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THE REPRESENTATIVE AGENT, OVERLAPPING GENERATIONS, AND ASSET PRICING

by

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I would like to thank Neil Wallace for helpful comments. Any remaining errors are my responsibility.
Introduction

Two popular theoretical constructs currently used in equilibrium modelling are the infinitely-lived representative agent framework, and the finite-lived overlapping generations framework. This paper will attempt to compare these two models and the different implications they have for asset pricing.

A frequently used strategy is to suppose that the economy is composed of a representative infinitely-lived agent who maximizes the discounted utility function

$$\sum_{t=0}^{\infty} \beta^t W(c_t)$$

subject to the budget constraint

$$c_t + P_t q_t \leq w_t + q_{t-1}(P_{t} + r_t)$$

where

$$\beta \in (0, 1)$$

$$c_t = \text{consumption in period (t)}.$$

$$q_t = \text{quantity of assets purchased by the consumer in period (t). In equilibrium we have } q_t = q_{t+1} = 1, \text{ say, for all } t.$$

$$P_t = \text{price of the asset in period (t)}.$$

$$r_t = \text{dividend paid by the asset in period (t)}.$$

$$w_t = \text{endowment received by the agent which is not paid by the asset.}$$

In some cases, as in Lucas [17] or LeRoy and LaCivita [15], the variable $w_t$ is assumed to be identically zero.

The solution to the problem (†) is then an exercise in dynamic programming. Existing methods can then be used to derive an equilibrium stationary solution for the price of capital as a function of the "state" variables ($w_t$) and ($r_t$). Also, equilibrium consumption will be $c_t = w_t + r_t$ if $q_t = 1$.

Lucas [17] has used this framework as an analytical device to study the behavior of the prices of assets. LeRoy and LaCivita [15] have
also used this construct to show that, within a certain class of utility functions, the variability of the asset price will be directly proportional to the variability of the dividends. Grossman and Shiller [4] have used the framework (†) to attempt to derive conditions on the function $U(c)$ which would enable the model to closely mimic the observed data on stock prices. The framework (†) is also being used as a basis for econometric estimation as in Hansen and Singleton [6], [7], and Hansen, Richard, and Singleton [5].

Although this representative agent construct has been studied and used, it has failed to rationalize:

(i) the excessive volatility of stock prices in the presence of low levels of risk aversion (Grossman and Shiller [4]). (See Huffman [9] for an explanation.)

(ii) the martingale property of stock prices (Lucas [17], Danthine [2], and Sargent [22]).

(iii) the excessive risk-premium possessed by equities (Prescott and Mehra [20]).

In a construct such as that used in (†), the representative agent will have the option to buy an asset which will yield a dividend in future periods. Therefore the agent will find it in his or her best interest to forecast and valuate all future dividends or rentals yielded by the security. In this model the equilibrium is repetitive in the sense that each period agents will choose to hold the same assets they held during the last period, although the security prices and consumption patterns may change over time. In short, agents will participate in the asset market the same way each period.

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1Those utility functions with constant relative risk aversion.
In contrast to the study of asset prices in a model such as \(†\), we propose to study the overlapping generations model with two-period lived agents. (See also Huberman [8].) This framework is useful in that it will permit heterogeneous participation in the asset market. For example, an agent who possesses an endowment of the homogeneous consumption good in the first period of his or her life may choose to exchange part of his or her endowment for a title to an asset which may yield a dividend the following period. Also, agents in the last period of their life are faced with a trivial decision problem: They will sell all securities in order to maximize utility. Therefore, there are two facts which should be noted about the environment. First, in any period there will be one group of individuals who are net purchasers of assets while there will be others who are net sellers of assets: not all agents participate in the asset market in the same way. Second, individuals in the second period of their life will supply all securities inelastically regardless of what forecasts of future dividends may be.

