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by

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ABSTRACT

In the present article sufficient conditions for global uniqueness and stability of fixed price equilibria are derived which refer to aggregate effective demands and supplies and which are therefore suited for application in macroeconomic analyses in fixed price models. The arguments are based on a new proof of strict monotonicity and global univalence of non-differentiable mappings. Stability is shown for continuous and for discrete processes of the tâtonnement type with respect to the adjustment of quantity constraints. It is also shown that the derived conditions have a strong economic intuition which may be regarded as a generalization of certain aspects of Keynesian macroeconomics.

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1. INTRODUCTION

During the last decade a microeconomic theory of non-Walrasian equilibria has developed which today can be regarded as a foundation of macroeconomic theory in a certain sense. In macroeconomics the idea of analyzing allocations which are not equilibria in the classical sense already has a long tradition. The concept of an equilibrium at arbitrarily given prices was developed by Benassy [1975], Drèze [1975], Malinvaud/Younès [1977] and Younès [1975]. These ideas are now widely accepted.

Although these contributions to non-Walrasian general equilibrium theory focus on the case of a pure exchange economy, it is in general possible to base on them the solution of the existence problem in macroeconomic models like those of Barro/Grossman [1976], Malinvaud [1977] and others. There is however no general solution to the uniqueness and stability problems which have to be dealt with in macroeconomic applications. Laroque [1981] discussed the local uniqueness and stability of fixed price equilibria at prices in a neighborhood of a competitive equilibrium and Schulz [1982] derived a sufficient condition for global uniqueness in the special case of a Benassy-equilibrium in a pure exchange economy.

The present article derives sufficient conditions for global uniqueness and stability of fixed price equilibria which refer directly to aggregate effective demands and supplies and which are therefore suited for direct application in macroeconomic analyses. Stability properties are discussed for tâtonnement processes with respect to the adjustment of quantity constraints. Continuous adjustment is analyzed for a rather wide class of processes while the discussion of discrete dynamics focuses on one particular adjustment rule. All conditions which are introduced have a common basis which is economically meaningful and intuitively clear.

In order to prove uniqueness a first step consists in the construction of a necessary condition which every fixed price equilibrium has to meet. It is then shown that under certain assumptions about the nature of spillover effects this condition can be met only once. This proves the uniqueness of the equilibrium, the existence of which will be assumed. The basic mathematical tool in the argument is a generalization of the Gale/Nikaido univalence theorem which has been presented in higher generality in Kosch [1982]. The arguments in the discussion of stability are the traditional ones: In the continuous case a Lyapunov-function is constructed and in the discrete case the results are concluded from the fact that the map which governs the dynamic process is a contraction if certain requirements are met.

2. A MACROECONOMIC CONSIDERATION OF FIXED PRICE EQUILIBRIA

The framework of the present analysis is as follows: The economy is characterized by m agents (engaged in production and in consumption, $m \geq 2$) and $n+1$ commodities ($n \geq 2$), indexed $0,\ldots,n$. The index 0 refers to money, whereas the other indices are left unspecified and are allowed to denote any good or factor. Prices are expressed in units of money and are exogenously given. They are fixed during the period under consideration and strictly positive. Market clearing is achieved by quantity rationing on the long side of each market. There is no rationing in the money market, i.e., consideration of trivial equilibria is excluded (see Grandmont [1977]). The process of quantity rationing is governed by a mechanism which divides aggregate into individual quantities and can be described by a piecewise differentiable mapping. Let $L^d := (L_1^d, \ldots, L_n^d)$ and $L^s := (L_1^s, \ldots, L_n^s)$ denote vectors of aggregate quantity constraints for supply and demand and let $\ell_i^d := (\ell_{i1}^d, \ldots, \ell_{in}^d)$ and $\ell_i^s := (\ell_{i1}^s, \ldots, \ell_{in}^s)$ denote vectors of individual quantity constraints for agent i $(i=1,\ldots,m)$ with respect to his net trades. We assume that all (ℓ_i^d, ℓ_i^s) $(i=1,\ldots,m)$ are continuous piecewise differentiable maps of (L^d, L^s) which satisfy

(1) $$\sum_{i=1}^m \ell_i^d = L^d, \quad \sum_{i=1}^m \ell_i^s = L^s$$

(2) $$\frac{\partial \ell_{ij}^d}{\partial L_j^d} \geq 0, \quad \frac{\partial \ell_{ij}^s}{\partial L_j^s} \geq 0$$ for all i,j in any point where these derivatives exist

