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ASPECTS OF STRONG SUMMABILITY ASSOCIATED WITH GENERALISED CESARO, RIESZ AND NÖRLUND SUMMABILITY

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by

Submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy

Faculty of Graduate Studies The University of Western Ontario London, Ontario, Canada March, 1975

Edward Hai-Wei Chang 1975

Generalised Cesàro Summability, Riesz Summability and Strong Riešz Summability have been extensively investigated by various authors. In this thesis a definition of Strong Generalised Cesàro Summability Method is proposed and the question of its equivalence with the Strong Riesz Summability Method is Established. In Chapter 3 some equivalence theorems between the Generalised Cesàro Methods and the Strong Generalised Cesàro Method are obtained. In Chapter 4 inclusion theorems between the Absolute Generalised Cesàro Methods and the Strong Generalised Cesàro Method are obtained.

ABSTRACT

We extend a result due to Kuttner, obtaining some strict inclusion theorems between Cesaro and Discrete Riesz Methods of Summability. And our investigation in this respect stems from Borwein and Cass's work on Strong Nörlund Summability.

In Chapter 6 we consider Nörlund Methods of Summability Associated with Polynomials which have been investigated by Borwein, and consider Strong and Absolute Nörlund Methods associated with them. We show, för example, that two polynomial Nörlund Methods are equivalent if and only if the associated Strong Methods are equivalent.

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I would like to express my sincere gratitude to my chief advisor, Dr. F. P. A. Cass for his kindness, encouragement and academic guidance.

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In this thesis, the symbols H, H_1 , H_2 , H_3 are used throughout to denote positive constants, but not necessarily having the same value at each occurrence.

The theorems, lemmata and corollaries are numbered by chapter. For example, Theorem 3.1 is the first theorem , in Chapter 3.

At the end of each proof we use the symbol /// to _ show that the proof is complete.

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CHAPTER 1

STRONG GENERALISED CESÀRO SUMMABILITY

§1.1 INTRODUCTION

We suppose throughout the thesis that $\lambda = \{\lambda_n\}$ is a sequence satisfying

(1.1) $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_n \rightarrow \infty.$

For the sake of convenience we take $\lambda_0 = 0$ in (1.1) instead of $\lambda_0 \ge 0$. By doing so we find that there is no loss of generality. This remark will be amplified on page 5.

We suppose also that p is a non-negative integer and for the series $\sum_{v=0}^{\infty} a_v$ we use the notation

 $s_n = \sum_{v=0}^{n} a_v, \quad n = 0, 1, 2, \dots$

In this chapter we introduce a definition of Strong Generalised Cesàro Summability and investigate some of its properties. We also give the definitions of several other summability methods whose properties and relations with the Strong Generalised Cesàro Summability are investigated in the later chapters.

\$1.2 SUMMABILITY METHODS

If a given summability method T assigns the sum s

to the series $\sum_{\nu=0}^{n} a_{\nu}$ with sequence of partial sums $\{s_n\}$, we

say that $\sum_{\nu=0}^{n} a_{\nu}$ is T-summable or $\{s_n\}$ is T-convergent to s.

We denote this by

 $\sum_{\nu=0}^{\infty} a_{\nu} = s (T)$

or by 🕤

A method of summability T is said to be regular, if $s_n \rightarrow s(T)$ whenever the sequence $\{s_n\}$ converges to s.

 $s_n \rightarrow s$ (T)

Let $Q = \{q_{n,r}\}$ (n,r = 0,1,2,...) be a (summability) matrix and let

(1.2) $\sigma_n = \sum_{r=0}^{\infty} q_{n,r} s_r.$

The sequence $\{s_n\}$ is said to be Q-convergent to the sum s if σ_n exists for n = 0, 1, 2, ... and tends to s as n tends to infinity.

The matrix $Q = \{q_{n,r}\}$ is regular if and only if

 $\sup_{n\geq 0} \sum_{r=0}^{\infty} |q_{n,r}| < \infty,$

(1.3)

(1.4)

 $\lim_{n \to \infty} q_{n,r} = 0, \text{ for } r = 0, 1, 2, ...,$

(1.5) $\lim_{n \to \infty} \sum_{r=0}^{\infty} q_{n_{i'}r} = 1.$

This is the Toeplitz Theorem for the regularity of the matrix Q.

The symbol P will be reserved for matrices $\{p_{n,r}\}$ with

 $p_{n,r} \ge -\theta$ (n,r = 0,1,2,...):

Such matrices will be called non-negative matrices. Let $\mu > 0$. The Strong Summability Methods $[P,Q]_{\mu}$ are defined as follows. We write $s_n \rightarrow s$ $[P,Q]_{\mu}$ if

3.1

(1.6) $\tau_{\mathbf{n}} = \sum_{\nu=0}^{\infty} p_{\mathbf{n},\mathbf{r}} |\sigma_{\mathbf{r}} - \mathbf{s}|^{\mu}$

exists for n = 0,1,2,... and tends to zero as n tends to infinity. Thus s is the $[P,Q]_{\mu}$ -limit of $\{s_n\}$ and the sequence is $[P,Q]_{\mu}$ -convergent to s.

If V and W are summability methods of any of the above types we shall say that W *includes* V, and use the notation $V \Rightarrow W$, if any sequence V-convergent to s is necessarily W-convergent to s. If W includes V but V does not include W, the inclusion V => W is said to be *strict*. If both $V \Rightarrow W$ and W => V, we say that V and W are *equivalent* and write V <=> W.

Let $\mu > 0$. We say that $\{s_n\}$ is absolutely (Q) -convergent or $|Q|_{\mu}$ -convergent if , (1.7)

§ %.3 RIESZ SUMMABILITY (R,λ,κ)

Let $\kappa \geq 0$ and $\lambda = \{\lambda_n\}$ satisfy (1.1). The Riesz Summability Method (R,λ,κ) is defined as follows.

 $\sum_{n=1}^{\infty} n^{\mu-1} |\sigma_n - \sigma_{n-1}|^{\mu} < \infty.$

Let
$$A_{\lambda}^{\kappa}(\tau) = \sum_{\substack{\lambda_{\mathcal{V}} \leq \tau \\ \lambda_{\mathcal{V}} \leq \tau}} for \kappa = 0,$$

 $A_{\lambda}^{\kappa}(\tau) = \sum_{\substack{\lambda_{\mathcal{V}} \leq \tau \\ \lambda_{\mathcal{V}} \leq \tau}} (\tau - \lambda_{\mathcal{V}})^{\kappa} a_{\mathcal{V}}, \text{ for } \kappa > 0,$
 $R_{\lambda}^{\kappa}(\tau) = A_{\lambda}^{\kappa}(\tau) = \sum_{\substack{\lambda_{\mathcal{V}} \leq \tau \\ \lambda_{\mathcal{V}} \leq \tau}} a_{\mathcal{V}}, \text{ for } \kappa = 0,$
and $R_{\lambda}^{\kappa}(\tau) = \sum_{\substack{\lambda_{\mathcal{V}} \leq \tau \\ \lambda_{\mathcal{V}} \leq \tau}} (1 - \frac{\lambda_{\mathcal{V}}}{\tau})^{\kappa} a_{\mathcal{V}}, \text{ for } \kappa > 0.$

The series $\sum_{\nu=0}^{\infty} a_{\nu}$ is said to be (R,λ,κ) -summable to s, if

$$R_{\lambda}^{\kappa}(\tau) \rightarrow s \quad as \ \tau' \rightarrow \infty.$$

(See Hardy and Riesz [12, pp. 21-22].) .

\$7.4 STRONG RIESZ SUMMABILITY $[R, \lambda, p+1]_{\mu}$

The series
$$\sum_{\nu=0}^{\infty}$$
 a is said to be strongly Riesz

Summable to s, with order p+1 and index $\mu > 0$; if

$$\mathbf{F}^{\mathbf{p+1}}(\hat{\boldsymbol{\omega}}) = \int_{0}^{\omega} |\mathbf{A}_{\lambda}^{\mathbf{p}_{\bullet}}(\tau) - \mathbf{s}\tau^{\mathbf{p}}|^{\mu} d\tau = .0 \{\boldsymbol{\omega}^{\mathbf{p}\mu+1}\}$$

We denote this by

$$\sum_{\nu=0}^{\infty} a_{\nu} = s [R, \lambda, p+1]_{\mu}.$$

The definition of the Strong Riesz Summability we have given here is due to Glatfeld [15]. Srivastava [24] and Boyd and Hyslop [8] have also given definitions of Strong Riesz Summability, but we shall not be concerned with them here.

We give now two examples to illustrate that no loss of generality is involved by taking $\lambda_0 = 0$ in (1.1).

Our first example deals with Riesz Summability. Let $\lambda = \{\lambda_n\}$ satisfy

 $0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n \rightarrow \infty$

and let
$$\delta = \{\delta_{-}\}$$
 satisfy

if $R_{1}^{K}(\tau)$, \rightarrow s, as $\tau \rightarrow \infty$,

 $\lambda_1 > \delta_0 > 0$ and $\delta_n = \lambda_n$ for $n \neq 0$.

Let $R_{\lambda}^{K}(t)$ be defined as in §1.3 and let

$$R_{\delta}^{\kappa}(\tau) = \sum_{\delta_{v} < \tau} (1 - \frac{\delta_{v}}{\tau})^{\kappa} a_{v}.$$

Then

. . .

$$R_{\delta}^{K}(\tau) - s = R_{\lambda}^{K}(\tau) - s + R_{\delta}^{K}(\tau) - R_{\lambda}^{K}(\tau)^{\circ}$$
$$= R_{\lambda}^{K}(\tau) - s + [(1 - \frac{\delta_{0}}{\tau})_{s} a_{0}^{\circ} - a_{0}]^{\circ}$$

Since $(1 - \frac{1}{\tau}) = a_0 + 0$ as $\tau \to \infty$, $R_{\delta}^{k}(\tau) \to s$ if and only.

 $A_{\delta}^{\mathbf{p}}(\tau) = \sum_{\mathbf{r}} (\tau - \delta_{\mathbf{v}})^{\mathbf{p}} \mathbf{a}_{\mathbf{v}}$

Our other example deals with Strong Riesz Summability. Let $A^p_{\lambda}(\tau)$ be defined as in §1.3 and let

Then
$$I_{1} = \int_{0}^{\omega} |\lambda_{0}^{p}(\tau) - s\tau^{p}|^{\mu} d\tau$$

$$= \int_{0}^{\omega} |\lambda_{0}^{p}(\tau) - s\tau^{p} + \lambda_{0}^{p}(\tau) - \lambda_{1}^{p}(\tau)|^{\mu} d\tau$$

$$\leq g^{\mu} (\int_{0}^{\omega} |\lambda_{0}^{p}(\tau) - s\tau^{p}|^{\mu} d\tau + \int_{0}^{\omega} |a_{0} d\tau + \delta_{0}|^{p} - a_{0} \tau^{p}|^{\mu} d\tau)$$

$$= 2^{\mu} (I_{2} + I_{3}^{p}).$$
Regradung a_{0} as the series $\sum_{\nu=0}^{\infty} b_{\nu}$ with $b_{0} = a_{0}$ and $b_{\nu} = 0$ for $\nu > 0$, we have $(f - \delta_{0})^{p}a_{0} = \sum_{0,\sqrt{\nu}} (\tau - \delta_{\nu})^{p}b_{\nu}.$
Since $\sum_{\nu=0}^{\infty} b_{\nu} = a_{0}$ and $[R, \lambda, p+1]_{\mu}$ is regular, (see Glatfeld
(15)), thus $I_{3} = o(\omega^{p\mu+1}).$ Hence $I_{2} = o(\omega^{p\mu+1}) => I_{1} = o(\omega^{p\mu+1}).$
Solve $(\lambda - \delta_{0})^{p}a_{0} = \sum_{0} - o(\omega^{p\mu+1}).$
Solve $(\lambda - \delta_{0})^{p}a_{0} = \sum_{0} - o(\omega^{p\mu+1}).$
Solve $(\lambda - \delta_{0})^{p}a_{0} = \sum_{0} - o(\omega^{p\mu+1}).$
Solve $(\lambda - \delta_{0})^{p}a_{0} = 0,$
 $(\mu^{p}a_{0})^{p}a_{0} = 0$

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If $t_n^p + s as n + \infty$, then $\sum_{\nu=0}^{\infty} a_{\nu}$ is said to be (C,λ,p) summable

to s and we write

(7

$$\sum_{\nu=0}^{\infty} a_{\nu} = s \quad (C, \lambda, p) .$$

Since (C,λ,p) is a matrix method in the sense described in \$1.2, we shall find it convenient to denote both the summability method and its associated matrix by (C,λ,p) . Since the entries in the matrix (C,λ,p) are zero above the main diagonal and non-zero on the main diagonal, it has an inverse.

\$1.6 STRONG GENERALISED CESARO SUMMABILITY [C, λ ,p+1]_u

Let $\lambda = {\lambda_n}$ satisfy (1.1). We define

 $E_{p}^{p}(\lambda) = E_{p}^{p} = 1$, for p = 0,

 $E_n^{\vec{p}}(\lambda) = E_n^p = \lambda_{n+1} \dots \lambda_{n+p}$, for $p = 1, 2, 3, \dots$,

and $n = 0, 1, 2, 3, \ldots$.

Since $\lambda_0 = 0$, we obtain

$$E_{m}^{p+1} = \sum_{n=0}^{m} (\lambda_{n+p+1} - \lambda_{n}) E_{n}^{p}.$$

We define

$$T_{m,\mu}^{1} = \sum_{n=0}^{m} (\lambda_{n+1} - \lambda_{n}) |t_{n}^{0} - s|^{\mu},$$

$$T_{m,\mu}^{p+1} = \sum_{n=0}^{m} (\lambda_{n+p+1} - \lambda_{n}) E_{n}^{p} |t_{n}^{p} - s|^{\mu},$$

$$\sigma_{m,\mu}^{1} = \lambda_{m+1}^{-1} \sum_{n=0}^{m} (\lambda_{n+1} - \lambda_{n}) |t_{n}^{0} - s|^{\mu},$$

We say that the series $\sum_{\nu=0}^{\infty} a_{\nu}$ is Strongly Generalised Cesaro

 $\sigma_{m,\mu}^{p+1} = \frac{\mathbf{T}_{m,\mu}^{p+1}}{\mathbf{E}_{m}^{p+1}} = \frac{1}{\mathbf{E}_{m}^{p+1}} \sum_{n=0}^{m} (\lambda_{n \neq p+1} - \lambda_{n}) \mathbf{E}_{n}^{p} |\mathbf{t}_{n}^{p} - \mathbf{s}|^{\mu}.$

Summable to s, with order p+1 and index μ , if

 $\sigma_{m,\mu}^{p+1} = o(1) \quad \text{as } m \to \infty.$

And we use the notation

 $\sum_{\nu=0}^{\infty} \tilde{a}_{\nu} = s [C, \lambda, p+1]_{\mu}.$

Generalised Cesaro Summability was first introduced by Jurkat, [16]. Burkill, [9], gave a different definition. The definition we use here is due to Burkill. The definition was extended to accommodate positive non-integral values of p by Borwein, [3]. We have not been able to formulate a suitable definition of $[\oplus, \lambda, p+1]_{\mu}$ with p non-integral.

Several persons have investigated relations between Riesz and Generalised Cesàro Summability. In particular, it is proved in Russell [23] that if λ is a sequence satisfying (1.1) and p is a non-negative integer then

 $(C,\lambda,p) => (R,\lambda,p), p = 0,1,2,3,...$

It is proved in Meir [20] that if λ is a sequence satisfying (1.1) and p is a non-negative integer then

 $(R,\lambda,p) \implies (C,\lambda,p), p = 0,1,2,3,...$

If in §1.5 we take $\lambda_n = n$, we recover the classical Cesàro Summability Method (C,p). (See Hardy [11].) 8

If in §1.6 we take $\lambda_n = n$, we obtain a summability method which although not equal to, is nevertheless equivalent, to the classical Strong Cesaro Summability Method [C, p+1]_µ. (See Borwein and Cass [6].)

We recall that $\sum_{\nu=0}^{\infty} a_{\nu} = s [C,p+1]_{\mu}$ if and only if

 $\frac{1}{n+1} \sum_{\nu=0}^{n} |s_{\nu}^{p} - s|^{\mu} = o(1),$

where $S_n^p = \frac{1}{\varepsilon_n^p} \sum_{\nu=0}^n \varepsilon_{n-\nu}^{p-1} s_{\nu}$ and $\varepsilon_n^p = \varepsilon \binom{n+p}{n}$.

In case where no confusion can arise, we omit the subscript μ from $\sigma_{m,\mu}^{p+1}$ and $T_{m,\mu}^{p+1}$.

\$1.7 SIMPLE INCLUSION THEOREMS

In order to simplify the notation and the proofs of theorems occurring later we introduce a matrix

$$\Lambda_{p+1} = \{\lambda_{m,n}^{p+1}\} = \{\lambda_{m,n}\}$$

which is defined as follows.

(1.8) For p = 0

and for p > 0

$$n = \begin{cases} \frac{1}{E^{p+1}} & (\lambda_{n+p+1} - \lambda_n) E^p_n, \text{ for } 0 \leq n \leq m, \\ m & & & \\ 0, & & & \text{for } n > m. \end{cases}$$

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It follows easily from the Toeplitz conditions (1.3);
(1.4), (1.5) that
$$\Lambda_{p+1}$$
 is regular.
We now establish some results pertaining to the Strong
Generalised Cesàro Summability.
 $\Box \text{ Let } \mathbb{C}_{n}^{p} \text{ and } t_{n}^{p} \text{ be defined aq in $1.5}.$ Then
 $\mathbb{C}_{n}^{p+1} - \mathbb{C}_{n-1}^{p+1} = (\lambda_{n+p+1} - \lambda_{n})\mathbb{C}_{n}^{p}$
so that
(1.9) $\mathbb{C}_{n}^{p+1} = \int_{\sqrt{-0}}^{p} (\lambda_{\nu+p+1} - \lambda_{\nu})\mathbb{C}_{\nu}^{p}.$
(See [23, p. 419].)
Hence
(1.10) $\frac{1}{\mathbb{E}_{m}^{p+1}} \int_{n=0}^{m} (\lambda_{n+p+1} - \lambda_{n})\mathbb{E}_{n}^{p} t_{n}^{p}.$
 $= \frac{1}{\mathbb{E}_{m}^{p+1}} \int_{n=0}^{m} (\lambda_{n+p+1} + \lambda_{n})\mathbb{C}_{n}^{p}$
 $= \mathbb{C}_{m}^{p+1} / \mathbb{E}_{m}^{p+1}$
 $= t_{m}^{p+1}.$
This means, in matrix notation,
(1.11) $(\mathbb{C}, \lambda, \mathbb{P}^{+1}) = \Lambda_{p+1} (\mathbb{C}, \lambda, \mathbb{P}).$
Moreover, \mathbb{P} ferring to (1.6), the definitions of
 $[\mathbb{C}, \lambda, \mathbb{P}^{+1}]_{\mu}$ and Λ_{p+1} , we have
(1.12) $[\mathbb{C}, \lambda, \mathbb{P}^{+1}]_{\mu} = [\Lambda_{p+1}, (\mathbb{C}, \lambda, \mathbb{P})]_{\mu}.$

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The following two theorems are given in Borwein, [1, Theorems 1 and 3]. We reproduce the proofs for the sake of completeness. 11

If Q is any matrix and $P = \{p_{n,r}\}, where p_{n,r} \ge 0$ for $n,r = 0,1,\ldots, \sum_{r=0}^{\infty} p_{n,r} \le M$ for $n = 0,1,\ldots$ and if $\mu_1 > \mu_2 > 0$ then $[P,Q]_{\mu_1} \Longrightarrow [P,Q]_{\mu_2}$. In particular, the conclusion holds if $\mu_1 > \mu_2 > 0$ and P is regular. PROOF.

By Hölder's inequality

 $\sum_{r=0}^{\infty} p_{n,r} |w_r|^{\mu_2} \leq (\sum_{r=0}^{\infty} p_{n,r} |w_r|^{\mu_1})^{\mu_2/\mu_1} M^{1-\mu_2/\mu_1}.$

for any sequence $\{w_n\}$. The required conclusion follows. /// THEOREM 1.2

If P is a regular (non-negative) matrix and Q is any matrix, then

$$(i) \quad Q \implies [P,Q]_{\mu'}, for \mu > 0, \Rightarrow$$

(ii)
$$[P,Q]_{\mu} \Rightarrow PQ, for \mu \geq 1.$$

PROOF

(i) If $\mathfrak{s}_n \neq \mathfrak{s}$, then, since P is regular

 $\sum_{r=0}^{n} p_{n,r} |s_r - s|^{\mu} = o(1), \text{ i.e., } I => [P,I]_{\mu} \text{ and inclusion}$

(i) follows.

(ii) Suppose that $s_n \rightarrow s[P,I]_{\mu}$. Then by Theorem 1.1, $s_n \rightarrow s[P,I]_1$ and so

12.

$$|\sum_{x=0}^{n} p_{n,x}(s_{x} - s)| \leq \sum_{r=0}^{n} p_{n,x}[s_{x} - s] \notin o(1)$$
Since P is regular, it follows that $s_{n} + s$ (P). Hence

$$[P, I]_{\mu} \Rightarrow P \text{ and inclusion (ii)} \text{ is an immediate consequence. ///
COROLLARY 1.1
If $\mu_{1} > \mu_{2} > 0$, then $[C,\lambda,p+1]_{\mu_{1}} \Rightarrow [C,\lambda,p+1]_{\mu_{2}}$.
PROOF
By (1.12), we know that $[C,\lambda,p+1]_{\mu} = [\Lambda_{p+1}, (C,\lambda,p)]_{\mu}$.
The inclusion is a consequence of Theorem 1.1 and the fact
that Λ_{p+1} is a regular and non-negative matrix. ///
COROLLARY 1.2
If $\mu > 0$, then $(C,\lambda,p) \Rightarrow [C,\lambda,p+1]_{\mu}$.
PROOF
Since $[C,\lambda,p+1]_{\mu} = [\Lambda_{p+1}, (C,\lambda,p)]_{\mu}$ and Λ_{p+q} is regular
and non-negative. The corollary is an immediate consequence
of Theorem 1.2 (i). ///
COROLLARY 1.3
If $\mu \ge 1$, then $[C,\lambda,p+1]_{\mu} \Rightarrow (C,\lambda,p+1)$.
The corollary is a consequence of Theorem 1.2 (ii). ///
COROLLARY 1.4
Suppose $\mu_{1} \ge 1$ and $\mu_{2} > 0$. Then
 $[C,\lambda,p+1]_{\mu_{1}} \Rightarrow [C,\lambda,p+2]_{\mu_{2}}$.
PROOF
This is a consequence of Corollary 1.3 and
Corollary 1.2. ///$$

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We mention two other properties of
$$[C, \lambda, p+1]_{\mu}$$
 here.
(1.13) If $\int_{v=0}^{\infty} a_{v} = s[C(\lambda, p+1]_{\mu} and \int_{v=0}^{\infty} a_{v} = s^{*}[C, \lambda, p+1]_{\mu}$
then $s = s^{*}$.
(I.14) If $\mu \ge 0$, then
 $\int_{v=0}^{\infty} a_{v} = a [(a, \lambda, p+1]_{\mu}$
and
 $\int_{v=0}^{\infty} b_{v} = b [C(\lambda, p+1]_{\mu}$.
 $\int_{v=0}^{\infty} c_{v} = \int_{v=0}^{\infty} (aa_{v} + \beta b_{v}) = \alpha a + \beta b [C, \lambda, p+1]_{\mu}$.

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CHAPTER 2

EQUIVALENCE BETWEEN STRONG GENERALISED_CESÀRO -SUMMABILITY AND STRONG RIESZ SUMMABILITY S

In this chapter we shall establish the equivalence between $[C,\lambda,p+1]_{\mu}$ and $[R,\lambda,p+1]_{\mu}$. We first prove a lemma. (Cf. Glatfeld [15].)

§2.1 A LEMMA

LEMMA 2.1

If $\chi(\tau) \geq 0$, continuous and Riemann integrable in [h,w], where h is any fixed positive real number and if $\alpha+\delta > 0$ and $\delta > 0$, then

$$\int_{h}^{\omega} \chi(\tau) d\tau = o(\omega^{\delta})$$

if and only if

$$\int_{h}^{\omega} \tau^{\alpha} \chi(\tau) d\tau = o(\omega^{\alpha+\delta}).$$

PROOF

Assume
$$\int_{h}^{\omega} \chi(\tau) d\tau = o(\omega^{\delta})$$
 and let $F(\omega) = \int_{h}^{\omega} \chi(\tau) d\tau$.

Then integrating by parts

$$\int_{h}^{\omega} \tau^{\alpha} \chi(\tau) d\tau = \left[\tau^{\alpha} F(\tau)\right]_{h}^{\omega} - \alpha \int_{h}^{\omega} \tau^{\alpha-1} F(\tau) d\tau$$

$$= \omega^{\alpha} F(\omega) - \alpha \left[\int_{h}^{\omega} \tau^{\alpha+\delta-1} \frac{F(\tau)}{\tau^{\delta}} d\tau\right]$$

$$= U - V,$$
and
$$U = c(\omega^{\alpha+\delta}) \text{ by hypothesis}.$$
Further
$$\frac{1}{\omega^{\alpha+\delta}} \int_{h}^{\omega} \tau^{\alpha+\delta-1} \frac{F(\tau)}{\tau^{\delta}} d\tau$$

$$= \int_{h}^{\omega} K(\omega, \tau) G(\tau) d\tau,$$
where
$$K(\omega, \tau) = \begin{cases} \frac{\alpha^{\alpha+\delta-1}}{\tau^{\delta}}, & 0 < \tau \leq -\omega, \end{cases}$$
Now
$$\int_{h}^{\infty} |K(\omega, \tau)| d\tau$$

$$= \frac{\omega^{\alpha+\delta} - h^{\alpha+\delta}}{(\alpha+\delta)\omega^{\alpha+\delta}}$$

$$= \frac{1}{\alpha+\delta} (1 - \frac{h^{\alpha+\delta}}{\omega^{\alpha+\delta}})$$
For every positive y
$$\lim_{\omega \to \infty} \int_{h}^{\infty} |K(\omega, \tau)| d\tau$$

$$= \lim_{\omega \to \infty} \frac{y^{\alpha+\delta} - h^{\alpha+\delta}}{(\alpha+\delta)\omega^{\alpha+\delta}}$$

$$= 0.$$

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Since $G(\tau) \rightarrow 0$ as $\tau \rightarrow \infty$, it follows from Hardy

[11, Theorem 6] that

$$\int_{h}^{\omega} K(\omega,\tau) G(\tau) d\tau \neq 0, \quad \text{as } \omega \neq \infty.$$

Thus $(V = O(\omega^{\alpha+\delta}).$

Hence

Conversely, if
$$\int_{b}^{\omega} \tau^{\alpha} \chi(\tau) d\tau \stackrel{\text{(a)}}{=} o(\omega^{\alpha+\delta})$$
, we take

 $\int_{a}^{\omega} \tau^{\alpha} \chi(\tau) d\tau = o(\omega^{\alpha+\delta}).$

 $\tau^{\alpha}\chi(\tau) = \chi(\tau)$ which is non-negative, continuous and integrable in [h, ω]. The result now follows from the first part by replacing δ by $\alpha+\delta$ and α by $-\alpha$. ///

Since $\lambda_0 = 0$, $R^p_{\lambda}(\mathcal{P}) + a_0$ as $\tau \to 0^+$, we conclude that as a consequence of Lemma 2.1.

(2.1)
$$\int_0^{\omega} |A_{\lambda}^{\mathbf{p}}(\tau) - s\tau^{\mathbf{p}}|^{\mu} d\tau = o(\omega^{\mathbf{p}\mu+1})$$

is equivalent to

$$\int_{0}^{\omega} |\mathbf{R}^{\mathbf{p}}_{\lambda}(\tau) - \mathbf{s}|^{\mu} d\tau = \mathbf{O}(\omega).$$

\$2.2 INCLUSION THEOREM FROM RIESZ TO CESARO THEOREM 2.1 Let $\mu > 0$ and λ satisfy (1.1). Then (i) $[R;\lambda,1]_{\mu} => [C,\lambda,1]_{\mu}$, (ii) If p > 0 and $\lambda_{n+1} = O(\lambda_n)$, then $[R,\lambda,p+1]_{\mu} => [C,\lambda,p+1]_{\mu}$. 16

PROOF
(i) Suppose
$$\sum_{\nu=0}^{\infty} a_{\nu} = s [R, \lambda, 1]_{\mu}$$
 where we may assume,
without loss of generality, that $s = 0$.

$$T_{m}^{1} = \sum_{n=0}^{m} (\lambda_{n+1} - \lambda_{n}) |\int_{\nu=0}^{n} a_{\nu}|^{\mu}$$

$$= \sum_{n=0}^{n} \int_{\lambda_{n}}^{\lambda_{n+1}} |\lambda_{\nu} \zeta_{\tau} a_{\nu}|^{\mu} d\tau$$

$$= \int_{0}^{\lambda_{m+1}} |R_{\lambda}^{0}(\tau)|^{\mu} d\tau$$

$$= o(\lambda_{m+1}), \text{ as } m + \infty.$$
Thus $\sum_{\nu=0}^{\infty} a_{\nu} = 0[C, \lambda, 1]_{\mu}$.
(ii) For the case $p > 0$, we assume that

$$\int_{\nu=0}^{\infty} a_{\nu} = 0[R, \lambda, p+1]_{\mu} \text{ so that}$$

$$\int_{0}^{\omega} |R_{\lambda}^{p}(\tau)|^{\mu} d\tau = o(\omega).$$
We are required to show that

$$\frac{\sigma_{m}^{p+1} = -\frac{1}{m} \sum_{m}^{m} (\lambda_{n+p+1} - \lambda_{n}) \sum_{n}^{p} |\frac{p}{2n}| + \frac{p}{2n}}{\sum_{m}^{p} (1)} = o(1), \text{ as } m^{2} + \infty.$$
We'divide the proof into four steps.
STEP I.
For every n, choose $q = q(n)$, a non-negative integer,
satisfying $q(n) \ge q(n-1)$ and

$$(2.2) \quad \frac{\lambda}{q+1} - \frac{\lambda}{q} = \max\{(\lambda_{i+1} - \lambda_i) \mid n \leq i \leq n+p\}.$$

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Fixing n we partition the interval $[\lambda_q, \lambda_{q+1}]$ into 2p+2 subintervals of length $\frac{\lambda_{q+1} - \lambda_{q}}{2p+2}$ with the points $\omega_{v} = \omega_{n,v} = \lambda_{q} + \frac{v}{2p+2} (\lambda_{q+1} - \lambda_{q}), \quad v = 0, 1, \dots, 2p+2.$ Since p > 0 and $\lambda_0 = 0$, $|R^p_{\lambda}(\tau)|^{\mu}$ is a continuous function of τ in the interval [0, ω]. Applying the Mean yalue Theorem of the alternate subintervals, we have, for $\langle j = 0, 1, 2, ..., p, mumbers$ $\theta_j = \theta_{n,j} \in [\omega_{2j+1}, \omega_{2j+2}]$ such that $\int_{\omega_{2j+1}}^{\omega_{2j+2}} |R_{\lambda}^{p}(\tau)|^{\mu} d\tau = (\omega_{2j+2} - \omega_{2j+1}) |R_{\lambda}^{p}(\theta_{j})|^{\mu}$ $= (\omega_{2j+2} - \omega_{2j+1}) | \sum_{\nu=0}^{q} (1 - \frac{\lambda_{\nu}}{\theta_{j}})^{p} a_{\nu}|^{\mu}.$ $\sum_{\substack{j=0\\j=0}}^{p} (\omega_{2j+2} - \omega_{2j+1}) \left| \sum_{\substack{j=0\\j=0}}^{q} (1 - \frac{\lambda_{j}}{\theta_{j}}) a_{j} \right|^{\mu}$ Thus $= \sum_{j=0}^{p} \int_{\omega_{2j+1}}^{\omega_{2j+2}} |R_{\lambda}^{p}(\tau)|^{\mu} d\tau$ $\leq \int_{\lambda}^{\lambda} q+1 |R_{\lambda}^{p}(\tau)|^{dt} d\tau.$ Since $\omega_{2j+2} - \omega_{2j+1} = \frac{1}{2p+2} (\lambda_{q+1} - \lambda_q)$, we have $(2.3) \sum_{n=0}^{m} \sum_{j=0}^{p} \frac{1}{2p+2} \left(\lambda_{q(n)+1} - \lambda_{q(n)} \right) \left[\sum_{\nu=0}^{q(n)} (1 - \frac{\lambda_{\nu}}{\theta_{n,j}})^{p} a_{\nu} \right]^{\mu}$ $\leq \sum_{n=0}^{m} \int_{\lambda_{q(n)}}^{\lambda_{q(n)}+1} |R_{\lambda}^{p}(\tau)|^{\mu} d\tau$

$\leq (p+1) \int_{-\pi}^{\pi+p+1} |R_{\lambda}^{p}(\tau)|^{\mu} d\tau,$

since q(n) is constant for at most p+1 different values of

STEP II.

n.

Using techniques similar to those used by Borwein [2] \checkmark we shall show that for every n, there are numbers

$$y_j = y_{n,j}$$
, for $j = 0, 1, 2, ..., p_r$,

such that the identity

(2.4)
$$\stackrel{p}{\prod} (x + b_{j}) = \sum_{j=0}^{p} y_{j} (x + \delta_{j})^{p}$$

holds for all real x, where

$$b_i = \frac{\lambda_{n+i} - \lambda_q}{\lambda_{q+1} - \lambda_q}, \quad \text{for } i = 1, 2, \dots, p$$

and •

$$\delta_{j} = \frac{\beta_{j} - \lambda_{q}}{\lambda_{q+1} - \lambda_{q}}, \quad \text{for } j = 0, 1, 2, \dots, p.$$

The identity (2.4) is equivalent to the system of linear equations

(2.5)
$$\sum_{j=0}^{p} \delta_{j}^{i} Y_{j} = \xi_{i}^{*}, \quad i = 0, 1, ...,$$

where

(2.6)
$$\xi_{i} = {\binom{p}{i}}^{-1} \cdot \sum_{b_{r_{1}} \ b_{r_{2}} \ b_{r_{1}} \ c_{2} \ b_{r_{1}}}^{b_{r_{2}} \ b_{r_{1}} \ b_{r_{2}} \ b_{r_{2}}$$

and where the sum in the expression for ξ_i is taken to be 1 when i = 0.

The determinant of the system (2.5) is the Vandermonde

determinant

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$$\Delta = \Pi \quad (\delta_{s} - \delta_{r})$$
$$0 \leq r < s \leq p$$

(See [25, p. 214].)

Now for s > r+

$$\delta_{\mathbf{s}} - \delta_{\mathbf{r}} = \frac{\partial_{\mathbf{s}} - \partial_{\mathbf{r}}}{\partial_{\mathbf{q}+1} - \partial_{\mathbf{q}}}$$

$$= \frac{\lambda_{q+1} - \lambda_{q}}{\lambda_{q+1} - \lambda_{q}}$$

$$= \frac{1}{2p+2} \cdot$$

 $\frac{\omega_{2s+1} - \omega_{2t+2}}{\omega_{2t+2}}$

$$\Delta \geq \frac{1}{(2p+2)^{p!}} > 0.$$

Using Cramer's rule, we have

$$\mathbf{y}_{\mathbf{r}} = \frac{\Delta_{\mathbf{r}}}{\Delta_{\mathbf{r}}},$$

where Δ_r is the determinant of the matrix $(d_{i,j})$, i, j = 0, 1, 2, ..., p, in which

$$d_{i,r} = \xi_i$$
 and $d_{i,j} = \delta_j^i$, $j \neq r$

STEP III.

Hence

We now show that the numbers $y_{n,r}$ are uniformly . bounded. Since

$$|\mathbf{b}_{r}| = |\frac{\lambda_{n+r} - \lambda_{q}}{\lambda_{q+1} - \lambda_{q}}|$$
$$\lambda_{n+p+1} - \lambda_{n}$$

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 $(2.8) \qquad \prod_{i=1}^{p} (1 - \frac{\lambda_{v}}{\lambda_{n+i}}) = \sum_{j=0}^{p} \frac{y_{j} \theta_{n,j}^{p}}{E_{n}^{p}} (1 - \frac{\lambda_{v}}{\theta_{n,j}})^{p}$ $= \sum_{j=0}^{p} C_{n,j} (1 - \frac{\lambda_{v}}{\theta_{n,j}})^{p},$

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where $C_{n,j} = \frac{Y_j \theta_{n,j}^p}{E_n^p}$.

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Since $\lambda_{n+1} = O(\lambda_n)$ and $y_{n,r}$ is uniformly bounded, we have

$$|C_{n,j}| \leq \frac{|Y_{n,j}|^{\lambda_{q+1}^{p}}}{\lambda_{n+1}^{p}} \leq H_{1}$$

 H_1 being independent of n and j.

Now it follows from (2.8) that,

$$t_{n}^{p} = \sum_{\nu=0}^{n} (1 - \frac{\lambda_{\nu}}{\lambda_{n+1}}) \cdots (1 - \frac{\lambda_{\nu}}{\lambda_{n+p}}) a$$

$$= \sum_{\nu=0}^{q} (1 - \frac{\lambda_{\nu}}{\lambda_{n+1}}) \cdots (1 - \frac{\lambda_{\nu}}{\lambda_{n+p}}) a$$

$$= \sum_{\nu=0}^{q} \sum_{j=0}^{p} C_{n,j} (1 - \frac{\lambda_{\nu}}{\theta_{n,j}})^{p} a_{\nu}$$

$$= \sum_{j=0}^{p} C_{n,j} \sum_{\nu=0}^{q} (1 - \frac{\lambda_{\nu}}{\theta_{n,j}})^{p} a_{\nu}.$$

$$\begin{split} p^{p+1}_{m} &= \frac{1}{E_{q}^{p+1}} \sum_{n=0}^{m} (\lambda_{n+p+1} - \lambda_{n}) E_{n}^{p} |t_{n}^{p}|^{\mu} \\ &\leq \frac{\lambda_{m+p+1}^{-1}}{m} \sum_{n=0}^{m} (\lambda_{n+p+1} - \lambda_{n}) \left\{ \sum_{j=0}^{p} |C_{n,j}| \sum_{\nu=0}^{q(n)} (1 - \frac{\lambda_{\nu}}{\theta_{n,j}})^{p} a_{\nu} | \right\}^{\mu} \\ &\leq \frac{\lambda_{m+p+1}^{-1}}{m} \sum_{n=0}^{m} (p+1) (\lambda_{q(n)+1} - \lambda_{q(n)}) (p+1)^{\mu} \sum_{j=0}^{p} |C_{n,j}|^{\mu} |\sum_{\nu=0}^{q(n)} (1 - \frac{\lambda_{\nu}}{\theta_{n,j}})^{p} a_{\nu} |^{\mu} \\ &\leq \frac{H_{2}}{\lambda_{m+p+1}} \sum_{n=0}^{m} (\lambda_{q(n)+1} - \lambda_{q(n)}) \sum_{j=0}^{p} |\sum_{\nu=0}^{q(n)} (1 - \frac{\lambda_{\nu}}{\theta_{n,j}})^{p} a_{\nu} |^{\mu} \\ &= \frac{H_{2} \times (2p+2)}{\lambda_{m+p+1}} \sum_{n=0}^{m} \sum_{j=0}^{p} \frac{1}{2p+2} (\lambda_{q(n)+1} - \lambda_{q(n)}) |\sum_{\nu=0}^{q(n)} (1 - \frac{\lambda_{\nu}}{\theta_{n,j}})^{p} a_{\nu} |^{\mu} \\ &\leq \frac{H_{3}}{\lambda_{m+p+1}} \int_{0}^{\lambda_{m+p+1}} |R_{\lambda}^{p}(\tau)|^{\mu} d\tau. \end{split}$$

The final inequality following from Step I.

Hence if
$$\sum_{\nu=0}^{\infty} a_{\nu} = 0 [R, \lambda, p+1]_{\mu}$$
, then ν
$$\frac{1}{\lambda_{m+p+1}} \int_{0}^{\lambda_{m+p+1}} |R_{\lambda}^{p}(\tau)|^{\mu} d\tau = o(1).$$

Thus $\sigma_m^{p+1} = o(1)$ so that $\sum_{\nu=0}^{\infty} a_{\nu} = 0 [C,\lambda,p+1]_{\mu}$.

\$2.3 INCLUSION THEOREM FROM CESARO TO RIESZ

We now investigate the inclusion in the opposite direction. And to facilitate the discussion we introduce the following notation. Given a function f defined in an interval [a,b], and distinct points x_i in this interval, we define

f[x] = f(x)

and
$$f[x_0, x_1, \dots, x_n] = \frac{f[x_0, \dots, x_{n-1}] - f[x_1, \dots, x_n]}{x_0 - x_n}$$

for n = 1, 2, 3,

The quantity $f[x_0, x_1, \ldots, x_n]$ is called the *divided* difference of f(x) of n arguments. For an exposition of the properties of divided differences see Milne-Thomson [21, Chapter 1].

In the proof of our next theorem we need the following results of Russell [23, pp. 425-428].

LEMMA 2.2

Let p be a non-négative integer.

