2021

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Research Report # 2021-3 **July 2021**

Western

Department of Economics

Research Report Series

Department of Economics
Social Science Centre
Western University
London, Ontario, N6A 5C2
Canada

SPECIFYING A GAME-THEORETIC EXTENSIVE FORM AS AN ABSTRACT 5-ARY RELATION

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Abstract. This paper specifies an extensive form as a 5-ary relation (i.e. set of quintuples) which satisfies certain abstract axioms. Each quintuple is understood to list a player, a situation (e.g. information set), a decision node, an action, and a successor node. Accordingly, the axioms are understood to specify abstract relationships between players, situations, nodes, and actions. Such an extensive form is called a "5-form", and a "5-form game" is defined to be a 5-form together with utility functions. The paper's main result is to construct a bijection between (a) those 5-form games with information-set situations and (b) **Gm** games (Streufert 2021). In this sense, 5-form games equivalently formulate almost all extensive-form games. An application weakens the tree axiom in the presence of the other axioms, which leads to a convenient decomposition of 5-forms.

1. Introduction

1.1. Definitions and main result

Suppose Q is a 5-ary relation. In other words,[1] suppose Q is a set of quintuples. Then the projection of Q onto any two of its coordinates is a set of pairs, and the projection of Q onto any three of its coordinates is a set of triples.

A quintuple in Q can be interpreted as listing a player, a situation,[2] a decision node, an action, and a successor node. With this interpretation, the above projections acquire game-theoretic meaning. For example, consider Q_{21}, which is the projection of Q onto its first two coordinates, with their order reversed. When interpreted, Q_{21} is a set of situation-player pairs which might be the graph of a function from situations to players. Similarly, Q_{32} might be the graph of a function from decision nodes to situations, and Q_{31} might be the graph of a function from decision

Date: July 22, 2021. *Keywords:* Situation, extensive-form game. *Classifications:* MSC 91A70, JEL C73. *Contact information:* pstreuf@uwo.ca, 519-661-2111x85384, Economics Department, University of Western Ontario, London, Ontario, N6A 5C2, Canada.

[1]This use of the term "5-ary relation" accords with Halmos 1974, Section 7. Alternatively, a set of quintuples could be regarded as the *graph* of a 5-ary relation (in which case a "5-ary relation" would be a sextuple listing five domains and a graph). This alternative would resemble the notion of correspondence in note 3 below.

[2]A "situation" can be either an information set (which is a set of nodes) or something else (such as the word "tomorrow"). In either case, a situation is a decision point. To clarify, consider an extensive-form game defined in a familiar way. Since there is a bijection between the game's decision points and the game's information sets, there is no loss of generality in identifying decision points with information sets. Yet this paper distinguishes between the two in order to view games more abstractly. In particular, a decision point will be called a "situation" (denoted j), and the bijection between situations and information sets will follow as a logical consequence of an axiom (called [Q2]). Incidentally, the conceptual distinction between decision points and information sets also appears elsewhere (e.g. Myerson 1991, page 40).

nodes to players. In a different vein, Q_{35}, when interpreted, is a set of decision-node-successor-node pairs which might be the edge set of an oriented tree, and Q_{354} is a set of triples which might be the graph of a labeling function taking a tree's edges to actions. Similarly, Q_{53} might be the graph of an immediate-predecessor function, Q_{34} might be the graph of a feasibility correspondence, and Q_{345} might be the graph of a next-node function taking decision-node-feasible-action pairs to successor nodes.

A "5-form" is defined to be a quintuple set Q which satisfies four abstract axioms. Although these axioms will be stated again in Section 2.1, they can be precisely stated here without interpretation or further notation. Axiom [Q1] states that Q_{21} is the graph of a function, axiom [Q2] states that Q_{32} is the graph of a function, and axiom [Q4] states that Q_{35} is the edge set of a nontrivial out-tree. Finally, axiom [Q3] states that each situation's slice of Q, projected onto coordinates 345, is the graph of a bijection from a two-dimensional Cartesian product.

A "5-form game" is constructed by combining a 5-form with a utility-function profile. The main result is Theorem 3.1, which shows that there is a bijection between (a) those 5-form games that have information-set situations[2] and (b) **Gm** games. **Gm** is a category of extensive-form games introduced by Streufert 2021 (henceforth `S21`). As is customary, a **Gm** game is defined as a tree adorned with actions, information sets, players, and utility functions. Further, `S21` Section 1.2 demonstrates that **Gm** games include almost all extensive-form games. Thus Theorem 3.1 shows that 5-form games can equivalently formulate almost all extensive-form games.

1.2. Motivation and an application

By construction, a 5-form embodies the abstract relationships between players, situations, nodes, and actions. This is the main motivation for this paper. To explore this a bit further, note that the various slices and projections of a 5-form lead simply and systematically to the various sets, partitions, functions, and correspondences used in game theory. This is illustrated not only by the projections above, but also by Theorem 3.2 below, which connects a number of 5-form derivatives to their **Gm** counterparts. In addition, note that customary game specifications begin with a tree, and then define subsequent components in terms of that tree. In contrast, the tree axiom here is treated like any other axiom.

As an application, Proposition 2.3 shows that the tree axiom [Q4] can be weakened in the presence of axioms [Q1]–[Q3]. Since the other axioms are more "local", this weakening of the tree axiom makes it easier to break apart an extensive form, which in turn promises to make it easier to calculate the equilibria of an extensive-form game. Relatedly, Corollary 2.4 characterizes 5-forms in a way that emphasizes their ability to be decomposed. Conveniently, decomposition can be expressed by partition because a 5-form is a set.

Finally, the reader might find that a 5-form is an easy way to formalize an extensive form, even for beginners. A quintuple set is just one thing rather than the customary list of things. Further, each quintuple is a tree edge (that is, a decision-node-successor-node pair) adorned with a player, situation, and action. Thus each quintuple transparently corresponds to an edge in a tree diagram. Consequently, a 5-form easily formalizes toy examples like the ones in the figures below, and seamlessly scales up to infinite games.

1.3. Literature

The rough idea of expressing an extensive form as an abstract relation appears in Streufert 2018. In particular, the set $\otimes$ of triples $\langle t, c, t^\sharp \rangle$ defined in its Section 3.1 is very close to set Q_{345} of triples $\langle w, a, y \rangle$ defined here (the earlier paper identified functions with their graphs). The preforms of Streufert 2018 supported the forms of Streufert 2020a, which in turn supported the games of Streufert 2020b. In retrospect, the Streufert 2020b specification was an uneasy compromise between the "standard" **Gm** specification of `S21` (Streufert 2021) and the abstract specification of the present paper. (Incidentally, the other papers in this paragraph also define categories, and a category for 5-form games is under development.)

The application to decomposition is related to the composable open games of Capucci, Ghani, Ledent, and Nordvall Forsberg 2021, Bolt, Hedges, and Zahn 2019, and Ghani, Kupke, Lambert, and Nordvall Forsberg 2018, 2020. The mathematics there is very different from the mathematics here, and more is said there about utility. Nonetheless it is clear that the application here addresses infinite as well as finite games, that it expresses decomposition and composition via the relatively straightforward operations of partition and union, and that it contributes the result of weakening the tree axiom.

1.4. Organization

Section 2 defines 5-forms and 5-form games. Its Proposition 2.3 weakens the tree axiom. Then Section 3 compares 5-form games to **Gm** games. Theorem 3.1 relates the two with a suitable bijection (the paper's main result), and Theorem 3.2 relates their components and derivatives. Finally, Appendix A supports Section 2, and Appendix B supports Section 3.

2. 5-Form Games

2.1. 5-Forms

Let Q be a set of quintuples, and let $\langle i, j, w, a, y \rangle$ denote an arbitrary member of Q. For any $m \in \{1,2,3,4,5\}$, let Q_m denote the projection of Q onto its m-th coordinate. Then define $I = Q_1$, $J = Q_2$, $W = Q_3$, $A = Q_4$, and $Y = Q_5$. Call I the set of *players*, call J the set of *situations*, call W the set of *decision nodes*, call A the set of *actions*, and call Y the set of *successor nodes*. Further, let $X = W \cup Y$, and call X the set of *nodes*.

Consider a situation $j \in J$. Let $Q_j = \{\, \langle i_*, j_*, w_*, a_*, y_* \rangle \in Q \,|\, j_* = j \,\}$. Thus Q_j is the slice of Q corresponding to the situation j. Then define $W_j = Q_{j,3}$, $A_j = Q_{j,4}$, and $Y_j = Q_{j,5}$. In other words, let W_j be the slice Q_j projected onto its third coordinate, let A_j be the slice Q_j projected onto its fourth coordinate, and let Y_j be the slice Q_j projected onto its fifth coordinate. Call W_j the set of *decision nodes for situation* j, or equivalently, the *information set for situation* j. Call A_j the set of *actions for situation* j, and call Y_j the set of *successor nodes for situation* j.

Further, consider a finite sequence in $\{1,2,3,4,5\}$. Let the appearance of such a sequence as a subscript denote projection onto the corresponding coordinates. An example is $Q_{21} = \{\, \langle j, i \rangle \,|\, (\exists w \in W, a \in A, y \in Y)\, \langle i, j, w, a, y \rangle \in Q \,\}$, which is Q projected onto the coordinate sequence 21. Notice that the coordinates in this projection are ordered as in the sequence 21. Another example is $Q_{j,345} = \{\, \langle w_*, a_*, y_* \rangle \,|\, (\exists i_* \in I)\, \langle i_*, j, w_*, a_*, y_* \rangle \in Q \,\}$, which is the slice Q_j projected onto the coordinate sequence 345.

A *5-form* (or *5-ary extensive form*) is a set Q of quintuples $\langle i, j, w, a, y\rangle$ such that

[Q1] Q_{21} is the graph of a function,[3]

[Q2] Q_{32} is the graph of a function,

[Q3] $(\forall j{\in}J)$ $Q_{j,345}$ is the graph of a bijection from $W_j{\times}A_j$ onto Y_j, and

[Q4] Q_{35} is the edge set of a nontrivial out-tree.[4]

Figures 2.1–2.3 provide some finite examples.[5] In general, the tree in [Q4] can have up to countably infinite height, and up to uncountably infinite degree.

i	j	w	a	y
P3	{3}	3	e	5
P3	{3}	3	f	6

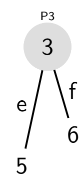

Figure 2.1. The 5-form Q^1, defined to be the set consisting of the table's non-header rows. The tree diagram provides the same data. In Q^1, the only situation ({3}) is equal to the only information set.

[3]To be clear, an arbitrary function $f{:}X{\to}Y$ is understood to be a triple (X, Y, f^{gr}) such that $f^{\mathsf{gr}} \subseteq X{\times}Y$ and $(\forall x{\in}X)(\exists! y{\in}Y)$ $\langle x, y\rangle \in f^{\mathsf{gr}}$. Relatedly, for future use, an arbitrary correspondence $F{:}X{\rightrightarrows}Y$ is understood to be a triple (X, Y, F^{gr}) such that $F^{\mathsf{gr}} \subseteq X{\times}Y$. The three components of a function or correspondence are called its domain, codomain, and graph.

