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Essays On The Dynamic Theory Of Optimal Policy For An Open Economy

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ESSAYS ON THE DYNAMIC THEORY OF OPTIMAL POLICY FOR AN OPEN ECONOMY

by

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ABSTRACT

This thesis is comprised of two distinct segments pertaining to the optimal policy choices of an economy interacting with the rest of the world. It is our purpose to attempt the construction of dynamic models which introduce the complexities of labour mobility and international externalities into the maze of factors determining government policy prescriptions. Consequently, we define an open economy as one which engages in international transactions be it product movements, factor movements or externalities.

The first part of the thesis deals with the role of labour mobility in the determination of a national policy plan. The models that have traditionally tackled this problem have been specified within a static framework. Manifestly, for a more complete characterization of the reality of the factor movement process one must undertake the intertemporal policy analysis relating to the allocation pattern concerning products and factors between nations.

We permit labour to be perfectly mobile between countries in the sense that at each time period the nation may or may not import or export labour. Yet although labour services are permitted to move, the owners of these services are tied to their home country. This means that the optimal policy is determined only for a given time interval, at the termination of the period all workers return home. Individuals, as a consequence, may never change nationalities even though they are able to work in foreign countries.

In our analysis the domestic country pursues its national welfare

over time which is defined by a social welfare functional. Our formalization of the functional is novel because we include not only consumption but also consideration of the externality effect that imported or exported labour may exert on welfare. The foreign country acts passively and is represented by a time invariant Mill-Marshall offer function. We are then able to derive temporal and intertemporal equilibria and comparative equilibria effects under various patterns of consumption, investment, production relations, product flows and labour flows between nations.

The last part of the thesis examines the implications of externalities, both international and intranational, in an economy interacting with the rest of the world in a game-theoretical environment.

We posit the existence of private and public sector capital stocks. The later jointly yields production benefits and consumption benefits to society. There are international externalities as formalized by the capital stock of a nation affecting not only welfare but also the technology of the other economy. We are then able to prove the existence, uniqueness and stability of world equilibrium under different assumptions concerning the behavioral interactions of the nations and formalize the optimal paths each economy should select.

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Finally, I wish to dedicate this thesis to my wife, Lidia. Words are truly inadequate when I state that this dissertation would not have been accomplished without her understanding, inspiration and love.

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LIST OF SYMBOLS

SYMBOL	NAME OF SYMBOL
Chapter 1.	Optimal Growth, International Trade And Labour Mobility
t	Time
Y_1	Output of consumption product
Y_2	Output of investment product
F_1	Production function of consumption product
F_2	Production function of investment product
K_1	Capital input of consumption product
K_2	Capital input of investment product
L_1	Labour input of consumption product
L_2	Labour input of investment product
X_1	Consumption expenditure of residents
X_2	Investment expenditure
Z_1	Excess demand for consumption
Z_2	Excess demand for investment
K	Capital endowment
L	Labour endowment
Z_L	Excess demand for labour
μ	Rate of capital depreciation
n	Rate of growth labour
K_o	Initial endowment of capital
L_o	Initial endowment of labour
z_L	Excess demand for labour in labour endowment form

SYMBOL	NAME OF SYMBOL
y_1	Output of consumption product in labour endowment form
y_2	Output of investment product in labour endowment form
f_1	Production function of consumption product in labour intensive form
f_2	Production function of investment product in labour intensive form
x_1	Consumption expenditure of residents in labour endowment form
x_2	Investment expenditure in labour endowment form
z_1	Excess demand for consumption in labour endowment form
z_2	Excess demand for investment in labour endowment form
k	Capital-labour ratio
k_1	Capital-labour input of consumption product
k_2	Capital-labour input of investment product
ℓ_1	Proportion of labour as an input in the consumption product
ℓ_2	Proportion of labour as an input in the investment product
k_o	Initial capital-labour ratio
λ	Rate of growth of labour plus rate of capital depreciation
g_1	Function comprising the foreign offer function
g_2	Function comprising the foreign offer function
ν	Social welfare function
U	Social welfare function
ϕ	Negative of consumption per imported or exported worker

SYMBOL	NAME OF SYMBOL
$\tilde{W}$	Social welfare functional
δ	Social rate of discount
X	Vector of non-excess demand commodities
Z	Vector of excess demands
V	Vector of prices
I	Objective functional
$L(\cdot)$	Lagrangian function
λ_1	Supply price of consumption
λ_2	Supply price of investment
p	Demand price of investment
w	The wage rate
r	The rental rate
ω	The wage/rental rate
y	Vector of outputs in labour endowment form
H	Hamiltonian function
q	Demand price of investment
ω_1	The wage/rental rate for the consumption product
ω_2	The wage/rental rate for the investment product
α	Rate of Harrod neutral technological change
β	Rate of labour endowment learning by doing
γ	Rate of labour input learning by doing
x_1^n	Consumption expenditure of nationals in labour endowment form
ξ_1, ξ_2	Characteristic roots of the steady state

SYMBOL	NAME OF SYMBOL
Chapter 2.	Externalities And Public Investment In A Two Country Differential Game Model
t	Time
Y_i	Output of country i = 1,2
F_i	Production function of country i = 1,2
H_i	Production function of country i = 1,2
K_{p_i}	Private capital of country i = 1,2
K_{g_i}	Public capital of country i = 1,2
L_i	Labour endowment of country i = 1,2
K_i	Capital endowment of country i = 1,2
C_i	Consumption in country i = 1,2
I_{p_i}	Private investment in country i = 1,2
I_{g_i}	Public investment in country i = 1,2
μ_i	Rate of capital depreciation in country i = 1,2
n	Rate of growth of labour in country 1 and 2
L_o	Initial endowment of labour in country 1 and 2
y_i	Output of country i = 1,2 in labour endowment form
k_{p_i}	Private capital of country i = 1,2 in labour endowment form
k_{g_i}	Public capital of country i = 1,2 in labour endowment form
c_i	Consumption in country i = 1,2 in labour endowment form
i_{p_i}	Private investment in country i = 1,2 in labour endowment form
i_{g_i}	Public investment in country i = 1,2 in labour endowment form

SYMBOL	NAME OF SYMBOL
υ_i	Social welfare function of country $i = 1,2$
U_i	Social welfare function of country $i = 1,2$
V_i	Social welfare function of country $i = 1,2$
W_i	Social welfare functional of country $i = 1,2$
B_i	Bliss point of country $i = 1,2$
f_i	Production function of country $i = 1,2$ in labour intensive form
h_i	Production function of country $i = 1,2$ in labour intensive form
k_i	Capital-labour ratio of country $i = 1,2$
k_{o_i}	Initial capital-labour ratio of country $i = 1,2$
λ_i	Rate of growth of labour plus rate of capital depreciation in country $i = 1,2$
$L_i(\cdot)$	Lagrangian function of country $i = 1,2$
p_i	Price of investment in country $i = 1,2$
q_i	Price of capital in country $i = 1,2$
Δ_i	Determinant in country $i = 1,2$
$\bar{c}_i$	Maximum level of consumption in labour endowment form in country $i = 1,2$
$\bar{k}_i$	Maximum level of the capital-labour ratio in country $i = 1,2$
$\underline{k}_i$	Minimum level of the capital-labour ratio in country $i = 1,2$
S_i	Stackleberg steady state reaction function for country $i = 1,2$

Chapter 1. Optimal Growth, International Trade And Labour Mobility

I. Introduction

The role of factor mobility in the determination of a national policy formulation has always been recognized as an important theoretical, as well as practical matter. The determinants of international factor movements, as in the causes of trade, depend essentially on national economic differentials. These differentials may be classified according to preference, technological or equilibrium conditions and national endowments.

We are able to discern two paradigms in the literature on the theory of factor mobility. Robert Mundell's work [13] exemplifies the first. He proved that in a Heckscher-Ohlin framework (endowments differ between countries) perfectly mobile capital is a substitute for product movements when a country imposes a prohibitive tariff. Secondly, Ronald Jones [10] and Murray Kemp [11] analysed the role of perfect capital mobility in models where the technology differs between nations and optimal taxes were derived for the capital movements. Finally, John Chipman [7] has developed a comprehensive examination of perfectly mobile capital and the various tariffs and taxes that may be imposed.

The importance of these models lies in their ability to determine capital flows with reference to fundamental international economic differences. In addition, they all stress the materiality of monopoly power by explicating the government's effect on the international equilibrium through tariffs and taxes. However, there are two significant directions that we can extend the previous models. Firstly, the results of these works have been gleaned from static structures. But, one must undertake a dynamic analysis of the allocation process for a more complete characterization of the reality of factor movements. Secondly, the role of the government in influencing the equilibrium conditions extends far beyond the imposition of various taxes and tariffs. Governments of factor importing and exporting countries may impose quota restrictions, queing procedures or may completely prohibit factor movements with certain nations. For instance, Canada only issues temporary work visas to named workers. An employer cannot order a group of anonymous labourers. The foreign government supplies the names to the Canadian Government who then enters into the transactions process.

Narrowing the focus to labour mobility alone, L. Hunter and G. Reid [9] state that governments may restrict the amount and composition of individual wealth internationally transferable. Migrants may be able to allocate their labour income into any form of savings, but before their return home they may be constrained to sell all their financial commodities of the foreign country. Political, social and economic institutional differences among countries are examples of further hindrances. Some of these differences will affect the migrant's behavior. For example, a policy objective of the Swiss Government was to encourage economic expansion yet at the same time protect their workers. Among the means of achieving this target was the restriction of "guest workers" to specific residential areas and types of employment, while at the same time denying them equal social security provisions. Manifestly, the implications of the dynamic nature of factor mobility and the government involvement in the transactions process must surely play a crucial part in determining a national policy plan.

Recently there has been theoretical work which has attempted to capture some of these complexities. James Melvin and James Markusen

[13] have dealt with the role of the government in a static Hechscher-Ohlin framework. They analyse the welfare effects of a government restricted capital movement from a large to a small economy.[1] On the dynamic side, Pranab Bardhan [5] has constructed a simple one-product, two-factor system in which a country, who influences the world rate of interest, may or may not import capital in a non-trading situation. Robin Bade [2] has extended the Bardhan model to allow for imperfect capital mobility. Bade and Bardhan focus on the dynamics and stress the role of the government by couching the problem in a centralized economic environment.

The purpose of this paper is to extend the two-factor, two-product, neoclassical optimal growth model to include trade and labour mobility. Bardhan [4] and Harl Ryder [15], among others, treat aspects of optimal growth and trade, but they assume, in particular, perfect factor immobility. In this essay we permit labour to be mobile between two countries, but under government control. However, a country not intertemporally constrained to only import or export labour brings to the forefront a fundamental distinction in discussions of population migration. The basic distinction pertains to permanent and temporary individual movements.

Our objective centres on temporary movements, which we will refer to as labour mobility or in everyday parlance as the "guest worker" or the "foreign contract labour" phenomenon. Formally, this means that although the government determined flow of labour services may move internationally, the owners of these services, the households, are tied to their home country. The government's optimal labour policy is effective for a given time interval; at the termination of the period all workers return home. Individuals, as a consequence, never change nationalities even though they are able to work in foreign countries. In this essay, then, we do not address any of the questions relating to permanent population migration such as the brain drain.

The significance of temporary labour movements cannot be underestimated from the viewpoint of recent historical and contemporary information describing the functioning of the world's economies. W. R. Böhning [6] presents an extensive institutional and quantitative description of the guest worker phenomenon on the European scene.[2] In particular, he discusses the E.E.C. countries, Switzerland, Austria, among others and the various traditional "feeder" countries such as Spain, Turkey, Yugoslavia and Portugal. Indeed, as is occasionally overlooked, the temporary foreign worker is present in the North American economies. Canada, for instance, imports guest workers from various nations. In particular, British Columbia imports labourers from Yugoslavia, the Prairie Provinces from Mexico and the resource and construction industries import from Britain, Poland and Scandinavia.

Having described the class of individual movements we are about to analyse we now need to specify the determinants of these flows. It is assumed that demand and/or production conditions differ between the countries such that there is the potential for international trade and labour transactions. In addition, as in Ryder [15], we suppose that one centralized nation determines the world prices of all the commodities. The fact that the price-setting nation is centrally planned implies that the monopoly power of this economy manifests itself in the government. These assumptions explicitly formalize the prominence of the government in the transactions process. In our model the large centralized economy maximizes, over time, its social welfare functional subject to technological and equilibrium conditions as well as the time-invariant offer function of the foreign country. The small foreign country is solely represented by its offer function because it behaves passively. This means that any set of prices determined by the large nation will bring about a definite quantity of product and labour movements between the two economies.[3]

Our formalization of the welfare functional is novel because we include not only consumption but also consideration of the externality effect that imported or exported labour may exert on welfare. To the receiving country there are the obvious benefits that foreign workers facilitate and accelerate the path to economic development. There is also the social gains from the intermingling of individuals from various cultures which enhances human understanding. Yet, "guest workers" are usually unskilled and therefore tend to have a relatively lower level of labour income. Hence by government edict or individual choice the foreign workers may gravitate to squalid ghettoes which may lead to crime and reinforce discrimination. The sending countries benefit from exporting surplus labour by maintaining full employment of their nationals even though their domestic production capabilities could not absorb the complete labour force. Finally, there are the social costs of having to separate families in order for workers to seek employment in foreign countries. Hence, we construct our model so that we are able to account for welfare external economies or diseconomies.

Among the major qualitative conclusions pertaining to the momentary or temporal equilibrium of the large nation is that in the region of production diversification we are able to prove the Generalized Rybczynski theorem, that an increase in the capital-labour endowment ratio increases (decreases) the capital-intensive (labour-intensive) product. Also the excess demand for labour is a function of different variables depending on the pattern of production. In production diversification an increase in the demand price of investment will increase (decrease) imports (or decrease (increase) exports) of labour when the investment product is labour-intensive (capital-intensive). When the economy completely specializes in the production of the investment product an increase in the investment price and the capital-labour ratio will increase imports (or decrease exports) of labour. Furthermore, an increase in the capital-labour ratio will increase imports (or decrease exports) of labour when the nation only produces the consumption product. Irrespective of the region of complete production specialization, the optimal value of imports (exports) of labour will be less than (greater than) the labour supply in labour endowment form divided by the capital-labour elasticity of labour imports (exports).

In terms of the optimal path one of the significant results is that the trajectory tracing the movement of the investment price as a function of the capital-labour endowment ratio may be upward sloping for values of the capital-labour ratio less than the steady state value. This implies that the country does not have to pay such a high price for being underdeveloped relative to nations without factor mobility. The reason for this is due to the fact that the government can increase the welfare of the nation by exporting more (or importing less) labour, thus increasing the consumption of their nationals remaining at home due to increased domestic consumption expenditure and increased repatriated labour income by the exported workers.

The basic model is set forth in section 2. In section 3, the conditions of temporal equilibrium are derived along with the alternative patterns of specialization of expenditure, production, trade and labour mobility. Section 4 deals with the intertemporal equilibrium and the nature of the optimal path. We then extend our analysis to allow for reversible investment, Harrod neutral technological change, for a certain class of learning by doing production functions and the competitive model.

2. The Model

We consider a two sector neoclassical growth model. In the model there are two countries, a domestic or home country and the foreign country. In each nation there exists three categories of agents, households, firms and a government. We have a consumption product, an investment product and the factors of production, labour and capital. ~~In the domestic country let~~

$$Y_i(t) = F_i(K_i(t), L_i(t)), \quad K_i(t), L_i(t) \geq 0 \quad i = 1, 2 \qquad (1)$$

where $Y_i(t)$ is the output of the i^{th} product ($i = 1$ is the consumption product, $i = 2$ is the investment product) in period t. F_i is the production function of the i^{th} product, $K_i(t)$ and $L_i(t)$ are the inputs of capital and labour, respectively in the i^{th} product in period t. We assume that the production functions have the following properties

F_i is homogeneous of degree one in $K_i(t)$, $L_i(t)$

$$\frac{\partial F_i}{\partial K_i(t)} > 0, \quad \frac{\partial F_i}{\partial L_i(t)} > 0$$

$$\frac{\partial^2 F_i}{\partial K^2_i(t)} < 0, \quad \frac{\partial^2 F_i}{\partial L_i^2(t)} < 0, \text{ for } 0 < (K_i(t), L_i(t)) < \infty \quad i = 1, 2.$$

$K_i(t)$ and $L_i(t)$ are piecewise continuous functions of time, $i = 1, 2.$[4]

Next we define consumption and investment as

$$X_i(t) = Y_i(t) + Z_i(t) \quad X_i(t) \geq 0, \quad Z_i(t) \varepsilon (-\infty, \infty), \; i = 1, 2. \qquad (2)$$

where $X_i(t)$ is consumption in period t, $X_2(t)$ is investment in period t and $Z_i(t)$ is the excess demand of the i^{th} product in period t with $Z_i(t) > 0$ signifying imports and $Z_i(t) < 0$ signifying exports. We also assume that $Z_i(t)$ is a piecewise continuous function time. Hence by the assumptions on $K_i(t)$, $L_i(t)$ and F_i we have $Y_i(t)$ is a piecewise continuous function and thus $X_i(t)$ is piecewise continuous.

The capital services market is delineated by

$$K_1(t) + K_2(t) = K(t) \tag{3}$$

where K(t) is the endowment of capital which is given in period t. K(t) is continuous and possesses piecewise continuous first derivatives. The equation for the rate of change of capital over time is given by

$$\dot{K}(t) = X_2(t) - \mu K(t) \tag{4}$$

where $K(t) \geq 0$, $0 \leq \mu < \infty$ and fixed with μ being the rate of depreciation of capital, and $K(0) = K_o$, $0 < K_o < \infty$, is the given endowment.

Finally, the labour services market is characterized as

$$L_1(t) + L_2(t) - Z_L(t) = L(t), \quad Z_L(t) \in (-\infty,\infty) \tag{5}$$

where L(t) is the endowment of labour in period t. In addition L(t) is continuous, has piecewise continuous first derivatives and is given by

$$L(t) = L_o e^{nt} \tag{6}$$

where $0 \leq n < \infty$ is the fixed rate of growth of labour and $0 < L_o < \infty$ is the given initial endowment. $Z_L(t)$ is the excess demand for labour, with $Z_L > 0$ signifying that the domestic country is an importer of labour in period t, and $Z_L(t) < 0$ means that the country exports labour in period t. $Z_L(t)$ is a piecewise continuous function of time. We must observe that equation (6) in conjunction with (5) means that although a country may or may not import or export labour the owners of the labour never change nationalities. At $t = 0$ once a unit of labour services is endowed to a household in the domestic country it may never become part of the endowment in the foreign country for $t \geq 0$. Thus, the labour endowment in period t does not include foreign labour. We may define $L(t) + Z_L(t)$ as the resident labour supply. If $Z_L(t) > 0$ then the resident labour supply is greater than the endowment of labour and if $Z_L(t) < 0$ then the resident labour supply is less. In the case where $Z_L(t) < 0$ then we may refer to $Z_L(t)$ as the nonresident labour supply.

For a more concise statement of the problem let us define the variables

$$\frac{Y_i(t)}{L(t)} = y_i(t), \quad \frac{X_i(t)}{L(t)} = x_i(t), \quad \frac{Z_i(t)}{L(t)} = z_i(t)$$

$$\frac{Z_L(t)}{L(t)} = z_L(t), \quad \frac{K(t)}{L(t)} = k(t), \quad \frac{L_i(t)}{L(t)} = \ell_i(t)$$

$$\frac{K_i(t)}{L_i(t)} = k_i(t), \quad i = 1, 2.$$

Thence, our model is tranformed to

$$y_i(t) = \ell_i(t)\ f_i(k_i(t)) \qquad i = 1, 2 \tag{7}$$

where $f_i(k_i(t))$ is defined for $k_i(t) \geq 0$. We also have

$$f_i'(k_i(t)) > 0, \ f_i''(k_i(t)) < 0, \ f_i'(0) = \infty, \ f_i'(\infty) = 0. \tag{8}$$

$$x_i(t) = y_i(t) + z_i(t) \qquad i = 1, 2 \tag{9}$$

$$k_1(t)\, \ell_1(t) + k_2(t)\, \ell_2(t) = k(t) \tag{10}$$

$$\ell_1(t) + \ell_2(t) - z_L(t) = 1 \tag{11}$$

$$\dot{k}(t) = x_2(t) - \lambda k(t) \tag{12}$$

where $\lambda = \mu + n$ and $k(0) = k_o$, $0 < k_o < \infty$.

Regarding the international commodity flows we assume that the two products and labour are perfectly mobile while capital is perfectly immobile. Defining the world market equilibrium conditions as the sum of world demand and supply for each product and labour equaling zero, the domestic demand equals the domestic supply of capital and the foreign demand equals the foreign supply of capital for each time period then equation (3) may be interpreted as the capital market equilibrium condition for the domestic country.

We must now describe the foreign country. This delineation is summarized by a Mill-Marshall offer function which relates the foreign excess demands of consumption, investment and labour. As a consequence of our definition of equilibrium we may specify the function in terms of domestic excess demands. Moreover, we find it convenient to define our variables in labour endowment or labour intensive form and this formulation brings to the forefront an important problem in dynamic international analysis. It is by now well known (Bardhan [3] and Kemp [12]) that if the rates of growth of labour differ between countries in a two country world then in the "limit" the nation with the higher rate will approximate a self-sufficient economy while the other country will also tend to a closed economy. Both these limiting cases are essentially special instances of the model where both countries are, indeed, open economies which implies, in particular, that the rates of growth of labour are the same. Therefore to treat the most general case we will assume that the rates are indeed equal. In the light of this assumption we may normalize, without loss of generality, the initial endowment in the two countries to be equal.

Consequently, for all time periods, the labour endowments in both countries are equal and $z_L(t) \in [-1,1]$, with $z_L(t) = -1$ if and only if nothing is produced in the home country and $z_L(t) = 1$ if and only if nothing is produced in the foreign country. We now may write the offer function and assume it is of the form:

$$z_1(t) = g_1(z_2(t)) + g_2(z_L(t)) \tag{13}$$

and $g_1(0) = 0 = g_2(0)$, $g_1'(z_2(t)) < 0$, $g_2'(z_L(t)) < 0$, $g_1''(z_2(t)) < 0$, $g_2''(z_L(t)) \gtreqless 0$. Then, of course, $0 = g_1(0) + g_2(0)$. Notice also that we are assuming the foreign capital-labour ratio to be fixed, that is why it does not appear in (13).[5] This implies that investment behavior in the foreign country is such that an increase in imports of the investment product must be accompanied by a proportional decrease in the production of the investment product. In addition for simplicity we assume (13) is additively separable which implies that demand and production conditions interact in the foreign countries (and any repatriations if the foreign country exports labour) such that the vertical distance between any two offer curves in (z_2, z_1) space is equal irrespective of the values of $z_2(t)$ and $z_1(t)$ (see figure 1). Obviously then imbedded in (13) are the demand and/or technological differences between the two nations.

Finally, since we are concerned with the optimal path, we must establish the appropriate objective functional for the government to maximize in order that the optimal trajectory may be selected. We postulate that the government attaches an increase in

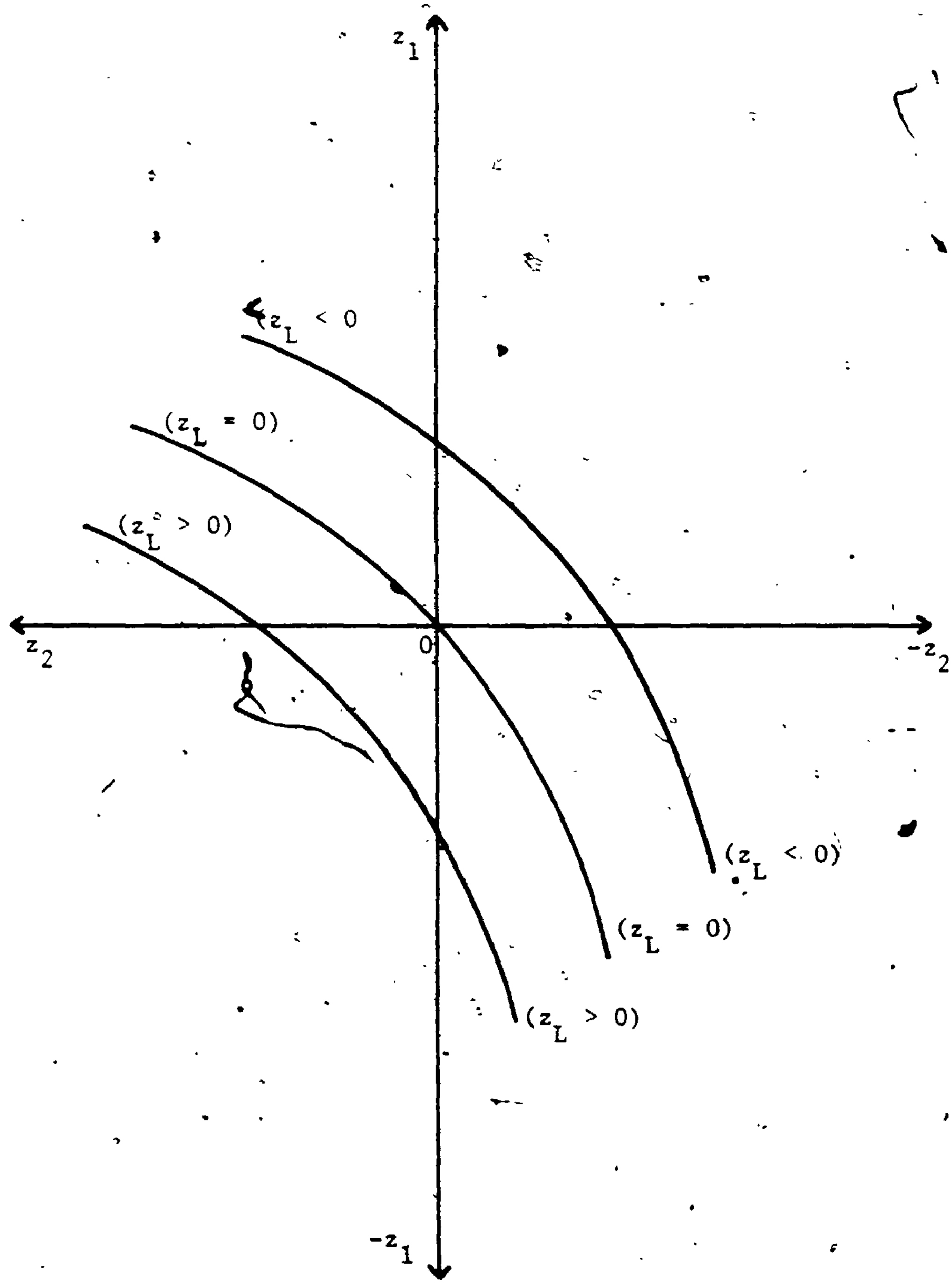

Figure 1. Foreign country's offer function with $z_L \gtreqless 0$

social welfare to increased consumption deflated by the labour endowment of the economy. But we must be aware that by the definition of demand any consumption of foreign workers in the domestic country is treated as home consumption. In a model purporting to analyze labour mobility, explicit recognition must be made of this phenomenon. If, for example, the social welfare function retained that portion of consumption attributable to imported workers then the home government posits that increases in consumption by these foreign labourers for a given labour endowment increases welfare. This may appear to be and indeed may be a plausible assumption. Nevertheless we shall not adopt this presupposition for it turns out that our specification generalizes the above mentioned.

To begin with, we assume that the consumption of imported workers in the domestic country is subtracted from the total consumption in the home country. This may be justified on the grounds that the country exhibits extreme nationalism with respect to its workers in particular and its population in general,in the sense that only consumption of its nationals not residents is enumerated for social welfare. We refer to this structuralization as Bardhan's extreme nationalism assumption as he was, I believe, the first (Bardhan [5]) to use a variant of it. What remains is the question concerning the treatment of the exported labour's consumption. Keeping in line with the nationalistic tendencies of the home government we assume that the consumption of the nonresident labour supply is added to the consumption of the resident population in evaluating national welfare. Given the posture of the problem, in particular that the nonresident labour supply always remain nationals of the domestic country, consistencey impels us to treat exported workers' consumption in this fashion.

We may translate the preceding into a social welfare function defined as, $\nu(x_1(t),\ z_L(t)) = U(x_1(t) + \phi(z_L(t))z_L(t))$ where $\phi(z_L(t))$ is the negative of the consumption per imported worker if $z_L(t) > 0$ or the negative of the consumption per exported worker if $z_L(t) < 0$. Therefore if $\phi(z_L(t))z_L(t)$ is negative it is defined as the negative of the total consumption of imported labour and is subtracted from $x_1(t)$. On the other hand if $\phi(z_L(t))\ z_L(t)$ is positive, it is defined as the total consumption of exported labour and is added to $x_1(t)$. Finally when $z_L(t) = 0$ the term $\phi(z_L(t))\ z_L(t)$ vanishes. In addition we assume $-\infty \leq \phi(z_L(t)) < 0$ for $z_L(t) \gtrless 0$.

Hence the domain of the welfare function is specified but nothing has been stated concerning the curvature of the function, which implies whether changes in the level of imported or exported workers lead to marginal social external economies or external diseconomies. Moreover if $z_L(t) > 0$ then it is only possible by definition to have $x_1(t) \geq -\phi(z_L(t))z_L(t) > 0$. If $x_1(t) = -\phi(z_L(t))z_L(t)$ then only the foreign workers in the domestic country consume. For any realistic economic model this is an untenable occurence. In addition when the country exports labour it is then possible for $x_1(t)$ to be equal to zero, so that any consumption must consist only of the consumption product produced and demanded in the foreign country. The introduction of international labour mobility precipitates the novel circumstance that $x_1(t) = 0$ does not imply zero consumption level for the population of the home country. However the economic process that arises because of $x_1(t) = 0$ is rather pathological, as we shall see in appendix 3. Consequently, in the main body of the paper we believe it wiser to rule out this case.

Previously, from equation 13, we have allowed the foreign offer function to adopt a myriad of shapes since $g_2'' \gtreqless 0$. To be more specific we now assume that the function $g_2(z_L(t)) + \phi(z_L(t))z_L(t)$ to be strictly concave and possess a non-positive first order derivative with respect to $z_L(t)$. This formulation is very general in that it allows for such phenomena as concave and/or convex foreign offer function and marginal social economies and diseconomies associated with changes in labour imports or exports. The economic meaning of the sign of the first order derivative may be easily explained. Suppose $z_L(t) > 0$ and $g_2' < 0 < -\phi' z_L - \phi$. This states that any decrease in imports of the consumption product must still permit sufficient domestic production to supply the increased consumption of the foreign workers due to an increase in imported labour. Similar meanings are apparent for the other cases that may occur. Summarizing we have

$$\begin{aligned}
&\nu(x_1(t),\ z_L(t)) = U(x_1(t) + \phi(z_L(t))z_L(t));\\
&-\infty \leqq \phi(z_L(t)) < 0 \text{ if } z_L(t) \gtrless 0,\ -\infty \leqq \phi(z_L(t)) \leqq +\infty \text{ if } z_L(t) = 0;\\
&U \epsilon C^2;\ U' > 0,\ U'' < 0,\ \text{if } \infty \geqq x_1(t) > \max\,(0, -\phi(z_L(t))z_L(t)),\\
&U' = \infty \text{ if } x_1(t) \leqq \max\,(0, -\phi(z_L(t))z_L(t)); \qquad (14)\\
&g_2'(z_L(t)) + \phi'(z_L(t))z_L(t) + \phi(z_L(t)) \leqq 0,\\
&g_2''(z_L(t)) + \phi''(z_L(t))z_L(t) + 2\phi'(z_L(t)) < 0.
\end{aligned}$$

Obviously the relations contained in (14) satisfy the prestated feasibility requirements. We must recognize that there is no reason to constrain $\phi(z_L(t)) = 0$ if $z_L(t) = 0$, because when $z_L(t) = 0$ the domestic country may experience an external diseconomy or economy depending on the sign of $\phi(z_L(t))$. However when $z_L(t) = 0$ we do not interpret $\phi(z_L(t))$ as the negative of consumption per imported or exported worker but only as a factor determining the externality associated with $z_L(t) = 0$.

Conjointly the term $\phi(z_L(t))z_L(t)$ has numerous attractive features. Firstly, it permits an abundant assortment of externality effects on social welfare associated with labour mobility. Secondly, in previous literature (e.g., Kemp [12]), dealing only with the simpler static models, the consumption of the domestic country's imported (or exported) labour was a constant. Moreover, this constant was the same whether the country imported or exported labour. Manifestly this is an unrealistic simplication which our model rectifies in a quite general manner. Thirdly, the inclusion of the term $\phi(z_L(t))\, z_L(t)$ as specified in (14) is the mathematical generalization of all cases where the home country evaluates the consumption of its residents and/or nationals in any combination as a single argument social welfare function; although the economic meanings are different in each case, such as the evaluation of the consumption of residents, i.e., $\phi(z_L(t))z_L(t) = 0$ for all values of $z_L(t)$.

Let us proceed to define the social welfare functional as,

$$W(x_1(t), z_L(t)) = \int_0^\infty e^{-\delta t}\, U(x_1(t) + \phi(z_L(t))z_L(t))dt \qquad (15)$$

where $0 < \delta < \infty$, is the constant social rate of discount. Imbedded δ, among other value judgments, is any appropriate weighting of each period's population.

We have now reached the stage where we can adequately define the problem confronting the domestic country. Define the vectors (we now drop the argument t from the relevant functions to simplify the notation),

$$X = (y_1, y_2, k_1, k_2, k, \ell_1\ \ell_2) \geq 0$$

$$Z = (z_2, z_L)$$

$$V = (\lambda_1, \lambda_2, p, w, r) \geq 0$$

where λ_1 is the supply price of consumption, λ_2 is the supply price of investment, p is the demand price of investment, w is the price of labour services and r is the rental on capital services. Notice that ℓ_1 and ℓ_2 may both be zero. In this case the domestic country does not produce anything but it exports all its labour and thus individuals derive their consumption and consumption imports from working in the foreign country. A solution of this type might be applicable for countries similar to Luxembourg or Gambia.

The economy has the following mini-max calculus of variations problem to solve,

$$\underset{\min\{V\}\ \max\{X,Z\}}{I}(X, Z, V) = \int_0^\infty e^{-\delta t} L(\cdot)dt \tag{16}$$

where

$$L(y_1, y_2, k_1, k_2, \ell_1, \ell_2, k, z_2, z_L, \lambda_1, \lambda_2, p, w, r) = U(y_1 + g_1(z_2) + g_2(z_L) + \phi(z_L)\ z_L) + \lambda_1(\ell_1 f_1(k_1) - y_1) + \lambda_2(\ell_2 f_2(k_2) - y_2) + w(1 + z_L - \ell_1 - \ell_2) + r(k - \ell_1 k_1 - \ell_2 k_2) + p(y_2 + z_2 - \lambda k - \dot{k}) \tag{17}$$

subject to, $X \geq 0$, $V \geq 0$, $0 < k_0 < \infty$.

Let,

$$\frac{w}{r} = \omega \qquad \omega_i(k_i) = \frac{f_i\,(k_i)}{f_i{'}(k_i)} - k_i \qquad i = 1,2$$

and for

$$\frac{k}{(1+z_L)} = k_i(\omega)$$

then

$$\omega_i = \omega_i \left(\frac{k}{(1+z_L)}\right) \qquad i = 1,2.$$

We also assume that,

$$\lim_{K_i \to o \quad L_i \to o} \left(\frac{K_i}{L_i}\right) = k_i \left(\omega_j \left(\frac{k}{(1+z_L)}\right)\right) \quad \text{if and only if}$$

$y_1 = 0,\ y_2 > 0$ or $y_1 > 0,\ y_2 = 0$ but not both $y_1 = 0 = y_2$,

$i, j = 1,2,\ i \neq j.$[6]

The last assumption (Hayek [8]) states that if the economy produces only one of the products then the capital labour ratio for the nonproduced product exists and is a function of the wage-rental ratio for the produced product. This assumtion is implicit in all neoclassical growth models and permits us to salvage the first order conditions and the sufficiency proof of optimality of the solution when the economy is completely specialized. Now the first order and transversality conditions of optimality are (all derivatives are evaluated at the optimum),[7]

$$\frac{\partial L}{\partial y_1} = U' - \lambda_1 \leq 0,\ y_1\,(U' - \lambda_1) = 0, \quad y_1 \geq 0,$$

$$\frac{\partial L}{\partial y_2} = p - \lambda_2 = 0,\ y_2(p - \lambda_2) = 0, \quad y_2 \geq 0,$$

$$\frac{\partial L}{\partial z_2} = U' g_1'(z_2) + p = 0, \quad \frac{\partial L}{\partial z_L} = U'(g_2'(z_L) + \phi'(z_L)\, z_L + \phi(z_L)) + w = 0,$$

$$\frac{\partial L}{\partial k_i} = \ell_i(\lambda_i f_i'(k_i) - r) \leq 0, \; k_i \ell_i(\lambda_i f_i'(k_i) - r) = 0, \; k_i \geq 0, \; i = 1,2$$

$$\frac{\partial L}{\partial \ell_i} = \lambda_i f_i(k_i) - w - rk_i \leq 0, \; \ell_i(\lambda_i f_i(k_i) - w - rk_i) = 0, \; \ell_i \geq 0, \; i = 1,2$$

$$\frac{\partial L}{\partial \lambda_i} = \ell_i f_i(k_i) - y_i = 0, \quad \lambda_i \geq 0, \; i = 1,2 \qquad (18)$$

$$\frac{\partial L}{\partial w} = 1 + z_L - \ell_1 - \ell_2 = 0, \; w \geq 0,$$

$$\frac{\partial L}{\partial r} = k - \ell_1 k_1 - \ell_2 k_2 = 0, \; r \geq 0,$$

$$\frac{\partial L}{\partial p} = y_2 + z_2 - \lambda k - \dot{k} = 0, \; p \geq 0,$$

$$\dot{p} = (\lambda + \delta)p - r,$$

$$\lim_{t\to\infty} e^{-\delta t} p(t) \geq 0, \quad \lim_{t\to\infty} e^{-\delta t} p(t)k(t) = 0, \quad 0 < k_o < \infty \; .$$

These conditions on the outputs, inputs and prices are quite standard when $y \geq 0$ for then the capital-labour ratios are well-defined in each of the production functions. Consequently we may make the appropriate substitutions for r in $\frac{\partial L}{\partial \ell_i}$, $i = 1,2$. However we allow $y_1 = 0 = y_2$, i.e., a feasible solution which may be optimal is for the country to export all of its labour, then $k = 0 = \ell_1 = \ell_2$. Although certain indéterminancies arise in this case, they play no part in the analysis as we shall observe later. Equation system (18) points out that at the optimum the demand price of the investment product in terms of the instantaneous marginal social welfare is equal to the negative of the slope of the foreign offer function with respect to z_2. The final interesting and important condition is the equation $\frac{\partial L}{\partial z_L} = 0$. This equation can be interpreted as saying that the domestic country determines the excess demand for labour according as the differential between the price of labour in terms of instantaneous marginal social welfare and the negative of the slope of the foreign offer function with respect to z_L is equal to the instantaneous marginal social welfare labour externality in terms of the instantaneous marginal social welfare.

From this juncture onward it is best to proceed by analyzing individually each of the different patterns that arise. In doing so, we shall find it more convenient for purposes of deriving temporal equilibria, comparative equilibria results and steady states to reformulate the model as an optimal control problem. The solution is also facilitated by initially setting $U' = 1$. We are then able to convert to the original and more general case where $U' > 0$, $U'' \neq 0$ by appropriate modifications. The disquieting aspect of letting $U' = 1$ for all values of the arguments of the social welfare function is that optimal policies of the form $x_1 = -\phi(z_L)z_L$ are feasible for $z_L > 0$. As stated earlier this obviously runs counter to any economically sententious formulation. In addition it is possible in the linear welfare function case to find that $x_1 = 0$ is a solution to the domestic country's problem. However because we are only interested in the linear case as a means to an end, and end which rules out $x_1 = 0$ solutions, we shall relegate policies of this form to appendix 3, where we include it for the sake of mathematical completeness. Finally we assume that the linear social welfare functional is only defined for $x_1 + \phi(z_L)z_L \geq 0$ which prohibits the completely meaningless solution $x_1 < -\phi(z_L)z_L$ if $z_L > 0$.

3. Temporal Equilibria and Patterns of Specialization

In this model we discern eight different specialization patterns of production and expenditure as specified in Table 1. In addition, within each of these patterns are the subclasses referring to the international flows of consumption, investment and labour.

3.1. Pattern $y_1 > 0, y_2 > 0$

We begin by assuming that factor intensity reversals never take place in the economy so that $k_1 - k_2$ is either greater than zero or less than zero over the horizon. Factor intensity reversals can be incorporated but this would only obfuscate the main thrust of the essay. Therefore we can solve for ℓ_1 and ℓ_2 from equation system (18) when $y_1 > 0$, $y_2 > 0$,

$$\ell_1 = \frac{k - k_2(1 + z_L)}{k_1 - k_2} , \tag{20}$$

$$\ell_2 = \frac{k_1(1 + z_L) - k}{k_1 - k_2} \tag{21}$$

Also

$$y_1 = \left(\frac{k - k_2(1 + z_L)}{k_1 - k_2}\right) f_1(k_1)$$

$$y_2 = \left(\frac{k_1(1 + z_L) - k}{k_1 - k_2}\right) f_2(k_2) . \tag{22}$$

Table 1. Patterns of Specialization

Expenditure	Production: $y_1 > 0,\ y_2 = 0$ (1)	$y_1 > 0,\ y_2 > 0$ (2)	$y_1 = 0,\ y_2 > 0$ (3)	$y_1 = 0 = y_2$ (4)
$x_1 > 0,\ x_2 = 0$ (1)	$y_1 > 0,\ y_2 = 0$ $-1 < z_L \leq 1$ $z_1 > -y_1,\ z_2 = 0$ $x_1 > 0,\ z_2 = 0$	$y_1 > 0,\ y_2 > 0$ $-1 < z_L \leq 1$ z_1 [illegible] $-y_1,\ z_2 = y_2$ $x_1 > 0,\ x_2 = 0$	$y_1 = 0,\ y_2 > 0$ $-1 < z_L < 1$ $z_1 > 0,\ z_2 = -y_2$ $x_1 = z_1,\ x_2 = 0$	$y_1 = 0 = y_2$ $z_L = -1$ $z_1 > 0,\ z_2 = 0$ $x_1 = z_1,\ x_2 = 0$
$x_1 > 0,\ x_2 > 0$ (2)	$y_1 > 0,\ y_2 = 0$ $-1 < z_L < 1$ $z_1 > -y_1,\ z_2 > 0$ $x_1 > 0,\ x_2 = z_2$	$y_1 > 0,\ y_2 > 0$ $-1 < z_L \leq 1$ $z_1 > -y_1,\ z_2 > -y_2$ $y_1 > 0,\ x_2 > 0$	$y_1 = 0,\ y_2 > 0$ $-1 < z_L < 1$ $z_1 > 0,\ z_2 > -y_2$ $x_1 = z_1,\ x_2 > 0$	$y_1 = 0 = y_2$ $z_L = -1$ $z_1 > 0,\ z_2 > 0$ $x_1 = z_1,\ x_2 = z_2$

The pattern (2,1) refers to the case where both products are produced and the economy only consumes; i.e., the first number refers to production and the second to expenditure.

Hence our problem is redefined as

$$W(z_L, z_2, k_1, k_2) = \int_o^{\infty} e^{-\delta t} \left[\left(\frac{k - k_2(1 + z_L)}{k_1 - k_2} \right) f_1(k_1) \right.$$
$$\max \{z_L, z_2, k_1; k_2\}$$

$$\left. + g_1(z_2) + g_2(z_L) + \phi(z_L) z_L \right] dt. \tag{23}$$

$$\text{subject to, } \dot{k} = \left[\frac{k_1(1 + z_L) - k}{k_1 - k_2} \right] f_2(k_2) + z_2 - \lambda k$$

$$k_1, k_2 \geq 0, \quad 0 < k_o < \infty.$$

Our first order, second order necessary and transversality conditions are derived from the following Hamiltonian,

$$H(z_L, z_2, k_1, k_2; k, q) = \left[\frac{k - k_2(1 + z_L)}{k_1 - k_2} \right] f_1(k_1) + g_1(z_2) + g_2(z_L)$$
$$+ \phi(z_L) z_L + q \left[\left(\frac{k_1(1 + z_L) - k}{k_1 - k_2} \right) f_2(k_2) \right.$$
$$\left. + z_2 - \lambda k \right]. \tag{24}$$

Then (all derivatives evaluated at the solution)

$$\frac{\partial H}{\partial z_2} = g_1'(z_2) + q = 0,$$

$$\frac{\partial H}{\partial z_L} = g_2'(z_L) + \phi'(z_L) z_L + \phi(z_L) - \frac{k_2 f_1(k_1)}{k_1 - k_2} + q \frac{k_1 f_2(k_2)}{k_1 - k_2} = 0,$$

$$\frac{\partial H}{\partial k_1} = \frac{k - k_2(1 + z_L)}{(k_1 - k_2)^2} \left[(k_1 - k_2) f_1'(k_1) - f_1(k_1) + q f_2(k_2) \right] = 0, \tag{25}$$

$$\frac{\partial H}{\partial k_2} = \frac{k_1(1 + z_L) - k}{(k_1 - k_2)^2} [-f_1(k_1) + q[f_2'(k_2)(k_1 - k_2) + f_2(k_2)]] = 0 ,$$

$$\dot{k} = (\frac{k_1(1 + z_L) - k}{k_1 - k_2}) f_2(k_2) + z_2 - \lambda k,$$

$$0 < k_o < \infty, \dot{q} = \delta q - \frac{\partial H}{\partial k}, \lim_{t \to \infty} e^{-\delta t} q(t) \geq 0, \lim_{t \to \infty} e^{-\delta t} q(t) k(t) = 0.$$

Furthermore the Legendre-Clebsch condition states that the Hamiltonian must be negative semi-definite with respect to the control variables, z_2, z_L, k_1, k_2. The Hessian of the Hamiltonian is comprised of the elements,

$$\frac{\partial^2 H}{\partial z_2^2} = g_1''(z_2) < 0, \quad \frac{\partial^2 H}{\partial z_2 \partial z_1} = 0 = \frac{\partial^2 H}{\partial z_L \partial z_2}, \quad \frac{\partial^2 H}{\partial z_2 \partial k_1} = 0 = \frac{\partial^2 H}{\partial k_1 \partial z_2}$$

$$\frac{\partial^2 H}{\partial z_2 \partial k_2} = 0 = \frac{\partial^2 H}{\partial k_2 \partial z_2}, \quad \frac{\partial^2 H}{\partial z_L^2} = g_2''(z_L) + \phi''(z_L) z_L + 2\phi'(z_L) < 0,$$

$$\frac{\partial^2 H}{\partial z_L \partial k_1} = - \frac{k_2 f_1'(k_1)}{k_1 - k_2} + \frac{k_2 f_1(k_1)}{(k_1 - k_2)^2} + \frac{q\ f_2(k_2)}{k_1 - k_2} - \frac{q\ k_1 f_2(k_2)}{(k_1 - k_2)^2} .$$

The sign of this expression is determined by setting $0 = \frac{\partial H}{\partial k_1} = \frac{\partial H}{\partial k_2}$.

