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Slutsky Matrix Symmetry: A New Behavioral Condition*

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Abstract The Slutsky matrix function encodes all the information about local variations in demand with respect to small (Slutsky) compensated price changes. When the demand function is the result of utility maximization the Slutsky matrix is symmetric. However, symmetry does not imply rationality. Here, we provide a necessary and sufficient condition for Slutsky symmetry. The new condition requires symmetric attention to compensated price-paths.

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1. Introduction

If a continuously differentiable demand function is the result of utility maximization, its Slutsky matrix is symmetric.[1] Comprising all derivatives of demand, the Slutsky matrix captures the changes in demanded quantities due to compensated changes in prices. The symmetry of the Slutsky matrix is also important in that it is a key ingredient of the solution to the "Integrability" problem (Hurwicz and Uzawa, 1971), i.e., to be able to derive the preferences underlying a given demand. Due to the importance of symmetry as a testable implication of rationality in applied work, an econometric test has been devised and applied to household consumption data (Haag, Hoderlein and Pendakur, 2009).

Yet, as Samuelson (1947) emphasizes, this property is somewhat mysterious and goes beyond what one would derive without the help of mathematics. In their authoritative monograph, Mas-Colell, Whinston and Green (1995) echo this opinion and highlight that the symmetry property is difficult to interpret in economic terms. The objective of this paper is to provide a behavioral restriction that is necessary and sufficient for Slutsky symmetry under mild regularity conditions.

While other existing foundations of the Slutsky matrix increase our understanding of the property, they do not hinge on a transparent revealed-preference condition. (In contrast, we showcase here such a condition, by focusing on the role of consumer attention to price changes.) For instance, the Ville Axiom of Revealed Preference (VARP, Hurwicz and Richter (1979)), a differential form of the Strong Axiom of Revealed preference, provides an impressive characterization of Slutsky symmetry, and it can be viewed as a technical restriction. Indeed, Jerison and Jerison (1993) define a Ville cycle as a situation where the consumption is the same at the beginning and after a rising real income situation, and VARP amounts to the

[1]The Slutsky matrix is also negative semidefinite and the price vector is in its eigenspace–i.e., it is associated with the null eigenvalue, from now on p-singularity. For a full study of rationality and its relation with the Slutsky matrix, see Aguiar and Serrano (2017).

nonexistence of Ville cycles. If the demand of a consumer exhibits a Ville cycle, the consumer is inattentive to wealth increases (a similar interpretation of Slutsky symmetry is given in Aguiar and Serrano (2017)). Jerison and Jerison (1996) provides a closely related condition using anti-symmetric income growth cycles, a discrete version of VARP. Finally, in a finite data setting, Aguiar and Serrano (2018) relate the asymmetry of the Slutsky matrix to the presence of revealed-preference cycles involving chains of at least three bundles.

Our new behavioral condition is *Symmetric Attention to Compensated Price-Paths* (SACPP). A compensation price-path describes a price change from a baseline situation to a target situation. The paths are called compensated because we impose the restriction that the consumer is always able to just afford the consumption bundle at baseline along the path (Slutsky compensation). This means that the consumer is kept "indifferent" within each path. But furthermore, SACPP requires that the consumer does not care about the particular compensation price-path leading to a given target. Hence, the wealth compensation required to move along one path or another should be always the same, introducing a more transparent revealed preference dimension on the space of compensated price paths.[2]

2. Preliminaries.

Consider a demand function $x : Z \mapsto X$, where $Z \equiv P \times W$ is a compact space of price-wealth pairs (p, w); $P \subset \mathbb{R}^L_{++}$; $W \subset \mathbb{R}_{++}$; and $X \equiv \mathbb{R}^L_+$ is the consumption set. This demand system is a generic function that maps price and wealth to consumption bundles.

