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Asymptotic Distribution Of Quantiles From Multivariate Population

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ASYMPTOTIC DISTRIBUTION OF QUANTILES FROM MULTIVARIATE POPULATION

by

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ABSTRACT

The present work deals with the asymptotic joint distribution of several quantiles from each components of a multivariate continuous random variable. It is shown that the joint distribution of the sample quantiles tends to a multivariate normal distribution.

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CHAPTER 1

INTRODUCTION

1.1 GENERAL INTRODUCTION

The present thesis is primarily concerned with the asymptotic joint distribution of quantiles from components of a multivariate continuous population.

Quantiles in univariate population have received a great deal of attention in the current literature. The limiting distribution of a sample quantile and limiting joint distribution of two sample quantiles are given by Cramer [6]. Mosteller [13] gave the limiting joint distribution of several sample quantiles.

The asymptotic joint distribution of the sample quantiles has been used in the estimation of location and scale parameters. Work in this area has been done by Mosteller [13], Ogawa [14], Ali [1] and others.

Quantiles in higher dimension have received relatively less attention. Mood [12] gave the asymptotic joint distribution of medians of components from bivariate population and gave the asymptotic joint distribution of medians of general dimension. Siddiqui [17] obtained the asymptotic joint distribution of quantiles one from each of the components of a bivariate population and some applications connected with the confidence limit on quantiles and confidence limit of ρ the correlation coefficient of the asymptotic distribution of quantiles from each component, were presented. Weiss [19] using a different method gave the asymptotic joint distribution of quantiles one from each component of a multivariate random variable. The present work may be thought of as extending the work of Mood, Siddiqui and Weiss.

SOME PRELIMINARIES

1.2 THE SAMPLE QUANTILES

Let $(X_1, X_2, \ldots, X_m)'$ be a continuous m-variate random variable $(m \geq 2)$ with strictly increasing known cumulative distribution function $F(x_1, x_2, \ldots, x_m)$ and probability density function $f(x_1, x_2, \ldots, x_m)$. Let $F_i(x_i)$, $f_i(x_i)$ denote respectively the marginal c.d.f. and p.d.f. of X_i, $i = 1, 2, \ldots, m$.

The equation

$$F_i(x_i) = \beta, \quad i = 1, 2, \ldots, m; \quad 0 < \beta < 1$$

has a unique solution in each x_i, say $x_i = \xi_{\beta_i}$, ξ_{β_i} is the population β_i-quantile of X_i .

Let $(X_{1j}, X_{2j}, \ldots, X_{mj})'$, $j=1,2,\ldots, N$ be a sample of size N from the m-variate variable $(X_1, X_2, \ldots, X_m)$.

The order statistics of the ith component are denoted by

$$X_{(i,1)} < X_{(i,2)} < \dots < X_{(i,N)}, \quad i=1,2,\dots, m.$$

For positive real β such that $0 < \beta < 1$, the sample β-quantile of the ith component X_i is $X_{(i,[N\beta]+1)}$, where [a] denotes the largest integer in a. Let α_{ij}, $j = 1, 2, \dots, r_i$; $i = 1, 2, \dots, m$ be set of real numbers such that

$$0 < \alpha_{ir_i} < \alpha_{i(r_i-1)} \dots < \alpha_{i1} < 1, \quad i=1,2,\dots, m$$

Corresponding to these real numbers, denote the r_i population quantiles of X_i by $\xi_{ir_i}, \xi_{i(r_i-1)}, \dots, \xi_{i1}$ with

$$\xi_{ir_i} < \xi_{i(r_i-1)} \dots < \xi_{i1}, \quad i=1,2,\dots, m .$$

The corresponding sample quantiles of X_i are Z_{ir_i}, $Z_{i(r_i-1)}, \dots, Z_{i1}$ with

$$Z_{ir_i} < Z_{i(r_i-1)} \dots < Z_{i1}, \quad i=1,2,\dots, m .$$

For the case $r_1 = r_2 = r_3 \dots = r_m = 1$ there is one quantile from each component. We will call $(Z_{11}, Z_{21}, \dots, Z_{m1})'$ the sample quantiles of order $(\alpha_{11}, \alpha_{21}, \dots, \alpha_{m1})$ or simply the sample $(\alpha_{11}, \alpha_{21}, \dots, \alpha_{m1})$-quantiles, and we write $(Z_{11}, Z_{21}, \dots, Z_{m1}) = (Z_1, Z_2, \dots, Z_m)$.

1.3 THE SYMBOL O

As usual, $f(x) = O(g(x))$ will mean that $f(x)/g(x)$ remains bounded as x tends to its limit. This may be read "f(x) is at most of the order of g(x)". Thus $A = O(1/\sqrt{N})$ as $N \to \infty$ means that $\lim_{N\to\infty} \sqrt{N}\, A$ remains bounded.

As a simple notation, the expression

$$u = v\,(\,1 + O(1/\sqrt{N})\,)$$

will be abbreviated to read

$$u \doteq v$$

where the dot after the equality signifies the omission of the factor $(\,1 + O(1/\sqrt{N})\,)$.

1.4 MULTINOMIAL DISTRIBUTION

A certain random experiment has r mutually exclusive events $E_1, E_2, \ldots, E_r$. The probability of the event E_i is $P(E_i) = p_i$, $p_i > 0$, $i = 1, 2, \ldots, r$ with $\sum_{i=1}^{r} p_i = 1$. In a series of N independent trials, let n_i represents the number of times that the event E_i occurs, $i = 1, 2, \ldots, r$. Then the probability of this occurrence is

$$\frac{N!}{n_1!\,n_2!\ldots n_r!}\, p_1^{n_1} p_2^{n_2} \ldots p_r^{n_r} \qquad \ldots .(1.4)$$

with

$$\sum_{n_1+\ldots+n_r=N} \frac{N!}{n_1!n_2!\ldots n_r!} p_1^{n_1} p_2^{n_2} \ldots p_r^{n_r} = 1$$

and

$$\sum_{i=1}^{r} n_i = N.$$

The distribution having probability function (1.4) is known as the multinomial distribution.

We shall use the following well-known normal approximation to the multinomial distribution [12],

$$\frac{N!}{n_1!n_2!\ldots n_r!} p_1^{n_1} p_2^{n_2} \ldots p_r^{n_r} .$$

$$\doteq [|A|/(2\pi)^{r-1}]^{1/2} \exp\{-\tfrac{1}{2} \Sigma A_{ij} t_i t_j\} \prod_{i=1}^{r-1} dt_i$$

where

$$t_i = \frac{n_i - Np_i}{\sqrt{N}} \qquad i = 1, 2, \ldots, r-1,$$

and A is the matrix $A = (A_{ij})$ with

$$A_{ii} = 1/p_i + 1/p_r , \quad i = 1, 2, \ldots, r-1,$$

$$A_{ij} = 1/p_r \quad \text{for} \quad i \neq j.$$

The matrix has determinant value $\prod_{i=1}^{r}(1/p_i)$.

Since, if

$$P = \frac{N!}{n_1!n_2!\ldots n_r!} p_1^{n_1} p_2^{n_2} \cdots p_r^{n_r}$$

then, by using Stirling's approximation

$$\ln P = \ln N! - \sum_{i=1}^{r} \ln n_i! + \sum_{i=1}^{r} n_i \ln p_i$$

$$= \frac{1}{2}\ln 2\pi + (N+\frac{1}{2})\ln N - N + O(\frac{1}{N})$$

$$- \sum [\frac{1}{2}\ln 2\pi + (n_i+\frac{1}{2})\ln n_i - n_i + O(\frac{1}{n_i})]$$

$$+ \sum n_i \ln p_i$$

$$= -(\frac{r-1}{2})\ln 2\pi + (N+\frac{1}{2})\ln N - \sum (n_i+\frac{1}{2})\ln n_i$$

$$+ \sum n_i \ln p_i + O(\frac{1}{N}) + \sum O(\frac{1}{n_i}) .$$

Let $n_i/N = p_i + e_i$, $i = 1, 2, \ldots, r$;

e_i is, by Chebyshev's inequality, of order $1/\sqrt{N}$, and $n_i = N(p_i + e_i)$. Now

$$(n_i+\frac{1}{2}) \ln n_i$$

$$= [N(p_i+e_i)+\frac{1}{2}] \ln Np_i(1+\frac{e_i}{p_i})$$

$$= [N(p_i+e_i)+\frac{1}{2}] [\ln Np_i + \frac{e_i}{p_i} - \frac{1}{2}(\frac{e_i}{p_i})^2 + O(\frac{1}{N^{3/2}})]$$

and $$\sum (n_i + \tfrac{1}{2}) \ln n_i$$

$$= \sum N(p_i+e_i) \ln N + \sum N(p_i+e_i) \ln p_i$$

$$+ \frac{1}{2} \sum \ln np_i + \frac{N}{2} \sum \frac{e_i^2}{p_i} + O(1/\sqrt{N})$$

with $e_1 + e_2 + e_3 + \ldots + e_r = 0$

Thus $$\ln P = -(\tfrac{r-1}{2}) \ln 2\pi + \tfrac{1}{2} \ln N - \tfrac{1}{2} \sum_{i=1}^{r} \ln Np_i - \tfrac{N}{2} \sum_{i=1}^{r} \frac{e_i^2}{p_i} + O(\tfrac{1}{\sqrt{N}})$$

$$= -(\tfrac{r-1}{2}) \ln 2\pi + \tfrac{1}{2} \ln N - \tfrac{1}{2} \ln N^r p_1 p_2 \cdots p_r$$

$$- \frac{N}{2}\left[\sum_{i=1}^{r-1} e_i^2 (\tfrac{1}{p_i} + \tfrac{1}{p_r}) + \sum_{i \neq j}^{r-1} e_i e_j / p_r \right] + O(\tfrac{1}{\sqrt{N}})$$

where we used the fact that $e_r = -e_1 - e_2 - \ldots - e_{r-1}$.

Therefore $$P = \left[\frac{(2\pi)^{r-1}}{N^{r-1} p_1 p_2 \cdots p_r}\right]^{1/2} \exp\{-\tfrac{1}{2} \sum A_{ij} e_i e_j\}$$

$$= \left[\frac{|A|}{(2\pi)^{r-1}}\right]^{1/2} \exp\{-\tfrac{1}{2} \sum A_{ij} t_i t_j\} \prod_{i=1}^{r-1} dt_i \qquad \ldots\ldots\ldots\ldots\ldots(1.4.1)$$

where $dt_i = 1/\sqrt{N}$, $i = 1, 2, \ldots, r-1$.