Yielding

$$(k_1 - k_2) f_1'(k_1) - f_1(k_1) + q f_2(k_2) = - f_1(k_1) + q[f_2'(k_2)(k_1 - k_2) + f_2(k_2)]. \qquad (26)$$

Solving for q, gives,

$$\frac{f_1(k_1) - (k_1 - k_2) f_1'(k_1)}{f_2(k_2)} = \frac{f_1(k_1)}{f_2(k_2) + f_2'(k_2)(k_1 - k_2)} \qquad (27)$$

$$f_1(k_1) f_2'(k_2) - f_2(k_2) f_1'(k_1) - f_1'(k_1) f_2'(k_2)\ (k_1 - k_2) = 0 \qquad (28)$$

and thus

$$\frac{f_1(k_1)}{f_1'(k_1)} - k_1 = \frac{f_2(k_2)}{f_2'(k_2)} - k_2 . \tag{29}$$

But from the definition of $\omega_i(k_i)$ $i = 1,2$ we then get

$$\omega := \omega_1(k_1) = \omega_2(k_2) . \tag{30}$$

Moreover $\frac{\partial H}{\partial k_1} = \frac{\partial H}{\partial k_2} = 0$ for $y_1 > 0$, $y_2 > 0$ implies that

$$q = \frac{f_1'(k_1)}{f_2'(k_2)} . \tag{31}$$

Returning to $\frac{\partial^2 H}{\partial z_L \partial k_1}$ we have with the appropriate substitutions for q and ω,

$$\frac{\partial^2 H}{\partial z_L \partial k_1} = \frac{f_1'(k_1)}{(k_1 - k_2)^2} [- k_2(k_1 - k_2) + k_2(\omega + k_1) + k_1(\omega + k_2)$$

$$- k_2(\omega + k_2) - k_1(\omega + k_2)] = 0 . \tag{32}$$

Now,

$$\frac{\partial^2 H}{\partial z_L \partial k_2} = - \frac{f_1(k_1)}{k_1 - k_2} - \frac{k_2 f_1(k_1)}{(k_1 - k_2)^2} + \frac{q k_1 f_2'(k_2)}{k_1 - k_2} + \frac{q k_1 f_2(k_2)}{(k_1 - k_2)^2} \tag{33}$$

$$\frac{\partial^2 H}{\partial z_L \partial k_2} = \frac{f_1'(k_1)}{(k_1 - k_2)^2} [- k_1(\omega + k_1) + k_1(k_1 - k_2) +$$

$$k_1(\omega + k_2)] = 0 \tag{34}$$

$$\frac{\partial^2 H}{\partial k_1 \partial z_L} = - \frac{k_2}{(k_1 - k_2)^2} \{(k_1 - k_2) f_1'(k_1) - f_1(k_1) +$$

$$q f_2(k_2)\} = 0 \tag{35}$$

$$\frac{\partial^2 H}{\partial k_1^2} = -2\left[\frac{k - k_2(1 + z_L)}{(k_1 - k_2)^3}\right]((k_1 - k_2)f_1'(k_1) - f_1(k_1)$$

$$+ qf_2(k_2)) + \left[\frac{k - k_2(1 + z_L)}{(k_1 - k_2)^2}\right](k_1 - k_2)f_1''(k_1) \quad (36)$$

$$\frac{\partial^2 H}{\partial k_1^2} = \left[\frac{k - k_2(1 + z_L)}{(k_1 - k_2)^2}\right](k_1 - k_2)f_1''(k_1)\ , \quad (37)$$

if $k_1 > k_2$ then $k - k_2(1 + z_L) > 0$ and $\frac{\partial^2 H}{\partial k_1^2} < 0$ if $k_2 > k_1$ then $k - k_2(1 + z_L) < 0$ and again $\frac{\partial^2 H}{\partial k_1^2} < 0$. Continuing we get,

$$\frac{\partial^2 H}{\partial k_1 \partial k_2} = \left[-\frac{(1 + z_L)}{(k_1 - k_2)^2} + 2\left[\frac{k - k_2(1 + z_L)}{(k_1 - k_2)^3}\right]\right]((k_1 - k_2)f_1'(k_1)$$

$$- f_1(k_1) + qf_2(k_2)) + \left(\frac{k - k_2(1 + z_L)}{(k_1 - k_2)^2}\right) \quad (38)$$

$$(- f_1'(k_1) + qf_2'(k_2)) = 0$$

$$\frac{\partial^2 H}{\partial k_2 \partial z_L} = \frac{k_1}{(k_1 - k_2)^2}\left[- f_1(k_1) + q(f_2'(k_2)(k_1 - k_2) + \right.$$

$$\left. + f_2(k_2))\right] = 0 \quad (39)$$

$$\frac{\partial^2 H}{\partial k_2^2} = \left[\frac{k_1(1 + z_L) - k}{(k_1 - k_2)^2}\right](k_1 - k_2)f_2''(k_2) < 0 \quad (40)$$

$$\frac{\partial^2 H}{\partial k_2 \partial k_1} = \left[\frac{(1 + z_L)}{(k_1 - k_2)^2} - 2\left[\frac{k_1(1 + z_L) - k}{(k_1 - k_2)^3}\right]\right](- f_1(k_1)$$

$$+ q(f_2'(k_2)(k_1 - k_2) + f_2(k_2)) + \left(\frac{k_1(1 + z_L) - k}{(k_1 - k_2)^2} \right)$$

$$(- f_1'(k_1) + qf_2'(k_2)) = 0 \; . \qquad (41)$$

The Hessian is then a diagonal matrix with all the diagonal elements negative and therefore the Hamiltonian is strictly concave in the controls. The complete explication of the Legendre-Clebsch necessary conditions illustrates the importance of the externality term and the acceptable forms it may take on while still satisfying the conditions of optimality.

We seek the solution of the first order conditions and the effect that changes in the capital-labour endowment ratio and the demand price of investment have on $k_1, k_2, z_2, z_L, z_1, y_1, y_2, x_1, x_2$. Before proceeding to prove the results we shall summarize them in the following theorem.

<u>Theorem 3.1.1.</u> If $y_1 > 0$; $y_2 > 0$ then $k_i = k_i(q)$, $k_i' \gtrless 0$ iff $k_1 \lessgtr k_2$, $i = 1,2$; $z_2 = \hat{z}_2(q)$, $\hat{z}_2' > 0$; $z_L = \hat{z}_L(q)$, if $k_1 > k_2$ then $\hat{z}_L' > 0$ if $k_2 > k_1$ then $\hat{z}_L' < 0$; $z_1 = \hat{z}_1(q)$, if $k_1 > k_2$ then $\hat{z}_1' < 0$, if $k_2 > k_1$, then $\hat{z}_1' \gtreqless 0$ iff $\frac{g_1'}{g_2'} \gtreqless -\frac{z_L'}{z_2'}$; $y_1 = \hat{y}_1(k,q)$, $\frac{\partial \hat{y}_1}{\partial k} \gtrless 0$ iff $k_1 \gtrless k_2$, $\frac{\partial \hat{y}_1}{\partial q} < 0$; $y_2 = \hat{y}_2(k,q)$, $\frac{\partial \hat{y}_2}{\partial k} \gtrless 0$ iff $k_2 \gtrless k_1$, $\frac{\partial \hat{y}_2}{\partial q} > 0$; $x_1 = \hat{x}_1(k,q)$, $\frac{\partial \hat{x}_1}{\partial k} \gtrless 0$ iff $k_1 \gtrless k_2$, if $k_1 > k_2$ then $\frac{\partial \hat{x}_1}{\partial q} < 0$, if $k_2 > k_1$ then $\frac{\partial \hat{x}_1}{\partial q} \gtreqless 0$ iff $\frac{\partial \hat{y}_1}{\partial q} \gtreqless -\frac{d\hat{z}_1}{dq}$; $x_2 = \hat{x}_2(k,q)$, $\frac{\partial \hat{x}_2}{\partial k} \gtrless 0$ iff $k_2 \gtrless k_1$, $\frac{\partial \hat{x}_2}{\partial q} > 0$.

Embarking on the proof, observe from the definition of $\omega_1(k_1)$,

$$\omega_i(k_i) = \frac{f_i(k_i)}{f_i'(k_i)} - k_i \qquad i = 1,2$$

$$\omega_i'(k_i) = -\frac{f_i(k_i)f_i''(k_i)}{(f_i'(k_i))^2} > 0 \qquad i = 1,2 \tag{42}$$

Since $\omega_i(k_i)$ is monotonic then,

$$k_i(\omega_i) = \omega_i^{-1}(\omega_i) \qquad i = 1,2 \tag{43}$$

and

$$k_i'(\omega_i) = \frac{1}{\omega_i'(k_i)} > 0 \qquad i = 1,2 \tag{44}$$

From $\frac{\partial H}{\partial k_1} = 0 = \frac{\partial H}{\partial k_2}$ we see that

$$q = p\ (\omega) = \frac{f_1'(k_1(\omega))}{f_2'(k_2(\omega))} > 0 \tag{45}$$

and

$$\frac{p'(\omega)}{p(\omega)} = \frac{k_1(\omega) - k_2(\omega)}{[k_2(\omega) + \omega][k_1(\omega) + \omega]} \tag{46}$$

with

$$\operatorname{sgn} \frac{p'(\omega)}{p(\omega)} = \operatorname{sgn}\ (k_1(\omega) - k_2(\omega)) . \tag{47}$$

Since $p(\omega)$ is monotonic then,

$$\omega = \omega(q) = p^{-1}(q) \tag{48}$$

and

$$\omega'(q) = \frac{1}{p'(\omega)}\ ,\ \operatorname{sgn} \omega'(q) = \operatorname{sgn}\ (k_1(\omega) - k_2(\omega)) . \tag{49}$$

Hence, the first order conditions yield $k_i(q)$ $i = 1,2$ with $\operatorname{sgn} k_i'(q) = \operatorname{sgn}\ (k_1(\omega) - k_2(\omega))$, $i = 1,2$. Next from $\frac{\partial H}{\partial z_2}$ we get,

$$\frac{dz_2}{dq} = -\frac{1}{g_1''(z_2)} > 0 \tag{50}$$

and thus,

$$z_2 = \hat{z}_2(q) \text{ , where } \hat{z}_2'(q) \text{ is given by (50).} \tag{51}$$

Substituting $k_i(q)$ $i = 1,2$ into $\frac{\partial H}{\partial z_L}$ and solving for $\frac{dz_L}{dq}$ yields,

$$[g_2''(z_L) + \phi''(z_L)z_L + 2\phi'(z_L)]\frac{dz_L}{dq} - \frac{k_2'(q)f_1(k_1)}{k_1(q) - k_2(q)} -$$

$$\frac{k_2(q)f_1'(k_1)k_1'(q)}{k_1(q) - k_2(q)} - \frac{k_2(q)f_1(k_1)}{(k_1(q) - k_2(q))^2}[-k_1'(q) + k_2'(q)] +$$

$$\frac{k_1(q)f_2(k_2)}{k_1(q) - k_2(q)} + \frac{qk_1'(q)f_2(k_2)}{k_1(q) - k_2(q)} + \frac{qk_1(q)f_2'(k_2)k_2'(q)}{k_1(q) - k_2(q)} + \tag{52}$$

$$\frac{qk_1(q)f_2'(k_2)k_2'(q)}{k_1(q) - k_2(q)} + \frac{qk_1(q)f_2(k_2)}{(k_1(q) - k_2(q))^2}[-k_1'(q) + k_2'(q)] = 0 .$$

Simplifying the notation and collecting $k_1'(q)$ terms we have,

$$\frac{k_1'}{(k_1 - k_2)^2}[-k_2f_1'(k_1)(k_1 - k_2) + k_2f_1(k_1) + qf_2(k_2)(k_1 - k_2) - qk_1f_2(k_2)] \tag{53}$$

and then using (45),

$$\frac{k_1'f_1'(k_1)}{(k_1 - k_2)^2}[k_2(k_2 + \omega) - k_2(\omega + k_2)] = 0 . \tag{54}$$

Collecting $k_2'(q)$ terms gives,

$$\frac{k_2'}{(k_1 - k_2)^2}[-f_1(k_1)(k_1 - k_2) - k_2f_1(k_1) + qk_1f_2'(k_2)(k_1 - k_2) + qk_1f_2(k_2)] . \tag{55}$$

Again using equation (45), (55) becomes,

$$\frac{k_2' f_2'(k_2)}{(k_1 - k_2)^2} [- qk_1(\omega + k_1) + qk_1(\omega + k_1)] = 0 . \tag{56}$$

Hence from (52)

$$\frac{dz_L}{dq} = \frac{- k_1 f_2(k_2)}{(k_1 - k_2)(g_2''(z_L) + \phi''(z_L)z_L + 2\phi'(z_L))} \tag{57}$$

and

$$\text{sgn}\ \frac{dz_L}{dq} = \text{sgn}\ (k_1 - k_2). \tag{58}$$

Therefore an increase in the demand price of investment increases the capital-labour ratio of the capital intensive product and decreases the ratio for the labour intensive one. An increase in q lowers exports (or raises imports) of the investment product and when the investment product is labour intensive then increases in q raise the imports (or lower exports) of labour. On the other hand, when the investment product is capital intensive, the opposite occurs with respect to z_L. So we can define the function

$$z_L = \hat{z}_L(q), \quad \text{with } \hat{z}_L' \text{ given by (57).} \tag{59}$$

Now by equation (13), the foreign offer function, we get,

$$z_1 = \hat{z}_1(q) \tag{60}$$

where

$$\hat{z}_1(q) = g_1(\hat{z}_2(q)) + g_2(\hat{z}_L(q)) \text{ and thus,}$$

$$\hat{z}_1'(q) < 0 \text{ when } k_1 > k_2 \text{ and} \tag{61}$$

$$\hat{z}_1'(q) \gtreqless 0 \text{ if and only if } \frac{g_1'(z_2)}{g_2'(z_L)} \gtreqless - \frac{\hat{z}_L'(q)}{\hat{z}_2'(q)} .$$

when $k_2 > k_1$.

An increase in q always lowers imports (or raises exports) of the consumption product when consumption is capital intensive due not only to the effect on investment but also because more foreign workers are entering the domestic country or more domestic workers are remaining in their home country. Yet when the consumption product is labour intensive, the effect of decreased imports (or increased exports) of labour may outweigh the effect on z_2, and thus, in this case, the sign of $\hat{z}_1'(q)$ is ambiguous. Continuing we have from equation (22)

$$y_1 = \hat{y}_1(k,q) = [\frac{k - k_2(q)(1 + \hat{z}_L(q))}{k_1(q) - k_2(q)}] f_1(k_1(q)) ,$$

$$y_2 = \hat{y}_2(k,q) = [\frac{k_1(q)(1 + \hat{z}_L(q)) - k}{k_1(q) - k_2(q)}] f_2(k_2(q)) . \tag{62}$$

Then,

$$\frac{\partial \hat{y}_1}{\partial k} = \frac{f_1(k_1)}{k_1 - k_2} = \frac{(k_1 + \omega) f_1'(k_1)}{k_1 - k_2}$$

$$\text{sgn} \frac{\partial \hat{y}_1}{\partial k} = \text{sgn} (k_1 - k_2) ,$$

$$\frac{\partial \hat{y}_2}{\partial k} = - \frac{f_2(k_2)}{k_1 - k_2} = - \frac{(k_2 + \omega) f_2'(k_2)}{k_1 - k_2} \tag{63}$$

$$\text{sgn} \frac{\partial \hat{y}_2}{\partial k} = \text{sgn} (k_2 - k_1) .$$

$$\frac{\partial \hat{y}_1}{\partial q} = - \frac{f_1'(k_1)}{(k_1 - k_2)^2} [k_1'(q)(k - k_2(1 + \hat{z}_L))(\omega + k_2) +$$

$$k_2'(q)(k_1(1 + \hat{z}_L) - k)(\omega + k_1)] - \frac{d\hat{z}_L}{dq} \frac{k_2 f_1(k_1)}{(k_1 - k_2)} < 0 ,$$

$$\frac{\partial \hat{y}_2}{\partial q} = \frac{f_2'(k_2)}{(k_1 - k_2)^2} [k_1'(q) \ (k - k_2(1 + \hat{z}_L))(\omega + k_2) + k_2'(q)(k_1(1 + \hat{z}_L) - k)(\omega + k_1)] + \frac{d\hat{z}_L}{dq} \frac{k_1 f_2(k_2)}{(k_1 - k_2)} > 0 \ . \tag{64}$$

These results state that an increase in the capital-labour endowment ratio increases the output per labour endowment of the capital intensive product and decreases the output per labour endowment of the labour intensive product. In addition, an increase in the demand price of investment always increases the output of investment and lowers the output of the consumption product.

By (63), (64) and equation (9) we get that

$$x_i = \hat{x}_i(q,k) = \hat{y}_i(k,q) + \hat{z}_i(q) \qquad i = 1,2 \tag{65}$$

and

$$\frac{\partial \hat{x}_1}{\partial k} = \frac{\partial \hat{y}_1}{\partial k}, \text{ with sgn } \frac{\partial \hat{x}_1}{\partial k} = \text{sgn } (k_1 - k_2) \ ,$$

$$\frac{\partial \hat{x}_2}{\partial k} = \frac{\partial \hat{y}_2}{\partial k}, \text{ with sgn } \frac{\partial \hat{x}_2}{\partial k} = \text{sgn } (k_2 - k_1) \ . \tag{66}$$

Also,

$$\frac{\partial \hat{x}_1}{\partial q} = \frac{\partial \hat{y}_1}{\partial q} + \frac{d\hat{z}_1}{dq} \ , \qquad \text{if } k_1 > k_2 \text{ then } \frac{\partial \hat{x}_1}{\partial q} < 0 \ ;$$

$$\text{if } k_2 > k_1 \text{ then } \frac{\partial \hat{x}_1}{\partial q} \gtreqless 0 \text{ if and only if } \frac{\partial \hat{y}_1}{\partial q} \gtreqless - \frac{d\hat{z}_1}{dq} \ , \tag{67}$$

$$\frac{\partial \hat{x}_2}{\partial q} = \frac{\partial \hat{y}_2}{\partial q} + \frac{d\hat{z}_1}{dq} > 0 \ .$$

3.1.1. Pattern (2,2) ($y_1 > 0$, $y_2 > 0$, $x_1 > 0$, $x_2 > 0$)

In section 3.1 we have only subdivided our model in relation to production specializations. At this time we are prepared to link the class $y_1 > 0$, $y_2 > 0$ to $x_1 > 0$, $x_2 > 0$ and $x_1 > 0$, $x_2 = 0$. It turns out that when the economy consumes and invests then all the results in part 3.1 do not have to be modified.

3.1.2. Pattern (2,1) ($y_1 > 0$, $y_2 > 0$, $x_1 > 0$, $x_2 = 0$)

Let us observe from Table 1 that when the economy is in pattern (2,1) we have

$$\hat{z}_2(q) = -\hat{y}_2(k,q) \ . \tag{68}$$

Hence we have an implicit function in k and q. Solving for q as a function of k yields,

$$\frac{dq}{dk} = \frac{-\dfrac{\partial \hat{y}_2}{\partial k}}{\dfrac{d\hat{z}_2}{dq} + \dfrac{\partial \hat{y}_2}{\partial q}} \tag{69}$$

with,

$$\text{sgn}\,\frac{dq}{dk} = \text{sgn}\,(k_1 - k_2) \ .$$

So we get the function $\hat{q}_2 = \hat{q}_2(k)$ where $\hat{q}_2'$ is defined by (69). Therefore in pattern (2,1),

$$\begin{aligned} k_i &= k_i(\hat{q}_2(k)) \quad i = 1,2; \quad z_2 = \hat{z}_2(\hat{q}_2(k)); \quad z_L = \hat{z}_L(\hat{q}_2(k)); \\ z_1 &= \hat{z}_1(\hat{q}_2(k)); \ y_1 = \hat{y}_1(k,\hat{q}_2(k)); \ y_2 = \hat{y}_2(k,\hat{q}_2(k)); \\ x_1 &= \hat{x}_1(k,\hat{q}_2(k)); \ x_2 = \hat{x}_2(k,\hat{q}_2(k)); \end{aligned} \tag{70}$$

$$q \leq \hat{q}_2(k) = p(\omega) \; .$$

3.2. Pattern $y_1 = 0,\ y_2 > 0$.

In this pattern $\ell_1 = 0$, $\ell_2 = 1 + z_L$ therefore, $k_2(1 + z_L) = k$ from equation (20) $\dfrac{k - k_2(1 + z_L)}{k_1 - k_2} = 0$, $\dfrac{k_1(1 + z_L) - k}{k_1 - k_2} = (1 + z_L)$.

Thus the domestic country's problem is redefined as,

$$W(z_2, z_L) = \int_0^\infty e^{-\delta t}(g_1(z_2) + g_2(z_L) + \phi(z_L)z_L)\, dt$$
$$\max \{z_2, z_L\}$$

subject to,

$$\dot{k} = (1 + z_L)f_2\left(\frac{k}{(1 + z_L)}\right) + z_2 - \lambda k \tag{71}$$

$$0 < k_o < \infty \; .$$

The Hamiltonian becomes,

$$H(z_2, z_L; q, k) = g_1(z_2) + g_2(z_L) + \phi(z_L)z_L + q((1 + z_L)\, f_2\left(\frac{k}{(1 + z_L)}\right) + z_2 - \lambda k) \; . \tag{72}$$

Now the necessary conditions of optimality are, (all derivatives evaluated at the solution)

$$\frac{\partial H}{\partial z_2} = g_1'(z_2) + q = 0 \; ,$$

$$\frac{\partial H}{\partial z_L} = g_2'(z_L) + \phi'(z_L)z_L + \phi(z_L) + q\left[\, f_2\left(\frac{k}{(1 + z_L)}\right) - \frac{k}{(1 + z_L)}\, f_2'\left(\frac{k}{(1 + z_L)}\right)\right] = 0 \; ,$$

$$\dot{k} = (1 + z_L)\, f_2\left(\frac{k}{(1 + z_L)}\right) + z_2 - \lambda k \; , \tag{73}$$

$$\dot{q} = \delta q - \frac{\partial H}{\partial k} ,$$

$$\lim_{t \to \infty} e^{-\delta t} q(t) \geq 0, \quad \lim_{t \to \infty} e^{-\delta t} q(t)k(t) = 0, \quad 0 < k_o < \infty ,$$

$$\frac{\partial^2 H}{\partial z_2^2} = g_1''(z_2) < 0, \quad \frac{\partial^2 H}{\partial z_2 \partial z_L} = \frac{\partial^2 H}{\partial z_L \partial z_2} = 0$$

$$\frac{\partial^2 H}{\partial z_L^2} = g_2''(z_L) + \phi''(z_L)z_L + 2\phi'(z_L) + \frac{qk^2}{(1 + z_L)^3} f_2''\left(\frac{k}{(1 + z_L)}\right) < 0 .$$

So H is strictly concave in the controls z_2 and z_L. By the assumption imposed on k_i $i = 1,2$, when the economy is specialized in production we have,

$$k_1 = k_1(\omega_2) , \quad k_2 = \frac{k}{(1 + z_L)} \tag{74}$$

and since,

$$\omega_2 = \omega_2 \left(\frac{k}{(1 + z_L)} \right) = \omega \tag{75}$$

then

$$k_1 = k_1\left(\frac{k}{(1 + z_L)} \right) . \tag{76}$$

Also,

$$p(\omega) = \frac{f_1'(k_1)}{f_2'(k_2)} = \frac{f_1'\left(k_1\left(\frac{k}{(1 + z_L)}\right)\right)}{f_2'\left(\frac{k}{(1 + z_L)}\right)} = p\left(\omega_2\left(\frac{k}{(1 + z_L)}\right)\right) . \tag{77}$$

If we substitute (74) and (76) into $\frac{\partial H}{\partial z_L} = 0$ we are able to find the effects of changes of k and q on z_2, z_L, z_1, y_1, y_2, x_1, x_2 .

Theorem 3.2.1. If $y_1 = 0$, $y_2 > 0$ then $z_2 = \tilde{z}_2(q)$, $\tilde{z}_2' > 0$; $z_L = \tilde{z}_L(k,q)$,

$$\frac{\partial \tilde{z}_L}{\partial q} > 0, \quad \frac{\partial \tilde{z}_L}{\partial k} > 0; \quad z_1 = \tilde{z}_1(k,q), \quad \frac{\partial \tilde{z}_1}{\partial k} < 0, \quad \frac{\partial \tilde{z}_1}{\partial q} < 0;$$

$$y_1 = 0; \quad y_2 = \tilde{y}_2(k,q), \quad \frac{\partial \tilde{y}_2}{\partial k} > 0, \quad \frac{\partial \tilde{y}_2}{\partial q} > 0; \quad x_1 = \tilde{x}_1(k,q)$$

$$\frac{\partial \tilde{x}_1}{\partial k} < 0, \quad \frac{\partial \tilde{x}_1}{\partial q} < 0; \quad x_2 = \tilde{x}_2(k,q), \quad \frac{\partial \tilde{x}_2}{\partial k} > 0, \quad \frac{\partial \tilde{x}_2}{\partial q} > 0 .$$

Advancing to prove the theorem we find initially that z_2 is defined by equation (51) in this pattern. Next from $\frac{\partial H}{\partial z_L} = 0$, z_L will be a function not only of q but also k. Solving for $\frac{\partial z_L}{\partial q}$ yields

$$[g_2''(z_L) + \phi''(z_L)z_L + 2\phi'(z_L)] \frac{\partial z_L}{\partial q} + f_2(k_2) - k_2 f_2'(k_2)$$
$$- \frac{q f_2'(k_2)k_2}{(1+z_L)} \frac{\partial z_L}{\partial q} + \frac{q k_2 f_2'(k_2)}{(1+z_L)} \frac{\partial z_L}{\partial q} + \frac{q k_2 f_2''(k_2) k_2}{(1+z_L)} \frac{\partial z_L}{\partial q} = 0 \tag{78}$$

and thus

$$\frac{\partial z_L}{\partial q} = \frac{-[f_2(k_2) - k_2 f_2'(k_2)](1+z_L)}{(1+z_L)[g_2''(z_L)+\phi''(z_L)z_L+2\phi'(z_L)]+qk_2^2 f_2''(k_2)} > 0 \tag{79}$$

Next computing $\frac{\partial z_L}{\partial k}$ gives,

$$[g_2''(z_L) + \phi''(z_L)z_L + 2\phi'(z_L)] \frac{\partial z_L}{\partial k} + q f_2'(k_2) \left[\frac{1}{(1+z_L)} - \right.$$
$$\left. \frac{k_2}{(1+z_L)} \frac{\partial z_L}{\partial k} \right] - \frac{q f_2''(k_2)}{(1+z_L)} + \frac{q k_2 f_2''(k_2)}{(1+z_L)} \frac{\partial z_L}{\partial k} - q k_2 f_2'(k_2)$$
$$\left(\frac{1}{(1+z_L)} - \frac{k_2}{(1+z_L)} \frac{\partial z_L}{\partial k} \right) = 0 . \tag{80}$$

Hence,

$$\frac{\partial z_L}{\partial k} = \frac{q k_2 f_2''(k_2)}{(1+z_L)(g_2''(z_L)+\phi''(z_L)z_L+2\phi'(z_L))+qk_2^2 f_2''(k_2)} > 0 . \tag{81}$$

Also, $k_2 \frac{\partial z_L}{\partial k} < 1$ as can be seen when both sides of (81) are multiplied by k_2. In the case when the economy specializes in the production of y_2(the investment product) an increase in q always increases imports of investment and labour (or decreases exports of both). Moreover an increase in the capital-labour endowment ratio increases imports of labour (or decreases exports). Accordingly, we may define the functions $z_2 = \tilde{z}_2(q)$ by equation (51) and $z_L = \tilde{z}_L(k,q)$ by equations (79) and (81). Next from the offer function we have

$$z_1 = \tilde{z}_1(k,q) = g_1(\tilde{z}_2(q)) + g_2(\tilde{z}_L(k,q)) \tag{82}$$

with

$$\frac{\partial \tilde{z}_1}{\partial k} = g_2' \frac{\partial \tilde{z}_L}{\partial k} < 0, \quad \frac{\partial \tilde{z}_1}{\partial q} = g_1' \frac{d\tilde{z}_2}{dq} + g_2' \frac{\partial \tilde{z}_L}{\partial q} < 0 \tag{83}$$

i.e., increases in k and q always decrease imports of the consumption product (or increase exports of it). We know that $y_1 = 0$ and that

$$y_2 = \tilde{y}_2(k,q) = (1 + \tilde{z}_L(k,q))f_2\left(\frac{k}{(1 + z_L(k,q))}\right) \tag{84}$$

with,

$$\frac{\partial \tilde{y}_2}{\partial k} = \frac{\partial \tilde{z}_L}{\partial k}(f_2(k_2) - k_2 f_2'(k_2)) + f_2'(k_2) > 0$$

$$\frac{\partial \tilde{y}_2}{\partial q} = \frac{\partial \tilde{z}_L}{\partial q}(f_2(k_2) - k_2 f_2'(k_2)) > 0. \tag{85}$$

Notice that an increase in the capital-labour endowment ratio and the demand price of investment always increases output of investment per labour endowment. In addition,

$$x_1 = \tilde{x}_1(k,q) = \tilde{z}_1(k,q)$$

$$\frac{\partial \tilde{x}_1}{\partial k} = \frac{\partial \tilde{z}_1}{\partial k} < 0, \quad \frac{\partial \tilde{x}_1}{\partial q} = \frac{\partial \tilde{z}_1}{\partial q} < 0 \quad . \tag{86}$$

and

$$x_2 = \tilde{x}_2(k,q) = \tilde{y}_2(k,q) + \tilde{z}_2(q)$$

$$\frac{\partial \tilde{x}_2}{\partial k} = \frac{\partial \tilde{y}_2}{\partial k} > 0, \quad \frac{\partial \tilde{x}_2}{\partial q} = \frac{\partial \tilde{y}_2}{\partial q} + \frac{d\tilde{z}_2}{dq} > 0 \quad . \tag{87}$$

As before we must recognize how the different specializations of expenditure alter our results.

3.2.1. Pattern (3,2) $(y_1 = 0,\ y_2 > 0,\ x_1 > 0,\ x_2 > 0)$

Our results do not change from the previous section except to note that from (77),

$$p(\omega) = p\left(\omega_2\left(\frac{k}{(1 + z_L)}\right)\right) \leq q \quad . \tag{88}$$

3.2.2. Pattern (3,1) $(y_1 = 0,\ y_2 > 0,\ x_1 = 0)$

Here we observe from Table 1 that,

$$\tilde{z}_2(q) = -\tilde{y}_2(k,q)$$

or

$$\tilde{z}_2(q) = -(1 + \tilde{z}_L(k,q))f_2\left(\frac{k}{(1 + \tilde{z}_L(k,q))}\right) \quad . \tag{89}$$

We have an implicit function in q and k. Solving for $\frac{dq}{dk}$ yields,

$$\frac{dq}{dk} = - \frac{\frac{\partial \tilde{z}_L}{\partial k}(f_2(k_2) - k_2 f_2'(k_2)) + f_2'(k_2)}{\frac{d\tilde{z}_2}{dq} + \frac{\partial \tilde{z}_L}{\partial q}(f_2(k_2) - k_2 f_2'(k_2))} < 0. \qquad (90)$$

Thus $\tilde{q}_2 = \tilde{q}_2(k)$, where $\tilde{q}_2'(k)$ is defined by (90). (91)

Furthermore, since we are specialized in y_2

$$\tilde{q}_2(k) \geqq q \geqq p(\omega_2(\frac{k}{(1 + z_L)})) . \qquad (92)$$

Consequently,

$$z_2 = \tilde{z}_2(\tilde{q}_2(k)), \quad z_L = \tilde{z}_L(k,\tilde{q}_2(k)), \quad z_1 = \tilde{z}_1(k,\tilde{q}_2(k))$$

$$y_2 = \tilde{y}_2(k,\tilde{q}_2(k)), \quad x_1 = \tilde{x}_1(k,\tilde{q}_2(k)), \quad x_2 = \tilde{x}_2(k,\tilde{q}_2(k)) .$$

3.3. Pattern $y_1 > 0,\ y_2 = 0$

In this pattern $\ell_2 = 0$, $\ell_1 = 1 + z_L$ therefore, $k_1(1 + z_L) = k$

from equation (20) and $\dfrac{k - k_2(1 + z_L)}{k_1 - k_2} = (1 + z_L)$, $\dfrac{k_1(1 + z_L) - k}{k_1 - k_2} = 0$.

Thus, the control problem is,

$$\max_{\{z_2, z_L\}} W(z_2, z_L) = \int_0^\infty e^{-\delta t}\Big((1 + z_L) f_1\Big(\frac{k}{(1 + z_L)}\Big) + g_1(z_2) + g_2(z_L) + \phi(z_L) z_L\Big)\, dt \tag{93}$$

subject to,

$$\dot{k} = z_2 - \lambda k$$

$$0 < k_o < \infty .$$

The Hamiltonian is,

$$H(z_2, z_L; k, q) = (1 + z_L) f_1\Big(\frac{k}{(1 + z_L)}\Big) + g_1(z_2) + g_2(z_L) + \phi(z_L) z_L + q(z_2 - \lambda k) . \tag{94}$$

This means that the necessary conditions are (all derivatives evaluated at the solution),

$$\frac{\partial H}{\partial z_2} = g_1'(z_2) + q = 0 ,$$

$$\frac{\partial H}{\partial z_L} = f_1(k_1) - k_1 f_1'(k_1) + g_2'(z_L) + \phi'(z_L) z_L + \phi(z_L) = 0 ,$$

$$\dot{k} = z_2 - \lambda k, \quad \dot{q} = \delta q - \frac{\partial H}{\partial k}, \quad 0 < k_o < \infty ,$$

$$\lim_{t \to \infty} e^{-\delta t} q(t) \geq 0, \quad \lim_{t \to \infty} e^{-\delta t} q(t)\, \dot{k}(t) = 0 , \tag{95}$$

$$\frac{\partial^2 H}{\partial z_2^2} = g_1''(z_2) < 0, \quad \frac{\partial^2 H}{\partial z_2 \partial z_L} = \frac{\partial^2 H}{\partial z_L \partial z_2} = 0 \ ,$$

$$\frac{\partial^2 H}{\partial z_L^2} = \frac{k_1^2 f_1''(k_1)}{(1 + z_L)} + g_2''(z_L) + \phi''(z_L) z_L + 2\phi'(z_L) < 0 \ .$$

Therefore H is strictly concave in the controls, z_2 and z_L. Furthermore, by the assumption imposed on $k_i \;\; i = 1,2$ when the economy is specialized in production we have,

$$k_2 = k_2(\omega_1), \quad k_1 = \frac{k}{(1 + z_L)} \tag{96}$$

and since,

$$\omega_1 = \omega_1 \left(\frac{k}{(1 + z_L)} \right) = \omega \tag{97}$$

then

$$k_2 = k_2 \left(\frac{k}{(1 + z_L)} \right) . \tag{98}$$

Also,

$$p(\omega) = \frac{f_1'(k_1)}{f_2'(k_2)} = \frac{f_1'\left(\frac{k}{(1 + z_L)} \right)}{f_2'\left(k_2\left(\frac{k}{(1+ z_L)}\right)\right)} = p\left(\omega_1\left(\frac{k}{(1 + z_L)} \right)\right) \tag{99}$$

If we substitute (96) and (98) into $\frac{\partial H}{\partial z_L} = 0$ we are able to find the effects of changes of k and q on $z_2, z_L, z_1, y_1, y_2, x_1, x_2$.

<u>Theorem 3.3.1.</u> If $y_1 > 0$, $y_2 = 0$ then $z_2 = \bar{z}_2(q)$, $\bar{z}_2' > 0$; $z_L = \bar{z}_L(k)$, $\bar{z}_L' > 0$; $z_1 = \bar{z}_1(k,q)$, $\frac{\partial \bar{z}_1}{\partial k} < 0$, $\frac{\partial \bar{z}_1}{\partial q} < 0$; $y_1 = \bar{y}_1(k)$, $\bar{y}_1' > 0$; $y_2 = 0$; $x_1 = \bar{x}_1(k,q)$, if $w \geq \Psi$ $\frac{\partial \bar{x}_1}{\partial k} > 0$, if $w < \Psi$ then $\frac{\partial \bar{x}_1}{\partial k} \gtreqless 0$ iff $\bar{z}_L'(w - \Psi) \gtreqless - f_1'(k_1)$, $\frac{\partial \bar{x}_1}{\partial q} < 0$; $x_2 = \bar{x}_2(q)$, $\bar{x}_2' > 0$.

Again z_2 is defined by equation (51) in this pattern, i.e., $z_2 = \bar{z}_2(q)$.

Now solving $\frac{\partial H}{\partial z_L} = 0$ for $\frac{\partial z_L}{\partial q}$ we find,

$$- \frac{f_1'(k_1)k_1}{(1 + z_L)} \frac{\partial z_L}{\partial q} + \frac{f_1'(k_1)k_1}{(1 + z_L)} \frac{\partial z_L}{\partial q} - \frac{k_1^2 f_1''(k_1)}{(1 + z_L)} \frac{\partial z_L}{\partial q}$$

$$+ [g_2''(z_L) + \phi''(z_L)z_L + 2\phi'(z_L)] \frac{\partial z_L}{\partial q} = 0 . \tag{100}$$

Hence $\frac{\partial z_L}{\partial q} = 0$. On the other hand, solving for $\frac{\partial z_L}{\partial k}$ yields,

$$\frac{k_1 f_1''(k_1)}{(1 + z_L)} + \frac{k_1^2 f_1''(k_1)}{(1 + z_L)} \frac{\partial z_L}{\partial k} + [g_2''(z_L) + \phi''(z_L)z_L$$

$$+ 2\phi'(z_L)] \frac{\partial z_L}{\partial k} = 0 \tag{101}$$

and therefore

$$\frac{\partial z_L}{\partial k} = \frac{k_1 f_1''(k_1)}{(1 + z_L)(g_2''(z_L) + \phi''(z_L)z_L + 2\phi'(z_L)) + k_1^2 f_1''(k_1)} > 0. \tag{102}$$

Notice that multiplying (102) by k_1 we find $k_1 \frac{\partial z_L}{\partial k} < 1$. Consequently, $z_L = \bar{z}_L(k)$ where $\bar{z}_L'$ is defined by (102). Next from the foreign offer function,

$$z_1 = \bar{z}_1(k,q) = g_1(\bar{z}_2(q)) + g_2(\bar{z}_L(k)) \tag{103}$$

where,

$$\frac{\partial \bar{z}_1}{\partial k} = g_2' \frac{d\bar{z}_L}{dk} < 0, \quad \frac{\partial \bar{z}_1}{\partial q} = g_1' \frac{d\bar{z}_2}{dq} < 0 . \tag{104}$$

These results state that as q increases imports of the investment product increase (or exports decrease); imports of the consumption product decrease (or exports increase); and imports (or exports) of labour are not affected. If k increases then as in the previous patterns imports (or exports) of the investment product are not affected. Nevertheless when the capital-labour endowment ratio increases, imports of labour increase (or exports decrease), and imports of consumption decrease (or exports increase). Concerning the production of consumption and investment we have that $y_2 = 0$ and y_1 is defined by,

$$y_1 = \bar{y}_1(k) = (1 + \bar{z}_L(k))f_1\left(\frac{k}{(1 + \bar{z}_L(k))}\right) \tag{105}$$

with,

$$\frac{d\bar{y}_1}{dk} = \frac{d\bar{z}_L}{dk}\,[f_1(k_1) - k_1 f_1'(k_1)] + f_1'(k_1) > 0\ , \tag{106}$$

i.e., an increase in the capital-labour endowment ratio always increases output of consumption per labour endowment.

Therefore,

$$x_1 = \bar{x}_1(k,q) = \bar{y}_1(k) + \bar{z}_1(k,q) \tag{107}$$

and

$$\frac{\partial \bar{x}_1}{\partial k} = \frac{d\bar{y}_1}{dk} + \frac{\partial \bar{z}_1}{\partial k}\,,\quad \frac{\partial \bar{x}_1}{\partial q} = \frac{\partial \bar{z}_1}{\partial q} < 0\ . \tag{108}$$

Substituting (106) and (104) into (108) and noting that $w = f_1(k_1) - k_1 f_1'(k_1)$ and $\Psi = -\,g_2'(z_L)$ we have,

$$\frac{\partial \bar{x}_1}{\partial k} = \frac{d\bar{z}_L}{dk}\,(w - \Psi) + f_1'(k_1) \tag{109}$$

Hence, if $w \geq \Psi$ then $\frac{\partial \bar{x}_1}{\partial k} > 0$. If $w < \Psi$ then $\frac{\partial \bar{x}_1}{\partial k} \gtreqless 0$ if and only if $\frac{d\bar{z}_L}{dk}(w - \Psi) \gtreqless -\,f_1(k_1)$. Notice that in the absence of any labour externality in social welfare, i.e., $w = \Psi$ then $\frac{\partial \bar{x}_1}{\partial k}$ is always positive. Equation (109) states that the sign of the effect of an increase of the capital-labour endowment ratio on consumption is determined by whether the marginal labour externality ($\Psi - w$, remembering that $U' = 1$) is less than, equal to or greater than the ratio of the marginal product of the produced commodity to the marginal increase in imports (or decrease in exports) of labour. Finally for expenditures on investment we define,

$$x_2 = \bar{x}_2(q) = \bar{z}_2(q)$$

with,

$$\frac{d\bar{x}_2}{dq} = \frac{d\bar{z}_2}{dq} > 0 \ . \tag{110}$$

3.3.1. Pattern (1,2) ($y_1 > 0$, $y_2 = 0$, $x_1 > 0$, $x_2 > 0$)

Our results are the same as in the prior section except that since $y_1 > 0$, $y_2 = 0$ we have

$$q \leq p(\omega_1(\frac{k}{(1 + z_L)})) \ . \tag{111}$$

3.3.2. Pattern (1,1) ($y_1 > 0$, $y_2 = 0$, $x_1 > 0$, $x_2 = 0$).

From Table 1 we have the following relationship between y_2 and z_2

$$y_2 = \bar{z}_2(q) = 0 \ . \tag{112}$$

Now since $g_1'(z_2) = q$, define $g_1'(o) = q_0$. Moreover because $\bar{z}_2' > 0$ if we do not want to enter into any other patterns we must have $q \leq q_0$. Therefore we have

$$q \lesseqgtr q_0 \lesseqgtr p(\omega_1(\frac{k}{(1 + \bar{z}_L)})) \quad . \tag{113}$$

Pattern $y_1 = 0 = y_2$ is somewhat of a degenerate case and it is more convenient to dispose of it during our discussion of the optimum path and the steady states. It is efficacious to take stock of the various relationships that we have derived and illustrate how the different patterns are connected in (k,q) space.

Table 2. Values of $z_2(k,q)$

Expenditure	Production (1)	Production (2)	Production (3)
(1)	0	$\hat{z}_2(\hat{q}_2(k)) = -\hat{y}_2(k,\hat{q}_2(k))$	$\tilde{z}_2(\tilde{q}_2(k)) = -\tilde{y}_2(k,\tilde{q}_2(k))$
(2)	$\bar{z}_2(q)$	$\hat{z}_2(q)$	$\tilde{z}_2(q)$

Table 3. Values of $z_L(k,q)$

Expenditure	Production (1)	Production (2)	Production (3)
(1)	$\bar{z}_L(k)$	$\hat{z}_L(\hat{q}_2(k))$	$\tilde{z}_L(k,\tilde{q}_2(k))$
(2)	$\bar{z}_L(k)$	$\hat{z}_L(q)$	$\tilde{z}_L(k,q)$

Table 4. Values of $z_1(k,q)$

Expenditure	Production (1)	Production (2)	Production (3)
(1)	$\bar{z}_1(k,q_0)$	$\hat{z}_1(\hat{q}_2(k))$	$\tilde{z}_1(k,\tilde{q}_2(k))$
(2)	$\bar{z}_1(k,q)$	$\hat{z}_1(q)$	$\tilde{z}_1(k,q)$

Table 5. Values of $y_1(k,q)$

Expenditure	Production (1)	(2)	(3)
(1)	$\bar{y}_1(k)=(1+\bar{z}_L(k))f_1(\frac{k}{(1+\bar{z}_L(K))})$	$\hat{y}_1(k,\hat{q}_2(k))$	0
(2)	$\bar{y}_1(k)=(1+\bar{z}_L(k))f_1(\frac{k}{(1+\bar{z}_L(k))})$	$\hat{y}_1(k,q)$	0

Table 6. Values of $y_2(k,q)$

Expenditure	Production (1)	(2)	(3)
(1)	0	$\hat{y}_2(k,q_2(k))$	$\tilde{y}_2(k,\tilde{q}_2(k))=(1+\tilde{z}_L(k,\tilde{q}_2(k)))f_2(\frac{k}{(1+\tilde{z}_L(k,\tilde{q}_2(k))})$
(2)	0	$\hat{y}_2(k,q)$	$\tilde{y}_2(k,q) = (1+\tilde{z}_L(k,q))f_2(\frac{k}{(1+\tilde{z}_L(k,q))})$

Table 7. Values of $x_1(k,q)$

Expenditure	Production (1)	(2)	(3)
(1)	$\bar{x}_1(k,q_0)$	$\hat{x}_1(k,\hat{q}_2(k))$	$\tilde{x}_1(k,\tilde{q}_2(k))$
(2)	$\bar{x}_1(k,q)$	$\hat{x}_1(k,q)$	$\tilde{x}_1(k,q)$

Table 8. Values of $x_2(k,q)$

Expenditure	Production (1)	Production (2)	Production (3)
(1)	$\bar{x}_2(q_0)$	$\hat{x}_2(k,\hat{q}_2(k))$	$\tilde{x}_2(k,\tilde{q}_2(k))$
(2)	$\bar{x}_2(q)$	$\hat{x}_2(k,q)$	$\tilde{x}_2(k,q)$

Table 9. Various Price Ranges

Expenditure	Production (1)	Production (2)	Production (3)
(1)	$q_0 \leq p(\omega_1(\frac{k}{(1+\bar{z}_L)}))$	$p(\omega_1(\frac{k}{(1+\bar{z}_L)}))<q\leq\hat{q}_2(k)$	$q\geq p(\omega_2(\frac{k}{(1+\tilde{z}_L)}))$
	$q \leq q_0$	$\hat{q}_2(k) < p(\omega_2(\frac{k}{(1+\tilde{z}_L)}))$	$\tilde{q}_2(k) \geq q$
		$\hat{q}_2(k) < q_o$	$\tilde{q}_2(k) < q_0$
(2)	$q \leq \omega_1(\frac{k}{(1+\bar{z}_L)}))$	$p(\omega_1(\frac{k}{(1+\bar{z}_L)})) < q$	$q \geq p(\omega_2(\frac{k}{(1+\tilde{z}_L)}))$
	$q > q_0$	$q < p(\omega_2(\frac{k}{(1+\tilde{z}_L)}))$	
		$q \gtreqless q_0$	$q \gtreqless q_0$

Hence from Tables 2, 4, 5, 6, 9 and the relevant slopes of the functions derived in the text we have the following diagrams.

