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Aspects Of Strong Summability Associated With Generalised Cesaro, Riesz And Noerlund Summability

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ASPECTS OF STRONG SUMMABILITY ASSOCIATED WITH GENERALISED CESÀRO, RIESZ AND NÖRLUND SUMMABILITY

by

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ABSTRACT

Generalised Cesàro Summability, Riesz Summability and Strong Riesz Summability have been extensively investigated by various authors. In this thesis a definition of Strong Generalised Cesàro Summability Method is proposed and the question of its equivalence with the Strong Riesz Summability Method is established. In Chapter 3 some equivalence theorems between the Generalised Cesàro Methods and the Strong Generalised Cesàro Method are obtained. In Chapter 4 inclusion theorems between the Absolute Generalised Cesàro Methods and the Strong Generalised Cesàro Method are obtained.

We extend a result due to Kuttner, obtaining some strict inclusion theorems between Cesàro and Discrete Riesz Methods of Summability. And our investigation in this respect stems from Borwein and Cass's work on Strong Nörlund Summability.

In Chapter 6 we consider Nörlund Methods of Summability Associated with Polynomials which have been investigated by Borwein, and consider Strong and Absolute Nörlund Methods associated with them. We show, for example, that two polynomial Nörlund Methods are equivalent if and only if the associated Strong Methods are equivalent.

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CONVENTIONS

In this thesis, the symbols H, H_1, H_2, H_3 are used throughout to denote positive constants, but not necessarily having the same value at each occurrence.

The theorems, lemmata and corollaries are numbered by chapter. For example, Theorem 3.1 is the first theorem in Chapter 3.

At the end of each proof we use the symbol /// to show that the proof is complete.

CHAPTER 1

STRONG GENERALISED CESÀRO SUMMABILITY

§1.1 INTRODUCTION

We suppose throughout the thesis that $\lambda = \{\lambda_n\}$ is a sequence satisfying

$$(1.1) \qquad 0 = \lambda_0 < \lambda_1 < \cdots\cdots < \lambda_n \to \infty.$$

For the sake of convenience we take $\lambda_0 = 0$ in (1.1) instead of $\lambda_0 \geq 0$. By doing so we find that there is no loss of generality. This remark will be amplified on page 5.

We suppose also that p is a non-negative integer and for the series $\sum_{\nu=0}^{\infty} a_\nu$ we use the notation

$$s_n = \sum_{\nu=0}^{n} a_\nu, \quad n = 0,1,2,\ldots\ldots .$$

In this chapter we introduce a definition of *Strong Generalised Cesàro Summability* and investigate some of its properties. We also give the definitions of several other summability methods whose properties and relations with the Strong Generalised Cesàro Summability are investigated in the later chapters.

§1.2 SUMMABILITY METHODS

If a given summability method T assigns the *sum* s to the series $\sum_{\nu=0}^{\infty} a_\nu$ with sequence of partial sums $\{s_n\}$, we say that $\sum_{\nu=0}^{\infty} a_\nu$ is T-*summable* or $\{s_n\}$ is T-*convergent* to s. We denote this by

$$\sum_{\nu=0}^{\infty} a_\nu = s \ (T)$$

or by

$$s_n \to s \ (T).$$

A method of summability T is said to be *regular*, if $s_n \to s(T)$ whenever the sequence $\{s_n\}$ converges to s.

Let $Q = \{q_{n,r}\}$ $(n, r = 0,1,2,\ldots)$ be a (summability) matrix and let

$$\sigma_n = \sum_{r=0}^{\infty} q_{n,r} s_r. \tag{1.2}$$

The sequence $\{s_n\}$ is said to be Q-*convergent* to the sum s if σ_n exists for $n = 0,1,2,\ldots$ and tends to s as n tends to infinity.

The matrix $Q = \{q_{n,r}\}$ is *regular* if and only if

$$\sup_{n\geq 0} \sum_{r=0}^{\infty} |q_{n,r}| < \infty, \tag{1.3}$$

(1.4) $$\lim_{n\to\infty} q_{n,r} = 0, \text{ for } r = 0,1,2,\ldots ,$$

(1.5) $$\lim_{n\to\infty} \sum_{r=0}^{\infty} q_{n,r} = 1.$$

This is the Toeplitz Theorem for the regularity of the matrix Q.

The symbol P will be reserved for matrices $\{p_{n,r}\}$ with

$$p_{n,r} \geq 0 \quad (n,r = 0,1,2,\ldots).$$

Such matrices will be called non-negative matrices.

Let $\mu > 0$. The *Strong Summability Methods* $[P,Q]_\mu$ are defined as follows. We write $s_n \to s\ [P,Q]_\mu$ if

(1.6) $$\tau_n = \sum_{\nu=0}^{\infty} p_{n,r}|\sigma_r - s|^{\mu}$$

exists for $n = 0,1,2,\ldots$ and tends to zero as n tends to infinity. Thus s is the $[P,Q]_\mu$-*limit* of $\{s_n\}$ and the sequence is $[P,Q]_\mu$-*convergent* to s.

If V and W are summability methods of any of the above types we shall say that W *includes* V, and use the notation V => W, if any sequence V-convergent to s is necessarily W-convergent to s. If W includes V but V does not include W, the inclusion V => W is said to be *strict*. If both V => W and W => V, we say that V and W are *equivalent* and write V <=> W.

Let $\mu > 0$. We say that $\{s_n\}$ is *absolutely* $(Q)_\mu$-*convergent* or $|Q|_\mu$-*convergent* if

(1.7) $$\sum_{n=1}^{\infty} n^{\mu-1}|\sigma_n - \sigma_{n-1}|^{\mu} < \infty.$$

§1.3 RIESZ SUMMABILITY (R,λ,κ)

Let $\kappa \geq 0$ and $\lambda = \{\lambda_n\}$ satisfy (1.1). The Riesz Summability Method (R,λ,κ) is defined as follows.

Let $$A_\lambda^\kappa(\tau) = \sum_{\lambda_\nu<\tau} a_\nu, \text{ for } \kappa = 0,$$

$$A_\lambda^\kappa(\tau) = \sum_{\lambda_\nu<\tau} (\tau - \lambda_\nu)^\kappa a_\nu, \text{ for } \kappa > 0,$$

$$R_\lambda^\kappa(\tau) = A_\lambda^\kappa(\tau) = \sum_{\lambda_\nu<\tau} a_\nu, \text{ for } \kappa = 0,$$

and $$R_\lambda^\kappa(\tau) = \sum_{\lambda_\nu<\tau} \left(1 - \frac{\lambda_\nu}{\tau}\right)^\kappa a_\nu, \text{ for } \kappa > 0.$$

The series $\sum_{\nu=0}^{\infty} a_\nu$ is said to be (R,λ,κ)-summable to s, if

$$R_\lambda^\kappa(\tau) \to s \quad \text{as } \tau \to \infty.$$

(See Hardy and Riesz [12, pp. 21-22].)

§1.4 STRONG RIESZ SUMMABILITY $[R,\lambda,p+1]_\mu$

The series $\sum_{\nu=0}^{\infty} a_\nu$ is said to be Strongly Riesz Summable to s, with order p+1 and index $\mu>0$, if

$$F^{p+1}(\omega) = \int_0^\omega |A_\lambda^p(\tau) - s\tau^p|^\mu \, d\tau = o(\omega^{p\mu+1}).$$

We denote this by

$$\sum_{\nu=0}^{\infty} a_\nu = s \; [R,\lambda,p+1]_\mu.$$

The definition of the Strong Riesz Summability we have given here is due to Glatfeld [15]. Srivastava [24] and Boyd and Hyslop [8] have also given definitions of Strong Riesz Summability, but we shall not be concerned with them here.

We give now two examples to illustrate that no loss of generality is involved by taking $\lambda_0 = 0$ in (1.1).

Our first example deals with Riesz Summability.

Let $\lambda = \{\lambda_n\}$ satisfy

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n \to \infty$$

and let $\delta = \{\delta_n\}$ satisfy

$$\lambda_1 > \delta_0 > 0 \text{ and } \delta_n = \lambda_n \text{ for } n \neq 0.$$

Let $R_\lambda^\kappa(\tau)$ be defined as in §1.3 and let

$$R_\delta^\kappa(\tau) = \sum_{\delta_\nu < \tau} \left(1 - \frac{\delta_\nu}{\tau}\right)^\kappa a_\nu .$$

Then
$$R_\delta^\kappa(\tau) - s = R_\lambda^\kappa(\tau) - s + R_\delta^\kappa(\tau) - R_\lambda^\kappa(\tau)$$
$$= R_\lambda^\kappa(\tau) - s + \left[\left(1 - \frac{\delta_0}{\tau}\right)^\kappa a_0 - a_0\right].$$

Since $\left(1 - \frac{\delta_0}{\tau}\right)^\kappa a_0 - a_0 \to 0$ as $\tau \to \infty$, $R_\delta^\kappa(\tau) \to s$ if and only if $R_\lambda^\kappa(\tau) \to s$, as $\tau \to \infty$.

Our other example deals with Strong Riesz Summability. Let $A_\lambda^p(\tau)$ be defined as in §1.3 and let

$$A_\delta^p(\tau) = \sum_{\delta_\nu < \tau} (\tau - \delta_\nu)^p a_\nu .$$

$$\text{Then } I_1 = \int_0^\omega |A_\delta^p(\tau) - s\tau^p|^\mu \, d\tau$$

$$= \int_0^\omega |A_\lambda^p(\tau) - s\tau^p + A_\delta^p(\tau) - A_\lambda^p(\tau)|^\mu \, d\tau$$

$$\leq 2^\mu \left(\int_0^\omega |A_\lambda^p(\tau) - s\tau^p|^\mu \, d\tau + \int_0^\omega |a_0(\tau - \delta_0)^p - a_0\tau^p|^\mu \, d\tau\right)$$

$$= 2^\mu (I_2 + I_3).$$

Regarding a_0 as the series $\sum_{\nu=0}^{\infty} b_\nu$ with $b_0 = a_0$ and $b_\nu = 0$ for $\nu > 0$, we have $(\tau - \delta_0)^p a_0 = \sum_{\delta_\nu < \tau} (\tau - \delta_\nu)^p b_\nu$. Since $\sum_{\nu=0}^{\infty} b_\nu = a_0$ and $[R,\lambda,p+1]_\mu$ is regular, (see Glatfeld [15]), thus $I_3 = o(\omega^{p\mu+1})$. Hence $I_2 = o(\omega^{p\mu+1}) \Rightarrow I_1 = o(\omega^{p\mu+1})$. Similarly $I_1 = o(\omega^{p\mu+1}) \Rightarrow I_2 \Rightarrow o(\omega^{p\mu+1})$.

§1.5 GENERALISED CESÀRO SUMMABILITY (C,λ,p)

Let $\lambda = \{\lambda_n\}$ satisfy (1.1).

Define $C_n^p = \sum_{\nu=0}^{n} a_\nu$, for $p = 0$,

$$C_n^p = \sum_{\nu=0}^{n} (\lambda_{n+1} - \lambda_\nu) \cdots (\lambda_{n+p} - \lambda_\nu) a_\nu, \text{ for } p = 1,2,3,\ldots,$$

$$t_n^p = C_n^p = \sum_{\nu=0}^{n} a_\nu, \text{ for } p = 0,$$

$$t_n^p = (\lambda_{n+1} \cdots \lambda_{n+p})^{-1} C_n^p$$

$$= \sum_{\nu=0}^{n} \left(1 - \frac{\lambda_\nu}{\lambda_{n+1}}\right) \cdots \left(1 - \frac{\lambda_\nu}{\lambda_{n+p}}\right) a_\nu, \text{ for } p = 1,2,3,\ldots$$

If $t_n^p \to s$ as $n \to \infty$, then $\sum_{\nu=0}^{\infty} a_\nu$ is said to be (C,λ,p) summable to s and we write

$$\sum_{\nu=0}^{\infty} a_\nu = s \quad (C,\lambda,p).$$

Since (C,λ,p) is a matrix method in the sense described in §1.2, we shall find it convenient to denote both the summability method and its associated matrix by (C,λ,p). Since the entries in the matrix (C,λ,p) are zero above the main diagonal and non-zero on the main diagonal, it has an inverse.

§1.6 STRONG GENERALISED CESÀRO SUMMABILITY $[C,\lambda,p+1]_\mu$

Let $\lambda = \{\lambda_n\}$ satisfy (1.1). We define

$$E_n^p(\lambda) = E_n^p = 1, \text{ for } p = 0,$$

$$E_n^p(\lambda) = E_n^p = \lambda_{n+1} \cdots \lambda_{n+p}, \text{ for } p = 1,2,3,\ldots,$$

and $n = 0,1,2,3,\ldots$.

Since $\lambda_0 = 0$, we obtain

$$E_m^{p+1} = \sum_{n=0}^{m} (\lambda_{n+p+1} - \lambda_n) E_n^p.$$

We define

$$T_{m,\mu}^1 = \sum_{n=0}^{m} (\lambda_{n+1} - \lambda_n) |t_n^0 - s|^\mu,$$

$$T_{m,\mu}^{p+1} = \sum_{n=0}^{m} (\lambda_{n+p+1} - \lambda_n) E_n^p |t_n^p - s|^\mu,$$

$$\sigma_{m,\mu}^1 = \lambda_{m+1}^{-1} \sum_{n=0}^{m} (\lambda_{n+1} - \lambda_n) |t_n^0 - s|^\mu,$$

$$\sigma^{p+1}_{m,\mu} = \frac{T^{p+1}_{m,\mu}}{E^{p+1}_m} = \frac{1}{E^{p+1}_m} \sum_{n=0}^{m} (\lambda_{n+p+1} - \lambda_n) E^p_n |t^p_n - s|^{\mu}.$$

We say that the series $\sum_{\nu=0}^{\infty} a_\nu$ is *Strongly Generalised Cesàro Summable* to s, with order p+1 and index μ, if

$$\sigma^{p+1}_{m,\mu} = o(1) \quad \text{as } m \to \infty.$$

And we use the notation

$$\sum_{\nu=0}^{\infty} a_\nu = s\ [C,\lambda,p+1]_\mu .$$

Generalised Cesàro Summability was first introduced by Jurkat, [16]. Burkill, [9], gave a different definition. The definition we use here is due to Burkill. The definition was extended to accommodate positive non-integral values of p by Borwein, [3]. We have not been able to formulate a suitable definition of $[C,\lambda,p+1]_\mu$ with p non-integral.

Several persons have investigated relations between Riesz and Generalised Cesàro Summability. In particular, it is proved in Russell [23] that if λ is a sequence satisfying (1.1) and p is a non-negative integer then

$$(C,\lambda,p) \Rightarrow (R,\lambda,p), \quad p = 0,1,2,3,\ldots .$$

It is proved in Meir [20] that if λ is a sequence satisfying (1.1) and p is a non-negative integer then

$$(R,\lambda,p) \Rightarrow (C,\lambda,p), \quad p = 0,1,2,3,\ldots .$$

If in §1.5 we take $\lambda_n = n$, we recover the classical Cesàro Summability Method (C,p). (See Hardy [11].)

If in §1.6 we take $\lambda_n = n$, we obtain a summability method which although not equal to, is nevertheless equivalent to the classical Strong Cesàro Summability Method $[C, p+1]_\mu$. (See Borwein and Cass [6].)

We recall that $\sum_{\nu=0}^{\infty} a_\nu = s\ [C,p+1]_\mu$ if and only if

$$\frac{1}{n+1} \sum_{\nu=0}^{n} |S_\nu^p - s|^\mu = o(1),$$

where $S_n^p = \frac{1}{\varepsilon_n^p} \sum_{\nu=0}^{n} \varepsilon_{n-\nu}^{p-1} s_\nu$ and $\varepsilon_n^p = \binom{n+p}{n}$.

In case where no confusion can arise, we omit the subscript μ from $\sigma_{m,\mu}^{p+1}$ and $T_{m,\mu}^{p+1}$.

§1.7 SIMPLE INCLUSION THEOREMS

In order to simplify the notation and the proofs of theorems occurring later we introduce a matrix

$$\Lambda_{p+1} = \{\lambda_{m,n}^{p+1}\} = \{\lambda_{m,n}\}$$

which is defined as follows.

(1.8) For $p = 0$

$$\lambda_{m,n} = \begin{cases} \frac{1}{E_m^1} (\lambda_{n+1} - \lambda_n) = \frac{1}{\lambda_{m+1}} (\lambda_{n+1} - \lambda_n), & \text{for } 0 \leq n \leq m, \\ 0, & \text{for } n > m; \end{cases}$$

and for $p > 0$

$$\lambda_{m,n} = \begin{cases} \frac{1}{E_m^{p+1}} (\lambda_{n+p+1} - \lambda_n) E_n^p, & \text{for } 0 \leq n \leq m, \\ 0, & \text{for } n > m. \end{cases}$$

It follows easily from the Toeplitz conditions (1.3), (1.4), (1.5) that Λ_{p+1} is regular.

We now establish some results pertaining to the Strong Generalised Cesàro Summability.

Let C_n^p and t_n^p be defined as in §1.5. Then

$$C_n^{p+1} - C_{n-1}^{p+1} = (\lambda_{n+p+1} - \lambda_n) C_n^p$$

so that

(1.9) $$C_n^{p+1} = \sum_{\nu=0}^{n} (\lambda_{\nu+p+1} - \lambda_\nu) C_\nu^p .$$

(See [23, p. 419].)

Hence

(1.10) $$\frac{1}{E_m^{p+1}} \sum_{n=0}^{m} (\lambda_{n+p+1} - \lambda_n) E_n^p \, t_n^p$$

$$= \frac{1}{E_m^{p+1}} \sum_{n=0}^{m} (\lambda_{n+p+1} - \lambda_n) C_n^p$$

$$= C_m^{p+1} / E_m^{p+1}$$

$$= t_m^{p+1} .$$

This means, in matrix notation,

(1.11) $$(C,\lambda,p+1) = \Lambda_{p+1} (C,\lambda,p) .$$

Moreover, referring to (1.6), the definitions of $[C,\lambda,p+1]_\mu$ and Λ_{p+1}, we have

(1.12) $$[C,\lambda,p+1]_\mu = [\Lambda_{p+1}, (C,\lambda,p)]_\mu .$$

The following two theorems are given in Borwein, [1, Theorems 1 and 3]. We reproduce the proofs for the sake of completeness.

THEOREM 1.1

If Q *is any matrix and* $P = \{p_{n,r}\}$, *where* $p_{n,r} \geq 0$ *for* $n,r = 0,1,\ldots,$ $\sum_{r=0}^{\infty} p_{n,r} < M$ *for* $n = 0,1,\ldots$ *and if* $\mu_1 > \mu_2 > 0$ *then* $[P,Q]_{\mu_1} \Rightarrow [P,Q]_{\mu_2}$. *In particular, the conclusion holds if* $\mu_1 > \mu_2 > 0$ *and* P *is regular.*

PROOF

By Hölder's inequality

$$\sum_{r=0}^{\infty} p_{n,r}|w_r|^{\mu_2} \leq \left(\sum_{r=0}^{\infty} p_{n,r}|w_r|^{\mu_1}\right)^{\mu_2/\mu_1} M^{1-\mu_2/\mu_1}.$$

for any sequence $\{w_n\}$. The required conclusion follows. ///

THEOREM 1.2

If P *is a regular (non-negative) matrix and* Q *is any matrix, then*

(i) $Q \Rightarrow [P,Q]_\mu$, *for* $\mu > 0$,

(ii) $[P,Q]_\mu \Rightarrow PQ$, *for* $\mu \geq 1$.

PROOF

(i) If $s_n \to s$, then, since P is regular

$\sum_{r=0}^{n} p_{n,r}|s_r - s|^{\mu} = o(1)$, i.e., $I \Rightarrow [P,I]_\mu$ and inclusion (i) follows.

(ii) Suppose that $s_n \to s\ [P,I]_\mu$. Then by Theorem 1.1, $s_n \to s\ [P,I]_1$ and so

$$\left|\sum_{r=0}^{n} p_{n,r}(s_r - s)\right| \leq \sum_{r=0}^{n} p_{n,r}|s_r - s| = o(1)$$

Since P is regular, it follows that $s_n \to s$ (P). Hence $[P,I]_\mu \Rightarrow P$ and inclusion (ii) is an immediate consequence. ///

COROLLARY 1.1

If $\mu_1 > \mu_2 > 0$, *then* $[C,\lambda,p+1]_{\mu_1} \Rightarrow [C,\lambda,p+1]_{\mu_2}$.

PROOF

By (1.12), we know that $[C,\lambda,p+1]_\mu = [\Lambda_{p+1}, (C,\lambda,p)]_\mu$. The inclusion is a consequence of Theorem 1.1 and the fact that Λ_{p+1} is a regular and non-negative matrix. ///

COROLLARY 1.2

If $\mu > 0$, *then* $(C,\lambda,p) \Rightarrow [C,\lambda,p+1]_\mu$.

PROOF

Since $[C,\lambda,p+1]_\mu = [\Lambda_{p+1}, (C,\lambda,p)]_\mu$ and Λ_{p+1} is regular and non-negative. The corollary is an immediate consequence of Theorem 1.2 (i). ///

COROLLARY 1.3

If $\mu \geq 1$, *then* $[C,\lambda,p+1]_\mu \Rightarrow (C,\lambda,p+1)$.

PROOF

By (1.11), we know that $(C,\lambda,p+1) = \Lambda_{p+1}(C,\lambda,p)$. The corollary is a consequence of Theorem 1.2 (ii). ///

COROLLARY 1.4

Suppose $\mu_1 \geq 1$ *and* $\mu_2 > 0$. *Then*

$$[C,\lambda,p+1]_{\mu_1} \Rightarrow [C,\lambda,p+2]_{\mu_2}.$$

PROOF

This is a consequence of Corollary 1.3 and Corollary 1.2. ///

We mention two other properties of $[C,\lambda,p+1]_\mu$ here.

(1.13) *If* $\sum_{\nu=0}^{\infty} a_\nu = s[C,\lambda,p+1]_\mu$ *and* $\sum_{\nu=0}^{\infty} a_\nu = s'[C,\lambda,p+1]_\mu$

then $s = s'$.

(1.14) *If* $\mu > 0$, *then*

$$\sum_{\nu=0}^{\infty} a_\nu = a\ [C,\lambda,p+1]_\mu$$

and

$$\sum_{\nu=0}^{\infty} b_\nu = b\ [C,\lambda,p+1]_\mu$$

implies

$$\sum_{\nu=0}^{\infty} c_\nu = \sum_{\nu=0}^{\infty} (\alpha a_\nu + \beta b_\nu) = \alpha a + \beta b\ [C,\lambda,p+1]_\mu .$$

CHAPTER 2

EQUIVALENCE BETWEEN STRONG GENERALISED CESÀRO SUMMABILITY AND STRONG RIESZ SUMMABILITY

In this chapter we shall establish the equivalence between $[C,\lambda,p+1]_\mu$ and $[R,\lambda,p+1]_\mu$. We first prove a lemma. (Cf. Glatfeld [15].)

§2.1 A LEMMA

LEMMA 2.1

If $\chi(\tau) \geq 0$, *continuous and Riemann integrable in* $[h,\omega]$, *where* h *is any fixed positive real number and if* $\alpha+\delta > 0$ *and* $\delta > 0$, *then*

$$\int_h^\omega \chi(\tau)\,d\tau = o(\omega^\delta)$$

if and only if

$$\int_h^\omega \tau^\alpha \chi(\tau)\,d\tau = o(\omega^{\alpha+\delta}).$$

PROOF

Assume $\int_h^\omega \chi(\tau)\,d\tau = o(\omega^\delta)$ and let $F(\omega) = \int_h^\omega \chi(\tau)\,d\tau$.

Then integrating by parts

$$\int_h^\omega \tau^\alpha \chi(\tau)\,d\tau = [\tau^\alpha F(\tau)]_h^\omega - \alpha\int_h^\omega \tau^{\alpha-1} F(\tau)\,d\tau$$

$$= \omega^\alpha F(\omega) - \alpha\int_h^\omega \tau^{\alpha+\delta-1} \frac{F(\tau)}{\tau^\delta}\,d\tau$$

$$= U - V,$$

and $U = o(\omega^{\alpha+\delta})$ by hypothesis.

Further

$$\frac{1}{\omega^{\alpha+\delta}} \int_h^\omega \tau^{\alpha+\delta-1} \frac{F(\tau)}{\tau^\delta}\,d\tau$$

$$= \int_h^\omega K(\omega,\tau)\; G(\tau)\; d\tau,$$

where

$$K(\omega;\tau) = \begin{cases} \dfrac{\tau^{\alpha+\delta-1}}{\omega^{\alpha+\delta}}, & 0 < \tau \leq \omega, \\ 0, & \tau > \omega, \end{cases}$$

and

$$G(\tau) = \frac{F(\tau)}{\tau^\delta}.$$

Now

$$\int_h^\infty |K(\omega,\tau)|\,d\tau$$

$$= \frac{\omega^{\alpha+\delta} - h^{\alpha+\delta}}{(\alpha+\delta)\,\omega^{\alpha+\delta}}$$

$$= \frac{1}{\alpha+\delta}\left(1 - \frac{h^{\alpha+\delta}}{\omega^{\alpha+\delta}}\right)$$

$$< \frac{1}{\alpha+\delta}.$$

For every positive y

$$\lim_{\omega\to\infty} \int_h^y K(\omega,\tau)\,d\tau$$

$$= \lim_{\omega\to\infty} \frac{y^{\alpha+\delta} - h^{\alpha+\delta}}{(\alpha+\delta)\,\omega^{\alpha+\delta}}$$

$$= 0.$$

Since $G(\tau) \to 0$ as $\tau \to \infty$, it follows from Hardy [11, Theorem 6] that

$$\int_h^\omega K(\omega,\tau)G(\tau)\,d\tau \to 0, \quad \text{as } \omega \to \infty.$$

Thus $$V = o(\omega^{\alpha+\delta}).$$

Hence $$\int_h^\omega \tau^\alpha \chi(\tau)\,d\tau = o(\omega^{\alpha+\delta}).$$

Conversely, if $\int_h^\omega \tau^\alpha \chi(\tau)\,d\tau = o(\omega^{\alpha+\delta})$, we take $\tau^\alpha\chi(\tau) = X(\tau)$ which is non-negative, continuous and integrable in $[h,\omega]$. The result now follows from the first part by replacing δ by $\alpha+\delta$ and α by $-\alpha$. ///

Since $\lambda_0 = 0$, $R^p_\lambda(\tau) \to a_0$ as $\tau \to 0^+$, we conclude that as a consequence of Lemma 2.1

(2.1) $$\int_0^\omega |A^p_\lambda(\tau) - s\tau^p|^\mu d\tau = o(\omega^{p\mu+1})$$

is equivalent to

$$\int_0^\omega |R^p_\lambda(\tau) - s|^\mu\, d\tau = o(\omega).$$

§2.2 INCLUSION THEOREM FROM RIESZ TO CESÀRO

THEOREM 2.1

Let $\mu > 0$ *and* λ *satisfy* (1.1). *Then*

(i) $[R,\lambda,1]_\mu \Rightarrow [C,\lambda,1]_\mu$,

(ii) *If* $p > 0$ *and* $\lambda_{n+1} = O(\lambda_n)$, *then*

$$[R,\lambda,p+1]_\mu \Rightarrow [C,\lambda,p+1]_\mu.$$

PROOF

(i) Suppose $\sum_{\nu=0}^{\infty} a_\nu = s\ [R,\lambda,1]_\mu$ where we may assume, without loss of generality, that $s = 0$.

$$T_m^1 = \sum_{n=0}^{m} (\lambda_{n+1} - \lambda_n) \left| \sum_{\nu=0}^{n} a_\nu \right|^\mu$$

$$= \sum_{n=0}^{m} \int_{\lambda_n}^{\lambda_{n+1}} \left| \sum_{\lambda_\nu < \tau} a_\nu \right|^\mu d\tau$$

$$= \int_0^{\lambda_{m+1}} |A_\lambda^0(\tau)|^\mu d\tau$$

$$= o(\lambda_{m+1}), \quad \text{as } m \to \infty.$$

Thus $\sum_{\nu=0}^{\infty} a_\nu = 0[C,\lambda,1]_\mu$.

(ii) For the case $p > 0$, we assume that

$\sum_{\nu=0}^{\infty} a_\nu = 0[R,\lambda,p+1]_\mu$ so that

$$\int_0^\omega |R_\lambda^p(\tau)|^\mu d\tau = o(\omega).$$

We are required to show that

$$\sigma_m^{p+1} = \frac{1}{E_m^{p+1}} \sum_{n=0}^{m} (\lambda_{n+p+1} - \lambda_n) E_n^p |t_n^p|$$

$$= o(1), \quad \text{as } m \to \infty.$$

We divide the proof into four steps.

STEP I.

For every n, choose $q = q(n)$, a non-negative integer, satisfying $q(n) \geq q(n-1)$ and

(2.2) $$\lambda_{q+1} - \lambda_q = \max\{(\lambda_{i+1} - \lambda_i) | n \leq i \leq n+p\}.$$

Fixing n we partition the interval $[\lambda_q, \lambda_{q+1}]$ into $2p+2$ subintervals of length $\frac{\lambda_{q+1} - \lambda_q}{2p+2}$ with the points

$$\omega_\nu = \omega_{n,\nu} = \lambda_q + \frac{\nu}{2p+2}(\lambda_{q+1} - \lambda_q), \quad \nu = 0, 1, \ldots, 2p+2.$$

Since $p > 0$ and $\lambda_0 = 0$, $|R_\lambda^p(\tau)|^\mu$ is a continuous function of τ in the interval $[0, \omega]$. Applying the Mean Value Theorem on the alternate subintervals, we have, for $j = 0, 1, 2, \ldots, p$, numbers

$$\theta_j = \theta_{n,j} \in [\omega_{2j+1}, \omega_{2j+2}]$$

such that

$$\int_{\omega_{2j+1}}^{\omega_{2j+2}} |R_\lambda^p(\tau)|^\mu d\tau = (\omega_{2j+2} - \omega_{2j+1})|R_\lambda^p(\theta_j)|^\mu$$

$$= (\omega_{2j+2} - \omega_{2j+1})\left|\sum_{\nu=0}^{q}\left(1 - \frac{\lambda_\nu}{\theta_j}\right)^p a_\nu\right|^\mu.$$

Thus
$$\sum_{j=0}^{p}(\omega_{2j+2} - \omega_{2j+1})\left|\sum_{\nu=0}^{q}\left(1 - \frac{\lambda_\nu}{\theta_j}\right)^p a_\nu\right|^\mu$$

$$= \sum_{j=0}^{p}\int_{\omega_{2j+1}}^{\omega_{2j+2}} |R_\lambda^p(\tau)|^\mu d\tau$$

$$\leq \int_{\lambda_q}^{\lambda_{q+1}} |R_\lambda^p(\tau)|^\mu d\tau.$$

Since $\omega_{2j+2} - \omega_{2j+1} = \frac{1}{2p+2}(\lambda_{q+1} - \lambda_q)$, we have

$$(2.3) \quad \sum_{n=0}^{m}\sum_{j=0}^{p}\frac{1}{2p+2}(\lambda_{q(n)+1} - \lambda_{q(n)})\left|\sum_{\nu=0}^{q(n)}\left(1 - \frac{\lambda_\nu}{\theta_{n,j}}\right)^p a_\nu\right|^\mu$$

$$\leq \sum_{n=0}^{m}\int_{\lambda_{q(n)}}^{\lambda_{q(n)+1}} |R_\lambda^p(\tau)|^\mu d\tau$$

$$\leq (p+1) \int_0^{\lambda_{n+p+1}} |R_\lambda^p(\tau)|^\mu d\tau,$$

since q(n) is constant for at most p+1 different values of n.

STEP II.

Using techniques similar to those used by Borwein [2], we shall show that for every n, there are numbers

$$y_j = y_{n,j}, \quad \text{for } j = 0,1,2,\ldots, p,$$

such that the identity

$$\prod_{i=1}^{p} (x + b_i) \equiv \sum_{j=0}^{p} y_j (x + \delta_j)^p \tag{2.4}$$

holds for all real x, where

$$b_i = \frac{\lambda_{n+i} - \lambda_q}{\lambda_{q+1} - \lambda_q}, \quad \text{for } i = 1,2,\ldots, p$$

and

$$\delta_j = \frac{\theta_j - \lambda_q}{\lambda_{q+1} - \lambda_q}, \quad \text{for } j = 0,1,2,\ldots, p.$$

The identity (2.4) is equivalent to the system of linear equations

$$\sum_{j=0}^{p} \delta_j^i \, y_j = \xi_i, \quad i = 0,1,\ldots, p, \tag{2.5}$$

where

$$\xi_i = \binom{p}{i}^{-1} \sum_{1 \leq r_1 < \cdots < r_i \leq p} b_{r_1} b_{r_2} \ldots b_{r_i} \tag{2.6}$$

and where the sum in the expression for ξ_i is taken to be 1 when i = 0.

The determinant of the system (2.5) is the Vandermonde determinant

$$\Delta = \prod_{0 \leq r < s \leq p} (\delta_s - \delta_r).$$

(See [25, p. 214].)

Now for $s > r$,

$$\begin{aligned}\delta_s - \delta_r &= \frac{\theta_s - \theta_r}{\lambda_{q+1} - \lambda_q} \\ &\geq \frac{\omega_{2s+1} - \omega_{2r+2}}{\lambda_{q+1} - \lambda_q} \\ &\geq \frac{\lambda_{q+1} - \lambda_q}{2p+2} \times \frac{1}{\lambda_{q+1} - \lambda_q} \\ &= \frac{1}{2p+2}.\end{aligned}$$

Hence $$\Delta \geq \frac{1}{(2p+2)^{p!}} > 0.$$

Using Cramer's rule, we have

$$y_r = \frac{\Delta_r}{\Delta},$$

where Δ_r is the determinant of the matrix $(d_{i,j})$, $i,j = 0,1,2,\ldots, p$, in which

$$d_{i,r} = \xi_i \text{ and } d_{i,j} = \delta_j^i, \; j \neq r.$$

STEP III.

We now show that the numbers $y_{n,r}$ are uniformly bounded. Since

$$\begin{aligned}|b_r| &= \left|\frac{\lambda_{n+r} - \lambda_q}{\lambda_{q+1} - \lambda_q}\right| \\ &\leq \frac{\lambda_{n+p+1} - \lambda_n}{\lambda_{q+1} - \lambda_q}\end{aligned}$$

$$\leq (p+1) \frac{\lambda_{q+1} - \lambda_q}{\lambda_{q+1} - \lambda_q}$$

$$= (p+1),$$

we see from (2.6) that for $i = 0,1,2,\ldots, p$

$$|\xi_i| \leq (p+1)^p.$$

Also $|\delta_j^i| = (\frac{\theta_{n,j} - \lambda_q}{\lambda_{q+1} - \lambda_q})^i \leq 1$, for $i,j = 0,1,2,\ldots p$.

Consequently

(2.7) $$|y_r| = |y_{n,r}| \leq (2p+2)^{p!} |\Delta_r| \leq H$$

where H is a constant independent of r and n.

STEP IV.

Here we establish an inequality between the $[C,\lambda,p+1]_\mu$-mean and the $[R,\lambda,p+1]_\mu$-mean of the series $\sum_{\nu=0}^{\infty} a_\nu$ which yields our result.

Let ν be any non-negative integer and put $x = \frac{\lambda_q - \lambda_\nu}{\lambda_{q+1} - \lambda_q}$ in (2.4), we obtain

$$\prod_{i=1}^{p} (\frac{\lambda_{n+i} - \lambda_\nu}{\lambda_{q+1} - \lambda_q}) = \sum_{j=0}^{p} y_j (\frac{\theta_{n,j} - \lambda_\nu}{\lambda_{q+1} - \lambda_q}).$$

Thus $$\prod_{i=1}^{p} (\lambda_{n+i} - \lambda_\nu) = \sum_{j=0}^{p} y_j (\theta_{n,j} - \lambda_\nu)^p.$$

Dividing by E_n^p, we have

$$(2.8)\qquad \prod_{i=1}^{p}\left(1-\frac{\lambda_\nu}{\lambda_{n+i}}\right)=\sum_{j=0}^{p}\frac{y_j\,\theta_{n,j}^{p}}{E_n^{p}}\left(1-\frac{\lambda_\nu}{\theta_{n,j}}\right)^{p}$$

$$=\sum_{j=0}^{p} C_{n,j}\left(1-\frac{\lambda_\nu}{\theta_{n,j}}\right)^{p},$$

where $C_{n,j}=\dfrac{y_j\ \theta_{n,j}^{p}}{E_n^{p}}$.

Since $\lambda_{n+1}=O(\lambda_n)$ and $y_{n,r}$ is uniformly bounded, we have

$$|C_{n,j}|\le\frac{|y_{n,j}|\lambda_{q+1}^{p}}{\lambda_{n+1}^{p}}\le H_1;$$

H_1 being independent of n and j.

Now it follows from (2.8) that

$$t_n^p=\sum_{\nu=0}^{n}\left(1-\frac{\lambda_\nu}{\lambda_{n+1}}\right)\cdots\left(1-\frac{\lambda_\nu}{\lambda_{n+p}}\right)a_\nu$$

$$=\sum_{\nu=0}^{q}\left(1-\frac{\lambda_\nu}{\lambda_{n+1}}\right)\cdots\left(1-\frac{\lambda_\nu}{\lambda_{n+p}}\right)a_\nu$$

$$=\sum_{\nu=0}^{q}\sum_{j=0}^{p}C_{n,j}\left(1-\frac{\lambda_\nu}{\theta_{n,j}}\right)^{p}a_\nu$$

$$=\sum_{j=0}^{p}C_{n,j}\sum_{\nu=0}^{q}\left(1-\frac{\lambda_\nu}{\theta_{n,j}}\right)^{p}a_\nu.$$

Thus

$$\sigma_m^{p+1} = \frac{1}{E_m^{p+1}} \sum_{n=0}^{m} (\lambda_{n+p+1} - \lambda_n)\, E_n^p\, |t_n^p|^{\mu}$$

$$\leq \lambda_{m+p+1}^{-1} \sum_{n=0}^{m} (\lambda_{n+p+1} - \lambda_n) \left\{ \sum_{j=0}^{p} |C_{n,j}| \sum_{\nu=0}^{q(n)} \left(1 - \frac{\lambda_\nu}{\theta_{n,j}}\right)^p a_\nu| \right\}^{\mu}$$

$$\leq \lambda_{m+p+1}^{-1} \sum_{n=0}^{m} (p+1)(\lambda_{q(n)+1} - \lambda_{q(n)})(p+1)^{\mu} \sum_{j=0}^{p} |C_{n,j}|^{\mu} \left| \sum_{\nu=0}^{q(n)} \left(1 - \frac{\lambda_\nu}{\theta_{n,j}}\right)^p a_\nu \right|^{\mu}$$

$$\leq \frac{H_2}{\lambda_{m+p+1}} \sum_{n=0}^{m} (\lambda_{q(n)+1} - \lambda_{q(n)}) \sum_{j=0}^{p} \left| \sum_{\nu=0}^{q(n)} \left(1 - \frac{\lambda_\nu}{\theta_{n,j}}\right)^p a_\nu \right|^{\mu}$$

$$= \frac{H_2 \times (2p+2)}{\lambda_{m+p+1}} \sum_{n=0}^{m} \sum_{j=0}^{p} \frac{1}{2p+2} (\lambda_{q(n)+1} - \lambda_{q(n)}) \left| \sum_{\nu=0}^{q(n)} \left(1 - \frac{\lambda_\nu}{\theta_{n,j}}\right)^p a_\nu \right|^{\mu}$$

$$\leq \frac{H_3}{\lambda_{m+p+1}} \int_0^{\lambda_{m+p+1}} |R_\lambda^p(\tau)|^{\mu}\, d\tau .$$

The final inequality following from Step I.