[4]To be clear, a nontrivial out-tree is defined as follows. As in Diestel 2010, Chapter 1, an *unoriented graph* is a pair $(X, \mathcal{E})$ such that X is a set and $\mathcal{E}$ is a collection of two-element subsets of X. The elements of X are called *nodes*, and the elements of $\mathcal{E}$ are called *edges*. A *path linking* x_0 *and* x_ℓ is an unoriented graph $(\bar{X}, \bar{\mathcal{E}})$ of the form $\bar{X} = \{x_0, x_1, x_2, \dots x_\ell\}$ and $\bar{\mathcal{E}} = \{\{x_0, x_1\}, \{x_1, x_2\}, \dots \{x_{\ell-1}, x_\ell\}\}$ in which distinct i and j satisfy $x_i \neq x_j$. Further, one graph $(X^o, \mathcal{E}^o)$ is said to be *in* another graph $(X, \mathcal{E})$ iff $X^o \subseteq X$ and $\mathcal{E}^o \subseteq \mathcal{E}$. An *unoriented tree* is an unoriented graph $(X, \mathcal{E})$ in which every two elements of X are linked by exactly one path in $(X, \mathcal{E})$.

Further, as in Bang-Jensen and Gutin 2009, Chapter 1, an *oriented graph* is a pair (X, E) such that X is a set and E is a collection of ordered pairs from X such that $(\forall x{\in}X, y{\in}X)$ $\langle x, y\rangle \in E \Rightarrow \langle y, x\rangle \notin E$ [this implies $(\forall x{\in}X)$ $\langle x, x\rangle \notin E$]. Denote the edges of an oriented graph by xy rather than $\langle x, y\rangle$. It is easily seen that each oriented graph (X, E) determines an unoriented graph $(X, \mathcal{E})$ by means of $E \ni xy \mapsto \{x, y\} \in \mathcal{E}$. An *oriented tree* is an oriented graph whose unoriented graph is an unoriented tree. Further, an *out-tree* is an oriented tree (X, E) such that $X{\setminus}E_2$ is a singleton, where E_2 is the projection of E onto its second coordinate. Finally, an out-tree (X, E) is *nontrivial* iff $E \neq \varnothing$.

[5]To tell a story matching the forms of Figures 2.2 and 2.3, suppose a student (called player P1) must decide, today, between the bad action of not doing her homework (called b) and the correct action of doing her homework (called c). Next, tonight, if the homework has been finished (node 1), a dog (player P2) must decide between the dumb action of eating the homework (d) and the good action of going back to sleep (g). Finally, tomorrow, without knowing whether the student chose bad (node 3) or the student chose correct and the dog chose dumb (node 4), the teacher (player P3) must decide between excusing the student (e) and failing the student (f).

i	j	w	a	y
P1	{0}	0	c	1
P1	{0}	0	b	3
P2	{1}	1	g	2
P2	{1}	1	d	4
P3	{3,4}	3	e	5
P3	{3,4}	3	f	6
P3	{3,4}	4	e	7
P3	{3,4}	4	f	8

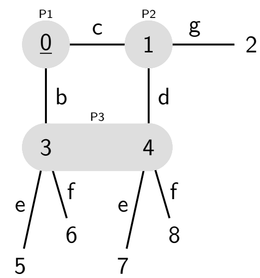

Figure 2.2. The 5-form Q^2, defined to be the set consisting of the table's non-header rows. The tree diagram provides the same data. In Q^2, each situation j is equal to its information set W_j^2.

i	j	w	a	y
P1	today	0	c	1
P1	today	0	b	3
P2	tonight	1	g	2
P2	tonight	1	d	4
P3	tomorrow	3	e	5
P3	tomorrow	3	f	6
P3	tomorrow	4	e	7
P3	tomorrow	4	f	8

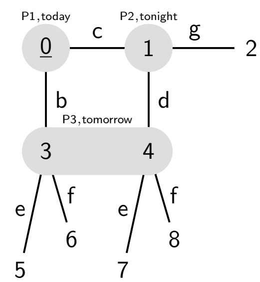

Figure 2.3. The 5-form Q^3, defined to be the set consisting of the table's non-header rows. The tree diagram provides the same data. In Q^3, situations are not identified with their information sets.

The remainder of this subsection will interpret [Q1]-[Q4], and derive a few entities for use later in the paper. To begin, axiom [Q1] states that each situation j is associated with exactly one player i.[6] This is interpreted to mean that exactly one player moves in each situation.

Axiom [Q2] states that each decision node w is associated with exactly one situation. This is equivalent to stating that W_{j_1} and W_{j_2} are disjoint for distinct j_1 and j_2. Alternatively, since both $\cup_{j\in J} W_j = W$ and $(\forall j{\in}J)\ W_j \neq \varnothing$ hold by construction, [Q2] is equivalent to $\langle W_j \rangle_{j\in J}$ being a bijectively indexed partition of W. Note that it is possible to identify situations j with their information sets W_j, as in Figures 2.1 and 2.2 (and much of the game theory literature). It is also possible to do otherwise, as in Figure 2.3 (and elsewhere such as Myerson 1991, page 40). Thus situations and information sets are conceptually distinct, as mentioned earlier in note 2.

To gather intuition for [Q3], say that an action $a \in A$ is *feasible* at a decision node $w \in W$ iff $\langle w, a\rangle \in Q_{34}$. Proposition 2.1 concludes, for each situation $j \in J$ and each decision node $w \in W_j$, that the set A_j equals the set of actions feasible at w. In this sense, each A_j is the set of actions that are feasible in situation j. Now consider [Q3] for a particular j. By Proposition 2.1, the set $W_j \times A_j$ consists

[6] In accord with note 3, axiom [Q1] is equivalent to $(\forall j{\in}J)(\exists! i{\in}I)\ \langle j, i\rangle \in Q_{21}$. Note that $(\forall j{\in}J)(\exists i{\in}I)\ \langle j, i\rangle \in Q_{21}$ holds by construction. Thus [Q1] is equivalent to stating that, for each $j \in J$, there is no more than one $i \in I$ such that $\langle j, i\rangle \in Q_{21}$. A similar observation can be made for [Q2], for the inverse function in [Q3], for [Q3b], and for [Q4b].

of pairs listing a decision node in j and a feasible action in j. Thus [Q3] states that each such decision-node-feasible-action pair uniquely determines a successor node in j, and that each successor node in j uniquely determines such a decision-node-feasible-action pair.

Proposition 2.1. *Suppose Q is a quintuple set which satisfies [Q2] and [Q3]. Take $j \in J$ and $w \in W_j$. Then $(\forall a{\in}A)$ $a \in A_j$ iff a is feasible at w. (Proof A.1.)*

To better understand [Q4], first recall (note 4) that the node set of an oriented tree is identical to the set of nodes that appear in an edge of the tree. Hence if Q_{35} is the edge set of an oriented tree, the tree's node set is $Q_3{\cup}Q_5$, which by definition is equal to X. Thus [Q4] is equivalent to (X, Q_{35}) being a nontrivial out-tree. Second recall that an out-tree is nontrivial iff its edge set is nonempty. Thus the nontriviality of (X, Q_{35}) is equivalent to the nonemptiness of Q. Third recall that an out-tree has exactly one node that does not appear as the second element of an edge. Thus [Q4] implies $Q_3{\smallsetminus}Q_5$ is a singleton, which by construction implies $W{\smallsetminus}Y$ is a singleton. Let r denote the sole element of $W{\smallsetminus}Y$, and call r the *root node*.

For future use, recall[7] that the paths in an out-tree can be identified with their node sets. Call $Y{\smallsetminus}W$ the set of *end nodes*, and let $\mathcal{Z}_{\mathsf{ft}}$ be the collection of (the node sets of) paths from r to some end node. Further, let $\mathcal{Z}_{\mathsf{inft}}$ be the collection of infinite paths from r. Finally, let $\mathcal{Z} = \mathcal{Z}_{\mathsf{ft}}{\cup}\mathcal{Z}_{\mathsf{inft}}$, and call $\mathcal{Z}$ the set of *runs* (often elsewhere "plays"). It is possible (i) that $\mathcal{Z} = \mathcal{Z}_{\mathsf{ft}}$ and $\mathcal{Z}_{\mathsf{inft}} = \varnothing$, as in Figures 2.1–2.3, or (ii) that $\mathcal{Z} = \mathcal{Z}_{\mathsf{inft}}$ and $\mathcal{Z}_{\mathsf{ft}} = \varnothing$, in which case $Y{\smallsetminus}W = \varnothing$, or (iii) that both $\mathcal{Z}_{\mathsf{ft}}$ and $\mathcal{Z}_{\mathsf{inft}}$ are nonempty.

2.2. An axiomatic overlap

Although axioms [Q1]–[Q4] are logically independent, [Q3] and [Q4] overlap in the presence of [Q2].

Proposition 2.2. *Suppose Q is a quintuple set which satisfies [Q2] and [Q4]. Then Q satisfies [Q3] iff*

[Q3a] $(\forall j{\in}J)$ $Q_{j,345}$ is the graph of a function from $W_j{\times}A_j$ to Y_j, and

[Q3b] Q_{54} is the graph of a function. (Proof A.3.)

Proposition 2.2 assumes [Q2] and [Q4] and then characterizes [Q3] by a set of conditions weaker than [Q3] itself. The proposition's assumption of the tree axiom [Q4] accords with customary extensive-form definitions in the sense that they begin with the definition of a tree.

Proposition 2.3. *Suppose Q is a quintuple set which satisfies [Q2] and [Q3]. Then Q satisfies [Q4] iff*

[Q4a] Q is nonempty,

[Q4b] Q_{52} is the graph of a function, and

[Q4c] $(\exists r{\in}W{\smallsetminus}Y)(\forall x{\in}X{\smallsetminus}\{r\})$ there is a path[7] in (X, Q_{35}) from r to x. (Proof A.4.)

---

[7] To be clear, as in Bang-Jensen and Gutin 2009, Chapter 1, a *path from x_0 to x_ℓ* is an oriented graph $(\bar{X}, \bar{E})$ of the form $\bar{X} = \{x_0, x_1, x_2, \ldots x_\ell\}$ and $\bar{E} = \{x_0x_1, x_1x_2, \ldots x_{\ell-1}x_\ell\}$ in which distinct i and j satisfy $x_i \neq x_j$. Similarly, an *infinite path from x_0* is an oriented graph $(\hat{X}, \hat{\mathcal{E}})$ of the form $\hat{X} = \{x_0, x_1, x_2, \ldots\}$ and $\hat{E} = \{x_0x_1, x_1x_2, \ldots\}$ in which distinct i and j satisfy $x_i \neq x_j$. If a path is in an out-tree, the path's indices and edge set are redundant. In particular, if $(\dot{X}, \dot{E})$ is a path in an out-tree (X, E), then $\dot{E} = \{\, xy{\in}E \mid \{x, y\}{\subseteq}\dot{X} \,\}$.

Proposition 2.3 assumes [Q2] and [Q3] and then characterizes [Q4] by a set of conditions weaker than [Q4] itself. This weakens the tree axiom. To explore this, recall [Q4] implies that the root node is connected to each non-root node by a unique path. Yet [Q4c] does not require its path to be unique. Essentially, Proposition 2.3 derives uniqueness from [Q3] and [Q4b]. Thus a significant portion of [Q4] is already present in [Q3].[8]

Since Proposition 2.3 weakens an axiom ([Q4]) which spans across situations, it increases the sense in which a 5-form can be decomposed by situation. To make this concrete, say that a *situation block* is a quintuple set $\hat{Q}$ that satisfies [Q1], [Q3], and $|\hat{Q}_2| = 1$. Then Proposition 2.3 leads to following characterization of a 5-form.