(3) $$\frac{\partial \ell_{ij}^d}{\partial L_k^d} = \frac{\partial \ell_{ij}^s}{\partial L_k^s} = 0$$ for all i,j and $k \neq j$

(4) $$\frac{\partial \ell_{ij}^s}{\partial L_k^d} = \frac{\partial \ell_{ij}^d}{\partial L_k^s} = 0$$ for all i, j, k

Each agent i is characterized by a set of feasible allocations X_i, initial endowments $\omega_i = (\omega_{i0}, \ldots, \omega_{in})$ and a utility function $u_i: X_i \to \mathbb{R}$. If the agent is engaged in production this function may of course depend on the given prices $p \in \mathbb{R}_+^{n+1}$. Given the constraint vectors (L^d, L^s) the agent's choice set is

$$\gamma_i := \{z \in X_i \mid p(z-\omega_i) \leq 0 \text{ and } -\ell_i^s \leq z-\omega_i \leq \ell_i^d\}$$

Let $z_i^* = (z_{i0}^*, \ldots, z_{in}^*)$ be the solution to the problem

(5) $$u_i \rightarrow \max_{z \in \gamma_i}$$

which is assumed to exist and to be unique. Then the effective demands x_{ij}^d and supplies x_{ij}^s of agent i for commodity j (j=1,...,n) are defined as

(6) $$x_{ij}^d := \max\{z_{ij}^* - \omega_{ij}, 0\} \ , \ x_{ij}^s := -\max\{\omega_{ij} - z_{ij}^*, 0\}$$

These are functions of ℓ_i^d and ℓ_i^s and in particular

(7) $$x_{ij}^d = x_{ij}^d(\ell_{ij}^d) \ , \ x_{ij}^s = x_{ij}^s(\ell_{ij}^s)$$

Rationing is assumed to be efficient in the sense that for j=1,...,n

(8) $$\exists\, i | x_{ij}^d < \ell_{ij}^d \Rightarrow \nexists\, k | x_{kj}^d(\ell_{kj}^d + \epsilon) > \ell_{kj}^d \qquad \forall \epsilon > 0$$

$$\exists\, i | x_{ij}^s < \ell_{ij}^s \Rightarrow \nexists\, k | x_{kj}^s(\ell_{kj}^s + \epsilon) > \ell_{kj}^s \qquad \forall \epsilon > 0$$

Let $X_j^d := \sum_{i=1}^m x_{ij}^d$ and $X_j^s := \sum_{i=1}^m x_{ij}^s$ denote aggregate effective demand and supply for commodity j (j=1,...,n) respectively and let $b \in \mathbb{R}_+$ denote an arbitrary real constant having the property to be larger than any possible binding aggregate constraint in any market.

In a framework with initial endowments of finite money value and continuous technologies the existence of such a constant is guaranteed.

Let $B^i \subset \mathbb{R}_+^i$ denote the cartesian product of i intervals [0,b], i.e., $B^i := X_{j=1}^i [0,b]$.

The equilibria this paper is concerned with are those in the sense of Drèze [1975] for the special case of completely fixed prices but expressed on a level of aggregates. A fixed price equilibrium is an allocation (x^*_{ij}) $(i=1,\ldots,m;\ j=0,\ldots,n)$ such that there are constraints (L^{d*},L^{s*}) satisfying

(9) i) $(x^*_{i0},\ldots,x^*_{in})$ is a maximizer to (5) $(i=1,\ldots,m)$

ii) $X^d_j = X^s_j$ $(j=1,\ldots,n)$

iii) For $j=1,\ldots,n$

$$\exists\, i \mid x^d_{ij}(\ell^d_{ij}+\varepsilon) > \ell^d_{ij} \qquad \forall\, \varepsilon > 0$$

$$\Rightarrow\ x^s_{kj}(\ell^s_{kj}+\delta) \le \ell^s_{kj} \qquad \forall\, \delta > 0 \quad \forall\, k \text{ and}$$

$$\exists\, i \mid x^s_{ij}(\ell^s_{ij}+\varepsilon) > \ell^s_{ij} \qquad \forall\, \varepsilon > 0$$

$$\Rightarrow\ x^d_{kj}(\ell^d_{kj}+\delta) \le \ell^d_{kj} \qquad \forall\, \delta > 0 \quad \forall\, k$$

Existence of equilibria of this type can be shown in a straightforward way on the basis of Drèze [1975] for the case of exchange economies or on the basis of Thore [1980] for an economy with production and immediate distribution of profits. In the following we shall assume that an equilibrium exists and concentrate on the exclusion of multiple and unstable equilibria. For this purpose fixed price equilibria will be characterized by certain necessary conditions. Proving uniqueness and stability for the point which meets these conditions then means that the same holds for the fixed price equilibrium.