Assume that x is continuously differentiable, and satisfies Walras' law ($p'x(p, w) = w$ for all $(p, w) \in Z$) and homogeneity of degree 0 (i.e., $x(\alpha p, \alpha w) = x(p, w)$ for all $\alpha > 0$). Let

[2]This paper takes as primitive the demand function which in principle requires infinite data, but as Aguiar and Serrano (2018) have shown it is always possible to interpolate a finite dataset of prices and demand quantities to recover a demand function in which one can test conditions such as VARP and the SACPP.

$\mathcal{X}(Z) \subset \mathcal{C}^1(Z)$ denote a set of functions that satisfy these characteristics, with $\mathcal{C}^1(Z)$ denoting the complete metric space of vector-valued functions $f : Z \mapsto \mathbb{R}^L$, that are continuously differentiable, uniformly bounded with compact domain $Z \subset \mathbb{R}^{L+1}_{++}$, equipped with a norm.

Next, we define the Slutsky matrix function:

Definition 1. For an arbitrary price-wealth pair (p, w) in Z, the Slutsky matrix function $S \in \mathcal{M}(Z)$ –the space of matrix-valued functions defined on Z– is defined pointwise:

$$S(p, w) = D_p x(p, w) + D_w x(p, w) x(p, w)' \in \mathbb{R}^{L \times L},$$

with entry

$$s_{l,k}(p, w) = \frac{\partial x_l(p, w)}{\partial p_k} + \frac{\partial x_l(p, w)}{\partial w} x_k(p, w).$$

The Slutsky matrix function is well-defined for all $x \in \mathcal{C}^1(Z)$. Restricted to the set of rational behaviors, the Slutsky matrix satisfies a number of regularity conditions, chiefly (for our purposes here) symmetry, i.e., $S(p, w) = S(p, w)'$.

3. The SACPP Condition and Symmetry of the Slutsky Matrix

Our aim in this section is to provide a behavioral condition that is equivalent to the symmetry of the Slutsky matrix. we call it *Symmetric Attention to Compensated Price-Paths* (SACPP). This condition is inspired by some informal discussion in the work of Russell (1997).

Fix a wealth level $w = \overline{w}$. Define the conditional demand function $x^{\overline{w}}(p) = x(p, \overline{w})$ and the associated conditional Slutsky matrix $S^{\overline{w}}(p) = D_p x^{\overline{w}}(p) - \frac{1}{\overline{w}} D_p x^{\overline{w}}(p) p x^{\overline{w}}(p)'$.[3] Following John (1995), the conditional Slutsky matrix can be defined as the Jacobian of the

[3]By homogeneity of degree 0, it follows that $-\frac{1}{\overline{w}} D_p x(p, \overline{w}) p = D_w x(p, \overline{w})$. In which case, the conditional Slutsky matrix is numerically equivalent to the Slutsky matrix for each wealth level.

price-level compensated conditional demand $f_q^{\overline{w}}(p) = x^{\overline{w}}((\overline{w}/p'q)p)$ at $q = x^{\overline{w}}(p)$. Then, $D_p f_q^{\overline{w}}(p)|_{q=x^{\overline{w}}(p)} = S^{\overline{w}}(p)$.

Define the correspondence $B_{\overline{w}}(p) = \{q \in P | q' x^{\overline{w}}(p) = \overline{w}\}$. The correspondence $B_{\overline{w}}(p)$ maps prices, for a fixed wealth, to the set of all price compensations that keep the original bundle just affordable (Slutsky compensation). The correspondence $B_{\overline{w}}(p)$ is lower hemicontinuous, convex-valued, and compact-valued, thanks to the compactness of Z. Hence, by Michael (1956), there are continuous selections of $B_{\overline{w}}(p)$. We further assume that there exist continuously differentiable selections.[4] Define $\mathcal{Q}$ as the set of all continuously differentiable selections of $B_{\overline{w}}(p)$, and note that any element of $\mathcal{Q}$ is a function of prices for a fixed wealth level. We denote $q, r \in \mathcal{Q}$ as typical elements of this set.

