as explained in Example 2.6.14. Let $V_1 = \{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid x_n > -\epsilon\}$ and $V_2 = \{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid x_n < -\tau\}$. Then $\{V_1, V_2\}$ forms an open cover of $\mathbb{R}^n$. So there exists a smooth partition of unity ρ_1, ρ_2 subordinate to this cover. Then $\rho_1 g + \rho_2 h$ is what we want.

We use this weak right lifting property of $\mathbb{R}^m$ repeatedly to get the desired smooth lift. Since $\mathbb{R}^n$ is not compact, we will try to manage on compact sets first, and then go further. For example, we will try on closed cubes, and then make the cubes bigger and bigger, so that eventually we reach the smooth lift for the whole $\mathbb{R}^n$. $\square$

As a direct result of Corollary 2.6.3, if we take a functorial cofibrant replacement

$\tilde{X}$ of a fibrant diffeological space X, then $\tilde{X}$ is both cofibrant and fibrant. In particular by the above corollary, if we take a functorial cofibrant replacement $\tilde{M}$ of a smooth manifold M, then $\tilde{M}$ is both cofibrant and fibrant.

Example 2.6.26. Let X be a topological space. Then $T(X)$ is a fibrant diffeological space, since $D(\Lambda^n) \to D(\mathbb{R}^n)$ has a retract in $\mathfrak{T}op$. However, if Y is a diffeological space, then the natural map $Y \to T(D(Y))$ is not always a weak equivalence in $\mathfrak{D}$iff. $Y = T^2_\theta$, the irrational torus of slope θ, is such an example (see Example 1.7.14).

Proposition 2.6.27. *Let A be a diffeological space which is a smooth retract of the diffeological realization of a simplicial set X, and let B be a fibrant diffeological space. If $i_n \times 1 : \Lambda^n \times |X|_{\tilde{D}} \to \mathbb{R}^n \times |X|_{\tilde{D}}$ is the diffeological realization of a cofibration in $\mathfrak{s}$Set for any $n \in \mathbb{Z}^+$, then $\mathfrak{D}$iff(A, B) is also a fibrant diffeological space.*

Proof. Since the class of fibrant diffeological spaces is closed under smooth retracts, it is enough to prove that $\mathfrak{D}$iff$(|X|_{\tilde{D}}, B)$ is fibrant. By cartesian closedness of $\mathfrak{D}$iff, it is equivalent to showing that $i_n \times 1 : \Lambda^n \times |X|_{\tilde{D}} \to \mathbb{R}^n \times |X|_{\tilde{D}}$ is a trivial cofibration for every $n \in \mathbb{Z}^+$. $i_n \times 1$ is a weak equivalence since i_n is a weak equivalence, and it is a cofibration by the assumption. □

Remark 2.6.28. It is not always true that $\mathbb{R}^n \times |X|_{\tilde{D}}$ is the diffeological realization of a simplicial set. For example, if $n = 1$ and X is the simplicial set whose non-degenerate simplices are

$$\bullet \rightrightarrows \bullet$$

then $\mathbb{R} \times |X|_{\tilde{D}}$ is not the diffeological realization of any simplicial set.

Corollary 2.6.29. *Let X be a fibrant diffeological space. Then $\mathfrak{D}$iff$(\mathbb{R}^m, X)$ is also fibrant for any $m \in \mathbb{N}$.*

Proof. This is because $\mathbb{R}^m = |\Delta^m|_{\tilde{D}}$, and for any $n \in \mathbb{Z}^+$, $\Lambda^n \times \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^m$ is the diffeological realization of the cofibration $\cup_{i=0}^{n-1} d^i(\Delta^{n+m-1}) \hookrightarrow \Delta^{n+m-1}$. □

Not every diffeological space is fibrant:

Example 2.6.30. Λ^n is not fibrant for any $n \geq 2$, since there is no smooth retract $\mathbb{R}^n \to \Lambda^n$ for the natural injective map $\Lambda^n \to \mathbb{R}^n$, using the definition of the coequalizer diffeology on Λ^n.

Λ^n_{sub} is not fibrant for any $n \geq 2$ as well. Otherwise, $i : \Lambda^n_{sub} \to \mathbb{R}^n$ has a smooth retract $f : \mathbb{R}^n \to \Lambda^n_{sub}$. Then the composition $i \circ f : \mathbb{R}^n \to \mathbb{R}^n$ is smooth in the usual sense, and $id = (i \circ f)_* : T_0\mathbb{R}^n \to T_0\mathbb{R}^n$. This implies that $i \circ f$ is a local diffeomorphism at 0 by the inverse function theorem, which is a contradiction.

For the same reasons, neither $\partial\mathbb{A}^n$ nor $\partial\mathbb{A}^n_{sub}$ is fibrant for any $n \geq 2$.

Example 2.6.31. Write $\hat{\mathbb{R}}^n$ for $\mathbb{R}^n$ with the diffeology generated by any set $S \subseteq \mathfrak{Diff}(\mathbb{R}^{n-1}, \mathbb{R}^n)$ which contains all the natural inclusions $\mathbb{R}^{n-1} \to \mathbb{R}^n$ into coordinate hyperplanes. Then $\hat{\mathbb{R}}^n$ is not fibrant for any $n \geq 1$. Otherwise, we have the following commutative diagram in $\mathfrak{Diff}$:

$$\begin{tikzcd}
\Lambda^n \arrow[r, "i"] \arrow[d, "i"'] & \hat{\mathbb{R}}^n \arrow[r] & \mathbb{R}^n \\
\mathbb{R}^n \arrow[ur, dotted, "F"] & & \\
U \arrow[u, hook] \arrow[r, "f"'] \arrow[uur, dotted, "F|_U"] & \mathbb{R}^{n-1}, \arrow[uu, "g"'] &
\end{tikzcd}$$

where $F \in \mathfrak{Diff}(\mathbb{R}^n, \hat{\mathbb{R}}^n)$, U is some neighborhood of $0 \in \mathbb{R}^n$, $f \in \mathfrak{Diff}(U, \mathbb{R}^{n-1})$, $g \in S$, and the unnamed horizontal arrow is the identity on the underlying set of $\mathbb{R}^n$. This implies that we have the following commutative diagram in $Vect_{\mathbb{R}}$:

$$\begin{tikzcd}
 & T_0\mathbb{R}^n \\
T_0U \arrow[ur, "(F|_U)_*"] \arrow[r, "f_*"'] & T_{f(0)}\mathbb{R}^{n-1}. \arrow[u, "g_*"']
\end{tikzcd}$$

Note that $(F|_U)_* = id$, which contradicts the commutativity of the above triangle.

The same method shows that for any $n, m \in \mathbb{N}$ with $n > m$, $\mathbb{R}^n$ with the diffeology generated by any set $S \subseteq \mathfrak{D}\mathrm{iff}(\mathbb{R}^m, \mathbb{R}^n)$ which contains all the natural inclusions $\mathbb{R}^m \to \mathbb{R}^n$ into coordinate m-planes, is not fibrant.

Example 2.6.32. For any pointed diffeological space (X, x), we can construct the path space $P(X, x) = \{f \in \mathfrak{D}\mathrm{iff}(\mathbb{R}, X) \mid f(0) = x\}$, with the sub-diffeology of the functional diffeology of $\mathfrak{D}\mathrm{iff}(\mathbb{R}, X)$. This diffeological space is always smoothly contractible since we have a smooth map $\alpha : P(X, x) \times \mathbb{R} \to P(X, x)$ defined by $\alpha(f, t)(s) = f(\theta(t)s)$, where $\theta : \mathbb{R} \to \mathbb{R}$ is some cut-off function. We also have a natural smooth map $ev_1 : P(X, x) \to X$ defined by $f \mapsto f(1)$. However, ev_1 is not always a fibration in $\mathfrak{D}\mathrm{iff}$. For example, take $X = \Lambda^2$ and $x = (0, 0) \in X$. Here is a modified proof using ideas of Gaohong Wang. We only need to show that the fiber of ev_1 at x, i.e. the loop space $\Omega(X, x) = \{f \in \mathfrak{D}\mathrm{iff}(\mathbb{R}, X) \mid f(0) = f(1) = x\}$ with the sub-diffeology of $P(X, x)$, is not fibrant. We can construct a smooth map $H : X \to \Omega(X, x)$ by $H(x, 0)(t) = (x\psi(t), 0)$ and $H(0, y)(t) = (0, y\psi(t))$, where $\psi : \mathbb{R} \to \mathbb{R}$ is a smooth function such that there exists $\epsilon > 0$ so that $\psi(t) = 0$ when $t < \epsilon$ or $t > 1 - \epsilon$ and $\psi(\frac{1}{2}) = 1$. Then the diagram

$$\begin{tikzcd}
\Lambda^2 \arrow[r, "H"] \arrow[d] & \Omega(X, x) \\
\mathbb{R}^2,
\end{tikzcd}$$

does not have a lift in $\mathfrak{D}\mathrm{iff}$, since otherwise $\Lambda^2 \to \mathbb{R}^2$ would have a smooth retract, which contradicts Example 2.6.30.

