1973

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Stefan Rinco

Recommended Citation
Rinco, Stefan, "Beta-expectation Tolerance Regions Based On The Structural Models" (1973). *Digitized Theses*. 734.
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BETA-EXPECTATION TOLERANCE REGIONS
BASED ON THE STRUCTURAL MODELS

by

Stefan Rinco

Department of Mathematics

Submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy

Faculty of Graduate Studies
The University of Western Ontario
London, Canada
July, 1973

ABSTRACT

Let $(\mathcal{X}, \mathcal{A}, P)$ be a probability space. The statistical tolerance region $Q(X)$ is defined as a statistic which maps the point X from $\mathcal{X}$ into a region $Q(X)$ belonging to $\mathcal{A}$. The probability content of $Q(X)$ is called the coverage of the tolerance region and is denoted by $C(Q)$. $Q(X)$ is a β-expectation tolerance region if the expected value of $C(Q)$ is equal to β.

The statistical tolerance regions in general and the β-expectation tolerance regions in particular are an important part of the statistical inference. They are used in quality control, life-testing and process reliability studies. So far in the literature they have been constructed by the standard methods and by using the Bayesian method of statistical inference. The present work deals primarily with the construction of the β-expectation tolerance regions using the structural method of statistical inference.

The structural method of inference, as developed by Fraser, re-examines most of the inference problems taking into account the internal structure of the response system. The analysis of this internal structure enables us to express our knowledge about the parameter (based on the data) in terms of its probability distribution, known

as the structural distribution. Using the structural distribution of parameters, the β-expectation tolerance regions are constructed for the following cases:

i) The samples from the normal distribution and the exponential distribution: the location-scale model.

ii) Difference of the samples from two normal distributions with different variances and equal variances.

iii) The regression model with the normal error variable.

iv) The samples from the multivariate normal distribution: the affine multivariate model.

v) The samples from q multivariate normal distributions: the generalized multivariate model.

vi) Pairwise difference of the samples from q multivariate normal distributions.

ACKNOWLEDGEMENTS

I wish to express my sincere gratitude to my research advisor Professor M. Safiul Haq for suggesting the topic of this research and for his active interest and close supervision throughout the course of this research.

I take this opportunity to thank Professor Mir M. Ali for his constant guidance, for reading the manuscript and for his valuable criticisms which improved the present thesis considerably.

I extend my thanks to the National Research Council of Canada, the Department of Mathematics and the Department of Applied Mathematics for their financial assistance for this research.

Miss Laura Barbe deserves special thanks for accepting the awful job of typing the thesis and doing it so well.

Last, but not least, I would like to thank my family for their encouragement throughout.

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CHAPTER 1

β-EXPECTATION TOLERANCE REGIONS BASED ON STRUCTURAL MODELS

1.1 Introduction. The present thesis primarily deals with the construction of the β-expectation tolerance region for the class of statistical models known as structural models. In particular, the β-expectation tolerance regions are constructed for the location-scale model, the regression model, the multivariate model and generalized multivariate model. The β-expectation tolerance regions are also constructed for the difference of the samples from two normal distributions and for the pairwise difference of the samples from q multivariate normal distributions.

1.2 β-expectation Tolerance Regions. Statistical tolerance regions are extensively used in problems of statistical inference such as life testing, quality control and process reliability studies. The theoretical basis of the tolerance regions drew the attention of statisticians from the early 1940's. The works of Wilks (1941), Paulson (1943), Wald and Wolfowitz (1946), Tukey (1947) may be mentioned. Fraser (1951 and 1953), Fraser and Guttman (1956), Guttman (1957 and 1959) and many others investigated the tolerance regions from the classical point of view.

Development of the Bayesian method of statistical inference helped to the further expansion of the theory of tolerance regions. The basic works in this area are due to Aitchison (1964), Aitchison and Sculthorpe (1965), Guttman (1969). The results of both approaches are put together in an excellent monograph on the tolerance regions by Guttman (1970).

Now we introduce the necessary terminology and notation and the definitions of the tolerance regions.

The set of all possible outcomes x, of an experiment is known as a sample space $\mathcal{X}$. The measurable space $\mathcal{X}(\mathcal{A})$ associates the sample space $\mathcal{X}$ with σ-algebra $\mathcal{A}$, which is defined on a class of subsets A of the sample space $\mathcal{X}$. The set of all n-tuples $(x_1, \ldots, x_n)$, where $x_i \in \mathcal{X}$ for all i is known as a product space and is denoted by $\mathcal{X}^n$. We will restrict ourselves to the n-dimensional Euclidean space, so that $\mathcal{X}^n = R^n$.

The class of probability measures over the space is denoted by $\{P_\theta/\theta \in \Omega\}$. $P_\theta(A)$ is the member of this class, where θ belongs to some indexing set Ω. We shall consider tolerance regions based on a sample of n from one of these probability measures. Now, for each value of the outcome $(x_1, \ldots, x_n)$ we wish to associate a subset of the space R^m. Accordingly, our first requirement is that a tolerance region $Q(x_1, \ldots, x_n)$ be a set function from R^n into some Borel field $\mathcal{B}$. The point of interest about the region $Q(x_1, \ldots, x_n)$ is the probability in the region as determined by the probability measure which gave rise to that outcome. The probability measure of $Q(x_1, \ldots, x_n)$ using P_θ is

$$P_\theta[Q(x_1, \ldots, x_n)];$$

which is called the coverage of the tolerance region. This function of the outcome has an induced probability distribution corresponding to the product measure of P_θ over R^m. It is this distribution that tells us how the probability content of $Q(x_1, \ldots, x_n)$ varies in repeated sampling from a given probability measure. We will be interested in the average or expected probability in a tolerance region $Q(x_1, \ldots, x_n)$.

Definition 1.2.1. $Q(x_1, \ldots, x_n)$ is a β-*expectation tolerance region* if

$$E_\theta\{P_\theta[Q(X_1, \ldots, X_n)]\} = \beta$$

for all $\theta \in \Omega$.

Thus, the statistical tolerance region is a statistic which maps the point $(X_1, \ldots, X_n) \in R^n$ into a region $Q(X_1,\ldots,X_n) \in A$, where Q is a random set function. The coverage of Q-abbreviated as C(Q)- is simply the probability content of the region Q for a given $\theta \in \Omega$. The coverage of the tolerance region Q, C(Q), is a random variable and has its own distribution, since Q is a random set function. Therefore it we construct the β-expectation tolerance region Q, we impose the condition that Q be such that the mean value of the distribution of its coverage C(Q) is β. The Definition 1.2.1 is very restrictive, because we search for suitable Q for all values of $\theta \in \Omega$. It should be also noted that such Q is not unique. The choice of position of the tolerance region Q is given by practical purposes. Sometimes we are interested in the middle part of the distribution, in other cases we might be interested in the right-hand (or left-hand) tail of the distribution. It should be noted also that because of practical purposes, the value of β is taken to be reasonably close to 1 (usually $\beta = .95$ or $\beta = .99$).

As an example of the tolerance region Q we might consider an interval $(X_{(k)}, X_{(n-k)}]$, where $X_{(k)} (1 \leq k < n)$ is the k-th order statistic of a sample $(X_1, \ldots, X_n)$, from a population having a continuous cumulative distribution function F(x). Then $F(X_{(n-k)}) - F(X_{(k)})$ is the coverage of the tolerance region Q, i.e. $C(Q) = F(X_{(n-k)}) - F(X_{(k)})$. Note that the coverage is a random variable with beta distribution $B(n - 2k, 2k + 1)$ (Wilks (1962), page 238), with the expectation being equal $(n-2k)(n+1)^{-1}$. Therefore our tolerance region Q will be the β-expectation tolerance region if we can find k and n such that

$$(n - 2k)(n + 1)^{-1} = \beta.$$

Let us take $\beta = .95$. Then if $n = 99$ and $k = 2$, the interval

$$Q = (X_{(2)}, X_{(97)}]$$

is the β-expectation tolerance region.

From the Bayesian point of view the coverage of the tolerance region is the function of the parameters involved, and the β-expectation tolerance regions are obtained by using the posterior distribution of the parameters. (Guttman (1970)).

The development of the structural method of statistical inference by Fraser (1968) puts a new light on the statistical philosophy which admits the distribution of parameters. The distribution of parameters for the structural models is called the structural distribution. In this thesis we construct the β-expectation tolerance regions based on the structural models. So a brief review of the structural method of statistical inference is given in the next section.

1.3 The Structural Method of Statistical Inference. This method gives special emphasis on the error variables associated with any system of observations as the basis of inference. The basic assumptions of this method are:

(i) The error variable $e \in \mathcal{E}$ has a known distribution on $\mathcal{E} \subset R^n$, which is denoted by

$$f(e)de. \tag{1.3.1}$$

(ii) The observation $x \in \mathcal{X}$ is the response generated from e by the application of a transformation θ. This is described by the structural equation

$$x = \theta e. \tag{1.3.2}$$

(iii) θ is a member of a unitary group of transformations G(A group of transformation G is unitary if $g_1 x = g_2 x \rightarrow g_1 = g_2$ for all $g_1, g_2 \in G$ and for all $x \in \mathcal{X}$).

Definition 1.3.1. A statistical model is a *structural model* if it satisfies assumptions (i), (ii) and (iii).

The structural model has two parts:

(a) the error variable e having known distribution on E and

(b) the structural equation describing the relationship of a realized value e from the error variable, the known response x, and the unknown quantity θ, taking values in the unitary group of transformations G on E.

The notation for the structural model is

$$\begin{cases} x = \theta e \\ f(e)de \end{cases} \qquad (1.3.3)$$

For the analysis of the structural model (1.3.3) the following is essential:

Definition 1.3.2. An *orbit* of e is a set Ge such that

$$Ge = \{ge/g \in G, e \in E\}.$$

Usually the orbit of e has dimension $1 \leq n$ and hence provides a basis for reduction. Also note that for every $x \in X$

$$e = \theta^{-1}x,$$

so

$$ge = g\theta^{-1}x = \tilde{g}x,$$

where $\tilde{g} \in G$, so therefore

$$Ge = Gx.$$

After obtaining the observation x, the orbit Ge = Gx is available to us as an event. That is, given the observation, the error variable e lies on the orbit Ge and so conditional probability statement of e on the orbit can be made.

Now, e on the orbit can be conveniently located from a reference point by the transformation

$$e = [e]D(e), \tag{1.3.4}$$

where D(e) is the reference point and [e] is an element of group G.

Definition 1.3.3. A function [e] from the space $\mathcal{E}$ to the group G is called a *transformation variable* if

$$[ge] = g[e] \quad \text{for all } g \in G \text{ and all } e \in \mathcal{E} \tag{1.3.5}$$

The [e]'s can be considered as new coordinates of the points on the orbit Ge. From (1.3.4) we have

$$D(e) = [e]^{-1}e.$$

Note that

$$D(e) = [e]^{-1}e = [e]^{-1}g^{-1}ge = (g[e])^{-1}ge = [ge]ge$$
$$= D(ge).$$

Thus reference point D(e) on each of the orbit Ge is uniquelly determined by the transformation variable [e], and so the set of all reference points indexes the class of all orbits.

Furthermore it follows that to find a conditional probability distribution of e on the orbit is equivalent to find the conditional probability distribution of [e] given the reference point D(e). Let $f^*([e]/D(e))d[e]$ be the conditional probability element of [e] given the reference point D(e) on the orbit Ge.

From the definition of the transformation variable we also have

$$[x] = [\theta e] = \theta[e] \;, \quad \theta \in G \tag{1.3.6}$$

and

$$\begin{aligned} D(e) &= [e]^{-1}e = [e]^{-1}\theta^{-1}\theta e = [\theta e]^{-1}\theta e \\ &= [x]^{-1}x = D(x). \end{aligned} \tag{1.3.7}$$

Thus, the model

$$\begin{cases} [x] = \theta[e] \\ f^*([e]/D(e))d[e] \end{cases} \tag{1.3.8}$$

satisfies Definition 1.3.1 and hence is a structural model. The model (1.3.8) is known as a *reduced structural model*, since it offers a reduction of the original model.

Probability elements. For derivation of $f^*([e]/D(e))$ the invariant measure is a very convenient tool. To define invariant measures on G we need one more assumption about G.

Assumption. G is a locally compact topological group.

This assumption assures us of the existence of at least one invariant measure on G (Halmos (1950), Hora and Buehler (1966)).

Definition 1.3.4. *An invariant measure* is a Borel measure μ in a locally compact topological group G such that $\mu(U) > 0$ for every nonempty

Borel open set U and $\mu(gB) = \mu(B)$ for every Borel set B and for all $g \in G$.

Let $\mu(\cdot)$ be the left invariant measure, $\nu(\cdot)$ be the right invariant measure and $\Delta(\cdot)$ be the modular function such that for all $g \in G$:

$$\mu(g) = \Delta(g)\nu(g).$$

Then the following properties hold:

$$\left.\begin{array}{l} \mu(gB) = \mu(B),\ \mu(Bg) = \Delta(g)\mu(B); \\ \nu(Bg) = \nu(B),\ \nu(gB) = \Delta(g^{-1})\nu(B); \\ \mu(B) = \nu(B^{-1}),\ \Delta(g) > 0; \\ \Delta(g_1 g_2) = \Delta(g_1)\Delta(g_2) \\ \text{and} \\ d\mu(g_1^{-1} g_2) = \Delta(g_2)d\nu(g_1) \end{array}\right\} \qquad (1.3.9)$$

for all g, g^{-1}, g_1, g_1^{-1} and g_2 belonging to G and for all Borel sets B of G. (Fraser (1968)).

Let m be an invariant measure defined on E such that $m(ge) = m(e)$ for all $g \in G$ and for all $e \in E$.

1) Conditional probability element on the orbit. Let $\bar{f}(e)dm(e)$ be the probability element of e with respect to the left invariant measure m on E. The probability element can then be expressed in terms of the reference point D(e) and the transformation variable [e] using (1.3.4) and the left invariant measure μ on G since two invariant measures m and μ differ by a constant only. Thus we have

$$\bar{f}(e)dm(e) = C\bar{f}([e]D(e))d\mu[e] \tag{1.3.10}$$

Therefore by normalizing (1.3.10) the conditional distribution of [e] given D(e) with respect to the invariant measure is obtained as

$$f^*([e]/D(e))d[e] = k(D)\bar{f}([e]D(e))d\mu[e]. \tag{1.3.11}$$

<u>2. The structural distribution of θ given x.</u> For an observed x we have the structural relation

$$[e] = \theta^{-1}[x],$$

which along with the conditional probability distribution (1.3.11) gives the structural distribution of θ, given x as

$$\begin{aligned} g(\theta/x)d\theta &= k(D)f(\theta^{-1}x)d\mu(\theta^{-1}[x]) \\ &= k(D)f(\theta^{-1}x)\Delta(x)d\mu(\theta^{-1}) \\ &= k(D)f(\theta^{-1}x)\Delta(x)d\nu(\theta). \end{aligned} \tag{1.3.12}$$

<u>Remarks</u>.

1) The structural distribution of θ does not depend on the choice of the transformation variable and is unique on the group space for a given structural model.

2) For particular structural models, the invariant measures will be determined by use of the invariant differentials on R^n. So for most cases the term invariant differential will be used in place of invariant measure. The construction of the invariant differentials has been discussed in detail by Fraser

(1968) and James (1954).

1.4 β-expectation Tolerance Regions Based on Structural Models.

From the structural point of view, the tolerance regions are constructed for the future responses (which might be denoted Y) from the system, based on actual data.

Consider the future response Y from the structural model (1.3.1). This future response will be generated from some error variable e* by relation

$$y = \theta e^*.$$

The realized value of e* is not known, however the probability element of e* is known and is equal to f(e)de (1.3.1), since the future response is generated by the same system. Therefore, using the relation

$$e^* = \theta^{-1} y,$$

the probability element of y is obtained as

$$f(\theta^{-1} y) J(e^* \rightarrow y) dy,$$

where $J(e^* \rightarrow y)$ is the Jacobian of transformation from $\mathcal{E}$ to $\mathcal{X}$. This probability element depends on the unknown value of the parameter θ. Let us denote this probability element by $p(y/\theta)dy$, so that

$$p(y/\theta) dy = f(\theta^{-1} y) J(e^* \rightarrow y) dy. \qquad (1.4.1)$$

Let us now define the structural tolerance region based on actual response x, as follows:

<u>Definition 1.4.1</u> The *structural tolerance region* is a statistic $Q(x)$ on R^n, the space of the future responses, based on data such that

$$C[Q(x)] = \int_{y \in Q(x)} p(y/\theta)dy \qquad (1.4.2)$$

<u>Definition 1.4.2</u> $Q(x)$ is a β-*expectation structural tolerance region* if

$$E_\Omega[C(Q(x))]$$
$$= \int_\Omega [\int_{Q(x)} p(y/\theta)dy]g(\theta/x)d\theta = \beta, \qquad (1.4.3)$$

where $g(\theta/x)$ is the structural distribution of parameters defined by (1.3.12).

Now, assuming that the conditions of Fubini's theorem hold, the left-hand side of (1.4.3) can be expressed as

$$E_\Omega[C(Q(x))] = \int_{Q(x)} \int_\Omega p(y/\theta)g(\theta/x)d\theta dy$$

$$= \int_{Q(x)} h(y/x)dy. \qquad (1.4.4)$$

The density $h(y/x)dy$, where

$$h(y/x) = \int_\Omega p(y/\theta)g(\theta/x)d\theta \qquad (1.4.5)$$

has been called the prediction distribution of the future response Y (Fraser and Haq (1969), (1970)).

Therefore to construct the β-expectation structural tolerance region for a particular structural model is equivalent to derive the prediction distribution of the future response from this model and then a region Q(x) such that

$$\int_{Q(x)} h(y/x)dy = \beta. \qquad (1.4.6)$$

It should be noted that such a Q(x) need not be unique.

Since in this thesis we will investigate this type of tolerance regions, namely the β-expectation structural tolerance regions, we will omit the word "structural" and simply call them the β-expectation tolerance regions.

1.5 Some Results from Matrix Algebra. For investigation of the structural models we will frequently use the results of matrix algebra. For terminology and some results we refer to any book on linear algebra or multivariate analysis (for example Anderson (1958), Morisson (1967)).