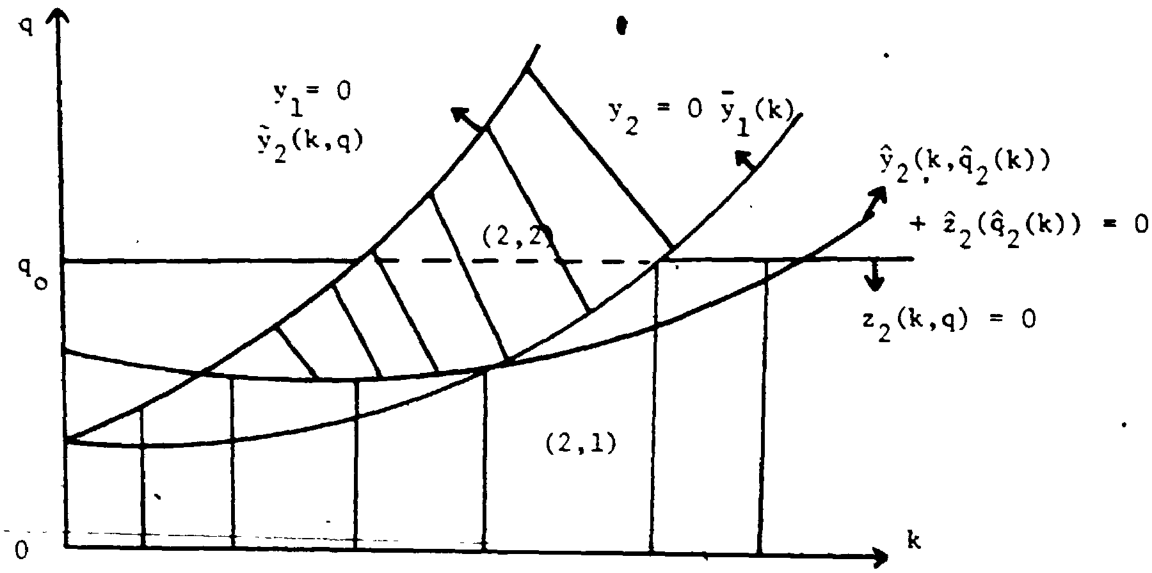

Figure 2. Patterns (2,2) and (2,1) when $k_1 > k_2$

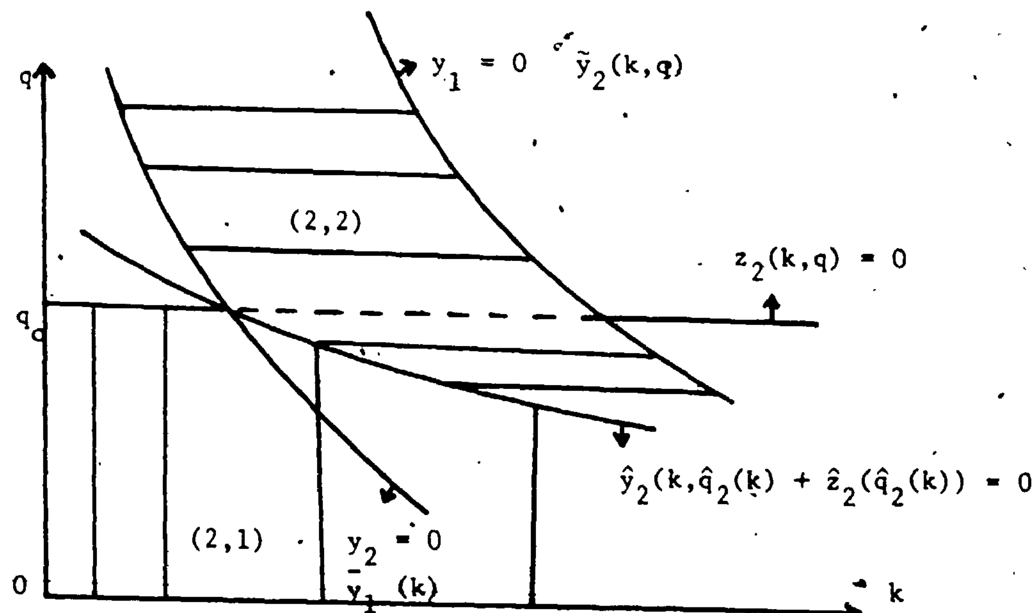

Figure 3. Patterns (2,2) and (2,1) when $k_2 > k_1$

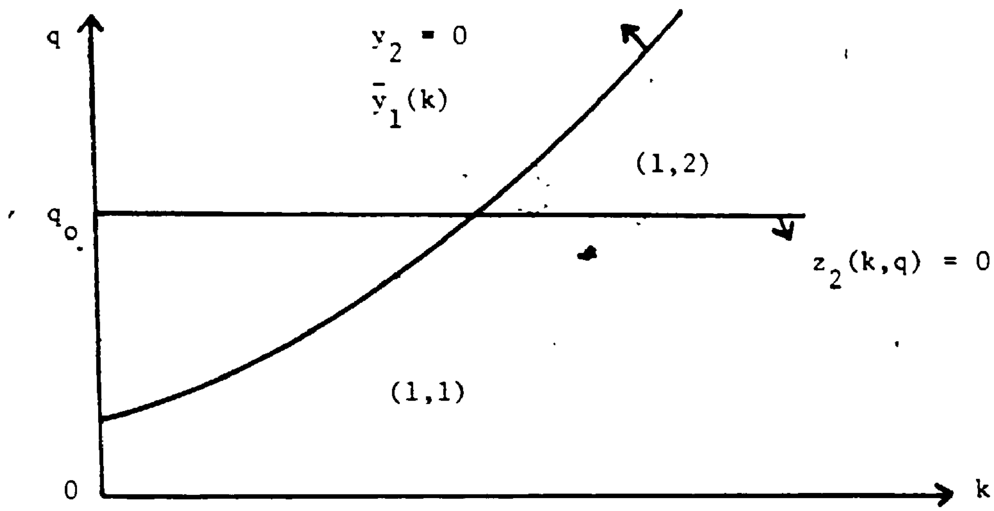

Figure 4. Patterns (1,2) and (1,1) when $k_1 > k_2$

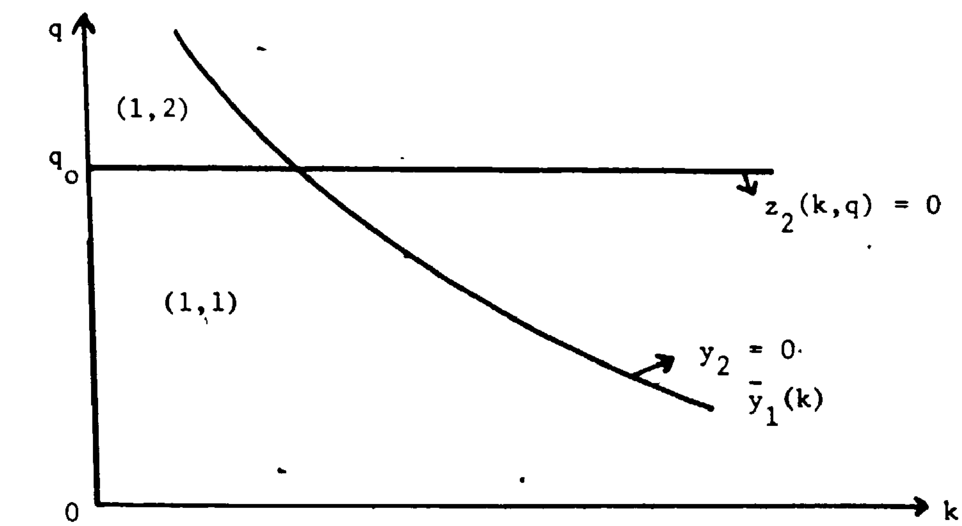

Figure 5. Patterns (1,2) and (1,1) when $k_2 > k_1$

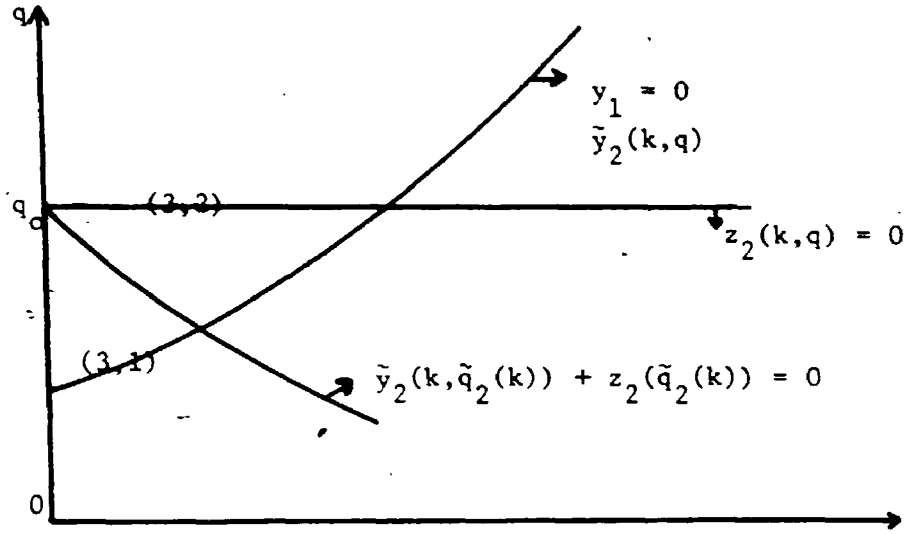

Figure 6. Patterns (3,2) and (3,1) when $k_1 > k_2$

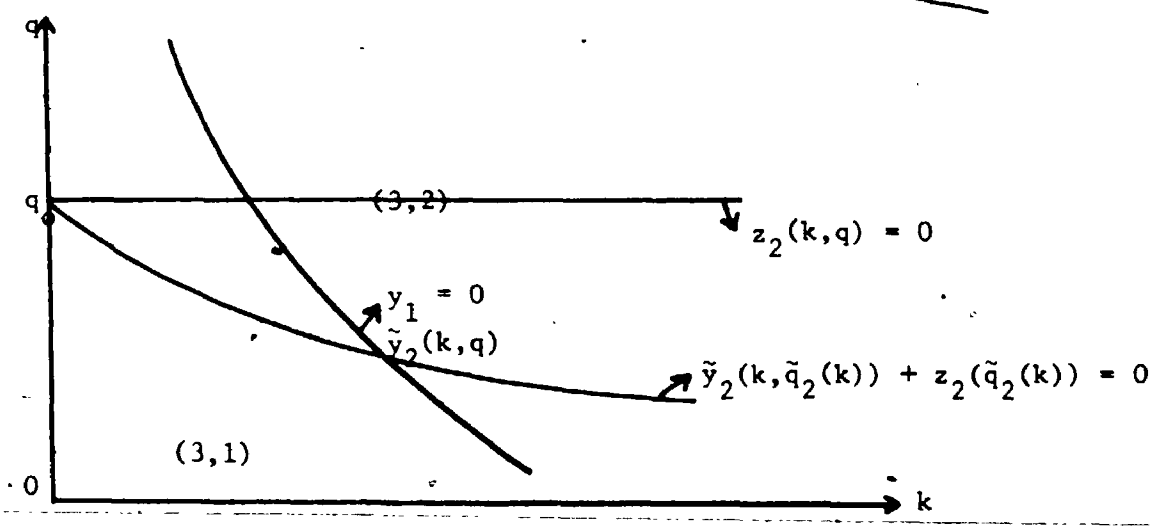

Figure 7. Patterns (3,2) and (3,1) when $k_2 > k_1$

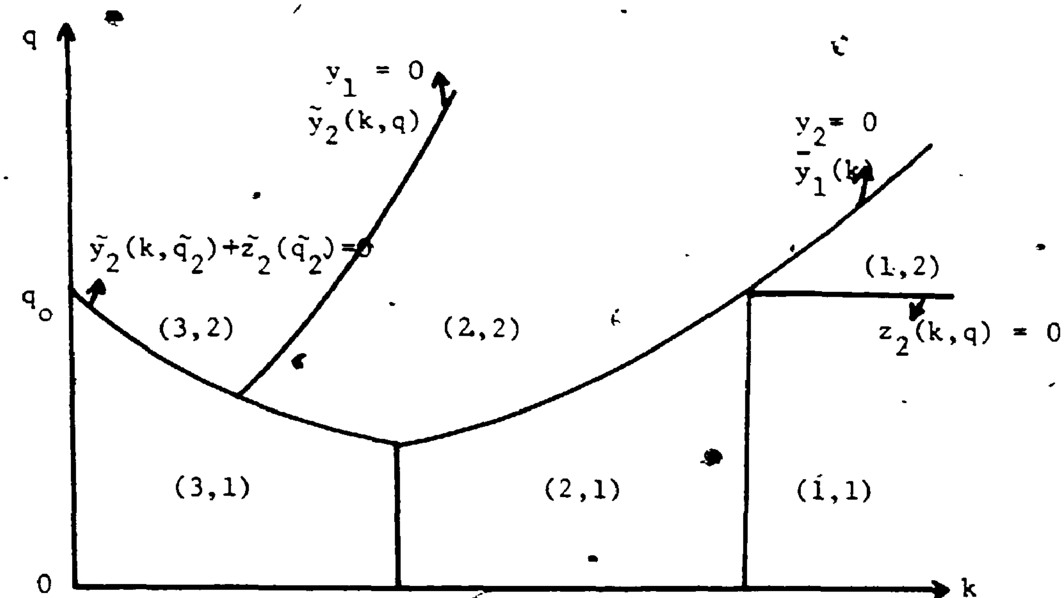

Figure 8. All patterns when $k_1 > k_2$

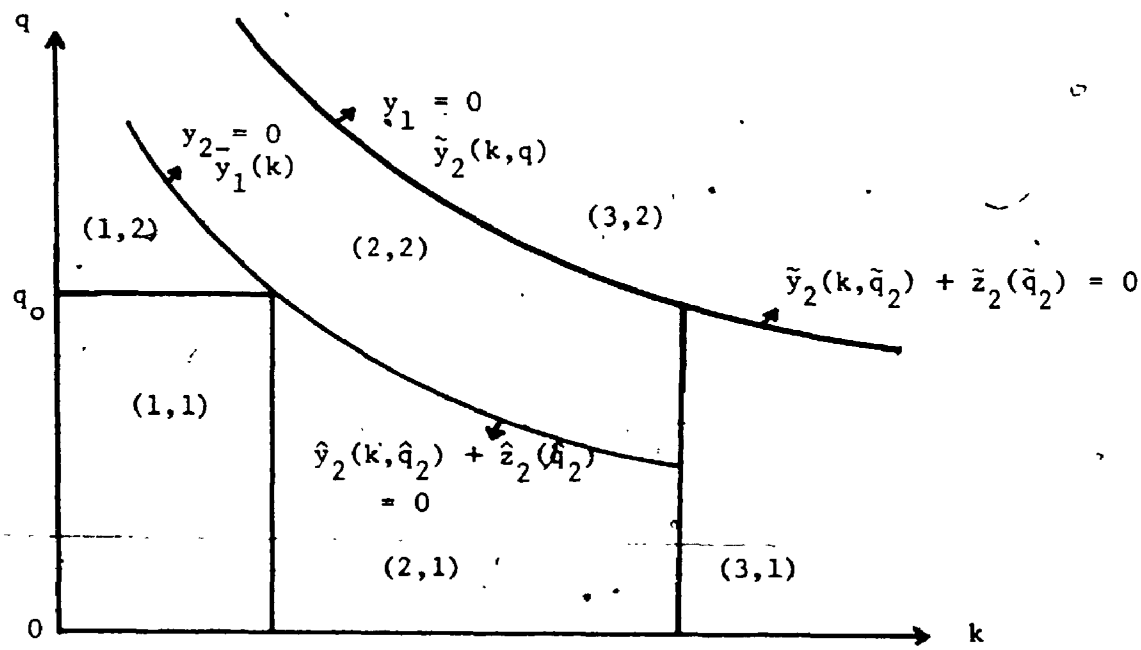

Figure 9. All patterns when $k_2 > k_1$

4. Intertemporal Equilibria and the Optimal Paths

We have amassed adequate information to examine the differential equations,

$$\dot{k} = y_2(k,q) + z_2(k,q) - \lambda k$$

$$\dot{q} = (\lambda+\delta)q - \frac{\partial y_1(k,q)}{\partial k} - \frac{\partial z_1(k,q)}{\partial k} - \frac{\partial[\phi(z_L(k,q))z_L(k,q)]}{\partial k} - q\frac{\partial y_2(k,q)}{\partial k} - q\frac{\partial z_2(k,q)}{\partial k} \tag{114}$$

where z_2, z_L, z_1, y_1, y_2, are defined by Tables 2, 3, 4, 5 and 6. We are interested in the steady state solution to (114), i.e., the solution where $\dot{k} = 0 = \dot{q}$.

4.1. $\dot{q} = 0$ Equation

4.1.1. Pattern (2,2)

From equation (63) and since $q = \frac{f_1'(k_1(q))}{f_2'(k_2(q))}$, $\dot{q}$ may be written as,

$$\dot{q} = (\lambda+\delta)\, q - f_2'(k_2(q))\, q \,. \tag{115}$$

Therefore

$$\frac{\partial \dot{q}}{\partial k} = 0 \,, \quad \frac{\partial(\frac{\dot{q}}{q})}{\partial q} = - f_2''(k_2)k_2'(q)$$

and

$$\operatorname{sgn} \frac{\partial(\frac{\dot{q}}{q})}{\partial q} = \operatorname{sgn}\,(k_1 - k_2) \,. \tag{116}$$

4.1.2. Pattern (2,1)

In this pattern we have

$$\dot{q} = (\lambda+\delta)\, q - \left[\frac{\partial \hat{y}_1}{\partial k} + \frac{\partial \hat{y}_1}{\partial \hat{q}_2}\frac{d\hat{q}_2}{dk}\right] - \frac{d\hat{z}_1}{d\hat{q}_2}\frac{d\hat{q}_2}{dk} - \frac{d\phi(\hat{z}_L)}{d\hat{z}_L}\frac{d\hat{z}_L}{d\hat{q}_2}\frac{d\hat{q}_2}{dk}\,\hat{z}_L - \phi(\hat{z}_L)\frac{d\hat{z}_L}{d\hat{q}_2}\frac{d\hat{q}_2}{dk}, \tag{117}$$

remembering that $\hat{z}_2(\hat{q}_2(k)) + \hat{y}_2(k,\hat{q}_2(k)) = 0$. Now from (61)

$$\frac{d\hat{z}_1}{d\hat{q}_2}\frac{d\hat{q}_2}{dk} = g_1'\frac{d\hat{z}_2}{d\hat{q}_2}\frac{d\hat{q}_2}{dk} + g_2'\frac{d\hat{z}_L}{d\hat{q}_2}\frac{d\hat{q}_2}{dk} \tag{118}$$

and from (68)

$$g_1'\frac{d\hat{z}_2}{d\hat{q}_2}\frac{d\hat{q}_2}{dk} = -g_1'\left[\frac{\partial \hat{y}_2}{\partial k} + \frac{\partial \hat{y}_2}{\partial \hat{q}_2}\frac{d\hat{q}_2}{dk}\right] \tag{119}$$

with $-g_1' = +\hat{q}_2(k)$, and (63)

$$g_1'\frac{d\hat{z}_2}{d\hat{q}_2}\frac{d\hat{q}_2}{dk} = \hat{q}_2(k)\cdot\left[-\frac{(k_2+\omega)f_2'(k_2)}{(k_1-k_2)} + \frac{\partial \hat{y}_2}{\partial \hat{q}_2}\frac{d\hat{q}_2}{dk}\right]. \tag{120}$$

Therefore since,

$$\frac{\partial \hat{y}_1}{\partial k} - \hat{q}_2(k)\frac{\partial \hat{y}_2}{\partial k} = \frac{(k_1+\omega)f_1'(k_1)}{k_1-k_2} - \hat{q}_2\,\frac{(k)f_2'(k_2)(k_2+\omega)}{k_1-k_2} = f_1'(k_1) \tag{121}$$

$\dot{q}$ becomes,

$$\dot{q} = (\lambda+\delta)\, q - f_1'(k) - \left[\frac{\partial \hat{y}_1}{\partial \hat{q}_2} + \frac{\partial \hat{y}_2}{\partial \hat{q}_2}\right]\frac{d\hat{q}_2}{dk} - \left[\phi'(\hat{z}_L)\hat{z}_L + \phi(\hat{z}_L) + g_2'\right]\frac{d\hat{z}_L}{d\hat{q}_2}\frac{d\hat{q}_2}{dk}. \tag{122}$$

Let us first deal with $\frac{\partial \hat{y}_1}{\partial \hat{q}_2} + \frac{\partial \hat{y}_2}{\partial \hat{q}_2}$. From (64),

$$\frac{\partial \hat{y}_2}{\partial \hat{q}_2} = - [\frac{\partial \hat{y}_1}{\partial \hat{q}_2} + \frac{d\hat{z}_L}{d\hat{q}_2} \frac{k_2 f_1(k_1)}{k_1 - k_2}] \frac{f_2'(k_2)}{f_1'(k_1)} + \frac{k_1 f_2(k_2)}{k_1 - k_2} \frac{d\hat{z}_L}{dq}. \quad (123)$$

Hence

$$\frac{\partial \hat{y}_1}{\partial \hat{q}_2} + \frac{\partial \hat{y}_2}{\partial \hat{q}_2} = \hat{q}_2 \frac{d\hat{q}_2}{dk} [\frac{d\hat{z}_L}{d\hat{q}_2} \frac{k_1 f_2(k_2)}{k_1 - k_2} - \frac{d\hat{z}_L}{d\hat{q}_2} \frac{k_2}{k_1 - k_2} \frac{f_1(k_1) f_2'(k_2)}{f_1'(k_1)}]. \quad (124)$$

Recalling that $f_2'(k_2)\omega = f_2(k_2) - k_2 f_2'(k_2)$ and $\hat{q}_2(k)(f_2(k_2) - k_2 f_2'(k_2)) = w$ we get,

$$\frac{\partial \hat{y}_1}{\partial \hat{q}_2} + \frac{\partial \hat{y}_2}{\partial \hat{q}_2} = \frac{d\hat{q}_2}{dk} \frac{d\hat{z}_L}{d\hat{q}_2} w. \quad (125)$$

But from the necessary conditions of optimality $(\phi'(z_L)z_L + \phi(z_L) + w + g_2'(z_L)) = 0$, so (122) becomes,

$$\dot{q} = (\lambda+\delta)\, q - f_1'(k_1(\hat{q}_2(k)). \quad (126)$$

Therefore using (49) and (69),

$$\frac{\partial \dot{q}}{\partial k} = - f_1''(k_1) k_1'(\hat{q}_2)\hat{q}_2' > 0$$

and

$$\frac{\partial \dot{q}}{\partial q} = \lambda + \delta > 0. \quad (127)$$

4.1.3. Pattern (3,2)

In this pattern,

$$\dot{q} = (\lambda+\delta) q - \frac{\partial \tilde{z}_1(k,q)}{\partial k} - \frac{\partial[\phi(\tilde{z}_L(k,q))\tilde{z}_L(k,q)]}{\partial k} - \frac{q \partial \tilde{y}_2(k,q)}{\partial k}. \quad (128)$$

Then from (73), (83) and (85),

$$\frac{\partial\tilde{z}_1(k,q)}{\partial k} + \frac{\partial[\phi(\tilde{z}_L(k,q))\tilde{z}_L(k,q)]}{\partial k} + \frac{q\,\partial\tilde{y}_2(k,q)}{\partial k} = [g_2' + \phi'(\tilde{z}_L)\tilde{z}_L + \phi(\tilde{z}_L)$$

$$+ q[f_2(k_2) - k_2 f_2'(k_2)]\frac{\partial\tilde{z}_L}{\partial k} + q\, f_2'(k_2). \quad (129)$$

Therefore, noting $k_2(1+\tilde{z}_L(k,q)) = k$,

$$\dot{q} = (\lambda+\delta)\, q - q\, f_2'\left(\frac{k}{(1+\tilde{z}_L(k,q)}\right). \quad (130)$$

Thus,

$$\frac{\partial\dot{q}}{\partial k} = \frac{q f_2''(k_2)}{(1+\tilde{z}_L(k,q)}\left[k_2 \cdot \frac{\partial\tilde{z}_L}{\partial k} - 1\right] > 0 \quad (131)$$

since $k_2 \dfrac{\partial\tilde{z}_L}{\partial k} - 1 < 0$ from (81).

Also from (79),

$$\frac{\partial(\frac{\dot{q}}{q})}{\partial q} = \frac{f_2''(k_2)k_2}{(1+\tilde{z}_L(k,q))}\frac{\partial\tilde{z}_L}{\partial q} < 0. \quad (132)$$

4.1.4. Pattern (3,1)

Next we have

$$\dot{q} = (\lambda+\delta)q - \frac{\partial\tilde{z}_1(k,\tilde{q}_2(k))}{\partial k} - \frac{\partial[\phi(\tilde{z}_L(k,\tilde{q}_2(k)))\tilde{z}_L(k,\tilde{q}_2(k))]}{\partial k}. \quad (133)$$

Moreover in this case,

$$\frac{\partial\tilde{z}_1(k,\tilde{q}_2(k))}{\partial k} = g_1'\frac{d\tilde{z}_2}{d\tilde{q}_2}\frac{d\tilde{q}_2}{dk} + g_2'\left[\frac{\partial\tilde{z}_L}{\partial\tilde{q}_2}\frac{d\tilde{q}_L}{\partial k} + \frac{\partial\tilde{z}_L}{\partial k}\right]. \quad (134)$$

In addition we know that,

$$g_1' \frac{d\tilde{z}_2}{d\tilde{q}_2} \frac{d\tilde{q}_2}{dk} = \tilde{q}_2(k) \left[\frac{\partial \tilde{y}_2}{\partial k} + \frac{\partial \tilde{y}_2}{\partial \tilde{q}_2} \frac{d\tilde{q}_2}{dk}\right] \tag{135}$$

and substituting (85) into (135) yields,

$$g_1' \frac{d\tilde{z}_2}{d\tilde{q}_2} \frac{d\tilde{q}_2}{dk} = \tilde{q}_2(k) \left[\frac{\partial \tilde{z}_L}{\partial k} (f_2(k_2) - k_2 f_2'(k_2)) + f_2'(k_2) + \frac{\partial \tilde{z}_L}{\partial \tilde{q}_2} \frac{d\tilde{q}_2}{dk} [f_2(k_2) - k_2 f_2'(k)]\right]. \tag{136}$$

Hence from the first order conditions (73), (136) and (133), equation (114) becomes

$$\dot{q} = (\lambda+\delta)\, q - \tilde{q}_2(k) f_2' \left(\frac{k}{(1+\tilde{z}_L(k,\tilde{q}_2(k)))}\right) . \tag{137}$$

Therefore differentiating (137) with respect to k and q gives,

$$\frac{\partial \dot{q}}{\partial k} = - \frac{d\tilde{q}_2}{dk} f_2'(k_2) + \frac{\tilde{q}_2(k) f_2''(k_2)}{(1 + \tilde{z}_L)} \left[k_2 \frac{\partial \tilde{z}_L}{\partial k} - 1\right] + \frac{\tilde{q}_2(k) f_2''(k_2) k_2}{(1 + \tilde{z}_L)} \frac{\partial \tilde{z}_L}{\partial \tilde{q}_2} \frac{d\tilde{q}_2}{dk} > 0 , \tag{138}$$

which is clearly positive since $k_2 \frac{\partial \tilde{z}_L}{\partial k} - 1 < 0$ from (81) and also from (79) and (90), $\frac{\partial \tilde{z}_L}{\partial \tilde{q}_2} > 0$, $\frac{d\tilde{q}_2}{dk} < 0$.

Finally,

$$\frac{\partial \dot{q}}{\partial q} = \lambda + \delta > 0. \tag{139}$$

4.1.5. <u>Pattern (1,2)</u>

Equation (114) in this section is defined as,

$$\dot{q} = (\lambda+\delta)\, q - \frac{d\bar{y}_1(k)}{dk} - \frac{\partial \bar{z}_1(k,q)}{\partial k} - \frac{d[\phi(\bar{z}_L(k))\bar{z}_L(k)]}{dk} \quad . \tag{140}$$

By equation (106), (104) and (95), (140) becomes,

$$\dot{q} = (\lambda+\delta)q - f_1'\left(\frac{k}{(1 + \bar{z}_L(k))}\right) . \tag{141}$$

Proceeding as before,

$$\frac{\partial \dot{q}}{\partial k} = \frac{f_1''(k_1)}{(1 + \bar{z}_L(k))} \left[k_1 \frac{d\bar{z}_L}{dk} - 1\right] > 0 \tag{142}$$

by equation (102).

Also,

$$\frac{\partial \dot{q}}{\partial q} = \lambda + \delta > 0. \tag{143}$$

4.1.6. Pattern (1,1)

Since there are no differences in the functions between patterns (1,2) and (1,1) except for the range of the demand price of investment, we have,

$$\dot{q} = (\lambda + \delta)q - f_1'\left(\frac{k}{(1 + \bar{z}_L(k))}\right) \tag{144}$$

$$\frac{\partial \dot{q}}{\partial k} = \frac{f_1''(k_1)}{(1+\bar{z}_L(k))} \left[k_1 \frac{d\bar{z}_L}{dk} - 1\right] > 0, \quad \frac{\partial \dot{q}}{\partial q} = \lambda + \delta > 0. \tag{145}$$

4.2. $\dot{k} = 0$ Equation

4.2.1. Pattern (2,2)

Now for the $\dot{k}$ equation in (2,2), (114) becomes,

$$\dot{k} = \hat{y}_2(k,q) + \hat{z}_2(q) - \lambda k. \tag{146}$$

Therefore,

$$\frac{\partial \dot{k}}{\partial k} = \frac{\partial \hat{y}_2}{\partial k} - \lambda \tag{147}$$

and if $k_1 > k_2$ then $\frac{\partial \hat{y}_2}{\partial k} < 0$ and (147) is negative. On the other hand for $k_2 > k_1$, $\frac{\partial \hat{y}_2}{\partial k} > 0$ and then (147) is indeterminate. Nevertheless from (115) we have for $\dot{q} = 0$, $f_2'(k_2) - \lambda = \delta > 0$. Hence

$$\left.\frac{\partial \dot{k}}{\partial k}\right|_{\dot{q}=0} = \frac{\partial \hat{y}_2}{\partial k} - \delta + f_2'(k_2) \tag{148}$$

and from (63)

$$\left.\frac{\partial \dot{k}}{\partial k}\right|_{\dot{q}=0} = - \frac{(k_2+\omega) f_2'(k_2)}{(k_1-k_2)} + \delta - f_2'(k_2) \tag{149}$$

$$\left.\frac{d\dot{k}}{dk}\right|_{\dot{q}=0} = - \left[\frac{(\omega+k_1)\lambda+\delta(\omega+k_2)}{(k_1-k_2)}\right] \tag{150}$$

Thus,

$$\operatorname{sgn} \left.\frac{d\dot{k}}{dk}\right|_{\dot{q}=0} = \operatorname{sgn} (k_2-k_1). \tag{151}$$

Moreover

$$\frac{\partial \dot{k}}{\partial q} = \frac{\partial \hat{y}_2}{\partial q} + \frac{d\hat{z}_2}{dq} > 0 \tag{152}$$

by (64) and (50).

4.2.2. Patterns (2,1), (3,1), (1,1)

Here we have,

$$\dot{k} = -\lambda k \tag{153}$$

$$\frac{\partial \dot{k}}{\partial k} = -\lambda < 0, \quad \frac{\partial \dot{k}}{\partial q} = 0.$$

Moreover $\dot{k} = 0$ if and only if $k = 0$.

4.2.3. Pattern (3,2)

The differential equation is

$$\dot{k} = \tilde{y}_2(k,q) + \tilde{z}_2(q) - \lambda k. \tag{154}$$

Therefore by (85)

$$\frac{\partial \dot{k}}{\partial k} = \frac{\partial \tilde{z}_L}{\partial k} (f_2(k_2) - k_2 f_2'(k_2)) + f_2'(k_2) - \lambda \tag{155}$$

and evaluating (155) at $q = 0$ $(f_2'(k_2) = \lambda + \delta)$,

$$\left.\frac{\partial \dot{k}}{\partial k}\right|_{q=0} = \frac{\partial \tilde{z}_L}{\partial k} (f_2(k_2) - k_2 f_2'(k_2)) + \delta > 0. \tag{156}$$

Differentiating (154) with respect to q yields

$$\frac{\partial \dot{k}}{\partial q} = \frac{\partial \tilde{z}_L}{\partial q} (f_2(k_2) - k_2 f_2'(k_2)) + \frac{d\tilde{z}_2}{dq} > 0 \tag{157}$$

by (51) and (79).

4.2.4. Pattern (1,2)

Equation (114) is defined as,

$$\dot{k} = \bar{z}_2(q) - \lambda k \tag{158}$$

with

$$\frac{\partial \dot{k}}{\partial k} = -\lambda < 0, \quad \frac{\partial \dot{k}}{\partial q} = \frac{d\bar{z}_2}{dq} > 0.$$

Reconciling our results we can observe that the $\dot{q} = 0$ curve is a continuous function with piecewise continuous first derivatives. The curve will be horizontal in (2,2), positively sloped in (3,2) and negatively sloped in (2,1), (3,1), (1,2), (1,1). To the left of the curve $\dot{q} < 0$ and to the right $\dot{q} > 0$ in (3,2), (2,1), (3,1), (1,2), (1,1). In (2,2) if $k_1 > k_2$ then $\dot{q} > 0$ above and $\dot{q} < 0$ below the horizontal $\dot{q} = 0$; if $k_2 > k_1$ then $\dot{q} < 0$ above and $\dot{q} > 0$ below. The $\dot{k} = 0$ curve is a continuous curve with piecewise continuous first derivatives. The curve is positively sloped in (1,2), in (2,2) and (3,2) the shape is unknown. Nevertheless when the $\dot{k} = 0$ curve intersects the $\dot{q} = 0$ in (2,2), the $\dot{k} = 0$ curve has the same slope as the value of $k_1 - k_2$. In (3,2) when the $\dot{k} = 0$ curve intersects the $\dot{q} = 0$ curve the $\dot{k} = 0$ curve is negatively sloped.

Finally the curve is the q - axis for $q \leq q_o$ in patterns (2,1), (3,1) and (1,1). Consequently steady states may only occur in (1,2), (2,2) and (3,2) because from the foreign offer function $0 < q < \infty$. Thus $\lim_{k \to 0} \dot{q} = -\infty$, $\lim_{k \to \infty} \dot{q} = (\lambda + \delta)q > 0$ and so $\dot{q} = \dot{k} = 0$ will never exist whenever the economy does not have any investment expenditure.

4.3. Patterns (4,2), (4,1) and the Steady State

In these degenerate cases $\ell_1 = \ell_2 = k = 0$ and therefore the problem facing the economy is,

$$\max_{\{z_2,\ z_L\}} W(z_2,z_L) = \int_0^\infty e^{-\delta t}(g_1(z_2) + g_2(z_L) + \phi(z_L)z_L)dt \qquad (159)$$

$$\text{subject to, } \dot{k} = \begin{cases} z_2 > 0 \text{ in } (4,2), \\ z_2 = 0 \text{ in } (4,1), \end{cases} \qquad 0 < k_0 < \infty.$$

Hence the Hamiltonian is defined as,

$$H(z_2,z_L;\ q) = g_1(z_2) + g_2(z_L) + \phi(z_L)z_L + q(z_2). \qquad (160)$$

Consequently (all derivatives are evaluated at the solution),

$$\frac{\partial H}{\partial z_2} = g_1'(z_2) + q = 0,\ \frac{\partial H}{\partial z_L} = g_2'(z_L) + \phi'(z_L)z_L + \phi(z_L) = 0, \qquad (161)$$

$$0 < k_0 < \infty,\quad \dot{k} = \begin{cases} z_2 > 0 \text{ in } (4,2), \\ z_2 = 0 \text{ in } (4,1), \end{cases} \dot{q} = \delta q,\ \lim_{t\to\infty} e^{-\delta t} q(t) \geq 0,\ \lim_{t\to\infty} e^{-\delta t}$$

$$q(t)k(t) = 0.$$

H is also strictly concave in z_2 and z_L. Now in pattern (4,2), $z_2 > 0$ so $\dot{k} = 0$ is not defined and $\dot{q} = 0$ if and only if $q = 0$. In (4,1) $\dot{k}$ is always equal to zero and $\dot{q} = 0$ if and only if $q = 0$. But since $-g_1'(z_2)$ is always positive and finite then by (161) $\infty > q > 0$ so $\dot{q} = 0$ is not defined for (4,2) and (4,1). Summarizing, we find that $\dot{k} = 0 = \dot{q}$ is not defined in (4,2), while in (4,1) $\dot{k} = 0$ but not $\dot{q} = 0$ is defined. Manifestly no steady states may appear in (4,2) and (4,1). Our story is not yet complete, however, because we have only dealt with $U' = 1$. As stated earlier in the essay we must convert our model to the original social welfare functional. To effectuate this define from equation (16) the Hamiltonian,

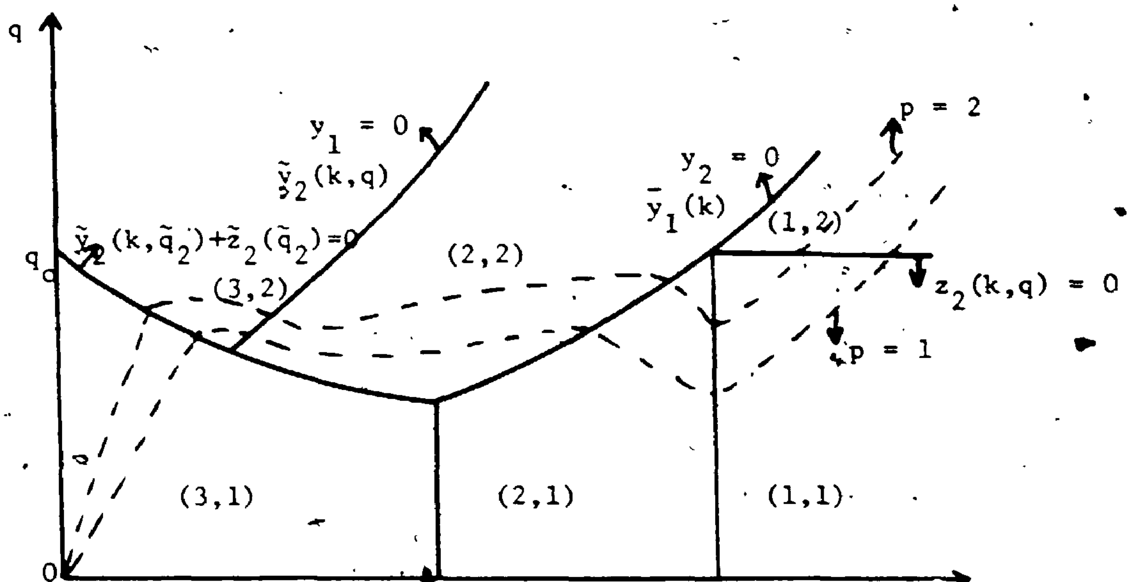

Figure 10. dp = 0 curve in (k,q) space when $k_1 > k_2$

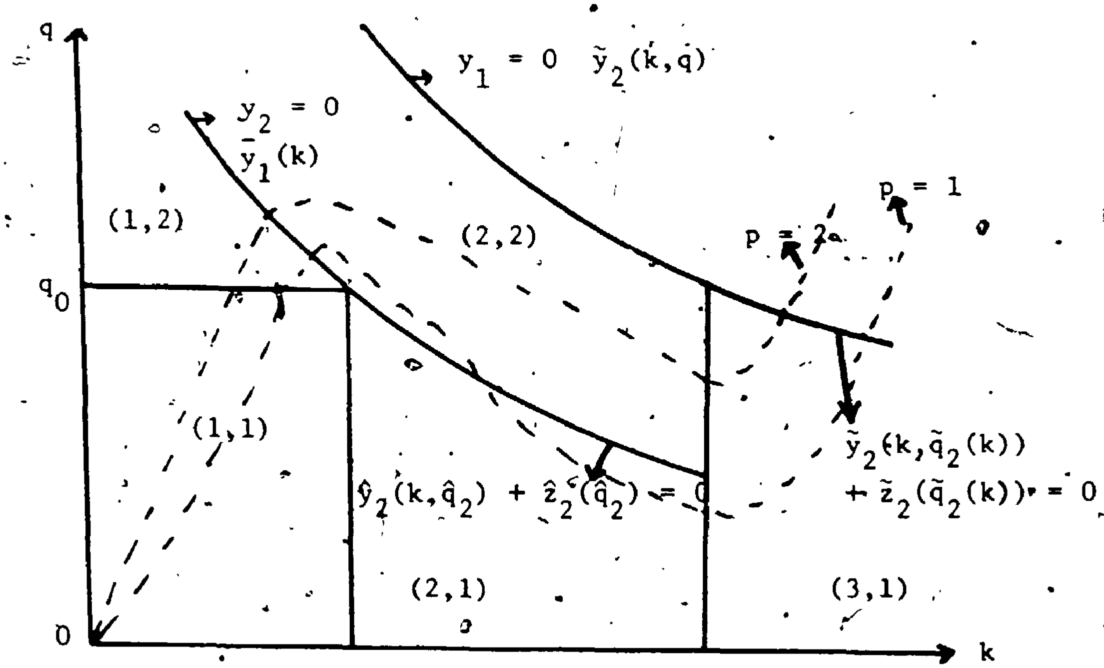

Figure 11. dp = 0 curve in (k,q) space when $k_2 > k_1$

function $\frac{\partial \bar{z}_1}{\partial q} + q \frac{d\bar{z}_2}{dq} = 0$, so (163) holds. Now equation (163) implies that for the Hamiltonian evaluated at the optimum,

$$\frac{\partial H(x_1(k,q), z_L(k,q), x_2(k,q); p,k)}{\partial q} = U' [\frac{\partial x_1}{\partial q} + (\phi' z_L + \phi)\frac{\partial z_L}{\partial q}]$$

$$+ p \frac{\partial x_2}{\partial q} = 0. \tag{167}$$

But (167) holds only when $qU'(x_1(k,q), z_L(k,q)) = p$. Hence our necessary conditions in each of our patterns are modified when $U' > 0$, $U'' \neq 0$ and p is the demand price of investment to include,

$$qU'(x_1(k,q), z_L(k,q)) = p$$

$$\dot{k} = x_2(k,q) - \lambda k$$

$$\dot{p} = (\lambda+\delta)p - U'[x_1(k,q), z_L(k,q)] [\frac{\partial x_1(k,q)}{\partial k} + (\phi'(z_L(k,q)) z_L(k,q) \tag{168}$$

$$+ \phi(z_L(k,q)))\frac{\partial z_L(k,q)}{\partial k}] - p \frac{\partial x_2(k,q)}{\partial k} .$$

From (167), (168) becomes

$$\dot{p} = U'(x_1(k,q), z_L(k,q))[(\lambda+\delta)q - \frac{\partial x_1(k,q)}{\partial k}$$

$$- \frac{\partial z_L(k,q)}{\partial k} [\phi'(z_L(k,q)) z_L(k,q) + \phi(z_L(k,q))] - q\frac{\partial x_2(k,q)}{\partial k} . \tag{169}$$

Therefore

$$\dot{p} = U'(x_1(k,q), z_L(k,q))\dot{q}. \tag{170}$$

Clearly $\dot{p} = 0$ if and only if $\dot{q} = 0$ since $U' > 0$ and because $\dot{k}$ is not amended, $\dot{k} = 0 = \dot{p}$ if and only if $\dot{k} = 0 = \dot{q}$. In other words not only is the $\dot{k} = 0$ locus the same but the $\dot{p} = 0$ locus is the same as the

$\dot{q} = 0$ locus for all patterns of specialization. Furthermore, we would find it convenient to represent the $\dot{p} = 0 = \dot{k}$ curves in (k,q) space and for this we need the function $\frac{dq}{dk}$ for every given value of p. So from (168)

$$\left.\frac{dq}{dk}\right|_{dp\,=\,0} = - \frac{qU'' \left[\frac{\partial y_1}{\partial k} + \frac{\partial z_1}{\partial k} + (\phi' z_L + \phi)\frac{\partial z_L}{\partial k}\right]}{U' + qU'' \left(\frac{\partial y_1}{\partial q} + \frac{\partial z_1}{\partial q} + (\phi' z_L + \phi)\frac{\partial z_L}{\partial q}\right)} . \tag{171}$$

To determine the sign of (171), recall (163),

$$\frac{\partial y_1}{\partial q} + \frac{\partial z_1}{\partial q} + (\phi' z_L + \phi)\frac{\partial z_L}{\partial q} = - q\left(\frac{\partial y_2}{\partial q} + \frac{\partial y_2}{\partial q}\right). \tag{172}$$

Evaluating the right side of (172) in all patterns we find that; in (2,2) from (51) and (64), (172) is negative, in (2,1) since we have $\hat{q}_2(k)$, (172) is zero, in (3,2) from (51) and (85), (172) is negative, in (3,1) we have $\tilde{q}_2(k)$, so (172) is zero, in (1,2) and (1,1) from (51), (172) is negative. Therefore the denominator in equation (171) is always positive. For the numerator, we must once more check all possible patterns. In (2,2) we only have $\frac{\partial \hat{y}_1}{\partial k} \gtrless 0$ if and only if $k_1 \gtrless k_2$, so $\left.\frac{dq}{dk}\right|_{dp=0} \gtrless 0$ if and only if $k_1 \gtrless k_2$. In (2,1) from (117) and (126), the numerator is $f_1'(k_1(\hat{q}_2(k))) > 0$, so $\left.\frac{dq}{dk}\right|_{dp=0} > 0$. In (3,2) from from (128) and (130) the numerator is $- w \frac{\partial \tilde{z}_L}{\partial k} < 0$, so $\left.\frac{dq}{dk}\right|_{dp=0} < 0$. In (3,1) from (133) and (137) the numerator is $\tilde{q}_2(k) f_2'(k_2(\tilde{q}_2(k))) > 0$, so $\left.\frac{dq}{dk}\right|_{dp=0} > 0$. In (1,2) and (1,1) from (141) and (142) the numerator is $f_1'(k_1) > 0$, so $\left.\frac{dq}{dk}\right|_{dp=0} > 0$. Therefore we now have the shape of the function in (k,q) space for a constant value of p. Moreover for $dk = 0$

$$\left.\frac{dq}{dp}\right|_{dk=0} = \frac{1}{\hat{U}' + qU''\left(\frac{\partial y_1}{\partial q} + \frac{\partial z_1}{\partial q} + (\phi' z_L + \phi)\frac{\partial z_L}{\partial q}\right)} > 0 \tag{173}$$

Clearly we have derived diagrams 10 and 11.

Finally, the conclusions pertaining to the existence of steady states in patterns (2,2), (3,2) or (1,2), are retained whether the social welfare function is linear or more generally just concave. It is also true that whether the intertemporal equilibrium exists in (2,2), (3,2) or (1,2) it is unique. Hence there is a unique intersection of the $\dot{k} = 0 = \dot{p}$ curves in (k,q) space. Let us denote the intersection by (k^*,p^*). This steady state is a saddle point[8] so there are two trajectories that converge to (k^*,p^*), one from $p \geq p^*$, $k \leq k^*$ and one from $p < p^*$, $k > k^*$. These two trajectories form a differentiable function $p(t) = p^*(k(t))$ for $t \in [0,\infty]$. In addition since $qU' = p$ must be satisfied along $p^*(k)$, $\infty > q > 0$ and $\infty > U' > 0$ then $\infty > p > 0$ for all values of k. In addition in patterns (2,2), (3,2), (1,2), since $\ell_1(k_1-k_2) + (1+z_L)k_2 = k$, if $k = 0$ then $k_1 = k_2 = 0$. By the assumptions on the production functions this implies that

$$\lim_{k\to 0} \dot{p} = (\lambda+\delta)\, p - \infty = -\infty \,. \tag{174}$$

On the other hand when $k = \infty$ in these patterns then $k_1 = k_2 = \infty$ and again from the assumptions on the production functions

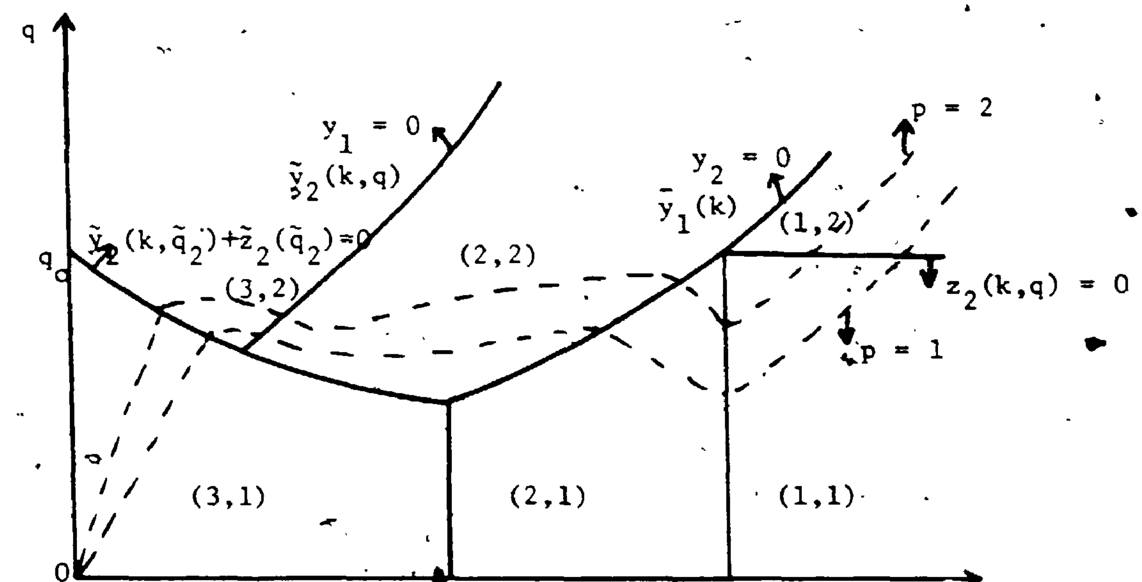

Figure 10. dp = 0 curve in (k,q) space when $k_1 > k_2$

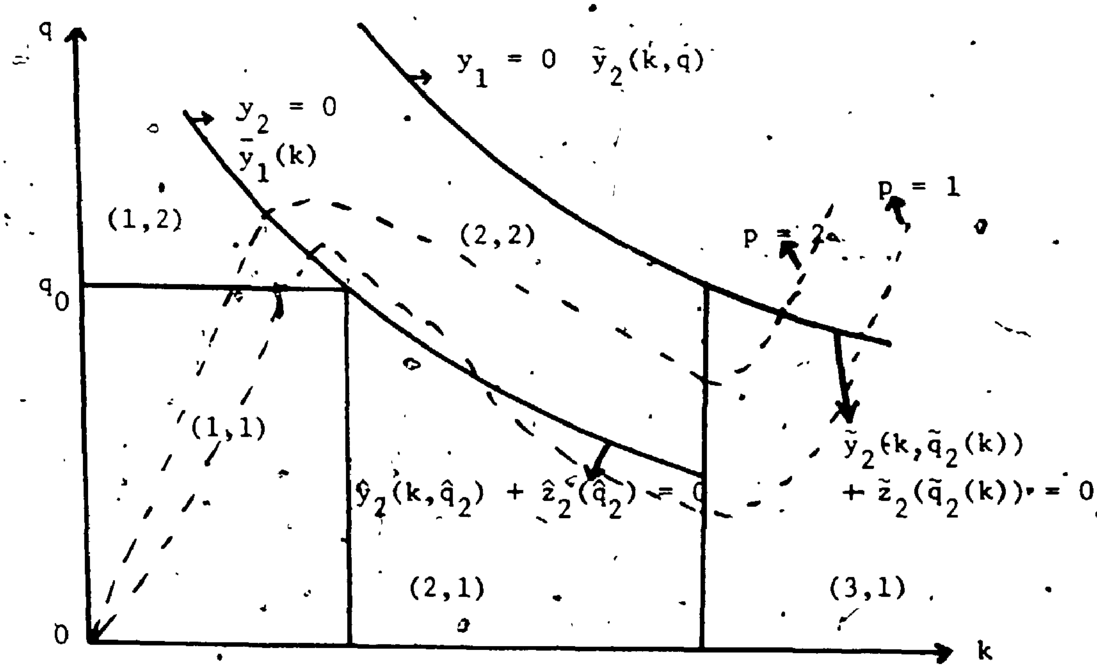

Figure 11. dp = 0 curve in (k,q) space when $k_2 > k_1$

$$\lim_{k\to\infty} \dot{p} = (\lambda+\delta)\, p > 0. \qquad (175)$$

Manifestly $\infty > p^* > 0$, $\infty > k^* > 0$. Representing the different equilibria geometrically in figures 12-17, it is important to perceive that in (k,q) space p^* is found at the point of intersection of the $\dot{p} = 0 = \dot{k}$ loci and a curve depicting a constant value of p (see figures 10 and 11).

