2022

2022-9 A Rationalization of the Weak Axiom of Revealed Preference

Victor H. Aguiar

Per Hjertstrand

Roberto Serrano

A Rationalization of the Weak Axiom of Revealed Preference

by

Victor H. Aguiar, Per Hjertstrand, and Roberto Serrano

Research Report # 2022-9 September 2022

Western

Department of Economics

Research Report Series

Department of Economics
Social Science Centre
Western University
London, Ontario, N6A 5C2
Canada

A Rationalization of the Weak Axiom of Revealed Preference*

Victor H. Aguiar† Per Hjertstrand‡ Roberto Serrano§

This version: September 2022

Abstract Samuelson's (1938) weak (generalized) axiom of revealed preference–WGARP–is a minimal and appealing consistency condition of choice. We offer a rationalization of WGARP in general settings. Our main result is an exact analog of the celebrated Afriat's theorem, but for WGARP. Its ordinal rationalization is in terms of an asymmetric and locally nonsatiated preference function. Its cardinal rationalization uses a coalitional multi-utility (CMU) maxmin representation with a coherency restriction on the coalition structure. Effectively, the CMU representation aggregates piecemeal preferences within the decision maker (multiple rationales without preference reversals that allow for transitivity violations). Basic consumer theory and welfare analysis are also developed. Extensions to the weak axiom of revealed preference–WARP–and choices obeying the law of demand are included.

JEL Classification: C60; D10.

Keywords: abstract consumer choice; weak axiom of revealed preference; Afriat's theorem; asymmetric preference function; coalitional multi-utility rationalization; welfare analysis.

*This paper supersedes our previous draft titled "The Theory of Weak Revealed Preference." We are grateful to Roy Allen, Tilman Börgers, Chris Chambers, Peter Caradonna, Pawel Dziewulski, Federico Echenique, Francoise Forges, Mira Frick, Charles Gauthier, Maria Goltsman, Reinhard John, Nail Kashaev, Bart Lipman, Rosa Matzkin, Yusufcan Masatlioglu, Hiroki Nishimura, Efe Ok, John Rehbeck, Phil Reny, Marciano Siniscalchi, Al Slivinsky, and Gerelt Tserenjigmid for useful comments and encouragement. We thank Judith Levi for her great editing work. Hjertstrand thanks Jan Wallander och Tom Hedelius stiftelse and Marianne och Marcus Wallenberg stiftelse for funding.

†Department of Economics, The University of Western Ontario, London, Ontario, Canada, vaguiar@uwo.ca.

‡Research Institute of Industrial Economics (IFN), Stockholm, Sweden, per.hjertstrand@ifn.se.

§Department of Economics, Brown University, Providence, Rhode Island, U.S.A., roberto_serrano@brown.edu.

1. Introduction

Rooted in the seminal work of Samuelson (1938), the weak (generalized) axiom of revealed preference–WGARP–has been seen as a minimal, normatively appealing, and potentially empirically robust consistency condition of choice. However, a general rationalization of WGARP remains unknown. Our paper fills this important gap in the literature. The axiom is built on the direct preference relation: bundle x^t is *(strictly) directly revealed preferred to* bundle x^s if a consumer chooses x^t whenever x^s is (strictly) affordable (at prices p^t), i.e., if it holds that $p^t(x^t - x^s) \geq (>)0$. WGARP states that there is no pair, x^t and x^s, such that x^t is directly revealed preferred to x^s, while at the same time x^s is strictly and directly revealed preferred to x^t.[1]

Standard utility maximization is characterized by the generalized axiom of revealed preference–GARP–(Afriat's theorem, Afriat 1967), which, in addition to being consistent with WGARP, requires transitivity of preferences. There is abundant experimental and field evidence against transitivity (Tversky 1969; Quah 2006). Our paper's second motivation is exploring a general consumer theory that, while accepting consistent binary choices, drops transitivity. In this endeavor, previous work does not provide a rationalization of WGARP without imposing additional restrictions. Indeed, the existing literature has mainly been focused on demand functions with infinite data and not demand correspondences with arbitrary (i.e., finite or infinite) data. For instance, the influential work of Kihlstrom et al. (1976)–henceforth KMS–, which essentially proposes to rewrite the entire theory of demand functions based on WGARP alone, falls short of providing a rationalization. The KMS paper (p.977) conjectures that WGARP can be rationalized by the model of consumer behavior in Shafer (1974). Shafer's model describes a nontransitive consumer who nevertheless satisfies WGARP. We discuss the conjecture in KMS and show its difficulties in finite data sets (because of the possibility of nonconvex preferences); see Subsection 3.2. More recently, Quah (2006) and Kim and Richter (1986) provide rationalizations of WGARP but rely on additional restrictions, such as a form of convexity in preferences and infinite data. Such convexity assumptions limit the scope of the results and their empirical interest. Our results, in contrast, do not impose convexity, are extended to general abstract choice setups, and can be used in both finite and infinite data.

[1] Samuelson (1938) focuses on demand functions, and studies the weak axiom of revealed preference (WARP–there is no pair of distinct observations such that x^t is directly revealed preferred to x^s, and at the same time x^s is directly revealed preferred to x^t), a special case of WGARP. Varian (1982) points out that it is empirically more convenient to work with demand correspondences, which, allowing for indifferences, provide a natural justification for WGARP. Our main focus is WGARP, but we extend our results to WARP (see our similar characterization of WARP in Section 6.1). In some early demand literature, WGARP is referred to as the weak weak axiom of revealed preference (Kihlstrom et al., 1976).

The modern revealed-preference analysis is interested in the study of WGARP due to its amenability to empirical work. Indeed, recognizing some difficulties in the computational complexity of standard utility maximization in setups of empirical interest (e.g., stochastic utility maximization, which is NP-hard to check; see Kitamura and Stoye 2018), there has been a renewed interest in using WGARP as a minimalist version of the standard model of rationality. (See, for example, Blundell et al. (2008), Blundell et al. (2014), Hoderlein and Stoye (2014), Cosaert and Demuynck (2018), and Cherchye et al. (2019).[2]) In addition, many results in general equilibrium, consumer theory, and measurement also rely on WGARP (e.g., Quah 2008).

Due to the potential lack of transitivity, we work with preference functions instead of utility functions.[3] Our paper provides an exact analog of the classic Afriat's Theorem, but for WGARP. Namely, (i) a minimal rationalization of WGARP in terms of a preference function with only ordinal properties (asymmetry and local nonsatiation); (ii) a characterization of WGARP in terms of Afriat and Varian inequalities (useful for empirical work); and (iii) a cardinal and tractable representation. In particular, the third item just mentioned provides a rationalization of WGARP based on a version of the coalitional multi-utility (CMU) model in Nishimura and Ok (2016). The CMU model allows us to develop a consumer theory for WGARP and provide a tractable way to model intransitive consumer behavior. We use a CMU model, which says that a bundle x is ranked over a bundle y whenever there is at least one coalition of utilities for which every utility inside it ranks x over y.

To motivate the CMU model, suppose the consumer is trying to figure out her true preferences over three alternatives A, B, and C. The consumer is endowed with three coalitions of utilities. Each coalition is associated with a different attribute or mood. There are three attributes U_1, U_2, and U_3. Coalition U_1 contains piecemeal utility u_{11} ($u_{11}(C) > u_{11}(B) > u_{11}(A)$), and piecemeal utility u_{12} ($u_{12}(A) > u_{12}(C) > u_{12}(B)$). Coalition U_2 contains piecemeal utility u_{21} ($u_{21}(B) > u_{21}(A) > u_{21}(C)$), and piecemeal utility u_{22} ($u_{22}(A) > u_{22}(B) > u_{22}(C)$). We need the CMU to satisfy coherency, which means that both coalitions must share a common utility u_* ($u_*(A) > u_*(C) > u_*(B)$). Finally, U_3 contains u_{31} ($u_{31}(B) > u_{31}(C) > u_{31}(A)$). In addition, it also contains the utilities u_{11} and u_{21}, yielding coherency.

When she compares A to C, all utilities in U_2 agree that A is better than C. As explained, according to our CMU model, the consumer prefers one alternative to another

[2]In all of these papers, WGARP is usually stated without indifference, because the object of interest is a demand function, not a demand correspondence.

[3]In doing so, we follow the tradition of Shafer (1974) and John (2001). Over the consumption set X, a preference function r is a mapping $r : X \times X \mapsto \mathbb{R}$. The inequality $r(x, y) \geq 0$ means that x is preferred to y.

whenever at least one coalition has unanimity in how the piecemeal utilities rank the two. Then this consumer declares that A is better than C ($A \succ C$). Similarly, when she compares C to B, all utilities in U_1 agree that C is preferred to B, and the consumer declares that $C \succ B$. Finally, when comparing A to B, all utilities in U_3 agree that B is better than A, and the consumer declares that $B \succ A$. This CMU consumer has produced a violation of transitivity, as $A \succ C \succ B \succ A$. However, the consumer is consistent in any pairwise choice. In particular, coherency guarantees that unanimity on one coalition implies that there is at least one piecemeal utility in every other coalition that can *veto* preference reversals.[4]

Thus, the CMU notion brings to the forefront the idea of preference formation when we think of each coalition as representing an attribute, state of mood, or some other hidden criterion, analogously to that in Richter and Rubinstein (2019). Standard utility maximization is a special case when the consumer only cares about one attribute (i.e., coalition), and there is only one piecemeal utility within the single coalition.

In the standard consumer setting, Theorem 1 in the current paper provides an exact analog of Afriat's Theorem for WGARP. In particular, this result states that a data set is consistent with WGARP *if and only if* it can be rationalized by a *coherent maxmin CMU preference function*.[5] A generalization to abstract choice settings, including nonlinear budgets and infinite data sets, is provided in Theorem 2.

To unpack the statement we just made, we define the notion of rationalization. We say that a finite data set $O^T = \{p^t, x^t\}_{t\in\mathbb{T}}$ (a collection of prices and commodity bundles) is rationalized by a preference function $r : X \times X \to \mathbb{R}$, where X is the consumption set, if, for all t, $r(x^t, y) \geq 0$ for any $y \in X$ that is affordable at price p^t (and wealth $p^t x^t$).

The maxmin CMU preference function is a representation of a preference function. Let Ω be a family of sets of utilities (coalitions), and let U be a typical element of Ω. Let $u : X \mapsto \mathbb{R}$ denote a piecemeal utility function that belongs to $U \in \Omega$. Coherency requires that every pair of coalitions has at least one common element, i.e., $U \cap V \neq \emptyset$ for all $U, V \in \Omega$. We say that a coherent maxmin CMU preference rationalizes a data set function r if, for any $x, y \in X$, we can write $r(x, y)$ as:

$$r(x, y) = \max_{U\in\Omega} \min_{u\in U}(u(x) - u(y)).$$

Thus, maxmin CMU rationalization can be interpreted as an aggregation of preferences of an individual with multiple piecemeal utility functions. When figuring out her preference

---

[4] This intransitive behavior is also described, for instance, in Hicks classic example, reproduced as Example 2.F.1 in Mas-Colell et al. (1995), which gives a data set satisfying WGARP (in its strict form) but is incompatible with the existence of underlying rational preferences.

[5] The result for WARP is similar, simply switching to strict rationalization (Section 6.1).

over two bundles, this consumer takes the maximum over the minimal difference among the piecemeal utilities of the two bundles. Hence, she justifies her preferences by saying that she prefers x over y whenever this ranking is unanimously true for all piecemeal utilities that belong to at least one coalition. Moreover, all pairs of coalitions share at least some piecemeal utility, thus limiting–to some extent–internal contradictions, but not enough to avoid intransitivity. While coherency is a minimal requirement on the CMU that we show to be empirically equivalent to choices being consistent with WGARP, we note that the CMU model without any restrictions has no empirical content (Nishimura and Ok 2016). Besides the additional restriction on the CMU, our characterization complements the results in Nishimura and Ok (2016) since they implicitly assume knowledge of the entire preference relation while our results are posed in a revealed-preference environment, which only assumes some knowledge of the consumer's choices. Indeed, we provide an extension result analogous to Afriat's Theorem, while Nishimura and Ok (2016) provide a representation result analogous to Debreu's classical utility representation theorem.[6] After making these remarks, we refer to the maxmin CMU model simply as the CMU model in the sequel.[7]

We show next that WGARP is compatible with the consumer having nonconvex preferences, which may lead to instances of indecisiveness, i.e., an empty-valued demand correspondence, in sharp contrast to Afriat's Theorem for the case of utility maximization. Nevertheless, we characterize the CMU's guaranteeing that the consumer problem always has a solution. The necessary and sufficient condition relies on the Nakamura number or the minimal number of coalitions in the CMU that share a common piecemeal utility. Suppose the Nakamura number, which measures the degree of agreement or collegiality among coalitions, is high enough. In that case, the associated demand correspondence will be nonempty.[8] In addition, we provide a sufficient condition, more closely tied to a relaxation of convexity. The new sufficient condition is called "total coherency in segments" and requires that when restricting the piecemeal utilities to any straight-line segment in the commodity space, all coalitions share a common piecemeal ranking. (Indeed, we also

[6] There is a way to connect both approaches: for instance, see Lemma 2.

[7] In a related model of choice under risk, Hara et al. (2019) is the first to define the coherence property over a collection of coalitions of utilities. However, they use a minimax coherent CMU representation, which implies completeness of the preference relation. In contrast, our maxmin coherent CMU model allows for both incomplete and intransitive preferences, but it imposes a critical regularity condition, i.e., asymmetry of the preference function (see Lemma 1.)

[8] Using another new result characterizing the Nakamura number condition in terms of k-acyclicity, a weakening of GARP, all is needed to guarantee nonemptiness of the demand correspondence is that the data not contain revealed-preference cycles of length smaller than the number of goods. The Nakamura number is an important measure of agreement in the literature on preference aggregation (Schofield, 1984).

show that the classical requirement of convexity in preferences, guaranteeing a solution to the consumer problem for the Shafer's (1974) nontransitive consumer implies total coherency in segments.)

The CMU model allows us to obtain standard textbook results for WGARP, such as homogeneity of degree 0 of the demand correspondence and the satisfaction of the compensated law of demand, and it offers some computational advantages due to its functional form. Beyond all these (positive) findings for consumer theory, we provide a new way to perform welfare analysis for the CMU. This welfare analysis bounds the set of alternatives that are ranked above a given commodity bundle. These new bounds are robust to failures of transitivity and convexity of preferences. Moreover, with a simple switch from weak to strict inequalities, we also demonstrate that the CMU model characterizes WARP, and by adding a quasilinearity restriction, the CMU delivers choices satisfying the law of demand.

The plan of the paper is as follows. After preliminaries are introduced in Section 2, Section 3 presents our characterizations of WGARP. Section 4 provides a textbook consumer theory for WGARP and the coherent CMU. Section 5 develops the corresponding welfare analysis. Section 6 notes the extensions of our main results to WARP and choices obeying the law of demand, as well as the extended literature review, all found in the online appendix. Section 7 concludes. All proofs are collected in the Appendix.

2. Preliminaries

Although our main result also covers abstract choice problems (as detailed at the end of the next section), we want to connect with the traditional revealed-preference literature and begin by presenting the classic consumer problem. Suppose that a consumer chooses bundles consisting of $L \geq 2$ goods. We assume that we have access to a *finite* number of observations, denoted by T, on the prices and chosen quantities of these goods, where observations are indexed by $\mathbb{T} = \{1, \ldots, T\}$. Let $x^t \in X \equiv \mathbb{R}^L_+ \setminus \{0\}$ denote the bundle of goods at observation $t \in \mathbb{T}$, which was purchased at prices $p^t \in P \equiv \mathbb{R}^L_{++}$. We impose Walras' law throughout: wealth at observation t is equivalent to $p^t x^t \in \mathbb{R}_{++}$, for all $t \in \mathbb{T}$.[9] We write $O^T = \{p^t, x^t\}_{t\in\mathbb{T}}$ to denote all price-quantity observations, and refer to O^T as

[9]We use the following notation: The inner product of two vectors $x, y \in \mathbb{R}^L$ is defined as $xy = \sum_{l=1}^{L} x_l y_l$. For all $(x, y) \in \mathbb{R}^L$, $x \geqq y$ if $x_i \geq y_i$ for all $i = 1, \ldots, L$; $x \geq y$ if $x \geqq y$ and $x \neq y$; and $x > y$ if $x_i > y_i$ for all $i = 1, \ldots, L$. As is customary, we denote $\mathbb{R}^L_+ = \{x \in \mathbb{R}^L : x \geqq (0, \ldots, 0)\}$ and $\mathbb{R}^L_{++} = \{x \in \mathbb{R}^L : x > (0, \ldots, 0)\}$.

the data. In practice, the data O^T describe a single consumer observed over time.

2.1. Revealed Preference

We begin by recalling some key definitions in the revealed-preference literature.

Definition 1. *(Direct revealed preferred relations) We say that x^t is directly revealed preferred to x^s, written $x^t \succeq^{R,D} x^s$, when $p^t x^t \geq p^t x^s$. Also, x^t is strictly and directly revealed preferred to x^s, written $x^t \succ^{R,D} x^s$, when $p^t x^t > p^t x^s$.*

If x^t is directly revealed preferred to x^s, the consumer chose x^t and not x^s, when both bundles were affordable. If x^t is strictly and directly revealed preferred to x^s, then she could also have saved money by choosing x^s. These definitions only compare pairs of bundles. We can extend them to compare any subset of bundles by using the transitive closure of the direct relation:

Definition 2. *(Revealed preferred relations) We say that x^t is revealed preferred to x^s, written $x^t \succeq^R x^s$, when there is a chain $(x^1, x^2, \ldots, x^n)$ with elements on X with $x^1 = x^t$ and $x^n = x^s$ such that $x^1 \succeq^{R,D} x^2 \succeq^{R,D} \ldots \succeq^{R,D} x^n$. Also, x^t is strictly revealed preferred to x^s, written $x^t \succ^R x^s$, when at least one of the directly revealed relations in the revealed preferred chain are strict.*

Next, we use these binary relations to define axioms that characterize different types of consistent consumer behavior. We begin with Samuelson's (1938) weak axiom of revealed preference:

Axiom 1. *(WARP) The weak axiom of revealed preference (WARP) holds if there is no pair of observations $s, t \in \mathbb{T}$ such that $x^t \succeq^{R,D} x^s$ and $x^s \succeq^{R,D} x^t$, with $x^t \neq x^s$.*

Kihlstrom et al. (1976) introduces a generalized version of WARP:

Axiom 2. *(WGARP) The weak generalized axiom of revealed preference (WGARP) holds if there is no pair of observations $s, t \in \mathbb{T}$ such that $x^t \succeq^{R,D} x^s$ and $x^s \succ^{R,D} x^t$.*

Samuelson (1948) shows how WARP can be used to construct a set of indifference curves in the $L = 2$ case, but also recognizes that WARP is not enough to characterize rationality when $L > 2$. Responding to this challenge, Houthakker (1950) introduces the strong axiom of revealed preference (SARP), which makes use of transitive comparisons between bundles as implied by the revealed-preference relation:

Axiom 3. *(SARP) The strong axiom of revealed preference (SARP) holds if there is no pair of observations $s, t \in \mathbb{T}$ such that $x^t \succeq^R x^s$ and $x^s \succeq^{R,D} x^t$, with $x^t \neq x^s$.*

Varian (1982) notes that SARP requires single-valued demand functions, and argues that it is empirically more convenient to work with demand correspondences and "flat" indifference curves. To accommodate these properties, Varian introduces the generalized axiom of revealed preference (GARP):

Axiom 4. *(GARP) The generalized axiom of revealed preference (GARP) holds if there is no pair of observations* $s, t \in \mathbb{T}$ *such that* $x^t \succeq^R x^s$ *and* $x^s \succ^{R,D} x^t$.

In the $L = 2$ case, as is well known, SARP is equivalent to WARP (Rose 1958), and GARP is equivalent to WGARP (Banerjee and Murphy 2006).

We focus on characterizing the weaker notion of demand correspondences (WGARP) in our principal analysis, leaving the characterization of the stricter notion of single-valued demand functions (WARP) to Section 6.1. But first, we recall the main results from the revealed-preference literature that are needed in order to introduce our contribution. Consider the following definition of rationalization:[10]

Definition 3. *(Utility rationalization) Consider a finite data set* $O^T = \{p^t, x^t\}_{t\in\mathbb{T}}$ *and a utility function* $u : X \mapsto \mathbb{R}$. *For all* $y \in X$ *and all* $t \in \mathbb{T}$ *such that* $p^t y \leq p^t x^t$, *the data* O^T *is rationalized by* u *if* $u(x^t) \geq u(y)$.