3. A CHARACTERIZATION OF FIXED PRICE EQUILIBRIA

If all agents' preference orderings are representable by strictly quasi-concave differentiable utility functions then individual effective demands and supplies are piecewise differentiable, continuous functions of the individual quantity constraints. Given the assumptions (1)-(6) the aggregate demands and supplies are piecewise differentiable, continuous functions of the aggregate quantity constraints:

$$X_i^d = X_i^d(L^d, L^s),\ X_i^s = X^s(L^d, L^s) \quad (i=1,\ldots,n)$$

In an equilibrium with constraints (L^{d*}, L^{s*}) no constraint can be binding if it is strictly greater than the traded quantity in the respective market. Since the variation of a constraint which is not binding can have no influence on the solution of the corresponding optimization problem we conclude

$$(10) \qquad X_i^d(L^{d*}, L^{s*}) = X_i^d(x^*, x^*) \quad (i=1,\ldots,n)$$

$$X_i^s(L^{d*}, L^{s*}) = X_i^s(x^*, x^*) \quad (i=1,\ldots,n)$$

where $x^* = (x_1^*, \ldots, x_n^*)$ is defined by

$$x_i^* := \min\{L_i^{d*}, L_i^{s*}, X_i^d(L^{d*}, L^{s*}), X_i^s(L^{d*}, L^{s*})\} \quad (i=1,\ldots,n)$$

Now define a map $\tilde{X} = B^n \to B^n$ by

(11) $\tilde{X} = (\tilde{X}_1,\ldots,\tilde{X}_n)$

$$\tilde{X}_i(x):=\min\{X_i^d(x,x),X_i^s(x,x)\} \qquad (i=1,\ldots,n)$$

According to (10) and (11), every equilibrium with constraints (L^{d*},L^{s*}) corresponds to a vector x* which is a fixed point of $\tilde{X}$.

Because of (9) the constraint x_i^* cannot be binding for D_i and S_i at the same time, therefore $\tilde{X}_i$ must be invariant with respect to an increase in x_i^* $(i=1,\ldots,n)$. This means that x* is also a fixed point of the map $X: B^n \to B^n$ defined by

(12) $X=(X_1,\ldots,X_n)$,

$$X_i(x):=\min\{X_i^d(x_{-i},x_{-i}),X_i^s(x_{-i},x_{-i})\} \quad ,$$

$$x_{-i}:=(x_1,\ldots,x_{i-1},b,x_{i+1},\ldots,x_n)$$

$$(i=1,\ldots,n) \; .$$

It should be pointed out that every equilibrium can be associated with a fixed point

(13) $x^* = X(x^*)$

of the mapping X, whereas the reverse is not true, i.e., (13) is in general not a representation of fixed price equilibria. For the present discussion, however, the only relevant point is the fact that (13) is a necessary condition for every fixed price equilibrium.

4. SOME PRELIMINARY RESULTS ON PIECEWISE DIFFERENTIABLE MAPPINGS

Lemma 1:

Let $I \subset \Re$ be an open interval and $f,g : I \to \Re$ be continuously differentiable in I. Then the set

$$U := \{x \in I \mid f(x) = g(x),\ f'(x) \neq g'(x)\}$$

contains no point of accumulation in I.

Proof:

Suppose that there is an $x \in U$ which is a point of accumulation of U in I. Then for all $\epsilon > 0$ there is an $x_\epsilon \in U$ having the properties $|x - x_\epsilon| < \epsilon$ and $(f-g)(x_\epsilon) := f(x_\epsilon) - g(x_\epsilon) = 0$. By Rolle's theorem there is an $\bar{x}_\epsilon \in U$ satisfying $|\bar{x}_\epsilon - x| < \epsilon$ and $(f-g)'(\bar{x}_\epsilon) := f'(\bar{x}_\epsilon) - g'(\bar{x}_\epsilon) = 0$. Consequently there is a sequence of points $\bar{x}_\epsilon$ in U converging to x and having this property. But since $(f-g)' = f' - g'$ is continuous, the derivatives of f and g cannot be different at x, contradicting $x \in U$.