There is a sense in which the infinitely lived agent model and the overlapping generations model with two-period lived agents can be considered extreme versions of models where agents live for many, but a finite number of periods. The crucial difference between these two frameworks would seem to be that in the overlapping generations model with, say, two-period lived agents and a constant population, each period half the consumers in the model will supply all stocks of assets inelastically irrespective of what future dividends may be. In contrast in the infinitely lived agent framework, each period every consumer will be concerned with what future dividends may be and will not simply supply their stocks of any asset inelastically. In between these two frameworks are a multitude of models where each agent lives for \(n\) periods, where \((2 < n < \infty)\).
In this case, in any period \((1/n)^{th}\) of the total population will be supplying their stocks inelastically and \((n-1)/n^{th}\) of the population will have their lifetime consumption affected by future dividends. Of course, for \(n = 2\) we have the two-period lived overlapping generations model, and as \(n\) approaches infinity we have the infinitely lived agent model. Clearly the longer each consumer lives, the more individuals who will be alive each period that will have a budget constraint similar to that of an infinitely lived agent.

In Huffman [9] it is shown that the excessive volatility of asset prices may be reconciled with the low variability in dividends and aggregate consumption, as well as existing measures of risk aversion, within the context of an overlapping generations model. If indeed the overlapping generations construct is more successful in explaining these phenomena, we may then ask what characteristics inherent in such a model are important in producing such results. The crucial difference between the two frameworks would seem to be that the budget sets of the consumers in the two models are different. This in turn leads to some agents in the overlapping generations model supplying their stocks of capital inelastically.

A natural question which arises at this point is: For a given pattern of aggregate consumption, which framework would require the least risk aversion in its utility function to produce a specified level of variability in asset prices? We may seek an answer to such a question by formulating both an overlapping generations model and an infinitely lived agent model which share such features as identical time paths for endowments, dividends, aggregate consumption, and asset prices. We may then "work backward" to recover the utility function in each model which would generate these time series. This will enable us to compare measures of risk aversion in the two models.
It is then possible to argue that there exists an "observational equivalence" between a stochastic overlapping generations model with two-period lived agents, and a model of infinitely lived agents who solve a stochastic discounted dynamic programming problem.¹ That is, we may have a time series for asset prices and aggregate consumption which results from a stochastic overlapping generations model with two-period lived agents and a specified probability and information structure. There may then exist a stochastic discounted dynamic programming model with infinitely lived agents, but with the same probability and information structure, which results in exactly the same time paths for asset prices and aggregate consumption. Similarly, given the time series for asset prices and aggregate consumption derived from a stochastic discounted dynamic programming model with infinitely lived agents and a specified probability and information structure, we may be able to derive a stochastic overlapping generations model, with the same probability and information structure, which results in exactly the same time series for asset prices and aggregate consumption. However, this is not to say that the same utility function can be used in both frameworks, which would result in the identical paths for the relevant time series.

In the next section we show that under certain circumstances this observational equivalence obtains.

The Observational Equivalence of an Infinitely-Lived Agent Model, and an Overlapping Generations Model

We will consider an overlapping generations model in which each period there is born a single two-period lived agent. This agent maximizes the utility function

\[ U(c_1^t) + V(c_2^t), \]

where \( c_i^t \) = \( i \)th period consumption of an agent born in period \( (t) \). An agent born

¹Aiyagari [1] has looked at a similar problem within a growth model context.
in period \((t)\) possesses an endowment \((w_t)\) of the homogeneous consumption good in the first period of his or her life, and no endowment in the second period of their life. The realization of \((w_t)\) occurs prior to the appearance of generation \((t+1)\). In the first period of their life, an agent born in period \((t)\) has the option of purchasing a share of perfectly durable non-producible capital, at a price \(P_t\) per unit, with a portion of their endowment. This unit of capital can then be sold in the last period of their life for the price \(P_{t+1}\). In addition, the agent collects an exogenous constant dividend \((r)\) yielded per unit of capital. The dividend is paid in units of the consumption good.\(^2\) The representative agent must then maximize the separable utility function

\[
U(c_1^t) + V(c_2^t)
\]

subject to

\[
\begin{aligned}
  c_1^t &= w_t - P_t q \\
  c_2^t &= (P_{t+1} + r) q
\end{aligned}
\]

(*)

Here \(q\) is the number of units of capital in the economy, and we shall normalize this so \(q = 1\) in an equilibrium.