A directed or directional Jacobian with respect to prices of a vector-valued function q in the direction of the vector $v \in \mathbb{R}^L$ is: $\lim_{t \to 0} \frac{q(p+tv)-q(p)}{t} = D_p q(p) v \in \mathbb{R}^L$ (i.e., the marginal change of q with respect to prices in the direction of v, with $D_p q(p)$ being the Jacobian of q). Define $q^* = q(p) - p$ for any $q \in \mathcal{Q}$.

Definition 2. (Directed price change compensation). For all $(p, \overline{w})$ and for selections $q, r \in \mathcal{Q}$, Let $m_{q,r}$ be the excess wealth amount that the consumer must be paid to accept a change from p in the direction of r^* within selection q, instead of a change from p in the direction of q^* within selection r:

$$\begin{aligned} m_{q,r}(\overline{w}, p) &= \lim_{t \to 0} [\frac{q(p+tr^*) - q(p)}{t} - \frac{r(p+tq^*) - r(p)}{t}]' x^{\overline{w}}(p) \\ &= [D_p q(p)(r(p) - p)]' x^{\overline{w}}(p) - [D_p r(p)(q(p) - p)]' x^{\overline{w}}(p). \end{aligned}$$

With this definition, we are ready to state the following behavioral restriction:

Definition 3. (Symmetric Attention to Compensated Price-Paths, SACPP) A demand

[4] This is mainly a technical assumption. For primitive conditions, see Dentcheva (1998).

$x \in \mathcal{X}(Z)$ is said to satisfy SACPP whenever for all $p, \overline{w}$ and for all $q, r \in \mathcal{Q}$: $m_{q,r}(p, \overline{w}) = 0$.

Note that, since the paths are constructed using Slutsky compensations, the consumer is always kept "indifferent" between any two discrete points in the path. Furthermore, comparing across paths, a consumer who obeys SACPP is said to be indifferent to the "direction" of price change. Intuitively, a rational consumer satisfies SACPP because she is a consequentialist: all paths or directions of compensation should be indifferent to her. We explore several examples next.

Proposition 1. A conditional demand $x^{\overline{w}}$ satisfies SACPP if and only if its conditional Slutsky matrix $S^{\overline{w}}$ is symmetric.

We can also state a immediate corollary of our main result.

Corollary 1. *For a given $\overline{w}$, and $q, r \in \mathcal{Q}$,*

$$m_{q,r}(p, \overline{w}) = q(p)'[S^{\overline{w}}(p)' - S^{\overline{w}}(p)]r(p).$$

The corollary above states that $m_{q,r}$ depends on the price paths and on the degree of asymmetry of the conditional Slutsky matrix. Hence, the results here also provide a foundation for the degree of asymmetry of the conditional Slutsky matrix. This line of research was pursued earlier in Aguiar and Serrano (2017).

The corollary also expands the intuition behind SACPP. A rational consumer does not mind the particular path as she perceives the price changes perfectly as is able to see that they give the same purchasing power. However, some boundedly rational consumers may satisfy SACPP without being fully rational that is, SACPP does not imply rationality.

4. Examples

This section illustrates the SACPP with several examples.

Example 1. (Compensation Price-Paths for a Cobb-Douglas) Without loss of generality, let $L = 2$ and let demand be a Cobb-Douglas $x(p, w) = (\frac{\alpha w}{p_1}, \frac{(1-\alpha)w}{p_2})'$. It has a symmetric Slutsky matrix, and has correspondence $B_{\overline{w}}(p) = \{q \in P : q_1 = \frac{p_1(p_2 - q_2 + \alpha q_2)}{\alpha p_2}\}$. We let $P = \{p \in \mathbb{R}^2_{++} : \frac{1}{K_l} \leq p_1, p_2 \leq K_h\}$, for fixed positive constants $K_h > \frac{1}{K_l}$. Fix $w = 1$. We illustrate SACPP with two simple compensation price-paths $q, r \in \mathcal{Q}$. Take compensation price-paths $q(p) = (\frac{p_1(\alpha - 1 + K_l p_2)}{K_l p_2 \alpha}, \frac{1}{K_l})'$ and $r(p) = (\frac{1}{K_l}, \frac{p_2(K_l p_1 - \alpha)}{K_l p_1 (1-\alpha)})'$. It is easy to verify that $r' x(p, 1) = q' x(p, 1) = 1$ and that q, r are continuously differentiable. Direct computation verifies that $m_{q,r} = 0$.