Afriat's (1967) fundamental theorem is well known:

Theorem A. (Afriat's theorem, Varian 1982) Consider a finite data set $O^T = \{p^t, x^t\}_{t\in\mathbb{T}}$. The following statements are equivalent:

(i) The data O^T can be rationalized by a locally nonsatiated utility function.

(ii) The data O^T satisfies GARP.

(iii) There exist numbers U^t and $\lambda^t > 0$ for all $t \in \mathbb{T}$ such that the Afriat inequalities:

$$U^t - U^s \geq \lambda^t p^t (x^t - x^s),$$

hold for all $s, t \in \mathbb{T}$.

(iv) There exist numbers V^t for all $t \in \mathbb{T}$ such that the Varian inequalities:

$$\text{if } p^t(x^t - x^s) \;\; \geq \;\; 0 \text{ then, } V^t - V^s \geq 0,$$

[10] We say that a utility function $u : X \mapsto \mathbb{R}$ is: *(i) continuous* if for any sequence (x^n) for $n \in \mathbb{N}_+$ such that $x^n \in X$ and $\lim_{n\to\infty} x^n = x$ with $x \in X$ implies $\lim_{n\to\infty} u(x^n) = u(x)$; *(ii) locally nonsatiated* if for any $x \in X$ and for any $\epsilon > 0$, there exists $y \in B(x, \epsilon)$ where $B(x, \epsilon) = \{z \in X| \quad ||z - x|| \leq \epsilon\}$ such that $u(y) > u(x)$; *(iii) strictly increasing* if for $x, y \in X$, $x \geq y$ implies $u(x) > u(y)$; and *(iv) concave* if for any $x, y \in X$, we have $u(x) - u(y) \geq \xi\,(x - y)$, for $\xi \in \partial u(y)$, where $\partial u(y)$ is the superdifferential of u.

$$\text{if } p^t(x^t - x^s) \;\; > \;\; 0 \text{ then, } V^t - V^s > 0,$$

hold for all $s, t \in \mathbb{T}$.

(v) The data O^T can be rationalized by a continuous, strictly increasing, and concave utility function.

There are three messages of Afriat's theorem. First, the equivalence of (i) and (ii) gives a minimal rationalization in terms of purely ordinal conditions (local nonsatiation, completeness, and transitivity of an underlying preference relation). Second, statements (ii), (iii), and (iv) give testable conditions that are easy to implement in practice. Moreover, third, statement (v) gives a rationalization in terms of a cardinal (concave) utility function. Their equivalence means, in particular, that continuity, monotonicity, and concavity are nontestable properties. In other words, separate violations of any of these properties cannot be detected in finite data sets.

Varian (1982) shows that the numbers U^t and λ^t in statement (iii) can be interpreted as measures of the utility level and marginal utility level of income at observation $t \in \mathbb{T}$. Analogously, the numbers V^t in statement (iv) can be interpreted as measures of the utility levels at the observed demands (Demuynck and Hjertstrand, 2019).

2.2. Preference Functions

We define a preference function as follows:

Definition 4. *(Preference function) A preference function, $r : X \times X \to \mathbb{R}$, maps ordered pairs of commodity bundles to real numbers.*

A preference function is a more flexible numerical representation of a consumer's preferences than a utility function, for example, with nontransitive preferences. If $r(x, y) \geq 0$ then the consumer prefers bundle x to y. Similarly, if $r(x, y) > 0$, x is strictly preferred to y. Next, we present two properties of preference functions that feature in our characterizations of WGARP:

Definition 5. *(Asymmetry) We say that a preference function $r : X \times X \mapsto \mathbb{R}$ is* asymmetric *if $r(x, y) \geq (>)0$ implies $r(y, x) \leq (<)0$ for all $x, y \in X$.*

Asymmetry is an ordinal concept stating that if the consumer prefers x over y, then she cannot simultaneously strictly prefer y over x. The notion of asymmetry can be further strengthened to obtain a cardinal version, called skew-symmetry:

Definition 6. *(Skew-symmetry) We say that a preference function* $r : X \times X \mapsto \mathbb{R}$ *is* skew-symmetric *if* $r(x, y) = -r(y, x)$ *for all* $x, y \in X$.

Skew-symmetry means that the preference function r induces a preference order on X that is complete and asymmetric. This property was studied by Shafer (1974) in his study of a theory of nontransitive consumers.

Next, we formally define some other important properties of preference functions:

Definition 7. *Consider a preference function* $r : X \times X \to \mathbb{R}$. *We say that:*

(i) r *is complete if for any* $x, y \in X$, *either* $r(x, y) \geq 0$ *or* $r(y, x) \geq 0$.

(ii) r *is* continuous *if for all* $y \in X$ *and any sequence* $\{x^n\}$ *of elements in* X *that converges to* $x \in X$ *it must be that* $\lim_{n\to\infty} r(x^n, y) = r(x, y)$.[11]

(iii) r *is* locally nonsatiated *if for any* $x, y \in X$ *such that* $r(x, y) = 0$ *and for any* $\epsilon > 0$, *there exists a* $y' \in B(y, \epsilon)$ *such that* $r(x, y') < 0$.[12]

(iv) r *is* strictly increasing *if for all* $x, y, z \in X$, $x \geq z$ *implies* $r(x, y) > r(z, y)$.

(v) r *is* quasiconcave *if for all* $x, y, z \in X$ *and any* $0 \leq \lambda \leq 1$ *we have* $r(\lambda x + (1-\lambda) z, y) \geq \min\{r(x, y), r(z, y)\}$, *and* strictly quasiconcave *if, for any* $0 < \lambda < 1$, *the inequality is strict whenever* $x \neq z$.

(vi) r *is* concave *if for all* $x, y, z \in X$ *and any* $0 \leq \lambda \leq 1$ *we have* $r(\lambda x + (1-\lambda) z, y) \geq \lambda r(x, y) + (1-\lambda) r(z, y)$, *and* strictly concave *if, for any* $0 < \lambda < 1$, *the inequality is strict whenever* $x \neq z$.

(vii) r *is* piecewise concave *if there is a set of concave functions in the first argument,* $f_t(x, y)$ *for* $t \in \mathbb{K}$, *where* $\mathbb{K}$ *is a compact set, such that* $r(x, y) = \max_{t\in\mathbb{K}}\{f_t(x, y)\}$, *and* strictly piecewise concave *if there is a similar set of strictly concave functions.*

Completeness is a weak condition to ensure that the preference function can rank all bundles. Continuity is a technical condition that is convenient to ensure the existence of a maximum in the constrained maximization of the preference function (Sonnenschein 1971). Local nonsatiation rules out thick indifference curves: if we take an arbitrarily small neighborhood of a bundle that is indifferent to a given bundle x, the neighborhood contains bundles that dominate x. Strict monotonicity means that "more is better".

[11]We state the weaker versions of these properties, as all we need is to work with movements in one of the arguments.

[12]The following stronger version of this assumption is the standard LNS of a preference relation: for any $x, y \in X$ such that $r(x, y) = 0$ and for any $\epsilon > 0$, there exists a $y' \in B(y, \epsilon)$ such that $r(y', x) > 0$. Thus, strictly speaking, a better name for our definition is perhaps *weak local nonsatiation*.

Quasiconcavity says that for any fixed $y \in X$, a mixture of two bundles $x, z \in X$ is at least as good as the worst of the two bundles, according to the preference function. Concavity is a cardinal version of quasiconcavity. Both are essential properties because they ensure well-behaved optimization problems. More precisely, quasiconcavity guarantees that a function defined on a compact set has a convex set of maxima, while a strictly concave function defined on a compact set always has a unique global maximum.

Piecewise concavity and its strict version are new properties, which are especially important for our cardinal characterization of WGARP.[13] The property implies that for a fixed $y \in X$, a mixture of two bundles $x, z \in X$ is at least as good as the worst one of the two bundles, but only if x, z are close enough. In other words, this is a local version of concavity, i.e., concavity implies piecewise concavity but not vice versa. This property provides tractability to the consumer maximization problem of the CMU (see Section 4).

It is helpful to translate the notions of quasiconcavity, concavity, and piecewise concavity of a preference function into properties of the underlying preference relation. Concavity or quasiconcavity of a preference function implies convexity of the preference relation –upper contour sets being convex sets. In turn, the preference relation's convexity implies the *star-shapedness of its upper contour set* (i.e., if x is preferred to y, any bundle in the convex combination of x and y is also preferred to y). Piecewise concavity of a preference function does not imply the preference relation's convexity but the star-shapedness property.

Finally, we define the notion of rationalization by a preference function, which is analogous to utility rationalization in Definition 3:

Definition 8. *(Preference function rationalization) Consider a finite data set* $O^T = \{p^t, x^t\}_{t\in\mathbb{T}}$ *and a preference function* $r : X \times X \mapsto \mathbb{R}$*. For all* $y \in X$ *and all* $t \in \mathbb{T}$ *such that* $p^t y \leq p^t x^t$*, the data* O^T *is rationalized by* r *if* $r(x^t, y) \geq 0$*.*

2.3. The Coalitional Multi-Utility Model

In this subsection, we introduce a particular representation of a preference function, which is essential for our cardinal characterization of WGARP.

Definition 9. *(Coalitional (strict) multi-utility model) We say that the preference function* $r(x, y)$ *is a coalitional (strict) multi-utility (CMU) function if, for any* $x, y \in X$*, it can be written as:*

$$r(x, y) = \max_{U\in\Omega} \min_{u\in U} (u(x) - u(y)),$$

[13] See Zangwill (1967) and Tsevendorj (2001) for a detailed discussion of piecewise-concave functions.

where Ω is a compact family of compact sets of piecemeal utility functions and any piecemeal utility function u is continuous, strictly increasing, and (strictly) concave.[14]

Note that the classical utility maximization model is a special case of the CMU model when the set of coalitions and sets of utilities are singletons, in which case, $r(x, y) = u(x) - u(y)$ for the single utility function u.

As presented in Definition 9, the CMU model is too general for our purposes, and we need to impose further structure to characterize WGARP. In particular, we will only consider CMU preference functions that are coherent:

Definition 10. *Consider the CMU preference function r in Definition 9. We say that r is coherent if the compact set Ω is such that for any $U, \hat{U} \in \Omega$, there exists a $u \in U \cap \hat{U}$.*

Coherency allows us to establish the following result:

Lemma 1. *(Coherency) If a preference function r is a coherent CMU function, then r is asymmetric.*

By Definitions 9 and 10, the coherent CMU preference function is continuous, strictly increasing, and piecewise concave (in the first argument). However, the coherent CMU model is neither skew-symmetric nor (quasi-)concave, which means that it does allow for preferences that are neither complete nor convex. This model is, therefore, different from the nontransitive (skew-symmetric) consumer model studied in Shafer (1974), which represents both complete and convex preferences. In the next section, we show that WGARP is empirically equivalent to a rationalization of a piecewise-concave CMU preference function but that this function generally is not concave. Moreover, this rationalizing function represents preferences with the star-shapedness property of upper contour sets, but it may not represent convex preferences. Of course, it is true that, if one considers sufficiently large finite data sets, the gap between the star-shapedness property and the convexity of the preference becomes very narrow.

While our main results offer characterizations of WGARP and WARP in terms of coherent CMU preference functions, there should be a way to translate them into the language of preference relations. Going in this direction, we close this section by offering such a connection. This general representation result answers the question of what preference relations can be represented by asymmetric preference functions, and it is an extension of the representation theorem in Nishimura and Ok (2016). Before establishing our connecting result, we need some preliminaries. A preference relation is denoted by

[14] For the standard consumer setting, we endow the space Ω with a suitable topology, while we use the usual topology for the finite-dimensional Euclidean space of commodities (see, for example, Nishimura and Ok (2016)). For more abstract settings, we work with an order structure, as detailed in Subsection 3.3.

$\succeq \subseteq X \times X$. We say that $\succeq$ is reflexive if for all $x \in X$, $x \succeq x$ and continuous if upper contour sets are closed subsets of $X \times X$. Moreover, for any bundles $x, y \in X$, $x \succ y$ if $x \succeq y$ and $\neg y \succeq x$, and $\succeq$ is asymmetric if: (i) $x \succeq y$ implies $\neg y \succ x$, and (ii) $x \succ y$ implies $\neg y \succeq x$.

Lemma 2. *(Representation) The following statements are equivalent:*

(i) $\succeq$ *is reflexive and asymmetric.*

(ii) For all $x, y \in X$, $x \succeq (\succ) y$ *if and only if* $r(x,y) = \sup_{U \in \Omega} \inf_{u \in U} (u(x) - u(y)) \geq (>) 0$, *where* Ω *is a nonempty collection of continuous utility functions satisfying coherency.*

In addition to reflexivity (as in Nishimura and Ok (2016)), we also have asymmetry of the preference relation. We note that a CMU preference function is always reflexive (i.e., $r(x,x) \geq 0$ for all $x \in X$).[15]

3. Complete characterizations of WGARP

This section presents characterizations of WGARP in consumer settings and in abstract choice settings, and an example illustrating how WGARP may be incompatible with convex preferences.

3.1. The Consumer Choice Setting

The next theorem provides a revealed-preference characterization of WGARP for finite data sets. This result mirrors Afriat's theorem, relying on preference-function rationalization (as opposed to utility rationalization):

Theorem 1. *Consider a finite data set* $O^T = \{p^t, x^t\}_{t \in \mathbb{T}}$. *The following statements are equivalent:*

(i) The data O^T *can be rationalized by a locally nonsatiated and asymmetric preference function.*

[15] We also note that if $\succeq$ is continuous (and X is compact), then we can strengthen the representation theorem and have that Ω be a compact collection of nonempty compact subsets of the set of continuous utility functions. Then, the sup and inf operators can be replaced by max and min.

(ii) The data O^T satisfies WGARP.

(iii) There exist numbers $R^{t,s}$ and $\lambda^t_{ts} > 0$ for all $s, t \in \mathbb{T}$ with $R^{t,s} = -R^{s,t}$ and $\lambda^t_{ts} = \lambda^t_{st}$ such that inequalities:

$$R^{t,s} \geq \lambda^t_{ts} p^t (x^t - x^s),$$

hold for all $s, t \in \mathbb{T}$.

(iv) There exist numbers $W^{t,s}$ for all $s, t \in \mathbb{T}$ with $W^{t,s} = -W^{s,t}$ such that inequalities:

$$\text{if } p^t(x^t - x^s) \geq 0 \text{ then, } W^{t,s} \geq 0,$$
$$\text{if } p^t(x^t - x^s) > 0 \text{ then, } W^{t,s} > 0,$$

hold for all $s, t \in \mathbb{T}$.

(v) The data O^T can be rationalized by a coherent CMU preference function (which, in particular, is asymmetric, continuous, monotonic, and piecewise concave).

This theorem is an exact analog of Afriat's theorem, but for WGARP and using preference-function rationalizations. First, the equivalence of (i) and (ii) identifies a minimal set of purely ordinal properties (asymmetry and local nonsatiation) of a preference function rationalizing data obeying WGARP. This equivalence is essentially an extension theorem of the revealed-preference relation satisfying WGARP (defined on the data set) to a preference function (defined over X) with the asymmetry and local nonsatiation properties.[16] Second, the equivalence of statements (ii), (iii), and (iv) provides practical inequalities for empirical work. Furthermore, third, the equivalence of (ii) and (v) is a cardinal rationalization, whose main difference with Afriat's rationalization of GARP is piecewise concavity instead of concavity. Since the CMU preference function in statement (v) also is asymmetric, continuous, monotonic, and piecewise concave, the equivalence of (i) and (v) shows that if the data can be rationalized by any nontrivial preference function at all, it can be rationalized by a preference function that satisfies continuity, monotonicity, and piecewise concavity. Put differently, separate violations of these properties cannot be detected in finite data sets.

The numbers $R^{t,s}$ and λ^t_{st} in statement (iii) have a similar interpretation as in Afriat's theorem for each pair of local utilities; that is, if we consider $t, s \in \mathbb{T}$, then $R^{t,s}$ is a measure of the utility difference $u_{ts}(x^t) - u_{ts}(x^s)$ for that particular pairwise data set, while λ^t_{st} is a measure of the marginal utility of income at observation $t \in \mathbb{T}$ in that pairwise data set.

[16] We will show later in a corollary that completeness can also be obtained in the extension result.

The proof of Theorem 1 shows that the property of asymmetry in statement (v) may be further strengthened to skew-symmetry, by establishing the following equivalence:

Corollary 1. *The data O^T can be rationalized by a continuous, monotonic, piecewise-concave, and skew-symmetric preference function if and only if O^T satisfies WGARP.*

This corollary helps to analyze a well-known claim in Kihlstrom et al. (1976). The claim conjectures that WGARP is empirically equivalent to a consumer maximizing an underlying preference function that is skew-symmetric, continuous, and *quasiconcave*.[17] Our characterization of WGARP shows that this conjecture would be false in finite data sets since a piecewise-concave preference function characterizes WGARP. Furthermore, again, we acknowledge that the gap between our star-shapedness implied property and convex preferences seem to disappear in infinite data sets, hence lending support to the KMS conjecture.[18]

3.2. On WGARP and Convexity of Preferences

In this subsection, we elaborate further on the last point made in the previous section by showing, employing a counterexample, that Kihlstrom et al.'s (1976) conjecture that the Shafer model can rationalize WGARP does not hold in finite data sets.

Example 1. *(Keiding and Tvede 2013, Example 1, p.467). Consider the data set O^3 with prices $p^1 = (4,1,5)'$, $p^2 = (5,4,1)'$, $p^3 = (1,5,4)'$, and bundles $x^1 = (4,1,1)'$, $x^2 = (1,4,1)'$, $x^3 = (1,1,4)'$. It is easy to verify that this data set satisfies WGARP. Consider a new unobserved commodity bundle, x^{T+1}:*

$$x^{T+1} = \frac{1}{3}(x^1 + x^2 + x^3) = (2,2,2)'.$$

The question is whether there exists a price-vector p^{T+1} such that the extended data set $O^3 \cup (p^{T+1}, x^{T+1})$ satisfies WGARP. Note that $p^t(x^t - x^{T+1}) = 2 > 0$, for all $t = 1,2,3$. But this implies that there is no $p \in P$ such that $p(x^{T+1} - x^t) < 0$ simultaneously for all $t = 1,2,3$. Hence, the extended data set violates WGARP.

[17] To be fair to KMS, the models are slightly different: KMS work with entire demand functions and does not use preference functions but preference relations.

[18] Note that removing the coherence restriction on the CMU means that the CMU can rationalize any data set. Indeed, following the arguments of Theorem 1, take a data set O^T, and break it into smaller data sets of size one such that $O_t^T = (p^t, x^t)$ for all t. By Afriat's theorem, any data set with one observation can always be rationalized by a continuous, concave, and, crucially, monotone utility. Note that this utility function is linear. Then we can build T coalitions containing one of these utilities. It can be shown that the CMU defined using these coalitions rationalizes the data set O^T. Although this always works, the constructed CMU is not coherent.