Example 2.6.33. $X = [0, \infty)$ as a sub-diffeological space of $\mathbb{R}$ is not fibrant. The following proof is due to G. Sinnamon. Let $f : \Lambda^3 \to X$ be defined by $f_i : \mathbb{R}^2 \to X$ with $f_i(x_j, x_k) = (x_j - x_k)^2$ for $\{i, j, k\} = \{1, 2, 3\}$. Assume that f has a smooth extension $G : \mathbb{R}^3 \to X$, that is, there exists a smooth function $F : \mathbb{R}^3 \to \mathbb{R}$

such that $\mathrm{Im}(F) \subseteq X$, $F(x_1, x_2, 0) = (x_1 - x_2)^2$, $F(0, x_2, x_3) = (x_2 - x_3)^2$ and $F(x_1, 0, x_3) = (x_1 - x_3)^2$. Consider the composition $h : \mathbb{R} \xrightarrow{g} \mathbb{R}^3 \xrightarrow{G} X$, with $g(t) = (t, t, t)$. Then by the chain rule, it is easy to calculate that $h(t) = -3t^2 + o(t^2)$, which contradicts that $\mathrm{Im}(F) \subseteq X$.

As an easy corollary of this result, any smooth manifold with boundary is not fibrant.

Hence this example together with Theorem 2.6.23 gives another proof showing that X_n and X_∞ (introduced in (2) of Example 1.8.1) are not diffeomorphic.

Remark 2.6.34. We call a diffeological space X *k-fibrant* if it has the right lifting property with respect to $\Lambda^n \to \mathbb{R}^n$ for all $n \in \{1, 2, \cdots, k\}$. Then every diffeological space is 1-fibrant since there is a smooth retract for $\Lambda^1 \to \mathbb{R}$, and a diffeological space is fibrant if and only if it is k-fibrant for all $k \in \mathbb{Z}^+$. It is not hard to prove that $[0, \infty)$ with the sub-diffeology of $\mathbb{R}$ is 2-fibrant, and the above example shows that it is not 3-fibrant. And $\hat{\mathbb{R}}^n$ in Example 2.6.31 is $(n-1)$-fibrant but not n-fibrant.

2.7 Comparison of smooth and simplicial homotopy groups for fibrant diffeological spaces

By Proposition 2.6.12, we know that $S_{\tilde{D}}(\mathbb{R})$ is a simplicial abelian group. So we can use Dold-Kan correspondence (see, for example, [GJ]) to calculate its simplicial homotopy groups. More precisely, given a simplicial group G, we can form a chain complex $N(G)$ with $(N(G))_n = \cap_{i=0}^{n-1} ker(d_i)$, where $d_i : G_n \to G_{n-1}$ is the boundary map, and the differential for the chain complex is given by $(-1)^n d_n : (N(G))_n \to (N(G))_{n-1}$. (This forms a chain complex due to the simplicial identities.) Moreover, $\pi_n^s(G, g) \cong H_n(N(G))$ for any $g \in G_0$.

In the case $G = S_{\tilde{D}}(\mathbb{R})$, $(N(G))_n = \{f \in \mathfrak{D}\mathrm{iff}(\mathbb{A}^n, \mathbb{R}) \mid d_i(f) = 0$ for all $i \in \{0, 1, \cdots, n-1\}\}$, and $(-1)^n d_n(f) = (-1)^n f(x_0, \cdots, x_{n-1}, 0)$. We claim that $H_n(G, 0) = 0$ for all $n \in \mathbb{N}$. In other words, if $f \in \mathfrak{D}\mathrm{iff}(\mathbb{A}^n, \mathbb{R})$ such that $d_i(f) = 0$ for all $i \in \{0, 1, \cdots, n\}$, then we can find some $g \in \mathfrak{D}\mathrm{iff}(\mathbb{A}^{n+1}, \mathbb{R})$ such that $d_i(g) = 0$ for all $i \in \{0, 1, \cdots, n\}$, and $f = (-1)^{n+1} d_{n+1}(g)$. In fact, $g(x_0, \cdots, x_{n+1}) = (-1)^{n+1}\phi(1 - x_{n+1}) f(\frac{x_0}{1-x_{n+1}}, \cdots, \frac{x_n}{1-x_{n+1}})$ with ϕ some cut-off function satisfies these properties.

Note that $\pi_n^D(\mathbb{R}, 0) = 0$, which implies that $\pi_n^s(S_{\tilde{D}}(\mathbb{R}), 0) \cong \pi_n^D(\mathbb{R}, 0)$. This is not a coincidence. We will show that the similar result holds for any pointed fibrant diffeological space.

Before giving the proof, we give two different characterizations of smooth homotopy groups of a pointed diffeological spaces as follows:

By the rescaling trick, we have the following observation.

Remark 2.7.1. Let (X, x) be a pointed diffeological space. Then $\pi_n^D(X, x)$ can also be characterized as $\{f \in \mathfrak{D}\mathrm{iff}(\mathbb{R}^n, X) \mid f|_{\mathbb{R}^n - \epsilon|\tilde{\Delta}^n|} = x$ for some $\epsilon > 0\}/\sim'$, where

$\epsilon|\tilde{\Delta}^n| = \{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid \sum_{i=1}^n x_i < 1 - \epsilon$ and $x_i > \epsilon$ for all $i \in \{1, 2, \cdots, n\}\}$, and $f \sim' g$ if and only if there exists $F \in \mathfrak{D}\mathrm{iff}(\mathbb{R}^{n+1}, X)$ such that $F(x_1, \cdots, x_n, 0) = f(x_1, \cdots, x_n)$, $F(x_1, \cdots, x_n, 1) = g(x_1, \cdots, x_n)$, and $F|_{(\mathbb{R}^n - \delta|\tilde{\Delta}^n|)\times\mathbb{R}} = x$ for some $\delta > 0$.

Proposition 2.7.2. *Let (X, x) be a pointed diffeological space. Then $\pi_n^D(X, x)$ can be characterized as $A = \{f \in \mathfrak{D}iff(\mathbb{R}^n, X) \mid f|_{\partial\mathbb{A}^n} = x\}/\sim$, where $f \sim g$ if and only if there exists $F \in \mathfrak{D}iff(\mathbb{R}^{n+1}, X)$ such that $F(x_1, \cdots, x_n, 0) = f(x_1, \cdots, x_n)$, $F(x_1, \cdots, x_n, 1) = g(x_1, \cdots, x_n)$, and $F|_{\partial\mathbb{A}^n\times\mathbb{R}} = x$.*

Proof. It is clear that this is true for $n = 0$.

For $n \geq 1$, define $i : \pi_n^D(X, x) \to A$ by sending $[f]$ to $[f]$, where f is chosen to satisfy the condition of Remark 2.7.1, that is, $f|_{\mathbb{R}^n - \epsilon|\tilde{\Delta}^n|} = x$ for some $\epsilon > 0$. Clearly, this map is well-defined.