Example 2. (Compensation Price-Paths for an Price-Inattentive Cobb-Douglas) We can change the previous example to add price inattention. Let demand be a price-inattentive Cobb-Douglas as defined in Gabaix (2014)

$$x(p, w) = \left(\frac{\alpha}{p_1^G} \frac{w}{\alpha(p_1/p_1^G) + (1-\alpha)(p_2/p_2^G)}, \frac{1-\alpha}{p_2^G} \frac{w}{\alpha(p_1/p_1^G) + (1-\alpha)(p_2/p_2^G)} \right)'$$

with $p_l^G = m_l p_l + (1 - m_l) p_l^d$ for $l \in \{1, 2\}$, and $m_l \in [0, 1]$. Fix Z to be some compact set of prices and $w = \overline{w}$. Let $E^{\overline{w}}(p) = S^{\overline{w}}(p) - S^{\overline{w}}(p)'$ and evaluate $p = p^d$ the default price, then

$$E^{\overline{w}}(p) = \frac{\overline{w}(\alpha - 1)\alpha(1 - m_2 + \alpha(m_1 + m_2 - 2))}{p_1 p_2} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

By Corollary 1 we conclude that

$$m_{q,r} = q(p)' E^{\overline{w}}(p) r(p) \neq 0,$$

generically when $m_1, m_2 \neq 1$. Recall that the case of perfect and symmetric attention to prices is $m_1 = m_2 = 1$. The expression above is further simplified if $\alpha = 1/2$, then

$$E^{\overline{w}}(p) = \frac{\overline{w}(m_1 - m_2)}{8p_1p_2}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

in this case SACPP holds also for the case of symmetric inattention $m_1 = m_2$. Notice that in this example HD0 fails, yet all our results in Corollary 1 hold for the compensated demand and compensated Slutsky matrix.

Example 3. (A demand that satisfies SACPP but fails WARP, Aguiar and Serrano (2021)) Let demand be $x(p, w) = (\frac{p_1 w}{p_1^2+p_2^2}, \frac{p_2 w}{p_1^2+p_2^2})'$. This demand has a symmetric Slutsky matrix, yet the Slutsky matrix is not negative semidefinite (i.e., it can be shown that this demand violates WARP).

The next example features a demand that fails SACPP, but whose conditional Slutsky matrix–and Slutsky matrix–is NSD. This means that NSD and SACPP are independent properties.

Example 4. (A demand that fails SACPP) Consider

$$x(p, w) = \left(\frac{w(p_1 + (1-c)p_2}{2p_1(p_1 + p_2)}, \frac{w(p_1 + (1-c)p_2)}{2p_2(p_1 + p_2)}, \frac{wcp_2}{p3(p1 + p2)}\right)'$$

for a constant $c \geq 0$. When $c = 0$, this demand is rational and has a Slutsky matrix that is symmetric, NSD, and singular in prices. When $0 < c < 1$ only the symmetry of the Slutsky matrix fails. Finally, when $c \geq 1$ the symmetry and the NSD of the Slutsky matrix fail. The correspondence

$$B_{\overline{w}}(p) = \{q \in P : q_1 = \frac{p_1(2p_1p_2p_3 + 2p_2^2p_3 - p_1p_3q_2 - p_2p_3q_2 + cp_2p_3q_2 - 2cp_2^2q_3)}{p_2(cp_2 - p_1 - p_2)p_3}\}.$$