Define

Then, for $\lambda_n < \tau \leq \lambda_{n+1}$

(**a**) $A_{\lambda}^{p}(\tau) = (-1)^{p+1} \sum_{\nu=n-p}^{n} c_{\tau} [\lambda_{\nu}, \lambda_{\nu+1}, \dots, \lambda_{\nu+p+1}] (\lambda_{\nu+p+1} - \lambda_{\nu}) c_{\nu}^{p}$

 $C_{\tau}(\mathbf{x}) = \begin{cases} (\tau - \mathbf{x})^{\mathbf{p}}, & \text{for } 0 \leq \mathbf{x} < \tau, \\ 0, & \text{for } \mathbf{x} \geq \tau. \end{cases}$

where we understand $C_{\nu}^{p} = 0$ whenever $\nu < 0$; and

(ii) for
$$\mathbf{n} - \mathbf{p} \leq \mathbf{v} \leq \mathbf{n}$$

 $|C_{\tau}[\lambda_{\mathbf{v}}, \lambda_{\mathbf{v}+1}, \dots, \lambda_{\mathbf{v}+\mathbf{p}+1}]|(\lambda_{\mathbf{v}+\mathbf{p}+1} - \lambda_{\mathbf{v}})| \leq \mathbf{n}$

where H is independent of n.

THEOREM 2.2

Let
$$\lambda$$
 satisfy (1.1). Then Q
(i) if $\mu > 0$, then $[C, \lambda, 1]_{\mu} \Rightarrow [R, \lambda, 1]_{\mu}$,
(ii) if $\dot{p} > 0$, $\mu \ge 1$ and $\lambda_{n+1} = O(\lambda_n)$, then
 $[C, \lambda, p+1]_{\mu} \Rightarrow [R, \lambda, p+1]_{\mu}$.

PROOF

(i) We suppose that
$$\sum_{\nu=0}^{\infty} a_{\nu} = 0 [C, \lambda, 1]_{\mu}$$
. Thus

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$$\sigma_{m}^{1} \doteq \frac{1}{\lambda_{m+1}} \sum_{n=0}^{m} (\lambda_{n+1} - \lambda_{n}) |\mathbf{s}_{n}|^{\mu} = o(1).$$

Hence

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(2.9)
$$\frac{\lambda_{m+1} - \lambda_m}{\lambda_{m+1}} |\mathbf{s}_m|^{\mu} = o(1).$$

Let $\omega > 0$ and suppose $\lambda_m < \omega \leq \lambda_{m+1}$. Then

$$\frac{1}{\omega}\int_{0}^{\omega}|A_{\lambda}^{0}(\tau)|^{\mu}d\tau = \frac{1}{\omega}\left\{\sum_{n=0}^{m-1}\int_{\lambda_{n}}^{\lambda_{n+1}}|\sum_{\nu=0}^{n}a_{\nu}|^{\mu}d\tau + \int_{\lambda_{m}}^{\omega}|\sum_{\nu=0}^{m}a_{\nu}|^{\mu}d\tau\right\}$$

$$= \frac{1}{\omega} \sum_{n=0}^{m-1} (\lambda_{n+1} - \lambda_n) |\mathbf{s}_n|^{\mu} + \frac{1}{\omega} (\omega - \lambda_m) |\mathbf{s}_m|^{\mu}$$
$$\leq \sigma_{m-1}^{1} + (1 - \frac{\lambda_m}{\lambda_{m+1}}) |\mathbf{s}_m|^{\mu}.$$

Now $\sigma_m^1 = o(1)$ which together with (2.9) yields

$$\frac{1}{\omega} \int_0^{\omega} |\mathbf{A}_{\lambda}^0(\tau)|^{\mu} d\tau = o(1).$$

Thus $\sum_{\nu=0}^{\infty} a_{\nu} = 0 [R, \lambda, 1]_{\mu}$.

(ii) Let $\tau > 0$ and suppose $\lambda_n < \tau \leq \lambda_{n+1}$.

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Then using Lemma 2.2 (i) and (ii) we see that

$$(2.10) |A_{\lambda}^{p}(\tau)|^{\mu} = \int_{\nu=n-p}^{n} C_{\tau} [\lambda_{\nu}, \lambda_{\nu+1}, \dots, \lambda_{\nu+p+1}] (\lambda_{\nu+p+1} - \lambda_{\nu}) C_{\nu}^{p}|^{\mu}$$
$$\leq (p+1)^{\mu} H \sum_{\nu=n-p}^{n} |C_{\nu}^{p}|^{\mu}.$$

Suppose $\omega > 0$ and $\lambda_m < \omega \leq \lambda_{m+1}$. Then

 $\int_{0}^{\omega} |\mathbf{A}_{\lambda}^{\mathbf{p}}(\tau)|^{\mu} d\tau$

$$\leq \sum_{n=0}^{m} \int_{\lambda_{n}}^{\lambda_{n+1}} |\mathbf{A}_{\lambda}^{\mathbf{p}}(\tau)|^{\mu} d\tau$$

$$\leq H_{1} \sum_{n=0}^{m} (\lambda_{n+1} - \lambda_{n}) \sum_{\nu=n-p}^{n} |C_{\nu}^{\mathbf{p}}|^{\mu}$$

$$= \sum_{n=0}^{m} (\lambda_{n+1} - \lambda_{n}) \sum_{\nu=n-p}^{n} |C_{\nu}^{\mathbf{p}}|^{\mu}$$

$$H_{1} \sum_{n=0}^{m} (\lambda_{n+1} - \lambda_{n}) \sum_{\nu=0}^{p} |c_{n-\nu}^{p}|^{\mu}$$

$$H_{1} \sum_{\nu=0}^{p} \sum_{n=\nu}^{m} (\lambda_{n+1} - \lambda_{n}) |c_{n-\nu}^{p}|^{\mu},$$

so that

(2.11)
$$\int_{0}^{\omega} |A_{\lambda}^{p}(\tau)|^{\mu} d\tau \leq H_{1} \sum_{\nu=0}^{p} \sum_{n=\nu}^{m} (\lambda_{n+1} - \lambda_{n}) |C_{n-\nu}^{p}|^{\mu}.$$

Now

$$\sigma_{m}^{p+1} \stackrel{*}{=} \frac{1}{E_{m}^{p+1}} \sum_{n=0}^{m} (\lambda_{n+p+1} - \lambda_{n}) E_{n}^{p} |\tau_{n}^{p}|^{\mu}$$
$$\stackrel{*}{=} \frac{1}{E_{m}^{p+1}} \sum_{n=0}^{m} (\lambda_{n+p+1} - \lambda_{n}) (E_{n}^{p})^{1-\mu} |c_{n}^{p}|^{\mu}$$

$$\geq \frac{1}{(\mathbf{E}_{m}^{\mathbf{p}})^{\mu}\lambda_{m+\mathbf{p}+1}} \sum_{n=0}^{m} (\lambda_{n+\mathbf{p}+1} - \lambda_{n}) |\mathbf{C}_{n}^{\mathbf{p}}|^{\mu}$$

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since $\mu \geq 1$.

Thus' for r = 0, 1, 2, ..., p

$$\sigma_{m}^{p+1} \geq \frac{1}{\lambda_{m+p+1}^{p\mu+1}} \sum_{n=0}^{m} (\lambda_{n+r+1} - \lambda_{n+r}) |c_{n}^{p}|^{\mu}$$

$$= \frac{1}{\lambda_{m+p+1}^{p\mu+1}} \sum_{\nu=r}^{m+r} (\lambda_{\nu+1} - \lambda_{\nu}) |C_{\nu-r}^{p}|^{\mu}$$

$$\stackrel{\simeq}{=} \frac{1}{\lambda_{m+p+1}^{p\mu+1}} \int_{\nu=r}^{m} (\lambda_{\nu+1} - \lambda_{\nu}) |c_{\nu-r}^{p}|^{\mu}.$$

If we now suppose $\sum_{\nu=0}^{\nu} a_{\nu} = 0 [C, \lambda, p+1]_{\nu}$, so that

 $\sigma_m^{p+1} = o(1)$, we have, for r = 0, 1, 2, ..., p

$$\frac{i}{\lambda_{m+p+1}^{p\mu+1}} \sum_{\nu=r}^{m} (\lambda_{\nu+1} - \lambda_{\nu}) |\mathcal{L}_{\nu-r}^{p}|^{\mu} = o(1),$$

as m → ∞.

Hence in view of (2.11) and the condition $\lambda_{m+1} = O(\lambda_m)$, we have

$$\int_{0}^{\omega} |A_{\lambda}^{p}(\tau)|^{\mu} d\tau = o(\lambda_{m+p+1}^{p\mu+1}) = o(\omega^{p\mu+1}).$$

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Hence $\sum_{\nu=0}^{\infty} a_{\nu} = 0 [R, \lambda, p+1]_{\mu}$ for $\mu \ge 1$.

Combining the results of Theorems 2.1 and 2.2, we have the following corollary.

THEOREM 2.3

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Let
$$\lambda = \{\lambda_n\}$$
 satisfy (1.1).
(i) If $\mu > 0$, then $[\mathbb{R}, \lambda, 1]_{\mu} \iff [\mathbb{C}, \lambda, 1]_{\mu}$.
(ii) If $p > 0$, $\mu \ge 1$ and $\lambda_{n+1} = O(\lambda'_n)$, then
 $[\mathbb{R}, \lambda, p+1]_{\mu} \iff [\mathbb{C}, \lambda, p+1]_{\mu}$.

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CHAPTER 3

SOME EQUIVALENCE THEOREMS

In this chapter we shall establish some equivalence theorems between various methods of Summability and Strong Summability.

§3.1 SOME LEMMAS

LEMMA 3.1

Let Λ_{p+1} be the matrix defined in §1.7. The inverse matrix $\Lambda'_{p+1} = \{\lambda'_{n,v}\}$ of Λ_{p+1} is given by

(3.1)
$$\lambda'_{n,n} = \frac{\lambda_{n+p+1}}{\lambda_{n+p+1} - \lambda_{n}},$$

$$\lambda_{n,n-1} = \frac{n}{\lambda_{n+p+1} - \lambda_{n}},$$
$$\lambda_{n,\nu} = 0 \text{ otherwise.}$$

PROOF

Let
$$C_{m,v} = \sum_{n=v}^{m} \lambda_{m,n} \lambda_{n,v}^{*}$$
, we show that $C_{m,v} = \delta_{m,v}^{*}$.

Referring to the definition of Λ_{p+1} , (1.8), we have for $\nu \neq m$

$$C_{m,v} = \lambda_{m,v}\lambda_{v,v}^{\lambda} + \lambda_{m,v+1}\lambda_{v+1,v}^{\lambda}$$

$$= \frac{1}{E_{m}^{p+1}} \cdot (\lambda_{v+p+1} - \lambda_{v}) E_{v}^{p} \frac{\lambda_{v+p+1}}{\lambda_{v+p+1} - \lambda_{v}}$$

$$- \frac{1}{E_{m}^{p+1}} (\lambda_{v+p+2} - \lambda_{v+1})E_{v+1}^{p} \frac{\lambda_{v+1}}{\lambda_{v+p+2} - \lambda_{v+1}}$$

$$= \frac{1}{E_{m}^{p+1}} (\lambda_{v+1} \cdots \lambda_{v+p} \cdot \lambda_{v+p+1} - \lambda_{v+1} \cdot \lambda_{v+2} \cdots \lambda_{v+p+1})$$

$$= 0,$$
and $c_{m,m} = \lambda_{m,m} \lambda_{m,m}$

$$= \frac{1}{E_{m}^{p+1}} (\lambda_{m+p+1} - \lambda_{m}) E_{m}^{p} \frac{\lambda_{m+p+1}}{\lambda_{m+p+1} - \lambda_{m}}$$

$$= 1.$$

$$+EMMA 3.2$$

$$\Lambda_{p+1} \iff 1 \text{ if and only if $\lim_{n \to \infty} \frac{\lambda_{n+p+1}}{\lambda_{n}} > 1.$
PROOF
$$I \implies \Lambda_{p+1} = 1 \text{ if and only if } \Lambda_{p+1}^{r} \text{ is regular. Referring}$$
to Lemma 3.1, we see that
$$(3.2) \qquad \lim_{n \to \infty} \lambda_{n,v}^{r} = \frac{-\lambda_{n}}{\lambda_{n+p+1} - \lambda_{n}} + \frac{\lambda_{n+p+1}}{\lambda_{n+p+1} - \lambda_{n}} = 1,$$$$

$$\sup_{n} \sum_{\nu=0}^{\infty} |\lambda_{n,\nu}^{\dagger}| \neq \sup_{n} \frac{\lambda_{n+p+1} + \lambda_{n}}{\lambda_{n+p+1} - \lambda_{n}}$$

(3.4)

$$\leq \sup_{n} \frac{2}{1 - \frac{\lambda_{n}}{\lambda_{n+p+1}}}$$

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This supremum is finite if and only if .

 $\lambda_{n+p+1} > 1$

 $\liminf_{n \to \infty} \frac{\lambda_{n+p+1}}{\lambda_n} > 1.$

Consequently,
$$\Lambda_{p+1} \stackrel{\sim}{\stackrel{\sim}{\stackrel{\sim}{\rightarrow}} 1$$
 if and only if

\$3.2 EQUIVALENCE THEOREMS THEOREM 3.°1

 $(C,\lambda,p) \iff (C,\lambda,p+1) \text{ if and only if } \liminf_{n \to \infty} \frac{\lambda_{n+p+1}}{\lambda_{n}} > 1.$

PROOF

By (1.11), we know that $(C,\lambda,p+1) = \Lambda_{p+1}(C,\lambda,p)$. Thus the result now follows from Lemma 3.2. / REMARK: In view of the fact $\{\lambda_n\}$ is an increasing sequence, so that $\liminf_{n \to \infty} \frac{\lambda_{n+p+1}}{\lambda_n} \geq 1$, we see that $\liminf_{n \to \infty} \frac{\lambda_{n+p+1}}{\lambda_n} = 0$

is necessary and sufficient for $(C,\lambda,p+1)$ to include strictly (C,λ,p) .

We now state a result of Borwein and Cass [5, Corollary 2] which yields an equivalence theorem between the methods (C, λ ,p) and [C, λ ,p+1]_µ.

THEOREM 3 🚜

Let $\mu > 0$. Let $\mu > 0$. (i) $p_{n,\nu} \ge 0$, for $n,\nu = 0,1,2,3...$, (ii) $\lim_{n \to \infty} p_{n,\nu} = 0$, for $\nu = 0,1,2,3,...$ 31

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Let
$$Q = \{q_{n,v}\}$$
 be a matrix such that for every
sequence $\{\sigma_v\}$ there is a sequence $\{s_v\}$ for which
 $\sigma_n = \int_{v=0}^{\infty} q_{n,v} q_v$
holds for $n = 0, 1, 2, 3, ...$
Then lim inf max $p_{n,v} = 0$
 $v \to \infty$ $n \ge 0$
is a necessary and sufficient condition for there to be a
sequence which is not Q-convergent, but which is
 $\{P, Q\}_u = convergent, to zerc$.
THEOREM 3.3
Let $u \ge 0$. Then $(C, \lambda, p) <=> [C, \lambda, p+1]_u$ if and only
if $\liminf_{n\to\infty} \frac{\lambda_{n+p+1}}{\lambda_n} > 1$.
PROOF
By Corollary 1.2, we have
 $(C, \lambda, p) = S^{-1}(C, \lambda, p+1]_u$, for $u \ge 0$.
Now $\Lambda_{p+1} = (\lambda_{n,v})$ satisfies
 $\lambda_{n,v} \ge 0$, for $n, v = 0, 1, 2, 3, ...$
and $\lim_{n\to\infty} \lambda_{n,v} = \lambda_{v,v}$, we have
 $\lim_{n\to\infty} \lim_{n\to\infty} \lambda_{n,v} = \lim_{v\to\infty} \lim_{v\to\infty} \lim_{v\to\infty} \lambda_{n,v} = \lim_{v\to\infty} \lim_{v\to\infty} \lim_{v\to\infty} \sum_{v\neq\infty} \lim_{v\to\infty} \lim_{n\to\infty} \lim_{v\to\infty} \lambda_{n,v} = \lim_{v\to\infty} \lim_$

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For the proof of the equivalence theorem between (C, λ ,p+1) and [C, λ ,p+1]_µ, we state another result of Borwein and Cass [5, Theorem 12]. **THEOREM 3.4** Let the matrix $P = \{p_{n,v}\}$ be regular and $p_{n,v}$ for v >'n. If(i) $p_{n,\nu} \geq p_{n+1,\nu}$, for $n \geq \nu$, $\nu = 0, 1, 2, ...$, (ii) $p_{n,n} \rightarrow 0$ (iii) $\sum_{\nu=0}^{n} p_{n,\nu} \leq \sum_{\nu=0}^{n+1} p_{n+1,\nu}, f^{\circ}r n = 0, l_{2}, 2, 3, ...$ then there is a divergent sequence of zeros and ones which is P-convergent to $\frac{1}{2}$, but not $[P,I]_{\mu}$ -convergent for any $\mu \geq 1$. (I denotes the identity matrix μ) THEOREM 3.5 If lim inf $\frac{\lambda_{n+p+1}}{\lambda_{n}} > 1$, then[°] . (i) $(C,\lambda,p+1) <=> [C,\lambda,p+1]_{\mu}, for \mu > 0_{g},$ (ii) If $\lim_{n \to \infty} \frac{\lambda_{n+p+1}}{\lambda_n} = 1$, then $(C, \lambda, p+1)$ strictly includes $[C,\lambda,p+1]_{\mu}$, for $\mu \geq 1$. PROOF Combining results of Theorem, 3.1 and Theorem 3.3 (i) we have $\liminf_{n \to \infty} \frac{\lambda_{n+p+1}}{\lambda_n} > 1$ implies that $(C,\lambda,p+1) \iff [C,\lambda,p+1]_{\mu}, \text{ for } \mu > 0.$ (ii) Since in the matrix λ_{p+1} , $\lambda_{n,n} = (1 - \frac{\lambda_n}{\lambda_{n+p+1}})$

and since the matrix
$$(C, \lambda, p)$$
 has an inverse, Theorem 3.4
shows that if $\lim_{n\to\infty} \frac{\lambda_{n+p+1}}{\lambda_n} = 1$, then there is a divergent
sequence $\{t_n^p\}$ of zeros and ones which is Λ_{p+1} -convergent to
 $\frac{1}{2}$, but not $[\Lambda_{p+1}, 1]_{\mu}$ -convergent for any $\mu \ge 1$. Since
 $\Lambda_{p+1}, \{t_n^p\} = \{t_n^{p+1}\}$ the result follows.
 Me now show that in Theorem 3.5 (ii) the condition
 $\lim_{n\to\infty} \frac{\lambda_{n+p+1}}{\lambda_n} = 1$ can not be replaced by $\lim_{n\to\infty} \inf \frac{\Lambda_{n+p+1}}{\Lambda_n} = 1$.
Let $P_0 > 0$ and $P_n \ge 0$, we say that
 $s_n + s (\overline{N}, P_n)$
if $\mu_n = \frac{1}{P_n} \sum_{\nu=0}^n P_{\nu} s_{\nu} + s$, where $P_n = \frac{n}{\nu_{20}} P_{\nu}$.
REMARK: (i) Λ_{p+1} is the method (\overline{N}, P_n) with $P_n = E_n^{p+1}$.
(ii) If (\overline{N}, P_n) is taken as P in Theorem 3.4, it
satisfies conditions (i) and (iii) of Theorem 3.4.
We shall now construct an (\overline{N}, P_n) method with
 $\lim_{n\to\infty} \frac{p_n}{P_n} = 0$ and with $[(\overline{N}, P_n), n]_1 <> (\overline{N}, P_n)$.
Let $C_n = \frac{P_n}{P_n}$, $0 < C_n < 1$, for $n \ge 1$.
Then $\mu_n P_n - \mu_{n-1} (1 - C_n) = c_n s_n$.
Now take $C_{2n} = I - \frac{1}{(n+1)^2}$, for $n \ge 1$.

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(3.5)
$$\lim_{n\to\infty} \inf \frac{P_n}{P_n} = 0,$$
(3.6)
$$u_{2n} - \frac{u_{2n-1}}{(n+1)^2} = (1 - \frac{1}{(n+1)^2}) s_{2n},$$
(3.7)
$$u_{2n+1} - u_{2n}(1 - \frac{1}{n+2}) = \frac{s_{2n+1}}{n+2}.$$
(consequently if $u_n + i$, then (3.6) and (3.7) give $s_{2n} + i$ and $s_{2n+1} = o(n)$.
On the other hand if $s_{2n} + i$ and $s_{2n+1} = o(n)$ if then $|\frac{u_n}{n+1}| \le \frac{1}{P_n} \int_{v=0}^{n} p_v \frac{|s_v|}{v+1} \le i$
wnd (3.6) and (3.7) imply that $i_n + i$.
Summarizing we have
 $s_n + s (\overline{N}, p_n)$ if and only if.
 $s_{2n} + s_{2n+1} = o(n).$
Thus $\langle \overline{N}, p_n \rangle$ is regular, not equivalent to convergence and $(\overline{N}, p_n) < s \ge ((\overline{N}, p_n), r_1)$.
Let $\lambda_0 = 0, \lambda_{n+1} = P_n$ for $n \ge 0$.
Then $\lambda_0 + \infty$, because (\overline{N}, p_n) is regular.
 $\lambda_1 = (\overline{N}, p_n),$
 $s_1 = i \min \frac{1}{h} \frac{h+1}{h} = 1$
and $[C, \lambda, 1]_T = (C, \lambda; 1)$.

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Combining the last example with Theorem 2.3 (i), we find that it is possible to have

 $\lim_{n \to \infty} \inf \frac{\lambda_{n+1}}{\lambda_n} = 1$

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and $[R, \lambda, 1]_1 \iff (R, \lambda, 1)$.

CHAPTER 4

ABSOLUTE GENERALISED CESÀRO SUMMABILITY

§4.1 DEFINITIONS

In this chapter we study the absolute methods of summability $|C_{\lambda},p|_{u}$ and $|R,\lambda,p|$.

Let $t_{\hat{n}}^{p}$ be defined as in §1.5 and $\mu > 0$. We define

 $\sum_{\nu=0}^{\Sigma} a_{\nu}$ to be summable $|C,\lambda,p|_{\mu}$ if

(4.1)
$$\sum_{n=1}^{\infty} \left(\frac{\lambda_n}{\lambda_{n+p+1} - \lambda_n} \right)^{\mu-1} |\mathbf{t}_n^p - \mathbf{t}_{n-1}^p|^{\mu} < \infty.$$

In §1.2, we defined $\sum_{\nu=0}^{\nu}$ a to be summable $|Q|_{\mu}$,

µ~> 0, if

(4.2)
$$\sum_{n=1}^{\infty} n^{\mu-1} |\sigma_n - \sigma_{n-1}|^{\mu} < \delta$$

where $\{\sigma_n\} = Q\{s_n\}$.

When $\mu = 1$, conditions (4.1) and (4.2) are equivalent. When $\mu \neq 1$, they may or may not differ.

For example, if
$$\lambda_n = n^{\alpha}$$
, $\alpha' > 0$, then

$$\frac{\lambda_n}{\lambda_{n+p+1} - \lambda_n} = \frac{n^{\alpha}}{(n+p+1)^{\alpha} - n^{\alpha}} = \frac{n^{\alpha}}{\alpha \theta_n^{\alpha-1}(p+1)}$$

where $n < \theta_n < n+p+1$.

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Let
$$\rho_n = \frac{\lambda_n}{n+p+1} - \lambda_n$$
.
Then $\frac{\rho_n}{n} + \frac{1}{a(p+1)}$, as $n \neq \infty$. So in this.case,
 $\sum_{n=1}^{\infty} \rho_n^{u-1} |t_n^p - t_{n-1}^p| < \infty$ if and only if
 $\frac{1}{n-1} n^{u-1} |t_n^p - t_{n-1}^p| < \infty$,
and the two conditions (4.1) and (4.2) are equivalent in
this case.
On the other hand, if $\lambda_n = \log(n+1)$, then
 $\rho_n = \frac{\log(n+1)}{\log(n+p+2) - \log(n+1)} = \frac{\theta_n \log(n+1)}{p+1}$,
where $n+1 < \theta_n < n+p+2$.
In this case $\frac{\rho_n}{n \log n} + \frac{1}{p+1}$, as $n + \frac{q}{p} = \frac{1}{p+1} = \frac{\theta_n}{p+1}$,
 $\sum_{n=1}^{\infty} \frac{\rho_n^{n-1}}{n} |t_n^p - t_{n-1}^p| < \infty$ if and only if
 $\sum_{n=1}^{\infty} \frac{\rho_n^{n-1}}{n} |t_n^p - t_{n-1}^p| < \infty$.
Let $\alpha_n = (t_n^p - t_{n-1}^p)$ and $\nu = 2$.
If we take $\alpha_n = \frac{1}{n \log n}$, then
 $\sum_{n=2}^{\infty} \frac{1}{n (\log n)^2} < \infty$.

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while
$$\sum_{n=2}^{\infty} n^{\mu-1} \log^{\mu-1} n |\alpha_n|^{\mu}$$

$$= \sum_{n=2}^{\infty} n \log n \left| \frac{1}{n \log n} \right|^2$$

$$= \sum_{n=2}^{\infty} \frac{1}{n \log n} = \infty.$$
This shows that the two conditions (4.1) and (4.2) are different in this case.

It is more natural to use condition (4.1) rather than condition (4.2) to define $|C,\lambda,p|_{\mu}$ summability. Thus

for the remainder of this chapter $\sum_{\nu=0}^{\nu} a_{\nu}$ is summable $\nu=0$

 $|C,\lambda,p|_{\mu}$ means condition (4.1) is satisfied.

We now give an example which shows that there are sequences λ for which $|C,\lambda,p|_{\mu} \neq \langle (C,\lambda,p)$. Let $\mu = 2$, $\lambda_n = \log(n+1)$ and

$$\alpha_n = t_n^p - t_{n-1}^p = \frac{1}{n \log n \log \log n}$$

Then

$$\sum_{n=2}^{n^{\mu-1}} \log^{\mu-1} n \Big| \frac{1}{n \log n \log \log n} \Big|^{\mu}$$

$$= \sum_{n=2}^{\infty} n \log n \frac{1}{n^2 (\log n)^2 (\log \log n)^2}$$
$$= \sum_{n=2}^{\infty} \frac{1}{n \log n (\log \log n)^2} < \infty.$$

But $\lim_{n \to \infty} t_n^p = t_1^p + \sum_{n=2}^{\infty} \alpha_n = t_1^p + \sum_{n=2}^{\infty} \frac{1}{n \log n \log \log n}$

 $\sum_{\nu=0}^{\infty} a_{\nu}$ is summable $|C, \lambda, \mu|_{1}$ means that

 $\sum_{n=1}^{\infty} |t_n^p - t_{n-1}^p| < \infty \text{ so that } \{t_n^p\} \text{ is convergent to s say. This means that } \sum_{\nu=0}^{\infty} a_{\nu} = s(C, \lambda, p). \text{ Hence we write and we have.}$

 $|C, \lambda, p|_1 = \langle (C, \lambda, p) \rangle$

Let $\mathbb{R}^{p}_{\lambda}(\tau)$ be defined as in §1.3. Then we say $\sum_{\nu=0}^{\infty} a_{\nu}$

is $|R, \lambda, p|$ summable, if

 $\mathbb{R}^{p}_{\lambda}(\tau) \rightarrow s \text{ as } \tau \rightarrow \infty,$

 $\int_{h}^{\infty} \left| dR_{\lambda}^{p}(\tau) \right| = \int_{h}^{\infty} \left| \frac{d}{d\tau} R_{\lambda}^{p}(\tau) \right| d\tau < \infty,$

and

where $h \ge \lambda_0$. (See Obrechkoff: Sur la sommation absolue des sèries de Dirichlet. C.R. 186, 1928.) We denote this by

$$\sum_{j=0}^{\infty} \mathbf{a}_{j} = \mathbf{s} | \mathbf{R}, \lambda, \mathbf{p} |.$$

§4.2 INCLUSION THEOREMS

The next lemma is a special case of a result due to Mears, [19, Theorem 1].

Let $Q = \{q_{n,v}\}$ be a regular matrix with $q_{n,v} = 0$ for

v > n. If $\sigma_n = \sum_{\nu=0}^n q_{n,\nu} s_{\nu}$, where $s_{\nu} = \sum_{\mu=0}^{\nu} a_{\mu}$, then a necessary

and sufficient condition for

$$\begin{split} & \sum_{n=1}^{\infty} |\sigma_n - \sigma_{n-1}| < \infty \\ & \sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty \\ & \sum_{n=1}^{\infty} |\sum_{n=1}^{n-1} |\alpha_n - \alpha_{n-1,n}| + \alpha_{n,n}| \le H \\ & (4.3) \qquad \sum_{n=k}^{\infty} |\sum_{\nu=k}^{n-1} (\alpha_{n,\nu} - \alpha_{n-1,\nu}) + \alpha_{n,n}| \le H \\ & \text{where H is independent of k.} \\ & \text{THEOREM 4.1} \\ & \text{For any non-negative integer p,} \\ & \sum_{\nu=0}^{\infty} \alpha_\nu = s |C,\lambda,p+1| |_{1} \text{ whenever } \sum_{\nu=0}^{\infty} \alpha_\nu = s |C,\lambda,p| |_{1} \\ & \text{PROOF} \\ & \text{We know that } (C,\lambda,p+1) = \Lambda_{p+k}(C,\lambda,p) \text{ where } \Lambda_{p+1} \text{ is } \\ & \text{defined in $1.7. By Lemma 4.1 if suffices to prove that} \\ & \sum_{n=k}^{\infty} \sum_{\nu=k}^{n-1} (\lambda_{n,\nu} - \lambda_{n-1,\nu}) + \lambda_{n,n}| \le H \\ & \text{where H is independent of k.} \\ & \text{Now, referring to (1.8)} \\ & \sum_{n=k}^{\infty} \sum_{\nu=k}^{n-1} (\lambda_{n,\nu} - \lambda_{n-1,\nu}) + \lambda_{n,n}| \\ & = \lambda_{k,k} + \sum_{n=k+1}^{\infty} (\sum_{\nu=k}^{n-1} (\lambda_{n,\nu} - \lambda_{n-1,\nu}) + \lambda_{n,n}| \\ & = \lambda_{k,k} + \sum_{n=k+1}^{\infty} (\sum_{\nu=k}^{n-1} (\lambda_{n,\nu} - \lambda_{n-1,\nu}) + \sum_{n=1}^{\infty} (\sum_{\nu=k}^{n-1} - \sum_{\nu=k}^{\infty} \sum_{\nu=k}^{n-1} \sum_{\nu=k}^{\infty} \sum_{\nu=k}^{n-1} \sum_{\nu=k}^{n-1} \sum_{\nu=k}^{\infty} \sum_{\nu=k}^{n-1} \sum_{\nu=k}^{\infty} \sum_{\nu=k}^{n-1} \sum_{\nu=k}^{\infty} \sum_{\nu=k}^{\infty} \sum_{\nu=k}^{n-1} \sum_{\nu=k}^{\infty} \sum_{\nu=k}^$$

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$$\frac{42}{\sum_{k=1}^{n} \left\{ \frac{1}{2} \sum_{k=1}^{n-1} \frac{(\lambda_{y+p+1} - \lambda_{y}) E_{y}^{p}}{\sum_{k=1}^{n-1} \frac{1}{\lambda_{n+p+1}} - \frac{1}{\lambda_{n}} \right\}} + \frac{\lambda_{n+p+1} - \lambda_{n}}{\lambda_{n+p+1}} \\
= \frac{\lambda_{k,k} + \sum_{n=k+1}^{\infty} \left| \left\{ \frac{1}{\lambda_{n+p+1}} - \frac{1}{\lambda_{n}} \right\} \left(\sum_{k=1}^{p-1} \frac{E_{k}^{p+1}}{E_{n}^{p}} - \sum_{k=1}^{n-1} \frac{E_{k+1}^{p+1}}{E_{k-1}^{p}} \right) \\
+ \frac{\lambda_{n+p+1} - \lambda_{n}}{\lambda_{n+p+1}} \\
= \frac{\lambda_{k,k} + \sum_{n=k+1}^{\infty} \left| \left\{ \frac{1}{\lambda_{n+p+1}} - \frac{1}{\lambda_{n}} \right\} \left(\sum_{k=1}^{p+1} - \frac{E_{k-1}^{p+1}}{E_{n}^{p}} + \frac{E_{k-1}^{p+1}}{\lambda_{n+p+1}} \right) \\
= \frac{\lambda_{k,k} + \sum_{n=k+1}^{\infty} \left| \frac{\lambda_{n}}{\lambda_{n+p+1}} - 1 - \frac{E_{k-1}^{p+1}}{E_{k-1}^{p+1}} + \frac{E_{k-1}^{p+1}}{E_{k-1}^{p+1}} + 1 - \frac{\lambda_{n}}{\lambda_{n+p+1}} \right| \\
= \frac{\lambda_{k,k} + E_{k-1}^{p+1}}{n} \sum_{n=k+1}^{\infty} \left| \frac{E_{k-1}^{p+1}}{E_{n}^{p+1}} + \frac{E_{k-1}^{p+1}}{E_{n-1}^{p+1}} + \frac{E_{k-1}^{p+1}}{E_{k-1}^{p+1}} \right| \\
= \frac{\lambda_{k,k} + E_{k-1}^{p+1}}{E_{k}^{p+1}} \sum_{n=k+1}^{\infty} \left| \frac{E_{k-1}^{p+1}}{E_{k}^{p+1}} + \frac{E_{k-1}^{p+1}}{E_{k-1}^{p+1}} \right| \\
= \frac{\lambda_{k,k} + E_{k-1}^{p+1}}{E_{k}^{p+1}} \sum_{n=k+1}^{\infty} \left| \frac{E_{k-1}^{p+1}}{E_{k}^{p+1}} + \frac{E_{k-1}^{p+1}}{E_{k}^{p+1}} \right| \\
= \frac{\lambda_{k,k} + E_{k-1}^{p+1}}{E_{k}^{p+1}} \sum_{n=k+1}^{\infty} \left| \frac{E_{k-1}^{p+1}}{E_{k}^{p+1}} + \frac{E_{k-1}^{p+1}}{E_{k}^{p+1}} \right| \\
= \frac{\lambda_{k,k} + E_{k-1}^{p+1}}{E_{k}^{p+1}} \sum_{n=k+1}^{\infty} \left| \frac{E_{k-1}^{p+1}}{E_{k}^{p+1}} + \frac{E_{k-1}^{p+1}}{E_{k}^{p+1}} \right| \\
= \frac{\lambda_{k,k} + E_{k-1}^{p+1}}{E_{k}^{p+1}} \sum_{n=k+1}^{\infty} \left| \frac{E_{k-1}^{p+1}}{E_{k}^{p+1}} + \frac{E_{k-1}^{p+1}}{E_{k}^{p+1}} \right| \\
= \frac{\lambda_{k,k} + E_{k-1}^{p+1}}{E_{k}^{p+1}} \sum_{n=k+1}^{\infty} \left| \frac{E_{k-1}^{p+1}}{E_{k}^{p+1}} + \frac{E_{k-1}^{p+1}}{E_{k}^{p+1}} \right| \\
= \frac{\lambda_{k-1}} + \frac{E_{k-1}^{p+1}}{E_{k}^{p+1}} \sum_{n=k+1}^{\infty} \left| \frac{E_{k-1}^{p+1}}{E_{k}^{p+1}} + \frac{E_{k-1}^{p+1}}{E_{k}^{p+1}} \right| \\
= 1.$$
Thus $\sum_{n=1}^{\infty} \left| \frac{E_{k-1}}{E_{k}^{p+1}} + \frac{E_{k-1}^{p+1}}{E_{k}^{p+1}} + \frac{E_{k-1}^{p+1}}{E_{k}^{p+1}} - \frac{E_{k-1}^{p+1}}{E_{k}^{p+1}} \right| \\
= 1.$

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Since
$$(C,\lambda;p) \Rightarrow (C,\lambda,p+1)$$
, $t_{n}^{p} + s$ implies $t_{n}^{p+1} + s$.
Consequently $\int_{v=0}^{\infty} a_{v} = s|C,\lambda,p+1|_{1}$ whenever $\int_{v=0}^{\infty} a_{v} = s|C,\lambda,p|_{1} / / /$
COROLLARY 4.1
 $\int_{v=0}^{\infty} a_{v} = s|C,\lambda,p|_{1}$ for $p \ge 1$, whenever $\int_{v=0}^{\infty} |a_{v}| < \infty$,
where $s = \int_{v=0}^{\infty} a_{v}$.
PROOF
Take $p = 0$ in Theorem 4.1 and proceed by induction. ///
THEOREM 4.2
 $\int_{v=0}^{\infty} a_{v} = s|C,\lambda,p|_{1}$ whenever $\int_{v=0}^{\infty} a_{v} = s|C,\lambda,p+1|_{1}$.
if and only if $\liminf_{n \neq \infty} \frac{\lambda_{n+p+1}}{\lambda_{n}} \ge 1$.
PROOF
 $(C,\lambda,p) = \Lambda_{p+1}(C,\lambda,p+1)$.
Referring to Lemma 3.1, we know $\inf_{n} \Lambda_{p+p} = \{\lambda_{n+v}^{p+1}\}$.
 $\lambda_{n,n-1}^{i} = \frac{-\lambda_{n}}{\lambda_{n+p+1} - \lambda_{n}} = 1 - \lambda_{n,n}^{i}$.
 $\lambda_{n,v}^{i} = 0$, otherwise.
By Lemma 3.2, we know Λ_{p+1}^{i} is regular if and only if
 $\lim_{n \to \infty} \frac{\lambda_{n+p+1}}{\lambda_{n}} > 1$.

Now
$$\sum_{n=k}^{\infty} |\sum_{v=k}^{n-1} (\lambda_{n,v}^{*} - \lambda_{n-\frac{1}{2},v}^{*}) + \lambda_{n,n}^{*}|$$

$$= \lambda_{k,k}^{*} + |\lambda_{k+1,k+1}^{*} + \lambda_{k+1,k}^{*} - \lambda_{k,k}^{*}| + \sum_{n=k+2}^{\infty} |\sum_{v=k}^{n-1} (\lambda_{n,\frac{1}{2}}^{*} - \lambda_{n-1,v}^{*})$$

$$+ \lambda_{n,n}^{*}|$$

$$= \lambda_{k+p+1}^{*} - \lambda_{k}^{*} + |1 - \lambda_{k,k}^{*}| + \sum_{n=k+2}^{\infty} |\lambda_{n,n}^{*} + \lambda_{n,n-1}^{*} - \lambda_{n-1,n-1}^{*}|$$

$$- \lambda_{n-1,n-2}^{*}|$$

$$= \frac{\lambda_{k+p+1} - \lambda_{k}}{\lambda_{k+p+1} - \lambda_{k}} + \sum_{n=k+2}^{\infty} |1 - 1|$$

$$= \frac{\lambda_{k+p+1} - \lambda_{k}}{\lambda_{k}} + \sum_{n=k+2}^{\infty} |1 - 1|$$
Thus it follows Lemma 4.1 that
$$\int_{v=0}^{\infty} a_{v} = s |C_{r}\lambda_{r}p|_{1} \text{ whenever } \int_{v=0}^{\infty} a_{v} = s |C_{r}\lambda_{r}p+T|_{1}$$
if and only if $\lim_{n \to \infty} \inf_{\lambda_{n}}^{\lambda_{n+p+1}} > 1.$

$$Mörile proved in (17) that $|R_{r}\lambda_{r}p| \ll hight expression (C_{r}\lambda_{r}p+1) |f end$

$$corollary.
COROLLARY 4.2
$$\int_{v=0}^{\infty} a_{v} = s |R_{r}\lambda_{r}p| = whenever \int_{v=0}^{\infty} a_{v} = s |R_{r}\lambda_{r}p+1| |f end$$

$$onty |f lim inf \frac{\lambda_{n+p+1}}{\lambda_{n}} > 1.$$$$$$

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We now turn our attention to the relationship between $|C,\lambda,p+1|_{\mu}$ and $[C,\lambda,p+1]_{\mu}$. To facilitate the discussion we use a result of Borwein, [1, Theorem 7], which we state as the next lemma. We include the proof for the sake of completeness.

LEMMA 4.2

If P is a regular matrix with non-negative entries, Q is a matrix and $\mu \ge 1$, then necessary and sufficient conditions for a series to be summable $[P,Q]_{\mu}$ to s are that it be PQ-summable to s and $[P,(I-P)Q]_{\mu}$ -summable to zero. PROOF

Let $\{\sigma_n\} = Q\{s_n\}$ and $\{\tau_n\} = P\{\sigma_n\}$. We have to prove

a)
$$\sum_{r=0}^{n} p_{n,r} |\sigma_r - s|^{\mu} = o(1)$$

if and only if

that

and

and

(b) $\tau_n \rightarrow s$

(c)
$$\sum_{r=0}^{\infty} p_{n,r} |\sigma_r - \tau_r|^{\mu} = o(1)$$
.