There are, however, three results which will be used more frequently in different chapters. So we will state them as Lemmas here for convenient references.

Lemma 1.5.1. (Anderson (1958) page 103). For the partitioned matrix

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

$$|A| = |A_{11}||A_{22} - A_{21}A_{11}^{-1}A_{12}|, \qquad (1.5.1)$$

if A_{11} is nonsingular and

$$|A| = |A_{22}||A_{11} - A_{12}A_{22}^{-1}A_{21}|, \qquad (1.5.2)$$

if A_{22} is nonsingular.

Lemma 1.5.2. (Goldberger (1964) page 27). For the partitioned matrix

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} ,$$

where A_{11} and A_{22} are both square and by their principal-minor nature nonsingular, its inverse

$$A^{-1} = \begin{bmatrix} A^{11} & A^{12} \\ A^{21} & A^{22} \end{bmatrix} , \qquad (1.5.3)$$

where

$$\left.\begin{aligned} A^{11} &= (A_{11}-A_{12}A_{22}^{-1}A_{21})^{-1} \\ A^{12} &= -(A_{11}-A_{12}A_{22}^{-1}A_{21})^{-1}A_{12}A_{22}^{-1} \\ A^{21} &= -A_{22}^{-1}A_{21}(A_{11}-A_{12}A_{22}^{-1}A_{21})^{-1} \\ A^{22} &= A_{22}^{-1}+A_{22}^{-1}A_{21}(A_{11}-A_{12}A_{22}^{-1}A_{21})^{-1}A_{12}A_{22}^{-1} \end{aligned}\right\} (1.5.4)$$

Lemma 1.5.3. The following rearrangement of the matrix expressions holds:

$$\begin{aligned} &(B-A)CC'(B-A)' + (D-AE)(D-AE)' \\ &= [A-(BCC'+DE')(CC'+EE')^{-1}](CC'+EE') \\ &\times [A-(BCC'+DE')(CC'EE')^{-1}]' \\ &+ (D-BE)(I-E'(CC'+EE')^{-1}E](D-BE)' \end{aligned} \qquad (1.5.5)$$

Proof of this lemma is given in appendix.

CHAPTER 2

THE LOCATION-SCALE MODEL

2.1 Introduction. In this chapter we will construct the β-expectation tolerance regions for the structural model which is known as the measurement model or location-scale model. This model has the form

$$\begin{cases} \underset{\sim}{x} = \mu\cdot\underset{\sim}{1} + \sigma\cdot\underset{\sim}{e} \\ \prod_{i=1}^{n} f(e_i)de_i \ , \end{cases} \qquad (2.1.1)$$

where $\underset{\sim}{x}' = (x_1, \ldots, x_n)$ is a vector of known responses, $\underset{\sim}{e}' = (e_1, \ldots, e_n)$ is realized, but unknown vector of error variables, σ is a scale factor applied to the error variable and μ is the general level of the response.

Then following Fraser (1968) the structural distribution of μ and σ, given the set of responses $\underset{\sim}{x}$ is

$$g(\mu,\ \sigma/\underset{\sim}{x})d\mu d\sigma = k(\underset{\sim}{d}) \prod_{i=1}^{n} f(\frac{x_i-\mu}{\sigma})\sigma^{-(n+1)} d\mu d\sigma, \qquad (2.1.2)$$

where $k(\underset{\sim}{d})$ is the normalizing constant:

$$k^{-1}(\underset{\sim}{d}) = \int\int_{\mu,\sigma} \prod_{i=1}^{n} f(\frac{x_i-\mu}{\sigma})\sigma^{-(n+1)} d\mu d\sigma.$$

For the rest of the derivations we will assume that the error variable follows normal distribution and exponential distribution. In other words, we will construct the

β-expectation tolerance regions for the samples from normal distribution and exponential distribution.

2.2 Normal Distribution.

Theorem 2.2.1. Let the error variable e have the normal distribution with 0 mean and variance 1, i.e.,

$$f(e)de = (2\pi)^{-\frac{1}{2}} \exp\{-\frac{e^2}{2}\}de.$$

Then for central 100β percent of the normal distribution being sampled, the region

$$Q = (\bar{x} - K_1 s_x/(n-1)^{\frac{1}{2}},\ \bar{x} + K_1 s_x/(n-1)^{\frac{1}{2}}] \quad (2.2.1)$$

is the β-expectation tolerance region, where

$$\bar{x} = n^{-1} \sum_{i=1}^{n} x_i \ ,\ s_x^2 = \sum_{i=1}^{n} (x_i - \bar{x})^2, \quad (2.2.2)$$

$$K_1 = (1 + n^{-1})^{\frac{1}{2}} t_{n-1;(1-\beta)/2} \quad (2.2.3)$$

and $t_{n-1;(1-\beta)/2}$ is the value of the t-distribution (n-1 degrees of freedom) exceeded with probability $(1-\beta)/2$.

Proof: Since the error variable e has standard normal distribution, the distribution of the realized errors for the location-scale model (2.1.1) is

$$\prod_{i-1}^{n} f(e_i)de_i = (2\pi)^{-\frac{n}{2}} \exp\{-\frac{1}{2} \sum_{i=1}^{n} e_i^2\} \prod_{i-1}^{n} de_i \ .$$

Then by (2.1.2) the structural distribution for μ and σ is

$$g(\mu,\sigma/\underset{\sim}{x})d\mu d\sigma = k(\underset{\sim}{d})\exp\{-\frac{1}{2\sigma^2}[n(\bar{x}-\mu)^2 + s_x^2]\}s_x^{n-1}\sigma^{-(n+1)}d\mu d\sigma,$$

where

$$k(\underset{\sim}{d}) = \frac{n^{\frac{1}{2}}}{2^{\frac{n}{2}-1}\pi^{\frac{1}{2}}\Gamma(\frac{n-1}{2})} .$$

For the future response variable Y, the distribution is

$$p(y/\mu,\sigma)dy = (2\pi\sigma^2)^{-\frac{1}{2}}\exp\{-\frac{(y-\mu)^2}{2\sigma^2}\}dy .$$

Therefore the joint distribution of Y, μ and σ is

$$p(y/\mu,\sigma)g(\mu,\sigma/\underset{\sim}{x})d\mu d\sigma dy = \frac{n^{\frac{1}{2}}}{2^{\frac{n-1}{2}}\pi\,\Gamma(\frac{n-1}{2})}$$

$$\times \exp -\frac{1}{2\sigma^2}[n(\bar{x}-\mu)^2+s_x^2+(y-\mu)^2]\ s_x^{n-1}\sigma^{-(n+2)}d\mu d\sigma dy.$$

Then by (1.4.5) the prediction distribution for Y is

$$h(y/\underset{\sim}{x})dy = \int_\Omega p(y/\mu,\sigma)g(\mu,\sigma/\underset{\sim}{x})d\mu d\sigma\cdot dy$$

$$= \frac{n^{\frac{1}{2}}}{2^{\frac{n-1}{2}}\pi\,\Gamma(\frac{n-1}{2})}\int_0^\infty\int_{-\infty}^\infty s_x^{n-1}\sigma^{-(n+2)}$$

$$\times \exp\{-\frac{1}{2\sigma^2}[(n+1)(\mu - \frac{n\bar{x}+y}{n+1})^2 + s_x^2 + \frac{n(y-\bar{x})^2}{n+1}]\}$$

$$\times\ d\mu d\sigma dy$$

$$= \frac{n^{\frac{1}{2}}}{2^{\frac{n-1}{2}} \pi \Gamma(\frac{n-1}{2})} \int_0^\infty \Big[\Big[\int_{-\infty}^\infty \exp\{-\frac{n+1}{2\sigma^2}(\mu - \frac{n\bar{x}+y}{n+1})^2\} d\mu\Big]$$

$$\times \exp\{-\frac{1}{2\sigma^2}[s_x^2 + \frac{n(y-\bar{x})^2}{n+1}]\} s_x^{n-1} \sigma^{-(n+2)} d\sigma \cdot dy$$

$$= \frac{n^{\frac{1}{2}}}{2^{\frac{n}{2}-1} (n+1)^{\frac{1}{2}} \pi^{\frac{1}{2}} \Gamma(\frac{n-1}{2})} \int_0^\infty \exp\{-\frac{1}{2\sigma^2}[s_x^2 + \frac{n(y-\bar{x})^2}{n+1}]\}$$

$$\times s_x^{n-1} \sigma^{-(n+1)} d\sigma \cdot dy$$

$$= \frac{n^{\frac{1}{2}} \Gamma(\frac{n}{2})}{(n+1)^{\frac{1}{2}} \Gamma(\frac{n-1}{2}) s_x \pi^{\frac{1}{2}} (n-1)^{\frac{1}{2}}} \left[1 + \frac{\left\{\frac{n^{\frac{1}{2}}(y-\bar{x})}{(n+1)^{\frac{1}{2}} s_x}\right]^2}{n-1}\right]^{-\frac{n}{2}} dy \quad (2.2.4)$$

That is, the prediction distribution of Y is such that

$$T_{n-1} = (\frac{n}{n+1})^{\frac{1}{2}} \frac{Y - \bar{x}}{s_x/(n-1)^{\frac{1}{2}}} \quad (2.2.5)$$

has Student's t-distribution with n-1 degrees of freedom.

Then by (1.4.6) the region Q defined by (2.2.1) is the β-expectation tolerance region if we take K_1 such that

$$K_1 = (1 + n^{-1})^{\frac{1}{2}} t_{n-1;(1-\beta)/2} .$$

This proves the theorem.

Corollary 2.2.1 For the left-hand and right-hand 100β per cent of the normal distribution being sampled the regions

$$Q_1 = (-\infty, \bar{x} + K_2 s_x/(n-1)^{\frac{1}{2}}] \qquad (2.2.7)$$

and

$$Q_2 = (\bar{x} - K_2 s_x/(n-1)^{\frac{1}{2}}, \infty) \qquad (2.2.8)$$

are respectively the β-expectation tolerance regions, where

$$K_2 = (1 + n^{-1})^{\frac{1}{2}} t_{n-1;1-\beta} \cdot \qquad (2.2.9)$$

Proof:

By using (1.4.6) and (2.2.5) it is readily seen that the regions Q_1 and Q_2 are the β-expectation tolerance regions if

$$K_2 = (1 + n^{-1})^{\frac{1}{2}} t_{n-1;1-\beta}.$$

2.3 Special Case: σ Known. Let us suppose that the conditions for the location-scale model are such that the scale factor applied to the error variable is known in advance and is equal to σ_0. Then we get the model which is called the simple measurement model or location model, which has the form

$$\begin{cases} \underset{\sim}{x} = \mu \cdot \underset{\sim}{1} + \sigma_0 \underset{\sim}{e} \\ \prod_{i=1}^{n} f(e_i) de_i \end{cases} , \qquad (2.3.1)$$

and for which the structural distribution of μ given the set of responses is reduced to

$$g(\mu/\underset{\sim}{x})d\mu = k(\underset{\sim}{d})g*(\frac{\bar{x}-\mu}{\sigma_0})d\mu, \qquad (2.3.2)$$

where g* is the conditional probability element for the location variable (transformation variable for the location model) given the orbit.

Theorem 2.3.1. Let the error variable e have the normal distribution with 0 mean and variance 1, i.e.

$$f(e)de = (2\pi)^{-\frac{1}{2}} \exp\{-\frac{e^2}{2}\}de.$$

Then for central 100β per cent of normal distribution being sampled the region

$$Q = (\bar{x} - K_3\sigma_0, \bar{x} + K_3\sigma_0] \qquad (2.3.3)$$

is the β-expectation tolerance region, where

$$\bar{x} = n^{-1} \sum_{i=1}^{n} x_i \qquad (2.3.4)$$

$$K_3 = (\frac{n+1}{n})^{\frac{1}{2}} z_{(1-\beta)/2} \qquad (2.3.5)$$

and $z_{(1-\beta)/2}$ is the value of the standard normal variable exceeded with probability (1-β)/2.

Proof:

Since the error variable e has standard normal distribution, the distribution of the realized errors for the location model (2.3.1) is

$$\prod_{i=1}^{n} f(e_i)de_i = (2\pi)^{-\frac{n}{2}} \exp\{-\frac{1}{2} \sum_{i=1}^{n} e_i^2\} \prod_{i=1}^{n} de_i \ .$$

Then by (2.3.2) the structural distribution for μ is

$$g(\mu/\underset{\sim}{x})d\mu = \left(\frac{n}{2\pi\sigma_0^2}\right)^{\frac{1}{2}} \exp\{-\frac{n}{2\sigma_0^2} (\bar{x}-\mu)^2\}d\mu.$$

For the future response variable Y, the distribution is

$$p(y/\mu)dy = (2\pi\sigma_0^2)^{-\frac{1}{2}} \exp\{-\frac{1}{2\sigma_0^2} (y-\mu)^2\}dy \ .$$

Therefore the joint distribution for Y and μ is

$$p(y/\mu)g(\mu/\underset{\sim}{x})d\mu dy = \frac{n^{\frac{1}{2}}}{2\pi\sigma_0^2} \exp\{-\frac{1}{2\sigma_0^2}[n(\bar{x}-\mu)^2 + (y-\mu)^2]\}d\mu dy.$$

Then by (1.4.5) the prediction distribution for Y is

$$h(y/\underset{\sim}{x})dy = \int_{\Omega} p(y/\mu)g(\mu/\underset{\sim}{x})d\mu\cdot dy$$

$$= \frac{n^{\frac{1}{2}}}{2\pi\sigma_0^2} \int_{-\infty}^{\infty} \exp\{-\frac{n+1}{2\sigma_0^2} (\mu - \frac{n\bar{x}+y}{n+1})^2\}d\mu$$

$$\times \exp\{-\frac{n}{2\sigma_0^2(n+1)} (y - \bar{x})^2\}dy$$

$$= \left(\frac{n}{2\pi\sigma_0^2(n+1)}\right)^{\frac{1}{2}} \exp\{-\frac{n}{2\sigma_0^2(n+1)} (y - \bar{x})^2\}dy \qquad (2.3.6)$$

That is, the prediction distribution of Y is such that

$$Z = (\frac{n}{n+1})^{\frac{1}{2}} \frac{Y - \bar{x}}{\sigma_0} \qquad (2.3.7)$$

is the standard normal variable.

Then by (1.4.6) the region Q defined by (2.3.2) is the β-expectation tolerance region if we take K_3 such that

$$K_3 = (\frac{n+1}{n})^{\frac{1}{2}} z_{(1-\beta)/2} ,$$

which was to be proved.

Corollary 2.3.1. For the left-hand and right-hand 100β per cent of normal distribution being sampled the regions

$$Q_1 = (-\infty, \bar{x} + K_4\sigma_0] \qquad (2.3.8)$$

and

$$Q_2 = (\bar{x} - K_4\sigma_0, \infty) \qquad (2.3.9)$$

are respectively the β-expectation tolerance regions if

$$K_4 = (\frac{n+1}{n})^{\frac{1}{2}} z_{1-\beta}. \qquad (2.3.10)$$

Proof:

By using (1.4.6) and (2.3.7) it is seen that the regions Q_1 and Q_2 are the β-expectation tolerance regions if

$$K_4 = (\frac{n+1}{n})^{\frac{1}{2}} z_{1-\beta}.$$

2.4 Exponential Distribution. We will now investigate the location-scale model again, but assume that the error variable has the exponential distribution. Since in the practical cases the main interest is the right-hand tail of the distribution, we will construct the tolerance regions of the type (a, ∞).

Theorem 2.4.1. Let the error variable e have the exponential distribution

$$f(e)de = \exp\{-e\}de, \quad e > 0.$$

Then for right-hand 100β per cent of exponential distribution being sampled, the region

$$Q = \begin{cases} (x_{(1)}+c_x d_{1;\beta}, \infty) & \text{for } \beta<n(n+1)^{-1} \\ (x_{(1)}, \infty) & \text{for } \beta=n(n+1)^{-1} \\ (x_{(1)}-n^{-1}c_x d_{2;\beta},\infty) & \text{for } \beta>n(n+1)^{-1} \end{cases} \tag{2.4.1}$$

where $x_{(i)}$ is ith ordered statistic and

$$c_x = \sum_{i=2}^{n} x_{(i)} - (n-1)x_{(1)} , \tag{2.4.2}$$

is the β-expectation tolerance region if $d_{1;\beta}$ and $d_{2;\beta}$ are as follows:

$$d_{1;\beta} = \left[\frac{n}{(n+1)\beta}\right]^{\frac{1}{n-1}} - 1 , \tag{2.4.3}$$

$$d_{2;\beta} = \left[\frac{1}{(n+1)(1-\beta)}\right]^{\frac{1}{n-1}} - 1. \qquad (2.4.4)$$

Proof:

Since the error variable e has the exponential distribution, the distribution of the realized errors for the location-scale model (2.1.1) is

$$\prod_{i=1}^{n} f(e_i)de_i = \exp\{-\sum_{i=1}^{n} e_i\} \prod_{i=1}^{n} de_i, \quad e_i > 0 \text{ for all } i.$$

Then by (2.1.2) the structural distribution for μ and σ is

$$g(\mu, \sigma/\underset{\sim}{x})d\mu d\sigma = \frac{n}{\Gamma(n-1)} \exp\{-\frac{1}{\sigma}[n(x_{(1)} - \mu) + c_x]\}$$

$$(\frac{c_x}{\sigma})^{n-1} \frac{d\mu d\sigma}{\sigma^2}, \qquad \text{for } \mu < x_{(1)} \text{ and } \sigma > 0.$$

For the future response variable Y, the distribution is

$$p(y/\mu,\sigma)dy = \frac{1}{\sigma} \exp\{-\frac{y-\mu}{\sigma}\}dy, \quad \text{for } y > \mu \text{ and } \sigma > 0.$$

Therefore the joint distribution of Y, μ and σ is

$$p(y/\mu,\sigma)g(\mu,\sigma/\underset{\sim}{x})dyd\mu d\sigma = \frac{n}{\Gamma(n-1)}\exp\{-\frac{1}{\sigma}[n(x_{(1)}-\mu)+(y-\mu)+c_x]\}$$

$$\times \frac{c_x^{n-1}}{\sigma^{n+2}} d\mu d\sigma dy, \quad \text{for } \mu<x_{(1)},$$

$$\mu < y \text{ and } \sigma > 0.$$

To find the prediction distribution for Y by (1.4.5) we have to consider two cases: $y < x_{(1)}$, and $y > x_{(1)}$, because of two conditions imposed on $\mu:\mu < x_{(1)}$ and $\mu < y$.