Note that there is a diffeomorphism of diffeological pairs $(\mathbb{R}^n, \partial\mathbb{A}^n)$ and $(\mathbb{A}^n, \partial'\mathbb{A}^n)$, where $\partial'\mathbb{A}^n = \{(x_0, \cdots, x_n) \in \mathbb{A}^n \mid x_i = 0$ for some $i \in \{0, 1, \cdots, n\}\}$ with the co-equalizer diffeology. For simplicity, we switch to use $(\mathbb{A}^n, \partial'\mathbb{A}^n)$ for the rest of the proof.

i is surjective. For any $[g] \in A$ with $g \in \mathfrak{D}\mathrm{iff}(\mathbb{A}^n, X)$ such that $g|_{\partial'\mathbb{A}^n} = x$, we need to find $F \in \mathfrak{D}\mathrm{iff}(\mathbb{A}^n \times \mathbb{R}, X)$ such that $F(x_0, \cdots, x_n, 1) = g(x_0, \cdots, x_n)$, $F|_{\partial'\mathbb{A}^n\times\mathbb{R}} = x$ and $F(x_0, \cdots, x_n, 0) = x$ if some $x_i < \epsilon$ for some $\epsilon > 0$. Let $\alpha : \mathbb{R} \to \mathbb{R}$ be the smooth function with $\alpha(y) = y\phi(y)$, where $\phi : \mathbb{R} \to \mathbb{R}$ be a cut-off function such that $\phi(y) = 0$ if $y < \epsilon$, $\phi(y) = 1$ if $y > 1 - \epsilon$ for some $0 < \epsilon < \frac{1}{n+1}$, and $\phi(y) > 0$ if $y \geq \frac{1}{n+1}$. Let $\alpha_t : \mathbb{R} \to \mathbb{R}$ be the smooth function with $\alpha_t(y) = \phi(t)y + (1 - \phi(t))\alpha(y)$, and let $p : \mathbb{R}^{n+1} - \{(x_0, \cdots, x_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^n x_i = 0\} \to \mathbb{A}^n$ be the smooth function with $p(x_0, \cdots, x_n) = (\frac{x_0}{\sum_{i=0}^n x_i}, \cdots, \frac{x_n}{\sum_{i=0}^n x_i})$. Define $F = g \circ p \circ \beta$ with $\beta : \mathbb{A}^n \times \mathbb{R} \to \mathbb{R}^{n+1} - \{(x_0, \cdots, x_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^n x_i = 0\}$ with $\beta(x_0, \cdots, x_n, t) = (\alpha_t(x_0), \cdots, \alpha_t(x_n))$, and it is easy to check that F has all the required properties.

i is injective. If $[f],[g] \in \pi_n^D(X,x)$ with both f and g chosen to satisfy the condition of Remark 2.7.1, such that $i([f]) = i([g])$, that is, there exists $F \in \mathfrak{D}\mathrm{iff}(\mathbb{A}^n \times \mathbb{R}, X)$ such that $F(x_0,\cdots x_n,0) = f(x_0,\cdots x_n)$, $F(x_0,\cdots x_n,1) = g(x_0,\cdots x_n)$ and $F|_{\partial'\mathbb{A}^n\times\mathbb{R}} = x$, then the composition $F\circ((p\circ\alpha^{n+1})\times 1_{\mathbb{R}})$ implies that $f\circ p\circ\alpha^{n+1} \sim' g\circ p\circ\alpha^{n+1}$. Clearly, $f \sim' f\circ p\circ\alpha^{n+1}$ and $g \sim' g\circ p\circ\alpha^{n+1}$, which implies that $[f] = [g]$ in $\pi_n^D(X,x)$. $\square$

Theorem 2.7.3. *Let (X,x) be a pointed diffeological space with X fibrant. Then $\pi_n^D(X,x) \cong \pi_n^s(S_{\tilde{D}}(X),x)$.*

Proof. Since X is a fibrant diffeological space, $S_{\tilde{D}}(X)$ is a Kan complex. The n^{th} simplicial homotopy group of $(S_{\tilde{D}}(X),x)$ is defined as follows [May]: $\pi_n^s(S_{\tilde{D}}(X),x) = \{f \in (S_{\tilde{D}}(X))_n \mid d_i(f) = x \text{ for all } i \in \{0,1,\cdots n\}\}/\sim_s$, with $f \sim_s g$ if there exists $h \in (S_{\tilde{D}(X)})_{n+1}$ such that $d_n(h) = f$, $d_{n+1}(h) = g$ and $d_i(h) = x$ for all $i \in \{0,1,\cdots n-1\}$.

Actually, $\pi_0^s(S_{\tilde{D}}(X))$ can also be described as the coequalizer of

$$(S_{\tilde{D}}(X))_1 \overset{d_0}{\underset{d_1}{\rightrightarrows}} (S_{\tilde{D}}(X))_0.$$

Therefore, $\pi_0^D(X) \cong \pi_0^s(S_{\tilde{D}}(X))$.

Now for $n \geq 1$, we define $\alpha : \pi_n^D(X,x) \to \pi_n^s(S_{\tilde{D}}(X),x)$ by $\alpha([f]) = [f]$, where $f \in \mathfrak{D}\mathrm{iff}(\mathbb{R}^n, X)$ is chosen to satisfy the condition of Proposition 2.7.2, that is, $f|_{\partial'\mathbb{A}^n} = x$. It is well-defined, since if $[f] = [g]$, that is, there exists $F \in \mathfrak{D}\mathrm{iff}(\mathbb{A}^n \times \mathbb{R}, X)$ such that $F(x_0,\cdots x_n,0) = f(x_0,\cdots x_n)$, $F(x_0,\cdots x_n,1) = g(x_0,\cdots x_n)$ and $F|_{\partial'\mathbb{A}^n\times\mathbb{R}} = x$, then divide $\mathbb{A}^n \times \mathbb{R}$ into $n+1$ $(n+1)$-simplices using the prism operators in [Ha]. Note that for each such $(n+1)$-simplex, exactly two faces are not contained in $\partial\mathbb{A}^n \times \mathbb{R}$. Since $\sim_s$ is an equivalence relation, we have $f \sim_s g$.

α is clearly surjective.

α is injective. Let $[f], [g] \in \pi_n^D(X, x)$ with both f and g chosen to satisfy the condition of Proposition 2.7.2, such that $\alpha([f]) = \alpha([g])$, that is, there exists $F \in \mathfrak{D}\mathrm{iff}(\mathbb{A}^{n+1}, X)$ such that $F(x_0, \cdots, x_{n-1}, 0, x_{n+1}) = f(x_0, \cdots, x_{n-1}, x_{n+1})$, $F(x_0, \cdots, x_n, 0) = g(x_0, \cdots, x_n)$ and $F(x_0, \cdots, x_{n+1}) = x$ if some other $x_i = 0$. Then the composition $F \circ \beta$ with $\beta : \mathbb{A}^n \times \mathbb{R} \to \mathbb{A}^{n+1}$ defined by

$$\beta(x_0, \cdots, x_n, t) = (x_0, \cdots, x_{n-1}, tx_n, (1 - t)x_n)$$

implies that $[f] = [g]$ in $\pi_n^D(X, x)$. $\square$

Appendix A

Basics of Model Categories

By axiomatizing the basic properties and relationship between $\mathfrak{Top}$ and Ho($\mathfrak{Top}$), D. Quillen introduced the concept of model categories in 1967 in his famous book [Q]. In the following 45 years, only slight changes have been made, and now model categories are a standard tool to do homotopy theory on any nice enough category.

We summarize some basics of model category theory in this appendix. The standard references are [DS], [GJ], [Hi], [Ho], [Q]. Due to limited space, many interesting topics related to model category theory are not covered, for example, the fact that the homotopy category of a (pointed) model category is a closed module over the homotopy category of (pointed) simplicial sets, that every homotopy category of a pointed model category is pre-triangulated, localizations of model categories, model category structures on simplicial (pre)sheaves [Ja1], D. Dugger's approach to A^1 homotopy theory [Du2], existence of non-cofibrantly generated model categories [CH], etc. All of these can be found in the references.

Only two uses of model category theory are explained: (1) the localization of a category with respect to arbitrary class of morphisms may not be a category again, but the localization of a model category with respect to the class of its weak equivalences is again a category; (2) model category theory unifies the idea of CW-approximations in $\mathfrak{Top}$ and projective resolutions of modules, and homological algebra can be recovered from a model category structure on the category of chain complexes.

In this thesis, model category theory will be used in Section 1.2.3 and most of Chapter 2. Readers can refer to this appendix for the notation and basic results (without proof) of model category theory whenever needed.

A.1 Kan extensions

We present the basics of Kan extensions in this section. A good reference is [Mac].

Definition A.1.1. Let A, B and C be three categories, and let $X : \mathsf{A} \to \mathsf{C}$, $F : \mathsf{A} \to \mathsf{B}$ be two functors between them. The *left Kan extension* $(LK_F(X), \mu)$ of X along F is a functor $LK_F(X) : \mathsf{B} \to \mathsf{C}$ together with a natural transformation $\mu : X \to LK_F(X) \circ F$ such that for any pair (G, ν) where $G : \mathsf{B} \to \mathsf{C}$ is a functor and $\nu : X \to G \circ F$ is a natural transformation, there is a unique natural transformation $\tau : LK_F(X) \to G$ that makes the following diagram commutative:

$$\begin{tikzcd} X \arrow[r, "\nu"] \arrow[d, "\mu"'] & G \circ F \\ LK_F(X) \circ F. \arrow[ur, "\tau_F"'] & \end{tikzcd}$$

Right Kan extensions are defined dually.