We note that the expression (1.4.1) implies that the multinomial probability converges uniformly to the multinormal density.

1.5 A CONVERGENCE THEOREM IN DISTRIBUTION

The following lemma is well-known [4], [18].

Lemma 1.5

Let X, $X^{(1)}$, $X^{(2)}$, ...be k-dimensional random variables and $X^{(n)}$ converges in distribution to X. Let $\phi_1(X)$, $\phi_2(X)$, ..., $\phi_m(X)$, $m \leq k$ be real continuous function on E^k then $[\phi_1(X^{(n)}), \phi_2(X^{(n)}), \ldots, \phi_m(X^{(n)})]'$ converges in distribution to $[\phi_1(X), \phi_2(X), \ldots, \phi_m(X)]'$.

Proof:

The function $T: E^k \longrightarrow E^m$

defined by

$$T(X) = [\phi_1(X), \phi_2(X), \ldots, \phi_m(X)]'$$

is a continuous function since each ϕ_i is, let $t = (t_1, t_2, \ldots, t_m)$ be a point in E^m,

$$E[\exp\{i(t, T(X^{(n)}))\}]$$

$$= \int \ldots \int \exp\{i(t,T(Z))dF_{X^{(n)}}(Z)$$

by Helly-Bray Theorem,

$$\xrightarrow{\text{as } N\to\infty} \int \ldots \int \exp\{i(t,T(Z))dF_X(Z)$$

therefore $[\phi_1(X^{(n)}), \phi_2(X^{(n)}), \ldots, \phi_m(X^{(n)})]'$ converges to $[\phi_1(X), \phi_2(X), \ldots, \phi_m(X)]'$ in distribution.

Corollary 1.5

Let X, $X^{(1)}$, $X^{(2)}$, ... be k-dimensional random variables and $X^{(n)}$ converges in distribution to X then $(X_i^{(n)}, X_j^{(n)})'$ converges in distribution to $(X_i, X_j)'$ where $X_h^{(n)}$ is the hth component of $X^{(n)}$ and X_h is the hth component of X, $h = 1, 2, \ldots, k$.

Our problem is to determine the asymptotic joint distribution of several quantiles from each component of a multivariate population.

CHAPTER 2

THE ASYMPTOTIC DISTRIBUTION OF QUANTILES ONE FROM EACH COMPONENT OF A MULTIVARIATE POPULATION

2.1 INTRODUCTION

This chapter deals with the asymptotic joint distribution of quantiles one from each component of a m-variate population. This problem has been dealt with by Weiss [19]. The method we use in proving this result is essentially an extension of the geometrical argument of Craig [5]. In Chapter 3 we will see that this method allows us to solve the more general problem of joint asymptotic distribution of several quantiles from each component of a multivariate population.

Assumption 2.1

$$f(x+\frac{1}{N}) = f(x) + O(\frac{1}{N})$$

if $f(x)$ is a continuous function with bounded first derivative the condition is satisfied.

We will prove the following theorem:

Theorem 2.1

Let $(Z_1, Z_2, \ldots, Z_m)'$ be the sample quantiles of order $(\alpha_1, \alpha_2, \ldots, \alpha_m)$, $0 < \alpha_i < 1$, of a m-variate continuous variable $(X_1, X_2, \ldots, X_m)'$ with strictly known c.d.f. $F(x_1, x_2, \ldots, x_m)$ and p.d.f. $f(x_1, x_2, \ldots, x_3)$ with marginal p.d.f.'s $f_1(x_1)$, $f_2(x_2)$, ..., $f_m(x_m)$ satisfying Assumption 2.1. Then the joint distribution of $W_i = \sqrt{N}\, f_i(\xi_i)(Z_i - \xi_i)$, $i = 1, 2, \ldots, m$; where $(\xi_1, \xi_2, \ldots, \xi_m)'$ is the corresponding population quantiles of $(X_1, X_2, \ldots, X_m)'$, tends to a m-variate normal distribution with means 0, 0, ..., 0 and variances and covariances

$$\text{Var } W_i = \alpha_i(1 - \alpha_i), \quad i = 1, 2, \ldots, m$$

$$\text{Cov}(W_i, W_j) = F_{ij}(\xi_i, \xi_j) - \alpha_i\alpha_j \quad \text{for } i \neq j$$

where $F_{ij}(x_i, x_j)$ is the joint marginal of X_i and X_j.

Without loss of generality, we assume that $(\xi_1, \xi_2, \ldots, \xi_m)' = (0, 0, \ldots, 0)'$. We follow essentially a method due to Mood [12] which consists of dividing the space into appropriate mutually disjoint regions. Multinomial consideration is then used to obtain appropriate probabilities and normal approximation is made. We use the same notations as in Mood [12].

Before proving the theorem, we first prove a special case of the theorem namely the case $m = 2$.

2.2 THE ASYMPTOTIC DISTRIBUTION OF QUANTILES ONE FROM EACH COMPONENT OF A MULTIVARIATE POPULATION

$m = 2$. Given a sample of size N from (X_1, X_2), let (Z_1, Z_2) be the sample quantiles of order (α_1, α_2). Consider the probability that (Z_1, Z_2) falls in the rectangle R"

$$z_1 - \tfrac{1}{2}dz_1 < x_1 < z_1 + \tfrac{1}{2}dz_1$$

$$z_2 - \tfrac{1}{2}dz_2 < x_2 < z_2 + \tfrac{1}{2}dz_2$$

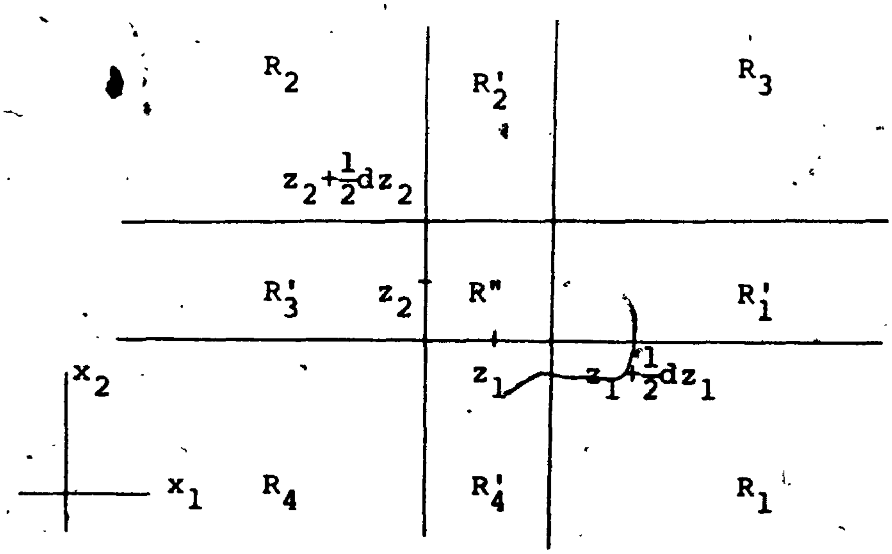

Figure 1

The remaining of the plane is divided by lines $x_1 = z_1 \pm \frac{1}{2}dz_1$, $x_2 = z_2 \pm \frac{1}{2}dz_2$ into regions R_1, R_2, R_3, R_4, R_1', R_2', R_3' and R_4' as indicated in Figure 1. Let $p_i^{(j)}$ denotes the probability that an element of the sample will fall in the region $R_i^{(j)}$

$$p_i^{(j)} = \iint_{R_i^{(j)}} f(x_1, x_2)dx_2dx_1$$

where $R_i^{(0)}$ should be realized as region R_i and $p_i^{(0)}$ is realized as p_i.

Neglecting terms involving differentials of higher order it is seen that

$$p_1 = \int_{z_1}^{\infty} \int_{-\infty}^{z_2} f(x_1, x_2)dx_2dx_1$$

$$p_2 = \int_{-\infty}^{z_1} \int_{z_2}^{\infty} f(x_1, x_2)dx_2dx_1$$

$$p_3 = \int_{z_1}^{\infty} \int_{z_2}^{\infty} f(x_1, x_2)\, dx_2dx_1$$

$$p_4 = \int_{-\infty}^{z_1} \int_{-\infty}^{z_2} f(x_1, x_2)dx_2dx_1$$

....(2.1.1)

$$p_1' = [\int_{z_1}^{\infty} f(x, z_2)dx_1]\, dz_2$$

$$p'_2 = [\int_{z_2}^{\infty} f(z_1, x_2)\, dx_2]\, dz_1$$

$$p'_3 = [\int_{-\infty}^{z_1} f(x_1, z_2)\, dx_1]\, dz_2$$

$$p'_4 = [\int_{-\infty}^{z_2} f(z_1, x_2)\, dx_2]\, dz_1$$

and $$p'' = f(z_1, z_2)\, dz_1 dz_2$$

With this set-up, we may consider that the sample is drawn from a multinomial population with probabilities $p_i^{(j)}$ falling in the region $R_i^{(j)}$. We will pick up those terms which give rise to the sample quantile (Z_1, Z_2).

There are two distinct cases namely Case (1): (Z_1, Z_2) is determined by one element of the sample; Case (2): (Z_1, Z_2) is determined by two different elements of the sample. We investigate the two cases separately.

Case (1). In this case, the sample quantile (Z_1, Z_2) is an element of the sample. It falls in region R" and the remaining of the elements of the sample fall in the regions R_1, R_2, R_3 and R_4 with n_i elements in R_i such that

$$n_3 + n_2 = N - ([N\alpha_2] + 1)$$

$$n_1 + n_4 = [N\alpha_2]$$

$$n_1 + n_3 = N - ([N\alpha_1] + 1)$$

$$n_2 + n_4 = [N\alpha_1]$$

with $$n_1 + n_2 + n_3 + n_4 = N-1 .$$

The probability that N observations can be divided into these groups is

$$C = \sum \frac{N!}{n_1!n_2!n_3!n_4!} p'' p_1^{n_1} p_2^{n_2} p_3^{n_3} p_4^{n_4}$$

where the summation sign means sum over all such possibilities.