(i) Suppose that (a) holds. Then by Theorem 1.2

(ii), (b) holds and so $\sum_{r=0}^{\infty} p_{n,r} |\tau_r - s|^{\mu} = o(1)$ since P is

regular. Hence by Minkowski's inequality and (a)

$$\left\{ \sum_{r=0}^{\infty} p_{n,r} |\sigma_{r} - \tau_{r}|^{\mu} \right\}^{1/\mu}$$

$$\leq \left\{ \sum_{r=0}^{\infty} p_{n,r} |\sigma_{r} - s|^{\mu} \right\}^{1/\mu} + \left\{ \sum_{r=0}^{\infty} p_{n,r} |\tau_{r} - s|^{\mu} \right\}^{1/\mu} = o(1)$$
(a) Eallows

(ii) Suppose that (b) and (c) hold. Since P is
regular, it follows from (b) that

$$\sum_{r=0}^{\infty} p_{n,r} |\tau_{r} - s|^{\mu} = o(1),$$
Hence by Minkowski's inequality and (c),
 $\left\{\sum_{r=0}^{\infty} p_{n,r} |\sigma_{r} - s|^{\mu}\right\}^{1/\mu}$
 $\leq \left\{\sum_{r=0}^{\infty} p_{n,r} |\sigma_{r} - \tau_{r}|^{\mu}\right\}^{1/\mu} + \left\{\sum_{r=0}^{\infty} p_{n,r} |\tau_{r} - s|^{\mu}\right\}^{1/\mu} = o(1),$
so that (a) holds.
The proof As thus complete.
THEOREM 4.3
Let $\mu \geq 1$. From
 $\sum_{v=0}^{\infty} a_{v} = s[C,\lambda, p\pm 1]_{\mu}$ if and only if
(4.4)
 $\sum_{v=0}^{\infty} a_{v} = s[C,\lambda, p\pm 1]_{\mu}$ if $p_{n} - sp^{\mu+1}|^{\mu} = o(1).$
Condition (4.5) means $|t_{p}^{\mu} - t_{n}^{\mu+1}|^{\mu} + 0(\Lambda_{p+1}).$
PROOF
In Lemma 4.2, take $P = \Lambda_{p+1}, 0 = (C,\lambda, p\pm 1)_{r}$.
THEOREM 4.4
 $\sum_{v=0}^{\infty} a_{v} = s[C,\lambda, p\pm 1]_{1}$ implies $\sum_{v=0}^{\infty} a_{v} = s[C;\lambda, p\pm 1]_{1}$.
THEOREM 4.4
 $\sum_{v=0}^{\infty} a_{v} = s[C,\lambda, p\pm 1]_{1}$ implies $\sum_{v=0}^{\infty} a_{v} = s[C;\lambda, p\pm 1]_{1}$.

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PROOF
Since
$$\sum_{n=1}^{\infty} |t_n^{p+1} - t_{n-1}^{p+1}| < \infty$$
 implies that t_n^{p+1} tends
to a limit, s say, we have $\sum_{\nu=0}^{\infty} a_{\nu} = s(C,\lambda,p+1)$. Hence to
prove the theorem it suffices to show condition (4.5) is
satisfied with $\mu = 1$.
Let $n \ge 1$.
 $|t_n^p - t_n^{p+1}|$
 $= |\frac{1}{E_n^p} \sum_{\nu=0}^n (\lambda_{n+1} - \lambda_{\nu}) \cdots (\lambda_{n+p} - \lambda_{\nu}) a_{\nu}$
 $- \frac{1}{E_n^{p+1}} \sum_{\nu=0}^n (\lambda_{n+1} - \lambda_{\nu}) \cdots (\lambda_{n+p+1} - \lambda_{\nu}) a_{\nu}^{o}|^{o}$
 $= |\frac{1}{E_n^{p+1}} \sum_{\nu=0}^n (\lambda_{n+1} - \lambda_{\nu}) \cdots (\lambda_{n+p-1} - \lambda_{\nu}) a_{\nu}|^{o}$
 $= |\frac{1}{E_n^{p+1}} \sum_{\nu=0}^n (\lambda_{n+1} - \lambda_{\nu}) \cdots (\lambda_{n+p} - \lambda_{\nu}) a_{\nu}|^{o}$
 $= |\frac{1}{E_n^{p+1}} \sum_{\nu=0}^n (\lambda_{n+1} - \lambda_{\nu}) \cdots (\lambda_{n+p} - \lambda_{\nu}) (x_{n+p+1} - \lambda_{n+p+1} + \lambda_{\nu}) a_{\nu}|^{o}$
 $= |\frac{1}{E_n^{p+1}} \sum_{\nu=0}^n (\lambda_{n+1} - \lambda_{\nu}) \cdots (\lambda_{n+p} - \lambda_{\nu}) \lambda_{\nu} a_{\nu}|^{o}$.
On the other hand
 $|t_n^{p+1} - t_{n-1}^{p+1}|$
 $= |\frac{1}{E_n^{p+1}} \sum_{\nu=0}^n (\lambda_n - \lambda_{\nu}) \cdots (\lambda_{n+p} - \lambda_{\nu}) a_{\nu}|^{o}$.

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$$\begin{split} & \sum_{n=1}^{n} \left(\lambda_{n+p+1} - \lambda_{n} \right) \cdots \left(\lambda_{n+p+1} - \lambda_{n} \right) a_{n} \\ & = \left| \frac{\lambda_{n}}{E_{n-1}^{p+2}} \prod_{\nu=0}^{n} (\lambda_{n} - \lambda_{\nu}) \cdots (\lambda_{n+p} - \lambda_{\nu}) a_{\nu} \right| \\ & = \left| \frac{1}{E_{n-1}^{p+2}} \prod_{\nu=0}^{n} (\lambda_{n+1} - \lambda_{\nu}) \cdots (\lambda_{n+p} - \lambda_{\nu}) \left\{ \lambda_{n} (\lambda_{n+p+1} - \lambda_{\nu}) - \lambda_{n+p+1} (\lambda_{n} - \lambda_{\nu}) \right\} a_{\nu} \right| \\ & = \left| \frac{1}{E_{n-1}^{p+2}} \prod_{\nu=0}^{n} (\lambda_{n+1} - \lambda_{\nu}) \cdots (\lambda_{n+p} - \lambda_{\nu}) (\lambda_{n+p+1} - \lambda_{n})^{\lambda} \phi a_{\nu} \right| \\ & = \left| \frac{1}{E_{n-1}^{p+2}} \prod_{\nu=0}^{n} (\lambda_{n+1} - \lambda_{\nu}) \cdots (\lambda_{n+p} - \lambda_{\nu}) (\lambda_{n+p+1} - \lambda_{n})^{\lambda} \phi a_{\nu} \right| \\ & = \frac{\lambda_{n+p+1} - \lambda_{n}}{\lambda_{n}} \left| \frac{1}{E_{n}^{p+1}} \prod_{\nu=0}^{n} (\lambda_{n+1} - \lambda_{\nu}) \cdots (\lambda_{n+p} - \lambda_{\nu}) \lambda_{\nu} a_{\nu} \right|, \\ & \text{for } n \geq 1. \\ \text{Hence} \\ & (4.6) \quad \left| t_{n}^{p+1} - t_{n-1}^{p+1} \right| = \frac{\lambda_{n+p+1} - \lambda_{n}}{\lambda_{n}} \left| t_{n}^{p} - t_{n}^{p+1} \right|, \text{ for } n \geq 1. \\ \text{Consequently multiplying } (4.6) \text{ by } E_{n-1}^{p+1}, \text{ we obtain} \\ & \int_{n=1}^{m} (\lambda_{n+p+1} - \lambda_{n}) E_{n}^{p} \left| t_{n}^{p} - t_{n}^{p+1} \right| \\ & = \int_{n=1}^{m} E_{n-1}^{p+1} \left| t_{n}^{p+1} - t_{n-1}^{p+1} \right|. \\ \text{Since } \lambda_{0} = 0, \quad \left| t_{0}^{p} - t_{0}^{p+1} \right| = \left| a_{0} - a_{0} \right| = 0. \\ \text{By taking } E_{n-1}^{p+1} \left| t_{n}^{p+1} - t_{n-1}^{p+1} \right| \\ & = \int_{n=0}^{m} \sum_{n=1}^{m} E_{n-1}^{p+1} \left| t_{n}^{p+1} - t_{n-1}^{p+1} \right| \\ & = \int_{n=0}^{m} \sum_{n=1}^{p+1} \left| t_{n}^{p+1} - t_{n-1}^{p+1} \right| \end{aligned}$$

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Let
$$\mathbf{b}_{\mathbf{r}} = \begin{vmatrix} \mathbf{t}_{\mathbf{r}}^{\mathbf{p+1}} - \mathbf{t}_{\mathbf{r-1}}^{\mathbf{p+1}} \end{vmatrix}$$
 and $\mathbf{B}_{\mathbf{n}} = \sum_{\mathbf{r}=0}^{\mathbf{n}} \mathbf{b}_{\mathbf{r}}$

Then from (4.7), we have

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$$\sum_{n=0}^{m} E_{n-1}^{p+1} b_{n}$$

$$= \sum_{n=0}^{m} E_{n-1}^{p+1} (B_{n} - B_{n-1})$$

$$= B_{m} E_{m}^{p+1} - \sum_{n=0}^{m} B_{n} (E_{n}^{p+1} - E_{n-1}^{p+1}).$$
Dividing by E_{m}^{p+1} , we obtain
$$B_{m} - \frac{i}{E_{m}^{p+1}} \sum_{n=0}^{m} (\lambda_{n+p+1} - \lambda_{n}) E_{n}^{p} B_{n} = o(1), \text{ as } m + \infty,$$
because of the regularity of Λ_{p+1} and the hypothesis
$$\sum_{\nu=0}^{\infty} a_{\nu} = s |C, \lambda, p+1|_{1} \text{ which means that } \{B_{n}\} \text{ is convergent.}$$
Thus the condition (4.5) is satisfied and the
theorem is proved.
(C.f. Borwein and Cass [6, Theorem 9].)
THEOREM 4.5
$$If \sum_{\nu=0}^{\infty} a_{\nu} = s (C, \lambda, p+1|_{\mu} \text{ implies that } \sum_{\nu=0}^{\infty} a_{\nu} = s [C, \lambda, p+1]_{\mu}$$
PROOF

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condition (4.5) is satisfied with μ > L

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Now referring to (4.6), we have

$$\left| \mathbf{t}_{n}^{p+1} - \mathbf{t}_{n-1}^{p+1} \right|^{\mu} = \left(\frac{\lambda_{n+p+1} - \lambda_{n}}{\lambda_{n}} \right)^{\mu} \left| \mathbf{t}_{n}^{p} - \mathbf{t}_{n}^{p+1} \right|^{\mu}$$

for $\mu > 1$ and $n \ge 1$. Thus

(4.8)
$$\left| t_{n}^{p} - t_{n}^{p+1} \right|^{\mu} = \left(\frac{\lambda_{n}}{\lambda_{n+p+1} - \lambda_{n}} \right)^{\mu} \left| t_{n}^{p+1} - t_{n-1}^{p+1} \right|^{\mu}$$

for $\mu > 1$, and $n \geq 1$.

Since
$$\left| t_{0}^{p} - t_{0}^{p+1} \right| = 0$$
 and $E_{-1^{n}}^{p+1} = 0$ and $t_{-1}^{p+1} = 0$,

we have, by (4.8),

$$\sum_{n=0}^{m} (\lambda_{n+p+1} - \lambda_n) E_n^p \left| t_n^p - t_n^{p+1} \right|^{\mu}$$

$$= \sum_{n=0}^{m} \sum_{n=1}^{p+1} \left(\frac{\lambda_n}{\lambda_{n+p+1} - \lambda_n} \right)^{\mu-1} \left[t_n^{p+1} - t_{n-1}^{p+1} \right]^{\mu}$$

Now let
$$b_r = \rho_r^{\mu-1} \left[t_r^{p+1} - t_{r-1}^{p+1} \right]^{\mu}$$

and
$$B_n = \sum_{r=0}^n b_r$$
,

and proceed as the last part of the proof of Theorem 4.4,

we have
$$\sum_{n=0}^{m} E_{n-1}^{p+1} \rho_n^{\mu-1} \left| t_n^{p+1} - t_{n-1}^{p+1} \right|^{\mu} = o(E_m^{p+1})$$

And hence $\sum_{n=0}^{m} (\lambda_{n+p+1} - \lambda_n) E_n^p | t_n^p - t_n^{p+1} |^{\mu} = o(E_m^{p+1}).$ ///

CHAPTER 5 SOME STRICT INCLUSION THEOREMS BETWEEN CESÀRO AND DISCRETE RIESZ METHODS OF SUMMABILITY

§5.1 DEFINITIONS

O

Suppose throughout this chapter that $\kappa > 0$,

 $s_n = \sum_{r=0}^{n} a_r$ $\epsilon_0^{\kappa} = 1$ $\varepsilon_n^{\kappa} = \binom{n+\kappa}{n}^* = \frac{(\kappa+1)(\kappa+2)\cdots(\kappa+n)}{n!} \quad \text{for } n > 0.$ and Let $\{p_n\}$ be a sequence with $p_n > 0$ for $n \ge 0$ and let $P_n = \sum_{r=0}^{n} p_r$. "Define (5.1) $t_n = \frac{1}{P_n} \sum_{r=0}^n p_{n-r} \dot{s}_r = \frac{1}{P_n} \sum_{r=0}^n p_{n-r} a_{r},$ (5.2) $t_n^{\Delta} = \frac{1}{p_n} \sum_{r=0}^n p_{n-r} a_r = \frac{1}{p_n} \sum_{r=0}^n (p_{n-r} - p_{n-1-r}) s_r, \quad (p_{-1} = 0).$... We say that the sequence $\{s_n\}$ is (N,p_n) -convergent to s if $t_n \rightarrow s$; and we write $s_n \rightarrow s (N, p_n)$. This is a Nörlund Summability Method. See for example Hardy [11, page 54].

(5.3)
$$\tau_{n} = \frac{1}{P_{n}} \sum_{r=0}^{n} p_{r} |t_{r}^{\Delta} - s|$$

We say that the sequence $\{s_n\}$ is $[N,p_n]$ -convergent to s if $\tau_n = o(1)$, and we write

$$p_n \rightarrow s [N, p_n].$$

(See Borwein and Cass [6].)

We say that the sequence $\{s_n\}$ is $|N,p_n|$ -convergent to s if

 $\sum_{n=1}^{\infty} |t_n - t_{n-1}| < \infty \quad \text{and } s = \lim t_n;$

and we write

 $\mathbf{s}_n \neq \mathbf{s} |\mathbf{N}, \mathbf{p}_n|$.

The Strong Summability Method $[N,p_n]$ is the method $[P,Q]_1$ (see §1.2) with $P = (\overline{N},p_n)$ (see §3.2) and Q the matrix associated with the transformation (5.2). We shall denote Q by $(N,\Delta p_n)$.

In the case of $[N,p_n]$ -summability, the method is interesting only if $P_n \rightarrow \infty$. This condition is satisfied by the summability methods we consider below.

If we take $p_n = \varepsilon_n^{\kappa-1}$, then (N,p_n) and $|N,p_n|$ are the Cesàro and Absolute Cesàro Summability Methods (C,κ) and $|C,\kappa|$ respectively.

The method $[N,p_n]$ with $p_n = \varepsilon_n^{\kappa-1}$ is equivalent (but not equal) to the Strong Cesàro Method $[C,\kappa]$ (See §1.6.) We shall denote this method $[N,p_n]$ also by $[C,\kappa]$. See Borwein and Cass [6, pages 98-99]. If

$$= \frac{1}{(n+1)^{\kappa}} \sum_{\nu=0}^{n} (n+1-\nu)^{\kappa} (s_{\nu} - s_{\nu-1})$$

$$= \frac{1}{(n+1)^{\kappa}} \sum_{\nu=0}^{n} [(n+1) - \nu)^{\kappa} - (n - \nu)^{\kappa}] s_{\nu},$$

then we say that the sequence $\{s_{v}\}$ is (R^*, n, κ) -convergent to s, if $\rho_n^{\kappa} \rightarrow s$ as $n \rightarrow \infty$. We denote this by

 $s_n + s (R^*, n, \kappa)$. Thus if we take $p_n = (n+1)^{\kappa} - n^{\kappa}$ for $n \ge 0$, then (N, p_n) and $|N, p_n|$ are the Discrete Reisz and Absolute Discrete Riesz Summability Methods (R^*, n, κ) and $|R^*, n, \kappa|$ respectively. We shall define the Strong Discrete Rissz Method of Summability $[R^*, n, \kappa]$ to be the method $[N, p_n]$ associated with this $\{p_n\}$.

\$5.2 KUTTNER'S THEOREM

In the definitions of (C,κ) and (R^*,n,κ) and the associated absolute methods, κ is usually allowed to 'satisfy $\kappa > -1$. The methods $[C,\kappa]$ and $[R^*,n,\kappa]$ make sense only when $\kappa > 0$ and it is for this reason we have so 'restricted κ .

THEOREM (Kuttner)

(i) If $-1 < \kappa < 2$, then (R^*, n, κ) is equivalent to (C, κ) and $|R^*, n, \kappa|$ is equivalent to $|C, \kappa|$.

(ii) There is a sequence (R*,n,2)-convergent but not

 $\rho_n^{\kappa} = \sum_{\nu=0}^n (1 - \frac{\nu}{n+1})^{\kappa} a_{\nu}$

(C,2)-convergent and a sequence $|R^*,n,2|$ -convergent but not |C,2|-convergent. But $|R^*,n,2| \Rightarrow (C,2)$.

(iii) If $\kappa > 2$, there is a sequence $|R^*,n,\kappa|$ -convergent but not (C,κ) -convergent. (See Kuttner [18].)

\$5.3 EXTENSION OF KUTTNER'S THEOREM AND OTHER RESULTS

For the proof of Theorem 5.1 we state two results of Borwein and Cass [6, Theorems 6 and 9] as our next two lemmas.

LEMMA 5.1

 $*[N,p_n] => (N,p_n).$

LEMMA 5.2

If $P_n \neq \infty$ and $\{s_n\}$ is $|N, p_n|$ -convergent, then

 $s_n \rightarrow s [N,p_n]$

where $s = \lim_{n \to \infty} t_n$ and t_n is defined as in (5.1).

THEOREM 5.1

If $\kappa > 0$, then $|\mathbf{R}^*, \mathbf{n}, \kappa| \Rightarrow [\mathbf{R}^*, \mathbf{n}, \kappa] \Rightarrow (\mathbf{R}^*, \mathbf{n}, \kappa)$. PROOF

That $[R^*,n,\kappa] \Rightarrow (R^*,n,\kappa)$ is a special case of Lemma 5.1. $|R^*,n,\kappa| \Rightarrow [R^*,n,\kappa]$ follows from Lemma 5.2. ///

The next theorem is known, but it also follows from Lemmas 5.1 and 5.2 as the Theorem 5.1.

THEOREM 5.2

 $|C,\kappa| \Rightarrow [C,\kappa] \Rightarrow (C,\kappa)$

THEOREM 5.3

Let $p_n > 0$ for $n \ge 0$ and suppose $P_n \to \infty$. Then there is a sequence which is $[N_n p_n]$ -convergent but not $|N_n p_n|$ convergent.

 $s_n \rightarrow s(N, p_n)$

PROOF

Borwein and Cass [6, Theorem 8] proved that $s_n \neq s[N,p_n]$ if and only if

and

(5,5)

Now

5.6)
$$\frac{1}{P_{n'}} \sum_{r=0}^{n} p_r |t_r^{\Delta} - t_r| = o(1)$$

where t_r and t_r^{Δ} are given by (5.1) and (5.2).

This is a special case of Lemma 4.2.

$$= \frac{\frac{1}{p_{r}} \sum_{\nu=0}^{r} (p_{r-\nu} - p_{r-1-\nu}) s_{\nu}}{\sum_{\nu=0}^{r} p_{r-\nu} s_{\nu}} = \frac{\frac{1}{p_{r}} \sum_{\nu=0}^{r} p_{r-\nu} s_{\nu}}{\sum_{\nu=0}^{r} p_{r-\nu} s_{\nu} - p_{r} \sum_{\nu=0}^{r} p_{r-\nu} s_{\nu}} - \frac{1}{p_{r}} \sum_{\nu=0}^{r} p_{r-\nu} s_{\nu}}{\sum_{\nu=0}^{r} p_{r-\nu} s_{\nu}} = \frac{p_{r} \sum_{\nu=0}^{r} p_{r-\nu} s_{\nu}}{\sum_{\nu=0}^{r} p_{r-\nu} s_{\nu}} + \frac{1}{p_{r}} \sum_{\nu=0}^{r} p_{r-\nu} s_{\nu} + \frac{1}{p_{r}} \sum_{\nu=0}^{r} p_{r-\nu} + \frac{1}{p_{r}} \sum_{\nu=0}^{r} p_$$

$$=\frac{\Pr[r-1] \sum_{\nu=0}^{n} \Pr[r-\nu] \sum_{\nu=0}^{n} \Pr[r] \sum_{\nu=0}^{n} \sum_{\nu=0}^{$$

so that

5.7)
$$p_r(t_r^{\Delta} - t_r) = P_{r-1}(t_r - t_{r-1}), r = 0, 1, 2, ...,$$

 $(P_{-1} = t_{-1} = 0).$
Choose $\{s_n\}$ so that $t_n^{\circ} - t_{n-1} = \frac{\delta_n p_n}{P_n D_n}$ where

 $D_n = \sum_{r=0}^n \frac{p_r}{p_r}$ and $\delta_n = \pm 1$ chosen in such a way that $\sum_{n=1}^{\infty} \frac{\delta_n p_n}{p_n D_n}$ converges. Then $\{t_n\}$ is convergent ensuring that (5.5) is satisfied. Also we have

$$\frac{1}{P_n} \sum_{r=0}^n P_r |t_r^{\Delta} - t_r| = \frac{1}{P_n} \sum_{r=0}^n P_{r-1} |t_r - t_{r-1}|$$
$$= \frac{1}{P_n} \sum_{r=0}^n \frac{P_{r-1} P_r}{P_r D_r}$$
$$= \sum_{r=0}^n a_{n,r} \frac{1}{D_r}$$

where $a_{n,r} = \frac{P_{r-1} P_r}{P_n P_r}$ for $0 \le r \le n$ and $a_{n,r} = 0$ for r > n.

Now $A = \{a_{n,r}\}$ is a matrix with zero column limits and

$$\sum_{r=0}^{n} |a_{n,r}| = \sum_{r=0}^{n} a_{n,r} \leq \frac{1}{P_n} \sum_{r=0}^{n} p_r = 1, \text{ for all } n,$$

so that it transforms null sequences into null sequences. Since by Abel-Dini Theorem $\lim_{n \to \infty} D_n = \infty, \frac{1}{D_r} \to 0$ as $r \to \infty$.

It follows that (5.6) is satisfied, so $s_n \rightarrow s [N,p_n]$. But by Abel-Dini Theorem again

$$\sum_{n=1}^{\infty} |\mathbf{t}_n - \mathbf{t}_{n-1}| = \sum_{n=1}^{\infty} \frac{\mathbf{p}_n}{\mathbf{p}_n \mathbf{D}_n} = \infty,$$

so $\{s_n\}$ is not $|N,p_n|$ -convergent.

COROLLARY 5.1

Let $\kappa > 0$. There is a sequence which is $[R^*, n, \kappa]$ convergent but not $|R^*, n, \kappa|$ -convergent. COROLLARY 5.2

Let $\kappa > 0$. There is a sequence which is $[C,\kappa]$ -convergent but not $|C,\kappa|$ -convergent.

THEOREM 5.4

Let $\kappa > 0$. There is a sequence which is (R^*, n, κ) convergent but not $[R^*, n, \kappa]$ -convergent. PR00F

Let $P = \{p_{n,\nu}^{\bullet}\}$, where $p_{n,\nu} = \frac{(\nu+1)^{\kappa} - \nu^{\kappa}}{(n+1)^{\kappa}}$ for $0 \leq \nu \leq n$ and $p_{n,\nu} = 0$ for $\nu > n$. It follows from Theorem 3.4 that \langle there is a sequence P-convergent but not [P,I]-convergent. Let $Q = \{q_{n,\nu}\}$ be the matrix such that

$$\sum_{\nu=0}^{n} q_{n,\nu} s_{\nu} = \frac{1}{p_n} \sum_{\nu=0}^{n} p_{n-\nu} a_{\nu}$$

where $p_n = (n+1)^{\kappa} - n^{\kappa}$. Then $[R^*, n, \kappa]$ -convergency is the same as [P,Q]-convergency and (R^*, n, κ) -convergency is the same as PQ-convergency. Since the matrix Q has an inverse our result now follows.

For the next theorem we state two results of Borwein and Cass [6, Theorem 1 and Corollary 1] as our next two lemmas.

LEMMA 5.4 ."

 $If(N,p_n) \Rightarrow (N,q_n)$ then $[N,p_n] \Rightarrow [N,q_n]$.

LEMMA 5.5

If $(N,p_n) \iff (N,q_n)$ then $[N,p_n] \iff [N,q_n]$.

THEOREM 5.6

- (i) If $\kappa > 0$, then $[C,\kappa] = [R^*,n,\kappa]$.
- (ii) If $0 < \kappa < 2$, then $[C,\kappa] <=> [R^*,n,\kappa]$.

PROOF

Since for $\kappa > 0$ we have $(C,\kappa) \Rightarrow (R^*,n,\kappa)$, (i) follows from Lemma 5.4. Since for $0 < \kappa < 2$ we have $(C,\kappa) <=> (R^*,n,\kappa)$ (ii) follows from Lemma 5.5. /// THEOREM 5.7 Sб

There is a sequence which is $|R^*,n,2|$ -convergent but not [C,2]-convergent. PROOF

For a given sequence $\{s_n\}$ we write

(5.8)
$$\sigma_{n} = \frac{1}{\varepsilon_{n}^{2}} \sum_{\nu=0}^{n} \varepsilon_{n-\nu}^{1} \mathbf{s}_{\nu} = \frac{s_{n}}{\varepsilon_{n}^{2}}$$

and

(5.9)
$$\xi_n = \frac{1}{(n+1)^2} \sum_{\nu=0}^n (n+1-\nu)^2 a_{\nu} = \frac{T_n}{(n+1)^2}$$

so that $\{\sigma_n\}$ and $\{\xi_n\}$ are respectively the (C,2) and (R*,n,2) transforms of the sequence $\{s_n\}$.

As in Kuttner [18, page 362] we have

(5.10) $T_0 = S_0; T_n = S_{n-1} + S_n, n = 1,2,3...$ and

(5.11)
$$S_n = \sum_{m=0}^n (-1)^{n-m} T_m.$$

Now take $S_n = (-1)^n so that T_n = (-1)^n$. Thus

 $\sum_{n=1}^{\infty} |\xi_n - \xi_{n-1}| < \infty \text{ and } \xi_n \to 0, \text{ so that if } \{s_n\} \text{ is the }$

sequence associated with this choice of S_n and T_n we have $s_n \neq 0 | R^*, n, 2 |$. To see that $\{s_n\}$ is not [C, 2]-convergent we notice first that by Theorem 5.1, $s_n \rightarrow 0 | R^*, n, 2|$ implies $s_n \rightarrow 0 [R^*]n, 2]$. Now by Theorem 5.6 [C,2] => [R*, n, 2], the only [C,2]-sum that $\{s_n\}$ could have is zero. $s_n - s_{n-1} = (-1)^n (2n - 1)$. But. $\frac{1}{m+1} \sum_{n=0}^{m} \left| \frac{(-1)^{n} (2n-1)}{n+1} \right|$ (5.12)Ť $=\frac{1}{m+1}\sum_{n=0}^{m}\frac{2n-1}{n+1}$. Since (C,1) is regular and $\frac{2n-1}{n+1} \neq 2$, (5.12) tends to 2 as $m \rightarrow \infty$. Thus $\{s_n\}$ is not [C,2]-convergent to zero. COROLLARY 5.3 . There is a sequence which is [R*,n,2]-convergent but not [C,2]-convergent. PROOF This follows from the fact that $|R^*,n,2| => [R^*,n,2]$. THEOREM 5.8 $[R^*, n, 2] \implies (C, 2).$ PROOF Referring to (5.9) we find that $T_r - T_{r-1} = \sum_{\nu=0}^{r} \{ (r+F-\nu)^2 - (r-\nu)^2 \} a_{\nu}.$ $s_n \neq 0$ [R*,n,2] if and only if $\frac{1}{P_{n}} \sum_{r=0}^{n} P_{r} \left| \frac{1}{P_{r}} \sum_{\nu=0}^{r} P_{r-\nu} a_{\nu} \right|^{*} = \frac{1}{P_{n}} \sum_{r=0}^{n} \left| \sum_{\nu=0}^{r} P_{r-\nu} a_{\nu} \right|^{*} = o(1), \quad \therefore$

where
$$p_{r-v} = (r+1-v)^2 - (r-v)^2$$
 and $P_n = (n+1)^2$. Hence $s_n + 0$
 $[R^*,n,2]$ if and only if
 $\frac{1}{(n+1)^2} \frac{n}{r+0} |T_r - T_{r-1}| = o(1)$, $(T_{-1} = 0)$
From (5.11) it follows that
 $|s_n| \le \frac{n}{r+0} |T_r - T_{r-1}|$.
Thus if $s_n + 0 [R^*,n,2]$, then $|s_n| = o(n^2)$ so that
 $s_n + 0(C,2)$,
Now if $s_n + s[R^*,n,2]$, then $s_n - s + 0[R^*,n,2]$ so
 $s_n - s + 0(C,2)$, i.e., $s_n + s(C,2)$.
THEOREM 5.9
There is a sequence which is $(C,2)$ -convergent but not
 $[R^*,n,2]$ -convergent.
PROOF
Choose $\{s_n\}$ so that
 $s'_{2n} = (-1)^n n^{3/2}$ and $s'_{2n+1} = 0$.
Then $s_n + o(C,2)$. But referring to (5.10)
 $T_{2r} - T_{2r-1} = s'_{2r} - s'_{2r}/2$
 $= (-1)^r (r^{3/2} + (r-1)^{3/2})$, $r = 1,2,...$
So if $2m \le n \le 2m+1$, then
 $\frac{n}{r=0} |T_r - T_{r-1}| \ge \frac{m}{r+1} |T_{2r} - T_{2r-1}|$
 $\ge \frac{m}{r+1} (r-1)^{3/2}$
 $\sim H m^{5/2}$
 $\sim H_1 n^{5/2}$

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where H, H₁ are independent of n.

Thus $\{s_n\}$ is not $[R^*, n, 2]$ -convergent to zero and our result follows.

THEOREM 5.10

Let $\kappa > 2$

(i) There is *a sequence which is [R*, n, κ]-convergent
 but not (C', κ)-convergent^δ.

(ii) There is a sequence which is $|R^*,n,\kappa|$ -convergent but not $[C,\kappa]$ -convergent.

- (iii) There is a sequence which is [R*,n,κ]-convergent but not [C,κ]-convergent. PROOF

Part (i) follows from Kuttner's Theorem (iii) and the fact that $|R^*,n,\kappa| \Rightarrow [R^*,n,\kappa]$.

Part (ii) follows from Kuttner's Theorem (iii) and the fact that $[C,\kappa] = (C,\kappa)$.

Part (iii) follows from part (ii) and the fact that $\frac{1}{2}$ $|R^*,n,\kappa| \Rightarrow [R^*,n,\kappa].$

The relations between the various summability methods discussed in this chapter are conveniently displayed in three figures below. In these figures the symbol \rightarrow denotes strict inclusion, the symbol \leftrightarrow denotes equivalence and the notation $P \leftrightarrow Q$ means that there is sequence which is P-convergent but not Q-convergent.

0 < к < 2⁶ ۴. $|\mathbf{R}^{\star};\mathbf{n},\kappa| \rightarrow [\mathbf{R}^{\star};\mathbf{n},\kappa] \rightarrow (\mathbf{R}^{\star},\mathbf{n},\kappa)$ Î $|C,\kappa| \rightarrow^{\alpha} \langle C,\kappa \rangle \rightarrow \langle C,\kappa \rangle$ Figure 1 **ب**ور . $|R^{*},n,2| \rightarrow [R^{*},n,2] \rightarrow (R^{*},n,2)$ 1 [C,2] + (C,2) C,2 → Figure 2 $|\mathbf{R}^{\star},\mathbf{n},\kappa| \rightarrow [\mathbf{R}^{\star},\mathbf{n},\kappa] \rightarrow (\mathbf{R}^{\star},\mathbf{n},\kappa)$ · . . ϕ_{i} |C,K| → [C,K] + (C,K) д Figure 3

CHAPTER 6

STRONG AND ABSOLUTE NÖRLUND METHODS OF SUMMABILITY ASSOCIATED WITH POLYNOMIALS

In this chapter our investigations stem from the results in D. Borwein [4]. We consider a Nörlund Method of Summability Associated with Polynomials and investigate the properties of an associated Strong Summability Method and of the Absolute Nörlund Method of Summability Associated with Polynomials.

§6.1 DEFINITIONS

and

Let s, s_n be arbitrary complex numbers, and whenever n < 0 we take s_n = 0. Let

 $p(z) = p_0 + p_1 z + \cdots + p_j z^j$ $q(z) = q_0 + q_1 z + \cdots + q_k z^k$

be polynomials with complex coefficients which satisfy the normalizing conditions

p(1) = 1 and q(1) = 1.

We suppose throughout that $p(0) \neq 0$, $q(0) \neq 0$, $p_n = 0$ for n > j and $q_n' = 0$ for n > k. We use the notations

- (6.1) $t_n = \sum_{\nu=0}^n p_{\nu} s_{n-\nu}, \quad n = 0, 1, 2, \dots,$
- (6.2) $u_n \bigoplus_{\nu=0}^n q_{\nu,n-\nu} n = 0, 1, 2, ...$

Associated with the polynomial p(z) is a Nörlund Method of Summability N_p which we call a *Polynomial Nörlund* Method and which is defined as follows.

The sequence $\{s_n\}$ is said to be N_p-convergent to s, and we write

(6.3)
$$s_n \rightarrow s(N_p)$$
, if $\lim_{n \rightarrow \infty} t_n = s$.

This definition is due to D. Borwein.

We define .

(6.4)
$$s_n \rightarrow s [C_1, N_p]$$

if $\frac{1}{n+1}\sum_{r=0}^{n} |t_r - s| = o(1)$, as $n \to \infty$.

This is the $[P,Q]_1$ defined in §1.2 with $P = C_1$ and

$$Q = N_{p}$$

Let
$$P_n = \sum_{\nu=0}^{n} p_{\nu}$$
 where P_n is non-zero for $n = 0, 1, 2, ...$

and $\tau_n = \frac{1}{p_n} \sum_{\nu=0}^{n} p_{\nu} s_{n-\nu}$. Then we say that the sequence $\{s_n\}$

Is (N,p_n) -convergent to s and we write

*(6.5) if $\lim_{n \to \infty} \tau_n = s.$ $s_n \to s(N, p_n)$

This is the Nörlund Summability Method given in §5.1, but here we allow p_v to be complex for all $v \ge 0$. Moreover, in this chapter we are only interested in the case where p_v 's are coefficients of a polynomial p(z) with p(1) = 1and we only use the (N, p_n) method in this sense. It is evident that in this sense (N,p_n) is equivalent to the Polynomial Nörlund Method N_n.

Let $P'_n = \sum_{r=0}^n |P_r|$ and $P'_n \neq 0$ for n = 0, 1, 2, ... Then (6.6) $s_n \neq s [N, P_n]$

if $\frac{1}{P_{r}} \sum_{r=0}^{n} |P_{r}| |\tau_{r} - s| = o(1)$, as $n \to \infty$.

This definition is analogous to the definition of $[N,p_n]$ given in §5.1, but we allow here p_v to be complex for $v \ge 0$. Moreover we let p_v 's be coefficients of a polynomial p(z) with p(1) = 1.

The Absolute Polynomial Nörlund Summability $|N_p|$ · is defined as follows.

(6.7) $s_n \rightarrow s |N|$ if $t_n \rightarrow s$ and $\sum_{n=0}^{\infty} |t_n - t_{n-1}| < \infty$, where $t_{-1} = 0$.

The method $[C_1, N_p]$ is a Strong Summability Method Associated with the Polynomial Norland Method. It is not the Strong Norland Summability Method defined in [6] which we considered in Chapter 5. Shortly we shall show that $[C_1, N_p]$ is equivalent to $[N, P_n]$. Thus $[C_1, N_p]$ is the Strong Norland Summability Method defined in [6] for (N, P_n) rather than for (N, P_n) .

We shall establish at first $[C_1, N_p] => [C_1, N_q]$ if and only if $N_p => N_q$ 00

It is shown in Borwein and Cass [6] that if (N, p_n) => (N,q_n) then $[N,p_n] => [N,q_n]$. We shall investigate the converse of this theorem in the case of the Polynomial. Nörlund Methods. Then we shall establish $|N_p| = |N_q|$ if and only if $N_p => N_q$. Finally we shall establish some minor results analogous to some of the results obtained in [4]. \$6.2 THE EQUIVALENCE OF $[C_1, N_p]$ AND $[N, P_n]$ THEOREM 6.1 $[C_1, N_p] \iff [N, P_n].$ PROOF The result is an elementary consequence of the fact that $P_n = \sum_{r=0}^{n} |P_r| = \sum_{r=0}^{j-1} |P_r| + n - j + 1 \sim n + 1$ which implies the equivalence of (\overline{N}, P_n) and (C, 1). /// THEOREMS ABOUT NÖRLUND METHODS OF SUMMABILITY §6.3 ASSOCIATED WITH POLYNOMIALS For completeness we shall quote without proof several results of Borwein [4]. The methods N_p and N_q mentioned in the following. theorems are Nörlund Methods associated with polynomials p(z) and q(z) as defined in §6.1. Evidently N_p and N_q are regular.

THEOREM 6.2"

The method N_f , associated with the polynomial f(z) = p(z)q(z), includes both N_p and N_q . (Borwein [4, Theorem 2].)

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THEOREM 6.3

The methods N_p and N_q are consistent, i.e., if $s_n \rightarrow s (N_p)$ and $s_n \rightarrow s' (N_q)$, then s = s'. (Borwein [4, Corollary].)

THEOREM 6.4

If h(z) is the highest common factor of p(z) and q(z), normalized so as to make h(i) = 1, then a necessary and sufficient condition for a sequence to be both N_p - and N_q -convergent is that it be N_h -convergent. (Borwein [4, Theorem 3].)

THEOREM 6.5

In order that N_q should include N_p it is necessary and sufficient that q(z)/p(z) should not have poles on or within the unit circle. (Borwein [4, Theorem I].)

THEOREM 6.6

If q(z)/p(z) has poles of maximum order m on the unit circle and does not have poles within the unit circle, then $(C,m)N_q$ includes N_p , but for any $\varepsilon > 0$, there is an N_p -convergent sequence which is not $(C,m-\varepsilon)N_q$ -convergent. (Borwein [4, Theorem II].) THEOREM 6.7 -

If q(z)/p(z) has a pole within the unit circle then there is an N_p -convergent sequence which is not AN_q -convergent. (Borwein [4, Theorem III].)

THEOREM 6.8

In order that N_p should be equivalent to (C,0) it is necessary and sufficient that p(z) should not have zeros on or within the unit circle. (Borwein [4, Theorem I⁺].)

THEOREM 6.9

 $\mathbf{t}_{\mathbf{n}} = \sum_{\mathbf{v}=0}^{n} \mathbf{p}_{\mathbf{v}} \mathbf{s}_{\mathbf{n}-\mathbf{v}},$

 $u_n = \sum_{\nu=0}^n q_\nu s_{n-\nu},$

If q(z)/p(z) has poles $\lambda_1, \lambda_2, \ldots, \lambda_k$, in the finite complex plane, of orders m_1, m_2, \ldots, m_k respectively, and if, for $n = 0, 1, 2, \ldots$,

then

$$u_{n} = \sum_{i=0}^{n} c_{v} t_{n-v} + \sum_{r=1}^{\ell} \sum_{\rho=1}^{m} c_{r,\rho} \sum_{\nu=0}^{n} \left(\frac{\nu+\rho-1}{\rho-1} \right) \lambda_{r}^{-\nu} t_{n-\nu}$$

where the C's are constants, depending only on p_0, p_1, \dots, p_j , q_0, q_1, \dots, q_k such that $c_n = 0$ for n > k - j and $C_{r'm_r} \neq 0$. (Borwein [4, Lemma 1].)

§6.4 [C₁,N_p] METHOD OF SUMMABILITY The following proposition is a special case of Theorem 1.2. νo

PROPOSITION 6.1

(i) $N_p => [C_1, N_p],$ (ii) $[C_1, N_p] => (C, 1)N_p.$

THEOREM 6.10

If q(z)/p(z) has no poles within or on the unit circle, then $[C_1, N_p] \Rightarrow [C_1, N_q]$.

Without loss of generality, we may assume $s_n \neq 0$ [C_1, N_p] and prove $s_n \neq 0$ [C_1, N_q].

Let
$$t_n = \sum_{\nu=0}^n p_\nu s_{n-\nu}$$
,
 $u_n = -\sum_{\nu=0}^n q_\nu s_{n-\nu}$.

If q(z)/p(z) has no poles within or on the unit circle, but has poles $\lambda_1, \lambda_2, \ldots, \lambda_\ell$ of order m_1, m_2, \ldots, m_ℓ outside the unit circle, then by Theorem 6.9

$$u_{n} = \sum_{\nu=0}^{n} C_{\nu} t_{n-\nu} + \sum_{r=1}^{\ell} \sum_{\rho=1}^{n} C_{r,\rho} \sum_{\nu=0}^{n} \left(\frac{\nu+\rho-1}{\rho-1} \right) \lambda_{r}^{-\nu} t_{n-\nu}$$

where the C's are constants, depending only on $p_0, p_1, ..., p_j, q_0, q_1, ..., q_k$, such that $c_n = 0$ for n > k - j and $c_{r'm_r} \neq 0$.