Definition A.1.2. Let A, B and C be three categories, and let $X : \mathsf{A} \to \mathsf{C}$, $F : \mathsf{A} \to \mathsf{B}$ be two functors between them. The *right Kan extension* $(RK_F(X), \mu)$ of X along F is a functor $RK_F(X) : \mathsf{B} \to \mathsf{C}$ together with a natural transformation $\mu : RK_F(X) \circ F \to X$ such that for any pair (G, ν) where $G : \mathsf{B} \to \mathsf{C}$ is a functor and $\nu : G \circ F \to X$ is a natural transformation, there is a unique natural transformation $\tau : G \to RK_F(X)$ that makes the following diagram commutative:

$$\begin{tikzcd} RK_F(X) \circ F \arrow[r, "\mu"] & X \\ G \circ F. \arrow[u, "\tau_F"] \arrow[ur, "\nu"'] & \end{tikzcd}$$

When a Kan extension exists, it is clearly unique up to a unique isomorphism.

Theorem A.1.3. *If A is small and C is cocomplete, then each left Kan extension exists, and $LK_F(X)(b) \cong \operatorname{colim}_{F(a) \to b} X(a)$. Dually, if A is small and C is complete, then each right Kan extension exists, and $RK_F(X)(b) \cong \lim_{b \to F(a)} X(a)$.*

Proof. To prove the first statement, define $\tau : LK_F(X)\ !\quad G$ by defining τ_b : $LK_F(X)(b)\ !\quad G(b)$ as follows. For any $f : F(a)\ !\quad b$ we have

$$X(a) \xrightarrow{\nu_a} G(F(a)) \xrightarrow{G(f)} G(b).$$

Hence we have the required map. The remaining parts are easy. □

Proposition A.1.4. *If all left Kan extensions exist, then we have the following adjoint pair:* $LK_F(?) : \mathsf{C}^{\mathcal{A}} \rightleftharpoons \mathsf{C}^{\mathcal{B}} : ? \pm F$. *If all right Kan extensions exist, then we have the following adjoint pair:* $? \pm F : \mathsf{C}^{\mathcal{B}} \rightleftharpoons \mathsf{C}^{\mathcal{A}} : RK_F(?)$.

Proof. This is direct from the definitions. □

We now give some examples of Kan extensions:

Example A.1.5. The colimit of a functor X from a small category A to a cocomplete category C is the left Kan extension of X along the unique functor from A to the terminal category 1 (that is, the category with one object and one morphism). Dually, The limit of a functor X from a small category A to a complete category C is the right Kan extension of X along the unique functor from A to the category 1.

Example A.1.6. Define functors $X : \Delta\ !\quad \mathfrak{Top}$ by $X(\underline{n}) = \mathsf{j}\Delta^n\mathsf{j}$, the standard n-simplex in $\mathfrak{Top}$, and $F : \Delta\ !\quad \mathfrak{s}\mathrm{Set}$ by $F(\underline{n}) = \Delta^n$, the standard n-simplex in $\mathfrak{s}\mathrm{Set}$. Then the left Kan extension of X along F is the usual geometric realization functor $\mathsf{j}?\mathsf{j}$.

Example A.1.7. Moreover, the left adjoint L in Theorem 2.2.1 is the left Kan extension of $F : \mathsf{C}\ !\ \mathsf{D}$ along the Yoneda embedding $\mathsf{C}\ !\quad \mathfrak{Pre}(\mathsf{C})$ defined by

$(f : c \to c') \mapsto (f_* : \mathsf{C}(?, c) \to \mathsf{C}(?, c'))$. We have natural isomorphisms:

$$
\begin{aligned}
\mathsf{D}(LX, Y) &= \mathsf{D}(\operatorname*{colim}_{\mathcal{C}(?,c)\to X} F(c), Y) \\
&\cong \lim_{\mathcal{C}(?,c)\to X} \mathsf{D}(F(c), Y) \\
&= \lim_{\mathcal{C}(?,c)\to X} RY(c) \\
&\cong \lim_{\mathcal{C}(?,c)\to X} \mathfrak{P}re(\mathsf{C})(\mathsf{C}(?, c), RY) \\
&\cong \mathfrak{P}re(\mathsf{C})(\operatorname*{colim}_{\mathcal{C}(?,c)\to X} \mathsf{C}(?, c), RY) \\
&\cong \mathfrak{P}re\mathsf{C}(X, RY),
\end{aligned}
$$

for any objects X in $\mathfrak{P}$re(C) and Y in D, and the adjointness follows.

A.2 Model categories

Good references for this section are [DS], [GJ], [Hi], [Ho], [Q]. However, the terminology is slightly different in these references. The purpose here is to summarize the theory with consistent terminology for the thesis.

A.2.1 Basic theory of model categories

The basics

Unless otherwise stated, the main source for this part is [Q].

Definition A.2.1. Let M be a category and let $f : A \to B$ and $g : C \to D$ be two morphisms in M. We say that f is a *retract* of g if we have the following commutative diagram in M

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & C \arrow[r] \arrow[d, "g"] & A \arrow[d, "f"] \\ B \arrow[r] & D \arrow[r] & B \end{tikzcd}$$

with the composition of the two horizontal morphisms being id_A and id_B, respectively.

Definition A.2.2. Let M be a category, let $f : A \to B$ be a morphism in M, and let I be a class of morphisms in M. We say that f has the *left lifting property with respect to I* if for any commutative solid diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r] \arrow[ur, dotted] & D \end{tikzcd}$$

with $g \in I$, the dotted arrow exists and makes the whole diagram commutative. Dually, we can define a morphism to have the *right lifting property* with respect to a class of morphisms.

The following definition is from [Ho]:

Definition A.2.3. Let M be a category. Define $\mathfrak{Map}(\mathsf{M})$ to be the category whose objects are maps in M and morphisms are commutative squares in M. A *functorial factorization* of maps in M is a pair of functors $\alpha, \beta : \mathfrak{Map}(\mathsf{M}) \to \mathfrak{Map}(\mathsf{M})$ such that $f = \alpha(f) \circ \beta(f)$ for any map f in M.

Definition A.2.4 ([Hi, Ho])**.** A *model category* is a category M together with three classes of morphisms (called *weak equivalences* W, *cofibrations* C and *fibrations* F), satisfying the following axioms:

(1) the category M is complete and cocomplete;

(2) (two out of three) let f and g be two morphisms in M such that $g \circ f$ is defined. If two of f, g and $g \circ f$ are weak equivalences, then so is the third;

(3) (retraction) the retract of a weak equivalence (cofibration, or fibration) is again a weak equivalence (cofibration, or fibration);

(4) (lifting) given a commutative solid diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & B \arrow[d, "g"] \\ C \arrow[r] \arrow[ur, dotted, "h"] & D \end{tikzcd}$$

in M with f a cofibration and g a fibration, then the dotted arrow h exists to make the whole diagram commutative, if either f or g is a weak equivalence;

(5) (factorization) every morphism in M can be functorially factored as a *trivial cofibration* (a morphism that is both a weak equivalence and a cofibration) followed by a fibration, and as a cofibration followed by a *trivial fibration* (a morphism that is both a weak equivalence and a fibration).

Remark A.2.5. In [DS, GJ], the definition of a model category (which is called a closed model category in [Q]) is defined similarly except (1) only finite limits and colimits of $\mathcal{M}$ are required to exist; (2) factorizations are not required to be functorial. However, most of the known examples of model categories satisfy all the above five axioms.

The definition of a model category is self-dual, in the following sense:

Lemma A.2.6. *If* $(\mathcal{M}, \mathcal{W}, \mathcal{C}, \mathcal{F})$ *is a model category, then* $(\mathcal{M}^{op}, \mathcal{W}^{op}, \mathcal{C}', \mathcal{F}')$ *with* $\mathcal{C}' = \mathcal{F}^{op}$ *and* $\mathcal{F}' = \mathcal{C}^{op}$ *is a model category.*

The following lemma can be found in [Hi]:

Lemma A.2.7. *Let* $\mathcal{M}$ *be a model category.*

(1) Let A be an object of $\mathcal{M}$*. Then the comma category* $A/\mathcal{M}$ *is a model category in which a map is a weak equivalence, fibration, or cofibration if it is one in* $\mathcal{M}$*.*

(2) Let A be an object of $\mathcal{M}$*. Then the comma category* $\mathcal{M}/A$ *is a model category in which a map is a weak equivalence, fibration, or cofibration if it is one in* $\mathcal{M}$*.*

Here are some properties of a model category:

Proposition A.2.8. *Let* $(\mathcal{M}, \mathcal{W}, \mathcal{C}, \mathcal{F})$ *be a model category. Then*

(1) $\mathcal{C} = \{f \in Mor(\mathcal{M}) \mid f$ *has the left lifting property with respect to* $\mathcal{F} \cap \mathcal{W}\}$*;*

(2) $\mathcal{C} \cap \mathcal{W} = \{f \in Mor(\mathcal{M}) \mid f$ *has the left lifting property with respect to* $\mathcal{F}\}$*;*

(3) both $\mathcal{C}$ *and* $\mathcal{C} \cap \mathcal{W}$ *are closed under pushouts;*

Dually,

(4) $\mathcal{F} = \{f \in Mor(\mathcal{M}) \mid f$ *has the right lifting property with respect to* $\mathcal{C} \cap \mathcal{W}\}$*;*

(5) $\mathcal{F} \cap \mathcal{W} = \{f \in Mor(\mathcal{M}) \mid f$ *has the right lifting property with respect to* $\mathcal{C}\}$

(6) both $\mathcal{F}$ *and* $\mathcal{F} \cap \mathcal{W}$ *are closed under pullbacks.*

These properties together with the model category axioms imply that any two of the three classes of morphisms W, C and F determine the other in a model category.