Case (2). The sample quantiles (Z_1, Z_2) is determined by two different elements of the sample. There are four different situations which give rise to this case; the two different elements of the sample are such that

(a) One in R_1' and one in R_2' ,

(b) One in R_1' and one in R_4' ,

(c) One in R_3' and one in R_2')

and (d) One in R_3' and one in R_4' .

For (a), the remaining of the elements of the sample must fall in regions R_1, R_2, R_3 and R_4 with n_i elements in R_i, $i = 1, 2, 3, 4$ in such a manner that

$$n_3 + n_2 = N - ([N\alpha_2] + 1) - 1$$

$$n_1 + n_4 = [N\alpha_2]$$

$$n_1 + n_3 = N - ([N\alpha_1] + 1) - 1$$

$$n_2 + n_4 = [N\alpha_1] .$$

The probability of such an occurrence is

$$B_1 = \sum_1 \frac{N!}{n_1!n_2!n_3!n_4!} p_1'p_2'p_1^{n_1} p_2^{n_2} p_3^{n_3} p_4^{n_4}$$

where the summation means sum over all such possible combinations of n_i .

Note that the n_i in this case should be differentiated from that of the former case. We use the same notations throughout for different cases, and when the normal approximations are made, the n_i 's are all immaterial. The corresponding probabilities associated with (b), (c) and (d) are respectively B_2, B_3, and B_4 with

$$B_2 = \sum_2 \frac{N!}{n_1!n_2!n_3!n_4!} p_1'p_4'p_1^{n_1} p_2^{n_2} p_3^{n_3} p_4^{n_4}$$

with $n_1 + n_2 + n_3 + n_4 = N-2$;

$$B_3 = \sum_3 \frac{N!}{n_1!n_2!n_3!n_4!} p_2'p_3'p_1^{n_1} p_2^{n_2} p_3^{n_3} p_4^{n_4}$$

with $n_1 + n_2 + n_3 + n_4 = N-2$;

and $$B_4 = \sum_4 \frac{N!}{n_1!n_2!n_3!n_4!} p_3'p_4'p_1^{n_1} p_2^{n_2} p_3^{n_3} p_4^{n_4}$$

with $n_1 + n_2 + n_3 + n_4 = N-2$.

If $g(z_1, z_2)$ is the density that gives the distribution of the sample quantiles (Z_1, Z_2), then

$$g(z_1, z_2)dz_1dz_2 = C + \sum_{i=1}^{4} B_i \qquad \text{....(2.1.2)}$$

The asymptotic distribution of the sample quantiles (Z_1, Z_2)

Examine C, B_1, B_2, B_3, B_4 closely, we note that the multinomial coefficient in C immediately after the summation sign, has one factor less in numerator than those of B_1, B_2, B_3 and B_4 since in C, $\Sigma n_i = N-1$ while in B_i's, $\Sigma n_i = N-2$. Thus the term C can be neglected in the asymptotic form as it is of order 1/N when compared to the B_i, i = 1, 2, 3, 4.

We shall use the normal approximation to approximate the multinomial distribution. Since they cannot be written in the finite form, we compute the sums Σ_1, Σ_2, Σ_3 and Σ_4 by integrations.

Consider the term B_1

$$B_1 = \sum_1 \frac{N!}{n_1!n_2!n_3!n_4!} p_1'p_2'p_1^{n_1} p_2^{n_2} p_3^{n_3} p_4^{n_4}$$

$$= N(N-1)p_1'p_2' \sum_1 \frac{(N-2)!}{n_1!n_2!n_3!n_4!} p_1^{n_1} p_2^{n_2} p_3^{n_3} p_4^{n_4} ,$$

since $n_1 + n_2 + n_3 + n_4 = N-2$, the expression after the summation sign is a multinomial coefficient and thus the normal approximation can be made. With r = 4,

$$B_1 \doteq N(N-1)p_1'p_2' \sum [|A|/(2\pi)^{r-1}]^{1/2}$$

$$\exp\{ -\tfrac{1}{2} \sum A_{ij}t_it_j \} \prod_{i=1}^{r-1} dt_i ;$$

after omitting a term of order 1/N ,

$$B_1 = N^2 p_1' p_2' \sum_1 \; [|A|/(2\pi)^3]^{1/2}$$

$$\exp\{-\tfrac{1}{2} \sum A_{ij} t_i t_j\} \prod_{i=1}^{3} dt_i \qquad \ldots.(2.1.3)$$

where

$$t_i = \frac{n_i - Np_i}{\sqrt{N}}, \quad i = 1, 2, 3;$$

and A is the matrix $A = [A_{ij}]$ with

$$A_{ii} = \frac{1}{p_i} + \frac{1}{p_4} \qquad \text{for } i = 1, 2, 3 \text{ and}$$

$$A_{ij} = \frac{1}{p_4} \qquad \text{for } i \neq j .$$

Let

$$u_1 = \sqrt{N} \left(\frac{n_1+n_3}{N} - (p_1+p_3) \right)$$

$$u_2 = \sqrt{N} \left(\frac{n_2+n_3}{N} - (p_2+p_3) \right)$$

then

$$t_1 = \sqrt{N} \left(\frac{n_1+n_3}{N} - (p_1+p_3) \right) - \frac{n_3-Np_3}{\sqrt{N}}$$

$$= u_1 - t_3 ,$$

$$t_2 = \sqrt{N} \left(\frac{n_2+n_3}{N} - (p_2+p_3) \right) - \frac{n_3-Np_3}{\sqrt{N}} = u_2 - t_3 .$$

Substitute $t_1 = u_1 - t_3$, $t_2 = u_2 - t_3$ in the quadratic form of the exponential in (2.1.3),

$$\sum A_{ij}t_i t_j$$

$$= (t_1, t_2, t_3) A (t_1, t_2, t_3)'$$

$$= (u_1-t_3, u_2-t_3, t_3) A (u_1-t_3, u_2-t_3, t_3)'$$

$$= t_3^2 \left(\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} + \frac{1}{p_4}\right) - 2t_3\left[\frac{u_1+u_2}{p_4} + \frac{u_1}{p_1} + \frac{u_2}{p_2}\right]$$

$$+ \frac{(u_1+u_2)^2}{p_4} + \frac{u_1^2}{p_1} + \frac{u_2^2}{p_2}$$

expression (2.1.3) becomes

$$B_1 = N^2 p_1' p_2' \sum_1 [|A|/(2\pi)^3]^{1/2}$$

$$\exp\{-\frac{1}{2}[t_3^2(\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}+\frac{1}{p_4})-2t_3[\frac{u_1+u_2}{p_4}+\frac{u_1}{p_1}+\frac{u_2}{p_2}]$$

$$+\frac{(u_1+u_2)^2}{p_4} + \frac{u_1^2}{p_1} + \frac{u_2^2}{p_2}]\}dt_1 dt_2 dt_3$$

................(2.1.4)

We note that in the normal approximation to the multinomial distribution, the factor dt_i corresponds to the factor $1/\sqrt{N}$, thus we may cancel a factor N with $dt_1 dt_2$, and B_1 in (2.1.4) becomes

$$B_1 \doteq N p_1' p_2' \sum_1 [|A|/(2\pi)^3]^{1/2}$$

$$\exp\{-\frac{1}{2}[t_3^2(\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}+\frac{1}{p_4})-2t_3[\frac{u_1+u_2}{p_4}+\frac{u_1}{p_1}+\frac{u_2}{p_2}]$$

$$+\frac{(u_1+u_2)^2}{p_4}+\frac{u_1^2}{p_1}+\frac{u_2^2}{p_2}]\} dt_3$$

In order to get rid of the summation sign Σ_1 we integrate t_3 through the range of entire real line. To within terms of order $1/\sqrt{N}$, this gives

$$B_1 \doteq \frac{N p_1' p_2' |A|^{1/2}}{2\pi(\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}+\frac{1}{p_4})^{1/2}}$$

$$\exp\{-\frac{1}{2}[\frac{(u_1+u_2)^2}{p_4}+\frac{u_1^2}{p_1}+\frac{u_2^2}{p_2}$$

$$-(\frac{u_1+u_2}{p_4}+\frac{u_1}{p_1}+\frac{u_2}{p_2})^2/(\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}+\frac{1}{p_4})]\}$$

................(2.1.5)

Replace z_1 and z_2 by zero in the integrals in (2.1.1) and let

$$q_1 = \int_0^\infty \int_{-\infty}^0 f(x_1, x_2)dx_2dx_1$$

$$q_2 = \int_{-\infty}^0 \int_0^\infty f(x_1, x_2)dx_2dx_1$$

$$q_3 = \int_0^{\infty} \int_0^{\infty} f(x_1, x_2)dx_2dx_1$$

$$q_4 = \int_{-\infty}^{0} \int_{-\infty}^{0} f(x_1, x_2)dx_2dx_1$$

$$q_1' = \int_0^{\infty} f(x_1, 0)dx_1 \ , \quad q_2' = \int_0^{\infty} f(0, x_2)dx_2$$

$$q_3' = \int_{-\infty}^{0} f(x_1, 0)dx_1 \ , \quad q_4' = \int_{-\infty}^{0} f(0, x_2)dx_2$$

it is seen that

$$q_4 = F(0, 0)$$

$$q_2 + q_4 = \alpha_1 \ , \qquad q_1 + q_4 = \alpha_2$$

$$q_1 + q_3 = 1-\alpha_1 \ , \qquad q_2 + q_3 = 1-\alpha_2$$

$$q_1' + q_3' = f_2(0) \ , \qquad q_2' + q_4' = f_1(0) \ .$$

and

$$p_i \doteq q_i \qquad i = 1, 2, 3, 4$$

$$p_i' \doteq q_i' dz_2 \ , \ i = 1, 3 \qquad \ldots\ldots\ldots(2.1.6)$$

$$p_i' \doteq q_i' dz_1 \ , \ i = 2, 4 \ .$$

Also we have

$$u_1 = \sqrt{N} \left(\frac{n_1+n_3}{N} - (p_1 + p_3) \right)$$

$$\doteq \sqrt{N}\ [\int_0^{\infty} \int_{-\infty}^{\infty} f(x_1, x_2)dx_2dx_1$$