Corollary 2.4. *Suppose Q is a quintuple set. Then Q is a 5-form iff it is the union of a nonempty collection of situation blocks which have distinct situations, decision nodes, and successor nodes, and whose union satisfies [Q4c]. (Proof A.5.)*

Recall that many large calculations can be simplified by breaking them apart. Thus the above tools for decomposing an extensive form promise to help simplify the calculation of equilibria in extensive-form games. The same principle motivates the literature's study of composable open games, as discussed in Section 1.3.

2.3. 5-Form Games

Suppose Q is a 5-form with its run collection $\mathcal{Z}$. A *utility function* for player i is a function of the form $U_i{:}\mathcal{Z}{\to}\mathbb{R}$. A *utility-function profile* is a $U = \langle U_i \rangle_{i \in I}$ which lists a utility function U_i for each player $i \in I$. A *5-form game* is a pair $G = (Q, U)$ consisting of a 5-form Q and a utility-function profile U. Figures 2.4 and 2.5[9] provide two finite examples.

i	j	w	a	y
P3	{3}	3	e	5
P3	{3}	3	f	6

Z	$U^1_{\mathsf{P3}}(Z)$
{3, 5}	2
{3, 6}	4

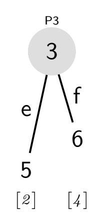

Figure 2.4. The 5-form game (Q^1, U^1). Q^1 is the set consisting of the upper table's rows, and $U^1_{\mathsf{P3}}{:}\mathcal{Z}^1{\to}\mathbb{R}$ is defined by the lower table. The tree diagram provides the same data.

[8]To put this another way, recall that a tree is characterized by connectedness and acyclicity. Connectedness is close to [Q4c]. Acyclicity is close to the combination of [Q3] and [Q4b], so in this sense, a large portion of acyclicity is already present in [Q3].

[9]Note 5 told a story about the Q^2 that appears in both Figure 2.2 and Figure 2.5. This note interprets Figure 2.5's U^2 within the context of note 5's story. In particular, the student most prefers being excused without doing the homework (run {0,3,5}), and least prefers failing after doing the homework (run {0,1,4,8}). The dog likes eating homework (runs {0,1,4,7} and {0,1,4,8}). Finally, the teacher does not want to excuse a badly behaving student (run {0,3,5}) or to fail a correctly behaving student (run {0,1,4,8}).

i	j	w	a	y
P1	{0}	0	c	1
P1	{0}	0	b	3
P2	{1}	1	g	2
P2	{1}	1	d	4
P3	{3,4}	3	e	5
P3	{3,4}	3	f	6
P3	{3,4}	4	e	7
P3	{3,4}	4	f	8

Z	$U^2_{\mathsf{P1}}(Z)$	$U^2_{\mathsf{P2}}(Z)$	$U^2_{\mathsf{P3}}(Z)$
{0, 3, 5}	1	0	2
{0, 3, 6}	0	0	4
{0, 1, 4, 7}	0	1	4
{0, 1, 4, 8}	-1	1	2
{0, 1, 2}	0	0	4

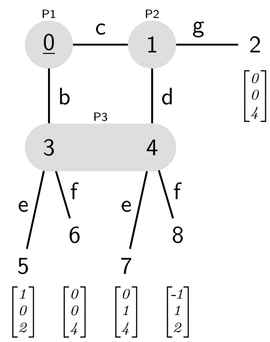

Figure 2.5. The 5-form game (Q^2, U^2). Q^2 is the set consisting of the upper table's rows, and $U^2 = \langle U^2_i{:}\mathcal{Z}^2{\rightarrow}\mathbb{R}\rangle_{i\in I^2}$ is defined by the lower table. The tree diagram provides the same data.

3. Equivalence with **Gm** Games

3.1. Review of **Gm** Games

The **Gm** games in `S21` are "standard" in the sense that they are defined as trees endowed with actions, information sets, players, and utility functions.[10] Further, **Gm** games are general in the sense that they include almost all extensive-form games. Notably included are games that specify nodes arbitrarily without restriction (e.g. Selten 1975, Myerson 1991, and Shoham and Leyton-Brown 2009); games that specify nodes as sequences of past actions (e.g. Osborne and Rubinstein 1994); and games that specify nodes as sets of future outcomes (e.g. von Neumann and Morgenstern 1944, Section 10, and Alós-Ferrer and Ritzberger 2016, Section 6.2). For further discussion, please see `S21` Section 1.2.

The remainder of this section recapitulates the definition of a **Gm** game and a few of its derivatives. First, let (X, E) be a nontrivial out-tree with node set X and edge set E (the tree can have up to countably infinite height, and up to uncountably infinite degree). Define W to be the projection of E onto its first coordinate (the members of W are called decision nodes). Further, define r to be the tree's root node, and define $\mathcal{Z}$ to be the tree's collection of runs.

Next, let $\mathcal{H}$ be a partition of W, and let λ be a surjective function from E. $\mathcal{H}$ generates the topology for W (the members of $\mathcal{H}$ are called information sets). Assume λ is *deterministic* in the sense that for any two edges of the form xy_1 and xy_2, the equality $\lambda(xy_1) = \lambda(xy_2)$ implies $y_1 = y_2$ (λ is called the labeling function). Define the set A to be the codomain of λ (the members of A are called actions). Also define $F{:}W{\rightrightarrows}A$ by $(\forall x{\in}W)\ F(x) = \{\ a{\in}A \mid (\exists y{\in}X{\smallsetminus}\{r\})\, \lambda(xy){=}a\ \}$ (F is called the feasibility correspondence). Assume $\langle F(x)\rangle_{x\in W}$ is continuous in the sense that $F(x_1) = F(x_2)$ for any two x_1 and x_2 in one member of $\mathcal{H}$. As in `S21`, a *continuously labeled tree* (or "CLT") is a tuple $(X, E, \mathcal{H}, \lambda)$ such that

[10] Also, the definition of **Gm** games is "standard" in the sense that its use of topology conforms with standard applications of category theory to subject areas other than game theory.

[C1] (X, E) is a nontrivial out-tree,

[C2] $\mathcal{H}$ is a partition of W,

[C3] λ is a deterministic surjective function from E, and

[C4] $\langle F(x) \rangle_{x \in W}$ is continuous from W.

Examples of continuously labeled trees can be found within the tree diagrams of Figures 2.1 and 2.2. Specifically, each of the two diagrams, without its players, depicts a continuously labeled tree.

Next, let μ be a continuous surjective function from W (called the move-assigning function). Define I to be the codomain of μ (the members of I are called players). Finally, provide each player $i \in I$ with a function $U_i{:}\mathcal{Z}{\rightarrow}\mathbb{R}$ (called player i's utility function). As in S21, a **Gm** *game* is a tuple $\mathit{\Gamma} = (X, E, \mathcal{H}, \lambda, \mu, U)$ such that [G1] $(X, E, \mathcal{H}, \lambda)$ is a continuously labeled tree,

[G2] μ is a continuous surjective function from W, and

[G3] $U = \langle U_i{:}\mathcal{Z}{\rightarrow}\mathbb{R} \rangle_{i \in I}$.

Examples of finite **Gm** games are depicted in the tree diagrams of Figures 2.4 and 2.5.

For future use, derive from a **Gm** game $\mathit{\Gamma}$ the function $p{:}X{\smallsetminus}\{r\}{\rightarrow}W$ whose graph is $\{\langle y, x \rangle | xy{\in}E\}$) (this is called the immediate-predecessor function). Also derive the function $n{:}F^{\mathsf{gr}}{\rightarrow}X{\smallsetminus}\{r\}$ that takes each $\langle x, a \rangle \in F^{\mathsf{gr}}$ to the unique $y \in X{\smallsetminus}\{r\}$ such that $\lambda(xy) = a$ (this is called the next-node function). These two functions are well-defined by S21 Sections 2.1 and 2.2.

3.2. Main result

This section compares the new 5-form games to the standard **Gm** games. To begin, recall that a 5-form's situations may or may not be information sets. In constrast, a **Gm** game has no concept of "situation" other than an information set. In this respect, a 5-form game is more general. To be precise, say that a 5-form has *info-situations* (short for "information-set situations") iff it identifies situations and information sets. In other words, a form Q has info-situations iff $(\forall j{\in}J)\ j = W_j$. Then say that a 5-form game (Q, U) has info-situations iff Q has info-situations. For example, Figure 2.4's game has info-situations. Similarly, Figure 2.5's game has info-situations. In contrast, any game built on Figure 2.3's form will not have info-situations.

The theorems below show that 5-form games with info-situations are equivalent to **Gm** games in a strong sense. The theorems concern a pair of operators. The operator S maps an info-situation 5-form game to a **Gm** game, and the operator T goes in the reverse direction. Briefly, S "Standardizes" and T "Tuplefies".

To be precise, let S be the operator that takes an info-situation 5-form game G to a **Gm** game by the rule

$$G \mapsto (\tilde{X}, \tilde{E}, \tilde{\mathcal{H}}, \tilde{\lambda}, \tilde{\mu}, \tilde{U}),$$

where $\tilde{X} = X$, $\tilde{E} = Q_{35}$, $\tilde{\mathcal{H}} = J$, $\tilde{\lambda} = (Q_{35}, A, Q_{354})$, $\tilde{\mu} = (W, I, Q_{31})$, and $\tilde{U} = U$ (the functions $\tilde{\lambda}$ and $\tilde{\mu}$ are defined as triples in accord with note 3). Conversely, let T be the operator that derives an info-situation 5-form game from a **Gm** game $\mathit{\Gamma}$ by the rule

$$(\dot{Q}, \dot{U}) \leftarrow\!\shortmid \mathit{\Gamma},$$

where $\dot{Q} = \{ \langle \mu(p(y)), H_{p(y)}, p(y), \lambda(p(y)y), y \rangle \,|\, y{\in}X{\smallsetminus}\{r\} \}$ and $\dot{U} = U$, where for each $x \in W$, H_x is the unique member of $\mathcal{H}$ that contains x. Alternatively, Lemma B.2 implies $X{\smallsetminus}\{r\} \ni y \mapsto \langle p(y), y \rangle \in E$ is a bijection, and thus $\dot{Q} = \{ \langle \mu(x), H_x, x, \lambda(xy), y \rangle \,|\, xy{\in}E \}$.

To illustrate T, please return to Figure 2.5. Regard the tree diagram on the figure's right-hand side as a **Gm** game $\mathit{\Gamma}$. Then regard the two tables on the figure's left-hand side as the 5-form game $\mathsf{T}\mathit{\Gamma} = (\dot{Q}, \dot{U})$. Transparently, the payoff vectors in $\mathit{\Gamma}$ have been re-arranged into the lower table of $\mathsf{T}\mathit{\Gamma}$. These are just two different ways of displaying the $\dot{U} = U$ that is part of both games. Less transparently, the upper table is the $\dot{Q}$ defined above. To see this, first note that the nodes in $X{\smallsetminus}\{r\}$ appear in the upper table's fifth column. Second consider $\mathsf{7} \in X{\smallsetminus}\{r\}$, for example. Since $p(\mathsf{7}) = \mathsf{4}$, the first four elements in the quintuple (i.e. row) for $\mathsf{7}$ are $\mu(p(\mathsf{7})) = \mu(\mathsf{4}) = \mathsf{P3}$, $H_{p(\mathsf{7})} = H_{\mathsf{4}} = \{\mathsf{3}, \mathsf{4}\}$, $p(\mathsf{7}) = \mathsf{4}$, and $\lambda(p(\mathsf{7})\mathsf{7})) = \lambda(\mathsf{47}) = \mathsf{e}$.