Hence if $\sum_{\nu=0}^{\infty} a_\nu = 0\ [R, \lambda, p+1]_\mu$, then

$$\frac{1}{\lambda_{m+p+1}} \int_0^{\lambda_{m+p+1}} | R_\lambda^p(\tau)|^{\mu}\, d\tau = o(1).$$

Thus $\sigma_m^{p+1} = o(1)$ so that $\sum_{\nu=0}^{\infty} a_\nu = 0\ [C, \lambda, p+1]_\mu$. ///

§2.3 INCLUSION THEOREM FROM CESARO TO RIESZ

We now investigate the inclusion in the opposite direction. And to facilitate the discussion we introduce the following notation.

Given a function f defined in an interval [a,b], and distinct points x_i in this interval, we define

$$f[x] = f(x)$$

and $$f[x_0,x_1,\ldots, x_n] = \frac{f[x_0,\ldots, x_{n-1}] - f[x_1,\ldots, x_n]}{x_0 - x_n}$$

for n = 1,2,3,... .

The quantity $f[x_0,x_1,\ldots x_n]$ is called the *divided difference* of f(x) of n arguments. For an exposition of the properties of divided differences see Milne-Thomson [21, Chapter 1].

In the proof of our next theorem we need the following results of Russell [23, pp. 425-428].

LEMMA 2.2

Let p *be a non-negative integer.*

Define $$C_\tau(x) = \begin{cases} (\tau-x)^p, & \text{for } 0 \leq x < \tau, \\ 0, & \text{for } x \geq \tau. \end{cases}$$

Then, for $\lambda_n < \tau \leq \lambda_{n+1}$

(i) $$A_\lambda^p(\tau) = (-1)^{p+1} \sum_{\nu=n-p}^{n} C_\tau[\lambda_\nu,\lambda_{\nu+1},\ldots, \lambda_{\nu+p+1}](\lambda_{\nu+p+1}-\lambda_\nu)C_\nu^p$$

where we understand $C_\nu^p = 0$ *whenever* $\nu < 0$; *and*

(ii) *for* $n - p \leq \nu \leq n$

$$|C_\tau[\lambda_\nu,\lambda_{\nu+1},\ldots, \lambda_{\nu+p+1}]|(\lambda_{\nu+p+1} - \lambda_\nu) \leq H .$$

where H *is independent of* n.

THEOREM 2.2

Let λ *satisfy* (1.1). *Then*

(i) *if* $\mu > 0$, *then* $[C,\lambda,1]_\mu \Rightarrow [R,\lambda,1]_\mu$,

(ii) *if* $p > 0$, $\mu \geq 1$ *and* $\lambda_{n+1} = O(\lambda_n)$, *then*

$[C,\lambda,p+1]_\mu \Rightarrow [R,\lambda,p+1]_\mu$.

PROOF

(i) We suppose that $\sum_{\nu=0}^{\infty} a_\nu = 0\ [C,\lambda,1]_\mu$. Thus

$$\sigma_m^1 = \frac{1}{\lambda_{m+1}} \sum_{n=0}^{m} (\lambda_{n+1} - \lambda_n)|s_n|^\mu = o(1).$$

Hence

$$\frac{\lambda_{m+1} - \lambda_m}{\lambda_{m+1}} |s_m|^\mu = o(1). \tag{2.9}$$

Let $\omega > 0$ and suppose $\lambda_m < \omega \leq \lambda_{m+1}$. Then

$$\frac{1}{\omega}\int_0^\omega |A_\lambda^0(\tau)|^\mu d\tau = \frac{1}{\omega}\left\{\sum_{n=0}^{m-1} \int_{\lambda_n}^{\lambda_{n+1}} |\sum_{\nu=0}^{n} a_\nu|^\mu d\tau + \int_{\lambda_m}^{\omega} |\sum_{\nu=0}^{m} a_\nu|^\mu d\tau\right\}$$

$$= \frac{1}{\omega}\sum_{n=0}^{m-1}(\lambda_{n+1} - \lambda_n)|s_n|^\mu + \frac{1}{\omega}(\omega - \lambda_m)|s_m|^\mu$$

$$\leq \sigma_{m-1}^1 + \left(1 - \frac{\lambda_m}{\lambda_{m+1}}\right)|s_m|^\mu.$$

Now $\sigma_m^1 = o(1)$ which together with (2.9) yields

$$\frac{1}{\omega}\int_0^\omega |A_\lambda^0(\tau)|^\mu d\tau = o(1).$$

Thus $\sum_{\nu=0}^{\infty} a_\nu = 0\ [R,\lambda,1]_\mu$.

(ii) Let $\tau > 0$ and suppose $\lambda_n < \tau \leq \lambda_{n+1}$.

Then using Lemma 2.2 (i) and (ii) we see that

$$(2.10)\qquad |A_\lambda^p(\tau)|^\mu = \Big|\sum_{\nu=n-p}^{n} C_\tau[\lambda_\nu, \lambda_{\nu+1}, \ldots, \lambda_{\nu+p+1}](\lambda_{\nu+p+1}-\lambda_\nu)C_\nu^p\Big|^\mu$$

$$\leq (p+1)^\mu H \sum_{\nu=n-p}^{n} |C_\nu^p|^\mu .$$

Suppose $\omega > 0$ and $\lambda_m < \omega \leq \lambda_{m+1}$. Then

$$\int_0^\omega |A_\lambda^p(\tau)|^\mu d\tau$$

$$\leq \sum_{n=0}^{m} \int_{\lambda_n}^{\lambda_{n+1}} |A_\lambda^p(\tau)|^\mu d\tau$$

$$\leq H_1 \sum_{n=0}^{m} (\lambda_{n+1} - \lambda_n) \sum_{\nu=n-p}^{n} |C_\nu^p|^\mu$$

$$= H_1 \sum_{n=0}^{m} (\lambda_{n+1} - \lambda_n) \sum_{\nu=0}^{p} |C_{n-\nu}^p|^\mu$$

$$= H_1 \sum_{\nu=0}^{p} \sum_{n=\nu}^{m} (\lambda_{n+1} - \lambda_n) |C_{n-\nu}^p|^\mu ,$$

so that

$$(2.11)\qquad \int_0^\omega |A_\lambda^p(\tau)|^\mu d\tau \leq H_1 \sum_{\nu=0}^{p} \sum_{n=\nu}^{m} (\lambda_{n+1} - \lambda_n) |C_{n-\nu}^p|^\mu .$$

Now

$$\sigma_m^{p+1} = \frac{1}{E_m^{p+1}} \sum_{n=0}^{m} (\lambda_{n+p+1} - \lambda_n)\, E_n^p\, |\tau_n^p|^\mu$$

$$= \frac{1}{E_m^{p+1}} \sum_{n=0}^{m} (\lambda_{n+p+1} - \lambda_n)(E_n^p)^{1-\mu} |C_n^p|^\mu$$

$$\geq \frac{1}{(E_m^p)^\mu \lambda_{m+p+1}} \sum_{n=0}^{m} (\lambda_{n+p+1} - \lambda_n) |C_n^p|^\mu ,$$

since $\mu \geq 1$.

Thus for $r = 0,1,2,\ldots, p$

$$\sigma_m^{p+1} \geq \frac{1}{\lambda_{m+p+1}^{p\mu+1}} \sum_{n=0}^{m} (\lambda_{n+r+1} - \lambda_{n+r}) |C_n^p|^\mu$$

$$= \frac{1}{\lambda_{m+p+1}^{p\mu+1}} \sum_{\nu=r}^{m+r} (\lambda_{\nu+1} - \lambda_\nu) |C_{\nu-r}^p|^\mu$$

$$\geq \frac{1}{\lambda_{m+p+1}^{p\mu+1}} \sum_{\nu=r}^{m} (\lambda_{\nu+1} - \lambda_\nu) |C_{\nu-r}^p|^\mu .$$

If we now suppose $\sum_{\nu=0}^{\infty} a_\nu = 0 \ [C,\lambda,p+1]_\mu$, so that $\sigma_m^{p+1} = o(1)$, we have, for $r = 0,1,2,\ldots, p$

$$\frac{1}{\lambda_{m+p+1}^{p\mu+1}} \sum_{\nu=r}^{m} (\lambda_{\nu+1} - \lambda_\nu) |C_{\nu-r}^p|^\mu = o(1),$$

as $m \to \infty$.

Hence in view of (2.11) and the condition $\lambda_{m+1} = O(\lambda_m)$, we have

$$\int_0^\omega |A_\lambda^p(\tau)|^\mu d\tau = o(\lambda_{m+p+1}^{p\mu+1}) = o(\omega^{p\mu+1}).$$

Hence $\sum_{\nu=0}^{\infty} a_\nu = 0 \ [R,\lambda,p+1]_\mu$ for $\mu \geq 1$. ///

Combining the results of Theorems 2.1 and 2.2, we have the following corollary.

THEOREM 2.3

Let $\lambda = \{\lambda_n\}$ *satisfy* (1.1).

(i) *If* $\mu > 0$, *then* $[R,\lambda,1]_\mu \iff [C,\lambda,1]_\mu$.

(ii) *If* $p > 0$, $\mu \geq 1$ *and* $\lambda_{n+1} = O(\lambda_n)$, *then*

$[R,\lambda,p+1]_\mu \iff [C,\lambda,p+1]_\mu$.

CHAPTER 3

SOME EQUIVALENCE THEOREMS

In this chapter we shall establish some equivalence theorems between various methods of Summability and Strong Summability.

§3.1 SOME LEMMAS

LEMMA 3.1

Let Λ_{p+1} *be the matrix defined in* §1.7. *The inverse matrix* $\Lambda'_{p+1} = \{\lambda'_{n,\nu}\}$ *of* Λ_{p+1} *is given by*

(3.1)
$$\lambda'_{n,n} = \frac{\lambda_{n+p+1}}{\lambda_{n+p+1} - \lambda_n},$$
$$\lambda'_{n,n-1} = \frac{-\lambda_n}{\lambda_{n+p+1} - \lambda_n},$$
$$\lambda'_{n,\nu} = 0 \textit{ otherwise}.$$

PROOF

Let $C_{m,\nu} = \sum_{n=\nu}^{m} \lambda_{m,n} \lambda'_{n,\nu}$, we show that $C_{m,\nu} = \delta_{m,\nu}$.

Referring to the definition of Λ_{p+1}, (1.8), we have for $\nu \neq m$

$$c_{m,\nu} = \lambda_{m,\nu}\lambda'_{\nu,\nu} + \lambda_{m,\nu+1}\lambda'_{\nu+1,\nu}$$

$$= \frac{1}{E_m^{p+1}}(\lambda_{\nu+p+1} - \lambda_\nu)\, E_\nu^p \frac{\lambda_{\nu+p+1}}{\lambda_{\nu+p+1} - \lambda_\nu}$$

$$- \frac{1}{E_m^{p+1}}(\lambda_{\nu+p+2} - \lambda_{\nu+1}) E_{\nu+1}^p \frac{\lambda_{\nu+1}}{\lambda_{\nu+p+2} - \lambda_{\nu+1}}$$

$$= \frac{1}{E_m^{p+1}}(\lambda_{\nu+1}\cdots\lambda_{\nu+p}\cdot\lambda_{\nu+p+1} - \lambda_{\nu+1}\cdot\lambda_{\nu+2}\cdots\lambda_{\nu+p+1})$$

$$= 0,$$

and $c_{m,m} = \lambda_{m,m}\lambda'_{m,m}$

$$= \frac{1}{E_m^{p+1}}(\lambda_{m+p+1} - \lambda_m)\, E_m^p \frac{\lambda_{m+p+1}}{\lambda_{m+p+1} - \lambda_m}$$

$$= 1.$$

LEMMA 3.2

$\Lambda_{p+1} \Leftrightarrow I$ *if and only if* $\liminf_{n\to\infty} \frac{\lambda_{n+p+1}}{\lambda_n} > 1.$

PROOF

$I \Rightarrow \Lambda_{p+1}$ follows from the regularity of Λ_{p+1}. $\Lambda_{p+1} \Rightarrow I$ if and only if Λ'_{p+1} is regular. Referring to Lemma 3.1, we see that

(3.2) $$\lim_{n\to\infty} \lambda'_{n,\nu} = 0, \text{ for every } \nu,$$

(3.3) $$\lim_{n\to\infty} \sum_{\nu=0}^{\infty} \lambda'_{n,\nu} = \frac{-\lambda_n}{\lambda_{n+p+1} - \lambda_n} + \frac{\lambda_{n+p+1}}{\lambda_{n+p+1} - \lambda_n} = 1,$$

(3.4) $$\sup_n \sum_{\nu=0}^{\infty} |\lambda'_{n,\nu}| = \sup_n \frac{\lambda_{n+p+1} + \lambda_n}{\lambda_{n+p+1} - \lambda_n}$$

$$\leq \sup_n \frac{2}{1 - \frac{\lambda_n}{\lambda_{n+p+1}}}.$$

This supremum is finite if and only if

$$\liminf_{n\to\infty} \frac{\lambda_{n+p+1}}{\lambda_n} > 1.$$

Consequently, Λ_{p+1} <=> I if and only if

$$\liminf_{n\to\infty} \frac{\lambda_{n+p+1}}{\lambda_n} > 1. \qquad ///$$

§3.2 EQUIVALENCE THEOREMS

THEOREM 3.1

(C,λ,p) <=> $(C,\lambda,p+1)$ *if and only if* $\liminf_{n\to\infty} \frac{\lambda_{n+p+1}}{\lambda_n} > 1.$

PROOF

By (1.11), we know that $(C,\lambda,p+1) = \Lambda_{p+1}\ (C,\lambda,p)$. Thus the result now follows from Lemma 3.2. ///

REMARK: In view of the fact $\{\lambda_n\}$ is an increasing sequence, so that $\liminf_{n\to\infty} \frac{\lambda_{n+p+1}}{\lambda_n} \geq 1$, we see that $\liminf_{n\to\infty} \frac{\lambda_{n+p+1}}{\lambda_n} = 1$ is necessary and sufficient for $(C,\lambda,p+1)$ to include strictly (C,λ,p).

We now state a result of Borwein and Cass [5, Corollary 2] which yields an equivalence theorem between the methods (C,λ,p) and $[C,\lambda,p+1]_\mu$.

THEOREM 3.2

Let $\mu > 0$.

Let $P = \{p_{n,\nu}\}$ *be a matrix with*

(i) $p_{n,\nu} \geq 0$, for $n,\nu = 0,1,2,3\ldots$,

(ii) $\lim_{n\to\infty} p_{n,\nu} = 0$, for $\nu = 0,1,2,3,\ldots$.

Let $Q = \{q_{n,\nu}\}$ *be a matrix such that for every sequence* $\{\sigma_\nu\}$ *there is a sequence* $\{s_\nu\}$ *for which*

$$\sigma_n = \sum_{\nu=0}^{\infty} q_{n,\nu}\, s_\nu$$

holds for $n = 0,1,2,3,\ldots$.

Then $\liminf_{\nu\to\infty} \max_{n\geq 0} p_{n,\nu} = 0$

is a necessary and sufficient condition for there to be a sequence which is not Q*-convergent, but which is* $[P,Q]_\mu$*-convergent to zero.*

THEOREM 3.3

Let $\mu > 0$. *Then* $(C,\lambda,p) \iff [C,\lambda,p+1]_\mu$ *if and only if* $\liminf_{n\to\infty} \frac{\lambda_{n+p+1}}{\lambda_n} > 1$.

PROOF

By Corollary 1.2, we have

$$(C,\lambda,p) \Rightarrow [C,\lambda,p+1]_\mu, \text{ for } \mu > 0.$$

Now $\Lambda_{p+1} = (\lambda_{n,\nu})$ satisfies

$$\lambda_{n,\nu} \geq 0, \text{ for } n,\nu = 0,1,2,\ldots$$

and

$$\lim_{n\to\infty} \lambda_{n,\nu} = 0, \text{ for } \nu = 0,1,2,3,\ldots .$$

And also since $\max_{n\geq 0} \lambda_{n,\nu} = \lambda_{\nu,\nu}$, we have

$$\liminf_{\nu\to\infty} \max_{n\geq 0} \lambda_{n,\nu} = \liminf_{\nu\to\infty} \lambda_{\nu,\nu}$$

$$= \liminf_{\nu\to\infty} \left(1 - \frac{\lambda_\nu}{\lambda_{\nu+p+1}}\right) .$$

Moreover (C,λ,p) has an inverse, so the result follows from Theorem 3.2 by taking $P = \Lambda_{p+1}$ and $Q = (C,\lambda,p)$. ///

For the proof of the equivalence theorem between $(C,\lambda,p+1)$ and $[C,\lambda,p+1]_\mu$, we state another result of Borwein and Cass [5, Theorem 12].

THEOREM 3.4

Let the matrix $P = \{p_{n,\nu}\}$ *be regular and* $p_{n,\nu} = 0$ *for* $\nu > n$. *If*

(i) $p_{n,\nu} \geq p_{n+1,\nu}$, *for* $n \geq \nu$, $\nu = 0,1,2,\ldots$,

(ii) $p_{n,n} \to 0$,

(iii) $\sum_{\nu=0}^{n} p_{n,\nu} \leq \sum_{\nu=0}^{n+1} p_{n+1,\nu}$, *for* $n = 0,1,2,3,\ldots$,

then there is a divergent sequence of zeros and ones which is P-convergent to $\frac{1}{2}$, *but not* $[P,I]_\mu$*-convergent for any* $\mu \geq 1$. *(I denotes the identity matrix.)*

THEOREM 3.5

(i) *If* $\liminf_{n\to\infty} \frac{\lambda_{n+p+1}}{\lambda_n} > 1$, *then*

$(C,\lambda,p+1) \Leftrightarrow [C,\lambda,p+1]_\mu$, *for* $\mu > 0$,

(ii) *If* $\lim_{n\to\infty} \frac{\lambda_{n+p+1}}{\lambda_n} = 1$, *then* $(C,\lambda,p+1)$ *strictly includes* $[C,\lambda,p+1]_\mu$, *for* $\mu \geq 1$.

PROOF

(i) Combining results of Theorem 3.1 and Theorem 3.3 we have $\liminf_{n\to\infty} \frac{\lambda_{n+p+1}}{\lambda_n} > 1$ implies that

$(C,\lambda,p+1) \Leftrightarrow [C,\lambda,p+1]_\mu$, for $\mu > 0$.

(ii) Since in the matrix Λ_{p+1}, $\lambda_{n,n} = \left(1 - \frac{\lambda_n}{\lambda_{n+p+1}}\right)$, and since the matrix (C,λ,p) has an inverse, Theorem 3.4 shows that if $\lim_{n\to\infty} \frac{\lambda_{n+p+1}}{\lambda_n} = 1$, then there is a divergent sequence $\{t_n^p\}$ of zeros and ones which is Λ_{p+1}-convergent to $\frac{1}{2}$, but not $[\Lambda_{p+1}, I]_\mu$-convergent for any $\mu \geq 1$. Since $\Lambda_{p+1}\{t_n^p\} = \{t_n^{p+1}\}$ the result follows. ///

We now show that in Theorem 3.5 (ii) the condition $\lim_{n\to\infty} \frac{\lambda_{n+p+1}}{\lambda_n} = 1$ can not be replaced by $\liminf_{n\to\infty} \frac{\lambda_{n+p+1}}{\lambda_n} = 1$.

Let $p_0 > 0$ and $p_n \geq 0$, we say that

$$s_n \to s \; (\bar{N}, p_n)$$

if $\mu_n = \frac{1}{P_n} \sum_{\nu=0}^{n} p_\nu s_\nu \to s$, where $P_n = \sum_{\nu=0}^{n} p_\nu$.

REMARK: (i) Λ_{p+1} is the method $(\bar{N},p_n)$ with $p_n = E_n^{p+1}$.

(ii) If $(\bar{N},p_n)$ is taken as P in Theorem 3.4, it satisfies conditions (i) and (iii) of Theorem 3.4.

We shall now construct an $(\bar{N},p_n)$ method with

$$\liminf_{n\to\infty} \frac{p_n}{P_n} = 0 \text{ and with } [(\bar{N},p_n),I]_1 \Leftrightarrow (\bar{N},p_n).$$

Let $c_n = \frac{p_n}{P_n}$, $\quad 0 < c_n < 1$, for $n \geq 1$.

Then $\mu_n P_n - \mu_{n-1} P_{n-1} = p_n s_n$,

and $\mu_n - \mu_{n-1}(1 - c_n) = c_n s_n$.

Now take $c_{2n} = 1 - \frac{1}{(n+1)^2}$, for $n \geq 1$,

$$c_{2n+1} = \frac{1}{n+2}, \text{ for } n \geq 0, \text{ so}$$

(3.5) $$\liminf_{n\to\infty} \frac{p_n}{P_n} = 0,$$

(3.6) $$\mu_{2n} - \frac{\mu_{2n-1}}{(n+1)^2} = \left(1 - \frac{1}{(n+1)^2}\right) s_{2n},$$

(3.7) $$\mu_{2n+1} - \mu_{2n}\left(1 - \frac{1}{n+2}\right) = \frac{s_{2n+1}}{n+2}.$$

Consequently if $\mu_n \to \ell$, then (3.6) and (3.7) give $s_{2n} \to \ell$ and $s_{2n+1} = o(n)$.

On the other hand if $s_{2n} \to \ell$ and $s_{2n+1} = o(n)$, then

$$\left|\frac{\mu_n}{n+1}\right| \le \frac{1}{P_n} \sum_{\nu=0}^{n} p_\nu \frac{|s_\nu|}{\nu+1} \le H$$

and (3.6) and (3.7) imply that $\mu_n \to \ell$.

Summarizing we have

$s_n \to s$ $(\bar{N},p_n)$ if and only if $s_{2n} \to s$ and $s_{2n+1} = o(n)$.

Thus $(\bar{N},p_n)$ is regular, not equivalent to convergence and $(\bar{N},p_n) \Leftrightarrow [(\bar{N},p_n),I]_1$.

Let $\lambda_0 = 0$, $\lambda_{n+1} = P_n$ for $n \ge 0$.

Then $\lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \lambda_{n+1} < \cdots$

and $\lambda_n \to \infty$, because $(\bar{N},p_n)$ is regular.

$$\Lambda_1 = (\bar{N},p_n),$$

$$\liminf_{n\to\infty} \frac{\lambda_{n+1}}{\lambda_n} = 1$$

and $[C,\lambda,1]_1 \Leftrightarrow (C,\lambda,1)$.

Combining the last example with Theorem 2.3 (i), we find that it is possible to have

$$\liminf_{n\to\infty} \frac{\lambda_{n+1}}{\lambda_n} = 1$$

and $[R,\lambda,1]_1 \Leftrightarrow (R,\lambda,1)$.

CHAPTER 4

ABSOLUTE GENERALISED CESÀRO SUMMABILITY

§4.1 DEFINITIONS

In this chapter we study the absolute methods of summability $|C,\lambda,p|_\mu$ and $|R,\lambda,p|$.

Let t_n^p be defined as in §1.5 and $\mu > 0$. We define $\sum_{\nu=0}^{\infty} a_\nu$ to be summable $|C,\lambda,p|_\mu$ if

$$(4.1) \qquad \sum_{n=1}^{\infty} \left(\frac{\lambda_n}{\lambda_{n+p+1} - \lambda_n}\right)^{\mu-1} |t_n^p - t_{n-1}^p|^\mu < \infty.$$

In §1.2, we defined $\sum_{\nu=0}^{\infty} a_\nu$ to be summable $|Q|_\mu$, $\mu > 0$, if

$$(4.2) \qquad \sum_{n=1}^{\infty} n^{\mu-1} |\sigma_n - \sigma_{n-1}|^\mu < \infty,$$

where $\{\sigma_n\} = Q\{s_n\}$.

When $\mu = 1$, conditions (4.1) and (4.2) are equivalent. When $\mu \neq 1$, they may or may not differ.

For example, if $\lambda_n = n^\alpha$, $\alpha > 0$, then

$$\frac{\lambda_n}{\lambda_{n+p+1} - \lambda_n} = \frac{n^\alpha}{(n+p+1)^\alpha - n^\alpha} = \frac{n^\alpha}{\alpha\theta_n^{\alpha-1}(p+1)}$$

where $n < \theta_n < n+p+1$.

Let $\rho_n = \dfrac{\lambda_n}{\lambda_{n+p+1} - \lambda_n}$.

Then $\dfrac{\rho_n}{n} \to \dfrac{1}{\alpha(p+1)}$, as $n \to \infty$. So in this case,

$$\sum_{n=1}^{\infty} \rho_n^{\mu-1} |t_n^p - t_{n-1}^p| < \infty \text{ if and only if}$$

$$\sum_{n=1}^{\infty} n^{\mu-1} |t_n^p - t_{n-1}^p| < \infty,$$

and the two conditions (4.1) and (4.2) are equivalent in this case.

On the other hand, if $\lambda_n = \log(n+1)$, then

$$\rho_n = \frac{\log(n+1)}{\log(n+p+2) - \log(n+1)} = \frac{\theta_n \log(n+1)}{p+1},$$

where $n+1 < \theta_n < n+p+2$.

In this case $\dfrac{\rho_n}{n \log n} \to \dfrac{1}{p+1}$, as $n \to \infty$ and

$$\sum_{n=1}^{\infty} \rho_n^{\mu-1} |t_n^p - t_{n-1}^p| < \infty \text{ if and only if}$$

$$\sum_{n=1}^{\infty} n^{\mu-1} \log^{\mu-1} n \; |t_n^p - t_{n-1}^p| < \infty.$$

Let $\alpha_n = t_n^p - t_{n-1}^p$ and $\mu = 2$.

If we take $\alpha_n = \dfrac{1}{n \log n}$, then

$$\sum_{n=2}^{\infty} n^{\mu-1} |\alpha_n|^{\mu} = \sum_{n=2}^{\infty} n |\alpha_n|^2$$

$$= \sum_{n=2}^{\infty} \frac{1}{n (\log n)^2} < \infty,$$

while $\sum_{n=2}^{\infty} n^{\mu-1} \log^{\mu-1} n |\alpha_n|^\mu$

$$= \sum_{n=2}^{\infty} n \log n \left|\frac{1}{n \log n}\right|^2$$

$$= \sum_{n=2}^{\infty} \frac{1}{n \log n} = \infty.$$

This shows that the two conditions (4.1) and (4.2) are different in this case.

It is more natural to use condition (4.1) rather than condition (4.2) to define $|C,\lambda,p|_\mu$ summability. Thus for the remainder of this chapter $\sum_{\nu=0}^{\infty} a_\nu$ is summable $|C,\lambda,p|_\mu$ means condition (4.1) is satisfied.

We now give an example which shows that there are sequences λ for which $|C,\lambda,p|_\mu \not\Rightarrow (C,\lambda,p)$.

Let $\mu = 2$, $\lambda_n = \log(n+1)$ and

$$\alpha_n = t_n^p - t_{n-1}^p = \frac{1}{n \log n \log \log n}.$$

Then
$$\sum_{n=2}^{\infty} n^{\mu-1} \log^{\mu-1} n \left|\frac{1}{n \log n \log \log n}\right|^\mu$$

$$= \sum_{n=2}^{\infty} n \log n \frac{1}{n^2 (\log n)^2 (\log \log n)^2}$$

$$= \sum_{n=2}^{\infty} \frac{1}{n \log n (\log \log n)^2} < \infty.$$

But $\lim_{n\to\infty} t_n^p = t_1^p + \sum_{n=2}^{\infty} \alpha_n = t_1^p + \sum_{n=2}^{\infty} \frac{1}{n \log n \log \log n} = \infty.$

$\sum_{\nu=0}^{\infty} a_\nu$ is summable $|C,\lambda,p|_1$ means that

$\sum_{n=1}^{\infty} |t_n^p - t_{n-1}^p| < \infty$ so that $\{t_n^p\}$ is convergent to s say. This means that $\sum_{\nu=0}^{\infty} a_\nu = s(C,\lambda,p)$. Hence we write and we have $|C,\lambda,p|_1 \Rightarrow (C,\lambda,p)$.

Let $R_\lambda^p(\tau)$ be defined as in §1.3. Then we say $\sum_{\nu=0}^{\infty} a_\nu$ is $|R,\lambda,p|$ summable, if

$$R_\lambda^p(\tau) \to s \text{ as } \tau \to \infty,$$

and

$$\int_h^\infty |dR_\lambda^p(\tau)| = \int_h^\infty \left|\frac{d}{d\tau} R_\lambda^p(\tau)\right| d\tau < \infty,$$

where $h \geqq \lambda_0$. (See Obrechkoff: Sur la sommation absolue des sèries de Dirichlet. C.R. 186, 1928.) We denote this by

$$\sum_{\nu=0}^{\infty} a_\nu = s|R,\lambda,p|.$$

§4.2 INCLUSION THEOREMS

The next lemma is a special case of a result due to Mears, [19, Theorem 1].

LEMMA 4.1

Let $Q = \{q_{n,\nu}\}$ *be a regular matrix with* $q_{n,\nu} = 0$ *for* $\nu > n$. *If* $\sigma_n = \sum_{\nu=0}^{n} q_{n,\nu} s_\nu$, *where* $s_\nu = \sum_{\mu=0}^{\nu} a_\mu$, *then a necessary and sufficient condition for*

$$\sum_{n=1}^{\infty} |\sigma_n - \sigma_{n-1}| < \infty$$

whenever $\sum_{n=1}^{\infty} |s_\nu - s_{\nu-1}| < \infty$ *is*

(4.3) $$\sum_{n=k}^{\infty} \left| \sum_{\nu=k}^{n-1} (q_{n,\nu} - q_{n-1,\nu}) + q_{n,n} \right| \leq H$$

where H *is independent of* k.

THEOREM 4.1

For any non-negative integer p,

$\sum_{\nu=0}^{\infty} a_\nu = s|C,\lambda,p+1|_1$ *whenever* $\sum_{\nu=0}^{\infty} a_\nu = s|C,\lambda,p|_1$.

PROOF

We know that $(C,\lambda,p+1) = \Lambda_{p+1}(C,\lambda,p)$ where Λ_{p+1} is defined in §1.7. By Lemma 4.1 it suffices to prove that

$$\sum_{n=k}^{\infty} \left| \sum_{\nu=k}^{n-1} (\lambda_{n,\nu} - \lambda_{n-1,\nu}) + \lambda_{n,n} \right| \leq H$$

where H is independent of k.

Now, referring to (1.8)

$$\sum_{n=k}^{\infty} \left| \sum_{\nu=k}^{n-1} (\lambda_{n,\nu} - \lambda_{n-1,\nu}) + \lambda_{n,n} \right|$$

$$= \lambda_{k,k} + \sum_{n=k+1}^{\infty} \left| \sum_{\nu=k}^{n-1} (\lambda_{n,\nu} - \lambda_{n-1,\nu}) + \lambda_{n,n} \right|$$

$$= \lambda_{k,k} + \sum_{n=k+1}^{\infty} \left| \sum_{\nu=k}^{n-1} \left[\frac{(\lambda_{\nu+p+1} - \lambda_\nu) E_\nu^p}{E_n^{p+1}} - \frac{(\lambda_{\nu+p+1} - \lambda_\nu) E_\nu^p}{E_{n-1}^{p+1}} \right] \right.$$

$$\left. + \frac{(\lambda_{n+p+1} - \lambda_n) E_n^p}{E_n^{p+1}} \right|$$

$$= \lambda_{k,k} + \sum_{n=k+1}^{\infty} \left| \sum_{\nu=k}^{n-1} \frac{(\lambda_{\nu+p+1} - \lambda_\nu) E_\nu^p}{E_n^p} \left(\frac{1}{\lambda_{n+p+1}} - \frac{1}{\lambda_n}\right) + \frac{\lambda_{n+p+1} - \lambda_n}{\lambda_{n+p+1}} \right|$$

$$= \lambda_{k,k} + \sum_{n=k+1}^{\infty} \left| \left(\frac{1}{\lambda_{n+p+1}} - \frac{1}{\lambda_n}\right) \left(\sum_{\nu=k}^{n-1} \frac{E_\nu^{p+1}}{E_n^p} - \sum_{\nu=k}^{n-1} \frac{E_{\nu-1}^{p+1}}{E_n^p} \right) + \frac{\lambda_{n+p+1} - \lambda_n}{\lambda_{n+p+1}} \right|$$

$$= \lambda_{k,k} + \sum_{n=k+1}^{\infty} \left| \left(\frac{1}{\lambda_{n+p+1}} - \frac{1}{\lambda_n}\right) \left(\frac{E_{n-1}^{p+1} - E_{k-1}^{p+1}}{E_n^p} \right) + \frac{\lambda_{n+p+1} - \lambda_n}{\lambda_{n+p+1}} \right|$$

$$= \lambda_{k,k} + \sum_{n=k+1}^{\infty} \left| \frac{\lambda_n}{\lambda_{n+p+1}} - 1 - \frac{E_{k-1}^{p+1}}{E_n^{p+1}} + \frac{E_{k-1}^{p+1}}{E_{n-1}^{p+1}} + 1 - \frac{\lambda_n}{\lambda_{n+p+1}} \right|$$

$$= \lambda_{k,k} + E_{k-1}^{p+1} \sum_{n=k+1}^{\infty} \left(\frac{-1}{E_n^{p+1}} + \frac{1}{E_{n-1}^{p+1}} \right)$$

$$= \lambda_{k,k} + \frac{E_{k-1}^{p+1}}{E_k^{p+1}}$$

$$= \frac{(\lambda_{k+p+1} - \lambda_k) E_k^p}{E_k^{p+1}} + \frac{E_{k-1}^{p+1}}{E_k^{p+1}}$$

$$= 1 - \frac{E_{k-1}^{p+1}}{E_k^{p+1}} + \frac{E_{k-1}^{p+1}}{E_k^{p+1}}$$

$$= 1.$$

Thus $\sum_{n=1}^{\infty} |t_n^p - t_{n-1}^p| < \infty \Rightarrow \sum_{n=1}^{\infty} |t_n^{p+1} - t_{n-1}^{p+1}| < \infty.$

Since $(C,\lambda,p) \Rightarrow (C,\lambda,p+1)$, $t_n^p \to s$ implies $t_n^{p+1} \to s$.

Consequently $\sum_{\nu=0}^{\infty} a_\nu = s|C,\lambda,p+1|_1$ whenever $\sum_{\nu=0}^{\infty} a_\nu = s|C,\lambda,p|_1$. ///

COROLLARY 4.1

$$\sum_{\nu=0}^{\infty} a_\nu = s|C,\lambda,p|_1 \textit{ for } p \geq 1, \textit{ whenever } \sum_{\nu=0}^{\infty} |a_\nu| < \infty,$$

where $s = \sum_{\nu=0}^{\infty} a_\nu$.

PROOF

Take p = 0 in Theorem 4.1 and proceed by induction. ///

THEOREM 4.2

$$\sum_{\nu=0}^{\infty} a_\nu = s|C,\lambda,p|_1 \textit{ whenever } \sum_{\nu=0}^{\infty} a_\nu = s|C,\lambda,p+1|_1$$

if and only if $\liminf_{n\to\infty} \frac{\lambda_{n+p+1}}{\lambda_n} > 1$.

PROOF

$$(C,\lambda,p) = \Lambda'_{p+1}(C,\lambda,p+1).$$

Referring to Lemma 3.1, we know in $\Lambda'_{p+1} = \{\lambda'_{n,\nu}\}$

$$\lambda'_{n,n} = \frac{\lambda_{n+p+1}}{\lambda_{n+p+1} - \lambda_n},$$

$$\lambda'_{n,n-1} = \frac{-\lambda_n}{\lambda_{n+p+1} - \lambda_n} = 1 - \lambda'_{n,n},$$

$$\lambda'_{n,\nu} = 0, \text{ otherwise.}$$

By Lemma 3.2, we know Λ'_{p+1} is regular if and only if

$\liminf_{n\to\infty} \frac{\lambda_{n+p+1}}{\lambda_n} > 1.$

Now $\sum_{n=k}^{\infty} \left| \sum_{\nu=k}^{n-1} (\lambda'_{n,\nu} - \lambda'_{n-1,\nu}) + \lambda'_{n,n} \right|$

$$= \lambda'_{k,k} + |\lambda'_{k+1,k+1} + \lambda'_{k+1,k} - \lambda'_{k,k}| + \sum_{n=k+2}^{\infty} \left| \sum_{\nu=k}^{n-1} (\lambda'_{n,\nu} - \lambda'_{n-1,\nu}) + \lambda'_{n,n} \right|$$

$$= \frac{\lambda_{k+p+1}}{\lambda_{k+p+1} - \lambda_k} + |1 - \lambda'_{k,k}| + \sum_{n=k+2}^{\infty} |\lambda'_{n,n} + \lambda'_{n,n-1} - \lambda'_{n-1,n-1} - \lambda'_{n-1,n-2}|$$

$$= \frac{\lambda_{k+p+1} + \lambda_k}{\lambda_{k+p+1} - \lambda_k} + \sum_{n=k+2}^{\infty} |1 - 1|$$

$$= \frac{\dfrac{\lambda_{k+p+1}}{\lambda_k} + 1}{\dfrac{\lambda_{k+p+1}}{\lambda_k} - 1}$$

Thus it follows Lemma 4.1 that

$$\sum_{\nu=0}^{\infty} a_\nu = s|C,\lambda,p|_1 \text{ whenever } \sum_{\nu=0}^{\infty} a_\nu = s|C,\lambda,p+1|_1$$

if and only if $\liminf_{n\to\infty} \frac{\lambda_{n+p+1}}{\lambda_n} > 1$. ///

Körle proved in [17] that $|R,\lambda,p| \iff |C,\lambda,p|_1$ for $p \geq 0$. Using this and the Theorem 4.2 we have the following corollary.

COROLLARY 4.2

$\sum_{\nu=0}^{\infty} a_\nu = s|R,\lambda,p|$ *whenever* $\sum_{\nu=0}^{\infty} a_\nu = s|R,\lambda,p+1|$ *if and only if* $\liminf_{n\to\infty} \frac{\lambda_{n+p+1}}{\lambda_n} > 1$.

We now turn our attention to the relationship between $|C,\lambda,p+1|_\mu$ and $[C,\lambda,p+1]_\mu$. To facilitate the discussion we use a result of Borwein, [1, Theorem 7], which we state as the next lemma. We include the proof for the sake of completeness.

LEMMA 4.2

If P *is a regular matrix with non-negative entries,* Q *is a matrix and* $\mu \geq 1$, *then necessary and sufficient conditions for a series to be summable* $[P,Q]_\mu$ *to* s *are that it be* PQ*-summable to* s *and* $[P,(I-P)Q]_\mu$*-summable to zero.*

PROOF

Let $\{\sigma_n\} = Q\{s_n\}$ and $\{\tau_n\} = P\{\sigma_n\}$. We have to prove that

$$\text{(a)} \quad \sum_{r=0}^{\infty} p_{n,r}|\sigma_r - s|^\mu = o(1)$$

if and only if

$$\text{(b)} \quad \tau_n \to s$$

and

$$\text{(c)} \quad \sum_{r=0}^{\infty} p_{n,r}|\sigma_r - \tau_r|^\mu = o(1).$$

(i) Suppose that (a) holds. Then by Theorem 1.2 (ii), (b) holds and so $\sum_{r=0}^{\infty} p_{n,r}|\tau_r - s|^\mu = o(1)$ since P is regular. Hence by Minkowski's inequality and (a)

$$\left\{\sum_{r=0}^{\infty} p_{n,r}|\sigma_r - \tau_r|^\mu\right\}^{1/\mu}$$

$$\leq \left\{\sum_{r=0}^{\infty} p_{n,r}|\sigma_r - s|^\mu\right\}^{1/\mu} + \left\{\sum_{r=0}^{\infty} p_{n,r}|\tau_r - s|^\mu\right\}^{1/\mu} = o(1),$$

and (c) follows.