$$- \int_{z_1}^{\infty} \int_{-\infty}^{\infty} f(x_1, x_2)dx_2dx_1\ ]$$

$$\doteq \sqrt{N} \int_0^{z_1} \int_{-\infty}^{\infty} f(x_1, x_2)dx_2dx_1$$

$$\doteq \sqrt{N} \int_0^{z_1} f_1(x_1)dx_1$$

$$\doteq \sqrt{N}\ z_1\ f_1(\theta x_1)\ , \quad 0 < \theta < 1\ ,$$

$$\doteq \sqrt{N}\ z_1\ f_1(0) \doteq w_1 \qquad \text{.............(2.1.7)}$$

similarly,

$$u_2 = \sqrt{N}\ (\ \frac{n_2+n_3}{N} - (p_2 + p_3)\ )$$

$$\doteq \sqrt{N}\ z_2\ f_2(0) \doteq w_2 \qquad \text{.............(2.1.8)}$$

Substitute (2.1.6),(2.1.7) and (2.1.8) in (2.1.5),

$$B_1 \doteq \frac{Nq_1'q_2'\ |A|^{1/2}}{2\pi(\frac{1}{q_1}+\frac{1}{q_2}+\frac{1}{q_3}+\frac{1}{q_4})^{1/2}} \exp\{-\frac{1}{2}[w_1^2(\frac{1}{q_4}+\frac{1}{q_1}-\frac{(\frac{1}{q_4}+\frac{1}{q_1})^2}{(\frac{1}{q_1}+\frac{1}{q_2}+\frac{1}{q_3}+\frac{1}{q_4})})$$

$$+ 2w_1w_2(\ \frac{1}{q_4} - \frac{(\frac{1}{q_4}+\frac{1}{q_1})(\frac{1}{q_4}+\frac{1}{q_2})}{\frac{1}{q_1}+\frac{1}{q_2}+\frac{1}{q_3}+\frac{1}{q_4}})$$

$$+ w_2^2\ (\ \frac{1}{q} + \frac{1}{q_2} - \frac{(\frac{1}{q_4}+\frac{1}{q_2})^2}{\frac{1}{q_1}+\frac{1}{q_2}+\frac{1}{q_3}+\frac{1}{q_4}})\ ]\ \}dz_1dz_2$$

$$\text{.............(2.1.9)}$$

The quadratic form in the exponent of the above expression is $(w_1, w_2)\ R\ (w_1, w_2)'$, where

$$R = \begin{bmatrix} d + a - \frac{(d+a)^2}{a+b+c+d} & d - \frac{(d+a)(d+b)}{a+b+c+d} \\ d - \frac{(d+a)(d+b)}{a+b+c+d} & d+b - \frac{(d+b)^2}{a+b+c+d} \end{bmatrix}$$

with $a = 1/q_1$, $b = 1/q_2$, $c = 1/q_3$, $d = 1/q_4$.

$$\det R = |R| = \frac{abc + abd + acd + bcd}{a + b + c + d}$$

and

$$R^{-1} = \frac{1}{|R|} \begin{bmatrix} d+b - \frac{(d+b)^2}{a+b+c+d} & \frac{(d+a)(d+b)}{a+b+c+d} - d \\ \frac{(d+a)(d+b)}{a+b+c+d} - d & d+a - \frac{(d+a)^2}{a+b+c+d} \end{bmatrix}$$

$$= \Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$$

with

$$\sigma_{11} = \frac{\frac{(d+b)(a+b+c+d) - (d+b)^2}{a + b + c + d}}{\frac{abc + abd + acd + bcd}{a + b + c + d}}$$

$$= \frac{(d+b)(a+c)}{abc+abd+acd+bcd}$$

$$= \frac{(1/q_4 + 1/q_2)(1/q_1 + 1/q_3)}{1/(q_1 q_2 q_3 q_4)}$$

$$= (q_1 + q_3)(q_2 + q_4)$$

$$= \alpha_1(1 - \alpha_1)$$

$$\sigma_{12} = \frac{\dfrac{(d+a)(d+b)-(a+b+c+d)d}{a+b+c+d}}{\dfrac{abc+abd+acd+bcd}{a+b+c+d}}$$

$$= \frac{ab - cd}{abc + abd + acd + bcd}$$

$$= \frac{[1/(q_1q_2) - 1/(q_3q_4)]}{1/(q_1q_2q_3q_4)}$$

$$= q_3q_4 - q_1q_2$$

$$= q_4(1 - q_1 - q_2 - q_4) - q_1q_2$$

$$= q_4 - (q_4+q_2)(q_4+q_1)$$

$$= F(0, 0) - \alpha_1\alpha_2$$

since $q_1 + q_2 + q_3 + q_4 = 1$. Similar calculation yields

$$\sigma_{22} = \alpha_2(1 - \alpha_2),$$

also

$$[|A|^{1/2}]/\left(\frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} + \frac{1}{q_4}\right)^{1/2}$$

$$= \left[\frac{1/(q_1q_2q_3q_4)}{\dfrac{q_1q_2q_3+q_1q_2q_4+q_1q_3q_4+q_2q_3q_4}{q_1q_2q_3q_4}}\right]^{1/2}$$

$$= [\ 1/(q_1q_2q_3+q_1q_2q_4+q_1q_3q_4+q_2q_3q_4)\ ]^{1/2}$$

$$= |R|^{1/2} ,$$

therefore the expression for B_1 in (2.1.9) becomes

$$B_1 \doteq \frac{Nq_1'q_2'}{2\pi} |\Sigma|^{-1/2} \exp\{-\tfrac{1}{2}(w_1,w_2)\Sigma^{-1}(w_1,w_2)'\}dz_1dz_2 \quad \text{.................(2.1.10)}$$

Following the same procedure of approximation as for B_1, the other terms B_2, B_3, and B_4 give rise to identical expressions as (2.1.10) except the factor $q_1'q_2'$, they are respectively

$$B_2 \doteq \frac{Nq_1'q_4'}{2\pi} |\Sigma|^{-1/2} \exp\{-\tfrac{1}{2}(w_1,w_2)\Sigma^{-1}(w_1,w_2)'\}dz_1dz_2$$

$$B_3 \doteq \frac{Nq_2'q_3'}{2\pi} |\Sigma|^{-1/2} \exp\{-\tfrac{1}{2}(w_1,w_2)\Sigma^{-1}(w_1,w_2)'\}dz_1dz_2$$

$$B_4 \doteq \frac{Nq_3'q_4'}{2\pi} |\Sigma|^{-1/2} \exp\{-\tfrac{1}{2}(w_1,w_2)\Sigma^{-1}(w_1,w_2)'\}dz_1dz_2 ,$$

and expression (2.1.2) becomes

$$g(z_1,z_2)dz_1dz_2 \doteq \frac{N(q_1'q_2'+q_1'q_4'+q_2'q_3'+q_3'q_4')}{2\pi} |\Sigma|^{-1/2}$$

$$\exp\{-\tfrac{1}{2}(w_1,w_2)\Sigma^{-1}(w_1,w_2)'\}dz_1dz_2$$

$$= \frac{Nf_1(0)f_2(0)|\Sigma|^{-1/2}}{2\pi} \exp\{-\tfrac{1}{2}(w_1,w_2)\Sigma^{-1}(w_1,w_2)'\}dz_1dz_2$$

$$= \frac{|\Sigma|^{-1/2}}{2\pi} \exp\{-\tfrac{1}{2}(w_1,w_2)\Sigma^{-1}(w_1,w_2)'\}dw_1dw_2 .$$

since

$$q_1'q_2' + q_1'q_4' + q_2'q_3' + q_3'q_4'$$

$$= (q_1'+q_3')(q_2'+q_4')$$

$$= f_1(0)\ f_2(0) .$$

That is, the joint distribution of W_1, W_2 tends to a bivariate normal distribution with means 0, 0 and variances and covariance

$$\text{Var } W_i = \alpha_i\ (1 - \alpha_i) , \quad i = 1, 2$$

and $$\text{Cov}(W_1,W_2) = F(0,0) - \alpha_1\alpha_2$$

where $F(x_1, x_2)$ is the distribution function of (X_1,X_2).

For general m $(m \geq 2)$, we will derive the form of the joint density of the variables rather than actually calculating it. Given a sample of size N from the variable $(X_1, X_2, \ldots, X_m)'$ let $(Z_1, Z_2, \ldots, Z_m)'$ be the sample quantiles of order $(\alpha_1, \alpha_2, \ldots, \alpha_m)$.

Without loss of generality, we assume that the population quantiles $(\xi_1, \xi_2, \ldots, \xi_m)' = (0, 0, \ldots, 0)'$.

Consider the probability that the sample quantiles $(Z_1, Z_2, \ldots, Z_m)$ lies in the hyperparallopiped $R^{(m)}$,

$$z_i - \tfrac{1}{2}dz_i < x_i < z_i + \tfrac{1}{2}dz_i \quad , \quad i = 1,2,\ldots,m .$$

The m-dimensional space is divided into 3^m regions by means of hyperplanes

$$x_i = z_i - \tfrac{1}{2}dz_i$$

$$x_i = z_i + \tfrac{1}{2}dz_i \quad , \qquad i = 1,2,\ldots,m ;$$

which are perpendicular to the x_i-axis. These regions are illustrated in Figure 2 for m = 3. There are 2^m primary regions $R_1, R_2, \ldots$, which correspond to the octants of 3-dimensional space as in Figure 2; $\binom{m}{1}2^{m-1}$ regions with one differential dimension, $R_1', R_2', \ldots$, which correspond to the slabs of Figure 2; $\binom{m}{2}2^{m-2}$ regions with two differential dimensions, $R_1'', R_2'', \ldots$; $\binom{m}{3}2^{m-3}$ regions with three differential dimensions and so forth; .....; $\binom{m}{m-1}2^{m-(m-1)}$ regions with (m-1) differential dimensions and $\binom{m}{m}2^{m-m}$ region with m differential dimensions, the last region being $R^{(m)}$.