Henceforth we assume that \(U(\cdot)\) and \(V(\cdot)\) are strictly concave, twice continuously differentiable and strictly increasing. Further we assume

\[
\begin{align*}
  & U'(c_1^t) \to \infty \text{ as } c_1^t \to 0 \\
  & V'(c_2^t) \to \infty \text{ as } c_2^t \to 0 \\
  & 0 \leq 1 + \frac{(c_2^t)V''(c_2^t)}{V'(c_2^t)} \equiv \alpha < 1 \\
  & 0 \leq 1 + \frac{(c_1^t)U''(c_1^t)}{U'(c_1^t)}
\end{align*}
\]

\(^2\) It may help to think of the capital as land which cannot be produced or depreciated.
We assume that the dividend \( r \) is constant but that the endowment \( w_t \) has the distribution \( f(w_t) \) over the interval \([\underline{w}, \overline{w}]\), where \( 0 < w < \overline{w} < \infty \). Aggregate consumption in period \( t \) then is equal to \( (w_t + r) \). With these restrictions, we have shown in Huffman [9] that there exists a pricing function \( P_t \equiv P(w_t) \) which is non-decreasing in its only argument.\(^3\) In fact, it can be shown in that \( P(w_t) \) will be strictly increasing and differentiable with the restrictions we have on the utility function.

Now we wish to show that there exists an infinitely lived agent model which is observationally equivalent to the overlapping generations model just described in (*)\(^4\). That is, we seek a \( \beta \) \((0 < \beta < 1)\) and a function \( W(\cdot) \) which is strictly increasing, continuously differentiable and concave such that the stochastic infinite horizon discounted dynamic programming problem,

\[
\max_{c_t} \sum_{t=0}^{\infty} \beta^t W(c_t) \quad \text{subject to} \quad c_t + \tilde{P}_t q_t = w_t + (\tilde{P}_t + r) q_{t-1}
\]

with \( q_t = 1 \forall t \) in equilibrium, yields the same time paths for asset prices and aggregate consumption as those derived from the overlapping generations model. The distribution function for \( w_t \) must be the same for both of these models.

**Proposition 1:** Given the time paths for aggregate consumption and asset prices derived from the stochastic overlapping generations model (*), there is a stochastic infinitely lived agent problem described by (†), with the same probability and information structure, which has as a result the identical time paths for aggregate consumption and asset prices.

\(^3\) To see this, set \( k_t = K, \delta = 0, N_t = 1 \forall t \). Then the equilibrium pricing function depends only on the endowment, and is a non-decreasing function.
\textbf{Proof:} Pick } \beta \in (0,1) \text{ such that } \\
\begin{equation}
\beta = \frac{1}{\left[ 1 + \int \frac{r}{P(w_{t+1})} dw_{t+1} \right]}
\end{equation}

We let } \bar{P}(w_t) \text{ denote the pricing function derived from the overlapping generations model. Define the utility function } \\
W(w_t) = \int w_t \left[ \frac{1}{P(s)} \right] ds
\begin{equation}
or, since \ c_t = w_t + r, \\
W(c_t - r) = \int^{w_t + r} \left[ \frac{1}{P(c_t - r)} \right] dc_t .
\end{equation}

Then clearly \\
\begin{equation}
\frac{\partial W(c_t - r)}{\partial c_t} = \frac{1}{P(c_t - r)} > 0
\end{equation}
and \\
\begin{equation}
\frac{\partial^2 W(c_t - r)}{\partial c_t^2} = \frac{-P'(c_t - r)}{P(c_t - r)^2} < 0 .
\end{equation}