Using Theorem 1, we can also clarify the source of this failure, traced to a violation of strict quasiconcavity of the preference function. Since the observed data satisfies WGARP, by Theorem 1, there is a preference function r rationalizing the data. Moreover, we have $r(x_1, x_2) \geq 0$, $r(x_2, x_3) \geq 0$, and $r(x_3, x_1) \geq 0$. In addition, we know that the new bundle, x^{T+1}, is a convex combination of the observed bundles and different from each of them. In our case, if the preference function is strictly quasiconcave in its first argument, then we must have $r(x^{T+1}, x^{T+1}) = 0 > \min_{t=1,2,3}\{r(x^t, x^{T+1})\}$. This implies that x^{T+1} must be revealed to be strictly better than at least one of the three observed bundles x^1, x^2, or x^3, i.e., $r(x^t, x^{T+1}) < 0$ for at least one $t = 1, 2, 3$. However, note that, for all $t = 1, 2, 3$, we have $p^t(x^t - x^{T+1}) = 2 > 0$. Thus, by the rationalization of Theorem 1, all observed commodity bundles must be weakly preferred to the new bundle, i.e., $r(x^t, x^{T+1}) \geq 0$ for all $t = 1, 2, 3$, a contradiction. Hence, the extended data set $O^3 \cup \{p, x^{T+1}\}$ cannot be rationalized by a strictly quasiconcave and asymmetric (or, of course, skew-symmetric) preference function. Also, it follows that it cannot be rationalized by a strictly concave and skew-symmetric preference function either, akin to the Shafer model of a concave nontransitive consumer (John, 2001). Nevertheless, as already pointed out, the gap between the piecewise concavity of the preference function in our characterization of WGARP and convex preferences seems to become negligible in infinite data sets, as the star-shapedness property would then "fill up" essentially the entire upper contour sets.[19]

Alternatively, suppose the new bundle added was $x^{T+2} = (2.1, 2.1, 2.1)$, using the same steps. In that case, we can argue that the extended data set would violate WGARP, and that there is no quasiconcave, monotonic, and asymmetric preference function that rationalizes the extended data. Alternatively, one could use local nonsatiation instead of monotonicity to arrive at the same conclusion. Monotonicity or local nonsatiation, just like strict quasiconcavity, are vehicles to generate the strict inequality for the preference function.[20]

Interestingly, this example shows that, along with local nonsatiation, quasiconcavity of the preference function is, in fact, a testable property in finite data sets. As such, this sheds light also on Samuelson's *eternal darkness conjecture*, saying that any finite data set can always be rationalized by a convex preference relation.

[19]We thank Phil Reny for raising a question that led us to this conjecture.

[20]There is a related example in Section 7.3 of Kim and Richter (1986). That example uses infinite data and is written for $L = 2$, where GARP and WGARP coincide, ruling out the intransitivity present in our example.

3.3. Abstract Choice Settings

In the remainder of this section, we generalize our characterization of WGARP to a classical abstract choice environment. This result extends our equivalence between WGARP and the coherent CMU model to a wide variety of different domains, including nonlinear budgets (Forges and Minelli, 2009) and the case of infinite data.

Consider a nonempty choice set X. Let this space be endowed with primitive order pair $(\geq, >)$. We assume that X is order $(\geq, >)$ dense separable, i.e., there is a countable set $Y = \{y_k\}_{k\in\mathbb{N}} \subseteq X$ such that, whenever $x > y$ with $x, z \in X$, there exist a $y_k \in Y$ such that $x > y_k > z$. In addition, we assume that the order-pair $(\geq, >)$ is acyclic and that $x > y \geq z$ implies $x > z$.

Let $\mathcal{A} \subseteq 2^X \setminus \emptyset$ be a collection of menus. Define a choice correspondence as

$$c : \mathcal{A} \to 2^X,$$

such that $c(A) \subseteq A$ and $c(A) \neq \emptyset$ for all $A \in \mathcal{A}$. A data set is an array $\boldsymbol{D} = (A, c(A))_{A\in\mathcal{A}}$.

We say a menu $A \in \mathcal{A}$ is comprehensive with respect to the pair $(\geq, >)$ whenever $x \in A$ and $x \geq y$ implies $y \in A$. Moreover, we let $x \succeq^D y$ if $x \in c(A)$ for some $A \in \mathcal{A}$ such that $y \in A$. We define the relations $\succeq^R$ as $x \succeq^R y$ if $x \succeq^D y$ and $x \succ^R y$ if there is a $B \in \mathcal{A}$ and $z \in B$ where $\{x, y, z\} \subseteq B$, $x \in c(B)$, and $z > y$. WGARP is defined as follows:

Definition 11. *A dataset $\boldsymbol{D}$ satisfies WGARP if $x \succeq^R y$ imply not $y \succ^R x$.*

Next, consider the following definition of rationalization:

Definition 12. *We say that a data set $\boldsymbol{D}$ is rationalized by a preference function $r : X \times X \to \mathbb{R}$ if*

$$c(A) \subseteq \{x \in A : r(x, y) \geq 0 \ \forall \ y \in A\}.$$

This notion of rationalization generalizes the traditional notion of rationalization for preference functions. We define the CMU model as follows:

Definition 13. *(General CMU) We say that a preference function is a general coalitional multi-utility (general CMU) representation, r, if there exists a family of collections of utility functions $u : X \to \mathbb{R}$, such that:*

$$r(x, y) = \sup_{U\in\Omega} \inf_{u\in U} (u(x) - u(y)),$$

for all $x, y \in X$, where every $u \in \cup_{U\in\Omega} U$ is monotonic on the order pair $(\geq, >)$ (i.e., if $x \geq (>) y$ then $u(x) \geq (>) u(y)$).

In contrast to the consumer choice setting previously considered, the general CMU in the classical abstract choice setting relaxes the concavity assumption of each utility function. Notably, Definition 13 does not require that Ω or U are compact. However, this definition maintains the monotonicity properties of each piecemeal utility with respect to a primitive order-pair $(\geq, >)$. The following theorem gives a complete characterization of WGARP for the classical case where the observed choice correspondence is assumed to be nonempty:

Theorem 2. *The following statements are equivalent:*

(i) The data $\boldsymbol{D}$ *can be rationalized by a coherent general CMU preference function.*

(ii) The data $\boldsymbol{D}$ *satisfies WGARP and all menus are comprehensive.*

We can contrast our results to the classical textbook characterization of WGARP in Mas-Colell et al. (1995). There, it is shown that if a dataset $\mathbf{D}$ contains all binary and ternary menus and the choice correspondence is nonempty in all of these (i.e., a rich dataset), then WGARP is equivalent to rationalization by a utility function. The rational model is a particular case of the coherent CMU model we consider here. Our result covers datasets that may not be rich. Importantly, we only assume that the choice correspondence is nonempty on $\mathcal{A}$, allowing for the correspondence to be empty elsewhere (outside of the collection of observed menus). If we assume that the choice correspondence is observed in infinite data and is always nonempty, our result will still hold, thus generalizing the results in Kim and Richter (1986). Note that even when the work of Kim and Richter (1986) does not assume comprehensive menus; our results can be directly extended to their setup. In closing this section, we note that the proof technique used to establish the results in this section could be adapted to obtain the general characterization of rational behavior in Nishimura et al. (2017) in order to provide a rationalization of WGARP in a broader collection of choice domains. Indeed, the techniques developed in this paper allow us to adapt existing results in revealed preference for rationality behavior for the case of WGARP because the existing results can be applied to the piecemeal utilities of the CMU.

4. Basic Consumer Theory for the Coherent CMU

In the previous section, we provided characterizations of WGARP based on the coherent CMU model. We also saw how WGARP might be incompatible with quasiconcave preference functions. In this section, we investigate the essential open question of the consumer preference maximization problem with a coherent CMU representation, focusing on counterfactual demand analysis. That is, for a new (possibly unobserved) price vector p^{T+1}, we study whether WGARP can predict demand generically. As such, we need to formalize what it means for WGARP to make out-of-sample predictions. The exercise is a sort of dual to that in Example 1, and our conclusions are similar.

We are interested in the following object:

Definition 14. *(W-Demand Set) Let a finite data set O^T satisfy WGARP. We define the W-demand set, or the set of all additional bundles compatible with WGARP given a new price, by*

$$D(p^{T+1}, w^{T+1}) = \{x \in X : O^T \cup \{p^{T+1}, x\} \text{ satisfies WGARP and } p^{T+1}x = w^{T+1}\}.$$

We obtain a negative result that seems to be new to the literature. In particular, the following lemma shows that, for some prices, the W-demand set may be empty, implying that, for these prices, it is impossible to predict demand using WGARP. That is, indecisiveness shows up, even in some pairwise comparisons of observations.

Lemma 3. *(Impossibility) There are data sets O^T such that for an open set of out-of-sample prices $p^{T+1} \in P$, the W-demand set is empty, i.e., $D(p^{T+1}, w^{T+1}) \equiv \emptyset$.*

We now illustrate Lemma 3 through a counterexample for a single price. The fact that there exists an open set of prices for which the W-demand set is empty shows that the counterexample does not constitute a degenerate case but is also robust to perturbations of the out-of-sample price.

Example 2. *(Empty-demand counterfactuals) Consider again the data set O^3 with prices $p^1 = (4,1,5)'$, $p^2 = (5,4,1)'$, $p^3 = (1,5,4)'$, and bundles $x^1 = (4,1,1)'$, $x^2 = (1,4,1)'$, $x^3 = (1,1,4)'$. Note that the income level in all observations is the same, i.e., $p^tx^t = 22$ for all $t \in \{1,2,3\}$. As noted in Example 1, this data set satisfies WGARP. Suppose the out-of-sample budget is: $p^{T+1} = \frac{1}{3}(p^1 + p^2 + p^3) = \frac{10}{3}(1,1,1)'$ and $w^{T+1} = 22$. Now, assume towards a contradiction that there exists a bundle x^{T+1} in the set $D(p^{T+1}, w^{T+1})$. Note that x^{T+1} is directly revealed preferred to x^t, because $22 = p^{T+1}x^{T+1} > p^{T+1}x^t = 20$ for all $t \in \{1,2,3\}$. By definition, it must be that $p^tx^t < p^tx^{T+1}$ for all $t \in \{1,2,3\}$, such that WGARP (and WARP) holds. However, averaging these inequalities, we get $22 = \frac{1}{3}(p^1x^1 + p^2x^2 + p^3x^3) < p^{T+1}x^{T+1} = 22$, where the right-hand side of the inequality follows from the definition of p^{T+1}. Hence, we obtain a contradiction, and can conclude that $D(p^{T+1}, w^{T+1}) = \emptyset$.*

4.1. Nonempty-Valued Demand Correspondences under WGARP

We have shown that the rationalization of a coherent CMU preference function is empirically equivalent to WGARP. The following result says that maximizing the coherent CMU preference function may lead to indecisiveness, i.e., an empty-valued demand correspondence. Defining $B(p, w)$ as the budget set, i.e., $B(p, w) = \{x \in X : px \leq w\}$, we have:

Lemma 4. *The demand correspondence* $x(p, w) \in \{x \in B(p, w) : [\max_{U \in \Omega} \min_{u \in U}(u(x) - u(y)) \geq 0 \; \forall \; y \in B(p, w)]\}$ *may be empty-valued for some prices and income pairs* (p, w)*, where* $p \in P$ *and* $w > 0$*.*

The proof of this lemma follows directly from Lemma 3 and our primary characterization result in Theorem 1. In contrast, if the consumer is a utility maximizer, there always exists a nonempty-valued demand correspondence. In other words, if observed prices and choices satisfy GARP, it is always possible to rank-order observed bundles by a utility.

Crucially, this negative result is *independent* of the CMU representation. In other words, indecisiveness is a property of WGARP, and not of the underlying representation. Indeed, the possibility of indecisiveness will carry on to any representation of WGARP, including the coherent CMU.

Indecisiveness is a real possibility in practice, and Gerasimou (2018) shows that it also features in other contexts. For our coherent CMU representation, when the demand correspondence image $x(p, w)$ is empty, it is obvious that no coalition of utilities $U \in \Omega$ have agreed on the best alternative that is also feasible. In fact, for every $x \in B(p, w)$, there is a feasible $y \in B(p, w)$ such that for some coalition of utilities, $U \in \Omega$ and for all $u \in U$ it follows that y is strictly preferred to x (i.e., $u(y) > u(x)$). That is, at least one coalition may "block" choosing some feasible object.

Without going all the way to convexity, given the negative result in Lemma 4, it is helpful to establish conditions that are weaker than transitivity but are enough to ensure that the demand correspondence associated with a coherent CMU preference function is nonempty-valued. These conditions require that the CMU preference function is complete, which we note, by the proof of Theorem 1, always can be assumed without loss of generality (given that the data satisfies WGARP).[21]

In the remainder of this subsection, we will first present a necessary and sufficient condition to guarantee that WGARP generates a nonempty demand correspondence.

[21] The proof of Theorem 1 is constructive and shows that a data set O^T satisfying WGARP can be rationalized by a complete, locally nonsatiated, and asymmetric preference function. The family of sets of utilities Ω of a complete CMU preference must contain, for each pair $x, y \in X$, a $U \in \Omega$ such that for any $u, v \in U$, $u(x) \geq u(y) \iff v(x) \geq v(y)$.

Then, we will present a weak sufficient condition, closer to a relaxation of convexity, that also gives nonempty demands.

4.1.a. *A Necessary and Sufficient Condition for Nonemptiness Based on the Nakamura Number.–* The coherent CMU can be viewed as an instance of preference aggregation in the simple voting games literature.[22] A coherent CMU is a *proper* simple voting game. This connection is valuable because it allows to exploit restrictions in the literature of simple games that guarantee the nonemptiness of the demand correspondence under the coherent CMU. We first define the Nakamura number, ν, as:

$$\nu = \min\{|\hat{\Omega}| \mid \hat{\Omega} \subseteq \Omega \text{ and } \cap_{U \in \hat{\Omega}} U = \emptyset\}.$$

When Ω has a collection of coalitions that share a common piecemeal utility, then ν is defined as ∞. The Nakamura number of a CMU captures the cardinality of the minimal number of coalitions that contain a common piecemeal utility. Applying the results in Schofield (1984), we can show the following:

Theorem 3. *If a complete and coherent CMU preference function has an associated Nakamura number, ν, such that $\nu - 1 \geq L$, then the demand correspondence is nonempty-valued for all $p \in P$ and $w > 0$.*

This result sheds light on the connection between the structure of the CMU, particularly the level of collegiality across coalitions, and the nonemptiness of the associated demand correspondence. The concavity of the piecemeal utilities plays a key role, as well as the compactness of the coalitions.

Due to the coherency restriction, the Nakamura number must be $\nu \geq 3$ for our CMU. This Nakamura number implies the nonemptiness of the demand correspondence for $L = 2$, and any such data set satisfying WGARP can be rationalized by a non-empty demand correspondence. Note that when Ω has a collection of coalitions that share a common piecemeal utility, then nonemptiness of the demand correspondence is guaranteed for any number of goods. Of course, the condition provided in Theorem 3 is much weaker than this form of total coherency. We remark that the restriction on the Nakamura number, $\nu - 1 \geq L$, is effectively a *necessary and sufficient* condition for the nonemptiness of the demand correspondence.[23] Finally, we highlight that Theorem 3 can be extended trivially

[22]See Martin and Salles (2013) for a recent survey on voting games as procedures to aggregate individual preferences. We are very grateful to Chris Chambers for pointing out the connection to the voting games literature.

[23]Schofield (1984) shows that the condition is necessary because examples have been built with empty demand correspondences when the condition above fails. Also, note that without completeness, we can only guarantee the existence of a nondominated alternative in a budget set.

to nonlinear and compact budgets.

4.1.b. *A Simple Testable Condition to Ensure Nonemptiness..–* The previous subsection established that a CMU with Nakamura number $\nu \geq L+1$ has a nonempty demand correspondence under completeness; based on this result, we will now formulate a condition to test if the demand correspondence associated with the rationalizing preference function is nonempty. This condition takes as primitive a finite data set O^T, and is a test if O^T admits a rationalization by a CMU whose Nakamura number weakly exceeds $L+1$.

To derive our new condition, we will establish a characterization of a CMU with Nakamura number, ν, satisfying $\nu \geq k+1$, where k is any positive integer number. This result will cover as a particular case $\nu \geq 3$, which corresponds to the coherent CMU. Our result illustrates that a bound on the Nakamura number of a CMU has observable implications on demand data. Thus, the testable implication of the bound on the Nakamura number of a CMU will guarantee nonempty demand correspondences whenever $k \geq L$ since completeness is not testable in finite data. We begin by introducing the axiom of k-acyclicity:

Axiom 5. *(k-acyclicity) For a fixed integer $k \geq 2$ there is no chain $(x^1, x^2, \cdots, x^k)$ with elements on X such that $x^1 \succeq^{R,D} x^2 \succeq^{R,D} \cdots \succeq^{R,D} x^k$ and $x^k \succ^{R,D} x^1$.*

The axiom k-acyclicity is a simplified version of GARP and checks that there are no cycles of length k in the data (GARP checks for cycles of any length up to T). k-acyclicity can be implemented using a simplified version of Warshall's algorithm to calculate transitive closures of binary relations and runs in polynomial time.[24] The following theorem gives our characterization:

Theorem 4. *Consider a finite data set $O^T = \{p^t, x^t\}_{t \in \mathbb{T}}$ and any number $k \geq 2$. The following statements are equivalent:*

(i) The data O^T can be rationalized by a coherent CMU preference function with Nakamura number $\nu \geq k+1$ (which in particular is asymmetric, continuous, monotonic, and piecewise concave).

(ii) The data O^T satisfies k-acyclicity.

This result shows that k-acyclicity exhausts the empirical content of a coherent CMU preference function with Nakamura number $\nu \geq k+1$. Thus, to ensure nonemptiness of the demand correspondence, all one needs to check is that the data O^T does not contain any cycles of length less than the number of goods in the data, L.

[24]See Varian (1982) for a discussion of Warshall's algorithm in the context of revealed preference.

4.1.c. *A Sufficient Condition for Nonemptiness: Total Coherency in Segments.*– Now, we present a sufficient condition that also guarantees a non-empty demand correspondence under WGARP. This condition is new to the literature and exploits the structure of linear budget sets. We also connect this condition with the convexity of preferences, used in the literature to guarantee nonemptiness of the demand correspondence without transitivity Shafer (1974).

Denote by $u|_{[a,b]}$ the restriction of a piecemeal utility $u \in U$ for any $U \in \Omega$ on a closed segment of points $[a,b] \subseteq X$, i.e., on the set of bundles $ta + (1-t)b$ for $a, b \in X$ and for any $t \in [0,1]$. Denote the set of $[a,b]$-restricted utilities corresponding to any $U \in \Omega$ as $U_{[a,b]}$. If two different utilities, u and v, have the same restriction in $[a,b]$, such that $u_{[a,b]} = v_{[a,b]}$, we eliminate duplicates and refer to this equivalence class representative as $u_{[a,b]}$. We define *total coherency in segments* as follows:

Definition 15. *(Total coherency in segments) A coherent CMU preference function r satisfies* total coherency in segments *if for all $a, b \in X$, there is a $u|_{[a,b]}$ in the intersection taken over $U \in \Omega$ of all $U|_{[a,b]}$, i.e., $\cap_{U \in \Omega} U|_{[a,b]} \neq \emptyset$.*

The next result formally states that a coherent CMU preference function satisfying total coherency in segments always generates a nonempty demand:

Theorem 5. *If a coherent and complete CMU preference function satisfies total coherency in segments, then the demand correspondence is nonempty-valued for all $p \in P$ and $w > 0$.*

This result highlights the local rationality nature of the coherent CMU model. To guarantee a nonempty-valued demand correspondence, we require that there is a point of agreement of all coalitions of utilities in Ω when restricted to any closed segment $[a,b]$. This restriction means that irrespective of the criterion that the consumer considers, there is always a utility that ranks all points in the interval $[a,b]$ in the same way. Note that the Nakamura number of a CMU that satisfies total coherency in segments does not need to be ∞ because the shared common utility in a segment $[a,b]$ may not be the same in another segment $[c,d]$.

The next result shows that total coherency in segments, is a weaker condition than the convexity of preferences:

Lemma 5. *If the CMU preference function is complete and quasiconcave, then it satisfies total coherency in segments.*

This result follows directly from Proposition 1 in Moldau (1996), showing that completeness and quasiconcavity of preference functions imply that the binary preferences represented by r are transitive in any closed segment $[a,b]$ for all $a, b \in X$. This result means that this property is implied by the restrictions of completeness and convexity of preferences imposed in the classical work of Shafer (1974) and Sonnenschein (1971) to guarantee nonemptiness of a demand correspondence without transitivity.