First for $y < x_{(1)}$:

$$h(y/\underset{\sim}{x})dy = \frac{n}{\Gamma(n-1)}\int_0^\infty\left[\int_{-\infty}^{y}\exp\{\frac{(n+1)\mu}{\sigma}\}d\mu\right]\exp\{-\frac{1}{\sigma}[nx_{(1)}+y+c_x]\}$$

$$\times\frac{c_x^{n-1}}{\sigma^{n+2}}\,d\sigma\cdot dy$$

$$= \frac{n}{(n+1)\Gamma(n-1)}\int_0^\infty\exp\{-\frac{1}{\sigma}[c_x+n(x_{(1)}-y)]\}\frac{c_x^{n-1}}{\sigma^{n+1}}\,d\sigma dy$$

$$= \frac{n(n-1)}{(n+1)c_x}\left[1+\frac{n(x_{(1)}-y)}{c_x}\right]^{-n}dy\ .$$

Secondly for $y \geq x_{(1)}$:

$$h(y/\underset{\sim}{x})dy = \frac{n}{\Gamma(n-1)}\int_0^\infty\left[\int_{-\infty}^{x_{(1)}}\exp\{\frac{(n+1)\mu}{\sigma}\}d\mu\right]\exp\{-\frac{1}{\sigma}[nx_{(1)}+y+c_x]\}$$

$$\times\frac{c_x^{n-1}}{\sigma^{n+1}}\,d\sigma\cdot dy$$

$$= \frac{n}{(n+1)\Gamma(n-1)}\int_0^\infty\exp\{-\frac{1}{\sigma}[c_x+y-x_{(1)}]\}\frac{c_x^{n-1}}{\sigma^{n+1}}\,d\sigma dy$$

$$= \frac{n(n-1)}{(n+1)c_x}\left[1+\frac{y-x_{(1)}}{c_x}\right]^{-n}dy\ .$$

Therefore the prediction distribution for Y is

$$h(y/\underset{\sim}{x})dy = \begin{cases}\dfrac{n(n-1)}{c_x(n+1)}\left[1+\dfrac{n(x_{(1)}-y)}{c_x}\right]^{-n}dy & \text{for } y < x_{(1)}\\[2ex] \dfrac{n(n-1)}{c_x(n+1)}\left[1+\dfrac{y-x_{(1)}}{c_x}\right]^{-n}dy & \text{for } y \geq x_{(1)}\end{cases} \qquad (2.4.5)$$

Then for the right-hand 100β per cent of exponential distribution being samples the relationship

$$\int_a^\infty h(y/\underset{\sim}{x})dy = \beta$$

should be fulfilled.

For that we first take $a = x_{(1)}$. Then

$$\frac{n(n-1)}{c_x(n+1)} \int_{x_{(1)}}^\infty [1 + \frac{y-x_{(1)}}{c_x}]^{-n} dy = \frac{n}{n+1}$$

Therefore for $\beta = \frac{n}{n+1}$ the region $Q = (x_{(1)}, \infty)$ is the β-expectation tolerance region.

Now for $\beta < n(n+1)^{-1}$ we have to find an "a" such that

$$\frac{n(n-1)}{c_x(n+1)} \int_a^\infty [1 + \frac{y-x_{(1)}}{c_x}]^{-n} dy = \beta.$$

For that

$$\frac{n(n-1)}{c_x(n+1)} \int_a^\infty [1 + \frac{y-x_{(1)}}{c_x}]^{-n} dy = \frac{n}{n+1}[1 + \frac{a-x_{(1)}}{c_x}]^{-(n-1)}$$

and therefore

$$\frac{n}{n+1}[1 + \frac{a-x_{(1)}}{c_x}]^{-(n-1)} = \beta,$$

from which we get

$$[1 + \frac{a-x_{(1)}}{c_x}]^{n-1} = \frac{n}{(n+1)\beta}$$

$$1 + \frac{a-x_{(1)}}{c_x} = [\frac{n}{(n+1)\beta}]^{\frac{1}{n-1}}$$

$$\frac{a-x_{(1)}}{c_x} = [\frac{n}{(n+1)\beta}]^{\frac{1}{n-1}} - 1 .$$

Let

$$d_{1;\beta} = [\frac{n}{(n+1)\beta}]^{\frac{1}{n-1}} - 1,$$

then

$$a = x_{(1)} + c_x d_{1;\beta} ,$$

so the region $Q = (x_{(1)} + c_x d_{1;\beta}, \infty)$ is the β-expectation tolerance region.

For $\beta > n(n+1)^{-1}$ we have to find an "a" such that

$$\frac{n(n-1)}{c_x(n+1)} \int_{\infty}^{a} [1 + \frac{n(x_{(1)}-y)}{c_x}]^{-n} dy = 1 - \beta.$$

For that

$$\frac{n(n-1)}{c_x(n+1)} \int_{-\infty}^{a} [1 + \frac{n(x_{(1)}-y)}{c_x}]^{-n} dy = \frac{1}{n+1}[1 + \frac{n(x_{(1)}-a)}{c_x}]^{-(n-1)}$$

and therefore

$$\frac{1}{n+1}[1 + \frac{n(x_{(1)}-a)}{c_x}]^{-(n-1)} = 1 - \beta,$$

from which

$$[1 + \frac{n(x_{(1)}-a)}{c_x}]^{n-1} = \frac{1}{(n+1)(1-\beta)}$$

$$1 + \frac{n(x_{(1)}-a)}{c_x} = [\frac{1}{(n+1)(1-\beta)}]^{\frac{1}{n-1}}$$

$$\frac{n(x_{(1)}-a)}{c_x} = [\frac{1}{(n+1)(1-\beta)}]^{\frac{1}{n-1}} - 1 .$$

Let

$$d_{2;\beta} = [\frac{1}{(n+1)(1-\beta)}]^{\frac{1}{n-1}} - 1 ,$$

then

$$x_{(1)} - a = n^{-1} c_x d_{2;\beta}$$

and

$$a = x_{(1)} - n^{-1} c_x d_{2;\beta},$$

so the region $Q = (x_{(1)} - n^{-1} c_x d_{2;\beta}, \infty)$ is the β-expectation tolerance region.

Combining all three results we see that the region Q defined at (2.4.1) is the β-expectation tolerance region for the right-hand 100β per cent of exponential distribution being sampled, which was to be proved.

Remark. The construction of the β-expectation tolerance regions for the left-hand side of the exponential distribution is equivalent to the construction of the $(1-\beta)$-expectation tolerance regions for the right-hand side of the exponential distribution.

To show this, let "a" be the point such that $Q_1 = (-\infty, a]$ and $Q_2 = (a, \infty)$. Then

$$Q_1 \cup Q_2 = R^1 \qquad \text{and } Q_1 \cap Q_2 = \emptyset.$$

It is evident, that

$$C(Q_1 \cup Q_2) = C(Q_1) + C(Q_2) = C(R^1) = 1.$$

Then

$$E_\Omega[C(Q_1) + C(Q_2)] = E_\Omega(1)$$

or

$$E_\Omega[C(Q_1)] + E_\Omega[C(Q_2)] = 1.$$

Assuming that Q_2 is the (1-β)-expectation tolerance region we get

$$E_\Omega[C(Q_1)] + 1 - \beta = 1$$

or

$$E_\Omega[C(Q_1)] = \beta,$$

which shows that $Q_1 = (-\infty, a]$ is the β-expectation tolerance region.

2.5. Special Case: $\mu = 0$ (Life Testing). Let us suppose that conditions for the location-scale model are now such, that the general level of the response is known in advance and is equal μ_0. Since $\mu_0 = 0$ is of great importance in statistics (so called life testing problem), without loss of generality we will investigate this case. Then we get the model which is called the scale model and which has the form

$$\begin{cases} \underset{\sim}{x} = \sigma \underset{\sim}{e} \\ \prod_{i=1}^{n} f(e_i) de_i \end{cases} \qquad (2.5.1)$$

Then the following theorem holds:

Theorem 2.5.1. Let the error variable e have the exponential distribution

$$f(e)de = \exp\{-e\}de, \qquad e > 0.$$

Then for the right-hand 100 per cent of exponential distribution being sampled the region

$$Q = (t_x \, d_{3;\beta}, \infty) , \tag{2.5.2}$$

where

$$t_x = \sum_{i=1}^{n} x_i \tag{2.5.3}$$

is the β-expectation tolerance region if

$$d_{3;\beta} = \beta^{-\frac{1}{n}} - 1. \tag{2.5.4}$$

Proof:

Since the error variable e has the exponential distribution, the distribution of the realized errors for the scale model (2.5.1) is

$$\prod_{i=1}^{n} f(e_i)de_i = \exp\{-\sum_{i=1}^{n} e_i\} \prod_{i=1}^{n} de_i, \quad e_i > 0 \text{ for all i.}$$

Then by (2.1.2) the structural distribution for σ (note that $\mu = 0$) is

$$g(\sigma/\underset{\sim}{x})d\sigma = \frac{1}{\Gamma(n)} \exp\{-\frac{t_x}{\sigma}\} \frac{t_x^n}{\sigma_x^{n+1}} d\sigma , \quad t_x > 0, \quad \sigma > 0.$$

For the future variable Y, the distribution is

$$p(y/\sigma)dy = \frac{1}{\sigma} \exp\{-\frac{y}{\sigma}\}dy, \qquad y > 0, \quad \sigma > 0.$$

Therefore the joint distribution of Y and σ is

$$p(y/\sigma)g(\sigma/\underset{\sim}{x})dyd\sigma = \frac{1}{\Gamma(n)} \exp\{-\frac{t_x+y}{\sigma}\}\frac{t_x^n}{\sigma^{n+2}} dyd\sigma,$$

$$t_x > 0,\ y > 0,\ \sigma > 0.$$

Then by (1.4.5) the prediction distribution for Y is

$$h(y/\underset{\sim}{x})dy = \frac{1}{\Gamma(n)} \int_0^\infty \exp\{-\frac{t_x+y}{\sigma}\}\frac{t_x^n}{\sigma^{n+2}} d\sigma \cdot dy$$

$$= \frac{n\Gamma(n)}{\Gamma(n)} \frac{t_x^n}{[t_x+y]^{n+1}} dy$$

$$= \frac{n}{t_x}[1 + \frac{y}{t_x}]^{-(n+1)}dy \qquad (2.5.5)$$

Now, we have to find an "a" such that

$$\int_a^\infty h(y/\underset{\sim}{x})dy = \beta,$$

to obtain the β-expectation tolerance region.

For that

$$\frac{n}{t_x}\int_a^\infty [1 + \frac{y}{t_x}]^{-(n+1)}dy = (1 + \frac{a}{t_x})^{-n}$$

and therefore

$$(1 + \frac{a}{t_x})^{-n} = \beta$$

from which

$$1 + \frac{a}{t_x} = (\frac{1}{\beta})^{\frac{1}{n}}$$

$$\frac{a}{t_x} = (\frac{1}{\beta})^{\frac{1}{n}} - 1$$

Let

$$d_{3;\beta} = \beta^{-\frac{1}{n}} - 1,$$

then

$$a = t_x d_{3;\beta} ,$$

and therefore the region Q defined at (2.5.1) is the β-expectation tolerance region, which was to be proved.

TABLE I

Values of $d_{1;\beta}$ for the Exponential Distribution

β n	0.01	0.05	0.1	0.9	0.95	0.99
2	65.6666667	12.3333333	5.6666667			
3	7.6602540	2.8729833	1.7386128			
4	3.3088694	1.5198421	1.0000000			
5	2.0213754	1.0205155	0.6990442			
6	1.4356262	0.7652924	0.5367762			
7	1.1070168	0.6112874	0.4354939			
8	0.8984833	0.5085299	0.3663112			
9	0.7550129	0.4351889	0.3160740	0.0000000		
10	0.6505285	0.3802562	0.2779443	0.0011173		
11	0.5711626	0.3375935	0.2480188	0.0018365		
12	0.5088914	0.3035127	0.2239083	0.0023042		
13	0.4587626	0.2756664	0.2040687	0.0026077		
14	0.4175595	0.2524903	0.1874579	0.0028014		
15	0.3831048	0.2329023	0.1733471	0.0029201		
16	0.3538734	0.2161302	0.1612117	0.0029868		
17	0.3287661	0.2016082	0.1506640	0.0030171		
18	0.3069706	0.1889126	0.1414116	0.0030218		
19	0.2878745	0.1777195	0.1332298	0.0030082	0.0000000	
20	0.2710069	0.1677773	0.1259429	0.0029818	0.0001317	
21	0.2560006	0.1588876	0.1194117	0.0029463	0.0002387	
22	0.2425641	0.1508918	0.1135244	0.0029046	0.0003258	
23	0.2304641	0.1436617	0.1081904	0.0028586	0.0003970	
24	0.2195109	0.1370923	0.1033351	0.0028099	0.0004553	
25	0.2095494	0.1310970	0.0988969	0.0027596	0.0005031	
26	0.2004508	0.1256039	0.0948242	0.0027084	0.0005422	
27	0.1921080	0.1205523	0.0910736	0.0026571	0.0005742	
28	0.1844306	0.1158912	0.0876085	0.0026059	0.0006002	
29	0.1773423	0.1115769	0.0843974	0.0025553	0.0006213	
30	0.1707778	0.1075722	0.0814133	0.0025055	0.0006382	
40	0.1246233	0.0791573	0.0601469	0.0020705	0.0006823	
50	0.0980973	0.0626154	0.0476896	0.0017476	0.0006428	
60	0.0808779	0.0517916	0.0395072	0.0015067	0.0005893	
70	0.0687995	0.0441580	0.0337213	0.0013222	0.0005379	
80	0.0598592	0.0384855	0.0294137	0.0011771	0.0004921	
90	0.0529748	0.0341044	0.0260819	0.0010602	0.0004522	
99	0.0480056	0.0309350	0.0236690	0.0009730	0.0004209	0.0000000
100	0.0475105	0.0306188	0.0234282	0.0009642	0.0004176	0.0000010

TABLE II

Values of $d_{2;\beta}$ for the Exponential Distribution

β n	0.9	0.95	0.99
2	2.3333333	5.6666666	23.3333333
3	0.5811388	1.2360679	4.0000000
4	0.2599210	0.5874010	1.7144176
5	0.1362193	0.3512001	1.0205155
6	0.0739409	0.2336341	0.7020816
7	0.0378908	0.1649930	0.5234153
8	0.0151653	0.1208334	0.4105676
9	0.0000000	0.0905077	0.3335214
10		0.0686822	0.2779442
11		0.0524097	0.2361804
12		0.0399390	0.2037896
13		0.0301690	0.1780290
14		0.0223760	0.1571178
15		0.0160665	0.1398522
16		0.0108935	0.1253909
17		0.0066067	0.1131289
18		0.0030218	0.1026209
19		0.0000000	0.0935324
20			0.0856070
21			0.0786458
22			0.0724916
23			0.0670191
24			0.0621271
25			0.0577331
26			0.0537690
27			0.0501785
28			0.0469144
29			0.0439368
30			0.0412122
40			0.0231248
50			0.0138365
60			0.0084130
70			0.0049759
80			0.0026709
90			0.0010602
99			0.0000000
100			

CHAPTER 3

DIFFERENCE OF SAMPLES FROM TWO NORMAL DISTRIBUTIONS

3.1 Introduction. In this chapter we will construct the β-expectation tolerance regions for the difference of samples from two normal distributions $N(\mu_1, \sigma_1^2)$ and $N(\mu_2,\sigma_2^2)$. In such a case for the corresponding structural method we consider two response variables x_1 and x_2, generated from two independent error variables e_1 and e_2, respectively, by the equations:

$$\begin{aligned} x_1 &= \mu_1 + \sigma_2 e_1 \\ x_2 &= \mu_2 + \sigma_2 e_2 , \end{aligned} \qquad (3.1.1)$$

where e_1 and e_2 have standard normal distributions, μ_1 and μ_2 are the general levels of the response variables x_1 and x_2, respectively and σ_1 and σ_2 are the scale factors applied to the error variables e_1 and e_2, respectively.

Since the corresponding structural model is given by Fraser (1968), Chapter 2 only as an exercise we will investigate it in more detail. First we will investigate the case $\sigma_1 \neq \sigma_2$ and call it model 1 and then we will consider the special case $\sigma_1 = \sigma_2$ and call it model 2.

3.2 The Model 1. Let $\underset{\sim}{x}_i' = (x_{i1}, x_{i2}, \ldots, x_{in_i})$ be the sequences of $n_i (i = 1, 2)$ observations of the response variables. Then the equations (3.1.1) lead to the model 1, which in convenient matrix notation is:

$$\left\{\begin{aligned} &\begin{pmatrix} \underset{\sim}{1}' & \underset{\sim}{0}' \\ \underset{\sim}{0}' & \underset{\sim}{1}' \\ \underset{\sim}{x}_1' & \underset{\sim}{0}' \\ \underset{\sim}{0}' & \underset{\sim}{x}_2' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \mu_1 & 0 & \sigma_1 & 0 \\ 0 & \mu_2 & 0 & \sigma_2 \end{pmatrix} \begin{pmatrix} \underset{\sim}{1}' & \underset{\sim}{0}' \\ \underset{\sim}{0}' & \underset{\sim}{1}' \\ \underset{\sim}{e}_1' & \underset{\sim}{0}' \\ \underset{\sim}{0}' & \underset{\sim}{e}_2' \end{pmatrix} \\ &\prod_{i=1}^{2} \prod_{j=1}^{n_i} f(e_{ij}) de_{ij} = (2\pi)^{-\frac{n_1+n_2}{2}} \exp\{-\frac{1}{2} \sum_{i=1}^{2} \sum_{j=1}^{n_i} e_{ij}^2\} \\ &\qquad \times \prod_{i,j} de_{ij}, \end{aligned}\right. \tag{3.2.1}$$

or

$$\left\{\begin{aligned} &X = \theta E \\ &f(E) dE \quad . \end{aligned}\right. \tag{3.2.2}$$

The transformation

$$\theta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \mu_1 & 0 & \sigma_1 & 0 \\ 0 & \mu_2 & 0 & \sigma_2 \end{pmatrix}$$

has positive scale factors σ_1 and σ_2 and relocations μ_1 and μ_2. Such a transformation is an element of the unitary positive-affine group

$$G = \left\{ g = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ a_1 & 0 & c_1 & 0 \\ 0 & a_2 & 0 & c_2 \end{bmatrix} \middle/ \begin{matrix} -\infty < a_1 < \infty \\ -\infty < a_2 < \infty \\ 0 < c_1 < \infty \\ 0 < c_2 < \infty \end{matrix} \right\}, \tag{3.2.3}$$

where the group operation is defined as a matrix multiplication rule.