The following theorem is the paper's main result.

Theorem 3.1. *(a) The operators* S *and* T *are well-defined. (b) If G is an info-situation 5-form game, then* $\mathsf{TS}\,G = G$. *(c) If $\mathit{\Gamma}$ is a* **Gm** *game, then* $\mathsf{ST}\mathit{\Gamma} = \mathit{\Gamma}$. *(Proof B.7.)*

The following theorem and corollary provide further connections between the two kinds of games.

Theorem 3.2. *Suppose that G is an info-situation 5-form game, and that* $\tilde{\mathit{\Gamma}} = \mathsf{S}G$. *Then the following hold. (a) $\tilde{I} = I$. (b) $\tilde{W} = W$. (c) $\tilde{A} = A$. (d) $\tilde{X} = X$. (e) $\tilde{r} = r$. (f) $\tilde{\mathcal{Z}} = \mathcal{Z}$. (g) $\tilde{U} = U$.*

(h) $\tilde{\mu} = (W, I, Q_{31})$.
(i) $\tilde{\mathcal{H}} = J$.
(j) $\tilde{F} = (W, A, Q_{34})$.
(k) $(\forall j{\in}J, w{\in}W_j)\ \tilde{F}(w) = A_j$.
(l) $(\forall H{\in}\tilde{\mathcal{H}}, x{\in}H)\ \tilde{F}(x) = A_H$.
(m) $\tilde{E} = Q_{35}$.
(n) $\tilde{X}{\smallsetminus}\{\tilde{r}\} = Y$.
(o) $\tilde{X}{\smallsetminus}\tilde{W} = Y{\smallsetminus}W$.
(p) $\tilde{p} = (Y, W, Q_{53})$.
(q) $\tilde{\lambda} = (Q_{35}, A, Q_{354})$.
(r) $\tilde{n} = (Q_{34}, Y, Q_{345})$. (Proof B.8.)

Corollary 3.3. *Suppose that $\tilde{\mathit{\Gamma}}$ is a* **Gm** *game, and that* $G = \mathsf{T}\tilde{\mathit{\Gamma}}$. *Then the conclusions of Theorem 3.2 hold. (Proof B.9.)*

APPENDIX A. FOR 5-FORM GAMES

Proof A.1 (for Proposition 2.1)**.** Take $a \in A$. First suppose $a \in A_j$. Then the definition of w implies $\langle w, a \rangle \in W_j{\times}A_j$, which by [Q3] implies there is $y \in Y_j$ such that $\langle w, a, y \rangle \in Q_{j,345}$, which implies $\langle w, a \rangle \in Q_{j,34}$, which implies $\langle w, a \rangle \in Q_{34}$, which by definition of feasibility implies a is feasible at w.

Conversely suppose a is feasible at x. Then the definition of feasibility implies $\langle w, a \rangle \in Q_{34}$, which by [Q2] and the definition of w implies $\langle w, a \rangle \in Q_{j,34}$, which implies $a \in A_j$. □

Lemma A.2. *Suppose Q is a quintuple set.*
(a) Assume [Q4]. Then Q_{53} is the graph of a function.
(b) Assume [Q2] and [Q4]. Then Q_{52} is the graph of a function.

Proof. (a). Take $y \in Y$. By construction, there is $w \in W$ such that [a] $\langle y, w\rangle \in Q_{53}$. Thus it suffices to show uniqueness. Toward that end, suppose there were also $w_* \in W$ such that [b] $\langle y, w_*\rangle \in Q_{53}$. Recall [Q4] implies there is $r \in W \smallsetminus Y$ such that

[c] $$(\forall x \in X) \text{ there is a unique path from } r \text{ to } x$$

(when $x = r$, the path is the trivial path from r to r). Fact [c] at $x = w$ implies there is a path from r to w, and thus [a] implies there is a path from r via w to y. Similarly, [c] at $x = w_*$ implies there is a path from r to w_*, and thus [b] implies there is a path from r via w_* to y. If w and w_* were distinct, the paths would be distinct, which would contradict [c] at $x = y$.

(b). Take $y \in Y$. By construction, there is $j \in J$ such that [a] $\langle y, j\rangle \in Q_{52}$. Thus it suffices to show uniqueness. Toward that end, suppose there were also $j_* \in J$ such that [b] $\langle y, j_*\rangle \in Q_{52}$. By construction, [a] implies there is $w \in W$ such that [c] $\langle j, w, y\rangle \in Q_{235}$. Similarly, [b] implies there is $w_* \in W$ such that [d] $\langle j_*, w_*, y\rangle \in Q_{235}$. By part (a), [c] and [d] imply $w = w_*$, which by [d] implies [e] $\langle j_*, w, y\rangle \in Q_{235}$. By [Q2], [c] and [e] imply $j = j_*$. □

Proof A.3 (for Proposition 2.2)**.**
Necessity of [Q3a]–[Q3b]. Assume [Q3]. [Q3a] follows easily from [Q3]. For [Q3b], take $y \in Y$. By construction, there are $j \in J$ and $a \in A$ such that [a] $\langle j, a, y\rangle \in Q_{245}$. This implies $\langle a, y\rangle \in Q_{45}$. Thus it suffices to show uniqueness. Toward that end, suppose there is also $a_* \in A$ such that $\langle a_*, y\rangle \in Q_{45}$. Then there would be $j_* \in J$ such that [b] $\langle j_*, a_*, y\rangle \in Q_{245}$. By Lemma A.2(b), [a] and [b] imply $j = j_*$, which by [b] implies $\langle j, a_*, y\rangle \in Q_{245}$. This and [a] imply that both $\langle a, y\rangle$ and $\langle a_*, y\rangle$ belong to $Q_{j,45}$. Thus [Q3]'s inverse function at j implies $a_* = a$.

Sufficiency of [Q3a]–[Q3b]. Assume [Q3a]–[Q3b]. Take $j \in J$. [Q3a] states $Q_{j,345}$ is the graph of a function from $W_j \times A_j$ to Y_j. Thus it suffices to show that this function is surjective and injective. Surjectivity holds by construction. For injectivity, it suffices that Q_{53} is the graph of a function by Lemma A.2(a), and that Q_{54} is the graph of a function by [Q3b]. □

Proof A.4 (for Proposition 2.3)**.**
Necessity of [Q4a]–[Q4c]. Assume [Q4]. For [Q4a] and [Q4c], see Section 2.1's paragraph discussing [Q4]. For [Q4b], see Lemma A.2(b).

Sufficiency of [Q4a]–[Q4c]. Assume [Q4a]–[Q4c]. By the definition of a nontrivial out-tree (note 4), it suffices to show that [A] (X, Q_{35}) is an oriented tree, [B] $W \smallsetminus Y$ is a singleton, and [C] Q_{35} is nonempty. For [B], condition [Q4c] states that $r \in W \smallsetminus Y$ and implies that any other $x \in W$ is in Y. Hence $W \smallsetminus Y$ is a singleton. For [C], condition [Q4a] implies that Q_{35} is nonempty.

For [A], define $\mathcal{E}$ by $\{\, \{w, y\} \,|\, \langle w, y\rangle \in Q_{53} \,\}$. By the definition of an oriented tree (note 4), it suffices to show that [a] $(\forall x_A \in X, x_B \in X)$ $\langle x_A, x_B\rangle \in Q_{35}$ implies $\langle x_B, x_A\rangle \notin Q_{35}$, [b] the unoriented graph $(X, \mathcal{E})$ is connected, and [c] the unoriented graph $(X, \mathcal{E})$ is acyclic. For [b], note that connectedness follows immediately from [Q4c].

For [a] and [c], define $p = (Y, W, Q_{53})$. To see that p is a function (note 3 discusses functions as triples), take $y \in Y$. The definitions of Y and W imply there is $w \in W$ such that [1] $\langle y, w \rangle \in Q_{53}$. Thus it suffices to show uniqueness. Toward that end, suppose $w_* \in W$ is such that [2] $\langle y, w_* \rangle \in Q_{53}$. Note [Q4b] implies there is a unique $j \in J$ such that $y \in Y_j$. Thus [1] and [2] imply that both $\langle w, y \rangle$ and $\langle w_*, y \rangle$ belong to $Q_{j,35}$. Thus the inverse function of [Q3] at j implies $w = w_*$. Thus p is well-defined. Further, note that [Q4c] implies there is $r \in W \setminus Y$ such that[11]

$$[3] \qquad (\forall x{\in}X{\setminus}\{r\})(\exists m{\geq}1)\ r = p^m(x).$$

For [a], suppose $x_A \in X$ and $x_B \in X$ were such that $\langle x_A, x_B \rangle \in Q_{35}$ and $\langle x_B, x_A \rangle \in Q_{35}$. The first implies $x_B \in Y$ and $x_A = p(x_B)$. The second implies $x_A \in Y$ and $x_B = p(x_A)$. Since both x_A and x_B are in Y, and since $r \in W \setminus Y$ by definition, neither x_A nor x_B is equal to r. Thus the equations $x_A = p(x_B)$ and $x_B = p(x_A)$ contradict [3] at $x = x_A$ (they also contradict [3] at $x = x_B$).

For [c], suppose there were a cycle in $(X, \mathcal{E})$. Then there would be a path of length greater than one whose tail and head constitute a pair in $\mathcal{E}$. More specifically, there would be $\ell \geq 2$ and an injective $\{x_0, x_1, ... x_\ell\}$ such that $(\forall n{\in}\{0, 1, ... \ell{-}1\})$ $\{x_n, x_{n+1}\} \in \mathcal{E}$, and such that $\{x_0, x_\ell\} \in \mathcal{E}$. Thus the definitions of $\mathcal{E}$ and p imply

$$(\forall n{\in}\{0, 1, ... \ell{-}1\})\ [\ x_n = p(x_{n+1}) \text{ or } x_{n+1} = p(x_n)\ ],$$
$$\text{and } [\ x_\ell = p(x_0) \text{ or } x_0 = p(x_\ell)\ ]\ .$$

Because p cannot take two values at one node, it must be that

$$[\ (\forall n{\in}\{0, 1, ... \ell{-}1\})\ x_n = p(x_{n+1}) \text{ and } x_\ell = p(x_0)\ ],$$
$$\text{or } [\ (\forall n{\in}\{0, 1, ... \ell{-}1\})\ x_{n+1} = p(x_n) \text{ and } x_0 = p(x_\ell)\ ].$$

In either contingency, $\{x_0, x_1, ... x_\ell\} \subseteq Y$, which by the definition of r implies that none of these nodes are equal to r. Thus either contingency contradicts [3] at $x = x_0$ (either contingency also contradicts [3] at any other node in $\{x_0, x_1, ... x_\ell\}$). □

Proof A.5 (for Corollary 2.4)**.** Suppose Q is a 5-form. Consider its slice collection $\{Q_j | j{\in}J\}$. By inspection, each Q_j is a situation block and $\cup_{j \in J} Q_j = Q$. Further, inspection shows that the situation blocks have distinct situations, and [Q2] implies that they have distinct decision nodes. Finally, the forward direction of Proposition 2.3 implies [Q4a]–[Q4c]. [Q4a] implies that the collection $\{Q_j | j{\in}J\}$ is nonempty, [Q4b] implies that the situation blocks have distinct successor nodes, and [Q4c] is employed as is.