Let P be a compact set that guarantees the existence of continuously differentiable paths and

$w = 1$. We take two arbitrary paths

$$q(p) = \left(\frac{p_1((p_1 + p_2)(2Kp_2 - 1)p_3 + cp_2(p_3 - 2p_2))}{Kp_2(p_1 + p_2 - cp_2)p_3}, \frac{1}{K}, \frac{1}{K} \right)',$$

and

$$r(p) = \left(\frac{1}{K}, \frac{p_2((2Kp_1 - 1)(p_1 + p_2)p_3 + cp_2(p_3 - 2p_1))}{Kp_1(p_1 + p_2 - cp_2)p_3}, \frac{1}{K} \right)'.$$

Direct computation gives us $m_{q,r}(c)$ as a function of c:

$$m_{q,r}(c) = \frac{2c(Kp_3 - 1)(-2cp_1p_2^2 + (p_1 + p_2)(-p_1 + (-1 + c + 2Kp_1)p_2)p_3)}{K^2(p_1 + p_2)(p_1 + p_2 - cp_2)p_3^2}.$$

For prices $p = (1, 2, 1)'$, and $K = 1$, we have $m_{q,r}(0) = 0$, $m_{q,r}(1/2) = \frac{7}{24}$, and $m_{q,r}(2) = \frac{11}{3}$. Importantly, $m_{q,r} \neq 0$ when $c = 1/2$, which maintains NSD of the Slutsky matrix, but breaks symmetry.

5. Conclusion

A new behavioral restriction is shown to be equivalent to the symmetry of the Slutsky matrix. The SACPP condition only requires continuous differentiability and knowledge of the function $x(p, w)$ for all $(p, w) \in Z$, making it as general as the VARP, but it is easier to interpret in economic terms.

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Appendix

Proof of Proposition 1

Proof. We prove that if the conditional demand $x^{\overline{w}}$ satisfies SACPP then its conditional Slutsky matrix $S^{\overline{w}}$ is symmetric. Fix $w = \overline{w}$. For any $q \in \mathcal{Q}$ it follows that $q(p)'x^{\overline{w}}(p) = \overline{w}$.

Then, by differentiating the former equality with respect to p:

$x^{\overline{w}}(p)'D_p q(p) + q(p)'D_p x^{\overline{w}}(p) = 0.$

Post multiplying by $r \in \mathcal{Q}$:

$x^{\overline{w}}(p)' D_p q(p) r(p) + q(p)' D_p x^{\overline{w}}(p) r(p) = 0$, or

$x^{\overline{w}}(p)' D_p q(p) r(p) = -q(p)' D_p x^{\overline{w}}(p) r(p)$.

Post multiplying by $-p$: $-x^{\overline{w}}(p)' D_p q(p) p = q(p)' D_p x^{\overline{w}}(p) p$.

Similarly, with any $r \in \mathcal{Q}$ it follows that $r(p)' x^{\overline{w}}(p) = \overline{w}$.

Differentiating with respect to p:

$x^{\overline{w}}(p)' D_p r(p) + r(p)' D_p x^{\overline{w}}(p) = 0$.

Post multiplying by $q \in \mathcal{Q}$:

$x^{\overline{w}}(p)' D_p r(p) q + r(p)' D_p x^{\overline{w}}(p) q(p) = 0$.

Transposing:

$q(p)' D_p r(p)' x^{\overline{w}} = -q' D_p x^{\overline{w}}(p)' r(p)$

Then, the quantity $m_{q,r}$ can be written pointwise as:

$m_{q,r}(\overline{w}, p) = [-q(p)' D_p x^{\overline{w}}(p) r(p) + q(p)' D_p x^{\overline{w}}(p) p + q(p)' D_p x^{\overline{w}}(p)' r(p) - p' D_p x^{\overline{w}}(p)' r(p)]$ for all $p \in P$.