It can be easily verified that

$$[E] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \bar{e}_1 & 0 & s_{e_1} & 0 \\ 0 & \bar{e}_2 & 0 & s_{e_2} \end{bmatrix} , \tag{3.2.4}$$

where

$$\bar{e}_i = n_i^{-1} \sum_{j=1}^{n_i} e_{ij} \quad i = 1, 2 \tag{3.2.5}$$

and

$$s_{e_i}^2 = \sum_{j=1}^{n_i} (e_{ij} - \bar{e}_i)^2 \quad i = 1, 2 \tag{3.2.6}$$

is a transformation variable for this model.

3.3 The Model 1:Distributions. In order to derive the conditional distribution on the orbit and the structural distribution of parameters the following invariant differentials based on the transformations may be helpful:

Consider first the error space E. Let us apply transformation $g \in G$ to the matrix of error variables E then

$$E^*=\begin{pmatrix} \underset{\sim}{1}' & \underset{\sim}{0}' \\ \underset{\sim}{0}' & \underset{\sim}{1}' \\ \underset{\sim}{e}_1^{*\prime} & \underset{\sim}{0}' \\ \underset{\sim}{0}' & \underset{\sim}{e}_2^{*\prime} \end{pmatrix}=gE=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ a_1 & 0 & c_1 & 0 \\ 0 & a_2 & 0 & c_2 \end{pmatrix}\begin{pmatrix} \underset{\sim}{1}' & \underset{\sim}{0}' \\ \underset{\sim}{0}' & \underset{\sim}{1}' \\ \underset{\sim}{e}_1' & \underset{\sim}{0}' \\ \underset{\sim}{0}' & \underset{\sim}{e}_2' \end{pmatrix}=\begin{pmatrix} \underset{\sim}{1}' & \underset{\sim}{0}' \\ \underset{\sim}{0}' & \underset{\sim}{1}' \\ a_1\underset{\sim}{1}'+c_1\underset{\sim}{e}_1' & \underset{\sim}{0}' \\ \underset{\sim}{0}' & a_2\underset{\sim}{1}'+c_2\underset{\sim}{e}_2' \end{pmatrix},$$

so

$$\underset{\sim}{e}_i^* = a_i\underset{\sim}{1} + c_i\underset{\sim}{e}_i \qquad i = 1,\ 2.$$

Then

$$J_{n_1+n_2}(g:E) = \left|\frac{\partial gE}{\partial E}\right| = c_1^{\,n_1} c_2^{\,n_2} \ ,$$

so

$$J_{n_1+n_2}(\underset{\sim}{e}_1,\ \underset{\sim}{e}_2) = s_{e_1}^{\ n_1} s_{e_2}^{\ n_2} \tag{3.3.1}$$

and

$$dm(\underset{\sim}{e}_1,\ \underset{\sim}{e}_2) = \frac{\prod\limits_{i,j} de_{ij}}{s_{e_1}^{\ n_1} s_{e_2}^{\ n_2}} = \frac{d\underset{\sim}{e}_1\ d\underset{\sim}{e}_2}{s_{e_1}^{\ n_1} s_{e_2}^{\ n_2}}$$

Now consider the invariant differentials on the group:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \tilde{a}_1 & 0 & \tilde{c}_1 & 0 \\ 0 & \tilde{a}_2 & 0 & \tilde{c}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ a_1 & 0 & c_1 & 0 \\ 0 & a_2 & 0 & c_2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ a_1^* & 0 & c_1^* & 0 \\ 0 & a_2^* & 0 & c_2^* \end{pmatrix} ,$$

which implies for i = 1, 2

$$\tilde{a}_i = a_i + c_i a_i^*$$

$$\tilde{c}_i = c_i c_i^* \ .$$

Therefore

$$J(g) = \left|\frac{\partial \tilde{g}}{\partial g}\right| = c_1^2\ c_2^2 \tag{3.3.2}$$

and

$$J^*(g) = \left|\frac{\partial \tilde{g}}{\partial g^*}\right| = c_1^*\ c_2^* \ . \tag{3.3.3}$$

This implies that

$$d\mu(\bar{e}_1,\ \bar{e}_2,\ s_{e_1},\ s_{e_2}) = s_{e_1}^{-2}\ s_{e_2}^{-2}\ d\bar{e}_1 d\bar{e}_2 ds_{e_1} ds_{e_2} \ ,$$

$$d\nu(\bar{e}_1,\ \bar{e}_2,\ s_{e_1},\ s_{e_2}) = s_{e_1}^{-1} s_{e_2}^{-1}\ d\bar{e}_1\ d\bar{e}_2\ ds_{e_1}\ ds_{e_2}$$

and

$$\Delta(\bar{e}_1,\ \bar{e}_2,\ s_{e_1},\ s_{e_2}) = s_{e_1}\ s_{e_2}\ s_{e_1}^{-2}\ s_{e_2}^{-2} = s_{e_1}^{-1}\ s_{e_2}^{-1} \ .$$

For this model the reference point D on the orbit by (1.4.6) is

$$D = \begin{pmatrix} \underset{\sim}{1}' & \underset{\sim}{0}' \\ \underset{\sim}{0}' & \underset{\sim}{1}' \\ \underset{\sim}{d}_1' & \underset{\sim}{0}' \\ \underset{\sim}{0}' & \underset{\sim}{d}_2' \end{pmatrix} = [E]^{-1}E = \begin{pmatrix} \underset{\sim}{1}' & \underset{\sim}{0}' \\ \underset{\sim}{0}' & \underset{\sim}{1}' \\ s_{e_1}^{-1}(\underset{\sim}{e}_1' - \bar{e}_1 \cdot \underset{\sim}{1}') & \underset{\sim}{0}' \\ \underset{\sim}{0}' & s_{e_2}^{-1}(\underset{\sim}{e}_2' - \bar{e}_2 \cdot \underset{\sim}{1}) \end{pmatrix} \ .$$

From this for i = 1, 2

$$\sum_{j=1}^{n_i} d_{ij} = \sum_{j=1}^{n_i} s_{e_i}^{-1}(e_{ij} - \bar{e}_i) = s_{e_i}^{-1} \sum_{j=1}^{n_i}(e_{ij} - \bar{e}_i) = 0 \tag{3.3.4}$$

and

$$\sum_{j=1}^{n} d_{ij}^2 = \sum_{j=1}^{n_i}\{s_{e_i}^{-1}(e_{ij} - \bar{e}_i)\}^2 = s_{e_i}^{-2} \sum_{j=1}^{n_i}(e_{ij} - \bar{e}_i)^2 = s_{e_i}^{-2} s_{e_i}^2 = 1. \tag{3.3.5}$$

Then by using (1.3.11), (3.3.4), (3.3.5) and the fact that the normalizing constant is

$$k(D) = (n_1\ n_2)^{\frac{1}{2}}\ A_{n_1-1}\ A_{n_2-1}\ ,$$

where

$$A_k = \frac{2\pi^{\frac{k}{2}}}{\Gamma(\frac{k}{2})}\ ;$$

the conditional distribution on the orbit is

$$f^*(\bar{e}_1,\ \bar{e}_2,\ s_{e_1},\ s_{e_2}/\underset{\sim}{d}_1,\ \underset{\sim}{d}_2)d\bar{e}_1 d\bar{e}_2 ds_{e_1} ds_{e_2} \tag{3.3.6}$$

$$= (n_1\ n_2)^{\frac{1}{2}}\ A_{n_1-1}\ A_{n_2-2}(2\pi)^{-\frac{n_1+n_2}{2}}$$

$$\times\ \exp\{-\frac{1}{2}\sum_{i=1}^{2}(n_i\bar{e}_i^2 + s_{e_i}^2)\}s_{e_1}^{n_1-2}\ s_{e_2}^{n_2-2}\ d\bar{e}_1 d\bar{e}_2 ds_{e_1} ds_{e_2}.$$

Now since [X] is a member of group G it follows, that

$$[X] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \bar{x}_1 & 0 & s_{x_1} & 0 \\ 0 & \bar{x}_2 & 0 & s_{x_2} \end{pmatrix}\ ,$$

where for i = 1, 2

$$\bar{x}_i = n_i^{-1}\sum_{j=1}^{n_i} x_{ij} \tag{3.3.7}$$

and

$$s_{x_i}^2 = \sum_{j=1}^{n_i}(x_{ij} - \bar{x}_i)^2. \tag{3.3.8}$$

The structural distribution of μ_1, μ_2, σ_1 and σ_2 is then (by (1.3.12) and (3.3.6))

$$g(\mu_1, \mu_2, \sigma_1, \sigma_2/\underset{\sim}{x}_1, \underset{\sim}{x}_2)d\mu_1 d\mu_2 d\sigma_1 d\sigma_2$$

$$= (n_1\ n_2)^{\frac{1}{2}} A_{n_1-1} A_{n_2-1} (2\pi)^{-\frac{n_1+n_2}{2}}$$

$$\times \exp\left\{-\frac{1}{2} \sum_{i=1}^{2} \left[\frac{n_i(\overline{x}_i-\mu_i)^2}{\sigma_i^2} - \frac{s_{x_i}^2}{\sigma_i^2}\right]\right\}$$

$$\times s_{x_1}^{n_1-1} s_{x_2}^{n_2-1} \sigma_1^{-(n_1+1)} \sigma_2^{-(n_2+1)}$$

$$\times d\mu_1 d\mu_2 d\sigma_1 d\sigma_2 \ . \qquad (3.3.9)$$

3.4 β-expectation Tolerance Region For the Variable $Z = X_1 - X_2$, Assuming $\sigma_1 \neq \sigma_2$.

Theorem 3.4.1. Let the independent error variables e_1 and e_2 have the normal distribution with 0 mean and variance 1, i.e.

$$f(e_i)de_i = (2\pi)^{-\frac{1}{2}} \exp\{-\tfrac{1}{2} e_i^2\}de_i \qquad i = 1, 2.$$

Then for the central 100β per cent of the distribution of the variable $Z = X_1 - X_2$ (where X_1 is $N(\mu_1, \sigma_1^2)$ and X_2 is $N(\mu_2, \sigma_2^2)$) being sampled, the region

$$Q=(\overline{x}_1-\overline{x}_2-d_{1-\beta}r;\ \overline{x}_1-\overline{x}_2+d_{1-\beta}r] \qquad (3.4.1)$$

is the β-expectation tolerance region, where

$$r^2 = s_{x_1}^2 \frac{n_1 + 1}{n_1(n_1-1)} + s_{x_2}^2 \frac{n_2 + 1}{n_2(n_2-1)} \qquad (3.4.2)$$

with $\bar{x}_1$, $\bar{x}_2$, s_{x_1} and s_{x_2} defined by (3.3.7)and (3.3.8) and where $d_{1-\beta}$ is the point exceeded with probability 1-β when using the Behrens-Fisher distribution with $(n_1 - 1)$ and $(n_2 - 1)$ degrees of freedom and the parameter δ, where δ is given by

$$\delta = \arctan\left(s_{x_2}\left[\frac{n_2+1}{n_2(n_2-1)}\right]^{\frac{1}{2}} \Big/ s_{x_1}\left[\frac{n_1+1}{n_1(n_1-1)}\right]^{\frac{1}{2}} \right) \qquad (3.4.3)$$

Proof:

Since the independent error variables e_1 and e_2 have the standard normal distribution, the distribution of the realized errors for the model 1 (3.2.1) is

$$\prod_{i=1}^{2} \prod_{j=1}^{n_i} f(e_{ij})de_{ij} = (2\pi)^{-\frac{n_1+n_2}{2}} \exp\{-\frac{1}{2} \sum_{i=1}^{2} \sum_{j=1}^{n_i} e_{ij}^2\}$$

$$\times \prod_{i,j} de_{ij} .$$

The structural distribution for μ_1, μ_2, σ_1 and σ_2 is then given by (3.3.9). For the independent future responses Y_1, Y_2 in the structural model (3.2.1) the distribution is

$$p(y_1,y_2/\mu_1,\mu_2,\sigma_1,\sigma_2)dy\ dy_2 = (2\pi\sigma_1\sigma_2)^{-1}$$

$$\times \exp\{-\frac{1}{2} \sum_{i=1}^{2} \left(\frac{y_i - \mu_i}{\sigma_i}\right)^2\} dy_1dy_2 .$$

Therefore the joint distribution of Y_1, Y_2, μ_1, μ_2, σ_1 and σ_2 is

$$p(y_1,y_2/\mu_1,\mu_2,\sigma_1,\sigma_2)g(\mu_1,\mu_2,\sigma_1,\sigma_2/\underset{\sim}{x}_1,\underset{\sim}{x}_2)$$

$$\times\ d\mu_1 d\mu_2 d\sigma_1 d\sigma_2 dy_1 dy_2$$

$$= (n_1\ n_2)^{\frac{1}{2}}\ A_{n_1-1}\ A_{n_2-2}(2\pi)^{-\frac{n_1+n_2+2}{2}}$$

$$\exp\left\{-\frac{1}{2}\sum_{i=1}^{2}\frac{n_i(\bar{x}_i-\mu_i)^2+s_{x_i}^2+(y_i-\mu_i)^2}{\sigma_i^2}\right\}$$

$$\times\ \frac{s_{x_1}^{n_1-1}}{\sigma_1^{n_1+2}}\ \frac{s_{x_2}^{n_2-1}}{\sigma_2^{n_2+2}}\ d\mu_1 d\mu_2 d\sigma_1 d\sigma_2 dy_1 dy_2\quad .$$

The term in the bracket in the exponent can be rearranged using the following result for i = 1, 2:

$$n_i(\bar{x}_i - \mu_i)^2 + (y_i - \mu_i)^2 = (n_i + 1)\left(\mu_i - \frac{n_i\bar{x}_i + y_i}{n_i+1}\right)^2$$

$$+ \frac{n_i}{n_i+1}(y_i - \bar{x}_i)^2 \qquad (3.4.4)$$

Then

$$p(y_1,y_2/\mu_1,\mu_2,\sigma_1,\sigma_2)g(\mu_1,\mu_2,\sigma_1,\sigma_2/x_1,x_2)$$

$$\times\ d\mu_1 d\mu_2 d\sigma_1 d\sigma_2 dy_1 dy_2$$

$$= (n_1\ n_2)^{\frac{1}{2}}\ A_{n_1-1}\ A_{n_2-1}(2\pi)^{-\frac{n_1+n_2+2}{2}}$$

$$\times\ \exp\{-\frac{1}{2}\ \sum_{i=1}^{2} \frac{n_i+1}{\sigma_i^2}\left(\mu_i\ -\ \frac{n_i\bar{x}_i\ +\ y_i}{n_i+1}\right)^2\}$$

$$\times\ \exp\{-\frac{1}{2}\ \sum_{i=1}^{2}\left[\frac{n_i}{n_i+1}\ \frac{(y_i-\bar{x}_i)^2}{\sigma_i^2}\ +\ \frac{s_{\bar{x}_i}^2}{\sigma_i^2}\right]\}$$

$$\times\ \frac{s_{x_1}^{n_2-1}}{{}^{n_1+2}_{1}}\ \frac{s_{x_2}^{n_2-1}}{{}^{n_2+2}_{1}}\ d\mu_1 d\mu_2 d\sigma_1 d\sigma_2 dy_1 dy_2\ \ .$$

Then by (1.4.5) the prediction distribution for Y_1 and Y_2 is

$h(y_1,\ y_2/\underset{\sim}{x}_1,\ \underset{\sim}{x}_2)dy_1,\ dy_2$

$$= (n_1\ n_2)^{\frac{1}{2}}\ A_{n_1-1}\ A_{n_2-1}(2\pi)^{-\frac{n_1+n_2+2}{2}} \int_0^{\infty}\!\!\int_0^{\infty} \frac{s_{x_1}^{n_1-1}}{\sigma_1^{n_1+2}}\ \frac{s_{x_2}^{n_2-1}}{\sigma_2^{n_2+2}}$$

$$\times\ \left[\int_{-\infty}^{\infty}\!\!\int_{-\infty}^{\infty} \exp\{-\frac{1}{2}\ \sum_{i=1}^{2}\ \frac{n_i+1}{\sigma_i^2}\left(\mu_i\ -\ \frac{n_i\bar{x}_i+y_i}{n_i+1}\right)^2\}d\mu_1 d\mu_2\right]$$

$$\times\ \exp\{-\frac{1}{2}\ \sum_{i=1}^{2} \frac{1}{\sigma_i^2}\left[\frac{n_i(y_i-\bar{x}_i)^2}{n_i+1}\ +\ s_{x_i}^2\right]\}d\sigma_1 d\sigma_2\cdot dy_1 dy_2$$

$$= \left[\frac{n_1\ n_2}{(n_1+1)(n_2+1)}\right]^{\frac{1}{2}} A_{n_1-1}\ A_{n_2-2}(2\pi)^{-\frac{n_1+n_2}{2}} \int_0^{\infty}\!\!\int_0^{\infty} \frac{s_{x_1}^{n_1-1}}{\sigma_1^{n_1+1}}\ \frac{s_{x_2}^{n_2-1}}{\sigma_2^{n_2+1}}$$

$$\times\ \exp\{-\frac{1}{2}\ \sum_{i=1}^{2}\ \frac{1}{\sigma_i^2}\left[\frac{n_i(y_i-\bar{x}_i)^2}{n_i+1}\ +\ s_{x_i}^2\right]\}d\sigma_1 d\sigma_2\cdot dy_1 dy_2$$

$$= \left[\frac{n_1\ n_2}{(n_1+1)(n_2+1)}\right]^{\frac{1}{2}} \frac{\Gamma(\frac{n_1}{2})\Gamma(\frac{n_2}{2})}{\Gamma(\frac{n_1-1}{2})\Gamma(\frac{n_2-1}{2})\ \pi s_{x_1}\ s_{x_2}}$$

$$\times \left[1 + \frac{n_1(y_1-\bar{x}_1)^2}{(n_1+1)s_{x_1}^2}\right]^{-\frac{n_1}{2}} \left[1 + \frac{n_2(y_2-\bar{x}_2)^2}{(n_2+1)s_{x_2}^2}\right]^{-\frac{n_1}{2}} dy_1 dy_2 .$$

Let us now introduce new variables for i = 1, 2:

$$T_i = \left[\frac{(n_i-1)n_i}{n_i+1}\right]^{\frac{1}{2}} \frac{Y_i - \bar{x}_i}{s_{x_i}} \quad ,$$

then T_1 and T_2 are variables having Student t-distribution with $(n_1 - 1)$ and $(n_2 - 2)$ degrees of freedom, respectively.