#

Corollary:

Let $I \subset \Re$ be an open interval and let $\{f_i\}$ be a finite set of continuously differentiable functions $f_i : I \to \Re$ $(i=1,\ldots,\ell)$.

Then the set

$$(14) \qquad U := \{x \in I \mid \exists\ i,j \in \{1,\ldots,\ell\},\ i \neq j \mid f_i(x) = f_j(x),\ f_i'(x) \neq f_j'(x)\}$$

contains no point of accumulation in I.

Lemma 2:

Let $T \subset \Re$ be a closed interval and let $f : T \to \Re$ be a piecewise differentiable map represented by a set of functions $\{f_i\}$ $(i=1,\ldots,\ell)$. For every $x \in T$ and for every representation f_i of f at x assume that

$$f_i'(x) > 0$$

Then f is strictly monotonically increasing.

Proof:

Let U be defined according to (14) for $I := \mathrm{int}(T)$. For each $x \in U$ there is an interval $I_\epsilon(x) = (x, x+\epsilon)$ such that $I_\epsilon(x) \cap U = \emptyset$; therefore f has a right derivative at x. In every $x \notin U$ f is differentiable.

Now the statement to be proved follows immediately from Bourbaki [1958], Chap. 1, §2, No. 2, Prop. 2. #

Lemma 3:

Let $T \subset \Re$ be a closed interval and let $f: T \to T$ be a piecewise differentiable map represented by a set of functions $\{f_i\}$ $(i=1,\ldots,\ell)$. For every $x \in T$ and for every representation f_i of f at x assume that

$$f_i'(x) < 1$$

Then there exists exactly one fixed point

$$f(x^*) = x^* \in T .$$

Proof:

The existence of x^* follows from Brouwer's fixed point theorem. Lemma 2 shows, that $(f - id_T)$ has at most one zero in T.

#

5. A SUFFICIENT CONDITION FOR GLOBAL UNIQUENESS

According to the assumptions with respect to rationing and individual decision making the function $X: B^n \to B^n$ is piecewise differentiable in the sense that its component functions can be represented as

$$X_i(x) := \min\{X_i^1(x),\dots,X_i^{n_i}(x)\} ,$$

where the numbers n_i $(i=1,\dots,n)$ are given and each $X_i^j : B^n \to B^1$ is a continuously differentiable function for all $i=1,\dots,n$ and all $j=1,\dots,n_i$.

Let $F^i : B^n \to B^i$ be defined as the mapping with components

$$F_j^i(x) := x_j - X_j(x) \qquad (j=1,\dots,i) \tag{15}$$

and define $(x^*)^i$ as the set of solutions to the system of equations

$$F^i(x_1,\dots,x_i,\bar{x}_{i+1},\dots,\bar{x}_n) = 0 \tag{16}$$

for given parameters $\bar{x}_{i+1},\dots,\bar{x}_n$. In each $x \in B^n$ the map F^n is represented by one or more continuously differentiable mappings. Let $\mathcal{J}(x)$ denote the set of Jacobians of these mappings at x. $\mathcal{J}(x)$ is non-empty for all $x \in B^n$.

Theorem 1:

Assume that for every $x \in B^n$ every Jacobian $(f_{ij}) \in \mathcal{J}(x)$ is a P-matrix, i.e., every principal minor of (f_{ij}) is strictly positive.

Then there exists at most one equilibrium.

Proof:

We show that there exists exactly one fixed point $x^* = X(x^*)$ in B^n. Using induction on i we show that for given parameters $\bar{x}_{i+1},\dots,\bar{x}_n$ each $(x^*)^i$ consists of exactly one element

(17) $$((x^*)^i_1,\ldots,(x^*)^i_1,\ \bar{x}_{i+1},\ldots,\bar{x}_n)$$

In the case $i = n$ in which there are no parameters this is the desired statement.