Remark 1. *One can construct examples that satisfy total coherency in segments and violate quasiconcavity. While having specific functional forms would be challenging; the example would be facilitated by observing that, for our purposes, the total agreement of preferences on intervals* $[a, b]$ *can be restricted to budget segments, i.e., for* $L = 2$*, segments where* $a = (a_1, 0)'$*,* $b = (0, b_2)'$*. Moreover, if one were to relax the assumption of concave piecemeal utilities, examples would be much easier to construct, simply by having a single utility in the intersection of all collections of utilities with an indifference map consisting of wavy curves.*

Remark 2. *We can state the sufficient conditions in Theorem 5 in terms of the underlying preference relation. The condition implies that the restriction of preferences to the closed interval* $[a, b]$ *is complete and transitive. Evidently, this condition is weaker than (global) completeness and transitivity, as violations of transitivity may occur across distinct closed intervals.*

4.2. Some Basic Textbook Material of Consumer Theory and WGARP

We include in this subsection three remarks of relevance to the basic textbook material of consumer theory:

Remark 3. *It is easy to show that if the observed data set satisfies WGARP, then the demand correspondence generated from the rationalizing preference function is homogeneous of degree zero and satisfies the compensated law of demand. The proof trivially follows from Theorems 1 and A1*[25] *–the latter in Section 6–and the fact that optimizing a CMU preference function over the budget set* $B(p, w)$ *is equivalent to optimizing the same preference function over* $B(\alpha p, \alpha w)$ *for any* $\alpha > 0$*.*[26] *Note that John (1995) established that WARP implies homogeneity of degree zero, but our results also cover the case of demand correspondences under WGARP.*[27]

Remark 4. *The CMU is also convenient when we want to formulate the expenditure minimization problem for the nontransitive consumer. The consumer's* expenditure function *is* $e(p, \overline{u}) = \min_{x \in X} px$ *subject to* $\max_{U \in \Omega} \min_{u \in U}(u(x) - \overline{u}) \geq 0$*. By continuity, this*

[25] Theorem A1 is in the Online Appendix.

[26] If the demand correspondence is also differentiable, then it is straightforward to show that its Slutsky matrix $S(p, w)$ is negative semidefinite for all $(p, w), p \in P, w > 0$.

[27] John's (1995) proof is geometric and does not rely on a representation of WARP.

constraint holds with equality, i.e., $e(p,\overline{u}) = \min_{x\in X} px$ *subject to* $\max_{U\in\Omega}\min_{u\in U}(u(x)-\overline{u}) = 0$*. Note that for any* $x^* \in \arg\min_{x\in X} px$ *subject to* $\max_{U\in\Omega}\min_{u\in U}(u(x)-\overline{u}) = 0$*, we have* $u(x^*) \geq \overline{u}$ *for some* $U \in \Omega$ *and all* $u \in U$*. In this sense, the CMU consumer always obtains a guaranteed level of "utility"* $\overline{u}$*. These solutions to the expenditure minimization problem comprise the* Hicksian demand *correspondence* $h(p,\overline{u}) = x^*$*, such that the compensated law of demand holds:* $(p'-p)(h(p',\overline{u}) - h(p,\overline{u})) \leq 0$*.*

Remark 5. *The CMU is useful for its piecewise concavity when we want to analyze or numerically solve the consumer's preference-function maximization problem. In particular, if we assume that every piecemeal utility is differentiable, then a maximizer* x^t *at prices* p^t *is such that* $r(x^t, y) \geq 0$ *for all* $y \in X$ *whenever* $w \geq p^t y$*. By asymmetry, we know that* $r(y, x^t) \leq 0$ *for all* y *such that* $w \geq p^t y$*. This means that* x^t *is the solution to the preference maximization problem, i.e.,* $x^t \in \arg\max_y r(y, x^t) + \lambda(p^t y - w)$*. The superdifferential of the CMU with locally differentiable and concave utilities can be computed following Tsevendorj (2001) and known facts about concave functions:*

$$\partial_1 r(x, x^t) = \bigcap_{U\in\Omega: r(x,x^t)=\min_{u\in U}(u(x)-u(x^t))} Co\left(\bigcup_{u\in U}\{\partial u(x) | u(x) - u(x^t) = \min_{u\in U}(u(x) - u(x^t))\}\right).$$

Hence, the supergradient of the CMU is the intersection across all active coalitions of the convex hull (Co) of the union of the gradients of the active piecemeal utility functions. Given this, the first-order conditions of the maximization problem above, assuming, without loss of generality, interiority and that $\partial_1 r(x^t, x^t)$ *is a singleton, are:*[28]

$$\partial_1 r(x^t, x^t) = \lambda p^t.$$

It directly follows that total coherency in segments guarantees that the supergradient of the CMU is nonempty. Schofield (1984) also demonstrate that nonemptiness of demand is roughly equivalent to the nonemptiness of the gradient under (quasi)concavity of the piecemeal utilities. When this gradient exists at a point, CMU preferences are locally acyclic Schofield (1984).

4.3. Examples

We conclude this section by illustrating WGARP in light of the CMU model, with the following examples:

[28] With some technicalities these first-order conditions are not only necessary, but also sufficient (Tsevendorj 2001).

Example 3. *Consider* $L = 2$ *goods, hamburger* (x_1) *and salad* (x_2)*. Suppose that the consumer's preferences consist of two attributes, health (h) and tastiness (t). The set of tasty utilities* U_t *has two elements,* $u_{t1}(x) = \overline{\beta}x_1 + x_2$ *and* $u_{t2}(x) = x_1 + x_2$*. The set of healthy utilities* U_h *also has two elements,* $u_{h1} = \underline{\beta}x_1 + x_2$*, and* $u_{h2} = x_1 + x_2$*, where* $\overline{\beta} > 1 > \underline{\beta}$*. The utilities* $u_{h2} = u_{t2}$ *are the same, making the set* $\Omega = \{U_h, U_t\}$ *coherent. The CMU preference function has a closed-form solution:*

$$r(x, y) = (x_1 + x_2) - (y_1 + y_2).$$

In this example, behavior is not only consistent with WGARP but also rational (consistent with GARP as $L = 2$) because the consumer endogenously decides to focus on both attributes to make her decision.

The next example illustrates a more complicated solution, where attributes are interpreted as consumer moods. It describes a situation where a threshold consumption level of some commodities may become a breaking point that triggers different moods.

Example 4. *Consider* $L = 3$ *goods, such that* x_1*,* x_2*,* x_3 *represent consumption of vegetables, chocolate, and meat, respectively. Suppose there are three moods* $\Omega = \{U_h, U_s, U_f\}$*, where* h *stands for hedonistic,* s *for stoic, and* f *for flexible. The utilities for the hedonistic mood are* $u_{h,\alpha}(x) = 0.5x_1 + \alpha x_2 + (1 - \alpha)x_3$ *for* $\alpha \in \{0.5, 0.6, 0.7\}$*. The utilities for the stoic mood are* $u_{s,\beta}(x) = \beta x_1 + 0.5x_2 + (1 - \beta)x_3$ *for* $\beta \in \{0.5, 0.6, 0.7\}$*. The utilities for the flexible mood are* $u_{f,1} = 0.5x_1 + 0.6x_2 + 0.4x_3$ *and* $u_{f,2} = 0.6x_1 + 0.5x_2 + 0.4x_3$*. This CMU satisfies coherence.*[29] *Now, we show that this CMU produces a violation of transitivity. Consider the bundles* $x^1 = (0, 20, 0)'$*,* $x^2 = (0, 10, 10)'$ *and* $x^3 = (20, 0, 0)'$*. Direct calculations show that* $r(x^1, x^2) = 1$*, while* $r(x^2, x^3) = 0$ *and* $r(x^3, x^1) = 0$*. However, the CMU is asymmetric (as well as continuous, monotone, and piecewise concave). The violation of transitivity is driven by the fact that different moods are active in each pairwise comparison. In fact, when comparing* x^1 *to* x^2*,* U_f *is active such that* $r(x^1, x^2) = \min(u_{f,1}(x^1) - u_{f,1}(x^2), u_{f,2}(x^1) - u_{f,2}(x^2))$*. Similarly, when comparing* x^2 *to* x^3*,* U_h *is active; and when comparing* x^3 *to* x^1*,* U_s *is active. Note that the CMU exhibits an endogenous mood switching behavior that depends on which pair of bundles is being compared.*

Our last example connects our work with the recent models of Hara et al. (2019) and Frick et al. (2019).

[29]Indeed, $U_h \cap U_s = u_{h,0.5}$ and $u_{h,0.5} = u_{s,0.5}$, and moreover, for U_f we have $u_{f,1} = u_{h,0.6}$ and $u_{f,2} = u_{s,0.6}$.

Example 5. *Consider a case where there is one physical good (money) and* 3 *states of the world (good, business-as-usual, bad). The Arrow-Debreu state-contingent securities are* x_l*, for* $l = 1, 2, 3$*. Suppose that the consumer knows that state* 1 *happens with probability* $1/2$*, and state* 2 *or* 3 *happens with probability* $1/2$*, but there is uncertainty about which of these two will occur. The consumer has two moods,* $U_r = \{u_{1,r}, u_{2,r}\}$ *and* $U_m = \{u_{1,m}, u_{2,m}\}$*, where* r *stands for realistic and* m *for pessimistic. The utility of an Arrow-Debreu security* x *associated with the different utilities* $i \in \{1, 2\}$ *and* $j \in \{r, m\}$ *is:*

$$u_{i,j} = 1/2v(x_1) + (1/2 - \pi_{i,j})v(x_2) + \pi_{i,j}v(x_3) = E_{\pi_{i,j}}[v(x)],$$

where $0 \leq \pi_{i,j} \leq 1/2$ *and* v *is a Bernoulli utility defined over money. The subjective probabilities are:* $\pi_{1,r} = 1/5$*,* $\pi_{1,m} = 1/3$*, and* $\pi_{2,r} = \pi_{2,m} = 1/4$*. Note that this model is similar to the ambiguity framework posed in Frick et al. (2019), where the "act" corresponds to choosing bundle* x *over bundle* y*, such that the utility of such "act" is given by:*

$$r(x, y) = \max_{j \in \{r,m\}} \min_{i \in \{1,2\}} (E_{\pi_{i,j}}(v(x) - v(y))).$$

5. Welfare Analysis and Recoverability of Preferences

The fact that convexity of preferences may fail for the coherent CMU implies that the methods in Varian (1982) need to be modified if one wishes to recover preferences consistent with WGARP. In this section, we discuss an approach in which all pairs of observations are used to provide new bounds on the true preferences. This effort results in a method to recover preferences that is robust to departures from the transitive or convexity properties of preferences. We begin by showing that recovering preferences using WGARP does not follow as a trivial corollary of the original approach proposed in Varian (1982). Subsequently, we propose an alternative method to recover bounds on preferences using WGARP.

It is useful to briefly recall the classical approach from Varian (1982), which finds upper and lower bounds to the true preferences of a consumer implied by her consumption choices. These are captured by the strict upper contour set of a commodity bundle x according to the true preference function r:

Definition 16. *(Set of strictly better alternatives) We define the set of strictly better* *alternatives than a (possibly unobserved) commodity bundle* $x \in X$ *as:*

$$U_r(x) = \{y \in X : r(y, x) > 0\},$$

for the true preference function r.

Varian (1982) defines the supporting set of prices for any new commodity bundle $x \in X$, so that the *extended data set*, $O^T \cup \{p, x\}$, satisfies GARP as:

$$S(x) = \{p \in P : O^T \cup \{p, x\} \text{ satisfies GARP}\}.$$

Varian then uses the set $S(x)$ to create upper and lower bounds for the set of interest $U_r(x)$. We need to define two new sets. The *revealed worse set* is:

$$RW(x) = \{y \in X : \forall p \in S(x), x \succ^{R,D}_{O^T \cup \{(p,x)\}} y\}$$

for $\succ^{R,D}$, defined on the extended data set $O^T \cup \{(p, x)\}$. The *nonrevealed worse set* $NRW(x)$ is the complement of $RW(x)$. The *revealed preferred set* is:

$$RP(x) = \{y \in X : \forall p \in S(y), y \succ^{R,D}_{O^T \cup \{(p,y)\}} x\}.$$

Varian (1982) shows that, in the case of utility maximization (i.e., $r(x, y) = u(x) - u(y)$ for some $u : X \to \mathbb{R}$ and all $x, y \in X$), we have:

$$RP(x) \subseteq U_r(x) \subseteq NRW(x).$$

One could be tempted to use the same construction for WGARP by replacing the definition of the supporting set $S(x)$ with one where the extended data set satisfies WGARP. Of course, when $L = 2$, this does not cause any problems since WGARP and GARP are equivalent in such a case. However, if $L > 2$, as we show, performing such an exercise is generally not advisable. In particular, we illustrate this through an example that, in some cases, yields an uninformative upper bound set $NRW(x)$.

Example 6. *Consider again the data set* O^T *and a new observed bundle* x^{T+1} *in Example 1. Given the observed behavior, suppose the goal is to recover the preferences of this consumer for the new commodity bundle. If one were to use the methods in Varian (1982), it is necessary to recover all prices* p^{T+1} *such that the extended data set* $O^3 \cup (p^{T+1}, x^{T+1})$ *satisfies WGARP. However, as shown in Example 1, there is no* $p \in P$ *in the supporting set. This fact creates a problem if the goal is to recover preferences using Varian (1982) approach because this method implicitly assumes that there always exists at least one such* *vector of prices satisfying WGARP.*

In this example, Varian's supporting set is empty, i.e., $S(x^{T+1}) = \emptyset$. Moreover, it directly follows that the set $U_r(x)$ may contain any monotonically dominated bundle such as $x^- = (1, 1, 1)$. Consequently, the upper bound of $U_r(x^{T+1})$ is uninformative, i.e., $NRW(x^{T+1}) = X \setminus x^{T+1}$. Thus, any analysis based on this approach is problematic since the observed behavior can be rationalized by a preference function that is strictly increasing (in the first argument). In other words, Varian's method to bound preferences does not provide any valuable information in Example 6. We can trace the source of this failure to a violation of the convexity of preferences. Summarizing these results, the lack of convexity of preferences, which can be inferred from behavior consistent with WGARP, limits the applicability of the tools developed in the classical treatment by Varian (1982).

In the rest of the section, we use the new notion of CMU preference rationalization as a way to provide new informative bounds on the true preferences. We show that these new bounds escape the problems associated with Varian's approach.

The proof of Theorem 1 shows that, without loss of generality, we can identify the set of coalitions of utilities in $\Omega = \cup_{t \in \mathbb{T}} U_t$ with the set of observations $\mathbb{T}$ (i.e., $u_{ts} \in U_t$ is the Afriat utility for the data set O^2_{st}).[30] Then, the true global preferences for any $x', x \in X$ are given by:

$$r(x', x) = \max_{U_t : t \in \mathbb{T}} \min_{u \in U_t} (u(x') - u(x)).$$

The proof also shows that any data set O^T satisfying WGARP can be broken down into T^2 pairwise data sets $O^2_{st} = \{(p^t, x^t), (p^s, x^s)\}$, and we argue that each one of these pairwise data sets satisfy GARP. For any pair of observations $s, t \in \mathbb{T}$, we define the local support set $S_{st}(x)$ for any $x \in X$ as in Varian (1982). Hence, for a data set of T observations, we have a collection of T^2 such local support sets. Note, by definition, that every one of these sets is nonempty. We follow a similar logic to that in Varian's approach, but in trying to construct the revealed preferred set to a given bundle, we do it "in chunks" based on each pair of observations (s, t). Thus, consider the following two definitions:

Definition 17. *(WGARP-robust revealed preferred set) For each $s, t \in \mathbb{T}$ let*

$$RP_{st}(x) = \{y \in X : \forall p \in S_{st}(y), py > px\}$$

be the pairwise revealed preferred set. We define the (WGARP-)robust revealed preferred set as:

$$RP^W(x) = \cup_{s \in \mathbb{T}} \cap_{t \in \mathbb{T}} RP_{st}(x).$$

[30] Indeed, there is a "canonical" set of attributes, which can be identified with $\mathbb{T}$. Within the attribute corresponding to observation x^t, there are utility functions u_{ts} for all s, where the preference on attribute t can be colored by any pairwise comparison. Coherency is attained by having $u_{ts} = u_{st}$.

That is, fixing observation s, we take the intersection over all observations t because we want to be confident that the "revealed preferred" relation holds for every pair (s, t). Then, we take the union over all s to define the robust revealed preferred set.

Next, we argue that the robust revealed preferred set is a lower bound of $U_r(x)$ for all $x \in X$. If $x' \in RP^W(x)$, this implies $x' \in RP_{st}(x)$ for all $t \in \mathbb{T}$ and for some $s^* \in \mathbb{T}$. Thus, it must be the case that, for s^* and for all $t \in \mathbb{T}$, the Afriat utilities of the data set $O^2_{s^*t}$ are such that $u_{s^*t}(x') > u_{s^*t}(x)$, which means that $r(x', x) \geq \min_{u \in U_{s^*}}(u(x') - u(x)) > 0$. Hence, if $x' \in RP^W(x)$, then we have $r(x', x) > 0$, which can be equivalently stated as: $RP^W(x) \subseteq U_r(x)$.

Definition 18. *(WGARP-robust (non)revealed worse set) For each $s, t \in \mathbb{T}$, let*

$$RW_{st}(x) = \{y \in X : \forall p \in S_{st}(x), px > py\}$$

be the pairwise revealed worse set. Let $NRW_{st}(x)$ be the complement of $RW_{st}(x)$. Define the (WGARP-)robust nonrevealed worse set as

$$NRW^W(x) = \cap_{s \in \mathbb{T}} \cup_{t \in \mathbb{T}} NRW_{st}(x).$$

From this definition, it directly follows that, if $r(x', x) > 0$, then $x' \in NRW^W(x)$. In particular, note that, if $r(x', x) > 0$, then there must be some $t^* \in \mathbb{T}$ (U_{t^*}) such that $u(x') > u(x)$ for all $u \in U_{t^*}$. By a direct application of the results in Varian (1982), we have $x' \in NRW_{st^*}(x)$ for all $s \in \mathbb{T}$. Then, by Definition 18, it follows that $x' \in NRW^W(x)$. Hence, this proves that $U_r(x) \subseteq NRW^W(x)$. The following theorem summarizes these steps, confirming that the bounds recovered using Varian's approach in this context are not sharp:

Theorem 6. *The upper contour set $U_r(x)$ of the true preferences at any given $x \in X$ is such that:*

$$RP^W(x) \subseteq U_r(x) \subseteq NRW^W(x).$$

Moreover, (i) the upper bound, $NRW(x)$, recovered using Varian's approach is not sharp, i.e., $NRW^W(x) \subseteq NRW(x)$ for all $x \in X$ (with strict containment for some $x \in X$); and (ii) the lower bound, $RP(x)$, recovered using Varian's approach is not sharp, i.e., $RP(x) \subseteq RP^W(x)$ for all $x \in X$ (with strict containment for some $x \in X$).

We note that, in the context of Example 6, $NRW^W(x^{T+1})$ does not contain the dominated bundle $x^- = (1\,1\,1)'$. In fact, $NRW^W(x^{T+1})$ excludes all commodity bundles that are monotonically dominated by x^{T+1}, which is a desirable property lacking in

Varian's analogous set $NRW(x^{T+1}) = X \setminus \{x^{T+1}\}$. Similar statements can be made about the $RP^W(x^-)$ set.

Thus, the first part of Theorem 6 shows that the new method of using subsets of data sets to calculate bounds on preferences yields informative bounds. The second part highlights that a naive application of the methodology in Varian (1982), when the assumption of convex preferences does not hold, is problematic.

6. Extensions of the Main Results and Discussions

6.1. WARP

In the Online Appendix, we provide a characterization of Samuelson's (1938) weak axiom of revealed preference (WARP).[31] There, we provide a complete revealed-preference characterization of WARP. This result mirrors Matzkin and Richter's (1991) theorem in terms of strict preference-function rationalization (as opposed to strict utility rationalization).

6.2. The Law of Demand

Following Brown and Calsamiglia (2007) and Allen and Rehbeck (2018), we can impose restrictions on the piecemeal utilities in the CMU preference function to provide a representation for choices obeying the *law of demand*–recall that WARP is essentially equivalent to its compensated version. This is done in the Online Appendix through a quasilinear restriction in Theorem A.1. This characterization may be of interest in its own right, due to the importance of the law of demand in both theoretical and applied literatures.[32]

[31] Recall from Section 2.1 that WARP holds if there is no pair of observations $s, t \in \mathbb{T}$ such that $x^t \succeq^{R,D} x^s$ and $x^s \succeq^{R,D} x^t$, with $x^t \neq x^s$.

[32] Specifically, demand functions satisfying the law of demand have downward-sloping demand curves, and allow the measurement of welfare changes in terms of consumer surplus for a given change in market prices (Brown and Calsamiglia 2007).

6.3. Extended Discussion of Related Literature

We extend the discussion of our results with the literature. We go beyond the connections established in the intro and throughout the manuscript in the Online Appendix.