A modification of the proof used by Lucas [17] shows that an equilibrium exists for the infinitely lived agent problem, and the equation for the pricing function is \\
\begin{equation}
\bar{P}(w_t) = \left[ \frac{1}{\bar{w}(c_t - r)} \right] \left[ \sum_{i=1}^{\infty} \beta^i \int \bar{w} \left( c_{t+1} - r \right) df(w_{t+1}) \right]
\end{equation}
or, since the } w_t \text{ are independent, \\
\begin{equation}
\bar{P}(w_t) = \left[ \frac{1}{\bar{w}(c_t - r)} \right] \left( \frac{1}{1 - \beta} \right) \int \bar{w} \left( c_{t+1} - r \right) df(w_{t+1}) .
\end{equation}

Using equations (1) and (2) we have
\[ \bar{P}(w_t) = \left( \frac{1}{w_t} \right) \cdot \frac{1}{1 - c_t} = \bar{P}(c_t - r) = P(w_t) . \]

Therefore, the pricing functions from the two models are identical. Further, the problem was formulated in such a way that aggregate consumption in the two models would be identical.

We note that the constant \( r \) which appears in the utility function \( W(c_t - r) \) is in fact the dividend yielded by the asset.

It is of interest to compare the measures of risk aversion in the utility functions of these two models. In the overlapping generations model the first order conditions for an optimum are

\[ P(w_t)U'(c^t_1) = E_t \{ P(w_{t+1}) + r \} U'(c^t_2) \]  \hspace{1cm} (4)

where

\[ c^t_1 = w_t - P(w_t)q \]

\[ c^t_2 = (P(w_{t+1}) + r)q. \]

Now taking the derivative of equation (4) with respect to \( w_t \) yields

\[ P'(w_t)U'(c^t_1) + P(w_t)U''(c^t_1)(1 - P'(w_t)) = 0 . \]

Therefore, we have that the measure of relative risk aversion is

\[ \frac{c^t_1 U''(c^t_1)}{U'(c^t_1)} = -\frac{(w_t - P(w_t))}{(1 - P'(w_t))} \cdot \frac{P'(w_t)}{P(w_t)} . \]  \hspace{1cm} (5)

From the infinitely lived agent problem, we have

\[ \bar{W}'(c_t) = \frac{1}{\bar{P}(w_t)} \]

and
\[ W''(c_t) = \frac{-\tilde{P}'(w_t)}{\tilde{P}(w_t)^2} \cdot \]

Therefore, the measure of relative risk aversion for the infinitely lived agent problem is

\[ \frac{c_t W''(c_t)}{W(c_t)} = \frac{- \left( w_t + r \right) \tilde{P}'(w_t)}{\tilde{P}(w_t)} \]

because \( c_t = w_t + r \).

Since \( \tilde{P}(w_t) \equiv P(w_t) \), any comparison of risk aversion in the two models rests on the relative sizes of the terms \( (w_t + r) \) and \( \left[ (w_t - P(w_t)) / (1 - P'(w_t)) \right] \).

**Example:** Consider a stochastic overlapping generations model, as described by (*) in which \( U(c_1^t) + V(c_2^t) = \lambda n(c_1^t) + \lambda n(c_2^t) \), and \( q = 1 \). The equilibrium pricing function will be \( P(w_t) = (w_t / 2) \), whilst aggregate consumption in period \( t \) will equal \( (w_t + r) \).

Now we seek an infinitely lived agent problem which implies the same pricing function and the same aggregate consumption. Let \( W(c_t) = \lambda n(c_t - r) \), and choose \( \beta \in (0, 1) \) such that

\[ \beta = \frac{1}{1 + 2r \int_0^{\tilde{w}} \frac{1}{w} df(w)} \]

As shown by Lucas [17], the equilibrium pricing function for the infinitely lived agent problem takes the form

\[ \tilde{P}_t = \left[ \frac{1}{W(c_t)} \right] \left[ \sum_{i=1}^{\infty} \beta^i \int_0^{\tilde{w}} rW(c_{t+i}) df(w_{t+i}) \right] \]

or
\[ \tilde{P}_t = \left[ \frac{1}{W(c_t)} \right] \left[ \frac{\beta}{1-\beta} \int_0^\infty W(c_{t+1}) df(w_{t+1}) \right] \]

since the \( w_t \) are independent.