So
$$|\mathbf{u}_{n}| \leq \sum_{\nu=0}^{n} |\mathbf{c}_{\nu}| |\mathbf{t}_{n-\nu}| + \sum_{r=1}^{\ell} \sum_{\rho=1}^{m_{r}} |\mathbf{c}_{r,\rho}| \sum_{\nu=0}^{n} |\binom{\nu+\rho-1}{\nu+\rho-1} \lambda_{r}^{-\nu}| |\mathbf{t}_{n-\nu}|$$

· · ·	
.	Thus $\frac{1}{m+1} \sum_{n=0}^{m} u_n $
· · ·	$ \leq \frac{1}{m+1} \sum_{n=0}^{m} \sum_{\nu=0}^{n} c_{\nu} t_{n-\nu} + \frac{1}{m+1} \sum_{n=0}^{m} \sum_{r=1}^{\ell} \sum_{\rho=1}^{m} c_{r,\rho} \sum_{\nu=0}^{n} \binom{\nu+\rho-1}{\rho-1} \lambda_{r}^{-\nu} t_{n-\nu} $
-	$= \sum_{\nu=0}^{m} c_{\nu} \frac{1}{m+1} \sum_{n=0}^{m-\nu} t_{n} + \sum_{r=1}^{\ell} \sum_{\rho=1}^{m} c_{r,\rho} \sum_{\nu=0}^{m} \binom{\nu+\rho-1}{\rho-1} \lambda_{r}^{-\nu} \frac{1}{m+1} \sum_{n=0}^{m-\nu} t_{n} ,$
, ,	where $c_v = 0$, for $v > \kappa - j$. Since the poles of $q(z)/p(z)$ are all outside the unit
, ,	circle, $ \lambda_r > 1$, for $r = 1, 2,, l$; and $\sum_{\nu=0}^{\infty} {\nu+\rho-1 \choose \rho-1} \lambda_r^{-\nu}$ is
~ .	thus absolutely convergent for each $r = 1, 2, \ldots, l$ and .
<i>.</i>	$\rho = 1, 2, \dots, m_r$. Consequently if $\frac{1}{m+1} \sum_{n=0}^m t_n \neq 0$ as $m \neq \infty$,
-	then $\frac{1}{m+1} \sum_{n=0}^{m} u_n \rightarrow 0 \text{ as } m \rightarrow \infty.$
-	If $q(z)/p(z)$ has no poles at all, then
-	$\frac{1}{m+1}\sum_{n=0}^{m} u_n \leq \sum_{\nu=0}^{m} c_{\nu} \frac{1}{m+1} \sum_{n=0}^{m-\nu} t_n , \text{ where } c_{\nu} = 0 \text{ for } \nu > k - j.$
	Hence the desired conclusion follows. ///
	THEOREM 6.11
•.	If (1) $q(z)/p(z)$ has a pole within the unit circle,
·	or (2) $q(z)/p(z)$ has no pole within the unit circle,
	but has poles of maximum order m on the unit circle, where
•	m > 1, then there is a # quence which is [C ₁ ,N _p]-convergent
	but not [C ₁ , N _q]-convergent. PROOF
¢	(1) $q(z)/p(z)$ has a pole within the unit circle.

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By Theorem 6.7 there is an N_p-convergent sequence which is not AN_q-convergent. Since (C,1) is regular, this sequence is $[C_1, N_p]$ -convergent. But, since it is not AN_q-convergent, it is not $(C,1)N_q$ -convergent. As a consequence of Proposition 6.1(ii) it is not $[C_1, N_q]$ -convergent.

(2) $\frac{q(z)}{p(z)}$ has no poles within the unit circle, but has poles of maximum order m on the unit circle, where m > 1. By Theorem 6.6 since m > 1, there is an N_p-convergent sequence which is not (C,1)N_q-convergent. Consequently, this sequence is $[C_1, N_p]$ -convergent, but, by Proposition 6.1(ii) it is not $[C_1, N_q]$ -convergent. ///

For the next theorem we need the following two lemmas. We use the notation $[C,1]_1$ to mean $[C_1,1]_1$. LEMMA 6.1.

Let $t_n = a\lambda^n$, $|\lambda| = 1$, $\lambda \neq 1$ and a is a non-zero complex number. Then $\{t_n\}$ is not $[C,1]_1$ -convergent. PROOF,

We know that $\{t_n\}$ is (C,1)-convergent. For

 $\frac{1}{m+1}\sum_{n=0}^{m} t_n = \frac{1}{m+1}\sum_{n=0}^{m} a\lambda^n = \frac{4}{m+1}\frac{1-\lambda^{m+1}}{1-\lambda}$

Since a is a constant and $\frac{1-\lambda^{m+1}}{1-\lambda} = 0(1)$, then $\frac{1}{m+1} \sum_{n=0}^{m} t_n \neq 0$,

as $m \rightarrow \infty$.

Thus if $\{t_n\}$ is $[C,1]_1$ -convergent, its sum has to be zero. But

$$\frac{1}{n+1} \sum_{n=0}^{m} |t_n| = \frac{1}{m+1} \sum_{n=0}^{m} |a| |\lambda^n| = |a|$$

which $+ \rightarrow 0$, since a $\neq 0$.

LEMMA 6.2

Let $\lambda_1, \lambda_2, \dots, \lambda_r$ be r distinct complex numbers, r > 1, with $|\lambda_v| = 1$, $\lambda_v \neq 1$ for $v = 1, 2, \dots, r$, and let a_1, a_2, \dots, a_r be non-zero complex numbers. If $t_n = a_1 \lambda_1^n + a_2 \lambda_2^n + \dots + a_r \lambda_r^n$, then $\{t_n\}$ is not $[C, 1]_1$ convergent.

PROQE

If $\{t_n\}$ is $[C,1]_1$ -convergent, its sum must be zero: $\frac{1}{m+1} \sum_{n=0}^{m} |t_n| = \frac{1}{m+1} \sum_{n=0}^{m} |a_1 + a_2 \left(\frac{\lambda_2}{\lambda_1}\right)^n + \cdots + a_r \left(\frac{\lambda_r}{\lambda_1}\right)^n |.$ If $t_n \neq 0 [C,1]_1$, then $\tau_n = a_1 + a_2 \left(\frac{\lambda_2}{\lambda_1}\right)^n + \cdots + a_r \left(\frac{\lambda_r}{\lambda_1}\right)^n |.$ $a_r \left(\frac{\lambda_r}{\lambda_1}\right)^n \neq o(C,1).$ But $\tau_n \neq a_1(C,1)$ and $a_1 \neq 0.$ ///

THEOREM 6.12

If $\frac{q(z)}{p(z)}$ has no poles within the unit circle, but has simple poles on the unit circle and has no poles of higher order on the unit circle, then there is a sequence which is $[C_1, N_p]$ -convergent but not $[C_1, N_q]$ -convergent. PROOF

Suppose $\frac{q(z)}{p(z)}$ has r poles of order 1, $\lambda_1, \lambda_2, \ldots, \lambda_r$, on the unit circle and $r \geq 1$, and suppose it has other poles, $\lambda_{r+1}, \ldots, \lambda_{\ell}$, outside the unit circle of order $m_{r+1}, \ldots, m_{\ell}$.

Since p(1) = 1, z = 1 cannot be a pole of $\frac{q(z)}{p(z)}$. i.e., $\lambda_{v} \neq 1$, for v = 1, 2, ..., r. Synce $p(0) \neq 0$, $\frac{1}{p(z)}$ is analytic in a neighbourhood U of the origin. There is a sequence $\{s_n\}$ such that, for in U, $\sum_{n=0}^{\infty} s_n z^n = \frac{1}{p(z)}$. Then, for z in U $\sum_{n=0}^{\infty} t_n z^n = p(z) \sum_{n=0}^{\infty} s_n z^n = 1,$ $\sum_{n=0}^{\infty} u_n z^n = q(z) \sum_{n=0}^{\infty} s_n z^n = \frac{q(z)}{p(z)} \sum_{n=0}^{\infty} t_n z^n$ Hence $t_0 = 1$, $t_n = 0$ for n > 0; and so $\{t_n\}$ is $[C,1]_1$ convergent to zero. That is $\{s_n\}$ is $[C_1, N_p]$ -convergent to zero. Now, by Theorem 6.9, $\mathbf{u}_{n}^{\prime} = \mathbf{c}_{n} + \sum_{\nu=r+1}^{2} \sum_{\rho=1}^{n\nu} \mathbf{c}_{\nu,\rho} \left\{ \begin{array}{c} n+\rho-1\\ \rho-1 \end{array} \right\} \lambda_{\nu}^{-n} + \sum_{\nu=1}^{r} \overline{\mathbf{c}}_{\nu,1}^{\prime} \lambda_{\nu}^{-n}$ $= u_n^1 \notin u_n^2,$ where $u_n^1 = c_n + \sum_{\nu=r+1}^{\ell} \sum_{\rho=1}^{m_{\nu}} c_{\nu,\rho} {n+\rho-1 \choose \rho-1} \lambda_{\nu}^{-n}$ and $u_n^2 = \sum_{\nu=1}^r c_{\nu,1} \lambda_{\nu}^{-n}$ Since $c_n = 0$ for $n > k^2 - j$, and $|\lambda_v| > 1$ for v = r+1, r+2, ... ℓ , $\{c_n\}$ and $\{c_{\nu,\rho} \begin{pmatrix} n+\rho-1\\ \rho-1 \end{pmatrix} \lambda_{\nu}^{-n} \}$ for $\nu = r+1, r+2, \ldots, \ell$, $\rho \neq 1, 2, \ldots, m$, are each convergent to zero. Since (C,1) is regular, u_n^1 is $[C,1]_1$ -convergent to zero. But

$$u_{n}^{2} = \int_{v=1}^{r} c_{v,1} \lambda_{v}^{-n} = \int_{v=1}^{r} c_{v,1} \frac{1}{\lambda_{v}^{-n}} = \int_{v=1}^{r} c_{v,1} \frac{1}{\lambda_{v}^{-n}}$$

and $\lambda_{1} \lambda_{2} \dots, \lambda_{r}$ are distinct and distinct from 2. And
 $|\lambda_{v}| = 1$, for $v = 1, 2, \dots, r$. Thus by Lemmas 6.F and 6.2,
we know that (u_{n}^{2}) is not $[C,1]_{1}$ -convergent for $r \ge 1$.
Consequently $\{u_{n}\}$ is not $[C,1]_{1}$ -convergent, that is
 (s_{n}) is not $_{n}^{2}(C_{1}, N_{q}]$ -convergent. ///
THEOREM 6.13
 $(C_{1}, N_{p}] \Rightarrow [C_{1}, N_{q}]$ if and only if $q(2)/p(2)$ has no
polee on or within the unit circle.
PROOF
The sufficiency part follows from Theorem 6.10.
The recessity part follows from Theorem 6.11 and 6.12. ///
THEOREM 6.14
 $(C_{1}, N_{p}] \Rightarrow [C_{1}, N_{q}]$ if and only if $N_{p} \Rightarrow N_{q}$.
PROOF?
This is a consequence of Theorems 6.13 and 5.5. ///
COROLLARY 6.1
 $.If (C_{1}, N_{p}] \le [C_{1}, N_{q}]$, then it is necessary and
sufficient that both $g(2)/p(2)$ and $p(2)/q(2)$ have no poles
to ar within the unit circle.
 $(CROULLARY 6.2)$
 $(C_{1}, N_{p}] <=> [C_{1}, N_{q}]$ if and only if $N_{p} <=> N_{q}$.

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Noting that N_q is identical with I when q(z) = 1(i.e., $q_0 = 1$, $q_n = 0$ for n > 0) and referring to Corollary 6.1 we obtain the following corollary. COROLLARY 6.3

In order that $[C_1, N_p] \iff [C_1, I]_1$ it is necessary and sufficient that p(z) should not have zeros on or within the unit circle.

COROLLARY 6.4

 $[C_1, N_p] \iff [C, 1]_1$ if and only if $N_p \iff I$.

For the following theorems and corollaries about the methods (N, p_n) , (N, q_n) , (N, P_n) , (N, Q_n) , $[N, P_n]$ and $[N, Q_n]$ we let p_v for $v = 0, 1, \dots, j$ and q_v for $v = 0, 1, \dots, k$ be the coefficients of the polynomials p(z) and q(z) respectively. We also let $P_r = \sum_{v=0}^{r} p_v \neq 0$ for $r = 0, 1, \dots, j - 1$ and $Q_r = \sum_{v=0}^{n} q_v \neq 0$ for $r = 0, 1, \dots, k-1$, and $P_n^* = \sum_{r=0}^{n} P_r \neq 0$ and $Q_n^* = \sum_{r=0}^{n} Q_r \neq 0$ for all $n \geq 0$, so that (N, p_n) , $(N, q_n) \neq (N, P_n)$, (N, Q_n) , $[N, P_n]$ and $[N, Q_n]$ are methods associated with p(z)

and q(z) respectively and are all well defined.

THEOREM 6.75

 $(N,p_n) \stackrel{\sim}{=} (N,q_n) \quad implies \quad that \quad (N,P_n) \stackrel{\sim}{=} (N,Q_n).$

► PROOF
Let
$$r_{\mathbf{r}}^{*} = \frac{1}{P_{\mathbf{r}}} \sum_{v=0}^{n} p_{\mathbf{r}-v} s_{v}$$
 and $\mu_{\mathbf{r}} = \frac{1}{Q_{\mathbf{r}}} \sum_{v=0}^{n} q_{\mathbf{r}+v} s_{v}$,
and let $W_{\mathbf{n}} = \frac{1}{P_{\mathbf{n}}} \sum_{x=0}^{n} p_{\mathbf{n}-\mathbf{r}} s_{\mathbf{r}}$ and $V_{\mathbf{n}} = \frac{1}{Q_{\mathbf{n}}} \sum_{v=0}^{n} q_{\mathbf{n}-\mathbf{r}} s_{\mathbf{r}}$.
• Let $\mathbf{k}(\mathbf{z}) = \frac{q(\mathbf{z})}{p(\mathbf{z})} = \frac{Q(\mathbf{z})}{p(\mathbf{z})}$ and $\mathbf{k}(\mathbf{z}) = \sum_{v=0}^{n} k_{v} s^{v}$.
We know that the necessary and sufficient conditions that
 $(\mathbf{x}, \mathbf{p}_{\mathbf{n}}) = > (\mathbf{M}, q_{\mathbf{n}})$
in this case are
 $(6.8) \qquad ||\mathbf{k}_{0}||\mathbf{P}_{\mathbf{n}}| + \cdots + ||\mathbf{k}_{\mathbf{n}}||\mathbf{P}_{0}| \leq \mathbf{B}^{T}|\mathbf{Q}_{\mathbf{n}}|$
where H is independent of n, and
 $(6.9) \qquad k_{n-\mathbf{r}}/Q_{\mathbf{n}} + 0$, for each r.
 $(\mathbf{c}, \mathbf{f}, [6, Proposition 11.)$
Thus, 'if $(\mathbf{N}, \mathbf{p}_{\mathbf{n}}) = (\mathbf{N}(q_{\mathbf{n}}), \text{ then } (6.8)$ and (6.9) are satisfied.
Now $\sum_{\mathbf{r}=0}^{n} ||\mathbf{k}_{\mathbf{n}-\mathbf{r}}|| \sum_{v=0}^{n} |\mathbf{r}_{\mathbf{r}-v}|| \leq \sum_{\mathbf{r}=0}^{n} ||\mathbf{k}_{\mathbf{n}-\mathbf{r}}|| \frac{p}{\mathbf{r}-v}|$
 $= \sum_{v=0}^{n} \sum_{\mathbf{r}=v}^{n} ||\mathbf{k}_{\mathbf{n}-\mathbf{r}}|| \frac{p}{\mathbf{r}-v}|$
 $\leq H \sum_{v=0}^{n} |\mathbf{0}_{\mathbf{n}-v}|$
 $= H \sum_{v=0}^{n} |\mathbf{0}_{\mathbf{n}-v}|$
 $= 0 (|\mathbf{0}_{\mathbf{n}}^{T}|),$
since $\mathbf{Q}_{v} = 1$ for $v \geq k$.

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And it is obvious that $k_{n-r}^{0*} \rightarrow 0$ as $n \rightarrow \infty$ for each r. Thus, by [6, Proposition 1] again, we have $(N, P_n) \implies (N, Q_n).$ COROLLARÝ 6.5 $(N, P_n) \iff (N, q_n)^* implies that <math>(N, P_n) \iff (N, Q_n)$. THEOREM 6.16 $[C_1, N_p] => [C_1, N_q]$ if and only if $(N, P_n) \stackrel{=>}{\underset{c}{=}} (N, Q_n)$. PROOF By Theorem 6.1 we know that $[C_1, N_p] \leq [N, P_n]$ and $[\mathcal{C}_1, \mathbb{N}_{\alpha}] \iff [\mathbb{N}, \mathbb{Q}_{n}^{\bullet}].$ By [6, Theorem 1], (c.f. Lemma 5.4), we have that if $(N, P_n) => (N, Q_n)$ then $[N, P_n] => [N, Q_n]$. Thus, if $(N,P_n) \Rightarrow (N,Q_n)$ then $[C_1, \hat{N}_p] \Rightarrow [C_1, N_q]$. Conversely, 'by Theorem 6.14, we have that if $[C_1, N_p] \Rightarrow [C_1, N_q], \text{ then } N_p \Rightarrow N_q.$ Hence, if $[C_1, N_p] = [C_1, N_q]$ then $(N, p_n) = (N, q_n)$. It follows from Theorem 6.15 that if $[C_1, N_p] \stackrel{=}{=}$ $[C_1, N_q]$ then $(N, P_n) => (N, Q_n)$. ||| COROLLARY 6.6 $[C_1, N_p] \iff [C_1, N_q]$ if and only if $(N, P_n)^* \iff (N, Q_n)$. , THEOREM 6.17 $[C_1, N_p] => [C_1, N_q]$ if and only if $(C, 1)N_p => (C, 1)N_q$. PROOF $(N,P_n) = (\overline{N},P_n) (N,P_n).$

From the proof of Theorem 6.1, we know that $(\overline{N}, P_n) \iff (C, 1)$. Thus $(N, P_n) \iff (C, 1) (N, P_n) \iff (C, 1) N_p$ and similarly we have $(N, Q_n) \iff (C, 1) (N, q_n) \iff (C, 1) N_q$.

It follows from Theorem 6.16 that $[C_1, N_p] \Rightarrow [C_1, N_q]$ if and only if $(C, 1)N_p \Rightarrow (C, 1)N_q$.

COROLLARY 6.7

 $[C_1, N_p] \iff [C_1, N_q]$ if and only if $(C, 1)N_p \iff (C, 1)N_q$.

\$6.5 ABSOLUTE POLYNOMIAL NÖRLUND METHODS OF SUMMABILITY THEOREM 6.18

If q(z)/p(z) has no poles on or within the unit circle, then $|N_p| \Rightarrow |N_q|$. PROOF

Suppose q(z)/p(z) has no poles on or within the unit circle, but has poles $\lambda_1, \lambda_2, \ldots, \lambda_k$ of orders m_1, m_2, \ldots, m_k outside the unit circle. Let

$$t_{n} = \sum_{\nu=0}^{n} p_{\nu} s_{n-\nu}$$
$$u_{n} = \sum_{\nu=0}^{n} q_{\nu} s_{n-\nu}, \quad \text{for } n = 0, 1, \dots$$

Then by Theorem 6.9

$$u_{n} = \sum_{\nu=0}^{n} c_{\nu} t_{n-\nu}^{*} + \sum_{r=1}^{\ell} \sum_{\rho=1}^{m_{r}} c_{r,\rho} \sum_{\nu=0}^{n} \left(\nu + \rho - F \right) \lambda_{r}^{-\nu} t_{n-\nu}$$

where c's are constants, depending only on p_0, p_1, \dots, p_j , q_0, q_1, \dots, q_k , such that $c_n = 0$ for n > k - j and $c_{r,m_r} \neq 0$. Hence

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(C,1) is regular, $s_n \neq s [C_1, N_p]$.

THEOREM 6.19

If q(z)/p(z) has a pole within the unit circle, then there is a sequence which is $|N_p|$ -convergent but not $|N_q|$ -convergent.

PROOF

U,

Let

Since $p(0) \neq 0$, $\frac{1}{p(z)}$ is analytic in a neighbourhood U of origin. There is a sequence $\{s_n\}$ such that for z in

 $\sum_{n=0}^{\infty} s_n z^n = \frac{1}{p(z)}$

 $t_n = \sum_{\nu=0}^{n} p_{\nu} s_{n-\nu},$

 $\mathbf{u}_{n} = \sum_{v=0}^{n} \mathbf{q}_{v} \mathbf{s}_{n-v}$

Then, for z in U,

 $\sum_{n=0}^{\infty} t_n z^n = p(z) \sum_{n=0}^{\infty} s_n z^n$

 $\sum_{n=0}^{\infty} u_n z^{n} = q(z) \qquad \sum_{n=0}^{\infty} s_n z^n = \frac{q(z)}{p(z)} .$

Hence $t_0 = 1$, $t_n = 0$ for n > 0, and so $\{s_n\}$ is $|N_p| - \infty$

convergent. On the other hand $\sum_{n=0}^{n} \mathbf{u}_{n} \mathbf{z}^{n}$ has a radius of

convergence less than unity, because by hypothesis q(z)/p(z)has a pole within the unit circle. Consequently $\{u_n\}$ is not A-convergent and so it is not (C, 1)-convergent. Hence $\{s_n\}$ is not (C, 1)N-convergent. By Proposition 6.1(ii), $\{s_n\}$ is not $[C_1, N_q]$ -convergent. Thus by Proposition 6.2 $\{s_n\}$ is not $|N_q|$ -convergent.

THEOREM 6.20

If $\frac{q(z)}{p(z)}$ has no poles within the unit circle, but has poles on the unit circle, then there is a sequence which is $|N_p|$ -convergent but not $|N_q|$ -convergent. PROOF

Let the poles of $\frac{q(z)}{p(z)}$ be $\lambda_1, \lambda_2, \ldots, \lambda_k$ of orders m_1, m_2, \ldots, m_k . Let the numbering be such that of these poles $\lambda_1, \lambda_2, \ldots, \lambda_k$, are on the unit circle, $\lambda_{k'+1}, \ldots, \lambda_k$ are outside the unit circle.

Since $p(0) \neq 0$, $\frac{1}{(1-z)p(z)}$ is analytic in a neighbourhood U of origin. There is a sequence $\{s_n\}$ such that, for z in U,

$$\sum_{n=0}^{\infty} s_n z^n = \frac{1}{(1-z)p(z)}$$

Then, for z in U;

$$\sum_{n=0}^{\infty} t_n z^n = p(z), \quad \sum_{n=0}^{\infty} s_n z^n = \frac{1}{1-z} = 1 + z + z^2 + z^3 + \sum_{n=0}^{\infty} u_n z^n = q(z), \quad \sum_{n=0}^{\infty} s_n z^n = \frac{q(z)}{p(z)} \sum_{n=0}^{\infty} t_n \tilde{z}^n.$$

Hence $t_n = 1$ for all $n \ge 0$ and so $\{s_n\}$ is $|N_p| - \frac{1}{2}$ convergent.

Now, by Theorem 6.9 since $t_n = 1$, for all $n \ge 0$,

$$u_{n} = \sum_{\nu=0}^{n} c_{\nu} + \sum_{r=1}^{\ell} \sum_{\rho=1}^{m} c_{r,\rho} \sum_{\nu=0}^{n} \left(\frac{\nu + \rho - 1}{\rho - 1} \right) \lambda_{r}^{-\nu}$$

where the c's are constants, depending only on $p_0, p_1, \ldots, p_j, q_0, q_1, \ldots, q_k$ such that $c_n = 0$ for n > k - jand $c_{r,m_{r}} \neq 0$. Thus $\mathbf{u}_{n-1} = \mathbf{c}_{n+1} + \sum_{r=1}^{k} \sum_{\rho=1}^{r} \mathbf{c}_{r,\rho} \begin{pmatrix} n+\rho-1\\ \rho-1 \end{pmatrix} \lambda_{r}^{-n}$ $= c_{n} + \sum_{r=\ell}^{\ell} \sum_{j=1}^{m_{r}} c_{r,\rho} {n+\rho-1 \choose \rho-1} \lambda_{r}^{-n} + \sum_{r=1}^{\ell} \sum_{\rho=1}^{m_{r}} c_{r,\rho} {n+\rho-1 \choose \rho-1} \lambda_{r}^{-n}$ $= w_n^1 + w_n^2,$ where $w_n^1 = c_n + \sum_{r=\ell'+1}^{\ell'} \sum_{\rho=1}^{m} c_{r,\rho} {n+\rho-1 \choose \rho-1} \lambda_r^{-n}$ $\gamma_{\mathbf{n}}^{2} = \sum_{\mathbf{r}=1}^{\ell} \sum_{\rho=1}^{\mathbf{m}} \mathbf{c}_{\mathbf{r},\rho} \left(\sum_{\rho=1}^{n+\rho-1} \lambda_{\mathbf{r}}^{-n} \right)$ Since $c_n = 0$ for n > k - j, $\sum_{n=0}^{\infty} c_n$ is absolutely convergent, and since $|\lambda_r| > 1$, for $r = l'+1, \ldots$, is absolutely convergent, for r = l + 1, ..., l, 1,2,..., m_r . Hence $\sum_{n=1}^{\infty} w_n^2 + is$ convergent. Now, for w_n^2 , if there are \Re " poles on the unit circle of maximum order m, where $1 \leq l'' \leq l'$ and $m \geq 1$, then we let the numbering be such that $\lambda_1, \lambda_2, \ldots, \lambda_{lu}$ have maximum order m. In this case,

$$\begin{split} |\mathbf{w}_{n}^{2}| &= |\sum_{r=1}^{k} \sum_{p=1}^{m} c_{r,p} \left[\frac{|\mathbf{r}|_{p-1}}{p-1} \right] \lambda_{r}^{-n} + \sum_{r=k^{T}+1}^{k} \sum_{p=1}^{m} c_{r,p} \left[\frac{|\mathbf{r}|_{p-1}}{p-1} \right] \lambda_{r}^{-n} \\ &= 0|c_{1,m} \left[\frac{|\mathbf{r}|_{m-1}}{m-1} \right] \lambda_{1}^{-n} + c_{2,m} \left[\frac{|\mathbf{r}|_{m-1}}{m-1} \right] \lambda_{2}^{-n} + \cdots + c_{k',m} \left[\frac{|\mathbf{r}|_{m-1}}{m-1} \right] \lambda_{k''}^{-n} | \\ &= 0 \left[\frac{|\mathbf{r}|_{m-1}}{m-1} \right] |c_{1,m} \overline{\lambda}_{1}^{-n} + c_{2,m} \overline{\lambda}_{2}^{-n} + \cdots + c_{k'',m} \overline{\lambda}_{k''}^{-n} | \\ &= 0 \left[\frac{|\mathbf{r}|_{m-1}}{m-1} \right] |c_{1,m} \overline{\lambda}_{1}^{-n} + c_{2,m} \overline{\lambda}_{2}^{-n} + \cdots + c_{k'',m} \overline{\lambda}_{k''}^{-n} | \\ &= 0 \left[\frac{|\mathbf{r}|_{m-1}}{m-1} \right] |c_{1,m} \overline{\lambda}_{1}^{-n} + c_{2,m} \overline{\lambda}_{2}^{-n} + \cdots + c_{k'',m} \overline{\lambda}_{k''}^{-n} | \\ &= 0 \left[\frac{|\mathbf{r}|_{m-1}}{m-1} \right] |c_{1,m} \overline{\lambda}_{1}^{-n} + c_{2,m} \overline{\lambda}_{2}^{-n} + \cdots + c_{k'',m} \overline{\lambda}_{k''}^{-n} | \\ &= 0 \left[\frac{|\mathbf{r}|_{m-1}}{m-1} \right] |c_{1,m} \overline{\lambda}_{1}^{-n} + c_{2,m} \overline{\lambda}_{2}^{-n} + \cdots + c_{k'',m} \overline{\lambda}_{k''}^{-n} | \\ &= 0 \left[\frac{|\mathbf{r}|_{m-1}}{m-1} \right] |c_{1,m} \overline{\lambda}_{1}^{-n} + c_{2,m} \overline{\lambda}_{2}^{-n} + \cdots + c_{k'',m} \overline{\lambda}_{k''}^{-n} | \\ &= 0 \left[\frac{|\mathbf{r}|_{m-1}}{m-1} \right] |c_{1,m} \overline{\lambda}_{1}^{-n} + c_{2,m} \overline{\lambda}_{k''}^{-n} + c_{k'',m} \overline{\lambda}_{k''}^{-n} | \\ &= 1, 2, \ldots, k^{m}; \text{ and } c_{\nu,m} \neq 0, \text{ for } \nu = 1, 2; \ldots, k'', \text{ by} \\ &\text{Lemma 6.1 and Lemma 6.2 we know that} \\ |c_{1,m} \overline{\lambda}_{1}^{-n} + c_{2,m} \overline{\lambda}_{2}^{-n} + \cdots + c_{k'',m} \overline{\lambda}_{k''}^{-n} | \\ &\text{cannot be convergent. A fore tow tat is does not convergent for k^{m} \geq 1. Thus \left\{ |c_{1,m} \overline{\lambda}_{1}^{-n} + c_{2,m} \overline{\lambda}_{k''}^{-n} | \right\} \\ &\text{cannot be convergent. A fore tow i to as n + \infty. This means that \sum_{n=0}^{\infty} ||w_{n}^{-}| diverges. Consequently \\ & \frac{n}{k} ||w_{n}^{-}| diverges, as n + \dots \\ & \frac{n}{n=0} ||w_{n}^{-} - d_{n-1}| \text{ diverges, as n + \dots } \\ &\text{If other words, } \{s_{n}\} \text{ is not } |N_{q}| \text{-convergent. } \\ & \text{THEOREN 6.21 \\ & \text{In order that } |N_{p}| = > |N_{q}|, \text{ it is necessary and sufficient that, \frac{q(z)}{p(z)} \text{ should not have poles on or within the unit or lose.} \\ \end{array}$$

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PROOF

The sufficiency part of the theorem follows from Theorem 6.18. The necessity part follows from Theorems 6,19 , and 6.20. COROLLARY 6.8 $|\mathbf{N}_{\mathbf{p}}| \Rightarrow |\mathbf{N}_{\mathbf{q}}| \text{ if and only if } \mathbf{N}_{\mathbf{p}} \Rightarrow \mathbf{N}_{\mathbf{q}}.$ °R00F This follows from Theorems 6.5 and 6.21. /// COROLLARY 6.9 $|N_p| \Rightarrow |N_q|$ if and only if $[C_1, N_p] \Rightarrow [C_1, N_q]$. PROOF This follows from Theorems 6.13 and 6.21. COROLLARY 6.10 $|N_p| \iff |N_q|$ if and only if $\frac{q(z)}{p(z)}$ and $\frac{p(z)}{q(z)}$ both have ••• no poles on or within the unit circle. COROLLARY 6.11 $|N_p| \ll |N_q|$ if and only if $N_p \ll N_q$. COROLLARY 6.12. $|N_p| \ll |N_q|$ if and only if $[C_1, N_p] \ll [C_1, N_q]$. Noting that N_a is identical with I when q(z) = 1, we have, as a consequence of Corollary 6.10, the following corollary. COROLLARY 6.13 In order that $\{s_n\}$ is $|N_p|$ convergent if and only if

that p(z) should not have zeros on or within the unit circle.

S6.6 SOME MINOR RESULTS

If f(z) = p(z)q(z), then

(i) $[C_1, N_p] \Rightarrow [C_1, N_f] and [C_1, N_q] \Rightarrow [C_1, N_f],$ (ii) $|N_p| \Rightarrow |N_f| and |N_q| \Rightarrow |N_f|.$ PROOF

(i) follows from Theorem 6.13 and (ii) follows from Theorem 6.21. ///

CORQLLARY 6.14

The methods $[C_1, N_p]$ and $[C_1, N_q]$ are consistent, i.e., if $s_n \stackrel{*}{\rightarrow} s [C_1, N_p]$ and $s_n \stackrel{*}{\rightarrow} s' [C_1, N_q]$, then s = s'.

THEOREM 6.23

If h(z) is the highest common factor of p(z) and q(z)normalized so as to make h(1) = 1, then

(i) a sequence is both $[C_1, N_p]$ - and $[C_1, N_q]$ - convergent if and only if it is, $[C_1, N_n]$ - convergent,

(ii) a sequence is both $|N_p|$ - and $|N_q|$ -convergent if and only if it is $|N_h|$ -convergent.

PROOF

(i) The sufficiency part follows from Theorem 6.22 (i).
 To prove the necessity part, we observe that there are polynomials

$$\begin{aligned} a(z) &= \sum_{n=0}^{l} a_n z^n \\ b(z) &= \sum_{n=0}^{l} b_n z^n \\ such that h(z) &= a(z)p(z) + b(z)q(z) \\ &= \sum_{n=0}^{l} b_n z^n, say, \\ \text{where } k_1, k_2, k_3 \text{ are non-negative integers.} \end{aligned}$$
Hence if $t_n &= \sum_{\nu=0}^{n} p_\nu s_{n-\nu}$ and $u_n &= \sum_{\nu=0}^{n} q_\nu s_{n-\nu}$, then $w_n &= \sum_{\nu=0}^{n} h_\nu s_{n-\nu} = \sum_{\nu=0}^{n} a_\nu t_{n-\nu} + \sum_{\nu=0}^{n} b_\nu u_{n-\nu}, \\ \text{where } a_\nu &= 0, \text{ for } \nu > k_1 \text{ and } b_\nu &= 0, \text{ for } \nu > k_2. \\ \text{Without loss of generality, we may assume } s_n + o[C_1, N_p] \\ \text{and } s_n + o[C_1, N_q]. Now \\ \hline \frac{1}{m+1} \sum_{n=0}^{m} \frac{1}{1} w_n! \leq \frac{1}{m+1} \sum_{n=0}^{m} \sum_{\nu=0}^{n} |a_\nu|!| t_{n-\nu}! + \frac{1}{m+1} \sum_{n=0}^{m} \sum_{\nu=0}^{n} |b_\nu|| u_{n-\nu}! \\ &\leq \frac{1}{m+1} \sum_{n=0}^{m} (\sum_{\nu=0}^{l} |a_\nu|!) |t_n| + \frac{1}{m+1} \sum_{n=0}^{m} \sum_{\nu=0}^{l} |b_\nu|| |u_n|. \\ &= o(1), \text{ as } m + \infty. \end{aligned}$
That is $s_n + o[C_1, N_n].$
(ii) The sufficiency part follows from Theorem 6.22(ii). As in the proof of (i), $w_n = \sum_{\nu=0}^{n} a_{n-\nu} t_\nu + \sum_{\nu=0}^{n} b_{n-\nu} u_\nu. \end{aligned}$

Hence $= \sum_{\nu=0}^{n} a_{n=\nu}^{\infty} (t_{\nu} - t_{\nu-1}) + \sum_{\nu=0}^{n} b_{n-\nu} (u_{\nu} - u_{\nu-1}),$ where $t_{1} = 0$, $u_{-1} = 0$, $w_{-1} = 0$ and $a_{n-v} = 0$ if $n - v > l_{1}$. $b_{n-\nu} = 0$, if $n - \nu > \ell_2$. $\sum_{n=0}^{\infty} |w_n - w_{n-1}|$ $\leq \sum_{n=0}^{m} \sum_{\nu=0}^{n} |a_{n+\nu}| |t_{\nu} - t_{\nu-1}| + \sum_{n=0}^{m} \sum_{\nu=0}^{n} |b_{n-\nu}| |u_{\nu} - u_{\nu-1}|$ $\leq \left(\sum_{\nu=0}^{\ell} |a_{\nu}|\right) \sum_{n=0}^{m} |t_{n} - t_{n-1}| + \left(\sum_{\nu=0}^{\ell} |b_{\nu}|\right) \sum_{n=0}^{m} |u_{n} - u_{n-1}|.$ Hence if $\sum_{n=0}^{m} |t_n - t_{n-1}| = O(1)$ and $\sum_{n=0}^{m} |u_n - u_{n-1}| = O(1)$, then $\sum_{n=0}^{m} |w_n - w_{n-1}| = O(1)$.

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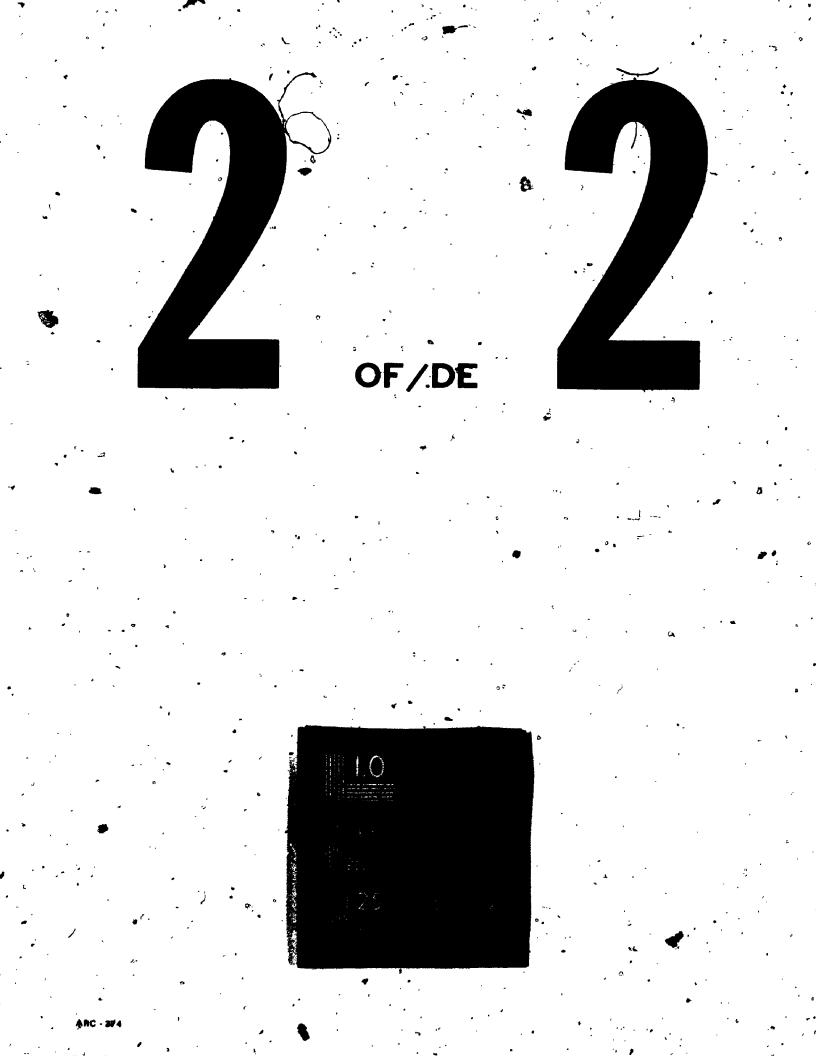
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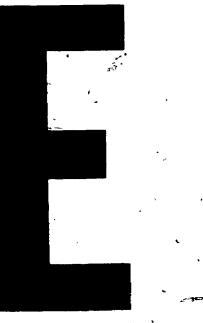
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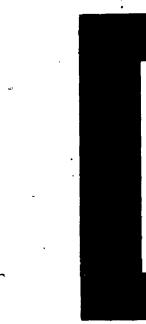


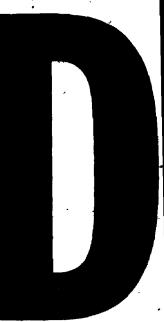


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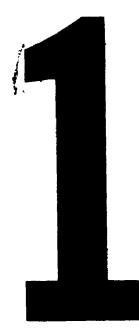


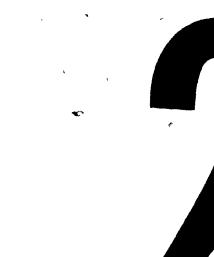












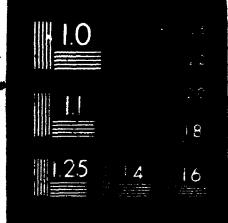


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ASPECTS OF STRONG SUMMABILITY ASSOCIATED WITH GENERALISED CESÀRO, "RIESZ AND NÖRLUND SUMMABILITY

> Edward Hai-Wei <u>Chang</u> Department of Mathematics

by

Submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy

Faculty of Graduate Studies The University of Western Ontario London, Ontario, Canada March, 1975

Edward Hai-Wei Chang 1975

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Generalised Cesàro Summability, Riesz Summability and Strong Riesz Summability have been extensively investigated by various authors. In this thesis a definition of Strong Generalised Cesàro Summability Method is proposed and the question of its equivalence with the Strong Riesz Summability Method is established. In Chapter 3 some equivalence theorems between the Generalised Cesàro Methods and the Strong Generalised Cesàro Method. In Chapter 4 inclusion theorems between the Absolute Generalised Cesàro Methods and the Strong Generalised Cesàro Method are obtained.

ABSTRACT

We extend a result due to Kuttner, obtaining some strict inclusion theorems between Cesaro and Discrete Riesz Methods of Summability. And our investigation in this respect stems from Borwein and Cass's work on Strong Nörlund Summability.

In Chapter 6 we consider Nörlund Methods of Summability Associated with Polynomials which have been investigated by Borwein, and consider Strong and Absolute Nörlund Methods associated with them. We show, for example, that two polynomial Nörlund Methods are equivalent if and only if the associated Strong Methods are equivalent.

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CONVENTIONS

In this thesis, the symbols H, H_1 , H_2 , H_3 are used throughout to denote positive constants, but not necessarily having the same value at each occurrence.

The theorems, lemmata and corollaries are numbered by chapter. For example, Theorem 3.1 is the first theorem in Chapter 3.

At the end of each proof we use the symbol /// to show that the proof is complete.