7. Conclusion

This paper offers Afriat-like theorems for WGARP and WARP. In particular, it shows that the coherent CMU preference function is equivalent to Samuelson's WGARP. We build a comprehensive theory of revealed preference based on the new rationalization. Our findings should be helpful for practitioners of revealed preference since, from an empirical perspective, WGARP is significantly more straightforward to work with than Varian's GARP. In applications, it is common for practitioners to use WGARP as synonymous with GARP. However, as shown in Cherchye et al. (2018), this is only true if price variation is limited.[33] For example, a finite data set of prices and observed consumption choices may be consistent with WGARP, but cannot be rationalized by a utility function. If this occurs, the interpretation of the direct revealed-preference relation is unclear, yet we show that meaningful welfare and counterfactual analysis are possible. We leave for future research the characterization of a coherent CMU that imposes convexity in the implied preferences.

[33] For a generalized treatment of when WARP implies rationality, see Caradonna (2018).

Appendix: Proofs

Proof of Lemma 1

Let r be a coherent CMU function and assume that $r(x, y) \geq 0$. By definition, $r(x, y) = \max_{U \in \Omega} \min_{u \in U}(u(x) - u(y)) \geq 0$, which means that there exists a $\hat{U}$ such that, for each $u \in \hat{U}$, $u(x) - u(y) \geq 0$. By coherency, this means that for all $U \in \Omega$, there is a $\hat{u} \in U \cap \hat{U}$ such that $\hat{u}(y) - \hat{u}(x) \leq 0$. Thus, $r(y, x) = \max_{U \in \Omega} \min_{u \in U}(u(y) - u(x)) \leq 0$, which is what we wanted to show. The same argument can be repeated with strict inequalities, in which case, it follows that if r is a coherent CMU and if $r(x, y) > 0$ then $r(y, x) < 0$.

Proof of Lemma 2

From Theorem 1.a. in Nishimura and Ok (2016), we know that $\succeq$ is reflexive if and only if $x \succeq y \iff r(x, y) = \sup_{U \in \Omega} \inf_{u \in U}(u(x) - u(y)) \geq 0$, where Ω is a non-empty collection of continuous utility functions. Now we will show that if $x \succ y$ then $r(x, y) > 0$ where r is defined above. But this holds since by definition $r(x, y) \geq 0$ and $r(y, x) < 0$, and by coherency this means that there exist a $U \in \Omega$ such that $\inf_{u \in U}(u(x) - u(y)) > 0$. Similarly, if $r(x, y) > 0$ we can show that $x \succ y$. Indeed, $r(x, y) > 0$ implies that $r(y, x) < 0$, which means that $x \succeq y$ and $\neg y \succeq x$. It remains to show that if $\succeq$ is asymmetric then Ω is such that for any $U, \hat{U} \in \Omega$, it must be that $U \cap \hat{U} \neq \emptyset$. Assume towards a contradiction that asymmetry holds and that coherency fails such that for some $\hat{U} \in \Omega \setminus \{U\}$ we have $U \cap \hat{U} \neq \emptyset$. This implies that if $x \succeq y$ then for all $u \in \hat{U}$ it must hold that $u(y) > u(x)$, and if $x \succ y$ then for all $u \in \hat{U}$ it must hold that $u(y) \geq u(x)$. But the previous statement is a violation of asymmetry of $\succeq$ and results in a contradiction. Lemma 1 shows that if Ω is coherent then $\succeq$ is asymmetric by the fact that this representation respects the strict part of $\succeq$.

Proof of Theorem 1

First, we prove the equivalence of WGARP with the systems of Afriat and Varian inequalities (statements (ii), (iii), and (iv)):

(ii) $\Longrightarrow$ *(iii).–* Suppose that WGARP holds. For every pair of observations in the data set O^T, we let O^2_{st} denote the data set consisting of the two observations $s, t \in \mathbb{T}$. Overall, we

have T^2 such data sets, which exhausts all possible pairwise comparisons in O^T. Obviously, for the two observations in each data set O^2_{ts}, WGARP is equivalent to GARP. By a direct application of Afriat's theorem, the following conditions are equivalent: *(i)* the data set O^2_{st} satisfies WGARP, *(ii)* there exist numbers U^k_{ts} and $\lambda^k_{ts} > 0$ for all $k \in \{t, s\}$ such that the Afriat inequalities: $U^k_{ts} - U^l_{ts} \geq \lambda^k_{ts} p^k(x^k - x^l)$ hold for all $k, l \in \{t, s\}$. Now, notice that the two data sets O^2_{ts} and O^2_{st} contain the same two bundles and that permuting the data is insignificant for Afriat's theorem. Thus, without loss of generality, we can set $U^k_{ts} = U^k_{st}$ and $\lambda^k_{ts} = \lambda^k_{st}$ for all $k \in \{t, s\}$. By defining $R^{t,s} = U^t_{ts} - U^s_{ts}$ and $R^{s,t} = U^s_{ts} - U^t_{ts}$, we get the inequalities in condition (iii).

(iii) $\Longrightarrow$ *(iv).*– Suppose that condition (iii) holds. Since $\lambda^t_{ts} > 0$, if $p^t(x^t - x^s) \geq 0$ then $R^{t,s} \geq 0$, and if $p^t(x^t - x^s) > 0$ then $R^{t,s} > 0$. Define $W^{t,s} = R^{t,s}$ for all $s, t \in \mathbb{T}$.

(iv) $\Longrightarrow$ *(ii).*– Suppose that condition (iv) holds, but that WGARP is violated, i.e., $p^t(x^t - x^s) \geq 0$ and $p^s(x^s - x^t) > 0$ for some $s, t \in \mathbb{T}$. Then $W^{t,s} \geq 0$ and $W^{s,t} > 0$, which violates asymmetry, and hence, condition (iv).Suppose that the inequalities in condition (iv) holds, but that WGARP is violated, i.e., $p^t(x^t - x^s) \geq 0$ and $p^s(x^s - x^t) > 0$ for some $s, t \in \mathbb{T}$. Then $W^{t,s} \geq 0$ and $W^{s,t} > 0$. Thus, $W^{t,s} + W^{s,t} > 0$, which violates the inequalities in condition (iv).

Next, we prove the ordinal characterization of WGARP (equivalence of statements (i) and (ii)):

(i) $\Longrightarrow$ *(ii).*– Let $r(x, y)$ be a locally nonsatiated asymmetric preference function that rationalizes the data. Suppose there is a violation of WGARP, so that $p^t x^t \geq p^t x^s$ and $p^s x^s > p^s x^t$ for some pair of observations $s, t \in \mathbb{T}$. Then, by rationalization in Definition 8, we have $r(x^t, x^s) \geq 0$. By asymmetry, $r(x^s, x^t) \leq 0$.

Also by rationalization, from $p^s x^s > p^s x^t$, we get that $r(x^s, x^t) \geq 0$. Putting both together, we get that $r(x^s, x^t) = 0$. But then, by local nonsatiation there exists $y \in B(x^t, \epsilon)$ for some small $\epsilon > 0$ such that $p^s x^s > p^s y$ with $r(x^s, y) < 0$, which contradicts that r rationalizes the data. Thus, there cannot exist an asymmetric locally nonsatiated function r rationalizing the data.

(ii) $\Longrightarrow$ *(i).*– This is a constructive proof. Suppose that WGARP in condition (ii) holds. Once again, for every pair of observations in the data set O^T, we let O^2_{st} denote the data set consisting of the two observations $s, t \in \mathbb{T}$. Hence, we have T^2 such data sets, which exhausts all possible pairwise comparisons in O^T. Obviously, for the two observations in each data set O^2_{ts}, WGARP is equivalent to GARP. For the two observations in every data set O^2_{st}, we define the Afriat function $u_{st} : X \to \mathbb{R}$ as in Afriat's theorem (See e.g.,

Varian 1982). From Afriat's theorem, we know that u_{st} is continuous, concave, and strictly increasing. Next, for all $x, y \in X$, we define the mapping: $r_{st} : X \times X \to \mathbb{R}$ as:

$$r_{st}(x, y) = u_{st}(x) - u_{st}(y).$$

Clearly, r_{st} is continuous in x and y, concave in x, and convex in y (since u_{st} is continuous and concave). Moreover, it is skew-symmetric, since $r_{st}(y, x) = u_{st}(y) - u_{st}(x) = -r_{st}(x, y)$. Notice that, since the function r_{st} is constructed for every $(s, t)-$ pair of observations in O^T, we have a collection of T^2 functions r_{st}.

Let the $T-1$ dimensional simplex be denoted as $\Delta = \{\lambda \in \mathbb{R}^T_+ | \sum_{t=1}^T \lambda_t = 1\}$. Define the preference function $r(x, y)$ for any $x, y \in X$ as:

$$\begin{aligned} r(x, y) &= \min_{\lambda \in \Delta} \max_{\mu \in \Delta} \sum_{s \in \mathbb{T}} \sum_{t \in \mathbb{T}} \lambda_s \mu_t r_{st}(x, y) \\ &= \max_{\mu \in \Delta} \min_{\lambda \in \Delta} \sum_{s \in \mathbb{T}} \sum_{t \in \mathbb{T}} \lambda_s \mu_t r_{st}(x, y). \end{aligned}$$

We first prove that the function r rationalizes the data set O^T. Consider $y \in X$ and some fixed $t \in \mathbb{T}$ such that $p^t x^t \geq p^t y$. Let $\mu^t \in \Delta$ be the vector such that $\mu^t_j = 0$ if $j \neq t$ and $\mu^t_j = 1$ if $j = t$. Then we have:

$$\begin{aligned} r(x^t, y) &= \max_{\mu \in \Delta} \min_{\lambda \in \Delta} \sum_{s \in \mathbb{T}} \sum_{t \in \mathbb{T}} \lambda_s \mu_t r_{st}(x^t, y) \\ &\geq \min_{\lambda \in \Delta} \sum_{i \in \mathbb{T}} \sum_{j \in \mathbb{T}} \lambda_i \mu^t_j r_{ij}(x^t, y) \\ &= \min_{\lambda \in \Delta} \sum_{i \in \mathbb{T}} \lambda_i r_{it}(x^t, y). \end{aligned}$$

It suffices to show that $r_{it}(x^t, y) \geq 0$ whenever $p^t x^t \geq p^t y$ for each data set O^2_{it}. But this follows directly from the definition of r_{it} and Afriat's theorem. Hence, $r(x^t, y) \geq 0$.

Now, we verify that the preference function r constructed is skew-symmetric (and hence asymmetric) and strictly increasing (and hence locally nonsatiated).

First, we show skew-symmetry. We have:

$$\begin{aligned} -r(x, y) &= -\min_{\lambda \in \Delta} \max_{\mu \in \Delta} \sum_{s \in \mathbb{T}} \sum_{t \in \mathbb{T}} \lambda_s \mu_t r_{st}(x, y) \\ &= \max_{\lambda \in \Delta} \min_{\mu \in \Delta} \sum_{s \in \mathbb{T}} \sum_{t \in \mathbb{T}} \lambda_s \mu_t (-r_{st}(x, y)), \end{aligned}$$

Since r_{st} is skew-symmetric (i.e., $-r_{st}(x, y) = r_{st}(y, x)$), we have (this follows directly from the classical von Neumann's minimax theorem because Δ is convex and compact, and the sum is linear in λ and μ):

$$\begin{aligned}
-r(x,y) &= \max_{\lambda\in\Delta}\min_{\mu\in\Delta}\sum_{s\in\mathbb{T}}\sum_{t\in\mathbb{T}}\lambda_s\mu_t(-r_{st}(x,y)) \\
&= \max_{\lambda\in\Delta}\min_{\mu\in\Delta}\sum_{s\in\mathbb{T}}\sum_{t\in\mathbb{T}}\lambda_s\mu_t r_{st}(y,x) \\
&= \min_{\mu\in\Delta}\max_{\lambda\in\Delta}\sum_{s\in\mathbb{T}}\sum_{t\in\mathbb{T}}\lambda_s\mu_t r_{st}(y,x) \\
&= r(y,x),
\end{aligned}$$

which proves that r is skew-symmetric.

And next, we show that r is strictly increasing. Consider any $x, y, z \in X$ such that $x > y$. Then:

$$\begin{aligned}
r_{st}(x,z) &= u_{st}(x) - u_{st}(z) \\
&> u_{st}(y) - u_{st}(z) \\
&= r_{st}(y,z),
\end{aligned}$$

where $u_{st}(x) > u_{st}(y)$ follows by Afriat's theorem. This implies:

$$\max_{\mu\in\Delta}\sum_{s\in\mathbb{T}}\sum_{t\in\mathbb{T}}\lambda_s\mu_t r_{st}(x,z) > \max_{\mu\in\Delta}\sum_{s\in\mathbb{T}}\sum_{t\in\mathbb{T}}\lambda_s\mu_t r_{st}(y,z),$$

for all $\lambda \in \Delta$. Thus, $r(x,z) > r(y,z)$.

And finally, we show the cardinal representation of WGARP by means of a coherent CMU function (equivalence of statements (ii) and (v)):

(v) $\Longrightarrow$ (ii).– Let r be a coherent CMU preference function that rationalizes the data. By Lemma 1, r is asymmetric. Suppose there is a violation of WGARP, so that $p^t x^t \geq p^t x^s$ and $p^s x^s > p^s x^t$ for some pair of observations $s, t \in \mathbb{T}$. Then, by rationalization in Definition 8, we have $r(x^t, x^s) \geq 0$ and $r(x^s, x^t) \geq 0$. We have two possible cases. Suppose first that $r(x^s, x^t) > 0$. This leads to a contradiction because by asymmetry $r(x^s, x^t) \leq 0$. So, suppose next that $r(x^s, x^t) = 0$, which means that for all $U \in \Omega$, there exists $u \in U$ such that $u(x^s) - u(x^t) \leq 0$. But there exists $y \in B(x^t, \epsilon)$ for some small $\epsilon > 0$ such that $y >> x^t$, and $p^s x^s > p^s y$. This means that for all $U \in \Omega$, there exists $u \in U$ such that $u(x^s) - u(y) < 0$. Then, $r(x^s, y) < 0$, which contradicts that r rationalizes the data, so this case is also impossible.

(ii) $\Longrightarrow$ (v).– This is again a constructive proof. Suppose that WGARP holds. Consider once again the data set O^2_{ts}, and recall that we have T^2 such data sets, which exhausts all

possible pairwise comparisons in the data set O^T. Again, for the two observations in every data set O^2_{st}, we define the Afriat function $u_{st} : X \to \mathbb{R}$ (Recall that u_{st} is continuous, concave, and strictly increasing).

Next, we define the set of utility functions as $U_t = \cup_{s\in\mathbb{T}}\{u_{ts}\}$, and the family of sets of utility functions as $\Omega_{\mathbb{T}} = \cup_{t\in\mathbb{T}}U_t$. It is trivial to verify that U_t is compact (finite and discrete) and that Ω also is finite and discrete. We begin by verifying that $\Omega_{\mathbb{T}}$ satisfies coherency. Indeed, $U_t \cap U_s = \{u_{ts}\} = \{u_{st}\}$. Coherency follows since $u_{ts} = u_{st}$ by Afriat's theorem.

Next, we define the coherent CMU preference function as:

$$r(x, y) = \max_{U\in\Omega_{\mathbb{T}}} \min_{u\in U}(u(x) - u(y)).$$

We prove that the function r rationalizes the data set O^T. Consider $y \in X$ and some fixed $t \in \mathbb{T}$ such that $p^t x^t \geq p^t y$. We have:

$$\begin{aligned} r(x^t, y) &= \max_{U\in\Omega_{\mathbb{T}}} \min_{u\in U}(u(x^t) - u(y)) \\ &\geq \min_{u\in U}(u(x^t) - u(y)). \end{aligned}$$

It suffices to show that $u(x^t) - u(y) \geq 0$ whenever $p^t x^t \geq p^t y$ for each $U_t \in \Omega_{\mathbb{T}}$. But this follows directly from the definition of U_t and Afriat's theorem. Hence, $r(x^t, y) \geq 0$.

Proof of Theorem 2

(i) $\Longrightarrow$ (ii).– If $x \succeq^R y$, then $x \in c(A)$ and $y \in A$ for some $A \in \mathcal{A}$. This means that there is a CMU preference function such that $r(x, y) \geq 0$. This is equivalent to saying that there is one $U^* \in \Omega$ such that $\inf_{u\in U^*}(u(x) - u(y)) \geq 0$. Assume towards a contradiction that $y \succ^R x$. This means that there is a set a $B \in \mathcal{A}$ and $z \in B$ where $\{y, x, z\} \subseteq B$, $y \in c(B)$ and $z > x$. If $y \in c(B)$ then there is a coalition $U^{**} \in \Omega$ such that $\inf_{u\in U^{**}}(u(y) - u(x)) \geq 0$. The first case corresponds to $\inf_{u\in U^{**}}(u(y) - u(x)) > 0$, which is a contradiction of coherency as this implies that $U^{**} \cap U^* = \emptyset$. The second case is $\inf_{u\in U^{**}}(u(y) - u(x)) = 0$, but in such a case we know that there exists a $z \in B$ such that for all $u \in U^{**}$ $u(z) > u(x)$. This means that $\inf_{u\in U^{**}}(u(y) - u(z)) < 0$. But this is a contradiction to the fact that the CMU rationalizes the data set $\boldsymbol{D}$.

(ii) $\Longrightarrow$ (i).– First, we need some preliminary results:

Theorem C. (Chambers and Echenique 2016; Theorem 2.19) Suppose that the acyclic order pair $(\geq, >)$ satisfies: $x > y \geq z$ implies $x > z$, that all $A \in \mathcal{A}$ are comprehensive, and X is $(\geq, >)$-dense separable. Then there exists a utility function which is monotone with respect to $(\geq, >)$ and that rationalizes the data set D if and only if $(\succeq^R, \succ^R)$ satisfies the generalized axiom of revealed preference (GARP).

Lemma 6. *If $\boldsymbol{D}$ satisfies WGARP and all menus are comprehensive then $\boldsymbol{D}$ is rationalized by a general coherent CMU preference function.*

Step 1: Define $\tilde{X} = \cup_{A \in \mathcal{A}} A$ as the set of observed alternatives. Take $x, y \in \tilde{X}$ and construct the direct preference relation constrained to $\{x, y\}$ as $\succeq^R |\{x, y\}$. By WGARP we know that the restriction $\succeq^R_{x,y} = \succeq^R |\{x, y\}$ is acyclic (i.e., satisfies GARP).

Step 2: Take $\succeq^R_{x,y}$ from Step 1 and add the following elements to this relation: $x \succeq^R_{x,y} z$ if $x \succeq^R z$, and $x \succ^R_{x,y} z$ if $x \succ^R z$ for all $z \in \tilde{X}$. Moreover, to $\succeq^R_{x,y}$ we add: $y \succeq^R_{x,y} z$ if $y \succeq^R z$ and $y \succ^R_{x,y} z$ if $y \succ^R z$.

Step 3: We show that the order-pair $(\succeq^R_{x,y}, \succ^R_{x,y})$ constructed in Step 2 is acyclic, i.e., satisfies GARP. Assume not, then $x^1 \succeq^R_{x,y} x^2, \cdots, x^{n-1} \succeq^R_{x,y} x^n$ and $x^n \succ^R_{x,y} x^1$. Without loss of generality assume that $x \succeq^R_{x,y} y$, in which case every element of this chain must involve $y \succeq^R_{x,t} x^k$ or $x \succeq^R_{x,t} x^r$. But then every chain has a sub chain $x \succeq^R_{x,y} x^k$ and $x^k \succ^R_{x,y} x$; or $y \succeq^R_{x,y} x^k$ and $x^k \succ^R_{x,y} x$. But this cannot hold because it implies that $x^k \succ^R_{x,y} x$. Hence, WGARP is violated by the construction of $(\succeq^R_{x,y}, \succ^R_{x,y})$ in Step 2.

Step 4: By Theorem C above and that we obtain a utility function $u_{x,y}$ that is monotone on $(\geq, >)$ and extends the order-pair $(\succeq^R_{x,y}, \succeq^R_{x,y})$ to X, we have that if $x \succeq^R_{x,y} y$ then $u_{x,y}(x) \geq u_{x,y}(y)$ (and $x \succ^R_{x,y} y$ then $u_{x,y}(x) > u_{x,y}(y)$). Without loss of generality, we let $u_{x,y} = u_{y,x}$ (this has to be set for the elements $x, y \in \tilde{X}$ that are not ranked).