Definition A.2.9. An object X in a model category is called *cofibrant* (or *fibrant*) if the natural map $0 \to X$ is a cofibration (or $X \to 1$ is a fibration). Here 0 denotes an initial object and 1 denotes a terminal object in the category.

Definition A.2.10. Let X be an object in a model category C.

(1) We say that $\tilde{X}$ is a *cofibrant replacement* of X if $\tilde{X}$ is cofibrant, and there is a weak equivalence $\tilde{X} \to X$.

(2) We say that X' is a *fibrant replacement* of X if X' is fibrant, and there is a weak equivalence $X \to X'$.

The following lemma is discussed in [Hi, Ho]:

Lemma A.2.11 (Ken Brown)**.** *Let M be a model category, and let N be a category with a subcategory N' that satisfies the two out of three property. Suppose the functor $F : \mathsf{M} \to \mathsf{N}$ takes every trivial cofibration between cofibrant objects to a morphism in N'. Then F takes every weak equivalence between cofibrant objects to a morphism in N'. Dually, if F takes every trivial fibration between fibrant objects to a morphism in N', then F takes every weak equivalence between fibrant objects to a morphism in N'.*

The following proposition is discussed in [Hi]:

Proposition A.2.12. *Let M be a model category, and let $f : X \to Y$ be a morphism in M.*

(1) If both X and Y are cofibrant, then f can be factored as $X \xrightarrow{g} Z \xrightarrow{h} Y$, where g is a cofibration, h is a trivial fibration, and there exists a trivial cofibration $k : Y \to Z$ such that $h \circ k = 1_Y$.

(2) If both X and Y are fibrant, then f can be factored as $X \xrightarrow{g} Z \xrightarrow{h} Y$, where g is a trivial cofibration, h is a fibration, and there exists a trivial fibration $k : Z \to X$ such that $k \circ g = 1_X$.

In general, given a category $\mathfrak{C}$ and a class of morphisms S of $\mathfrak{C}$, we can formally form the localization $\mathsf{S}^{-1}\mathfrak{C}$ in which the maps in S have been inverted. However, the localization may not be a category in the usual sense, since the target of the hom-functors may not land in $\mathfrak{S}$et any more. For example, let C be a category with a proper class of objects, and let a, b be two distinct objects in C. Besides the identity maps, the only morphisms are maps from a (and b) to all the other objects except b (a), exactly one map for each such object. Let S be the class of all the morphisms from b. Then $S^{-1}\mathsf{C}$ is not a category. However, model categories don't have this problem:

Theorem A.2.13. *Let* M *be a model category with weak equivalences* W*. Then the localization* $\mathsf{W}^{-1}\mathsf{M}$ *is again a category (that is, all hom-functors have targets in* $\mathfrak{S}$*et).*

Definition A.2.14. We call the localization in the above theorem the *homotopy category* of M, and denote it by $\mathrm{Ho}(\mathsf{M})$.

The above theorem is true for the following reasons:

Let M be a model category. Let's denote by M_c (M_f, M_{cf}) the full subcategory of M consisting of cofibrant objects (fibrant objects, objects that are both fibrant and cofibrant). Then the inclusion functors induce equivalences of categories $Ho(\mathsf{M}_{cf}) \to Ho(\mathsf{M}_c) \to Ho(\mathsf{M})$ and $Ho(\mathsf{M}_{cf}) \to Ho(\mathsf{M}_f) \to Ho(\mathsf{M})$.

For $Ho(\mathsf{M}_{cf})$, there is another description as follows:

Definition A.2.15. Let M be a model category, and let $f, g : X \to Y$ be two morphisms in M.

(1) A *cylinder object* of X is a factorization of the fold map $X \coprod X \to X$ into a cofibration $i_0 + i_1 : X \coprod X \to X'$ followed by a weak equivalence $X' \to X$.

(2) A *path object* of Y is a factorization of the diagonal map $Y \to Y \times Y$ into a weak equivalence $Y \to Y'$ followed by a fibration $(p_0, p_1) : Y' \to Y \times Y$.

(3) A *left homotopy* from f to g is a morphism $F : X' \to Y$ in M for some cylinder object X' of X such that $F \circ i_0 = f$ and $F \circ i_1 = g$. In this case, we write $f \sim_l g$.

(4) A *right homotopy* from f to g is a morphism $G : X \to Y'$ in M for some path object Y' of Y such that $p_0 \circ G = f$ and $p_1 \circ G = g$. In this case, we write $f \sim_r g$

(5) We say that f and g are *homotopic* if they are both left and right homotopic. In this case, we write $f \sim g$.

(6) We say that f is a *homotopy equivalence* if there is a morphism $h : Y \to X$ such that $h \circ f \sim 1_X$ and $f \circ h \sim 1_Y$.

Here is a summary of the basic properties of the above defined terms:

Proposition A.2.16. *Let M be a model category, and let $f, g : X \to Y$ be two morphisms in M.*

(1) If $f \sim_l g$ and $h : Y \to Z$ is a morphism in M, then $h \circ f \sim_l h \circ g$; dually, if $f \sim_r g$ and $h : Z \to X$ is a morphism in M, then $f \circ h \sim_r g \circ h$.

(2) If Y is fibrant, $f \sim_l g$, and $h : Z \to X$ is a morphism in M, then $f \circ h \sim_l g \circ h$; dually, if X is cofibrant, $f \sim_r g$, and $h : Y \to Z$ is a morphism in M, then $h \circ f \sim_r h \circ g$.

(3) If X is cofibrant, then left homotopy is an equivalence relation on $\mathsf{M}(X, Y)$; dually, if Y is fibrant, then right homotopy is an equivalence relation on $\mathsf{M}(X, Y)$.

(4) If X is cofibrant and $h : Y \to Z$ is a trivial fibration or a weak equivalence between fibrant objects, then h induces an isomorphism $\mathsf{M}(X, Y)/\sim_l \to \mathsf{M}(X, Z)/\sim_l$; dually, if Y is fibrant and $h : Z \to X$ is a trvial cofibration or a weak equivalence between cofibrant objects, then h induces an isomorphism $\mathsf{M}(X, Y)/\sim_r \to \mathsf{M}(Z, Y)/\sim_r$.

(5) If X is cofibrant, then $f \sim_l g$ implies $f \sim_r g$. Furthermore, if Y' is any path

object of Y, then there is a right homotopy $X \to Y'$ from f to g; dually, if Y is fibrant, then $f \sim_r g$ implies $f \sim_l g$. Furthermore, if X' is any cylinder object of X, then there is a left homotopy $X' \to Y$ from f to g.

(6) If X is cofibrant and Y is fibrant, then left homotopy and right homotopy on $\mathsf{M}(X, Y)$ coincide, and both are equivalence relations. Furthermore, if $f \sim g$, then there is a left (right) homotopy from f to g through any cylinder object of X (path object of Y).

(7) The homotopy relation on the morphisms of M_{cf} is an equivalence relation which is compatible with composition. Hence the quotient category $\mathsf{M}_{cf}/\sim$ exists.

Proposition A.2.17. *Let M be a model category, and let $f : X \to Y$ be a morphism in M.*

(1) If both X and Y are cofibrant, then f is a weak equivalence if and only if for every fibrant object Z of M, the induced map $f^* : \mathsf{M}(Y, Z)/\sim \to \mathsf{M}(X, Z)/\sim$ is a bijection.

(2) If both X and Y are fibrant, then f is a weak equivalence if and only if for every cofibrant object Z of M, the induced map $f_* : \mathsf{M}(Z, X)/\sim \to \mathsf{M}(Z, Y)/\sim$ is a bijection.