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CHAPTER 1

STRONG GENERALISED CESARO SUMMABILITY

§1.1 INTRODUCTION

We suppose throughout the thesis that λ = $\{\lambda_n\}$ is a sequence satisfying

(1.1) $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_n \neq \infty$

For the sake of convenience we take $\lambda_0 = 0$ in (1.1) instead of $\lambda_0 \ge 0$. By doing so we find that there is no loss of generality. This remark will be amplified on page 5.

We suppose also that p is a non-negative integer and for the series $\sum_{v=0}^{\infty} a_v$ we use the notation

 $s_n = \sum_{\nu=0}^{n} a_{\nu}, \quad n = 0, 1, 2, \dots$

In this chapter we introduce a definition of Strong Generalised Cesàro Summability and investigate some of its properties. We also give the definitions of several other summability methods whose properties and relations with the Strong Generalised Cesàro Summability are investigated in the later chapters. 'If a given summability method T assigns the sum s

to the series $\sum_{\nu=0}^{n} a_{\nu}$ with sequence of partial sums $\{s_n\}$, we

say that $\sum_{\nu=0}^{n} a_{\nu}$ is T-summable or $\{s_n\}$ is T-convergent to s.

 $\sum_{\nu=0}^{1} a_{\nu} = s (T)$

We denote this by

or by

 $s_n \rightarrow s$ (T).

A method of summability T is said to be *regular*, if $s_n \rightarrow s(T)$ whenever the sequence $\{s_n\}$ converges to s.

Let $Q = \{q_{n,r}\}$ (n, r = 0, 1, 2, ...) be a (summability) matrix and let

(1.2)
$$\sigma_n = \sum_{r=0}^{\infty} q_{n,r} s_r,$$

The sequence $\{s_n\}$ is said to be Q-convergent to the sum s if σ_n exists for n = 0, 1, 2, ... and tends to s as n tends to infinity.

The matrix $Q = \{q_{n,r}\}$ is regular if and only if

 $\sup_{n\geq 0} \sum_{r=0}^{\infty} |q_{n,r}|^{*} < \infty,$

(1.3)

(1.4) $\lim_{n \to \infty} q_{n,r} = 0$, for r = 0, 1, 2, ...,

(1.5) $\lim_{n\to\infty} \sum_{r=0}^{\infty} q_{n_i, r} = 1.$

This is the Toeplitz Theorem for the regularity of the matrix Q.

The symbol P will be reserved for matrices $\{{\tt p}_{\tt n,r}\}$ with

$$p_{n,r} \ge -\theta$$
 (n,r = 0,1,2,...):

Such matrices will be called non-negative matrices.

Let $\mu > 0$. The Strong Summability Methods $[P,Q]_{\mu}$ are defined as follows. We write $s_n \neq s [P,Q]_{\mu}$ if

(1.6)
$$\tau_n = \sum_{\nu=0}^{\infty} p_{n,r} |\sigma_r - s|^{\nu}$$

exists for n = 0, 1, 2, ... and tends to zero as n tends to infinity. Thus s is the $[P,Q]_{\mu}$ -limit of $\{s_n\}$ and the sequence is $[P,Q]_{\mu}$ -convergent to s.

If V and W are summability methods of any of the above types we shall say that W *includes* V, and use the notation V => W, if any sequence V-convergent to s is necessarily W-convergent to s. If W includes V but V does not include W, the inclusion V => W is said to be *strict*. If both V => W and W => V, we say that V and W are *equivalent* and write V <=> W.

Let $\mu > 0$. We say that $\{s_n\}$ is absolutely (Q)_u-convergent or $|Q|_{\mu}$ -convergent if , (1.7)

§ 7.3 RIESZ SUMMABILITY $(\mathbf{R}, \lambda', \kappa)$

Let $\kappa \geq 0$ and $\lambda = \{\lambda_n\}$ satisfy (1.1). The Riesz \cdot Summability Method (R,λ,κ) is defined as follows.

 $\sum_{n=1}^{\infty} n^{\mu-1} |\sigma_n - \sigma_{n-1}|^{\mu} < \infty.$

Let
$$A_{\lambda}^{\kappa}(\tau) = \sum_{\substack{\lambda_{\mathcal{V}} < \tau \\ \lambda_{\mathcal{V}} < \tau}} \mathbf{a}_{\mathcal{V}}$$
, for $\kappa = 0$,
 $A_{\lambda}^{\kappa}(\tau) = \sum_{\substack{\lambda_{\mathcal{V}} < \tau \\ \lambda_{\mathcal{V}} < \tau}} (\tau - \lambda_{\mathcal{V}})^{\kappa} \mathbf{a}_{\mathcal{V}}$, for $\kappa > 0$,
 $R_{\lambda}^{\kappa}(\tau) = A_{\lambda}^{\kappa}(\tau) = \sum_{\substack{\lambda_{\mathcal{V}} < \tau \\ \lambda_{\mathcal{V}} < \tau}} \mathbf{a}_{\mathcal{V}}$, for $\kappa = 0$,
and $R_{\lambda}^{\kappa}(\tau) = \sum_{\substack{\lambda_{\mathcal{V}} < \tau \\ \lambda_{\mathcal{V}} < \tau}} (1 - \frac{\lambda \mathcal{V}}{\tau})^{\kappa} \mathbf{a}_{\mathcal{V}}$, for $\kappa > 0$.

The series $\sum_{\nu=0}^{3} a_{\nu}$ is said to be (R,λ,κ) -summable to s, if

$$R^{\kappa}_{\lambda}(\tau) \rightarrow s \quad as \tau \rightarrow \infty$$
.

(See Hardy and Riesz [12, pp. 21-22].) .

§7.4 STRONG RIESZ SUMMABILITY $[R, \lambda, p+1]_{\mu}$

The series
$$\sum_{\nu=0}^{\infty} a$$
 is said to be strongly Riesz

Summable to s, with order p+1 and index $\mu > 0$, if

$$\mathbf{F}^{\mathbf{p+1}}(\omega) = \int_{0}^{\omega} |\mathbf{A}_{\lambda}^{\mathbf{p}}(\tau) - \mathbf{s}\tau^{\mathbf{p}}|^{\mu} d\tau = o(\omega^{\mathbf{p}\mu+1})$$

We denote this by

$$\sum_{\nu=0}^{\infty} a_{\nu} = s [R, \lambda, p+1]_{\mu}.$$

The definition of the Strong Riesz Summability we have given here is due to Glatfeld [15]. Srivastava [24] and Boyd and Hyslop [8] have also given definitions of Strong Riesz Summability, but we shall not be concerned with them here. 5

We give now two examples to illustrate that no loss of generality is involved by taking $\lambda_0 = 0$ in (1.1).

Our first example deals with Riesz Summability.

Let $\lambda = \{\lambda_n\}$ satisfy $0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n \rightarrow \infty$

and let $\delta = \{\delta_n\}$ satisfy

 $\lambda_1 > \delta_0 > 0$ and $\delta_n = \lambda_n$ for $n \neq 0$. Let $R_{\lambda}^{K}(t)$ be defined as in §1.3 and let

Since $(1 - \frac{a_0}{2}) = a_0 + 0$ as $\tau + \infty$, $R_0^k(\tau) + s$ if and only

 $A^{\mathbf{p}}_{\delta \mathbf{v}}(\tau) = \sum_{\delta_{v} < \tau} (\tau - \delta_{v})^{\mathbf{p}} a_{v}.$

 $R_{\delta}^{\kappa}(\tau) = \sum_{\delta < \tau} (1 - \frac{\upsilon}{\tau})^{\kappa} a_{\upsilon}.$

 $R_{\delta}^{\kappa}(\tau) - s = R_{\lambda}^{\kappa}(\tau) - s + R_{\delta}^{\kappa}(\tau) - R_{\lambda}^{\kappa}(\tau)^{\circ}$

 $= R_{\lambda}^{\kappa}(\tau) - s + [(1 - \frac{\delta_0}{\tau})]_{\alpha} a_{0} - a_{0}]^{\alpha}.$

Then

Ĵ

If $R_{\lambda}^{K}(\tau) \neq s$, as $\tau \neq \infty$. Our other example deals with Strong Riesz Summability. Let $A_{\lambda}^{p}(\tau)$ be defined as in §1.3 and let $\neq \infty$.

Then
$$I_{1,z} = \int_{0}^{\omega} [\lambda_{0}^{p}(\tau) - s\tau^{p}]^{\mu} d\tau$$

$$= \int_{0}^{\omega} [\lambda_{1}^{p}(\tau) - s\tau^{p} + \lambda_{0}^{p}(\tau) - \lambda_{1}^{p}(\tau)]^{\mu} d\tau$$

$$\leq 2^{\mu} (\int_{0}^{\omega} [\lambda_{1}^{p}(\tau) - s\tau^{p}]^{\mu} dt + \int_{0}^{\omega} [a_{0} dt + \delta_{0}]^{p} - a_{0} \tau^{p}]^{\mu} d\tau$$

$$= 2^{\mu} (I_{2} + I_{3})^{\mu}$$
Regarding a_{0} as the series $\int_{0}^{\infty} b_{v}$ with $b_{0} = a_{0}$ and $b_{v} = 0$ for $v > 0$, we have $(f - \delta_{0})^{p}a_{0} = \int_{0}^{\infty} (\tau - \delta_{v})^{p}b_{v}$.
Since $\int_{v=0}^{\infty} b_{v} = a_{0}$ and $(R, \lambda, p+1)_{\mu}$ is regular. (see Glatfeld
(I5)A, thus $I_{3} = o(\omega^{p\mu+1})$. Hence $I_{2} = o(\omega^{p\mu+1}) => I_{1} = o(\omega^{p\mu+1})$.
Since $\int_{v=0}^{\infty} b_{v} = a_{0} (r \tau - b_{v})^{p}dt$
 $i = t \lambda = (\lambda_{n})$ satisfy (1.1).
Define $C_{n}^{p} = \int_{v=0}^{p} a_{v}$, for $p = 0$,
 $t_{n}^{p} = C_{n}^{p} = \int_{v=0}^{p} a_{v} (for p = 0)$,
 $t_{n}^{p} = C_{n}^{p} = \int_{v=0}^{p} a_{v} (for p = 0)$,
 $t_{n}^{p} = C_{n}^{p} = \int_{v=0}^{p} a_{v} (for p = 0)$,
 $t_{n}^{p} = (\lambda_{n+1} \dots \lambda_{n+p})^{-1} C_{n}^{p}$
 $= \int_{v=0}^{n} (1 - \frac{\lambda_{v}}{\lambda_{n+1}}) \dots (1 - \frac{\lambda_{v}}{\lambda_{n+p}})a_{v}$, for $p = 1, 2, 3, \dots$.

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If $t_n^p \rightarrow s as n \rightarrow \infty$, then $\sum_{\nu=0}^{n} a_{\nu}$ is said to be (C, λ, p) summable

to s and we write

0

$$\sum_{\nu=0}^{\infty} a_{\nu} = s \quad (C, \lambda, p).$$

' ? '

Since (C,λ,p) is a matrix method in the sense described in §1.2, we shall find it convenient to denote both the summability method and its associated matrix by (C,λ,p) . Since the entries in the matrix (C,λ,p) are zero above the main diagonal and non-zero on the main diagonal, it has an inverse.

\$1.6 STRONG GENERALISED CESÀRO SUMMABILITY [C, λ ,p+1]_u

Let $\lambda = \{\lambda_n\}$ satisfy (1.1). We define

 $E_n^p(\lambda) = E_n^p = 1, \text{ for } p = 0,$

 $E_n^{\hat{p}}(\lambda) = E_n^p = \lambda_{n+1} \dots \lambda_{n+p}, \text{ for } p = 1,2,3,\dots,$ and n = 0,1,2,3,...

Since $\lambda_0 = 0$, we obtain

$$\mathbf{E}_{\mathbf{m}}^{\mathbf{p+1}} = \sum_{n=0}^{\mathbf{m}} (\lambda_{n+\mathbf{p+1}} - \lambda_n) \mathbf{E}_{\mathbf{n}}^{\mathbf{p}}.$$

We define

$$T_{m,\mu}^{1} = \sum_{n=0}^{m} (\lambda_{n+1} - \lambda_{n}) |t_{n}^{0} - s|^{\mu},$$

$$T_{m,\mu}^{p+1} = \sum_{n=0}^{m} (\lambda_{n+p+1} - \lambda_{n}) E_{n}^{p} |t_{n}^{p} - s|^{\mu},$$

$$\sigma_{m,\mu}^{1} = \lambda_{m+1}^{-1} \sum_{n=0}^{m} (\lambda_{n+1} - \lambda_{n}) |t_{n}^{0} - s|^{\mu},$$

 $\sigma_{m,\mu}^{p+1} = \frac{T_{m,\mu}^{p+1}}{E_{m}^{p+1}} = \frac{1}{E_{m}^{p+1}} \sum_{n=0}^{m} (\lambda_{n \neq p+1} - \lambda_{n}) E_{n}^{p} |t_{n}^{p} - s|^{\mu}.$

We say that the series $\sum_{\nu=0}^{\nu} a_{\nu}$ is Strongly Generalised Cesaro

Summable to s, with order p+1 and index μ , if

 $\sigma_{m,\mu}^{p+1} = o(1) \quad \text{as } m \to \infty.$

And we use the notation

$$\sum_{\nu=0}^{\infty} a_{\nu} = s [C, \lambda, p+1]_{\mu}.$$

Generalised Cesaro Summability was first introduced by Jurkat, [16]. Burkill, [9], gave a different definition. The definition we use here is due to Burkill. The definition was extended to accommodate positive non-integral values of p by Borwein, [3]. We have not been able to formulate a suitable definition of $[\underline{C}, \lambda, p+1]_{\mu}$ with p non-integral.

Several persons have investigated relations between Riesz and Generalised Cesàro Summability. In particular, it is proved in Russell [23] that if λ is a sequence satisfying (1.1) and p is a non-negative integer then

 $(C, \lambda, p) => (R, \lambda, p), p = 0, 1, 2, 3, ..., p$

It is proved in Meir [20] that if λ is a sequence satisfying (1.1) and p is a non-negative integer then

 $(R,\lambda,p) \implies (C,\lambda,p), p = 0,1,2,3,...$

If in §1.5 we take $\lambda_n = n$, we recover the classical Cesàro Summability Method (C,p). (See Hardy [11].)

If in §1,6 we take $\lambda_n = n^2$, we obtain a summability. method which although not equal to, is nevertheless equivalent to the classical Strong Cesàro Summability Method [C, p+1] $_{11}$. (See Borwein and Cass [6].) We recall that $\sum_{\nu=0}^{\infty} a_{\nu} = s [C,p+1]_{\mu}$ if and only if $\frac{1}{n+1} \sum_{\nu=0}^{n} |s_{\nu}^{p} - s|^{\mu} = o(1),$ where $S_n^p = \frac{1}{\varepsilon_n^p} \sum_{\nu=0}^n \varepsilon_{n-\nu}^{p-1} s_{\nu}$ and $\varepsilon_n^p = {n+p \choose n}$. In case where no confusion can arise, we omit the subscript μ from $\sigma_{m'_{\mu}\mu}^{p+1}$ and $T_{m'_{\mu}\mu}^{p+1}$ SIMPLE INCLUSION THEOREMS §1.7 In order to simplify the notation and the proofs of theorems occurring later we introduce a matrix $\Lambda_{p+1} = \{\lambda_{m,n}^{p+1}\} = \{\lambda_{m,n}\}$ which is defined as follows. (1.8)For p = 0for n > m; and for p >for n

It follows easily from the Toeplitz conditions (1.3); (1.4), (1.5) that Λ_{p+1} is regular. We now establish some résults pertaining to the Strong Generalised Cesaro Summability. Let C_n^p and t_n^p be defined as in §1.5. Then $C_n^{p+1} - C_{n-1}^{p+1} = (\lambda_{n+p+1} - \lambda_n) C_n^p$ so that $C_n^{p+1} = \sum_{\nu=0}^n (\lambda_{\nu+p+1} - \lambda_{\nu}) C_{\nu}^p.$ `(1.9) (See [23, p. 419].) Hènce $\frac{1}{E^{p+1}} \sum_{n=0}^{m} (\lambda_{n+p+1} - \lambda_n) E^p_n t^p_n$ (1.10) . $= \frac{1}{E^{p+1}} \sum_{n=0}^{m} (\lambda_{n+p+1} + \lambda_n) C_n^p$ $= C_{m}^{p+1} / E_{m}^{p+1}$ $= t_m^{p+1}$. This means, in matrix notation, it $(C,\lambda,p+1) = \Lambda_{p+1} (C,\lambda,p).$ (1.11) Moreover, referring to (1.6), the definitions of $[C, \lambda, p+1]_{\mu}$ and Λ_{p+1} , we have $[C,\lambda,p+1]_{\mu} = [\Lambda_{p+1}, (C,\lambda,p)]_{\mu}.$ (1.12)

The following two theorems are given in Borwein, [1, Theorems 1 and 3]. We reproduce the proofs for the sake of completeness. 11,

If Q is any matrix and $P = \{p_{n,r}\}, where p_{n,r} \ge 0$ for $n,r = 0,1,\ldots, \sum_{r=0}^{\infty} p_{n,r} < M$ for $n = 0,1,\ldots$ and if $\mu_1 > \mu_2 > 0$ then $[P,Q]_{\mu_1} \Longrightarrow [P,Q]_{\mu_2}$. In particular, the conclusion holds if $\mu_1 > \mu_2 > 0$ and P is regular. PROOF

By Hölder's inequality

 $\sum_{r=0}^{\infty} p_{n,r} |w_r|^{\mu_2} \leq (\sum_{r=0}^{\infty} p_{n,r} |w_r|^{\mu_1}) \frac{2^{\mu_1}}{M} M^{1-\mu_2/\mu_1}.$

for any sequence $\{w_n\}$. The required conclusion follows. /// THEOREM 1.2

If P is a regular (non-negative) matrix and Q is any matrix, then

(i)
$$Q \Rightarrow [P,Q]_{\mu'}$$
 for $\mu > 0$,
(ii) $[P,Q]_{\mu} \Rightarrow PQ$, for $\mu \ge 1$.

PROOF

(i) If $s_n \neq s$, then, since P is regular

 $\sum_{r=0}^{n} p_{n,r} |s_r - s|^{\mu} = o(1), \text{ i.e., } I => [P,I]_{\mu} \text{ and inclusion}$

(i) follows.

(ii) Suppose that $s_n + s [P,I]_{\mu}$. Then by Theorem 1.1, $s_n + s [P,I]_1$ and so

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$$\left| \frac{1}{2} p_{n,x}(s_x - s) \right| \leq \sum_{z=0}^{n} p_{n,x}[s_x - s] \quad o(1).$$
Since P is regular, it follows that $s_n + s$ (P). Hence
 $\left[p, \Pi_{\mu} \rightarrow P$ and inclusion (ii) is an immediate consequence. ///
COROLARY 1.1
If $\mu_1 > \mu_2 > 0$, then $[C_i\lambda, p+1]_{\mu_1} \Rightarrow [C_i\lambda, p+1]_{\mu_2}.$
PROOF
By (1.12), we know that $[C,\lambda, p+1]_{\mu} = [\Lambda_{p+1}, (C,\lambda, p)]_{\mu}.$
The inclusion is a consequence of Theorem 1.1 and the fact
that \hbar_{p+1} is a regular and non-negative matrix. ///
COROLLARY 1.2
If $\mu > 0$, then $(C,\lambda, p) \Rightarrow [C,\lambda; p+1]_{\mu}.$
PROOF
Since $[C,\lambda, p+1]_{\mu} = [\Lambda_{p+1}, (C,\lambda, p)]_{\mu}$ and Λ_{p+2} is regular
and non-negative. The corollary is an immediate consequence
of Theorem 1.2 (4). ///
COROLLARY 1.3
If $\mu \ge 1$, then $[C,\lambda, p+1]_{\mu} \Rightarrow (C,\lambda, p+1).$
PROOF
By (1.11), we know that $(C,\lambda, p+1) = \Lambda_{p+1} (C,\lambda, p).$
The corollary is a consequence of Theorem 1.2 (ii). ///
COROLLARY 1.4
Suppose $\mu_1 \ge 1$ and $\mu_2 > 0$. Then
 $[C,\lambda, p+1]_{\mu_1} \Rightarrow [C,\lambda, p+2]_{\mu_2}.$
PROOF
* This is a consequence of Corollary 1.3 and
Corollary 1.2. ///

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We mantion two other properties of
$$[C, \lambda, p+1]_{\mu}$$
 here
(1.13) If $\int_{v=0}^{\infty} a_{v} = s[0, \lambda, p+1]_{\mu}$ and $\int_{v=0}^{\infty} a_{v} = s^{*}(C, \lambda, p+1]_{\mu}$
then $s = s^{*}$
(I.14) If $\mu \ge 0$, then
 $\int_{v=0}^{\infty} b_{v} = s^{*}[C, \lambda, p+1]_{\mu}$
implies
 $\int_{u=0}^{\infty} c_{v} = \int_{v=0}^{\infty} (ua_{v} + \beta b_{v})^{*} = qa + \beta b [C, \lambda, p+1]_{\mu}$.

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CHAPTER 2

EQUIVALENCE BETWEEN STRONG GENERALISED CESÀRO -SUMMABPLITY AND STRONG RIESZ SUMMABILITY

In this chapter we shall establish the equivalence between $[C, \lambda, p+1]_{\mu}$ and $[R, \lambda, p+1]_{\mu}$. We first prove a lemma. (Cf. Glatfeld [15].)

§2.1 A LEMMA

LEMMA 2.1

If $\chi(\tau) \geq 0$, continuous and Riemann integrable in [h, ω], where h is any fixed positive real number and if $\alpha+\delta > 0$ and $\delta > 0$, then

 $\int_{h}^{\omega} \chi(\tau) d\tau = o(\omega^{\delta})$

if and only if

$$= \int_{h}^{\omega} \tau^{\alpha} \chi(\tau) d\tau = o(\omega^{\alpha+\delta}).$$

PROOF

Assume
$$\int_{h}^{\omega} \chi(\tau) d\tau = o(\omega^{\delta})$$
 and let $F(\omega) = \int_{h}^{\omega} \chi(\tau) d\tau$

Then integrating by parts

 $\int_{h}^{\omega} \tau^{\alpha} \chi(\tau) d\tau = [\tau^{\alpha} F(\tau)]_{h}^{\omega} - \alpha \int_{h}^{\omega} \tau^{\alpha-1} F(\tau) d\tau$ $= \omega^{\alpha} \mathbf{F}(\omega) - \alpha \int_{\mathbf{b}}^{\omega} \tau^{\alpha+\delta-1} \frac{\mathbf{F}(\tau)}{\tau^{\delta}} d\tau$ = U - V, $U = o(\omega^{\alpha+\delta})$ by hypothesis. and $\frac{1}{\omega^{\alpha+\delta}} \int_{h}^{\omega} \tau^{\alpha+\delta-1} \frac{F(\tau)}{\tau^{\delta}} d\tau$ Further $= \int_{h}^{\omega} K(\omega,\tau) G(\tau) d\tau,$ $K(\omega,\tau) = \begin{cases} \frac{\tau^{\alpha+\delta-1}}{\omega^{\alpha+\delta}}, & 0 < \tau \leq \omega, \\ 0, & \tau > \omega, \end{cases}$ where $G(\tau) = \frac{F(\tau)}{\tau^{\delta}}$. -and $\int_{b}^{\infty} |K(\omega,\tau)| d\tau$ Now $= \frac{\omega^{\alpha+\delta} - h^{\alpha+\delta}}{(\alpha+\delta)\omega^{\alpha+\delta}}$ $= \frac{1}{\alpha+\delta} (1 - \frac{h^{\alpha+\delta}}{\omega^{\alpha+\delta}})$ $< \frac{1}{\alpha+\delta}$. For every positive y $\lim_{\omega \to \infty} \int_{h}^{\infty} K(\omega, \tau) d\tau$ $= \lim_{\omega \to \infty} \frac{y^{\alpha+\delta} - h^{\dot{\alpha}+\delta}}{(\alpha+\delta)\omega^{\alpha+\delta}}$

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Since G(τ) \rightarrow 0 as $\tau \rightarrow \infty$, it follows from Hardy

[11, Theorem 6] that

$$\int_{h}^{\omega} K(\omega,\tau)G(\tau)d\tau \rightarrow 0, \quad as \ \omega \rightarrow \infty.$$

Thus $(V = o(\omega^{\alpha+\delta}) \cdot p_{\alpha+\delta}^{\alpha+\delta}$

Hence

Conversely, if
$$\int_{h}^{\omega} \tau^{\alpha} \chi(\tau) d\tau = o(\omega^{\alpha+\delta})$$
, we take

 $\int_{-b}^{\omega} \tau^{\alpha} \chi(\tau) d\tau = o(\omega^{\alpha+\delta}).$

 $\tau^{\alpha}\chi(\tau) = \chi(\tau)$ which is non-negative, continuous and integrable in $[h, \omega]$. The result now follows from the first part by replacing δ by $\alpha + \delta$ and α by $-\alpha$. ///

Since $\lambda_0 = 0$, $\mathbb{R}^p_{\lambda}(\mathcal{P}) + a_0$ as $\tau \to 0^+$, we conclude that as a consequence of Lemma 2.1.

(2.1)
$$\int_{0}^{\omega} |A_{\lambda}^{p}(\tau) - s\tau^{p}|^{\mu} d\tau = o(\omega^{p\mu+1})$$

is equivalent to

$$\int_{0}^{\omega} \left| R_{\lambda}^{p}(\tau) - s \right|^{\mu} d\tau = \mathbf{b}(\omega).$$

§2.2 INCLUSION THEOREM FROM RIESZ TO CESARO THEOREM 2.] Let $\mu > 0$ and λ satisfy (1.1). Then (i) $[\mathbf{R}, \lambda, 1]_{\mu} => [\mathbf{C}, \lambda, 1]_{\mu}$, (ii) If $\mathbf{p} > 0$ and $\lambda_{n+1} = O(\lambda_n)$, then $[\mathbf{R}, \lambda, p+1]_{\mu} => [\mathbf{C}, \lambda, p+1]_{\mu}$.

PROF
(1) Suppose
$$\sum_{\nu=0}^{\infty} A_{\nu} = s [R, \lambda, L]_{\mu}$$
 where we may assume,
without loss of generality, that $s = 0$.
 $T_{m}^{1} = \sum_{n=0}^{m} (\lambda_{n+1} - \lambda_{n}\lambda) \sum_{\nu=0}^{n} a_{\nu} |^{\mu}$.
 $= \prod_{n=0}^{m} \int_{\lambda_{n}}^{\lambda_{n+1}} |\lambda_{\nu} \leq a_{\nu}|^{\mu} d\tau$
 $= \int_{0}^{\lambda_{m+1}} |A_{\lambda}^{0}(\tau)|^{\mu} d\tau$
 $= o(\lambda_{m+1})$, as $m + \infty$.
Thus $\sum_{\nu=0}^{\infty} a_{\nu} = 0 [C, \lambda, 1]_{\mu}$.
(ii)' For the case $p \ge 0$, we assume that
 $\int_{0}^{\omega} |R_{\lambda}^{p}(\tau)|^{\mu} d\tau = o(\omega)$.
We are required to show that
 $= o(1)$, as $m^{t+\infty}$.
We divide the proof into four steps.
STEP 'I.
For every n, choose $q = q(n)$, a non-negative integer,
satisfying $q(n) \ge q(n-1)$ and
 $(2\cdot2) - \lambda_{q+1} - \lambda_q = \max((\lambda_{1+1} - \lambda_{1}) |n \le 1 \le n+p)$.

Fixing n we partition the interval $[\lambda_q, \lambda_{q+1}]$ into $\frac{\lambda_{q+1} - \lambda_{q}}{2p+2}$ subintervals of length $\frac{\lambda_{q+1} - \lambda_{q}}{2p+2}$ with the points $\omega_{v} = \omega_{n,v} = \lambda_{q} + \frac{v}{2p+2} (\lambda_{q+1} - \lambda_{q}), \quad v = 0, 1, \dots, 2p+2.$ Since p > 0 and $\lambda_0 = 0$, $|R_{\lambda}^{p}(\tau)|^{\mu}$ is a continuous function of τ in the interval [0, ω]. Applying the Mean yalue Theorem of the alternate subintervals, we have, for $\langle j = 0, 1, 2, ..., p, gumbers$ $\theta_{j} = \theta_{n,j} \in [\omega_{2j+1}, \omega_{2j+2}]$ such that $\int_{\substack{\omega_{2j+2} \\ \omega_{2j+1}}}^{\omega_{2j+2}} |R_{\lambda}^{p}(\tau)|^{\mu} d\tau = (\omega_{2j+2} - \omega_{2j+1}) |R_{\lambda}^{p}(\theta_{j})|^{\mu}$ $= \left(\omega_{2j+2} - \omega_{2j+1} \right) \left| \sum_{\nu=0}^{q} \left(1 - \frac{\lambda_{\nu}}{\theta_{j}} \right)^{p} a_{\nu} \right|^{\mu}.$ $\sum_{j=0}^{p} (\omega_{2j+2} - \omega_{2j+1}) \left| \sum_{\nu=0}^{q} (1 - \frac{\lambda_{\nu}}{\theta_{j}})^{p} a_{\nu} \right|^{\mu}$ ſhus $= \sum_{j=0}^{p} \int_{\omega_{2j+1}}^{\omega_{2j+2}} |R_{\lambda}^{p}(\tau)|^{\mu} d\tau$ $\leq \int_{\lambda}^{\lambda} q+1 |R_{\lambda}^{p}(\tau)|^{di} d\tau.$ Since $\omega_{2j+2} - \omega_{2j+1} = \frac{1}{2p+2} (\lambda_{q+1} - \lambda_q)$, we have $(2.3) \sum_{\substack{n=0 \ j=0}}^{\mathbf{m}} \sum_{\substack{2p+2 \ q(n)+1}}^{\mathbf{p}} \frac{1}{\sqrt{q(n)+1}} \sum_{\substack{n=0 \ q(n)}}^{\mathbf{q(n)}} \frac{q(n)}{\sqrt{q(n)}} \sum_{\substack{n=0 \ q(n)}}^{\mathbf{q(n)}} \frac{1}{\sqrt{q(n)}} a_{\mathbf{v}}$ $\leq \sum_{n=0}^{m} \int_{\lambda_{q(n)}}^{\lambda_{q(n)+1}} |\dot{R}_{\lambda}^{p}(\tau)|^{\mu} d\tau$

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 $\leq (p+1) \int_{-\infty}^{\infty} \frac{1}{|R_{\lambda}^{p}(\tau)|^{\mu} d\tau} d\tau,$ since q(n) is constant for at most p+1 different values of STEP II. Using techniques similar to those used by Borwein [2] we shall show that for every n, there are numbers $y_j = y_{n,j}$, for $j = 0, 1, 2, ..., p_i$ such that the identity $\sum_{i=1}^{p} (x + b_i) \equiv \sum_{i=0}^{p} y_i (x + \delta_i)^{p}$ (2.4) holds for all real x, where $b_i = \frac{\lambda_{n+i} - \lambda_q}{\lambda_{q+1} - \lambda_q}$, for i = 1, 2, ..., p $\delta_{j} = \frac{\theta_{j} - \lambda_{q}}{\lambda_{q+1} - \lambda_{q}}, \quad \text{for } j = 0, 1, 2, \dots, p.$ and The identity (2.4) is equivalent to the system of linear equations ~ $\sum_{j=0}^{\xi} \delta_{j}^{i} y_{j} = \xi_{i}, \quad i = 0, 1, \dots, p,$ (2.5) where $\xi_{i} = {\binom{p}{i}}^{-1} \sum_{b_{r_{1}} b_{r_{2}} \cdots b_{r_{i}}} b_{r_{i}} \\ 1 \leq r_{1} < \cdots < r_{i} \leq p \neq$ (2.6)

and where the sum in the expression for ξ_i is taken to be 1 when i = 0.

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The determinant of the system (2.5) is the Vandermonde

determinant

$$\Delta = \Pi (\delta_{\mathbf{s}} - \delta_{\mathbf{r}})$$
$$0 \leq \mathbf{r} < \mathbf{s} \leq \mathbf{p}$$

(See [25, p. 214].)

Now for s > r

$$s - \delta_{r} = \frac{\theta_{s} - \theta_{r}}{\lambda_{q+1} - \lambda_{q}}$$
$$\geq \frac{\omega_{2s+1} - \omega_{2r+2}}{\lambda_{q+1} - \lambda_{q}}$$

Hence

Using Cramer's rule, we have

$$\mathbf{y}_{\mathbf{r}} = \frac{\Delta_{\mathbf{r}}}{\Delta_{\mathbf{r}}},$$

where Δ_r is the determinant of the matrix $(d_{i,j})$, i,j = 0,1,2,..., p, in which

$$d_{i,r} = \xi_i$$
 and $d_{i,j} = \delta_j^i$, $j \neq r$

SZEP III.

We now show that the numbers $y_{n,r}$ are uniformly bounded. Since

$$|\mathbf{b}_{\mathbf{r}}| = \left|\frac{\lambda_{\mathbf{n}+\mathbf{r}} - \lambda_{\mathbf{q}}}{\lambda_{\mathbf{q}+1} - \lambda_{\mathbf{q}}}\right|$$
$$\leq \frac{\lambda_{\mathbf{n}+\mathbf{p}+1} - \lambda_{\mathbf{n}}}{\lambda_{\mathbf{q}+1} - \lambda_{\mathbf{q}}}$$

$$\leq (p+1) \frac{\lambda_{q+1} - \lambda_{q}}{\lambda_{q+1} - \lambda_{q}}$$

= $(p+1)$,
we see from (2.6) that for $i = 0, 1, 2, ..., p$
 $|\xi_{1}| \leq (p+1)^{p}$.

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Also
$$|\delta_j^i| = (\frac{\theta_{n,j} - \lambda_q}{\lambda_{q+1} - \lambda_q})^i \leq 1$$
, for $i, j = 0, 1, 2, ... p$

Consequently

(2.7)
$$|y_r|^{t} = |y_{n,r}| \leq (2p+2)^{p!} |\Delta_r| \leq H$$

where H is a constant independent of r and n.

Here we establish an inequality between the $[C,\lambda,p+1]_{\mu} - mean and the [R,\lambda,p+1]_{\mu} - mean of the series \sum_{\nu=0}^{\infty} a_{\nu}$

which yields our result.

Let v be any non-negative integer and put

$$x = \frac{\lambda_{q} - \lambda_{v}}{\lambda_{q+1} - \lambda_{q}} \text{ in } (2.4), \text{ we obtain}$$

$$\prod_{\substack{i=1\\j \neq 0}}^{p} \frac{\lambda_{n+i} - \lambda_{v}}{\lambda_{q+1} - \lambda_{q}} = \sum_{\substack{j=0\\j \neq 0}}^{p} y_{j} \frac{\theta_{n,j} - \lambda_{v}}{\lambda_{q+1} - \lambda_{q}}$$
Thus
$$\prod_{\substack{i=1\\j \neq 0}}^{p} (\lambda_{n+i} - \lambda_{v}) = \sum_{\substack{j=0\\j \neq 0}}^{p} y_{j} (\theta_{n,j} - \lambda_{v})^{p}.$$
Dividing by E_{n}^{p} , we have

 $(2.8), \qquad \prod_{i=1}^{p} (1 - \frac{\lambda_{v}}{\lambda_{n+i}}) = \sum_{j=0}^{p} \frac{y_{j} \theta_{n,j}^{p}}{E_{n}^{p}} (1 - \frac{\lambda_{v}}{\theta_{n,j}})^{p}$ $= \sum_{j=0}^{p} C_{n,j} (1 - \frac{\lambda_{v}}{\theta_{n,j}})^{p},$

where $C_{n,j} = \frac{Y_j \frac{\theta_{n,j}^p}{\theta_{n,j}^p}}{E_n^p}$.

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Since $\lambda_{n+1} = O(\lambda_n)$ and $y_{n,r}$ is uniformly bounded, we have

$$|c_{n,j}| \leq \frac{|Y_{n,j}|^{\lambda_{q+1}^p}}{\lambda_{n+1}^p} \leq H_1,$$

 H_1 being independent of n and j.

Now it follows from (2.8) that

$$t_{n}^{p} = \sum_{\nu=0}^{n} (1 - \frac{\lambda_{\nu}}{\lambda_{n+1}}) \cdots (1 - \frac{\lambda_{\nu}}{\lambda_{n+p}}) a$$
$$= \sum_{\nu=0}^{q} (1 - \frac{\lambda_{\nu}}{\lambda_{n+1}}) \cdots (1 - \frac{\lambda_{\nu}}{\lambda_{n+p}}) a$$
$$= \sum_{\nu=0}^{q} \sum_{j=0}^{p} c_{n,j} (1 - \frac{\lambda_{\nu}}{\theta_{n,j}})^{p} a_{\nu}$$
$$= \sum_{j=0}^{p} c_{n,j} \sum_{\nu=0}^{q} (1 - \frac{\lambda_{\nu}}{\theta_{n,j}})^{p} a_{\nu}.$$

Thus

$$\begin{split} \sigma_{m}^{p+1} &= \frac{1}{E_{m}^{p+1}} \sum_{n=0}^{m} (\lambda_{n+p+1} - \lambda_{n}) E_{n}^{p} \left| \epsilon_{n}^{p} \right|^{\mu} \\ &\leq \lambda_{m+p+1}^{-1} \sum_{n=0}^{m} (\lambda_{n+p+1} - \lambda_{n}) \left\{ \sum_{j=0}^{p} |c_{n,j}| \sum_{\nu=0}^{q(n)} (1 - \frac{\lambda_{\nu}}{\theta_{n,j}})^{p} a_{\nu} | \right\}^{\mu} \\ &\leq \lambda_{m+p+1}^{-1} \sum_{n=0}^{m} (p+1) (\lambda_{q(n)+1} - \lambda_{q(n)}) (p+1)^{\mu} \sum_{j=0}^{p} |c_{n,j}|^{\mu} | \sum_{\nu=0}^{q(n)} (1 - \frac{\lambda_{\nu}}{\theta_{n,j}})^{p} a_{\nu} \\ &\leq \frac{H_{2}}{\lambda_{m+p+1}} \sum_{n=0}^{m} (\lambda_{q(n)+1} - \lambda_{q(n)}) \sum_{j=0}^{p} |\sum_{\nu=0}^{q(n)} (1 - \frac{\lambda_{\nu}}{\theta_{n,j}})^{p} a_{\nu} |^{\mu} \\ &= \frac{H_{2} \times (2p+2)}{\lambda_{m+p+1}} \sum_{n=0}^{m} \sum_{j=0}^{p} \frac{1}{2p+2} (\lambda_{q(n)+1} - \lambda_{q(n)}) |\sum_{\nu=0}^{q(n)} (1 - \frac{\lambda_{\nu}}{\theta_{p,j}})^{p} a_{\nu} |^{\mu} \\ &\leq \frac{H_{3}}{\lambda_{m+p+1}} \int_{0}^{\lambda_{m+p+1}} |R_{\lambda}^{p}(\tau)|^{\mu} d\tau. \end{split}$$
The final inequality following from Step I.

Thus $c_{m}^{p+1} = o(1)$ so that $\sum_{\nu=0}^{\infty} a_{\nu} = 0 \ [C,\lambda,p+1]_{\mu}. \end{split}$

\$2.3 INCLUSION THEOREM FROM CESARO TO RIESZ

We now investigate the inclusion in the opposite direction. And to facilitate the discussion we introduce the following notation.

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Given a function f defined in an interval [a,b], and distinct points x_i in this interval, we define

f[x] = f(x)

and
$$f[x_0, x_1, \dots, x_n] = \frac{f[x_0, \dots, x_{n-1}] - f[x_1, \dots, x_n]}{x_0 - x_n}$$

for n = 1, 2, 3,

The quantity $f[x_0, x_1, \dots, x_n]$ is called the *divided* difference of f(x) of n arguments. For an exposition of the properties of divided differences see Milne-Thomson [21, Chapter 1].

In the proof of our next theorem we need the following results of Russell [23, pp. 425-428]. LEMMA 2.2

Let p be a non-negative integer.

Define $C_{\tau}(x) = \begin{cases} (\tau - x)^{p}, & \text{for } 0 \leq x < \tau, \\ 0, & \text{for } x \geq \tau. \end{cases}$ Then, for $\lambda_{n} < \tau \leq \lambda_{n+1}$

(i)
$$A_{\lambda}^{p}(\tau) = (-1)^{p+1} \sum_{\nu=n-p}^{n} C_{\tau} [\lambda_{\nu}, \lambda_{\nu+1}, \dots, \lambda_{\nu+p+1}] (\lambda_{\nu+p+1} - \lambda_{\nu}) C_{\nu}^{p}$$

where we understand $C_{\nu}^{p} = 0$ whenever $\nu < 0$; and

(ii) $for \mathbf{n} - \mathbf{p} \leq \mathbf{v} \leq \mathbf{\tilde{n}}$

$$\left|C_{\tau}\left[\lambda_{\nu},\lambda_{\nu+1},\ldots,\lambda_{\nu+p+1}\right]\right|\left(\lambda_{\nu+p+1}-\lambda_{\nu}\right) \leq H$$

where H is independent of n.

THEOREM 2.2

Let
$$\lambda$$
 satisfy (1.1). Then
(i) if $\mu > 0$, then $[C, \lambda, 1]_{\mu} => [R, \lambda, 1]_{\mu}$,
(ii) if $p > 0$, $\mu \ge 1$ and $\lambda_{n+1} = O(\lambda_n)$, then
 $[C, \lambda, p+1]_{\mu} => [R, \lambda, p+1]_{\mu}$.