Then, dividing and multiplying the second term and the fourth term by $\overline{w}$ and using the fact that $x^{\overline{w}}(p)' r = x^{\overline{w}}(p)' q = \overline{w}$:

$$\begin{aligned} m_{q,r}(\overline{w}, p) = [&-q(p)' D_p x^{\overline{w}}(p) r(p) + \frac{1}{\overline{w}} q(p)' D_p x^{\overline{w}}(p) p x^{\overline{w}}(p)' r(p) \\ &+ q(p)' D_p x^{\overline{w}}(p)' r(p) - q(p)' x^{\overline{w}}(p) p' D_p x^{\overline{w}}(p)' r(p) \frac{1}{\overline{w}}]. \end{aligned}$$

Rearranging terms, it follows that:

$m_{q,r}(\overline{w}, p) = q(p)'[S^{\overline{w}}(p)' - S^{\overline{w}}(p)] r(p) = 0$ for all $p \in P$ and $\overline{w} \in W$.

In particular, for $q \neq p$, $r \neq p$ and $q \neq r$, this implies that:

$S^{\overline{w}}(p) = S^{\overline{w}}(p)'$ (symmetry).

If the conditional Slutsky matrix is symmetric then SACPP holds trivially. ■

Connection with VARP

The Ville Axiom of Revealed Preference (VARP) due to Hurwicz and Richter (1979)) is roughly equivalent to the symmetry of the Slutsky matrix. First, we need some preliminaries.

Definition 1. (Real income path). A real income path consists of both a wealth path $w : [0, b] \mapsto W$, and a price path $p : [0, b] \mapsto P$, having that $(w(\tau), p(\tau))$ is a piecewise continuously differentiable path in $\mathcal{Z}$.

Next, we define a Ville Cycle:

Definition 2. (Ville cycle) A Ville Cycle $C^{V(S),b}$ is a pair of functions $(p(\tau), x(\tau))$ for $\tau \in [0, b]$ for some $b > 0$ where x is a S continuously differentiable commodity path $x : [0, b] \to \mathbb{R}^L_+$ such that $x(0) = x(b)$ and $x \in \mathcal{C}^S([0, b]; \mathbb{R}^L_+)$ for $S \geq 1$ and $p(\tau)\frac{\partial x(\tau)}{\partial \tau} > 0$ almost everywhere in $\tau \in [0, b]$, for any piecewise continuous price path $p : [0, b] \to \mathbb{R}^L_{++}$.

The Ville Axiom of revealed preference (VARP) requires that a demand function $x \in \mathcal{X}(Z)$ does not have a Ville cycle $C^{V(S),b}$ for all $S \geq 1$ and $b > 0$.

Observe that if there is a rising real income situation $\frac{\partial w}{\partial \tau}(\tau) > \frac{\partial p}{\partial \tau}(\tau)' x(p(\tau), w(\tau))$, and the price-wealth path forms a cycle $(p(0), w(0)) = (p(b), w(b))$, then the situation is equivalent (almost everywhere) to $p(\tau)\frac{\partial x(\tau)}{\partial \tau} > 0$ for all $\tau \in [0, b]$ since $\frac{\partial w}{\partial \tau}(\tau) = p(\tau)'\frac{\partial x(\tau)}{\partial \tau} + \frac{\partial p}{\partial \tau}(\tau)' x(p(\tau), w(\tau))$ (Jerison and Jerison (1993)). A Ville cycle means that the demand is the same at the beginning and at the et of a path but the consumer faces a rising real income situation.

The VARP is hard to interpret in economic terms. The main reason is that a differentiable commodity path does not contain any revealed preference information. This is in contrast with a discrete commodity path. In fact, a discrete cycle would say that $p^{t\prime}(x^t - x^{t-1}) > 0$ implies that x^t is revealed preferred strictly to x^{t-1}. However, $p(\tau)' \frac{\partial x(\tau)}{\partial \tau} > 0$ for any $\tau \in [0, b]$ does not contain any such preference revelation.