Now define

$$r^2 = s_{x_1}^2 \frac{n_1+1}{n_1(n_1-1)} + s_{x_2}^2 \frac{n_2+1}{n_2(n_2-1)}$$

and

$$\tan \delta = s_{x_2}\left[\frac{n_2+1}{n_2(n_2-1)}\right]^{\frac{1}{2}} \Bigg/ \left[s_{x_1} \frac{n_1+1}{n_1(n_1-1)}\right]^{\frac{1}{2}} .$$

Then we have

$$r \cos \delta = s_{x_1}\left[\frac{n_1+1}{n_1(n_1-1)}\right]^{\frac{1}{2}}$$

and

$$r \sin \delta = s_{x_2}\left[\frac{n_2+1}{n_2(n_2-1)}\right]^{\frac{1}{2}}$$

Hence the prediction distribution for $Z = Y_1 - Y_2$ can be represented in the form (for references see for example Fisher (1939))

$$Z = Y_1 - Y_2 = (\bar{x}_1 - \bar{x}_2) + r(T_1 \cos \delta - T_2 \sin \delta)$$

$$= (\bar{x}_1 - \bar{x}_2) + rU, \qquad (3.4.5)$$

where the distribution for the variable

$$U = T_1 \cos \delta - T_2 \sin \delta$$

is known as the Behrens-Fisher distribution with (n_1-1) and (n_2-1) degrees of freedom and the parameter δ defined by (3.4.3). Then by (1.4.6) the region Q defined by (3.4.1) is the β-expectation tolerance region, which was to be proved.

3.5 The Model 2. Let us now investigate the special case of the previous problem. We will now assume that the scale factors applied to the error variables are the same, namely $\sigma_1 = \sigma_2 = \sigma$, say. Then the two response variables are generated from two error variables by the equations

$$x_1 = \mu_1 + \sigma e_1$$
$$x_2 = \mu_2 + \sigma e_2 \quad . \qquad (3.5.1)$$

Since the structure of the system has changed we have to construct the new structural model for this system. Let us again assume that two error variables have standard normal distributions. Let μ_1 and μ_2 be the general levels of two response variables x_1 and x_2 respectively. Let σ be the common scale factor applied to the error variables e_1 and e_2. Let $\underset{\sim}{x}'_i = (x_{i1}, \ldots, x_{in_i})$ be the sequences of $n_i (i = 1, 2)$ observations of the response variables. This leads to the model 2:

$$\left\{\begin{array}{l} \begin{pmatrix} \underset{\sim}{1}' & \underset{\sim}{0}' \\ \underset{\sim}{0}' & \underset{\sim}{1}' \\ \underset{\sim}{x}'_1 & \underset{\sim}{0}' \\ \underset{\sim}{0}' & \underset{\sim}{x}'_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \mu_1 & 0 & \sigma & 0 \\ 0 & \mu_2 & 0 & \sigma \end{pmatrix} \begin{pmatrix} \underset{\sim}{1}' & \underset{\sim}{0}' \\ \underset{\sim}{0}' & \underset{\sim}{1}' \\ \underset{\sim}{e}'_1 & \underset{\sim}{0}' \\ \underset{\sim}{0}' & \underset{\sim}{e}'_2 \end{pmatrix} \\ \prod_{i=1}^{2} \prod_{j=1}^{n_i} f(e_{ij}) de_{ij} = (2\pi)^{-\frac{n_1+n_2}{2}} \exp\{-\frac{1}{2} \sum_{i=1}^{2} \sum_{j=1}^{n_i} e_{ij}^2\} \prod_{i,j} de_{ij}, \end{array}\right. \tag{3.5.2}$$

or

$$\left\{\begin{array}{l} X = \theta E \\ f(E)dE \ . \end{array}\right. \tag{3.5.3}$$

The transformation

$$\theta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \mu_1 & 0 & \sigma & 0 \\ 0 & \mu_2 & 0 & \sigma \end{pmatrix}$$

has a positive scaling factor σ and relocations μ_1 and μ_2. Such a transformation is an element of the unitary positive-affine group

$$G = \left\{ g = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ a_1 & 0 & c & 0 \\ 0 & a_2 & 0 & c \end{bmatrix} \middle/ \begin{array}{c} -\infty < a_1 < \infty \\ -\infty < a_2 < \infty \\ 0 < c < \infty \end{array} \right\} , \qquad (3.5.4)$$

where the group operation is defined as a matrix multiplication rule.

It can be easily verified that

$$[E] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \bar{e}_1 & 0 & s_e & 0 \\ 0 & \bar{e}_2 & 0 & s_e \end{bmatrix} , \qquad (3.5.5)$$

where

$$\bar{e}_i = n_i^{-1} \sum_{j=1}^{n_i} e_{ij} \qquad i = 1, 2 \qquad (3.5.6)$$

and

$$s_e = \sum_{i=1}^{2} \sum_{j=1}^{n_i} (e_{ij} - \bar{e}_i)^2 \qquad (3.5.7)$$

is a transformation variable for this model.

3.6 The Model 2: Distributions.

In order to derive the conditional distribution on the orbit and the structural distributions of parameters, the following invariante differentials based on the transformations may be helpful:

Consider first the error space E. Let us apply transformation $g \in G$ to the matrix of error variables E.

Then

$$E^* = \begin{pmatrix} \underset{\sim}{1}' & \underset{\sim}{0}' \\ \underset{\sim}{0}' & \underset{\sim}{1}' \\ \underset{\sim}{e}_1^{*\prime} & \underset{\sim}{0}' \\ \underset{\sim}{0}' & \underset{\sim}{e}_2^{*\prime} \end{pmatrix} = gE = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ a_1 & 0 & c & 0 \\ 0 & a_2 & 0 & c \end{pmatrix} \begin{pmatrix} \underset{\sim}{1}' & \underset{\sim}{0}' \\ \underset{\sim}{0}' & \underset{\sim}{1}' \\ \underset{\sim}{e}_1' & \underset{\sim}{0}' \\ \underset{\sim}{0}' & \underset{\sim}{e}_2' \end{pmatrix} = \begin{pmatrix} \underset{\sim}{1}' & \underset{\sim}{0}' \\ \underset{\sim}{0}' & \underset{\sim}{1}' \\ a_1\underset{\sim}{1}'+c\underset{\sim}{e}_2' & \underset{\sim}{0}' \\ \underset{\sim}{0}' & a_2\underset{\sim}{1}'+c\underset{\sim}{e}_2' \end{pmatrix}$$

so

$$\underset{\sim}{e}_i^* = a_i\underset{\sim}{1} + c\underset{\sim}{e}_i \qquad i = 1,\ 2\ .$$

Then

$$J_{n_1+n_2}(g:E) = \left|\frac{\partial gE}{\partial E}\right| = c^{n_1+n_2}\ , \qquad (3.6.1)$$

and

$$dm(\underset{\sim}{e}_1,\ \underset{\sim}{e}_2) = \frac{\prod_{i,j} d\underset{\sim}{e}_{ij}}{s_e^{n_1+n_2}} = \frac{d\underset{\sim}{e}_1\ d\underset{\sim}{e}_2}{s_e^{n_1+n_2}}\ .$$

Now consider the invariant differentials on the group

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \tilde{a}_1 & 0 & \tilde{c} & 0 \\ 0 & \tilde{a}_2 & 0 & \tilde{c} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ a_1 & 0 & c & 0 \\ 0 & a_2 & 0 & c \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ a_1^* & 0 & c^* & 0 \\ 0 & a_2^* & 0 & c^* \end{pmatrix}\ ,$$

which implies

$$\tilde{a}_i = a_i + ca_i^* \qquad i = 1,\ 2$$

$$\tilde{c} = cc^*.$$

Therefore

$$J(g) = \left|\frac{\partial\tilde{g}}{\partial g}\right| = c^3 \qquad (3.6.2)$$

and

$$J^*(g) = \left|\frac{\partial \tilde{g}}{\partial g^*}\right| = c^* \qquad (3.6.3)$$

This implies that

$$d\mu(\bar{e}_1, \bar{e}_2, s_e) = s_e^{-3} d\bar{e}_1 d\bar{e}_2 ds_e,$$

$$d\nu(\bar{e}_1, \bar{e}_2, s_e) = s_e^{-1} d\bar{e}_1 d\bar{e}_2 ds_e$$

and

$$\Delta(\bar{e}_1, \bar{e}_2, s_e) = s_e s_e^{-3} = s_e^{-2}$$

The reference point D on the orbit by (1.3.6) is

$$D = \begin{pmatrix} \underset{\sim}{1}' & \underset{\sim}{0}' \\ \underset{\sim}{0}' & \underset{\sim}{1}' \\ \underset{\sim}{d}_1' & \underset{\sim}{0} \\ \underset{\sim}{0}' & \underset{\sim}{d}_2' \end{pmatrix} = [E]^{-1}E = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\bar{e}_1 s_e^{-1} & 0 & s_e^{-1} & 0 \\ 0 & -\bar{e}_2 s_e^{-1} & 0 & s_e^{-1} \end{pmatrix} \begin{pmatrix} \underset{\sim}{1}' & \underset{\sim}{0}' \\ \underset{\sim}{0}' & \underset{\sim}{1}' \\ \underset{\sim}{e}_1' & \underset{\sim}{0}' \\ \underset{\sim}{0}' & \underset{\sim}{e}_2' \end{pmatrix}$$

$$\begin{pmatrix} \underset{\sim}{1}' & \underset{\sim}{0}' \\ \underset{\sim}{0}' & \underset{\sim}{1}' \\ s_e^{-1}(\underset{\sim}{e}_1' - \bar{e}_1 \underset{\sim}{1}') & \underset{\sim}{0}' \\ \underset{\sim}{0}' & s_e^{-1}(\underset{\sim}{e}_2' - \bar{e}_2 \underset{\sim}{1}') \end{pmatrix}$$

From this

$$\sum_{j=1}^{n_i} d_{ij} = \sum_{j=1}^{n_i} s_e^{-1}(e_{ij} - \bar{e}_i) = s_e^{-1} \sum_{j=1}^{n_i} (e_{ij} - \bar{e}_i) = 0, \qquad (3.6.4)$$

$$i = 1,\ 2$$

and

$$\sum_{i=1}^{2} \sum_{j=1}^{n_i} d_{ij}^2 = \sum_{i=1}^{2} \sum_{j=1}^{n_i} [s_e^{-1}(e_{ij} - \bar{e}_i)]^2$$

$$= s_e^{-2} \sum_{i=1}^{2} \sum_{j=1}^{n_i} (e_{ij} - \bar{e}_i)^2 = s_e^{-2} s_e^{2} = 1. \qquad (3.6.5)$$

Then, by using (1.3.11), (3.6.4), (3.6.5) and the fact that the normalizing constant is

$$k(D) = (n_1 \; n_2)^{\frac{1}{2}} \; A_{n_1+n_2-2}$$

the conditional distribution on the orbit is

$$f^*(\bar{e}_1, \bar{e}_2, s_e / \text{đ}_1, \text{đ}_2) d\bar{e}_1 d\bar{e}_2 ds_e$$

$$= (n_1 \; n_2)^{\frac{1}{2}} \; A_{n_1+n_2-2} (2\pi)^{-\frac{n_1+n_2}{2}} \exp\{-\tfrac{1}{2}(n_1 \bar{e}_1^{\,2} + n_2 \bar{e}_2^{2} + s_e^{2})\}$$

$$\times \; s_e^{n_1+n_2-3} \; d\bar{e}_1 d\bar{e}_2 ds_e \; . \qquad (3.6.6)$$

Now since [X] is a member of group G it follows that

$$[X] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \bar{x}_1 & 0 & s_x & 0 \\ 0 & \bar{x}_2 & 0 & s_x \end{pmatrix} ,$$

where

$$\bar{x}_i = n_i^{-1} \sum_{j=1}^{n_i} x_{ij} \qquad i = 1, 2 \qquad (3.6.7)$$

and

$$s_x^2 = \sum_{i=1}^{2} \sum_{j=1}^{n_i} (x_{ij} - \bar{x}_i)^2 \qquad (3.6.8)$$

The structural distribution for μ_1, μ_2, σ is then (using (1.3.12) and (3.6.6))

$$g(\mu_1, \mu_2, \sigma/\underset{\sim}{x}_1, \underset{\sim}{x}_2)d\mu_1 d\mu_2 d\sigma$$

$$= (n_1 n_2)^{\frac{1}{2}} A_{n_1+n_2-2}(2\pi)^{-\frac{n_1+n_2}{2}}$$

$$\times \exp\{-\frac{1}{2\sigma^2}[n_1(\bar{x}_1 - \mu_1)^2 + n_2(\bar{x}_2 - \mu_2)^2 + s_x^2]\}$$

$$\times s_x^{n_1+n_2-2} \sigma^{-(n_1+n_2+1)} d\mu_1 d\mu_2 d\sigma \ . \qquad (3.6.9)$$

3.7 β-expectation Tolerance Region For the Variable $Z = X_1 - X_2$, Assuming $\sigma_1 = \sigma_2$.

Before proceeding with the main theorem of this section we will state a Lemma, which will be helpful in proving later developments.

Lemma 3.7.1. (Cornish (1954)) If the distribution of the random variable $\underset{\sim}{Y} = (Y_1, \ldots, Y_{n-1})'$ is

$$h(\underset{\sim}{y})d\underset{\sim}{y} = \frac{\Gamma(\frac{\nu+n-1}{2})|R|^{-\frac{1}{2}}}{\pi^{\frac{n-1}{2}} \Gamma(\frac{\nu}{2})}$$

$$\times (1 + \underset{\sim}{y}'R^{-1}\underset{\sim}{y})^{-\frac{\nu+n-1}{2}} d\underset{\sim}{y}, \ -\infty<y_i<\infty \qquad (3.7.1)$$

then the distribution of the random variable $Z = (Z_1, \ldots, Z_p)'$, which is the linear combination of $\underset{\sim}{Y}$ given by the relation

$$\underset{\sim}{Z} = H\underset{\sim}{Y} \ , \qquad (3.7.2)$$

where H is such that $HH' \neq 0$, is

$$h(\underset{\sim}{z})d\underset{\sim}{z} = \frac{\Gamma(\frac{\nu+p}{2})|HRH'|^{-\frac{1}{2}}}{\pi^{\frac{p}{2}}\Gamma(\frac{\nu}{2})}$$

$$\times\ (1+\underset{\sim}{z}'(HRH')^{-1}\underset{\sim}{z})^{-\frac{\nu+p}{2}}\ d\underset{\sim}{z} \qquad (3.7.3)$$

Now we state the main theorem of this section.

<u>Theorem 3.5.1</u>. Let the independent error variables e_1 and e_2 have the normal distribution with 0 mean and variance 1, i.e.

$$f(e_i)de_i = (2\pi)^{-\frac{1}{2}}\exp\{-\frac{e_i^2}{2}\}de_i \quad i = 1,\ 2.$$

Then for central 100β per cent of the distribution of the variable $Z = X_1 - X_2$ (where X_1 is $N(\mu_1,\ \sigma^2)$ and X_2 is $N(\mu_2,\sigma^2)$) being sampled, the region

$$Q = (\bar{x}_1-\bar{x}_2-K_5s_x/(n_1+n_2-2)^{\frac{1}{2}},$$

$$\bar{x}_1-\bar{x}_2+K_5s_x/(n_1+n_2-2)^{\frac{1}{2}}] \qquad (3.7.4)$$

is β-expectation tolerance region, where $\bar{x}_1$ and $\bar{x}_2$ are defined by (3.6.7), s_x is defined by (3.6.8) and

$$K_5 = (2+n_1^{-1}+n_2^{-1})^{\frac{1}{2}}\ t_{n_1+n_2-2;(1-\beta)/2},$$

where $t_{n_1+n_2-2;(1-\beta)/2}$ is the value of the t-distribution (n_1+n_2-2 degrees of freedom) exceeded with probability (1-β)/2.