Conversely, suppose $\{\hat{Q}_j | j{\in}\hat{J}\}$ is a nonempty collection of situation blocks which have distinct situations, decision nodes, and successor nodes, and whose union satisfies [Q4c]. It suffices to show that $\cup_{j \in \hat{J}} \hat{Q}_j$ satisfies [Q1]–[Q4]. [Q1] holds because the situation blocks have distinct situations and because each situation block satisfies [Q1]. [Q2] holds because the situation blocks have distinct decision nodes. [Q3] holds because the situation blocks have distinct situations and because each situation block satisfies [Q3]. Finally, consider [Q4]. By reverse direction of Proposition 2.3 and the assumption of [Q4c], it suffices to show that $\cup_{j \in \hat{J}} \hat{Q}_j$ satisfies [Q4a] and [Q4b]. [Q4a] holds because the collection of situation blocks is

[11] Incidentally, Knuth 1997, page 373, uses a condition like [3] to define oriented trees. Another appearance is Streufert 2018, equation (1).

nonempty. [Q4b] holds because the situation blocks have distinct successor nodes. □

Lemma A.6. *Suppose Q satisfies [Q4]. Then $Y = X \smallsetminus \{r\}$.*

Proof. [Q4] implies the well-definition of r. In steps, $X \smallsetminus \{r\}$ by the definition of X is equal to $(W \cup Y) \smallsetminus \{r\}$, which by the definition of r is equal to $(W \cup Y) \smallsetminus (W \smallsetminus Y)$, which reduces to Y. □

APPENDIX B. FOR EQUIVALENCE WITH **Gm** GAMES

Lemma B.1. *Suppose G is an info-situation 5-form game. Let $\tilde{\Gamma} = \mathsf{S}G$. Then (a) $\tilde{\Gamma}$ is a* **Gm** *game. Also, (b) $\tilde{r} = r$, (c) $\tilde{W} = W$, (d) $\tilde{\mathcal{Z}} = \mathcal{Z}$, (e) $\tilde{X} \smallsetminus \tilde{r} = Y$, (f) $\tilde{A} = A$, (g) $\tilde{I} = I$,*
(h) $\tilde{F} = (W, A, Q_{34})$,
(i) $(\forall j \in J, w \in W_j)\ \tilde{F}(w) = A_j$, and
(j) $\tilde{n} = (Q_{34}, Y, Q_{345})$.

Proof. Part (a) holds by Claim 13, parts (b)–(d) by Claim 1(b–d), part (e) by Claim 2, part (f) by Claim 6(b), part (g) by Claim 9(b), part (h) by Claim 10, part (i) by Claim 14, and part (j) by Claim 15.

Claim 1: (a) $(\tilde{X}, \tilde{E})$ is a nontrivial out-tree. (b) $\tilde{r} = r$. (c) $\tilde{W} = W$. (d) $\tilde{\mathcal{Z}} = \mathcal{Z}$. First, note $\tilde{X}$ by definition is X, which by definition is $Q_3 \cup Q_5$. Second, note $\tilde{E}$ by definition is Q_{35}. Thus [1] $(\tilde{X}, \tilde{E}) = (Q_3 \cup Q_5, Q_{35})$. Now recall (note 4) that the edges of an out-tree determine its nodes. Thus [Q4] implies that $(Q_3 \cup Q_5, Q_{35})$ is a nontrivial out-tree, which by [1] implies (a). Further, since $(\tilde{X}, \tilde{E})$ and $(Q_3 \cup Q_5, Q_{35})$ are the same out-tree, they have the same root node, the same decision-node set, and the same run collection. Hence (b)–(d) hold.

Claim 2: $\tilde{X} \smallsetminus \{\tilde{r}\} = Y$. To see this, note $\tilde{X} \smallsetminus \{\tilde{r}\}$ by the definition of $\tilde{X}$ is equal to $X \smallsetminus \{\tilde{r}\}$, which by Claim 1(b) is equal to $X \smallsetminus \{r\}$, which by Lemma A.6 is equal to Y.

Claim 3: $\tilde{\mathcal{H}}$ partitions $\tilde{W}$. [Q2] implies $\{W_j | j \in J\}$ is pairwise-disjoint. Thus, since both $\cup_{j \in J} W_j = W$ and $(\forall j \in J)\ W_j \neq \varnothing$ hold by inspection, $\{W_j | j \in J\}$ partitions W. Meanwhile, the assumption of info-situations implies $J = \{W_j | j \in J\}$. The last two sentences imply that J partitions W. Thus it suffices to note that $J = \tilde{\mathcal{H}}$ by the definition of $\tilde{\mathcal{H}}$, and that $W = \tilde{W}$ by Claim 1(c).

Claim 4: (Q_{35}, A, Q_{354}) is a well-defined function. Take $\langle w, y \rangle \in Q_{35}$. Then there are j and a such that [1] $\langle w, a, y \rangle \in Q_{j,345}$. Hence [Q3] at j implies that [2] this a is the unique element of A_j such that $\langle w, a, y \rangle \in Q_{j,345}$. Thus, since $Q_{j,345} \subseteq Q_{345}$ and since $A_j \subseteq A$, this a satisfies $a \in A$ and $\langle w, a, y \rangle \in Q_{345}$. Thus it remains to show that there is not another $a_* \in A$ such that $\langle w, a_*, y \rangle \in Q_{345}$. If there were, [2] implies there is $j_* \neq j$ such that $\langle w, a_*, y \rangle \in Q_{j_*,345}$. This implies $w \in W_{j_*}$. But [1] implies $w \in W_j$. The last two sentences and $j_* \neq j$ contradict [Q2].

Claim 5: (Q_{35}, A, Q_{354}) is surjective. Take $a \in A$. Then there are w and y such that $\langle w, a, y \rangle \in Q_{354}$, which implies a is the image of $\langle w, y \rangle \in Q_{35}$.

Claim 6: (a) $\tilde{\lambda}$ is a surjective function from $\tilde{E}$. (b) $\tilde{A} = A$. By definition, $\tilde{E} = Q_{35}$ and $\tilde{\lambda} = (Q_{35}, A, Q_{354})$. Thus Claims 4 and 5 imply (a). For (b), $\tilde{A}$ by definition is the codomain of $\tilde{\lambda}$, which by inspection is A.

Claim 7: (W, I, Q_{31}) is a well-defined function. Take $w \in W$. Then there are i and j such that [a] $\langle i, j, w\rangle \in Q_{123}$. Hence $i \in I$ and $\langle w, i\rangle \in Q_{31}$. Thus it suffices to show that there is not another $i_* \in I$ such that $\langle w, i_*\rangle \in Q_{31}$. Suppose there were. Then there would be j_* such that [b] $\langle i_*, j_*, w\rangle \in Q_{123}$. [a], [b], and [Q2] imply $j = j_*$, which by [a], [b], and [Q1] imply $i = i_*$, which contradicts the distinctness of i and i_*.

Claim 8: (W, I, Q_{31}) is surjective. Take $i \in I$. Then there is w such that $\langle i, w\rangle \in Q_{13}$, which implies i is the image of $w \in W$.

Claim 9: (a) $\tilde{\mu}$ is a surjective function from $\tilde{W}$. (b) $\tilde{I} = I$. By definition, $\tilde{\mu} = (W, I, Q_{31})$. Thus Claims 7 and 8 imply $\tilde{\mu}$ is a surjective function from W. Thus Claim 1(c) implies (a). For (b), $\tilde{I}$ by definition is the codomain of $\tilde{\mu}$, which by inspection is I.

Claim 10: $\tilde{F} = (W, A, Q_{34})$. By inspection, (W, A, Q_{34}) is a well-defined correspondence (recall the definition of a correspondence in note 3). By definition, the domain of $\tilde{F}$ is $\tilde{W}$, which by Claim 1(c) is W. By definition, the codomain of $\tilde{F}$ is $\tilde{A}$, which by Claim 6(b) is A. Thus it suffices to show

$$\begin{aligned}
\tilde{F}^{\mathsf{gr}} &= \{\, \langle w,a\rangle \in \tilde{W}\times\tilde{A} \mid (\exists y \in \tilde{X} \smallsetminus \{\tilde{r}\})\, \tilde{\lambda}(wy) = a \,\} \\
&= \{\, \langle w,a\rangle \in W\times A \mid (\exists y \in Y)\, \tilde{\lambda}(wy) = a \,\} \\
&= \{\, \langle w,a\rangle \in W\times A \mid (\exists y \in Y)\, \langle w,a,y\rangle \in Q_{345} \,\} \\
&= Q_{34}.
\end{aligned}$$

The first equality holds by the definition of $\tilde{F}$, the second by Claims 1(c), 6(b), and 2, the third by the definition of $\tilde{\lambda}$, and the fourth by inspection.

Claim 11: $(\tilde{X}, \tilde{E}, \tilde{\mathcal{H}}, \tilde{\lambda})$ is a continuously labeled tree (CLT). It suffices to show [C1]–[C4]. [C1] follows from Claim 1(a). [C2] follows from Claim 3.

For [C3], Claim 6(a) shows $\tilde{\lambda}$ is a surjective function from $\tilde{E}$. Thus it suffices to show that $\tilde{\lambda}$ is deterministic. Toward that end, take $xy_1 \in \tilde{E}$ and $xy_2 \in \tilde{E}$ such that $\tilde{\lambda}(xy_1) = \tilde{\lambda}(xy_2)$. Let a be this common value of $\tilde{\lambda}$. Then the definition of $\tilde{\lambda}$ implies [a] $\langle x, a, y_1\rangle \in Q_{345}$ and [b] $\langle x, a, y_2\rangle \in Q_{345}$. Further, [Q2] implies that there is exactly one $j \in J$ such that $x \in W_j$. Thus [a] and [b] imply $\langle x, a, y_1\rangle \in Q_{j,345}$ and $\langle x, a, y_2\rangle \in Q_{j,345}$. Thus [Q3] implies $y_1 = y_2$.