(ii) Suppose that (b) and (c) hold. Since P is regular, it follows from (b) that

$$\sum_{r=0}^{\infty} p_{n,r}|\tau_r - s|^{\mu} = o(1).$$

Hence by Minkowski's inequality and (c),

$$\left\{\sum_{r=0}^{\infty} p_{n,r}|\sigma_r - s|^{\mu}\right\}^{1/\mu}$$

$$\leq \left\{\sum_{r=0}^{\infty} p_{n,r}|\sigma_r - \tau_r^*|^{\mu}\right\}^{1/\mu} + \left\{\sum_{r=0}^{\infty} p_{n,r}|\tau_r - s|^{\mu}\right\}^{1/\mu} = o(1),$$

so that (a) holds.

The proof is thus complete. ///

THEOREM 4.3

Let $\mu \geq 1$. *Then*

$$\sum_{\nu=0}^{\infty} a_\nu = s[C,\lambda,p+1]_\mu \text{ if and only if}$$

(4.4) $$\sum_{\nu=0}^{\infty} a_\nu = s(C,\lambda,p+1),$$

and

(4.5) $$\frac{1}{E_m^{p+1}} \sum_{n=0}^{m} (\lambda_{n+p+1} - \lambda_n) E_n^p |t_n^p - t_n^{p+1}|^{\mu} = o(1).$$

Condition (4.5) *means* $|t_n^p - t_n^{p+1}|^{\mu} \to 0(\Lambda_{p+1})$.

PROOF

In Lemma 4.2, take $P = \Lambda_{p+1}$, $Q = (C,\lambda,p)$ and observe that

$$(I - P)Q = (C,\lambda,p) - (C,\lambda,p+1). \quad ///$$

THEOREM 4.4

$$\sum_{\nu=0}^{\infty} a_\nu = s|C,\lambda,p+1|_1 \text{ implies } \sum_{\nu=0}^{\infty} a_\nu = s[C,\lambda,p+1]_1.$$

PROOF

Since $\sum_{n=1}^{\infty} |t_n^{p+1} - t_{n-1}^{p+1}| < \infty$ implies that t_n^{p+1} tends to a limit, s say, we have $\sum_{\nu=0}^{\infty} a_\nu = s(C,\lambda,p+1)$. Hence to prove the theorem it suffices to show condition (4.5) is satisfied with $\mu = 1$.

Let $n \geq 1$.

$$|t_n^p - t_n^{p+1}|$$

$$= \left|\frac{1}{E_n^p} \sum_{\nu=0}^{n} (\lambda_{n+1} - \lambda_\nu) \cdots (\lambda_{n+p} - \lambda_\nu) a_\nu - \frac{1}{E_n^{p+1}} \sum_{\nu=0}^{n} (\lambda_{n+1} - \lambda_\nu) \cdots (\lambda_{n+p+1} - \lambda_\nu) a_\nu\right|$$

$$= \left|\frac{\lambda_{n+p+1}}{E_n^{p+1}} \sum_{\nu=0}^{n} (\lambda_{n+1} - \lambda_\nu) \cdots (\lambda_{n+p} - \lambda_\nu) a_\nu - \frac{1}{E_n^{p+1}} \sum_{\nu=0}^{n} (\lambda_{n+1} - \lambda_\nu) \cdots (\lambda_{n+p+1} - \lambda_\nu) a_\nu\right|$$

$$= \left|\frac{1}{E_n^{p+1}} \sum_{\nu=0}^{n} (\lambda_{n+1} - \lambda_\nu) \cdots (\lambda_{n+p} - \lambda_\nu)(\lambda_{n+p+1} - \lambda_{n+p+1} + \lambda_\nu) a_\nu\right|$$

$$= \left|\frac{1}{E_n^{p+1}} \sum_{\nu=0}^{n} (\lambda_{n+1} - \lambda_\nu) \cdots (\lambda_{n+p} - \lambda_\nu) \lambda_\nu a_\nu\right|.$$

On the other hand

$$\left|t_n^{p+1} - t_{n-1}^{p+1}\right|$$

$$= \left|\frac{1}{E_n^{p+1}} \sum_{\nu=0}^{n} (\lambda_{n+1} - \lambda_\nu) \cdots (\lambda_{n+p+1} - \lambda_\nu) a_\nu - \frac{1}{E_{n-1}^{p+1}} \sum_{\nu=0}^{n} (\lambda_n - \lambda_\nu) \cdots (\lambda_{n+p} - \lambda_\nu) a_\nu\right|.$$

$$= \left| \frac{\lambda_n}{E_{n-1}^{p+2}} \sum_{\nu=0}^{n} (\lambda_{n+1} - \lambda_\nu) \cdots (\lambda_{n+p+1} - \lambda_\nu) a_\nu \right.$$

$$\left. - \frac{\lambda_{n+p+1}}{E_{n-1}^{p+2}} \sum_{\nu=0}^{n} (\lambda_n - \lambda_\nu) \cdots (\lambda_{n+p} - \lambda_\nu) a_\nu \right|$$

$$= \left| \frac{1}{E_{n-1}^{p+2}} \sum_{\nu=0}^{n} (\lambda_{n+1} - \lambda_\nu) \cdots (\lambda_{n+p} - \lambda_\nu) \left\{ \lambda_n (\lambda_{n+p+1} - \lambda_\nu) \right. \right.$$

$$\left. \left. - \lambda_{n+p+1} (\lambda_n - \lambda_\nu) \right\} a_\nu \right|$$

$$= \left| \frac{1}{E_{n-1}^{p+2}} \sum_{\nu=0}^{n} (\lambda_{n+1} - \lambda_\nu) \cdots (\lambda_{n+p} - \lambda_\nu)(\lambda_{n+p+1} - \lambda_n) \lambda_\nu a_\nu \right|$$

$$= \frac{\lambda_{n+p+1} - \lambda_n}{\lambda_n} \left| \frac{1}{E_n^{p+1}} \sum_{\nu=0}^{n} (\lambda_{n+1} - \lambda_\nu) \cdots (\lambda_{n+p} - \lambda_\nu) \lambda_\nu a_\nu \right|,$$

for $n \geq 1$.

Hence

$$(4.6) \qquad \left| t_n^{p+1} - t_{n-1}^{p+1} \right| = \frac{\lambda_{n+p+1} - \lambda_n}{\lambda_n} \left| t_n^p - t_n^{p+1} \right|, \text{ for } n \geq 1.$$

Consequently multiplying (4.6) by E_{n-1}^{p+1}, we obtain

$$\sum_{n=1}^{m} (\lambda_{n+p+1} - \lambda_n) E_n^p \left| t_n^p - t_n^{p+1} \right|$$

$$= \sum_{n=1}^{m} E_{n-1}^{p+1} \left| t_n^{p+1} - t_{n-1}^{p+1} \right|.$$

Since $\lambda_0 = 0$, $\left| t_0^p - t_0^{p+1} \right| = |a_0 - a_0| = 0.$

By taking $E_{-1}^{p+1} = 0$ and $t_{-1}^{p+1} = 0$, we have

$$(4.7) \qquad \sum_{n=0}^{m} (\lambda_{n+p+1} - \lambda_n) E_n^p \left| t_n^p - t_n^{p+1} \right|$$

$$= \sum_{n=0}^{m} E_{n-1}^{p+1} \left| t_n^{p+1} - t_{n-1}^{p+1} \right|.$$

Let $b_r = \left| t_r^{p+1} - t_{r-1}^{p+1} \right|$ and $B_n = \sum_{r=0}^{n} b_r$.

Then from (4.7), we have

$$\sum_{n=0}^{m} E_{n-1}^{p+1} b_n$$

$$= \sum_{n=0}^{m} E_{n-1}^{p+1} (B_n - B_{n-1})$$

$$= B_m E_m^{p+1} - \sum_{n=0}^{m} B_n (E_n^{p+1} - E_{n-1}^{p+1}).$$

Dividing by E_m^{p+1}, we obtain

$$B_m - \frac{1}{E_m^{p+1}} \sum_{n=0}^{m} (\lambda_{n+p+1} - \lambda_n) E_n^p B_n = o(1), \text{ as } m \to \infty,$$

because of the regularity of Λ_{p+1} and the hypothesis $\sum_{\nu=0}^{\infty} a_\nu = s|C,\lambda,p+1|_1$ which means that $\{B_n\}$ is convergent.

Thus the condition (4.5) is satisfied and the theorem is proved. ///

(C.f. Borwein and Cass [6, Theorem 9].)

THEOREM 4.5

If $\sum_{\nu=0}^{\infty} a_\nu = s \ (C,\lambda,p+1)$ *then, for* $\mu > 1$,

$\sum_{\nu=0}^{\infty} a_\nu = s|C,\lambda,p+1|_\mu$ *implies that* $\sum_{\nu=0}^{\infty} a_\nu = s[C,\lambda,p+1]$.

PROOF

Since $\sum_{\nu=0}^{\infty} a_\nu = s(C,\lambda,p+1)$, it suffices to show that condition (4.5) is satisfied with $\mu > 1$.

Now referring to (4.6), we have

$$\left|t_n^{p+1} - t_{n-1}^{p+1}\right|^{\mu} = \left(\frac{\lambda_{n+p+1} - \lambda_n}{\lambda_n}\right)^{\mu}\left|t_n^p - t_n^{p+1}\right|^{\mu}$$

for $\mu > 1$ and $n \geq 1$. Thus

$$\left|t_n^p - t_n^{p+1}\right|^{\mu} = \left(\frac{\lambda_n}{\lambda_{n+p+1} - \lambda_n}\right)^{\mu}\left|t_n^{p+1} - t_{n-1}^{p+1}\right|^{\mu} \tag{4.8}$$

for $\mu > 1$ and $n \geq 1$.

Since $\left|t_0^p - t_0^{p+1}\right| = 0$ and $E_{-1}^{p+1} = 0$ and $t_{-1}^{p+1} = 0$, we have, by (4.8),

$$\sum_{n=0}^{m} (\lambda_{n+p+1} - \lambda_n) E_n^p \left|t_n^p - t_n^{p+1}\right|^{\mu}$$

$$= \sum_{n=0}^{m} E_{n-1}^{p+1} \left(\frac{\lambda_n}{\lambda_{n+p+1} - \lambda_n}\right)^{\mu-1}\left|t_n^{p+1} - t_{n-1}^{p+1}\right|^{\mu}.$$

Now let $b_r = \rho_r^{\mu-1}\left|t_r^{p+1} - t_{r-1}^{p+1}\right|^{\mu}$

and $B_n = \sum_{r=0}^{n} b_r$,

and proceed as the last part of the proof of Theorem 4.4, we have $\sum_{n=0}^{m} E_{n-1}^{p+1}\rho_n^{\mu-1}\left|t_n^{p+1} - t_{n-1}^{p+1}\right|^{\mu} = o(E_m^{p+1})$.

And hence $\sum_{n=0}^{m} (\lambda_{n+p+1} - \lambda_n) E_n^p \left|t_n^p - t_n^{p+1}\right|^{\mu} = o(E_m^{p+1})$. ///

CHAPTER 5

SOME STRICT INCLUSION THEOREMS BETWEEN CESÀRO AND DISCRETE RIESZ METHODS OF SUMMABILITY

§5.1 DEFINITIONS

Suppose throughout this chapter that $\kappa > 0$,

$$s_n = \sum_{r=0}^{n} a_r,$$

$$\varepsilon_0^{\kappa} = 1,$$

and

$$\varepsilon_n^{\kappa} = \binom{n+\kappa}{n} = \frac{(\kappa+1)(\kappa+2)\cdots(\kappa+n)}{n!} \quad \text{for } n > 0.$$

Let $\{p_n\}$ be a sequence with $p_n > 0$ for $n \geq 0$ and let

$$P_n = \sum_{r=0}^{n} p_r.$$

Define

$$(5.1) \quad t_n = \frac{1}{P_n} \sum_{r=0}^{n} p_{n-r} s_r = \frac{1}{P_n} \sum_{r=0}^{n} P_{n-r} a_r,$$

$$(5.2) \quad t_n^{\Delta} = \frac{1}{P_n} \sum_{r=0}^{n} p_{n-r} a_r = \frac{1}{P_n} \sum_{r=0}^{n} (p_{n-r} - p_{n-1-r}) s_r, \quad (p_{-1} = 0).$$

We say that the sequence $\{s_n\}$ is (N,p_n)-convergent to s if $t_n \to s$; and we write

$$s_n \to s \ (N,p_n).$$

This is a Nörlund Summability Method. See for example Hardy [11, page 54].

Let

(5.3) $\tau_n = \frac{1}{P_n} \sum_{r=0}^{n} p_r |t_r^{\Delta} - s|$.

We say that the sequence $\{s_n\}$ is $[N,p_n]$-convergent to s if $\tau_n = o(1)$, and we write

$$s_n \to s\ [N,p_n].$$

(See Borwein and Cass [6].)

We say that the sequence $\{s_n\}$ is $|N,p_n|$-convergent to s if

$$\sum_{n=1}^{\infty} |t_n - t_{n-1}| < \infty \quad \text{and } s = \lim t_n;$$

and we write

$$s_n \to s\ |N,p_n|.$$

The Strong Summability Method $[N,p_n]$ is the method $[P,Q]_1$ (see §1.2) with $P = (\overline{N},p_n)$ (see §3.2) and Q the matrix associated with the transformation (5.2). We shall denote Q by $(N,\Delta p_n)$.

In the case of $[N,p_n]$-summability, the method is interesting only if $P_n \to \infty$. This condition is satisfied by the summability methods we consider below.

If we take $p_n = \varepsilon_n^{\kappa-1}$, then (N,p_n) and $|N,p_n|$ are the Cesàro and Absolute Cesàro Summability Methods (C,κ) and $|C,\kappa|$ respectively.

The method $[N,p_n]$ with $p_n = \varepsilon_n^{\kappa-1}$ is equivalent (but not equal) to the Strong Cesàro Method $[C,\kappa]$ (See §1.6.) We shall denote this method $[N,p_n]$ also by $[C,\kappa]$. See Borwein and Cass [6, pages 98-99].

If

$$\rho_n^\kappa = \sum_{\nu=0}^{n} \left(1 - \frac{\nu}{n+1}\right)^\kappa a_\nu \tag{5.4}$$

$$= \frac{1}{(n+1)^\kappa} \sum_{\nu=0}^{n} (n+1-\nu)^\kappa (s_\nu - s_{\nu-1})$$

$$= \frac{1}{(n+1)^\kappa} \sum_{\nu=0}^{n} [((n+1)-\nu)^\kappa - (n-\nu)^\kappa] s_\nu ,$$

then we say that the sequence $\{s_\nu\}$ is (R^*,n,κ)-convergent to s, if $\rho_n^\kappa \to s$ as $n \to \infty$. We denote this by

$$s_n \to s \ (R^*,n,\kappa).$$

Thus if we take $p_n = (n+1)^\kappa - n^\kappa$ for $n \geq 0$, then (N,p_n) and $|N,p_n|$ are the Discrete Reisz and Absolute Discrete Riesz Summability Methods (R^*,n,κ) and $|R^*,n,\kappa|$ respectively. We shall define the *Strong Discrete Riesz Method of Summability* $[R^*,n,\kappa]$ to be the method $[N,p_n]$ associated with this $\{p_n\}$.

§5.2 KUTTNER'S THEOREM

In the definitions of (C,κ) and (R^*,n,κ) and the associated absolute methods, κ is usually allowed to satisfy $\kappa > -1$. The methods $[C,\kappa]$ and $[R^*,n,\kappa]$ make sense only when $\kappa > 0$ and it is for this reason we have so restricted κ.

THEOREM (Kuttner)

(i) If $-1 < \kappa < 2$, *then* (R^*,n,κ) *is equivalent to* (C,κ) *and* $|R^*,n,\kappa|$ *is equivalent to* $|C,\kappa|$.

(ii) There is a sequence $(R^*,n,2)$*-convergent but not*

$(C,2)$-convergent and a sequence $|R^*,n,2|$-convergent but not $|C,2|$-convergent. But $|R^*,n,2| \Rightarrow (C,2)$.

(iii) If $\kappa > 2$, there is a sequence $|R^*,n,\kappa|$-convergent but not (C,κ)-convergent.

(See Kuttner [18].)

§5.3 EXTENSION OF KUTTNER'S THEOREM AND OTHER RESULTS

For the proof of Theorem 5.1 we state two results of Borwein and Cass [6, Theorems 6 and 9] as our next two lemmas.

LEMMA 5.1

$[N,p_n] \Rightarrow (N,p_n)$.

LEMMA 5.2

If $P_n \to \infty$ and $\{s_n\}$ is $|N,p_n|$-convergent, then

$$s_n \to s \ [N,p_n]$$

where $s = \lim_{n\to\infty} t_n$ *and* t_n *is defined as in* (5.1).

THEOREM 5.1

If $\kappa > 0$, then $|R^*,n,\kappa| \Rightarrow [R^*,n,\kappa] \Rightarrow (R^*,n,\kappa)$.

PROOF

That $[R^*,n,\kappa] \Rightarrow (R^*,n,\kappa)$ is a special case of Lemma 5.1. $|R^*,n,\kappa| \Rightarrow [R^*,n,\kappa]$ follows from Lemma 5.2. ///

The next theorem is known, but it also follows from Lemmas 5.1 and 5.2 as the Theorem 5.1.

THEOREM 5.2

$|C,\kappa| \Rightarrow [C,\kappa] \Rightarrow (C,\kappa)$.

THEOREM 5.3

Let $p_n > 0$ *for* $n \geq 0$ *and suppose* $P_n \to \infty$. *Then there is a sequence which is* $[N,p_n]$*-convergent but not* $|N,p_n|$*-convergent.*

PROOF

Borwein and Cass [6, Theorem 8] proved that $s_n \to s[N,p_n]$ if and only if

(5.5) $$s_n \to s(N,p_n)$$

and

(5.6) $$\frac{1}{P_n}\sum_{r=0}^{n} p_r|t_r^{\Delta} - t_r| = o(1)$$

where t_r and t_r^{Δ} are given by (5.1) and (5.2).

This is a special case of Lemma 4.2.

Now

$$t_r^{\Delta} - t_r = \frac{1}{p_r}\sum_{\nu=0}^{r}(p_{r-\nu} - p_{r-1-\nu})s_\nu - \frac{1}{P_r}\sum_{\nu=0}^{n} p_{r-\nu}s_\nu$$

$$= \frac{P_r\sum_{\nu=0}^{r} p_{r-\nu}s_\nu - P_r\sum_{\nu=0}^{r-1} p_{r-1-\nu}s_\nu - p_r\sum_{\nu=0}^{r} p_{r-\nu}s_\nu}{p_r P_r}$$

$$= \frac{P_{r-1}\sum_{\nu=0}^{r} p_{r-\nu}s_\nu - P_r\sum_{\nu=0}^{r-1} p_{r-1-\nu}s_\nu}{p_r P_r},$$

so that

(5.7) $$p_r(t_r^{\Delta} - t_r) = P_{r-1}(t_r - t_{r-1}), \quad r = 0,1,2,\ldots,$$

$(P_{-1} = t_{-1} = 0)$.

Choose $\{s_n\}$ so that $t_n - t_{n-1} = \frac{\delta_n}{P_n}\frac{p_n}{D_n}$ where

$D_n = \sum_{r=0}^{n} \frac{p_r}{P_r}$ and $\delta_n = \pm 1$ chosen in such a way that $\sum_{n=1}^{\infty} \frac{\delta_n\, p_n}{P_n\, D_n}$ converges. Then $\{t_n\}$ is convergent ensuring that (5.5) is satisfied. Also we have

$$\frac{1}{P_n} \sum_{r=0}^{n} p_r \left|t_r^{\Delta} - t_r\right| = \frac{1}{P_n} \sum_{r=0}^{n} P_{r-1} \left|t_r - t_{r-1}\right|$$

$$= \frac{1}{P_n} \sum_{r=0}^{n} \frac{P_{r-1}\, p_r}{P_r\, D_r}$$

$$= \sum_{r=0}^{n} a_{n,r} \frac{1}{D_r}$$

where $a_{n,r} = \frac{P_{r-1}\, p_r}{P_n\, P_r}$ for $0 \leq r \leq n$ and $a_{n,r} = 0$ for $r > n$.

Now $A = \{a_{n,r}\}$ is a matrix with zero column limits and

$$\sum_{r=0}^{n} |a_{n,r}| = \sum_{r=0}^{n} a_{n,r} \leq \frac{1}{P_n} \sum_{r=0}^{n} p_r = 1, \quad \text{for all } n,$$

so that it transforms null sequences into null sequences. Since by Abel-Dini Theorem $\lim_{n\to\infty} D_n = \infty$, $\frac{1}{D_r} \to 0$ as $r \to \infty$.

It follows that (5.6) is satisfied, so $s_n \to s\ [N,p_n]$. But by Abel-Dini Theorem again

$$\sum_{n=1}^{\infty} |t_n - t_{n-1}| = \sum_{n=1}^{\infty} \frac{p_n}{P_n\, D_n} = \infty,$$

so $\{s_n\}$ is not $|N,p_n|$-convergent. ///

COROLLARY 5.1

Let $\kappa > 0$. There is a sequence which is $[R^*,n,\kappa]$-convergent but not $|R^*,n,\kappa|$-convergent.

COROLLARY 5.2

Let $\kappa > 0$. *There is a sequence which is* $[C,\kappa]$-*convergent but not* $|C,\kappa|$-*convergent.*

THEOREM 5.4

Let $\kappa > 0$. *There is a sequence which is* (R^*,n,κ)-*convergent but not* $[R^*,n,\kappa]$-*convergent.*

PROOF

Let $P = \{p_{n,\nu}\}$, where $p_{n,\nu} = \dfrac{(\nu+1)^{\kappa} - \nu^{\kappa}}{(n+1)^{\kappa}}$ for $0 \leq \nu \leq n$ and $p_{n,\nu} = 0$ for $\nu > n$. It follows from Theorem 3.4 that there is a sequence P-convergent but not [P,I]-convergent. Let $Q = \{q_{n,\nu}\}$ be the matrix such that

$$\sum_{\nu=0}^{n} q_{n,\nu} s_{\nu} = \frac{1}{P_n} \sum_{\nu=0}^{n} p_{n-\nu} a_{\nu}$$

where $p_n = (n+1)^{\kappa} - n^{\kappa}$. Then $[R^*,n,\kappa]$-convergency is the same as [P,Q]-convergency and (R^*,n,κ)-convergency is the same as PQ-convergency. Since the matrix Q has an inverse our result now follows. ///

For the next theorem we state two results of Borwein and Cass [6, Theorem 1 and Corollary 1] as our next two lemmas.

LEMMA 5.4

If $(N,p_n) \Rightarrow (N,q_n)$ *then* $[N,p_n] \Rightarrow [N,q_n]$.

LEMMA 5.5

If $(N,p_n) \Leftrightarrow (N,q_n)$ *then* $[N,p_n] \Leftrightarrow [N,q_n]$.

THEOREM 5.6

(i) *If* $\kappa > 0$, *then* $[C,\kappa] \Rightarrow [R^*,n,\kappa]$.

(ii) *If* $0 < \kappa < 2$, *then* $[C,\kappa] \Leftrightarrow [R^*,n,\kappa]$.

PROOF

Since for $\kappa > 0$ we have $(C,\kappa) \Rightarrow (R^*,n,\kappa)$, (i) follows from Lemma 5.4. Since for $0 < \kappa < 2$ we have $(C,\kappa) \Leftrightarrow (R^*,n,\kappa)$ (ii) follows from Lemma 5.5. ///

THEOREM 5.7

There is a sequence which is $|R^*,n,2|$*-convergent but not* $[C,2]$*-convergent.*

PROOF

For a given sequence $\{s_n\}$ we write

$$(5.8) \qquad \sigma_n = \frac{1}{\varepsilon_n^2} \sum_{\nu=0}^{n} \varepsilon_{n-\nu}^1 s_\nu = \frac{S_n}{\varepsilon_n^2}$$

and

$$(5.9) \qquad \xi_n = \frac{1}{(n+1)^2} \sum_{\nu=0}^{n} (n+1-\nu)^2 a_\nu = \frac{T_n}{(n+1)^2}$$

so that $\{\sigma_n\}$ and $\{\xi_n\}$ are respectively the $(C,2)$ and $(R^*,n,2)$ transforms of the sequence $\{s_n\}$.

As in Kuttner [18, page 362] we have

$$(5.10) \qquad T_0 = S_0; \quad T_n = S_{n-1} + S_n, \quad n = 1,2,3\ldots$$

and

$$(5.11) \qquad S_n = \sum_{m=0}^{n} (-1)^{n-m} T_m.$$

Now take $S_n = (-1)^n n$ so that $T_n = (-1)^n$. Thus $\sum_{n=1}^{\infty} |\xi_n - \xi_{n-1}| < \infty$ and $\xi_n \to 0$, so that if $\{s_n\}$ is the sequence associated with this choice of S_n and T_n we have $s_n \to 0|R^*,n,2|$. To see that $\{s_n\}$ is not $[C,2]$-convergent we notice first that by Theorem 5.1, $s_n \to 0|R^*,n,2|$ implies $s_n \to 0\ [R^*,n,2]$. Now by Theorem 5.6 $[C,2] \Rightarrow [R^*,n,2]$, the only $[C,2]$-sum that $\{s_n\}$ could have is zero.

But
$$s_n - s_{n-1} = (-1)^n(2n-1).$$

(5.12)
$$\frac{1}{m+1}\sum_{n=0}^{m}\left|\frac{(-1)^n(2n-1)}{n+1}\right|$$

$$= \frac{1}{m+1}\sum_{n=0}^{m}\frac{2n-1}{n+1}.$$

Since (C,1) is regular and $\frac{2n-1}{n+1} \to 2$, (5.12) tends to 2 as $m \to \infty$. Thus $\{s_n\}$ is not $[C,2]$-convergent to zero. ///

COROLLARY 5.3

There is a sequence which is $[R^*,n,2]$*-convergent but not* $[C,2]$*-convergent.*

PROOF

This follows from the fact that $|R^*,n,2| \Rightarrow [R^*,n,2]$.

THEOREM 5.8

$[R^*,n,2] \Rightarrow (C,2)$.

PROOF

Referring to (5.9) we find that

$$T_r - T_{r-1} = \sum_{\nu=0}^{r}\{(r+1-\nu)^2 - (r-\nu)^2\}a_\nu .$$

$s_n \to 0\ [R^*,n,2]$ if and only if

$$\frac{1}{P_n}\sum_{r=0}^{n}p_r\left|\frac{1}{p_r}\sum_{\nu=0}^{r}p_{r-\nu}a_\nu\right| = \frac{1}{P_n}\sum_{r=0}^{n}\left|\sum_{\nu=0}^{r}p_{r-\nu}^{\circ}a_\nu\right| = o(1),$$

where $p_{r-\nu} = (r+1-\nu)^2 - (r-\nu)^2$ and $P_n = (n+1)^2$. Hence $s_n \to 0$ $[R^*,n,2]$ if and only if

$$\frac{1}{(n+1)^2} \sum_{r=0}^{n} |T_r - T_{r-1}| = o(1), \quad (T_{-1} = 0).$$

From (5.11) it follows that

$$|S_n| \leq \sum_{r=0}^{n} |T_r - T_{r-1}| .$$

Thus if $s_n \to 0$ $[R^*,n,2]$, then $|S_n| = o(n^2)$ so that $s_n \to 0(C,2)$.

Now if $s_n \to s[R^*,n,2]$, then $s_n - s \to 0[R^*,n,2]$ so $s_n - s \to 0(C,2)$, i.e., $s_n \to s(C,2)$. ///

THEOREM 5.9

There is a sequence which is (C,2)*-convergent but not* $[R^*,n,2]$*-convergent.*

PROOF

Choose $\{s_n\}$ so that

$$S_{2n} = (-1)^n n^{3/2} \text{ and } S_{2n+1} = 0.$$

Then $s_n \to o(C,2)$. But referring to (5.10)

$$\begin{aligned} T_{2r} - T_{2r-1} &= S_{2r} - S_{2r-2} \\ &= (-1)^r\{r^{3/2} + (r-1)^{3/2}\}, \quad r = 1,2,\ldots . \end{aligned}$$

So if $2m \leq n \leq 2m+1$, then

$$\begin{aligned} \sum_{r=0}^{n} |T_r - T_{r-1}| &\geq \sum_{r=1}^{m} |T_{2r} - T_{2r-1}| \\ &\geq \sum_{r=1}^{m} (r-1)^{3/2} \\ &\sim H\, m^{5/2} \\ &\sim H_1 n^{5/2} \end{aligned}$$

where H, H_1 are independent of n.

Thus $\{s_n\}$ is not $[R^*,n,2]$-convergent to zero and our result follows. ///

THEOREM 5.10

Let $\kappa > 2$

(i) *There is a sequence which is* $[R^*,n,\kappa]$*-convergent but not* (C,κ)*-convergent.*

(ii) *There is a sequence which is* $|R^*,n,\kappa|$*-convergent but not* $[C,\kappa]$*-convergent.*

(iii) *There is a sequence which is* $[R^*,n,\kappa]$*-convergent but not* $[C,\kappa]$*-convergent.*

PROOF

Part (i) follows from Kuttner's Theorem (iii) and the fact that $|R^*,n,\kappa| \Rightarrow [R^*,n,\kappa]$.

Part (ii) follows from Kuttner's Theorem (iii) and the fact that $[C,\kappa] \Rightarrow (C,\kappa)$.

Part (iii) follows from part (ii) and the fact that $|R^*,n,\kappa| \Rightarrow [R^*,n,\kappa]$.

The relations between the various summability methods discussed in this chapter are conveniently displayed in three figures below. In these figures the symbol $\rightarrow$ denotes strict inclusion, the symbol $\leftrightarrow$ denotes equivalence and the notation $P \not\leftrightarrow Q$ means that there is sequence which is P-convergent but not Q-convergent.

$0 < \kappa < 2$

$$\begin{array}{ccccc} |R^*,n,\kappa| & \rightarrow & [R^*,n,\kappa] & \rightarrow & (R^*,n,\kappa) \\ \updownarrow & & \updownarrow & & \updownarrow \\ |C,\kappa| & \rightarrow & [C,\kappa] & \rightarrow & (C,\kappa) \end{array}$$

Figure 1

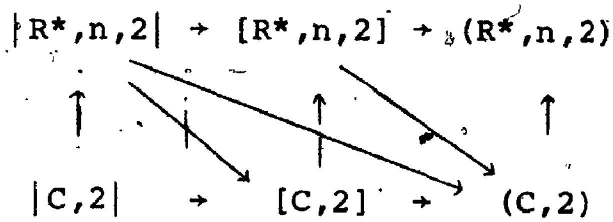

Figure 2

$\kappa > 2$

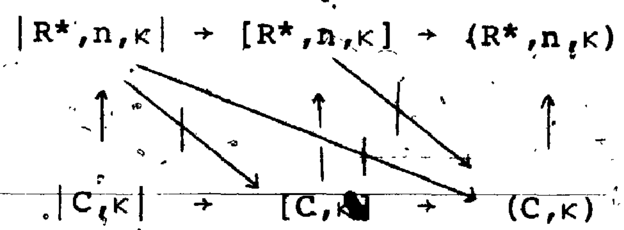

Figure 3

CHAPTER 6

STRONG AND ABSOLUTE NÖRLUND METHODS OF SUMMABILITY ASSOCIATED WITH POLYNOMIALS

In this chapter our investigations stem from the results in D. Borwein [4]. We consider a Nörlund Method of Summability Associated with Polynomials and investigate the properties of an associated Strong Summability Method and of the Absolute Nörlund Method of Summability Associated with Polynomials.

§6.1 DEFINITIONS

Let s, s_n be arbitrary complex numbers, and whenever $n < 0$ we take $s_n = 0$. Let

$$p(z) = p_0 + p_1 z + \cdots + p_j z^j$$

and

$$q(z) = q_0 + q_1 z + \cdots + q_k z^k$$

be polynomials with complex coefficients which satisfy the normalizing conditions

$$p(1) = 1 \text{ and } q(1) = 1.$$

We suppose throughout that $p(0) \neq 0$, $q(0) \neq 0$, $p_n = 0$ for $n > j$ and $q_n = 0$ for $n > k$. We use the notations

$$(6.1) \qquad t_n = \sum_{\nu=0}^{n} p_\nu s_{n-\nu}, \quad n = 0,1,2,\ldots,$$

$$(6.2) \qquad u_n = \sum_{\nu=0}^{n} q_\nu s_{n-\nu}, \quad n = 0,1,2,\ldots .$$

Associated with the polynomial $p(z)$ is a Nörlund Method of Summability N_p which we call a *Polynomial Nörlund Method* and which is defined as follows.

The sequence $\{s_n\}$ is said to be N_p-convergent to s, and we write

$$s_n \to s(N_p), \text{ if } \lim_{n\to\infty} t_n = s. \tag{6.3}$$

This definition is due to D. Borwein.

We define

$$s_n \to s\ [C_1, N_p] \tag{6.4}$$

if
$$\frac{1}{n+1}\sum_{r=0}^{n} |t_r - s| = o(1), \quad \text{as } n \to \infty.$$

This is the $[P,Q]_1$ defined in §1.2 with $P = C_1$ and $Q = N_p$.

Let $P_n = \sum_{\nu=0}^{n} p_\nu$ where P_n is non-zero for $n = 0,1,2,\ldots$ and $\tau_n = \frac{1}{P_n}\sum_{\nu=0}^{n} p_\nu s_{n-\nu}$. Then we say that the sequence $\{s_n\}$ is (N,p_n)-convergent to s and we write

$$s_n \to s(N,p_n) \tag{6.5}$$

if $\lim_{n\to\infty} \tau_n = s$.

This is the Nörlund Summability Method given in §5.1, but here we allow p_ν to be complex for all $\nu \geq 0$. Moreover, in this chapter we are only interested in the case where p_ν's are coefficients of a polynomial $p(z)$ with $p(1) = 1$ and we only use the (N,p_n) method in this sense. It is evident that in this sense (N,p_n) is equivalent to the Polynomial Nörlund Method N_p.

Let $P'_n = \sum_{r=0}^{n} |P_r|$ and $P'_n \neq 0$ for $n = 0,1,2,\ldots$. Then

(6.6) $$s_n \to s\ [N,P_n]$$

if $\frac{1}{P'_n} \sum_{r=0}^{n} |P_r|\,|\tau_r - s| = o(1)$, as $n \to \infty$.

This definition is analogous to the definition of $[N,p_n]$ given in §5.1, but we allow here p_ν to be complex for $\nu \geqq 0$. Moreover we let p_ν's be coefficients of a polynomial $p(z)$ with $p(1) = 1$.

The Absolute Polynomial Nörlund Summability $|N_p|$ is defined as follows.

(6.7) $$s_n \to s\ |N_p|$$

if $t_n \to s$ and $\sum_{n=0}^{\infty} |t_n - t_{n-1}| < \infty$, where $t_{-1} = 0$.

The method $[C_1,N_p]$ is a Strong Summability Method Associated with the Polynomial Nörlund Method. It is not the Strong Nörlund Summability Method defined in [6] which we considered in Chapter 5. Shortly we shall show that $[C_1,N_p]$ is equivalent to $[N,P_n]$. Thus $[C_1,N_p]$ is the Strong Nörlund Summability Method defined in [6] for (N,P_n) rather than for (N,p_n).

We shall establish at first $[C_1,N_p] \Rightarrow [C_1,N_q]$ if and only if $N_p \Rightarrow N_q$.

It is shown in Borwein and Cass [6] that if $(N,p_n) \Rightarrow (N,q_n)$ then $[N,p_n] \Rightarrow [N,q_n]$. We shall investigate the converse of this theorem in the case of the Polynomial Nörlund Methods.

Then we shall establish $|N_p| \Rightarrow |N_q|$ if and only if $N_p \Rightarrow N_q$.

Finally we shall establish some minor results analogous to some of the results obtained in [4].

§6.2 THE EQUIVALENCE OF $[C_1,N_p]$ AND $[N,P_n]$

THEOREM 6.1

$$[C_1,N_p] \Leftrightarrow [N,P_n].$$

PROOF

The result is an elementary consequence of the fact that $P'_n = \sum_{r=0}^{n} |P_r| = \sum_{r=0}^{j-1} |P_r| + n - j + 1 \sim n + 1$ which implies the equivalence of $(\bar{N},P_n)$ and $(C,1)$. ///

§6.3 THEOREMS ABOUT NÖRLUND METHODS OF SUMMABILITY ASSOCIATED WITH POLYNOMIALS

For completeness we shall quote without proof several results of Borwein [4].

The methods N_p and N_q mentioned in the following theorems are Nörlund Methods associated with polynomials $p(z)$ and $q(z)$ as defined in §6.1. Evidently N_p and N_q are regular.

THEOREM 6.2

The method N_f, *associated with the polynomial* $f(z) = p(z)q(z)$, *includes both* N_p *and* N_q. (Borwein [4, Theorem 2].)

THEOREM 6.3

The methods N_p *and* N_q *are consistent, i.e., if* $s_n \to s$ (N_p) *and* $s_n \to s'$ (N_q), *then* $s = s'$. (Borwein [4, Corollary].)

THEOREM 6.4

If $h(z)$ *is the highest common factor of* $p(z)$ *and* $q(z)$, *normalized so as to make* $h(1) = 1$, *then a necessary and sufficient condition for a sequence to be both* N_p- *and* N_q-*convergent is that it be* N_h-*convergent.* (Borwein [4, Theorem 3].)

THEOREM 6.5

In order that N_q *should include* N_p *it is necessary and sufficient that* $q(z)/p(z)$ *should not have poles on or within the unit circle.* (Borwein [4, Theorem I].)

THEOREM 6.6

If $q(z)/p(z)$ *has poles of maximum order* m *on the unit circle and does not have poles within the unit circle, then* $(C,m)N_q$ *includes* N_p, *but for any* $\varepsilon > 0$, *there is an* N_p-*convergent sequence which is not* $(C,m-\varepsilon)N_q$-*convergent.* (Borwein [4, Theorem II].)

THEOREM 6.7

If $q(z)/p(z)$ *has a pole within the unit circle then there is an* N_p*-convergent sequence which is not* AN_q*-convergent.* (Borwein [4, Theorem III].)

THEOREM 6.8

In order that N_p *should be equivalent to* $(C,0)$ *it is necessary and sufficient that* $p(z)$ *should not have zeros on or within the unit circle.* (Borwein [4, Theorem I+].)

THEOREM 6.9

If $q(z)/p(z)$ *has poles* $\lambda_1, \lambda_2, \ldots, \lambda_\ell$, *in the finite complex plane, of orders* $m_1, m_2, \ldots, m_\ell$ *respectively, and if, for* $n = 0,1,2,\ldots,$

$$t_n = \sum_{\nu=0}^{n} p_\nu s_{n-\nu},$$

$$u_n = \sum_{\nu=0}^{n} q_\nu s_{n-\nu},$$

then

$$u_n = \sum_{\nu=0}^{n} C_\nu t_{n-\nu} + \sum_{r=1}^{\ell} \sum_{\rho=1}^{m_r} C_{r,\rho} \sum_{\nu=0}^{n} \binom{\nu+\rho-1}{\rho-1} \lambda_r^{-\nu} t_{n-\nu}$$

where the C's are constants, depending only on $p_0, p_1, \ldots, p_j$, $q_0, q_1, \ldots, q_k$ *such that* $c_n = 0$ *for* $n > k - j$ *and* $C_{r,m_r} \neq 0$. (Borwein [4, Lemma 1].)