Proof:

Since the independent error variables e_1 and e_2 have the standard normal distributions, the distribution of the realized errors for the model 2 (3.5.1) is

$$\prod_{i=1}^{2} \prod_{j=1}^{n_i} f(e_{ij})de_{ij} = (2\pi)^{-\frac{n_1+n_2}{2}}$$

$$\times \exp\{-\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n_i} e_{ij}^2\} \prod_{i,j} de_{ij} .$$

The structural distribution for μ_1, μ_2 and σ is then given by (3.6.9). For the independent future responses Y_1 and Y_2 the distribution is

$$p(y_1,y_2/\mu_1,\mu_2,\sigma)dy_1dy_2 = (2\pi\sigma^2)^{-1}$$

$$\times \exp\{-\frac{1}{2\sigma^2}[(y_1 - \mu_1)^2 + (y_2 - \mu_2)^2]\}dy_1dy_2$$

Therefore the joint distribution of Y_1, Y_2, μ_1, μ_2 and σ is

$$p(y_1,y_2/\mu_1,\mu_2,\sigma)g(\mu_1,\mu_2,\sigma/\underset{\sim}{x}_1,\underset{\sim}{x}_2)dy_1dy_2d\mu_1d\mu_2d\sigma$$

$$= (n_1\ n_2)^{\frac{1}{2}} A_{n_1+n_2-2}(2\pi)^{-\frac{n_1+n_2+2}{2}}$$

$$\times \exp\{-\frac{1}{2\sigma^2}[n_1(\bar{x}_1-\mu_1)^2+n_2(\bar{x}_2-\mu_2)^2+(y_1-\mu_1)^2+(y_2-\mu_2)^2+s_x^2]\}$$

$$\times \frac{s_x^{n_1+n_2-2}}{n_1+n_2+3} dy_1dy_2d\mu_1d\mu_2d\sigma .$$

The term in the bracket in the exponent can be rearranged, using result (3.4.4) for i = 1, 2. Then

$$p(y_1,\ y_2/\mu_1,\ \mu_2,\ \sigma)g(\mu_1,\ \mu_2,\ \sigma/\underset{\sim}{x}_1\ \underset{\sim}{x}_2)dy_1dy_2d\mu_1d\mu_2d\sigma$$

$$= (n_1\ n_2)^{\frac{1}{2}}A_{n_1+n_2-2}(2\pi)^{-\frac{n_1+n_2+2}{2}}$$

$$\times \exp\{-\frac{1}{2\sigma^2}\sum_{i=1}^{2}(n_i+1)(\mu_i - \frac{n_i\ \bar{x}_i + y_i}{n_i + 1})^2\}$$

$$\times \exp\{-\frac{1}{2\sigma^2}[s_x^2 + \sum_{i=1}^{2}\frac{n_i}{n_i+1}(y_i - \bar{x}_i)^2]\}$$

$$\times \frac{s_x^{n_1+n_2-2}}{\sigma^{n_1+n_2+3}}\ dy_1dy_2d\mu_1d\mu_2d\sigma\ .$$

Then by (1.4.5) the prediction distribution for Y_1, Y_2 is

$$h(y_1,y_2/\underset{\sim}{x}_1,\underset{\sim}{x}_2)dy_1dy_2 = \int_0^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty p(y_1,y_2/\mu_1,\mu_2,\sigma)$$

$$\times\ g(\mu_1,\mu_2,\sigma/\underset{\sim}{x}_1,\underset{\sim}{x}_2)d\mu_1d\mu_2d\sigma\cdot dy_1dy_2$$

$$= (n_1\ n_2)^{\frac{1}{2}}A_{n_1+n_2-2}(2\pi)^{-\frac{n_1+n_2+2}{2}}$$

$$\times \int_0^\infty\left[\int_{-\infty}^\infty\int_{-\infty}^\infty \exp\{-\frac{1}{2\sigma^2}\sum_{i=1}^{2}(n_i+1)(\mu_i+\frac{n_i\ \bar{x}_i+y_i}{n_i+1})^2\}d\mu_1d\mu_2\right]$$

$$\times \exp\{-\frac{1}{2\sigma^2}\left[s_x^2 + \sum_{i=1}^{2}\frac{n_i}{n_i+1}(y_i - \bar{x}_i)^2\right]\}\frac{s_x^{n_1+n_2-2}}{\sigma^{n_1+n_2+3}}\ d\sigma dy_1dy_2$$

$$= \left[\frac{n_1\ n_2}{(n_1+1)(n_2+1)}\right]^{\frac{1}{2}}\ A_{n_1+n_2-2}(2\pi)^{-\frac{n_1+n_2}{2}}$$

$$\int_0^\infty \exp\{-\frac{1}{2\sigma^2}[s_x^2 + \sum_{i=1}^{2} \frac{n_i}{n_i+1}(y_i - \bar{x}_i)^2]\} \frac{s_x^{n_1+n_2-2}}{\sigma^{n_1+n_2+1}} d\sigma dy_1 dy_2$$

$$= \left[\frac{n_1\ n_2}{(n_1+1)(n_2+1)}\right]^{\frac{1}{2}} \frac{\Gamma(\frac{n_1+n_2}{2})}{\pi\ s_x^2\ \Gamma(\frac{n_1+n_2-2}{2})}$$

$$\times \left[1 + \frac{n_1(y_1 - \bar{x}_1)^2}{(n_1+1)s_x^2} + \frac{n_2(y_2 - \bar{x}_2)^2}{(n_2+1)s_x^2}\right]^{-\frac{n_1+n_2}{2}} dy_1 dy_2$$

$$= \left[\frac{n_1\ n_2}{(n_1+1)(n_2+1)}\right]^{\frac{1}{2}} \frac{\Gamma(\frac{n_1+n_2}{2})}{\pi\ s_x^2\ \ \Gamma(\frac{n_1+n_2-2}{2})}$$

$$\left[1 + (\frac{y_1-\bar{x}_1}{s_x}\ \frac{y_2-\bar{x}_2}{s_x}) \begin{pmatrix} \frac{n_1+1}{n_1} & 0 \\ 0 & \frac{n_2+1}{n_2} \end{pmatrix} \begin{pmatrix} \frac{y_1-\bar{x}_1}{s_x} \\ \frac{y_2-\bar{x}_2}{s_x} \end{pmatrix}\right]^{-\frac{n_1+n_2}{2}} dy_1 dy_2 \ .$$

Let us now make a linear transformation

$$Z = (1\ -1) \begin{pmatrix} \frac{Y_1-\bar{x}_1}{s_x} \\ \frac{Y_2-\bar{x}_2}{s_x} \end{pmatrix} = \frac{Y_1-Y_2-(\bar{x}_1-\bar{x}_2)}{s_x} \ . \qquad (3.7.5)$$

Then

$$(1\ -1) \begin{pmatrix} \frac{n_1+1}{n_1} & 0 \\ 0 & \frac{n_2+1}{n_2} \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{n_1+1}{n_1} + \frac{n_2+1}{n_2} = \frac{n_1 + n_2 + 2n_1n_2}{n_1\ n_2}$$

Then by using Lemma 3.7.1 we get

$$h(z/\underset{\sim}{x}_1,\underset{\sim}{x}_2)\,dz = \left(\frac{n_1\ n_2}{n_1+n_2+2n_1n_2}\right)^{\frac{1}{2}} \frac{\Gamma(\frac{n_1+n_2-1}{2})}{\sqrt{\pi}\ s_x\ \Gamma(\frac{n_1+n_2-2}{2})}$$

$$\times \{1 + \frac{n_1n_2[z-(\bar{x}_1-\bar{x}_2)]^2}{(n_1+n_2+2n_1n_2)s_x^2}\}^{-\frac{n_1+n_2-1}{2}} \quad dz \qquad (3.7.6)$$

That is we have, that the prediction distribution of Z is such that

$$T_{n_1+n_2-2} = \left(\frac{n_1n_2}{n_1+n_2+2n_1n_2}\right)^{\frac{1}{2}} \frac{Z - (\bar{x}_1-\bar{x}_2)}{s_x/(n_1+n_2-2)^{\frac{1}{2}}} \qquad (3.7.7)$$

has the Student's t-distribution with $n_1 + n_2 - 2$ degrees of freedom.

Then by (1.4.6) the region Q defined by (3.7.4) is the β-expectation tolerance region if we take K_5 such that

$$K_5 = (2 + n_1^{-1} + n_2^{-1})^{\frac{1}{2}}\ t_{n_1+n_2-2;(1-\beta)/2}\ .$$

This proves the theorem.

CHAPTER 4

THE REGRESSION MODEL

4.1 Introduction. In this chapter we will investigate the construction of the β-expectation tolerance regions for the regression model:

$$\underset{\sim}{x} = V'\underset{\sim}{\beta} + \sigma\underset{\sim}{e}, \tag{4.1.1}$$

where $\underset{\sim}{x}' = (x_1 \;\dots\; x_n)$ is the vector of n response variables, V is a p × n matrix of known elements usually called the design matrix, $\underset{\sim}{\beta}' = (\beta_1 \;\dots\; \beta_p)$ is the vector of regression coefficients, $\underset{\sim}{e}' = (e_1 \;\dots\; e_n)$ is the vector of error variables and σ is the scale factor applied to the error variable.

For the structural regression model the response variable $\underset{\sim}{x}$ may be considered as generated by the response generators β_j operated on the controllable variables v_{ji} and σ operated on the error variable $\underset{\sim}{e}$ as follows:

$$x_i = \sum_{j=1}^{p} \beta_j v_{ji} + \sigma e_i, \quad i = 1, \dots, n \tag{4.1.2}$$

Also for a set of responses, the error pattern in this system in some arbitrary units has the form of independent realization of the error variable e with the probability element f(e)de on the real line R^1. The regression model can then be conveniently expressed in the following form:

$$\begin{cases} \begin{pmatrix} V \\ \underset{\sim}{x}' \end{pmatrix} = \begin{pmatrix} I & 0 \\ \beta' & \sigma \end{pmatrix} \begin{pmatrix} V \\ \underset{\sim}{e}' \end{pmatrix} \\ \prod_{i=1}^{n} f(e_i)de_i \quad , \end{cases} \tag{4.1.3}$$

or

$$\begin{cases} X = \theta E \\ f(E)dE \end{cases} \tag{4.1.4}$$

The transformation θ is an element of the regression-scale group

$$G = \left\{ g = \begin{pmatrix} 1 & \dots 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & \dots 1 & 0 \\ a_1 & \dots a_p & c \end{pmatrix} = \begin{pmatrix} I & \underset{\sim}{0} \\ \underset{\sim}{a}' & c \end{pmatrix} \middle/ \begin{matrix} -\infty < a_j < \infty, \ j = 1,\dots,p \\ 0 < c < \infty \end{matrix} \right\}$$

where the group operation is defined as matrix multiplication.

Then following Fraser (1968), Chapter III, the structural distribution of $\underset{\sim}{\beta}$ and σ, given the set of responses, is

$$g(\underset{\sim}{\beta},\sigma/\underset{\sim}{x})d\underset{\sim}{\beta}d\sigma = k(D) \prod_{i=1}^{n} f\left(\frac{x_i - \sum_{j=1}^{p} \beta_j v_{ji}}{\sigma} \right) \frac{s^{n-p}(x)}{\sigma^{n+1}} \, d\underset{\sim}{\beta}d\sigma, \tag{4.1.5}$$

where

$$\begin{aligned} s^2(\underset{\sim}{x}) &= (\underset{\sim}{x} - V'\underset{\sim}{b}(\underset{\sim}{x}))'(\underset{\sim}{x} - V'\underset{\sim}{b}(\underset{\sim}{x})) \\ \underset{\sim}{b}(\underset{\sim}{x}) &= (VV')^{-1}V\underset{\sim}{x} \quad . \end{aligned} \tag{4.1.6}$$

We will construct the β-expectation tolerance region, assuming normal distribution of error variable.

4.2 Normal Distribution. Before proceeding with the main Theorem in this chapter we state a Lemma, which will be helpful in proving the Theorem.

Lemma 4.2.1 (Tiao and Guttman (1965)). If the random variable $\underset{\sim}{z}$ has a multivariate T-distribution with 1 degrees of freedom and quadratic form R, then the random variable $k^{-1}\underset{\sim}{z}'R^{-1}\underset{\sim}{z}$ has an F-distribution with k and 1 degrees of freedom (k is a positive integer).

Theorem 4.2.1 Let the error variable e have the normal distribution with 0 mean and variance 1, i.e.

$$f(e)de = (2\pi)^{-\frac{1}{2}} \exp\{-\tfrac{1}{2}e^2\}de.$$

Then for central 100β per cent of normal distribution being sampled, the ellipsoidal region

$$Q=\{\underset{\sim}{y}/(\underset{\sim}{y}-W'\underset{\sim}{b}(\underset{\sim}{x}))'\left[\frac{S}{n-p}\right]^{-1}(\underset{\sim}{y}-W'\underset{\sim}{b}(\underset{\sim}{x}))\leq pF_{p;n-p;1-\beta}\} \qquad (4.2.1)$$

is the β-expectation tolerance region, where W is the design matrix for future responses,

$$S^{-1} = s^{-2}(\underset{\sim}{x})(I - W'(VV' + WW')^{-1}W),$$

$\underset{\sim}{b}(\underset{\sim}{x})$ and $s^2(\underset{\sim}{x})$ are defined as in (4.1.6) and $F_{p;n-p;1-\beta}$ is the point exceeded with probability 1-β when using the F distribution with p and n - p degrees of freedom.

Proof:

Since the error variable e has standard normal distribution, the distribution of the realized errors for the regression model (4.1.3) is

$$\prod_{i=1}^{n} f(e_i)de_i = (2\pi)^{-\frac{n}{2}} \exp\{-\frac{1}{2} \sum_{i=1}^{n} e_i^2\} \prod_{i=1}^{n} de_i \ .$$

Then by (4.1.5) the structural distribution for $\underset{\sim}{\beta}$ and σ is

$$g(\underset{\sim}{\beta},\sigma/\underset{\sim}{x})d\underset{\sim}{\beta}d\sigma = |VV'|^{\frac{1}{2}}(2\pi)^{-\frac{n}{2}}\exp\{-\frac{1}{2}(\underset{\sim}{\beta}-\underset{\sim}{b}(\underset{\sim}{x}))' \frac{VV'}{\sigma^2}(\underset{\sim}{\beta}-\underset{\sim}{b}(\underset{\sim}{x}))\}$$

$$\times A_{n-p} \exp\{-\frac{s^2(\underset{\sim}{x})}{2\sigma^2}\} \frac{s^{n-p}(\underset{\sim}{x})}{\sigma^{n+1}} d\underset{\sim}{\beta}d\sigma \ .$$

For the n' future responses $\underset{\sim}{Y}$, with design matrix W, the distribution is

$$p(\underset{\sim}{y}/\underset{\sim}{\beta},\sigma)d\underset{\sim}{y} = (2\pi\sigma^2)^{-\frac{n'}{2}} \exp\{-\frac{1}{2\sigma^2}(\underset{\sim}{y}-W'\underset{\sim}{\beta})'(\underset{\sim}{y}-W'\underset{\sim}{\beta})\ d\underset{\sim}{y} \ .$$

Therefore the joint distribution of $\underset{\sim}{Y}$, $\underset{\sim}{\beta}$ and σ is

$$p(\underset{\sim}{y}/\underset{\sim}{\beta},\sigma)g(\underset{\sim}{\beta},\sigma/\underset{\sim}{x})d\underset{\sim}{y}d\underset{\sim}{\beta}d\sigma$$

$$= |VV'|^{\frac{1}{2}}(2\pi)^{-\frac{n+n'}{2}} A_{n-p}$$

$$\times \exp\{-\frac{1}{2\sigma^2}\left[(\underset{\sim}{b}'(\underset{\sim}{x})-\underset{\sim}{\beta}')VV'(\underset{\sim}{b}'(\underset{\sim}{x})-\underset{\sim}{\beta}')' + (\underset{\sim}{y}'-\underset{\sim}{\beta}'W)(\underset{\sim}{y}'-\underset{\sim}{\beta}'W)'\right]\}$$

$$\times \exp\{-\frac{s^2(\underset{\sim}{x})}{2\sigma^2}\} \frac{s^{n-p}(\underset{\sim}{x})}{\sigma^{n+n'+1}} d\underset{\sim}{y}d\underset{\sim}{\beta}d\sigma \ .$$

The matrix expression in the exponential can be rearranged as follows from Lemma 1.5.3:

$$(\underset{\sim}{b}'(\underset{\sim}{x})-\underset{\sim}{\beta}')VV'(\underset{\sim}{b}'(\underset{\sim}{x})-\underset{\sim}{\beta}')' + (\underset{\sim}{y}'-\underset{\sim}{\beta}'W)(\underset{\sim}{y}'-\underset{\sim}{\beta}'W)'$$

$$= (\underset{\sim}{\beta}'-D')(VV'+WW')(\underset{\sim}{\beta}'-D')' + (\underset{\sim}{y}'-\underset{\sim}{b}'(\underset{\sim}{x})W)S_1^{-1}(\underset{\sim}{y}'-\underset{\sim}{b}'(\underset{\sim}{x})W)' ,$$

where

$$D' = (\underset{\sim}{b}'(\underset{\sim}{x})VV' + \underset{\sim}{y}'W')(VV' + WW')^{-1}$$

and

$$S_1^{-1} = I - W'(VV' + WW')^{-1}W.$$

Then

$$p(\underset{\sim}{y}/\underset{\sim}{\beta},\sigma)g(\underset{\sim}{\beta},\sigma/\underset{\sim}{x})d\underset{\sim}{y}d\underset{\sim}{\beta}d\sigma$$

$$= |VV'|^{\frac{1}{2}}(2\pi)^{-\frac{n+n'}{2}} A_{n-p} \exp\{-\frac{1}{2\sigma^2}(\underset{\sim}{\beta}-D)'(VV' + WW')(\underset{\sim}{\beta}-D)\}$$

$$\times \exp\{-\frac{1}{2\sigma^2}[(\underset{\sim}{y}-W'\underset{\sim}{b}(\underset{\sim}{x}))'S_1^{-1}(\underset{\sim}{y}-W'\underset{\sim}{b}(\underset{\sim}{x})) + s^2(\underset{\sim}{x})]\}$$

$$\times \frac{s^{n-p}(\underset{\sim}{x})}{\sigma^{n+n'+1}} d\underset{\sim}{\beta}d\sigma d\underset{\sim}{y} .$$

Then by (1.4.5) the prediction distribution for $\underset{\sim}{Y}$ is

$$h(\underset{\sim}{y}/\underset{\sim}{x})d\underset{\sim}{y}$$

$$= |VV'|^{\frac{1}{2}}(2\pi)^{-\frac{n+n'}{2}} A_{n-p} \int_0^\infty \sigma^{-p}\int_\beta \exp\{-\frac{1}{2\sigma^2}(\underset{\sim}{\beta}-D)'(VV'+WW')(\underset{\sim}{\beta}-D)\}d\underset{\sim}{\beta}$$

$$\times \exp\{-\frac{1}{2\sigma^2}[(\underset{\sim}{y}-W'\underset{\sim}{b}(\underset{\sim}{x}))'S_1^{-1}(\underset{\sim}{y}-W'\underset{\sim}{b}(\underset{\sim}{x}))+s^2(\underset{\sim}{x})] \frac{s^{n-p}(\underset{\sim}{x})}{\sigma^{n+n'-p+1}} d\sigma\cdot d\underset{\sim}{y}$$

$$= \frac{|VV'|^{\frac{1}{2}} A_{n-p} s^{n-p}(\underset{\sim}{x})}{|VV'+WW'|^{\frac{1}{2}} (2\pi)^{\frac{n+n'-p}{2}}}$$

$$\times \int_0^\infty \exp\{-\frac{1}{2\sigma^2}[(\underset{\sim}{y}-W'\underset{\sim}{b}(\underset{\sim}{x}))' S_1^{-1}(\underset{\sim}{y}-W'\underset{\sim}{b}(\underset{\sim}{x}) + s^2(\underset{\sim}{x})]\}$$

$$\times \sigma^{-(n+n'-p+1)} d\sigma \cdot dy$$

$$= \frac{|VV'|^{\frac{1}{2}} 2\pi^{\frac{n-p}{2}} s^{n-p}(\underset{\sim}{x}) 2^{\frac{n+n'-p-2}{2}} \Gamma(\frac{n+n'-p}{2})}{|VV'+WW'|^{\frac{1}{2}} (2\pi)^{\frac{n+n'-p}{2}} \Gamma(\frac{n-p}{2})}$$

$$\times |s^2(\underset{\sim}{x}) + (\underset{\sim}{y}-W'\underset{\sim}{b}(\underset{\sim}{x}))' S_1^{-1}(\underset{\sim}{y}-W'\underset{\sim}{b}(\underset{\sim}{x}))|^{-\frac{n+n'-p}{2}} d\underset{\sim}{y} .$$