Total number of regions is

$$2^m + \binom{m}{1} 2^{m-1} + \binom{m}{2} 2^{m-2} + \binom{m}{3} 2^{m-3} + \ldots + \binom{m}{m-1} 2 + 1$$

$$= (2 + 1)^m = 3^m .$$

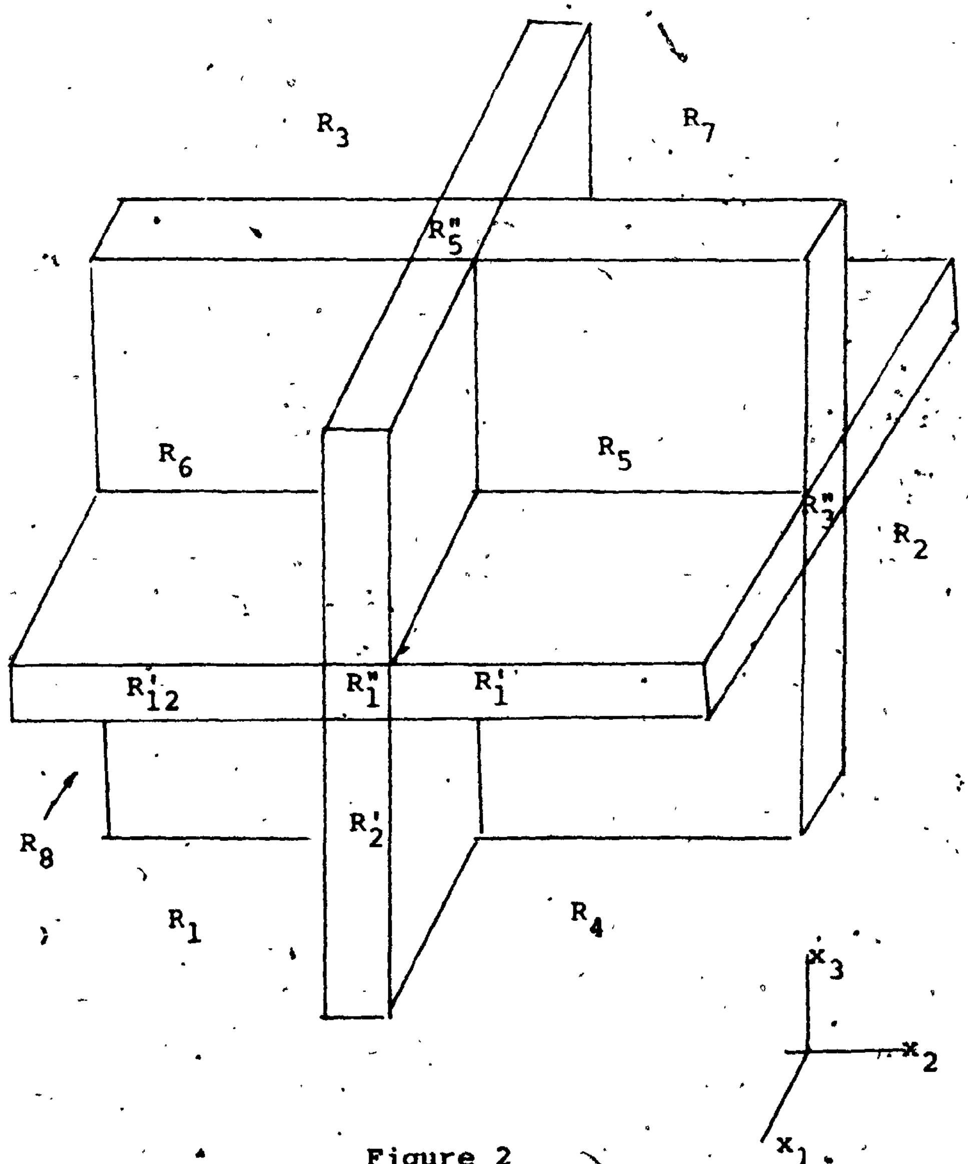

Figure 2

Let $p_i^{(j)}$ be the probability that an element falls in the region $R_i^{(j)}$,

$$p_i^{(j)} = \int_{R_i^{(j)}} f(x_1,x_2,\ldots,x_m)dx_1dx_2\ldots dx_m \; .$$

Neglecting terms involving differentials of higher order, it is seen that

$$p_i = \int_{R_i^*} f(x_1,x_2,\ldots,x_m)dx_1dx_2\ldots dx_m$$

$$p_i' = [\int_{R_i'^*} f(x_1,x_2,\ldots,x_m)\prod^* dx_i]dz_\beta$$

where R_i^* is the region R_i with its possible boundaries $z_i \pm \frac{1}{2}dz_i$ replaced by z_i and $R_i'^*$ is one-dimension-less region obtained from R_i' by omitting the differential dimension. $\prod^*$ indicates that one of dx_i's is omitted. If the differential dimension is dz_β, then x_β is replaced by z_β in $f(x_1, x_2, \ldots, x_m)$.

With these set-ups, we may consider that the sample is drawn from a multinomial population with probabilities $p_i^{(j)}$ falling in the region $R_i^{(j)}$. We will pick up those terms which give rise to the sample quantiles $(Z_1, Z_2, \ldots, Z_m)$.

There are two distinct cases namely case (a): the sample quantiles $(Z_1, Z_2, \ldots, Z_m)'$ is determined by m different elements of the sample; case (b): the sample quantiles $(Z_1, Z_2, \ldots, Z_m)'$ is determined by less than m elements of the sample. We study the two cases separately.

Case (a): the sample quantiles $(Z_1, Z_2, \ldots, Z_m)'$ is determined by m different elements of the sample, then there is one of these m elements in each of m regions R_i''s whose differential dimensions are mutually perpendicular, and the remaining of the elements of the sample fall in the primary regions, $R_1, R_2, \ldots, R_{2^m}$ with n_i elements respectively. The n_i's are subjected to the independent restrictions of the following type,

$$\sum{}' n_i = N' - C' \qquad \ldots\ldots\ldots \text{(2.1.11)}$$

where N' is one of $N-([N\alpha_i]+1)$, $[N\alpha_i]$, $i = 1, 2, \ldots, m$ and $0 \leq C' < m$ as in the case for $m = 2$, depending on on which side of which hyperplane $x_i = z_i$. $\sum{}'$ indicates that the sum is to be taken over all n_i's on the same side of a hyperplane. In addition to these restrictions,

$$\sum_{i=1}^{2^m} n_i = N - m .$$

The probability of this occurence for a particular choice of m regions R_i''s is

$$B = \prod_{\gamma=1}^{m} p'_{i_\gamma} \sum [N!/(\prod_{i=1}^{2^m} n_i!)] \prod_{i=1}^{2^m} p_i^{n_i}$$

$$= N(N-1)\ldots(N-m+1) \prod p'_{i_\gamma} \sum \frac{(N-m)!}{\Pi\, n_i!} \prod p_i^{n_i}$$

where the summation means sum over all such possible combinations. Note that the term after the summation sign is a multinomial probability since $\Sigma\, n_i = N-m$. There are altogether $2^{m(m-1)}$ such B's. In order to include all ways in which the sample quantiles are determined by m different elements of the sample, we add together those B's.

Case (b): if the sample quantiles are determined by less than m elements, say m-h elements, $0 < h < m$, the probability of this occurrence for a particular choice is of the form

$$C = \prod_{\delta} p_{i_\delta}^{(j)} \sum \frac{N!}{\Pi\, n_i!} \prod_{i=1}^{2^m} p_i^{n_i}$$

$$= N(N-1)\ldots(N-m+h-1) \prod_{\delta} p_{i_\delta}^{(j)} \sum \frac{(N-m+h)!}{\Pi\, n_i!} \prod p_i^{n_i}$$

with those $p_{i_\delta}^{(j)}$ such that $\Sigma j = m$. But now $\Sigma n_i = N-m+h$ and C is of lower power in N as compared to B, and thus C may be omitted in obtaining the asymptotic expression and we are thus left to find only the asymptotic form for those B's.

If $g(z_1, z_2, \ldots, z_m)$ is the density that gives the distribution of $Z_1, Z_2, \ldots, Z_m$, then

$$\int g(z_1, z_2, \ldots, z_m) dz_1 \ldots dz_m$$

$$= \sum_1 C + \sum_2 B \qquad \ldots\ldots\ldots\ldots(2.1.12)$$

where $\sum_1$ means sum of all such C's which arise from the case where $(Z_1, Z_2, \ldots, Z_m)'$ is determined by $m-h$ elements of the sample and $\sum_2$ means sum of all such B's which arise from the case where $(Z_1, Z_2, \ldots, Z_m)'$ is determined by m different elements of the sample.

Consider the term B. Neglecting terms of lower power in N

$$B = N^m \prod_{\gamma=1}^{m} p'_{i_\gamma} \sum \frac{(N-m)!}{\Pi\, n_i!} \prod p_i^{n_i}$$

using the normal approximation with $r = 2^m$, the above B becomes

$$B = N^m \prod_{\gamma=1}^{m} p_{i_\gamma}^{!} \sum [|A|/(2\pi)^{r-1}]^{1/2}$$

$$\exp\{ -\frac{1}{2} \sum A_{ij} t_i t_j \} \prod_{i=1}^{r-1} dt_i \quad \ldots .(2.1.13)$$

where

$$t_i = \frac{n_i - Np_i}{\sqrt{N}}, \qquad i = 1, 2, \ldots, r-1,$$

and A is the matrix $A = (A_{ij})$ with

$$A_{ii} = \frac{1}{p_i} + \frac{1}{p_r}, \qquad i = 1, 2, \ldots, r-1,$$

and

$$A_{ij} = \frac{1}{p_r} \qquad \text{for } i \neq j .$$

Now we define

$$u_1 = \sqrt{N} \left[\frac{\sum_1 n_i}{N} - \sum_1 p_i \right]$$

where the summation is taken over all n_i's on the positive side of the coordinate hyperplane $x_1 = z_1$, with n_1 being one of the n_i's.

Similarly, we define

$$u_2 = \sqrt{N} \left[\frac{\sum_2 n_i}{N} - \sum_2 p_i \right]$$

$$\vdots$$

$$u_m = \sqrt{N} \left[\frac{\sum_m n_i}{N} - \sum_m p_i \right] .$$

where the summation $\sum_m$ means sum over all n_i's on the positive side of the coordinate hyperplane $x_m = z_m$, with n_m being one of the n_i's .

It is seen that

$$t_1 = \frac{n_1 - Np_1}{\sqrt{N}}$$

$$= u_1 - \sum_{-1} t_i$$

where $\sum_{-1}$ sums over the same indices as $\sum_1$ except the index 1 , and

$$t_2 = u_2 - \sum_{-2} t_i$$

$$\vdots$$

$$t_m = u_m - \sum_{-m} t_i$$

where $\sum_{-m}$ sums over the same indices as $\sum_m$ except the index m .