§6.4 $[C_1, N_p]$ METHOD OF SUMMABILITY

The following proposition is a special case of Theorem 1.2.

PROPOSITION 6.1

(i) $N_p \Rightarrow [C_1, N_p]$,

(ii) $[C_1, N_p] \Rightarrow (C,1)N_p$.

THEOREM 6.10

If $q(z)/p(z)$ *has no poles within or on the unit circle, then* $[C_1, N_p] \Rightarrow [C_1, N_q]$.

PROOF

Without loss of generality, we may assume $s_n \to 0$ $[C_1, N_p]$ and prove $s_n \to 0$ $[C_1, N_q]$.

Let $t_n = \sum_{\nu=0}^{n} p_\nu s_{n-\nu}$,

$$u_n = \sum_{\nu=0}^{n} q_\nu s_{n-\nu}.$$

If $q(z)/p(z)$ has no poles within or on the unit circle, but has poles $\lambda_1, \lambda_2, \ldots, \lambda_\ell$ of order $m_1, m_2, \ldots, m_\ell$ outside the unit circle, then by Theorem 6.9

$$u_n = \sum_{\nu=0}^{n} C_\nu t_{n-\nu} + \sum_{r=1}^{\ell} \sum_{\rho=1}^{m_r} C_{r,\rho} \sum_{\nu=0}^{n} \binom{\nu+\rho-1}{\rho-1} \lambda_r^{-\nu} t_{n-\nu}$$

where the C's are constants, depending only on $p_0, p_1, \ldots, p_j, q_0, q_1, \ldots, q_k$, such that $c_n = 0$ for $n > k - j$ and $c_{r,m_r} \neq 0$.

So $|u_n| \leq \sum_{\nu=0}^{n} |c_\nu||t_{n-\nu}| + \sum_{r=1}^{\ell} \sum_{\rho=1}^{m_r} |c_{r,\rho}| \sum_{\nu=0}^{n} \left|\binom{\nu+\rho-1}{\rho-1} \lambda_r^{-\nu}\right| |t_{n-\nu}|$.

Thus $\frac{1}{m+1}\sum_{n=0}^{m}|u_n|$

$$\leq \frac{1}{m+1}\sum_{n=0}^{m}\sum_{\nu=0}^{n}|c_\nu||t_{n-\nu}| + \frac{1}{m+1}\sum_{n=0}^{m}\sum_{r=1}^{\ell}\sum_{\rho=1}^{m_r}|c_{r,\rho}|\sum_{\nu=0}^{n}\left|\binom{\nu+\rho-1}{\rho-1}\lambda_r^{-\nu}\right||t_{n-\nu}|$$

$$= \sum_{\nu=0}^{m}|c_\nu|\frac{1}{m+1}\sum_{n=0}^{m-\nu}|t_n| + \sum_{r=1}^{\ell}\sum_{\rho=1}^{m_r}|c_{r,\rho}|\sum_{\nu=0}^{m}\left|\binom{\nu+\rho-1}{\rho-1}\lambda_r^{-\nu}\right|\frac{1}{m+1}\sum_{n=0}^{m-\nu}|t_n|,$$

where $c_\nu = 0$, for $\nu > \kappa - j$.

Since the poles of $q(z)/p(z)$ are all outside the unit circle, $|\lambda_r| > 1$, for $r = 1,2,\ldots,\ell$; and $\sum_{\nu=0}^{\infty}\binom{\nu+\rho-1}{\rho-1}\lambda_r^{-\nu}$ is thus absolutely convergent for each $r = 1,2,\ldots,\ell$ and $\rho = 1,2,\ldots,m_r$. Consequently if $\frac{1}{m+1}\sum_{n=0}^{m}|t_n| \to 0$ as $m \to \infty$, then $\frac{1}{m+1}\sum_{n=0}^{m}|u_n| \to 0$ as $m \to \infty$.

If $q(z)/p(z)$ has no poles at all, then

$\frac{1}{m+1}\sum_{n=0}^{m}|u_n| \leq \sum_{\nu=0}^{m}|c_\nu|\frac{1}{m+1}\sum_{n=0}^{m-\nu}|t_n|$, where $c_\nu = 0$ for $\nu > k - j$.

Hence the desired conclusion follows. ///

THEOREM 6.11

If (1) $q(z)/p(z)$ *has a pole within the unit circle,*

or (2) $q(z)/p(z)$ *has no pole within the unit circle, but has poles of maximum order* m *on the unit circle, where* $m > 1$, *then there is a sequence which is* $[C_1,N_p]$*-convergent but not* $[C_1,N_q]$*-convergent.*

PROOF

(1) $q(z)/p(z)$ has a pole within the unit circle.

By Theorem 6.7 there is an N_p-convergent sequence which is not AN_q-convergent. Since (C,1) is regular, this sequence is $[C_1,N_p]$-convergent. But, since it is not AN_q-convergent, it is not $(C,1)N_q$-convergent. As a consequence of Proposition 6.1(ii) it is not $[C_1,N_q]$-convergent.

(2) $\frac{q(z)}{p(z)}$ has no poles within the unit circle, but has poles of maximum order m on the unit circle, where $m > 1$. By Theorem 6.6 since $m > 1$, there is an N_p-convergent sequence which is not $(C,1)N_q$-convergent. Consequently, this sequence is $[C_1,N_p]$-convergent, but, by Proposition 6.1(ii) it is not $[C_1,N_q]$-convergent. ///

For the next theorem we need the following two lemmas. We use the notation $[C,1]_1$ to mean $[C_1,I]_1$.

LEMMA 6.1 .

Let $t_n = a\lambda^n$, $|\lambda| = 1$, $\lambda \neq 1$ *and* a *is a non-zero complex number. Then* $\{t_n\}$ *is not* $[C,1]_1$*-convergent.*

PROOF .

We know that $\{t_n\}$ is (C,1)-convergent. For

$$\frac{1}{m+1}\sum_{n=0}^{m} t_n = \frac{1}{m+1}\sum_{n=0}^{m} a\lambda^n = \frac{a}{m+1}\,\frac{1-\lambda^{m+1}}{1-\lambda}.$$

Since a is a constant and $\frac{1-\lambda^{m+1}}{1-\lambda} = O(1)$, then $\frac{1}{m+1}\sum_{n=0}^{m} t_n \to 0$, as $m \to \infty$.

Thus if $\{t_n\}$ is $[C,1]_1$-convergent, its sum has to be zero. But

$$\frac{1}{m+1}\sum_{n=0}^{m}|t_n| = \frac{1}{m+1}\sum_{n=0}^{m}|a||\lambda^n| = |a|$$

which $\not\to 0$, since $a \neq 0$. ///

LEMMA 6.2.

Let $\lambda_1, \lambda_2, \ldots, \lambda_r$ be r distinct complex numbers, $r > 1$, with $|\lambda_\nu| = 1$, $\lambda_\nu \neq 1$ for $\nu = 1,2,\ldots r$, and let $a_1, a_2, \ldots, a_r$ be non-zero complex numbers. If $t_n = a_1\lambda_1^n + a_2\lambda_2^n + \cdots + a_r\lambda_r^n$, then $\{t_n\}$ is not $[C,1]_1$-convergent.

PROOF

If $\{t_n\}$ is $[C,1]_1$-convergent, its sum must be zero:

$$\frac{1}{m+1}\sum_{n=0}^{m}|t_n| = \frac{1}{m+1}\sum_{n=0}^{m}\left|a_1 + a_2\left(\frac{\lambda_2}{\lambda_1}\right)^n + \cdots + a_r\left(\frac{\lambda_r}{\lambda_1}\right)^n\right|.$$

If $t_n \to 0[C,1]_1$, then $\tau_n = a_1 + a_2\left(\frac{\lambda_2}{\lambda_1}\right)^n + \cdots + a_r\left(\frac{\lambda_r}{\lambda_1}\right)^n \to o(C,1)$. But $\tau_n \to a_1(C,1)$ and $a_1 \neq 0$. ///

THEOREM 6.12

If $\frac{q(z)}{p(z)}$ has no poles within the unit circle, but has simple poles on the unit circle and has no poles of higher order on the unit circle, then there is a sequence which is $[C_1,N_p]$-convergent but not $[C_1,N_q]$-convergent.

PROOF

Suppose $\frac{q(z)}{p(z)}$ has r poles of order 1, $\lambda_1, \lambda_2, \ldots, \lambda_r$, on the unit circle and $r \geq 1$, and suppose it has other poles, $\lambda_{r+1}, \ldots, \lambda_\ell$, outside the unit circle of order $m_{r+1}, \ldots, m_\ell$.

Since $p(1) = 1$, $z = 1$ cannot be a pole of $\frac{q(z)}{p(z)}$. i.e., $\lambda_\nu \neq 1$, for $\nu = 1, 2, \ldots r$.

Since $p(0) \neq 0$, $\frac{1}{p(z)}$ is analytic in a neighbourhood U of the origin. There is a sequence $\{s_n\}$ such that, for z in U,

$$\sum_{n=0}^{\infty} s_n z^n = \frac{1}{p(z)}.$$

Then, for z in U

$$\sum_{n=0}^{\infty} t_n z^n = p(z) \sum_{n=0}^{\infty} s_n z^n = 1,$$

$$\sum_{n=0}^{\infty} u_n z^n = q(z) \sum_{n=0}^{\infty} s_n z^n = \frac{q(z)}{p(z)} \sum_{n=0}^{\infty} t_n z^n .$$

Hence $t_0 = 1$, $t_n = 0$ for $n > 0$; and so $\{t_n\}$ is $[C,1]_1$-convergent to zero. That is $\{s_n\}$ is $[C_1, N_p]$-convergent to zero.

Now, by Theorem 6.9,

$$u_n = c_n + \sum_{\nu=r+1}^{\ell} \sum_{\rho=1}^{m_\nu} c_{\nu,\rho} \binom{n+\rho-1}{\rho-1} \lambda_\nu^{-n} + \sum_{\nu=1}^{r} c_{\nu,1} \lambda_\nu^{-n}$$

$$= u_n^1 + u_n^2,$$

where $u_n^1 = c_n + \sum_{\nu=r+1}^{\ell} \sum_{\rho=1}^{m_\nu} c_{\nu,\rho} \binom{n+\rho-1}{\rho-1} \lambda_\nu^{-n}$

and $u_n^2 = \sum_{\nu=1}^{r} c_{\nu,1} \lambda_\nu^{-n}$

Since $c_n = 0$ for $n > k - j$, and $|\lambda_\nu| > 1$ for $\nu = r+1, r+2, \ldots \ell$, $\{c_n\}$ and $\left\{c_{\nu,\rho} \binom{n+\rho-1}{\rho-1} \lambda_\nu^{-n}\right\}$ for $\nu = r+1, r+2, \ldots, \ell$, $\rho = 1, 2, \ldots, m_\nu$ are each convergent to zero. Since (C,1) is regular, u_n^1 is $[C,1]_1$-convergent to zero. But

$$u_n^2 = \sum_{\nu=1}^{r} c_{\nu,1}\lambda_\nu^{-n} = \sum_{\nu=1}^{r} c_{\nu,1}\frac{\overline{\lambda}_\nu^n}{\lambda_\nu^n\overline{\lambda}_\nu^n} = \sum_{\nu=1}^{r} c_{\nu,1}\overline{\lambda}_\nu^n ,$$

and $\overline{\lambda}_1, \overline{\lambda}_2, \ldots, \overline{\lambda}_r$ are distinct and distinct from 1. And $|\overline{\lambda}_\nu| = 1$, for $\nu = 1,2,\ldots, r$. Thus by Lemmas 6.1 and 6.2, we know that $\{u_n^2\}$ is not $[C,1]_1$-convergent for $r \geq 1$.

Consequently $\{u_n\}$ is not $[C,1]_1$-convergent, that is $\{s_n\}$ is not $[C_1,N_q]$-convergent. ///

THEOREM 6.13

$[C_1,N_p] \Rightarrow [C_1,N_q]$ *if and only if* $q(z)/p(z)$ *has no poles on or within the unit circle.*

PROOF

The sufficiency part follows from Theorem 6.10. The necessity part follows from Theorems 6.11 and 6.12. ///

THEOREM 6.14

$[C_1,N_p] \Rightarrow [C_1,N_q]$ *if and only if* $N_p \Rightarrow N_q$.

PROOF

This is a consequence of Theorems 6.13 and 6.5. ///

COROLLARY 6.1

If $[C_1,N_p] \Leftrightarrow [C_1,N_q]$, *then it is necessary and sufficient that both* $q(z)/p(z)$ *and* $p(z)/q(z)$ *have no poles on or within the unit circle.*

COROLLARY 6.2

$[C_1,N_p] \Leftrightarrow [C_1,N_q]$ *if and only if* $N_p \Leftrightarrow N_q$.

Noting that N_q is identical with I when $q(z) = 1$ (i.e., $q_0 = 1$, $q_n = 0$ for $n > 0$) and referring to Corollary 6.1 we obtain the following corollary.

COROLLARY 6.3

In order that $[C_1, N_p] \Leftrightarrow [C_1, I]_1$ *it is necessary and sufficient that* p(z) *should not have zeros on or within the unit circle.*

COROLLARY 6.4

$[C_1, N_p] \Leftrightarrow [C, 1]_1$ *if and only if* $N_p \Leftrightarrow I$.

For the following theorems and corollaries about the methods (N, p_n), (N, q_n), (N, P_n), (N, Q_n), $[N, P_n]$ and $[N, Q_n]$ we let p_ν for $\nu = 0, 1, \ldots, j$ and q_ν for $\nu = 0, 1, \ldots, k$ be the coefficients of the polynomials p(z) and q(z) respectively. We also let $P_r = \sum_{\nu=0}^{r} p_\nu \neq 0$ for $r = 0, 1, \ldots, j-1$ and

$Q_r = \sum_{\nu=0}^{r} q_\nu \neq 0$ for $r = 0, 1, \ldots, k-1$, and $P_n^* = \sum_{r=0}^{n} P_r \neq 0$ and

$Q_n^* = \sum_{r=0}^{n} Q_r \neq 0$ for all $n \geq 0$, so that (N, p_n), (N, q_n), (N, P_n), (N, Q_n), $[N, P_n]$ and $[N, Q_n]$ are methods associated with p(z) and q(z) respectively and are all well defined.

THEOREM 6.15

$(N, p_n) \Rightarrow (N, q_n)$ *implies that* $(N, P_n) \Rightarrow (N, Q_n)$.

PROOF

$$\text{Let } \tau_r = \frac{1}{P_r} \sum_{\nu=0}^{r} p_{r-\nu} s_\nu \text{ and } \mu_r = \frac{1}{Q_r} \sum_{\nu=0}^{r} q_{r-\nu} s_\nu ,$$

$$\text{and let } W_n = \frac{1}{P_n^*} \sum_{r=0}^{n} P_{n-r} s_r \text{ and } V_n = \frac{1}{Q_n^*} \sum_{r=0}^{n} Q_{n-r} s_r .$$

$$\text{Let } k(z) = \frac{q(z)}{p(z)} = \frac{Q(z)}{P(z)} \text{ and } k(z) = \sum_{\nu=0}^{\infty} k_\nu z^\nu .$$

We know that the necessary and sufficient conditions that

$$(N,p_n) \Rightarrow (N,q_n)$$

in this case are

$$|k_0| |P_n| + \cdots + |k_n| |P_0| \le H |Q_n| \tag{6.8}$$

where H is independent of n, and

$$k_{n-r}/Q_n \to 0, \quad \text{for each } r. \tag{6.9}$$

(c.f. [6, Proposition 1].)

Thus, if $(N,p_n) \Rightarrow (N,q_n)$, then (6.8) and (6.9) are satisfied.

$$\begin{aligned}
\text{Now} \quad \sum_{r=0}^{n} |k_{n-r}| \left| \sum_{\nu=0}^{r} P_{r-\nu} \right| &\le \sum_{r=0}^{n} |k_{n-r}| \sum_{\nu=0}^{r} |P_{r-\nu}| \\
&= \sum_{\nu=0}^{n} \sum_{r=\nu}^{n} |k_{n-r}| |P_{r-\nu}| \\
&\le H \sum_{\nu=0}^{n} |Q_{n-\nu}| \\
&= H \sum_{\nu=0}^{n} |Q_\nu| \\
&= O(|\sum_{\nu=0}^{n} Q_\nu|) \\
&= O(|Q_n^*|),
\end{aligned}$$

since $Q_\nu = 1$ for $\nu \ge k$.

And it is obvious that $k_{n-r}/Q_n^* \to 0$ as $n \to \infty$ for each r.

Thus, by [6, Proposition 1] again, we have

$$(N,P_n) \Rightarrow (N,Q_n). \qquad ///$$

COROLLARY 6.5

$(N,p_n) \Leftrightarrow (N,q_n)$ *implies that* $(N,P_n) \Leftrightarrow (N,Q_n)$.

THEOREM 6.16

$[C_1,N_p] \Rightarrow [C_1,N_q]$ *if and only if* $(N,P_n) \Rightarrow (N,Q_n)$.

PROOF

By Theorem 6.1 we know that $[C_1,N_p] \Leftrightarrow [N,P_n]$ and $[C_1,N_q] \Leftrightarrow [N,Q_n]$.

By [6, Theorem 1], (c.f. Lemma 5.4), we have that if $(N,P_n) \Rightarrow (N,Q_n)$ then $[N,P_n] \Rightarrow [N,Q_n]$.

Thus, if $(N,P_n) \Rightarrow (N,Q_n)$ then $[C_1,N_p] \Rightarrow [C_1,N_q]$.

Conversely, by Theorem 6.14, we have that if $[C_1,N_p] \Rightarrow [C_1,N_q]$, then $N_p \Rightarrow N_q$.

Hence, if $[C_1,N_p] \Rightarrow [C_1,N_q]$ then $(N,p_n) \Rightarrow (N,q_n)$.

It follows from Theorem 6.15 that if $[C_1,N_p] \Rightarrow [C_1,N_q]$ then $(N,P_n) \Rightarrow (N,Q_n)$. ///

COROLLARY 6.6

$[C_1,N_p] \Leftrightarrow [C_1,N_q]$ *if and only if* $(N,P_n) \Leftrightarrow (N,Q_n)$.

THEOREM 6.17

$[C_1,N_p] \Rightarrow [C_1,N_q]$ *if and only if* $(C,1)N_p \Rightarrow (C,1)N_q$.

PROOF

$$(N,P_n) = (\overline{N},P_n)(N,p_n).$$

From the proof of Theorem 6.1, we know that $(\bar{N},P_n) \Leftrightarrow (C,1)$. Thus $(N,P_n) \Leftrightarrow (C,1)(\bar{N},p_n) \Leftrightarrow (C,1)N_p$ and similarly we have $(N,Q_n) \Leftrightarrow (C,1)(N,q_n) \Leftrightarrow (C,1)N_q$.

It follows from Theorem 6.16 that $[C_1,N_p] \Rightarrow [C_1,N_q]$ if and only if $(C,1)N_p \Rightarrow (C,1)N_q$. ///

COROLLARY 6.7

$[C_1,N_p] \Leftrightarrow [C_1,N_q]$ if and only if $(C,1)N_p \Leftrightarrow (C,1)N_q$.

§6.5 ABSOLUTE POLYNOMIAL NÖRLUND METHODS OF SUMMABILITY

THEOREM 6.18

If $q(z)/p(z)$ has no poles on or within the unit circle, then $|N_p| \Rightarrow |N_q|$.

PROOF

Suppose $q(z)/p(z)$ has no poles on or within the unit circle, but has poles $\lambda_1,\lambda_2,\ldots,\lambda_\ell$ of orders $m_1,m_2,\ldots,m_\ell$ outside the unit circle. Let

$$t_n = \sum_{\nu=0}^{n} p_\nu s_{n-\nu}$$

$$u_n = \sum_{\nu=0}^{n} q_\nu s_{n-\nu}, \quad \text{for } n = 0,1,\ldots .$$

Then by Theorem 6.9

$$u_n = \sum_{\nu=0}^{n} c_\nu t_{n-\nu} + \sum_{r=1}^{\ell} \sum_{\rho=1}^{m_r} c_{r,\rho} \sum_{\nu=0}^{n} \binom{\nu+\rho-1}{\rho-1} \lambda_r^{-\nu} t_{n-\nu},$$

where c's are constants, depending only on $p_0,p_1,\ldots p_j$, $q_0,q_1,\ldots q_k$, such that $c_n = 0$ for $n > k - j$ and $c_{r,m_r} \neq 0$.

Hence

$$u_n - u_{n-1} = \sum_{\nu=0}^{n} c_\nu (t_{n-\nu} - t_{n-1-\nu}) + \sum_{r=1}^{\ell} \sum_{\rho=1}^{m_r} c_{r,\rho} \sum_{\nu=0}^{n} \binom{\nu+\rho-1}{\rho-1} \lambda_r^{-\nu} (t_{n-\nu} - t_{n-1-\nu}),$$

by taking $t_{-1} = 0$, $u_{-1} = 0$. And

$$\sum_{n=0}^{m} |u_n - u_{n-1}|$$

$$\leq \sum_{n=0}^{m} \sum_{\nu=0}^{n} |c_\nu| |t_{n-\nu} - t_{n-1-\nu}| + \sum_{n=0}^{m} \sum_{r=1}^{\ell} \sum_{\rho=1}^{m_r} |c_{r,\rho}| \sum_{\nu=0}^{n} \left| \binom{\nu+\rho-1}{\rho-1} \lambda_r^{-\nu} \right| |t_{n-\nu} - t_{n-1-\nu}|$$

$$\leq \sum_{\nu=0}^{m} |c_\nu| \sum_{n=0}^{m} |t_n - t_{n-1}| + \sum_{r=1}^{\ell} \sum_{\rho=1}^{m_r} |c_{r,\rho}| \sum_{\nu=0}^{m} \left| \binom{\nu+\rho-1}{\rho-1} \lambda_r^{-\nu} \right| \sum_{n=0}^{m} |t_n - t_{n-1}|.$$

Since $|\lambda_r| > 1$, for $r = 1, 2, \ldots, \ell$,

$\sum_{\nu=0}^{\infty} \left| \binom{\nu+\rho-1}{\rho-1} \lambda_r^{-\nu} \right|$ is convergent for $r = 1, 2, \ldots, \ell$, $\rho = 1, \ldots, m_r$.

$c_\nu = 0$, for $\nu > k - j$.

Thus we have $\sum_{n=0}^{m} |t_n - t_{n-1}| = O(1) \Rightarrow \sum_{n=0}^{m} |u_n - u_{n-1}| = O(1)$.

If $q(z)/p(z)$ has no poles at all, then it is readily seen that $\sum_{n=0}^{m} |u_n - u_{n-1}| \leq \sum_{\nu=0}^{m} |c_\nu| \sum_{n=0}^{m} |t_n - t_{n-1}|$. Since $c_\nu = 0$ for $\nu > k - j$, we have $\sum_{n=0}^{m} |t_n - t_{n-1}| = O(1)$ $\Rightarrow \sum_{n=0}^{m} |u_n - u_{n-1}| = O(1)$. By Theorem 6.3, N_p and N_q are consistent. Thus $|N_p| \Rightarrow |N_q|$. ///

PROPOSITION 6.2

$|N_p| \Rightarrow [C_1, N_p]$.

PROOF

$s_n \in |N_p| \Rightarrow t_n \to s$, as $n \to \infty$ for some s. Since

(C,1) is regular, $s_n \to s\ [C_1, N_p]$. ///

THEOREM 6.19

If $q(z)/p(z)$ *has a pole within the unit circle, then there is a sequence which is* $|N_p|$*-convergent but not* $|N_q|$*-convergent.*

PROOF

Since $p(0) \neq 0$, $\frac{1}{p(z)}$ is analytic in a neighbourhood U of origin. There is a sequence $\{s_n\}$ such that for z in U,

$$\sum_{n=0}^{\infty} s_n z^n = \frac{1}{p(z)} .$$

Let

$$t_n = \sum_{\nu=0}^{n} p_\nu s_{n-\nu} ,$$

$$u_n = \sum_{\nu=0}^{n} q_\nu s_{n-\nu} .$$

Then, for z in U,

$$\sum_{n=0}^{\infty} t_n z^n = p(z) \sum_{n=0}^{\infty} s_n z^n = 1,$$

$$\sum_{n=0}^{\infty} u_n z^n = q(z) \sum_{n=0}^{\infty} s_n z^n = \frac{q(z)}{p(z)} .$$

Hence $t_0 = 1$, $t_n = 0$ for $n > 0$, and so $\{s_n\}$ is $|N_p|$-convergent. On the other hand $\sum_{n=0}^{\infty} u_n z^n$ has a radius of convergence less than unity, because by hypothesis $q(z)/p(z)$ has a pole within the unit circle. Consequently $\{u_n\}$ is not A-convergent and so it is not (C,1)-convergent. Hence $\{s_n\}$ is not $(C,1)N_q$-convergent. By Proposition 6.1(ii), $\{s_n\}$

is not $[C_1,N_q]$-convergent. Thus by Proposition 6.2 $\{s_n\}$ is not $|N_q|$-convergent. ///

THEOREM 6.20

If $\frac{q(z)}{p(z)}$ *has no poles within the unit circle, but has poles on the unit circle, then there is a sequence which is* $|N_p|$*-convergent but not* $|N_q|$*-convergent.*

PROOF

Let the poles of $\frac{q(z)}{p(z)}$ be $\lambda_1,\lambda_2,\ldots, \lambda_\ell$ of orders $m_1,m_2,\ldots, m_\ell$. Let the numbering be such that of these poles $\lambda_1,\lambda_2,\ldots, \lambda_{\ell'}$ are on the unit circle, $\lambda_{\ell'+1},\ldots, \lambda_\ell$ are outside the unit circle.

Since $p(0) \neq 0$, $\frac{1}{(1-z)p(z)}$ is analytic in a neighbourhood U of origin. There is a sequence $\{s_n\}$ such that, for z in U,

$$\sum_{n=0}^{\infty} s_n z^n = \frac{1}{(1-z)p(z)}$$

Then, for z in U,

$$\sum_{n=0}^{\infty} t_n z^n = p(z) \sum_{n=0}^{\infty} s_n z^n = \frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots ,$$

$$\sum_{n=0}^{\infty} u_n z^n = q(z) \sum_{n=0}^{\infty} s_n z^n = \frac{q(z)}{p(z)} \sum_{n=0}^{\infty} t_n z^n .$$

Hence $t_n = 1$ for all $n \geq 0$ and so $\{s_n\}$ is $|N_p|$-convergent.

Now, by Theorem 6.9 since $t_n = 1$, for all $n \geq 0$,

$$u_n = \sum_{\nu=0}^{n} c_\nu + \sum_{r=1}^{\ell} \sum_{\rho=1}^{m_r} c_{r,\rho} \sum_{\nu=0}^{n} \binom{\nu+\rho-1}{\rho-1} \lambda_r^{-\nu},$$

where the c's are constants, depending only on $p_0, p_1, \ldots, p_j, q_0, q_1, \ldots, q_k$ such that $c_n = 0$ for $n > k - j$ and $c_{r,m_r} \neq 0$.

Thus

$$u_n - u_{n-1} = c_n + \sum_{r=1}^{\ell} \sum_{\rho=1}^{m_r} c_{r,\rho} \binom{n+\rho-1}{\rho-1} \lambda_r^{-n}$$

$$= c_n + \sum_{r=\ell'+1}^{\ell} \sum_{\rho=1}^{m_r} c_{r,\rho} \binom{n+\rho-1}{\rho-1} \lambda_r^{-n} + \sum_{r=1}^{\ell'} \sum_{\rho=1}^{m_r} c_{r,\rho} \binom{n+\rho-1}{\rho-1} \lambda_r^{-n}$$

$$= w_n^1 + w_n^2,$$

where $w_n^1 = c_n + \sum_{r=\ell'+1}^{\ell} \sum_{\rho=1}^{m_r} c_{r,\rho} \binom{n+\rho-1}{\rho-1} \lambda_r^{-n}$,

$$w_n^2 = \sum_{r=1}^{\ell'} \sum_{\rho=1}^{m_r} c_{r,\rho} \binom{n+\rho-1}{\rho-1} \lambda_r^{-n}.$$

Since $c_n = 0$ for $n > k - j$, $\sum_{n=0}^{\infty} c_n$ is absolutely convergent, and since $|\lambda_r| > 1$, for $r = \ell'+1, \ldots, \ell$, $\sum_{n=0}^{\infty} \binom{n+\rho-1}{\rho-1} \lambda_r^{-n}$ is absolutely convergent, for $r = \ell'+1, \ldots, \ell$, $\rho = 1, 2, \ldots, m_r$. Hence $\sum_{n=0}^{\infty} \{w_n^1\}$ is convergent.

Now, for w_n^2, if there are ℓ'' poles on the unit circle of maximum order m, where $1 \leq \ell'' \leq \ell'$ and $m \geq 1$, then we let the numbering be such that $\lambda_1, \lambda_2, \ldots, \lambda_{\ell''}$ have maximum order m. In this case,

$$|w_n^2| = \left|\sum_{r=1}^{\ell''}\sum_{\rho=1}^{m} c_{r,\rho}\binom{n+\rho-1}{\rho-1}\lambda_r^{-n} + \sum_{r=\ell''+1}^{\ell'}\sum_{\rho=1}^{m_r} c_{r,\rho}\binom{n+\rho-1}{\rho-1}\lambda_r^{-n}\right|$$

$$= O\left|c_{1,m}\binom{n+m-1}{m-1}\lambda_1^{-n} + c_{2,m}\binom{n+m-1}{m-1}\lambda_2^{-n} + \cdots + c_{\ell'',m}\binom{n+m-1}{m-1}\lambda_{\ell''}^{-n}\right|$$

$$= O\binom{n+m-1}{m-1}\left|c_{1,m}\lambda_1^{-n} + c_{2,m}\lambda_2^{-n} + \cdots + c_{\ell'',m}\lambda_{\ell''}^{-n}\right|$$

$$= O\binom{n+m-1}{m-1}\left|c_{1,m}\overline{\lambda}_1^{n} + c_{2,m}\overline{\lambda}_2^{n} + \cdots + c_{\ell'',m}\overline{\lambda}_{\ell''}^{n}\right| .$$

Since $p(1) = 1$, $z = 1$ is not a pole of $\frac{q(z)}{p(z)}$; and since $\overline{\lambda}_\nu$ are distinct and distinct from 1, and $|\overline{\lambda}_\nu| = 1$, for $\nu = 1,2,\ldots,\ell''$; and $c_{\nu,m} \neq 0$, for $\nu = 1,2,\ldots,\ell''$, by Lemma 6.1 and Lemma 6.2 we know that $\{c_{1,m}\overline{\lambda}_1^n + c_{2,m}\overline{\lambda}_2^n + \cdots + c_{\ell'',m}\overline{\lambda}_{\ell''}^n\}$ is not $[C,1]_1$-convergent for $\ell'' \geq 1$. Thus $\{|c_{1,m}\overline{\lambda}_1^n + c_{2,m}\overline{\lambda}_2^n + \cdots + c_{\ell'',m}\overline{\lambda}_{\ell''}^n|\}$ cannot be convergent. *A fortiori* it does not converge to zero. Hence $|w_n^2|$ does not tend to zero as $n \to \infty$. This means that $\sum_{n=0}^{\infty}|w_n^2|$ diverges. Consequently

$$\sum_{n=0}^{m}|u_n - u_{n-1}| \text{ diverges, as } n \to \infty.$$

In other words, $\{s_n\}$ is not $|N_q|$-convergent. ///

THEOREM 6.21

In order that $|N_p| \Rightarrow |N_q|$, *it is necessary and sufficient that* $\frac{q(z)}{p(z)}$ *should not have poles on or within the unit circle.*

PROOF

The sufficiency part of the theorem follows from Theorem 6.18. The necessity part follows from Theorems 6.19 and 6.20. ///

COROLLARY 6.8

$|N_p| \Rightarrow |N_q|$ if and only if $N_p \Rightarrow N_q$.

PROOF

This follows from Theorems 6.5 and 6.21. ///

COROLLARY 6.9

$|N_p| \Rightarrow |N_q|$ if and only if $[C_1, N_p] \Rightarrow [C_1, N_q]$.

PROOF

This follows from Theorems 6.13 and 6.21. ///

COROLLARY 6.10

$|N_p| \Leftrightarrow |N_q|$ if and only if $\frac{q(z)}{p(z)}$ and $\frac{p(z)}{q(z)}$ both have no poles on or within the unit circle.

COROLLARY 6.11

$|N_p| \Leftrightarrow |N_q|$ if and only if $N_p \Leftrightarrow N_q$.

COROLLARY 6.12

$|N_p| \Leftrightarrow |N_q|$ if and only if $[C_1, N_p] \Leftrightarrow [C_1, N_q]$.

Noting that N_q is identical with I when $q(z) = 1$, we have, as a consequence of Corollary 6.10, the following corollary.

COROLLARY 6.13

In order that $\{s_n\}$ is $|N_p|$ convergent if and only if

$\sum_{r=0}^{\infty} a_\nu$ *is absolutely convergent it is necessary and sufficient that* p(z) *should not have zeros on or within the unit circle.*

§6.6 SOME MINOR RESULTS

THEOREM 6.22

If $f(z) = p(z)q(z)$, *then*

(i) $[C_1,N_p] \Rightarrow [C_1,N_f]$ *and* $[C_1,N_q] \Rightarrow [C_1,N_f]$,

(ii) $|N_p| \Rightarrow |N_f|$ *and* $|N_q| \Rightarrow |N_f|$.

PROOF

(i) follows from Theorem 6.13 and (ii) follows from Theorem 6.21. ///

COROLLARY 6.14

The methods $[C_1,N_p]$ *and* $[C_1,N_q]$ *are consistent, i.e., if* $s_n \to s\ [C_1,N_p]$ *and* $s_n \to s'\ [C_1,N_q]$, *then* $s = s'$.

THEOREM 6.23

If $h(z)$ *is the highest common factor of* $p(z)$ *and* $q(z)$ *normalized so as to make* $h(1) = 1$, *then*

(i) *a sequence is both* $[C_1,N_p]$- *and* $[C_1,N_q]$-*convergent if and only if it is* $[C_1,N_h]$-*convergent,*

(ii) *a sequence is both* $|N_p|$- *and* $|N_q|$-*convergent if and only if it is* $|N_h|$-*convergent.*

PROOF

(i) The sufficiency part follows from Theorem 6.22 (i). To prove the necessity part, we observe that there are polynomials

$$a(z) = \sum_{n=0}^{\ell_1} a_n z^n$$

$$b(z) = \sum_{n=0}^{\ell_2} b_n z^n$$

such that $h(z) = a(z)p(z) + b(z)q(z)$

$$= \sum_{n=0}^{\ell_3} h_n z^n, \text{ say,}$$

where ℓ_1, ℓ_2, ℓ_3 are non-negative integers.

Hence if $t_n = \sum_{\nu=0}^{n} p_\nu s_{n-\nu}$ and $u_n = \sum_{\nu=0}^{n} q_\nu s_{n-\nu}$, then

$$w_n = \sum_{\nu=0}^{n} h_\nu s_{n-\nu} = \sum_{\nu=0}^{n} a_\nu t_{n-\nu} + \sum_{\nu=0}^{n} b_\nu u_{n-\nu},$$

where $a_\nu = 0$, for $\nu > \ell_1$ and $b_\nu = 0$, for $\nu > \ell_2$.

Without loss of generality, we may assume $s_n \to o[C_1, N_p]$ and $s_n \to o[C_1, N_q]$. Now

$$\frac{1}{m+1} \sum_{n=0}^{m} |w_n| \leq \frac{1}{m+1} \sum_{n=0}^{m} \sum_{\nu=0}^{n} |a_\nu| |t_{n-\nu}| + \frac{1}{m+1} \sum_{n=0}^{m} \sum_{\nu=0}^{n} |b_\nu| |u_{n-\nu}|$$

$$\leq \frac{1}{m+1} \sum_{n=0}^{m} \left(\sum_{\nu=0}^{\ell_1} |a_\nu| \right) |t_n| + \frac{1}{m+1} \sum_{n=0}^{m} \left(\sum_{\nu=0}^{\ell_2} |b_\nu| \right) |u_n|$$

$$= o(1), \quad \text{as } m \to \infty.$$

That is $s_n \to o[C_1, N_h]$.

(ii) The sufficiency part follows from Theorem 6.22(ii). As in the proof of (i),

$$w_n = \sum_{\nu=0}^{n} h_\nu s_{n-\nu} = \sum_{\nu=0}^{n} a_{n-\nu} t_\nu + \sum_{\nu=0}^{n} b_{n-\nu} u_\nu.$$

Hence $w_n - w_{n-1}$

$$= \sum_{\nu=0}^{n} a_{n-\nu}(t_\nu - t_{\nu-1}) + \sum_{\nu=0}^{n} b_{n-\nu}(u_\nu - u_{\nu-1}),$$

where $t_{-1} = 0$, $u_{-1} = 0$, $w_{-1} = 0$ and $a_{n-\nu} = 0$ if $n - \nu > \ell_1$, $b_{n-\nu} = 0$, if $n - \nu > \ell_2$.

$$\sum_{n=0}^{m} |w_n - w_{n-1}|$$

$$\leq \sum_{n=0}^{m} \sum_{\nu=0}^{n} |a_{n-\nu}||t_\nu - t_{\nu-1}| + \sum_{n=0}^{m} \sum_{\nu=0}^{n} |b_{n-\nu}||u_\nu - u_{\nu-1}|$$

$$\leq \left(\sum_{\nu=0}^{\ell_1} |a_\nu|\right) \sum_{n=0}^{m} |t_n - t_{n-1}| + \left(\sum_{\nu=0}^{\ell_2} |b_\nu|\right) \sum_{n=0}^{m} |u_n - u_{n-1}|.$$

Hence if $\sum_{n=0}^{m} |t_n - t_{n-1}| = O(1)$ and $\sum_{n=0}^{m} |u_n - u_{n-1}| = O(1)$,

then $\sum_{n=0}^{m} |w_n - w_{n-1}| = O(1)$. ///

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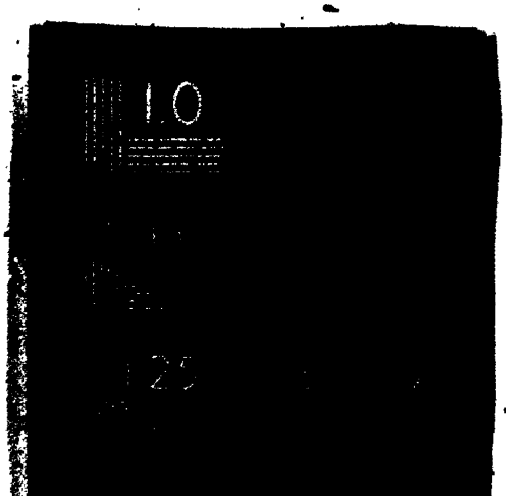

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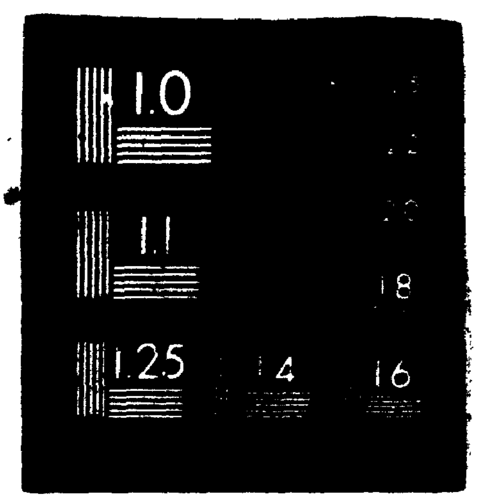

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ASPECTS OF STRONG SUMMABILITY ASSOCIATED WITH GENERALISED CESARO, RIESZ AND NÖRLUND SUMMABILITY

by

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Department of Mathematics

Submitted in partial fulfillment
of the requirements for the degree of
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ABSTRACT

Generalised Cesàro Summability, Riesz Summability and Strong Riesz Summability have been extensively investigated by various authors. In this thesis a definition of Strong Generalised Cesàro Summability Method is proposed and the question of its equivalence with the Strong Riesz Summability Method is established. In Chapter 3 some equivalence theorems between the Generalised Cesàro Methods and the Strong Generalised Cesàro Method are obtained. In Chapter 4 inclusion theorems between the Absolute Generalised Cesàro Methods and the Strong Generalised Cesàro Method are obtained.