By the Lemma 1.5.1

$$|S_1^{-1}| = |I-W'(VV' + WW')^{-1}W| = \frac{1}{|VV'+WW'|}\begin{vmatrix} VV'+WW' & W \\ W' & I \end{vmatrix}$$

$$= \frac{|I|}{|VV'+WW'|} |VV' + WW' - WI^{-1}W'| = \frac{|VV'|}{|VV'+WW'|} ,$$

and therefore

$$\frac{|VV'|^{\frac{1}{2}}}{|VV'+WW'|^{\frac{1}{2}}} = |S_1|^{-\frac{1}{2}} .$$

Then

$$h(\underset{\sim}{y}/\underset{\sim}{x}) d\underset{\sim}{y} = \frac{|S_1|^{-\frac{1}{2}} \Gamma(\frac{n+n'-p}{2})}{\pi^{\frac{n'}{2}} s^{n'}(\underset{\sim}{x}) \Gamma(\frac{n-p}{2})}$$

$$\times \left|1 + (\underset{\sim}{y}-W'\underset{\sim}{b}(\underset{\sim}{x}))' \frac{S_1^{-1}}{s^2(\underset{\sim}{x})} (\underset{\sim}{y}-W'\underset{\sim}{b}(\underset{\sim}{x}))\right|^{-\frac{n+n'-p}{2}} d\underset{\sim}{y} .$$

Denote

$$S^{-1} = s^{-2}(\underset{\sim}{x})S_1^{-1} ,$$

then

$$s^{-n'}(\underset{\sim}{x})|S_1|^{-\frac{1}{2}} = |S|^{-\frac{1}{2}} ,$$

and therefore

$$h(\underset{\sim}{y}/\underset{\sim}{x})d\underset{\sim}{y} = \frac{|S|^{-\frac{1}{2}}\,\Gamma(\frac{n+n'-p}{2})}{\pi^{\frac{n'}{2}}\,\Gamma(\frac{n-p}{2})}$$

$$\times \left|1 + (\underset{\sim}{y}-W'\underset{\sim}{b}(\underset{\sim}{x}))'S^{-1}(\underset{\sim}{y}-W'\underset{\sim}{b}(\underset{\sim}{x}))\right|^{-\frac{n+n'-p}{2}} dy \qquad (4.2.2)$$

Now if we let

$$\underset{\sim}{z} = \sqrt{n-p}\;(\underset{\sim}{y}-W'\underset{\sim}{b}(\underset{\sim}{x}))$$

we get

$$h(\underset{\sim}{z}/\underset{\sim}{x})d\underset{\sim}{z} = \frac{|S|^{-\frac{1}{2}}\,\Gamma(\frac{n-p+n'}{2})}{[\pi(n-p)]^{-\frac{n'}{2}}\,\Gamma(\frac{n-p}{2})}\left|1 + \frac{\underset{\sim}{z}'S^{-1}\underset{\sim}{z}}{n-p}\right|^{-\frac{n-p+n'}{2}} d\underset{\sim}{z} \quad (4.2.3)$$

That is we have that

$$\underset{\sim}{Z} = \sqrt{n-p}\;(\underset{\sim}{Y} - W'\underset{\sim}{b}(\underset{\sim}{x}))$$

is a multivariate T-variable with n-p degrees of freedom and quadratic form S. By Lemma 4.2.1 it means that

$$\frac{\underset{\sim}{Z}'S^{-1}\underset{\sim}{Z}}{n-p} = (\underset{\sim}{Y}-W'\underset{\sim}{b}(\underset{\sim}{x}))'S^{-1}(\underset{\sim}{Y}-W'\underset{\sim}{b}(\underset{\sim}{x})) = \frac{p}{n-p}\,F_{p,n-p} .$$

Then by (1.4.6) the region Q defined by (4.2.1) is the β-expectation tolerance region, which was to be proved.

CHAPTER 5

THE AFFINE MULTIVARIATE MODEL

5.1 Introduction. In this chapter we will investigate the construction of β-expectation tolerance region for affine multivariate model. For this model, consider a system with p response variables $x_1, \ldots, x_p$, which are generated from p error variables $e_1, \ldots, e_p$ with a known distribution on R^p, by the relations:

$$\begin{aligned} x_1 &= \mu_1 + \gamma_{11}e_1 + \cdots + \gamma_{1p}e_p \\ &\vdots \\ x_p &= \mu_p + \gamma_{p1}e_1 + \cdots + \gamma_{pp}e_p \quad . \end{aligned}$$

The characteristics μ_i, γ_{jk} $(i, j, k = 1, \ldots, p)$ can be viewed as follows: μ_i is the general level for the corresponding response variable and γ_{jk} is the coefficient applied to the k-th error variable as its contribution towards the linear distortion of the j-th response variable.

Now consider n performances of the system and let $x_i' = (x_{i1} \cdots x_{in})$ be the observations for the i-th response variable $(i = 1, \ldots, p)$. In matrix notation let

$$\begin{pmatrix} 1 & \cdots & 1 \\ x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{p1} & \cdots & x_{pn} \end{pmatrix} = \begin{pmatrix} \underset{\sim}{1}' \\ \underset{\sim}{x}_1' \\ \vdots \\ \underset{\sim}{x}_p' \end{pmatrix} = \begin{pmatrix} \underset{\sim}{1}' \\ \underline{X} \end{pmatrix} = X \ ,$$

$$\begin{pmatrix} 1 & \dots & 1 \\ e_{11} & \dots & e_{1n} \\ \vdots & & \vdots \\ e_{p1} & \dots & e_{pn} \end{pmatrix} = \begin{pmatrix} \underset{\sim}{1}' \\ \underset{\sim}{e}_1' \\ \vdots \\ \underset{\sim}{e}_p' \end{pmatrix} = \begin{pmatrix} \underset{\sim}{1}' \\ \underline{E} \end{pmatrix} = E$$

and

$$\begin{pmatrix} 1 & 0 & \dots & 0 \\ \mu_1 & \gamma_{11} & \dots & \gamma_{1p} \\ \vdots & \vdots & & \vdots \\ \mu_p & \gamma_{p1} & \dots & \gamma_{pp} \end{pmatrix} = \begin{pmatrix} 1 & \underset{\sim}{0}' \\ \underset{\sim}{\mu} & \Gamma \end{pmatrix} = \theta$$

The system and the n performances can then be described by the *Affine Multivariate Model:*

$$\left\{ \begin{array}{l} \begin{pmatrix} \underset{\sim}{1}' \\ \underset{\sim}{x}_1' \\ \vdots \\ \underset{\sim}{x}_p' \end{pmatrix} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ \mu_1 & \gamma_{11} & \dots & \gamma_{1p} \\ \vdots & \vdots & & \vdots \\ \mu_p & \gamma_{p1} & \dots & \gamma_{pp} \end{pmatrix} \begin{pmatrix} \underset{\sim}{1}' \\ \underset{\sim}{e}_1' \\ \vdots \\ \underset{\sim}{e}_p' \end{pmatrix} \\ \\ \prod_{i=1}^{n} f(e_{1i}, \dots, e_{pi}) de_{1i} \dots de_{pi} \quad , \end{array} \right. \tag{5.1.1}$$

or

$$\left\{ \begin{array}{l} X = \theta E \\ f(E) dE \ . \end{array} \right. \tag{5.1.2}$$

The transformation θ is an element of the positive affine group on R^p:

$$G=\left\{ g=\begin{pmatrix} 1 & 0 & \cdots & 0 \\ a_1 & c_{11} & \cdots & c_{1p} \\ \vdots & \vdots & & \vdots \\ a_p & c_{p1} & \cdots & c_{pp} \end{pmatrix} = \begin{pmatrix} 1 & \underset{\sim}{0}' \\ \underset{\sim}{a} & C \end{pmatrix} \middle/ \begin{matrix} -\infty < a_j < \infty \ j = 1,\ldots,p \\ -\infty < c_{jk} < \infty \ j,k = 1,\ldots,p \\ |C| > 0 \end{matrix} \right\},$$

where the group operation is defined as a matrix multiplication rule.

To avoid the degeneracy for this model, it is assumed $n \geq p + 1$.

If the error variables are standardized such that their variance-covariance matrix is I, then,the variance-covariance matrix for the possible response variables is $\Gamma\Gamma' = \Sigma$ (say).

Consider now a transformation g applied to the error matrix E

$$\tilde{E} = gE.$$

Vectors $\underset{\sim}{e}_1, \ldots, \underset{\sim}{e}_p$ are carried into vectors $\tilde{\underset{\sim}{e}}_1, \ldots, \tilde{\underset{\sim}{e}}_p$; in fact vectors $\underset{\sim}{e}_1, \ldots, \underset{\sim}{e}_p$ in R^n are carried into vectors $\tilde{\underset{\sim}{e}}_1, \ldots, \tilde{\underset{\sim}{e}}_p$ in the linear subspace $L(\underset{\sim}{1}, \underset{\sim}{e}_1, \ldots, \underset{\sim}{e}_p)$ of R^n. The transformations g in G produce arbitrary $\tilde{\underset{\sim}{e}}_1, \ldots, \tilde{\underset{\sim}{e}}_p$ in $L(\underset{\sim}{1}, \underset{\sim}{e}_1, \ldots, \underset{\sim}{e}_p)$ except that the orientation of $\underset{\sim}{1}, \tilde{\underset{\sim}{e}}_1, \ldots, \tilde{\underset{\sim}{e}}_p$ must be the same as the orientation of $\underset{\sim}{1}, \underset{\sim}{e}_1, \ldots, \underset{\sim}{e}_p$.

Let us now take any $g \in G$. It is evident that g can be factored as follows:

$$g = {}_T g \; g_0 ,$$

where

$${}_T g = \begin{pmatrix} 1 & \underset{\sim}{0}' \\ \underset{\sim}{a} & T \end{pmatrix}$$

with T a positive lower triangular matrix and

$$g_0 = \begin{pmatrix} 1 & \underset{\sim}{\ell}' \\ \underset{\sim}{\ell} & \mathcal{O} \end{pmatrix}$$

with $\mathcal{O}$ an orthogonal matrix.

Let ${}_TG$ be a group of all elements ${}_Tg$:

$${}_TG = \left\{ {}_Tg = \begin{pmatrix} 1 & 0 & 0 & \dots 0 \\ a_1 & c_{(1)} & 0 & \dots 0 \\ a_2 & b_{21} & c_{(2)} & \dots 0 \\ \vdots & \vdots & \vdots & \vdots \\ a_p & b_{p1} & b_{p2} & \dots c_{(p)} \end{pmatrix} = \begin{pmatrix} 1 & \underset{\sim}{0}' \\ \underset{\sim}{a} & T \end{pmatrix} \middle/ \begin{matrix} -\infty < a_j < \infty \quad j=1,\dots,p \\ 0 < c_{(j)} < \infty \quad j=1,\dots,p \\ -\infty < b_{jk} < \infty \quad j\ k, k=1,\dots,p \\ |T| > 0 \end{matrix} \right\}$$

(5.1.3)

This group is known as the location-progression group (Fraser (1968) page 141) and it is a subgroup of G. In our application of the affine multivariate model we will restrict ourselves to the error variable having multivariate normal distribution, so the analysis of the model will mainly depend on ${}_TG$, since the multivariate normal distribution is rotationally symetric (Fraser (1968), Chapter 5). We will also need the transformation variable for the location-progression group ${}_TG$ to construct the transformation variable of the positive affine group G. The transformation variable for the group ${}_TG$ has been derived by Fraser (1968). We will derive this transformation variable in the different way. The difference is that our elements of the transformation variable are given by explicit formula. We will also prove that the variable, defined in such a way is the transformation variable for ${}_TG$. The advantage of introducing the transformation variable for ${}_TG$ this way is that it will help us in the construction of the transformation variable for the generalized multivariate model in the next chapter.

<u>Lemma 5.1.1</u>

$$\underset{T}{[E]}=\begin{pmatrix} 1 & \underset{\sim}{0}' \\ \underset{\sim}{m}(E) & T(E) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ m_1(E) & s_{(1)}(E) & 0 & \dots & 0 \\ m_2(E) & t_{21}(E) & s_{(2)}(E) & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ m_p(E) & t_{p1}(E) & t_{p2}(E) & \dots & s_{(p)}(E) \end{pmatrix} \tag{5.1.4}$$

is a transformation variable for the location-progression group ${}_TG$ (5.1.3), where non-zero, non-diagonal elements of the $(i + 1)$-st row of matrix (5.1.4) for $i = 1, \dots, p$ are given by (denoting $m_i(E) = t_{io}(E)$)

$$\begin{aligned} \underset{\sim}{t}_i(E) &= (t_{io}(E)t_{i1}(E)\dots t_{i\ i-1}(E))' \\ &= N^{-1}_{i-1}D^{*}_{i-1}(E)\underset{\sim}{e}_i\ , \end{aligned} \tag{5.1.5}$$

the diagonal elements are given by

$$s^2_{(i)}(E) = (\underset{\sim}{e}_i-D^{*}_{i-1}(E)\underset{\sim}{t}_i(E))'(\underset{\sim}{e}_i-D^{*}_{i-1}(E)\underset{\sim}{t}_i(E)), \tag{5.1.6}$$

$$N^{-1}_{i-1} = \begin{pmatrix} n^{-1} & \underset{\sim}{0}' \\ \underset{\sim}{0} & I_{i-1} \end{pmatrix}$$

and

$$D^*_{i-1}(E) = \begin{pmatrix} \underset{\sim}{d}_o^{*'}(E) \\ \underline{D}^*_{i-1}(E) \end{pmatrix} = \begin{pmatrix} \underset{\sim}{d}_o^{*'}(E) \\ \underset{\sim}{d}_1^{*'}(E) \\ \vdots \\ \underset{\sim}{d}_{i-1}^{*'}(E) \end{pmatrix}$$

with $\underset{\sim}{d}_j^*(E)$, for $j = 1, \ldots, i-1$ given by recurrence formula

$$\begin{aligned} \underset{\sim}{d}_j^*(E) &= s^{-1}_{(j)}(E)(\underset{\sim}{e}_j - D^{*'}_{j-1}(E)\underset{\sim}{t}_j(E)) \qquad (5.1.7) \\ &= s^{-1}_{(j)}(E)(\underset{\sim}{e}_j - \sum_{k=0}^{j-1} t_{jk}(E)\underset{\sim}{d}_k^*(E)) \ , \end{aligned}$$

where

$$\underset{\sim}{d}_o^*(E) = \underset{\sim}{1}.$$

Proof:

Let us first prove few simple facts about the inner products of the vectors $\underset{\sim}{e}_i$ and $\underset{\sim}{d}_i^*$ $(i = 1, \ldots, p)$.

i) From (5.1.5) we see that

$$\underset{\sim}{t}_i = \begin{pmatrix} t_{io} \\ t_{i1} \\ \vdots \\ t_{ii-1} \end{pmatrix} = \begin{pmatrix} n^{-1} & 0\ldots0 \\ 0 & 1\ldots0 \\ \vdots & \vdots \quad \vdots \\ 0 & 0\ldots1 \end{pmatrix} \begin{pmatrix} \underset{\sim}{1}' \\ \underset{\sim}{d}_1^{*'} \\ \vdots \\ \underset{\sim}{d}_{i-1}^{*'} \end{pmatrix} \underset{\sim}{e}_i = \begin{pmatrix} n^{-1}\underset{\sim}{1}' \\ \underset{\sim}{d}_1^{*'} \\ \vdots \\ \underset{\sim}{d}_{i-1}^{*'} \end{pmatrix} \underset{\sim}{e}_i = \begin{pmatrix} n^{-1}\underset{\sim}{1}'\underset{\sim}{e}_i \\ \underset{\sim}{d}_1^{*'}\underset{\sim}{e}_i \\ \vdots \\ \underset{\sim}{d}_{i-1}^{*'}\underset{\sim}{e}_i \end{pmatrix},$$

so by comparing

$$\underset{\sim}{1}'\underset{\sim}{e}_i = nt_{io} = n\bar{e}_i \quad \text{for } i = 1, \ldots, p \qquad (5.1.8)$$

and

$$\underset{\sim}{d}_i^{*'}\underset{\sim}{e}_i = t_{ij} \quad \text{for } j < i,\ j = 1, \ldots, p-1 \ . \qquad (5.1.9)$$

ii) Using (5.1.6) and (5.1.7), we get

$$\underset{\sim}{d}_i^{*'} d_i^{*} = s_{(i)}^{-1}(\underset{\sim}{e}_i - D_{i-1}^{*'}\underset{\sim}{t}_i)' s_{(i)}^{-1}(\underset{\sim}{e}_i - D_{i-1}^{*'}\underset{\sim}{t}_i)$$

$$= s_{(i)}^{-2}(\underset{\sim}{e}_i - D_{i-1}^{*1}\underset{\sim}{t}_i)'(\underset{\sim}{e}_i - D_{i-1}^{*1}\underset{\sim}{t}_i) = s_{(i)}^{-2} s_{(i)}^{2} = 1,$$

so

$$\underset{\sim}{d}_i^{*'}\underset{\sim}{d}_i^{*} = 1 \qquad \text{for } i = 1, \ldots, p. \qquad (5.1.10)$$

iii) For the inner product $\underset{\sim}{d}_j^{*'}\underset{\sim}{d}_i^{*}$, $j \neq i$ we will use the principle of the mathematical induction.