PROOF

(i) We suppose that
$$\sum_{\nu=0}^{\infty} a_{\nu} = 0 [C, \lambda, 1]_{\mu}$$
. Thus

$$\sigma_{\mathbf{m}}^{\mathbf{l}} \stackrel{*}{=} \frac{1}{\lambda_{\mathbf{m}+1}} \sum_{\mathbf{n}=0}^{\mathbf{m}} (\lambda_{\mathbf{n}+1} - \lambda_{\mathbf{n}}) |\mathbf{s}_{\mathbf{n}}|^{\mu} = o(1).$$

Hence

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(2.9)
$$\frac{\lambda_{m+1} - \lambda_m}{\lambda_{m+1}} |\mathbf{s}_m|^{\mu} = o(1).$$

Let $\omega > 0$ and suppose $\lambda_m < \omega \leq \lambda_{m+1}$. Then

$$\frac{1}{\omega}\int_{0}^{\omega}|\mathbf{A}_{\lambda}^{0}(\tau)|^{\mu}d\tau = \frac{1}{\omega}\left\{\sum_{n=0}^{m-1}\int_{\lambda_{n}}^{\lambda_{n+1}}|\sum_{\nu=0}^{n}\mathbf{a}_{\nu}|^{\mu}d\tau + \int_{\lambda_{m}}^{\omega}|\sum_{\nu=0}^{m}\mathbf{a}_{\nu}|^{\mu}d\tau\right\}$$

$$= \frac{1}{\omega} \sum_{n=0}^{m-1} (\lambda_{n+1} - \lambda_n) |\mathbf{s}_n|^{\mu} + \frac{1}{\omega} (\omega - \lambda_m) |\mathbf{s}_m|^{\mu}$$
$$\leq \sigma_{m-1}^1 + (1 - \frac{\lambda_m}{\lambda_{m+1}}) |\mathbf{s}_m|^{\mu}.$$

Now $\sigma_m^1 = o(1)$ which together with (2.9) yields

$$\frac{1}{\omega}\int_0^\omega |\mathbf{A}^0_\lambda(\tau)|^{\mu}d\tau = o(1).$$

Thus $\sum_{\nu=0}^{\infty} a_{\nu} = 0 [R, \lambda, 1]_{\mu}$.

(ii) Let $\tau > 0$ and suppose $\lambda_n < \tau \leq \lambda_{n+1}$.

Then using Lemma 2.2 (i) and (ii) we see that

$$\begin{split} \lambda^{(2,10)} & \left[\lambda_{\lambda}^{\mathbf{p}}(\tau)\right]^{\mu} \stackrel{=}{=} \left[\sum_{\nu=n-p}^{n} c_{\tau} \left[\lambda_{\nu}, \lambda_{\nu+1}, \dots, \lambda_{\nu+p+1}\right] \left(\lambda_{\nu+p+1} - \lambda_{\nu}\right) c_{\nu}^{\mathbf{p}}\right]^{\mu} \\ & \leq (p+1)^{\mu} H \sum_{\nu=n-p}^{n} \left[c_{\nu}^{\mathbf{p}}\right]^{\mu} \\ & \text{Suppose } \omega > 0 \text{ and } \lambda_{\mathbf{m}} < \tilde{\omega} \leq \lambda_{\mathbf{m}+1} \text{. Then} \\ & \int_{0}^{\omega} \left[\lambda_{\lambda}^{\mathbf{p}}(\tau)\right]^{\mu} d\tau \\ & \leq \prod_{n=0}^{m} \int_{\lambda_{n}^{-1}}^{\lambda_{n+1}} \left[\lambda_{\lambda}^{\mathbf{p}}(\tau)\right]^{\mu} d\tau \\ & \leq H_{1} \sum_{n=0}^{m} \left(\lambda_{n+1} - \lambda_{n}\right) \sum_{\nu=n-p}^{n} \left[c_{\nu}^{\mathbf{p}}\right]^{\mu} \\ & = H_{1} \sum_{\nu=0}^{m} \sum_{n=\nu}^{m} \left(\lambda_{n+1} - \lambda_{n}\right) \sum_{\nu=0}^{n} \left[c_{\mathbf{p}+\nu}^{\mathbf{p}}\right]^{\mu} \\ & = H_{1} \sum_{\nu=0}^{m} \sum_{n=\nu}^{m} \left(\lambda_{n+1} - \lambda_{n}\right) \sum_{\nu=0}^{p} \left[c_{\mathbf{p}+\nu}^{\mathbf{p}}\right]^{\mu} \\ & \text{so that} \\ & (2,11) \int_{0}^{\omega} \left[\lambda_{\lambda}^{\mathbf{p}}(\tau)\right]^{\mu} d\tau \leq H_{1} \sum_{\nu=0}^{p} \sum_{n=\nu}^{m} \left(\lambda_{n+1} - \lambda_{n}\right) \left[c_{\mathbf{p}-\nu}^{\mathbf{p}}\right]^{\mu} \\ & = \sum_{\mathbf{p}=1}^{\frac{1}{p}} \sum_{n=0}^{m} \left(\lambda_{n+p+1} - \lambda_{n}\right) \sum_{\nu=0}^{p} \left[\tau_{\mathbf{p}}^{\mathbf{p}}\right]^{\mu} \\ & = \sum_{\mathbf{p}=1}^{\frac{1}{p}} \sum_{n=0}^{m} \left(\lambda_{n+p+1} - \lambda_{n}\right) \left[c_{\mathbf{p}}^{\mathbf{p}}\right]^{\mu} \\ & = \frac{1}{E_{\mathbf{p}}^{\mathbf{p}+1}} \sum_{n=0}^{m} \left(\lambda_{n+p+1} - \lambda_{n}\right) \left[c_{\mathbf{p}}^{\mathbf{p}}\right]^{\mu} \\ & = \frac{1}{\left(\overline{E_{\mathbf{p}}^{\mathbf{p}}\right)^{\nu} \lambda_{\mathbf{m}+p+1}} \prod_{n=0}^{m} \left(\lambda_{n+p+1} - \lambda_{n}\right) \left[c_{\mathbf{p}}^{\mathbf{p}}\right]^{\mu} \\ & \geq \frac{1}{\left(\overline{E_{\mathbf{p}}^{\mathbf{p}}\right)^{\nu} \lambda_{\mathbf{m}+p+1}} \prod_{n=0}^{m} \left(\lambda_{n+p+1} - \lambda_{n}\right) \left[c_{\mathbf{p}}^{\mathbf{p}}\right]^{\mu} \\ & \geq \frac{1}{\left(\overline{E_{\mathbf{p}}^{\mathbf{p}}\right)^{\nu} \lambda_{\mathbf{m}+p+1}}} \prod_{n=0}^{m} \left(\lambda_{n+p+1} - \lambda_{n}\right) \left[c_{\mathbf{p}}^{\mathbf{p}}\right]^{\mu} \\ & = \frac{1}{\left(\overline{E_{\mathbf{p}}^{\mathbf{p}}\right)^{\nu} \lambda_{\mathbf{m}+p+1}}} \prod_{n=0}^{m} \left(\lambda_{n+p+1} - \lambda_{n}\right) \left[c_{\mathbf{p}}^{\mathbf{p}}\right]^{\mu} \\ & \geq \frac{1}{\left(\overline{E_{\mathbf{p}}^{\mathbf{p}}\right)^{\nu} \lambda_{\mathbf{m}+p+1}}} \prod_{n=0}^{m} \left(\lambda_{n+p+1} - \lambda_{n}\right) \left[c_{\mathbf{p}}^{\mathbf{p}}\right]^{\mu} \\ & = \frac{1}{\left(\overline{E_{\mathbf{p}}^{\mathbf{p}}\right)^{\nu} \lambda_{\mathbf{m}+p+1}}} \prod_{n=0}^{m} \left(\lambda_{n+p+1} - \lambda_{n}\right) \left[c_{\mathbf{p}}^{\mathbf{p}}\right]^{\mu} \\ & \geq \frac{1}{\left(\overline{E_{\mathbf{p}}^{\mathbf{p}}\right)^{\nu} \lambda_{\mathbf{p}+p+1}}} \prod_{n=0}^{m} \left(\lambda_{n+p+1} - \lambda_{n}\right) \left[c_{\mathbf{p}}^{\mathbf{p}}\right]^{\mu} \\ & = \frac{1}{\left(\overline{E_{\mathbf{p}}^{\mathbf{p}}\right)^{\nu} \lambda_{\mathbf{p}+p+1}}} \prod_{n=0}^{m} \left(\lambda_{n+p+1} - \lambda_{n}\right) \left[c_{\mathbf{p}}^{\mathbf{p}}\right]^{\mu} \\ & = \frac{1}{\left(\overline{E_{\mathbf{p}}^{\mathbf{p}}\right)^{\nu} \lambda_{\mathbf{p}+p+1}}} \prod_{n=0}^{m} \left(\lambda_{n+p+1} - \lambda_{n}\right) \left[c_{\mathbf{p}}^{\mathbf{p}}\right]^{\mu} \\$$

since $\mu \geq 1$.

Thus for
$$\mathbf{r} = 0, 1, 2, ..., p$$

$$\sigma_{m}^{p+1} \geq \frac{1}{\lambda_{m+p+1}^{p\mu+1}} \int_{n=0}^{m} (\lambda_{n+r+1} - \lambda_{n+r}) |C_{n}^{p}|^{\mu}$$

$$= \frac{1}{\lambda_{m+p+1}^{p\mu+1}} \int_{\nu=r}^{m+r} (\lambda_{\nu+1} - \lambda_{\nu}) |C_{\nu-r}^{p}|^{\mu}$$

$$\geq \frac{1}{\lambda_{m+p+1}^{p\mu+1}} \int_{\nu=r}^{m} (\lambda_{\nu+1} - \lambda_{\nu}) |C_{\nu-r}^{p}|^{\mu}.$$
If we now suppose $\sum_{\nu=0}^{\infty} a_{\nu} = 0 [C, \lambda, \overline{p+1}]_{\mu}$, so that

$$\sigma_{m}^{p+1} = o(1), \text{ we have, for } \mathbf{r} = 0, 1, 2, ..., p$$

$$\frac{1}{\lambda_{m+p+1}^{p\mu+1}} \int_{\nu=r}^{m} (\lambda_{\nu+1} - \lambda_{\nu}) |C_{\nu-r}^{p}|^{\mu} = o(1),$$
as $m + \infty$.
Hence in view of (2.11) and the condition $\lambda_{m+1} = O(\lambda_{m}),$,
we have

$$\int_{0}^{\omega} |\lambda_{\lambda}^{p}(\tau)|^{\mu} d\tau = o(\lambda_{m+p+1}^{p\mu+1}) = o(\omega^{p\mu+1}).$$
Hence $\sum_{\nu=0}^{\infty} a_{\nu} = 0 [R, \lambda, p+1]_{\mu}$ for $\mu \ge 1.$ ///
Combining the results of Theorems 2.1 and 2.2,) we

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have the following corollary.

THEOREM 2.3

Let $\lambda = \{\lambda_n\}$ satisfy (1.1). If $\mu > 0$, then $[R, \lambda, 1]_{\mu} \iff [C, \lambda, 1]_{\mu}$. (i) If p > 0, $\mu \ge 1$ and $\lambda_{n+1} = O(\lambda'_n)$, then (ií) $[R,\lambda,p+1]_{\mu} \iff [C,\lambda,p+1]_{\mu}.$

CHAPTER 3

SOME EQUIVALENCE THEOREMS

In this chapter we shall establish some equivalence theorems between various methods of Summability and Strong Summability.

§3.1 SOME LEMMAS

LEMMA 3.1

Let Λ_{p+1} be the matrix defined in §1.7. The inverse matrix $\Lambda'_{p+1} = \{\lambda'_{n,v}\}$ of Λ_{p+1} is given by

(3.1) $\lambda_{n,n}^{\dagger} = \frac{\lambda_{n+p+1}}{\lambda_{n+p+1} - \lambda_{n}},$ $\lambda_{n,n-1}^{\dagger} = \frac{-\lambda_{n}}{\lambda_{n+p+1} - \lambda_{n}},$ $\lambda_{n,\nu}^{\dagger} = 0 \text{ otherwise.}$

PROOF

Let $C_{m,\nu} \stackrel{m}{=} \sum_{n=\nu}^{m} \lambda_{m,n} \lambda_{n,\nu}^{\dagger}$, we show that $C_{m,\nu} \stackrel{m}{=} \delta_{m,\nu}$.

Referring to the definition of Λ_{p+1} , (1.8), we have for $\nu \neq m$

$$C_{m,v} = \lambda_{m,v}\lambda_{v,v}^{1} + \lambda_{m,v+1}^{1} \lambda_{v+1,v}^{1}$$

$$= \frac{1}{\mathbb{E}_{m}^{p+1}} \cdot (\lambda_{v+p+1} - \lambda_{v}) \mathbb{E}_{v}^{p} \frac{\lambda_{v+p+1}}{\lambda_{v+p+1} - \lambda_{v}}$$

$$= \frac{1}{\mathbb{E}_{m}^{p+1}} \cdot (\lambda_{v+p+2} - \lambda_{v+1}) \mathbb{E}_{v+1}^{p} \frac{\lambda_{v+1}}{\lambda_{v+p+2} - \lambda_{v+1}}$$

$$= \frac{1}{\mathbb{E}_{m}^{p+1}} \cdot (\lambda_{v+1} \cdots \lambda_{v+p} \cdot \lambda_{v+p+1} - \lambda_{v+1} \cdot \lambda_{v+2} \cdots \lambda_{v+p+1})$$

$$= 0,$$
and $c_{m,m} = \lambda_{m,m} \lambda_{m,m}^{1}$

$$= \frac{1}{\mathbb{E}_{m}^{p+1}} \cdot (\lambda_{m+p+1} - \lambda_{m}) \mathbb{E}_{m}^{p} \frac{\lambda_{m+p+1}}{\lambda_{m+p+1} - \lambda_{m}}$$

$$= 1.$$
LEMMA 3.2
$$\Lambda_{p+1} \iff i \text{ if and only if } \lambda_{p+1}^{i} \cdot i \text{ sregularity of } \Lambda_{p+1}^{i}$$

$$= \lambda_{p+1} \text{ follows from the regularity of } \Lambda_{p+1}^{i}$$

$$= \lambda_{p+1} = \text{ if and only if } \Lambda_{p+1}^{i} \cdot i \text{ sregular. Referring}$$
to Lemma 3.1, we see that
$$(3.2) \qquad \lim_{n \to \infty} \lambda_{n,v}^{i} = 0, \text{ for every } v,$$

$$(3.3) \qquad \lim_{n \to \infty} \sum_{v=0}^{\infty} |\lambda_{n,v}^{i}| = \sup_{v=0}^{-\lambda_{n}} \frac{\lambda_{n+p+1} + \lambda_{n}}{\lambda_{n+p+1} - \lambda_{n}}$$

$$\leq \sup_{n} \frac{2}{1 - \frac{\lambda_{n}}{\lambda_{n+p+1}}}$$

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This supremum is finite if and only if

$$\lim_{n \to \infty} \inf \frac{\lambda_{n+p+1}}{\lambda_{n}} > 1.$$

 $\liminf_{n\to\infty}\frac{\lambda_{n+p+1}}{\lambda_n} > 1.$

Consequently, $\Lambda_{p+1} \leq I$ if and only if

\$3.2 EQUIVALENCE THEOREMS

 $(C,\lambda,p) \iff (C,\lambda,p+1) \text{ if and only if } \liminf_{n \to \infty} \frac{\lambda_{n+p+1}}{\lambda_n} > 1.$ PROOF

By (1.11), we know that $(C,\lambda,p+1) = \Lambda_{p+1}(C,\lambda,p)$. Thus the result now follows from Lemma 3.2. REMARK: In view of the fact $\{\lambda_n\}$ is an increasing sequence,

so that $\liminf_{n \to \infty} \frac{\lambda_{n+p+1}}{\lambda_n} \ge 1$, we see that $\liminf_{n \to \infty} \frac{\lambda_{n+p+1}}{\lambda_n} = 01^{\circ}$

is necessary and sufficient for $(C,\lambda,p+1)$ to include strictly (C,λ,p) .

We now state a result of Borwein and Cass [5, Corollary 2] which yields an equivalence theorem between the methods (C, λ ,p) and [C, λ ,p+1]_µ.

THEOREM 3.2

Let $\mu > 0$. Let $P = \{p_{n,\nu}\}$ be a matrix with (i) $p_{n,\nu} \ge 0$, for $n, \nu = 0, 1, 2, 3, ...,$ (ii) $\lim_{n \to \infty} p_{n,\nu} = 0$, for $\nu = 0, 1, 2, 3, ...$ 31

Let
$$Q = \{Q_{n,v}\}$$
 be a matrix such that for every
sequence $\{\sigma_{v}\}$ there is a sequence $\{s_{v}\}$ for which
 $\sigma_{n} = \int_{v=0}^{\infty} q_{n,v} s_{v}$
holds for $n = 0, 1, 2, 3, ...$
Then lim inf max $p_{n,v} = 0$
is a necessary and sufficient condition for there to be a
sequence which is not Q-convergent, but which is
 $[P,Q]_{v}$ -convergent, to zero.
THEOREM 3.3
Let $\mu > 0$. Then $(C,\lambda,p) <=> [C,\lambda,p+1]_{\mu}$ if and only
if lim inf $\frac{\lambda_{n+p+1}}{\lambda_{n}} > 1$.
PROOF
By Corollary 1.2, we have
 $(C,\lambda,p) = >^{<} (C,\lambda,p+1]_{v}$, for $\mu > 0$.
Now $\Lambda_{p+1} = (\lambda_{n,v})$ satisfies
 $\lambda_{n,v} \ge 0$, for $n,v = 0, 1, 2, ...$
and $\lim_{n\to\infty} \lambda_{n,v} = \lambda_{v,v}$, we have
 $\frac{n_{2}0}{\sum_{v=\infty} \Lambda_{n,v}} = \lim_{n\to\infty} \inf \lambda_{v,v}$
 $\lambda_{n,v} = \lim_{n\to\infty} \inf \lambda_{v,v}$ = $\lim_{v\to\infty} \inf \lambda_{v,v}$
 $\lim_{v\to\infty} \lim_{n\to\infty} \sum_{n>0} n, v = \lim_{v\to\infty} \inf \lambda_{v,v}$
Horeover (C,λ,p) has an inverse, so the result follows from
Theorem 3.2 by taking $P = \Lambda_{p+1}$ and $Q = .(C,\lambda,p)$, ///

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For the proof of the equivalence theorem between (C, λ ,p+1) and [C, λ ,p+1]_µ, we state another result of Borwein and Cass [5, Theorem 12]. JHEOREM 3.4

Let the matrix $P = \{P_{n,v}\}$ be regular and $P_{n,v} = 0$ for v > n. If

- (i) $p_{n,\nu} \ge p_{n+1,\nu}$, for $n \ge \nu, \nu = 0, 1, 2, ...,$ (ii) $p_{n,n} \to 0,$
- (iii) $\sum_{\nu=0}^{n} p_{n,\nu} \leq \sum_{\nu=0}^{n+1} p_{n+1,\nu}, f \circ r = 0, 1, 2, 3, ...$

then there is a divergent sequence of zeros and ones which is P-convergent to $\frac{1}{2}$, but not $[P,L]_{\mu}$ -convergent for any $\mu \ge 1$. (I denotes the identity matrix.)

THEOREM 3.5

(i) If $\liminf_{n \to \infty} \frac{\lambda_{n+p+1}}{\lambda_n} > 1$, then

 $(C,\lambda,p+1) \iff [C,\lambda,p+1]_{\mu}, for \mu > 0_{g},$

(ii) If $\lim_{n\to\infty} \frac{\lambda_{n+p+1}}{\lambda_n} = 1$, then $(C,\lambda,p+1)$ strictly includes $[C,\lambda,p+1]_{\mu}$, for $\mu \geq 1$.

PROOF

(i) Combining results of Theorem 3.1 and Theorem 3.3° we have $\liminf_{n \to \infty} \frac{\lambda_{n+p+1}}{\lambda_{n}} > 1$ implies that

$$(C,\lambda,p+1) \iff [C,\lambda,p+1]_{\mu}, \text{ for } \mu > 0.$$

(ii) Since in the matrix Λ_{p+1} , $\lambda_{n,n} = (1 - \frac{\lambda_n}{\lambda_{n+p+1}})$.

and since the matrix
$$(C, \lambda, p)$$
 has an inverse, Theorem 3.4
shows that if $\lim_{n\to\infty} \frac{\lambda_n + p+1}{\lambda_n} = 1$, then there is a divergent
sequence $\{t_n^p\}$ of zeros and ones which is Λ_{p+1} -convergent to
 $\frac{1}{2}$, but not $[\Lambda_{p+1}, 1]_{\mu}$ -convergent for any $\mu \ge 1$. Since
 $\Lambda_{p+1}, \{t_n^p\} = \{t_n^{p+1}\}$ the result follows.
 $///$
We now show that in Theorem 3.5 (ii) the condition
 $\lim_{n\to\infty} \frac{\lambda_n + p+1}{\lambda_n} = 1$ can not be replaced by $\lim_{n\to\infty} \inf \frac{\lambda_n + p+1}{\lambda_n} = 1$.
Let $P_0 > 0$ and $P_n \ge 0$, we say that
 $M_{p+1} = \frac{1}{2} \prod_{n\to0}^{n} P_{\nu} s_{\nu} + s$, where $P_n = \prod_{\nu=0}^{n} P_{\nu}$.
REMARK: (i) Λ_{p+1} is the method (\overline{N}, P_n) with $P_n = \mathbb{E}_n^{p+1}$.
(ii) If (\overline{N}, P_n) is taken as P in Theorem 3.4, it
satisfies conditions (i) and (iii) of Theorem 3.4.
We shall now construct an (\overline{N}, P_n) method with
 $\lim_{n\to\infty} \inf \frac{p_n}{P_n} = 0$ and with $[(\overline{N}, P_n), I]_1 \iff (\overline{N}, P_n)$.
Let $C_{\mu} = \frac{P_n}{P_n}$, $0 < C_n < 1$, for $n \ge 1$.
Then $\mu_n P_n - \mu_{n-1} (1 - C_n) = c_n \cdot s_n$.
Now take $C_{2n} = 1 - \frac{1}{(n+1)^2}$, for $n \ge 1$.

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$$\lim_{n \to \infty} \inf \frac{P_n}{P_n} = 0,$$

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(3.6)
$$\mu_{2n} - \frac{\mu_{2n-1}}{(n+1)^2} = (1 - \frac{1}{(n+1)^2}) s_{2n}$$

(3.7)
$$\mu_{2n+1} - \mu_{2n} (1 - \frac{1}{n+2}) = \frac{s_{2n+1}}{n+2}$$

Consequently if $\mu_n \neq l$, then (3.6) and (3.7) give $s_{2n} \neq l$ and $s_{2n+1} = o(n)$. 35

On the other hand if $s_{2n} \neq k$ and $s_{2n+1} = o(n) \land$ then

$$\left|\frac{\overset{\mu}{n}}{\overset{n}{n+1}}\right| \leq \frac{1}{\overset{\mu}{P}_{n}} \sum_{\nu=0}^{n} \overset{\mu}{P}_{\nu} \frac{\left|\mathbf{s}_{\nu}\right|}{\overset{\nu}{\nu+1}} \leq H$$

and (3.6) and (3.7) imply that $\mu_n \rightarrow \ell$. Summarizing we have

$$s_n \rightarrow s \ (\overline{N}, p_n)$$
 if and only if
 $s_{2n} \rightarrow s \ and \ s_{2n+1} = o(n)$.

Thus (\overline{N}, p_n) is regular, not equivalent to convergence and $(\overline{N}, p_n) <=> [(\overline{N}, p_n), I]_1$.

Let
$$\lambda_0 = 0$$
, $\lambda_{n+1} = P_n$ for $n \ge 0$.
Then $\lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \lambda_{n+1} < \cdots$

and $\lambda_n \rightarrow \infty$, because (\overline{N}, p_n) is regular.

$$\Lambda_{1} = (\overline{N}, p_{n}),$$

$$\lim_{n \to \infty} \inf \frac{\lambda_{n+1}}{\lambda_{n}} = 1$$

and
$$[C,\lambda,1]_{1 < =} (C,\lambda,1)$$
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Combining the last example with Theorem 2.3 (i), we find that it is possible to have

 $\lim_{n \to \infty} \inf \frac{\lambda_{n+1}}{\lambda_n} = 1$

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and $[R, \lambda, 1]_{1} \iff (R, \lambda, 1)$.

CHAPTER 4

ABSOLUTE GENERALISED CESÀRO SUMMABILITY

§4.1 DEFINITIONS

In this chapter we study the absolute methods of summability $|C_r\lambda,p|_{\mu}$ and $|\hat{R},\lambda,p|$.

Let t_n^p be defined as in §1.5 and $\mu > 0$. We define $\sum_{\nu=0}^{\infty} a_{\nu}$ to be summable $|C,\lambda,p|_{\mu}$ if

(4.1)

$$\sum_{n=1}^{\infty} \left(\frac{\lambda_n}{\lambda_{n+p+1} - \lambda_n} \right)^{\mu-1} |t_n^p - t_{n-1}^p|^{\mu}$$

In §1.2, we defined $\sum_{\nu=0}^{\nu} a_{\nu}$ to be summable $|Q|_{\mu}$,

μ-> 0, if ...

(4.2)
$$\sum_{n=1}^{\infty} n^{\mu-1} |\sigma_n - \sigma_{n-1}|^{\mu} < \infty,$$

where $\{\sigma_n\} = Q\{s_n\}$.

When $\mu = 1$, conditions (4.1) and (4.2) are equivalent. When $\mu \neq 1$, they may or may not differ.

$$\checkmark$$
 For example, if $\lambda_n = n^{\alpha}$, $\alpha > 0$; then

$$\frac{\lambda_{n}}{\lambda_{n+p+1} - \lambda_{n}} = \frac{n^{\alpha}}{(n+p+1)^{\alpha} - n^{\alpha}} = \frac{n^{\alpha}}{\alpha \theta_{n}^{\alpha-1}(p+1)}$$

where $n < \theta_n < n+p+1$.

$$Jet \rho_{n} = \frac{\lambda_{n}}{\lambda_{n+p+1} - \lambda_{n}}.$$
Then $\frac{\rho_{n}}{n} + \frac{1}{\alpha(p+1)}$, as $n \neq \infty$. So in this case,

$$\int_{n=1}^{\infty} \rho_{n}^{\mu-1} |t_{n}^{p} - t_{n-1}^{p}| < \infty \text{ if and only if}$$

$$\int_{n=1}^{\infty} n^{\mu-1} |t_{n}^{p} - t_{n-1}^{p}| < \infty,$$
and the two conditions (4.1) and (4.2) are equivalent in this case.
On the other hand, if $\lambda_{n} = \log(n+1)$, then

$$\rho_{n} = \frac{\log(n+1)}{\log(n+p+2) - \log(n+1)} = \frac{\theta_{n} \log(n+1)}{p+1},$$
where $n+1 < \theta_{n} < n+p+2$.
In this case $\frac{\rho_{n}}{n \log(n} + \frac{1}{p+1}$, as $n + \frac{p}{2} + \frac{2}{2} +$

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while
$$\sum_{n=2}^{\infty} n^{\mu-1} \log^{\mu-1} n |\alpha_n|^{\mu}$$

$$= \sum_{n=2}^{\infty} n \log n \left| \frac{1}{n \log n} \right|$$
$$= \sum_{n=2}^{\infty} \frac{1}{n \log n} = \infty.$$

This shows that the two conditions (4.1) and (4.2) are different in this case.

It is more natural to use condition (4.1) rather than condition (4.2) to define $|C,\lambda,p|_{u}$ summábility. Thus

for the remainder of this chapter $\sum_{\nu=0}^{\infty} a_{\nu}$ is summable

 $|C,\lambda,p|_{u}$ means condition (4.1) is satisfied.

We now give an example which shows that there are sequences λ for which $|\mathcal{L}, \lambda, p|_{\mu} \neq \langle (\mathcal{L}, \lambda, p)$. Let $\mu = 2$, $\lambda_n = \log(n+1)$ and

$$\alpha_n = t_n^p - t_{n-1}^p = \frac{1}{n \log n \log \log n}$$

 $\sum_{n=2}^{\infty} n^{\mu-1} \log^{\mu-1} n \left| \frac{1}{n \log n \log \log n} \right|^{\mu}$

Then

$$= \sum_{n=2}^{\infty} n \log n \frac{1}{n^2 (\log n)^2 (\log \log n)^2}$$
$$= \sum_{n=2}^{\infty} \frac{1}{n \log n (\log \log n)^2} < \infty.$$

But $\lim_{n \to \infty} t_n^p = t_1^p + \sum_{n=2}^{\infty} \alpha_n = t_1^p + \sum_{n=2}^{n} \frac{1}{n \log n \log \log n}$

 $\sum_{\nu=0}^{\infty} a_{\nu}$ is summable $|C, \lambda, p|_{1}$ means that

 $\sum_{n=1}^{\infty} |t_n^p - t_{n-1}^p| < \infty \text{ so that } \{t_n^p\} \text{ is convergent to s say. This}$ means that $\sum_{\nu=0}^{\infty} a_{\nu} = s(C,\lambda,p).$ Hence we write and we have $|C,\lambda,p|_1 => (C,\lambda,p).$

Let $\mathbb{R}^{p}_{\lambda}(\tau)$ be defined as in §1.3. Then we say $\sum_{\nu=0}^{\infty} a_{\nu}$

is $|R,\lambda,p|$ summable, if

 $R^{\mathbf{p}}_{\lambda}(\tau) + \mathbf{s} \text{ as } \tau + \infty,$ $\int_{\mathbf{h}}^{\infty} \left| dR^{\mathbf{p}}_{\lambda}(\tau) \right| = \int_{\mathbf{h}}^{\infty} \left| \frac{d}{d\tau} R^{\mathbf{p}}_{\lambda}(\tau) \right| d\tau < \infty,$

and

where $h \ge \lambda_0$. (See Obrechkoff: Sur la sommation absolue des sèries de Dirichlet. C.R. 186, 1928.) We denote this by

$$\sum_{\nu=0}^{\infty} a_{\nu} = s | R, \lambda, p |.$$

§4.2 INCLUSION THEOREMS

The next lemma is a special case of a result due to Mears, [19, Theorem 1].

Let $Q = \{q_{n,v}\}$ be a regular matrix with $q_{n,v} = 0$ for v > n: If $\sigma_n = \sum_{\nu=0}^{n} q_{n,\nu} s_{\nu}$, where $s_{\nu} = \sum_{\mu=0}^{\nu} a_{\mu}$, then a necessary

and sufficient condition for

$$\begin{split} &\sum_{n=1}^{\infty} |\sigma_{n} - \sigma_{n-1}| < \infty \\ &\sum_{n=1}^{\infty} |s_{v} - s_{v-1}| < \infty \\ &(4.3) \qquad \sum_{n=k}^{\infty} |\sum_{v=k}^{n-1} (q_{n,v} - q_{n-1,v}) + q_{n,n}| \le H \\ &\text{where H is independent of k.} \\ &\text{THEOREM 4.1} \\ &\text{For any non-negative integer p,} \\ &\sum_{v=0}^{\infty} a_{v} = s|C,\lambda,p+1|_{1} \text{ whenever } \sum_{v=0}^{\infty} a_{v} = s|C,\lambda,p|_{1}. \\ &\text{PROOF} \\ &\text{We know that } (C,\lambda,p+1) = \Lambda_{p+1}(C,\lambda,p) \text{ where } \Lambda_{p+1} \text{ is defined in $1.7. By Lemma 4.1 it suffices to prove that} \\ &\sum_{n=k}^{\infty} |\sum_{v=k}^{n-1} (\lambda_{n,v} - \lambda_{n-1,v}) + \lambda_{n,n}| \le H \\ &\text{where H is independent of k.} \\ &\text{Now, referring to (1.8)} \\ &\sum_{n=k}^{\infty} |\sum_{v=k}^{n-1} (\lambda_{n,v} - \lambda_{n-1,v}) + \lambda_{n,n}| \\ &= \lambda_{k,k} + \sum_{n=k+1}^{\infty} |\sum_{v=k}^{n-1} (\lambda_{n,v} - \lambda_{n-1,v}) + \lambda_{n,n}| \\ &= \lambda_{k,k} + \sum_{n=k+1}^{\infty} |\sum_{v=k}^{n-1} (\lambda_{n,v} - \lambda_{n-1,v}) + \lambda_{n,n}| \\ &= \lambda_{k,k} + \sum_{n=k+1}^{\infty} |\sum_{v=k}^{n-1} (\lambda_{n,v} - \lambda_{n-1,v}) + \lambda_{n,n}| \\ &= \lambda_{k,k} + \sum_{n=k+1}^{\infty} |\sum_{v=k}^{n-1} (\lambda_{n,v} - \lambda_{n-1,v}) + \lambda_{n,n}| \\ &= \lambda_{k,k} + \sum_{n=k+1}^{\infty} |\sum_{v=k}^{n-1} (\lambda_{n,v} - \lambda_{n-1,v}) + \sum_{v=k}^{n-1} (\lambda_{v+p+1} - \lambda_{v}) E_{n}^{p} \\ &= \lambda_{k,k} + \sum_{n=k+1}^{\infty} |\sum_{v=k}^{n-1} (\lambda_{n,v} - \lambda_{n-1,v}) + \sum_{v=k}^{n-1} (\lambda_{v+p+1} - \lambda_{v}) E_{n}^{p} \\ &= \lambda_{k,k} + \sum_{n=k+1}^{\infty} |\sum_{v=k}^{n-1} (\lambda_{n,v} - \lambda_{n-1,v}) + \sum_{v=k}^{n-1} (\lambda_{v+p+1} - \lambda_{v}) E_{n}^{p} \\ &= \lambda_{k,k} + \sum_{n=k+1}^{\infty} |\sum_{v=k}^{n-1} (\lambda_{n,v} - \lambda_{n-1,v}) + \sum_{v=k}^{n-1} (\lambda_{v+p+1} - \lambda_{v}) E_{n}^{p} \\ &= \lambda_{k,k} + \sum_{n=k+1}^{\infty} |\sum_{v=k}^{n-1} (\lambda_{v+p+1} - \lambda_{v}) E_{n}^{p} \\ &= \sum_{v=k}^{n-1} (\lambda_{v+p+1} - \lambda_{v}) E_{n}^{p} \\ &= \sum_{v=k$$

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$$\begin{split} &= \lambda_{k,k}^{\perp} + \sum_{\substack{n=k+1 \\ k \neq k}}^{\infty} \left| \sum_{\substack{\nu=k \\ n=k+1 \\ \nu=k}}^{n-1} \frac{(\lambda_{\nu+p+1} - \lambda_{\nu})E_{\nu}^{p}}{-E_{n}^{p}} \left(\frac{1}{\lambda_{n+p+1}} - \frac{1}{\lambda_{n}} \right) \right| \\ &+ \frac{\lambda_{n+p+1} - \lambda_{n}}{\lambda_{n+p+1}} \right| \\ &= \lambda_{k,k} + \sum_{\substack{n=k+1 \\ n=k+1 \\ k \neq k}}^{\infty} \left| \left(\frac{1}{\lambda_{n+p+1}} - \frac{1}{\lambda_{n}} \right) \left(\sum_{\substack{\nu=k \\ n=k-1 \\ k \neq k}}^{p+1} - \frac{1}{\lambda_{n+p+1}} \right) \right| \\ &= \lambda_{k,k} + \sum_{\substack{n=k+1 \\ n=k+1 \\ k \neq k}}^{\infty} \left| \left(\frac{1}{\lambda_{n+p+1}} - \frac{1}{\lambda_{n}} \right) \left(\frac{E_{n-1}^{p+1} - E_{k-1}^{p+1}}{E_{n}^{p}} \right) + \frac{\lambda_{n+p+1} - \lambda_{n}}{\lambda_{n+p+1}} \right| \\ &= \lambda_{k,k} + \sum_{\substack{n=k+1 \\ k=k}}^{\infty} \left| \left(\frac{\lambda_{n}}{\lambda_{n+p+1}} - 1 - \frac{E_{k-1}^{p+1}}{E_{n}^{p+1}} + \frac{E_{k-1}^{p+1}}{E_{n-1}^{p+1}} \right) \right| \\ &= \lambda_{k,k} + E_{k-1}^{p+1} \sum_{\substack{n=k+1 \\ n=k+1 \\ k \neq k}}^{\infty} \left\{ \frac{E_{k+1}^{p+1}}{E_{k}^{p+1}} + \frac{E_{k-1}^{p+1}}{E_{n-1}^{p+1}} \right\} \\ &= \lambda_{k,k} + \frac{E_{k+1}^{p+1}}{E_{k}^{p+1}} + \frac{E_{k-1}^{p+1}}{E_{n-1}^{p+1}} \\ &= \lambda_{k,k} + \frac{E_{k+1}^{p+1}}{E_{k}^{p+1}} \\ &= 1 - \frac{E_{k+1}^{p+1} + E_{k}^{p+1}}{E_{k}^{p+1}} \\ &= 1. \end{split}$$
Thus $\sum_{n=1}^{\infty} |t_{n}^{p} - t_{n-1}^{p}| < \infty = \sum_{n=1}^{\infty} |t_{n}^{p+1} - t_{n-1}^{p+1}| < \infty. \end{split}$

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$$\int_{n+\infty}^{\infty} \left\{ \frac{\lambda_{n+1}}{\lambda_{n+1}} + \frac{\lambda_{n+1}}{\lambda_{n+1}} + \frac{\lambda_{n+1}}{\lambda_{n+1}} + \frac{\lambda_{n+1}}{\lambda_{n+1}} \right\}$$
Since $(\zeta, \lambda_{q}p) \Rightarrow (\zeta, \lambda_{q}p+1), t_{n+\infty}^{p} \Rightarrow \lim_{\lambda \to 0} \left\{ \frac{\lambda_{q}}{\lambda_{q}} + \frac{\lambda_{q}}{\lambda_{q}} + \frac{\lambda_{q}}{\lambda_{q}} \right\}$
Consequently $\int_{n+\infty}^{\infty} a_{\nu} = s |\zeta, \lambda, p|_{1}$ for $p \ge 1$, whenever $\int_{n+0}^{\infty} a_{\nu} = s |\zeta, \lambda, p|_{1}$ (constant)
where $s = \int_{\nu=0}^{\infty} a_{\nu}$.
PROOF
Take $p = 0$ in Theorem 4.1 and proceed by induction. ///
THEOREM 4.2.
 $\int_{\nu=0}^{\infty} a_{\nu} = s |\zeta, \lambda, p|_{1}$ whenever $\int_{\nu=0}^{\infty} a_{\nu} = s |\zeta, \lambda, p+1|_{1}$
if and only if $\lim_{n\to\infty} \inf_{\lambda=1}^{n+p+1} \frac{\lambda_{n+1}}{\lambda_{n+1}} > 1$.
PROOF
($\zeta, \lambda, p = \lambda_{p+1}^{1}(\zeta, \lambda, p+1)$.
Referring to Lemma 3.1, we know in $\Lambda_{p+1}^{1} = {\lambda_{n+1}^{1}}, \lambda_{n+1}^{1} = {$

Now
$$\prod_{n=k}^{\infty} \lfloor \frac{n-1}{\sqrt{n+k}} (\lambda_{n,v}^{*} - \lambda_{n-1,v}^{*})^{*} + \lambda_{n,n}^{*} \rfloor$$

$$= \lambda_{k,k}^{*} + \lfloor \lambda_{k+1,k+1}^{*} + \lambda_{k+1,k}^{*} - \lambda_{k,k}^{*} \rfloor + \prod_{n=k+2}^{\infty} \lfloor \frac{n-1}{\sqrt{n+k}} (\lambda_{n,k}^{*} - \lambda_{n-1,v}^{*})^{*} + \frac{\lambda_{n,n}^{*}}{\sqrt{n+k}} \rfloor$$

$$= \frac{\lambda_{k+p+1}^{*}}{\beta_{k+p+1}^{*} + \lambda_{k}^{*}} + \lfloor 1 - \lambda_{k,k} \rfloor + \prod_{n=k+2}^{\infty} \lfloor \lambda_{n,n}^{*} + \lambda_{n,n-1}^{*} - \lambda_{n-1,n-1}^{*} - \frac{\lambda_{n-1,n-2}^{*}}{\sqrt{n+k}} \rfloor$$

$$= \frac{\lambda_{k+p+1}^{*} + \lambda_{k}^{*}}{\gamma_{k+p+1}^{*} - \lambda_{k}^{*}} + \prod_{n=k+2}^{\infty} \lfloor 1 - 1 \rfloor$$

$$= \frac{\lambda_{k+p+1}^{*} + \lambda_{k}^{*}}{\gamma_{k}^{*}} + \prod_{n=k+2}^{\infty} \lfloor 1 - 1 \rfloor$$
Thus it follows Lemma 4.1 that
$$\int_{v=0}^{\infty} a_{v} = s \lfloor C_{v}\lambda_{v}p \rfloor_{1} \text{ whenever } \int_{v=0}^{\infty} a_{v} = s \lfloor C_{v}\lambda_{v}p+1 \rfloor_{1}$$
if and only if $\liminf_{n+\infty}^{*} \frac{\lambda_{n+p+1}^{*}}{\lambda_{n}} > 1$.
$$(CROILARY 4.2)$$

$$\int_{v=0}^{\infty} a_{v} = s \lfloor R_{v}\lambda_{v}p \rfloor \text{ whenever } \int_{v=0}^{\infty} a_{v} = s \lfloor R_{v}\lambda_{v}p+1 \rfloor \text{ if and}$$

$$on iy \text{ if } \lim_{n+\infty} \ln \frac{\lambda_{n+p+1}}{\lambda_{n}} > 1$$
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We now turn our attention to the relationship between $|C,\lambda,p+1|_{\mu}$ and $[C,\lambda,p+1]_{\mu}$. To facilitate the discussion we use a result of Borwein, [1, Theorem 7], which we state as the next lemma. We include the proof for the sake of completeness.