In concluding this section we are able to deduce certain characteristics of the motion of the economy through time defined by,

$$t \in [0, \infty],\ 0 < k_o = k(o) < \infty,\ 0 < p(o) = p^*(k_o) < \infty, \qquad (176)$$

$$\dot{k}(t) = x_2(k(t),q(t)) - \lambda k(t),$$

$$p(t) = p^*(k(t)),\ k(\infty) = k^*,\ p(\infty) = p^*.$$

If $k_1 > k_2$ and the steady state occurs in (2,2) then for $k_o < k^*$ the domestic country specializes in the production of the investment product, expenditure consists of consumption and investment with imports of consumption, i.e., the economy moves from (3,2) to (2,2). The path that the country follows is positively sloped which is in contrast to the usual one and two sector models where the optimal trajectory is negatively sloped. The reason for the difference originates with the fact that labour is perfectly mobile between nations. In pattern (3,2) with $k_o < k^*$ the country endeavors to raise its capital-labour endowment ratio in order to reach steady state. However in (3,2) the domestic country is unable to decrease the production of the consumption product and allocate factors to the investment product because

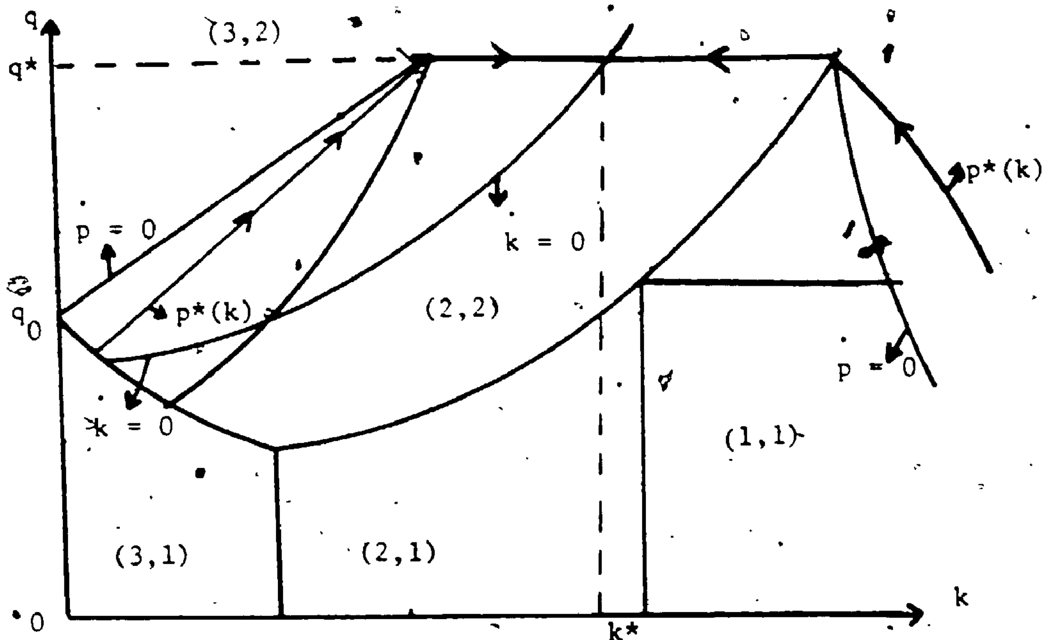

Figure 12. Optimal path when steady state occurs in (2,2) and $k_1 > k_2$

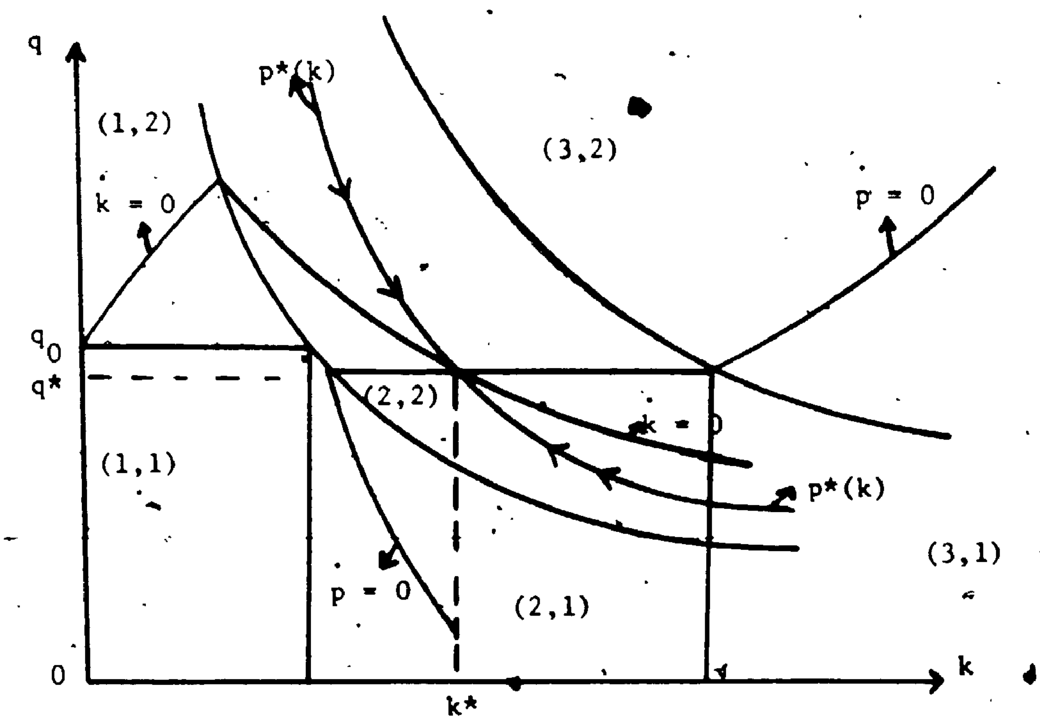

Figure 13. Optimal path when steady state occurs in (2,2) and $k_2 > k_1$

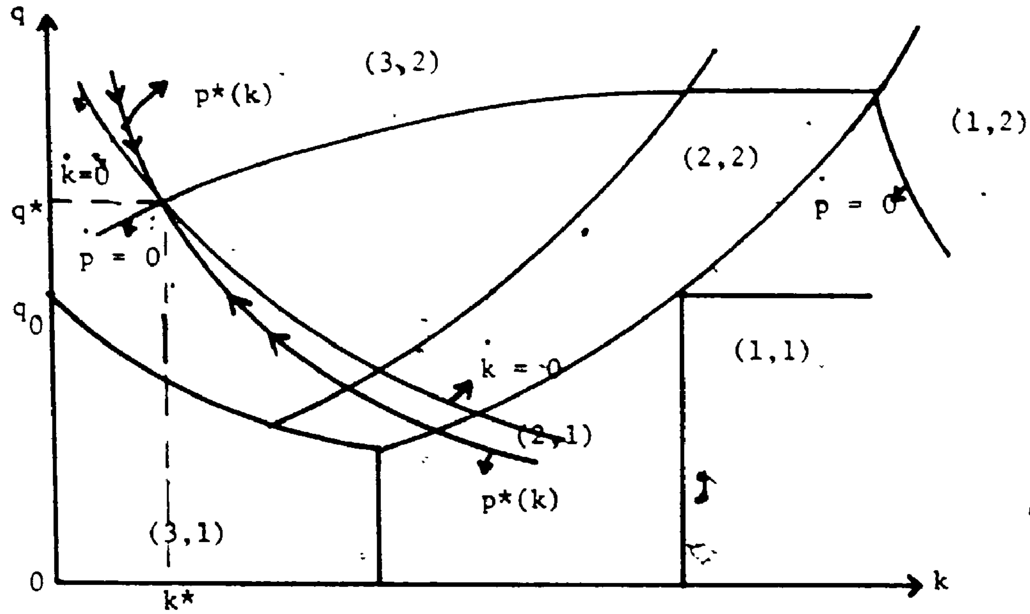

Figure 14. Optimal path when steady state occurs in (3,2) and $k_1 > k_2$

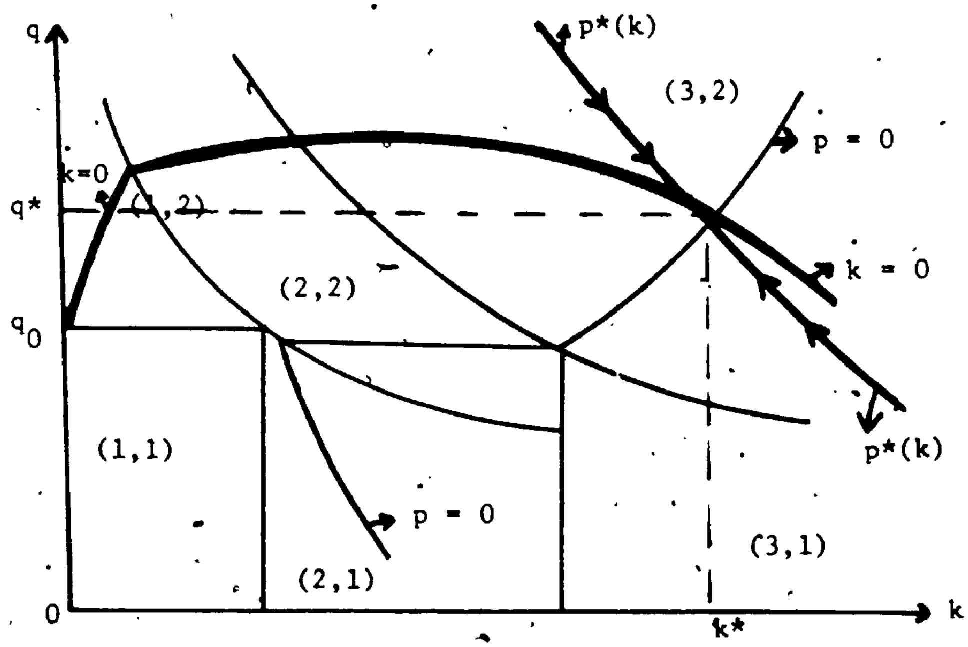

Figure 15. Optimal path when steady state occurs in (3,2) and $k_2 > k_1$

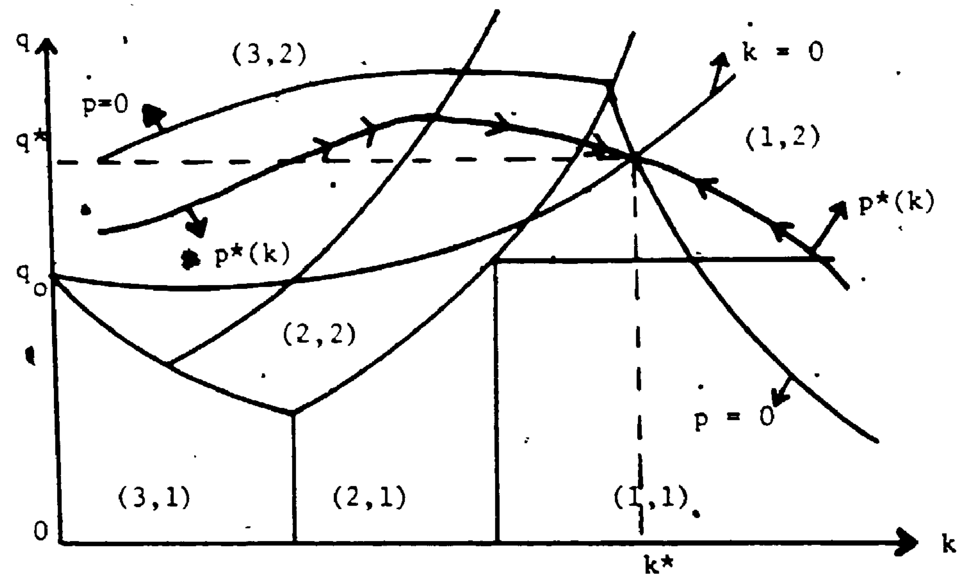

Figure 16. Optimal path when steady state occurs in (1,2) and $k_1 > k_2$

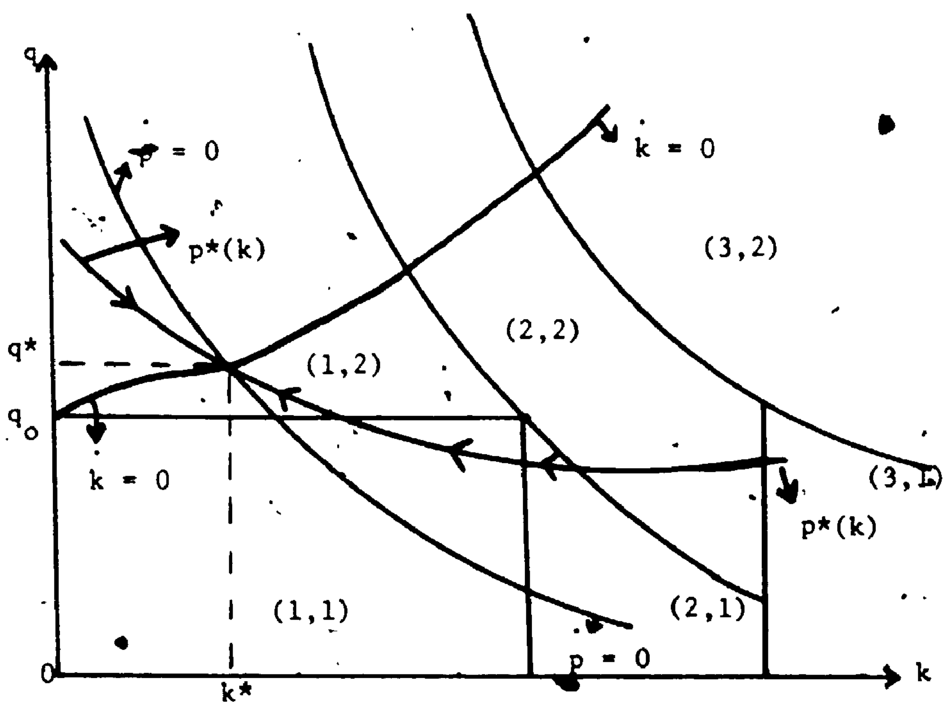

Figure 17. Optimal path when steady state occurs in (1,2) and $k_2 > k_1$

it is already specialized in y_2. The economy not only raises the value of k but it does so optimally, which entails maximizing social welfare evaluated in terms of the consumption of nationals. Consequently in our model there is a trade-off available to the nation which is not feasible in closed or open economies with perfect factor immobility. The trade-off entails producing less of y_2 but still none of y_1 but exporting more labour or importing less labour by lowering the demand price of investment. This leads to increased consumption for the nationals and therefore higher social welfare. Translating these conclusions into geometric terms it implies that the economy selects the minimum feasible value of q (and thus p by equation (173)) while increasing the capital-labour endowment ratio. In our model this behaviour is consistent with an upward sloping trajectory in (3,2). To prove that increases in q always lowers social welfare in (3,2), recall equation (166). We get that for any value of $k = \tilde{k}$ in (3,2)

$$dU\Big|_{k=\tilde{k}} = U'\left(\frac{\partial x_1}{\partial q} + (\phi' z_L + \phi)\frac{\partial z_L}{\partial q}\, dq\right|_{k=\tilde{k}} = -U' q \frac{\partial x_2}{\partial q}\, dq\Big|_{k=\tilde{k}} < 0.$$

Next define the time interval $\tau \in [\tau_1, \tau_2]$ that the economy is in (3,2). Thus in discrete terms

$$dW = \sum_{t=\tau_1}^{\tau_2} -U' q \frac{\partial x_2}{\partial q}\, dq.\Big|_{k(t) = \tilde{k}(t)} < 0$$

is the loss in welfare for not selecting the minimum q along a trajectory in (3,2). If $k_o > k^*$ the country may start from either (2,1) or (1,1). If the economy begins from (2,1), i.e., diversified in production consumes and exports investment, then it proceeds directly to (2,2). If the economy begins in (1,1), $y_2 = 0 = z_2 = x_2$, then it may pass through (2,1) to (2,2) or (1,2) to (2,2). In addition because $\dot{p} > 0$ above the horizontal line $\dot{p} = 0$ and $\dot{p} < 0$ below the line for $k_1 > k_2$ the optimal trajectory must be the $\dot{p} = 0$ locus while the home country is in pattern (2,2). On the other hand if $k_2 > k_1$ and the steady state appears in (2,2) then for $k_o < k^*$ the movement is from (1,2) to (2,2) where (1,2) is defined by specialization in the production of the consumption product, diversification in expenditure and importation of investment. If $k_o > k^*$ then the economy may start from (2,1) and proceed to (2,2) or from (3,1) to (2,1) to (2,2) or from (3,1) to (2,2) directly or finally from (3,1) to (3,2) to (2,2).

Now when the intertemporal equilibrium exists in (3,2), for $k_1 > k_2$ and $k_o < k^*$ the domestic country will always remain specialized in the production of the investment product, diversified in expenditure and importing consumption. However when $k_o > k^*$ the nation may initially be in (2,1), (3,1) or (1,1) and thus a myriad of possibilities exist. The only qualification is that the economy never penetrates pattern (1,2). This is because in (1,2) the $\dot{k} = 0$ curve has a positive slope, the $\dot{p} = 0$ curve has a negative slope and with $k_1 > k_2$ the $\dot{k} = 0$ locus intersects the $\dot{p} = 0$ locus from below. Hence for the steady state to be in (3,2) and since it is unique the $\dot{k} = 0$ curve must not enter (1,2). Furthermore the optimal trajectory for $k_o > k^*$ lies below $\dot{k} = 0$ locus. Therefore the path avoids (1,2) (see figure 14). Next when $k_2 > k_1$ and $k_o < k^*$ then the country traverses (1,2) to (2,2) to (3,2). It always remains diversified in expenditure but initially specializes in the production of the consumption product and imports investment. If $k_o > k^*$ the economy moves from (3,1), i.e., specialized in the production of investment, exporting it and importing consumption which is the only expenditure, to (3,2).

The last pattern that the steady state may appear in is (1,2). If $k_1 > k_2$ and $k_o < k^*$ the nation passes from (3,2) to (2,2) to (1,2). Moreover while the economy is in pattern (2,2) the optimal values of p and k must be such that the trajectory is on or below the $\dot{p} = 0$ locus; with $k_o > k^*$ the economy moves from (1,1) to (1,2). Initially the economy specializes in expenditure and the production of consumption. Finally when $k_2 > k_1$ the country remains forever in (1,2) if $k_o < k^*$, if $k_o > k^*$ many cases are feasible since the initial point may be in (1,1), (2,1) or (3,1).

5. Reversible Investment

Previously we have constrained investment expenditure to be non-negative. Suppose that we now permit $x_2 \gtreqless 0$, i.e., disinvestment is feasible. This means that even if the rate of growth of the labour endowment is zero and there is no depreciation on the capital stock, the country could still decumulate capital. The home country transforms this negative investment expenditure into increased exports of investment. Hence, we have a new expenditure pattern to analyze, $x_1 > 0$, $x_2 < 0$ which is represented by Table 10.

Manifestly, in each of the patterns with $x_2 < 0$ the functions of the appropriate variables are identical to the cases where $x_2 > 0$. In particular $z_2 = \hat{z}_2(q)$ in patterns (2,2) and (2,3) because both are derived from the control problem defined by equation (23). The reasons for the differences in expenditure pattern $x_1 > 0$, $x_2 = 0$ is that we are able to define implicit functions $y_2 + z_2 = 0$ in the variables k and q which we cannot do in expenditure patterns $x_1 > 0$, $x_2 > 0$ and $x_1 > 0$, $x_2 < 0$.

Geometrically in (k,q) phase space the patterns $x_1 > 0$, $x_2 < 0$ replace $x_1 > 0$, $x_2 = 0$. Furthermore, expenditure pattern (1) shrinks to the boundary curves between $x_1 > 0$, $x_2 > 0$ and $x_1 > 0$, $x_2 < 0$. Therefore, we have the following diagrams.

Proceeding, the shape of the $\dot{q} = 0$ locus is the same in expenditure patterns 2 and 3, since the functions of the variables are alike. In addition, when $x_1 > 0$, $x_2 < 0$ then $\dot{k} = 0$ if and only if $k = 0$; in (1,3), $\dot{k} = z_2 - \lambda k < 0$ for $k > 0$ since $z_2 < 0$; in (2,3) and (3,3), $\dot{k} = y_2 + z_2 - \lambda k < 0$ for $k > 0$ since $z_2 + y_2 < 0$; in

Table 10. $x_2 < 0$ Patterns

Expenditure	Production (1)	(2)	(3)	(4)
	$y_1 > 0, y_2 = 0$	$y_1 > 0, y_2 > 0$	$y_1 = 0, y_2 > 0$	$y_1 = 0$ [illegible] y_2
$x_1 > 0, x_2 < 0$ (3)	$-1 < z_L \leq 1$	$-1 < z_L \leq 1$	$-1 < z_L < 1$	$z_L = -1$
	$z_1 > -y_1, z_2 < 0$	$z_1 > -y_1, z_2 < -y_2$	$z_1 > 0, z_2 < -y_2$	$z_1 > 0, z_2 < 0$
	$x_1 > 0, x_2 = z_2$	$x_1 > 0, x_2 < 0$	$x_1 = z_1, x_2 \leq 0$	$x_1 = z_1, x_2 = z_2$

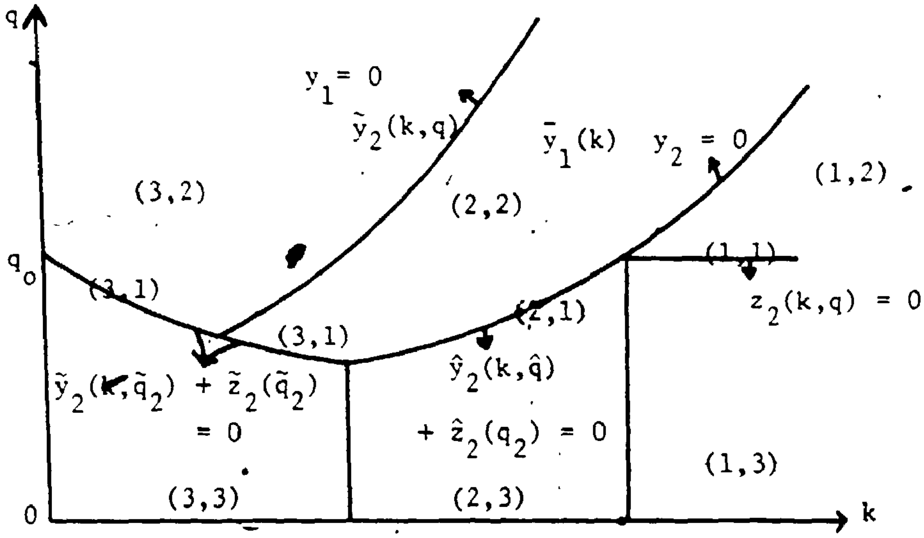

Figure 18. All patterns when $k_1 > k_2$

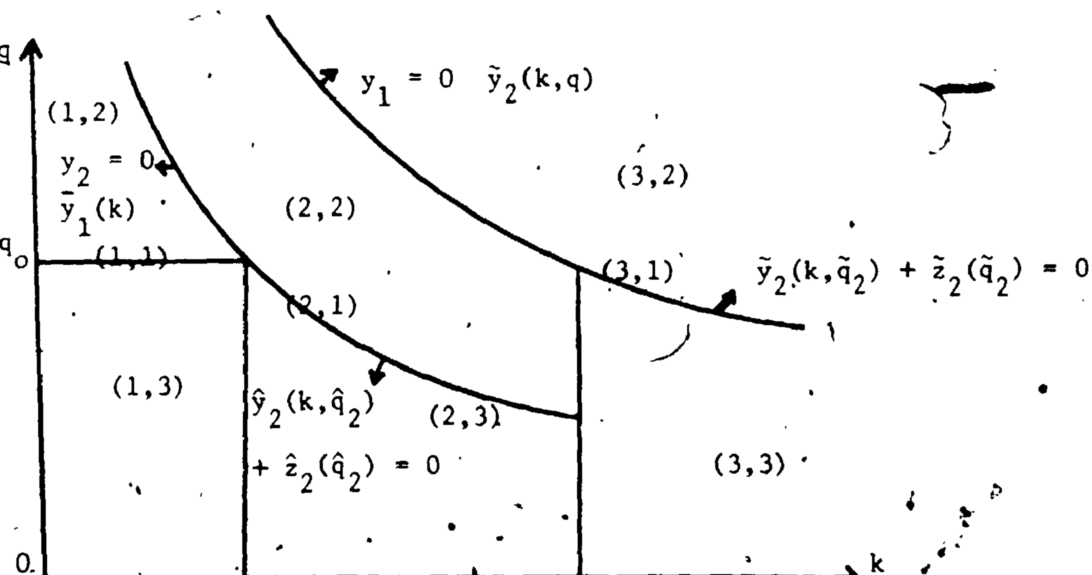

Figure 19. All patterns when $k_2 > k_1$

(4,3), $\dot{k} = z_2 < 0$. Therefore, intertemporal equilibria do not exist in expenditure pattern $\bar{3}$. Consequently, in depicting the steady state and optimal path for $k_1 > k_2$ figures 12, 14 and 16 are still applicable. However, when $k_2 > k_1$, figures 13, 15 and 17 must be altered to figures 20, 21 and 22 respectively.

Finally in viewing the optimal path we are able to discern that for $k_o \leq k^*$ the description of the path is the same as the irreversible investment model. On the other hand, for $k_o > k^*$ the economy will always traverse one of the patterns with disinvestment.[10]

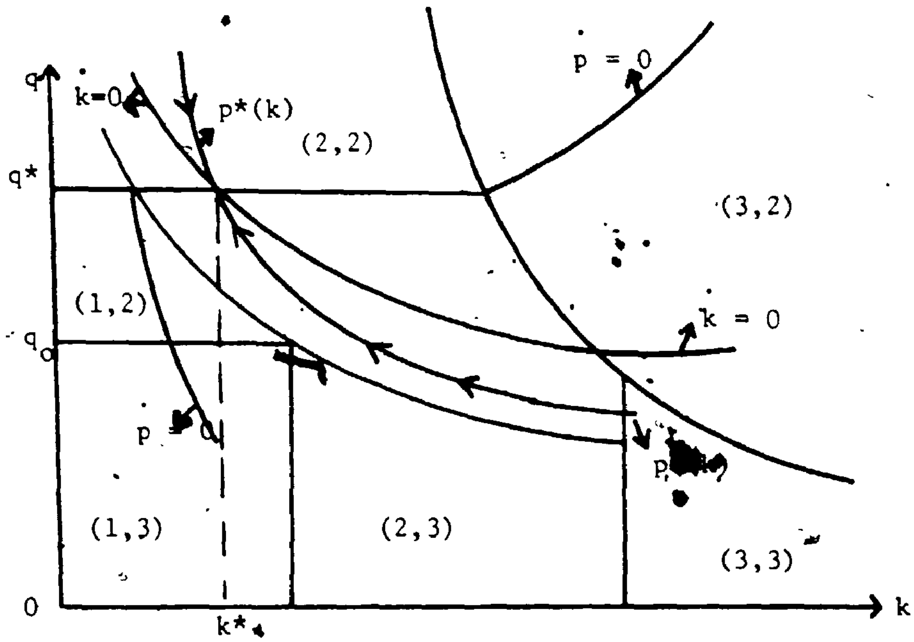

Figure 20. Optimal path when steady state occurs in (2,2) and $k_2 > k_1$

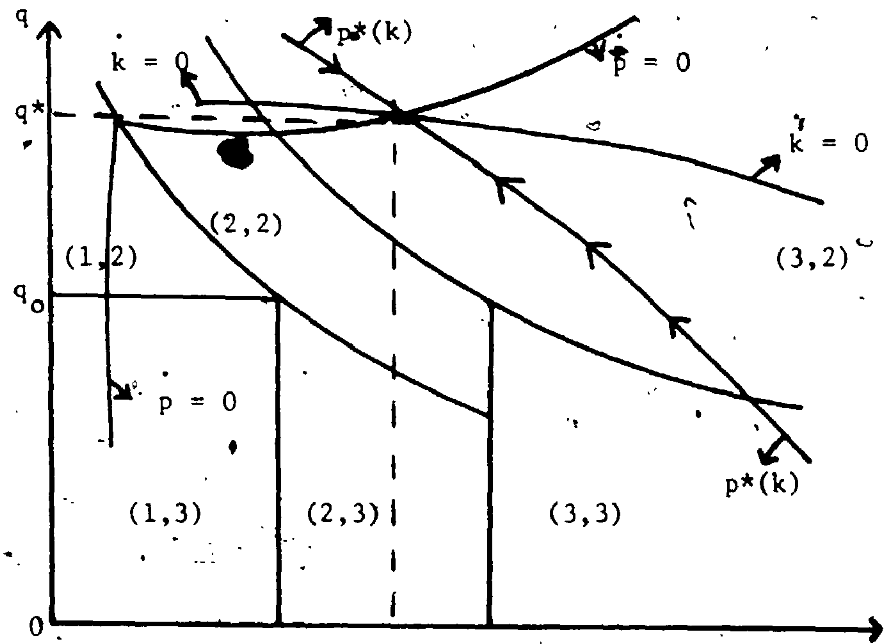

Figure 21. Optimal path when steady state occurs in (3,2) and $k_2 > k_1$

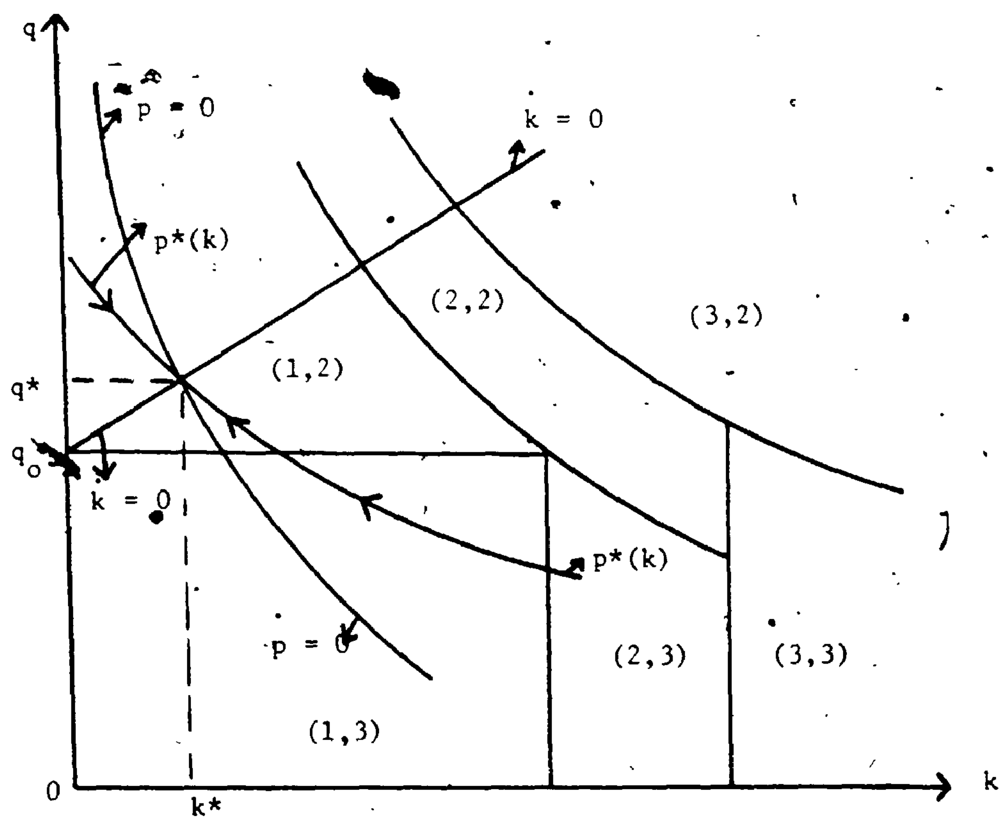

Figure 22. Optimal path when steady state occurs in (1,2) and $k_2 > k_1$

6. Harrod Neutral Technological Change

Our model, as most other optimal growth models, readilv handles Harrod neutral technological change. Adapting the model to allow for this phenomenon, redefine the production functions for the consumption and investment products as,

$$Y_i = F_i(K_i, e^{\alpha t} L_i) \qquad i = 1,2 \tag{177}$$

where $0 < \alpha < \infty$ is the rate of Harrod neutral technological change and F_i, $i = 1,2$, has the properties specified for equation (1). We assume that the rate of technological change is the same across products and countries. This implies, in particular, that the domain of the foreign offer function is growing exponentially at the rate $\alpha + n$. Whereas in sections 2 - 5 it was growing at the rate n. Hence, we are still in the medium size country framework and will not approach the case where one nation becomes small and the other large.

Next define the variables in labour efficiency units,

$$k_i = \frac{K_i}{e^{\alpha t}L_i}, \quad k = \frac{K}{e^{\alpha t}L},$$

$$y_i = \frac{Y_i}{e^{\alpha t}L}, \quad x_i = \frac{X_i}{e^{\alpha t}L},$$

$$z_L = \frac{Z_L}{e^{\alpha t}L}, \quad z_i = \frac{Z_i}{e^{\alpha t}L}, \qquad i = 1,2$$

Moreover, from the rate of change of the capital endowment equation, (4) and (177) we get, 2 OF/DE 2

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$$\dot{k} = y_2 + z_2 - (\lambda + \alpha)k \ . \tag{178}$$

Recall that the evaluation of welfare is in terms of variables in labour endowment not efficient labour endowment form. Therefore, the welfare functional becomes.

$$W(e^{\alpha t} x_1,\ e^{\alpha t} z_L) = \int_0^\infty e^{-\delta t} U(e^{\alpha t}(x_1 + \phi(z_L)z_L))\ dt. \tag{179}$$

Consequently, the saddle point problem confronting the domestic economy is

$$I(X, Z, V) = \int_0^\infty e^{-\delta t} L(\cdot)\ dt \tag{180}$$

$$\min \{V\},\ \max \{X,Z\}$$

where

$$L(y_1,y_2,k_1,k_2,k,\ell_1,\ell_2,z_2,z_L,\lambda_1,\lambda_2,p,w,r) \tag{181}$$

$$= U(e^{\alpha t}(y_1 + g_1(z_2) + g_2(z_L) + \phi(z_L)z_L)) +$$

$$\sum_{i=1}^{2} \lambda_i(\ell_i f_i(k_i) - y_i) + w(1 + e^{\alpha t}z_L - \ell_1 - \ell_2) +$$

$$r(k - k_1\ell_1 - k_2\ell_2) + p(y_2 + z_2 - (\lambda + \alpha)k - \dot{k}) \ .$$

The first order and transversality conditions are,

$$\frac{\partial L}{\partial y_1} = U'e^{\alpha t} - \lambda_1 \leq 0,\quad y_1(U'e^{\alpha t} - \lambda_1) = 0,\quad y_1 \geq 0,$$

$$\frac{\partial L}{\partial y_2} = -\lambda_2 + p \leq 0,\quad y_2(-\lambda_2 + p) = 0,\ y_2 \geq 0,$$

$$\frac{\partial L}{\partial z_2} = U'e^{\alpha t}\, g_1'(z_2) + p = 0, \quad \frac{\partial L}{\partial z_L} = U'e^{\alpha t}(g_2'(z_L)$$

$$+ \phi'(z_L)z_L + \phi(z_L)) + we^{\alpha t} = 0 ,$$

$$\frac{\partial L}{\partial k_i} = \lambda_i \ell_i (f_i'(k_i) - r \leq 0, \quad k_i \lambda_i \ell_i (f_i'(k_i) - r) = 0,$$

$$k_i \geq 0, \quad i = 1,2$$

$$\frac{\partial L}{\partial \ell_i} = \lambda_i f_i(k_i) - w - rk_i \leq 0, \quad \ell_i(\lambda_i f_i(k_i) - w - rk_i) = 0 ,$$

$$\ell_i \geq 0, \quad i = 1,2$$

$$\frac{\partial L}{\partial \lambda_i} = \ell_i f_i(k_i) - y_i = 0, \quad \lambda_i \geq 0, \quad i = 1,2 \tag{182}$$

$$\frac{\partial L}{\partial w} = 1 + e^{\alpha t}z_L - \ell_1 - \ell_2 = 0, \quad w \geq 0, \quad \frac{\partial L}{\partial r} = k - k_1\ell_1$$

$$- k_2\ell_2 = 0, \quad r \geq 0,$$

$$\frac{\partial L}{\partial p} = y_2 + z_2 - (\lambda + \alpha)k - \dot{k} = 0, \; p \geq 0, \; \dot{p} = (\delta + \lambda + \alpha)p - r ,$$

$$\lim_{t \to \infty} e^{-\delta t}\, p(t) \geq 0, \quad \lim_{t \to \infty} e^{-\delta t}\, p(t)k(t) = 0, \quad 0 < k_o < \infty .$$

The crucial observation is that all prices except the wage rate, w, are evaluated in labour endowment terms not efficient labour endowment terms. To rectify this, define

$$p_1 = pe^{-\alpha t} , \quad r_1 = re^{-\alpha t} \tag{183}$$

Therefore,

$$\dot{p}_1 = \dot{p}e^{-\alpha t} - \alpha e^{-\alpha t}p$$
$$\dot{r}_1 = \dot{r}e^{-\alpha t} - \alpha e^{-\alpha t}r \; . \tag{184}$$

Dividing by p_1 and r_1 respectively and substituting $\dot{p} = (\delta + \lambda + \alpha)p - r$ yields

$$\dot{p}_1 = (\lambda + \delta)p_1 - r_1 \; . \tag{185}$$

All our results derived in sections 3 - 5 are basically applicable to section 6, except that we must redefine the variables to be in efficient labour endowment and intensive forms. The only difference is that there is an extra positive parameter in the $\dot{k}$ equation which does not alter the qualitative conclusions.

7. Learning by Doing

Just as labour-augmenting technological change creates few analytical difficulties but requires reinterpretation, so appropriate modifications of the manner labour enters the production functions only introduce some interpretative complications. Proceeding to our first example of learning by doing we assume that output is generated by the following production functions,

$$Y_i = F_i(K_i, L^{\beta t}L_i) \qquad i = 1,2 \tag{186}$$

where $0 < \beta < \infty$ and F_i, $i = 1,2$ has the properties specified in equation (1). The term L^β reflects the economic fact that as the labour endowment increases it produces "knowledge" or "experience" as a by-product. This stock of knowledge increases the efficiency of each worker in the fashion of a Samuelson collective commodity. This implies that the efficiency of any number of workers can be improved equally, which is reflected by L^β not depending on the number of workers and β being the same for the consumption and investment products. In the presence of labour-augmenting technological change (186) becomes

$$Y_i = F_i(K_i, L^{\beta t}e^{\alpha t}L_i) \qquad i = 1,2 \tag{187}$$

Recalling that $L = L_o e^{nt}$, (187) can be rewritten as,

$$Y_i = F_i, e^{(\alpha + n\beta)t}L_o L_i) \qquad i = 1,2 \tag{188}$$

Once more to retain the nature of our model we assume β is the same across countries. Therefore by substituting for α in section 6 the expression $(\alpha + n\beta)$ our first model of learning by doing and Harrod neutral technological change is complete.

Our second example is a generalization of an Arrow-Kurz [1] formulation,

$$Y_i = F_i(K_i,\ L_i^{\gamma}) \qquad i = 1,2 \tag{189}$$

where $0 < \gamma < \infty$ and F_i has the usual properties, in particular F_i is homogeneous of degree one in K_i and L_i^{γ}. Hence, as $\gamma \gtreqless 1$, F_i, $i = 1, 2$ exhibits increasing, constant or decreasing returns to scale in K_i and L_i. The economic interpretation of (189) is straightforward if we think of L_i, $i = 1,2$ as the labour input but L_i^{γ} as the "experienced" labour input. The domestic country does not only import or export labour but experienced labour. Again, for the aforementioned reasons γ is assumed to be the same in both nations. Combining (189) with labour-augmented technological change we get,

$$Y_i = F_i(K_i,\ e^{\alpha t}L_i^{\gamma}) \qquad i = 1,2 \tag{190}$$

Next define the variables in experienced labour efficiency units,

$$k_i = \frac{K_i}{e^{\alpha t}L_i^{\gamma}},\ k = \frac{K}{e^{\alpha t}L^{\gamma}},$$

$$y_i = \frac{Y_i}{e^{\alpha t}L^{\gamma}},\ x_i = \frac{X_i}{e^{\alpha t}L^{\gamma}},$$

$$z_L = \frac{Z_L}{e^{\alpha t}L^{\gamma}},\ z_i = \frac{Z_i}{e^{\alpha t}L^{\gamma}}, \qquad i = 1,2$$

Furthermore, from equations (4) and (190) we have,

$$\dot{k} = y_2 + z_2 - (\mu + \gamma n + \alpha)k . \tag{191}$$

Again noting that welfare is evaluated in labour endowment terms[11] and that $L^{\gamma-1} = e^{n(\gamma - 1)t}$, the social welfare function is

$$W(e^{(\alpha + n(\gamma - 1))t} x_1, e^{(\alpha + n(\gamma - 1))t} z_L) = \int_0^\infty e^{-\delta t} U(e^{(\alpha + n(\gamma - 1))t}(x_1 + \phi(z_L)z_L)) \, dt . \tag{192}$$

Thus, the optimizing program becomes,

$$I(X, Z, V) = \int_0^\infty e^{-\delta t} \hat{L}(\cdot) \, dt \qquad \min \{V\}, \max \{X,Z\} \tag{193}$$

where

$$\begin{aligned} \hat{L}(y_1,y_2,k_1,k_2,\ell_1,\ell_2,k,z_2,z_L,\lambda_1,\lambda_2,p,w,r) \\ = U(e^{(\alpha + n(\gamma - 1))t}(y_1 + g_1(z_2) + g_2(z_L) + \phi(z_L)z_L)) \\ + \sum_{i=1}^{2} \lambda_i(\ell_i^{\gamma} f_i(k_i) - y_i) + w(1 + e^{(\alpha + n(\gamma - 1))t} z_L - \ell_1 - \ell_2) \\ + r(k - k_1\ell_1^{\gamma} - k_2\ell_2^{\gamma}) + p(y_2 + z_2 - (\mu + \gamma n + \alpha)k - \dot{k}) . \end{aligned} \tag{194}$$

The conditions of optimality are quite manifest; suffice it to note that

$$\dot{p} = (\delta + \mu + \gamma n + \alpha)p - r . \tag{195}$$

Define

$$\begin{aligned} p_2 &= pe^{-(\alpha + n(\gamma - 1))t} \\ r_2 &= re^{-(\alpha + n(\gamma - 1))t} . \end{aligned} \tag{196}$$

Therefore,

$$\dot{p}_2 = \dot{p}e^{-(\alpha + n(\gamma - 1))t} - p(\alpha + n(\gamma - 1))\, e^{-(\alpha + n(\gamma - 1))t}$$

$$\dot{r}_2 = \dot{r}e^{-(\alpha + n(\gamma - 1))t} - r(\alpha + n(\gamma - 1))\, e^{-(\alpha + n(\gamma - 1))t} , \tag{197}$$

consequently,

$$\dot{p}_2 = (\delta + \lambda)p_2 - r_2 . \tag{198}$$

Thus, our results of sections 3 - 5 are satisfied except to notice that the interpretation of the variables is different and the k equation is trivially modified.

8. Competitive Economy

Up to this stage, we have been analyzing the allocation mechanism of a centralized economy which maximizes a social welfare functional. However, it is altogether clear that the model may be applied to a competitive economy. If we assume that utility maximizing households all have the identical intertemporal utility functional of the form $\int_0^\infty e^{-\delta t} U(x_1)dt$ and profit maximizing firms with production functions given by equation (1) behave competitively, then in equilibrium equation system (18) is satisfied except for $\frac{\partial L}{\partial z_2} = 0 = \frac{\partial L}{\partial z_L}$. The equations are not fulfilled because in international equilibrium under competitive markets the world price ratio of the investment product in terms of the consumption product is $-\frac{z_1}{z_2}$. The optimal domestic price ratio is equal to the negative of the marginal social welfare times the slope of the foreign offer function with respect to z_2. The two prices differ due to the circumstance that the home country has some degree of monopoly power in trade which cannot be exploited by competitive behavior. We are confronted with a similar issue when encountering the labour market. In world equilibrium the wage rate in terms of the consumption product is $-\frac{z_1}{z_L}$. Yet the optimal domestic wage rate is equal to the negative of the marginal social welfare times the slope of the foreign offer function with respect to z_L adjusted for the marginal social labour externality. The divergence is effectuated because the domestic country does not utilize its monopoly power or perceive the externality effect of imported or exported workers on social welfare under a competitive labour market structure. In our model, there are essentially two distinct ways to handle the pricing problem. Firstly, government can control trade and labour movements directly while the remaining processes of the economic system function in a decentralized competitive manner. Secondly, the government may impose a tariff on the prices of the products and labour to which competitive firms engaging in trade and demanding foreign labour are subject. This interjacence in the markets by the government will align the competitive prices to the optimal prices found in equation system (18).

Appendix 1. Uniqueness of Optimal Path

Theorem Al.1. If a solution (X^o, Z^o, V^o) of the equation system (18) exists then (X^o, Z^o, V^o) is a saddle point of $I(X, Z, V)$ and (X^o, Z^o) is the unique optimal policy.