For $i = 1$ the statement is trivially true since X_1 is assumed to be a function which is constant with respect to x_1. Assume that the statement holds for $i = m$ $(m < n)$. This means that $(x^*)^m$ is a function of $\bar{x}_{m+1},\ldots,\bar{x}_n$ and in particular for fixed $\bar{x}_{m+2},\ldots,\bar{x}_n$ it is a function

(18) $$(x^*)^m = g^m(\bar{x}_{m+1})$$

which is differentiable at any point where no change of representation occurs in (16). Using the implicit function theorem the derivative of any representation of (18) can be calculated by solving the system

(19) $$\begin{bmatrix} 1 & -\frac{\partial X_1}{\partial x_2} & \cdots & -\frac{\partial X_1}{\partial x_m} \\ \vdots & \ddots & & \\ -\frac{\partial X_m}{\partial x_1} & \cdots & & 1 \end{bmatrix} \begin{bmatrix} \frac{d(x^*)^m_1}{d\bar{x}_{m+1}} \\ \\ \frac{d(x^*)^m_m}{d\bar{x}_{m+1}} \end{bmatrix} = \begin{bmatrix} \frac{\partial X_1}{\partial x_{m+1}} \\ \\ \frac{\partial X_m}{\partial x_{m+1}} \end{bmatrix}$$

Writing

(20) $$f^m := (f_{ij})_{(i,j=1,\ldots,m)}$$

for the above matrix,

(21) $\Delta^m := \det(f^m)$

for its determinant, and

(22) $\mathrm{cof}^{ij}(f^m)$

for the cofactor of the (i,j)-element, Cramer's rule yields for i=1,...,m

(23)
$$\left(\frac{dg^m}{d\bar{x}_{m+1}}\right)_i = \frac{1}{\Delta^m} \Sigma^m_{j=1} (-f_{jm+1}) \mathrm{cof}^{ji}(f^m)$$
$$= \frac{1}{\Delta^m} \mathrm{cof}^{m+1i}(f^{m+1})$$

where the f_{jm+1} are coefficients of the matrix f^{m+1} or order $(m+1) \chi (m+1)$ which corresponds to that in (19) for m augmented by one.

Turning now to the composite map

(24) $h^m : \bar{x}_{m+1} \rightarrow g^m(\bar{x}_{m+1}) \rightarrow X_{m+1}((x^*)^m)$

and differentiating using the chain rule one obtains

(25)
$$\frac{dh^m}{d\bar{x}_{m+1}} = \frac{1}{\Delta^m} \Sigma^m_{i=1} - f_{m+1i} \ \mathrm{cof}^{m+1i}(f_{m+1})$$

But this expression is part of the Laplace expansion of $\det(f^{m+1})$ with respect to the last row of f^{m+1}

(26)
$$\Delta^{m+1} := \det(f^{m+1})$$
$$= \Sigma^m_{i=1} f_{m+1i} \ \mathrm{cof}^{m+1i}(f^{m+1}) + \Delta^m$$

Therefore we conclude that

(27)
$$\frac{dh^m}{d\bar{x}_{m+1}} = 1 - \frac{\Delta^{m+1}}{\Delta^m} < 1$$

since both determinants, Δ^m and Δ^{m+1}, are assumed to be positive.

Since there are only finitely many representations for X, the number of representations for h^m is also finite. Therefore Lemma 3 can be applied to show that h^m has exactly one fixed point $y^* = h^m(y^*)$ in B^1. But this shows that the system

$$F^{m+1}(x_1,\ldots,x_{m+1},\bar{x}_{m+2},\ldots,\bar{x}_n) = 0 \tag{28}$$

has for given parameters $\bar{x}_i$ $(i=m+2,\ldots,n)$ exactly one solution which is given by

$$(g_1^m(y^*),\ldots,g_m^m(y^*),y^*,\bar{x}_{m+2},\ldots,\bar{x}_n) \tag{29}$$

#

Remark:

The use of conditions which refer to the P-matrix property is rather common in the literature and this property can be used for giving an economic meaning to the above theorem. From the proof, however, it should be clear that only strict positivity of the leading principal minors is required, which is in fact a weaker condition. Consequently the statement of Theorem 1 can be strengthened.

6. A SUFFICIENT CONDITION FOR GLOBAL STABILITY WITH CONTINUOUS ADJUSTMENT

In the following we shall be concerned with tâtonnement processes only. In particular we shall focus on the following general adjustment rule

$$\dot{x}_i = G_i(X_i(x) - x_i) \qquad (i=1,\ldots,n) \tag{30}$$

where each G_i is some sign-preserving, differentiable and strictly monotone function.