Step 5: Define $U_x = \{u | \, u_{x,y} \; \exists y \; \in \tilde{X}\}$.

Now we show that the CMU with $\Omega = \{U | \, U_x \; \exists x \; \in \; \tilde{X}\}$ rationalizes the data set:

Step 6: Fix an x such that $x \in c(A)$ for some $A \in \mathcal{A}$. Then by Step 3, $x \succeq_{x,y} b$ for all $b \in A$ and all $y \in \tilde{X}$. By the definition of U_x, we have $\inf_{u \in U_x}(u(x) - u(b)) \geq 0$ for all $b \in A$. This implies $r(x, b) \geq 0$ for all $b \in A$, which means that $x \in \{a \in A \mid r(a, b) \geq 0 \, \forall \, b \in A\}$.

We conclude that the constructed CMU is coherent, because, by construction, there is always an element $u_{x,y} \in U_x \cap U_y$. In addition, the piecemeal utilities are monotone with respect to the order-pair $(\geq, >)$ by Theorem A.

Proof of Lemma 3

Consider the data set O^3 with prices $p^1 = (4\;1\;5)'$, $p^2 = (5\;4\;1)'$, and $p^3 = (1\;5\;4)'$, and bundles $x^1 = (4\;1\;1)'$, $x^2 = (1\;4\;1)'$, $x^3 = (1\;1\;4)'$. This data set satisfies WGARP. Notice that $p^t x^t = 22$ for all $t = 1, 2, 3$. Define the out-of-sample price: $p^{T+1} = \frac{22}{k}(p^1 + p^2 + p^3)$ for some $k \geq 60$, and the income level $w^{T+1} = p^{T+1}x^{T+1} = 22$. Then we have:

$$p^{T+1} = \frac{220}{k}(1 \quad 1 \quad 1)',$$

$$(x_1^{T+1} + x_2^{T+1} + x_3^{T+1}) = \frac{k}{10}.$$

More important, we observe that:

$$22 = p^{T+1}x^{T+1} \geq p^{T+1}x^t = \frac{22 \cdot 60}{k}.$$

Assume towards contradiction that x^{T+1} is in $D(p^{T+1}, w^{T+1})$, then it must be that, $p^t x^t < p^t x^{T+1}$ for $t = 1, 2, 3$. Adding up inequalities we obtain, $66 = (p^1x^1 + p^2x^2 + p^3x^3) < 10(x_1^{T+1} + x_2^{T+1} + x_3^{T+1}) = k$. This produces a contradiction whenever $60 \leq k < 66$ for WGARP, and WARP. There is a continuum of examples.

Proof of Theorem 3

First, notice that due to local nonsatiation of the coherent CMU, r given p and $w > 0$ we can focus on a compact and convex choice set of dimension $L - 1$, $W \subseteq \mathbb{R}^{L-1}$, that will contain the maximizer of $r$constrained to $B(p, w)$. This choice set is the budget constraint boundary.

Note that each piecemeal utility is assumed to be continuous and concave. Also, $\nu - 1 > L$ implies that $\nu - 2 \geq L - 1$. This means that all assumptions in Theorem 4 in Schofield (1984) are satisfied for the case of a finite number of piecemeal utilities.

Note that the same proof of Theorem 4 in Schofield (1984) works for the case of compact coalitions of piecemeal utilities (possibly an infinite number of them), when we replace the finite version of the Helly's theorem used in the proof of Theorem 4 in Schofield (1984), for the infinite version of Helly's theorem in Eckhoff (1993).

Therefore, Theorem 4 in Schofield (1984) allows us to conclude that there is an $x \in W$ such that there is no $y \in W$ such that $r(y, x) > 0$, this means that $r(y, x) \leq 0$ for all $y \in W$. Invoking completeness and asymmetry of the CMU we conclude that $r(y, x) \geq 0$ for all $y \in W$.

Proof of Theorem 4

(i) $\Longrightarrow$ (ii).– Fix k and let r be a coherent CMU preference function with Nakamura number $\nu \geq k+1$ that rationalizes the data O^T. Suppose towards contradiction that the data O^T violates of k-acyclicity. Then there is a chain $(x^1, x^2, \cdots, x^k)$ with elements on X such that $x^1 \succeq^{R,D} x^2 \succeq^{R,D} \cdots \succeq^{R,D} x^k$ and $x^k \succ^{R,D} x^1$. Then by rationalization in Definition 8, we have $r(x^1, x^2) \geq 0, \ldots, r(x^2, x^3) \geq 0, \ldots, r(x^{k-1}, x^k) \geq 0$, but $r(x^k, x^1) \geq 0$. There are two cases. First, $r(x^k, x^1) > 0$. But this is a contradiction since there is a $U^* \in \Omega$ such that for all $u \in U^*$, $u(x^k) - u(x^1) > 0$. Indeed, we have coalitions U_s such that for all $u \in U_s$ and all $s = \{1, \cdots, k-1\}$, $u(x^s) - u(x^{s+1}) \geq 0$. Since $\nu \geq k+1$ this means that $\cap_{s\in\{1,\cdots,k-1\}} U_s \cap U^* \neq \emptyset$. But this is a contradiction because this implies the existence of a $u \in \cap_{s\in\{1,\cdots,k-1\}} U_s \cap U^*$ such that $u(x^1) \geq u(x^k) > u(x^1)$, which is impossible.

Second, $r(x^k, x^1) = 0$, which means that for all $U \in \Omega$, there exists $u \in U$ such that $u(x^k) - u(x^1) \leq 0$. But there exists $y \in B(x^1, \epsilon)$ for some small $\epsilon > 0$ such that $y >> x^1$, and $p^k x^k > p^k y$ (because by assumption $p^k x^k > p^k x^1$). This means that for all $U \in \Omega$, there exists $u \in U$ such that $u(x^k) - u(y) < 0$. Then, $r(x^k, y) < 0$, which contradicts that r rationalizes the data, so this case is also impossible.

(ii) $\Longrightarrow$ (i).– We begin by constructing datasets $O^k_{t,s,\cdots,w}$ of size $k \geq 2$. For these k observations k-acyclicity implies that GARP holds on $O^k_{t,s,\cdots,w}$. Then, using Afriat's theorem we define the utility function $u_{t,s,\cdots,w} : X \to \mathbb{R}$ (recall that $u_{t,s,\cdots,w}$ is continuous, concave, and strictly increasing).

Next, we define the set of utility functions as $U_t = \cup_{s\in\mathbb{T}} \cdots \cup_{w\in\mathbb{T}} \{u_{t,s,\cdots,w}\}$ and the family of sets of utility functions as $\Omega_{\mathbb{T}} = \cup_{t\in\mathbb{T}} U_t$. It is trivial to verify that U_t is compact (finite and discrete).

Next, we verify that the Nakamura number associated with $\Omega_{\mathbb{T}}$ is $\nu \geq k+1$. Indeed, $\cap_{t'\in\{t,s,\cdots,w\}} U_{t'} = \{u_{t,s,\cdots,w}\}$ for all sequences $\{t, s, \cdots, w\}$ of size k. This is true because Afriat's theorem produces the same utility for any permutation of the subdataset $O^k_{t,s,\cdots,w}$. Hence, $\nu \geq k+1$.

Define the coherent CMU preference function as:

$$r(x, y) = \max_{U\in\Omega_{\mathbb{T}}} \min_{u\in U} (u(x) - u(y)).$$

Finally, we prove that the function r rationalizes the data set O^T. Consider $y \in X$ and some fixed $t \in \mathbb{T}$ such that $p^t x^t \geq p^t y$. We have:

$$r(x^t, y) = \max_{U\in\Omega_{\mathbb{T}}} \min_{u\in U} (u(x^t) - u(y))$$

$$\geq \min_{u \in U}(u(x^t) - u(y)).$$

It suffices to show that $u(x^t) - u(y) \geq 0$ whenever $p^t x^t \geq p^t y$ for each $U_t \in \Omega_{\mathbb{T}}$. But this follows directly from the definition of U_t and Afriat's theorem. Hence, $r(x^t, y) \geq 0$.

Proof of Theorem 5

The existence of $u|_{[a,b]}$ in the intersection of all $U|_{[a,b]}$, means that the restriction of the preference function r to $[a, b]$, denoted by $r_{[a,b]}$, represents a complete order on $[a, b]$ that only allows for weak cycles. Formally, $r|_{[a,b]}$ allows only for weak cycles if and only if, for $x^1, x^2, \cdots, x^n \in [a, b]$, we have that $r|_{[a,b]}(x^1, x^2) \geq r|_{[a,b]}(x^2, x^3) \geq \cdots \geq r|_{[a,b]}(x^{n-1}, x^n) \geq 0$ does not imply $r|_{[a,b]}(x^n, x^1) > 0$. To see this is true, notice that the existence of a $u|_{[a,b]}$ in the intersection of all $U|_{[a,b]}$ means that $u|_{[a,b]}(x^1) \geq u|_{[a,b]}(x^2) \geq \cdots \geq u|_{[a,b]}(x^n)$ implies, for all $U|_{[a,b]}$, that there is a $u|_{[a,b]}$ such that $u|_{[a,b]}(x^n) - u|_{[a,b]}(x^1) \leq 0$. This, in turn, implies $r|_{[a,b]}(x^n, x^1) \leq 0$. Because r is a complete, monotone and coherent CMU, the order $r_{[a,b]}$ is complete, continuous and strictly monotone, and moreover, it is transitive. To see the latter, note that for $x, y, z \in [a, b]$, if $r_{[a,b]}(x, y) \geq 0$ and $r_{[a,b]}(y, z) \geq 0$ then by completeness either $r_{[a,b]}(x, z) \geq 0$ or $r_{[a,b]}(z, x) \geq 0$ holds, but by weak cycles, it can only be that $r_{[a,b]}(x, z) \geq 0$ holds. Finally, we apply Theorem 2 in Moldau (1996) to conclude that for all $(p, w), p \in P$ and $w > 0$, there exists a $x \in B(p, w)$ for any complete and coherent CMU such that $[\max_{U \in \Omega} \min_{u \in U}(u(x) - u(y))] \geq 0$ for all $y \in B(p, w)$.

Proof of Theorem 6

The proof of the first part of the theorem follows from the discussion before the statement of the theorem. We prove the second part. Since $RW_{st}(x) \supseteq RW(x)$, we have that $NRW_{st}(x) \subseteq NRW(x)$. Thus, by construction $NRW^W(x) \subseteq NRW(x)$. We are going to show that $NRW^w(x) \subset NRW(x)$ for some $x \in X$. We will do this in the context of Example 1. Clearly, the bundle $x^- = (1\,1\,1)'$ is monotonically dominated by x^{T+1}. First, note that the upper bound using Varian's method contains this dominated option, i.e., $x^- \in NRW(x) = X \setminus \{x^{T+1}\}$. Second, note that for all $s, t \in \mathbb{T}$ and by strict monotonicity, we must have $u_{st}(x^{T+1}) > u_{st}(x^-)$ (this follows by Afriat's theorem applied to the data O^2_{st}). This implies that $x^- \notin NRW_{st}(x^{T+1})$ for all $t, s \in \mathbb{T}$. It also follows that $RP(w) \subseteq RP^W(x)$, since $RP(x) \subseteq RP_{st}(x)$ holds for all $s, t \in \mathbb{T}$.

Consider again, in the context of Example 1, the bundle $x^- = (1\,1\,1)'$. We are going to show that $x^{T+1} = (2\,2\,2)'$ is not in $RP(x^-)$, but that it is in $RP^W(x^-)$. From Example

1, we know that $S(x^{T+1}) = \emptyset$, which implies $x^{T+1} \notin RP(x^-)$. However, we also have that $p(x^{T+1} - x^-) > 0$ for all $p \in P$, which means that for the rationalizing piecemeal utility function, we have $u_{st}(x^{T+1}) > u_{st}(x^-)$ for all $s, t \in \mathbb{T}$. This means that $x^{T+1} \in RP_{st}(x^-)$ for all $s, t \in \mathbb{T}$. Hence, $x^{T+1} \in RP^W(x^-)$.

References

Afriat, S. N. (1967). The construction of utility functions from expenditure data. *International Economic Review*, 8(1):67–77.

Allen, R. and Rehbeck, J. (2018). Assessing Misspecification and Aggregation for Structured Preferences. *Unpublished working paper*.

Banerjee, S. and Murphy, J. (2006). A simplified test for preference rationality of two-commodity choice. *Experimental Economics*, 9(1):67–75.

Blundell, R., Browning, M., and Crawford, I. (2008). Best Nonparametric Bounds on Demand Responses. *Econometrica*, 76(6):1227–1262.

Blundell, R., Kristensen, D., and Matzkin, R. (2014). Bounding quantile demand functions using revealed preference inequalities. *Journal of Econometrics*, 179(2):112–127.

Brown, D. and Calsamiglia, C. (2007). The nonparametric approach to applied welfare analysis. *Economic Theory*, 31(1):183–188.

Caradonna, P. (2018). How strong is the weak axiom? Technical report, Georgetown University, Department of Economics.

Chambers, C. P. and Echenique, F. (2016). *Revealed Preference Theory*. Econometric Society Monograph, Cambridge University Press.

Cherchye, L., Demuynck, T., and De Rock, B. (2018). Transitivity of preferences: when does it matter? *Theoretical Economics*, 13(3):1043–1076.

Cherchye, L., Demuynck, T., and De Rock, B. (2019). Bounding counterfactual demand with unobserved heterogeneity and endogenous expenditures. *Journal of Econometrics*, forthcoming.

Cosaert, S. and Demuynck, T. (2018). Nonparametric welfare and demand analysis with unobserved individual heterogeneity. *Review of Economics and Statistics*, 100(2):349–361.

Demuynck, T. and Hjertstrand, P. (2019). Samuelson's approach to revealed preference theory: Some recent advances. In Anderson, R., Barnett, W., and Cord, R., editors, *Paul Samuelson: Master of Modern Economics*. Palgrave MacMillan, London.

Eckhoff, J. (1993). Chapter 2.1 - helly, radon, and caratheodory type theorems. In Gruber, P. and Wills, J., editors, *Handbook of Convex Geometry*, pages 389–448. North-Holland, Amsterdam.

Forges, F. and Minelli, E. (2009). Afriat's theorem for general budget sets. *Journal of Economic Theory*, 144:135–145.

Frick, M., Iijima, R., and Le Yaouanq, Y. (2019). Boolean expected utility. Working paper.

Gerasimou, G. (2018). Indecisiveness, undesirability and overload revealed through rational choice deferral. *The Economic Journal*, 128(614):2450–2479.

Hara, K., Ok, E. A., and Riella, G. (2019). Coalitional expected multi-utility theory. *Econometrica*, 87(3):933–980.

Hoderlein, S. and Stoye, J. (2014). Revealed preferences in a heterogeneous population. *Review of Economics and Statistics*, 96(2):197–213.

Houthakker, H. (1950). Revealed preference and the utility function. *Economica*, 17(66):159–174.

John, R. (1995). The weak axiom of revealed preference and homogeneity of demand functions. *Economics Letters*, 47(1):11–16.

John, R. (2001). The concave nontransitive consumer. *Journal of Global Optimization*, 20(3-4):297–308.

Keiding, H. and Tvede, M. (2013). Revealed smooth nontransitive preferences. *Economic Theory*, 54(3):463–484.

Kihlstrom, R., Mas-Colell, A., and Sonnenschein, H. (1976). The demand theory of the weak axiom of revealed preference. *Econometrica*, 44(5):971–978.

Kim, T. and Richter, M. K. (1986). Nontransitive-nontotal consumer theory. *Journal of Economic Theory*, 38(2):324–363.

Kitamura, Y. and Stoye, J. (2018). Nonparametric analysis of random utility models. *Econometrica*, 86(6):1883–1909.

Martin, M. and Salles, M. (2013). Social choice and cooperative game theory: Voting games as social aggregation functions. *International Game Theory Review*, 15(03):1340012.

Mas-Colell, A., Whinston, M. D., and Green, J. R. (1995). *Microeconomic Theory*. Oxford University Press Oxford, UK.

Matzkin, R. L. and Richter, M. K. (1991). Testing strictly concave rationality. *Journal of Economic Theory*, 53(2):287–303.

Moldau, J. H. (1996). A simple existence proof of demand functions without standard transitivity. *Journal of Mathematical Economics*, 25(3):325–333.

Nishimura, H. and Ok, E. A. (2016). Utility representation of an incomplete and nontransitive preference relation. *Journal of Economic Theory*, 166:164–185.

Nishimura, H., Ok, E. A., and Quah, J. K.-H. (2017). A comprehensive approach to revealed preference theory. *American Economic Review*, 107(4):1239–63.

Quah, J. K.-H. (2006). Weak axiomatic demand theory. *Economic Theory*, 29(3):677–699.

Quah, J. K.-H. (2008). The existence of equilibrium when excess demand obeys the weak axiom. *Journal of Mathematical Economics*, 44(3-4):337–343.

Richter, M. and Rubinstein, A. (2019). Convex preferences: A new definition. *Theoretical Economics*, 14(4):1169–1183.

Rose, H. (1958). Consistency of preference: The two-commodity case. *Review of Economic Studies*, 25(2):124–125.

Samuelson, P. A. (1938). A note on the pure theory of consumer's behaviour. *Economica*, 5(17):61–71.

Samuelson, P. A. (1948). Consumption theory in terms of revealed preference. *Economica*, 15(60):243–253.

Schofield, N. (1984). Social equilibrium and cycles on compact sets. *Journal of Economic Theory*, 33(1):59–71.

Shafer, W. J. (1974). The nontransitive consumer. *Econometrica*, 42(5):913–919.

Sonnenschein, H. (1971). Demand theory without transitive preferences, with applications to the theory of competitive equilibrium. In Chipman, J., Hurwich, L., and Richter, M., S.-H., editors, *Preferences, Utility and Demand*. Harcourt Brace Jovanovich, New York.

Tsevendorj, I. (2001). Piecewise-convex maximization problems. *Journal of Global Optimization*, 21(1):1–14.

Tversky, A. (1969). Intransitivity of preferences. *Psychological Review*, 76(1):31–48.

Varian, H. R. (1982). The nonparametric approach to demand analysis. *Econometrica*, 50(4):945–973.

Zangwill, W. (1967). The piecewise concave function. *Management Science*, 13(11):900–912.

Online Appendix to "A Rationalization of the Weak Axiom of Revealed Preference" NOT FOR PUBLICATION

Victor H. Aguiar* Per Hjertstrand† Roberto Serrano‡

This version: September 2022

A. Extended Discussion of Related Literature

In this section, we extend our discussion of the relationship with previous works. Afriat (1967) and Varian (1982) show that the classical notion of rationality is equivalent to the Generalized Axiom of Revealed Preference (GARP). In the current study, we show that a data set that satisfies WGARP, but perhaps not GARP, is consistent with a weaker notion of rationalization. Of course, classical utility maximization is a special case, when there is a (global) utility function u that is capable of rationalizing the data set O^T, in which case there exists a utility function u such that $r(x, y) = u(x) - u(y)$ for all $x, y \in X$.

Our results for the consumer setting are derived under the assumption that the researcher only observes a finite number of choices. In the original formulation of revealed-preference theory (Samuelson 1938 and Houthakker 1950), it is implicitly assumed that

*Department of Economics, The University of Western Ontario, London, Ontario, Canada, vaguiar@uwo.ca.

†Research Institute of Industrial Economics (IFN), Stockholm, Sweden, per.hjertstrand@ifn.se.