If P is a regular matrix with non-negative entries, Q is a matrix and $\mu \ge 1$, then necessary and sufficient conditions for a series to be summable $[P,Q]_{\mu}$ to s are that it be PQ-summable to s and $[P,(I-P)Q]_{\mu}$ -summable to zero. PROOF

Let $\{\sigma_n\} = Q\{s_n\}$ and $\{\tau_n\} = P\{\sigma_n\}$. We have to prove

(a)
$$\sum_{r=0}^{\infty} p_{n,r} |\sigma_r - s|^{\mu} = o(1)$$

if and only if

that

and

(b) τ → s

(c)
$$\sum_{r=0}^{\infty} p_{n,r} |\sigma_r - \tau_r|^{\mu} = o(1)$$
.

 $\left\{ \sum_{r=0}^{\infty} p_{n,r} |\sigma_r - \tau_r|^{\mu} \right\}^{1/\mu}$

(i) Suppose that (a) holds. Then by Theorem 1.2

(ii), (b) holds and so $\sum_{r=0}^{\infty} p_{n,r} |\tau_r - s|^{\mu} = o(1)$ since P is regular. Hence by Minkowski's inequality and (a)

 $\leq \left\{ \sum_{r=0}^{\infty} p_{n,r} |\sigma_r - s|^{\mu} \right\}^{1/\mu} + \left\{ \sum_{r=0}^{\infty} p_{n,r} |\tau_r - s|^{\mu} \right\}^{1/\mu} = o(1),$

and (c) follows.

(ii) Suppose that (b) and (c) hold. Since P is
regular, it follows from (b) that

$$\int_{x=0}^{\infty} p_{n,x} \int_{x=0}^{\infty} r_{x} = s|^{\mu} = o(1),$$
Hence by Minkowski's inequality and (c),
 $\left\{\frac{\pi}{2}, 0, p_{n,x} | \sigma_{x} = s|^{\mu}\right\}^{1/\mu}$, $\left\{\frac{\pi}{2}, 0, p_{n,x} | \sigma_{x} = s|^{\mu}\right\}^{1/\mu}$
 $\leq \left\{\frac{\pi}{2}, 0, p_{n,x} | \sigma_{x} = r_{x}^{-1}|^{\mu}\right\}^{1/\mu} + \left\{\frac{\pi}{2}, 0, p_{n,x} | \tau_{x} = s|^{\mu}\right\}^{1/\mu} = o(1),$
so that (a) holds.
The proof is thus complete.
THEOREM 4.3
Let $\mu \geq 1$?. Then
 $\int_{\nu=0}^{\infty} a_{\nu} = s(C, \lambda, p+1)_{\mu}$ if and only if
(4.4)
 $\int_{\mu=0}^{\infty} a_{\nu} = s(C, \lambda, p+1)_{\mu}$ if $p_{n}^{\mu} = \int_{\mu=0}^{\mu-1} \int_{\mu=0}^{\mu} e_{n}^{\mu-1} = o(1).$
Condition (4.5) means $|t_{n}^{\mu} = t_{n}^{\mu+1}|^{\mu} + 0(\Lambda_{p+1}).$
PROOF
In Lemma 4.2, take P = Λ_{p+1} , $O = (C, \lambda, p)$ and observe that
 $(1 + \tilde{p})Q = (C, \lambda, p) = (C, \lambda, p+1).$
THEOREM 4.4
 $\int_{\nu=0}^{\infty} a_{\nu} = s[C, \lambda, p+1]_{1}$ implies $\int_{\nu=0}^{\infty} a_{\nu} = s[C_{j}\lambda, p+1]_{1}.$

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PROOF

Since $\sum_{n=1}^{\infty} |t_n^{p+1} - t_{n-1}^{p+1}| < \infty$ implies that t_n^{p+1} tends to a limit, s say, we have $\sum_{\nu=0}^{\infty} a_{\nu} = s(C,\lambda,p+1)$. Hence to prove the theorem it suffices to show condition (4.5) is satisfied with μ Let $n \geq 1$. $|t_{n}^{p} - t_{n}^{p+1}|$ $= \left| \frac{1}{E^{p}} \sum_{\nu=0}^{n} (\lambda_{n+1} - \lambda_{\nu}) \cdots (\lambda_{n+p} - \lambda_{\nu}) a_{\nu} \right|$ $-\frac{1}{E_{\nu}^{p+1}}\sum_{\nu=0}^{n}(\lambda_{n+1}-\lambda_{\nu})\cdots(\lambda_{n+p+1}-\lambda_{\nu})a_{\nu}^{\omega}|^{\omega}$ $= \left| \frac{\lambda_{n+p+1}}{E^{p+1}} \sum_{\nu=0}^{n} (\lambda_{n+1} - \lambda_{\nu}) \cdots (\lambda_{n+p} - \lambda_{\nu}) a_{\nu} \right|$ $-\frac{1}{E^{p+1}} \int_{\frac{1}{\sqrt{2}}0}^{n} (\lambda_{n+1} - \lambda_{\nu}) \cdots (\lambda_{n+p+1} - \lambda_{\nu}) a_{\nu}$ $= \left| \frac{1}{E^{p+1}} \sum_{\nu=0}^{n} (\lambda_{n+1} - \lambda_{\nu}) \cdots (\lambda_{n+p} - \lambda_{\nu}) (X_{n+p+1} - \lambda_{n+p+1} + \lambda_{\nu}) a_{\nu} \right|$ $= \left| \frac{1}{E_{p+1}^{p+1}} \sum_{\nu=0}^{n} (\lambda_{n+1}^{\lambda} - \lambda_{\nu}) \cdots (\lambda_{n+p} - \lambda_{\nu}) \lambda_{\nu} a_{\nu} \right|.$ On the other hand $t_{n}^{p+1} - t_{n-1}^{p+1}$ $= \left| \frac{1}{E^{p+1}} \int_{\nu=0}^{n} (\lambda_{n+1} - \lambda_{\nu}) \cdot (\lambda_{n+p+1} - \lambda_{\nu}) a_{\nu} \right|$ $-\frac{1}{E_{n+1}^{p+1}}\sum_{\nu=0}^{n}(\lambda_{n}-\lambda_{\nu})\cdots(\lambda_{n+p}-\lambda_{\nu})a_{\nu}$

$$\begin{aligned} &= \left| \frac{\lambda_{n}}{E_{n-1}^{p+2}} \sum_{\nu=0}^{n} (\lambda_{n+1} - \lambda_{\nu}^{\nu}) \cdots (\lambda_{n+p+1} - \lambda_{\nu})^{a} \right| \\ &= \left| \frac{\lambda_{n}}{E_{n-1}^{p+2}} \sum_{\nu=0}^{n} (\lambda_{n} - \lambda_{\nu}) \cdots (\lambda_{n+p} - \lambda_{\nu}) \left\{ \lambda_{n} (\lambda_{n+p+1} - \lambda_{\nu}) - \lambda_{n+p+1} (\lambda_{n} - \lambda_{\nu}) \right\}^{a} \right| \\ &= \left| \frac{1}{E_{n-1}^{p+2}} \sum_{\nu=0}^{n} (\lambda_{n+1} - \lambda_{\nu}) \cdots (\lambda_{n+p} - \lambda_{\nu}) \left\{ \lambda_{n} (\lambda_{n+p+1} - \lambda_{n}) \lambda_{\nu}^{a} \right\} \right| \\ &= \left| \frac{1}{E_{n-1}^{p+2}} \sum_{\nu=0}^{n} (\lambda_{n+1} - \lambda_{\nu}) \cdots (\lambda_{n+p} - \lambda_{\nu}) (\lambda_{n+p+1} - \lambda_{n}) \lambda_{\nu}^{a} \right| \\ &= \left| \frac{1}{E_{n-1}^{p+2}} \sum_{\nu=0}^{n} (\lambda_{n+1} - \lambda_{\nu}) \cdots (\lambda_{n+p} - \lambda_{\nu}) (\lambda_{n+p+1} - \lambda_{n}) \lambda_{\nu}^{a} \right| \\ &= \left| \frac{1}{E_{n-1}^{p+1}} - \frac{1}{2E_{n-1}^{p+1}} \right| = \left| \frac{1}{2E_{n-1}^{p+1}} \sum_{\nu=0}^{n} (\lambda_{n+1} - \lambda_{\nu}) \cdots (\lambda_{n+p} - \lambda_{\nu}) \lambda_{\nu}^{a} \right| \\ &= \left| \frac{1}{E_{n-1}^{p+1}} - \frac{1}{E_{n-1}^{p+1}} \right| = \left| \frac{1}{2E_{n-1}^{p+1}} \sum_{\nu=0}^{n} (\lambda_{n+1} - \lambda_{\nu}) \cdots (\lambda_{n+p} - \lambda_{\nu}) \lambda_{\nu}^{a} \right| \\ &= \left| \frac{1}{E_{n-1}^{p+1}} - \frac{1}{E_{n-1}^{p+1}} \right| = \left| \frac{1}{2E_{n-1}^{p+1}} - \frac{1}{2E_{n-1}^{p+1}} \right| \\ &= \left| \frac{1}{E_{n-1}^{p+1}} - \frac{1}{E_{n-1}^{p+1}} \right| = \left| \frac{1}{2E_{n-1}^{p+1}} \right| \\ &= \left| \frac{1}{E_{n-1}^{p+1}} + \left| \frac{1}{2E_{n-1}^{p+1}} - \frac{1}{2E_{n-1}^{p+1}} \right| \\ &= \left| \frac{1}{E_{n-1}^{p+1}} + \frac{1}{2E_{n-1}^{p+1}} - \frac{1}{2E_{n-1}^{p+1}} \right| \\ &= \left| \frac{1}{E_{n-1}^{p+1}} + \frac{1}{2E_{n-1}^{p+1}} - \frac{1}{2E_{n-1}^{p+1}} \right| \\ &= \left| \frac{1}{E_{n-1}^{p+1}} + \frac{1}{2E_{n-1}^{p+1}} - \frac{1}{2E_{n-1}^{p+1}} \right| \\ &= \left| \frac{1}{E_{n-1}^{p+1}} + \frac{1}{2E_{n-1}^{p+1}} + \frac{1}{2E_{n-1}^{p+1}} \right| \\ &= \left| \frac{1}{E_{n-1}^{p}} + \frac{1}{2E_{n-1}^{p+1}} + \frac{1}{2E_{n-1}^{p+1}} + \frac{1}{2E_{n-1}^{p+1}} \right| \\ &= \left| \frac{1}{E_{n-1}^{p}} + \frac{1}{2E_{n-1}^{p+1}} + \frac{1}{2E_{n-1}^{p+1}} + \frac{1}{2E_{n-1}^{p+1}} \right| \\ &= \left| \frac{1}{E_{n-1}^{p}} + \frac{1}{2E_{n-1}^{p+1}} + \frac{1}{2E_{n-1}^{p+1}} + \frac{1}{2E_{n-1}^{p+1}} + \frac{1}{2E_{n-1}^{p+1}} + \frac{1}{2E_{n-1}^{p+1}} \right| \\ &= \left| \frac{1}{E_{n-1}^{p}} + \frac{1}{2E_{n-1}^{p+1}} + \frac{1}{2E_{n-1}^{p+1}}$$

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Let
$$\mathbf{b}_r = \begin{vmatrix} \mathbf{t}_r^{p+1} - \mathbf{t}_{r-1}^{p+1} \end{vmatrix}$$
 and $\mathbf{B}_n = \sum_{r=0}^n \mathbf{b}_r$.

Then from (4.7), we have

 $\sum_{n=0}^{\infty} E_{n-1}^{p+1} b_n$ $= \sum_{n=0}^{m} E_{n-1}^{p+1} (B_{n} - B_{n-1})$ $= \dot{B}_{m} E_{m}^{p+1} - \sum_{n=0}^{m} B_{n} (E_{n}^{p+1} - E_{n-1}^{p+1}).$ Dividing by E_m^{p+1} , we obtain $B_{m} - \frac{1}{E_{m}^{p+1}} \sum_{n=0}^{m} (\lambda_{n+p+1} - \lambda_{n}) E_{n}^{p} B_{n} = o(1), \text{ as } m + \infty,$ because of the regularity of Λ_{p+1} and the hypothesis $\sum_{\nu=0}^{n} a_{\nu} = s |C, \lambda, p \neq 1|_{1} \text{ which means that } \{B_{n}\} \text{ is convergent.}$ Thus the condition (4.5) is satisfied and the theorem is proved. (C.f. Borwein and Cass [6, Theorem 9].) THEOREM 4.5 If $\sum_{\nu=0}^{\infty} a_{\nu} = s$ (C, λ ,p+1) then, for $\mu > 1$, $\sum_{\nu=0}^{\infty} a_{\nu} = s | C, \lambda, p+1 |_{\mu} \text{ implies that } \sum_{\nu=0}^{\infty} a_{\nu} = s [C, \lambda, p+1]_{\mu}$ PROOF Since $\sum_{\nu=0}^{\lambda} a_{\nu} = s(C,\lambda,p+1)$, it suffices to show that condition (4.5) is satisfied with $\mu > 1$.

Now referring to (4.6), we have

$$\left| \mathbf{t}_{n}^{p+1} - \mathbf{t}_{n-1}^{p+1} \right|^{\mu} = \left(\frac{\lambda_{n+p+1} - \lambda_{n}}{\lambda_{n}} \right)^{\mu} \left| \mathbf{t}_{n}^{p} - \mathbf{t}_{n}^{p+1} \right|^{\mu}$$

for $\mu > 1$ and $n \ge 1$. Thus

(4.8)
$$|t_n^p - t_n^{p+1}|^{\mu} = \left(\frac{\lambda_n}{\lambda_{n+p+1} - \lambda_n}\right)^{\mu} |t_n^{p+1} - t_{n-1}^{p+1}|^{\mu}$$

for $\mu > 1$ and $n \ge 1$.

Since
$$|t_0^p - t_0^{p+1}| = 0$$
 and $E_{-1}^{p+1} = 0$ and $t_{-1}^{p+1} = 0$,

we have, by (4.8),

$$\sum_{n=0}^{m} (\lambda_{n+p+1} - \lambda_n) E_n^p \left| t_n^p - t_n^{p+1} \right|^\mu$$
$$= \sum_{n=0}^{m} E_{n-1}^{p+1} \left(\frac{\lambda_n}{\lambda_{n+p+1} - \lambda_n} \right)^{\mu-1} \left| t_n^{p+1} - t_{n-1}^{p+1} \right|^\mu$$

Now let
$$b_r = \rho_r^{\mu-1} | t_r^{p+1} - t_{r-1}^{p+1} |^{\mu}$$

and
$$B_n = \sum_{r=0}^n b_r$$

and proceed as the last part of the proof of Theorem 4.4, we have $\sum_{n=0}^{m} \left| \frac{p+1}{n-1} \rho_n^{\mu-1} \right| t_n^{p+1} - \left| t_{n-1}^{p+1} \right|^{\mu} = o(E_m^{p+1}).$

And hence $\sum_{n=0}^{m} (\lambda_{n+p+1} - \lambda_n) E_n^p | t_n^p - t_n^{p+1} |^{\mu} = o(E_m^{p+1}).$

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CHAPTER 5 SOME STRICT INCLUSION THEOREMS BETWEEN

CESÀRO AND DISCRETE RIESZ METHODS OF SUMMABILITY

§5.1 DEFINITIONS

 $s_n = \sum_{r=0}^n a_r'$

 $\varepsilon_0^{\kappa} = 1,$

Suppose throughout this chapter that $\kappa > 0$,

and

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$$\varepsilon_{n}^{\kappa} = {\binom{n+\kappa}{n}}^{\circ} = \frac{(\kappa+1)(\kappa+2)\cdots(\kappa+n)}{n!} \quad \text{for } n > 0.$$

Let $\{p_n\}$ be a sequence with $p_n > 0$ for $n \ge 0$ and let

 $P_n = \sum_{r=0}^n P_r.$

Define

(5.1)
$$t_n = \frac{1}{p_n} \sum_{r=0}^{n} p_{n-r} \dot{s}_r = \frac{1}{p_n} \sum_{r=0}^{n'} p_{n-r} a_{r'}$$

(5.2) $t_n^{\Delta} = \frac{1}{p_n} \sum_{r=0}^{n} p_{n-r} a_{r} = \frac{1}{p_n} \sum_{r=0}^{n} (p_{n-r} - p_{n-1-r}) s_{r'} (p_{-1} = 0)$.

We say that the sequence $\{s_n\}$ is (N,p_n) -convergent to s f t_n + s; and we write

$$s_n \rightarrow s (N, p_n)$$
.

This is a Nörlund Summability Method. See for example Hardy [11, page 54].

(5.3)
$$\tau_n = \frac{1}{P_n} \sum_{r=0}^n p_r |t_r^{\Delta} - s|.$$

We say that the sequence $\{s_n\}$ is $[N,p_n]$ -convergent to s if $\tau_n = o(1)$, and we write

$$s_n + s [N, p_n].$$

(See Borwein and Cass [6].)

We say that the sequence $\{s_n\}$ is $|N,p_n|$ -convergent to s if

 $\sum_{n=1}^{\infty} |t_n - t_{n-1}| < \infty \text{ and } s = \lim t_n;$

and we write

 $s_n \rightarrow s |N, p_n|.$

The Strong Summability Method $[N,p_n]$ is the method $[P,Q]_1$ (see §1.2) with $P = (\overline{N},p_n)$ (see §3.2) and Q the matrix associated with the transformation (5.2). We shall denote Q by $(N,\Delta p_n)$.

In the case of $[N,p_n]$ -summability, the method is interesting only if $P_n \neq \infty$. This condition is satisfied by the summability methods we consider below.

If we take $p_n = \varepsilon_n^{\kappa-1}$, then (N,p_n) and $|N,p_n|$ are the Cesàro and Absolute Cesàro Summability Methods (C, κ) and

|C, K| respectively.

The method $[N,p_n]$ with $p_n = \varepsilon_n^{\kappa-1}$ is equivalent (but not equal) to the Strong Cesàro Method $[C,\kappa]$ (See §1.6.) We shall denote this method $[N,p_n]$ also by $[C,\kappa]$. See Borwein and Cass [6, pages 98-99].

$$\rho_n^{\kappa} = \sum_{\nu=0}^n (1 - \frac{\nu}{n+1})^{\kappa} a_{\nu}$$

$$\frac{1}{(n+1)^{\kappa}} \sum_{\nu=0}^{n} (n+1-\nu)^{\kappa} (s_{\nu} - s_{\nu-1})$$

$$= \frac{1}{(n+1)^{\kappa}} \sum_{\nu=0}^{n} [(n+1) - \nu]^{\kappa} - (n - \nu)^{\kappa}]s_{\nu}$$

then we say that the sequence $\{s_{v}\}$ is (R^*, n, κ) -convergent to s, if $\rho_{n}^{\kappa} \rightarrow s$ as $n \rightarrow \infty$. We denote this by

 $s_n \rightarrow s (R^*, n_{\kappa})$.

Thus if we take $p_n = (n+1)^{\kappa} - n^{\kappa}$ for $n \ge 0$, then (N,p_n) and $|N,p_n|$ are the Discrete Reisz and Absolute Discrete Riesz Summability Methods (R^*,n,κ) and $|R^*,n,\kappa|$ respectively. We shall define the Strong Discrete Riesz Method of Summability $[R^*,n,\kappa]$ to be the method $[N,p_n]$ associated with this $\{p_n\}$.

\$5.2 KUTTNER'S THEOREM

In the definitions of (C,κ) and (R^*,n,κ) and the associated absolute methods, κ is usually allowed to satisfy $\kappa > -1$. The methods $[C,\kappa]$ and $[R^*,n,\kappa]$ make sense only when $\kappa > 0$ and it is for this reason we have so restricted κ .

THEOREM (Kuttner)

(i) If $-1 < \kappa < 2$, then (R^*, n, κ) is equivalent to (C, κ) and $|R^*, n, \kappa|$ is equivalent to $|C, \kappa|$.

(ii) There is a sequence (R*,n,2)-convergent but not

If

(C,2)-convergent and a sequence $|R^*,n,2|$ -convergent but not |C,2|-convergent. But $|R^*,n,2| \Rightarrow (C,2)$.

(iii) If $\kappa > 2$, there is a sequence $|R^*,n,\kappa|$ -convergent but not (C,κ) -convergent.

\$5.3 EXTENSION OF KUTTNER'S THEOREM AND OTHER RESULTS

For the proof of Theorem 5.1 we state two results of Borwein and Cass [6, Theorems 6 and 9] as our next two lemmas.

LEMMA 5.1

(See Kuttner [18].)

$$[N,p_n] => (N,p_n).$$

LEMMA 5.2

If $P_n \rightarrow \infty$ and $\{s_n\}$ is $|N, p_n|$ -convergent, then

$$s_n \rightarrow s [N,p_n]$$

where $s = \lim_{n \to \infty} t_n$ and t_n is defined as in (5.1).

THEOREM 5.1

If $\kappa > 0$, then $|\mathbf{R}^{\star}, \mathbf{n}, \kappa| \Rightarrow [\mathbf{R}^{\star}, \mathbf{n}, \kappa] \Rightarrow (\mathbf{R}^{\star}, \mathbf{n}, \kappa)$. PROOF

That $[R^*,n,\kappa] \Rightarrow (R^*,n,\kappa)$ is a special case of Lemma 5.1. $|R^*,n,\kappa| \Rightarrow [R^*,n,\kappa]$ follows from Lemma 5.2. ///

The next theorem is known, but it also follows from Lemmas 5.1 and 5.2 as the Theorem 5.1.

THEOREM 5.2

 $|C,\kappa| => [C,\kappa] => (C,\kappa).$

THEOREM 5.3

Let $p_n > 0$ for $n \ge 0$ and suppose $P_n \to \infty$. Then there is a sequence which is $[N, p_n]$ -convergent but not $|N, p_n|$ convergent.

PROOF

Borwein and Cass [6, Theorem 8] proved that $s_n \neq s[N,p_n]$ if and only if $s_n \rightarrow s(N,p_n)$ (5, 5)and $\frac{1}{P} \sum_{r=0}^{n} p_r |t_r^{\Delta} - t_r| = o(1)$ (5.6)where t_r and t_r^{Δ} are given by (5.1) and (5.2). This is a special case of Lemma 4.2. Now $t_{r}^{\Delta} - t_{r} = \frac{1}{p_{r}} \sum_{\nu=0}^{r} (p_{r-\nu} - p_{r-1-\nu}) s_{\nu} - \frac{1}{p_{r}} \sum_{\nu=0}^{n} p_{r-\nu} s_{\nu}$ $=\frac{\Pr[\sum_{\nu=0}^{r} p_{r-\nu} s_{\nu} - P_{r} \sum_{\nu=0}^{r-1} p_{r-1-\nu} s_{\nu} - P_{r} \sum_{\nu=0}^{r} p_{r-\nu} s_{\nu}}{p_{r}}$ $= \frac{\Pr_{r-1} \sum_{\nu=0}^{r} \Pr_{r-\nu} s_{\nu} - \Pr_{r} \sum_{\nu=0}^{r-1} \Pr_{r-1-\nu} s_{\nu}}{\Pr_{r-1-\nu} s_{\nu}}$ so that $p_r(t_r^{\Delta} - t_r) = P_{r-1}(t_r - t_{r-1}), r = 0, 1, 2, ...$ (5.7) $(P_{-1} = t_{-1} = 0)$. Choose $\{s_n\}$ so that $t_n - t_{n-1} \neq \frac{n p_n}{p_n p_n}$ where

 $D_n = \sum_{r=0}^n \frac{p_r}{p_r}$ and $\delta_n = \pm 1$ chosen in such a way that $\sum_{n=1}^\infty \frac{\delta_n \frac{p_n}{p_n D_n}}{n=1}$ converges. Then $\{t_n\}$ is convergent ensuring that (5.5) is satisfied. Also we have

$$\frac{1}{P_n} \sum_{r=0}^n P_r |t_r^{\Delta} - t_r| = \frac{1}{P_n} \sum_{r=0}^n P_{r-1} |t_r - t_{r-1}|$$
$$= \frac{1}{P_n} \sum_{r=0}^n \frac{P_{r-1} P_r}{P_r D_r}$$
$$= \sum_{r=0}^n a_{n,r} \frac{1}{D_r}$$

where $a_{n,r} = \frac{P_{r-1} P_r}{P_r}$ for $0 \le r \le n$ and $a_{n,r} = 0$ for r > n. Now $A = \{a_{n,r}\}$ is a matrix with zero column limits and

$$\sum_{r=0}^{n} |a_{n,r}| = \sum_{r=0}^{n} a_{n,r} \leq \frac{1}{p_{n}} \sum_{r=0}^{n} p_{r} = 1, \text{ for all } n,$$

so that it transforms null sequences into null sequences. Since by Abel-Dini Theorem $\lim_{n \to \infty} D_n = \infty$, $\frac{1}{D_r} \to 0$ as $r \to \infty$.

It follows that (5.6) is satisfied, so $s_n \rightarrow s [N,p_n]$. But by Abel-Dihi Theorem again

$$\sum_{n=1}^{\infty} |t_n - t_{n-1}| = \sum_{n=1}^{\infty} \frac{P_n}{P_n D_n} = \infty$$

so $\{s_n\}$ is not $|N,p_n|$ -convergent.

COROLLARY 5.1

Let $\kappa > 0$. There is a sequence which is $[R^*, n, \kappa]$ convergent but not $|R^*, n, \kappa|$ -convergent. COROLLARY '5.2

Let $\kappa > 0$. There is a sequence which is $[C,\kappa]$ -convergent but not $|C,\kappa|$ -convergent.

THEOREM 5.4

• Let $\kappa > 0$. There is a sequence which is (R^*, n, κ) convergent but not $[R^*, n, \kappa]$ -convergent.

PROOF

Let $P = \{p_{n,\nu}^{\bullet}\}$, where $p_{n,\nu} = \frac{(\nu+1)^{\kappa} - \nu^{\kappa}}{(n+1)^{\kappa}}$ for $0 \le \nu \le n$

and $p_{n,v} = 0$ for v > n. It follows from Theorem 3.4 that there is a sequence P-convergent but not [P,I]-convergent. Let $Q = \{q_{n,v}\}$ be the matrix such that

$$\sum_{\nu=0}^{n} q_{n,\nu} s_{\nu} = \frac{1}{p_n} \sum_{\nu=0}^{n} p_{n-\nu} a_{\nu}$$

where $p_n = (n+1)^{\kappa} - n^{\kappa}$. Then $[R^*, n, \kappa]$ -convergency is the same as [P,Q]-convergency and (R^*, n, κ) -convergency is the same as PQ-convergency. Since the matrix Q has an inverse our result now follows.

For the next theorem we state two results of Borwein and Cass [6, Theorem 1 and Corollary 1] as our next two . lemmas.

LEMMA 5.4

 $If (N,p_n) \implies (N,q_n) then [N,p_n] \implies [N,q_n].$

LEMMA 5.5

If $(N, p_n) \iff (N, q_n)$ then $[N, p_n] \iff [N, q_n]$.

THEOREM 5.6

- (i) If $\kappa > 0$, then $[C,\kappa] \Rightarrow [R^*,n,\kappa]$.
- (ii) If $0 < \kappa < 2$, then $[C,\kappa] <=> [R^*,n,\kappa]$.

PROOF

Since for $\kappa > 0$ we have $(C,\kappa) \Rightarrow (R^*,n,\kappa)$, (i) follows from Lemma 5.4. Since for $0 < \kappa < 2$ we have $(C,\kappa) \iff (R^*,n,\kappa)$ (ii) follows from Lemma 5.5. ///

THEOREM 5.7

There is a sequence which is $|R^*,n,2|$ -convergent but not [C,2]-convergent. PROOF

For a given sequence $\{s_n\}$ we write

(5.8)
$$\sigma_{n} = \frac{1}{\varepsilon_{n}^{2}} \sum_{\nu=0}^{n} \varepsilon_{n-\nu}^{1} s_{\nu} = \frac{n}{\varepsilon_{n}^{2}}$$

and

(5.9)
$$\xi_{n} = \frac{1}{(n+1)^{2}} \sum_{\nu=0}^{n} (n+1 - \nu)^{2} a_{\nu} = \frac{T_{n}}{(n+1)^{2}}$$

so that $\{\sigma_n\}$ and $\{\xi_n\}$ are respectively the (C,2) and (R*,n,2) transforms of the sequence $\{s_n\}$.

As in Kuttner [18, page 362] we have

(5.10)		$T_0 = \vec{s}_0;$	$\mathbf{T}_{n} = \mathbf{S}_{n-1} + \mathbf{S}_{n},$	n = 1,2,3	4
and a	(•			۲,

(5.11)
$$S_n = \sum_{m=0}^n (-1)^{n-m} T_m.$$

Now take $S_n = (-1)^n$ so that $T_n = (-1)^n$. Thus

 $\sum_{n=1}^{\infty} |\xi_n - \xi_{n-1}| < \infty \text{ and } \xi_n \neq 0, \text{ so that if } \{s_n\} \text{ is the }$

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sequence associated with this choice of S_n and T_n we have $s_n \rightarrow 0 | R^*, h, 2 |$. To see that $\{s_n\}$ is not [C, 2]-convergent we notice first that by Theorem 5.1, $s_n \neq 0 | \mathbb{R}^*, n, 2 |$ implies $s_n^* \rightarrow 0 \ [R^*, n, 2]$. Now by Theorem 5.6 [C,2] => [R*, n, 2], the only [C,2]-sum that $\{s_n\}$ could have is zero. $s_n - s_{n-1} = (-1)^n (2n - 1)$. But. $\frac{1}{m+1} \sum_{n=0}^{m} \left| \frac{(-1)^{n} (2n-1)}{n+1} \right|$ (5.12) $=\frac{1}{m+1}\sum_{n=0}^{m}\frac{2n-1}{n+1}$. Since (C,1) is regular and $\frac{2n-1}{n+1} \neq 2$, (5.12) tends to 2 as $m \rightarrow \infty$, Thus $\{s_n\}$ is not [C,2]-convergent to zero. 1// COROLLARY 5.3 · There is a sequence which is [R*,n,2]-convergent but not [C,2]-convergent. PROOF This follows from the fact that $|R^*,n,2| \Rightarrow [R^*,n,2]$. THEOREM 5.8 $[R^*, n, 2] \implies (C, 2).$ PROOF Referring to (5.9) we find that $T_r - T_{r-1} = \sum_{\nu=0}^{r} \{ (r+\overline{r}-\nu)^2 - (r-\nu)^2 \} a_{\nu}.$ $s_n \neq 0$ [R*,n,2] if and only if $\frac{1}{P_{p}} \sum_{r=0}^{n} P_{r} \left| \frac{1}{P_{p}} \sum_{\nu=0}^{r} P_{r-\nu} a_{\nu} \right| = \frac{1}{P_{p}} \sum_{r=0}^{n} \left| \sum_{\nu=0}^{r} P_{r-\nu}^{*} a_{\nu} \right| = o(1), \cdots$

where
$$P_{r-v} = (r+1-v)^2 - (r-v)^2$$
 and $P_n = (n+1)^2$. Hence $s_n + 0$
 $[R^*,n,2]$ if and only if
 $\frac{1}{(n+1)^2} \frac{n}{r_v} |T_r - T_{r-1}| = o(1), (T_{-1} = 0).$
From (5.11) it follows that
 $|S_n| \le \frac{n}{r_v} |T_r - T_{r-1}|.$
Thus if $s_n + 0$ $[R^*,n,2]$, then $|S_n| = o(n^2)$ so that
 $s_n + 0 (C,2).$
Now if $s_n + s[R^*,n,2]$, then $|S_n| = o(n^2)$ so that
 $s_n - s = 0(C,2), \text{ i.e., } s_n + s(C,2).$
THEOREM 5.9
There is a sequence which is $(C,2)$ -convergent but not
 $[R^*,n,2]$ -convergent.
PROOF
Choose (s_n) so that
 $S_{2n} = (-1)^n n^{3/2}$ and $S_{2n+1} = 0.$
Then $s_n^- + o(C,2)$. But referring to (5.10)
 $T_{2r} - T_{2r-1} = S_{2r} - S_{2r-0}.$
 $= (-1)^r (r^{3/2} + (r-1)^{3/2}), r = 1,2,...$
So if $2m \le n \le 4m+1$, then
 $\frac{n}{r=0} |T_r - T_{r-1}| \ge \prod_{r=1}^m |T_{2r} - T_{2r-1}|$
 $\ge -\prod_{r=1}^m (r-1)^{3/2}$
 $\sim H_1 n^{5/2}$

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where H, H_1 are independent of n.

Thus $\{s_n\}$ is not $[R^*, n, 2]$ -convergent to zero and our result follows.

THEOREM 5.10

Let $\kappa > 2$

(i) There is a sequence which is [R*,n,κ]-convergent
 but not (C,,κ)-convergent^δ.

(ii) There is a sequence which is $|R^*,n,\kappa|$ -convergent but not $[C,\kappa]$ -convergent.

(iii) There is a sequence which is [R*,n,K]-convergent but not [C,K]-convergent.

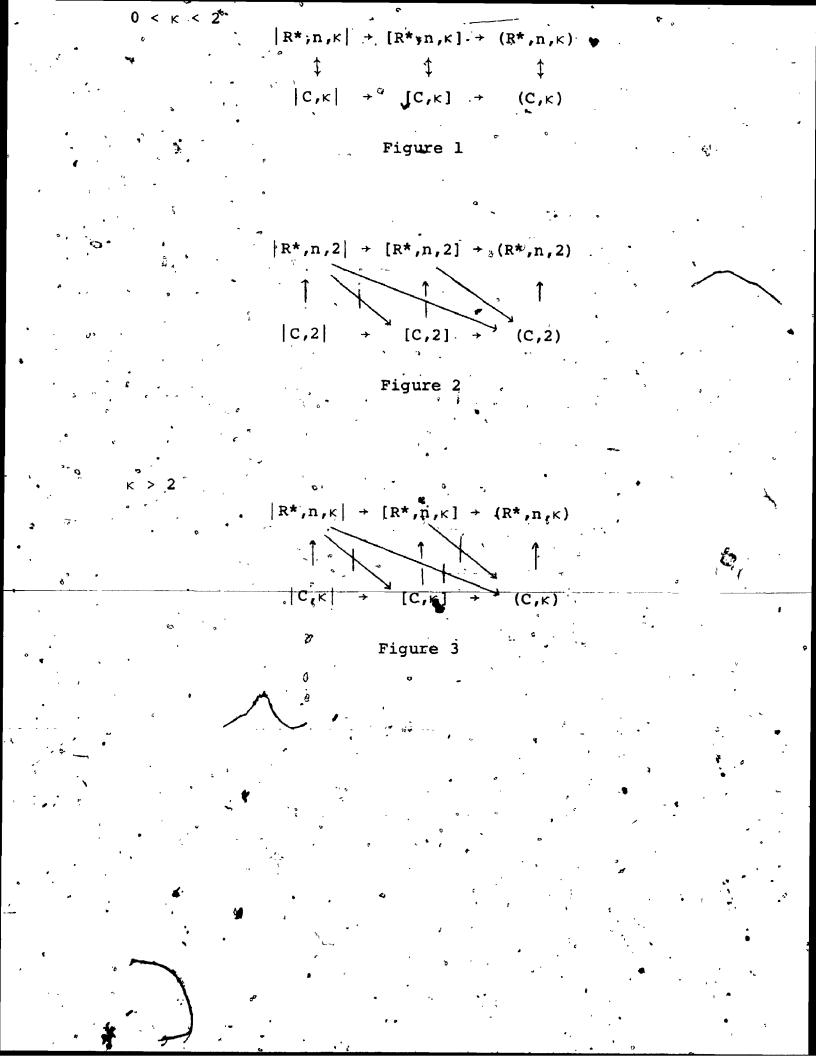
PROOF

Part (i) follows from Kuttner's Theorem (iii) and the fact that $|R^*,n,\kappa| \Rightarrow [R^*,n,\kappa]$.

Part (ii) follows from Kuttner's Theorem (iii) and the fact that $[C,\kappa] => (C,\kappa)$.

. Part (iii) follows from part (ii) and the fact that $\frac{1}{4}$ $|R^*,n,\kappa| \Rightarrow [R^*,n,\kappa].$

The relations between the various summability methods discussed in this chapter are conveniently displayed in three figures below. In these, figures the symbol \rightarrow denotes strict inclusion, the symbol \leftrightarrow denotes equivalence and the notation $P \leftrightarrow Q$ means that there is sequence which is P-convergent but not Q-convergent.



• CHAPTER 6

STRONG AND ABSOLUTE NÖRLUND METHODS OF SUMMABILITY ASSOCIATED WITH POLYNOMIALS

In this chapter our investigations stem from the results in D. Borwein [4]. We consider a Nörlund Method of Summability Associated with Polynomials and investigate the properties of an associated Strong Summability Method and of the Absolute Nörlund Method of Summability Associated with Polynomials.

\$6.1 DEFINITIONS

and

Let s, s_n be arbitrary complex numbers, and whenever n < 0 we take s_n = 0. Let

 $p(z) = p_0 + p_1 z + \cdots + p_j z^j$ $q(z) = q_0 + q_1 z + \cdots + q_k z^k$

be polynomials with complex coefficients which satisfy the normalizing conditions

p(1) = 1 and q(1) = 1.

We suppose throughout that $p(0) \neq 0$, $q(0) \neq 0$, $p_n = 0$ for n > 'j and $q_n' = 0$ for n > k. We use the notations

(6.1)
$$t_n = \sum_{\nu=0}^n p_{\nu} s_{n-\nu}, \quad n = 0, 1, 2, \dots,$$

(6.2) $u_n \bigoplus_{\nu=0}^{n} q_{\nu} s_{n-\nu}, \quad n = 0, 1, 2, ...$

Associated with the polynomial p(z) is a Nörlund Method of Summability N_p which we call a *Polynomial Nörlund*[®] Method and which is defined as follows.

The sequence $\{s_n\}$ is said to be N_p-convergent to s, and we write

(6.3)
$$s_n \rightarrow s(N_p)$$
, if $\lim_{n \to \infty} t_n = s$.

This definition is due to D. Borwein.

We define
$$s_n \rightarrow s [C_1, N_n]$$

if
$$\frac{1}{n+1}\sum_{r=0}^{n} |t_r - s| = o(1)$$
, as $n \to \infty$.

This is the $[P,Q]_1$ defined in §1.2 with $P = C_1$ and

$$Q = N$$

Let $P_n = \sum_{\nu=0}^{n} p_{\nu}$ where P_n is non-zero for n = 0, 1, 2, ...

and $\tau_n = \frac{1}{\frac{p}{n}} \sum_{\nu=0}^{n} p_{\nu} s_{n-\nu}$. Then we say that the sequence $\{s_n\}$

is (N,p_n)-convergent to s and we write

 $s_n \neq s(N,p_n)$ if lim $\tau_n = s$.

This is the Nörlund Summability Method given in §5.1, but here we allow p_v to be complex for all $v \ge 0$. Moreover, in this chapter we are only interested in the case where p_v 's are coefficients of a polynomial p(z) with p(1) = 1and we only use the (N, p_n) method in this sense. It is evident that in this sense (N,p_n) is equivalent to the Polynomial Nörlund Method N_n.

Let
$$P'_n = \sum_{r=0}^n |P_r|$$
 and $P'_n \neq 0$ for $n = 0, 1, 2, ...$ Then
(6.6) $s_n \neq s [N, P_n]$
if $\frac{1}{P'_n} \sum_{r=0}^n |P_r| |\tau_r - s| = o(1)$, as $n \neq \infty$.
This definition is analogous to the definition of

 $[N,p_n]$ given in §5.1, but we allow here p_v to be complex for $v \ge 0$. Moreover we let p_v 's be coefficients of a polynomial p(z) with p(1) = 1.

The Absolute Polynomial Nörlund Summability $|N_p|$ is defined as follows.

(6.7)
$$s_n \neq s |N_p|$$

if $t_n \neq s$ and $\sum_{n=0}^{\infty} |t_n - t_{n-1}| < \infty$, where $t_{-1} = 0$.

The method $[C_1, N_p]$ is a Strong Summability Method Associated with the Polynomial Nörlund Method. It is not the Strong Nörlund Summability Method defined in [6] which we considered in Chapter 5. Shortly we shall show that $[C_1, N_p]$ is equivalent to $[N, P_n]$. Thus $[C_1, N_p]$ is the Strong Nörlund Summability Method defined in [6] for (N, P_n) , rather than for (N, P_n) .

We shall establish at first $[C_1, N_p] => [C_1, N_q]$ if and only if $N_p => N_q$. It is shown in Borwein and Cass [6] that if (N,p_n) => (N,q_n) then $[N,p_n]$ => $[N,q_n]$. We shall investigate the converse of this theorem in the case of the Polynomial Nörlund Methods.

. Then we shall establish $|{\tt N}_p|$ => $|{\tt N}_q|$ if and only if ${\tt N}_p$ => ${\tt N}_q.$

Finally we shall establish some minor results analogous to some of the results obtained in [4].

\$6.2 THE EQUIVALENCE OF $[C_1, N_p]$ AND $[N, P_n]$ THEOREM 6.1

 $[C_1, N_p] <=> [N, P_n].$

PROOF

The result is an elementary consequence of the fact that $P_n = \sum_{r=0}^{n} |P_r| = \sum_{r=0}^{j-1} |P_r| + n - j + 1 \sqrt{n} + 1$ which implies the equivalence of $\overline{(N, P_n)}$ and (C,1). ///

\$6.3 THEOREMS ABOUT NÖRLUND METHODS OF SUMMABILITY ASSOCIATED WITH POLYNOMIALS

For completeness we shall quote without proof several results of Borwein [4].