Now using \( W(c_t) = \Delta n(c_t - r) \), with \( c_t = w_t + r \), we have

\[ \tilde{P}_t = (w_t) \left[ \frac{\beta}{1-\beta} \int_0^\infty \left( \frac{1}{w_t + r} \right) df(w_{t+1}) \right] \]

or

\[ \tilde{P}_t \equiv \tilde{P}(w_t) = \left( \frac{w_t}{2} \right) = P(w_t), \]

as was to be shown.

We may investigate the measures of risk aversion in each of these models which could result in the specified degree of variability for asset prices.

For the overlapping generations model the measure of relative risk aversion for first period utility is

\[ \frac{(c_1^t)U''(c_1^t)}{U'(c_1^t)} = \frac{(c_1^t)\left(\frac{1}{c_1^t}\right)}{\left(\frac{1}{c_1^t}\right)} = -1 \]

This is also the measure of relative risk aversion for second period utility.

For the infinitely lived agent framework, the measure of relative risk aversion is

\[ \frac{c_tW''(c_t)}{W'(c_t)} = \frac{(c_t)\left(\frac{1}{c_t-r}\right)^2}{\left(\frac{1}{c_t-r}\right)} = \left(\frac{c_t}{c_t-r}\right) = -\left(\frac{w_t+r}{w_t}\right) < -1. \]

Therefore, in order to account for the variability in asset prices, the use of the infinitely lived agent framework would necessitate the use of a utility function which had a higher measure of relative risk aversion.

We can also state that the converse of Proposition 1 is true.
Proposition 2: Assume that in the equilibrium which results from the infinitely lived agent problem we have that:

(i) the pricing function $\tilde{\Pi}(w_t)$ from the infinitely lived agent problem is differentiable,

(ii) $[w_t - \tilde{\Pi}(w_t)] > 0$, for all $w_t \in [\underline{w}, \overline{w}]$,

(iii) $0 \leq \frac{\partial \tilde{\Pi}(w_t)}{\partial w_t} < 1$, for all $w_t \in [\underline{w}, \overline{w}]$.

Then given the time paths for aggregate consumption and asset prices derived from the stochastic infinitely lived agent problem described by (I), there exists a stochastic overlapping generations model of the form (II), with the same probability and information structure, which has as a result the identical time paths for aggregate consumption and asset prices.

Proof: Let

$$V(c^t_2) = \ln(c^t_2)$$

$$c^t_1 = w_t - \Pi(w_t)q$$

$$c^t_2 = [\Pi(w_t) + \delta]q$$

with $q = 1$ in equilibrium. Define

$$g(w_t) = w_t - \tilde{\Pi}(w_t).$$

By the third qualification of Proposition 2 we have that $0 < g'(w_t) \leq 1$. Then $g(w_t)$ is a one-to-one mapping (of what will turn out to be, first period endowments into first period consumption allocations; see the diagram below). Therefore, there is an inverse function, $h = g^{-1}$ (which will map first period consumption allocations into first period endowments). Further, since $g(\cdot)$ is
differentiable, by the inverse function theorem we have that \( h \) is differentiable and \( h' \geq 1 \). Now define

\[
\varphi(c_1^t) = \left. \frac{1}{h(c_1^t) - c_1^t} \right|_{c_1^t} = \left. \frac{1}{w_t - c_1^t} \right|_{c_1^t}
\]

and let

\[
U(c_1^t) = \int_{c}^{c_1^t} \varphi(s)ds
\]

where

\[
c = \inf_{w \in [\bar{w}, \tilde{w}]} \{ w - \tilde{P}(w)q \}.
\]

Clearly,

\[
U'(c_1^t) = \varphi(c_1^t) > 0
\]

and

\[
U''(c_1^t) = \frac{-[h'(c_1^t) - 1]}{[h(c_1^t) - c_1^t]^2} \leq 0.
\]