Proof. Let (X^o, Z^o, V^o) be a solution to equation system (18). Let (X^1, Z^1) be a set of feasible vectors (i.e., satisfying equations (8), (9), (10), (11), (12), (13). Then,

$$
\begin{aligned}
I(X^o, Z^o, V^o) - I(X^1, Z^1, V^o) = \int_o^\infty e^{-\delta t}[U(y_1^o + g_1(z_2^o) + g_2(z_L{}^o) - g_2(z_L{}^1) \\
+ \phi(z_L^o)z_L^o) - U(y_1^1 + g_1(z_2^1) + \phi(z_L^1)z_L^1) - \lambda_1^o(y_1^o - y_1^1) \\
+ \sum_{i=1}^{2} \lambda_i^o(\ell_i^o\, f_i(k_i^o) - \ell_i^1 f_i(k_i^1)) - (\lambda_2^o - p^o)(y_2^o - y_1^1) \\
+ w^o(z_L^o - z_L^1) - w^o(\sum_{i=1}^{2} (\ell_i^o - \ell_i^1)) - r^o(\sum_{i=1}^{2} (\ell_i^o\, k_i^o - \ell_i^1\, k_i^1)) \\
+ (r^o - p^o\, \lambda)(k^o - k^1) + p^o(z_2^o - z_2^1) - p^o(\dot{k}^o - \dot{k}^1)]dt. \quad (A1.1)
\end{aligned}
$$

Next adding and subtracting $U'(y_1^o - y_1^1)$, $U'g_1'(z_2^o - z_2^1)$, $U'(g_2' + \phi' z_L + \phi)$ $(z_L^o - z_L^1)$ yields,

$$
\begin{aligned}
I(X^o, Z^o, V^o) - I(X^1, Z^1, V^o) = \int_o^\infty e^{-\delta t}[U(y_1^o + g_1(z_2^o) + g_2(z_L{}^o) - g_2(z_L{}^1) \\
+ \phi(z_L^o)z_L^o) - U(y_1^1 + g_1(z_2^1) + \phi(z_L^1)z_L^1) - U'(y_1^o - y_1^1) \\
+ (U' - \lambda_1^o)(y_1^o - y_1^1) - (\lambda_2^o - p^o)(y_2^o - y_2^1) - U'g_1'(z_2^o - z_2^1) \\
+ (U'g_1' + p^o)(z_2^o - z_2^1) - U'(g_2' + \phi' z_L + \phi)(z_L^o - z_L^1) \\
+ (U'(g_2' + \phi' z_L + \phi) + w^o)(z_L^o - z_L^1) + \sum_{i=1}^{2} \ell_i^o(\lambda_i^o\, f_i(k_i^o)
\end{aligned}
$$

$$- w^o - r^o k_1^o) - \sum_{i=1}^{2} \lambda_i^1(\lambda_i^1 f_i(k_i^1) - w^o - r^o k_i^1)$$

$$+ (r^o - p^o \lambda)(k^o - k^1) - p^o(\dot{k}^o - \dot{k}^1)]dt. \qquad (A1.2)$$

Recalling (18), in particular that $\dot{p} = (\lambda + \delta)p - r$, (A1.2) becomes

$$I(X^o, Z^o, V^o) - I(X^1, Z^1, V^o) = \int_o^\infty e^{-\delta t}[U(y_1^o + g_1(z_2^o) + g_2(z_L{}^o) - g_2(z_L{}^1)$$

$$+ \phi(z_L^o)z_L^o) - U(y_1^1 + g_1(z_2^1) + \phi(z_L^1)z_L^1) - U'(y_1^o - y_1^1)$$

$$- U'g_1'(z_2^o - z_2^1) - U'(g_2' + \phi' z_L + \phi)(z_L^o - z_L^1)$$

$$+ (U' - \lambda_1^o)(y_1^o - y_1^1) + (p^o - \lambda_2^o)(y_2^o - y_2^1) \qquad (A1.3)$$

$$+ (\delta p^o - \dot{p})(k^o - k^1) - p^o(\dot{k}^o - \dot{k}^1)]dt.$$

Integrating we find,

$$\int_o^\infty e^{-\delta t} p^o(\dot{k}^o - \dot{k}^1)dt = p^o(k^o - k^1)e^{-\delta t}\Big|_o^\infty - \int_o^\infty e^{-\delta t}$$

$$(\dot{p} - \delta p^o)(k^o - k^1)dt. \qquad (A1.4)$$

By the transversality condition $-p^o(k^o - k^1)e^{-\delta t}\Big|_o^\infty \geq 0$. So substituting (A1.4) into (A1.3) gives $I(X^o, Z^o, V^o) - I(X^1, Z^1, V^o) > 0$ by the strict concavity of the social welfare function, the first order and transversality conditions. Thus (X^o, Z^o) is the unique maximum, i.e., the unique optimal policy of $I(X, Z, V)$.

Furthermore, because all the indirect constraints are equalities, for any $V^1 \geq 0$ we have

$$I(X^o, Z^o, V^o) = I(X^o, Z^o, V^1) \qquad (A1.5)$$

Hence (X^o, Z^o, V^o) is a saddle point of $I(X, Z, V)$ Q.E.D.

Appendix 2. Local Stability of the Intertemporal Equilibria

Suppose that the steady state occurs in pattern (2,2). We know that from equation (170), $\dot{p} = U'\dot{q}$ and therefore when we linearize $\dot{p}$ around the equilibrium point (k^*,p^*) we get

$$\frac{\partial \dot{p}}{\partial q}\bigg|_{(k^*,\,p^*)} = U''\left(\frac{\partial x_1}{\partial q} + (\phi' z_L + \phi)\frac{\partial z_L}{\partial q}\right)\dot{q} + U'\frac{\partial \dot{q}}{\partial q}\bigg|_{(k^*,p^*)} . \quad \text{(A2.1)}$$

Since $\dot{q} = 0 = \dot{k}$ at (k^*,p^*) then

$$\frac{\partial \dot{p}}{\partial q}\bigg|_{(k^*,p^*)} = U'\frac{\partial \dot{q}}{\partial q}\bigg|_{(k^*,p^*)} \quad \text{(A2.2)}$$

and similarly,

$$\frac{\partial \dot{p}}{\partial k}\bigg|_{(k^*,p^*)} = U'\frac{\partial \dot{q}}{\partial q}\bigg|_{(k^*,p^*)} . \quad \text{(A2.3)}$$

Hence because $U' > 0$ the signs of $\frac{\partial \dot{p}}{\partial q}$ and $\frac{\partial \dot{p}}{\partial k}$ are determined by the signs of $\frac{\partial \dot{q}}{\partial q}$ and $\frac{\partial \dot{q}}{\partial k}$ respectively, with the derivatives evaluated at the steady state. Equation (116) shows that $\text{sgn}\frac{\partial \dot{q}}{\partial q}\Big|_{(k^*,p^*)} = \text{sgn}$ $(k_1 - k_2)$ and $\frac{\partial \dot{q}}{\partial k}\Big|_{(k^*,p^*)} = 0$. In addition for the linearization of the $\dot{k}$ equation we have from equations (150) and (152) that, sgn $\frac{\partial \dot{k}}{\partial k}\Big|_{(k^*,p^*)} = \text{sgn}(k_2 - k_1)$ and $\frac{\partial \dot{k}}{\partial q}{}_{(k^*,p^*)} > 0$. Consequently the characteristic roots of the system are,

$$\S_1,\S_2 = 1/2\left[-\left(U'\frac{\partial \dot{q}}{\partial q} + \frac{\partial \dot{k}}{\partial k}\right) \pm \sqrt{\left(U'\frac{\partial \dot{q}}{\partial q} + \frac{\partial \dot{k}}{\partial k}\right)^2 - 4\left(U'\frac{\partial \dot{q}}{\partial q}\frac{\partial \dot{k}}{\partial k}\right)}\ \right].$$

(A2.4)

The roots are real and opposite in sign, so if the steady state occurs in pattern (2,2) it is a local saddle point.

Next assume that the intertemporal equilibrium exists in pattern (2,3). From equations (132), (131), (A2.2) and (A2.3),

$$\frac{\partial \dot{p}}{\partial q}\bigg|_{(k^*,p^*)} = U' \frac{\partial \dot{q}}{\partial q}\bigg|_{(k^*,\ p^*)} < 0 \qquad \text{(A2.5)}$$

$$\frac{\partial \dot{p}}{\partial k}\bigg|_{(k^*,p^*)} = U' \frac{\partial \dot{q}}{\partial k}\bigg|_{(k^*,p^*)} > 0 .$$

Moreover in conjunction with (A2.5), (156) and (157) the characteristic roots are

$$\S_1,\S_2 = 1/2\ [-(U'\frac{\partial \dot{q}}{\partial q} + \frac{\partial \dot{k}}{\partial k}) \pm \sqrt{(U'\frac{\partial \dot{q}}{\partial q} + \frac{\partial \dot{k}}{\partial k})^2 + 4(U'\frac{\partial \dot{q}}{\partial q}\frac{\partial \dot{k}}{\partial k} - \frac{\partial \dot{q}}{\partial k}\frac{\partial \dot{k}}{\partial q})}\ ] \qquad \text{(A2.6)}$$

Again the roots are real and opposite in sign and thus the steady state in (3,2) is a local saddle point.

Finally if the steady state occurs in (1,2) we have from equations (142), (143) and (158) that $U' \frac{\partial \dot{q}}{\partial q}\big|_{(k^*,p^*)} > 0$, $U' \frac{\partial \dot{q}}{\partial k}\big|_{(k^*,p^*)} > 0$, $\frac{\partial \dot{k}}{\partial k}\big|_{(k^*,p^*)} < 0$ and $\frac{\partial \dot{k}}{\partial q}\big|_{(k^*,q^*)} > 0$. Therefore the roots are defined by (A2.6) and once more the steady state is a local saddle point.

Appendix 3. Patterns with Consumption Expenditure Equated to Zero

At the outset it is convenient to define the consumption of nationals of the domestic country's consumption product in labour endowment form by

$$x_1^n = \begin{cases} x_1 + \phi(z_L)z_L \geq 0 & \text{if } z_L < 0, \\ x_1 \geq 0 & \text{if } z_L \geq 0. \end{cases} \tag{A3.1}$$

This definition facilitates the derivation of the number of combinations that are feasible when $x_1 = 0$. This is because when $x_1 = 0$ and since by (14), $-\infty \leq \phi(z_L) \leq 0$ for $z_L > 0$, then by (A3.1), $z_L \leq 0$ when $x_1 = 0$.[12] Consequently since the social welfare functional in the linear case is defined only when $x_1 + \phi(z_L)z_L \geq 0$, whenever $x_1 = 0$ the domestic country now exports labour. The different cases are found in Table 11. Proceeding as in the text we will first analyse the temporal equilibria and then the motion of the economy through time. In pattern (2,4) we have,

$$\begin{aligned} &y_2 = \hat{y}_2(k,q), \; z_2 = \hat{z}_2(q), \; z_L = \hat{z}_L(q), \\ &x_2 = \hat{x}_2(k,q), \; x_1 = 0, \\ &\hat{z}_1(q) = -\hat{y}_1(k,q). \end{aligned} \tag{A3.2}$$

Solving q as a function of k from (A3.2) yields

$$\frac{dq}{dk} = -\frac{\dfrac{\partial \hat{y}_1}{\partial k}}{\dfrac{d\hat{z}_1}{dq} + \dfrac{\partial \hat{y}_1}{\partial q}}. \tag{A3.3}$$

Table 11. $x_1 = 0$ patterns

Expenditure	Production (1)	(2)	(3)	(4)
	$y_1 > 0,\ y_2 = 0$	$y_1 > 0,\ y_2 > 0$	$y_1 = 0,\ y_2 > 0$	$y_1 = 0 = y_2$
	$z_L < 0$	$z_L < 0$	$z_L < 0$	$z_L = -1$
$x_1 = 0,\ x_2 > 0$ (4)	$z_1 = -y_1,\ z_2 > 0$	$z_1 = -y_1,\ z_2 > 0$	$z_1 = 0,\ z_2 > 0$	$z_1 = 0,\ z_2 > 0$
	$x_1 = 0,\ x_2 = z_2$	$x_1 = 0,\ x_2 > 0$	$x_1 = 0,\ x_2 > 0$	$x_1 = 0,\ x_2 = z_2$

If $k_1 > k_2$ then $\frac{dq}{dk} > 0$ by equations (61), (63) and (64). Investigating the case $k_2 > k_1$, recall that

$$\frac{\partial \hat{y}_1}{\partial q} + \frac{d\hat{z}_1}{dq} = -q\left(\frac{\partial \hat{y}_2}{\partial q} + \frac{d\hat{z}_2}{dq}\right) - (\phi'\hat{z}_L + \phi)\frac{d\hat{z}_L}{dq}, \tag{A3.4}$$

with $-q\left(\frac{\partial \hat{y}_2}{dq} + \frac{d\hat{z}_2}{dq}\right) < 0$ and $\frac{d\hat{z}_L}{dq} < 0$ if $k_2 > k_1$. Noting that $z_L < 0$ and when the elasticity of consumption in the foreign country per unit of exported labour is

$$\frac{\phi'\hat{z}_L}{\phi} \leq -1 \tag{A3.5}$$

i.e., not inelastic then $\frac{dq}{dk} < 0$. However when the elasticity measure is inelastic then from (A3.4)

$$\frac{\partial \hat{y}_1}{\partial q} + \frac{d\hat{z}_1}{dq} \gtreqless 0 \quad \text{if and only if} \quad \frac{q}{\psi - w} \gtreqless - \frac{\frac{d\hat{z}_L}{dq}}{\frac{\partial \hat{x}_2}{\partial q}}. \tag{A3.6}$$

Consequently,

$$\frac{dq}{dk} \gtrless 0 \text{ if and only if } \frac{q}{\psi - w} \gtrless \frac{\frac{d\hat{z}_L}{dq}}{\frac{\partial \hat{x}_2}{dq}} .$$

(A3.7)

$$\frac{dq}{dk} = \infty \text{ if and only if } \frac{q}{\psi - w} = - \frac{\frac{d\hat{z}_L}{dq}}{\frac{\partial \hat{x}_2}{\partial q}} .$$

We are able to define $\hat{q}_1 = \hat{q}_1(k)$ by (A3.3).

Now in pattern (3,4) we have,

$$\tilde{y}_2(k,q) = (1 + \tilde{z}_L(k,q))f_2\left(\frac{k}{1+\tilde{z}_L(k,q)}\right) ;$$

$$y_1 = 0 = z_1 = x_1;\ x_2 = \text{[illegible]}(k,q),\ z_2 = \tilde{z}_2(q). \tag{A3.8}$$

Once more we have that q is a function of k. From the foreign offer function

$$0 = g_1(\tilde{z}_2(q)) + g_2(\tilde{z}_L(q,k)) . \tag{A3.9}$$

Thus

$$\frac{dq}{dk} = - \frac{g_2' \frac{\partial \tilde{z}_L}{dk}}{g_1' \frac{d\tilde{z}_2}{dq} + g_2' \frac{\partial \tilde{z}_L}{\partial q}} < 0 \tag{A3.10}$$

by equations (51), (79), and (81). Let us then define $\tilde{q}_1 = \tilde{q}_1(k)$ by (A3.10).

In case (1,4),

$$y_2 = 0,\ \bar{y}_1(k) = (1 + \bar{z}_L(k))\, f_1\left(\frac{k}{(1+\bar{z}_L(k))}\right)$$

$$z_2 = \bar{z}_2(q),\ x_2 = \bar{x}_2(k,q)$$

$$x_1 = \bar{x}_1(k,q) \tag{A3.11}$$

$$\bar{z}_1(k,q) = -\bar{y}_1(k) .$$

Therefore,

$$\frac{dq}{dk} = - \frac{\frac{d\bar{y}_1}{dk} + \frac{\partial \bar{z}_1}{\partial k}}{\frac{\partial \bar{z}_1}{\partial q}} . \tag{A3.12}$$

Also,

$$\frac{d\bar{y}_1}{dk} = \frac{d\bar{z}_L}{dk}\,(f_1(k_1) - k_1 f_1'(k)) + f_1'(k_1)$$

$$\frac{\partial \bar{z}_1}{\partial k} = g_2' \frac{d\bar{z}_L}{dk} \qquad \text{(A3.13)}$$

and so

$$\frac{d\bar{y}_1}{dk} + \frac{d\bar{z}_1}{\partial k} = -\frac{d\bar{z}_L}{dk}\,(\phi'\bar{z}_L + \phi) + f_1'(k_1)\ . \qquad \text{(A3.14)}$$

If $\frac{\phi'\bar{z}_L}{\phi} \leq -1$ then (A3.14) is positive and $\frac{dq}{dk} > 0$. If $\frac{\phi'\bar{z}_L}{\phi} > -1$ then (A3.14) is positive, zero or negative depending on $(\psi - w) \lesseqgtr \frac{f_1'(k_1)}{\frac{d\bar{z}_L}{dk}}$. This says that when labour is exported and $\frac{\phi'\bar{z}_L}{\phi} > -1$ then the effect of a change in the capital-labour endowment ratio on the demand price of investment depends on the magnitude of the differential between the price of labour services with and without the externality and the ratio of the marginal product of y_1 with respect to k to the increases in labour exports due to increases in k. Obviously it is the externality effect which is initiating the somewhat indeterminate signs of $\frac{dq}{dk}$ in (2,4) and (1,4). In symbols for (1,4) we get

$$\frac{dq}{dk} \gtreqless 0 \text{ if and only if } (\psi - w) \lesseqgtr \frac{f_1'(k_1)}{\frac{d\bar{z}_L}{dk}}\ . \qquad \text{(A3.15)}$$

Consequently we define $\bar{q}_1 = \bar{q}_1(k)$ by (A3.12) with the appropriate slopes.

We shall for the moment leave aside the degenerate case (4,4). Moreover the price ranges are; (2,4), $\hat{q}_1 \gtreqless q_0, p(\omega_1(\frac{k}{(1+\bar{z}_L)})) < \hat{q}_1(k) \leq q < p(\omega_2(\frac{k}{(1+\bar{z}_L)}))$; (3,4), $\tilde{q}_1(k) \gtrless q_0$, $q \geq \tilde{q}_1(k) \geq p(\omega_2(\frac{k}{(1+\bar{z}_L)}))$; (1,4), $q_0 < \bar{q}_1(k) \leq q \leq p(\omega_1(\frac{k}{(1+\bar{z}_L)}))$.

Turning to the solution of the differential equations (114) to determine the optimal path when $x_1 = 0$ we observe in (2,4) that from (114) and (A3.2),

$$\dot{q} = (\lambda+\delta)q - (\frac{d\phi(\hat{z}_L)\hat{z}_L}{d\hat{q}_1} \frac{d\hat{q}_1}{dk}) - q\,[\,\frac{\partial\hat{y}_2}{\partial k} + \frac{\partial\hat{y}_2}{\partial\hat{q}_1}\frac{d\hat{q}_1}{dk} + \frac{d\hat{z}_2}{d\hat{q}_1}\frac{d\hat{q}_1}{dk}\,]\,. \qquad \text{(A3.16)}$$

From the foreign offer function and the necessary condition that

$$-\,g_1'(\hat{z}_2) = \hat{q}_1(k),$$

$$q\,\frac{d\hat{z}_2}{d\hat{q}_1}\frac{d\hat{q}_1}{dk} = -\frac{q}{\hat{q}_1}\frac{d\hat{z}_1}{d\hat{q}_1}\frac{d\hat{q}_1}{dk} + \frac{q}{\hat{q}_1}\,g_2'\,\frac{d\hat{z}_L}{d\hat{q}_1}\frac{d\hat{q}_1}{dk}\,. \qquad \text{(A3.17)}$$

Also,

$$\frac{q}{\hat{q}_1}\frac{d\hat{z}_L}{d\hat{q}_1}\frac{d\hat{q}_1}{dk} = -\frac{q}{\hat{q}_1}(\frac{\partial\hat{y}_1}{\partial k} + \frac{\partial\hat{y}_1}{\partial\hat{q}_1}\frac{d\hat{q}_1}{dk})\,. \qquad \text{(A3.18)}$$

Hence

$$\dot{q} = (\lambda+\delta)q - (\phi'\hat{z}_L + \phi + \frac{q}{\hat{q}_1}\,g_2'\lambda\,\frac{d\hat{z}_L}{d\hat{q}_1}\frac{d\hat{q}_1}{dk}$$

$$- q\left(\frac{\partial \hat{y}_1}{\partial k}\ \frac{1}{\hat{q}_1} + \frac{\partial \hat{y}_2}{\partial k} + \left(\frac{\partial \hat{y}_1}{\partial \hat{q}_1}\ \frac{1}{\hat{q}_1} + \frac{\partial \hat{y}_2}{\partial \hat{q}_1}\right) \frac{d\hat{q}_1}{dk}\right) . \tag{A3.19}$$

From (63) and the fact that $\hat{q}_1(k) = \dfrac{f_1'(k_1(\hat{q}_1))}{f_2'(k_2(\hat{q}_2))}$ we have,

$$q\ \frac{\partial \hat{y}_2}{\partial k} = -\frac{q}{\hat{q}_1}\ \frac{\partial \hat{y}_1}{\partial k}\ \left(\frac{k_2 + \omega}{k_1 + \omega}\right) . \tag{A3.20}$$

Therefore

$$-\left(q\frac{\partial \hat{y}_2}{\partial k} + \frac{q}{\hat{q}_1}\ \frac{\partial \hat{y}_1}{\partial k} = -\left(\frac{-q}{\hat{q}_1}\ \frac{\partial \hat{y}_1}{\partial k}\ \left(\frac{k_2 + \omega}{k_1 + \omega}\right)\right.\right.$$

$$+ \frac{q}{\hat{q}_1}\ \frac{\partial \hat{y}_1}{\partial k}\ ) = \frac{-q}{\hat{q}_1}\ f_1'\ (k_1(\hat{q}_1)). \tag{A3.21}$$

Next we see that from (64)

$$-q\ \frac{\partial \hat{y}_2}{\partial \hat{q}_1}\ \frac{d\hat{q}_1}{dk} = -\left[-\frac{\partial \hat{y}_1}{\partial \hat{q}_1}\frac{q}{\hat{q}_1} - \frac{d\hat{z}_L}{d\hat{q}_1}\frac{q}{\hat{q}_1}\ f_1(k_1(\hat{q}_1))\ \frac{k_2}{k_1 - k_2}\right.$$

$$\left. + k_1\ \frac{d\hat{z}_L}{d\hat{q}_1}\ q\ f_2(k_2(\hat{q}_2)\ \frac{k_1}{k_1 - k_2}\ \right] \frac{d\hat{q}_1}{dk} . \tag{A3.22}$$

This yields,

$$\left(\frac{-q}{\hat{q}_1}\ \frac{\partial \hat{y}_1}{\partial \hat{q}_1} - q\ \frac{\partial \hat{y}_2}{\partial \hat{q}_1}\right) \frac{d\hat{q}_1}{dk} = -\left(\frac{d\hat{z}_L}{d\hat{q}_1}\ f_2\ (k_2(\hat{q}_1))^q\ \frac{k_1}{k_1 - k_2}\right.$$

$$\left. - \frac{d\hat{z}_L}{d\hat{q}_1}\ \frac{k_2}{k_1 - k_2}\ \frac{q}{\hat{q}_1} f_1(k_1(\hat{q}_1))\right)\frac{d\hat{q}_1}{dk} , \tag{A3.23}$$

and remembering $\hat{q}_1(k)(f_2(k_2) - k_2 f_2'(k_2)) = w$,

$$- q \left(\frac{\partial \hat{y}_1}{\partial \hat{q}_1} \frac{1}{\hat{q}_1} + \frac{\partial \hat{y}_2}{\partial \hat{q}_1}\right) \frac{d\hat{q}_1}{dk} = \frac{d\hat{z}_L}{d\hat{q}_1} \frac{d\hat{q}_1}{dk} \frac{q}{\hat{q}_1} w. \tag{A3.24}$$

This transforms (A3.19) to

$$\dot{q} = (\lambda+\delta) q - (\phi'\hat{z}_L + \phi + \frac{q}{\hat{q}_1} (g_2' + w)) \frac{d\hat{z}_L}{d\hat{q}_1} \frac{d\hat{q}_1}{dk}$$

$$- \frac{q}{\hat{q}_1} f_1' (k_1(\hat{q}_1)). \tag{A3.25}$$

Recalling that $\psi = -g_2'$ and the necessary condition $\phi'\hat{z}_L + \phi + g_2' + w = 0$ we get

$$\dot{q} = (\lambda+\delta)q - \frac{q}{\hat{q}_1} f_1'(k_1(\hat{q}_1)) + (\frac{q}{\hat{q}_1} - 1)(\psi-w) \frac{d\hat{z}_L}{d\hat{q}_1} \frac{d\hat{q}_1}{dk} . \tag{A3.26}$$

If $q = \hat{q}_1$ then $\dot{q} = (\lambda+\delta) q - f_1'(k_1(\hat{q}_1))$ and

$$\frac{\partial \dot{q}}{\partial k} = - f_1''(k_1)k_1'(\hat{q}_1) \frac{d\hat{q}_1}{dk} . \tag{A3.27}$$

Therefore if $k_1 > k_2$ then $k_1'(\hat{q}_1) < 0$, $\frac{d\hat{q}_1}{dk} > 0$ and $\frac{\partial \dot{q}}{\partial k} > 0$. If $k_2 > k_1$, $\frac{\phi'\hat{z}_L}{\phi} \leq -1$ then $k_1'(\hat{q}_1) < 0$, $\frac{d\hat{q}_1}{dk} < 0$ and $\frac{\partial \dot{q}}{\partial k} > 0$; if $\frac{\phi' z_L}{\phi} > -1$ then $\text{sgn} \frac{\partial \dot{q}}{\partial k} = \text{sgn} - \frac{d\hat{q}_1}{dk}$ as defined by (A3.7). On the other hand when $q > \hat{q}_1$ then evaluating (A3.26) at $\dot{q} = 0$,

$$\frac{q}{\hat{q}_1} f_1'(k_1(\hat{q}_1)) - (\lambda+\delta)q = (\frac{q}{\hat{q}_1} - 1)(\psi-w) \frac{d\hat{z}_L}{d\hat{q}_1} \frac{d\hat{q}_1}{dk} . \tag{A3.28}$$

At $\dot{q} = 0$ and $q = \hat{q}_1$, $f_1'(k_1(\hat{q}_1)) = (\lambda+\delta)q$. Therefore $q > \hat{q}_1$ the left side of (A3.28) is positive. This means that for $k_1 > k_2$ by (58) and $\frac{d\hat{q}_1}{dk} > 0$ at $q = 0$ it must be true that $\frac{\phi'\hat{z}_L}{\phi} < -1$, i.e., the elasticity of consumption in the foreign country per exported worker is elastic ($\psi - w > 0$). Now when $k_2 > k_1$ and since $0 < k_1 < \infty$ at $q = 0$ by the assumptions on the production functions then $\frac{d\hat{q}_1}{dk} \neq \infty$. Thus the left side of (A3.28) is positive and finite for $k_2 > k_1$. In addition when $k_2 > k_1$ and $\frac{d\hat{q}_1}{dk} < 0$ then $\psi - w > 0$ or $\frac{\phi'\hat{z}_L}{\phi} < -1$; for $\frac{d\hat{q}_1}{dk} > 0$ then $\psi - w < 0$ or $\frac{\phi'\hat{z}_L}{\phi} > -1$. Finally notice that ψ may never equal w when $q > \hat{q}_1$ and $\dot{q} = 0$ implying that the elasticity of consumption in the foreign country per unit of exported labour must not be unity. Continuing we find that when $q = \hat{q}_1$ then

$$\frac{\partial \dot{q}}{\partial q} = \lambda + \delta > 0 \, , \tag{A3.29}$$

and when $q > \hat{q}_1$ then

$$\frac{\partial \dot{q}}{\partial q} = (\lambda+\delta) - \frac{1}{\hat{q}_1} \left[f_1'(k_1(\hat{q}_1)) - (\psi - w) \frac{d\hat{z}_L}{d\hat{q}_1} \frac{d\hat{q}_1}{dk} \right] \tag{A3.30}$$

the sign of which is indeterminate. Proceeding to the $\dot{q}$ equation for (3,4) we get from (114) and (A3.8)

$$\begin{aligned} \dot{q} = \; & (\lambda+\delta)q - (\phi'\hat{z}_L + \phi)\left(\frac{\partial \tilde{z}_L}{\partial k} + \frac{\partial \tilde{z}_L}{\partial \tilde{q}_1} \frac{d\tilde{q}_1}{dk}\right) \\ & - q\left(\frac{\partial \tilde{y}_2}{\partial k} + \frac{\partial \tilde{y}_2}{\partial \tilde{q}_1} \frac{d\tilde{q}_1}{dk} + \frac{d\tilde{z}_2}{d\tilde{q}_1} \frac{d\tilde{q}_1}{dk}\right) . \end{aligned} \tag{A3.31}$$

Next from (85)

$$-q\left(\frac{\partial \tilde{y}_2}{\partial k} + \frac{\partial \tilde{y}_2}{\partial \tilde{q}_1} \frac{d\tilde{q}_1}{dk}\right) = -q\left(\frac{\partial \tilde{z}_L}{\partial k}(f_2(k_2) - k_2 f_2'(k_2)) + f_2'(k_2)\right.$$

$$\left. + \frac{\partial \tilde{z}_L}{\partial \tilde{q}_1} \frac{d\tilde{q}_1}{dk}(f_2(k_2) - k_2 f_2'(k_2))\right). \qquad \text{(A3.32)}$$

Also from the foreign country's offer function (82),

$$-g_2' \frac{q}{\tilde{q}_1}\left(\frac{\partial \tilde{z}_L}{\partial k} + \frac{\partial \tilde{z}_L}{\partial \tilde{q}_1} \frac{d\tilde{q}_1}{dk}\right) = -q \frac{d\tilde{z}_2}{d\tilde{q}_1} \frac{d\tilde{q}_1}{dk}. \qquad \text{(A3.33)}$$

Hence (A3.31) becomes,

$$\dot{q} = (\lambda+\delta)q - qf_2'(k_2) + \left(\frac{q}{\tilde{q}_1} - 1\right)(\psi - w)\left(\frac{\partial \tilde{z}_L}{\partial k} + \frac{\partial \tilde{z}_L}{\partial \tilde{q}_1} \frac{d\tilde{q}_1}{dk}\right). \qquad \text{(A3.34)}$$

If $q = \tilde{q}_1$ and recalling $k_2 = \frac{k}{(1+\tilde{z}_L(k, \tilde{q}_1(k)))}$

$$\frac{\partial \dot{q}}{\partial k} = \frac{qf_2''(k_2)}{(1+\tilde{z}_L)}\left(\frac{\partial \tilde{z}_L}{\partial k} k_2 - 1\right) \pm \frac{qf_2''(k_2)}{(1+\tilde{z}_L)} \frac{\partial \tilde{z}_L}{\partial \tilde{q}_1} \frac{d\tilde{q}_1}{dk} k_2 > 0 \qquad \text{(A3.35)}$$

by equations (79), (81) and (A3.10). Moreover observe $\frac{\partial(\frac{q}{\tilde{q}})}{\partial q} = 0$.

If $q > \tilde{q}_1$ the expressions become more intricate. We do know for $q = \tilde{q}_1$ at $\dot{q} = 0$, $f_2'(k_2) = \lambda+\delta$. Hence at $\dot{q} = 0$, $q > \tilde{q}_1$ and since k_2 is solely a function of k, $f_2'(k_2) = \lambda+\delta$. This produces

$$\left(\frac{q}{\tilde{q}_1} - 1\right)(\psi - w)\left(\frac{\partial \tilde{z}_L}{\partial k} + \frac{\partial \tilde{z}_L}{\partial \tilde{q}_1} \frac{d\tilde{q}_1}{dk}\right) = 0. \qquad \text{(A3.36)}$$

The only way (A3.36) holds with $q > \tilde{q}_1$ and $\frac{\partial \tilde{z}_L}{\partial k} + \frac{\partial \tilde{z}_L}{\partial \tilde{q}_1} \frac{d\tilde{q}_1}{dk} > 0$ by

(A3.9), (A3.10) is when $\psi = w$ or $\frac{\phi'\tilde{z}_L}{\phi} = -1$.

In pattern (1,4) the q equation is

$$\dot{q} = (\lambda+\delta)q - (\phi'\bar{z}_L + \phi)\frac{d\bar{z}_L}{dk} - q\frac{d\bar{z}_2}{d\bar{q}_1}\frac{d\bar{q}_1}{dk} \quad . \qquad \text{(A3.37)}$$

From the foreign offer function, (A3.11) and (106)

$$-q\frac{d\bar{z}_2}{d\bar{q}_1}\frac{d\bar{q}_1}{dk} = -\frac{q}{\bar{q}_1}f_1'(k_1) + \frac{q}{\bar{q}_1}(\psi - w)\frac{d\bar{z}_L}{dk} \quad . \qquad \text{(A3.38)}$$

Therefore (A3.37) becomes

$$\dot{q} = (\lambda+\delta)q - \frac{q}{\bar{q}_1}f_1'(k_1) + (\frac{q}{\bar{q}_1} - 1)(\psi - w)\frac{d\bar{z}_L}{dk} \quad . \qquad \text{(A3.39)}$$

If $q = \bar{q}_1$ and recalling $k_1 = \frac{k}{(1+\bar{z}_L(k))}$,

$$\frac{\partial\dot{q}}{\partial k} = \frac{f_1''(k_1)}{(1+\bar{z}_L)}(k_1\frac{d\bar{z}_L}{dk} - 1) > 0 \ , \quad \frac{\partial\dot{q}}{\partial q} = \lambda + \delta > 0, \qquad \text{(A3.40)}$$

by equation (102). At $\dot{q} = 0$ and $q = \bar{q}_1$, $f_1'(k_1) = \lambda + \delta$. Thus with $q > \bar{q}_1$ at $\dot{q} = 0$, $\frac{q}{\bar{q}_1}f_1'(k_1) > (\lambda + \delta)q$ and

$$(\frac{q}{\bar{q}_1} - 1)(\psi - w)\frac{d\bar{z}_L}{dk} > 0. \qquad \text{(A3.41)}$$

This implies that $\psi > w$ of $\frac{\phi'\bar{z}_L}{\phi} < -1$. So far we have gleaned some rather interesting implications pertaining to the pattern of international labour movements. Although we do not in general know the slope of the $\dot{q} = 0$ curve or if steady states may exist when $x_1 = 0$.

To complete the answer to the existence question we must investigate the role of the $\dot{k}$ equation.

In pattern (2,4)

$$\dot{k} = \hat{y}_2(k,\hat{q}_1(k)) + \hat{z}_2(\hat{q}_1(k)) - \lambda k. \tag{A3.42}$$

Differentiating with respect to k yields,

$$\frac{\partial\dot{k}}{\partial k} = \frac{\partial\hat{y}_2}{\partial k} + \frac{\partial\hat{y}_2}{\partial\hat{q}_1}\frac{d\hat{q}_1}{dk} + \frac{d\hat{z}_2}{d\hat{q}_1}\frac{d\hat{q}_1}{dk} - \lambda \tag{A3.43}$$

and with equations (51), (63), and (64), (A3.43) becomes

$$\frac{\partial\dot{k}}{\partial k} = \frac{f_1'(k_1)}{\hat{q}_1} - \frac{d\hat{z}_L}{d\hat{q}_1}\frac{d\hat{q}_1}{dk}\frac{(\psi - w)}{\hat{q}_1} - \lambda . \tag{A3.44}$$

Next from (A3.25)

$$\frac{\partial\dot{k}}{\partial k} = \delta - \frac{\dot{q}}{q} - \frac{(\psi - w)}{q}\frac{d\hat{z}_L}{d\hat{q}_1}\frac{d\hat{q}_1}{dk} . \tag{A3.45}$$

Also $\frac{\partial\dot{k}}{\partial k} = 0$, $\dot{k} = 0$ when $k = 0$ by the assumptions on the production functions and for $\dot{k} = 0$ we must have $\frac{\partial\dot{k}}{\partial k} = 0$. Consequently $\frac{\partial\dot{k}}{\partial k} = 0$ at at $\dot{q} = 0$ if and only if

$$\delta q = (\psi - w)\frac{d\hat{z}_L}{d\hat{q}_1}\frac{d\hat{q}_1}{dk} . \tag{A3.46}$$

(A3.46) is satisfied when for $\frac{d\hat{q}_1}{dk}$, $\psi > w$ we have $\frac{d\hat{q}_1}{dk} < 0$, $\psi - w < 0$.

In pattern (3,4),

$$\dot{k} = \tilde{y}_2(k,\tilde{q}_1(k)) + \tilde{z}_2(\tilde{q}_1(k)) - \lambda k. \tag{A3.47}$$

Obviously $\frac{\partial \dot{k}}{\partial q} = 0$ and differentiating (A3.47) with respect to k using (A3.32) and (A3.33) yields

$$\frac{\partial \dot{k}}{\partial k} = f_2'(k_2) - \lambda - \frac{(\psi - w)}{\tilde{q}_1}\left(\frac{\partial \tilde{z}_L}{\partial k} + \frac{\partial \tilde{z}_L}{\partial \tilde{q}_1}\frac{d\tilde{q}_1}{dk}\right). \tag{A3.48}$$

From (A3.34)

$$\frac{\partial \dot{k}}{\partial k} = \delta - \frac{\dot{q}}{q} - \frac{(\psi - w)}{q}\left(\frac{\partial \tilde{z}_L}{\partial k} + \frac{\partial \tilde{z}_L}{\partial \tilde{q}_1}\frac{d\tilde{q}_1}{dk}\right) \tag{A3.49}$$

Since $\ddot{k} = 0$ when $\frac{\partial \dot{k}}{\partial k} = 0$ then $\dot{k} = \dot{q} = 0$ if and only if

$$\delta q = (\psi - w)\left(\frac{\partial \tilde{z}_L}{\partial k} + \frac{\partial \tilde{z}_L}{\partial \tilde{q}_1}\frac{d\tilde{q}_1}{dk}\right), \tag{A3.50}$$

which is satisfied when $\psi - w > 0$. But if $q > \tilde{q}_1$ then from (A3.36) $\psi - w$ so a steady state may only exist in (3,4) when $\psi - w > 0$ and $q = \tilde{q}_1$.

Finally in pattern (1,4),

$$\dot{k} = \bar{z}_2(\bar{q}_1(k)) - \lambda k. \tag{A3.51}$$

By (A3.38) and (A3.39)

$$\frac{\partial \dot{k}}{\partial k} = \delta - \frac{\dot{q}}{q} - \frac{(\psi - w)}{q}\frac{d\bar{z}_L}{dk}. \tag{A3.52}$$

Again $\ddot{k} = 0$ when $\dot{k} = 0$, $\frac{\partial \dot{k}}{\partial q} = 0$ and $\dot{k} = 0 = \dot{q}$ if and only if

$$\delta q = (\psi - w)\frac{d\bar{z}_L}{dk}, \tag{A3.53}$$

which is true when $\psi - w > 0$. Lastly in pattern (4,4) from (161) $q = \delta q$, $\dot{k} = z_2$ therefore no steady state exists here.

These complex calculations illustrate the fundamental point that steady states may exist in cases when $x_1 = 0$. This conclusion is contrary to models exhibiting factor immobility between nations and ostensibly counter to "economic intuition". However careful examination resolves the apparent deleterious result. If consumption is nil in the domestic country then the country must export labour. The exported labour is able to consume in the foreign country by the amount $\phi(z_L)z_L$. In addition to the remainder of the income earned the exported labour may repatriate part of $\phi(z_L)z_L$ thus making it feasible for the rest of the population of the home country to consume. This mechanism appears to be rather pathological and therefore we believe that economic palpability dictates the extrication of this case. Nevertheless, as can be observed by appendix 3, our model easily deals with $x_1 = 0$ solutions.

Footnotes

[1]A small country is defined as one which takes world prices as given. A large country influences, not only international quantities, but prices as well.

[2]Böhning [6], provides an extensive bibliography of the empirical literature on labour mobility. Chipman [7], provides an excellent bibliography of the theoretical literature on factor movements.

[3]This assumption was utilized by Bardhan [4] and Ryder [15].

[4]In addition, we assume that

$$\lim_{K_i \to 0} \frac{\partial F_i}{\partial K_i} = \infty = \lim_{L_i \to 0} \frac{\partial F_i}{\partial L_i} \text{ and } \lim_{K_i \to \infty} \frac{\partial F_i}{\partial K_i} = 0 = \lim_{L_i \to \infty} \frac{\partial F_i}{\partial L_i}, \quad i=1,2.$$

These derivative conditions are in fact excessive in the sense that they are only required to be larger (for $K_i = 0 = L_i$, $i=1,2$) than or smaller (for $K_i = \infty = L_i$, $i=1,2$) than certain parameters. In actuality this is the way these conditions should be interpreted.

[5]As in Chipman [7], we can define a foreign transformation function in implicit form as

$$T\left(\frac{Y_1^f}{L}, \frac{Y_2^f}{L}, \frac{K^f}{L}, \frac{Z_L^f}{L}\right) = 0$$

(when f denotes the foreign country and L is identical in the foreign and domestic countries). Notice that the excess demand for labour appears in the transformation function because it affects production conditions. Utilizing this transformation function with the foreign demand conditions and the definitions of the excess demands, yields an offer function

$$G\left(\frac{Z_1^f}{L}, \frac{Z_2^f}{L}, \frac{Z_L^f}{L}, \frac{K^f}{L}\right) = 0.$$

Z_L^f appears in the offer function because it affected production. In addition, if the country exports labour then any repatriated income will influence the manner in which Z_L^f enters the later function. In our context we specialise the form and domain of the offer function. We assume that $\frac{K^f}{L}$ is constant i.e. the foreign country is always in steady state and the function is additively separable i.e. demand conditions, production conditions and any repartriations interact such that

$$\frac{Z_1^f}{L} = g_1\left(\frac{Z_2^f}{L}\right) + g_2\left(\frac{Z_L^f}{L}\right).$$

[6] $y = (y_1, y_2) \geq 0$ means $y_1 \geqq 0$, $y_2 \geqq 0$ but not both zero.

[7] If a solution exists these conditions, equation system (18), are sufficient for a unique optimum. For a proof see Appendix 1.

[8] For the stability analysis see Appendix 2. Moreover, because the optimal path is unique and all the steady states are saddle points then only one of the intertemporal equilibria may exist.

[9] The description of the optimal path was predicated on the basis that k_o was "sufficiently" less than k^* and k_o was "sufficiently" larger than k^*.

[10] The discussion concerning the optimal path for $k_o > k^*$ in the irreversible investment case is applicable here if we replace (1,1), (2,1), (3,1) by (1,3), (2,3), (3,3) respectively and note that the ingression from expenditure pattern 3 to 2 must always entail a penetration of pattern $x_1 > 0$, $x_2 = 0$.

[11] Without loss of generality normalize initial labour endowment to unity.

[12] The case when $z_L = 0 = x_1$ is found in Ryder [15]. He proves that the steady state cannot exist in this pattern. Therefore we shall deal with the $z_L < 0$ case.

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Chapter 2. Externalities And Public Investment In A Two Country Differential Game Model

I. Introduction

The concept of collective commodities or more generally externalities has, in recent years, been incorporated into dynamic optimizing analyses of closed economies. These models have frequently dealt with problems pertaining to the environment or natural resources, as exemplified in Forster [7], Keeler, Spence and Zeckhauser [9] and Plourde [14], [15].

The perplexing questions relating to the dynamic study of external effects between nations have yet to be fully explored. In a static framework Michael Connally [5] and [6] has developed a theory of international public commodities. Recently James Markusen [10] and [11] has derived optimal taxes and cooperative equilibria solutions for international externalities affecting national welfare. In addition his analyses have been carried out in an atemporal structure.

The first objective of this paper is to examine the implications for formulating an intertemporal national policy program when there exists international and intranational externalities.

In a dynamic setting, all optimal policy prescriptions for an open economy have assumed the "small" country hypothesis or the "passivity" of the rest of the world.[1] These suppositions are quite understandable, as a starting point, for they enable one to focus solely on the policies of a single nation.

This brings us to the second objective of this essay. We develop a model in which economies engage in international transactions in a more sophisticated fashion than the small country or passivity assumptions permit. We postulate various international dependences and derive the optimal allocations of outputs, factors and expenditure for each nation and for the world as a whole under different behavioral environments. In so doing, we find that the theory of non-cooperative differential games as developed by Berkovitz [2], [3] and Starr and Ho [16] is particularly revealing as it enables us to frame the analysis as an application, extension and modification of the static Stackleberg theory of duopoly.[2]

The externalities that we will discuss affect not only social welfare but also national technologies. These externalities are assumed to be stocks, so that accumulation and decumulation implies temporal behavioral dependence within and between countries. Hence, past transactions are affecting present and will affect future behavior. We envision non-human real capital as the vintage or characteristic of stocks which give rise to the externalities.

Examples illustrating the severity and importance of the phenomena are the following. Suppose at least one nation engages in the production of armaments. In the process of research and development of arms we can distinguish three basic stages; construction, testing and stockpiling. Any or all of these phases of armament accumulation could affect foreigner's national welfare. In addition, these stages could conceivably influence the foreign country's production. For instance, the fallout from atmospheric testing of various weaponry may have deleterious consequences on the dairy and non-dairy products of the agricultural and related industries in the foreign nations. Presumably, then, production as well as welfare considerations led to the protestations by different factions in Japan, Australia, New Zealand, Canada and the U.S.A. pertaining to nuclear testing in the Pacific.

Secondly, suppose a country is comprised of two distinct regions which are separated by a foreign country; for example the forty-eight adjacent states of the U.S.A. and Alaska. Needed natural resources, such as petroleum and minerals, may be found in one region and must be transported to the other part of the nation, through or near the foreign country. Any unforeseen shipping accident or other form of transportation, such as a pipeline through a sensitive northern environment, may cause irrevocable damage to the foreign country. Clearly the type of international externalities that influence welfare and technology are, in general, transactions affecting the environment and, in particular, pollution.

The structure of this undertaking builds on the classic work of Kenneth Arrow and Mordecai Kurz [1] and D. L. Brito [4]. We posit the existence of public and private capital stocks, with public capital jointly yielding production and consumption benefits. The international externalities are formalized by assuming the capital stock of a nation affects the welfare and technology of the other "player". Finally, each country is viewed as maximizing a social welfare functional subject to the various expenditure, production and equilibrium relations which in turn depend on the intertemporal behavior of the foreign economy.

The analysis is divided into two parts. In the first segment the countries behave as Stackleberg followers; that is, they presume each other's capital stock is exogeneous to their program. Among the various qualitative results is that autonomous increases in the level of foreign capital affects domestic consumption and private domestic capital to the degree that public capital is influenced by the change in foreign capital. This illustrates the interdependencies that may exist between international and intranational externalities within a nation. For example, the spill-over effects on domestic consumption of a foreign caused oil leak will depend, in part, on the degree to which the government must divert its resources from generating consumption and production benefits, such as schools and highways, to rectifying a disaster which was originally not part of the environment. We derive a steady state Stackleberg reaction function which traces the effect that changes in foreign capital exert on domestic capital while the domestic country remains in steady state. Finally, we prove that given a fundamental sufficient condition on the limits to the value of the production externality, a unique world equilibrium exists and is globally stable.

In the second part of the paper we assume that the nations are Stackleberg leaders; that is, they each presume the other economy obeys its reaction function. Now, the accumulation of domestic capital and the rate of change of the investment price depend explicitly on the form of the foreign country's reaction function and indicates the international dynamic interdependence. The steady state values of capital and the price of investment under different leader-follower situations are compared and turn out to be highly sensitive to the various shapes of these reaction functions.

2. The Model

Let us consider a model with two countries denoted by the subscripts 1 and 2. Each country has three categories of agents, households, firms and a government. There is one output which is both a consumption and an investment product. The three factors of production are labour, private capital and public capital. In addition the foreign capital exerts an externality on the output. Hence, define the technological conditions by the expression

$$Y_i(t) = F_i(K_{pi}(t), K_{gi}(t), L_i(t)) + H_i(K_{gi}(t), K_j(t), L_i(t)) \quad i,j=1,2 \quad i \neq j \quad (1)$$

where $Y_i(t)$ is the output in the i^{th} country in period t, F_i and H_i are the production functions for the i^{th} country, $K_{pi}(t)$ is the endowment of private capital in period t for the i^{th} country, $K_{gi}(t)$ is the endowment of public capital for the i^{th} country in period t, $L_i(t)$ is the labour endowment of the i^{th} country in period t and $K_j(t)$ is the capital encowment of the j^{th} country, $i \neq j$, in period t.

We assume the production functions have the following properties:

> F_i is defined for $(K_{pi}(t), K_{gi}(t), L_i(t)) \geq 0$, H_i is defined for $(K_{gi}(t), K_j(t), L_i(t)) \geq 0$, F_i and H_i are strictly concave, homogeneous of degree 1 and twice continuously differentiable.

$$\frac{\partial F_i}{\partial K_{pi}(t)} > 0, \ \frac{\partial F_i}{\partial K_{gi}(t)} + \frac{\partial H_i}{\partial K_{gi}(t)} > 0, \ \frac{\partial F_i}{\partial L_i(t)} + \frac{\partial H_i}{\partial L_i(t)} > 0, \ \frac{\partial H_i}{\partial K_j(t)} \gtreqless 0.$$

Equation (5) illustrates the fact that in determining the technology of the nation only the public capital stock is affected by alterations in the quantity of foreign capital. This means when public capital is complemented with or substituted for foreign capital, the role of the government in production is changed. However, due to the nature of the separability in equation (1), the private sector's role is never affected by changes in $K_j(t)$.