We also have the following generalization of Whitehead's theorem from $\mathfrak{Top}$, which implies $Ho(\mathsf{M}_{cf}) \cong \mathsf{M}_{cf}/\sim$ for any model category M:

Theorem A.2.18. *Let M be a model category. Then a map in M_{cf} is a weak equivalence if and only if it is a homotopy equivalence.*

We summarize the above theory in the following theorem:

Theorem A.2.19. *Let M be a model category, and let $\gamma : \mathsf{M} \to Ho(\mathsf{M})$ be the canonical functor. Write Q (R) for the cofibrant (fibrant) replacement functor of M.*

(1) The inclusion functor $\mathcal{M}_{cf} \to \mathcal{M}$ *induces an equivalence of categories*

$$\mathcal{M}_{cf}/\sim \ \to Ho(\mathcal{M}_{cf}) \to Ho(\mathcal{M}).$$

(2) There are natural isomorphisms

$$\mathcal{M}(QRX, QRY)/\sim \ \cong Ho(\mathcal{M})(\gamma(X), \gamma(Y)) \cong \mathcal{M}(RQX, RQY)/\sim .$$

In addition, there is a natural isomorphism $Ho(\mathcal{M})(\gamma(X), \gamma(Y)) \cong \mathcal{M}(QX, RY)/\sim$.

(3) γ *identifies left or right homotopic maps.*

(4) If f *is a morphism in* $\mathcal{M}$ *such that* $\gamma(f)$ *is an isomorphism in* $Ho(\mathcal{M})$*, then* f *is a weak equivalence in* $\mathcal{M}$.

We consider functors between model categories in the following proposition. The last two conditions are due to D. Dugger (see [Hi]):

Proposition A.2.20. *Let* $F : \mathcal{M} \rightleftarrows \mathcal{N} : G$ *be an adjoint pair between model categories. Then the followings are equivalent:*

(1) F *preserves both cofibrations and trivial cofibrations.*

(2) G *preserves both fibrations and trivial fibrations.*

(3) F *preserves cofibrations and* G *preserves fibrations.*

(4) F *preserves trivial cofibrations and* G *preserves trivial fibrations.*

(5) F *preserves cofibrations between cofibrant objects and all trivial cofibrations.*

(6) G *preserves fibrations between fibrant objects and all trivial fibrations.*

If one of the above conditions holds, we call the adjoint pair a *Quillen pair*. We also call F a *left Quillen functor* and G a *right Quillen functor*.

Definition A.2.21. Given a functor $F : \mathcal{M} \to \mathcal{N}$ between two model categories. The *total left derived functor* $LF : Ho(\mathcal{M}) \to Ho(\mathcal{N})$ of F is defined to be $RK_{\gamma_{\mathcal{M}}}(\gamma_{\mathcal{N}} \circ F)$, and the *total right derived functor* $RF : Ho(\mathcal{M}) \to Ho(\mathcal{N})$ of F is defined to be $LK_{\gamma_{\mathcal{M}}}(\gamma_{\mathcal{N}} \circ F)$.

Theorem A.2.22. *Let $F : \mathsf{M} \rightleftharpoons \mathsf{N} : G$ be a Quillen pair between two model categories. Then the total left derived functor LF of F and the total right derived functor RG of G both exist, and they form an adjoint pair $LF : Ho(\mathsf{M}) \rightleftharpoons Ho(\mathsf{N}) : RG$. Moreover, LF is the composition $Ho(\mathsf{M}) \xrightarrow{Ho(Q)} Ho(\mathsf{M}_c) \xrightarrow{Ho(F)} Ho(\mathsf{N})$, and RG is the composition $Ho(\mathsf{N}) \xrightarrow{Ho(R)} Ho(\mathsf{N}_f) \xrightarrow{Ho(G)} Ho(\mathsf{M})$.*

Definition A.2.23. A Quillen pair $F : \mathsf{M} \rightleftharpoons \mathsf{N} : G$ between two model categories is called a *Quillen equivalence*, if for any cofibrant object M in M and any fibrant object N in N, $M \to GN$ is a weak equivalence in M if and only if its adjoint $FM \to N$ is a weak equivalence in N.

Theorem A.2.24. *Let $F : \mathsf{M} \rightleftharpoons \mathsf{N} : G$ be a Quillen equivalence between two model categories. Then the adjoint pair $LF : Ho(\mathsf{M}) \rightleftharpoons Ho(\mathsf{N}) : RG$ is an equivalence of categories.*

The small object argument

We talk about the small object argument in this part. Good references for the material in this part are [Hi, Ho]:

An *ordinal* is an isomorphism class of well-ordered sets. Or we define the empty set to be an ordinal named 0, and a non-zero ordinal is the well ordered set of all lesser ordinals.

Definition A.2.25. (1) The *successor* of an ordinal β is the smallest ordinal greater than β, and a *limit ordinal* is an ordinal that is neither 0 nor a successor ordinal.

(2) A *cardinal* is an ordinal that is of greater cardinality than any lesser ordinal.

Definition A.2.26. Let M be a cocomplete category, let λ be an ordinal, and let I be a class of morphisms in M. Then

(1) a λ-*sequence* in M is a functor $X : \lambda \to \mathsf{M}$ such that for every limit ordinal $\gamma < \lambda$, the natural morphism $\operatorname{colim}_{\beta<\gamma} X_\beta \to X_\gamma$ is an isomorphism;

(2) a *transfinite composition* of maps in I is $X_0 \to \operatorname{colim}_{\beta<\lambda} X_\beta$ for a λ-sequence X with each $X_\beta \to X_{\beta+1}$ in I for all ordinals β with $\beta + 1 < \lambda$.

Definition A.2.27. Let I be a class of morphisms in a category M.

(1) A morphism in M is I-*injective* if it has the right lifting property with respect to every element in I. The class of all I-injective maps will be denoted by I-inj.

(2) A morphism in M is an I-*cofibration* if it has the left lifting property with respect to all I-injective maps. The class of all I-cofibrations will be denoted by I-cof.

Definition A.2.28. Let I be a set of morphisms in a cocomplete category M. A *relative I-cell complex* is a transfinite composition of pushouts of elements in I. The class of all relative I-cell complexes will be denoted by I-cell.

Definition A.2.29. Let γ be a cardinal. An ordinal α is γ-*filtered* if it is a limit ordinal and, if $A \subseteq \alpha$ and $|A| < \gamma$, then $sup(A) < \alpha$.

Definition A.2.30. Let I be a class of morphisms in a cocomplete category M. An object M in M is *small relative to* I if there exists a cardinal κ, such that for every κ-filtered ordinal λ and every λ-sequence $X : \lambda \to \mathsf{M}$ with $X_\beta \to X_{\beta+1}$ in I for every ordinal β with $\beta + 1 < \lambda$, the natural set map $\operatorname{colim}_{\beta<\lambda} \mathsf{M}(M, X_\beta) \to \mathsf{M}(M, \operatorname{colim}_{\beta<\lambda} X_\beta)$ is a bijection. M is *small* if it is small relative to $\mathrm{Mor}(\mathsf{M})$.

Definition A.2.31. Let M be a cocomplete category, and let I be a set of morphisms in M. We say that I *permits the small object argument* if the domain of any element in I is small relative to I-cell.

Now we can state the famous small object argument:

Theorem A.2.32. *Let* M *be a cocomplete category, and let* I *be a set of maps in* M*, which permits the small object argument. Then there is a functorial factorization of maps in* M*. More precisely, any map* f *in* M *can be written as* $f = \alpha(f) \pm \beta(f)$ *with* $\alpha(f)$ 2 I*-inj and* $\beta(f)$ 2 I*-cell.*

Related concepts for model categories

We are going to talk about cofibrant generation, properness and simplicial model categories in this part. A good source for this material is [Hi]:

Definition A.2.33. A *cofibrantly generated model category* is a model category such that

(1) there exists a set I of morphisms (called *a set of generating cofibrations*) that permits the small object argument and such that a morphism is a trivial fibration if and only if it has the right lifting property with respect to every element of I, and

(2) there exists a set J of morphisms (called *a set of generating trivial cofibrations*) that permits the small object argument and such that a morphism is a fibration if and only if it has the right lifting property with respect to every element of J.