The methods N_p and N_q mentioned in the following theorems are Nörlund Methods associated with polynomials p(z) and q(z) as defined in §6.1. Evidently N_p and N_q are regular. THEOREM 6.2

The method N_f , associated with the polynomial f(z) = p(z)q(z), includes both N_p and N_q . (Borwein [4, Theorem 2].)

THEOREM 6.3

The methods N_p and N_q are consistent, i.e., if $s_n \rightarrow s (N_p)$ and $s_n \rightarrow s' (N_q)$, then s = s'. (Borwein [4, Corollary].)

THEOREM 6.4

If h(z) is the highest common factor of p(z) and q(z), normalized so as to make h(1) = 1, then a necessary and sufficient condition for a sequence to be both N_p - and N_q -convergent is that it be N_h -convergent. (Borwein [4, Theorem 3].)

THEOREM 6.5

In order that N_q should include N_p it is necessary and sufficient that q(z)/p(z) should not have poles on or within the unit circle. (Borwein [4, Theorem I].)

THEOREM 6.6

If q(z)/p(z) has poles of maximum order m on the unit circle and does not have poles within the unit circle, then $(C,m)N_q$ includes N_p ; but for any $\varepsilon > 0$, there is an N_p -convergent sequence which is not $(C,m-\varepsilon)N_q$ -convergent. (Borwein [4, Theorem II].) THEOREM 6.7

If q(z)/p(z) has a pole within the unit circle then there is an N_p -convergent sequence which is not AN_q -convergent. (Borwein [4, Theorem III].)

UTHEOREM 6.8

In order that N_p should be equivalent to (C,0) it is necessary and sufficient that p(z) should not have zeros on or within the unit circle. (Borwein [4, Theorem I⁺].)

THEOREM 6.9

 $t_{n} = \sum_{\nu=0}^{n} p_{\nu} s_{n-\nu},$

 $u_n = \sum_{\nu=0}^n \overline{q_{\nu}} s_{n-\nu'}$

If q(z)/p(z) has poles $\lambda_1, \lambda_2, \ldots, \lambda_k$, in the finite complex plane, of orders m_1, m_2, \ldots, m_k respectively, and if, for $n = 0, 1, 2, \ldots$,

then

$$\mathbf{u}_{n} = \sum_{\nu=0}^{n} \mathbf{c}_{\nu} \mathbf{t}_{n-\nu} + \sum_{r=1}^{\ell} \sum_{\rho=1}^{m_{r}} \mathbf{c}_{r,\rho} \sum_{\nu=0}^{n} \begin{pmatrix} \nu+\rho-1\\ \rho-1 \end{pmatrix} \lambda_{r}^{-\nu} \mathbf{t}_{n-\nu}$$

where the C are constants, depending only on p_0, p_1, \dots, p_j , q_0, q_1, \dots, q_k such that $c_n = 0$ for n > k - j and $C_{r'm} \neq 0$. (Borwein [4, Lemma 1].)

§6.4 [C₁,N_p] METHOD OF SUMMABILITY The following proposition is a special case of Theorem 1.2. PROPOSITION 6.1

(i)
$$N_p => [C_1, N_p],$$

(ii) $[C_1, N_p] => (C, 1)N_p.$

THEOREM 6.10

PPOOF

If $q(z)/\dot{p}(z)$ has no poles within or on the unit circle, then $[C_1, N_p] => [C_1, N_q]$.

Without loss of generality, we may assume $s_n \neq 0$ [C_1 , N_p] and prove $s_n \neq 0$ [C_1 , N_q].

Let
$$t_n = \sum_{\nu=0}^{n} p_{\nu} s_{n-\nu}$$

$$\sum_{n=1}^{\infty} q_{\nu} s_{n-\nu}.$$

If q(z)/p(z) has no poles within or on the unit circle, but has poles $\lambda_1, \lambda_2, \ldots, \lambda_l$ of order m_1, m_2, \ldots, m_l outside the unit circle, then by Theorem 6.9

$$u_{n} = \sum_{\nu=0}^{n} C_{\nu} t_{n-\nu} + \sum_{r=1}^{\ell} \sum_{\rho=1}^{n} C_{r,\rho} \sum_{\nu=0}^{n} \left(\frac{\nu+\rho-1}{\rho-1} \right) \lambda_{r}^{-\nu} t_{n-\nu}$$

where the C's are constants, depending only on $p_0, p_1, \dots, p_j, q_0, q_1, \dots, q_k$, such that $c_n = 0$ for n > k - j and $c_{r'm} \neq 0$.

So
$$|u_n| \leq \sum_{\nu=0}^{n} |c_{\nu}| |t_{n-\nu}| + \sum_{r=1}^{\ell} \sum_{\rho=1}^{m_r} |c_{r,\rho}| \sum_{\nu=0}^{n} | \begin{pmatrix} \nu + \rho - 1 \\ \rho - 1 \end{pmatrix} \lambda_r^{-\nu} | |t_{n-\nu}|$$

	Thus $\frac{1}{m+1} \sum_{n=0}^{m} u_n $
	$\leq \frac{1}{m+1} \sum_{n=0}^{m} \sum_{\nu=0}^{n} c_{\nu} t_{n-\nu} + \frac{1}{m+1} \sum_{n=0}^{m} \sum_{r=1}^{\ell} \sum_{\rho=1}^{m} c_{r,\rho} \sum_{\nu=0}^{n} \binom{\nu+\rho-1}{\rho-1} \lambda_{r}^{-\nu} t_{n-\nu} $
	$= \sum_{\nu=0}^{m} c_{\nu} \frac{1}{m+1} \sum_{n=0}^{m-\nu} t_{n} + \sum_{r=1}^{\ell} \sum_{\rho=1}^{m} c_{r,\rho} \sum_{\nu=0}^{m} \binom{\nu+\rho-1}{\rho-1} \lambda_{r}^{-\nu} \frac{1}{m+1} \sum_{n=0}^{m-\nu} t_{n} ,$
4	where $c_v = 0$, for $v > \kappa - j$.
	Since the poles of $q(z)/p(z)$ are all outside the unit
	circle, $ \lambda_r > 1$, for $r = 1, 2,, \ell$, and $\sum_{\nu=0}^{\infty} {\nu+\rho-1 \choose \rho-1} \lambda_r^{-\nu}$ is
۰.	thus absolutely convergent for each $r = 1, 2,, l$ and
	$\rho = 1, 2, \dots, m_r^{\bullet}$. Consequently if $\frac{1}{m+1} \sum_{n=0}^{m} t_n \neq 0$ as $m \to \infty$,
-	then $\frac{1}{m+1}\sum_{n=0}^{m} u_n \rightarrow 0$ as $m \rightarrow \infty$.
	If $q(z)/p(z)$ has no poles at all, then
•	$\frac{1}{m+1}\sum_{n=0}^{m} u_n \leq \sum_{\nu=0}^{m} c_{\nu} \frac{1}{m+1} \sum_{n=0}^{m-\nu} t_n , \text{ where } c_{\nu} = 0 \text{ for } \nu > k - j.$
	Hence the desired conclusion follows. ///
-	THEOREM 6.11
	If (1) $q(z)/p(z)$ has a pole within the unit circle,
· · · · · · · · · · · · · · · · · · ·	or (2) $q(z)/p(z)$ has no pole within the unit circle,
	but has poles of maximum order m on the unit circle, where
,	$m > 1$, then there is a sequence which is $[C_1, N_p]$ -convergent
	but not [C ₁ , N _q]-convergent.
C	PROOF
	(1) $q(z)/p(z)$ has a pole within the unit circle.

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By Theorem 6.7 there is an N_p-convergent sequence which is not AN_q-convergent. Since (C,1) is regular, this sequence is $[C_1, N_p]$ -convergent. But, since it is not AN_q-convergent, it is not $(C,1)N_q$ -convergent. As a consequence of Proposition 6.1(ii) it is not $[C_1, N_q]$ -convergent.

(2) $\frac{q(z)}{p(z)}$ has no poles within the unit circle, but has poles on maximum order m on the unit circle, where m > 1. By Theorem 6.6 since m > 1, there is an N_p-convergent sequence which is not (C,1)N_q-convergent. Consequently, this sequence is $[C_1, N_p]$ -convergent, but, by Proposition 6.1(ii) it is not $[C_1, N_q]$ -convergent. ///

For the next theorem we need the following two lemmas. We use the notation $[C,1]_1$ to mean $[C_1,1]_1$. LEMMA 6.1

Let $t_n = a\lambda^n$, $|\lambda| = 1$, $\lambda \neq 1$ and a is a non-zero complex number. Then $\{t_n\}$ is not $[C,1]_1$ -convergent.

We know that $\{t_n\}$ is (C,1)-convergent. For

$$\frac{1}{m+1} \sum_{n=0}^{m} t_n = \frac{1}{m+1} \sum_{n=0}^{m} a\lambda^n = \frac{1}{m+1} \frac{1-\lambda^{m+1}}{1-\lambda}$$

Since a is a constant and $\frac{1-\lambda^{m+1}}{1-\lambda} = 0(1)$, then $\frac{1}{m+1} \sum_{n=0}^{m} t_n \neq 0$,

as m → ∞.

PROOF

Thus if $\{t_n\}$ is $[C,1]_1$ -convergent, its sum has to be zero. But

$$\frac{1}{m+1} \sum_{n=0}^{m} |t_n| = \frac{1}{m+1} \sum_{n=0}^{m} |a| |\lambda^n| = |a|$$

which \mapsto 0, since a \neq 0.

LEMMA 6.2

Let $\lambda_1, \lambda_2, \dots, \lambda_r$ be r distinct complex numbers, r > 1, with $|\lambda_v| = 1$, $\lambda_v \neq 1$ for $v = 1, 2, \dots$ r, and let a_1, a_2, \dots, a_r be non-zero complex numbers. If $t_n = a_1 \lambda_1^n + a_2 \lambda_2^n + \dots + a_r \lambda_r^n$, then $[t_n]$ is not $[C,1]_1$ convergent.

PROOF

If $\{t_n\}$ is $[C,1]_1$ -convergent, its sum must be zero: $\frac{1}{m+1} \sum_{n=0}^{m} |t_n| = \frac{1}{m+1} \sum_{n=0}^{m} |a_1 + a_2 \left(\frac{\lambda_2}{\lambda_1}\right)^n + \cdots + a_r \left(\frac{\lambda_r}{\lambda_1}\right)^n |.$ If $t_n \neq 0[C,1]_1$, then $\tau_n = a_1 + a_2 \left(\frac{\lambda_2}{\lambda_1}\right)^n + \cdots + a_r \left(\frac{\lambda_r}{\lambda_1}\right)^n |.$ $a_r \left(\frac{\lambda_r}{\lambda_1}\right)^n \neq o(C,1)$. But $\tau_n \neq a_1(C,1)$ and $a_1 \neq 0$. ///

THEOREM 6.12

If $\frac{q(z)}{p(z)}$ has no poles within the unit circle, but has simple poles on the unit circle and has no poles of higher order on the unit circle, then there is a sequence which is $[C_1, N_p]$ -convergent but not $[C_1, N_q]$ -convergent. PROOF

Suppose $\frac{q(z)}{p(z)}$ has r poles of order 1, $\lambda_1, \lambda_2, \ldots, \lambda_r$, on the unit circle and $r \geq 1$, and suppose it has other poles, $\lambda_{r+1}, \ldots, \lambda_{\ell}$, outside the unit circle of order $m_{r+1}, \ldots, m_{\ell}$.

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Since p(1) = 1, z = 1 cannot be a pole of $\frac{q(z)}{p(z)}$. i.e., $\lambda_{v} \neq 1$, for v = 1, 2, ..., r. Spince $p(0) \neq 0$, $\frac{1}{p(z)}$ is analytic in a neighbourhood There is a sequence $\{s_n\}$ such that, for U of the origin. z in U, $\sum_{n=0}^{\infty} s_n z^n = \frac{1}{p(z)} .$ Then, for z in U $\sum_{n=0}^{\infty} t_n z^n = p(z) \sum_{n=0}^{\infty} s_n z^n = 1,$ $\sum_{n=0}^{\infty} u_n z^n = q(z) \sum_{n=0}^{\infty} s_n z^n = \frac{q(z)}{p(z)} \sum_{n=0}^{\infty} t_n z^n$ Hence $t_{0_n} = 1$, $t_n = 0$ for n > 0, and so $\{t_n\}$ is $[C,1]_1$. convergent to zero. That is $\{s_n\}$ is $[C_1, N_p]$ -convergent to. zero. Now, by Theorem 6.9, $u_{n}' = c_{n} + \sum_{\nu=r+1}^{\ell} \sum_{\rho=1}^{m_{\nu}} c_{\nu,\rho} \left(\frac{n+\rho-1}{\rho-1} \right) \frac{1}{\nu} + \sum_{\nu=1}^{r} c_{\nu,1}' \frac{1}{\nu} \frac{1}{\nu}$ $= u_n^1 + u_n^2,$ where $u_n^1 = c_n + \sum_{\nu=r+1}^{\ell} \sum_{\rho=1}^{m_{\nu}} c_{\nu,\rho} {n+\rho-1 \choose \rho-1} \lambda_{\nu}^{-n}$ and $u_n^2 = \sum_{\nu=1}^n c_{\nu,1} \lambda_{\nu}^{-n}$. Since $c_n = 0$ for n > k - j, and $|\lambda_v| > 1$ for v = r+1, r+2, ... ℓ , $\{c_n\}$ and $\{c_{\nu,\rho} \begin{pmatrix} n+\rho-1\\ \rho-1 \end{pmatrix} \lambda_{\nu}^{-n} \}$ for $\nu = r+1, r+2, \ldots, \ell$, $\rho = 1, 2, \dots, m_{v}$ are each convergent to zero. Since (C,1) is regular, u_n^1 is $[C,1]_1$ -convergent to zero. But

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 $u_{n}^{2} = \sum_{\nu=1}^{r} c_{\nu,1} \lambda_{\nu}^{-n} = \sum_{\nu=1}^{r} c_{\nu,1} \frac{\overline{\lambda}^{n}}{\overline{\lambda}^{n}} = \sum_{\nu=1}^{r} c_{\nu,1} \overline{\lambda}^{n}_{\nu},$ and $\overline{\lambda}_1, \overline{\lambda}_2, \ldots, \overline{\lambda}_r$ are distinct and distinct from 1. And $|X_{\nu}| = 1$, for $\nu = 1, 2, ..., r$. Thus by Lemmas 6.1 and 6.2, we know that $\{u_n^2\}$ is not $[C,1]_1$ -convergent for $r \ge 1$. Consequently $\{u_n\}$ is not $[C,1]_1$ -convergent, that is $\{s_n\}$ is not $[C_1, N_{\alpha}]$ -convergent. /// THEOREM 6.13 $[C_1, N_p] \implies [C_1, N_q]$ if and only if q(z)/p(z) has no poles on or within the unit circle. PROOF The sufficiency part follows from Theorem 6.10. The necessity part follows from Theorems 6.11 and 6.12. /// THEOREM 6.14 $[C_1, N_p] \implies [C_1, N_q] if and only if N_p \implies N_q$. PROOF This is a consequence of Theorems 6.13 and 6.5. /// COROLLARY 6.1 If $[C_1, N_{q}] \leq [C_1, N_{q}]$, then it is necessary and sufficient, that both q(z)/p(z) and p(z)/q(z) have no poles on or within the unit circle. COROLLARY 6.2 $[C_1, N_p] \iff [C_1, N_q]$ if and only if $N_p \iff N_q$.

Noting that N_q is identical with I when q(z) = 1(i.e., $q_0 = 1$, $q_n = 0$ for n > 0) and referring to Corollary 6.1 we obtain the following corollary. COROLLARY 6.3

In order that $[C_1, N_p] \iff [C_1, I]_1$ it is necessary and sufficient that p(z) should not have zeros on or within the unit circle.

COROLLARY 6.4

 $[C_1, N_p] \iff [C, 1]_1$ if and only if $N_p \iff 1$.

For the following theorems and corollaries about the methods (N, p_n) , (N, q_n) , (N, P_n) , (N, Q_n) , $[N, P_n]$ and $[N, Q_n]$ we let p_v for v = 0, 1, ..., j and q_v for v = 0, 1, ..., k be the coefficients of the polynomials p(z) and q(z) respectively. We also let $P_r = \sum_{v=0}^r p_v \neq 0$ for r = 0, 1, ..., j-1 and $Q_r = \sum_{v=0}^r q_v \neq 0$ for r = 0, 1, ..., k-1, and $P_n^{\star} = \sum_{r=0}^n P_r \neq 0$ and $Q_n^{\star} = \sum_{v=0}^n Q_r \neq 0$ for all $n \geq 0$, so that (N, p_n) , $(N, q_n) \in (N, P_n)$, (N, Q_n) , $[N, P_n]$ and $[N, Q_n]$ are methods associated with p(z)and q(z) respectively and are all well defined. THEOREM 6.15

 $(N,P_n) => :(N,q_n)$ implies that $(N,P_n) => (N,Q_n)$.

► PROOF
Let
$$r_{\mathbf{r}} = \frac{1}{|\mathbf{r}_{\mathbf{r}}|} \sum_{v=0}^{T} p_{\mathbf{r}-v} \mathbf{s}_{v}$$
 and $\mathbf{y}_{\mathbf{r}} = \frac{1}{|\mathbf{r}_{\mathbf{r}}|} \sum_{v=0}^{T} \mathbf{g}_{\mathbf{r}-v} \mathbf{s}_{v}$,
and let $\mathbf{W}_{\mathbf{n}} = \frac{1}{|\mathbf{r}_{\mathbf{n}}|} \sum_{x=0}^{T} \mathbf{p}_{\mathbf{n}-x} \mathbf{s}_{x}$ and $\mathbf{V}_{\mathbf{n}} = \frac{1}{|\mathbf{r}_{\mathbf{n}}|} \sum_{x=0}^{T} \mathbf{0}_{\mathbf{n}-x} \mathbf{s}_{x}^{*}$.
Let $\mathbf{k}(z) = \frac{q(z)}{p(z)} = \frac{Q(z)}{p(z)}$ and $\mathbf{k}(z) = \sum_{v=0}^{T} \mathbf{k}_{v} \mathbf{z}^{v}$.
We know that the necessary and sufficient conditions that
 $(\mathbf{N}, \mathbf{p}_{\mathbf{n}}) \Rightarrow (\mathbf{N}, \mathbf{q}_{\mathbf{n}})$
in this case are:
 $(6.8) ||\mathbf{k}_{0}||\mathbf{P}_{\mathbf{n}}| + \cdots + ||\mathbf{k}_{\mathbf{n}}||\mathbf{P}_{0}| \leq \mathbf{H}^{*}|\mathbf{0}_{\mathbf{n}}|$
where H is independent of n, and
 $(6.9) ||\mathbf{k}_{\mathbf{n}-x}|| \sum_{v=0}^{T} ||\mathbf{r}_{\mathbf{n}-v}| \leq \sum_{v=0}^{T} ||\mathbf{k}_{\mathbf{n}-x}| \sum_{v=0}^{T} ||\mathbf{r}_{\mathbf{n}-v}|$
Thus, 'if $(\mathbf{N}, \mathbf{p}_{\mathbf{n}}) \Rightarrow (\mathbf{N}/\mathbf{q}_{\mathbf{n}})$, then (6.8) and (6.9) are satisfied.
Now $\sum_{x=0}^{T} ||\mathbf{k}_{\mathbf{n}-x}|| \sum_{v=0}^{T} ||\mathbf{r}_{\mathbf{n}-v}| \leq \sum_{v=0}^{T} ||\mathbf{k}_{\mathbf{n}-x}| ||\mathbf{P}_{\mathbf{n}-v}|$
 $\leq \mathbf{H} \sum_{v=0}^{T} ||\mathbf{n}_{\mathbf{n}-v}|$
 $= \sum_{v=0}^{T} \sum_{v=0}^{T} ||\mathbf{k}_{\mathbf{n}-x}| ||\mathbf{P}_{\mathbf{n}-v}|$
 $\leq \mathbf{H} \sum_{v=0}^{T} ||\mathbf{Q}_{\mathbf{n}-v}|$
 $= \mathbf{H} \sum_{v=0}^{T} ||\mathbf{Q}_{\mathbf{n}-v}|$
 $= \mathbf{0}(||\mathbf{Q}_{\mathbf{n}}|)$,
since $\mathbf{Q}_{v} = 1$ for $v \geq k$.

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And it is obvious that
$$k_{n-1}/Q_n^* + 0$$
 as $n + \infty$ for each r.
Thus, by [6, Proposition 1] again, we have
 $(N, P_n) \Rightarrow (N, Q_n)$. ////
COROLLARY 6.5
 $[(N, P_n) \iff (N, Q_n)' implies that $(N, P_n) \iff (N, Q_n)$.
THEOREM 6.16
 $[C_1, N_p] \Rightarrow [C_1, N_q] if and only if $(N, P_n) \Rightarrow (N, Q_n)$.
PROOF
By Theorem 6.1 we know that $[C_1, N_p] \iff (N, P_n)$ and
 $[C_1, N_q] \iff (N, Q_n)$.
By [6, Theorem 1], (c.f. Lemma 5.4), we have that if
 $(N, P_n) \Rightarrow (N, Q_n)$ then $[N, P_n] \Rightarrow (N, Q_n)$.
Thus, if $(N, P_n) \Rightarrow (N, Q_n)$ then $[C_1, N_p] \Rightarrow [C_1, N_q]$.
Conversely, by Theorem 6.14, we have that if
 $[C_1, N_q] \Rightarrow [C_1, N_q]$ then $N_p \Rightarrow N_q$.
Hence, if $[C_1, N_p] \Rightarrow (C_1, N_q]$ then $(N, P_n) = (N, q_n)$.
It follows from Theorem 6.15 that if $[C_1, N_p] \Rightarrow [C_1, N_q]$.
COROLLARY 6.6
 $[C_1, N_p] \Rightarrow [C_1, N_q] if and only if $(N, P_n), \iff (N, Q_n)$.
THEOREM 6.17
 $[C_1, N_p] = > [C_1, N_q] if and only if $(C, 1)N_p \Rightarrow r(C, 1)N_q$.
PROOF
 $(N, P_n) = (\overline{N}, P_n)(N, P_n)$.$$$$

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From the proof of Theorem 6.1, we know that $(\overline{N}, P_n) \iff (C, 1)$. Thus $(N, P_n) \iff (C, 1) (N, P_n) \iff (C, 1) N_p$ and similarly we have $(N, Q_n) \iff (C, 1) (N, q_n) \iff (C, 1) N_q$.

It follows from Theorem 6.16 that $[C_1, N_p] \Rightarrow [C_1, N_q]$ if and only if $(C, 1)N_p \Rightarrow (C, 1)N_q^\circ$. COROLLARY 6.7

 $[C_1, N_p] \iff [C_1, N_q] \text{ if and only if } (C, 1) N_p \iff (C, 1) N_q.$

\$6.5 ABSOLUTE POLYNOMIAL NÖRLUND METHODS OF SUMMABILITY THEOREM 6.18

If q(z)/p(z) has no poles on or within the unit circle, then $|N_p| \Rightarrow |N_q|$. PROOF

 $t_n = \sum_{\nu=0}^{n} p_{\nu} s_{\vec{n}-\nu}$

Suppose q(z)/p(z) has no poles on or within the unit circle, but has poles $\lambda_1, \lambda_2, \ldots, \lambda_l$ of orders m_1, m_2, \ldots, m_l outside the unit circle. Let

$$u_n = \sum_{\nu=0}^{n} q_{\nu} s_{n-\nu}, \quad \text{for } n = 0, 1, \dots$$

Then by Theorem 6.9

$$u_{n} = \sum_{\nu=0}^{n} c_{\nu} t_{n-\nu} + \sum_{r=1}^{\ell} \sum_{\rho=1}^{m} c_{r,\rho} \sum_{\nu=0}^{n} (\nu+\rho-1) \lambda_{r} t_{n-\nu}$$

where c's are constants, depending only on p_0, p_1, \dots, p_j , q_0, q_1, \dots, q_k , such that $c_n = 0$ for n > k - j and $c_{r,m} \neq 0$ Hence

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(C,1) is regular, $s_n \neq s [C_1, N_p]$.

THEOREM 6.19

If q(z)/p(z) has a pole within the unit circle, then there is a sequence which is $|N_p|$ -convergent but not $|N_q|$ -convergent. PROOF

Since $p(0) \neq 0$, $\frac{1}{p(z)}$ is analytic in a neighbourhood U of origin. There is a sequence $\{s_n\}$ such that for z in U,

 $\sum_{n=0}^{\infty} s_n z^n = \frac{1}{p(z)} .$

 $t_n = \sum_{\nu=0}^{n} p_{\nu} s_{n-\nu},$

 $u_n = \sum_{\nu=0}^{n} q_{\nu} s_{n-\nu}$

Then, for z in U,

Let

 $\sum_{n=0}^{\infty} t_n z^n = p(z) \sum_{n=0}^{\infty} s_n z^n = 1,$

 $\sum_{n=0}^{\infty} u_n z^{n'} = q(z) \sum_{n=0}^{\infty} s_n z^n = \frac{q(z)}{p(z)}.$ Hence $t_0 = 1$, $t_n = 0$ for n > 0, and so $\{s_n\}$ is $|N_p| = *$

convergent. On the other hand $\sum_{n=0}^{\infty} u_n z^n$ has a radius of

convergence less than unity, because by hypothesis q(z)/p(z)has a pole within the unit circle. Consequently $\{u_n\}$ is not A-convergent and so it is not (C, 1)-convergent. Hence $\{s_n\}$ is not (C, 1) -convergent. By Proposition 6.1(ii), $\{s_n\}$ is not $[C_1, N_q]$ -convergent. Thus by Proposition 6.2 $\{s_n\}$ is not $|N_q|$ -convergent.

THEOREM 6.20

If $\frac{q(z)}{p(z)}$ has no poles within the unit circle, but has poles on the unit circle, then there is a sequence which is $\frac{|N_p|}{p}$ -convergent but not $|N_q|$ -convergent. •PROOF

Let the poles of $\frac{q(z)}{p(z)}$ be $\lambda_1, \lambda_2, \ldots, \lambda_{\ell}$ of orders $m_1, m_2, \ldots, m_{\ell}$. Let the numbering be such that of these poles $\lambda_1, \lambda_2, \ldots, \lambda_{\ell}$ are on the unit circle, $\lambda_{\ell'+1}, \ldots, \lambda_{\ell'}$ are outside the unit circle.

Since $p(0) \neq 0$, $\frac{1}{(1-z)p(z)}$ is analytic in a neighbourhood U of origin. There is a sequence $\{s_n\}$ such that, for z in U,

$$\sum_{n=0}^{\infty} s_n z^n = \frac{1}{(1-z)p(z)}$$

Then, for z in U,

$$\sum_{n=0}^{\infty} t_n z^n = p(z) \quad \sum_{n=0}^{\infty} s_n z^n = \frac{1}{1-z} = 1 + z + z^2 + z$$

$$\sum_{n=0}^{\infty} u_n \overline{z}^n = q(z) \quad \sum_{n=0}^{\infty} s_n z^n = \frac{q(z)}{p(z)} \quad \sum_{n=0}^{\infty} t_n z^n.$$

Hence $t_n = 1$ for all $n \ge 0$ and so $\{s_n\}$ is $|N_p|$ convergent.

Now, by Theorem 6.9 since $t_n = 1$, for all $n \ge 0$,

$$\mathbf{u}_{n} = \sum_{\nu=0}^{n} \mathbf{c}_{\nu} + \sum_{r=1}^{\ell} \sum_{\rho=1}^{m_{r}} \mathbf{c}_{r,\rho} \sum_{\nu=0}^{n} \left(\frac{\nu+\rho-1}{\rho-1} \right) \lambda_{r}^{-\nu},$$

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where the c's are constants, depending only on

 $p_0, p_1, \dots, p_j, q_0, q_1, \dots, q_k$ such that $c_n = 0'$ for $n > k - q_k$ and $c_{r_{j}m_{r}}^{*} \neq 0$.

Thus

$$-\mathbf{u}_{n-1} = \mathbf{c}_{n} + \sum_{r=1}^{\ell} \sum_{\rho=1}^{m_{r}} \mathbf{c}_{r,\rho} \begin{pmatrix} n+\rho-1\\ \rho-1 \end{pmatrix} \lambda_{r}^{-n}$$

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$$w_{n}^{2} = \sum_{r=1}^{\ell} \sum_{\rho=1}^{m} c_{r,\rho} {n+\rho-1 \choose \rho-1} \lambda_{r}^{-n}.$$

Since $c_n = 0$ for n > k - j, $\sum_{n=0}^{\infty} c_n$ is absolutely

convergent, and since $|\lambda_r| > 1$, for $r = l'+1, \ldots, l$,

 $\sum_{n=0}^{\infty} {n+\rho-1 \choose \rho-1} \lambda_r^{-n} \text{ is absolutely convergent, for } r = \ell +1, \ldots, \ell,$

 $p = 1, 2, \dots, m_r$. Hence $\sum_{n=0}^{\infty} |w_n^2|$, is convergent.

Now, for w_n^2 , if there are ℓ poles on the unit circle of maximum order m, where $1 \leq \ell' \leq \ell'$ and $m \geq 1$, then we let the numbering be such that $\lambda_1, \lambda_2, \ldots, \lambda_{\ell'}$ have maximum order m. In this case,

 $|\mathbf{w}_{n}^{2}| = |\sum_{\mathbf{r}=1}^{\mathbf{r}''} \sum_{\rho=1}^{\mathbf{m}} c_{\mathbf{r},\rho} {n+\rho-1 \choose \rho-1} \tilde{\lambda}_{\mathbf{r}}^{-n} + \sum_{\mathbf{r}=\boldsymbol{\ell}''+1}^{\mathbf{r}} \sum_{\rho=1}^{\mathbf{r}} c_{\mathbf{r},\rho} {n+\rho-1 \choose \rho-1} \lambda_{\mathbf{r}}^{-n}|$ $= O | c_{1,m} \binom{n+m-1}{m-1} \lambda_1^{-n} + c_{2,m} \binom{n+m-1}{m-1} \lambda_2^{-n} + \cdots + c_{\ell'',m} \binom{n+m-1}{m-1} \lambda_{\ell''}^{-n} |$ $= O\left[{n+m-1 \atop m-1} \right] |c_{1,m}^{n} \lambda_1^{-n} + c_{2,m}^{n} \lambda_2^{-n} + \cdots + c_{\ell'',m}^{n} \lambda_{\ell''}^{-n}|^{-n}$ $= O\left(\frac{n+m-1}{m-1}\right) |c_{1,m}\overline{\lambda_1^n} + c_{2,m}\overline{\lambda_2^n} + \cdots + c_{\ell'',m}\overline{\lambda_{\ell''}^n}|.$ Since $p(1) = 1, x^2 = 1$ is not a pole of $\frac{q(z)}{p(z)}$; and since $\overline{\lambda}_{0}$ are distinct and distinct from 1, and $|\overline{\lambda}_{_{\mathrm{U}}}|$ = 1, for v = 1, 2, ..., l''; and $c_{v,m} \neq 0$, for v = 1, 2, ..., l'', byLemma 6.1 and Lemma 6.2 we know that $\{c_{1,m}\overline{\lambda}_{1}^{n}+c_{2,m}\overline{\lambda}_{2}^{n}+\cdots+c_{\ell},m\overline{\lambda}_{\ell}^{n}\}$ is not $[C,1]_{1}$ -convergent for $l'' \geq 1$. Thus $\{|c_{1,m} \overline{\lambda}_1^n + c_{2,m} \overline{\lambda}_2^n + \cdots + c_{l'',m} \overline{\lambda}_{l''}^n|\}$ cannot be convergent. A fortiori it does not converge to zero. Hence $|w_n^2|$ does not tend to zero as $n \rightarrow \infty$. This means that $\sum_{n=0}^{\infty} |\psi_n^2|$ diverges. Consequently $\sum_{n=0}^{\infty} |u_n - u_{n-1}| \text{ diverges, as } n \neq \infty$ In other words, $\{s_n\}$ is not $|N_q|$ -convergent. THEORE 6.21 In order that $|N_p| => |N_q|$, it is necessary and sufficient that $\frac{q(z)}{p(z)}$ should not have poles on or within the unit circle.

PROOF

The sufficiency part of the theorem follows from Theorem 6.18. The necessity part follows from Theorems 6,19 and 6.20. COROLLARY 6.8 $|\mathbf{N}_{\mathbf{p}}| => |\mathbf{N}_{\mathbf{q}}|$ if and only if $\mathbf{\tilde{N}}_{\mathbf{p}} => \mathbf{N}_{\mathbf{q}}$. This follows, from Theorems 6.5 and 6.21. -/// COROLLARY 6.9 $|N_p| \Rightarrow |N_q|$ is and only if $[C_1, N_p] \Rightarrow [C_1, N_q]$. PROOF This follows from Theorems 6.13 and 6.21. COROLLARY 6.10 $|N_p| \iff |N_q|$ if and only if $\frac{q(z)}{p(z)}$ and $\frac{p(z)}{q(z)}$ both have no poles on or within the unit circle. COROLLARY 6.11 $|N_p| <=> |N_q|$ if and only if $N_p <=> N_q$. COROLLARY 6.12 $|N_p| \ll |N_q|$ if and only if $[C_1, N_p] \ll [C_1, N_q]$. Noting that N_{α} is identical with I when q(z) = 1, we have, as a consequence of Corollary 6.10, the following corollary. COROLLARY 6.13 In order that $\{s_n\}$ is $|N_p|$ convergent if and only if

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a, is absolutely convergent is necessary and sufficient that p(z) should not have zeros on or within the unit circle. SOME MINOR RESULTS -\$6.6 THEOREM 6.22 · . ' If f(z) = p(z)q(z), then (i) $[C_1, N_p] => [C_1, N_f]$ and $[C_1, N_q] => [C_1, N_f]$, (ii) $|N_p| \Rightarrow |N_f| \cdot and |N_q| \Rightarrow |N_f|$. PROOF (i) follows from Theorem 6.13 and (ii) follows from Theorem 6.21. CORQLLARY 6.14 The methods $[C_1, N_p]$ and $[C_1, N_q]$ are consistent, i.e., if $s_n \rightarrow s [C_1, N_p]$ and $s_n \rightarrow s' [C_1, N_q]$, then s = s'. THEOREM 6.23 If h(z) is the highest common factor of p(z) and q(z)normalized so as to make h(1) = 1, then (i) a sequence is both $[C_1, N_p]$ - and $[C_1, N_q]$ -convergent if and only if it is $[C_1, N_h]$ -convergent, (ii) a sequence is both $|N_p|$ - and $|N_q|$ -convergent if and only if it is $|N_{h}|$ -convergent. PROOF The sufficiency part follows from Theorem 6.22 (i). (i) To prove the necessity part, we observe that there are polynomials

$$\begin{aligned} d(z) &= \sum_{n=0}^{l} a_n z^n \\ b(z) &= \sum_{n=0}^{l} b_n z^n \end{aligned}$$

such that $h(z) = a(z)p(z) + b(z)q(z)$
$$&= \sum_{n=0}^{l} a_n z^n, \text{ say,} \end{aligned}$$

where l_1, l_2, l_3 are non-negative integers.
Hence if $t_n = \sum_{\nu=0}^{n} p_\nu s_{n-\nu}$, and $u_n = \sum_{\nu=0}^{n} q_\nu s_{n-\nu}$, then.
 $w_n = \sum_{\nu=0}^{n} h_\nu s_{n-\nu} = \sum_{\nu=0}^{n} a_\nu t_{n-\nu} + \sum_{\nu=0}^{n} b_\nu u_{n-\nu}$,
where $a_\nu = 0$, for $\nu > l_1$ and $b_\nu = 0$, for $\nu > l_2$.
Without loss of generality, we may assume $s_n + o[C_1/N_p]$
and $s_n + o[C_1, N_q]$. Now
$$\frac{1}{m+1} \prod_{n=0}^{m} + w_n| \le \frac{1}{m+1} \prod_{n=0}^{m} \sum_{\nu=0}^{n} |a_\nu|| |t_n| + \frac{1}{m+1} \prod_{n=0}^{m} \sum_{\nu=0}^{n} |b_\nu|| |u_{n-\nu}| \\ \le \frac{1}{m+1} \prod_{n=0}^{m} (\sum_{\nu=0}^{l} |a_\nu||) |t_n| + \frac{1}{m+1} \prod_{n=0}^{m} (\sum_{\nu=0}^{l} |b_\nu||) |u_n| \\ = o(1), \quad \text{as } m + \infty. \end{aligned}$$

That is $s_n + o[C_1, N_h]$.
(ii) The sufficiency part follows from Theorem 6.22(ii).
As in the proof of (i),

$$w_{n} = \sum_{\nu=0}^{n} h_{\nu} = \sum_{\nu=0}^{n} (-\nu t_{\nu} + \sum_{\nu=0}^{n} b_{n-\nu} u_{\nu})$$

Hence
$$\mathbf{w}_{n} - \mathbf{w}_{n-1}$$

 $\mathbf{v}_{n-0} = \sum_{v=0}^{n} a_{n-v} (\mathbf{t}_{v} - \mathbf{t}_{v-1}) + \sum_{v=0}^{n} b_{n-v} (\mathbf{u}_{v} - \mathbf{u}_{v-1}),$
 $\mathbf{v}_{v}^{where} \mathbf{t}_{2} = 0, \mathbf{u}_{-1} = 0, \mathbf{w}_{-1} = 0 \text{ and } \mathbf{u}_{n-v} = 0 \text{ if } n - v > s_{1},$
 $\mathbf{b}_{n-v} = 0, \text{ if } n - v > s_{2}.$
 $\sum_{n=0}^{m} |\mathbf{w}_{n} - \mathbf{w}_{n-1}|$
 $\leq \sum_{v=0}^{m} \sum_{v=0}^{n} |\mathbf{u}_{n-v}| + \mathbf{t}_{v-1}| + \sum_{n=0}^{m} \sum_{v=0}^{n} |\mathbf{b}_{n-v}| |\mathbf{u}_{v} - \mathbf{u}_{v-1}|,$
 $\leq \sum_{v=0}^{l} |\mathbf{u}_{v}| \sum_{n=0}^{m} |\mathbf{t}_{n} - \mathbf{t}_{n-1}| + (\sum_{v=0}^{2} |\mathbf{b}_{v}|) \sum_{n=0}^{m} |\mathbf{u}_{n} - \mathbf{u}_{n-1}|.$
Hence if $\sum_{n=0}^{m} |\mathbf{t}_{n} - \mathbf{t}_{n-1}| = O(1),$ and $\sum_{n=0}^{m} |\mathbf{u}_{n} - \mathbf{u}_{n-1}| = O(1),$
then $\sum_{n=0}^{m} |\mathbf{w}_{n} - \mathbf{w}_{n-1}| = O(1).$

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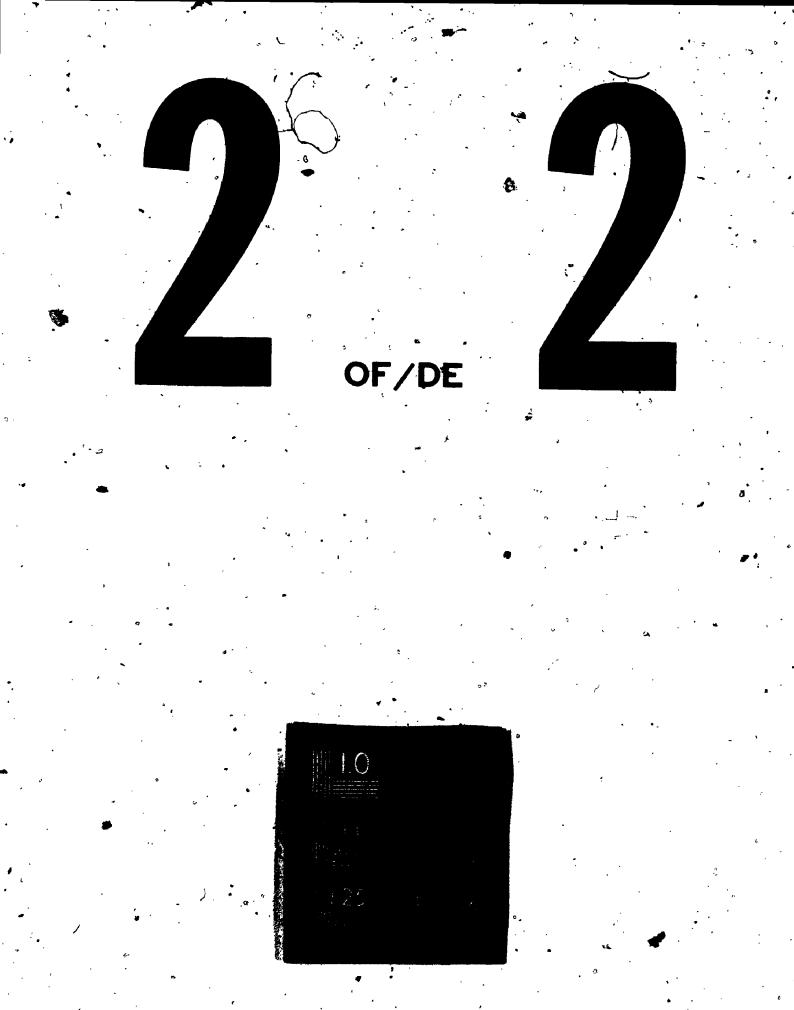
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