The primary regions R_i, $i = 1, 2, \ldots, 2^m$, can be so labelled that it will result in each $\sum_{-1} t_i$, $\sum_{-2} t_i$, ..., $\sum_{-m} t_i$ being a sum of certain subsets of $t_{m+1}, t_{m+2}, \ldots, t_{2^m-1}$. The following is a way to accomplish this:

On positive side of $x_i = z_i$ label the region where $x_k < z_k$ $k \neq i$ as R_i , $i = 1, 2, \ldots, m$.

Label arbitrarily the rest of the regions as R_{m+1}, R_{m+2}, ..., R_{2^m-1} reserving the region where $x_i \leq z_i$ for all i to be labelled as R_{2^m}. For examples, the labellings are shown in Figure 3 for m = 2 and in Figure 4 for m=3.

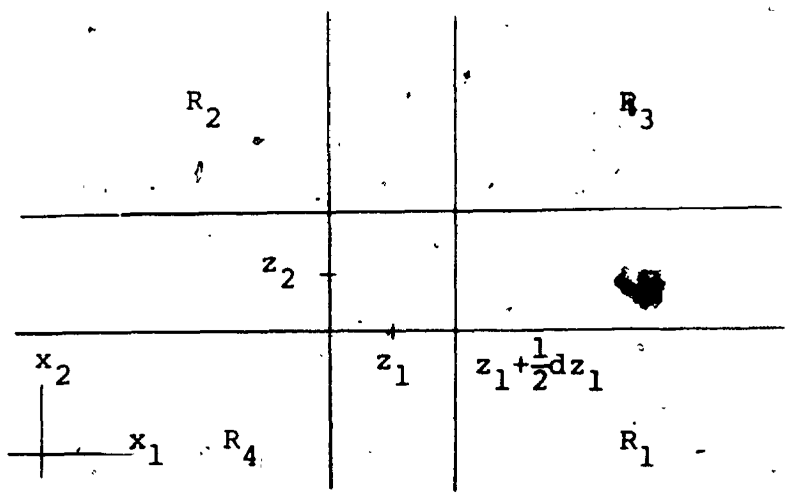

Figure 3.

According to Figure 3,

$$u_1 = \sqrt{N}\ [\frac{n_1+n_3}{N} - (p_1+p_3)]$$

$$u_2 = \sqrt{N}\ [\frac{n_2+n_3}{N} - (p_2+p_3)]$$

and

$$t_1 = u_1 - t_3$$

$$t_2 = u_2 - t_3\ .$$

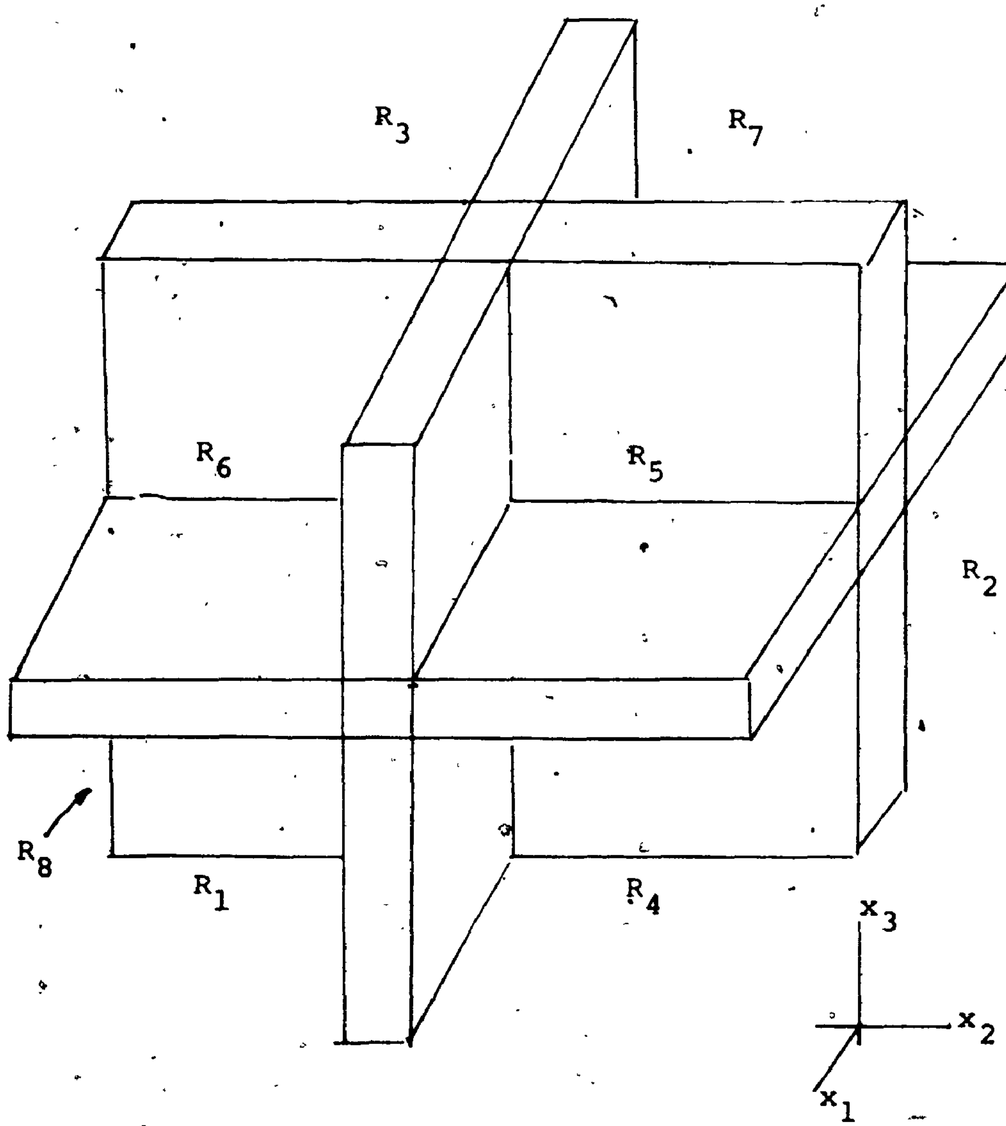

Figure 4

According to Figure 4 ,

$$u_1 = \sqrt{N}\ [\ \frac{n_1+n_4+n_5+n_6}{N} - (p_1+ p_4+ p_5+ p_6)]$$

$$u_2 = \sqrt{N}\ [\ \frac{n_2+n_4+n_5+n_7}{N} - (p_2+ p_4+ p_5+ p_7)]$$

$$u_3 = \sqrt{N}\ [\ \frac{n_3+n_5+n_6+n_7}{N} - (p_3+ p_5+ p_6+ p_7)]$$

and $t_1 = u_1 - (t_4 + t_5 + t_6) = u_1 - \sum_{-1} t_i$

$t_2 = u_2 - (t_4 + t_5 + t_7) = u_2 - \sum_{-2} t_i$

$t_3 = u_3 - (t_5 + t_6 + t_7) = u_3 - \sum_{-3} t_i$.

Substitute $t_i = u_i - \sum_{-i} t_j$, $i = 1, 2, \ldots, m$ in the expression for B in (2.1.13), the quadratic form in the exponential in t_k , $k = 1, 2, \ldots, 2^{m-1}$, will become quadratic form in $u_1, u_2, \ldots, u_m, t_{m+1}, t_{m+2}, \ldots, t_{2^m-1}$. Since t_k, $k = 1, 2, \ldots, 2^m-1$, are joint normal and u_i, $i = 1, 2, \ldots, m$, are certain linear combinations of the t_k, hence $u_1, u_2, \ldots, u_m, t_{m+1}, t_{m+2}, \ldots, t_{2^m-1}$ are joint normal.

Recall that dt_i corresponds to $1/\sqrt{N}$, thus cancelling $N^{m/2}$ with $dt_1 dt_2 \ldots dt_m$, (2.1.13) becomes

$$B = N^{m/2} \prod_{\gamma=1}^{m} p_{i_\gamma}^{!} \sum [|A| / (2\pi)^{r-1}]^{1/2}$$

$$\exp\{-\frac{1}{2} Q(u_1, \ldots, u_m, t_{m+1}, \ldots, t_{2^m-1})\} \prod_{i=m+1}^{2^m-1} dt_i$$

where Q is quadratic form in $u_1, \ldots, u_m, t_{m+1}, \ldots, t_{2^m-1}$. Q will be used generically to denote quadratic form and is not the same from equation to equation.

In order to get rid of the summation sign, we integrate $t_{m+1}, \ldots, t_{2^m-1}$ each from $-\infty$ to ∞. This is equivalent to finding the joint marginal of $u_1, u_2, \ldots, u_m$. We get

$$B = K|A|^{1/2} \left[\frac{N}{2\pi}\right]^{m/2} \prod_{\gamma=1}^{m} p'_{i_\gamma} \exp\{-\tfrac{1}{2}Q(u_1, \ldots, u_m)\} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.1.14)$$

where the remaining constant of integration is absorbed in K, and $u_1, u_2, \ldots, u_m$ are joint normal.

Define

$$q_i = \int_{\bar{R}_i} f(x_1, x_2, \ldots, x_m) \prod_{j=1}^{m} dx_j$$

$$q'_i = \int_{\bar{R}'_i} f(x_1, x_2, \ldots, x_m) \prod{}^{*} dx_j$$

$\bar{R}_i$ corresponds to R_i and are regions bounded by the coordinate hyperplanes $x_s = 0$, $s = 1, 2, \ldots, m$; and $\bar{R}'_i$ corresponds to R'_i and are regions into which the coordinate hyperplanes are divided by the remaining coordinate hyperplanes; for example, when $m = 3$, the four parts of the plane $x_3 = 0$, which is resulted when the plane $x_3 = 0$ is divided by the planes $x_1 = 0$, and $x_2 = 0$.

It is seen that

$$p_i \doteq q_i \quad , \quad i=1,2,\ldots, 2^m$$

$$\ldots\ldots\ldots(2.1.15)$$

and $$\prod_{\gamma=1}^{m} p'_{i_\gamma} = \prod_{\gamma=1}^{m} q'_{i_\gamma} dz_\gamma \; .$$

Substitute (2.1.15) in (2.1.14), B becomes

$$B \doteq K \left[\frac{N}{2\pi}\right]^{m/2} \exp\{-\tfrac{1}{2}Q(u_1,u_2,\ldots,u_m)\} \prod_{i=1}^{m} dz_i$$

$$\ldots\ldots\ldots\ldots\ldots\ldots(2.1.16)$$

where the rest of the constant is absorbed in K .