Let $G : B^n \to B^n$ be defined as

$$G(x) := (G_1(x_1), \ldots, G_n(X_n))$$

Then (30) can be rewritten as

(31) $$\dot{x} = G(x) .$$

Theorem 2:

Assume that for every $x \in B^n$ every Jacobian $(f_{ij}) \in \mathcal{J}(x)$ and the Jacobian (g_{ij}) of G are such that the matrix

(32) $$[((g_{ij})(f_{ij}))^T + ((g_{ij})(f_{ij}))]$$

is positive definite.

Then the equilibrium is globally stable.

Proof:

(32) implies that all the matrices $[(g_{ij})(f_{ij})]$ are P-matrices and this in turn implies that all the matrices (f_{ij}) are P-matrices since each (g_{ij}) is a diagonal matrix with positive entries multiplication by which cannot reverse the sign of any principal minor. Therefore the equilibrium, the existence of which is assumed, is unique according to Theorem 1.

In order to show stability we have to construct a Lyapunov function. In the present context this can be done in a very straightforward way and we can use an argument which is very similar to one used by Arrow and Hurwicz [1958] in stability analysis of Walrasian equilibria. Obviously the function

(33) $$V(x(t)) := \| G(X(x(t)) - x(t) \|^2$$
$$= [G(X(x(t)) - x(t)]^T [G(X(x(t)) - x(t)]$$

attains its minimum value in the uniquely determined equilibrium x* only. It remains to be shown that V(x(t)) is strictly decreasing along any path

$\{x(t)\}$ which does not start in x^*.

In every point in B^n and for every representation of X in that point we calculate the derivative

$$(34)\qquad \frac{dV(x(t))}{dt} = 2[G(X(x(t)) - x(t))]^T (g_{ij})(-f_{ij})\dot{x}$$

$$= [G(X(x(t)) - x(t))]^T [(g_{ij})(-f_{ij}) + ((g_{ij})(-f_{ij}))^T]$$

$$[G(X(x(t)) - x(t)]$$

$$< 0$$

Therefore Lemma 2 is applicable and we conclude the strict monotonicity of V along any path $\{x(t)\}$ which does not start in the equilibrium point.

#

7. A SUFFICIENT CONDITION FOR GLOBAL STABILITY WITH DISCRETE ADJUSTMENT

In the following we focus on one particular class of adjustment rules only which, however, seems to be a very natural one:

(35) $$x^{t+1} = \lambda X(x^t) + (1 - \lambda)x^t \qquad (t=0,1,\ldots)$$

where $\lambda \in [0,1]$ is an arbitrary constant.

On the basis of (35) we can derive a rather simple stability condition and generalizations to processes of the type

(36) $$x^{t+1} = x^t + G(X(x^t) - x^t) \qquad (t=0,1,\ldots)$$

are straightforward.

The process (35) will converge for every initial point $x^o \in B^n$ if the map

(37) $$x \to \lambda X(x) + (1 - \lambda)x$$

is a contraction, which means that there is a positive constant $K < 1$ such that

(38) $$K\|x - y\| \geq \|(\lambda X(x) + (1 - \lambda)x) - (\lambda X(y) + (1 - \lambda)y)\| \quad \forall x, y \in B^n$$

We first show that without loss of generality we may focus on one particular process.

Lemma 4:

If the map (37) is a contraction for $\lambda = 1$, then the same holds true for any $\lambda \in (0,1]$.

Proof:

Suppose there is a constant $K < 1$ such that condition (7) is fulfilled for $\lambda = 1$. Then for any λ we conclude

$$\begin{aligned}
&\|((1-\lambda)x + \lambda X(x)) - ((1-\lambda)y + \lambda X(y))\| \\
&\quad = \|(1-\lambda)(x-y) + \lambda(X(x) - X(y))\| \\
&\quad \le (1-\lambda)\|x-y\| + \lambda\|X(x) - X(y)\| \\
&\quad \le (1-\lambda)\|x-y\| + \lambda K\|x-y\| \\
&\quad = (1-\lambda(1-K))\|x-y\|
\end{aligned}$$

Clearly $(1-\lambda(1-K))$ is a positive constant smaller than 1. #

Theorem 3:

Assume that for every $x \in B^n$ every Jacobian $(f_{ij}) \in \mathcal{J}(x)$ has the property that the matrix

(39) $$[((f_{ij}) + (f_{ij})^T) - (f_{ij})(f_{ij})^T]$$

is positive definite. Then the equilibrium is globally stable.