Given that we define product and factor market equilibria where demand does not exceed supply and that we assume capital is perfectly mobile between the private and public sectors we have,

$$\begin{aligned} &C_i(t) + I_{pi}(t) + I_{gi}(t) \leq Y_i(t) \\ &K_{pi}(t) + K_{gi}(t) \leq K_i(t) \qquad i=1,2 \end{aligned} \tag{2}$$

where $C_i(t)$, $I_{pi}(t)$, $I_{gi}(t)$ are the i^{th} country's consumption, private investment and public investment respectively, in period t.[3] The rate of change of the capital stocks is denoted by

$$\begin{aligned} &\dot{K}_{pi}(t) = I_{pi}(t) - \mu_i K_{pi}(t) \\ &\dot{K}_{gi}(t) = I_{gi}(t) - \mu_i K_{gi}(t) \end{aligned} \tag{3}$$

where $0 \leq \mu_i < \infty$ is the fixed rate of depreciation on public and private capital. The labour endowment in each period is

$$L_i(t) = L_o e^{nt} \qquad i=1,2 \tag{4}$$

We assume that each country's endowment of labour is a constant proportion of the world's endowment in each period. This implies that the rate of growth of labour is the same for each country, i.e. by (4) $0 \leq n < \infty$ is the fixed rate for i and j. Moreover we can then, without loss of generality, normalize the initial endowment in both countries to be equal, so that $L_i(t) = L_j(t)$ and therefore it is feasible to define the following variables,

$$y_i(t) = \frac{Y_i(t)}{L_i(t)},\ k_{pi}(t) = \frac{K_{pi}(t)}{L_i(t)},\ k_{gi}(t) = \frac{K_{gi}(t)}{L_i(t)}$$

$$c_i(t) = \frac{C_i(t)}{L_i(t)},\ i_{pi}(t) = \frac{I_{pi}(t)}{L_i(t)},\ i_{gi}(t) = \frac{I_{gi}(t)}{L_i(t)} \qquad i=1,2$$

We have yet to specify the social welfare function and the intertemporal objective functional of the government. Let social welfare by given by the function

$$\nu_i(c_i(t),k_{gi}(t),k_j(t)) = U_i(c_i(t),k_{gi}(t)) + V_i(k_{gi}(t),k_j(t)) \qquad i=1,2. \quad (5)$$

Assume equation (5) possesses the following properties:

> U_i and V_i are each defined for $(c_i(t),k_{gi}(t),k_j(t)) \geq 0$, strictly concave and twice continuously differentiable,
>
> $$\frac{\partial U_i}{\partial c_i(t)} > 0,\ \frac{\partial U_i}{\partial k_{gi}(t)} + \frac{\partial V_i}{\partial k_{gi}(t)} > 0,\ \frac{\partial V_i}{\partial k_j(t)} \gtreqless 0.$$

The form which has been adopted for the welfare function is to some degree one of separability. Domestic consumption and foreign capital do not interact in the sense that the respective cross partials are zero. Just as in the specified technology, only the role of the government in determining national welfare is influenced by the level of foreign capital.

Thus the welfare functional is

$$W_i(c_i(t),k_{gi}(t),k_j(t)) = \int_0^\infty [B_i - U_i(c_i(t),k_{gi}(t),k_j(t)) - V_i(k_{gi}(t),k_j(t))]dt \quad (6)$$

where $0 < B_i < \infty$ is the fixed bliss point for the i^{th} economy. It is important to appreciate that we have purposely not constrained the foreign capital to provide either positive or negative marginal benefits; in order that we do not constrict the solution and unnecessarily limit the applicability of the model.

We may now delineate the program confronting the i^{th} nation. The government selects the optimal values of the controls, consumption, public and private investment that minimizes the difference between bliss and social welfare subject to the market and technology conditions which in turn depend on the policy choices of the foreign country. Formally the problem is to

$$\min_{\{c_i(t),k_{pi}(t),k_{gi}(t)\}} \int_0^\infty [B - U_i(c_i(t),k_{gi}(t)) - V_i(k_{gi}(t),k_j(t))]dt$$

subject to,

$$k_{pi}(t) + k_{gi}(t) \leq k_i(t)$$

$$\dot{k}_i(t) = f_i(k_{pi}(t),k_{gi}(t)) + h_i(k_{gi}(t),k_j(t)) - c_i(t) - \lambda_i k_i(t), \quad (7)$$

$$c_i(t) \geq 0, \quad 0 < k_{oi} < \infty, \quad i=1,2$$

where $\lambda_i = \mu_i + n$, k_{oi} is the initial capital-labour endowment ratio in country i, f_i and h_i are derived from F_i and H_i utilizing the homogeneity assumption and $\dot{k}_i(t)$ is calculated from equations (1), (2), (3).

3. The Stackleberg Follower Case

The Stackleberg follower case defines behavior according to the supposition that when each of the countries executes its policies none of the choices have any effect on the other player in the game. In the context of our analysis this means that both nations perceive the foreign capital in their welfare and production functions as given. Hence let $k_j(t) = \bar{k}_j(t)$ in the i^{th} economy's program. Following Starr and Ho [6] we may apply the Lagrangian concept to find the necessary conditions for a noncooperative Nash equilibrium. Thus, define the Lagrangian of equation system (7) by

$$\begin{aligned} L_i(c_i(t), k_{pi}(t), k_{gi}(t), k_j(t), p_i(t), q_i(t)) &= U_i(c_i(t), k_{gi}(t)) + \\ &V_i(k_{gi}(t), \bar{k}_j(t)) - B_i + p_i(t)(f_i(k_{pi}(t), k_{gi}(t)) + \\ &h_i(k_{gi}(t), \bar{k}_j(t)) - c_i(t) - \lambda_i k_i(t)) + q_i(t) \\ &(k_i(t) - k_{pi}(t) - k_{gi}(t)) \qquad i,j=1,2 \quad i \neq j. \end{aligned} \tag{8}$$

The first order, canonical and transversality conditions are (we drop the argument (t) from the relevant functions for notational convenience)

$$\begin{aligned} &\frac{\partial L_i}{\partial c_i} = \frac{\partial U_i}{\partial c_i} - p_i = 0, \quad \frac{\partial L_i}{\partial k_{pi}} = p_i \frac{\partial f_i}{\partial k_{pi}} - q_i = 0, \\ &\frac{\partial L_i}{\partial k_{gi}} = \frac{\partial U_i}{\partial k_{gi}} + \frac{\partial V_i}{\partial k_{gi}} + p_i \left(\frac{\partial f_i}{\partial k_{gi}} + \frac{\partial h_i}{\partial k_{gi}}\right) - q_i = 0 \\ &\frac{\partial L_i}{\partial q_i} = k_i - k_{pi} - k_{gi} = 0, \\ &\dot{k}_i = f_i(k_{pi}, k_{gi}) + h_i(k_{gi}, \bar{k}_j) - c_i - \lambda_i k_i, \end{aligned} \tag{9}$$

$$\dot{p} = \lambda_i p_i - q_i, \; 0 < k_{oi} < \infty,$$

$$\lim_{t \to \infty} p_i(t) \geq 0, \; \lim_{t \to \infty} p_i(t) k_i(t) = 0, \quad i,j=1,2, \quad i \neq j.$$

Simplifying equation system (9) yields,

$$\frac{\partial U_i}{\partial c_i} - p_i = 0, \; k - k_{pi} - k_{gi} = 0$$

$$\frac{\partial U_i}{\partial k_{gi}} + \frac{\partial V_i}{\partial k_{gi}} + p_i\left(\frac{\partial f_i}{\partial k_{gi}} + \frac{\partial h_i}{\partial k_{gi}} - \frac{\partial f_i}{\partial k_{pi}}\right) = 0 \qquad (10)$$

$$\dot{k}_i = f_i(k_{pi}, k_{gi}) + h_i(k_{gi}, \bar{k}_j) - c_i - \lambda_i k_i,$$

$$\dot{p}_i = \left(\lambda_i - \frac{\partial f_i}{\partial k_{pi}}\right) p_i \qquad i,j=1,2, \quad i \neq j.$$

Notice that from the assumptions on the form of the production and welfare conditions, given by (1) and (5) respectively, that in equilibrium the marginal product of private capital is greater than the marginal product of public capital. Consequently from $k_{gi}(0)$ and $k_{pi}(0)$ an initial jump may be necessary to correct the initial allocation of capital between the private and public sectors in order to satisfy (10) at t=0. This is the reason why $k_{pi}(t)$ and $k_{gi}(t)$ need not be continuous at t=0. Furthermore once capital is allocated to the private sector it is never optimal to transfer some or all of the stock to the public sector due to the relationship between their respective marginal products.

3.1 Temporal Equilibria

It is well known [1, chapter 4] that by the restrictions imposed on the social welfare and production functions that the temporal equilibrium — the solution to the first three equations of (10) for each country given k_i, k_j, p_i — exists and is unique. Thus we have the following functions $c_i = c_i(k_i,k_j,p_i)$, $k_{pi} = k_{pi}(k_i,k_j,p_i)$ and $k_{gi} = k_{gi}(k_i,k_j,p_i)$. It is interesting to solve for the relevant comparative equilibrium effects, not only for their self importance, for the clarification they bring to bear on the dynamic analysis. Hence totally differentiating the equations comprising the temporal equilibrium yields

$$\begin{bmatrix} \frac{\partial^2 U_i}{\partial c_i^2} & \frac{\partial^2 U_i}{\partial c_i \partial k_{gi}} & 0 \\ \frac{\partial^2 U_i}{\partial k_{gi}\partial c_i} & [\frac{\partial^2 U_i}{\partial k_{gi}^2} + \frac{\partial^2 V_i}{\partial k_{gi}^2} + p_i(\frac{\partial^2 f_i}{\partial k_{gi}^2} + \frac{\partial^2 h_i}{\partial k_{gi}^2} - \frac{\partial^2 f_i}{\partial k_{pi}\partial k_{gi}})] & p_i(\frac{\partial^2 f_i}{\partial k_{gi}\partial k_{pi}} - \frac{\partial^2 f_i}{\partial k_{pi}^2}) \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} dc_i \\ dk_{gi} \\ dk_{pi} \end{bmatrix}$$

$$= \begin{bmatrix} dp_i \\ -(\frac{\partial^2 V_i}{\partial k_{gi}\partial k_j} + p_i \frac{\partial^2 h_i}{\partial k_{gi}\partial k_j} dk_j - dp_i (\frac{\partial f_i}{\partial k_{gi}} + \frac{\partial h_i}{\partial k_{gi}} - \frac{\partial f_i}{\partial k_{pi}}) \\ dk_i \end{bmatrix}. \qquad (11)$$

The determinant of the matrix is (henceforth all relationship will apply to both nations)

$$\Delta_i = [\frac{\partial^2 U_i}{\partial k_{gi}^2} + \frac{\partial^2 V_i}{\partial k_{gi}^2} + p_i(\frac{\partial^2 f_i}{\partial k_{gi}^2} + \frac{\partial^2 h_i}{\partial k_{gi}^2} + \frac{\partial^2 f_i}{\partial k_{pi}^2} - 2\frac{\partial^2 f_i}{\partial k_{pi}\partial k_{gi}})] \frac{\partial^2 U_i}{\partial c_i^2} - (\frac{\partial^2 U_i}{\partial c_i \partial k_{gi}})^2. \qquad (12)$$

To determine the sign of (12) we assume that

$$- \frac{\dfrac{\partial^2 f_i}{\partial k_{gi}^2} + \dfrac{\partial^2 h_i}{\partial k_{gi}^2}}{\dfrac{\partial f_i}{\partial k_{gi}} + \dfrac{\partial h_i}{\partial k_{gi}}} k_{gi} > - \frac{\dfrac{\partial^2 f_i}{\partial k_{pi} \partial k_{gi}}}{\dfrac{\partial f_i}{\partial k_{pi}}} k_{gi}$$

and (13)

$$- \frac{\dfrac{\partial^2 f_i}{\partial k_{pi}^2}}{\dfrac{\partial f_i}{\partial k_{pi}}} k_{pi} > - \frac{\dfrac{\partial^2 f_i}{\partial k_{gi} \partial k_{pi}}}{\dfrac{\partial f_i}{\partial k_{gi}} + \dfrac{\partial h_i}{\partial k_{gi}}} k_{pi}.$$

In economic terms these assumptions state that the own elasticity of the marginal product of private capital is greater than the cross elasticity of the marginal product of public capital with respect to private capital and similarly if we interchange the two types of capital. Intuitively, we are permitting small increases in public or private capital to have a "stronger" effect on their own marginal product than on the other factor's marginal product. In addition, from the necessary conditions $\frac{\partial f_i}{\partial k_{pi}} > \frac{\partial f_i}{\partial k_{gi}} + \frac{\partial h_i}{\partial k_{gi}}$ and this implies that

$$+ \frac{\partial^2 f_i}{\partial k_{gi}^2} + \frac{\partial^2 h_i}{\partial k_{gi}^2} - \frac{\partial^2 f_i}{\partial k_{pi} \partial k_{gi}} < 0$$

and

$$\frac{\partial^2 f_i}{\partial k_{pi}^2} - \frac{\partial^2 f_i}{\partial k_{gi}\partial k_{pi}} < 0.$$

If private and public capital weakly complement each other, i.e. $\frac{\partial^2 f_i}{\partial k_{gi}\partial k_{pi}} \geq 0$, then from the strict concavity of F_i and H_i the assumptions are satisfied. On the other hand, if $\frac{\partial^2 f_i}{\partial k_{gi}\partial k_{pi}} < 0$ we should still expect the marginal product of private capital to fall more when a unit of private rather than public capital is added and of course the relation is anticipated to be retained when we interchange the types of capital. The determinant given by (12) is therefore positive.

Proceeding we find from equation set (11) that

$$\frac{\partial c_i}{\partial p_i} = \frac{1}{\Delta_i}\Big[\frac{\partial^2 U_i}{\partial k_{gi}^2} + \frac{\partial^2 V_i}{\partial k_{gi}^2} + p_i\Big(\frac{\partial^2 f_i}{\partial k_{gi}^2} + \frac{\partial^2 h_i}{\partial k_{gi}^2} + \frac{\partial^2 f_i}{\partial k_{pi}^2} - 2\frac{\partial^2 U_i}{\partial k_{pi}\partial k_{gi}}\Big)$$

$$- \frac{\partial^2 U_i}{\partial c_i\partial k_{gi}}\Big(\frac{\partial f_i}{\partial k_{pi}} - \frac{\partial f_i}{\partial k_{gi}} - \frac{\partial h_i}{\partial k_{gi}}\Big)\Big]. \qquad (14)$$

In order to be able to determine the sign of (14) let

$$- \frac{\dfrac{\partial^2 U_i}{\partial k_{gi}^2} + \dfrac{\partial^2 V_i}{\partial k_{gi}^2}}{\dfrac{\partial U_i}{\partial k_{gi}} + \dfrac{\partial V_i}{\partial k_{gi}}}\, k_{gi} > - \frac{\dfrac{\partial^2 U_i}{\partial c_i\partial k_{gi}}}{\dfrac{\partial U_i}{\partial c_i}}$$

and (15)

$$-\frac{\frac{\partial^2 U_i}{\partial c_i^2}}{\frac{\partial U_i}{\partial c_i}} c_i > -\frac{\frac{\partial^2 U_i}{\partial k_{gi}\partial c_i}}{\frac{\partial U_i}{\partial k_{gi}} + \frac{\partial V_i}{\partial k_{gi}}} c_i .$$

Equation set (15) states that the own elasticity or marginal welfare of public capital is greater than the cross elasticity of marginal welfare of consumption with respect to public capital. In addition the own elasticity of marginal welfare of consumption is greater than the cross elasticity of marginal welfare of public capital with respect to consumption. Substituting from (10) for the last bracketed term in (14) and rearranging yields

$$\frac{\partial c_i}{\partial p_i} = \frac{1}{\Delta_i} p_i \left(\frac{\partial^2 f_i}{\partial k_{gi}^2} + \frac{\partial^2 h_i}{\partial k_{gi}^2} + \frac{\partial^2 f_i}{\partial k_{pi}^2} - 2\frac{\partial^2 f_i}{\partial k_{pi}\partial k_{gi}}\right)$$

$$+ \frac{1}{\Delta_i}\left[\frac{\partial^2 U_i}{\partial k_{gi}^2} + \frac{\partial^2 V_i}{\partial k_{gi}^2} - \frac{\partial^2 U_i}{\partial c_i \partial k_{gi}}\left(\frac{\partial U_i}{\partial k_{gi}} + \frac{\partial V_i}{\partial k_{gi}}\right) / \frac{\partial U_i}{\partial c_i}\right]. \qquad (16)$$

The second bracketed term is then negative by equation (15). Moreover from (13) the first bracketed term is negative and thus $\frac{\partial c_i}{\partial p_i} < 0$. Next from (11) we find

$$\frac{\partial c_i}{\partial k_i} = \frac{\partial^2 U_i}{\partial c_i \partial k_{gi}} \frac{p_i}{\Delta_i} \left(\frac{\partial^2 f_i}{\partial k_{gi} \partial k_{pi}} - \frac{\partial^2 f_i}{\partial k_{pi}^2}\right). \tag{17}$$

By (13) we have sgn $\frac{\partial c_i}{\partial k_i}$ = sgn $\frac{\partial^2 U_i}{\partial c_i \partial k_{gi}}$. Finally for the consumption function we get,

$$\frac{\partial c_i}{\partial k_j} = \frac{1}{\Delta_i} \frac{\partial^2 U_i}{\partial c_i \partial k_{gi}} \left[\frac{\partial^2 V_i}{\partial k_{gi} \partial k_j} + p_i \frac{\partial^2 h_i}{\partial k_{gi} \partial k_j}\right]. \tag{18}$$

The interpretation of the bracketed term in (18) is relatively straightforward. The total benefits from a small increase in public capital is

$$\frac{\partial U_i}{\partial k_{gi}} + \frac{\partial V_i}{\partial k_{gi}} + p_i\left(\frac{\partial f_i}{\partial k_{gi}} + \frac{\partial h_i}{\partial k_{gi}}\right). \tag{19}$$

Now if we pose the question what are the total benefits from a small increase in foreign capital on the marginal benefits of public capital the answer is simply the bracketed term in equation (18). We assume that this term may never be zero, i.e.

$$\frac{\partial^2 V_i}{\partial k_{gi} \partial k_j} + p_i \frac{\partial^2 h_i}{\partial k_{gi} \partial k_j} \gtrless 0. \tag{20}$$

Consequently sgn $\frac{\partial c_i}{\partial k_j}$ = sgn $\frac{\partial^2 U_i}{\partial c_i \partial k_{gi}}$ if $\frac{\partial^2 V_i}{\partial k_{gi} \partial k_j} + p_i \frac{\partial^2 h_i}{\partial k_{gi} \partial k_j} > 0$ and

sgn $\frac{\partial c_i}{\partial k_j}$ = sgn $-\frac{\partial^2 U_i}{\partial c_i \partial k_{gi}}$ if $\frac{\partial^2 V_i}{\partial k_{gi} \partial k_j} + p_i \frac{\partial^2 h_i}{\partial k_{gi} \partial k_j} < 0$. Expression (18) illustrates that although foreign capital does not interact with domestic consumption in the welfare function, k_j does exert a modification on c_i. The effect surfaces through the responsiveness of public to foreign capital in consumption and production; then proceeding to the connection between public capital and domestic consumption.

Next solving for the effects on public capital we get,

$$\frac{\partial k_{gi}}{\partial p_i} = \frac{1}{\Delta_i}\left[\frac{\partial^2 U_i}{\partial c_i^2}\left(\frac{\partial f_i}{\partial k_{pi}} - \frac{\partial f_i}{\partial k_{gi}} - \frac{\partial h_i}{\partial k_{gi}}\right) - \frac{\partial^2 U_i}{\partial k_{gi}\partial c_i}\right] < 0, \tag{21}$$

by (15).

$$\frac{\partial k_{gi}}{\partial k_i} = \frac{\partial^2 U_i}{\partial c_i^2}\ \frac{p_i}{\Delta_i}\left(\frac{\partial^2 f_i}{\partial k_{pi}^2} - \frac{\partial^2 f_i}{\partial k_{gi}\partial k_{pi}}\right) > 0, \tag{22}$$

by the strict concavity of U_i and equation (13).

$$\frac{\partial k_{gi}}{\partial k_j} = -\frac{1}{\Delta_i}\ \frac{\partial^2 U_i}{\partial c_i^2}\left(\frac{\partial^2 V_i}{\partial k_{gi}\partial k_j} + p_i\frac{\partial^2 h_i}{\partial k_{gi}\partial k_j}\right), \tag{23}$$

with $\operatorname{sgn}\dfrac{\partial k_{gi}}{\partial k_j} = \operatorname{sgn}\left(\dfrac{\partial^2 V_i}{\partial k_{gi}\partial k_j} + p_i\dfrac{\partial^2 h_i}{\partial k_{gi}\partial k_j}\right).$

For the private capital stock function,

$$\frac{\partial k_{pi}}{\partial p_i} = -\frac{\partial k_{gi}}{\partial p_i} = -\frac{1}{\Delta_i}\left[\frac{\partial^2 U_i}{\partial c_i^2}\left(\frac{\partial f_i}{\partial k_{pi}} - \frac{\partial f_i}{\partial k_{gi}} - \frac{\partial h_i}{\partial k_{gi}}\right) - \frac{\partial^2 U_i}{\partial k_{gi}\partial c_i}\right] > 0, \tag{24}$$

$$\frac{\partial k_{pi}}{\partial k_i} = 1 - \frac{\partial k_{gi}}{\partial k_i}$$

$$= \frac{1}{\Delta_i}\left[\frac{\partial^2 U_i}{\partial c_i^2}\left(\frac{\partial^2 U_i}{\partial k_{gi}^2} + \frac{\partial^2 V_i}{\partial k_{gi}^2} + p_i\left(\frac{\partial^2 f_i}{\partial k_{gi}^2} + \frac{\partial^2 h_i}{\partial k_{gi}^2} - \frac{\partial^2 f_i}{\partial k_{pi}\partial k_{gi}}\right)\right) - \left(\frac{\partial^2 U_i}{\partial c_i \partial k_{gi}}\right)^2\right]$$

$$> 0 \tag{25}$$

$$\frac{\partial k_{pi}}{\partial k_j} = -\frac{\partial k_{gi}}{\partial k_j} = \frac{1}{\Delta_i}\frac{\partial^2 U_i}{\partial c_i^2}\left(\frac{\partial^2 V_i}{\partial k_{gi}\partial k_j} + p_i \frac{\partial^2 h_i}{\partial k_{gi}\partial k_j}\right),$$

with $\operatorname{sgn} \frac{\partial k_{pi}}{\partial k_j} = \operatorname{sgn} -\left(\frac{\partial^2 V_i}{\partial k_{gi}\partial k_j} + p_i \frac{\partial^2 h_i}{\partial k_{gi}\partial k_j}\right)$.

Summarizing our results yields,

<u>Theorem 3.1.</u>

If the functions $c_i = c_i(k_i,k_j,p_i)$, $k_{gi} = k_{gi}(k_i,k_j,p_i)$, $k_{pi} = k_{pi}(k_i,k_j,p_i)$ are defined by the first order equations in (10) and the assumptions as defined by (13), (15) and (20) hold then $\frac{\partial c_i}{\partial p_i} < 0$; $\operatorname{sgn} \frac{\partial c_i}{\partial k_i} = \operatorname{sgn} \frac{\partial^2 U_i}{\partial c_i \partial k_{gi}}$; if $\frac{\partial^2 V_i}{\partial k_{gi}\partial k_j} + p_i \frac{\partial^2 h_i}{\partial k_{gi}\partial k_j} > 0$ then $\operatorname{sgn} \frac{\partial c_i}{\partial k_j} =$ $\operatorname{sgn} \frac{\partial^2 U_i}{\partial c_i \partial k_{gi}}$. If $\frac{\partial^2 V_i}{\partial k_{gi}\partial k_j} + p_i \frac{\partial^2 h_i}{\partial k_{gi}\partial k_j} < 0$ then $\operatorname{sgn} \frac{\partial c_i}{\partial k_j} = \operatorname{sgn} -\frac{\partial^2 U_i}{\partial c_i \partial k_{gi}}$:

$$\frac{\partial k_{gi}}{\partial p_i} < 0,\ \frac{\partial k_{gi}}{\partial k_i} > 0,\ \text{sgn}\ \frac{\partial k_{gi}}{\partial k_j} = \text{sgn}\ \left(\frac{\partial^2 v_i}{\partial k_{gi}\partial k_j} + p_i \frac{\partial^2 h_i}{\partial k_{gi}\partial k_j}\right):\ \frac{\partial k_{pi}}{\partial p_i} > 0,$$

$$\frac{\partial k_{pi}}{\partial k_i} > 0,\ \text{sgn}\ \frac{\partial k_{pi}}{\partial k_j} = \text{sgn} - \left(\frac{\partial^2 v_i}{\partial k_{gi}\partial k_j} + p_i \frac{\partial^2 h_i}{\partial k_{gi}\partial k_j}\right).$$

3.2 Intertemporal Equilibria

Let us return to the equations in (10) which define the rate of change of the state and costate variables. Since we are interested in the steady state solution substitute in the functions defined by theorem 3.1 and set $\dot{k}_i = \dot{p}_i = 0$. Define the equations by

$$\phi_{1i}(k_i,k_j,p_i) = f_i(k_{pi}(k_i,k_j,p_i),\ k_{gi}(k_i,k_j,p_i)) + h_i(k_{gi}(k_i,k_j,p_i),k_j) - c_i(k_i,k_j,p_i) - \lambda_i k_i \quad (27)$$

$$\phi_{2i}(k_i,k_j,p_i) = (\lambda_i - \frac{\partial f_i}{\partial k_{pi}})\ p_i.$$

It is obvious from equation system (27) that we may solve for p_i and k_i as functions of k_j. In so doing we need $\left.\frac{\partial p_i}{\partial k_i}\right|_{\phi_{1i}(k_i,k_j,p_i)=0}$ and $\left.\frac{\partial p_i}{\partial k_i}\right|_{\phi_{2i}(k_i,k_j,p_i)=0}$ with $k_j = \bar{k}_j$. First differentiate $\phi_{1i}(k_i,k_j,p_i)$ with respect to p_i,

$$\frac{\partial \phi_{1i}}{\partial p_i} = \frac{\partial f_i}{\partial k_{pi}}\frac{\partial k_{pi}}{\partial p_i} + \frac{\partial f_i}{\partial k_{gi}}\frac{\partial k_{gi}}{\partial p_i} + \frac{\partial h_i}{\partial k_{gi}}\frac{\partial k_{gi}}{\partial p_i} - \frac{\partial c_i}{\partial p_i} \quad (28)$$

$$= \left(\frac{\partial f_i}{\partial k_{gi}} + \frac{\partial h_i}{\partial k_{gi}} - \frac{\partial f_i}{\partial k_{pi}}\right) \frac{\partial k_{gi}}{\partial p_i} - \frac{\partial c_i}{\partial p_i} ,$$

from theorem 3.1 $\frac{\partial \phi_{1i}}{\partial p_i} > 0$. Continuing we have that

$$\frac{\partial \phi_{1i}}{\partial k_i} = \frac{\partial f_i}{\partial k_{pi}} \frac{\partial k_{pi}}{\partial k_i} + \frac{\partial f_i}{\partial k_{gi}} \frac{\partial k_{gi}}{\partial k_i} + \frac{\partial h_i}{\partial k_{gi}} \frac{\partial k_{gi}}{\partial k_i} - \frac{\partial c_i}{\partial k_i} - \lambda_i$$

(29)

$$= \frac{\partial f_i}{\partial k_{pi}} - \lambda_i + \left(\frac{\partial f_i}{\partial k_{gi}} + \frac{\partial h_i}{\partial k_{gi}} - \frac{\partial f_i}{\partial k_{pi}}\right) \frac{\partial k_{gi}}{\partial k_i} - \frac{\partial c_i}{\partial k_i} .$$

To determine the sign of (29) recall that if $0 < p \leq \infty$ in steady state, then $\frac{\partial f_i}{\partial k_{pi}} = \lambda_i$. Next from (17) and (22) substitute for $\frac{\partial k_{gi}}{\partial k_i}$, $\frac{\partial c_i}{\partial k_i}$, note equation $\frac{\partial L_i}{\partial k_{gi}} = 0$ and then (29) becomes,

$$\left.\frac{\partial \phi_{1i}}{\partial k_i}\right|_{\phi_{2i}(k_i,k_j,p_i)=0} = \frac{1}{\Delta_i} \left(\frac{\partial f_i}{\partial k_{gi}} + \frac{\partial h_i}{\partial k_{gi}} - \frac{\partial f_i}{\partial k_{pi}}\right) \left(\frac{\partial^2 U_i}{\partial c_i^2} p_i \left(\frac{\partial^2 f_i}{\partial k_{pi}^2} - \frac{\partial^2 f_i}{\partial k_{gi} \partial k_{pi}}\right)\right)$$

$$+ \frac{\partial^2 U_i}{\partial c_i \partial k_{gi}} \frac{p_i}{\Delta_i} \left(\frac{\partial^2 f_i}{\partial k_{pi}^2} - \frac{\partial^2 f_i}{\partial k_{gi} \partial k_{pi}}\right)$$

$$= \frac{p_i}{\Delta_i} \left(\frac{\partial^2 f_i}{\partial k_{gi} \partial k_{pi}} - \frac{\partial^2 f_i}{\partial k_{pi}^2}\right) \left(\frac{\partial V_i}{\partial k_{gi}} + \frac{\partial U_i}{\partial k_{gi}}\right)$$

$$\left[\frac{\frac{\partial^2 U_i}{\partial c_i^2}}{\frac{\partial U_i}{\partial c_i}} - \frac{\frac{\partial^2 U_i}{\partial c_i \partial k_{gi}}}{\frac{\partial U_i}{\partial k_{gi}} + \frac{\partial V_i}{\partial k_{gi}}}\right] < 0. \tag{30}$$

Hence $\left.\frac{\partial \phi_{1i}}{\partial k_i}\right|_{\phi_{2i}(k_i,k_j,p_i)=0} < 0$ and $\left.\frac{\partial p_i}{\partial k_i}\right|_{\substack{\phi_{1i}(k_i,\bar{k}_j,p_i)=0 \\ \phi_{2i}(k_i,\bar{k}_j,p_i)=0}} = -\frac{\partial \phi_{1i}}{\partial k_i} \Big/ \frac{\partial \phi_{1i}}{\partial p_i} > 0.$

We have the result that the shape of the $\phi_{1i}(k_i,\bar{k}_j,p_i) = 0$ locus is in general indeterminate. Nevertheless at any intersection with the $\phi_{2i}(k_i,\bar{k}_j,p_i)=0$ curve $\phi_{1i}(k_i,\bar{k}_j,p_i)=0$ must be positively sloped.

Proceeding to $\phi_{2i}(k_i,k_j,p_i)$ and differentiating with respect to p_i,

$$\frac{\partial \phi_{2i}}{\partial p_i} = \left[-\frac{\partial^2 f_i}{\partial k_{pi}^2}\frac{\partial k_{pi}}{\partial p_i} - \frac{\partial^2 f_i}{\partial k_{pi}\partial k_{gi}}\frac{\partial k_{pi}}{\partial p_i}\right] p_i$$

$$= \left(\frac{\partial^2 f_i}{\partial k_{pi}^2} - \frac{\partial^2 f_i}{\partial k_{pi}\partial k_{gi}}\right)\frac{\partial k_{gi}}{\partial p_i} p_i > 0, \tag{31}$$

from theorem 3.1. Continuing we get,

$$\frac{\partial \phi_{2i}}{\partial k_i} = \left[-\frac{\partial^2 f_i}{\partial k_{pi}^2}\frac{\partial k_{pi}}{\partial k_i} - \frac{\partial^2 f_i}{\partial k_{pi}\partial k_{gi}}\frac{\partial k_{gi}}{\partial k_i}\right] p_i$$

$$= \left[-\frac{\partial^2 f_i}{\partial k_{pi}^2} + \left[\frac{\partial^2 f_i}{\partial k_{pi}^2} - \frac{\partial^2 f_i}{\partial k_{pi}\partial k_{gi}}\right]\frac{\partial k_{gi}}{\partial k_i}\right] p_i$$

$$= [-\frac{\partial^2 f_i}{\partial k_{pi}^2}\frac{\Delta_i}{\Delta_i} + [\frac{\partial^2 f_i}{\partial k_{pi}^2} - \frac{\partial^2 f_i}{\partial k_{pi}\partial k_{gi}}] (+\frac{\partial^2 U_i}{\partial c_i^2}\frac{p_i}{\Delta_i} (\frac{\partial^2 f_i}{\partial k_{pi}^2} - \frac{\partial^2 f_i}{\partial k_{pi}\partial k_{gi}}))]p_i$$

$$= \frac{p_i}{\Delta_i} [-\frac{\partial^2 f_i}{\partial k_{pi}^2} (\frac{\partial^2 U_i}{\partial c_i^2} p_i (\frac{\partial^2 f_i}{\partial k_{gi}^2} + \frac{\partial^2 h_i}{\partial k_{gi}^2} + \frac{\partial^2 f_i}{\partial k_{pi}^2} - 2\frac{\partial^2 f_i}{\partial k_{pi}\partial k_{gi}})$$

$$- \frac{\partial^2 f_i}{\partial k_{pi}^2} [\frac{\partial^2 U_i}{\partial c_i^2} (\frac{\partial^2 U_i}{\partial k_{gi}^2} + \frac{\partial^2 V_i}{\partial k_{gi}^2}) - (\frac{\partial^2 U_i}{\partial k_{gi}\partial c_i})^2]$$

$$+ (\frac{\partial^2 f_i}{\partial k_{pi}^2} - \frac{\partial^2 f_i}{\partial k_{pi}\partial k_{gi}})^2 p_i \frac{\partial^2 U_i}{\partial c_i^2}]$$

$$= \frac{p_i}{\Delta_i} [\frac{\partial^2 U_i}{\partial c_i^2} p_i (\frac{\partial^2 f_i}{\partial k_{pi}^2} (\frac{\partial^2 f_i}{\partial k_{gi}^2} + \frac{\partial^2 h_i}{\partial k_{gi}^2}) - (\frac{\partial^2 f_i}{\partial k_{pi}\partial k_{gi}})^2)$$

$$- \frac{\partial^2 f_i}{\partial k_{pi}^2} (\frac{\partial^2 U_i}{\partial c_i^2} (\frac{\partial^2 U_i}{\partial k_{gi}^2} + \frac{\partial^2 V_i}{\partial k_{gi}^2}) - (\frac{\partial^2 U_i}{\partial k_{gi}\partial c_i})^2)] > 0, \qquad (32)$$

by the strict concavity of U_i, V_i, F_i and H_i. Therefore $\left.\frac{\partial p_i}{\partial k_i}\right|_{\phi_{2i}(k_i,\bar{k}_j,p_i)=0}$

$= -\frac{\partial \phi_{2i}}{\partial k_i} / \frac{\partial \phi_{2i}}{\partial p_i} < 0$, i.e., the $\phi_{2i}(k_i,\bar{k}_j,p_i)=0$ locus is negatively sloped.

To guarantee that the stationary solution exists for positive, finite values of c_i^* and k_i we assume that

$$\lim_{c\to 0} \frac{\partial U_i}{\partial c_i}(c_i,k_{gi}) \Big/ \frac{\partial U_i}{\partial k_{gi}}(c_i,k_{gi}) + \frac{\partial V_i}{\partial k_{gi}}(k_{gi},k_j) = \infty,$$

$$\lim_{k_{pi}\to 0} \frac{\partial f_i}{\partial k_{pi}}(k_{pi},k_{gi}) = \lim_{k_{gi}\to 0} \frac{\partial f_i}{\partial k_{gi}}(k_{pi},k_{gi}) + \frac{\partial h_i}{\partial k_{gi}}(k_{gi},k_j) = \infty \quad (33)$$

$$\lim_{k_{pi}\to \infty} \frac{\partial f_i}{\partial k_{pi}}(k_{pi},k_{gi}) = \lim_{k_{gi}\to \infty} \frac{\partial f_i}{\partial k_{gi}}(k_{pi},k_{gi}) + \frac{\partial h_i}{\partial k_{gi}}(k_{gi},k_j) = 0.$$ [4]

We also make the assumptions that

$$\bar{c}_i > f_i(k_{pi},k_{gi}) + h_i(k_{gi},k_j) - \lambda_i k_i, \text{ where}$$
$$\lim_{p_i\to 0} \sup c_i(k_i,k_j,p_i) = \bar{c}_i. \quad (34)$$

Equation set (34) states that at the level of maximum consumption i.e. where $\frac{\partial U_i}{\partial c_i} = 0$, net investment is less than consumption. In addition,

for any $\lambda_i > 0$ there exists an $\eta_i > 0$ such that

$$k_{pi} + k_{gi} > \eta_i \text{ implies } \frac{f_i(k_{pi},k_{gi}) + h_i(k_{gi},k_j)}{k_{pi} + k_{gi}} < \lambda_i. \quad (35)$$

The assumption given by (35) states that for a sufficiently large capital stock the output capital ratio can be made arbitrarily small irrespective of how the capital is allocated between the public and private sectors. We may now prove,

<u>Lemma 3.1.</u> If (33), (34) and (35) are satisfied then there is a unique stationary solution, $0 < p_i^* < \infty$, $0 < k_i^* < \infty$ for $k_j = \bar{k}_j$, to equation system (27). This solution is a saddle point.

Proof. Firstly from the $\phi_{2i}(k_i,\bar{k}_j,p_i) = 0$ equation we are able to solve for $k_i = \psi_{2i}(\bar{k}_j, p_i)$ and by (31) and (32) $\frac{\partial \psi_{2i}}{\partial p_i} < 0$. Define

$\lim\limits_{p_i \to \infty} \inf \psi_{2i}(\bar{k}_j,p_i) = \underline{k}_i$, i.e. as $p_i \to \infty$ then $k_i \to \underline{k}_i$. We want to show that $\underline{0} < \underline{k}_i < \infty$. By the definition of $\phi_{2i}(k_i,k_j,p_i)$, $\phi_{2i}(\underline{k}_i,\bar{k}_j,\bar{\infty}) = 0$ must satisfy, in particular, the first order conditions in (9). Thus as $p_i \to \infty$ $c_i \to 0$ and from (33) and (10), $\frac{\partial f_i}{\partial k_{pi}} = \frac{\partial f_i}{\partial k_{gi}} + \frac{\partial h_i}{\partial k_{gi}}$. In addition because $\phi_{2i}(\underline{k}_i, \bar{k}_j,\infty) = 0$ then by (27) $\frac{\partial f_i}{\partial k_{pi}} = \lambda_i = \frac{\partial f_i}{\partial k_{gi}} + \frac{\partial h_i}{\partial k_{gi}}$. But from (33) this is only true when $0 < k_{pi} < \infty$ and $0 < k_{gi} < \infty$. Consequently we have $0 < k_{pi}(\underline{k}_i,\bar{k}_j,) < \infty$, $0 < k_{gi}(\underline{k}_i,\bar{k}_j,\infty) < \infty$ and from (10) $0 < \underline{k}_i < \infty$.

We now want to show that $\phi_{1i}(k_i,k_j,p_i) > 0$ at $k_i = \underline{k}_i$, $k_j = \bar{k}_j$ and $p_i = \infty$. From (27) and (33)

$$\phi_{1i}(\underline{k}_i,\bar{k}_j,\infty) = f_i(k_{pi}(\underline{k}_i,\bar{k}_j,\infty), k_{gi}(\underline{k}_i,\bar{k}_j,\infty)) + h_i(k_{gi}(\underline{k}_i,\bar{k}_j,\infty),\bar{k}_j) - \lambda_i\underline{k}_i - c_i(\underline{k}_i,\bar{k}_j,\infty)$$

$$\phi_{1i}(\underline{k}_i,\bar{k}_j,\infty) = f_i(k_{pi}(\underline{k}_i,\bar{k}_j,\infty), k_{gi}(\underline{k}_i,\bar{k}_j,\infty)) + h_i(k_{gi}(\underline{k}_i,\bar{k}_j,\infty),\bar{k}_j) - \lambda_i\underline{k}_i.$$

Now $\phi_{1i}(\underline{k}_i,\bar{k}_j,\infty)$ is a strictly concave function, due to the assumptions imposed on the production functions, and is nonegative at $k_i = 0$. Moreover at $k_i = 0$

$$\frac{\partial \phi_{1i}(0,\bar{k}_j,\infty)}{\partial k_i} = \frac{\partial f_i(k_{pi}(0,\bar{k}_j,\infty), k_{gi}(0,\bar{k}_j,\infty))}{\partial k_{pi}} = \infty,$$

and at $k_i = \underline{k}_i$

$$\frac{\partial \phi_{1i}(\underline{k}_i,\bar{k}_j,\infty)}{\partial k_i} = \frac{\partial f_i(kp_i(\underline{k}_i,\bar{k}_j,\infty), k_{gi}(\underline{k}_i,\bar{k}_j,\infty)}{\partial k_{pi}} - \lambda_i = 0.$$

Therefore at $\underline{k}_i$, $\phi_{1i}(\underline{k}_i,\overline{k}_j,\infty) > 0$ and so $p_i = \infty$, $k_i = \underline{k}_i$ is not a steady state.

Next we want to show that at $p_i = 0$, $k_i = \underline{k}_i$, $\phi_{1i}(\underline{k}_i,\overline{k}_j,0) < 0$. When $p_i = 0$ by (34) $c_i = \overline{c}_i$ and trivially $\phi_{1i}(\underline{k}_i,\overline{k}_j,0) < 0$. Therefore there exists a positive finite p_i for which $\phi_{1i}(\underline{k}_i,\overline{k}_j,p_i) = 0$ and the p_i is unique by (28). Since $\phi_{2i}(\underline{k}_i,\overline{k}_j,\infty) = 0$, it follows that $\phi_{1i}(k_i,\overline{k}_j,p_i)$ lies below the locus $\phi_{2i}(k_i,\overline{k}_j,p_i) = 0$ for k_i in a right hand neighbourhood of $\underline{k}_i$.

Finally, for existence it is now sufficient to show that for large values of k_i, $\phi_{1i}(k_i,\overline{k}_j,p_i) = 0$ lies above $\phi_{2i}(k_i,\overline{k}_j,p_i) = 0$. We know by (34) that $\phi_{1i}(k_i,\overline{k}_j,0) < 0$ for all k_i and now we want $\phi_{1i}(k_i,\overline{k}_j,p_i) < 0$ for all p_i and k_i sufficiently large. By (35) we have

$$\phi_{1i}(k_i,\overline{k}_j,p_i) = f_i(k_{pi}(k_i,\overline{k}_j,p_i),\ k_{gi}(k_i,\overline{k}_j,p_i)) + h_i(k_{gi}(k_i,\overline{k}_j,p_i),\overline{k}_j) - \lambda_i k_i - c_i(k_i,\overline{k}_j,p_i)$$

$$\phi_{1i}(k_i,\overline{k}_j,p_i) \leq k_i\left[\left[\frac{f_i(k_{pi}(k_i,\overline{k}_j,p_i),k_{gi}(k_i,\overline{k}_j,p_i)) + h_i(k_{gi}(k_i,\overline{k}_j,p_i),\overline{k}_j)}{k_{pi}(k_i,\overline{k}_j,p_i) + k_{gi}(k_i,\overline{k}_j,p_i)}\right] - \lambda_i\right] < 0,$$

for $k_i = k_{pi}(k_i,\overline{k}_j,p_i) + k_{gi}(k_i,\overline{k}_j,p_i)$ sufficiently large. Thus p_i approaches infinity along $\phi_{1i}(k_i,\overline{k}_j,p_i) = 0$ for sufficiently large k_i. Hence with the fact that along $\phi_{2i}(k_i,\overline{k}_j,p_i) = 0$ p_i approaches zero for sufficiently large k_i we have proved that at least one stationary solution, $0 < p_i^* < \infty$, $0 < k_i^* < \infty$, exists.

Furthermore because of (30) at (k_i^*, p_i^*), $\phi_{1i}(k_i,\overline{k}_j,p_i) = 0$ is upward sloping and since $\phi_{2i}(k_i,\overline{k}_j,p_i)$ is always negatively sloped there cannot be any stationary solutions for $k_i > k_i^*$. On the other hand at $\underline{k}_i$, $\phi_{1i}(\underline{k}_i,\overline{k}_j,p_i) = 0$ lies below $\phi_{2i}(\underline{k}_i,\overline{k}_j,p_i)=0$ and along $\phi_{1i}(k_i,\overline{k}_j,p_i) = 0$

there is, by (28), a unique p_i for any value of k_i. Consequently as k_i increases from $\underline{k}_i$ once $\phi_{1i}(k_i,\bar{k}_j,p_i) = 0 = \phi_{2i}(k_i,\bar{k}_j,p_i)$ the loci can never intersect again.

Lastly, from (28) and (31) the steady state is a saddle point. Q.E.D.

Since (k_i^*, p_i^*) is a saddle point there is a function $p_i = p_i(k_i, \bar{k}_j)$ satisfying equation set (9) passing through (k_i^*, p_i^*). This function is the optimal trajectory and satisfies the differential equation

$$\left.\frac{\partial p_i}{\partial k_i}\right|_{k_j = \bar{k}_j} = (\lambda_i - \frac{\partial f_i}{\partial k_{pi}})\, p_i \,/\, f_i(k_{pi},k_{gi}) + h_i(k_{gi},\bar{k}_j) - \lambda_i k_i - c_i.$$

In figure 1 we have depicted the intertemporal equilibrium for $k_j = \bar{k}_j$.

Clearly we are closer to our goal of finding the world steady state. The following step is to differentiate (27) with respect to k_j and solve for $\frac{dp_i}{dk_j}$ and $\frac{dk_i}{dk_j}$ where $\phi_{1i}(k_i,k_j,p_i) = \phi_{2i}(k_i,k_j,p_i) = 0$.