But we have

$$u_j = \sqrt{N} \left[\sum_j \frac{n_i}{N} - \sum_j p_i \right]$$

$$\doteq \sqrt{N} \left[\int_0^\infty \left(\int_\infty \ldots \int_\infty f(x_1,\ldots,x_m) \prod^{(j)} dx_i \right) dx_j \right.$$

$$\left. - \int_{z_j}^\infty \left(\int_{-\infty}^\infty \quad \int_{-\infty}^\infty f(x_1,\ldots,x_m) \prod^{(j)} dx_i \right) dx_j \right]$$

where $\prod^{(j)}$ means dx_j is missing ;

$$\doteq \sqrt{N} \left[\int_0^\infty f_j(x_j)dx_j - \int_{z_j}^\infty f_j(x_j)dx_j \right]$$

$$\doteq \sqrt{N} f_j(0)\, z_j \doteq w_j \; , \qquad \ldots\ldots\ldots\ldots(2.1.17)$$

$j = 1, 2, \ldots, m$ as in the case for $m = 2$.

Substitute (2.1.17) in (2.1.16),

$$B_{\cdot} = K\left[\frac{N}{2\pi}\right]^{m/2}\exp\{-\tfrac{1}{2}Q(w_1,w_2,\ldots,w_m)\}\prod_{i=1}^{m} dz_i \quad .$$

................(2.1.18)

The other B 's will give the same expression as in (2.1.18) except that the constant K will be different. Thus summing up all those B 's (2,1.12) becomes

$$g(z_1,z_2,\ldots,z_m)dz_1dz_2\ldots dz_m$$

$$= K^*\left[\frac{N}{2\pi}\right]^{m/2}\exp\{-\tfrac{1}{2}Q(w_1,w_2,\ldots,w_m)\}\prod_{i=1}^{m} dz_i \,,$$

$$= K^*\left[\frac{1}{2\pi}\right]^{m/2}\exp\{-\tfrac{1}{2}Q(w_1,w_2,\ldots,w_m)\}\prod_{i=1}^{m} dw_i \,.$$

K^* can be determined by integrating the right-hand-side and equate it to one. W_1, W_2, ..., W_m are joint normal since u_1, u_2, ..., u_m are joint normal.

In view of Lemma 1.4 , to specify the asymptotic form of W_1, W_2, ..., W_m, only asymptotic means, variances and covariances of the W_i 's are needed. However, that can be done by considering the marginal distribution of the bivariate (W_i, W_j) $i \neq j$, and this has been done. This establishes Theorem 2.1 .

CHAPTER 3

THE ASYMPTOTIC DISTRIBUTION OF SEVERAL QUANTILES FROM EACH COMPONENT OF A MULTIVARIATE POPULATION

3.1 INTRODUCTION

The technique used in the last chapter can be extended to the case where one or more quantiles are taken from each component.

Let $F(x_1, x_2, \ldots, x_m)$, $f(x_1, x_2, \ldots, x_m)$ be respectively the known c.d.f. and p.d.f. of the m-variate continuous random variable $(X_1, X_2, \ldots, X_m)'$. Denote the marginal c.d.f. of X_i by $F_i(x_i)$ and the marginal p.d.f. by $f_i(x_i)$. Let α_{ij}, $j = 1, 2, \ldots, r_i$, $i = 1, 2, \ldots, m$ be set of real numbers such that

$$0 < \alpha_{ir_i} < \alpha_{i(r_i-1)} < \cdots < \alpha_{i1} < 1, \; i=1,2,\ldots,m.$$

Corresponding to these real numbers, denote the r_i population quantiles of X_i by $\xi_{ir_i}, \xi_{i(r_i-1)}, \ldots, \xi_{i1}$ with

$$\xi_{ir_i} < \xi_{i(r_i-1)} \cdots < \xi_{i1}, \; i = 1, 2, \ldots, m .$$

A sample of size N is taken from $(X_1, X_2, \ldots, X_m)'$. Denote the corresponding r_i sample quantiles of X_i by $Z_{ir_i}, Z_{i(r_i-1)}, \ldots, Z_{i1}$ with

$$Z_{ir_i} < Z_{i(r_i-1)} < \ldots < Z_{i1} , \quad i=1,2,\ldots, m .$$

We will establish the following theorem:

Theorem 3.1

Let $\{ Z_{ij} \mid j=1,2,\ldots,r_i;\ i=1,2,\ldots,m \}$ be set of several quantiles from each component of the m-variate continuous variable $(X_1, X_2, \ldots, X_m)'$ with strictly increasing known c.d.f. $F(x_1, x_2, \ldots, x_m)$ and p.d.f. $f(x_1, x_2, \ldots, x_m)$. Let $f_i(x_i)$ be the marginal p.d.f. of X_i $i=1,2,\ldots, m$; satisfying Assumption 2.1. Then the joint distribution of $W_{ij} = \sqrt{N}\, f_i(\xi_{ij})\ (Z_{ij} - \xi_{ij})$, $j=1,2,\ldots, r_i$; $i=1,2,\ldots, m$; tends to a Σr_i-dimensional normal distribution with means 0, 0, ..., 0 and variances and covariances

$$\text{Var } W_{ij} = \alpha_{ij}(1-\alpha_{ij}) \ , \ j=1,2,\ldots,r_i;\ i=1,2,\ldots, m$$

$$\text{Cov}(W_{ij},W_{kl}) = F_{ik}(\xi_{ij},\xi_{kl}) - F_i(\xi_{ij})F_k(\xi_{kl}),\ i\neq k ,$$

$$\text{Cov}(W_{ij},W_{il}) = \alpha_{ij}(\ 1 - \alpha_{il}) \quad \text{with } \alpha_{ij} < \alpha_{il}$$

where F_{ik}, F_s are respectively the c.d.f. of (X_i,X_k), X_s.

3.2 THE ASYMPTOTIC JOINT DISTRIBUTION OF SEVERAL QUANTILES FROM EACH COMPONENT OF A m-VARIATE POPULATION

Consider the probability of the following event

$$z_{ij} - \frac{1}{2}dz_{ij} < Z_{ij} < z_{ij} + \frac{1}{2}dz_{i}, \quad j=1,2,\ldots, r_i \quad i=1,2,\ldots, m .$$

Divid the m-dimensional space into different regions by hyperplanes

$$x_i = z_{ij} - \frac{1}{2}\, dz_{ij} \qquad j = 1, 2, \ldots, r_i$$

$$x_i = z_{ij} + \frac{1}{2}\, dz_{ij} , \qquad i = 1, 2, \ldots, m .$$

Let R_i, $i = 1, 2, \ldots, \prod_{i=1}^{m} (r_i+1)$, denote the primary regions without differential dimension; let R_i' 's denote the regions with one differential dimension and let p_i, p_i' be the probabilities that an element falls in R_i and R_i' respectively.

We label the primary regions R_i as follow: Label the region where $x_i < z_{ir_i}$ for all i, $i=1,2,\ldots, m$, as $R_{\Pi(r_i+1)}$. On the positive side of $x_1 = z_{11}$, label the region where $x_i < z_{ir_i}$ $i \neq 1$, as R_1, and label the (r_1-1) regions on the negative side of $x_1 = z_{11}$ where $x_i < z_{ir_i}$ $i \neq 1$ as $R_2, R_3, \ldots, R_{r_1}$.

On the positive side of $x_2 = z_{21}$ label the region where $x_i < z_{ir_i}$ $i \neq 2$ as R_{r_1+1} and label the (r_2-1) regions on the negative side of $x_2 = z_{21}$ where $x_i < z_{ir_i}$ $i \neq 2$ as $R_{r_1+2}, R_{r_1+3}, \ldots, R_{r_1+r_2}$.

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On the positive side of $x_m = z_{m1}$ label the region where $x_i < z_{ir_i}$ $i \neq m$ as $R_{\sum_{i=1}^{m-1} r_i + 1}$ and label the r_m-1 regions on the negative side of $x_m = z_{m1}$ where $x_i < z_{ir_i}$ $i \neq m$ as $R_{\sum_{i=1}^{m-1} r_i + 2}, \ldots, R_{\sum_{i=1}^{m} r_i}$. The remaining of the R_i's are arbitrarily labelled.

Let

$$P_i = \int_{R_i} f(x_1, x_2, \ldots, x_m) \prod_{i=1}^{m} dx_i$$

$$P_i' = \int_{R_i'} f(x_1, x_2, \ldots, x_m) \prod_{i=1}^{m} dx_i$$

Neglecting terms involving differentials of higher order it is seen that,

$$p_i = \int_{R_i^*} f(x_1, x_2, \ldots, x_m) \prod_{i=1}^{m} dx_i$$

$$p_i' = \int_{R_i'^*} f(x_1, x_2, \ldots, x_m) \prod^* dx_i \, dz_{\beta\gamma}$$

where R_i^* is the region R_i with its possible boundaries $z_{ij} \pm \frac{1}{2} dz_{ij}$ replaced by z_{ij}, and $R_i'^*$ is one-dimensionless region obtained from R_i' by omitting the differential dimension. $\prod^*$ indicates that one of the dx_i 's is omitted. If the differential dimension is $dz_{\beta\gamma}$, and is parallel to the x_i- axis, then x_i is replace by $z_{\beta\gamma}$ in $f(x_1, x_2, \ldots, x_m)$.

If $\{ Z_{ij} \mid j=1,2,\ldots,r_i;\ i=1,2,\ldots,m \}$ is determined by less than D ($D = \sum_{i=1}^{m} r_i$) elements of the sample, the terms arised from this case can be neglected in the asymptotic expression as in two-dimensional situation.