For [C4], note that the first of the following statements is equivalent to [C4]. Further, the last of the following statements holds because Proposition 2.1 implies the equivalence of [a] a is feasible at x_1, [b] a is feasible at x_2, and [c] $a \in A_j$. Thus it suffices to show the following equivalences.[12]

$$\begin{aligned}
&(\forall H \in \tilde{\mathcal{H}}, x_1 \in H, x_2 \in H)\ \tilde{F}(x_1) = \tilde{F}(x_2) \\
\text{iff } &(\forall j \in J, x_1 \in j, x_2 \in j)\ \tilde{F}(x_1) = \tilde{F}(x_2) \\
\text{iff } &(\forall j \in J, x_1 \in W_j, x_2 \in W_j)\ \tilde{F}(x_1) = \tilde{F}(x_2) \\
\text{iff } &(\forall j \in J, w_1 \in W_j, w_2 \in W_j, a \in A)\ a \in \tilde{F}(x_1) \Leftrightarrow a \in \tilde{F}(x_2) \\
\text{iff } &(\forall j \in J, w_1 \in W_j, w_2 \in W_j, a \in A)\ \langle x_1, a\rangle \in Q_{34} \Leftrightarrow \langle x_2, a\rangle \in Q_{34} \\
\text{iff } &(\forall j \in J, w_1 \in W_j, w_2 \in W_j, a \in A)\ a \text{ is feasible at } x_1 \Leftrightarrow a \text{ is feasible at } x_2.
\end{aligned}$$

[12] This argument is a fortiori in the sense that it would suffice to show that the last statement implies the first.

The first equivalence holds by the definition of $\tilde{\mathcal{H}}$, the second holds because G has info-situations by assumption, the third because the codomain of $\tilde{F}$ is A by Claim 10, the fourth by Claim 10 again, and the fifth by the definition of feasibility.

Claim 12: $(\forall j{\in}J, w_A{\in}W_j, w_B{\in}W_j, i{\in}I)\ \langle i,w_A\rangle{\in}Q_{13} \Rightarrow \langle i,w_B\rangle{\in}Q_{13}$. To show this, take $j \in J$, $w_A \in W_j$, $w_B \in W_j$, and $i \in I$, and assume $\langle i, w_A\rangle \in Q_{13}$. The assumption on i implies there is $j_A \in J$ such that [a] $\langle i, j_A, w_A\rangle \in Q_{123}$. Meanwhile, the definition of w_A implies $\langle j, w_A\rangle \in Q_{23}$. Also [a] implies $\langle j_A, w_A\rangle \in Q_{23}$. The last two sentences and [Q2] imply $j = j_A$, which by [a] implies [b] $\langle i, j, w_A\rangle \in Q_{123}$. Meanwhile, the definition of w_B implies that there is $i_B \in I$ such that [c] $\langle i_B, j, w_B\rangle \in Q_{123}$. Note [b], [c], and [Q1] imply $i_B = i$. Thus [c] implies $\langle i, j, w_B\rangle \in Q_{123}$, which implies $\langle i, w_B\rangle \in Q_{13}$.

Claim 13: $\tilde{\Gamma}$ is a **Gm** game. Recall $\tilde{\Gamma} = (\tilde{X}, \tilde{E}, \tilde{\mathcal{H}}, \tilde{\lambda}, \tilde{\mu}, \tilde{U})$ by definition. It suffices to show [G1]–[G3]. [G1] follows from Claim 11.

For [G2], note that $\tilde{\mu}$ is a surjective function from $\tilde{W}$ by Claim 9(a). Thus it suffices to show that $\tilde{\mu}$ is continuous. Equivalently, it suffices to show the first of the following statements. Note that the last of the following statements holds by Claim 12. Thus it suffices to show the following equivalences.[12]

$$\begin{aligned}
&(\forall H{\in}\tilde{\mathcal{H}}, x_1{\in}H, x_2{\in}H)\ \tilde{\mu}(x_1) = \tilde{\mu}(x_2)\\
\text{iff }&(\forall j{\in}J, x_1{\in}j, x_2{\in}j)\ \tilde{\mu}(x_1) = \tilde{\mu}(x_2)\\
\text{iff }&(\forall j{\in}J, x_1{\in}W_j, x_2{\in}W_j)\ \tilde{\mu}(x_1) = \tilde{\mu}(x_2)\\
\text{iff }&(\forall j{\in}J, x_1{\in}W_j, x_2{\in}W_j, i{\in}I)\ \tilde{\mu}(x_1){=}i \Rightarrow \tilde{\mu}(x_2){=}i\\
\text{iff }&(\forall j{\in}J, x_1{\in}W_j, x_2{\in}W_j, i{\in}I)\ \langle i,x_1\rangle{\in}Q_{13} \Rightarrow \langle i,x_2\rangle{\in}Q_{13}
\end{aligned}$$

The first equivalence holds by the definition of $\tilde{\mathcal{H}}$, the second holds because G has info-situations by assumption, the third holds by symmetry and because the codomain of $\tilde{\mu}$ is I by the definition of $\tilde{\mu}$, and the fourth because the graph of μ is Q_{31} by the definition of $\tilde{\mu}$.

For [G3], recall that the definition of a 5-form game implies that $(\forall i{\in}I)\ U_i{:}\mathcal{Z}{\to}\mathbb{R}$. This suffices for [G3] by the definition of $\tilde{U}$, Claim 9(b), and Claim 1(d).

Claim 14: $(\forall j{\in}J, w{\in}W_j)\ \tilde{F}(w) = A_j$. Take $j \in J$ and $w \in W_j$. The last of the following statements holds by Proposition 2.1. Thus it suffices[12] to show

$$\begin{aligned}
&\tilde{F}(w) = A_j\\
\text{iff }&(\forall a{\in}A)\ a{\in}\tilde{F}(w) \Leftrightarrow a{\in}A_j\\
\text{iff }&(\forall a{\in}A)\ \langle w,a\rangle{\in}Q_{34} \Leftrightarrow a{\in}A_j\\
\text{iff }&(\forall a{\in}A)\ a \text{ is feasible at } w \Leftrightarrow a{\in}A_j.
\end{aligned}$$

The first equivalence holds because A is the codomain of $\tilde{F}$ by Claim 10 and because $A_j \subseteq A$. The second holds by Claim 10 again. The third holds by the definition of feasibility.

Claim 15: $\tilde{n} = (Q_{34}, Y, Q_{345})$. Claim 13 and S21 Lemma A.1(b) imply the well-definition of $\tilde{n}{:}\tilde{F}^{\mathsf{gr}}{\to}\tilde{X}{\smallsetminus}\{\tilde{r}\}$, which is defined to take each $\langle x, a\rangle \in \tilde{F}^{\mathsf{gr}}$ to the unique $y \in \tilde{X}{\smallsetminus}\{\tilde{r}\}$ such that $\tilde{\lambda}(xy) = a$. In accord with note 3, it suffices to show that [a] $\tilde{F}^{\mathsf{gr}} = Q_{34}$, [b] $\tilde{X}{\smallsetminus}\{\tilde{r}\} = Y$, and [c] $\tilde{n}^{\mathsf{gr}} = Q_{345}$. [a] holds by Claim 10. [b] holds by Claim 2. For [c],

$$\begin{aligned}\tilde{n}^{\mathsf{gr}} &= \{\ \langle x,a,y\rangle{\in}\tilde{F}^{\mathsf{gr}}{\times}(\tilde{X}\smallsetminus\{\tilde{r}\})\mid \tilde{\lambda}(xy){=}a\ \}\\ &= \{\ \langle x,a,y\rangle{\in}Q_{34}{\times}Y \mid \tilde{\lambda}(xy){=}a\ \}\\ &= \{\ \langle x,a,y\rangle{\in}Q_{34}{\times}Y \mid \langle x,a,y\rangle{\in}Q_{345}\ \}\\ &= Q_{345},\end{aligned}$$

where the first equality follows from the definition of $\tilde{n}$, the second follows from [a] and [b], the third follows from the definition of $\tilde{\lambda}$, and the fourth holds by inspection. □

Lemma B.2. [Formalizes part of `S21` note 8, and concerns only **Gm** games.] *Suppose (X,E) is a nontrivial out-tree. Then $X\smallsetminus\{r\} \in y \mapsto \langle p(y),y\rangle \in E$ is a bijection.*

Proof. Section 3.1 notes that the function $p = (X\smallsetminus\{r\}, W, \{\langle y,x\rangle|xy{\in}E\})$ is well-defined. Thus $\{\langle p(y),y\rangle|y{\in}X\smallsetminus\{r\}\} = E$, which implies that the lemma's function is well-defined and surjective. It is injective by inspection. □

Lemma B.3. [Extends `S21` Lemma A.1, and concerns only **Gm** games.] *Suppose that (X,E) is a nontrivial out-tree, and that $\lambda{:}E{\to}A$ is surjective and deterministic. Then n is a bijection. Its inverse is*

$$F^{\mathsf{gr}} \ni \langle\, p(y),\ \lambda(p(y)y)\,\rangle \hookleftarrow y \in X\smallsetminus\{r\}.$$

Proof. Let $n^*{:}X\smallsetminus\{r\}{\to}F^{\mathsf{gr}}$ denote the claimed inverse. Since $n{:}F^{\mathsf{gr}}{\to}X\smallsetminus\{r\}$ is well-defined by `S21` Lemma A.1(b), the lemma holds by the following three claims.

Claim 1: n^* is well-defined. Take $y \in X\smallsetminus\{r\}$. The definition of p implies that $p(y)y \in E$, which implies that $\lambda(p(y)y)$ is well-defined. Thus it suffices to show that $\langle\, p(y),\ \lambda(p(y)y)\,\rangle \in F^{\mathsf{gr}}$. The definition of p implies $p(y)y \in E$, which implies $y \in \{\, y_+ \,|\, p(y)y_+{\in}E\,\}$, which by the inverse function in `S21` Lemma A.1(c) at $x_o = p(y)$ implies $F(p(y)) \ni \lambda(p(y)y)$, which implies $\langle\, p(y),\ \lambda(p(y)y)\,\rangle \in F^{\mathsf{gr}}$.

Claim 2: $(\forall\langle x,a\rangle{\in}F^{\mathsf{gr}})\ n^*(n(\langle x,a\rangle)) = \langle x,a\rangle$. Take $\langle x,a\rangle \in F^{\mathsf{gr}}$. By definition $n(\langle x,a\rangle)$ is the unique $y \in X\smallsetminus\{r\}$ such that [a] $\lambda(xy) = a$. Since the domain of λ is E, this implies $xy \in E$, which by the definition of p implies [b] $x = p(y)$. [a] and [b] imply [c] $\lambda(p(y)y) = a$. In conclusion, $n^*(n(\langle x,a\rangle))$ by the definition of y is $n^*(y)$, which by the definition of n^* is $\langle\, p(y),\ \lambda(p(y)y)\,\rangle$, which by [b] and [c] is $\langle x,a\rangle$.

Claim 3: $(\forall y{\in}X\smallsetminus\{r\})\ n(n^*(y)) = y$. Take $y \in X\smallsetminus\{r\}$. By definition, $n^*(y) = \langle\, p(y),\ \lambda(p(y)y)\,\rangle$. Thus the definition of n implies that $n(n^*(y))$ is the unique $\hat{y} \in X\smallsetminus\{r\}$ such that $\lambda(p(y)\hat{y}) = \lambda(p(y)y)$. Thus, since λ is deterministic by assumption, $\hat{y} = y$. Hence the definition of $\hat{y}$ implies $n(n^*(y)) = y$. □

Lemma B.4. *Suppose Γ is a* **Gm** *game. Let $\dot{G} = \mathsf{T}\Gamma$. Then (a) $\dot{G}$ is a well-defined 5-form game with info-situations. Also, (b) $\dot{I} = I$, (c) $\dot{J} = \mathcal{H}$, (d) $\dot{W} = W$, (e) $\dot{A} = A$, (f) $\dot{X} = X$, and (g) $\dot{Q}_{35} = E$.*

Proof. Part (a) holds by Claim 17, parts (b)–(e) by Claim 6(b–e), part (f) by Claim 18, and part (g) by Claim 14(a).