Proof:

We show that the map X is a contraction. First we concentrate on two points $x,y \in B^n$ having the property that the whole line segment connecting these points lies in a subset of the convex set B^n on which the mapping X is differentiable. From the mean value theorem we conclude

$$\begin{aligned}
&(x-y)^T(x-y) - (X(x) - X(y))^T(X(x) - X(y)) \\
&\ge (x-y)^T(x-y) - ((\tfrac{\partial X}{\partial x}(z))(x-y))^T((\tfrac{\partial X}{\partial x}(z))(x-y))
\end{aligned}$$

for some $z \in \{\lambda x + (1-\lambda)y \mid \lambda \in [0,1]\}$

where $\frac{\partial X}{\partial x}(z)$ means the Jacobian of the respective representation of X in $z \in B^n$. This expression, however, equals

$$(40) \qquad (x - y)^T [I - (\frac{\partial X}{\partial x}(z))^T (\frac{\partial X}{\partial x}(z))] (x - y)$$

$$= (x - y)^T [(f_{ij}) + (f_{ij})^T - (f_{ij})(f_{ij})^T] (x - y)$$

$$> 0$$

and therefore we conclude

$$(41) \qquad \|X(x) - X(y)\| < \|x - y\|$$

Now look at two arbitrarily chosen points $x, y \in B^n$. Without loss of generality we may assume $\|x - y\| = 1$. All points on the line segment connecting x and y are of the form $z_\lambda = (1 - \lambda)x + \lambda y$ with $\lambda \in [0,1]$ and $\|x - z_\lambda\| = \lambda$. Look at the map $f: \lambda \to (\|X(x) - X(z_\lambda)\| - \lambda)$. It is piecewise differentiable; because of (41) there is no point in (0,1) where its right derivative, if it exists, can be strictly positive and there are points where the right derivative is strictly negative. Again lemma 1 can be applied and using the argument of lemma 2 we conclude $f(1) < f(0) = 0$, i.e.,

$$(42) \qquad \|X(x) - X(y)\| < \|x - y\|$$

Since $B^n \times B^n$ is a compact set the function $d = B^n \times B^n \to \Re$ defined by

$$d(x,y) = \|x - y\| - \|X(x) - X(y)\|$$

has a minimum which is strictly positive because of (42). This means, however, that there is an $\epsilon > 0$ such that

(43) $\|X(x) - X(y)\| \leq \|x - y\| - \epsilon \quad \forall\, x,y \in B^n$

Now let $K \in \mathbb{R}_+$ be an arbitrary constant such that

(44) $K\epsilon > \max_{B^n \times B^n} \|x - y\|$

then

(45) $\|X(x) - X(y)\| \leq \|x - y\| - \frac{\max\|x - y\|}{K}$

$$\leq \|x - y\| (1 - \frac{1}{K}) \quad \forall\, x,y \in B^n$$

Since $0 < (1 - \frac{1}{K}) < 1$ this means that X is a contraction as was to be shown. #

8. AN ECONOMIC INTERPRETATION

The following proposition holds (see McKenzie [1959]):

Let A = : (a_{ij}) be a nxn-matrix with a positive dominant diagonal, i.e.,

$a_{ii} > 0$

$\exists\, d_i > 0 \mid |d_i a_{ii}| > \sum_{j \neq i} |d_j a_{ij}| \quad (i=1,\ldots,n).$

Then A is a P-matrix.

McKenzie analyzed matrices of derivates with respect to variations in prices whereas here derivates of aggregate spill-overs are considered.

The off diagonal elements in any column j of the matrices involved in the conditions of Theorems 1 - 3 are the values of sign-preserving and strictly monotone functions of the respective coefficients of (f_{ij}). These express the aggregate agents' reactions upon an increase in their trading possibilities on market j. Diagonal dominance of these matrices means therefore that effective demands and supplies are sufficiently inelastic with respect to variations in quantity constraints.