‡Department of Economics, Brown University, Providence, Rhode Island, U.S.A., roberto_serrano@brown.edu.

the entire demand function, or a demand correspondence, is observed. We provide a characterization for the abstract setting which covers finite and infinite data sets, and also nonlinear budgets.[1]

The closest works to our paper are Kim and Richter (1986) and Quah (2006). Both works provide rationalizations of demand correspondences, or functions, consistent with WGARP or WARP, using additional conditions on the invertibility of demand, where preferences are assumed convex (in a certain sense).[2] Our paper generalizes these contributions by (i) providing a rationalization of WGARP/WARP for finite data sets, and (ii) relaxing the invertibility requirement for every commodity bundle in X.[3]

Preference functions with the skew-symmetry property were introduced by Shafer (1974). We show that rationalization by the weaker notion of asymmetric preference functions is essentially equivalent to WGARP; the proof of Theorem 1 also reveals that WGARP implies that one can always find a skew-symmetric preference-function rationalization. Moreover, we also show that WGARP is equivalent to rationalization by a new kind of preference function, the CMU preference function, and our results answer in the negative the conjecture posed in Kihlstrom et al. (1976), concerning the equivalence between Shafer's skew-symmetric preference functions and WGARP.[4]

Krauss (1985) provides a representation of 2-monotone operators (effectively equivalent to the law of demand), by means of a skew-symmetric preference function. To our knowledge, our results regarding WGARP are new in the mathematical literature on monotone operators as well, extending the contribution of Krauss (1985) to 2-cyclical consistent operators (effectively equivalent to WGARP). We also provide an extension for the original representation of the law of demand, connecting it with CMU quasilinear rationalization, as in Brown and Calsamiglia (2007), as well as covering the case of limited data sets. John (2001) studies a generalization of the law of demand that allows weights. This condition is stronger than WGARP. He provides a revealed-preference characterization of this condition, and establishes the equivalence of the weighted law

[1]The only other work we are aware of that has provided a revealed-preference characterization of nontransitive consumers in abstract settings is Dziewulski (2021).

[2]The notions of convexity of preferences in Kim-Richter and Quah are strictly weaker than standard convexity of preferences.

[3]Mariotti (2008) provides a study of WGARP in abstract environments with complete data sets (all choice sets are observed), and shows that WARP is equivalent to the maximization (in a new sense that he calls justified, of a binary preference relation that is asymmetric). Mariotti (2008) does not provide a representation theorem for WGARP nor does he deal with limited data sets.

[4]The main result in Kihlstrom et al. (1976) shows that if the demand function is differentiable and satisfies WGARP at every point in its domain, then the Slutsky substitution matrix derived from the demand function is negative semidefinite at every point.

of demand and a special case of Shafer's (1974) preference function that is concave in the first argument. The weighted law of demand condition is cardinal and cannot be translated to abstract settings.

Some papers have extended the Varian (1982) method to recover preferences to different types of consumer demand models. Two notable examples are, Blundell et al. (2003, 2008) which show that it is possible to substantially enhance recovery and prediction results by combining revealed-preference theory with nonparametric estimation of Engel curves. However, their analysis is based on WARP, which we have shown may be problematic in a setup with more than two goods (Blundell et al. (2003) considers 22 goods and Blundell et al. (2008) uses three goods in their empirical applications). Blundell et al. (2015) shows how the methods in Blundell et al. (2003, 2008) can be modified to derive sharp bounds on welfare measures under SARP (i.e., global rationality).

Finally, Halevy et al. (2017) shows that Varian's method to recover preferences under GARP does not apply to nonconvex preferences, and suggests an alternative method based on monotonicity. However, when GARP holds, concavity is not a testable restriction. Our analysis provides a different solution based on piecewise concavity, which, being an alternative to monotonicity, is also robust to the possible lack of convexity of preferences, when the data satisfies WGARP but violates GARP. Note that, in our setup, convexity of preferences is, in fact, a testable condition.

B. WARP

In this subsection, we provide a characterization of Samuelson's (1938) weak axiom of revealed preference (WARP).[5] As discussed in Section 2.1, the strong axiom of revealed preference (SARP) is the transitive counterpart to WARP as it allows for transitive comparisons between bundles.[6] Matzkin and Richter (1991) provides a complete characterization of SARP by showing that it is a necessary and sufficient condition for a data set O^T to be strictly rationalized by a continuous, strictly increasing, and strictly concave utility function. Before stating their result, we define the notion of strict rationalization as follows:

[5]Recall from Section 2.1 that WARP holds if there is no pair of observations $s, t \in \mathbb{T}$ such that $x^t \succeq^{R,D} x^s$ and $x^s \succeq^{R,D} x^t$, with $x^t \neq x^s$.

[6]Recall that SARP holds if there is no pair of observations $s, t \in \mathbb{T}$ such that $x^t \succeq^{R} x^s$ and $x^s \succeq^{R,D} x^t$, with $x^t \neq x^s$.

Definition 1. *(Strict utility rationalization) Consider a finite data set* $O^T = \{p^t, x^t\}_{t\in\mathbb{T}}$ *and a utility function* $u : X \mapsto \mathbb{R}$*. For all* $y \in X$ *and all* $t \in \mathbb{T}$ *such that* $p^t y \leq p^t x^t$*, the data* O^T *is strictly rationalized by* u *if* $u(x^t) > u(y)$ *whenever* $y \neq x^t$*.*

Theorem B. (Matzkin and Richter 1991) Consider a finite data set $O^T = \{p^t, x^t\}_{t\in\mathbb{T}}$. The following statements are equivalent:

(i) The data O^T can be strictly rationalized by a utility function.

(ii) The data O^T satisfies SARP.

(iii) There exist numbers U^t and $\lambda^t > 0$ for all $t \in \mathbb{T}$ such that the inequalities:

$$\begin{aligned} \text{if } x^t &\neq x^s \text{ then, } U^t - U^s > \lambda^t p^t \left(x^t - x^s\right), \\ \text{if } x^t &= x^s \text{ then, } U^t - U^s = 0, \end{aligned}$$

hold for all $s, t \in \mathbb{T}$.

(iv) There exist numbers V^t for all $t \in \mathbb{T}$ such that the inequalities:

$$\begin{aligned} \text{if } x^t &\neq x^s \text{ and } p^t(x^t - x^s) \geq 0 \text{ then, } V^t - V^s > 0, \\ \text{if } x^t &= x^s \text{ then, } V^t - V^s = 0, \end{aligned}$$

hold for all $s, t \in \mathbb{T}$.

(v) The data O^T can be strictly rationalized by a continuous, strictly increasing, and strictly concave utility function.

Matzkin and Richter's theorem is analogous to Afriat's theorem in terms of strict rationalization. We note that the equivalence of statements (i) and (ii) is the content of Houthakker's theorem, since a utility function must represent a complete and transitive preference relation. Moreover, their result shows that continuity, monotonicity, and strict concavity are empirically nontestable properties.[7]

Next, we provide a complete revealed-preference characterization of WARP. This result mirrors Matzkin and Richter's theorem in terms of strict preference-function rationalization (as opposed to strict utility rationalization). In this context, we define strict preference function rationalization as follows:

[7] Matzkin and Richter's original formulation consists of statements (i), (ii), (iii), and (v). Talla Nobibon et al. (2016) proves the equivalence of statements (ii) and (iv).

Definition 2. *(Strict preference-function rationalization) Consider a finite data set* $O^T = \{p^t, x^t\}_{t \in \mathbb{T}}$ *and a preference function* $r : X \times X \mapsto \mathbb{R}$. *For all* $y \in X$ *and all* $t \in \mathbb{T}$ *such that* $p^t y \leq p^t x^t$, *the data* O^T *is strictly rationalized by* r *if* $r(x^t, y) > 0$ *whenever* $y \neq x^t$.

The next result contains our characterization:

Theorem A1. *Consider a finite data set* $O^T = \{p^t, x^t\}_{t \in \mathbb{T}}$. *The following statements are equivalent:*

(i) The data O^T *can be strictly rationalized by an asymmetric preference function.*

(ii) The data O^T *satisfies WARP.*

(iii) There exist numbers $R^{t,s}$ *and* $\lambda_{ts}^t > 0$ *for all* $t, s \in \mathbb{T}$ *with* $R^{t,s} = -R^{s,t}$ *and* $\lambda_{ts}^t = \lambda_{st}^t$ *such that inequalities:*

$$\begin{aligned} \text{if } x^t &\neq x^s \text{ then, } R^{t,s} > \lambda_{ts}^t p^t \left(x^t - x^s\right), \\ \text{if } x^t &= x^s \text{ then, } R^{t,s} = 0, \end{aligned}$$

hold for all $t, s \in \mathbb{T}$.

(iv) There exist numbers $W^{t,s}$ *for all* $t, s \in \mathbb{T}$ *with* $W^{t,s} = -W^{s,t}$ *such that inequalities:*

$$\begin{aligned} \text{if } x^t &\neq x^s \text{ and } p^t(x^t - x^s) \geq 0 \text{ then, } W^{t,s} > 0, \\ \text{if } x^t &= x^s \text{ then, } W^{t,s} = 0, \end{aligned}$$

hold for all $t, s \in \mathbb{T}$.

(v) The data O^T *can be strictly rationalized by a coherent strict CMU preference function (which in particular satisfies asymmetry, continuity, monotonicity, and strict piecewise concavity).*

Analogous to our characterization of WGARP in Theorem 1, this result offers: (a) a minimal strict rationalization of WARP on the basis of an ordinal property of the preference function; (b) characterizations of WARP in terms of workable inequalities; and (c) a cardinal representation in terms of a coherent CMU. This third part shows that separate violations of continuity, monotonicity, and strict piecewise concavity cannot be detected in finite data sets.

C. The Law of Demand and the Quasilinear Preference Maximization Model

As described in Section 6.2 of the paper, we can impose restrictions on the piecemeal utilities in the CMU preference function to provide a representation for choices obeying the law of demand. This section derives necessary and sufficient conditions for a finite data set to be rationalized by a continuous, strictly increasing, skew-symmetric, concave, and quasilinear preference function. Interestingly, we show that one such condition is the law of demand, and consequently, this is equivalent to rationalization by a CMU preference function with quasilinear piecemeal utilities. Before presenting these results, we briefly recall the revealed-preference characterization for quasilinear-utility maximization.

Definition 3. *(Quasilinear utility maximization) Consider a locally nonsatiated utility function $u(x)$. We say that a consumer facing prices $p \in P$ and income $w \in W$ is a quasilinear utility maximizer if she solves*

$$\max_{x \in X} u(x) + w - px \iff \max_{x \in X, y \in \mathbb{R}} u(x) + y \ \ s.t.\ px + y \leq w.$$

As in standard applications of quasilinear utility maximization, we allow the numeraire y to be negative in order to avoid technicalities related to corner solutions.[8] Brown and Calsamiglia (2007) shows that the axiom of cyclical monotonicity is a necessary and sufficient condition for a data set to be rationalized by a continuous, strictly increasing, concave, and quasilinear utility function.

Axiom 1. *(Cyclical monotonicity) Cyclical monotonicity holds if, for all distinct choices of indices $(1, 2, \ldots, n) \in \mathbb{T}$:*

$$p^1(x^1 - x^2) + p^2(x^2 - x^3) + \cdots + p^n(x^n - x^1) \leq 0.$$

The next theorem recalls the revealed-preference characterization of quasilinear utility maximization from Brown and Calsamiglia (2007) and Allen and Rehbeck (2018):

Theorem D. (Brown and Calsamiglia 2007; Allen and Rehbeck 2018) Consider a finite data set $O^T = \{p^t, x^t\}_{t \in \mathbb{T}}$. The following statements are equivalent:

[8] Allen and Rehbeck (2018) shows the equivalence of the unconstrained quasilinear maximization and the constrained version with a numeraire in Definition 3.

(i) The data O^T can be rationalized by a locally nonsatiated and quasilinear utility function.

(ii) The data O^T satisfies cyclical monotonicity.

(iii) There exist numbers U^t for all $t \in \mathbb{T}$ such that the inequalities:

$$U^t - U^s \geq p^t(x^t - x^s),$$

hold for all $s, t \in \mathbb{T}$.

(iv) The data O^T can be rationalized by a continuous, strictly increasing, concave, and quasilinear utility function.

Next, we state our main result in this section by giving a characterization of the law of demand. The axiom of the law of demand is defined as:

Axiom 2. *(Law of demand) The law of demand holds if, for all observations $s, t \in \mathbb{T}$:*

$$(p^t - p^s)(x^t - x^s) \leq 0.$$

For any sequence consisting of only two (distinct) observations $s, t \in \mathbb{T}$, it is easy to see that cyclical monotonicity and the law of demand are equivalent. We say that a CMU preference function is a CMU *quasilinear* preference function if, for any $u \in U$, and any $U \in \Omega$ the piecemeal utility function u is continuous, strictly increasing, concave, and quasilinear. The next theorem shows that the law of demand is equivalent to rationalization of a CMU quasilinear preference function:

Theorem B1. *Consider a finite data set $O^T = \{p^t, x^t\}_{t\in\mathbb{T}}$. The following statements are equivalent:*

(i) The data O^T can be rationalized by a locally nonsatiated, skew-symmetric, and quasilinear preference function.

(ii) The data O^T satisfies the law of demand.

(iii) There exist numbers $R^{t,s}$, for all $s, t \in \mathbb{T}$, with $R^{t,s} = -R^{s,t}$, such that inequalities:

$$R^{t,s} \geq p^t(x^t - x^s),$$

hold for all $s, t \in \mathbb{T}$.

(iv) The data O^T can be rationalized by a coherent CMU quasilinear preference function.

The proof of Theorem B1 shows that continuity and strict monotonicity can be imposed on the rationalizing quasilinear preference function in statement (i) without loss of generality (i.e., these properties are separately nontestable).

Proofs of Section B: WARP

Proof of Theorem A1

As in the proof of our main result, we organize this proof in three blocks. First, we prove the equivalence of WARP with the appropriate systems of inequalities. Next, we prove the ordinal characterization of WARP. And finally, we provide the cardinal representation of WARP by means of a coherent strict CMU preference function.

We begin with the equivalence of statements (ii), (iii), and (iv):

(ii) $\Longrightarrow$ (iii).– Suppose that WARP holds. For all $s, t \in \mathbb{T}$, we let the data set O^2_{st} consist of the two observations $s, t \in \mathbb{T}$. Overall, this gives T^2 such data sets. By a direct application of Matzkin and Richter's (1991) theorem, the following conditions are equivalent: *(i)* the data set O^2_{st} satisfies WARP, *(ii)* there exist numbers U^k_{ts} and $\lambda^k_{ts} > 0$ for all $k \in \{t, s\}$ such that the inequalities: *if* $x^k \neq x^l$ *then,* $U^k_{ts} - U^l_{ts} > \lambda^k_{ts} p^k \left(x^k - x^l\right)$, *and if* $x^k = x^l$ *then,* $U^k_{ts} - U^l_{ts} = 0$ hold for all $k, l \in \{t, s\}$. Since permuting the data is insignificant for Matzkin and Richter's (1991) theorem, we can without loss of generality set $U^k_{ts} = U^k_{st}$ and $\lambda^k_{ts} = \lambda^k_{st}$ for all $k \in \{t, s\}$. We obtain the inequalities in condition (iii) by defining $R^{t,s} = U^t_{ts} - U^s_{ts}$ and $R^{s,t} = U^s_{ts} - U^t_{ts}$.

(iii) $\Longrightarrow$ (iv).– Suppose that condition (iii) holds. If $x \neq x^t$ and $p^t(x^t - x^s) \geq 0$ then $R^{t,s} > 0$. We obtain condition (iv) by defining $W^{t,s} = R^{t,s}$ for all $s, t \in \mathbb{T}$.

(iv) $\Longrightarrow$ (ii).– Suppose that condition (iv) holds, but that WARP is violated, i.e., $p^t(x^t - x^s) \geq 0$ and $p^s(x^s - x^t) \geq 0$ with $x^t \neq x^s$ for some $s, t \in \mathbb{T}$. Then $W^{t,s} > 0$ and $W^{s,t} > 0$. Thus, (iv) is violated.

Next, we show the equivalence of statements (i) and (ii):

(i) $\implies$ (ii).– Let $r(x, y)$ be an asymmetric preference function that strictly rationalizes the data. Suppose there is a violation of WARP, so that $p^t x^t \geq p^t x^s$ and $p^s x^s \geq p^s x^t$ with $x^s \neq x^t$ for some pair of observations $s, t \in \mathbb{T}$. Then, by strict rationalization in Definition 2, we have $r(x^t, x^s) > 0$ and $r(x^s, x^t) > 0$. But this violates asymmetry.

(ii) $\implies$ (i).– Since this proof is very similar to the similar step in the proof of Theorem 1, we only give the main parts (and the parts that differ).

Suppose that WARP in condition (ii) holds. Consider the T^2 data sets O^2_{ts} for every pair of observations $s, t \in \mathbb{T}$. For the two observations in each data set O^2_{st}, we define the function $u_{st} : X \to \mathbb{R}$ as in Matzkin and Richter (1991). From this, we know that each function u_{st} is continuous, strictly concave and strictly increasing. Next, for all $x, y \in X$, we define the mapping: $r_{st} : X \times X \to \mathbb{R}$ as:

$$r_{st}(x, y) = \begin{cases} u_{st}(x) - u_{st}(y) & \text{if} \quad s \neq t, \\ p^t(x - y) - \varepsilon(g(x - x^t) - g(y - x^t)), & \text{if} \quad s = t. \end{cases}$$

for some small $\varepsilon > 0$ and where the function g is defined in Matzkin and Richter (1991). Clearly, each function $r_{s,t}$ is continuous, strictly concave and skew-symmetric.

Define the preference function $r(x, y)$ for any $x, y \in X$ as:

$$\begin{aligned} r(x, y) &= \min_{\lambda \in \Delta} \max_{\mu \in \Delta} \sum_{s \in \mathbb{T}} \sum_{t \in \mathbb{T}} \lambda_s \mu_t r_{st}(x, y) \\ &= \max_{\mu \in \Delta} \min_{\lambda \in \Delta} \sum_{s \in \mathbb{T}} \sum_{t \in \mathbb{T}} \lambda_s \mu_t r_{st}(x, y). \end{aligned}$$

First, we prove that the function r strictly rationalizes the data set O^T. Consider $y \in X$ and some fixed $t \in \mathbb{T}$ such that $x^t \neq y$ and $p^t x^t \geq p^t y$. Let $\mu^t \in \Delta$ be such that $\mu^t_j = 0$ if $j \neq t$ and $\mu^t_j = 1$ if $j = t$. By the same argument as in the proof of Theorem 1, we have

$$\begin{aligned} r(x^t, y) &\geq \min_{\lambda \in \Delta} \sum_{i \in \mathbb{T}} \sum_{j \in \mathbb{T}} \lambda_i \mu^t_j r_{ij}(x^t, y) \\ &= \min_{\lambda \in \Delta} \sum_{i \in \mathbb{T}} \lambda_i r_{it}(x^t, y). \end{aligned}$$

It suffices to show that $r_{it}(x^t, y) > 0$ whenever $x^t \neq y$ and $p^t x^t \geq p^t y$ for each data set O^2_{it}. But this follows directly from the definition of r_{it} and Matzkin and Richter's (1991) theorem. Hence, $r(x^t, y) > 0$.

By the exact same arguments as in the proof of Theorem 1, it can be shown that the function $r(x, y)$ is skew-symmetric, (and hence, asymmetric).

And finally, we show the equivalence of statements (ii) and (v):

(v) $\Longrightarrow$ *(ii).–* Let r be a coherent CMU preference function that strictly rationalizes the data. Suppose there is a violation of WARP, so that $p^t x^t \geq p^t x^s$ and $p^s x^s \geq p^s x^t$ for some pair of distinct observations $s, t \in \mathbb{T}$. Then, by strict rationalization in Definition 2, we have $r(x^t, x^s) > 0$ and $r(x^s, x^t) > 0$. But this violates asymmetry (by Lemma 1).

(ii) $\Longrightarrow$ *(v).–* We only give the main parts (and the parts that differ from Theorem 1).

Suppose that WARP in condition (ii) holds. Consider the T^2 data sets O^2_{ts} for every pair of observations $s, t \in \mathbb{T}$. For the two observations in each data set O^2_{st}, we define the function $u_{st} : X \to \mathbb{R}$ as in Matzkin and Richter (1991), where each u_{st} is continuous, strictly concave, and strictly increasing.

Next, as in the proof of Theorem 1, we define $U_t = \cup_{s\in\mathbb{T}}\{u_{ts}\}$ and $\Omega_{\mathbb{T}} = \cup_{t\in\mathbb{T}} U_t$. Of course, U_t and Ω are compact (finite and discrete). By the same arguments as in the proof of Theorem 1, we have that $\Omega_{\mathbb{T}}$ satisfies coherency. Finally, we define the coherent strict CMU preference function as:

$$r(x, y) = \max_{U\in\Omega_{\mathbb{T}}} \min_{u\in U}(u(x) - u(y)).$$

Consider $y \in X$ and some fixed $t \in \mathbb{T}$ such that $p^t x^t \geq p^t y$. We have:

$$\begin{aligned} r(x^t, y) &= \max_{U\in\Omega_{\mathbb{T}}} \min_{u\in U}(u(x^t) - u(y)) \\ &\geq \min_{u\in U}(u(x^t) - u(y)). \end{aligned}$$

To show that our strict CMU function strictly rationalizes the data, i.e., $r(x^t, y) > 0$, it suffices to show that $u(x^t) - u(y) > 0$ whenever $x^t \neq y$ and $p^t x^t \geq p^t y$ for each data set O^2_{it}. But this follows directly from the definition of $\Omega_{\mathbb{T}}$ and Matzkin and Richter's (1991) theorem.