Claim 1: $X\smallsetminus\{r\} \ni y \mapsto p(y) \in W$ is a well-defined surjective function. `S21` Section 2.1 defines $p{:}X\smallsetminus\{r\}{\to}W$ and notes that it is surjective.

Claim 2: $W \ni x \mapsto H_x \in \mathcal{H}$ is a well-defined surjective function. This follows from [C2] and the definition of $\langle H_x \rangle_{x \in W}$.

Claim 3: $X\smallsetminus\{r\} \ni y \mapsto H_{p(y)} \in \mathcal{H}$ is a well-defined surjective function. This follows from Claims 1 and 2.

Claim 4: $X\smallsetminus\{r\} \ni y \mapsto \mu\circ p(y) \in I$ is a well-defined surjective function. [G2] and the definition of I imply that $W \ni w \mapsto \mu(w) \in I$ is a well-defined surjective function. Thus Claim 1 implies the present result.

Claim 5: $X\smallsetminus\{r\} \ni y \mapsto \lambda(p(y)y) \in A$ is a well-defined surjective function. Lemma B.2 implies $X\smallsetminus\{r\} \ni y \mapsto p(y)y \in E$ is a well-defined surjective function. Further, [C3] and the definition of A imply $E \ni xy \mapsto \lambda(xy) \in A$ is a well-defined surjective function. The previous two sentences imply the present result.

Claim 6: (a) $\dot{Q}$ is well-defined. (b) $\dot{I} = I$. (c) $\dot{J} = \mathcal{H}$. (d) $\dot{W} = W$. (e) $\dot{A} = A$. (f) $\dot{Y} = X\smallsetminus\{r\}$. (a) follows from the definition of $\dot{Q}$ and the well-definitions shown in Claims 4, 3, 1, and 5. (b) follows from the definition of $\dot{Q}$ and the surjectivity of Claim 4. (c) follows from the definition of $\dot{Q}$ and the surjectivity of Claim 3. (d) follows from the definition of $\dot{Q}$ and the surjectivity of Claim 1. (e) follows from the definition of $\dot{Q}$ and the surjectivity of Claim 5. (f) follows from the definition of $\dot{Q}$ by inspection.

Claim 7: $\dot{Q}$ satisfies [Q1]. By the definition of $\dot{Q}$, it suffices to show that $(\forall y_1{\in}X\smallsetminus\{r\}, y_2{\in}X\smallsetminus\{r\})\ H_{p(y_1)} = H_{p(y_2)} \Rightarrow \mu\circ p(y_1) = \mu\circ p(y_2)$. Toward that end, take $y_1 \in X\smallsetminus\{r\}$ and $y_2 \in X\smallsetminus\{r\}$ and assume $H_{p(y_1)} = H_{p(y_2)}$. Then there is $H \in \mathcal{H}$ which contains both $p(y_1)$ and $p(y_2)$. Thus [G2] implies $\mu\circ p(y_1) = \mu\circ p(y_2)$.

Claim 8: $\dot{Q}$ satisfies [Q2]. By the definition of $\dot{Q}$, it suffices to show that $(\forall y_1{\in}X\smallsetminus\{r\}, y_2{\in}X\smallsetminus\{r\})\ p(y_1) = p(y_2) \Rightarrow H_{p(y_1)} = H_{p(y_2)}$. Toward that end, take $y_1 \in X\smallsetminus\{r\}$ and $y_2 \in X\smallsetminus\{r\}$. Claim 1 implies $p(y_1) \in W$ and $p(y_2) \in W$. Thus Claim 2 implies $p(y_1) = p(y_2) \Rightarrow H_{p(y_1)} = H_{p(y_2)}$.

Claim 9: $(\dot{W}, \dot{A}, \dot{Q}_{34}) = F$. The definition of F states that its domain is W and its codomain is A. These equal $\dot{W}$ and $\dot{A}$ by Claim 6(d,e). Thus it suffices to show

$$\begin{aligned} F^{\mathsf{gr}} &= \{\, \langle x,a\rangle{\in}W{\times}A \mid (\exists y{\in}X\smallsetminus\{r\})\, a{=}\lambda(xy) \,\} \\ &= \{\, \langle x,a\rangle{\in}W{\times}A \mid (\exists y{\in}X\smallsetminus\{r\})\, xy{\in}E,\ a{=}\lambda(xy) \,\} \\ &= \{\, \langle x,a\rangle{\in}W{\times}A \mid (\exists y{\in}X\smallsetminus\{r\})\, x{=}p(y),\ a{=}\lambda(xy) \,\} \\ &= \{\, \langle x,a\rangle{\in}W{\times}A \mid (\exists y{\in}X\smallsetminus\{r\})\, x{=}p(y),\ a{=}\lambda(p(y)y) \,\} \\ &= \{\, \langle p(y),\lambda(p(y)y)\rangle{\in}W{\times}A \mid y{\in}X\smallsetminus\{r\} \,\} \\ &= \{\, \langle p(y),\lambda(p(y)y)\rangle \mid y{\in}X\smallsetminus\{r\} \,\} \\ &= \dot{Q}_{34}. \end{aligned}$$

The first equality holds by the definition of F, the second because the domain of λ is E by [C3], and the third by the definition of p. The fourth and fifth hold by manipulation. The sixth holds because the codomain of p is W by the definition of p, and because the codomain of λ is A by the definition of A. The seventh holds by the definition of $\dot{Q}$.

Claim 10: $(\forall j{\in}\dot{J}, w{\in}\dot{W})\ w \in \dot{W}_j$ iff $w \in j$. To see this, take $j \in \dot{J}$ and $w \in \dot{W}$.

For the forward direction, suppose $w \in \dot{W}_j$. Then $\langle j,w\rangle \in \dot{Q}_{23}$, which by the definition of $\dot{Q}$ implies there is [a] $y \in X\smallsetminus\{r\}$ such that [b] $j = H_{p(y)}$ and [c] $w =$

$p(y)$. Claim 1 and [a] imply $p(y) \in W$, which by the definition of $\langle H_x \rangle_{x \in W}$ implies $p(y) \in H_{p(y)}$, which by [c] implies $w \in H_{p(y)}$, which by [b] implies $w \in j$.

For the reverse direction, suppose $w \in j$. The definition of w implies there is [1] $y_w \in W \smallsetminus \{r\}$ such that [2] $w = p(y_w)$. Also, the definition of j implies there is $y_j \in X \smallsetminus \{r\}$ such that [3] $j = H_{p(y_j)}$. Since $w \in j$ by assumption, [2] and [3] imply [4] $p(y_w) \in H_{p(y_j)}$. Meanwhile, Claim 1 and [1] imply $p(y_w) \in W$, which by the definition of $\langle H_x \rangle_{x \in W}$ implies [5] $p(y_w) \in H_{p(y_w)}$. Claim 3 and [C2] imply that $H_{p(y_w)}$ and $H_{p(y_j)}$ are partition elements. Thus [4] and [5] imply that $H_{p(y_w)} = H_{p(y_j)}$, which by [3] implies [6] $j = H_{p(y_w)}$. By the definition of $\dot{Q}$, facts [1], [2], and [6] imply that $\langle j, w \rangle \in \dot{Q}_{23}$. This is equivalent to $w \in \dot{W}_j$.

Claim 11: $(\forall j {\in} \dot{J}, w {\in} \dot{W}_j, a {\in} \dot{A})$ $\langle w, a \rangle \in \dot{Q}_{34}$ *iff* $a \in \dot{A}_j$. Take $j \in \dot{J}$, $w \in \dot{W}_j$, and $a \in \dot{A}$. For the forward direction, suppose $\langle w, a \rangle \in \dot{Q}_{34}$. Then the definition of w and Claim 8 imply $\langle w, a \rangle \in \dot{Q}_{j,34}$, which implies $a \in \dot{A}_j$.

For the reverse direction, suppose $a \in \dot{A}_j$. Then there exists w_a such that [a] $\langle w_a, a \rangle \in \dot{Q}_{j,34}$. Thus $\langle w_a, a \rangle \in \dot{Q}_{34}$, which by Claim 9 implies [b] $a \in F(w_a)$. Fact [a] also implies $w_a \in \dot{W}_j$, which by Claim 10 implies $w_a \in j$. Similarly the definition of w by Claim 10 implies $w \in j$. By the definition of j and Claim 6(c), the last two sentences imply that w_a and w belong to the same information set in $\mathcal{H}$. Thus [b] and [C4] imply that $a \in F(w)$. Hence Claim 9 implies $\langle w, a \rangle \in \dot{Q}_{34}$.

Claim 12: $(\forall j {\in} \dot{J})$ $\dot{Q}_{j,34} = \dot{W}_j {\times} \dot{Q}_j$. To see this, take $j \in \dot{J}$. The forward inclusion is immediate. For the reverse inclusion, suppose $\langle w, a \rangle \in \dot{W}_j {\times} \dot{A}_j$. Then [a] $w \in \dot{W}_j$ and [b] $a \in \dot{A}_j$. By Claim 11, [a] and [b] imply $\langle w, a \rangle \in \dot{Q}_{34}$. Thus [a] and Claim 8 imply $\langle w, a \rangle \in \dot{Q}_{j,34}$.

Claim 13: $\dot{Q}$ satisfies [Q3]. The definition of $\dot{Q}$, and the definition of the inverse function in Lemma B.3, together imply that $\dot{Q}_{345}$ is the graph of a bijection from its first two coordinates to its third. Now take $j \in \dot{J}$. Since any subset of the graph of a bijection is the graph of a bijection, $\dot{Q}_{j,345}$ is the graph of a bijection from its first two coordinates to its third. Note that the projection of $\dot{Q}_{j,345}$ on its first two coordinates is $\dot{Q}_{j,34}$, and that the projection of $\dot{Q}_{j,345}$ on its third coordinate is $\dot{Y}_j$. Thus $\dot{Q}_{j,345}$ is the graph of a bijection from $\dot{Q}_{j,34}$ onto $\dot{Y}_j$. Claim 12 completes the proof.

Claim 14: (a) $\dot{Q}_{35} = E$. (b) $\dot{Q}$ satisfies [Q4]. (c) $\dot{\mathcal{Z}} = \mathcal{Z}$. For (a), note the definition of $\dot{Q}$ implies $\dot{Q}_{35} = \{ \langle p(y), y \rangle \,|\, y {\in} X {\smallsetminus} \{r\} \}$. Thus, since $p = (X {\smallsetminus} \{r\}, W, \{\langle y, x \rangle | xy {\in} E\})$ by definition, $\dot{Q}_{35} = E$. For (b), note [C1] implies (X, E) is a nontrivial out-tree. Thus (a) implies $\dot{Q}_{35}$ is the edge set of a nontrivial out-tree. For (c), recall that the edge set of a nontrivial out-tree determines its node set. Thus (a) implies (X, E) is identical to the out-tree with edge set $\dot{Q}_{35}$. Hence their run collections are identical.

Claim 15: $\dot{Q}$ is a 5-form. It suffices to show that $\dot{Q}$ satisfies [Q1]–[Q4]. These follow from Claims 7, 8, 13, and 14(b).