The first order necessary condition for the two-period agent's optimization problem is

\[
P(w_t)U'(c_1^t) = E_t\{ (P(w_{t+1}) + r)V'(c_2^t) \},
\]

or, with the necessary substitutions,

\[
P(w_t)\phi(c_1^t) = \frac{P(w_t)}{P(w_t)} = 1 = \frac{P(w_{t+1}) + r}{P(w_{t+1}) + r}.
\]

Now letting \( P(w_t) = \tilde{P}(w_t) \) from the infinitely lived agent problem, we have that the above stochastic overlapping generations model will have the same pricing function and time path for aggregate consumption as the infinitely lived agent problem.
Note that the form of the function $h(\cdot)$, and therefore $U(\cdot)$, depends only on the form of the pricing function $\tilde{P}(\cdot)$ derived from the infinitely lived agent model. We also note that the second qualification of Proposition 2 merely ensures that first period consumption for the two-period lived agent is positive.

Again we wish to compare the measures of relative risk aversion in the utility functions of these two models. From the infinitely lived agent problem, we rewrite equation (3) as

$$\tilde{P}(w_t) = \frac{\alpha}{W(c_t)}$$

where

$$\alpha = \left( \frac{\beta}{1-\beta} \right) \int \frac{rW(c_{t+1})df(w_{t+1})}{w}.$$  

Then we have

$$W''(c_t) = \frac{\alpha \tilde{P}'(w_t)}{\tilde{P}(w_t)^2},$$

and the measure of relative risk aversion for the infinitely lived agent problem is

$$\frac{c_t W''(c_t)}{W(c_t)} = -\frac{-(w_t+r)\tilde{P}'(w_t)}{\tilde{P}(w_t)}.$$  

This is identical to equation (6).

In the overlapping generations model we have

$$P(w_t) = \frac{1}{U'(c^t_1)}$$

with $c^t_1 = w_t - P(w_t)$. Differentiating with respect to $(w_t)$ yields

$$U''(c^t_1)(1 - P'(w_t)) = \frac{-P'(w_t)}{P(w_t)^2}.$$  

The measure of relative risk aversion for first period utility in the overlapping generations model is
\[
\frac{c_1^T U''(c_1^T)}{U'(c_1^T)} = \frac{-(w_t - P(w_t))}{(1 - P'(w_t))} \left[ \frac{P'(w_t)}{P(w_t)} \right].
\]

This is identical to equation (5). Therefore, it seems that comparisons of measures of relative risk aversion in the two models lie in a comparison of the terms in equations (5) and (6).

However, it may be important to compare the risk aversion of the infinitely lived agent model to that of second period consumption for the two-period lived agent. For a two-period lived agent, it is actual second period consumption which is uncertain at the time when the agent makes his or her consumption decision for the first period. Intuitively, one might think that this uncertainty would lead to asset price variability being sensitive to risk aversion for second period utility. However, it is straightforward to verify that the measure of relative risk aversion for second period utility for the overlapping generations model just constructed is (-1). Therefore, given the pricing function derived from an infinitely lived agent problem in which the utility function had "high" risk aversion, to derive this same pricing function for an overlapping generations model we would not need excessively "high" risk aversion for second period utility.

Lastly, it may be deemed inappropriate to compare measures of risk aversion in the two models because the construction of the utility function for the overlapping generations model, in Proposition 2, does not lead to a single utility function, of the form \( U(c_1^T) + \beta U(c_2^T) \), which is comparable to the discounted utility of the infinitely lived model.
Conclusion

We have shown that, given a time series for aggregate consumption and the price of an asset, there may exist both a two-period lived overlapping generations model, and a representative infinitely-lived agent model, which will both mimic the aforementioned time series. We then compared the measures of relative risk aversion in the two models. However, we could not say that one model would invariably require more risk aversion than the other to produce a specified amount of asset price variability with a fixed level of consumption variability.

Lastly, we have shown that this observational equivalence obtains for a specific physical environment. It is not clear, however, that such an equivalence obtains for every such overlapping generations, or infinitely-lived agent model.
REFERENCES