Proofs of Section C: The Law of Demand and the Quasilinear Preference Maximization Model

Proof Of Theorem D.

(i) $\Longrightarrow$ *(ii).*– By the definition of quasilinear rationalization, we have for any observation $s \in \mathbb{T}$ with $x = x^s$,

$$u(x^t) - p^t x^t \geq u(x^s) - p^t x^s.$$

Thus, after rearranging terms, for any sequence of distinct choices of indices $(1, 2, 3, ..., n) \in \mathbb{T}$, we have:

$$\begin{aligned} p^1x^2 - p^1x^1 &\geq u(x^2) - u(x^1), \\ p^2x^3 - p^2x^2 &\geq u(x^3) - u(x^2), \\ &\vdots \\ p^nx^1 - p^nx^n &\geq u(x^1) - u(x^n). \end{aligned}$$

Adding up both sides, we get:

$$\begin{aligned} & (p^1x^2 - p^1x^1) + (p^2x^3 - p^2x^2) + \cdots + (p^nx^1 - p^nx^n) \\ \geq \; & (u(x^2) - u(x^1)) + (u(x^3) - u(x^2)) + \cdots + (u(x^1) - u(x^n)) \\ = \; & 0. \end{aligned}$$

Thus,

$$p^1(x^1 - x^2) + p^2(x^2 - x^3) + \cdots + p^n(x^n - x^1) \leq 0,$$

which is cyclical monotonicity.

(ii) $\Longrightarrow$ *(iii).*– Suppose that condition (ii) holds and define:

$$U^t = \min_{\{1,2,3,...,n,t\} \in \mathbb{T}} \{p^1(x^2 - x^1) + p^2(x^3 - x^2) + \cdots + p^n(x^t - x^n)\},$$

for all $t \in \mathbb{T}$. That is, U^t is a minimum of the given expression over all sequences starting anywhere and terminating at t. Note that there are only finitely many sequences because their elements are distinct. Hence, the minimum always exists. To show that the numbers U^t do satisfy the inequalities in statement (iii), suppose that:

$$\begin{aligned} U^t &= p^1(x^2 - x^1) + p^2(x^3 - x^2) + \cdots + p^n(x^t - x^n), \\ U^s &= p^a(x^b - x^a) + p^b(x^c - x^b) + \cdots + p^m(x^s - x^m), \end{aligned}$$

for some distinct sequences $\{1, 2, 3, ...n, t\} \in \mathbb{T}$ and $\{a, b, c, ..., m, s\} \in \mathbb{T}$. Then:

$$\begin{aligned} U^t &= p^1(x^2 - x^1) + p^2(x^3 - x^2) + \cdots + p^n(x^t - x^n) \\ &\leq p^a(x^b - x^a) + p^b(x^c - x^b) + \cdots + p^m(x^s - x^m) + p^s(x^t - x^s) \\ &= U^s + p^s(x^t - x^s), \end{aligned}$$

since the value on the left-hand side of the inequality is a minimum over all paths to t. Hence,

$$U^t \leq U^s + p^s(x^t - x^s),$$

for all $s, t \in \mathbb{T}$, which are the inequalities in statement (iii).

(iii) $\Longrightarrow$ *(iv).–* Suppose that condition (iii) holds. For all $x \in X$, define the function:

$$u(x) = \min_{s \in \mathbb{T}} \{U^s + p^s(x - x^s)\}$$

Since u is defined as the lower envelope of a set of linear functions, it is continuous, strictly increasing and concave. Moreover, it is easy to show that $u(x^t) = U^t$ for all $t \in \mathbb{T}$. Finally, for all $x \in X$ and all $t \in \mathbb{T}$:

$$\begin{aligned} u(x) - p^t x &= \min_{s \in \mathbb{T}} \{U^s + p^s(x - x^s)\} - p^t x \\ &\leq U^t + p^t(x - x^t) - p^t x \\ &= U^t - p^t x^t \\ &= u(x^t) - p^t x^t. \end{aligned}$$

Thus, u rationalizes the data set O^T.

(iv) $\Longrightarrow$ *(i).–* Trivial.

Proof of Theorem B1

(i) $\implies$ (ii).– If the data set O^T can be rationalized by a skew-symmetric and quasilinear preference function, then for all $t \in \mathbb{T}$ and all $y \in X$,

$$r(x^t, y) \geq p^t(x^t - y).$$

In particular, it must be that for $y = x^s$, $r(x^t, x^s) \geq p^t(x^t - x^s)$. Analogously, we have $r(x^s, x^t) \geq p^s(x^s - x^t)$ for all $s, t \in \mathbb{T}$. Adding these inequalities, and by skew-symmetry, we have:

$$0 = r(x^t, x^s) + r(x^s, x^t) \geq p^t(x^t - x^s) + p^s(x^s - x^t).$$

Rearranging terms, we get:

$$(p^t - p^s)(x^t - x^s) \leq 0,$$

for all $s, t \in \mathbb{T}$, which is the law of demand.

(ii) $\implies$ (iii).– Assume that condition (ii) holds and define:

$$R^{s,t} = \frac{1}{2}(p^s(x^s - x^t) - p^t(x^t - x^s)).$$

Clearly, $R^{s,t} = -R^{t,s}$ for all $s, t \in \mathbb{T}$. Moreover,

$$\begin{aligned} R^{s,t} &= \frac{1}{2}(p^s(x^s - x^t) - p^t(x^t - x^s)) \\ &= \frac{1}{2}(p^s(x^s - x^t) + p^t(x^s - x^t)) \\ &= \frac{1}{2}(p^s(x^s - x^t) - p^t(x^s - x^t) + 2p^t(x^s - x^t)). \end{aligned}$$

By condition (ii), we have

$$p^s(x^s - x^t) - p^t(x^s - x^t) = (p^s - p^t)(x^s - x^t) \leq 0.$$

Hence,

$$\begin{aligned} R^{s,t} &= \frac{1}{2}(p^s(x^s - x^t) - p^t(x^s - x^t) + 2p^t(x^s - x^t) \\ &= \frac{1}{2}(p^s(x^s - x^t) - p^t(x^s - x^t)) + p^t(x^s - x^t) \iff \\ R^{s,t} - p^t(x^s - x^t) &= \frac{1}{2}(p^s(x^s - x^t) - p^t(x^s - x^t)) \end{aligned}$$

$$\leq 0,$$

implying:

$$\begin{aligned} -R^{s,t} &\geq -p^t\left(x^s - x^t\right) \Longleftrightarrow \\ R^{t,s} &\geq p^t\left(x^t - x^s\right), \end{aligned}$$

which are the inequalities in statement (iii).

(iii) $\Longrightarrow$ *(i).–* Suppose that condition (iii) holds and define for all $x, y \in X$ the functions:

$$r_{st}(x,y) = R^{s,t} + p^s(x - x^s) - p^t\left(y - x^t\right).$$

Clearly, the function r_{st} is continuous, strictly increasing, concave in x, and convex in y. Since $R^{s,t} = -R^{t,s}$, we have:

$$\begin{aligned} -r_{st}(x,y) &= -\left(R^{s,t} + p^s(x - x^s) - p^t\left(y - x^t\right)\right) \\ &= R^{t,s} + p^t\left(y - x^t\right) - p^s(x - x^s) \\ &= r_{ts}(y,x). \end{aligned}$$

Let the $T-1$ dimensional simplex be denoted $\Delta = \{\lambda \in \mathbb{R}^T_+ | \sum_{t=1}^T \lambda_t = 1\}$. Define the preference function $r(x,y)$ for any $x, y \in X$ as:

$$\begin{aligned} r(x,y) &= \min_{\lambda \in \Delta} \max_{\mu \in \Delta} \sum_{s \in \mathbb{T}} \sum_{t \in \mathbb{T}} \lambda_s \mu_t r_{st}(x,y) \\ &= \max_{\mu \in \Delta} \min_{\lambda \in \Delta} \sum_{s \in \mathbb{T}} \sum_{t \in \mathbb{T}} \lambda_s \mu_t r_{st}(x,y). \end{aligned}$$

We show that the function r is skew-symmetric, continuous, strictly increasing, and concave. First, we show skew-symmetry:

$$\begin{aligned} -r(x,y) &= -\min_{\lambda \in \Delta} \max_{\mu \in \Delta} \sum_{s \in \mathbb{T}} \sum_{t \in \mathbb{T}} \lambda_s \mu_t r_{st}(x,y) \\ &= \max_{\lambda \in \Delta} \min_{\mu \in \Delta} \sum_{s \in \mathbb{T}} \sum_{t \in \mathbb{T}} -\lambda_s \mu_t r_{st}(x,y) \\ &= \max_{\lambda \in \Delta} \min_{\mu \in \Delta} \sum_{s \in \mathbb{T}} \sum_{t \in \mathbb{T}} -\lambda_s \mu_t r_{ts}(y,x) \\ &= r(y,x), \end{aligned}$$

since $-r_{st}(x,y) = r_{ts}(y,x)$.

Second, we show that r is continuous. The simplex Δ consists of a finite number of elements and is therefore compact. Moreover, from above, we know that r_{st} is continuous. Hence, for any $\lambda, \mu \in \Delta$, the function

$$f(x, y; \lambda, \mu) = \sum_{s\in\mathbb{T}} \sum_{t\in\mathbb{T}} \lambda_s \mu_t r_{st}(x, y),$$

is continuous. By a direct application of Berge's maximum theorem it follows that $r(x, y) = \min_{\lambda\in\Delta} \max_{\mu\in\Delta} f(x, y; \lambda, \mu)$ is a continuous function of $x, y \in X$.

Third, we show that r is strictly increasing. Consider $x, y, z \in X$ such that $x > y$. Since each function r_{st} is strictly increasing we have:

$$\max_{\mu\in\Delta} \sum_{s\in\mathbb{T}} \sum_{t\in\mathbb{T}} \lambda_t \mu_s r_{st}(x, z) > \max_{\mu\in\Delta} \sum_{s\in\mathbb{T}} \sum_{t\in\mathbb{T}} \lambda_t \mu_s r_{st}(y, z),$$

for all Δ. Hence,

$$\min_{\lambda\in\Delta} \max_{\mu\in\Delta} \sum_{s\in\mathbb{T}} \sum_{t\in\mathbb{T}} \lambda_t \mu_s r_{st}(x, z) > \min_{\lambda\in\Delta} \max_{\mu\in\Delta} \sum_{s\in\mathbb{T}} \sum_{t\in\mathbb{T}} \lambda_t \mu_s r_{st}(y, z),$$

which shows that r is strictly increasing in the first argument x.

Fourth, we show that $r(x, y)$ is concave in x. Fix y and $\lambda \in \Delta$, and consider the function:

$$r_\lambda(x) = \max_{\mu\in\Delta} \sum_{s\in\mathbb{T}} \sum_{t\in\mathbb{T}} \lambda_s \mu_t r_{st}(x, y).$$

We have:

$$\begin{aligned}
r_\lambda(x) &= \max_{\mu\in\Delta} \sum_{s\in\mathbb{T}} \sum_{t\in\mathbb{T}} \lambda_s \mu_t r_{st}(x, y) \\
&= \max_{\mu\in\Delta} \sum_{s\in\mathbb{T}} \sum_{t\in\mathbb{T}} \lambda_s \mu_t \left(R^{s,t} + p^s (x - x^s) - p^t \left(y - x^t \right) \right) \\
&= \max_{\mu\in\Delta} \sum_{s\in\mathbb{T}} \lambda_s \left(\sum_{t\in\mathbb{T}} \left(\mu_t R^{s,t} + \mu_t p^s (x - x^s) - \mu_t p^t \left(y - x^t \right) \right) \right) \\
&= \max_{\mu\in\Delta} \sum_{s\in\mathbb{T}} \lambda_s \left(p^s (x - x^s) + \sum_{t\in\mathbb{T}} \left(\mu_t R^{s,t} - \mu_t p^t \left(y - x^t \right) \right) \right) \\
&= \sum_{s\in\mathbb{T}} \lambda_s p^s (x - x^s) + \max_{\mu\in\Delta} \sum_{s\in\mathbb{T}} \sum_{t\in\mathbb{T}} \left(\mu_t R^{s,t} - \mu_t p^t \left(y - x^t \right) \right).
\end{aligned}$$

Clearly, $r_\lambda(x)$ is linear in x and, as such, concave. Hence, $r(x, y) = \min_{\lambda\in\Delta} r_\lambda(x)$ is the minimum over a set of linear functions and is therefore also concave.

Finally, we show that r is a quasilinear preference function that rationalizes the data.

For all $y \in X$ and all $t \in \mathbb{T}$:

$$\begin{aligned} r\left(x^t, y\right) &= \min_{\lambda \in \Delta} \max_{\mu \in \Delta} \sum_{s \in \mathbb{T}} \sum_{t \in \mathbb{T}} \lambda_s \mu_t r_{st}\left(x^t, y\right) \\ &\geq \min_{\lambda \in \Delta} \sum_{s \in \mathbb{T}} \sum_{v \in \mathbb{T}} \lambda_s \mu_v^t r_{sv}\left(x^t, y\right) \\ &= \min_{\lambda \in \Delta} \sum_{t \in \mathbb{T}} \lambda_s r_{st}\left(x^t, y\right), \end{aligned}$$

where $\mu_v^t = 1$ when $v = t$ and zero otherwise. Note that the term $p^t(y - x^t)$ does not depend on s, which implies:

$$\begin{aligned} r(x^t, y) &\geq \min_{\lambda \in \Delta} \sum_{s \in \mathbb{T}} \lambda_s r_{st}\left(x^t, y\right) \\ &= \min_{\lambda \in \Delta} \sum_{s \in \mathbb{T}} \lambda_s \left(R^{s,t} + p^s\left(x^t - x^s\right) - p^t\left(y - x^t\right)\right) \\ &= -\sum_{s \in \mathbb{T}} \lambda_s p^t\left(y - x^t\right) + \min_{\lambda \in \Delta} \sum_{s \in \mathbb{T}} \lambda_s \left(R^{s,t} + p^s\left(x^t - x^s\right)\right) \\ &= -p^t\left(y - x^t\right) + \min_{\lambda \in \Delta} \sum_{s \in \mathbb{T}} \lambda_s \left(R^{s,t} + p^s\left(x^t - x^s\right)\right). \end{aligned}$$

Thus, r is a quasilinear preference function that rationalizes the data O^T since:

$$\begin{aligned} r\left(x^t, y\right) - p^t\left(x^t - y\right) &= \min_{\lambda \in \Delta} \sum_{s \in \mathbb{T}} \lambda_s \left(R^{s,t} + p^s\left(x^t - x^s\right)\right) \\ &\geq 0, \end{aligned}$$

because $R^{s,t} + p^s\left(x^t - x^s\right) \geq 0$ by condition (iii) and $\lambda_s \geq 0$ for all $s, t \in \mathbb{T}$.

(iii) $\Longrightarrow$ *(iv).–* Consider the T^2 data sets O_{ts}^2 for every pair of observations $s, t \in \mathbb{T}$. For any O_{st}^2, condition (ii) implies that the law of demand holds, which, in turn, implies that cyclical monotonicity holds (since the law of demand and cyclical monotonicity are equivalent for O_{st}^2). By directly applying Theorem D and Theorem 2.2 in Brown and Calsamiglia (2007), we can construct piecemeal utility functions u_{st} that rationalize O_{st}^2 and are continuous, strictly increasing, and quasilinear.

Using the piecemeal utilities u_{st}, we construct a quasilinear CMU preference function, r, as:

$$r(x, y) = \max_{U \in \Omega_{\mathbb{T}}} \min_{u \in U}(u(x) - u(y)).$$

By the same arguments as in Theorem 1 we have that r rationalizes the data set O^T.

(iv) $\Longrightarrow$ *(iii).–* Consider the number:

$$r(x^t, x^s) = \max_{U \in \Omega_{\mathbb{T}}} \min_{u \in U} (u(x^t) - u(x^s)),$$

where the utility functions u are continuous, strictly increasing, concave, and quasilinear. The property of quasilinearity implies that we can without loss of generality impose the restriction that the marginal utility of income is equal to unity (Brown and Calsamiglia 2007). Then, the piecemeal utility function, $\overline{u}$, that solves the CMU maximization problem is also quasilinear, where:

$$\begin{aligned} r(x^t, x^s) &= \max_{U \in \Omega_{\mathbb{T}}} \min_{u \in U} (u(x^t) - u(x^s)) \\ &= \overline{u}(x^t) - \overline{u}(x^s). \end{aligned}$$

By concavity of $\overline{u}$, we have for any $s, t \in \mathbb{T}$,

$$\overline{u}(x^t) - \overline{u}(x^s) \geq p^t(x^t - x^s),$$

We get the inequalities in condition (iii) by defining the numbers $R^{ts} = \overline{u}(x^t) - \overline{u}(x^s)$.

References

Afriat, S. N. (1967). The construction of utility functions from expenditure data. *International Economic Review*, 8(1):67–77.

Allen, R. and Rehbeck, J. (2018). Assessing Misspecification and Aggregation for Structured Preferences. *Unpublished working paper.*

Blundell, R., Browning, M., Cherchye, L., Crawford, I., De Rock, B., and Vermuelen, F. (2015). Sharp for SARP: Nonparametric bounds on counterfactual demands. *American Economic Journal: Microeconomics*, 7(1):43–60.

Blundell, R., Browning, M., and Crawford, I. (2003). Nonparametric engel curves and revealed preference. *Econometrica*, 71(1):1227–1262.

Blundell, R., Browning, M., and Crawford, I. (2008). Best Nonparametric Bounds on Demand Responses. *Econometrica*, 76(6):1227–1262.

Brown, D. and Calsamiglia, C. (2007). The nonparametric approach to applied welfare analysis. *Economic Theory*, 31(1):183–188.

Dziewulski, P. (2021). A comprehensive revealed preference approach to approximate utility maximisation and non-transitive indifferences. Technical report.

Halevy, Y., Persitz, D., and Zrill, L. (2017). Non-parametric bounds for non-convex preferences. *Journal of Economic Behavior and Organization*, 137:105–112.

Houthakker, H. (1950). Revealed preference and the utility function. *Economica*, 17(66):159–174.

John, R. (2001). The concave nontransitive consumer. *Journal of Global Optimization*, 20(3-4):297–308.

Kihlstrom, R., Mas-Colell, A., and Sonnenschein, H. (1976). The demand theory of the weak axiom of revealed preference. *Econometrica*, 44(5):971–978.

Kim, T. and Richter, M. K. (1986). Nontransitive-nontotal consumer theory. *Journal of Economic Theory*, 38(2):324–363.

Krauss, E. (1985). A representation of arbitrary maximal monotone operators via subgradients of skew-symmetric saddle functions. *Nonlinear Analysis: Theory, Methods & Applications*, 9(12):1381–1399.

Mariotti, M. (2008). What kind of preference maximization does the weak axiom of revealed preference characterize? *Economic Theory*, 35(2):403–406.

Matzkin, R. L. and Richter, M. K. (1991). Testing strictly concave rationality. *Journal of Economic Theory*, 53(2):287–303.

Quah, J. K.-H. (2006). Weak axiomatic demand theory. *Economic Theory*, 29(3):677–699.

Samuelson, P. A. (1938). A note on the pure theory of consumer's behaviour. *Economica*, 5(17):61–71.

Shafer, W. J. (1974). The nontransitive consumer. *Econometrica*, 42(5):913–919.

Talla Nobibon, F., Cherchye, L., Crama, Y., Demuynck, T., De Rock, B., and Spieksma, F. (2016). Revealed preference tests of collectively rational consumption behavior: Formulations and algorithms. *Operations Research*, 64(6):1197–1216.

Varian, H. R. (1982). The nonparametric approach to demand analysis. *Econometrica*, 50(4):945–973.