We are only concerned with terms which arise from the case where $\{ Z_{ij} \mid j=1,2,\ldots, r_i;\ i=1,2,\ldots, m \}$ is determined by D different elements of the sample. If this is so, then there is one element in each of the r_i slides , $i = 1, 2, \ldots, m$

$$z_{ij} - \frac{1}{2} dz_{ij} < x_i < z_{ij} + \frac{1}{2} dz_{ij} \ , \ j=1,2,\ldots,r_i .$$

Consider one of these possibilities where one element is in each of those slides $z_{ij}-\frac{1}{2}dz_{ij} < x_i < z_{ij}+\frac{1}{2}dz_{ij}$, with $x_k > z_{k1}$, $k \neq i$; the probability of this occurrence is, with $r = \prod_{i=1}^{m} (r_i+1)$

$$B = \prod_{\gamma=1}^{D} p_{i_\gamma}^{!} \sum \frac{N!}{\Pi n_i!} \prod_{i=1}^{r} p_i^{n_i} \qquad \ldots\ldots\ldots\ldots(3.2.1)$$

where n_i are number of elements in R_i and $\sum_{i=1}^{r} n_i = N-D$.

If $g(z_{11}, z_{12}, \ldots, z_{mr_m})$ is the density that gives the distribution of $z_{11}, z_{12}, \ldots, z_{m1}, \ldots, z_{mr_m}$,

$$g(z_{11}, z_{12}, \ldots, z_{mr_m}) \prod_{i=1}^{m} \prod_{j=1}^{r_i} dz_{ij}$$

$$= \sum_1 C + \sum_2 B \qquad \ldots\ldots\ldots\ldots(3.2.2)$$

where $\sum_1$ means sum of all such C which arises from the case where $\{ Z_{ij} \mid j=1,2,\ldots,r_i;\ i=1,2,\ldots, m \}$ is determined by D-h ($D > h > 0$) elements of the sample and $\sum_2$ means sum of all such B which arises from the case that it is determined by D distinct elements.

Consider the term B in (3.2.1),

$$B = N(N-1)\ldots(N-D+1) \prod_{\gamma=1}^{D} p'_{i_\gamma} \sum \frac{(N-D)!}{\Pi\, n_i!} \prod_{i=1}^{r} p_i^{n_i} ,$$

neglecting terms of lower power in N and applying the normal approximation,

$$B \doteq N^{D} \prod_{\gamma=1}^{D} p'_{i_\gamma} \sum [|A|/(2\pi)^{r-1}]^{1/2}$$

$$\exp\{-\tfrac{1}{2} \sum A_{ij} t_i t_j\} \prod_{i=1}^{r-1} dt_i \quad \ldots\ldots(3.2.3)$$

where

$$t_i = \frac{n_i - Np_i}{\sqrt{N}} , \quad i = 1, 2, \ldots, r-1 ,$$

and $A = (A_{ij})$ with

$$A_{ii} = \frac{1}{p_i} + \frac{1}{p_r} , \qquad i = 1, 2, \ldots, r-1 ,$$

and $$A_{ij} = \frac{1}{p_r} \qquad \text{for } i \neq j .$$

Define

$$u_1 = \sqrt{N}\, [\textstyle\sum_1 \frac{n_i}{N} - \sum_1 p_i]$$

where $\sum_1$ indicates sum over the regions R_i on the positive side of $x_1 = z_{11}$.

$$u_2 = \ldots$$

$$\vdots$$

$$u_{r_1} = \sqrt{N} \; [\; \textstyle\sum_{r_1} \frac{n_i}{N} - \sum_{r_1} p_i \;]$$

where $\sum_{r_1}$ indicates sum over the regions R_i on the positive side of $x_1 = z_{1r_1}$,

$$u_{r_1+1} = \ldots$$

$$\vdots$$

$$u_{r_1+r_2}$$

$$\vdots$$

$$u_D = \sqrt{N} \; [\; \textstyle\sum_D \frac{n_i}{N} - \sum_D p_i \;]$$

where $\sum_D$ indicates sum over the regions R_i on the positive side of $x_m = z_{mr_m}$.

We see that

$$t_1 = u_1 - \textstyle\sum_{-1} t_j \, ,$$

$$t_2 = u_2 - \textstyle\sum_{-2} t_j \, , \qquad \ldots\ldots\ldots (3.2.4)$$

$$\vdots$$

$$t_D = u_D - \textstyle\sum_{-D} t_j \; ;$$

where $\sum_{-i}$ indicates sum over the same indices as in $\sum_i$ except the index i .

It is seen that by the way we label the primary regions R_i's , $t_1, t_2, \ldots, t_D$ are linear functions of $u_1, u_2, \ldots, u_D, t_{D+1}, \ldots, t_{r-1}$. Since $t_1, \ldots, t_{r-1}$ are joint normal, $u_1, \ldots, u_D, t_{D+1}, \ldots, t_{r-1}$ are joint normal .

Substitute (3.2.4) in (3.2.3),

$$B = N^D \prod p'_{i_\gamma} \sum{}' [|A|/(2\pi)^{r-1}]^{1/2}$$

$$\exp\{-\tfrac{1}{2}Q(u_1,\ldots,u_D,t_{D+1},\ldots,t_{r-1})\} \prod_{i=1}^{r-1} dt_i$$

$$= N^{D/2} \prod p'_{i_\gamma} \sum [|A|/(2\pi)^{r-1}]^{1/2}$$

$$\exp\{-\tfrac{1}{2} Q(u_1,\ldots,u_D,t_{D+1},\ldots,t_{r-1})\} \prod_{i=D+1}^{r-1} dt_i$$

where Q is quadratic form in $u_1, \ldots, u_{D+1}, t_{D+1}, \ldots, t_{r-1}$. Q is used generically to denote quadratic form and is not the same from equation to equation.

In order to get rid of the summation sign, we integrate t_i , $i = D + 1, \ldots, r-1$ each from $-\infty$ to ∞ .

This is equivalent to finding the joint marginal of u_1, $u_2, \ldots, u_D$; we get

$$B \doteq K \, |A|^{1/2} [\frac{N}{2\pi}]^{D/2} \prod p'_{i_\gamma} \exp\{-\frac{1}{2} Q(u_1, u_2, \ldots, u_D)\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3.2.5)$$

where the rest of the constant of integration is absorbed in K .

Define

$$q_i = \int_{R_i^{**}} f(x_1, x_2, \ldots, x_m) \prod_{i=1}^{m} dx_i$$

$$q'_i = \int_{R_i^{'**}} f(x_1, x_2, \ldots, x_m) \prod{}^{*} dx_i$$

where R_i^{**}, $R_i^{'**}$ are respectively R_i^* ; $R_i^{'*}$ with possible boundaries z_{ij} replaced by ξ_{ij} .

We see that

$$p_i \doteq q_i ,$$

and
$$\prod_{\gamma=1}^{D} p'_{i_\gamma} = \prod_{\gamma=1}^{D} q'_{i_\gamma} \prod_{i=1}^{m} \prod_{j=1}^{r_i} dz_{ij} \quad \ldots\ldots\ldots (3.2.6)$$

Substitute (3.2.6) in (3.2.5), we get

$$B \doteq K \left[\frac{N}{2\pi}\right]^{D/2} \exp\{-\tfrac{1}{2}Q(u_1, u_2, \ldots, u_D)\} \prod_{i=1}^{m} \prod_{j=1}^{r_i} dz_{ij}$$

................(3.2.7)

with the rest of the constant absorbed in K .

But we have

$$u_1 = \sqrt{N}\ [\ \textstyle\sum_1 \frac{n_i}{N} - \sum_1 p_i\ ]$$

$$\doteq \sqrt{N}\ [\ \int_{\xi_{11}}^{\infty} (\ \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, \ldots, x_m) \prod_{i=2}^{m} dx_i) dx_1$$

$$- \int_{z_{11}}^{\infty} (\ \int_{-\infty}^{\infty} \quad \int_{-\infty}^{\infty} f(x_1, \ldots, x_m) \prod_{i=2}^{m} dx_i) dx_1]$$

$$\doteq \sqrt{N}\ [\ \int_{\xi_{11}}^{\infty} f_1(x_1) dx_1 - \int_{z_{11}}^{\infty} f_1(x_1) dx_1\ ]$$

$$\doteq \sqrt{N}\ (z_{11} - \xi_{11})\ f_1(\xi_{11})$$

$$\doteq w_{11}$$

similarly,

$$u_2 \doteq w_{12} .$$

$$\vdots$$

................(3.2.8)

$$u_D \doteq w_{mr_m} .$$

Substitute (3.2.8) in (3.2.7), B becomes

$$B \doteq K\left[\frac{N}{2\pi}\right]^{D/2} \exp\{-\tfrac{1}{2}Q(w_{11}, w_{12}, \ldots, w_{mr_m})\} \prod_{i=1}^{m} \prod_{j=1}^{r_i} dz_{ij}$$

..................(3.2.9)

Other B 's will give rise the identical asymptotic expression as in (3.2.9) except that the factor K will be different; it is clear then that

$$g(z_{11}, z_{12}, \ldots, z_{mr_m}) \prod_{i=1}^{m} \prod_{j=1}^{r_i} dz_{ij}$$

$$\doteq K^* \left[\frac{N}{2\pi}\right]^{D/2} \exp\{-\tfrac{1}{2}Q(w_{11}, w_{12}, \ldots, w_{mr_m})\} \prod_{i=1}^{m} \prod_{j=1}^{r_i} dz_{ij}$$

$$\doteq \frac{K^*}{(2\pi)^{D/2}} \exp\{-\tfrac{1}{2}Q(w_{11}, w_{12}, \ldots, w_{mr_m})\} \prod_{i=1}^{m} \prod_{j=1}^{r_i} dw_{ij}$$

where $D = \sum_{i=1}^{m} r_i$, the constant K^* can be determined by integrating the right-hand-side and equate to one. W_{ij} , $j = 1, 2, \ldots, r_i$; $i = 1, 2, \ldots, m$ are joint normal since $u_1, u_2, \ldots, u_D$ are joint normal and each w_{ij} is a linear function of the u_i 's.

In view of lemma 2.4 , to specify the asymptotic distribution of $W_{11}, W_{12}, \ldots, W_{mr_m}$ only the asymptotic means and variances and covariances between the variables are needed. However that can be done by considering the bivariate distribution of any two of the W_{ij}'s, say W_{ij}, W_{kc} as in the last chapter if $i \neq k$; if $i = k$, then the sample quantiles comes from the same component and this is well-known. Therefore the joint distribution of $W_{11}, W_{12}, \ldots, W_{mr_m}$ tends to a $\sum_{i=1}^{m} r_i$ -dimensional normal distribution with means and variances and covariances as mentioned in the theorem. This establishes Theorem 3.1 .

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