Proposition A.2.34. *Let* $(\mathsf{M}, \mathsf{W}, \mathsf{C}, \mathsf{F})$ *be a cofibrantly generated model category with generating cofibrations* I *and generating trivial cofibrations* J*. Then*

(1) $\mathsf{C} = I$*-cof = the class of all retracts of* I*-cell;*

(2) $\mathsf{F} = J$*-inj;*

(3) $\mathsf{C} \backslash \mathsf{W} = J$*-cof = the class of all retracts of* J*-cell;*

(4) $\mathsf{F} \backslash \mathsf{W} = J$*-inj.*

Here is a way to recognize a cofibrantly generated model category:

Theorem A.2.35 ([Hi, Ho]). *Let M be a category, let W be a subcategory of M, and let I and J be two sets of maps in M, such that*

(1) M is complete and cocomplete;

(2) W has the two out of three property and is closed under retracts;

(3) both I and J permit the small object argument;

(4) J-cell $\subseteq \mathsf{W} \cap$ I-cof;

(5) I-inj $\subseteq \mathsf{W} \cap$ J-inj;

(6) either $\mathsf{W} \cap$ I-cof $\subseteq$ J-cof or $\mathsf{W} \cap$ J-inj $\subseteq$ I-inj.

Then there is a cofibrantly generated model category structure on M with W the weak equivalences, I-cof the cofibrations, J-inj the fibrations, I the generating cofibrations and J the generating trivial cofibrations.

Here is another way to recognize a cofibrantly generated model category through an adjoint pair:

Theorem A.2.36 (D.M. Kan). *Let C be a cofibrantly generated model category with generating cofibrations I and generating trivial cofibrations J. Let D be a category that is closed under small limits and colimits, and let $F : \mathsf{C} \rightleftharpoons \mathsf{D} : G$ be an adjoint pair. Write $FI = \{Fu \mid u \in I\}$ and $FJ = \{Fv \mid v \in J\}$. If*

(1) both of the sets FI and FJ permit the small object argument, and

(2) G takes relative FJ-cell complexes to weak equivalences,

then there is a cofibrantly generated model category structure on D in which FI is a set of generating cofibrations, FJ is a set of generating trivial cofibrations, and the weak equivalences are the maps that G takes into a weak equivalence in C. Furthermore, with respect to this model category structure, (F, G) is a Quillen pair.

Definition A.2.37. If the conditions of Kan's theorem hold, we say that we can *lift the model category structure from C to D.*

Definition A.2.38. A model category is said to be

(1) *right proper* if the class of weak equivalences is closed under pullback along fibrations;

(2) *left proper* if the class of weak equivalences is closed under pushout along cofibrations;

(3) *proper* if it is both right and left proper.

Proposition A.2.39. *Let M be a model category.*

(1) If every object of M is cofibrant, then M is left proper;

(2) If every object of M is fibrant, then M is right proper;

(3) If every object of M is both cofibrant and fibrant, then M is proper.

Proposition A.2.40. *Let M be a right proper model category. If*

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[d] & Z \arrow[l] \arrow[d] \\
X' \arrow[r] & Y' & Z' \arrow[l]
\end{tikzcd}$$

is a commutative diagram in M with all vertical maps being weak equivalences, and at least one map in each row being a fibration, then the natural map of pullbacks $X \times_Y Z \to X' \times_{Y'} Z'$ is also a weak equivalence.

Dually, let M be a left proper model category. If

$$\begin{tikzcd}
X \arrow[d] & Y \arrow[l] \arrow[r] \arrow[d] & Z \arrow[d] \\
X' & Y' \arrow[l] \arrow[r] & Z'
\end{tikzcd}$$

is a commutative diagram in M with all vertical maps being weak equivalences, and at least one map in each row being a cofibration, then the natural map of pushouts $X \coprod_Y Z \to X' \coprod_{Y'} Z'$ is also a weak equivalence.

Definition A.2.41. A *simplicial category* A is a category enriched in simplicial sets. That is, the following conditions hold:

(1) for any two objects X, Y, there is a simplicial set $\underline{\mathsf{A}}(X, Y)$;

(2) (simplicial composition) for any three objects X, Y, Z, there is a simplicial map $c_{X,Y,Z} : \underline{\mathsf{A}}(Y, Z) \times \underline{\mathsf{A}}(X, Y) \to \underline{\mathsf{A}}(X, Z)$;

(3) (simplicial identity) for any object X, there is a simplicial map $\eta : \Delta^0 \to \underline{\mathsf{A}}(X, X)$;

(4) simplicial composition for $\underline{\mathsf{A}}$ is associative, and η provides left and right simplicial identity;

(5) for any two objects X and Y, there is an isomorphism $(\underline{\mathsf{A}}(X, Y))_0 \cong \mathsf{A}(X, Y)$ which is compatible with the composition rules.

Definition A.2.42. A *simplicial model category* M is a model category that is also a simplicial category, satisfyings the following axioms:

(1) for any two objects X, Y in M and any simplicial set K, there are objects $X \otimes K$ and Y^K in M such that there are isomorphisms of simplicial sets

$$\underline{\mathsf{M}}(X \otimes K, Y) \cong \mathfrak{s}\underline{Set}(K, \underline{\mathsf{M}}(X, Y)) \cong \underline{\mathsf{M}}(X, Y^K)$$

that are natural in X, Y and K;

(2) for any cofibration $i : A \to B$ and any fibration $p : X \to Y$ in M, $\underline{\mathsf{M}}(B, X) \to \underline{\mathsf{M}}(A, X) \times_{\underline{\mathcal{M}}(A,Y)} \underline{\mathsf{M}}(B, Y)$ is a fibration in $\mathfrak{s}$Set, which is also a weak equivalence whenever i or p is.

Reedy model structure

This part is only used in the proof of Proposition 2.5.3. A good reference for this part is [Hi]:

Definition A.2.43. A *Reedy category* $(\mathsf{B}, \mathsf{B}_+, \mathsf{B}_-, d)$ is a small category B together with two subcategories B_+ and B_-, and a function (called the *degree function*) $d : \mathrm{Obj}(\mathsf{B}) \to \mathbb{N}$, such that

(1) every non-identity morphism of B_+ raises degree;

(2) every non-identity morphism of B_- lowers degree;

(3) every morphism g in B has a unique factorization $g = g_+ \circ g_-$ with g_+ in B_+ and g_- in B_-.

Example A.2.44. Let Δ be a category with objects $\underline{n} = \{0, 1, \cdots, n\}$ for all $n \in \mathbb{N}$ and morphisms $\Delta(\underline{m}, \underline{n})$ the set of all non-decreasing maps $\underline{m} \to \underline{n}$. Or if we view each $\underline{n}$ as an ordered set (hence a category) $0 < 1 < \cdots < n$, then Δ is the small category with objects the ordered sets $\underline{n}$ for all $n \in \mathbb{N}$ and morphisms functors between them. In fact, $(\Delta, \Delta_1, \Delta_2; d)$ is a Reedy category, where Δ_1 and Δ_2 are the subcategories of Δ with the same objects as Δ and morphisms injective and surjective maps in Δ, respectively, and $d : \mathrm{Obj}(\Delta) \to \mathbb{N}$ is given by $\underline{n} \mapsto n$.

Let B be a category, and let b be an object of B. We write B_b and B^b for the full subcategory of B/b and b/B without $id_b : b \to b$, respectively.

Let C be a model category, let B be a Reedy category, and let b be an object of B. Write L_b and M_b for the composition of functors $\mathsf{C}^{\mathcal{B}} \longrightarrow \mathsf{C}^{\mathcal{B}_+} \longrightarrow \mathsf{C}^{(\mathcal{B}_+)_b} \xrightarrow{\mathrm{colim}} \mathsf{C}$ and $\mathsf{C}^{\mathcal{B}} \longrightarrow \mathsf{C}^{\mathcal{B}_-} \longrightarrow \mathsf{C}^{(\mathcal{B}_-)^b} \xrightarrow{\lim} \mathsf{C}$, respectively, where both first functors are restrictions, and second functors are forgetful functors. Therefore, we have natural maps $L_b\mathbf{X} \to \mathbf{X}_b$ and $\mathbf{X}_b \to M_b\mathbf{X}$ for any object $\mathbf{X}$ of $\mathsf{C}^{\mathcal{B}}$.