We extend a result due to Kuttner, obtaining some strict inclusion theorems between Cesàro and Discrete Riesz Methods of Summability. And our investigation in this respect stems from Borwein and Cass's work on Strong Nörlund Summability.

In Chapter 6 we consider Nörlund Methods of Summability Associated with Polynomials which have been investigated by Borwein, and consider Strong and Absolute Nörlund Methods associated with them. We show, for example, that two polynomial Nörlund Methods are equivalent if and only if the associated Strong Methods are equivalent.

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TABLE OF CONTENTS

CONVENTIONS

In this thesis, the symbols H, H_1, H_2, H_3 are used throughout to denote positive constants, but not necessarily having the same value at each occurrence.

The theorems, lemmata and corollaries are numbered by chapter. For example, Theorem 3.1 is the first theorem in Chapter 3.

At the end of each proof we use the symbol /// to show that the proof is complete.

CHAPTER 1

STRONG GENERALISED CESÀRO SUMMABILITY

§1.1 INTRODUCTION

We suppose throughout the thesis that $\lambda = \{\lambda_n\}$ is a sequence satisfying

$$(1.1) \qquad 0 = \lambda_0 < \lambda_1 < \cdots\cdots < \lambda_n \to \infty .$$

For the sake of convenience we take $\lambda_0 = 0$ in (1.1) instead of $\lambda_0 \geqq 0$. By doing so we find that there is no loss of generality. This remark will be amplified on page 5.

We suppose also that p is a non-negative integer and for the series $\sum_{\nu=0}^{\infty} a_\nu$ we use the notation

$$s_n = \sum_{\nu=0}^{n} a_\nu, \quad n = 0,1,2,\ldots\ldots .$$

In this chapter we introduce a definition of *Strong Generalised Cesàro Summability* and investigate some of its properties. We also give the definitions of several other summability methods whose properties and relations with the Strong Generalised Cesàro Summability are investigated in the later chapters.

If a given summability method T assigns the *sum* s to the series $\sum_{\nu=0}^{\infty} a_\nu$ with sequence of partial sums $\{s_n\}$, we say that $\sum_{\nu=0}^{\infty} a_\nu$ is T-*summable* or $\{s_n\}$ is T-*convergent* to s. We denote this by

$$\sum_{\nu=0}^{\infty} a_\nu = s \ (T)$$

or by

$$s_n \to s \ (T).$$

A method of summability T is said to be *regular*, if $s_n \to s(T)$ whenever the sequence $\{s_n\}$ converges to s.

Let $Q = \{q_{n,r}\}$ $(n, r = 0,1,2,\ldots)$ be a (summability) matrix and let

(1.2) $$\sigma_n = \sum_{r=0}^{\infty} q_{n,r} \, s_r.$$

The sequence $\{s_n\}$ is said to be Q-*convergent* to the sum s if σ_n exists for $n = 0,1,2,\ldots$ and tends to s as n tends to infinity.

The matrix $Q = \{q_{n,r}\}$ is *regular* if and only if

(1.3) $$\sup_{n\geq 0} \sum_{r=0}^{\infty} |q_{n,r}| < \infty,$$

(1.4) $$\lim_{n\to\infty} q_{n,r} = 0, \text{ for } r = 0,1,2,\ldots ,$$

(1.5) $$\lim_{n\to\infty} \sum_{r=0}^{\infty} q_{n,r} = 1.$$

This is the Toeplitz Theorem for the regularity of the matrix Q.

The symbol P will be reserved for matrices $\{p_{n,r}\}$ with

$$p_{n,r} \geqq 0 \qquad (n,r = 0,1,2,\ldots).$$

Such matrices will be called non-negative matrices.

Let $\mu > 0$. The *Strong Summability Methods* $[P,Q]_\mu$ are defined as follows. We write $s_n \to s\ [P,Q]_\mu$ if

(1.6) $$\tau_n = \sum_{\nu=0}^{\infty} p_{n,r}|\sigma_r - s|^\mu$$

exists for $n = 0,1,2,\ldots$ and tends to zero as n tends to infinity. Thus s is the $[P,Q]_\mu$-*limit* of $\{s_n\}$ and the sequence is $[P,Q]_\mu$-*convergent* to s.

If V and W are summability methods of any of the above types we shall say that W *includes* V, and use the notation V => W, if any sequence V-convergent to s is necessarily W-convergent to s. If W includes V but V does not include W, the inclusion V => W is said to be *strict*. If both V => W and W => V, we say that V and W are *equivalent* and write V <=> W.

Let $\mu > 0$. We say that $\{s_n\}$ is *absolutely* $(Q)_\mu$-*convergent* or $|Q|_\mu$-*convergent* if

(1.7) $$\sum_{n=1}^{\infty} n^{\mu-1}|\sigma_n - \sigma_{n-1}|^{\mu} < \infty.$$

§1.3 RIESZ SUMMABILITY (R,λ,κ)

Let $\kappa \geq 0$ and $\lambda = \{\lambda_n\}$ satisfy (1.1). The Riesz Summability Method (R,λ,κ) is defined as follows.

Let $A_\lambda^\kappa(\tau) = \sum_{\lambda_\nu<\tau} a_\nu$, for $\kappa = 0$,

$A_\lambda^\kappa(\tau) = \sum_{\lambda_\nu<\tau} (\tau - \lambda_\nu)^\kappa a_\nu$, for $\kappa > 0$,

$R_\lambda^\kappa(\tau) = A_\lambda^\kappa(\tau) = \sum_{\lambda_\nu<\tau} a_\nu$, for $\kappa = 0$,

and $R_\lambda^\kappa(\tau) = \sum_{\lambda_\nu<\tau} (1 - \frac{\lambda_\nu}{\tau})^\kappa a_\nu$, for $\kappa > 0$.

The series $\sum_{\nu=0}^{\infty} a_\nu$ is said to be (R,λ,κ)-summable to s, if

$$R_\lambda^\kappa(\tau) \to s \quad \text{as } \tau \to \infty.$$

(See Hardy and Riesz [12, pp. 21-22].)

§1.4 STRONG RIESZ SUMMABILITY $[R,\lambda,p+1]_\mu$

The series $\sum_{\nu=0}^{\infty} a_\nu$ is said to be Strongly Riesz Summable to s, with order p+1 and index $\mu>0$, if

$$F^{p+1}(\omega) = \int_0^\omega |A_\lambda^p(\tau) - s\tau^p|^\mu \, d\tau = o(\omega^{p\mu+1}).$$

We denote this by

$$\sum_{\nu=0}^{\infty} a_\nu = s \ [R,\lambda,p+1]_\mu.$$

The definition of the Strong Riesz Summability we have given here is due to Glatfeld [15]. Srivastava [24] and Boyd and Hyslop [8] have also given definitions of Strong Riesz Summability, but we shall not be concerned with them here.

We give now two examples to illustrate that no loss of generality is involved by taking $\lambda_0 = 0$ in (1.1).

Our first example deals with Riesz Summability.

Let $\lambda = \{\lambda_n\}$ satisfy

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n \to \infty$$

and let $\delta = \{\delta_n\}$ satisfy

$$\lambda_1 > \delta_0 > 0 \text{ and } \delta_n = \lambda_n \text{ for } n \neq 0.$$

Let $R^{\kappa}_{\lambda}(\tau)$ be defined as in §1.3 and let

$$R^{\kappa}_{\delta}(\tau) = \sum_{\delta_\nu < \tau} \left(1 - \frac{\delta_\nu}{\tau}\right)^{\kappa} a_\nu .$$

Then

$$R^{\kappa}_{\delta}(\tau) - s = R^{\kappa}_{\lambda}(\tau) - s + R^{\kappa}_{\delta}(\tau) - R^{\kappa}_{\lambda}(\tau)$$

$$= R^{\kappa}_{\lambda}(\tau) - s + \left[\left(1 - \frac{\delta_0}{\tau}\right)^{\kappa} a_0 - a_0\right].$$

Since $\left(1 - \frac{\delta_0}{\tau}\right)^{\kappa} a_0 - a_0 \to 0$ as $\tau \to \infty$, $R^{\kappa}_{\delta}(\tau) \to s$ if and only if $R^{\kappa}_{\lambda}(\tau) \to s$, as $\tau \to \infty$.

Our other example deals with Strong Riesz Summability. Let $A^{p}_{\lambda}(\tau)$ be defined as in §1.3 and let

$$A^{p}_{\delta}(\tau) = \sum_{\delta_\nu < \tau} (\tau - \delta_\nu)^{p} a_\nu .$$

Then $I_1 = \int_0^{\omega} |A_{\delta}^{p}(\tau) - s\tau^p|^{\mu} \, d\tau$

$$= \int_0^{\omega} |A_{\lambda}^{p}(\tau) - s\tau^p + A_{\delta}^{p}(\tau) - A_{\lambda}^{p}(\tau)|^{\mu} \, d\tau$$

$$\leq 2^{\mu}\left(\int_0^{\omega} |A_{\lambda}^{p}(\tau) - s\tau^p|^{\mu} \, d\tau + \int_0^{\omega} |a_0(\tau - \delta_0)^p - a_0\tau^p|^{\mu} \, d\tau\right)$$

$$= 2^{\mu}(I_2 + I_3).$$

Regarding a_0 as the series $\sum_{\nu=0}^{\infty} b_{\nu}$ with $b_0 = a_0$ and $b_{\nu} = 0$ for $\nu > 0$, we have $(\tau - \delta_0)^p a_0 = \sum_{\delta_{\nu}<\tau} (\tau - \delta_{\nu})^p b_{\nu}$. Since $\sum_{\nu=0}^{\infty} b_{\nu} = a_0$ and $[R,\lambda,p+1]_{\mu}$ is regular, (see Glatfeld [15]), thus $I_3 = o(\omega^{p\mu+1})$. Hence $I_2 = o(\omega^{p\mu+1}) \Rightarrow I_1 = o(\omega^{p\mu+1})$. Similarly $I_1 = o(\omega^{p\mu+1}) \Rightarrow I_2 \Rightarrow o(\omega^{p\mu+1})$.

§1.5 GENERALISED CESÀRO SUMMABILITY (C,λ,p)

Let $\lambda = \{\lambda_n\}$ satisfy (1.1).

Define $C_n^p = \sum_{\nu=0}^{n} a_{\nu}$, for $p = 0$,

$$C_n^p = \sum_{\nu=0}^{n} (\lambda_{n+1} - \lambda_{\nu}) \cdots (\lambda_{n+p} - \lambda_{\nu}) a_{\nu}, \text{ for } p = 1,2,3,\ldots,$$

$$t_n^p = C_n^p = \sum_{\nu=0}^{n} a_{\nu}, \text{ for } p = 0,$$

$$t_n^p = (\lambda_{n+1} \cdots \lambda_{n+p})^{-1} C_n^p$$

$$= \sum_{\nu=0}^{n} \left(1 - \frac{\lambda_{\nu}}{\lambda_{n+1}}\right) \cdots \left(1 - \frac{\lambda_{\nu}}{\lambda_{n+p}}\right) a_{\nu}, \text{ for } p = 1,2,3,\ldots .$$

If $t_n^p \to s$ as $n \to \infty$, then $\sum_{\nu=0}^{\infty} a_\nu$ is said to be (C,λ,p) summable to s and we write

$$\sum_{\nu=0}^{\infty} a_\nu = s \ (C,\lambda,p).$$

Since (C,λ,p) is a matrix method in the sense described in §1.2, we shall find it convenient to denote both the summability method and its associated matrix by (C,λ,p). Since the entries in the matrix (C,λ,p) are zero above the main diagonal and non-zero on the main diagonal, it has an inverse.

§1.6 STRONG GENERALISED CESÀRO SUMMABILITY $[C,\lambda,p+1]_\mu$

Let $\lambda = \{\lambda_n\}$ satisfy (1.1). We define

$$E_n^p(\lambda) = E_n^p = 1, \text{ for } p = 0,$$

$$E_n^p(\lambda) = E_n^p = \lambda_{n+1} \cdots \lambda_{n+p}, \text{ for } p = 1,2,3,\ldots,$$

and $n = 0,1,2,3,\ldots$.

Since $\lambda_0 = 0$, we obtain

$$E_m^{p+1} = \sum_{n=0}^{m} (\lambda_{n+p+1} - \lambda_n) E_n^p.$$

We define

$$T_{m,\mu}^1 = \sum_{n=0}^{m} (\lambda_{n+1} - \lambda_n) |t_n^0 - s|^\mu,$$

$$T_{m,\mu}^{p+1} = \sum_{n=0}^{m} (\lambda_{n+p+1} - \lambda_n) E_n^p |t_n^p - s|^\mu,$$

$$\sigma_{m,\mu}^1 = \lambda_{m+1}^{-1} \sum_{n=0}^{m} (\lambda_{n+1} - \lambda_n) |t_n^0 - s|^\mu,$$

$$\sigma^{p+1}_{m,\mu} = \frac{T^{p+1}_{m,\mu}}{E^{p+1}_m} = \frac{1}{E^{p+1}_m} \sum_{n=0}^{m} (\lambda_{n+p+1} - \lambda_n) E^p_n |t^p_n - s|^\mu .$$

We say that the series $\sum_{\nu=0}^{\infty} a_\nu$ is *Strongly Generalised Cesàro Summable* to s, with order p+1 and index μ, if

$$\sigma^{p+1}_{m,\mu} = o(1) \quad \text{as } m \to \infty.$$

And we use the notation

$$\sum_{\nu=0}^{\infty} a_\nu = s \; [C,\lambda,p+1]_\mu .$$

Generalised Cesàro Summability was first introduced by Jurkat, [16]. Burkill, [9], gave a different definition. The definition we use here is due to Burkill. The definition was extended to accommodate positive non-integral values of p by Borwein, [3]. We have not been able to formulate a suitable definition of $[C,\lambda,p+1]_\mu$ with p non-integral.

Several persons have investigated relations between Riesz and Generalised Cesàro Summability. In particular, it is proved in Russell [23] that if λ is a sequence satisfying (1.1) and p is a non-negative integer then

$$(C,\lambda,p) \Rightarrow (R,\lambda,p), \quad p = 0,1,2,3,\ldots .$$

It is proved in Meir [20] that if λ is a sequence satisfying (1.1) and p is a non-negative integer then

$$(R,\lambda,p) \Rightarrow (C,\lambda,p), \quad p = 0,1,2,3,\ldots .$$

If in §1.5 we take $\lambda_n = n$, we recover the classical Cesàro Summability Method (C,p). (See Hardy [11].)

If in §1.6 we take $\lambda_n = n$, we obtain a summability method which although not equal to, is nevertheless equivalent to the classical Strong Cesàro Summability Method $[C, p+1]_\mu$. (See Borwein and Cass [6].)

We recall that $\sum_{\nu=0}^{\infty} a_\nu = s \; [C,p+1]_\mu$ if and only if

$$\frac{1}{n+1} \sum_{\nu=0}^{n} |s_\nu^p - s|^\mu = o(1),$$

where $s_n^p = \frac{1}{\varepsilon_n^p} \sum_{\nu=0}^{n} \varepsilon_{n-\nu}^{p-1} s_\nu$ and $\varepsilon_n^p = \binom{n+p}{n}$.

In case where no confusion can arise, we omit the subscript μ from $\sigma_{m,\mu}^{p+1}$ and $T_{m,\mu}^{p+1}$.

§1.7 SIMPLE INCLUSION THEOREMS

In order to simplify the notation and the proofs of theorems occurring later we introduce a matrix

$$\Lambda_{p+1} = \{\lambda_{m,n}^{p+1}\} = \{\lambda_{m,n}\}$$

which is defined as follows.

(1.8) For $p = 0$

$$\lambda_{m,n} = \begin{cases} \frac{1}{E_m^1} (\lambda_{n+1} - \lambda_n) = \frac{1}{\lambda_{m+1}} (\lambda_{n+1} - \lambda_n), & \text{for } 0 \le n \le m, \\ 0, & \text{for } n > m; \end{cases}$$

and for $p > 0$

$$\lambda_{m,n} = \begin{cases} \frac{1}{E_m^{p+1}} (\lambda_{n+p+1} - \lambda_n) E_n^p, & \text{for } 0 \le n \le m, \\ 0, & \text{for } n > m. \end{cases}$$

It follows easily from the Toeplitz conditions (1.3), (1.4), (1.5) that Λ_{p+1} is regular.

We now establish some results pertaining to the Strong Generalised Cesàro Summability.

Let C_n^p and t_n^p be defined as in §1.5. Then

$$C_n^{p+1} - C_{n-1}^{p+1} = (\lambda_{n+p+1} - \lambda_n) C_n^p$$

so that

$$C_n^{p+1} = \sum_{\nu=0}^{n} (\lambda_{\nu+p+1} - \lambda_\nu) C_\nu^p. \tag{1.9}$$

(See [23, p. 419].)

Hence

$$\begin{aligned} \frac{1}{E_m^{p+1}} \sum_{n=0}^{m} (\lambda_{n+p+1} - \lambda_n) E_n^p t_n^p &= \frac{1}{E_m^{p+1}} \sum_{n=0}^{m} (\lambda_{n+p+1} - \lambda_n) C_n^p \\ &= C_m^{p+1} / E_m^{p+1} \\ &= t_m^{p+1}. \end{aligned} \tag{1.10}$$

This means, in matrix notation,

$$(C,\lambda,p+1) = \Lambda_{p+1} (C,\lambda,p). \tag{1.11}$$

Moreover, referring to (1.6), the definitions of $[C,\lambda,p+1]_\mu$ and Λ_{p+1}, we have

$$[C,\lambda,p+1]_\mu = [\Lambda_{p+1}, (C,\lambda,p)]_\mu. \tag{1.12}$$

The following two theorems are given in Borwein, [1, Theorems 1 and 3]. We reproduce the proofs for the sake of completeness.

THEOREM 1.1

If Q *is any matrix and* $P = \{p_{n,r}\}$, *where* $p_{n,r} \geq 0$ *for* $n,r = 0,1,\ldots,$ $\sum_{r=0}^{\infty} p_{n,r} < M$ *for* $n = 0,1,\ldots$ *and if* $\mu_1 > \mu_2 > 0$ *then* $[P,Q]_{\mu_1} \Rightarrow [P,Q]_{\mu_2}$. *In particular, the conclusion holds if* $\mu_1 > \mu_2 > 0$ *and* P *is regular.*

PROOF

By Hölder's inequality

$$\sum_{r=0}^{\infty} p_{n,r}|w_r|^{\mu_2} \leq \left(\sum_{r=0}^{\infty} p_{n,r}|w_r|^{\mu_1}\right)^{\mu_2/\mu_1} M^{1-\mu_2/\mu_1}.$$

for any sequence $\{w_n\}$. The required conclusion follows. ///

THEOREM 1.2

If P *is a regular (non-negative) matrix and* Q *is any matrix, then*

(i) $Q \Rightarrow [P,Q]_\mu$, *for* $\mu > 0$,

(ii) $[P,Q]_\mu \Rightarrow PQ$, *for* $\mu \geq 1$.

PROOF

(i) If $s_n \to s$, then, since P is regular

$\sum_{r=0}^{n} p_{n,r}|s_r - s|^\mu = o(1)$, i.e., $I \Rightarrow [P,I]_\mu$ and inclusion (i) follows.

(ii) Suppose that $s_n \to s$ $[P,I]_\mu$. Then by Theorem 1.1, $s_n \to s$ $[P,I]_1$ and so

$$\left|\sum_{r=0}^{n} p_{n,r}(s_r - s)\right| \leq \sum_{r=0}^{n} p_{n,r}|s_r - s| = o(1)$$

Since P is regular, it follows that $s_n \to s$ (P). Hence $[P,I]_\mu \Rightarrow P$ and inclusion (ii) is an immediate consequence. ///

COROLLARY 1.1

If $\mu_1 > \mu_2 > 0$, *then* $[C,\lambda,p+1]_{\mu_1} \Rightarrow [C,\lambda,p+1]_{\mu_2}$.

PROOF

By (1.12), we know that $[C,\lambda,p+1]_\mu = [\Lambda_{p+1}, (C,\lambda,p)]_\mu$. The inclusion is a consequence of Theorem 1.1 and the fact that Λ_{p+1} is a regular and non-negative matrix. ///

COROLLARY 1.2

If $\mu > 0$, *then* $(C,\lambda,p) \Rightarrow [C,\lambda,p+1]_\mu$.

PROOF

Since $[C,\lambda,p+1]_\mu = [\Lambda_{p+1}, (C,\lambda,p)]_\mu$ and Λ_{p+1} is regular and non-negative. The corollary is an immediate consequence of Theorem 1.2 (i). ///

COROLLARY 1.3

If $\mu \geq 1$, *then* $[C,\lambda,p+1]_\mu \Rightarrow (C,\lambda,p+1)$.

PROOF

By (1.11), we know that $(C,\lambda,p+1) = \Lambda_{p+1}(C,\lambda,p)$. The corollary is a consequence of Theorem 1.2 (ii). ///

COROLLARY 1.4

Suppose $\mu_1 \geq 1$ *and* $\mu_2 > 0$. *Then*

$$[C,\lambda,p+1]_{\mu_1} \Rightarrow [C,\lambda,p+2]_{\mu_2}.$$

PROOF

This is a consequence of Corollary 1.3 and Corollary 1.2. ///

We mention two other properties of $[C,\lambda,p+1]_\mu$ here.

(1.13) *If* $\sum_{\nu=0}^{\infty} a_\nu = s[C,\lambda,p+1]_\mu$ *and* $\sum_{\nu=0}^{\infty} a_\nu = s'[C,\lambda,p+1]_\mu$ *then* $s = s'$.

(1.14) *If* $\mu > 0$, *then*

$$\sum_{\nu=0}^{\infty} a_\nu = a\ [C,\lambda,p+1]_\mu$$

and

$$\sum_{\nu=0}^{\infty} b_\nu = b\ [C,\lambda,p+1]_\mu$$

implies

$$\sum_{\nu=0}^{\infty} c_\nu = \sum_{\nu=0}^{\infty} (\alpha a_\nu + \beta b_\nu) = \alpha a + \beta b\ [C,\lambda,p+1]_\mu .$$

CHAPTER 2

EQUIVALENCE BETWEEN STRONG GENERALISED CESÀRO SUMMABILITY AND STRONG RIESZ SUMMABILITY

In this chapter we shall establish the equivalence between $[C,\lambda,p+1]_\mu$ and $[R,\lambda,p+1]_\mu$. We first prove a lemma. (Cf. Glatfeld [15].)

§2.1 A LEMMA

LEMMA 2.1

If $\chi(\tau) \geq 0$, *continuous and Riemann integrable in* $[h,\omega]$, *where* h *is any fixed positive real number and if* $\alpha+\delta > 0$ *and* $\delta > 0$, *then*

$$\int_h^\omega \chi(\tau)\,d\tau = o(\omega^\delta)$$

if and only if

$$\int_h^\omega \tau^\alpha \chi(\tau)\,d\tau = o(\omega^{\alpha+\delta}).$$

PROOF

Assume $\int_h^\omega \chi(\tau)\,d\tau = o(\omega^\delta)$ and let $F(\omega) = \int_h^\omega \chi(\tau)\,d\tau$.

Then integrating by parts

$$\int_h^\omega \tau^\alpha \chi(\tau)\,d\tau = [\tau^\alpha F(\tau)]_h^\omega - \alpha\int_h^\omega \tau^{\alpha-1}F(\tau)\,d\tau$$

$$= \omega^\alpha F(\omega) - \alpha\int_h^\omega \tau^{\alpha+\delta-1}\,\frac{F(\tau)}{\tau^\delta}\,d\tau$$

$$= U - V,$$

and $U = o(\omega^{\alpha+\delta})$ by hypothesis.

Further

$$\frac{1}{\omega^{\alpha+\delta}}\int_h^\omega \tau^{\alpha+\delta-1}\,\frac{F(\tau)}{\tau^\delta}\,d\tau$$

$$= \int_h^\omega K(\omega,\tau)\;G(\tau)\;d\tau,$$

where
$$K(\omega,\tau) = \begin{cases} \dfrac{\tau^{\alpha+\delta-1}}{\omega^{\alpha+\delta}}, & 0 < \tau \leq \omega, \\ 0, & \tau > \omega, \end{cases}$$

and
$$G(\tau) = \frac{F(\tau)}{\tau^\delta}.$$

Now

$$\int_h^\infty |K(\omega,\tau)|\,d\tau$$

$$= \frac{\omega^{\alpha+\delta} - h^{\alpha+\delta}}{(\alpha+\delta)\,\omega^{\alpha+\delta}}$$

$$= \frac{1}{\alpha+\delta}\left(1 - \frac{h^{\alpha+\delta}}{\omega^{\alpha+\delta}}\right)$$

$$< \frac{1}{\alpha+\delta}.$$

For every positive y

$$\lim_{\omega\to\infty}\int_h^y K(\omega,\tau)\,d\tau$$

$$= \lim_{\omega\to\infty}\frac{y^{\alpha+\delta} - h^{\alpha+\delta}}{(\alpha+\delta)\,\omega^{\alpha+\delta}}$$

$$= 0.$$

Since $G(\tau) \to 0$ as $\tau \to \infty$, it follows from Hardy [11, Theorem 6] that

$$\int_h^\omega K(\omega,\tau)G(\tau)\,d\tau \to 0, \quad \text{as } \omega \to \infty.$$

Thus $$V = o(\omega^{\alpha+\delta}).$$

Hence $$\int_h^\omega \tau^\alpha \chi(\tau)\,d\tau = o(\omega^{\alpha+\delta}).$$

Conversely, if $\int_h^\omega \tau^\alpha \chi(\tau)\,d\tau = o(\omega^{\alpha+\delta})$, we take $\tau^\alpha \chi(\tau) = X(\tau)$ which is non-negative, continuous and integrable in $[h,\omega]$. The result now follows from the first part by replacing δ by $\alpha+\delta$ and α by $-\alpha$. ///

Since $\lambda_0 = 0$, $R^p_\lambda(\tau) \to a_0$ as $\tau \to 0^+$, we conclude that as a consequence of Lemma 2.1

$$(2.1) \qquad \int_0^\omega |A^p_\lambda(\tau) - s\tau^p|^\mu d\tau = o(\omega^{p\mu+1})$$

is equivalent to

$$\int_0^\omega |R^p_\lambda(\tau) - s|^\mu \, d\tau = o(\omega).$$

§2.2 INCLUSION THEOREM FROM RIESZ TO CESÀRO

THEOREM 2.1

Let $\mu > 0$ *and* λ *satisfy* (1.1). *Then*

(i) $[R,\lambda,1]_\mu \Rightarrow [C,\lambda,1]_\mu$,

(ii) *If* $p > 0$ *and* $\lambda_{n+1} = O(\lambda_n)$, *then*

$$[R,\lambda,p+1]_\mu \Rightarrow [C,\lambda,p+1]_\mu.$$

PROOF

(i) Suppose $\sum_{\nu=0}^{\infty} a_\nu = s\ [R,\lambda,1]_\mu$ where we may assume, without loss of generality, that $s = 0$.

$$T_m^1 = \sum_{n=0}^{m} (\lambda_{n+1} - \lambda_n) \left| \sum_{\nu=0}^{n} a_\nu \right|^\mu$$

$$= \sum_{n=0}^{m} \int_{\lambda_n}^{\lambda_{n+1}} \left| \sum_{\lambda_\nu < \tau} a_\nu \right|^\mu d\tau$$

$$= \int_0^{\lambda_{m+1}} |A_\lambda^0(\tau)|^\mu d\tau$$

$$= o(\lambda_{m+1}), \quad \text{as } m \to \infty.$$

Thus $\sum_{\nu=0}^{\infty} a_\nu = 0\,[C,\lambda,1]_\mu$.

(ii) For the case $p > 0$, we assume that $\sum_{\nu=0}^{\infty} a_\nu = 0\,[R,\lambda,p+1]_\mu$ so that

$$\int_0^\omega |R_\lambda^p(\tau)|^\mu d\tau = o(\omega).$$

We are required to show that

$$\sigma_m^{p+1} = \frac{1}{E_m^{p+1}} \sum_{n=0}^{m} (\lambda_{n+p+1} - \lambda_n) E_n^p |t_n^p|$$

$$= o(1), \quad \text{as } m \to \infty.$$

We divide the proof into four steps.

STEP I.

For every n, choose $q = q(n)$, a non-negative integer, satisfying $q(n) \geq q(n-1)$ and

(2.2) $\qquad \lambda_{q+1} - \lambda_q = \max\{(\lambda_{i+1} - \lambda_i) \mid n \leq i \leq n+p\}.$

Fixing n we partition the interval $[\lambda_q, \lambda_{q+1}]$ into $2p+2$ subintervals of length $\frac{\lambda_{q+1} - \lambda_q}{2p+2}$ with the points

$$\omega_\nu = \omega_{n,\nu} = \lambda_q + \frac{\nu}{2p+2} (\lambda_{q+1} - \lambda_q), \qquad \nu = 0,1,\ldots, 2p+2.$$

Since $p > 0$ and $\lambda_0 = 0$, $|R_\lambda^p(\tau)|^\mu$ is a continuous function of τ in the interval $[0, \omega]$. Applying the Mean Value Theorem on the alternate subintervals, we have, for $j = 0,1,2,\ldots, p$, numbers

$$\theta_j = \theta_{n,j} \in [\omega_{2j+1}, \omega_{2j+2}]$$

such that

$$\int_{\omega_{2j+1}}^{\omega_{2j+2}} |R_\lambda^p(\tau)|^\mu d\tau = (\omega_{2j+2} - \omega_{2j+1}) |R_\lambda^p(\theta_j)|^\mu$$

$$= (\omega_{2j+2} - \omega_{2j+1}) \left| \sum_{\nu=0}^{q} \left(1 - \frac{\lambda_\nu}{\theta_j}\right)^p a_\nu \right|^\mu .$$

Thus

$$\sum_{j=0}^{p} (\omega_{2j+2} - \omega_{2j+1}) \left| \sum_{\nu=0}^{q} \left(1 - \frac{\lambda_\nu}{\theta_j}\right)^p a_\nu \right|^\mu$$

$$= \sum_{j=0}^{p} \int_{\omega_{2j+1}}^{\omega_{2j+2}} |R_\lambda^p(\tau)|^\mu d\tau$$

$$\leq \int_{\lambda_q}^{\lambda_{q+1}} |R_\lambda^p(\tau)|^\mu d\tau .$$

Since $\omega_{2j+2} - \omega_{2j+1} = \frac{1}{2p+2} (\lambda_{q+1} - \lambda_q)$, we have

$$(2.3) \quad \sum_{n=0}^{m} \sum_{j=0}^{p} \frac{1}{2p+2} (\lambda_{q(n)+1} - \lambda_{q(n)}) \left| \sum_{\nu=0}^{q(n)} \left(1 - \frac{\lambda_\nu}{\theta_{n,j}}\right)^p a_\nu \right|^\mu$$

$$\leq \sum_{n=0}^{m} \int_{\lambda_{q(n)}}^{\lambda_{q(n)+1}} |R_\lambda^p(\tau)|^\mu d\tau$$

$$\leq (p+1) \int_0^{\lambda_{m+p+1}} |R_\lambda^p(\tau)|^\mu d\tau,$$

since q(n) is constant for at most p+1 different values of n.

STEP II.

Using techniques similar to those used by Borwein [2] we shall show that for every n, there are numbers

$$y_j = y_{n,j}, \quad \text{for } j = 0,1,2,\ldots, p,$$

such that the identity

(2.4) $$\prod_{i=1}^{p} (x + b_i) \equiv \sum_{j=0}^{p} y_j (x + \delta_j)^p$$

holds for all real x, where

$$b_i = \frac{\lambda_{n+i} - \lambda_q}{\lambda_{q+1} - \lambda_q}, \quad \text{for } i = 1,2,\ldots, p$$

and $$\delta_j = \frac{\theta_j - \lambda_q}{\lambda_{q+1} - \lambda_q}, \quad \text{for } j = 0,1,2,\ldots, p.$$

The identity (2.4) is equivalent to the system of linear equations

(2.5) $$\sum_{j=0}^{p} \delta_j^i \, y_j = \xi_i, \quad i = 0,1,\ldots, p,$$

where

(2.6) $$\xi_i = \binom{p}{i}^{-1} \sum_{1 \leq r_1 < \cdots < r_i \leq p} b_{r_1} b_{r_2} \cdots b_{r_i}$$

and where the sum in the expression for ξ_i is taken to be 1 when i = 0.

The determinant of the system (2.5) is the Vandermonde determinant

$$\Delta = \prod_{0 \le r < s \le p} (\delta_s - \delta_r).$$

(See [25, p. 214].)

Now for $s > r$

$$\delta_s - \delta_r = \frac{\theta_s - \theta_r}{\lambda_{q+1} - \lambda_q}$$

$$\ge \frac{\omega_{2s+1} - \omega_{2r+2}}{\lambda_{q+1} - \lambda_q}$$

$$\ge \frac{\lambda_{q+1} - \lambda_q}{2p+2} \times \frac{1}{\lambda_{q+1} - \lambda_q}$$

$$= \frac{1}{2p+2}.$$

Hence $$\Delta \ge \frac{1}{(2p+2)^{p!}} > 0.$$

Using Cramer's rule, we have

$$y_r = \frac{\Delta_r}{\Delta},$$

where Δ_r is the determinant of the matrix $(d_{i,j})$, $i,j = 0,1,2,\ldots, p$, in which

$$d_{i,r} = \xi_i \text{ and } d_{i,j} = \delta_j^i,\ j \ne r.$$

STEP III.

We now show that the numbers $y_{n,r}$ are uniformly bounded. Since

$$|b_r| = \left|\frac{\lambda_{n+r} - \lambda_q}{\lambda_{q+1} - \lambda_q}\right|$$

$$\le \frac{\lambda_{n+p+1} - \lambda_n}{\lambda_{q+1} - \lambda_q}$$

$$\leq (p+1) \frac{\lambda_{q+1} - \lambda_q}{\lambda_{q+1} - \lambda_q}$$

$$= (p+1),$$

we see from (2.6) that for $i = 0,1,2,\ldots, p$

$$|\xi_i| \leq (p+1)^p.$$

Also $|\delta_j^i| = \left(\frac{\theta_{n,j} - \lambda_q}{\lambda_{q+1} - \lambda_q}\right)^i \leq 1$, for $i,j = 0,1,2,\ldots p$.

Consequently

(2.7) $$|y_r| = |y_{n,r}| \leq (2p+2)^{p!} |\Delta_r| \leq H$$

where H is a constant independent of r and n.

STEP IV.

Here we establish an inequality between the $[C,\lambda,p+1]_\mu$-mean and the $[R,\lambda,p+1]_\mu$-mean of the series $\sum_{\nu=0}^{\infty} a_\nu$ which yields our result.

Let ν be any non-negative integer and put $x = \frac{\lambda_q - \lambda_\nu}{\lambda_{q+1} - \lambda_q}$ in (2.4), we obtain

$$\prod_{i=1}^{p} \left(\frac{\lambda_{n+i} - \lambda_\nu}{\lambda_{q+1} - \lambda_q}\right) = \sum_{j=0}^{p} y_j \left(\frac{\theta_{n,j} - \lambda_\nu}{\lambda_{q+1} - \lambda_q}\right)^p.$$

Thus $$\prod_{i=1}^{p} (\lambda_{n+i} - \lambda_\nu) = \sum_{j=0}^{p} y_j (\theta_{n,j} - \lambda_\nu)^p.$$

Dividing by E_n^p, we have

$$(2.8) \qquad \prod_{i=1}^{p} \left(1 - \frac{\lambda_\nu}{\lambda_{n+i}}\right) = \sum_{j=0}^{p} \frac{y_j\, \theta_{n,j}^p}{E_n^p} \left(1 - \frac{\lambda_\nu}{\theta_{n,j}}\right)^p$$

$$= \sum_{j=0}^{p} C_{n,j} \left(1 - \frac{\lambda_\nu}{\theta_{n,j}}\right)^p ,$$

where $C_{n,j} = \dfrac{y_j\, \theta_{n,j}^p}{E_n^p}$.

Since $\lambda_{n+1} = O(\lambda_n)$ and $y_{n,r}$ is uniformly bounded, we have

$$|C_{n,j}| \leq \frac{|y_{n,j}|\lambda_{q+1}^p}{\lambda_{n+1}^p} \leq H_1,$$

H_1 being independent of n and j.

Now it follows from (2.8) that

$$t_n^p = \sum_{\nu=0}^{n} \left(1 - \frac{\lambda_\nu}{\lambda_{n+1}}\right) \cdots \left(1 - \frac{\lambda_\nu}{\lambda_{n+p}}\right) a_\nu$$

$$= \sum_{\nu=0}^{q} \left(1 - \frac{\lambda_\nu}{\lambda_{n+1}}\right) \cdots \left(1 - \frac{\lambda_\nu}{\lambda_{n+p}}\right) a_\nu$$

$$= \sum_{\nu=0}^{q} \sum_{j=0}^{p} C_{n,j} \left(1 - \frac{\lambda_\nu}{\theta_{n,j}}\right)^p a_\nu$$

$$= \sum_{j=0}^{p} C_{n,j} \sum_{\nu=0}^{q} \left(1 - \frac{\lambda_\nu}{\theta_{n,j}}\right)^p a_\nu .$$

Thus

$$\sigma_m^{p+1} = \frac{1}{E_m^{p+1}} \sum_{n=0}^{m} (\lambda_{n+p+1} - \lambda_n)\, E_n^p \,|t_n^p|^{\mu}$$

$$\leq \lambda_{m+p+1}^{-1} \sum_{n=0}^{m} (\lambda_{n+p+1} - \lambda_n)\left\{ \sum_{j=0}^{p} |C_{n,j}| \sum_{\nu=0}^{q(n)} (1 - \frac{\lambda_\nu}{\theta_{n,j}})^p a_\nu| \right\}^{\mu}$$

$$\leq \lambda_{m+p+1}^{-1} \sum_{n=0}^{m} (p+1)(\lambda_{q(n)+1} - \lambda_{q(n)})(p+1)^{\mu} \sum_{j=0}^{p} |C_{n,j}|^{\mu} \Big| \sum_{\nu=0}^{q(n)} (1 - \frac{\lambda_\nu}{\theta_{n,j}})^p a_\nu \Big|^{\mu}$$

$$\leq \frac{H_2}{\lambda_{m+p+1}} \sum_{n=0}^{m} (\lambda_{q(n)+1} - \lambda_{q(n)}) \sum_{j=0}^{p} \Big| \sum_{\nu=0}^{q(n)} (1 - \frac{\lambda_\nu}{\theta_{n,j}})^p a_\nu \Big|^{\mu}$$

$$= \frac{H_2 \times (2p+2)}{\lambda_{m+p+1}} \sum_{n=0}^{m} \sum_{j=0}^{p} \frac{1}{2p+2} (\lambda_{q(n)+1} - \lambda_{q(n)}) \Big| \sum_{\nu=0}^{q(n)} (1 - \frac{\lambda_\nu}{\theta_{n,j}})^p a_\nu \Big|^{\mu}$$

$$\leq \frac{H_3}{\lambda_{m+p+1}} \int_0^{\lambda_{m+p+1}} |R_\lambda^p(\tau)|^{\mu}\, d\tau .$$

The final inequality following from Step I.