These calculations yield,

$$\begin{bmatrix} (\frac{\partial f_i}{\partial k_{gi}} + \frac{\partial h_i}{\partial k_{gi}} - \frac{\partial f_i}{\partial k_{pi}}) \frac{\partial k_{gi}}{\partial p_i} - \frac{\partial c_i}{\partial p_i} & (\frac{\partial f_i}{\partial k_{gi}} + \frac{\partial h_i}{\partial k_{gi}} - \frac{\partial f_i}{\partial k_{pi}}) \frac{\partial k_{gi}}{\partial k_i} - \frac{\partial c_i}{\partial k_i} \\ (\frac{\partial^2 f_i}{\partial k_{pi}^2} - \frac{\partial^2 f_i}{\partial k_{pi}\partial k_{gi}}) \frac{\partial k_{gi}}{\partial p_i} p_i & p_i[- \frac{\partial^2 f_i}{\partial k_{pi}^2} + (\frac{\partial^2 f_i}{\partial k_{pi}^2} - \frac{\partial^2 f_i}{\partial k_{pi}\partial k_{gi}}) \frac{\partial k_{gi}}{\partial k_i}] \end{bmatrix}$$

$$\begin{bmatrix} \frac{dp_i}{dk_j} \\ \frac{dk_i}{dk_j} \end{bmatrix} = \begin{bmatrix} - \frac{\partial h_i}{\partial k_j} - (\frac{\partial f_i}{\partial k_{gi}} + \frac{\partial h_i}{\partial k_{gi}} - \frac{\partial f_i}{\partial k_{pi}}) \frac{\partial k_{gi}}{\partial k_j} + \frac{\partial c_i}{\partial k_j} \\ - (\frac{\partial^2 f_i}{\partial k_{pi}^2} - \frac{\partial^2 f_i}{\partial k_{pi}\partial k_{gi}}) \frac{\partial k_{gi}}{\partial k_j} p_i \end{bmatrix}. \quad (36)$$

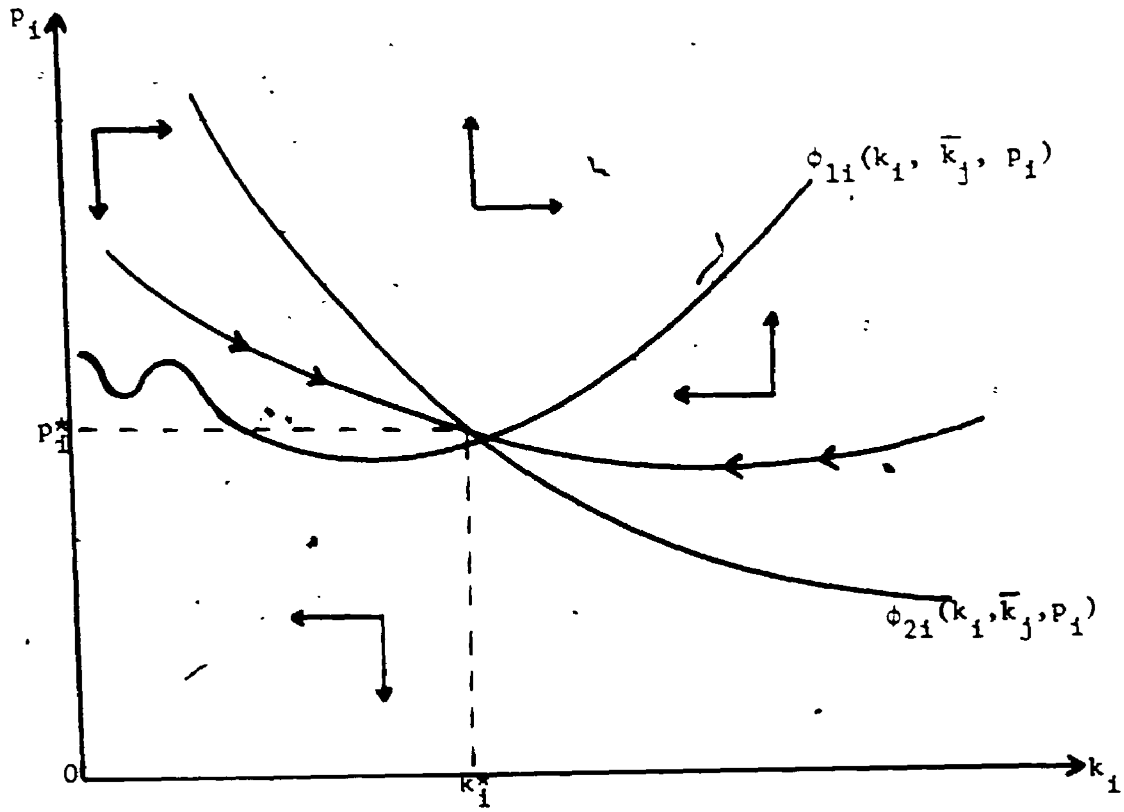

Figure 1. Steady state for $k_j = \hat{k}_j$ in country i

The determinant of the matrix in (36) is positive by (28),(30), (31) and (32). In order to unambiguously determine the sign of the vector on the right side of the equality sign of (36) we posit that

$$\text{if } \frac{\partial^2 v_i}{\partial k_{gi}\partial k_j} + p_i \frac{\partial^2 h_i}{\partial k_{gi}\partial k_j} > 0 \text{ then } -\frac{\partial k_{gi}}{\partial p_i}^2 \left(\frac{\partial^2 v_i}{\partial k_{gi}\partial k_j} + p_i \frac{\partial^2 h_i}{\partial k_{gi}\partial k_j}\right) \leq \frac{\partial k_{gi}}{\partial p_i}\frac{\partial h_i}{\partial k_j} < \left(\frac{\partial^2 v_i}{\partial k_{gi}\partial k_j} + p_i \frac{\partial^2 h_i}{\partial k_{gi}\partial k_j}\right),$$

$$\text{if } \frac{\partial^2 v_i}{\partial k_{gi}\partial k_j} + p_i \frac{\partial^2 h_i}{\partial k_{gi}\partial k_j} < 0 \text{ then } -\frac{\partial k_{gi}}{\partial p_i}^2 \left(\frac{\partial^2 v_i}{\partial k_{gi}\partial k_j} + p_i \frac{\partial^2 h_i}{\partial k_{gi}\partial k_j}\right) \geq \frac{\partial k_{gi}}{\partial p_i}\frac{\partial h_i}{\partial k_j} > \left(\frac{\partial^2 v_i}{\partial k_{gi}\partial k_j} + p_i \frac{\partial^2 h_i}{\partial k_{gi}\partial k_j}\right). \tag{37}$$

Condition (37) places bounds on the magnitude of the production externality. The assumption states that the production externality must not exceed the absolute value of the effect of an increase of foreign capital on the marginal benefits of public capital weighted by the negative of the effect of a change in the price of investment on public capital. In addition the production externality weighted by $\frac{\partial k_{gi}}{\partial p_i}$ must not exceed the absolute value of the effect of changes in foreign capital on the marginal benefits of public capital. Recalling that $\frac{\partial k_{gi}}{\partial p_i} < 0$ implies that the production externality may still be an economy $\left(\frac{\partial h_i}{\partial k_j} > 0\right)$ or a diseconomy $\left(\frac{\partial h_i}{\partial k_j} < 0\right)$ and clearly if there is no production externality then (37) is satisfied. Furthermore as we shall observe later, restrictions on the magnitude of the production externality play a crucial part in finding sufficient conditions for the existence and uniqueness of a world steady state.

Upon investigating the sign of the first element of the vector on the right side of the equality sign in (36), we find after substituting in (18), (23) and utilizing (21) that

$$-\frac{\partial h_i}{\partial k_j} - \left(\frac{\partial f_i}{\partial k_{gi}} + \frac{\partial h_i}{\partial k_{gi}} - \frac{\partial f_i}{\partial k_{pi}}\right)\frac{\partial k_{gi}}{\partial k_j} + \frac{\partial c_i}{\partial k_j}$$

$$= -\frac{\partial h_i}{\partial k_j} - \frac{\partial k_{gi}}{\partial p_i}\left(\frac{\partial^2 v_i}{\partial k_{gi}\partial k_j} + p_i\frac{\partial^2 h_i}{\partial k_{gi}\partial k_j}\right). \qquad (38)$$

The sign of which by (37) depends directly on the sign of $\frac{\partial^2 v_i}{\partial k_{gi}\partial k_j} + p_i\frac{\partial^2 h_i}{\partial k_{gi}\partial k_j}$. For the second element in the vector by (13) and (23) if $\frac{\partial^2 v_i}{\partial k_{gi}\partial k_j} + p_i\frac{\partial^2 h_i}{\partial k_{gi}\partial k_j} \gtrless 0$ then the term is positive or negative. Manifestly from (36)

$$\text{sgn}\frac{dp_i}{dk_j} = \text{sgn}\left[\frac{\partial^2 v_i}{\partial k_{gi}\partial k_j} + p_i\frac{\partial^2 h_i}{\partial k_{gi}\partial k_j}\right]. \qquad (39)$$

The sign of $\frac{dk_i}{dk_j}$ is not quite so obvious because the elements of the matrix whose determinant comprises the numerator are all positive. Solving from (36) for this determinant and substituting in (14), (18), (21) and (23) we find

$$p_i\left(\frac{\partial^2 f_i}{\partial k_{pi}^2} - \frac{\partial^2 f_i}{\partial k_{pi}\partial k_{gi}}\right)\frac{\partial k_{gi}}{\partial k_j}\frac{\partial c_i}{\partial p_i} + \frac{\partial h_i}{\partial k_j}p_i\frac{\partial k_{gi}}{\partial p_i}\left(\frac{\partial^2 f_i}{\partial k_{pi}^2} - \frac{\partial^2 f_i}{\partial k_{gi}\partial k_{pi}}\right)$$

$$-p_i\left(\frac{\partial^2 f_i}{\partial k_{pi}^2} - \frac{\partial^2 f_i}{\partial k_{pi}\partial k_{gi}}\right)\frac{\partial k_{gi}}{\partial p_i}\frac{\partial c_i}{\partial k_j} = p_i\left(\frac{\partial^2 f_i}{\partial k_{pi}^2} - \frac{\partial^2 f_i}{\partial k_{pi}\partial k_{gi}}\right) \tag{40}$$

$$\left[\frac{\partial h_i}{\partial k_j}\frac{\partial k_{gi}}{\partial p_i} - \left(\frac{\partial^2 V_i}{\partial k_{gi}\partial k_j} + p_i\frac{\partial^2 h_i}{\partial k_{gi}\partial k_j}\right)\right].$$

Equation (40) represents the numerator and given (37) the sign is positive or negative according to whether $\frac{\partial^2 V_i}{\partial k_{gi}\partial k_j} + p_i \frac{\partial^2 h_i}{\partial k_{gi}\partial k_j} \gtrless 0$. Hence we may state

$$\operatorname{sgn}\frac{dk_i}{dk_j} = \operatorname{sgn}\left(\frac{\partial^2 V_i}{\partial k_{gi}\partial k_j} + p_i\frac{\partial^2 h_i}{\partial k_{gi}\partial k_j}\right). \tag{41}$$

With the information collected in (39) and (41) we can define the functions,

$$\left.\begin{aligned} & k_i = S_i(k_j), \\ & k_j \geq 0,\ S_i \in C^2,\ \frac{dS_i}{dk_j} \text{ given by (41) and } k_i \in (0,\infty); \\ & p_i = P_i(k_j), \\ & k_j \geq 0,\ P_i \in C^2,\ \frac{dP_i}{dk_j} \text{ given by (39) and } p_i \in (0,\infty). \end{aligned}\right\} \tag{42}$$

At this juncture let us define an international steady state when $S_i(k_j) = S_j^{-1}(k_j)$ with S_i called the Stackleberg steady state reaction for country i. Yet there is no indication that one exists, for example if $\frac{dS_i}{dk_j}$ and $\frac{dS_j}{dk_i}$ are greater than one. This fact portends that a sufficient condition for a unique equilibrium is that the previous derivatives are less than one in absolute value.

Theorem 3.2. If equation set (37) is satisfied and $\frac{\partial^2 f_i}{\partial k_{pi}^2}(\frac{\partial^2 f_i}{\partial k_{gi}^2} + \frac{\partial^2 h_i}{\partial k_{gi}^2})$

$- (\frac{\partial^2 f_i}{\partial k_{gi}\partial k_{pi}})^2 > |(\frac{\partial^2 v_i}{\partial k_{gi}\partial k_j} + p_i \frac{\partial^2 h_i}{\partial k_{gi}\partial k_j})|(\frac{\partial^2 f_i}{\partial k_{pi}\partial k_{gi}} - \frac{\partial^2 f_i}{\partial k_{pi}^2})$ then there exists a unique international steady state.

Proof: Firstly this new condition states in economic terms that the determinant of the hessian of the functions f_i and h_i must not only be positive (which it is by the strict concavity of F_i and H_i) but must be sufficiently positive so that it is greater than the difference between the cross partial of public and private capital and the second partial of private capital in production, weighted by the absolute value of the effect of increases of foreign capital on the marginal benefits of public capital.

Now we want to prove that $|\frac{dS_i}{dk_j}| < 1$. It is sufficient to show that when $\frac{\partial^2 v_i}{\partial k_{gi}\partial k_j} + p_i \frac{\partial^2 h_i}{\partial k_{gi}\partial k_j} > 0$ the last term in the matrix in (36) is greater than the second element of the vector on the right side of the equality sign of (36). Hence using (12), (22) and (23) we find

$$- \frac{\partial^2 f_i}{\partial k_{pi}^2} \frac{\Delta_i}{\Delta_i} p_i + \frac{p_i}{\Delta_i} (\frac{\partial^2 f_i}{\partial k_{pi}^2} - \frac{\partial^2 f_i}{\partial k_{gi}\partial k_{pi}}) \tag{43}$$

$$[\frac{\partial^2 U_i}{\partial c_i^2} p_i (\frac{\partial^2 f_i}{\partial k_{pi}^2} - \frac{\partial^2 f_i}{\partial k_{gi}\partial k_{pi}}) - \frac{\partial^2 U_i}{\partial c_i^2} p_i (\frac{\partial^2 v_i}{\partial k_{gi}\partial k_j} + p_i \frac{\partial^2 h_i}{\partial k_{gi}\partial k_j})].$$

Collecting terms yields

$$- \frac{p_i}{\Delta_i} \frac{\partial^2 f_i}{\partial k_{pi}^2} [\frac{\partial^2 U_i}{\partial c_i^2} (\frac{\partial^2 U_i}{\partial k_{gi}^2} + \frac{\partial^2 v_i}{\partial k_{gi}^2}) - (\frac{\partial^2 U_i}{\partial k_{gi}\partial c_i})^2]$$

$$- \frac{p_i}{\Delta_i} \frac{\partial^2 U_i}{\partial c_i^2} (\frac{\partial^2 f_i}{\partial k_{pi}^2} (\frac{\partial^2 f_i}{\partial k_{gi}^2} + \frac{\partial^2 h_i}{\partial k_{gi}^2}) - (\frac{\partial^2 f_i}{\partial k_{pi}\partial k_{gi}})^2 + (\frac{\partial^2 f_i}{\partial k_{pi}^2} - \frac{\partial^2 f_i}{\partial k_{gi}\partial k_{pi}})$$

$$(\frac{\partial^2 v_i}{\partial k_{gi}\partial k_j} + p_i \frac{\partial^2 h_i}{\partial k_{gi}\partial k_j}) > 0$$

by the strict concavity of U_i and V_i and the condition in the statement of the theorem. Consequently

$\left|\frac{dS_i}{dk_j}\right| < 1$ for $0 < k_i < \infty$ and $\left|\frac{dS_j}{dk_i}\right| < 1$ for $0 < k_j < \infty$. Next define the set

$$A = \{k_i, k_j : k_i \in (0,\infty),\ k_j \in (0,\infty)\}.$$

The function

$$\begin{bmatrix} k_i \\ k_j \end{bmatrix} = \begin{bmatrix} S_i(k_j) \\ S_j(k_i) \end{bmatrix} \quad (45)$$

is a contraction mapping for the Euclidian norm in A. Let

$$B = \{k_i, k_j : \ S_i(k_j) - S_j^{-1}(k_j) \geq 0,\ k_i \geq S_j^{-1}(k_j)\}$$

Clearly, since the absolute value of the slopes of the Stackleberg reaction functions for country i and j is less than one over the entire domain which is the positive real numbers, then $A \cap B \neq \phi$. Let $(\hat{k}_i, \hat{k}_j)$ be an element in $A \cap B$ such that $\hat{k}_i = S_j^{-1}(\hat{k}_j)$. $S_i(k_j)$ and $S_j(k_j)$ are monotonic so $S_j(k_i) > \hat{k}_j$ for all $k_i > \hat{k}_i$ and $S(k_j) > \hat{k}_i$ for all $k_j > \hat{k}_j$. Hence define

$$C = \{k_i, k_j : k_i \geq \hat{k}_i, k_j \geq \hat{k}_j\}.$$

Therefore S: C —> C implies that S is a contraction mapping in C which is a metric space for the Euclidean norm. Consequently by the Borel-Cantelli Fixed Point Theorem, there exists a unique point in C. Q.E.D.

To illustrate the theorem somewhat more intuitively suppose that $\frac{\partial^2 V_i}{\partial k_{gi} \partial k_j} + p_i \frac{\partial^2 h_i}{\partial k_{gi} \partial k_j} < 0$, $i, j = 1, 2$ $i \neq j$. In country j from (41), as k_i increases k_j decreases. However, for $k_i \geq 0$ $k_j \in (0,\infty)$, by (42) therefore we have

$\lim_{k_i \to \infty}$ inf $S_j(k_i) = \underline{k}_j$ where $0 < k_j < \infty$ and $S_j^{-1}(\underline{k}_j) = \infty$. For country i we have $\lim_{k_j \to 0} \sup S_i(k_j) = \bar{k}_i$ where $0 < \bar{k}_i < \infty$ and consequently at $\underline{k}_j$ $S_i(\underline{k}_j) < S_j^{-1}(\underline{k}_j)$. Similarly for $k_i = 0$ we have $\lim_{k_i \to 0} \sup S_j(k_i) = \bar{k}_j$ $\lim_{k_j \to \infty} \inf S_i(k_j) = \underline{k}_i$, $0 < \bar{k}_j < \infty$, $0 < \underline{k}_i < \infty$ so $0 = S_j^{-1}(\bar{k}_j) < S_i(\bar{k}_j)$. Hence $S_i(k_j)$ and $S_j^{-1}(k_j)$ must intersect and since in this example $-1 < \frac{dS_i}{dk_j} < 0$ and $\frac{dS_j^{-1}}{dk_j} < -1$, the intersection occurs only once. Geometrical illustrations of theorem 3.2 are depicted in figures 2-5.

Proceeding to the stability analysis we are able to demonstrate

Proposition 3.1. The steady state given by theorem 3.2 is globally stable.

Firstly for the case $0 < \frac{dk_i}{dk_j} < 1$ for $i, j = 1, 2, i \neq j$, observe figure 2. At point A $\tilde{k}_i < \hat{k}_i$ and from (27), (30), (32) and noting that at $\phi_{2i}(k_i,k_j,p_i) = 0$ we have p_i as a function of k_i and k_j then

$$\left.\frac{\partial\phi_{1i}}{\partial k_i}\right|_{\phi_{2i}(k_i,k_j,p_i)=0} = \left(\frac{\partial f_i}{\partial k_{gi}} + \frac{\partial h_i}{\partial k_{gi}} - \frac{\partial f_i}{\partial k_{pi}}\right)\left(\frac{\partial k_{gi}}{\partial p_i}\frac{\partial p_i}{\partial k_i} + \frac{\partial k_{gi}}{\partial k_i}\right) - \frac{\partial c_i}{\partial k_i} < 0$$

Hence at A $\dot{k}_i > 0$. In country j at A $\tilde{k}_j > \hat{k}_j$ and with $\left.\frac{\partial\phi_{1j}}{\partial k_j}\right|_{\phi_{2j}(k_j,k_i,p_j)=0} < 0$ then $\dot{k}_j < 0$. The analysis is identical for the three other cases and we see in figures 3-5 that the steady state is globally stable. Finally, figures 6-9 depict the various optimal paths for the four patterns that may arise.

4. Stackleberg Leader Case

The behavior in this section is characterized by the hypothesis that country i assumes country j reacts according to equation (42). In other words country i expects country j to be a follower. The Lagrangian for country i becomes

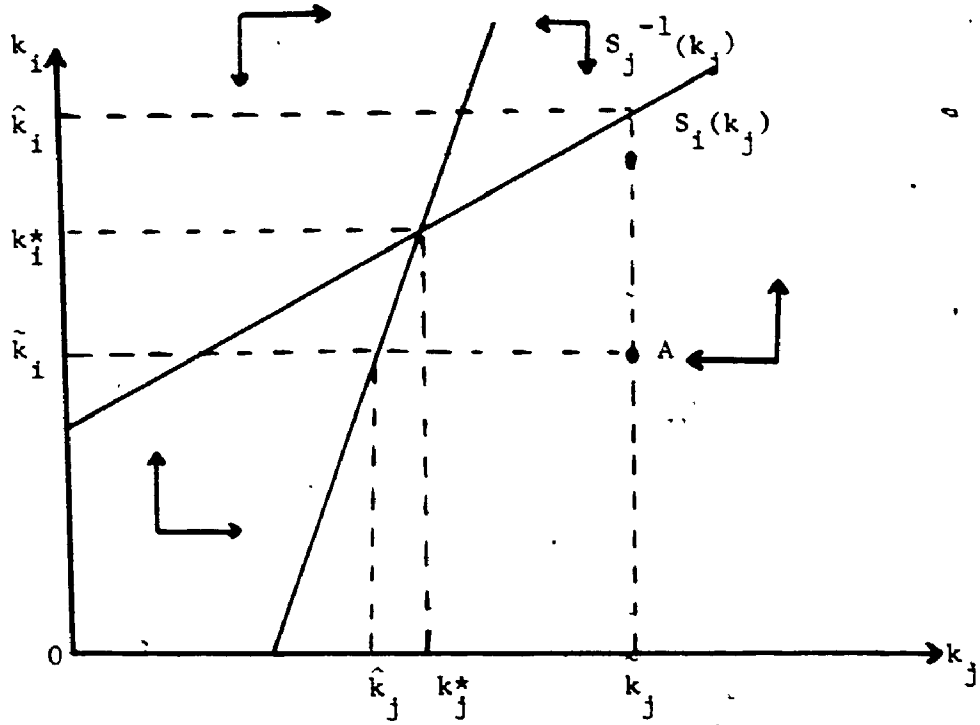

Figure 2. $0 < \frac{dS_i}{dk_j} < 1, \frac{dS_j^{-1}}{dk_j} > 1$

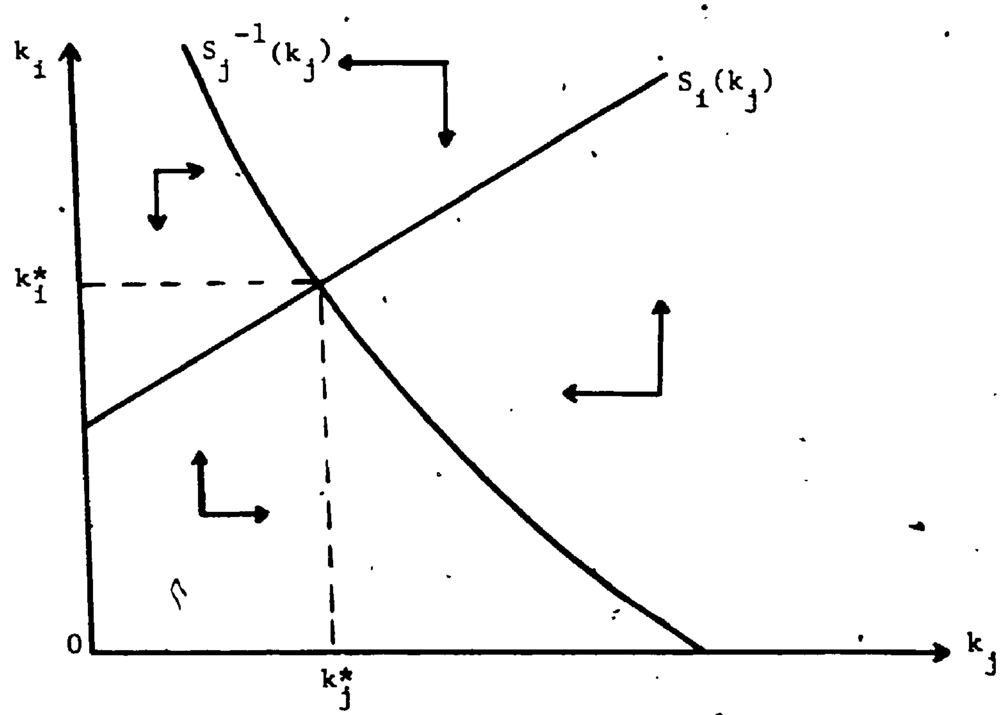

Figure 3. $0 < \frac{dS_i}{dk_j} < 1, \frac{dS_j^{-1}}{dk_j} < -1$

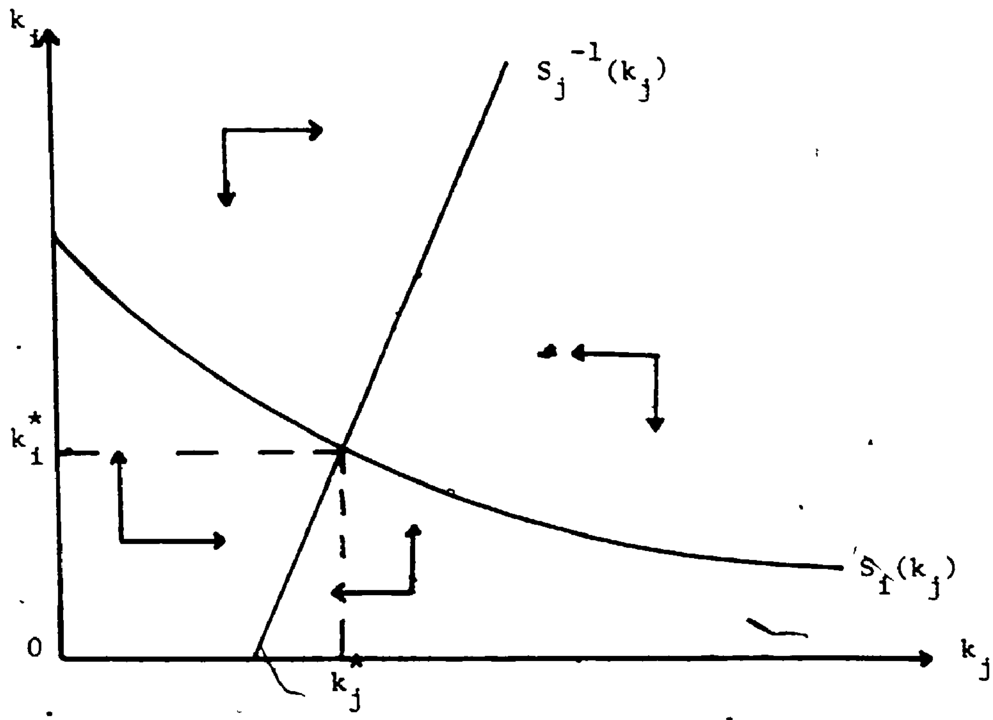

Figure 4. $-1 < \frac{dS_i}{dk_j} < 0, \frac{dS_j^{-1}}{dl_k} > 1$

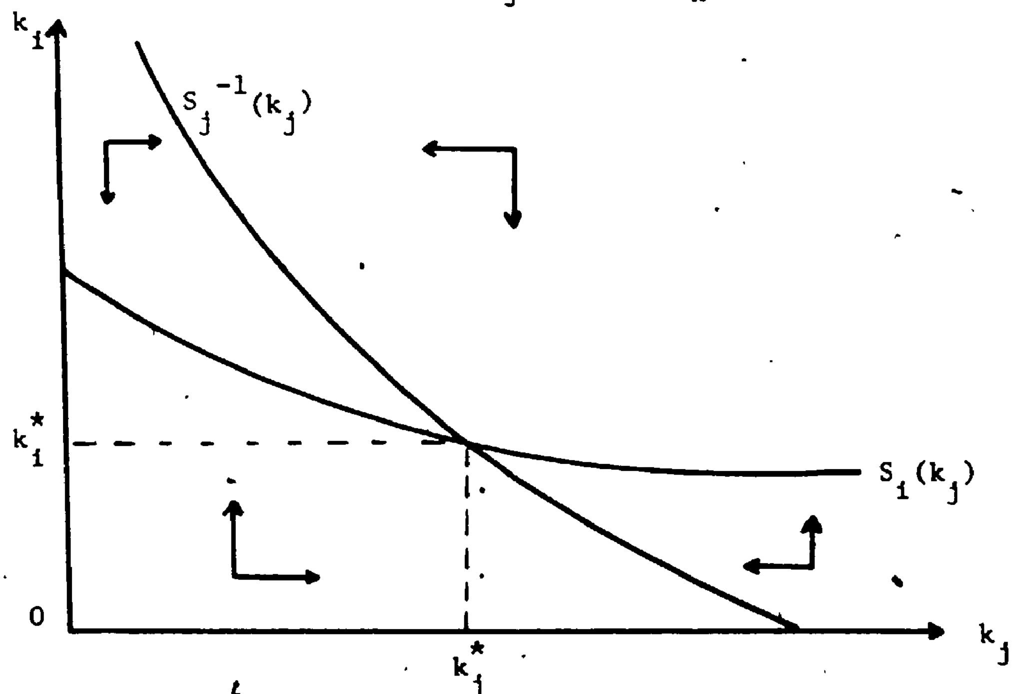

Figure 5. $-1 < \frac{dS_i}{dk_j} < 0, \frac{dS_j^{-1}}{dk_j} < 1$

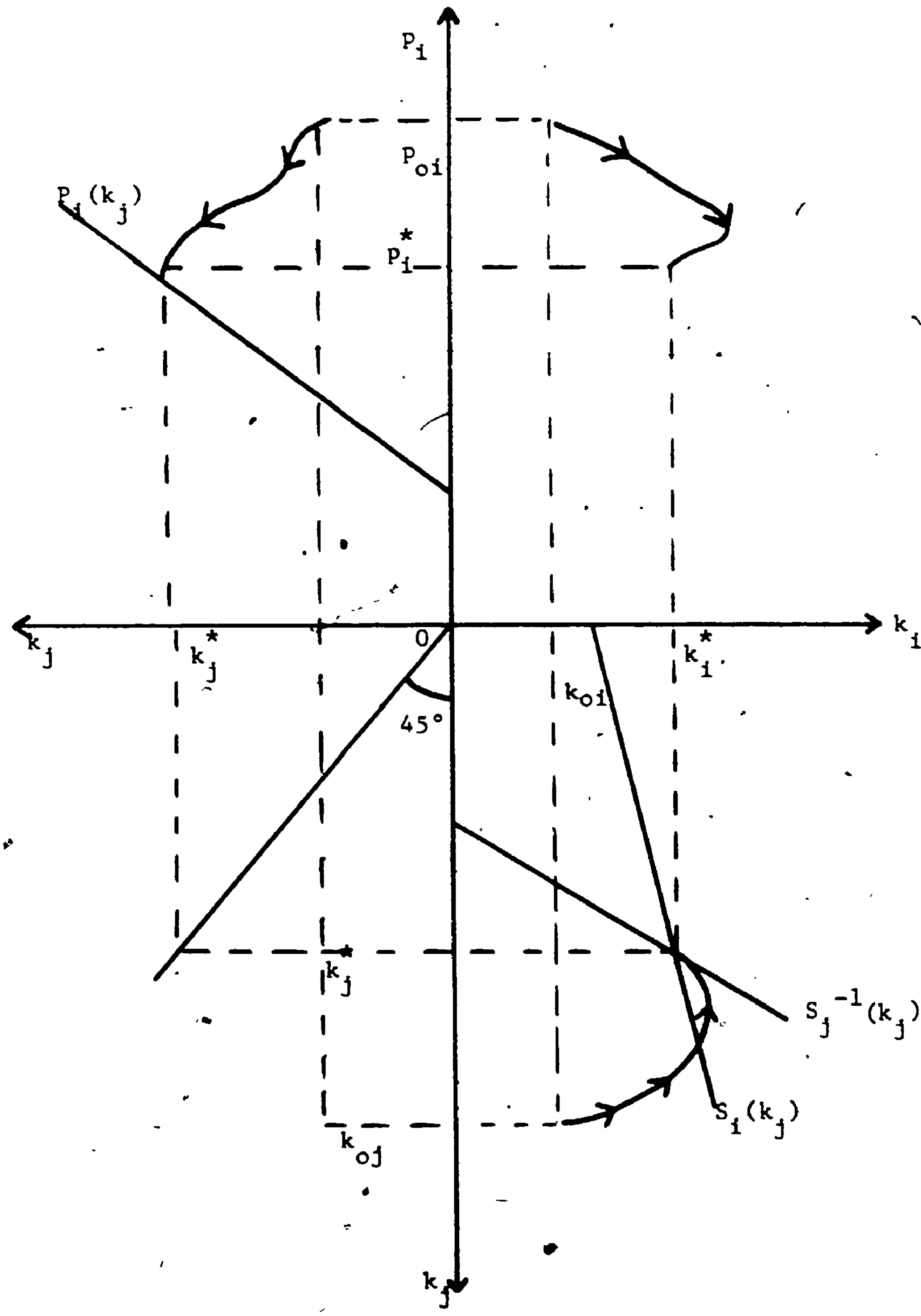

Figure 6. International steady state with

$$0 < \frac{dS_i}{dk_j} < 1, \frac{dS_i^{-1}}{dk_j} > 1$$

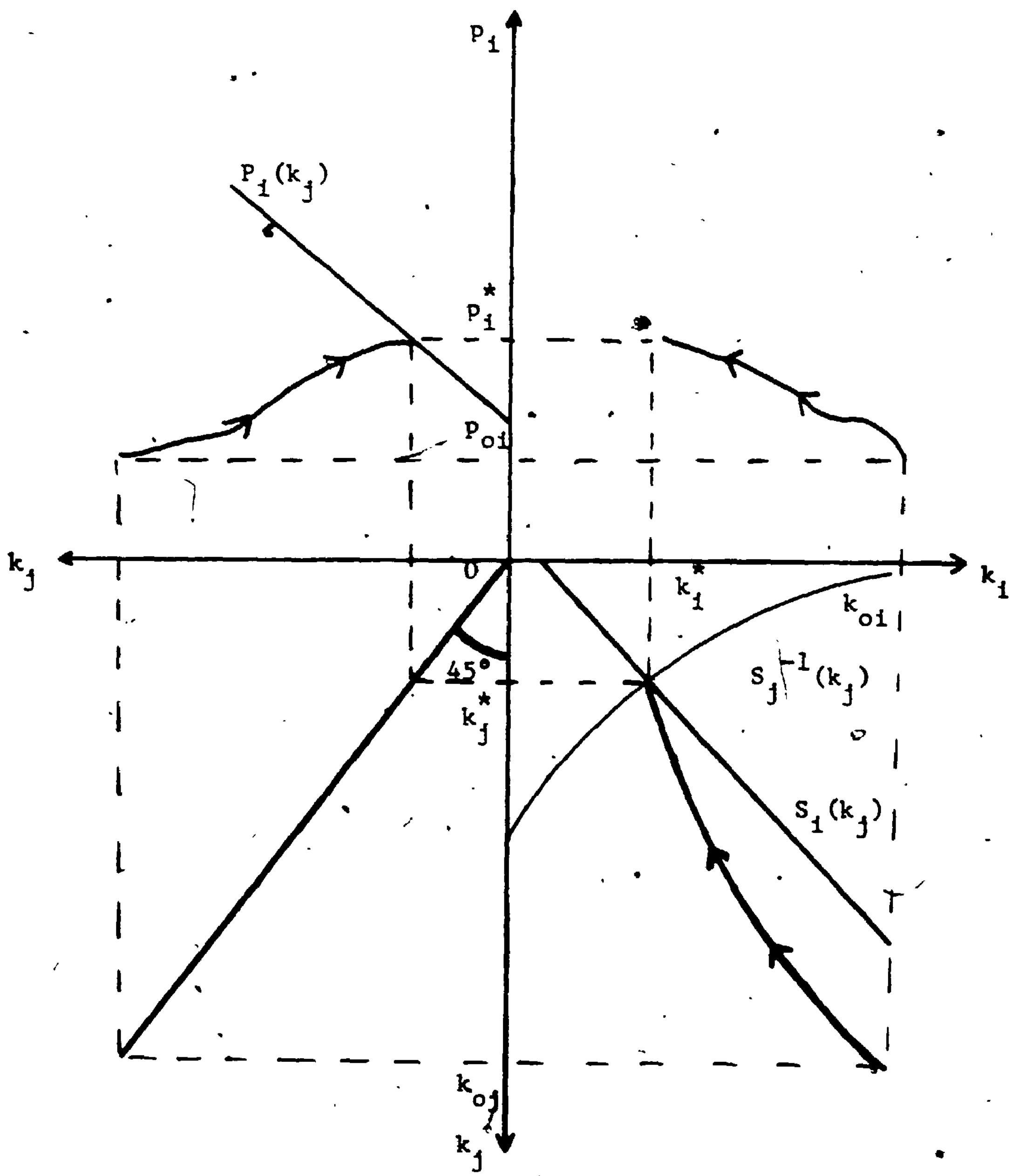

Figure 7. International Steady State with

$$0 < \frac{dS_i}{dk_j} < 1, \ \frac{dS_i^{-1}}{dk_j} < -1$$

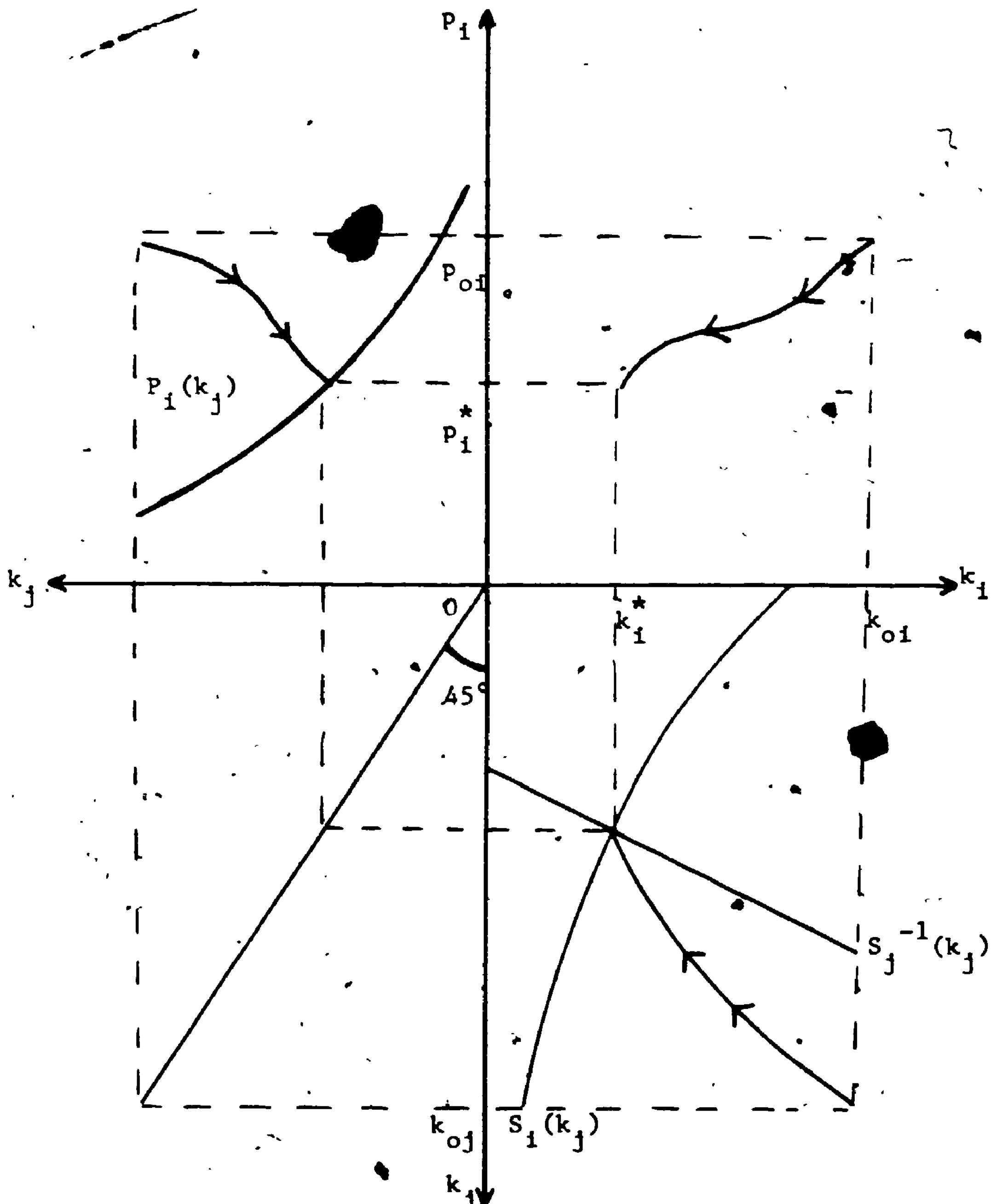

Figure 8. International Steady State with

$$-1 < \frac{dS_i}{dk_j} < 0,\ \frac{dS_i^{-1}}{dk_j} > 1$$

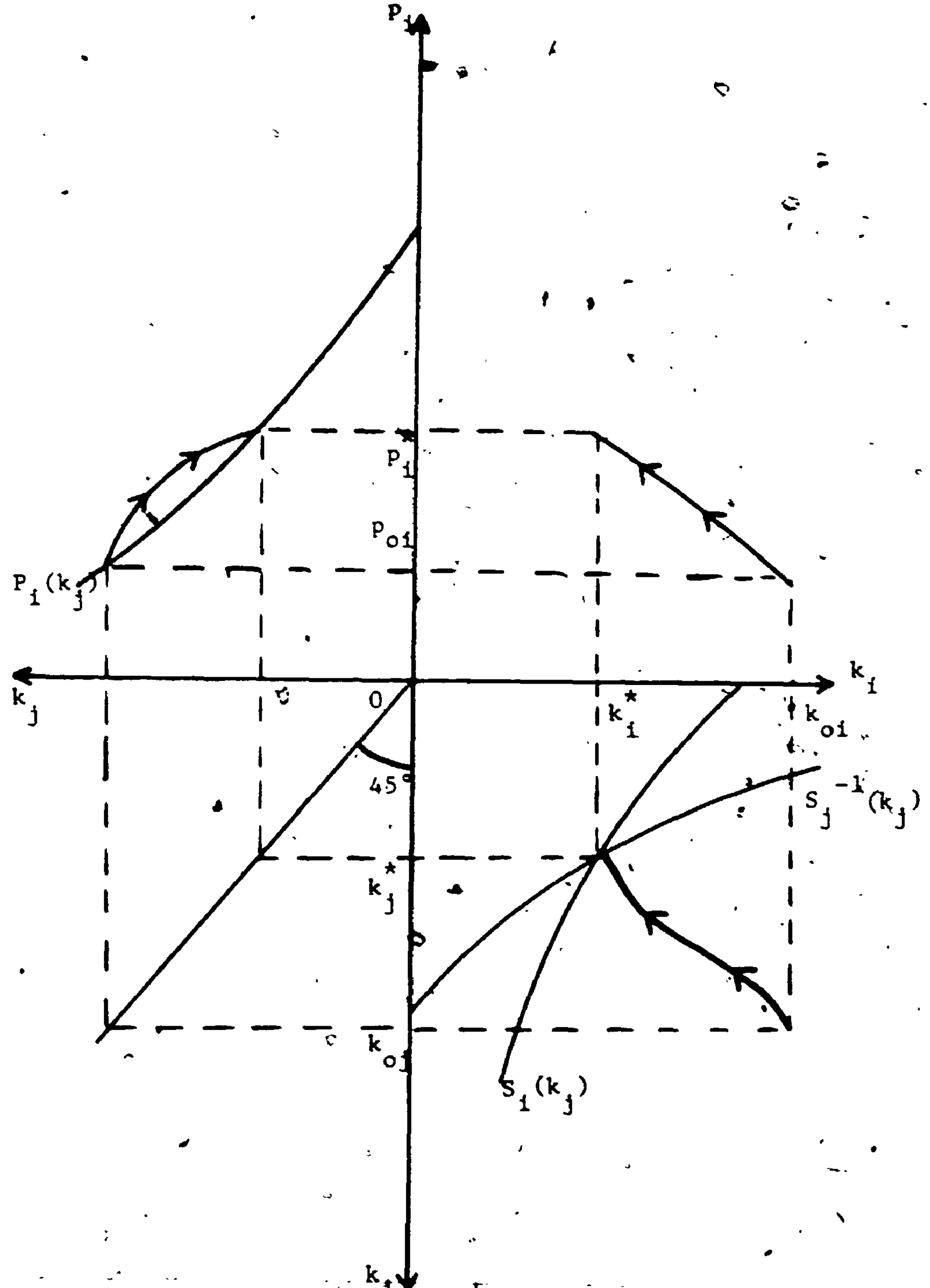

Figure 9. International steady state with

$$-1 < \frac{dS_i}{dk_j} < 0, \quad \frac{dS_j^{-1}}{dk_j} < 1$$

$$L_i(c_i,k_{gi},k_{pi},k_i,p_i,q_i) = U_i(c_i,k_{gi}) + V_i(k_{gi},S_j(k_j))$$

$$- B_i + p_i(f_i(k_{pi},k_{gi}) + h_i(k_{gi},S_j(k_i)) - c_i - \lambda_i k_i) \qquad (46)$$

$$+ q_i(k_i - k_{pi} - k_{gi})$$

The first order and transversality conditions are identical to (9). However the canonical equations are now

$$\dot{k}_i = f_i(k_{pi},k_{gi}) + h_i(k_{gi},S_j(k_i)) - c_i - \lambda_i k_i \quad ,$$

$$\dot{p}_i = (\lambda_i - \frac{\partial f_i}{\partial k_{pi}})\, p_i - [\frac{\partial V_i}{\partial S_j} + p_i \frac{\partial h_i}{\partial S_j}] \frac{dS_j}{dk_i} \qquad (47)$$

where $[\frac{\partial V_i}{\partial S_j} + p_i \frac{\partial h_i}{\partial S_j}] \frac{dS_j}{dk_i}$ is the interaction term indicating the interchange of the strategy of country j on the Lagrangian of country i. This term is comprised of the slope of j's reaction function weighted by the total benefits (positive or negative) to country i of a small increase in foreign capital.

Once again we may solve for c_i, k_{gi} and k_{pi} as functions of only k_i and p_i. Equation system (11) is applicable except that $-[\frac{\partial^2 V_i}{\partial k_{gi}\partial k_j} + p_i \frac{\partial^2 h_i}{\partial k_{gi}\, k_j}]\, dk_j$ is now replaced by $- [\frac{\partial^2 V_i}{\partial k_{gi}\,\partial S_j} + p_i \frac{\partial^2 h_i}{\partial k_{gi}\partial S_j}] \frac{dS_j}{dk_i}\, dk_i$. Clearly then $\frac{\partial c_i}{\partial p_i}$, $\frac{\partial k_{gi}}{\partial p_i}$ and $\frac{\partial k_{pi}}{\partial p_i}$ are identical to the respective results in theorem 3.1. The effect of a change in the state variable is more ambiguous because it is the composition of the relative magnitudes of of changes in k_i^c and k_j given by theorem 3.1. We get

$$\frac{\partial c_i}{\partial k_i} = \frac{1}{\Delta_i}\frac{\partial^2 U_i}{\partial c_i \partial k_{gi}}\left[\left(\frac{\partial^2 V_i}{\partial k_{gi}\partial S_j} + p_i \frac{\partial^2 h_i}{\partial k_{gi}\partial S_j}\right)\frac{dS_j}{dk_i} + p_i\left(\frac{\partial^2 f_i}{\partial k_{gi}\partial k_{pi}} - \frac{\partial^2 f_i}{\partial k_{pi}^2}\right)\right] . \quad (48)$$

If $\left(\frac{\partial^2 V_i}{\partial k_{gi}\partial S_j} + p_i \frac{\partial^2 h_i}{\partial k_{gi}\partial S_j}\right)$ and $\frac{dS_j}{dk_i}$ are of the same sign then by (13),

$\operatorname{sgn} \frac{\partial c_i}{\partial k_i} = \operatorname{sgn} \frac{\partial^2 U_i}{\partial c_i \partial k_{gi}}$. Otherwise the sign of (48) is indeterminate.

$$\frac{\partial k_{gi}}{\partial k_i} = -\frac{1}{\Delta_i}\frac{\partial^2 U_i}{\partial c_i^2}\left[\left(\frac{\partial^2 V_i}{\partial k_{gi}\partial S_j} + p_i \frac{\partial^2 h_i}{\partial k_{gi}\partial S_j}\right)\frac{dS_j}{dk_i} + p_i\left(\frac{\partial^2 f_i}{\partial k_{gi}\partial k_{pi}} - \frac{\partial^2 f_i}{\partial k_{pi}^2}\right)\right]. \quad (49)$$

Once again if $\frac{dS_j}{dk_i}$ and the first bracketed term of (49) have the same sign then $\frac{\partial k_{gi}}{\partial k_i} > 0$. These conclusions state that if the direct effect and the indirect effect through the interaction with the j^{th} country of the changes in the i^{th} country's capital-labour endowment ratio pull in the same direction then the changes in consumption and public capital are relatively unambiguous. Proceeding,

$$\frac{\partial k_{pi}}{\partial k_i} = \frac{1}{\Delta_i}\frac{\partial^2 U_i}{\partial c_i^2}\left[\frac{\partial^2 U_i}{\partial k_{gi}^2} + \frac{\partial^2 V_i}{\partial k_{gi}} + \left(\frac{\partial^2 V_i}{\partial k_{gi}\partial S_j} + p_i \frac{\partial^2 h_i}{\partial k_{gi}\partial S_j}\right)\frac{dS_j}{dk_i}\right.$$

$$\left. + p_i\left(\frac{\partial^2 f_i}{\partial k_{gi}^2} + \frac{\partial^2 h_i}{\partial k_{gi}^2} - \frac{\partial^2 f_i}{\partial k_{pi}\partial k_{gi}}\right)\right] - \frac{1}{\Delta_i}\left(\frac{\partial^2 U_i}{\partial c_i \partial k_{gi}}\right)^2 , \quad (50)$$

the sign of which is ambiguous. Let us summarize by

<u>Theorem 4.1.</u> If $c_i = c_i(k_i,p_i)$, $k_{gi} = k_{gi}(k_i,p_i)$ and $k_{pi} = k_{pi}(k_i,p_i)$ are defined from the first order conditions of (46) and given (13), (15)

and (20) then $\frac{\partial c_i}{\partial p_i}$, $\frac{\partial k_{gi}}{\partial p_i}$, $\frac{\partial k_{pi}}{\partial p_i}$ are defined by theorem 3.1 and $\frac{\partial c_i}{\partial k_i}$, $\frac{\partial k_{gi}}{\partial k_i}$ and $\frac{\partial k_{pi}}{\partial k_i}$ are given by equations (48), (49) and (50) respectively.