Theorem A.2.45. *Suppose C is a model category, and B is a Reedy category. Then there is a model category structure (called the* Reedy model category structure*) on the functor category $\mathsf{C}^{\mathcal{B}}$, with $f : \mathbf{X} \to \mathbf{Y}$ a weak equivalence if f_b is a weak equivalence for every $b \in Obj(B)$; $f : \mathbf{X} \to \mathbf{Y}$ a (trivial) cofibration if $\mathbf{X}_b \coprod_{L_b\mathbf{X}} L_b\mathbf{Y} \to \mathbf{Y}_b$ is a (trivial) cofibration for every $b \in Obj(B)$; and $f : \mathbf{X} \to \mathbf{Y}$ a (trivial) fibration if*

$\mathbf{X}_b \to \mathbf{Y}_b \times_{M_b\mathbf{Y}} M_b\mathbf{X}$ *is a (trivial) fibration for every* $b \in Obj(B)$*. The Reedy model structure on* $\mathsf{C}^{\mathcal{B}}$ *is left proper, right proper, proper, or simplicial, if* C *is.*

Definition A.2.46. Let C be a model category. A *cosimplicial resolution* of an object c in C is a cofibrant replacement of the constant cosimplicial object c (by abuse of notation) in the Reedy model category C^{Δ}.

Proposition A.2.47. *Any two cosimplicial resolutions of an object in a model category are connected by an essentially unique zig-zag of weak equivalences between cosimplicial resolutions of this object, where essentially uniqueness means that any such zig-zag of weak equivalences connecting the two cosimplicial resolutions can be reached from any other such zig-zag of weak equivalences connecting the two cosimplicial resolutions by composing two adjacent arrows pointing in the same direction, or deleting two adjacent identical arrows with different directions, or their inverses.*

Proposition A.2.48. *Let* C *be a model category. If* $\mathbf{c}$ *is a cosimplicial object in* C*, then it is a cosimplicial resolution if and only if it is Reedy cofibrant and all of the coface and codegeneracy operators of* $\mathbf{c}$ *are weak equivalences.*

Proposition A.2.49. *Let* C *be a model category. If* $i : \mathbf{a} \to \mathbf{b}$ *is a Reedy weak equivalence of cosimplicial resolutions in* C*, and* c *is a fibrant object of* C*, then* $i^* : \mathsf{C}(\mathbf{b}, c) \to \mathsf{C}(\mathbf{a}, c)$ *is a weak equivalence of fibrant simplicial sets.*

Definition A.2.50. Let C be a model category, let $\mathbf{c}$ be a cosimplicial object in C, and let K be a simplicial set. We define $\mathbf{c} \otimes K = \operatorname{colim}_{\Delta^n \to K} \mathbf{c}(\underline{n})$.

Proposition A.2.51. *Let* C *be a model category. If* $\mathbf{c}$ *is a cosimplicial object in* C*, then* $L_{\underline{n}}\mathbf{c} \to \mathbf{c}(\underline{n})$ *is naturally isomorphic to* $\mathbf{c} \otimes \partial\Delta^n \to \mathbf{c} \otimes \Delta^n$ *for all* $n \in \mathbb{N}$*.*

Theorem A.2.52. *Let* C *be a model category and let* $\mathbf{c}$ *be a cosimplicial object in* C*. Then there is an adjoint pair* $\mathbf{c} \otimes ? : \mathfrak{s}Set \rightleftharpoons \mathsf{C} : \mathsf{C}(\mathbf{c}, ?)$*.*

A.2.2 Examples of model categories

The following standard examples are taken from [Q, DS]:

Example A.2.53. There is a cofibrantly generated proper simplicial model category structure on $\mathfrak{T}$op with weak equivalences the weak homotopy equivalences (for all points of the domain), fibrations the Serre fibrations, and cofibrations the maps having the left lifting property with respect to all trivial fibrations. We call it the *standard model structure* on $\mathfrak{T}$op.

Under this model structure, every topological space is fibrant. The generating cofibrations are $I = \{S^{n-1} \to D^n \mid n \geq 0\}$, and the generating trivial cofibrations are $J = \{D^n \times \{0\} \to D^n \times I \mid n \geq 0\}$. Every cofibration is a retract of a generalized relative CW inclusion. Therefore, CW approximation is a (non-functorial) cofibrant replacement.

Example A.2.54. The category $\mathfrak{s}$Set of simplicial sets is defined to be the functor category $\mathfrak{S}\mathrm{et}^{\Delta^{op}} = \mathfrak{P}\mathrm{re}(\Delta)$. There is a cofibrantly generated proper simplicial model category structure on $\mathfrak{s}$Set with weak equivalences the maps whose geometric realization is a weak homotopy equivalences in $\mathfrak{T}$op, cofibrations the monomorphisms, and fibrations the Kan fibrations. The generating cofibrations are $I = \{\partial\Delta^n \to \Delta^n \mid n \in \mathbb{N}\}$ and the generating trivial cofibrations are $J = \{\Lambda^n_k \to \Delta^n \mid n \in \mathbb{Z}^+, 0 \leq k \leq n\}$. Here $\Delta^n(\underline{m}) = \Delta(\underline{m}, \underline{n})$. We call it the *standard model structure* on $\mathfrak{s}$Set. Under this model structure, every simplicial set is cofibrant.

Given a simplicial set X, we use X_n to denote $X(\underline{n})$, which is isomorphic to $\mathfrak{s}\mathrm{Set}(\Delta^n, X)$ by the Yoneda lemma.

The usual adjoint pair of geometric realization and singular functors $|-| : \mathfrak{s}\mathrm{Set} \rightleftarrows \mathfrak{T}\mathrm{op} : s$ gives a Quillen equivalence. The unit and counit maps for the adjunction $X \to s(|X|)$ and $|s(Y)| \to Y$ are both weak equivalences for any simplicial set X and any topological space Y, respectively.

Example A.2.55. Let R be a unital ring. Then there is a model category structure on $\mathrm{Ch}^{\geq 0}(R)$ with weak equivalences the quasi-isomorphisms, fibrations the degreewise epimorphisms for degree > 0, cofibrations the degreewise monomorphisms with degreewise projective cokernels. This model category is cofibrantly generated by $I = \{S^{n-1}(R) \to D^n(R) \mid n \in \mathbb{N}\}$ and $J = \{0 \to D^n(R) \mid n \in \mathbb{Z}^+\}$, where for $n \in \mathbb{Z}^+$ and any (left) R-module M, $S^{n-1}(M)$ is the cochain complex with M in degree $n-1$ and 0 elsewhere, and $D^n(M)$ is the cochain complex with M in degrees $n-1$ and n, and 0 elsewhere, with $d_n = 1_M$; $S^{-1}(M) = 0$ and $D^0(M) = S^0(M)$. Under this model structure, every cochain complex is fibrant, and cofibrant objects are exactly cochain complexes with degreewise projective R-modules. Therefore, a cofibrant replacement of $S^0(M)$ is exactly a projective resolution of M. Furthermore, $\mathrm{Ho}(\mathrm{Ch}^{\geq 0}(R))(S^0(M), S^n(N)) \cong \mathrm{Ext}^n_R(M, N)$.

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Curriculum Vitae

Enxin Wu

Education:

- **The University of Western Ontario:** Canada, 2008-2012.

  Ph.D in Mathematics, expected August 2012.

- **The University of Western Ontario:** Canada, 2007-2008.

  M.Sc. in Mathematics, August 2008.

- **Zhejiang University:** P.R. China, 2005-2007.

  M.Sc. in Mathematics, June 2007.

- **Zhejiang University:** P.R. China, 2001-2005.

  B.Sc. in Mathematics, June 2005.

Teaching Experience:

- **The University of Western Ontario:** September - December 2011.

  Course Instructor for Methods of Finite Mathematics (Mathematics 1228A).

- **The University of Western Ontario:** 2007 - 2012.

  Teaching Assistant.

- **Zhejiang University:** 2005-2007.

  Teaching Assistant.

Selected Scholarships and Awards:

- **Graduate Thesis Research Fund:** The University of Western Ontario, 2011.
- **Santoni Graduate Scholarship:** The University of Western Ontario, 2009.
- **Western Graduate Research Scholarship:** The University of Western Ontario, 2007-2012.
- **Meritorious Winner of the American Mathematical Contest in Modeling**, 2005.
- **Second Prize for Innovation Scholarship in the academic year 2003-2004:** Zhejiang University, 2004.
- **Second Prize for the Mathematical Modeling Contest among all undergraduates in China**, 2004.
- **First Prize for the 2nd Mathematical Modeling Contest for Undergraduates in Zhejiang University:** Zhejiang University, 2004.
- **Excellent All-round Student in the academic year 2002-2003:** Zhejiang University, 2003.
- **First Prize for Excellent Undergraduate Scholarship in the academic year 2002-2003:** Zhejiang University, 2003.

- **Chern's Scholarship in the academic year 2002-2003:** Zhejiang University, 2003.
- **Excellent All-round Student in the academic year 2001-2002:** Zhejiang University, 2002.
- **First Prize for Excellent Undergraduate Scholarship in the academic year 2001-2002:** Zhejiang University, 2002.