Hence if $\sum_{\nu=0}^{\infty} a_\nu = 0\ [R,\lambda,p+1]_\mu$, then

$$\frac{1}{\lambda_{m+p+1}} \int_0^{\lambda_{m+p+1}} |R_\lambda^p(\tau)|^{\mu}\, d\tau = o(1).$$

Thus $\sigma_m^{p+1} = o(1)$ so that $\sum_{\nu=0}^{\infty} a_\nu = 0\ [C,\lambda,p+1]_\mu$. ///

§2.3 INCLUSION THEOREM FROM CESARO TO RIESZ

We now investigate the inclusion in the opposite direction. And to facilitate the discussion we introduce the following notation.

Given a function f defined in an interval [a,b], and distinct points x_i in this interval, we define

$$f[x] = f(x)$$

and $$f[x_0,x_1,\ldots, x_n] = \frac{f[x_0,\ldots, x_{n-1}] - f[x_1,\ldots, x_n]}{x_0 - x_n}$$

for $n = 1,2,3,\ldots$.

The quantity $f[x_0,x_1,\ldots x_n]$ is called the *divided difference* of f(x) of n arguments. For an exposition of the properties of divided differences see Milne-Thomson [21, Chapter 1].

In the proof of our next theorem we need the following results of Russell [23, pp. 425-428].

LEMMA 2.2

Let p *be a non-negative integer.*

Define $$C_\tau(x) = \begin{cases} (\tau-x)^p, & \text{for } 0 \leq x < \tau, \\ 0, & \text{for } x \geq \tau. \end{cases}$$

Then, for $\lambda_n < \tau \leq \lambda_{n+1}$

(i) $$A_\lambda^p(\tau) = (-1)^{p+1} \sum_{\nu=n-p}^{n} C_\tau[\lambda_\nu,\lambda_{\nu+1},\ldots, \lambda_{\nu+p+1}](\lambda_{\nu+p+1}-\lambda_\nu)C_\nu^p$$

where we understand $C_\nu^p = 0$ *whenever* $\nu < 0$; *and*

(ii) *for* $n - p \leq \nu \leq n$

$$|C_\tau[\lambda_\nu,\lambda_{\nu+1},\ldots, \lambda_{\nu+p+1}]|(\lambda_{\nu+p+1} - \lambda_\nu) \leq H .$$

where H *is independent of* n.

THEOREM 2.2

Let λ satisfy (1.1). Then

(i) *if* $\mu > 0$, *then* $[C,\lambda,1]_\mu \Rightarrow [R,\lambda,1]_\mu$,

(ii) *if* $p > 0$, $\mu \geq 1$ *and* $\lambda_{n+1} = O(\lambda_n)$, *then*

$[C,\lambda,p+1]_\mu \Rightarrow [R,\lambda,p+1]_\mu$.

PROOF

(i) We suppose that $\sum_{\nu=0}^{\infty} a_\nu = 0\ [C,\lambda,1]_\mu$. Thus

$$\sigma_m^1 = \frac{1}{\lambda_{m+1}} \sum_{n=0}^{m} (\lambda_{n+1} - \lambda_n)|s_n|^\mu = o(1).$$

Hence

$$\frac{\lambda_{m+1} - \lambda_m}{\lambda_{m+1}} |s_m|^\mu = o(1). \tag{2.9}$$

Let $\omega > 0$ and suppose $\lambda_m < \omega \leq \lambda_{m+1}$. Then

$$\frac{1}{\omega}\int_0^\omega |A_\lambda^0(\tau)|^\mu d\tau = \frac{1}{\omega}\left\{\sum_{n=0}^{m-1} \int_{\lambda_n}^{\lambda_{n+1}} |\sum_{\nu=0}^{n} a_\nu|^\mu d\tau + \int_{\lambda_m}^{\omega} |\sum_{\nu=0}^{m} a_\nu|^\mu d\tau\right\}$$

$$= \frac{1}{\omega}\sum_{n=0}^{m-1}(\lambda_{n+1} - \lambda_n)|s_n|^\mu + \frac{1}{\omega}(\omega - \lambda_m)|s_m|^\mu$$

$$\leq \sigma_{m-1}^1 + \left(1 - \frac{\lambda_m}{\lambda_{m+1}}\right)|s_m|^\mu.$$

Now $\sigma_m^1 = o(1)$ which together with (2.9) yields

$$\frac{1}{\omega}\int_0^\omega |A_\lambda^0(\tau)|^\mu d\tau = o(1).$$

Thus $\sum_{\nu=0}^{\infty} a_\nu = 0\ [R,\lambda,1]_\mu$.

(ii) Let $\tau > 0$ and suppose $\lambda_n < \tau \leq \lambda_{n+1}$.

Then using Lemma 2.2 (i) and (ii) we see that

$$(2.10)\qquad |A_\lambda^p(\tau)|^\mu = \Big|\sum_{\nu=n-p}^{n} C_\tau[\lambda_\nu, \lambda_{\nu+1}, \ldots, \lambda_{\nu+p+1}](\lambda_{\nu+p+1}-\lambda_\nu)C_\nu^p\Big|^\mu$$

$$\leq (p+1)^\mu H \sum_{\nu=n-p}^{n} |C_\nu^p|^\mu .$$

Suppose $\omega > 0$ and $\lambda_m < \omega \leq \lambda_{m+1}$. Then

$$\int_0^\omega |A_\lambda^p(\tau)|^\mu d\tau$$

$$\leq \sum_{n=0}^{m} \int_{\lambda_n}^{\lambda_{n+1}} |A_\lambda^p(\tau)|^\mu d\tau$$

$$\leq H_1 \sum_{n=0}^{m} (\lambda_{n+1} - \lambda_n) \sum_{\nu=n-p}^{n} |C_\nu^p|^\mu$$

$$= H_1 \sum_{n=0}^{m} (\lambda_{n+1} - \lambda_n) \sum_{\nu=0}^{p} |C_{n-\nu}^p|^\mu$$

$$= H_1 \sum_{\nu=0}^{p} \sum_{n=\nu}^{m} (\lambda_{n+1} - \lambda_n) |C_{n-\nu}^p|^\mu ,$$

so that

$$(2.11)\qquad \int_0^\omega |A_\lambda^p(\tau)|^\mu d\tau \leq H_1 \sum_{\nu=0}^{p} \sum_{n=\nu}^{m} (\lambda_{n+1} - \lambda_n) |C_{n-\nu}^p|^\mu .$$

Now

$$\sigma_m^{p+1} = \frac{1}{E_m^{p+1}} \sum_{n=0}^{m} (\lambda_{n+p+1} - \lambda_n)\, E_n^p\, |\tau_n^p|^\mu$$

$$= \frac{1}{E_m^{p+1}} \sum_{n=0}^{m} (\lambda_{n+p+1} - \lambda_n)(E_n^p)^{1-\mu} |C_n^p|^\mu$$

$$\geq \frac{1}{(E_m^p)^\mu \lambda_{m+p+1}} \sum_{n=0}^{m} (\lambda_{n+p+1} - \lambda_n) |C_n^p|^\mu ,$$

since $\mu \geqq 1$.

Thus for $r = 0,1,2,\ldots, p$

$$\sigma_m^{p+1} \geqq \frac{1}{\lambda_{m+p+1}^{p\mu+1}} \sum_{n=0}^{m} (\lambda_{n+r+1} - \lambda_{n+r})|C_n^p|^\mu$$

$$= \frac{1}{\lambda_{m+p+1}^{p\mu+1}} \sum_{\nu=r}^{m+r} (\lambda_{\nu+1} - \lambda_\nu)|C_{\nu-r}^p|^\mu$$

$$\geqq \frac{1}{\lambda_{m+p+1}^{p\mu+1}} \sum_{\nu=r}^{m} (\lambda_{\nu+1} - \lambda_\nu)|C_{\nu-r}^p|^\mu .$$

If we now suppose $\sum_{\nu=0}^{\infty} a_\nu = 0\ [C,\lambda,p+1]_\mu$, so that $\sigma_m^{p+1} = o(1)$, we have, for $r = 0,1,2,\ldots, p$

$$\frac{1}{\lambda_{m+p+1}^{p\mu+1}} \sum_{\nu=r}^{m} (\lambda_{\nu+1} - \lambda_\nu)|C_{\nu-r}^p|^\mu = o(1),$$

as $m \to \infty$.

Hence in view of (2.11) and the condition $\lambda_{m+1} = O(\lambda_m)$, we have

$$\int_0^\omega |A_\lambda^p(\tau)|^\mu d\tau = o(\lambda_{m+p+1}^{p\mu+1}) = o(\omega^{p\mu+1}).$$

Hence $\sum_{\nu=0}^{\infty} a_\nu = 0\ [R,\lambda,p+1]_\mu$ for $\mu \geqq 1$. ///

Combining the results of Theorems 2.1 and 2.2, we have the following corollary.

THEOREM 2.3

Let $\lambda = \{\lambda_n\}$ *satisfy* (1.1).

(i) *If* $\mu > 0$, *then* $[R,\lambda,1]_\mu \Longleftrightarrow [C,\lambda,1]_\mu$.

(ii) *If* $p > 0$, $\mu \geqq 1$ *and* $\lambda_{n+1} = O(\lambda'_n)$, *then*

$[R,\lambda,p+1]_\mu \Longleftrightarrow [C,\lambda,p+1]_\mu$.

CHAPTER 3

SOME EQUIVALENCE THEOREMS

In this chapter we shall establish some equivalence theorems between various methods of Summability and Strong Summability.

§3.1 SOME LEMMAS

LEMMA 3.1

Let Λ_{p+1} be the matrix defined in §1.7. *The inverse matrix $\Lambda'_{p+1} = \{\lambda'_{n,\nu}\}$ of Λ_{p+1} is given by*

(3.1)
$$\lambda'_{n,n} = \frac{\lambda_{n+p+1}}{\lambda_{n+p+1} - \lambda_n},$$

$$\lambda'_{n,n-1} = \frac{-\lambda_n}{\lambda_{n+p+1} - \lambda_n},$$

$$\lambda'_{n,\nu} = 0 \text{ otherwise.}$$

PROOF

Let $C_{m,\nu} = \sum_{n=\nu}^{m} \lambda_{m,n} \lambda'_{n,\nu}$, we show that $C_{m,\nu} = \delta_{m,\nu}$.

Referring to the definition of Λ_{p+1}, (1.8), we have for $\nu \neq m$

$$c_{m,\nu} = \lambda_{m,\nu}\lambda'_{\nu,\nu} + \lambda_{m,\nu+1}\,\lambda'_{\nu+1,\nu}$$

$$= \frac{1}{E_m^{p+1}}(\lambda_{\nu+p+1} - \lambda_\nu)\, E_\nu^p \frac{\lambda_{\nu+p+1}}{\lambda_{\nu+p+1} - \lambda_\nu}$$

$$- \frac{1}{E_m^{p+1}}(\lambda_{\nu+p+2} - \lambda_{\nu+1}) E_{\nu+1}^p \frac{\lambda_{\nu+1}}{\lambda_{\nu+p+2} - \lambda_{\nu+1}}$$

$$= \frac{1}{E_m^{p+1}}(\lambda_{\nu+1}\cdots\lambda_{\nu+p}\cdot\lambda_{\nu+p+1} - \lambda_{\nu+1}\cdot\lambda_{\nu+2}\cdots\lambda_{\nu+p+1})$$

$$= 0,$$

and $c_{m,m} = \lambda_{m,m}\,\lambda'_{m,m}$

$$= \frac{1}{E_m^{p+1}}(\lambda_{m+p+1} - \lambda_m)\, E_m^p \frac{\lambda_{m+p+1}}{\lambda_{m+p+1} - \lambda_m}$$

$$= 1.$$

LEMMA 3.2

$\Lambda_{p+1} \Longleftrightarrow I$ *if and only if* $\liminf_{n\to\infty} \frac{\lambda_{n+p+1}}{\lambda_n} > 1.$

PROOF

$I \Rightarrow \Lambda_{p+1}$ follows from the regularity of Λ_{p+1}.

$\Lambda_{p+1} \Rightarrow I$ if and only if Λ'_{p+1} is regular. Referring to Lemma 3.1, we see that

(3.2) $$\lim_{n\to\infty} \lambda'_{n,\nu} = 0, \text{ for every } \nu,$$

(3.3) $$\lim_{n\to\infty} \sum_{\nu=0}^{\infty} \lambda'_{n,\nu} = \frac{-\lambda_n}{\lambda_{n+p+1} - \lambda_n} + \frac{\lambda_{n+p+1}}{\lambda_{n+p+1} - \lambda_n} = 1,$$

(3.4) $$\sup_n \sum_{\nu=0}^{\infty} |\lambda'_{n,\nu}| = \sup_n \frac{\lambda_{n+p+1} + \lambda_n}{\lambda_{n+p+1} - \lambda_n}$$

$$\le \sup_n \frac{2}{1 - \frac{\lambda_n}{\lambda_{n+p+1}}}.$$

This supremum is finite if and only if

$$\liminf_{n\to\infty} \frac{\lambda_{n+p+1}}{\lambda_n} > 1.$$

Consequently, Λ_{p+1} <=> I if and only if

$$\liminf_{n\to\infty} \frac{\lambda_{n+p+1}}{\lambda_n} > 1. \qquad ///$$

§3.2 EQUIVALENCE THEOREMS

THEOREM 3.1

(C,λ,p) <=> $(C,\lambda,p+1)$ *if and only if* $\liminf_{n\to\infty} \frac{\lambda_{n+p+1}}{\lambda_n} > 1$.

PROOF

By (1.11), we know that $(C,\lambda,p+1) = \Lambda_{p+1}\,(C,\lambda,p)$. Thus the result now follows from Lemma 3.2. ///

REMARK: In view of the fact $\{\lambda_n\}$ is an increasing sequence, so that $\liminf_{n\to\infty} \frac{\lambda_{n+p+1}}{\lambda_n} \geq 1$, we see that $\liminf_{n\to\infty} \frac{\lambda_{n+p+1}}{\lambda_n} = 1$ is necessary and sufficient for $(C,\lambda,p+1)$ to include strictly (C,λ,p).

We now state a result of Borwein and Cass [5, Corollary 2] which yields an equivalence theorem between the methods (C,λ,p) and $[C,\lambda,p+1]_\mu$.

THEOREM 3.2

Let $\mu > 0$.

Let $P = \{p_{n,\nu}\}$ *be a matrix with*

(i) $p_{n,\nu} \geq 0$, for $n,\nu = 0,1,2,3\ldots$,

(ii) $\lim_{n\to\infty} p_{n,\nu} = 0$, for $\nu = 0,1,2,3,\ldots$.

Let $Q = \{q_{n,\nu}\}$ *be a matrix such that for every sequence* $\{\sigma_\nu\}$ *there is a sequence* $\{s_\nu\}$ *for which*

$$\sigma_n = \sum_{\nu=0}^{\infty} q_{n,\nu} s_\nu$$

holds for $n = 0,1,2,3,\ldots$.

Then $\liminf_{\nu\to\infty} \max_{n\geq 0} p_{n,\nu} = 0$

is a necessary and sufficient condition for there to be a sequence which is not Q-convergent, but which is $[P,Q]_\mu$*-convergent to zero.*

THEOREM 3.3

Let $\mu > 0$. *Then* $(C,\lambda,p) \iff [C,\lambda,p+1]_\mu$ *if and only if* $\liminf_{n\to\infty} \frac{\lambda_{n+p+1}}{\lambda_n} > 1$.

PROOF

By Corollary 1.2, we have

$$(C,\lambda,p) \Rightarrow [C,\lambda,p+1]_\mu, \text{ for } \mu > 0.$$

Now $\Lambda_{p+1} = (\lambda_{n,\nu})$ satisfies

$$\lambda_{n,\nu} \geq 0, \text{ for } n,\nu = 0,1,2,\ldots$$

and $$\lim_{n\to\infty} \lambda_{n,\nu} = 0, \text{ for } \nu = 0,1,2,3,\ldots$$

And also since $\max_{n\geq 0} \lambda_{n,\nu} = \lambda_{\nu,\nu}$, we have

$$\liminf_{\nu\to\infty} \max_{n\geq 0} \lambda_{n,\nu} = \liminf_{\nu\to\infty} \lambda_{\nu,\nu}$$

$$= \liminf_{\nu\to\infty} \left(1 - \frac{\lambda_\nu}{\lambda_{\nu+p+1}}\right).$$

Moreover (C,λ,p) has an inverse, so the result follows from Theorem 3.2 by taking $P = \Lambda_{p+1}$ and $Q = (C,\lambda,p)$. ///

For the proof of the equivalence theorem between $(C,\lambda,p+1)$ and $[C,\lambda,p+1]_\mu$, we state another result of Borwein and Cass [5, Theorem 12].

THEOREM 3.4

Let the matrix $P = \{p_{n,\nu}\}$ *be regular and* $p_{n,\nu} = 0$ *for* $\nu > n$. *If*

(i) $p_{n,\nu} \geq p_{n+1,\nu}$, *for* $n \geq \nu$, $\nu = 0,1,2,\ldots$,

(ii) $p_{n,n} \to 0$,

(iii) $\sum_{\nu=0}^{n} p_{n,\nu} \leq \sum_{\nu=0}^{n+1} p_{n+1,\nu}$, *for* $n = 0,1,2,3,\ldots$,

then there is a divergent sequence of zeros and ones which is P-convergent to $\frac{1}{2}$, *but not* $[P,I]_\mu$*-convergent for any* $\mu \geq 1$. *(I denotes the identity matrix.)*

THEOREM 3.5

(i) *If* $\liminf_{n\to\infty} \frac{\lambda_{n+p+1}}{\lambda_n} > 1$, *then*

$$(C,\lambda,p+1) \Leftrightarrow [C,\lambda,p+1]_\mu, \text{ for } \mu > 0,$$

(ii) *If* $\lim_{n\to\infty} \frac{\lambda_{n+p+1}}{\lambda_n} = 1$, *then* $(C,\lambda,p+1)$ *strictly includes* $[C,\lambda,p+1]_\mu$, *for* $\mu \geq 1$.

PROOF

(i) Combining results of Theorem 3.1 and Theorem 3.3 we have $\liminf_{n\to\infty} \frac{\lambda_{n+p+1}}{\lambda_n} > 1$ implies that

$$(C,\lambda,p+1) \Leftrightarrow [C,\lambda,p+1]_\mu, \text{ for } \mu > 0.$$

(ii) Since in the matrix Λ_{p+1}, $\lambda_{n,n} = (1 - \frac{\lambda_n}{\lambda_{n+p+1}})$.

and since the matrix (C,λ,p) has an inverse, Theorem 3.4 shows that if $\lim_{n\to\infty} \frac{\lambda_{n+p+1}}{\lambda_n} = 1$, then there is a divergent sequence $\{t_n^p\}$ of zeros and ones which is Λ_{p+1}-convergent to $\frac{1}{2}$, but not $[\Lambda_{p+1}, I]_\mu$-convergent for any $\mu \geq 1$. Since $\Lambda_{p+1}\{t_n^p\} = \{t_n^{p+1}\}$ the result follows. ///

We now show that in Theorem 3.5 (ii) the condition $\lim_{n\to\infty} \frac{\lambda_{n+p+1}}{\lambda_n} = 1$ can not be replaced by $\liminf_{n\to\infty} \frac{\lambda_{n+p+1}}{\lambda_n} = 1$.

Let $P_0 > 0$ and $p_n \geq 0$, we say that

$$s_n \to s \ (\bar{N}, p_n)$$

if $\mu_n = \frac{1}{P_n} \sum_{\nu=0}^{n} p_\nu s_\nu \to s$, where $P_n = \sum_{\nu=0}^{n} p_\nu$.

REMARK: (i) Λ_{p+1} is the method $(\bar{N},p_n)$ with $P_n = E_n^{p+1}$.

(ii) If $(\bar{N},p_n)$ is taken as P in Theorem 3.4, it satisfies conditions (i) and (iii) of Theorem 3.4.

We shall now construct an $(\bar{N},p_n)$ method with $\liminf_{n\to\infty} \frac{p_n}{P_n} = 0$ and with $[(\bar{N},p_n),I]_1 \Leftrightarrow (\bar{N},p_n)$.

Let $\quad C_n = \frac{p_n}{P_n}, \quad 0 < C_n < 1$, for $n \geq 1$.

Then $\mu_n P_n - \mu_{n-1} P_{n-1} = p_n s_n$,

and $\mu_n - \mu_{n-1}(1 - C_n) = c_n s_n$.

Now take $C_{2n} = 1 - \frac{1}{(n+1)^2}$, for $n \geq 1$,

$$C_{2n+1} = \frac{1}{n+2}, \text{ for } n \geq 0, \text{ so}$$

(3.5) $$\liminf_{n\to\infty} \frac{p_n}{P_n} = 0,$$

(3.6) $$\mu_{2n} - \frac{\mu_{2n-1}}{(n+1)^2} = \left(1 - \frac{1}{(n+1)^2}\right) s_{2n},$$

(3.7) $$\mu_{2n+1} - \mu_{2n}\left(1 - \frac{1}{n+2}\right) = \frac{s_{2n+1}}{n+2}.$$

Consequently if $\mu_n \to \ell$, then (3.6) and (3.7) give $s_{2n} \to \ell$ and $s_{2n+1} = o(n)$.

On the other hand if $s_{2n} \to \ell$ and $s_{2n+1} = o(n)$, then

$$\left|\frac{\mu_n}{n+1}\right| \leq \frac{1}{P_n} \sum_{\nu=0}^{n} p_\nu \frac{|s_\nu|}{\nu+1} \leq H$$

and (3.6) and (3.7) imply that $\mu_n \to \ell$.

Summarizing we have

$s_n \to s$ $(\overline{N},p_n)$ if and only if

$s_{2n} \to s$ and $s_{2n+1} = o(n)$.

Thus $(\overline{N},p_n)$ is regular, not equivalent to convergence and $(\overline{N},p_n) \Leftrightarrow [(\overline{N},p_n),I]_1$.

Let $\lambda_0 = 0$, $\lambda_{n+1} = P_n$ for $n \geq 0$.

Then $\lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \lambda_{n+1} < \cdots$

and $\lambda_n \to \infty$, because $(\overline{N},p_n)$ is regular.

$$\Lambda_1 = (\overline{N},p_n),$$

$$\liminf_{n\to\infty} \frac{\lambda_{n+1}}{\lambda_n} = 1$$

and $[C,\lambda,1]_1 \Leftrightarrow (C,\lambda,1)$.

Combining the last example with Theorem 2.3 (i), we find that it is possible to have

$$\liminf_{n\to\infty} \frac{\lambda_{n+1}}{\lambda_n} = 1$$

and $[R,\lambda,1]_1 \Leftrightarrow (R,\lambda,1)$.

CHAPTER 4

ABSOLUTE GENERALISED CESÀRO SUMMABILITY

§4.1 DEFINITIONS

In this chapter we study the absolute methods of summability $|C,\lambda,p|_\mu$ and $|R,\lambda,p|$.

Let t_n^p be defined as in §1.5 and $\mu > 0$. We define $\sum_{\nu=0}^{\infty} a_\nu$ to be summable $|C,\lambda,p|_\mu$ if

$$(4.1) \qquad \sum_{n=1}^{\infty} \left(\frac{\lambda_n}{\lambda_{n+p+1} - \lambda_n}\right)^{\mu-1} |t_n^p - t_{n-1}^p|^\mu < \infty.$$

In §1.2, we defined $\sum_{\nu=0}^{\infty} a_\nu$ to be summable $|Q|_\mu$, $\mu > 0$, if

$$(4.2) \qquad \sum_{n=1}^{\infty} n^{\mu-1} |\sigma_n - \sigma_{n-1}|^\mu < \infty,$$

where $\{\sigma_n\} = Q\{s_n\}$.

When $\mu = 1$, conditions (4.1) and (4.2) are equivalent. When $\mu \neq 1$, they may or may not differ.

For example, if $\lambda_n = n^\alpha$, $\alpha > 0$, then

$$\frac{\lambda_n}{\lambda_{n+p+1} - \lambda_n} = \frac{n^\alpha}{(n+p+1)^\alpha - n^\alpha} = \frac{n^\alpha}{\alpha\theta_n^{\alpha-1}(p+1)}$$

where $n < \theta_n < n+p+1$.

Let $\rho_n = \dfrac{\lambda_n}{\lambda_{n+p+1} - \lambda_n}$.

Then $\dfrac{\rho_n}{n} \to \dfrac{1}{\alpha(p+1)}$, as $n \to \infty$. So in this case,

$$\sum_{n=1}^{\infty} \rho_n^{\mu-1} \, |t_n^p - t_{n-1}^p| < \infty \text{ if and only if}$$

$$\sum_{n=1}^{\infty} n^{\mu-1} \, |t_n^p - t_{n-1}^p| < \infty,$$

and the two conditions (4.1) and (4.2) are equivalent in this case.

On the other hand, if $\lambda_n = \log(n+1)$, then

$$\rho_n = \frac{\log(n+1)}{\log(n+p+2) - \log(n+1)} = \frac{\theta_n \log(n+1)}{p+1},$$

where $n+1 < \theta_n < n+p+2$.

In this case $\dfrac{\rho_n}{n \log n} \to \dfrac{1}{p+1}$, as $n \to \infty$, and

$$\sum_{n=1}^{\infty} \rho_n^{\mu-1} |t_n^p - t_{n-1}^p| < \infty \text{ if and only if}$$

$$\sum_{n=1}^{\infty} n^{\mu-1} \log^{\mu-1} n \, |t_n^p - t_{n-1}^p| < \infty.$$

Let $\alpha_n = t_n^p - t_{n-1}^p$ and $\mu = 2$.

If we take $\alpha_n = \dfrac{1}{n \log n}$, then

$$\sum_{n=2}^{\infty} n^{\mu-1} |\alpha_n|^{\mu} = \sum_{n=2}^{\infty} n |\alpha_n|^2$$

$$= \sum_{n=2}^{\infty} \frac{1}{n (\log n)^2} < \infty,$$

$$\text{while } \sum_{n=2}^{\infty} n^{\mu-1} \log^{\mu-1} n |\alpha_n|^{\mu}$$

$$= \sum_{n=2}^{\infty} n \log n \left|\frac{1}{n \log n}\right|^2$$

$$= \sum_{n=2}^{\infty} \frac{1}{n \log n} = \infty.$$

This shows that the two conditions (4.1) and (4.2) are different in this case.

It is more natural to use condition (4.1) rather than condition (4.2) to define $|C,\lambda,p|_{\mu}$ summability. Thus for the remainder of this chapter $\sum_{\nu=0}^{\infty} a_{\nu}$ is summable $|C,\lambda,p|_{\mu}$ means condition (4.1) is satisfied.

We now give an example which shows that there are sequences λ for which $|C,\lambda,p|_{\mu} \not\Rightarrow (C,\lambda,p)$.

Let $\mu = 2$, $\lambda_n = \log(n+1)$ and

$$\alpha_n = t_n^p - t_{n-1}^p = \frac{1}{n \log n \log \log n}.$$

$$\text{Then} \quad \sum_{n=2}^{\infty} n^{\mu-1} \log^{\mu-1} n \left|\frac{1}{n \log n \log \log n}\right|^{\mu}$$

$$= \sum_{n=2}^{\infty} n \log n \frac{1}{n^2 (\log n)^2 (\log \log n)^2}$$

$$= \sum_{n=2}^{\infty} \frac{1}{n \log n (\log \log n)^2} < \infty.$$

$$\text{But } \lim_{n\to\infty} t_n^p = t_1^p + \sum_{n=2}^{\infty} \alpha_n = t_1^p + \sum_{n=2}^{\infty} \frac{1}{n \log n \log \log n} = \infty.$$

$\sum_{\nu=0}^{\infty} a_\nu$ is summable $|C,\lambda,p|_1$ means that

$\sum_{n=1}^{\infty} |t_n^p - t_{n-1}^p| < \infty$ so that $\{t_n^p\}$ is convergent to s say. This means that $\sum_{\nu=0}^{\infty} a_\nu = s(C,\lambda,p)$. Hence we write and we have $|C,\lambda,p|_1 \Rightarrow (C,\lambda,p)$.

Let $R_\lambda^p(\tau)$ be defined as in §1.3. Then we say $\sum_{\nu=0}^{\infty} a_\nu$ is $|R,\lambda,p|$ summable, if

$$R_\lambda^p(\tau) \to s \text{ as } \tau \to \infty,$$

and
$$\int_h^\infty |dR_\lambda^p(\tau)| = \int_h^\infty \left|\frac{d}{d\tau} R_\lambda^p(\tau)\right| d\tau < \infty,$$

where $h \geq \lambda_0$. (See Obrechkoff: Sur la sommation absolue des sèries de Dirichlet. C.R. 186, 1928.) We denote this by

$$\sum_{\nu=0}^{\infty} a_\nu = s|R,\lambda,p|.$$

§4.2 INCLUSION THEOREMS

The next lemma is a special case of a result due to Mears, [19, Theorem 1].

LEMMA 4.1

Let $Q = \{q_{n,\nu}\}$ *be a regular matrix with* $q_{n,\nu} = 0$ *for* $\nu > n$. *If* $\sigma_n = \sum_{\nu=0}^{n} q_{n,\nu} s_\nu$, *where* $s_\nu = \sum_{\mu=0}^{\nu} a_\mu$, *then a necessary and sufficient condition for*

$$\sum_{n=1}^{\infty} |\sigma_n - \sigma_{n-1}| < \infty$$

whenever $\sum_{n=1}^{\infty} |s_\nu - s_{\nu-1}| < \infty$ *is*

(4.3) $$\sum_{n=k}^{\infty} \left| \sum_{\nu=k}^{n-1} (q_{n,\nu} - q_{n-1,\nu}) + q_{n,n} \right| \leq H$$

where H *is independent of* k.

THEOREM 4.1

For any non-negative integer p,

$\sum_{\nu=0}^{\infty} a_\nu = s|C,\lambda,p+1|_1$ *whenever* $\sum_{\nu=0}^{\infty} a_\nu = s|C,\lambda,p|_1$.

PROOF

We know that $(C,\lambda,p+1) = \Lambda_{p+1}(C,\lambda,p)$ where Λ_{p+1} is defined in §1.7. By Lemma 4.1 it suffices to prove that

$$\sum_{n=k}^{\infty} \left| \sum_{\nu=k}^{n-1} (\lambda_{n,\nu} - \lambda_{n-1,\nu}) + \lambda_{n,n} \right| \leq H$$

where H is independent of k.

Now, referring to (1.8)

$$\sum_{n=k}^{\infty} \left| \sum_{\nu=k}^{n-1} (\lambda_{n,\nu} - \lambda_{n-1,\nu}) + \lambda_{n,n} \right|$$

$$= \lambda_{k,k} + \sum_{n=k+1}^{\infty} \left| \sum_{\nu=k}^{n-1} (\lambda_{n,\nu} - \lambda_{n-1,\nu}) + \lambda_{n,n} \right|$$

$$= \lambda_{k,k} + \sum_{n=k+1}^{\infty} \left| \sum_{\nu=k}^{n-1} \left[\frac{(\lambda_{\nu+p+1} - \lambda_\nu)E_\nu^p}{E_n^{p+1}} - \frac{(\lambda_{\nu+p+1} - \lambda_\nu)E_\nu^p}{E_{n-1}^{p+1}} \right] \right.$$

$$\left. + \frac{(\lambda_{n+p+1} - \lambda_n)E_n^p}{E_n^{p+1}} \right|$$

$$= \lambda_{k,k} + \sum_{n=k+1}^{\infty} \left| \sum_{\nu=k}^{n-1} \frac{(\lambda_{\nu+p+1} - \lambda_\nu) E_\nu^p}{E_n^p} \left(\frac{1}{\lambda_{n+p+1}} - \frac{1}{\lambda_n} \right) \right.$$

$$\left. + \frac{\lambda_{n+p+1} - \lambda_n}{\lambda_{n+p+1}} \right|$$

$$= \lambda_{k,k} + \sum_{n=k+1}^{\infty} \left| \left(\frac{1}{\lambda_{n+p+1}} - \frac{1}{\lambda_n} \right) \left(\sum_{\nu=k}^{n-1} \frac{E_\nu^{p+1}}{E_n^p} - \sum_{\nu=k}^{n-1} \frac{E_{\nu-1}^{p+1}}{E_n^p} \right) \right.$$

$$\left. + \frac{\lambda_{n+p+1} - \lambda_n}{\lambda_{n+p+1}} \right|$$

$$= \lambda_{k,k} + \sum_{n=k+1}^{\infty} \left| \left(\frac{1}{\lambda_{n+p+1}} - \frac{1}{\lambda_n} \right) \left(\frac{E_{n-1}^{p+1} - E_{k-1}^{p+1}}{E_n^p} \right) + \frac{\lambda_{n+p+1} - \lambda_n}{\lambda_{n+p+1}} \right|$$

$$= \lambda_{k,k} + \sum_{n=k+1}^{\infty} \left| \frac{\lambda_n}{\lambda_{n+p+1}} - 1 - \frac{E_{k-1}^{p+1}}{E_n^{p+1}} + \frac{E_{k-1}^{p+1}}{E_{n-1}^{p+1}} + 1 - \frac{\lambda_n}{\lambda_{n+p+1}} \right|$$

$$= \lambda_{k,k} + E_{k-1}^{p+1} \sum_{n=k+1}^{\infty} \left(\frac{-1}{E_n^{p+1}} + \frac{1}{E_{n-1}^{p+1}} \right)$$

$$= \lambda_{k,k} + \frac{E_{k-1}^{p+1}}{E_k^{p+1}}$$

$$= \frac{(\lambda_{k+p+1} - \lambda_k) E_k^p}{E_k^{p+1}} + \frac{E_{k-1}^{p+1}}{E_k^{p+1}}$$

$$= 1 - \frac{E_{k-1}^{p+1}}{E_k^{p+1}} + \frac{E_{k-1}^{p+1}}{E_k^{p+1}}$$

$$= 1.$$

Thus $\sum_{n=1}^{\infty} |t_n^p - t_{n-1}^p| < \infty \Rightarrow \sum_{n=1}^{\infty} |t_n^{p+1} - t_{n-1}^{p+1}| < \infty.$

Since $(C,\lambda,p) \Rightarrow (C,\lambda,p+1)$, $t_n^p \to s$ implies $t_n^{p+1} \to s$.

Consequently $\sum_{\nu=0}^{\infty} a_\nu = s|C,\lambda,p+1|_1$ whenever $\sum_{\nu=0}^{\infty} a_\nu = s|C,\lambda,p|_1$. ///

COROLLARY 4.1

$$\sum_{\nu=0}^{\infty} a_\nu = s|C,\lambda,p|_1 \textit{ for } p \geq 1, \textit{ whenever } \sum_{\nu=0}^{\infty} |a_\nu| < \infty,$$

where $s = \sum_{\nu=0}^{\infty} a_\nu$.

PROOF

Take $p = 0$ in Theorem 4.1 and proceed by induction. ///

THEOREM 4.2

$$\sum_{\nu=0}^{\infty} a_\nu = s|C,\lambda,p|_1 \textit{ whenever } \sum_{\nu=0}^{\infty} a_\nu = s|C,\lambda,p+1|_1$$

if and only if $\liminf_{n\to\infty} \frac{\lambda_{n+p+1}}{\lambda_n} > 1$.

PROOF

$$(C,\lambda,p) = \Lambda'_{p+1}(C,\lambda,p+1).$$

Referring to Lemma 3.1, we know in $\Lambda'_{p+1} = \{\lambda'_{n,\nu}\}$

$$\lambda'_{n,n} = \frac{\lambda_{n+p+1}}{\lambda_{n+p+1} - \lambda_n},$$

$$\lambda'_{n,n-1} = \frac{-\lambda_n}{\lambda_{n+p+1} - \lambda_n} = 1 - \lambda'_{n,n},$$

$$\lambda'_{n,\nu} = 0, \text{ otherwise.}$$

By Lemma 3.2, we know Λ'_{p+1} is regular if and only if

$\liminf_{n\to\infty} \frac{\lambda_{n+p+1}}{\lambda_n} > 1.$

Now $\sum_{n=k}^{\infty} \left| \sum_{\nu=k}^{n-1} (\lambda'_{n,\nu} - \lambda'_{n-1,\nu}) + \lambda'_{n,n} \right|$

$$= \lambda'_{k,k} + |\lambda'_{k+1,k+1} + \lambda'_{k+1,k} - \lambda'_{k,k}| + \sum_{n=k+2}^{\infty} \left| \sum_{\nu=k}^{n-1} (\lambda'_{n,\nu} - \lambda'_{n-1,\nu}) + \lambda'_{n,n} \right|$$

$$= \frac{\lambda_{k+p+1}}{\lambda_{k+p+1} - \lambda_k} + |1 - \lambda'_{k,k}| + \sum_{n=k+2}^{\infty} |\lambda'_{n,n} + \lambda'_{n,n-1} - \lambda'_{n-1,n-1} - \lambda'_{n-1,n-2}|$$

$$= \frac{\lambda_{k+p+1} + \lambda_k}{\lambda_{k+p+1} - \lambda_k} + \sum_{n=k+2}^{\infty} |1 - 1|$$

$$= \frac{\frac{\lambda_{k+p+1}}{\lambda_k} + 1}{\frac{\lambda_{k+p+1}}{\lambda_k} - 1}$$

Thus it follows Lemma 4.1 that

$$\sum_{\nu=0}^{\infty} a_\nu = s|C,\lambda,p|_1 \text{ whenever } \sum_{\nu=0}^{\infty} a_\nu = s|C,\lambda,p+1|_1$$

if and only if $\liminf_{n\to\infty} \frac{\lambda_{n+p+1}}{\lambda_n} > 1.$ ///

Körle proved in [17] that $|R,\lambda,p| \iff |C,\lambda,p|_1$ for $p \geq 0$. Using this and the Theorem 4.2 we have the following corollary.

COROLLARY 4.2

$\sum_{\nu=0}^{\infty} a_\nu = s|R,\lambda,p|$ whenever $\sum_{\nu=0}^{\infty} a_\nu = s|R,\lambda,p+1|$ if and only if $\liminf_{n\to\infty} \frac{\lambda_{n+p+1}}{\lambda_n} > 1$.

We now turn our attention to the relationship between $|C,\lambda,p+1|_\mu$ and $[C,\lambda,p+1]_\mu$. To facilitate the discussion we use a result of Borwein, [1, Theorem 7], which we state as the next lemma. We include the proof for the sake of completeness.

LEMMA 4.2

If P is a regular matrix with non-negative entries, Q is a matrix and $\mu \geq 1$, then necessary and sufficient conditions for a series to be summable $[P,Q]_\mu$ to s are that it be PQ-summable to s and $[P,(I-P)Q]_\mu$-summable to zero.