Take the simple three commodity macro model as an example for illustration. Denote the effective demands and supplies of households and producers as C,L and Y,Z respectively and the effectively traded quantities of labour and consumption goods as ℓ and x. Then the functions X_ℓ and X_x (the effective quantities of ℓ and x) look like those in the figure below which shows a situation of Keynesian unemployment. Taking $G = \mathrm{id}_{\mathbb{R}^n}$ as a special case of (30), uniqueness as well as stability of the equilibrium under discrete and continuous adjustment is implied by the three conditions

(46) $$1 - X_{\ell c} X_{c\ell} > 0$$

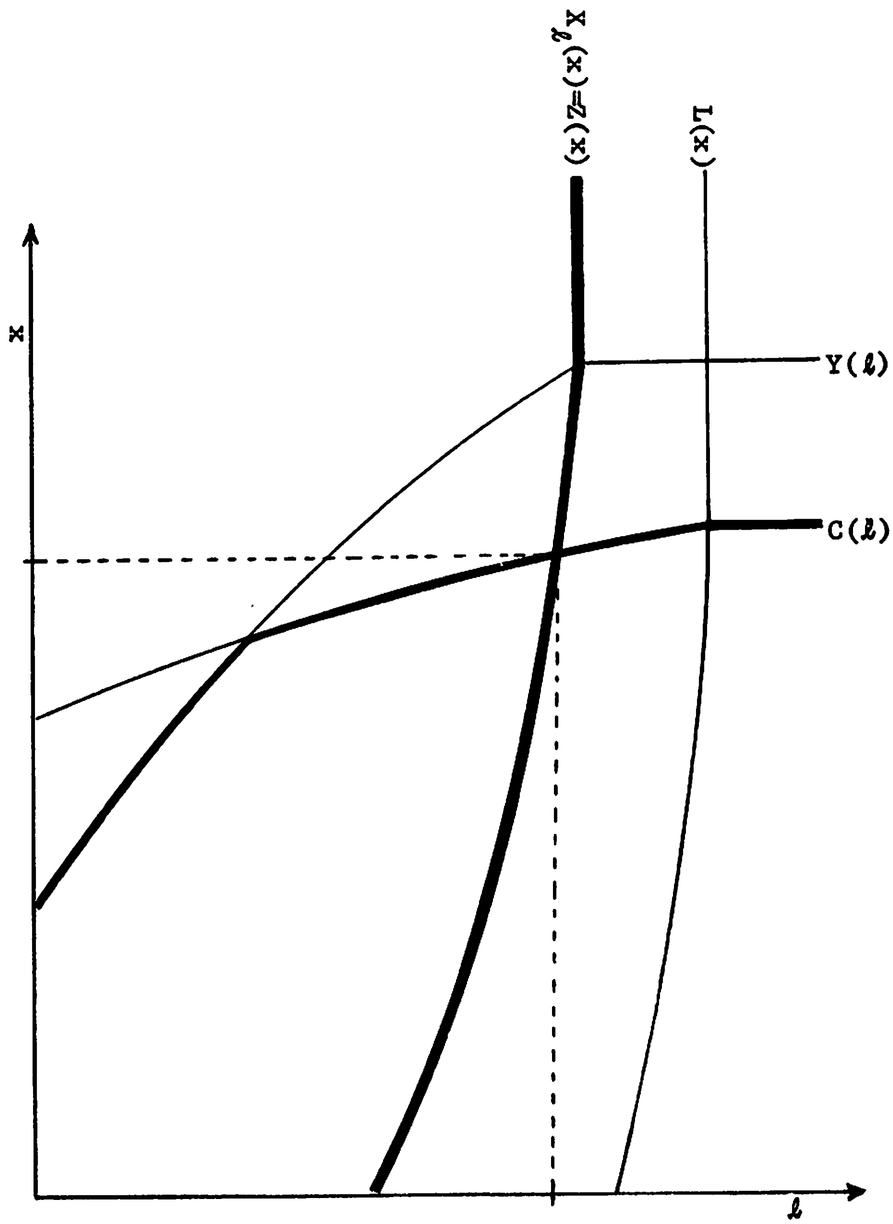
(47) $$4 - X_{\ell c}^2 - X_{c\ell}^2 - 2X_{\ell c} X_{c\ell} > 0$$

(48) $$1 - (X_{c\ell} X_{\ell c})^2 > 0$$

where the additional subscript means partial differentiation. All these follow from the conditions

(49) $\{C_\ell, Y_\ell, L_x, Z_x\} \subset [0,1)$ for all admissable values of x and ℓ

Remembering the traditional analysis of the Keynesian case, (49) is recognized as the straightforward generalization of the familiar uniqueness and stability condition "marginal rate of savings $\in (0,1]$".


x
$X_\ell(x)=Z(x)$
L(x)
Y(ℓ)
C(ℓ)
ℓ

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