Turning to the international steady state solution, as before, define

$$\phi_{1i}(k_i,p_i) = f_i(k_{pi}(k_i,p_i),k_{gi}(k_i,p_i))+h_i(k_{gi}(k_i,p_i),S_j(k_i))-c_i(k_i,p_i) - \lambda_i k_i) \tag{51}$$

$$\phi_{2i}(k_i,p_i) = (\lambda_i - \frac{\partial f_i}{\partial k_{pi}})\, p_i - [\frac{\partial V_i}{\partial S_j} + p_i \frac{\partial h_i}{\partial S_j}] \frac{dS_j}{dk_i}.$$

Differentiating ϕ_{2i} with respect to k_i yields

$$\frac{\partial \phi_{2i}}{\partial k_i} = -p_i \frac{\partial^2 f_i}{\partial k_{pi}^2} + [\frac{\partial^2 f_i}{\partial k_{pi}^2} - \frac{\partial^2 f_i}{\partial k_{pi}\partial k_{gi}}]\, p_i \frac{\partial k_{gi}}{\partial k_i} - [\frac{\partial^2 V_i}{\partial k_{gi}\partial S_j} + p_i \frac{\partial^2 h_i}{\partial k_{gi}\partial S_j}]$$

$$\frac{dS_j}{dk_i} \frac{\partial k_{gi}}{\partial k_i} - [(\frac{\partial^2 V_i}{\partial S_j^2} + p_i \frac{\partial^2 h_i}{\partial S_j^2})(\frac{\partial S_j}{\partial k_i})^2 + (\frac{\partial V_i}{\partial S_j} + p_i \frac{\partial h_i}{\partial S_j}) \frac{d^2 S_j}{dk_i^2}].$$

Substituting for $\frac{\partial k_{gi}}{\partial k_i}$ from (49) and from the definition of Δ_i found in (12) we have,

$$\frac{\partial \phi_{2i}}{\partial k_i} = -\, p_i \frac{\partial^2 f_i}{\partial k_{pi}^2} [\frac{\partial^2 U_i}{\partial c_i^2} (\frac{\partial^2 U_i}{\partial k_{gi}^2} + \frac{\partial^2 V_i}{\partial k_{gi}^2} + p_i (\frac{\partial^2 f_i}{\partial k_{gi}^2} + \frac{\partial^2 h_i}{\partial k_{gi}^2} +$$

$$\frac{\partial^2 f_i}{\partial k_{pi}^2} - 2 \frac{\partial^2 f_i}{\partial k_{pi}\partial k_{gi}})) - (\frac{\partial^2 U_i}{\partial k_{gi}\partial c_i})^2] + \frac{\partial^2 U_i}{\partial c_i^2}$$

$$((\frac{\partial^2 V_i}{\partial k_{gi}\partial S_j} + p_i \frac{\partial^2 h_i}{\partial k_{gi}\partial S_j})\frac{dS_j}{dk_i} + p_i (\frac{\partial^2 f_i}{\partial k_{pi}\partial k_{gi}} - \frac{\partial^2 f_i}{\partial k_{pi}^2}))^2 -$$

$$[(\frac{\partial^2 V_i}{\partial S_j^2} + p_i \frac{\partial^2 h_i}{\partial S_j^2})(\frac{dS_j}{dk_i})^2 + (\frac{\partial V_i}{\partial S_j} + p_i \frac{\partial h_i}{\partial S_j}) \frac{d^2 S_j}{dk_i^2}]$$

$$[\frac{\partial^2 U_i}{\partial c_i^2} (\frac{\partial^2 U_i}{\partial k_{gi}^2} + \frac{\partial^2 V_i}{\partial k_{gi}^2} + p_i(\frac{\partial^2 f_i}{\partial k_{gi}^2} + \frac{\partial^2 h_i}{\partial k_{gi}^2} + \frac{\partial^2 f_i}{\partial k_{pi}^2} - 2 \frac{\partial^2 f_i}{\partial k_{pi}\partial k_{gi}}))$$

$$- (\frac{\partial^2 U_i}{\partial k_{gi}\partial c_i})^2].$$

Collecting terms yields,

$$\frac{\partial \phi_{2i}}{\partial k_i} = - p_i \frac{\partial^2 f_i}{\partial k_{pi}^2} [\frac{\partial^2 U_i}{\partial c_i^2} (\frac{\partial^2 U_i}{\partial k_{gi}^2} + \frac{\partial^2 V_i}{\partial k_{gi}^2}) - (\frac{\partial^2 U_i}{\partial k_{gi}\ c_i})^2] -$$

$$p_i^2 \frac{\partial^2 U_i}{\partial c_i^2} [\frac{\partial^2 f_i}{\partial k_{pi}^2} (\frac{\partial^2 f_i}{\partial k_{gi}^2} + \frac{\partial^2 h_i}{\partial k_{gi}^2}) - (\frac{\partial^2 f_i}{\partial k_{pi}\partial k_{gi}})^2]$$

$$- [\frac{\partial^2 U_i}{\partial c_i^2} [(\frac{\partial^2 U_i}{\partial k_{gi}^2} + \frac{\partial^2 V_i}{\partial k_{gi}^2} + p_i \frac{\partial^2 h_i}{\partial k_{gi}^2}) [\frac{\partial^2 V_i}{\partial S_j^2} + p_i \frac{\partial^2 h_i}{\partial S_j^2}) \qquad (52)$$

$$(\frac{dS_j}{dk_i})^2 + (\frac{\partial V_i}{\partial S_j} + p_i \frac{\partial h_i}{\partial S_j}) \frac{d^2 S_j}{dk_i^2}] + (\frac{\partial^2 V_i}{\partial k_{gi}\partial S_j} + p_i \frac{\partial^2 h_i}{\partial k_{gi}\partial S_j})^2 (\frac{dS_j}{dk_i})^2]$$

$$+ (\frac{\partial^2 U_i}{\partial k_{gi}\partial c_i})^2 [(\frac{\partial^2 V_i}{\partial S_j^2} + p_i \frac{\partial^2 h_i}{\partial S_j^2})(\frac{dS_j}{dk_i})^2 + (\frac{\partial V_i}{\partial S_j} + p_i \frac{\partial h_i}{\partial S_j}) \frac{\partial^2 S_j}{\partial k_i^2}]]$$

$$+ \frac{\partial^2 U_i}{\partial c_i^2} p_i (2 \frac{\partial^2 f_i}{\partial k_{pi} \partial k_{gi}} - \frac{\partial^2 f_i}{\partial k_{pi}^2} - \frac{\partial^2 f_i}{\partial k_{gi}^2}) [(\frac{\partial^2 V_i}{\partial S_j^2} + p_i \frac{\partial^2 h_i}{\partial S_j^2}) (\frac{dS_i}{dk_i})^2 +$$

$$(\frac{\partial V_i}{\partial S_j} + p_i \frac{\partial h_i}{\partial S_j}) \frac{\partial^2 S_i}{\partial k_i^2}] + \frac{\partial^2 U_i}{\partial c_i^2} p_i \left(2 \frac{\partial^2 f_i}{\partial k_{pi} \partial k_{gi}} - 2 \frac{\partial^2 f_i}{\partial k_{pi}^2}\right)$$

$$(\frac{\partial^2 V_i}{\partial k_{gi} \partial S_j} + p_i \frac{\partial^2 h_i}{\partial k_{gi} \partial S_j}) \frac{dS_i}{dk_i}].$$

Before determining the sign of $\frac{\partial \phi_{2i}}{\partial k_i}$ it is necessary, at this point, to somewhat strengthen the assumptions found in (13). By equation (13) and in conjunction with (9) (the first order conditions) it was shown that $\frac{\partial^2 f_i}{\partial k_{gi}^2} + \frac{\partial^2 h_i}{\partial k_{gi}^2} - \frac{\partial^2 f_i}{\partial k_{pi} \partial k_{gi}} < 0$. Now we assume

$$\frac{\partial^2 f_i}{\partial k_{gi}^2} - \frac{\partial^2 f_i}{\partial k_{gi} \partial k_{pi}} < 0. \qquad (53)$$

Equation (53) states, an increase in private capital has less of an effect than an increase in public capital on the marginal product of public capital given by the f_i segment of the production function. Notice with (53) and by the strict concavity of H_i that (13) is satisfied. Returning to (52) observe that the first bracketed term is positive from the strict concavity of U_i and V_i. The second bracketed term is positive from the strict concavity of F_i and H_i and the third bracketed term is negative because it is the determinant of the Hessian of the strictly concave function

$U_i + V_i + p_iH_i$. The fourth term is positive due to (13), (53) and the strict concavity of $V_i + p_iH_i$. Hence if $(\frac{\partial^2 V_i}{\partial k_{gi}\partial S_j} + p_i \frac{\partial^2 h_i}{\partial k_{gi}\partial S_j})$ and $\frac{dS_j}{dk_i}$ are of opposite signs then $\frac{\partial\phi_{2i}}{\partial k_i} > 0$, otherwise if they have the same signs then $\frac{\partial\phi_{2i}}{\partial k_i} \gtreqless 0$. Next

$$\frac{\partial\phi_{2i}}{\partial p_i} = [\frac{\partial V_i}{\partial S_j}\frac{dS_j}{dk_i}/\frac{\partial U_i}{\partial c_i}] - \frac{\partial k_{gi}}{\partial p_i}[(\frac{\partial^2 V_i}{\partial k_{gi}\partial S_j} + p_i \frac{\partial^2 h_i}{\partial k_{gi}\partial S_j})\frac{dS_j}{dk_i} + p_i(\frac{\partial^2 f_i}{\partial k_{gi}\partial k_{pi}} - \frac{\partial^2 f_i}{\partial k_{pi}^2})], \tag{54}$$

the sign of which is ambiguous for the moment. For the $\phi_{1i}(k_i,p_i)$ equation, $\frac{\partial\phi_{1i}}{\partial p_i}$ is given by (28) and

$$\left.\frac{\partial\phi_{1i}}{\partial k_i}\right|_{\phi_{2i}(k_i,p_i)=0} = (-\frac{\partial V_i}{\partial S_j}\frac{\partial S_j}{\partial k_i}/\frac{\partial U_i}{\partial c_i}) + (\frac{\partial f_i}{\partial k_{gi}} + \frac{\partial h_i}{\partial k_{gi}} - \frac{\partial f_i}{\partial k_{pi}})\frac{\partial k_{gi}}{\partial k_i} - \frac{\partial c_i}{\partial k_i}$$

Substitute equations (48) and (49) for $\frac{\partial c_i}{\partial k_i}$ and $\frac{\partial k_{gi}}{\partial k_i}$ respectively.

$$\left.\frac{\partial\phi_{1i}}{\partial k_i}\right|_{\phi_{2i}(k_i,p_i)=0} = (-\frac{\partial V_i}{\partial S_j}\frac{dS_j}{dk_j}/\frac{\partial U_i}{\partial c_i}) + (\frac{\partial U_i}{\partial k_{gi}} + \frac{\partial V_i}{\partial k_{gi}})[\frac{\partial^2 U_i}{\partial c_i^2}/\frac{\partial U_i}{\partial c_i} - \frac{\partial^2 U_i}{\partial k_{gi}\partial c_i}/\frac{\partial U_i}{\partial k_{gi}} + \frac{\partial V_i}{\partial k_{gi}}]$$

$$[(\frac{\partial^2 V_i}{\partial k_{gi}\partial S_j} + p_i \frac{\partial^2 h_i}{\partial k_{gi}\partial S_j})\frac{dS_j}{dk_i} + p_i(\frac{\partial^2 f_i}{\partial k_{gi}\partial k_{pi}} - \frac{\partial^2 f_i}{\partial k_{pi}^2})]. \tag{55}$$

The sign of (55) is ambiguous. Although there are certain indeterminancies at present we are able to discern six important cases which are found in Table 1. These patterns are deemed consequential because they convey decisive results and are demarcated in terms of the basic equations of the model as set forth in section 2.

4.1 Special Cases

4.1.1 The first pattern is denoted by the following relationships;

$$\frac{\partial^2 v_i}{\partial k_{gi}\partial s_j} + p_f \frac{\partial^2 h_i}{\partial k_{gi}\partial s_j} > 0,\quad \frac{\partial v_i}{\partial s_j} \geqq 0,\quad \frac{\partial h_i}{\partial s_j} \leqq 0,\quad \frac{\partial v_i}{\partial s_j} + p_i \frac{\partial h_i}{\partial s_j} \gtreqless 0,\quad \frac{\partial s_i}{\partial k_j} > 0.$$

In this case the foreign reaction is positively sloped, the consumption extremality is not a diseconomy, the production externality is not an economy and so the total benefits from small increases in foreign capital may be any sign. Therefore from (52) $\frac{\partial \phi_{2i}}{\partial k_i} \gtreqless 0$, from (54) $\frac{\partial \phi_{2i}}{\partial p_i} > 0$, from (28) $\frac{\partial \phi_{1i}}{\partial p_i} > 0$ and from (55) $\left.\frac{\partial \phi_{1i}}{\partial k_i}\right|_{\phi_{2i}(k_i,p_i)=0} < 0$. Consequently, when $\frac{\partial \phi_{2i}}{\partial k_i} > 0$ there exists a unique steady state which is a saddle point, when $\frac{\partial \phi_{2i}}{\partial k_i} = 0$ again there is a unique saddle point steady state and the optimal trajectory is the $\phi_{2i}(k_i,p_i) = 0$ locus. Finally, for $\frac{\partial \phi_{2i}}{\partial k_i} < 0$ there may be multiple steady states, all of them totally unstable. These conclusions immediately illustrate the fact that when we discard assumptions allowing us to focus on the policies of one nation, either because it is a price taker or a price setter, optimal behavior of the country may be radically altered. In the extreme we could find ~~there is indeed only a degenerate optimal~~ path in the sense that the

Table 1: Special Cases

Relations	1	2	3	4	5	6
$\frac{\partial^2 v_i}{\partial k_{gi} \partial s_j} + p_i \frac{\partial^2 h_i}{\partial k_{gi} \partial s_j}$	> 0	> 0	< 0	< 0	< 0	< 0
$\frac{\partial v_i}{\partial s_j}$	$\geqq 0$	$\leqq 0$	$\geqq 0$	$\leqq 0$	$\leqq 0$	$\geqq 0$
$\frac{\partial h_i}{\partial s_j}$	$\lesseqgtr 0$	$\lesseqgtr 0$	$\gtreqless 0$	$\gtreqless 0$	$\gtreqless 0$	$\gtreqless 0$
$\frac{ds_j}{dk_i}$	> 0	< 0	> 0	> 0	< 0	< 0
$(\frac{\partial^2 v_i}{\partial k_{gi} \partial s_j} + p_i \frac{\partial^2 h_i}{\partial k_{gi} \partial s_j})\frac{ds_j}{dk_i} + p_i(\frac{\partial^2 f_i}{\partial k_{gi} \partial k_{pi}} - \frac{\partial^2 f_i}{\partial k_{pi}^2})$	> 0	> 0	> 0	< 0	> 0	< 0

country will only reach intertemporal equilibrium if its initial capital labour ratio is the steady state value.

In comparing this case with the Stackleberg follower case defined by the characteristics of 4.1.1, i.e., positively sloped reaction functions for both countries let (k^*_{iF}, p^*_{iF}) and (k^*_{iL}, p^*_{iL}) respectively denote the follower and leader steady state values of k_i and p_i. Clearly, k^*_{iF} and k^*_{iL} satisfy $\phi_{1i}(k_i,p_i) = 0$ but

$$\phi_{2i}(k^*_{iL},p^*_{iF}) = (\lambda_i - \frac{\partial f_i}{\partial k_{pi}})\, p^*_{iF} - [\frac{\partial V_i}{\partial S_j} + p^*_{iF}\frac{\partial h_i}{\partial S_j}]\frac{dS_j}{dk^*_{iF}} = - [\frac{\partial V_i}{\partial S_j} + p^*_{iF}\frac{\partial h_i}{\partial S_j}]\frac{dS_j}{dk^*_{iF}}, \quad (56)$$

because $\lambda_i = \frac{\partial f_i}{\partial k_{pi}}$ at (k^*_{iF}, p^*_{iF}). Thus, if the bracketed term in (56) is positive and since $\frac{ds_j}{dk_i} > 0$ then $\phi_{2i}(k^*_{iF}, p^*_{iF}) < 0$. For the cases $\frac{\partial \phi_{2i}}{\partial k_i} \gtrless 0$ this means that we must increase k_i and p_i to get $\phi_{2i}(k_i,p_i) = 0$ rather than less than zero and still maintain $\phi_{2i}(k_i,p_i) = 0$. These conclusions are illustrated in figure 10 for the case $\frac{\partial \phi_{2i}}{\partial k_i} > 0$. If the bracketed term in (56) is negative then $k^*_{iL} \gtrless k^*_{iF}$ and $p^*_{iL} < p^*_{iF}$; if the term is zero then the steady states coincide. In figure 10 we assumed that country i was the leader and j the follower. Figure 11 depicts i and j as leaders when $\frac{\partial \phi_{2i}}{\partial k_i} > 0$ and the bracketed term in (56) is positive. Other diagrams for different values of $\frac{\partial V_i}{\partial S_j} + p_i \frac{\partial h_i}{\partial S_j}$ and the subclasses when $\frac{\partial \phi_{2i}}{\partial k_i} \lesseqgtr 0$ may be similarly derived making allowances for the preceeding discussion.

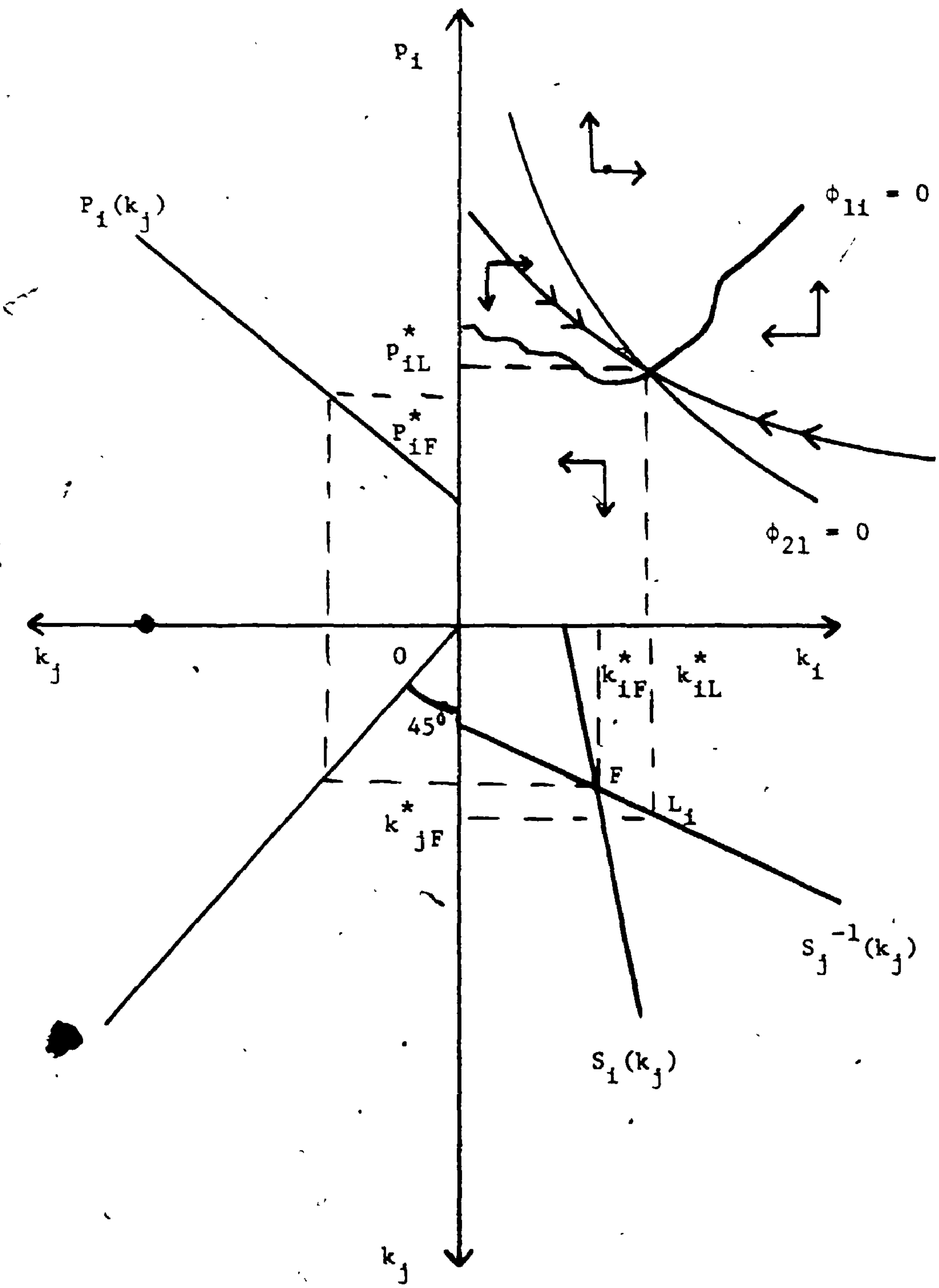

Figure 10. Case 1, $\frac{\partial\phi_{2i}}{\partial k_i} > 0$, $\phi_{2i}(k^*_{iF}, p^*_{iF}) < 0$, i leader and j follower

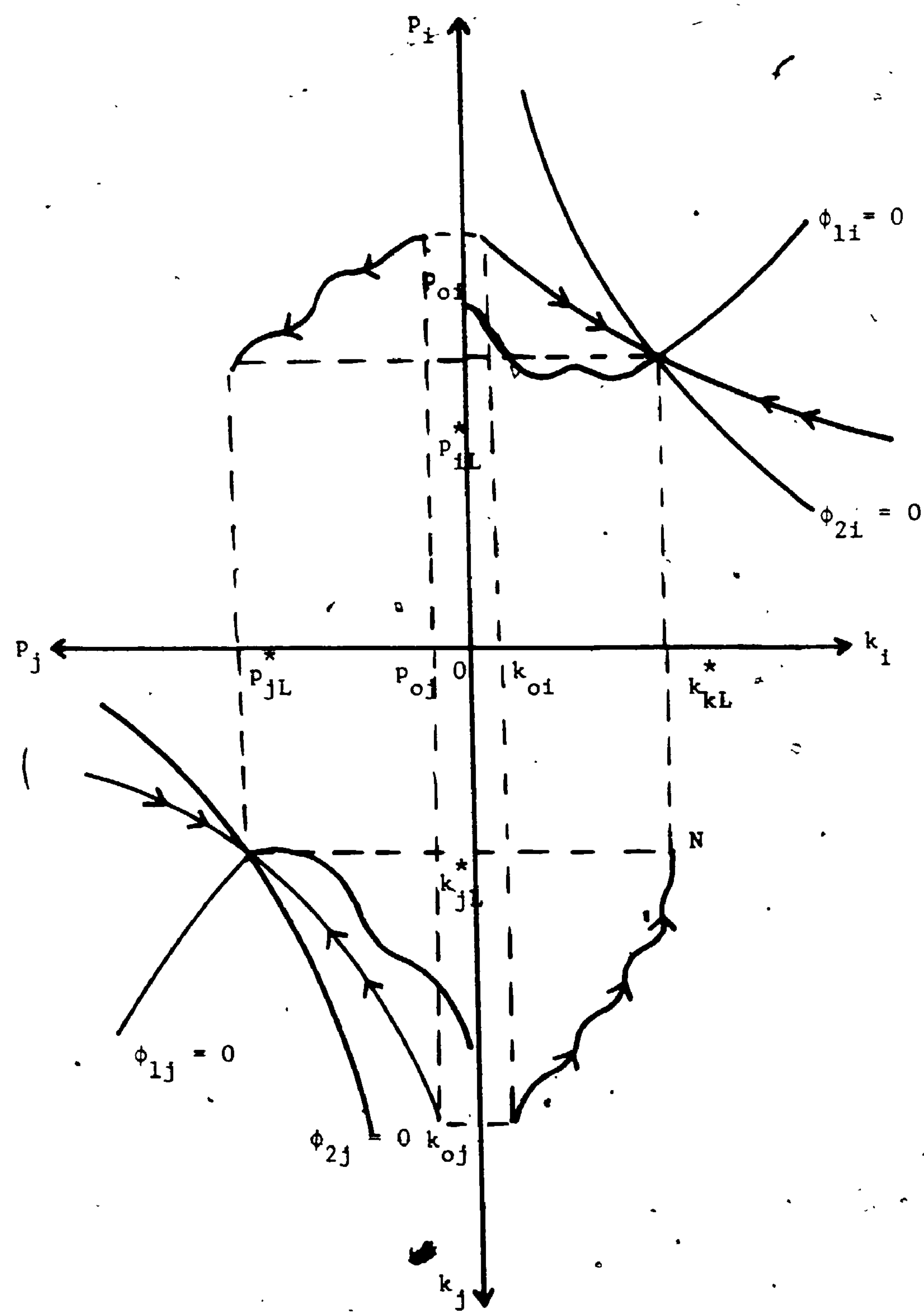

Figure 11. Case 1, $\frac{\partial \phi_{2i}}{\partial k_i} > 0$, i and j leaders

4.1.2 For case 2 given in Table 1, from equations (52), (54), (28) and (55) we have $\frac{\partial \phi_{2i}}{\partial k_i} > 0$, $\frac{\partial \phi_{2i}}{\partial p_i} > 0$, $\frac{\partial \phi_{1i}}{\partial p_i} > 0$, $\left.\frac{\partial \phi_{1i}}{\partial k_i}\right|_{\phi_{2i}(k_i,p_i) = 0} < 0$

Hence this pattern is identical to 4.1.1 when $\frac{\partial \phi_{2i}}{\partial k_i} > 0$ and $\frac{\partial V_i}{\partial S_j} + p_i \frac{\partial h_i}{\partial S_j} \gtreqless 0$. The reason why the cases are identical when the marginal benefits of foreign capital are of opposite sign is because the foreign country's reaction function is negatively sloped while previously it was positively sloped. This yields the same sign for equation (56).

4.1.3 In case 3, the relations given by (52), (54), (28) and (55) are identical to case 2. In addition, the results concerning the relative magnitudes of the steady state values of (k_i, p_i) for the leader and follower classes are the same as in case 1 when $\frac{\partial \phi_{2i}}{\partial k_i} > 0$ and $\frac{\partial V_i}{\partial S_j} + p_i \frac{\partial h_i}{\partial S_j} \gtreqless 0$. It is important to notice that the slopes of the reaction function and the function defined by (39), $P_i(k_j)$, are always the same sign. Consequently, although the i^{th} country's reaction function, in case 3, is negatively sloped when we trace out the solutions we follow the $P_i(k_j)$ function, as well, which is also decreasing. So if k_j increases then k_i and p_i always move in the same direction along $S_i(k_j)$ and $P_i(k_j)$. Hence, the slope of the i^{th} country's reaction function is immaterial to the outcome and therefore in case 3 $k^*_{iL} \gtreqless k^*_{iF}$, $P^*_{iL} \gtreqless P^*_{iF}$ according to

$$\frac{\partial V_i}{\partial S_j} + p_i \frac{\partial h_i}{\partial S_j} \gtreqless 0.$$

4.1.4 This pattern yields $\frac{\partial \phi_{2i}}{\partial k_i} > 0$, $\frac{\partial \phi_{2i}}{\partial p_i} < 0$, $\frac{\partial \phi_{1i}}{\partial p_i} > 0$, $\left.\frac{\partial \phi_{1i}}{\partial k_i}\right|_{\phi 2i(k_i, p_i) = 0} > 0$. There is a unique steady state which is a saddle point. However, the $\phi_{2i}(k_i, p_i) = 0$ locus is upward sloping and $\phi_{1i}(k_i, p_i) = 0$ locus is downward sloping at the steady state. In addition, due to the partial derivatives previously written that when $\frac{\partial V_i}{\partial S_j} + p_i \frac{\partial h_i}{\partial S_j} > 0$ so that $\phi_{2i}(k^*_{iF}, p^*_{iF}) < 0$ there will be a decrease in p_i and increase in k_i in the leader steady state. Conversely, if $\phi_{2i}(k^*_{iF}, p^*_{iF}) > 0$ then p_i increases and k_i decreases. Finally, when $\frac{\partial V_i}{\partial S_j} + p_i \frac{\partial h_i}{\partial S_j} = 0$ then the follower and leader steady state coincide. Figure 12 illustrates the case when $\phi_{21}(k^*_{1F}, p^*_{1F}) > 0$ and country i is the leader while country j is the follower. In figure 13 the same pattern is depicted with both nations behaving as leaders.

4.1.5 Pattern 5 is identical to pattern 1 except for one detail. Because the reaction functions of both countries are negatively sloped then $\phi_{21}(k^*_{1F}, p^*_{1F}) \gtreqless 0$ when $\frac{\partial V_i}{\partial S_j} + p_i \frac{\partial h_i}{\partial S_j} \gtreqless 0$. Whereas in case 1 $\phi_{21}(k^*_{1F}, p^*_{1F}) \gtreqless 0$ when $\frac{\partial V_i}{\partial S_j} + \frac{\partial h_i}{\partial S_j} \gtreqless 0$. Hence now $k^*_{iL} \gtreqless k^*_{iF}$ and $p^*_{iL} \gtreqless p^*_{iF}$ as the marginal benefits of foreign capital are negative, zero or positive.

4.1.6 Now we find that for case 6, $\frac{\partial \phi_{2i}}{\partial k_i} \gtreqless 0$, $\frac{\partial \phi_{2i}}{\partial p_i} < 0$, $\frac{\partial \phi_{1i}}{\partial p_i} > 0$ and $\frac{\partial \phi_{1i}}{\partial p_i} > 0$ and $\left.\frac{\partial \phi_{1i}}{\partial k_i}\right|_{\phi_{2i}(k_i, p_i)=0} > 0$. Clearly, with $\frac{\partial \phi_{2i}}{\partial k_i} > 0$

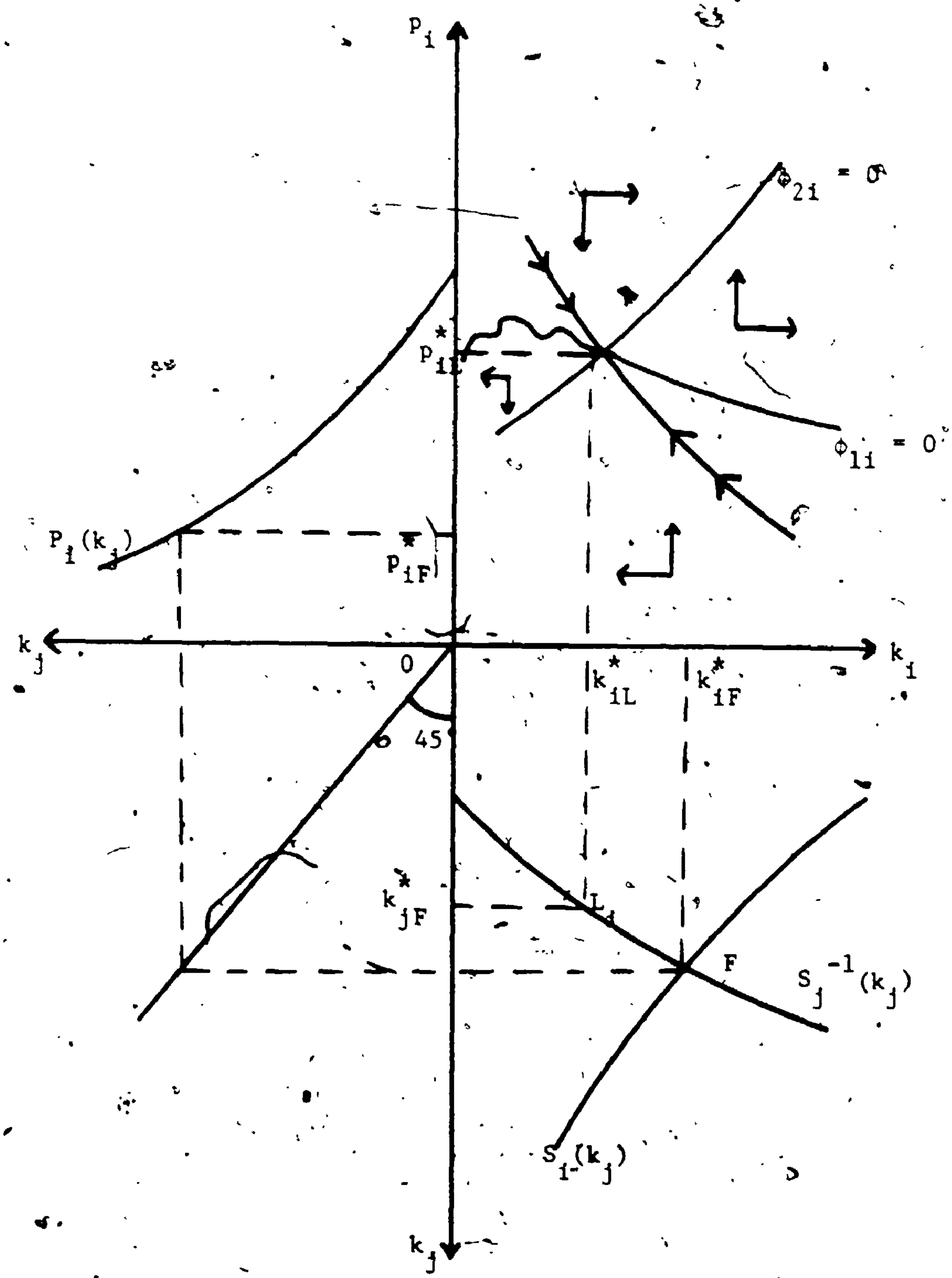

Figure 12. Case 4, $\phi_{2i}(k^*_{iF}, p^*_{iF}) > 0$,
i leader and j follower

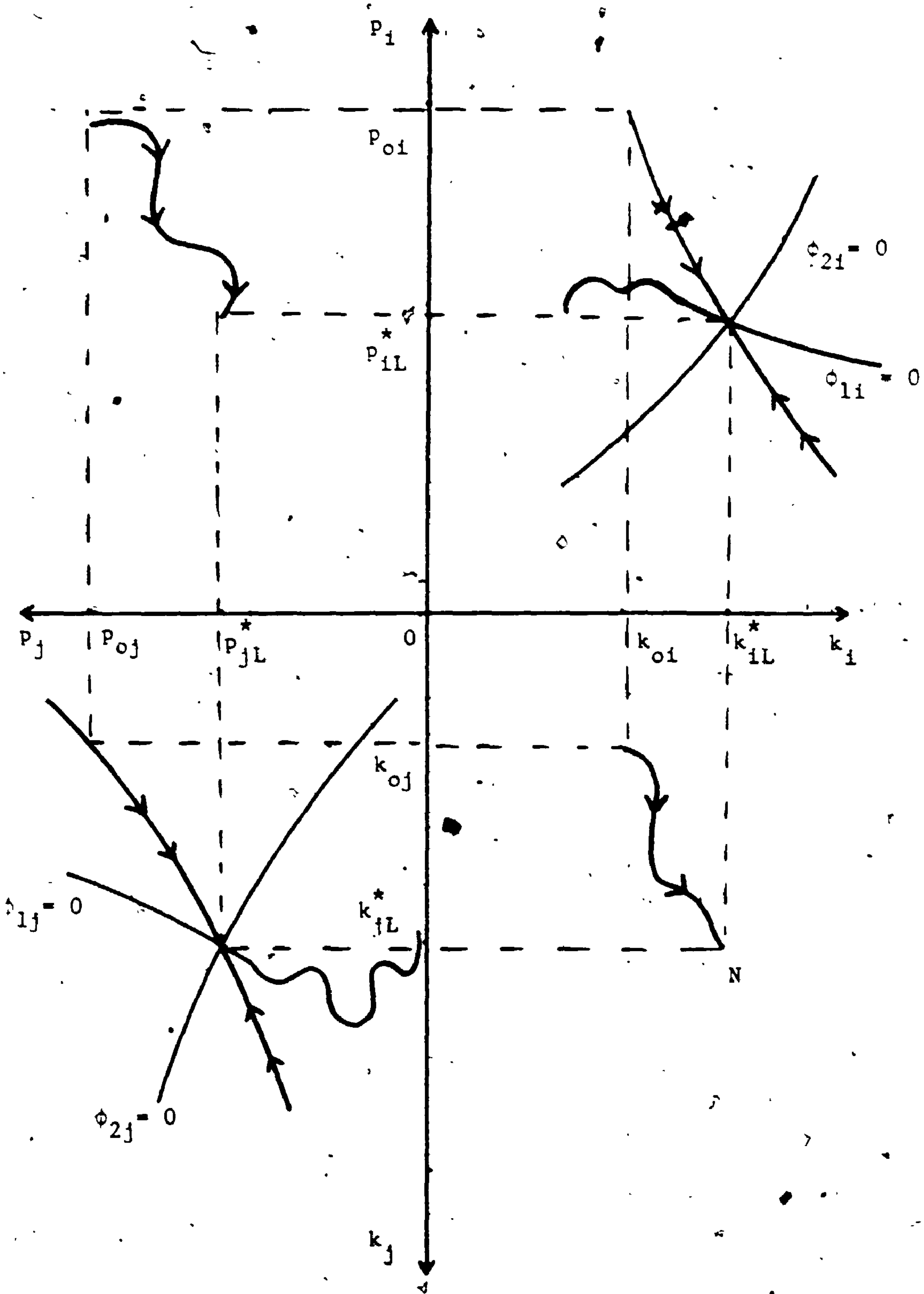

Figure 13. Case 4, i and j leaders

the $\phi_{1i}(k_i,p_i)$ and $\phi_{2i}(k_i,p_i)$ curves are the same shape as found in pattern 4 yielding a unique saddle point steady state. Although here $\phi_{2i}(k^*_{iF}, p^*_{iF}) \gtreqless 0$ because $\frac{\partial V_i}{\partial S_j} + p_i \frac{\partial h_i}{\partial S_j} \gtreqless 0$ and $\frac{dS_j}{dk_i} < 0$ so $k^*_{iL} \lesseqgtr k^*_{iF}$, $p^*_{iL} \gtreqless p^*_{iF}$ according to whether the marginal benefits of foreign capital are nonzero or zero. When $\frac{\partial \phi_{2i}}{\partial k_i} = 0$ there is a unique steady state which is totally unstable and so nothing more will be discussed concerning this case.

Finally, we come to $\frac{\partial \phi_{2i}}{\partial k_i} < 0$. There will be generally a multiplicity of steady states with the ones defined by the $\phi_{2i}(k_i,p_i) = 0$ locus cutting the $\phi_{1i}(k_i,p_i) = 0$ locus from above as the saddle points and the other steady states as totally unstable. Moreover, since $\frac{\partial \phi_{2i}}{\partial k_i} < 0$ then although $\phi_{2i}(k^*_{iF}, p^*_{iF}) \gtreqless 0$ it is still feasible when the marginal benefits of foreign countries are nonzero to have $k^*_{iL} < k^*_{iF}$, $p^*_{iL} > p^*_{iF}$ or $k^*_{iL} > k^*_{iF}$, $p^*_{iL} < p^*_{iF}$. Intuitively this argument can be explained in the following manner. Suppose the follower steady state coincided with an unstable equilibrium in the leader model by initially letting $\frac{\partial V_i}{\partial S_j} + p_i \frac{\partial h_i}{\partial S_j} = 0$. Now let $\frac{\partial V_i}{\partial S_j} + p_i \frac{\partial h_i}{\partial S_j}$ increase. Consequently, this unstable steady state no longer consitutes an equilibrium of the system because $\phi_{1i}(k_i,p_i) = \phi_{2i}(k_i,p_i) = 0$ loci have shifted. Accompanying these shifts the economy abandons the previous equilibrium and traverses to a saddle point which can lie on either side of the unstable steady state. Due to the curvature of the loci one saddle point is represented by a higher p_i and lower k_i while the other saddle point has the converse configuration. These points are

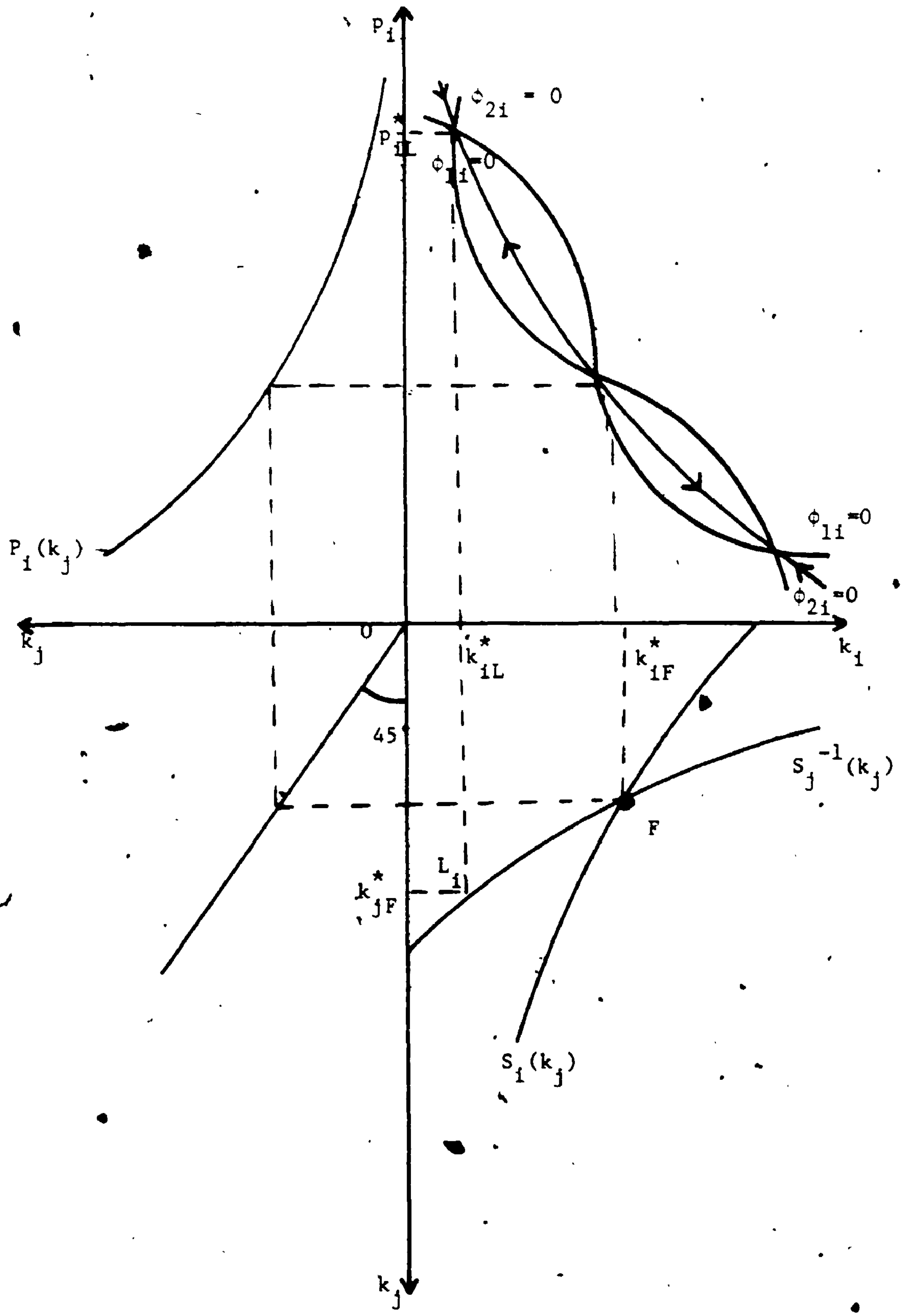

Figure 14. Case 6, $\frac{\partial \phi_{2i}}{\partial k_i} < 0$,

i leader and j follower

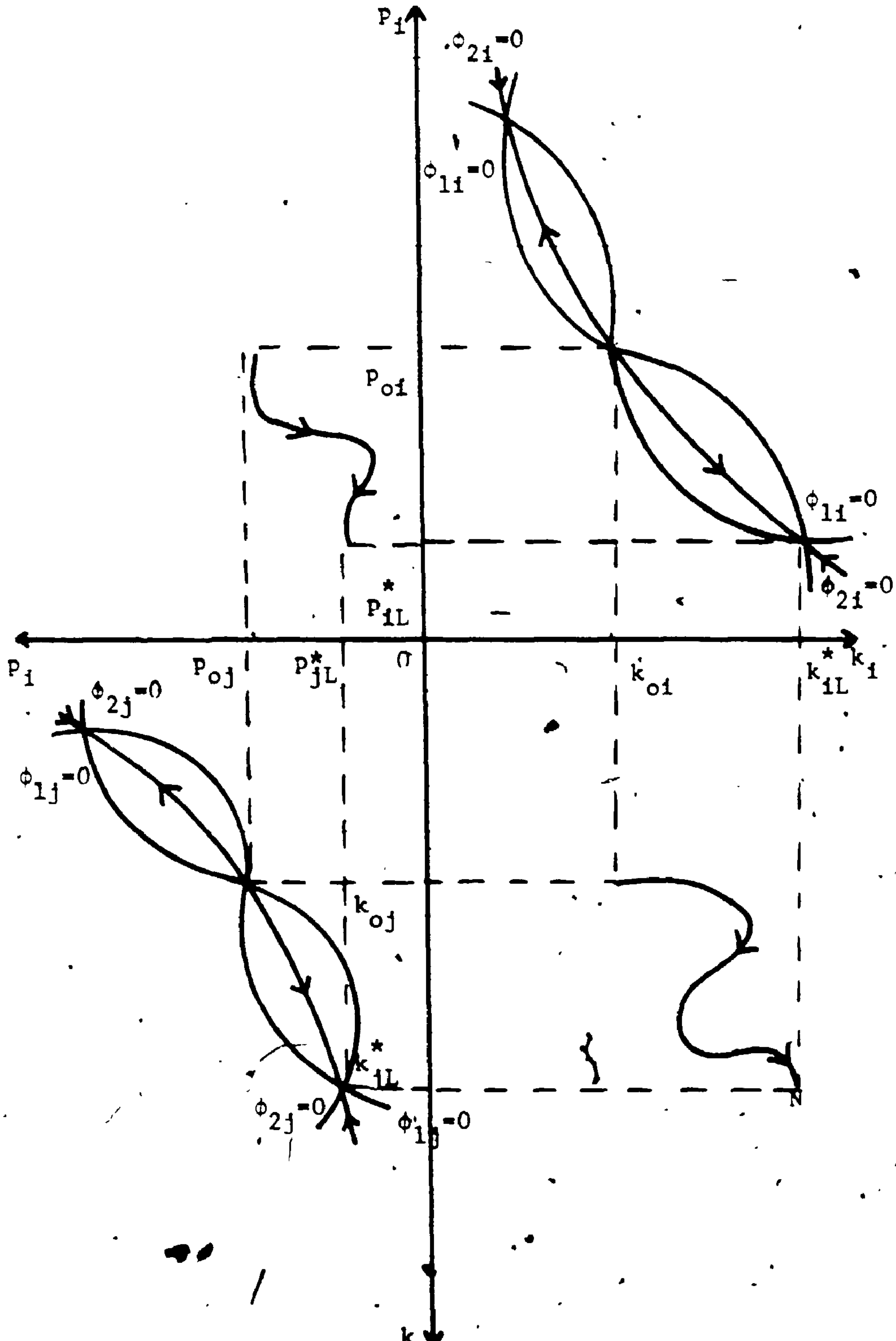

Figure 15. Case 6, $\frac{\partial \phi_{2i}}{\partial k_i} < 0$, i and j leaders

illustrated in figure 14 when country i is the leader and j the follower and in figure 15 when both are leaders.[5]

It appears then that to introduce complex interaction behavior between nations both multiplies the number and alters the qualitative nature of the feasible solutions. Yet, given the intricacies of the model, in particular the general formulation of the intertemporal externalities, the results, as developed and expounded within the text, remain relatively unambiguous. These significant circumstances tend to limit the purview of policy prescriptions for nations in the world which do not behave as either complete price takers or price setters.

Footnotes

1. A small country takes world prices as given and the passivity assumption means that the non-passive country is the price setter.

2. For a complete account of Cournot and Stackleberg duopoly models one can consult Intrilligator [6] or Mayberry, Nash and Shubik [9].

3. All variables except the stocks of capital and labour are piecewise continuous functions of time. The stocks are continuous and have piecewise continuous first derivatives with respect to time for $t > 0$.

4. The conditions on the production functions are excessive, for example all we need is that

$$\frac{\partial f_i}{\partial k_{pi}} > \lambda_i \text{ for } k_{pi} = 0 \text{ and } \frac{\partial f_i}{\partial k_{pi}} < \lambda_i \text{ for } k_{pi} = \infty$$

as the conditions on the marginal product of private capital. But since in the following sections the condition changes we decided simply to make one set of assumptions covering all cases in this paper. Obviously, then we just interpret these conditions as stating that the derivatives of the production functions at zero and infinite values are respectively larger than and smaller than certain parameters.

5. In dealing with the cases when both nations are leaders we have always illustrated figures with the two countries in case 1 or 4 etc. However, we can just as easily assume country i is a leader with the relations given by say case 1 and country j is a leader with the relations found in case 6 for example. All this entails is the combining of the relevant leader follower diagrams to show the optimal paths that arise in these cases. Except for the obvious that in general the optimal paths will change no new important results occur.

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