Claim 16: $\dot{Q}$ uses info-situations. Take $j \in \dot{J}$. Claim 10 implies [a] $(\forall w {\in} \dot{W})$ $w \in j$ iff $w \in \dot{W}_j$. Meanwhile, Claim 6(c) and [C2] imply $j \subseteq W$, which by Claim 6(d) implies $j \subseteq \dot{W}$. Further, $\dot{W}_j \subseteq \dot{W}$ by construction. The last two sentences and [a] imply $j = \dot{W}_j$.

Claim 17: $\dot{G}$ is a well-defined 5-form game with info-situations. Recall $\dot{G} = (\dot{Q},\dot{U})$ by definition. Because of Claims 15 and 16, it suffices to show that $\dot{U}$ is a well-defined utility-function profile. This follows from the definition of $\dot{U}$, Claim 6(b), and Claim 14(c).

Claim 18: $\dot{X} = X$. In steps, $\dot{X}$ by definition is equal to $\dot{W}\cup\dot{Y}$, which by Claim 6(d,f) is equal to $W\cup(X\smallsetminus\{r\})$, which by $r \in W$ is equal to X. □

Lemma B.5. *Suppose G is a 5-form game with info-situations. Then* $\mathsf{TS}\,G = G$.

Proof. Let $\tilde{\varGamma} = \mathsf{S}G$ and $\dot{G} = \mathsf{T}\tilde{\varGamma}$. It suffices to show $\dot{G} = G$. The definitions of S and T imply $\dot{U} = \tilde{U} = U$. Thus it suffices to show $\dot{Q} = Q$. This is proved by Claims 2 and 4 below.

Claim 1: $Q \subseteq \{\, \langle \tilde{\mu}\circ\tilde{p}(y), \tilde{H}_{\tilde{p}(y)}, \tilde{p}(y), \tilde{\lambda}(\tilde{p}(y)y), y\rangle \,|\, y{\in}Y \,\}$.

Take $\langle i,j,w,a,y\rangle \in Q$. Note [a] $y \in Y$. Since $\langle w,y\rangle \in Q_{35}$, the definition of $\tilde{E}$ implies $\langle w,y\rangle \in \tilde{E}$, which by the definition of $\tilde{p}$ implies [b] $w = \tilde{p}(y)$. Since $\langle w,y,a\rangle \in Q_{354}$, the definition of $\tilde{\lambda}$ implies $a = \tilde{\lambda}(wy)$, which by [b] implies [c] $a = \tilde{\lambda}(\tilde{p}(y)y)$. Since $\langle w,i\rangle \in Q_{31}$, the definition of $\tilde{\mu}$ implies $i = \tilde{\mu}(w)$, which by [b] implies [d] $i = \tilde{\mu}\circ\tilde{p}(y)$.

Note $j \in J$, which by the definition of $\tilde{\mathcal{H}}$ implies [e] $j \in \tilde{\mathcal{H}}$. Also note $w \in W_j$, which by Q using info-situations implies [f] $w \in j$. Meanwhile, $\tilde{\mathcal{H}}$ is a partition by [C2] for $\tilde{\varGamma}$. Thus [e] and [f] imply j is the unique element of $\tilde{\mathcal{H}}$ containing w. Thus the definition of $\langle H_x\rangle_{x\in W}$ implies $j = H_w$, which by [b] implies [g] $j = H_{\tilde{p}(y)}$.

Finally, $\langle i,j,w,a,y\rangle = \langle \tilde{\mu}\circ\tilde{p}(y), \tilde{H}_{\tilde{p}(y)}, \tilde{p}(y), \tilde{\lambda}(\tilde{p}(y)y), y\rangle$ by [d], [g], [b], and [c]. Further, $y \in Y$ by [a].

Claim 2: $Q \subseteq \dot{Q}$. To see this, take $\langle i,j,w,a,y\rangle \in Q$. Note [a] $y \in Y$. Also, Claim 1 implies [b] $\langle i,j,w,a,y\rangle = \langle \tilde{\mu}\circ\tilde{p}(y), \tilde{H}_{\tilde{p}(y)}, \tilde{p}(y), \tilde{\lambda}(\tilde{p}(y)y), y\rangle$. Lemma B.1(e) and [a] imply $y \in \tilde{X}\smallsetminus\{\tilde{r}\}$. Thus the definition of $\dot{Q}$ and [b] imply $\langle i,j,w,a,y\rangle \in \dot{Q}$.

Claim 3: $Q = \{\ \langle \tilde{\mu}\circ\tilde{p}(y), \tilde{H}_{\tilde{p}(y)}, \tilde{p}(y), \tilde{\lambda}(\tilde{p}(y)y), y\rangle \mid y{\in}Y\ \}$. Claim 1 shows the forward inclusion. From another perspective, Claim 1 shows that the set Q is a subset of the graph of a function from Y (to be clear, the function's argument $y \in Y$ appears in the fifth coordinate, and the function takes each $y \in Y$ to the quadruple $\langle \tilde{\mu}\circ\tilde{p}(y), \tilde{H}_{\tilde{p}(y)}, \tilde{p}(y), \tilde{\lambda}(\tilde{p}(y)y)\rangle$). Thus the set Q and the graph of the function are equal if the projection of Q onto its fifth coordinate is Y. This holds by the definition of Y.

Claim 4: $Q \supseteq \dot{Q}$. Take $\langle i,j,w,a,y\rangle \in \dot{Q}$. Then the definition of $\dot{Q}$ implies [a] $y \in \tilde{X}\smallsetminus\{\tilde{r}\}$ and [b] $\langle i,j,w,a,y\rangle = \langle \tilde{\mu}\circ\tilde{p}(y), \tilde{H}_{\tilde{p}(y)}, \tilde{p}(y), \tilde{\lambda}(\tilde{p}(y)y), y\rangle$. Fact [a] and Lemma B.1(e) imply $y \in Y$. Thus [b] and Claim 3 imply $\langle i,j,w,a,y\rangle \in Q$. □

Lemma B.6. *Suppose $\varGamma$ is a* **Gm** *game. Then* $\mathsf{ST}\varGamma = \varGamma$.

Proof. Let $\dot{G} = \mathsf{T}\varGamma$, and $\tilde{\varGamma} = \mathsf{S}\dot{G}$. It suffices to show $\tilde{\varGamma} = \varGamma$. In other words, it suffices to show $(\tilde{X}, \tilde{E}, \tilde{\mathcal{H}}, \tilde{\lambda}, \tilde{\mu}, \tilde{U}) = (X, E, \mathcal{H}, \lambda, \mu, U)$. This is done, one component at a time, by Claims 1, 2, 3, 5, 7, and 8.

Claim 1: $\tilde{X} = X$. $\tilde{X}$ by definition is $\dot{X}$, which by Lemma B.4(f) is X.

Claim 2: $\tilde{E} = E$. $\tilde{E}$ by definition is $\dot{Q}_{35}$, which by Lemma B.4(g) is E.

Claim 3: $\tilde{\mathcal{H}} = \mathcal{H}$. $\tilde{\mathcal{H}}$ by definition is $\dot{J}$, which by Lemma B.4(c) is $\mathcal{H}$.

Claim 4: $(\dot{Q}_{35}, \dot{A}, \dot{Q}_{354}) = \lambda$. [C3] and the definition of A imply [a] $\lambda = (E, A, \lambda^{\mathsf{gr}})$. Lemma B.4(g) implies $\dot{Q}_{35} = E$, and Lemma B.4(e) implies $\dot{A} = A$. Thus it suffices that

$$\begin{aligned}\dot{Q}_{354} &= \{\, \langle p(y), y, \lambda(p(y)y) \rangle \,|\, y{\in}X{\setminus}\{r\} \,\} \\ &= \{\, \langle x, y, \lambda(xy) \rangle \,|\, xy{\in}E \,\} = \lambda^{\mathsf{gr}},\end{aligned}$$

where the first equality holds by the definition of $\dot{Q}$, the second by Lemma B.2, and the last because the domain of λ is E by [a].

Claim 5: $\tilde{\lambda} = \lambda$. $\tilde{\lambda}$ by definition is $(\dot{Q}_{35}, \dot{A}, \dot{Q}_{354})$, which by Claim 4 is λ.

Claim 6: $(\dot{W}, \dot{I}, \dot{Q}_{31}) = \mu$. [G2] and the definition of I imply [a] $\mu = (W, I, \mu^{\mathsf{gr}})$. Lemma B.4(d,b) imply $\dot{W} = W$ and $\dot{I} = I$. Thus it suffices that

$$\begin{aligned}\dot{Q}_{31} &= \{\, \langle \mu{\circ}p(y), p(y) \rangle \,|\, y{\in}X{\setminus}\{r\} \,\} \\ &= \{\, \langle \mu(w), w \rangle \,|\, w{\in}W \,\} = \mu^{\mathsf{gr}},\end{aligned}$$

where the first equality holds by the definition of $\dot{Q}$, the second because p is onto W by `S21` Section 2.1, and the third because the domain of μ is W by [a].

Claim 7: $\tilde{\mu} = \mu$. $\tilde{\mu}$ by definition is $(\dot{W}, \dot{I}, \dot{Q}_{31})$, which by Claim 6 is μ.

Claim 8: $\tilde{U} = U$. $\tilde{U}$ by definition is $\dot{U}$, which by definition is U. □

Proof B.7 (for Theorem 3.1)**.** Part (a) follows from Lemmas B.1(a) and B.4(a). Part (b) follows from Lemma B.5. Part (c) follows from Lemma B.6. □

Proof B.8 (for Theorem 3.2)**.** The definition of $\tilde{\Gamma} = (\tilde{X}, \tilde{E}, \tilde{\mathcal{H}}, \tilde{\lambda}, \tilde{\mu}, \tilde{U})$ implies, respectively, parts (d), (m), (i), (q), (h), and (g). Lemma B.1(b-j) imply, respectively, parts (e), (b), (f), (n), (c), (a), (j), (k), and (r). Thus parts (l), (o), and (p) remain.

For (l), consider part (k). Since G is assumed to have info-situations, part (k) is equivalent to $(\forall j{\in}J, w{\in}j)\ \tilde{F}(x) = A_j$. Thus part (i) implies part (l).

For (o), $\tilde{X}{\setminus}\tilde{W}$ by $\tilde{r} \in \tilde{W}$ is equal to $(\tilde{X}{\setminus}\tilde{r}){\setminus}\tilde{W}$, which by parts (n) and (b) is equal to $Y{\setminus}W$.

For (p), the definition of $\tilde{p}$ implies that $\tilde{p}$ is equal to $(\tilde{X}{\setminus}\tilde{r}, \tilde{W}, \{\langle y, x\rangle | xy{\in}\tilde{E}\}$, which by parts (n) and (b) is equal to $(Y, W, \{\langle y, x\rangle | xy{\in}\tilde{E}\})$, which by part (m) is equal to (Y, W, Q_{53}). □

Proof B.9 (for Corollary 3.3)**.** Theorem 3.1(a) implies G is an info-situation 5-form game. Further, $\tilde{\Gamma}$ by Theorem 3.1(c) is equal to $\mathsf{ST}\tilde{\Gamma}$, which by the definition of G is equal to $\mathsf{S}\,G$. Thus the assumptions of Theorem 3.2 are met. □

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