PROOF

Let $\{\sigma_n\} = Q\{s_n\}$ and $\{\tau_n\} = P\{\sigma_n\}$. We have to prove that

$$\text{(a)} \quad \sum_{r=0}^{\infty} p_{n,r}|\sigma_r - s|^\mu = o(1)$$

if and only if

$$\text{(b)} \quad \tau_n \to s$$

and

$$\text{(c)} \quad \sum_{r=0}^{\infty} p_{n,r}|\sigma_r - \tau_r|^\mu = o(1).$$

(i) Suppose that (a) holds. Then by Theorem 1.2 (ii), (b) holds and so $\sum_{r=0}^{\infty} p_{n,r}|\tau_r - s|^\mu = o(1)$ since P is regular. Hence by Minkowski's inequality and (a)

$$\left\{\sum_{r=0}^{\infty} p_{n,r}|\sigma_r - \tau_r|^\mu\right\}^{1/\mu}$$

$$\leq \left\{\sum_{r=0}^{\infty} p_{n,r}|\sigma_r - s|^\mu\right\}^{1/\mu} + \left\{\sum_{r=0}^{\infty} p_{n,r}|\tau_r - s|^\mu\right\}^{1/\mu} = o(1),$$

and (c) follows.

(ii) Suppose that (b) and (c) hold. Since P is regular, it follows from (b) that

$$\sum_{r=0}^{\infty} p_{n,r}|\tau_r - s|^{\mu} = o(1).$$

Hence by Minkowski's inequality and (c),

$$\left\{\sum_{r=0}^{\infty} p_{n,r}|\sigma_r - s|^{\mu}\right\}^{1/\mu}$$

$$\leq \left\{\sum_{r=0}^{\infty} p_{n,r}|\sigma_r - \tau_r^*|^{\mu}\right\}^{1/\mu} + \left\{\sum_{r=0}^{\infty} p_{n,r}|\tau_r - s|^{\mu}\right\}^{1/\mu} = o(1),$$

so that (a) holds.

The proof is thus complete. ///

THEOREM 4.3

Let $\mu \geq 1$. *Then*

$\sum_{\nu=0}^{\infty} a_\nu = s[C,\lambda,p+1]_\mu$ *if and only if*

(4.4) $$\sum_{\nu=0}^{\infty} a_\nu = s(C,\lambda,p+1),$$

and

(4.5) $$\frac{1}{E_m^{p+1}} \sum_{n=0}^{m} (\lambda_{n+p+1} - \lambda_n) E_n^p |t_n^p - t_n^{p+1}|^{\mu} = o(1).$$

Condition (4.5) *means* $|t_n^p - t_n^{p+1}|^{\mu} \to 0(\Lambda_{p+1})$.

PROOF

In Lemma 4.2, take $P = \Lambda_{p+1}$, $Q = (C,\lambda,p)$ and observe that

$$(I - P)Q = (C,\lambda,p) - (C,\lambda,p+1). \qquad ///$$

THEOREM 4.4

$$\sum_{\nu=0}^{\infty} a_\nu = s|C,\lambda,p+1|_1 \textit{ implies } \sum_{\nu=0}^{\infty} a_\nu = s[C,\lambda,p+1]_1.$$

PROOF

Since $\sum_{n=1}^{\infty} |t_n^{p+1} - t_{n-1}^{p+1}| < \infty$ implies that t_n^{p+1} tends to a limit, s say, we have $\sum_{\nu=0}^{\infty} a_\nu = s(C,\lambda,p+1)$. Hence to prove the theorem it suffices to show condition (4.5) is satisfied with $\mu = 1$.

Let $n \geq 1$.

$$|t_n^p - t_n^{p+1}|$$

$$= \Big|\frac{1}{E_n^p} \sum_{\nu=0}^{n} (\lambda_{n+1} - \lambda_\nu) \cdots (\lambda_{n+p} - \lambda_\nu) a_\nu$$

$$- \frac{1}{E_n^{p+1}} \sum_{\nu=0}^{n} (\lambda_{n+1} - \lambda_\nu) \cdots (\lambda_{n+p+1} - \lambda_\nu) a_\nu\Big|$$

$$= \Big|\frac{\lambda_{n+p+1}}{E_n^{p+1}} \sum_{\nu=0}^{n} (\lambda_{n+1} - \lambda_\nu) \cdots (\lambda_{n+p} - \lambda_\nu) a_\nu$$

$$- \frac{1}{E_n^{p+1}} \sum_{\nu=0}^{n} (\lambda_{n+1} - \lambda_\nu) \cdots (\lambda_{n+p+1} - \lambda_\nu) a_\nu\Big|$$

$$= \Big|\frac{1}{E_n^{p+1}} \sum_{\nu=0}^{n} (\lambda_{n+1} - \lambda_\nu) \cdots (\lambda_{n+p} - \lambda_\nu)(\lambda_{n+p+1} - \lambda_{n+p+1} + \lambda_\nu) a_\nu\Big|$$

$$= \Big|\frac{1}{E_n^{p+1}} \sum_{\nu=0}^{n} (\lambda_{n+1} - \lambda_\nu) \cdots (\lambda_{n+p} - \lambda_\nu)\lambda_\nu a_\nu\Big|.$$

On the other hand

$$\left|t_n^{p+1} - t_{n-1}^{p+1}\right|$$

$$= \Big|\frac{1}{E_n^{p+1}} \sum_{\nu=0}^{n} (\lambda_{n+1} - \lambda_\nu) \cdots (\lambda_{n+p+1} - \lambda_\nu) a_\nu$$

$$- \frac{1}{E_{n-1}^{p+1}} \sum_{\nu=0}^{n} (\lambda_n - \lambda_\nu) \cdots (\lambda_{n+p} - \lambda_\nu) a_\nu\Big|.$$

$$= \left| \frac{\lambda_n}{E_{n-1}^{p+2}} \sum_{\nu=0}^{n} (\lambda_{n+1} - \lambda_\nu) \cdots (\lambda_{n+p+1} - \lambda_\nu) a_\nu \right.$$

$$\left. - \frac{\lambda_{n+p+1}}{E_{n-1}^{p+2}} \sum_{\nu=0}^{n} (\lambda_n - \lambda_\nu) \cdots (\lambda_{n+p} - \lambda_\nu) a_\nu \right|$$

$$= \left| \frac{1}{E_{n-1}^{p+2}} \sum_{\nu=0}^{n} (\lambda_{n+1} - \lambda_\nu) \cdots (\lambda_{n+p} - \lambda_\nu) \left\{ \lambda_n (\lambda_{n+p+1} - \lambda_\nu) \right. \right.$$

$$\left. \left. - \lambda_{n+p+1} (\lambda_n - \lambda_\nu) \right\} a_\nu \right|$$

$$= \left| \frac{1}{E_{n-1}^{p+2}} \sum_{\nu=0}^{n} (\lambda_{n+1} - \lambda_\nu) \cdots (\lambda_{n+p} - \lambda_\nu)(\lambda_{n+p+1} - \lambda_n) \lambda_\nu a_\nu \right|$$

$$= \frac{\lambda_{n+p+1} - \lambda_n}{\lambda_n} \left| \frac{1}{E_n^{p+1}} \sum_{\nu=0}^{n} (\lambda_{n+1} - \lambda_\nu) \cdots (\lambda_{n+p} - \lambda_\nu) \lambda_\nu a_\nu \right|,$$

for $n \geq 1$.

Hence

$$(4.6) \quad \left| t_n^{p+1} - t_{n-1}^{p+1} \right| = \frac{\lambda_{n+p+1} - \lambda_n}{\lambda_n} \left| t_n^p - t_n^{p+1} \right|, \text{ for } n \geq 1.$$

Consequently multiplying (4.6) by E_{n-1}^{p+1}, we obtain

$$\sum_{n=1}^{m} (\lambda_{n+p+1} - \lambda_n) E_n^p \left| t_n^p - t_n^{p+1} \right|$$

$$= \sum_{n=1}^{m} E_{n-1}^{p+1} \left| t_n^{p+1} - t_{n-1}^{p+1} \right|.$$

Since $\lambda_0 = 0$, $\left| t_0^p - t_0^{p+1} \right| = |a_0 - a_0| = 0$.

By taking $E_{-1}^{p+1} = 0$ and $t_{-1}^{p+1} = 0$, we have

$$(4.7) \quad \sum_{n=0}^{m} (\lambda_{n+p+1} - \lambda_n) E_n^p \left| t_n^p - t_n^{p+1} \right|$$

$$= \sum_{n=0}^{m} E_{n-1}^{p+1} \left| t_n^{p+1} - t_{n-1}^{p+1} \right|.$$

Let $b_r = \left| t_r^{p+1} - t_{r-1}^{p+1} \right|$ and $B_n = \sum_{r=0}^{n} b_r$.

Then from (4.7), we have

$$\sum_{n=0}^{m} E_{n-1}^{p+1} b_n$$

$$= \sum_{n=0}^{m} E_{n-1}^{p+1} (B_n - B_{n-1})$$

$$= B_m E_m^{p+1} - \sum_{n=0}^{m} B_n (E_n^{p+1} - E_{n-1}^{p+1}).$$

Dividing by E_m^{p+1}, we obtain

$$B_m - \frac{1}{E_m^{p+1}} \sum_{n=0}^{m} (\lambda_{n+p+1} - \lambda_n) E_n^p B_n = o(1), \text{ as } m \to \infty,$$

because of the regularity of Λ_{p+1} and the hypothesis $\sum_{\nu=0}^{\infty} a_\nu = s|C,\lambda,p+1|_1$ which means that $\{B_n\}$ is convergent.

Thus the condition (4.5) is satisfied and the theorem is proved. ///

(C.f. Borwein and Cass [6, Theorem 9].)

THEOREM 4.5

If $\sum_{\nu=0}^{\infty} a_\nu = s \ (C,\lambda,p+1)$ *then, for* $\mu > 1$,

$\sum_{\nu=0}^{\infty} a_\nu = s|C,\lambda,p+1|_\mu$ *implies that* $\sum_{\nu=0}^{\infty} a_\nu = s[C,\lambda,p+1]_\mu$.

PROOF

Since $\sum_{\nu=0}^{\infty} a_\nu = s(C,\lambda,p+1)$, it suffices to show that condition (4.5) is satisfied with $\mu > 1$.

Now referring to (4.6), we have

$$\left|t_n^{p+1} - t_{n-1}^{p+1}\right|^\mu = \left(\frac{\lambda_{n+p+1} - \lambda_n}{\lambda_n}\right)^\mu \left|t_n^p - t_n^{p+1}\right|^\mu$$

for $\mu > 1$ and $n \geq 1$. Thus

$$\left|t_n^p - t_n^{p+1}\right|^\mu = \left(\frac{\lambda_n}{\lambda_{n+p+1} - \lambda_n}\right)^\mu \left|t_n^{p+1} - t_{n-1}^{p+1}\right|^\mu \tag{4.8}$$

for $\mu > 1$ and $n \geq 1$.

Since $\left|t_0^p - t_0^{p+1}\right| = 0$ and $E_{-1}^{p+1} = 0$ and $t_{-1}^{p+1} = 0$, we have, by (4.8),

$$\sum_{n=0}^{m} (\lambda_{n+p+1} - \lambda_n) E_n^p \left|t_n^p - t_n^{p+1}\right|^\mu$$

$$= \sum_{n=0}^{m} E_{n-1}^{p+1} \left(\frac{\lambda_n}{\lambda_{n+p+1} - \lambda_n}\right)^{\mu-1} \left|t_n^{p+1} - t_{n-1}^{p+1}\right|^\mu .$$

Now let $b_r = \rho_r^{\mu-1}\left|t_r^{p+1} - t_{r-1}^{p+1}\right|^\mu$

$$\text{and } B_n = \sum_{r=0}^{n} b_r ,$$

and proceed as the last part of the proof of Theorem 4.4,

we have $\displaystyle\sum_{n=0}^{m} E_{n-1}^{p+1}\rho_n^{\mu-1}\left|t_n^{p+1} - t_{n-1}^{p+1}\right|^\mu = o(E_m^{p+1})$.

And hence $\displaystyle\sum_{n=0}^{m} (\lambda_{n+p+1} - \lambda_n) E_n^p \left|t_n^p - t_n^{p+1}\right|^\mu = o(E_m^{p+1})$. ///

CHAPTER 5

SOME STRICT INCLUSION THEOREMS BETWEEN CESÀRO AND DISCRETE RIESZ METHODS OF SUMMABILITY

§5.1 DEFINITIONS

Suppose throughout this chapter that $\kappa > 0$,

$$s_n = \sum_{r=0}^{n} a_r,$$

$$\varepsilon_0^\kappa = 1,$$

and

$$\varepsilon_n^\kappa = \binom{n+\kappa}{n} = \frac{(\kappa+1)(\kappa+2)\cdots(\kappa+n)}{n!} \quad \text{for } n > 0.$$

Let $\{p_n\}$ be a sequence with $p_n > 0$ for $n \geq 0$ and let

$$P_n = \sum_{r=0}^{n} p_r.$$

Define

$$(5.1) \quad t_n = \frac{1}{P_n} \sum_{r=0}^{n} p_{n-r} s_r = \frac{1}{P_n} \sum_{r=0}^{n} P_{n-r} a_r,$$

$$(5.2) \quad t_n^\Delta = \frac{1}{P_n} \sum_{r=0}^{n} p_{n-r} a_r = \frac{1}{P_n} \sum_{r=0}^{n} (p_{n-r} - p_{n-1-r}) s_r, \quad (p_{-1} = 0).$$

We say that the sequence $\{s_n\}$ is (N,p_n)-convergent to s if $t_n \to s$; and we write

$$s_n \to s \ (N,p_n).$$

This is a Nörlund Summability Method. See for example Hardy [11, page 54].

Let

$$(5.3) \quad \tau_n = \frac{1}{P_n} \sum_{r=0}^{n} p_r |t_r^{\Delta} - s|.$$

We say that the sequence $\{s_n\}$ is $[N,p_n]$-convergent to s if $\tau_n = o(1)$, and we write

$$s_n \to s \ [N,p_n].$$

(See Borwein and Cass [6].)

We say that the sequence $\{s_n\}$ is $|N,p_n|$-convergent to s if

$$\sum_{n=1}^{\infty} |t_n - t_{n-1}| < \infty \quad \text{and } s = \lim t_n;$$

and we write

$$s_n \to s \ |N,p_n|.$$

The Strong Summability Method $[N,p_n]$ is the method $[P,Q]_1$ (see §1.2) with $P = (\overline{N},p_n)$ (see §3.2) and Q the matrix associated with the transformation (5.2). We shall denote Q by $(N,\Delta p_n)$.

In the case of $[N,p_n]$-summability, the method is interesting only if $P_n \to \infty$. This condition is satisfied by the summability methods we consider below.

If we take $p_n = \varepsilon_n^{\kappa-1}$, then (N,p_n) and $|N,p_n|$ are the Cesàro and Absolute Cesàro Summability Methods (C,κ) and $|C,\kappa|$ respectively.

The method $[N,p_n]$ with $p_n = \varepsilon_n^{\kappa-1}$ is equivalent (but not equal) to the Strong Cesàro Method $[C,\kappa]$ (See §1.6.) We shall denote this method $[N,p_n]$ also by $[C,\kappa]$. See Borwein and Cass [6, pages 98-99].

If

$$(5.4) \qquad \rho_n^\kappa = \sum_{\nu=0}^{n} \left(1 - \frac{\nu}{n+1}\right)^\kappa a_\nu$$

$$= \frac{1}{(n+1)^\kappa} \sum_{\nu=0}^{n} (n+1-\nu)^\kappa (s_\nu - s_{\nu-1})$$

$$= \frac{1}{(n+1)^\kappa} \sum_{\nu=0}^{n} [((n+1)-\nu)^\kappa - (n-\nu)^\kappa] s_\nu,$$

then we say that the sequence $\{s_\nu\}$ is (R^*,n,κ)-convergent to s, if $\rho_n^\kappa \to s$ as $n \to \infty$. We denote this by

$$s_n \to s \ (R^*,n,\kappa).$$

Thus if we take $p_n = (n+1)^\kappa - n^\kappa$ for $n \geq 0$, then (N,p_n) and $|N,p_n|$ are the Discrete Reisz and Absolute Discrete Riesz Summability Methods (R^*,n,κ) and $|R^*,n,\kappa|$ respectively. We shall define the *Strong Discrete Riesz Method of Summability* $[R^*,n,\kappa]$ to be the method $[N,p_n]$ associated with this $\{p_n\}$.

§5.2 KUTTNER'S THEOREM

In the definitions of (C,κ) and (R^*,n,κ) and the associated absolute methods, κ is usually allowed to satisfy $\kappa > -1$. The methods $[C,\kappa]$ and $[R^*,n,\kappa]$ make sense only when $\kappa > 0$ and it is for this reason we have so restricted κ.

THEOREM (Kuttner)

(i) If $-1 < \kappa < 2$, *then* (R^*,n,κ) *is equivalent to* (C,κ) *and* $|R^*,n,\kappa|$ *is equivalent to* $|C,\kappa|$.

(ii) There is a sequence $(R^*,n,2)$*-convergent but not*

(C,2)-convergent and a sequence $|R^*,n,2|$*-convergent but not* $|C,2|$*-convergent. But* $|R^*,n,2| \Rightarrow (C,2)$.

(iii) If $\kappa > 2$, *there is a sequence* $|R^*,n,\kappa|$*-convergent but not* (C,κ)*-convergent.*

(See Kuttner [18].)

§5.3 EXTENSION OF KUTTNER'S THEOREM AND OTHER RESULTS

For the proof of Theorem 5.1 we state two results of Borwein and Cass [6, Theorems 6 and 9] as our next two lemmas.

LEMMA 5.1

$[N,p_n] \Rightarrow (N,p_n)$.

LEMMA 5.2

If $P_n \to \infty$ *and* $\{s_n\}$ *is* $|N,p_n|$*-convergent, then*

$$s_n \to s\ [N,p_n]$$

where $s = \lim_{n\to\infty} t_n$ *and* t_n *is defined as in* (5.1).

THEOREM 5.1

If $\kappa > 0$, *then* $|R^*,n,\kappa| \Rightarrow [R^*,n,\kappa] \Rightarrow (R^*,n,\kappa)$.

PROOF

That $[R^*,n,\kappa] \Rightarrow (R^*,n,\kappa)$ is a special case of Lemma 5.1. $|R^*,n,\kappa| \Rightarrow [R^*,n,\kappa]$ follows from Lemma 5.2. ///

The next theorem is known, but it also follows from Lemmas 5.1 and 5.2 as the Theorem 5.1.

THEOREM 5.2

$|C,\kappa| \Rightarrow [C,\kappa] \Rightarrow (C,\kappa)$.

THEOREM 5.3

Let $p_n > 0$ *for* $n \geq 0$ *and suppose* $P_n \to \infty$. *Then there is a sequence which is* $[N,p_n]$*-convergent but not* $|N,p_n|$*-convergent.*

PROOF

Borwein and Cass [6, Theorem 8] proved that $s_n \to s[N,p_n]$ if and only if

(5.5) $$s_n \to s(N,p_n),$$

and

(5.6) $$\frac{1}{P_n}\sum_{r=0}^{n} p_r |t_r^{\Delta} - t_r| = o(1)$$

where t_r and t_r^{Δ} are given by (5.1) and (5.2).

This is a special case of Lemma 4.2.

Now

$$t_r^{\Delta} - t_r = \frac{1}{p_r}\sum_{\nu=0}^{r}(p_{r-\nu} - p_{r-1-\nu})s_\nu - \frac{1}{P_r}\sum_{\nu=0}^{n} p_{r-\nu}s_\nu$$

$$= \frac{P_r \sum_{\nu=0}^{r} p_{r-\nu}s_\nu - P_r \sum_{\nu=0}^{r-1} p_{r-1-\nu}s_\nu - p_r \sum_{\nu=0}^{r} p_{r-\nu}s_\nu}{p_r P_r}$$

$$= \frac{P_{r-1} \sum_{\nu=0}^{r} p_{r-\nu}s_\nu - P_r \sum_{\nu=0}^{r-1} p_{r-1-\nu}s_\nu}{p_r P_r},$$

so that

(5.7) $$p_r(t_r^{\Delta} - t_r) = P_{r-1}(t_r - t_{r-1}), \quad r = 0,1,2,\ldots,$$

$(P_{-1} = t_{-1} = 0)$.

Choose $\{s_n\}$ so that $t_n - t_{n-1} = \frac{\delta_n}{P_n}\frac{p_n}{D_n}$ where

$D_n = \sum_{r=0}^{n} \frac{p_r}{P_r}$ and $\delta_n = \pm 1$ chosen in such a way that $\sum_{n=1}^{\infty} \frac{\delta_n \, p_n}{P_n \, D_n}$ converges. Then $\{t_n\}$ is convergent ensuring that (5.5) is satisfied. Also we have

$$\frac{1}{P_n} \sum_{r=0}^{n} p_r \left| t_r^{\Delta} - t_r \right| = \frac{1}{P_n} \sum_{r=0}^{n} P_{r-1} \left| t_r - t_{r-1} \right|$$

$$= \frac{1}{P_n} \sum_{r=0}^{n} \frac{P_{r-1} \, p_r}{P_r \, D_r}$$

$$= \sum_{r=0}^{n} a_{n,r} \frac{1}{D_r}$$

where $a_{n,r} = \frac{P_{r-1} \, p_r}{P_n \, P_r}$ for $0 \leq r \leq n$ and $a_{n,r} = 0$ for $r > n$.

Now $A = \{a_{n,r}\}$ is a matrix with zero column limits and

$$\sum_{r=0}^{n} |a_{n,r}| = \sum_{r=0}^{n} a_{n,r} \leq \frac{1}{P_n} \sum_{r=0}^{n} p_r = 1, \quad \text{for all } n,$$

so that it transforms null sequences into null sequences. Since by Abel-Dini Theorem $\lim_{n\to\infty} D_n = \infty$, $\frac{1}{D_r} \to 0$ as $r \to \infty$.

It follows that (5.6) is satisfied, so $s_n \to s \; [N,p_n]$. But by Abel-Dini Theorem again

$$\sum_{n=1}^{\infty} |t_n - t_{n-1}| = \sum_{n=1}^{\infty} \frac{p_n}{P_n \, D_n} = \infty,$$

so $\{s_n\}$ is not $|N,p_n|$-convergent. ///

COROLLARY 5.1

Let $\kappa > 0$. *There is a sequence which is* $[\hat{R}^*,n,\kappa]$-*convergent but not* $|R^*,n,\kappa|$-*convergent.*

COROLLARY 5.2

Let $\kappa > 0$. There is a sequence which is $[C,\kappa]$-convergent but not $|C,\kappa|$-convergent.

THEOREM 5.4

Let $\kappa > 0$. There is a sequence which is (R^*,n,κ)-convergent but not $[R^*,n,\kappa]$-convergent.

PROOF

Let $P = \{p_{n,\nu}\}$, where $p_{n,\nu} = \dfrac{(\nu+1)^\kappa - \nu^\kappa}{(n+1)^\kappa}$ for $0 \leq \nu \leq n$ and $p_{n,\nu} = 0$ for $\nu > n$. It follows from Theorem 3.4 that there is a sequence P-convergent but not $[P,I]$-convergent. Let $Q = \{q_{n,\nu}\}$ be the matrix such that

$$\sum_{\nu=0}^{n} q_{n,\nu} s_\nu = \frac{1}{P_n} \sum_{\nu=0}^{n} p_{n-\nu} a_\nu$$

where $p_n = (n+1)^\kappa - n^\kappa$. Then $[R^*,n,\kappa]$-convergency is the same as $[P,Q]$-convergency and (R^*,n,κ)-convergency is the same as PQ-convergency. Since the matrix Q has an inverse our result now follows. ///

For the next theorem we state two results of Borwein and Cass [6, Theorem 1 and Corollary 1] as our next two lemmas.

LEMMA 5.4

If $(N,p_n) \Rightarrow (N,q_n)$ then $[N,p_n] \Rightarrow [N,q_n]$.

LEMMA 5.5

If $(N,p_n) \Leftrightarrow (N,q_n)$ then $[N,p_n] \Leftrightarrow [N,q_n]$.

THEOREM 5.6

(i) *If* $\kappa > 0$, *then* $[C,\kappa] \Rightarrow [R^*,n,\kappa]$.

(ii) *If* $0 < \kappa < 2$, *then* $[C,\kappa] \Leftrightarrow [R^*,n,\kappa]$.

PROOF

Since for $\kappa > 0$ we have $(C,\kappa) \Rightarrow (R^*,n,\kappa)$, (i) follows from Lemma 5.4. Since for $0 < \kappa < 2$ we have $(C,\kappa) \Leftrightarrow (R^*,n,\kappa)$ (ii) follows from Lemma 5.5. ///

THEOREM 5.7

There is a sequence which is $|R^*,n,2|$*-convergent but not* $[C,2]$*-convergent.*

PROOF

For a given sequence $\{s_n\}$ we write

$$\sigma_n = \frac{1}{\varepsilon_n^2} \sum_{\nu=0}^{n} \varepsilon_{n-\nu}^1 s_\nu = \frac{S_n}{\varepsilon_n^2} \tag{5.8}$$

and

$$\xi_n = \frac{1}{(n+1)^2} \sum_{\nu=0}^{n} (n+1-\nu)^2 a_\nu = \frac{T_n}{(n+1)^2} \tag{5.9}$$

so that $\{\sigma_n\}$ and $\{\xi_n\}$ are respectively the $(C,2)$ and $(R^*,n,2)$ transforms of the sequence $\{s_n\}$.

As in Kuttner [18, page 362] we have

$$T_0 = S_0; \quad T_n = S_{n-1} + S_n, \quad n = 1,2,3\ldots \tag{5.10}$$

and

$$S_n = \sum_{m=0}^{n} (-1)^{n-m} T_m. \tag{5.11}$$

Now take $s_n = (-1)^n n$ so that $T_n = (-1)^n$. Thus $\sum_{n=1}^{\infty} |\xi_n - \xi_{n-1}| < \infty$ and $\xi_n \to 0$, so that if $\{s_n\}$ is the sequence associated with this choice of S_n and T_n we have $s_n \to 0|R^*,n,2|$. To see that $\{s_n\}$ is not $[C,2]$-convergent we notice first that by Theorem 5.1, $s_n \to 0|R^*,n,2|$ implies $s_n \to 0\ [R^*,n,2]$. Now by Theorem 5.6 $[C,2] \Rightarrow [R^*,n,2]$, the only $[C,2]$-sum that $\{s_n\}$ could have is zero.

But,
$$s_n - s_{n-1} = (-1)^n(2n-1).$$

(5.12)
$$\frac{1}{m+1}\sum_{n=0}^{m}\left|\frac{(-1)^n(2n-1)}{n+1}\right|$$

$$= \frac{1}{m+1}\sum_{n=0}^{m}\frac{2n-1}{n+1}.$$

Since $(C,1)$ is regular and $\frac{2n-1}{n+1} \to 2$, (5.12) tends to 2 as $m \to \infty$. Thus $\{s_n\}$ is not $[C,2]$-convergent to zero. ///

COROLLARY 5.3

There is a sequence which is $[R^*,n,2]$*-convergent but not* $[C,2]$*-convergent.*

PROOF

This follows from the fact that $|R^*,n,2| \Rightarrow [R^*,n,2]$.

THEOREM 5.8

$[R^*,n,2] \not\Rightarrow (C,2)$.

PROOF

Referring to (5.9) we find that

$$T_r - T_{r-1} = \sum_{\nu=0}^{r}\{(r+1-\nu)^2 - (r-\nu)^2\}a_\nu.$$

$s_n \to 0\ [R^*,n,2]$ if and only if

$$\frac{1}{P_n}\sum_{r=0}^{n}p_r\left|\frac{1}{P_r}\sum_{\nu=0}^{r}p_{r-\nu}a_\nu\right| = \frac{1}{P_n}\sum_{r=0}^{n}\left|\sum_{\nu=0}^{r}p^\circ_{r-\nu}a_\nu\right| = o(1),$$

where $p_{r-\nu} = (r+1-\nu)^2 - (r-\nu)^2$ and $P_n = (n+1)^2$. Hence $s_n \to 0$ $[R^*,n,2]$ if and only if

$$\frac{1}{(n+1)^2}\sum_{r=0}^{n}|T_r - T_{r-1}| = o(1), \quad (T_{-1} = 0).$$

From (5.11) it follows that

$$|S_n| \leq \sum_{r=0}^{n}|T_r - T_{r-1}| .$$

Thus if $s_n \to 0$ $[R^*,n,2]$, then $|S_n| = o(n^2)$ so that $s_n \to 0(C,2)$.

Now if $s_n \to s[R^*,n,2]$, then $s_n - s \to 0[R^*,n,2]$ so $s_n - s \to 0(C,2)$, i.e., $s_n \to s(C,2)$. ///

THEOREM 5.9

There is a sequence which is (C,2)*-convergent but not* $[R^*,n,2]$*-convergent.*

PROOF

Choose $\{s_n\}$ so that

$$s_{2n} = (-1)^n n^{3/2} \text{ and } s_{2n+1} = 0.$$

Then $s_n \to o(C,2)$. But referring to (5.10)

$$T_{2r} - T_{2r-1} = s_{2r} - s_{2r-2}$$

$$= (-1)^r\{r^{3/2} + (r-1)^{3/2}\}, \quad r = 1,2,\ldots .$$

So if $2m \leq n \leq 2m+1$, then

$$\sum_{r=0}^{n}|T_r - T_{r-1}| \geq \sum_{r=1}^{m}|T_{2r} - T_{2r-1}|$$

$$\geq \sum_{r=1}^{m}(r-1)^{3/2}$$

$$\sim H\, m^{5/2}$$

$$\sim H_1 n^{5/2}$$

where H, H_1 are independent of n.

Thus $\{s_n\}$ is not [R*,n,2]-convergent to zero and our result follows. ///

THEOREM 5.10

Let $\kappa > 2$

(i) *There is a sequence which is* [R*,n,κ]*-convergent but not* (C,κ)*-convergent.*

(ii) *There is a sequence which is* |R*,n,κ|*-convergent but not* [C,κ]*-convergent.*

(iii) *There is a sequence which is* [R*,n,κ]*-convergent but not* [C,κ]*-convergent.*

PROOF

Part (i) follows from Kuttner's Theorem (iii) and the fact that |R*,n,κ| => [R*,n,κ].

Part (ii) follows from Kuttner's Theorem (iii) and the fact that [C,κ] => (C,κ).

Part (iii) follows from part (ii) and the fact that |R*,n,κ| => [R*,n,κ].

The relations between the various summability methods discussed in this chapter are conveniently displayed in three figures below. In these figures the symbol → denotes strict inclusion, the symbol ↔ denotes equivalence and the notation P ↛ Q means that there is sequence which is P-convergent but not Q-convergent.

$0 < \kappa < 2$

$$\begin{array}{ccccc} |R^*,n,\kappa| & \rightarrow & [R^*,n,\kappa] & \rightarrow & (R^*,n,\kappa) \\ \updownarrow & & \updownarrow & & \updownarrow \\ |C,\kappa| & \rightarrow & [C,\kappa] & \rightarrow & (C,\kappa) \end{array}$$

Figure 1

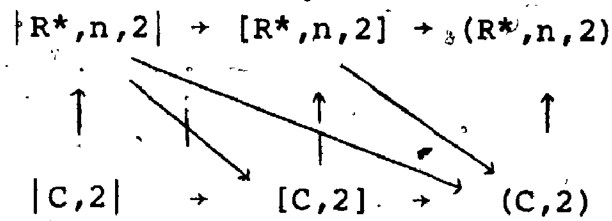

Figure 2

$\kappa > 2$

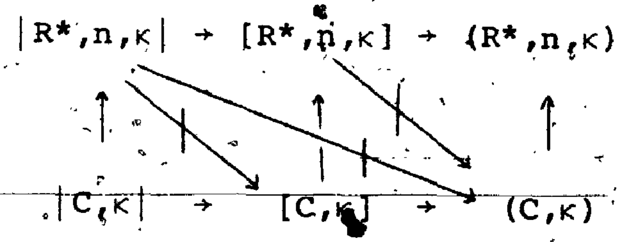

Figure 3

CHAPTER 6

STRONG AND ABSOLUTE NÖRLUND METHODS OF SUMMABILITY ASSOCIATED WITH POLYNOMIALS

In this chapter our investigations stem from the results in D. Borwein [4]. We consider a Nörlund Method of Summability Associated with Polynomials and investigate the properties of an associated Strong Summability Method and of the Absolute Nörlund Method of Summability Associated with Polynomials.

§6.1 DEFINITIONS

Let s, s_n be arbitrary complex numbers, and whenever $n < 0$ we take $s_n = 0$. Let

$$p(z) = p_0 + p_1 z + \cdots + p_j z^j$$

and

$$q(z) = q_0 + q_1 z + \cdots + q_k z^k$$

be polynomials with complex coefficients which satisfy the normalizing conditions

$$p(1) = 1 \text{ and } q(1) = 1.$$

We suppose throughout that $p(0) \neq 0$, $q(0) \neq 0$, $p_n = 0$ for $n > j$ and $q_n = 0$ for $n > k$. We use the notations

(6.1) $$t_n = \sum_{\nu=0}^{n} p_\nu s_{n-\nu}, \qquad n = 0,1,2,\ldots,$$

(6.2) $$u_n = \sum_{\nu=0}^{n} q_\nu s_{n-\nu}, \qquad n = 0,1,2,\ldots .$$

Associated with the polynomial $p(z)$ is a Nörlund Method of Summability N_p which we call a *Polynomial Nörlund Method* and which is defined as follows.

The sequence $\{s_n\}$ is said to be N_p-convergent to s, and we write

(6.3) $$s_n \to s(N_p), \text{ if } \lim_{n\to\infty} t_n = s.$$

This definition is due to D. Borwein.

We define

(6.4) $$s_n \to s\ [C_1, N_p]$$

if $$\frac{1}{n+1} \sum_{r=0}^{n} |t_r - s| = o(1), \quad \text{as } n \to \infty.$$

This is the $[P,Q]_1$ defined in §1.2 with $P = C_1$ and $Q = N_p$.

Let $P_n = \sum_{\nu=0}^{n} p_\nu$ where P_n is non-zero for $n = 0,1,2,...$ and $\tau_n = \frac{1}{P_n} \sum_{\nu=0}^{n} p_\nu s_{n-\nu}$. Then we say that the sequence $\{s_n\}$ is (N,p_n)-convergent to s and we write

(6.5) $$s_n \to s(N,p_n)$$

if $\lim_{n\to\infty} \tau_n = s$.

This is the Nörlund Summability Method given in §5.1, but here we allow p_ν to be complex for all $\nu \geq 0$. Moreover, in this chapter we are only interested in the case where p_ν's are coefficients of a polynomial $p(z)$ with $p(1) = 1$ and we only use the (N,p_n) method in this sense. It is evident that in this sense (N,p_n) is equivalent to the Polynomial Nörlund Method N_p.

Let $P'_n = \sum_{r=0}^{n} |P_r|$ and $P'_n \neq 0$ for $n = 0,1,2,\dots$. Then

(6.6) $$s_n \to s\ [N,P_n]$$

if $\frac{1}{P'_n} \sum_{r=0}^{n} |P_r|\,|\tau_r - s| = o(1)$, as $n \to \infty$.

This definition is analogous to the definition of $[N,p_n]$ given in §5.1, but we allow here p_ν to be complex for $\nu \geqq 0$. Moreover we let p_ν's be coefficients of a polynomial $p(z)$ with $p(1) = 1$.

The Absolute Polynomial Nörlund Summability $|N_p|$ is defined as follows.

(6.7) $$s_n \to s\ |N_p|$$

if $t_n \to s$ and $\sum_{n=0}^{\infty} |t_n - t_{n-1}| < \infty$, where $t_{-1} = 0$.

The method $[C_1,N_p]$ is a Strong Summability Method Associated with the Polynomial Nörlund Method. It is not the Strong Nörlund Summability Method defined in [6] which we considered in Chapter 5. Shortly we shall show that $[C_1,N_p]$ is equivalent to $[N,P_n]$. Thus $[C_1,N_p]$ is the Strong Nörlund Summability Method defined in [6] for (N,P_n) rather than for (N,p_n).

We shall establish at first $[C_1,N_p] \Rightarrow [C_1,N_q]$ if and only if $N_p \Rightarrow N_q$.

It is shown in Borwein and Cass [6] that if $(N,p_n) \Rightarrow (N,q_n)$ then $[N,p_n] \Rightarrow [N,q_n]$. We shall investigate the converse of this theorem in the case of the Polynomial Nörlund Methods.

Then we shall establish $|N_p| \Rightarrow |N_q|$ if and only if $N_p \Rightarrow N_q$.

Finally, we shall establish some minor results analogous to some of the results obtained in [4].

§6.2 THE EQUIVALENCE OF $[C_1,N_p]$ AND $[N,P_n]$

THEOREM 6.1

$$[C_1,N_p] \Leftrightarrow [N,P_n].$$

PROOF

The result is an elementary consequence of the fact that $P_n' = \sum_{r=0}^{n} |P_r| = \sum_{r=0}^{j-1} |P_r| + n - j + 1 \sim n + 1$ which implies the equivalence of $(\overline{N},P_n)$ and $(C,1)$. ///

§6.3 THEOREMS ABOUT NÖRLUND METHODS OF SUMMABILITY ASSOCIATED WITH POLYNOMIALS

For completeness we shall quote without proof several results of Borwein [4].

The methods N_p and N_q mentioned in the following theorems are Nörlund Methods associated with polynomials $p(z)$ and $q(z)$ as defined in §6.1. Evidently N_p and N_q are regular.

THEOREM 6.2

The method N_f*, associated with the polynomial* $f(z) = p(z)q(z)$*, includes both* N_p *and* N_q. (Borwein [4, Theorem 2].)

THEOREM 6.3

The methods N_p *and* N_q *are consistent, i.e., if* $s_n \to s$ (N_p) *and* $s_n \to s'$ (N_q)*, then* $s = s'$. (Borwein [4, Corollary].)

THEOREM 6.4

If $h(z)$ *is the highest common factor of* $p(z)$ *and* $q(z)$*, normalized so as to make* $h(1) = 1$*, then a necessary and sufficient condition for a sequence to be both* N_p*- and* N_q*-convergent is that it be* N_h*-convergent.* (Borwein [4, Theorem 3].)

THEOREM 6.5

In order that N_q *should include* N_p *it is necessary and sufficient that* $q(z)/p(z)$ *should not have poles on or within the unit circle.* (Borwein [4, Theorem I].)

THEOREM 6.6

If $q(z)/p(z)$ *has poles of maximum order* m *on the unit circle and does not have poles within the unit circle, then* $(C,m)N_q$ *includes* N_p*; but for any* $\varepsilon > 0$*, there is an* N_p*-convergent sequence which is not* $(C,m-\varepsilon)N_q$*-convergent.* (Borwein [4, Theorem II].)

THEOREM 6.7

If $q(z)/p(z)$ *has a pole within the unit circle then there is an* N_p*-convergent sequence which is not* AN_q*-convergent.* (Borwein [4, Theorem III].)

THEOREM 6.8

In order that N_p *should be equivalent to* $(C,0)$ *it is necessary and sufficient that* $p(z)$ *should not have zeros on or within the unit circle.* (Borwein [4, Theorem I$^+$].)

THEOREM 6.9

If $q(z)/p(z)$ *has poles* $\lambda_1, \lambda_2, \ldots, \lambda_\ell$*, in the finite complex plane, of orders* $m_1, m_2, \ldots, m_\ell$ *respectively, and if, for* $n = 0, 1, 2, \ldots,$

$$t_n = \sum_{\nu=0}^{n} p_\nu s_{n-\nu},$$

$$u_n = \sum_{\nu=0}^{n} q_\nu s_{n-\nu},$$

then

$$u_n = \sum_{\nu=0}^{n} c_\nu t_{n-\nu} + \sum_{r=1}^{\ell} \sum_{\rho=1}^{m_r} C_{r,\rho} \sum_{\nu=0}^{n} \binom{\nu+\rho-1}{\rho-1} \lambda_r^{-\nu} t_{n-\nu}$$

where the C's *are constants, depending only on* $p_0, p_1, \ldots p_j,$ $q_0, q_1, \ldots, q_k$ *such that* $c_n = 0$ *for* $n > k - j$ *and* $C_{r,m_r} \neq 0$. (Borwein [4, Lemma 1].)

§6.4 $[C_1, N_p]$ METHOD OF SUMMABILITY

The following proposition is a special case of Theorem 1.2.

PROPOSITION 6.1

(i) $N_p \Rightarrow [C_1, N_p]$,

(ii) $[C_1, N_p] \Rightarrow (C,1)N_p$.

THEOREM 6.10

If $q(z)/p(z)$ *has no poles within or on the unit circle, then* $[C_1, N_p] \Rightarrow [C_1, N_q]$.

PROOF

Without loss of generality, we may assume $s_n \to 0$ $[C_1, N_p]$ and prove $s_n \to 0$ $[C_1, N_q]$.

Let $t_n = \sum_{\nu=0}^{n} p_\nu s_{n-\nu}$,

$$u_n = \sum_{\nu=0}^{n} q_\nu s_{n-\nu}.$$

If $q(z)/p(z)$ has no poles within or on the unit circle, but has poles $\lambda_1, \lambda_2, \ldots, \lambda_\ell$ of order $m_1, m_2, \ldots, m_\ell$ outside the unit circle, then by Theorem 6.9

$$u_n = \sum_{\nu=0}^{n} C_\nu t_{n-\nu} + \sum_{r=1}^{\ell} \sum_{\rho=1}^{m_r} C_{r,\rho} \sum_{\nu=0}^{n} \binom{\nu+\rho-1}{\rho-1} \lambda_r^{-\nu} t_{n-\nu}$$

where the C's are constants, depending only on $p_0, p_1, \ldots, p_j, q_0, q_1, \ldots, q_k$, such that $c_n = 0$ for $n > k - j$ and $c_{r,m_r} \neq 0$.

So $|u_n| \leq \sum_{\nu=0}^{n} |c_\nu| |t_{n-\nu}| + \sum_{r=1}^{\ell} \sum_{\rho=1}^{m_r} |c_{r,\rho}| \sum_{\nu=0}^{n} \left|\binom{\nu+\rho-1}{\rho-1}\lambda_r^{-\nu}\right| |t_{n-\nu}|$.

Thus $\frac{1}{m+1}\sum_{n=0}^{m}|u_n|$

$$\leq \frac{1}{m+1}\sum_{n=0}^{m}\sum_{\nu=0}^{n}|c_\nu||t_{n-\nu}| + \frac{1}{m+1}\sum_{n=0}^{m}\sum_{r=1}^{\ell}\sum_{\rho=1}^{m_r}|c_{r,\rho}|\sum_{\nu=0}^{n}\left|\binom{\nu+\rho-1}{\rho-1}\lambda_r^{-\nu}\right||t_{n-\nu}|$$

$$= \sum_{\nu=0}^{m}|c_\nu|\frac{1}{m+1}\sum_{n=0}^{m-\nu}|t_n| + \sum_{r=1}^{\ell}\sum_{\rho=1}^{m_r}|c_{r,\rho}|\sum_{\nu=0}^{m}\left|\binom{\nu+\rho-1}{\rho-1}\lambda_r^{-\nu}\right|\frac{1}{m+1}\sum_{n=0}^{m-\nu}|t_n|,$$

where $c_\nu = 0$, for $\nu > \kappa - j$.

Since the poles of $q(z)/p(z)$ are all outside the unit circle, $|\lambda_r| > 1$, for $r = 1,2,\ldots,\ell$, and $\sum_{\nu=0}^{\infty}\binom{\nu+\rho-1}{\rho-1}\lambda_r^{-\nu}$ is thus absolutely convergent for each $r = 1,2,\ldots,\ell$ and $\rho = 1,2,\ldots,m_r$. Consequently if $\frac{1}{m+1}\sum_{n=0}^{m}|t_n| \to 0$ as $m \to \infty$, then $\frac{1}{m+1}\sum_{n=0}^{m}|u_n| \to 0$ as $m \to \infty$.

If $q(z)/p(z)$ has no poles at all, then

$\frac{1}{m+1}\sum_{n=0}^{m}|u_n| \leq \sum_{\nu=0}^{m}|c_\nu|\frac{1}{m+1}\sum_{n=0}^{m-\nu}|t_n|$, where $c_\nu = 0$ for $\nu > k - j$.

Hence the desired conclusion follows. ///

THEOREM 6.11

If (1) $q(z)/p(z)$ *has a pole within the unit circle,*

or (2) $q(z)/p(z)$ *has no pole within the unit circle, but has poles of maximum order* m *on the unit circle, where* m > 1, *then there is a sequence which is* $[C_1,N_p]$*-convergent but not* $[C_1,N_q]$*-convergent.*

PROOF

(1) $q(z)/p(z)$ has a pole within the unit circle.

By Theorem 6.7 there is an N_p-convergent sequence which is not AN_q-convergent. Since (C,1) is regular, this sequence is $[C_1,N_p]$-convergent. But, since it is not AN_q-convergent, it is not $(C,1)N_q$-convergent. As a consequence of Proposition 6.1(ii) it is not $[C_1,N_q]$-convergent.

(2) $\frac{q(z)}{p(z)}$ has no poles within the unit circle, but has poles of maximum order m on the unit circle, where $m > 1$. By Theorem 6.6 since $m > 1$, there is an N_p-convergent sequence which is not $(C,1)N_q$-convergent. Consequently, this sequence is $[C_1,N_p]$-convergent, but, by Proposition 6.1(ii) it is not $[C_1,N_q]$-convergent. ///

For the next theorem we need the following two lemmas. We use the notation $[C,1]_1$ to mean $[C_1,I]_1$.

LEMMA 6.1

Let $t_n = a\lambda^n$, $|\lambda| = 1$, $\lambda \neq 1$ *and* a *is a non-zero complex number. Then* $\{t_n\}$ *is not* $[C,1]_1$*-convergent.*

PROOF

We know that $\{t_n\}$ is (C,1)-convergent. For

$$\frac{1}{m+1}\sum_{n=0}^{m} t_n = \frac{1}{m+1}\sum_{n=0}^{m} a\lambda^n = \frac{a}{m+1}\,\frac{1-\lambda^{m+1}}{1-\lambda}.$$

Since a is a constant and $\frac{1-\lambda^{m+1}}{1-\lambda} = O(1)$, then $\frac{1}{m+1}\sum_{n=0}^{m} t_n \to 0$, as $m \to \infty$.

Thus if $\{t_n\}$ is $[C,1]_1$-convergent, its sum has to be zero. But

$$\frac{1}{m+1}\sum_{n=0}^{m}|t_n| = \frac{1}{m+1}\sum_{n=0}^{m}|a||\lambda^n| = |a|$$

which $\not\to 0$, since $a \neq 0$. ///

LEMMA 6.2

Let $\lambda_1, \lambda_2, \ldots, \lambda_r$ be r distinct complex numbers, $r > 1$, with $|\lambda_\nu| = 1$, $\lambda_\nu \neq 1$ for $\nu = 1,2,\ldots r$, and let $a_1, a_2, \ldots, a_r$ be non-zero complex numbers. If $t_n = a_1\lambda_1^n + a_2\lambda_2^n + \cdots + a_r\lambda_r^n$, then $\{t_n\}$ is not $[C,1]_1$-convergent.

PROOF

If $\{t_n\}$ is $[C,1]_1$-convergent, its sum must be zero:

$$\frac{1}{m+1}\sum_{n=0}^{m}|t_n| = \frac{1}{m+1}\sum_{n=0}^{m}\left|a_1 + a_2\left(\frac{\lambda_2}{\lambda_1}\right)^n + \cdots + a_r\left(\frac{\lambda_r}{\lambda_1}\right)^n\right|.$$

If $t_n \to 0[C,1]_1$, then $\tau_n = a_1 + a_2\left(\frac{\lambda_2}{\lambda_1}\right)^n + \cdots + a_r\left(\frac{\lambda_r}{\lambda_1}\right)^n \to o(C,1)$. But $\tau_n \to a_1(C,1)$ and $a_1 \neq 0$. ///

THEOREM 6.12

If $\frac{q(z)}{p(z)}$ has no poles within the unit circle, but has simple poles on the unit circle and has no poles of higher order on the unit circle, then there is a sequence which is $[C_1,N_p]$-convergent but not $[C_1,N_q]$-convergent.

PROOF

Suppose $\frac{q(z)}{p(z)}$ has r poles of order 1, $\lambda_1, \lambda_2, \ldots, \lambda_r$, on the unit circle and $r \geq 1$, and suppose it has other poles, $\lambda_{r+1}, \ldots, \lambda_\ell$, outside the unit circle of order $m_{r+1}, \ldots, m_\ell$.

Since $p(1) = 1$, $z = 1$ cannot be a pole of $\frac{q(z)}{p(z)}$. i.e., $\lambda_\nu \neq 1$, for $\nu = 1,2,\ldots r$.

Since $p(0) \neq 0$, $\frac{1}{p(z)}$ is analytic in a neighbourhood U of the origin. There is a sequence $\{s_n\}$ such that, for z in U,

$$\sum_{n=0}^{\infty} s_n z^n = \frac{1}{p(z)} .$$

Then, for z in U

$$\sum_{n=0}^{\infty} t_n z^n = p(z) \sum_{n=0}^{\infty} s_n z^n = 1,$$

$$\sum_{n=0}^{\infty} u_n z^n = q(z) \sum_{n=0}^{\infty} s_n z^n = \frac{q(z)}{p(z)} \sum_{n=0}^{\infty} t_n z^n .$$

Hence $t_0 = 1$, $t_n = 0$ for $n > 0$, and so $\{t_n\}$ is $[C,1]_1$-convergent to zero. That is $\{s_n\}$ is $[C_1,N_p]$-convergent to zero.

Now, by Theorem 6.9,

$$u_n = c_n + \sum_{\nu=r+1}^{\ell} \sum_{\rho=1}^{m_\nu} c_{\nu,\rho} \binom{n+\rho-1}{\rho-1} \lambda_\nu^{-n} + \sum_{\nu=1}^{r} c_{\nu,1} \lambda_\nu^{-n}$$

$$= u_n^1 + u_n^2,$$

where $u_n^1 = c_n + \sum_{\nu=r+1}^{\ell} \sum_{\rho=1}^{m_\nu} c_{\nu,\rho} \binom{n+\rho-1}{\rho-1} \lambda_\nu^{-n}$,

and $u_n^2 = \sum_{\nu=1}^{r} c_{\nu,1} \lambda_\nu^{-n}$.

Since $c_n = 0$ for $n > k - j$, and $|\lambda_\nu| > 1$ for $\nu = r+1, r+2, \ldots \ell$, $\{c_n\}$ and $\left\{c_{\nu,\rho} \binom{n+\rho-1}{\rho-1} \lambda_\nu^{-n}\right\}$ for $\nu = r+1, r+2, \ldots, \ell$, $\rho = 1,2,\ldots, m_\nu$ are each convergent to zero. Since $(C,1)$ is regular, u_n^1 is $[C,1]_1$-convergent to zero. But

$$u_n^2 = \sum_{\nu=1}^{r} c_{\nu,1}\lambda_\nu^{-n} = \sum_{\nu=1}^{r} c_{\nu,1}\frac{\bar{\lambda}_\nu^{n}}{\lambda_\nu^{n}\bar{\lambda}_\nu^{n}} = \sum_{\nu=1}^{r} c_{\nu,1}\bar{\lambda}_\nu^{n},$$

and $\bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_r$ are distinct and distinct from 1. And $|\bar{\lambda}_\nu| = 1$, for $\nu = 1,2,\ldots, r$. Thus by Lemmas 6.1 and 6.2, we know that $\{u_n^2\}$ is not $[C,1]_1$-convergent for $r \geq 1$.

Consequently $\{u_n\}$ is not $[C,1]_1$-convergent, that is $\{s_n\}$ is not $[C_1,N_q]$-convergent. ///

THEOREM 6.13

$[C_1,N_p] \Rightarrow [C_1,N_q]$ *if and only if* $q(z)/p(z)$ *has no poles on or within the unit circle.*

PROOF

The sufficiency part follows from Theorem 6.10. The necessity part follows from Theorems 6.11 and 6.12. ///

THEOREM 6.14

$[C_1,N_p] \Rightarrow [C_1,N_q]$ *if and only if* $N_p \Rightarrow N_q$.

PROOF

This is a consequence of Theorems 6.13 and 6.5. ///

COROLLARY 6.1

If $[C_1,N_p] \Leftarrow\Rightarrow [C_1,N_q]$, *then it is necessary and sufficient that both* $q(z)/p(z)$ *and* $p(z)/q(z)$ *have no poles on or within the unit circle.*

COROLLARY 6.2

$[C_1,N_p] \Leftarrow\Rightarrow [C_1,N_q]$ *if and only if* $N_p \Leftarrow\Rightarrow N_q$.

Noting that N_q is identical with I when $q(z) = 1$ (i.e., $q_0 = 1$, $q_n = 0$ for $n > 0$) and referring to Corollary 6.1 we obtain the following corollary.

COROLLARY 6.3

In order that $[C_1, N_p] \Leftrightarrow [C_1, I]_1$ *it is necessary and sufficient that* $p(z)$ *should not have zeros on or within the unit circle.*

COROLLARY 6.4

$[C_1, N_p] \Leftrightarrow [C, 1]_1$ *if and only if* $N_p \Leftrightarrow I$.

For the following theorems and corollaries about the methods (N,p_n), (N,q_n), (N,P_n), (N,Q_n), $[N,P_n]$ and $[N,Q_n]$ we let p_ν for $\nu = 0,1,\ldots, j$ and q_ν for $\nu = 0,1,\ldots, k$ be the coefficients of the polynomials $p(z)$ and $q(z)$ respectively. We also let $P_r = \sum_{\nu=0}^{r} p_\nu \neq 0$ for $r = 0,1,\ldots, j-1$ and

$Q_r = \sum_{\nu=0}^{r} q_\nu \neq 0$ for $r = 0,1,\ldots, k-1$, and $P_n^* = \sum_{r=0}^{n} P_r \neq 0$ and

$Q_n^* = \sum_{r=0}^{n} Q_r \neq 0$ for all $n \geq 0$, so that (N,p_n), (N,q_n), (N,P_n), (N,Q_n), $[N,P_n]$ and $[N,Q_n]$ are methods associated with $p(z)$ and $q(z)$ respectively and are all well defined.

THEOREM 6.15

$(N,p_n) \Rightarrow (N,q_n)$ *implies that* $(N,P_n) \Rightarrow (N,Q_n)$.

PROOF

Let $\tau_r = \frac{1}{P_r} \sum_{\nu=0}^{r} p_{r-\nu} s_\nu$ and $\mu_r = \frac{1}{Q_r} \sum_{\nu=0}^{r} q_{r-\nu} s_\nu$,

and let $W_n = \frac{1}{P_n^*} \sum_{r=0}^{n} P_{n-r} s_r$ and $V_n = \frac{1}{Q_n^*} \sum_{r=0}^{n} Q_{n-r} s_r$.

Let $k(z) = \frac{q(z)}{p(z)} = \frac{Q(z)}{P(z)}$ and $k(z) = \sum_{\nu=0}^{\infty} k_\nu z^\nu$.

We know that the necessary and sufficient conditions that

$$(N,p_n) \Rightarrow (N,q_n)$$

in this case are

(6.8) $$|k_0||P_n| + \cdots + |k_n||P_0| \leq H |Q_n|$$

where H is independent of n, and

(6.9) $$k_{n-r}/Q_n \to 0, \quad \text{for each } r.$$

(c.f. [6, Proposition 1].)

Thus, if $(N,p_n) \Rightarrow (N,q_n)$, then (6.8) and (6.9) are satisfied.

Now
$$\sum_{r=0}^{n} |k_{n-r}| \left|\sum_{\nu=0}^{r} p_{r-\nu}\right| \leq \sum_{r=0}^{n} |k_{n-r}| \sum_{\nu=0}^{r} |p_{r-\nu}|$$

$$= \sum_{\nu=0}^{n} \sum_{r=\nu}^{n} |k_{n-r}||p_{r-\nu}|$$

$$\leq H \sum_{\nu=0}^{n} |Q_{n-\nu}|$$

$$= H \sum_{\nu=0}^{n} |Q_\nu|$$

$$= O\left(\left|\sum_{\nu=0}^{n} Q_\nu\right|\right)$$

$$= O(|Q_n^*|),$$

since $Q_\nu = 1$ for $\nu \geq k$.

And it is obvious that $k_{n-r}/Q_n^* \to 0$ as $n \to \infty$ for each r.

Thus, by [6, Proposition 1] again, we have

$$(N,P_n) \Rightarrow (N,Q_n). \qquad ///$$

COROLLARY 6.5

$(N,p_n) \Leftrightarrow (N,q_n)$ *implies that* $(N,P_n) \Leftrightarrow (N,Q_n)$.

THEOREM 6.16

$[C_1,N_p] \Rightarrow [C_1,N_q]$ *if and only if* $(N,P_n) \Rightarrow (N,Q_n)$.

PROOF

By Theorem 6.1 we know that $[C_1,N_p] \Leftrightarrow [N,P_n]$ and $[C_1,N_q] \Leftrightarrow [N,Q_n]$.

By [6, Theorem 1], (c.f. Lemma 5.4), we have that if $(N,P_n) \Rightarrow (N,Q_n)$ then $[N,P_n] \Rightarrow [N,Q_n]$.

Thus, if $(N,P_n) \Rightarrow (N,Q_n)$ then $[C_1,N_p] \Rightarrow [C_1,N_q]$.

Conversely, by Theorem 6.14, we have that if $[C_1,N_p] \Rightarrow [C_1,N_q]$, then $N_p \Rightarrow N_q$.

Hence, if $[C_1,N_p] \Rightarrow [C_1,N_q]$ then $(N,p_n) \Rightarrow (N,q_n)$.

It follows from Theorem 6.15 that if $[C_1,N_p] \Rightarrow [C_1,N_q]$ then $(N,P_n) \Rightarrow (N,Q_n)$. ///

COROLLARY 6.6

$[C_1,N_p] \Leftrightarrow [C_1,N_q]$ *if and only if* $(N,P_n) \Leftrightarrow (N,Q_n)$.

THEOREM 6.17

$[C_1,N_p] \Rightarrow [C_1,N_q]$ *if and only if* $(C,1)N_p \Rightarrow (C,1)N_q$.

PROOF

$$(N,P_n) = (\bar{N},P_n)(N,p_n).$$

From the proof of Theorem 6.1, we know that $(\bar{N},P_n) \Leftrightarrow (C,1)$. Thus $(N,P_n) \Leftrightarrow (C,1)(N,p_n) \Leftrightarrow (C,1)N_p$ and similarly we have $(N,Q_n) \Leftrightarrow (C,1)(N,q_n) \Leftrightarrow (C,1)N_q$.

It follows from Theorem 6.16 that $[C_1,N_p] \Rightarrow [C_1,N_q]$ if and only if $(C,1)N_p \Rightarrow (C,1)N_q$. ///

COROLLARY 6.7

$[C_1,N_p] \Leftrightarrow [C_1,N_q]$ *if and only if* $(C,1)N_p \Leftrightarrow (C,1)N_q$.

§6.5 ABSOLUTE POLYNOMIAL NÖRLUND METHODS OF SUMMABILITY

THEOREM 6.18

If $q(z)/p(z)$ *has no poles on or within the unit circle, then* $|N_p| \Rightarrow |N_q|$.

PROOF

Suppose $q(z)/p(z)$ has no poles on or within the unit circle, but has poles $\lambda_1, \lambda_2, \ldots, \lambda_\ell$ of orders $m_1, m_2, \ldots, m_\ell$ outside the unit circle. Let

$$t_n = \sum_{\nu=0}^{n} p_\nu s_{n-\nu}$$

$$u_n = \sum_{\nu=0}^{n} q_\nu s_{n-\nu}, \quad \text{for } n = 0,1,\ldots .$$

Then by Theorem 6.9

$$u_n = \sum_{\nu=0}^{n} c_\nu t_{n-\nu} + \sum_{r=1}^{\ell} \sum_{\rho=1}^{m_r} c_{r,\rho} \sum_{\nu=0}^{n} \binom{\nu+\rho-1}{\rho-1} \lambda_r^{-\nu} t_{n-\nu},$$

where c's are constants, depending only on $p_0, p_1, \ldots p_j$, $q_0, q_1, \ldots q_k$, such that $c_n = 0$ for $n > k - j$ and $c_{r,m_r} \neq 0$.

Hence

$$u_n - u_{n-1} = \sum_{\nu=0}^{n} c_\nu (t_{n-\nu} - t_{n-1-\nu}) + \sum_{r=1}^{\ell} \sum_{\rho=1}^{m_r} c_{r,\rho} \sum_{\nu=0}^{n} \binom{\nu+\rho-1}{\rho-1} \lambda_r^{-\nu} (t_{n-\nu} - t_{n-1-\nu}),$$

by taking $t_{-1} = 0$, $u_{-1} = 0$. And

$$\sum_{n=0}^{m} |u_n - u_{n-1}|$$

$$\leq \sum_{n=0}^{m} \sum_{\nu=0}^{n} |c_\nu| |t_{n-\nu} - t_{n-1-\nu}| + \sum_{n=0}^{m} \sum_{r=1}^{\ell} \sum_{\rho=1}^{m_r} |c_{r,\rho}| \sum_{\nu=0}^{n} \left| \binom{\nu+\rho-1}{\rho-1} \lambda_r^{-\nu} \right| |t_{n-\nu} - t_{n-1-\nu}|$$

$$\leq \sum_{\nu=0}^{m} |c_\nu| \sum_{n=0}^{m} |t_n - t_{n-1}| + \sum_{r=1}^{\ell} \sum_{\rho=1}^{m_r} |c_{r,\rho}| \sum_{\nu=0}^{m} \left| \binom{\nu+\rho-1}{\rho-1} \lambda_r^{-\nu} \right| \sum_{n=0}^{m} |t_n - t_{n-1}|.$$

Since $|\lambda_r| > 1$, for $r = 1,2,\ldots,\ell$,

$\sum_{\nu=0}^{\infty} \left| \binom{\nu+\rho-1}{\rho-1} \lambda_r^{-\nu} \right|$ is convergent for $r = 1,2,\ldots,\ell$, $\rho = 1,\ldots,m_r$.

$c_\nu = 0$, for $\nu > k - j$.

Thus we have $\sum_{n=0}^{m} |t_n - t_{n-1}| = O(1) \Rightarrow \sum_{n=0}^{m} |u_n - u_{n-1}| = O(1)$.

If $q(z)/p(z)$ has no poles at all, then it is readily seen that $\sum_{n=0}^{m} |u_n - u_{n-1}| \leq \sum_{\nu=0}^{m} |c_\nu| \sum_{n=0}^{m} |t_n - t_{n-1}|$. Since $c_\nu = 0$ for $\nu > k - j$, we have $\sum_{n=0}^{m} |t_n - t_{n-1}| = O(1)$.

$\Rightarrow \sum_{n=0}^{m} |u_n - u_{n-1}| = O(1)$. By Theorem 6.3, N_p and N_q are consistent. Thus $|N_p| \Rightarrow |N_q|$. ///

PROPOSITION 6.2

$|N_p| \Rightarrow [C_1, N_p]$.

PROOF

$s_n \in |N_p| \Rightarrow t_n \to s$, as $n \to \infty$ for some s. Since

(C,1) is regular, $s_n \to s\ [C_1, N_p]$. ///

THEOREM 6.19

If $q(z)/p(z)$ *has a pole within the unit circle, then there is a sequence which is* $|N_p|$*-convergent but not* $|N_q|$*-convergent.*

PROOF

Since $p(0) \neq 0$, $\frac{1}{p(z)}$ is analytic in a neighbourhood U of origin. There is a sequence $\{s_n\}$ such that for z in U,

$$\sum_{n=0}^{\infty} s_n z^n = \frac{1}{p(z)} .$$

Let

$$t_n = \sum_{\nu=0}^{n} p_\nu s_{n-\nu},$$

$$u_n = \sum_{\nu=0}^{n} q_\nu s_{n-\nu}.$$

Then, for z in U,

$$\sum_{n=0}^{\infty} t_n z^n = p(z) \sum_{n=0}^{\infty} s_n z^n = 1,$$

$$\sum_{n=0}^{\infty} u_n z^n = q(z) \sum_{n=0}^{\infty} s_n z^n = \frac{q(z)}{p(z)} .$$

Hence $t_0 = 1$, $t_n = 0$ for $n > 0$, and so $\{s_n\}$ is $|N_p|$-convergent. On the other hand $\sum_{n=0}^{\infty} u_n z^n$ has a radius of convergence less than unity, because by hypothesis $q(z)/p(z)$ has a pole within the unit circle. Consequently $\{u_n\}$ is not A-convergent and so it is not (C,1)-convergent. Hence $\{s_n\}$ is not $(C,1)N_q$-convergent. By Proposition 6.1(ii), $\{s_n\}$

is not $[C_1, N_q]$-convergent. Thus by Proposition 6.2 $\{s_n\}$ is not $|N_q|$-convergent. ///

THEOREM 6.20

If $\frac{q(z)}{p(z)}$ *has no poles within the unit circle, but has poles on the unit circle, then there is a sequence which is* $|N_p|$*-convergent but not* $|N_q|$*-convergent.*

PROOF

Let the poles of $\frac{q(z)}{p(z)}$ be $\lambda_1, \lambda_2, \ldots, \lambda_\ell$ of orders $m_1, m_2, \ldots, m_\ell$. Let the numbering be such that of these poles $\lambda_1, \lambda_2, \ldots, \lambda_{\ell'}$ are on the unit circle, $\lambda_{\ell'+1}, \ldots, \lambda_\ell$ are outside the unit circle.

Since $p(0) \neq 0$, $\frac{1}{(1-z)p(z)}$ is analytic in a neighbourhood U of origin. There is a sequence $\{s_n\}$ such that, for z in U,

$$\sum_{n=0}^{\infty} s_n z^n = \frac{1}{(1-z)p(z)}$$

Then, for z in U,

$$\sum_{n=0}^{\infty} t_n z^n = p(z) \sum_{n=0}^{\infty} s_n z^n = \frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots ,$$

$$\sum_{n=0}^{\infty} u_n z^n = q(z) \sum_{n=0}^{\infty} s_n z^n = \frac{q(z)}{p(z)} \sum_{n=0}^{\infty} t_n z^n .$$

Hence $t_n = 1$ for all $n \geq 0$ and so $\{s_n\}$ is $|N_p|$-convergent.

Now, by Theorem 6.9 since $t_n = 1$, for all $n \geq 0$,

$$u_n = \sum_{\nu=0}^{n} c_\nu + \sum_{r=1}^{\ell} \sum_{\rho=1}^{m_r} c_{r,\rho} \sum_{\nu=0}^{n} \binom{\nu+\rho-1}{\rho-1} \lambda_r^{-\nu} ,$$

where the c's are constants, depending only on $p_0, p_1, \ldots, p_j, q_0, q_1, \ldots, q_k$ such that $c_n = 0$ for $n > k - j$ and $c_{r,m_r} \neq 0$.

Thus

$$u_n - u_{n-1} = c_n + \sum_{r=1}^{\ell} \sum_{\rho=1}^{m_r} c_{r,\rho} \binom{n+\rho-1}{\rho-1} \lambda_r^{-n}$$

$$= c_n + \sum_{r=\ell'+1}^{\ell} \sum_{\rho=1}^{m_r} c_{r,\rho} \binom{n+\rho-1}{\rho-1} \lambda_r^{-n} + \sum_{r=1}^{\ell'} \sum_{\rho=1}^{m_r} c_{r,\rho} \binom{n+\rho-1}{\rho-1} \lambda_r^{-n}$$

$$= w_n^1 + w_n^2,$$

where $w_n^1 = c_n + \sum_{r=\ell'+1}^{\ell} \sum_{\rho=1}^{m_r} c_{r,\rho} \binom{n+\rho-1}{\rho-1} \lambda_r^{-n}$,

$$w_n^2 = \sum_{r=1}^{\ell'} \sum_{\rho=1}^{m_r} c_{r,\rho} \binom{n+\rho-1}{\rho-1} \lambda_r^{-n}.$$

Since $c_n = 0$ for $n > k - j$, $\sum_{n=0}^{\infty} c_n$ is absolutely convergent, and since $|\lambda_r| > 1$, for $r = \ell'+1, \ldots, \ell$, $\sum_{n=0}^{\infty} \binom{n+\rho-1}{\rho-1} \lambda_r^{-n}$ is absolutely convergent, for $r = \ell'+1, \ldots, \ell$, $\rho = 1, 2, \ldots, m_r$. Hence $\sum_{n=0}^{\infty} |w_n^1|$ is convergent.

Now, for w_n^2, if there are ℓ'' poles on the unit circle of maximum order m, where $1 \leq \ell'' \leq \ell'$ and $m \geq 1$, then we let the numbering be such that $\lambda_1, \lambda_2, \ldots, \lambda_{\ell''}$ have maximum order m. In this case,

$$|w_n^2| = \left| \sum_{r=1}^{\ell''} \sum_{\rho=1}^{m} c_{r,\rho} \binom{n+\rho-1}{\rho-1} \lambda_r^{-n} + \sum_{r=\ell''+1}^{\ell'} \sum_{\rho=1}^{m_r} c_{r,\rho} \binom{n+\rho-1}{\rho-1} \lambda_r^{-n} \right|$$

$$= O\left| c_{1,m}\binom{n+m-1}{m-1}\lambda_1^{-n} + c_{2,m}\binom{n+m-1}{m-1}\lambda_2^{-n} + \cdots + c_{\ell'',m}\binom{n+m-1}{m-1}\lambda_{\ell''}^{-n} \right|$$

$$= O\binom{n+m-1}{m-1} \left| c_{1,m}\lambda_1^{-n} + c_{2,m}\lambda_2^{-n} + \cdots + c_{\ell'',m}\lambda_{\ell''}^{-n} \right|$$

$$= O\binom{n+m-1}{m-1} \left| c_{1,m}\overline{\lambda}_1^{n} + c_{2,m}\overline{\lambda}_2^{n} + \cdots + c_{\ell'',m}\overline{\lambda}_{\ell''}^{n} \right| .$$

Since $p(1) = 1$, $z = 1$ is not a pole of $\frac{q(z)}{p(z)}$; and since $\overline{\lambda}_\nu$ are distinct and distinct from 1, and $|\overline{\lambda}_\nu| = 1$, for $\nu = 1,2,\ldots, \ell''$; and $c_{\nu,m} \neq 0$, for $\nu = 1,2,\ldots, \ell''$, by Lemma 6.1 and Lemma 6.2 we know that $\{c_{1,m}\overline{\lambda}_1^n + c_{2,m}\overline{\lambda}_2^n + \cdots + c_{\ell'',m}\overline{\lambda}_{\ell''}^n\}$ is not $[C,1]_1$-convergent for $\ell'' \geqq 1$. Thus $\{|c_{1,m}\overline{\lambda}_1^n + c_{2,m}\overline{\lambda}_2^n + \cdots + c_{\ell'',m}\overline{\lambda}_{\ell''}^n|\}$ cannot be convergent. *A fortiori* it does not converge to zero. Hence $|w_n^2|$ does not tend to zero as $n \to \infty$. This means that $\sum_{n=0}^{\infty} |w_n^2|$ diverges. Consequently

$$\sum_{n=0}^{m} |u_n - u_{n-1}| \text{ diverges, as } n \to \infty.$$

In other words, $\{s_n\}$ is not $|N_q|$-convergent. ///

THEOREM 6.21

In order that $|N_p| \Rightarrow |N_q|$, *it is necessary and sufficient that* $\frac{q(z)}{p(z)}$ *should not have poles on or within the unit circle.*

PROOF

The sufficiency part of the theorem follows from Theorem 6.18. The necessity part follows from Theorems 6.19 and 6.20. ///

COROLLARY 6.8

$|N_p| \Rightarrow |N_q|$ *if and only if* $N_p \Rightarrow N_q$.

PROOF

This follows from Theorems 6.5 and 6.21. ///

COROLLARY 6.9

$|N_p| \Rightarrow |N_q|$ *if and only if* $[C_1, N_p] \Rightarrow [C_1, N_q]$.

PROOF

This follows from Theorems 6.13 and 6.21. ///

COROLLARY 6.10

$|N_p| \Leftrightarrow |N_q|$ *if and only if* $\frac{q(z)}{p(z)}$ *and* $\frac{p(z)}{q(z)}$ *both have no poles on or within the unit circle.*

COROLLARY 6.11

$|N_p| \Leftrightarrow |N_q|$ *if and only if* $N_p \Leftrightarrow N_q$.

COROLLARY 6.12

$|N_p| \Leftrightarrow |N_q|$ *if and only if* $[C_1, N_p] \Leftrightarrow [C_1, N_q]$.

Noting that N_q is identical with I when $q(z) = 1$, we have, as a consequence of Corollary 6.10, the following corollary.

COROLLARY 6.13

In order that $\{s_n\}$ *is* $|N_p|$ *convergent if and only if*

$\sum_{r=0}^{\infty} a_\nu$ *is absolutely convergent it is necessary and sufficient that* p(z) *should not have zeros on or within the unit circle.*

§6.6 SOME MINOR RESULTS

THEOREM 6.22

If $f(z) = p(z)q(z)$, *then*

(i) $[C_1,N_p] \Rightarrow [C_1,N_f]$ *and* $[C_1,N_q] \Rightarrow [C_1,N_f]$,

(ii) $|N_p| \Rightarrow |N_f|$ *and* $|N_q| \Rightarrow |N_f|$.

PROOF

(i) follows from Theorem 6.13 and (ii) follows from Theorem 6.21. ///

COROLLARY 6.14

The methods $[C_1,N_p]$ *and* $[C_1,N_q]$ *are consistent, i.e., if* $s_n \to s$ $[C_1,N_p]$ *and* $s_n \to s'$ $[C_1,N_q]$, *then* $s = s'$.

THEOREM 6.23

If h(z) *is the highest common factor of* p(z) *and* q(z) *normalized so as to make* h(1) = 1, *then*

(i) *a sequence is both* $[C_1,N_p]$- *and* $[C_1,N_q]$-*convergent if and only if it is* $[C_1,N_h]$-*convergent,*

(ii) *a sequence is both* $|N_p|$- *and* $|N_q|$-*convergent if and only if it is* $|N_h|$-*convergent.*

PROOF

(i) The sufficiency part follows from Theorem 6.22 (i).

To prove the necessity part, we observe that there are polynomials

$$a(z) = \sum_{n=0}^{\ell_1} a_n z^n$$

$$b(z) = \sum_{n=0}^{\ell_2} b_n z^n$$

such that $h(z) = a(z)p(z) + b(z)q(z)$

$$= \sum_{n=0}^{\ell_3} h_n z^n, \text{ say,}$$

where ℓ_1, ℓ_2, ℓ_3 are non-negative integers.

Hence if $t_n = \sum_{\nu=0}^{n} p_\nu s_{n-\nu}$ and $u_n = \sum_{\nu=0}^{n} q_\nu s_{n-\nu}$, then

$$w_n = \sum_{\nu=0}^{n} h_\nu s_{n-\nu} = \sum_{\nu=0}^{n} a_\nu t_{n-\nu} + \sum_{\nu=0}^{n} b_\nu u_{n-\nu},$$

where $a_\nu = 0$, for $\nu > \ell_1$ and $b_\nu = 0$, for $\nu > \ell_2$.

Without loss of generality, we may assume $s_n \to o[C_1, N_p]$ and $s_n \to o[C_1, N_q]$. Now

$$\frac{1}{m+1} \sum_{n=0}^{m} |w_n| \leq \frac{1}{m+1} \sum_{n=0}^{m} \sum_{\nu=0}^{n} |a_\nu| |t_{n-\nu}| + \frac{1}{m+1} \sum_{n=0}^{m} \sum_{\nu=0}^{n} |b_\nu| |u_{n-\nu}|$$

$$\leq \frac{1}{m+1} \sum_{n=0}^{m} \left(\sum_{\nu=0}^{\ell_1} |a_\nu| \right) |t_n| + \frac{1}{m+1} \sum_{n=0}^{m} \left(\sum_{\nu=0}^{\ell_2} |b_\nu| \right) |u_n|$$

$$= o(1), \quad \text{as } m \to \infty.$$

That is $s_n \to o[C_1, N_h]$.

(ii) The sufficiency part follows from Theorem 6.22(ii). As in the proof of (i),

$$w_n = \sum_{\nu=0}^{n} h_\nu s_{n-\nu} = \sum_{\nu=0}^{n} a_{n-\nu} t_\nu + \sum_{\nu=0}^{n} b_{n-\nu} u_\nu.$$

Hence $w_n - w_{n-1}$

$$= \sum_{\nu=0}^{n} a_{n-\nu}(t_\nu - t_{\nu-1}) + \sum_{\nu=0}^{n} b_{n-\nu}(u_\nu - u_{\nu-1}),$$

where $t_{-1} = 0$, $u_{-1} = 0$, $w_{-1} = 0$ and $a_{n-\nu} = 0$ if $n - \nu > \ell_1$, $b_{n-\nu} = 0$, if $n - \nu > \ell_2$.

$$\sum_{n=0}^{m} |w_n - w_{n-1}|$$

$$\leq \sum_{n=0}^{m} \sum_{\nu=0}^{n} |a_{n-\nu}||t_\nu - t_{\nu-1}| + \sum_{n=0}^{m} \sum_{\nu=0}^{n} |b_{n-\nu}||u_\nu - u_{\nu-1}|$$

$$\leq \left(\sum_{\nu=0}^{\ell_1} |a_\nu|\right) \sum_{n=0}^{m} |t_n - t_{n-1}| + \left(\sum_{\nu=0}^{\ell_2} |b_\nu|\right) \sum_{n=0}^{m} |u_n - u_{n-1}|.$$

Hence if $\sum_{n=0}^{m} |t_n - t_{n-1}| = O(1)$ and $\sum_{n=0}^{m} |u_n - u_{n-1}| = O(1)$,

then $\sum_{n=0}^{m} |w_n - w_{n-1}| = O(1)$. ///

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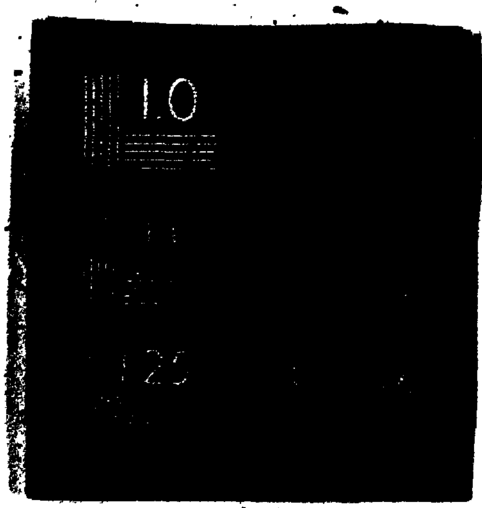

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