1°) $\underset{\sim}{d}_1^{*'}\underset{\sim}{1} = s_{(1)}^{-1}(\underset{\sim}{e}_1 - t_{10}\underset{\sim}{1})'\underset{\sim}{1} = s_{(1)}^{-1}(\underset{\sim}{e}_1'\underset{\sim}{1} - \bar{e}_1\underset{\sim}{1}'\underset{\sim}{1}) = s_{(1)}^{-1}(n\bar{e}_1 - n\bar{e}_1)$

$$= 0.$$

2°) Let us assume that up to $i = j - 1$ $\underset{\sim}{d}_i^{*'}\underset{\sim}{1} = 0$, then by using (5.1.7) we get

$$\underset{\sim}{d}_j^{*'}\underset{\sim}{1} = s_{(j)}^{-1}(\underset{\sim}{e}_j - D_{j-1}^{*'}\underset{\sim}{t}_j)'\underset{\sim}{1} = s_{(j)}^{-1}(\underset{\sim}{e}_j'\underset{\sim}{1} - \underset{\sim}{t}_j' D_{j-1}^{*}\underset{\sim}{1})$$

$$= s_{(j)}^{-1}(n\bar{e}_j - n\bar{e}_j) = 0.$$

3°) Let us now assume that up to $i = j - 1$ $\underset{\sim}{d}_i^{*}\underset{\sim}{d}_k^{*} = 0$. Without loss of generality we can assume that $0 < k < j - 1$. Then by using (5.1.7), (5.1.9) and (5.1.10) we get

$$\underset{\sim}{d}_j^{*'}\underset{\sim}{d}_k^{*} = s_{(j)}^{-1}(\underset{\sim}{e}_j - D_j^{*'}\underset{\sim}{t}_j)'\underset{\sim}{d}_k^{*} = s_{(j)}^{-1}(\underset{\sim}{e}_j'\underset{\sim}{d}_k^{*} - \sum_{l=0}^{j-1} t_{jl}\underset{\sim}{d}_l^{*'}\underset{\sim}{d}_k^{*})$$

$$= s_{(j)}^{-1}(\underset{\sim}{d}_k^{*'}\underset{\sim}{e}_j - t_{jk}\underset{\sim}{d}_k^{*'}\underset{\sim}{d}_k^{*}) = s_{(j)}^{-1}(t_{jk} - t_{jk}) = 0,$$

so

$$\underset{\sim}{d}_j^{*\prime}\,\underset{\sim}{d}_i = 0 \quad \text{for } j \neq i \quad i = 0, 1, \ldots, p. \qquad (5.1.11)$$

iv) By using (5.1.7), (5.1.10) and (5.1.11) we get

$$\underset{\sim}{d}_i^{*\prime}\,\underset{\sim}{e}_i = \underset{\sim}{d}_i^{*\prime}(\underset{\sim}{e}_i - D_{i-1}^{*\prime}\underset{\sim}{t}_i + D_{i-1}^{*\prime}\underset{\sim}{t}_i) = \underset{\sim}{d}_i^{*\prime}(\underset{\sim}{e}_i - D_{i-1}^{*\prime}\underset{\sim}{t}_i) + \underset{\sim}{d}_i^{*\prime}\,D_{i-1}^{*\prime}\underset{\sim}{t}_i$$

$$= s_{(i)}\underset{\sim}{d}_i^{*\prime}\,\underset{\sim}{d}_i^{*} + \sum_{k=0}^{i-1} t_{ik}\underset{\sim}{d}_i^{*\prime}\,\underset{\sim}{d}_k = s_{(i)} ,$$

so

$$\underset{\sim}{d}_i^{*\prime}\,\underset{\sim}{e}_i = s_{(i)} \quad \text{for } i = 1, \ldots, p. \qquad (5.1.12)$$

v) By using (5.1.7) and (5.1.11) for j > i we get

$$\underset{\sim}{d}_j^{*\prime}\,\underset{\sim}{e}_i = \underset{\sim}{d}_j^{*\prime}(\underset{\sim}{e}_i - D_{i-1}^{*\prime}\underset{\sim}{t}_i + D_{i-1}^{*\prime}\underset{\sim}{t}_i) = \underset{\sim}{d}_j^{*\prime}(\underset{\sim}{e}_i - D_{i-1}^{*\prime}\underset{\sim}{t}_i) + \underset{\sim}{d}_j^{*\prime}\,D_{i-1}^{*\prime}\underset{\sim}{t}_i$$

$$= s_{(i)}\underset{\sim}{d}_j^{*\prime}\,\underset{\sim}{d}_i^{*} + (D_{i-1}^{*}\underset{\sim}{d}_j^{*})'\underset{\sim}{t}_i = 0,$$

so

$$\underset{\sim}{d}_j^{*\prime}\,\underset{\sim}{e}_i = 0 \quad \text{for } j > i \quad i = 1, \ldots, p-1. \qquad (5.1.13)$$

Let us now denote

$$E_i' = (\underset{\sim}{1} \;\; \underset{\sim}{e}_1 \cdots \underset{\sim}{e}_i)$$

and

$$\underset{T}{[E]}_i = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ m_1(E) & s_{(1)}(E) & 0 & \cdots & 0 \\ m_2(E) & t_{21}(E) & s_{(2)}(E) & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ m_i(E) & t_{i1}(E) & t_{i2}(E) & \cdots & s_{(i)}(E) \end{pmatrix} ,$$

for i = 1, ..., p.

Then we can prove few relationships between $\underset{T}{[E]}_i$ and E_i.

vi) $$\underset{T}{[E]}_i = N_i^{-1} D_i^* E_i' \ . \qquad (5.1.14)$$

For that

$$N_i^{-1} D_i^* E_i' = \begin{pmatrix} n^{-1} & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} \begin{pmatrix} \underset{\sim}{1}' \\ \underset{\sim}{d}_1^{*\prime} \\ \underset{\sim}{d}_2^{*\prime} \\ \vdots \\ \underset{\sim}{d}_i^{*\prime} \end{pmatrix} (\underset{\sim}{1} \quad \underset{\sim}{e}_1 \quad \underset{\sim}{e}_2 \ \cdots \ \underset{\sim}{e}_i)$$

$$= \begin{pmatrix} n_1^{-1} \underset{\sim}{1}' \\ \underset{\sim}{d}_1^{*\prime} \\ \underset{\sim}{d}_2^{*\prime} \\ \vdots \\ \underset{\sim}{d}_i^{*\prime} \end{pmatrix} (\underset{\sim}{1} \ \underset{\sim}{e}_1 \ \underset{\sim}{e}_2 \cdots \underset{\sim}{e}_i) = \begin{pmatrix} n^{-1}\underset{\sim}{1}'\underset{\sim}{1} & n^{-1}\underset{\sim}{1}'\underset{\sim}{e}_1 & n^{-1}\underset{\sim}{1}'\underset{\sim}{e}_2 \cdots & n^{-1}\underset{\sim}{1}'\underset{\sim}{e}_i \\ \underset{\sim}{d}_1^{*\prime}\underset{\sim}{1} & \underset{\sim}{d}_1^{*\prime}\underset{\sim}{e}_1 & \underset{\sim}{d}_1^{*\prime}\underset{\sim}{e}_2 & \cdots \underset{\sim}{d}_1^{*\prime}\underset{\sim}{e}_i \\ \underset{\sim}{d}_2^{*\prime}\underset{\sim}{1} & \underset{\sim}{d}_i^{*}\underset{\sim}{e}_1 & \underset{\sim}{d}_2^{*\prime}\underset{\sim}{e}_2 & \cdots \underset{\sim}{d}_i^{*\prime}\underset{\sim}{e}_i \\ \vdots & \vdots & \vdots & \vdots \\ \underset{\sim}{d}_i^{*\prime}\underset{\sim}{1} & \underset{\sim}{d}_i^{*\prime}\underset{\sim}{e}_1 & \underset{\sim}{d}_i^{*\prime}\underset{\sim}{e}_2 & \cdots \underset{\sim}{d}_i^{*\prime}\underset{\sim}{e}_i \end{pmatrix}$$

$$= \begin{pmatrix} 1 & m_1 & m_2 & \dots m_i \\ 0 & s_{(1)} & t_{21} & \dots t_{i1} \\ 0 & 0 & s_{(2)} & \dots t_{i2} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots s_{(i)} \end{pmatrix} = \underset{T}{[E]}_i \ .$$

vii) $$E_i = \underset{T}{[E]}_i D_i^* \ . \qquad (5.1.15)$$

For that

$$ {}_T[E]_i D_i^* = \begin{pmatrix} 1 & 0 & \dots 0 \\ t_{10} & s_{(1)} & \dots 0 \\ \vdots & \vdots & \\ t_{i0} & t_{i1} & \dots s_{(i)} \end{pmatrix} \begin{pmatrix} \underset{\sim}{1}' \\ \underset{\sim}{d}_1^{*'} \\ \vdots \\ \underset{\sim}{d}_i^{*'} \end{pmatrix} = \begin{pmatrix} \underset{\sim}{1}' \\ t_{10}\underset{\sim}{1}' + s_{(1)}\underset{\sim}{d}_1^{*'} \\ \vdots \\ \sum_{k=0}^{i-1} t_{ik}\underset{\sim}{d}_k^{*'} + s_{(i)}\underset{\sim}{d}_i^{*'} \end{pmatrix} $$

$$ \begin{pmatrix} \underset{\sim}{1}' \\ \bar{e}_1\underset{\sim}{1}' + s_{(1)} s_{(1)}^{-1}(\underset{\sim}{e}_1' - \bar{e}_1\underset{\sim}{1}') \\ \vdots \\ \sum_{k=0}^{i-1} t_{ik}\underset{\sim}{d}_k^{*} + s_{(i)} s_{(i)}^{-1}(\underset{\sim}{e}_i' - \sum_{k=0}^{i-1} t_{ik}\underset{\sim}{d}_k^{*}) \end{pmatrix} = \begin{pmatrix} \underset{\sim}{1}' \\ \underset{\sim}{e}_1' \\ \vdots \\ \underset{\sim}{e}_i' \end{pmatrix} = E_i . $$

Now we can prove our lemma. For this we will again use the principle of the mathematical induction.

1°) Let us assume p = 1. Then

$$ {}_T[E]_1 = \begin{pmatrix} 1 & 0 \\ m_1(E) & s_{(1)}(E) \end{pmatrix} , $$

where

$$ m_1(E) = \underset{\sim}{t}_1(E) = (t_{10}(E))' = N_0^{-1} D_0^*(E)\underset{\sim}{e}_1 = n^{-1}\underset{\sim}{1}'\underset{\sim}{e}_1 = \bar{e}_1 , $$

and

$$ s_{(1)}^2(E) = (\underset{\sim}{e}_1 - \underset{\sim}{\bar{e}}_1 1)' = (\underset{\sim}{e}_1 - \underset{\sim}{\bar{e}}_1 1) = \sum_{i=1}^{n} (e_{1i} - \bar{e}_1)^2 . $$

The transformation ${}_T\theta \in {}_T G$ in this case (using the notation ${}_T\theta_i$ for i = 1, 1, ..., p) is

$$ {}_T\theta_1 = \begin{pmatrix} 1 & 0 \\ a_1 & c_{(1)} \end{pmatrix} . $$

Then

$$ {}_{T}\theta_1 {}_{T}[E]_1 = \begin{pmatrix} 1 & 0 \\ a_1 & c_{(1)} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ m_1(E) & s_{(1)}(E) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a_1 + c_{(1)} m_1(E) & c_{(1)} s_{(1)}(E) \end{pmatrix} $$

and

$$ {}_{T}\theta_1 E_1 = \begin{pmatrix} 1 & 0 \\ a_1 & c_{(1)} \end{pmatrix} \begin{pmatrix} \underset{\sim}{1}' \\ \underset{\sim}{e}_1' \end{pmatrix} = \begin{pmatrix} \underset{\sim}{1}' \\ a_1 \underset{\sim}{1}' + c_{(1)} \underset{\sim}{e}_1' \end{pmatrix} . $$

It follows from the location-scale model (Chapter 2) that

$$ m_1({}_{T}\theta_1 E_1) = t_{10}({}_{T}\theta_1 E_1) = a_1 + c_{(1)} m_1(E_1) $$

and

$$ s_{(1)}({}_{T}\theta_1 E_1) = c_{(1)} s_{(1)}(E_1) , $$

so ${}_{T}[E]_1$ is the transformation variable. Also

$$ \underset{\sim}{d}_1^{*}({}_{T}\theta_1 E_1) = s_{(1)}^{-1}({}_{T}\theta_1 E_1)(\underset{\sim}{e}_1^{*} - t_{10}({}_{T}\theta_1 E_1)\underset{\sim}{1}) $$

$$ = c_{(1)}^{-1} s_{(1)}^{-1}(E_1)(a_1\underset{\sim}{1} + c_{(1)}\underset{\sim}{e}_1 - a_1\underset{\sim}{1} - c_{(1)} t_{10}(E_1)\underset{\sim}{1}) $$

$$ = c_{(1)}^{-1} s_{(1)}^{-1}(E_1) c_{(1)}(\underset{\sim}{e}_1 - t_{10}(E_1)\underset{\sim}{1}) = \underset{\sim}{d}_1^{*}(E_1) , $$

so

$$ D_1^{*}({}_{T}\theta_1 E_1) = D_1^{*}(E_1) . $$

2°) Let us now assume that up to $p = i - 1$ ${}_{T}\theta_{i-1} {}_{T}[E]_{i-1}$ $= {}_{T}[{}_{T}\theta_{i-1} E_{i-1}]_{i-1}$ and $D_{i-1}^{*}({}_{T}\theta_{i-1} E_{i-1}) = D_{i-1}^{*}(E_{i-1})$ and let us show that this is true for $p = i$. For that

$${}_T\theta_i = \begin{pmatrix} {}_T\theta_{i-1} & \underset{\sim}{0} \\ \\ \underset{\sim}{b}_i' & c_{(i)} \end{pmatrix} , \quad {}_T[E]_i = \begin{pmatrix} {}_T[E]_{i-1} & \underset{\sim}{0} \\ \\ \underset{\sim}{t}_i'(E_i) & s_{(i)}(E_i) \end{pmatrix} ,$$

so

$${}_T\theta_i\,{}_T[E]_i = \begin{pmatrix} {}_T\theta_{i-1} & \underset{\sim}{0} \\ \\ \underset{\sim}{b}_i' & c_{(i)} \end{pmatrix} \begin{pmatrix} {}_T[E]_{i-1} & \underset{\sim}{0} \\ \\ \underset{\sim}{t}_i'(E_i) & s_{(i)}(E_i) \end{pmatrix}$$

$$= \begin{pmatrix} {}_T\theta_{i-1}\,{}_T[E]_{i-1} & \underset{\sim}{0} \\ \\ \underset{\sim}{b}_i'\,{}_T[E]_{i-1}+c_{(i)}\underset{\sim}{t}_i'(E_i) & c_{(i)}s_{(i)}(E_i) \end{pmatrix}$$

Also

$$E_i^* = {}_T\theta_i E_i = \begin{pmatrix} {}_T\theta_{i-1} & \underset{\sim}{0} \\ \\ \underset{\sim}{b}_i' & c_{(i)} \end{pmatrix} \begin{pmatrix} E_{i-1} \\ \\ \underset{\sim}{e}_i' \end{pmatrix} = \begin{pmatrix} {}_T\theta_{i-1}E_{i-1} \\ \\ \underset{\sim}{b}_i'E_{i-1}+c_{(i)}\underset{\sim}{e}_i' \end{pmatrix} ,$$

therefore

$$\underset{\sim}{e}_i^{*\prime} = \underset{\sim}{b}_i'E_{i-1} + c_{(i)}\underset{\sim}{e}_i' .$$

Then by using (5.1.14) we get

$$\underset{\sim}{t}_i({}_T\theta_iE_i)=N_{i-1}^{-1}D_{i-1}^*({}_T\theta_{i-1}E_{i-1})\underset{\sim}{e}_i=N_{i-1}^{-1}D_{i-1}^*(E_{i-1})(E_{i-1}'\underset{\sim}{b}_i+c_{(i)}\underset{\sim}{e}_i)$$

$$=N_{i-1}^{-1}D_{i-1}^*E_{i-1}'\underset{\sim}{b}_i + c_{(i)}N_{i-1}^{-1}D_{i-1}^*(E_{i-1})\underset{\sim}{e}_i$$

$$={}_T[E]_{i-1}'\underset{\sim}{b}_i + c_{(i)}\underset{\sim}{t}_i(E_i) ,$$

so

$$\underset{\sim}{t}_i'({}_T\theta_iE_i) = \underset{\sim}{b}_i'\,{}_T[E]_{i-1} + c_{(i)}\underset{\sim}{t}_i'(E_i) . \qquad (5.1.16)$$

Also by using (5.1.15) and (5.1.16) we get

$$s^2_{(i)}({}_T\theta_iE_i)$$

$$=(\underset{\sim}{e}^*_i-D^{*'}_{i-1}({}_T\theta_{i-1}E_{i-1})\underset{\sim}{t}_i({}_T\theta_iE_i))'(\underset{\sim}{e}^*_i-D^{*'}_{i-1}({}_T\theta_{i-1}E_{i-1})\underset{\sim}{t}_i({}_T\theta_iE_i))$$

$$=(E'_{i-1}\underset{\sim}{b}_i+c_{(i)}\underset{\sim}{b}_i-D^{*'}_{i-1}(E_{i-1}){}_T[E]'_{i-1}\underset{\sim}{b}_i-c_{(i)}D^{*'}_{i-1}(E_{i-1})\underset{\sim}{t}_i(E_i))'$$

$$\times(E'_{i-1}\underset{\sim}{b}_i+c_{(i)}\underset{\sim}{e}_i-D^{*'}_{i-1}(E_{i-1}){}_T[E]'_{i-1}\underset{\sim}{b}_i-c_{(i)}D^{*'}_{i-1}(E_{i-1})\underset{\sim}{t}_i(E_i))$$

$$=[E'_{i-1}\underset{\sim}{b}_i-E'_{i-1}b_i+c_{(i)}(\underset{\sim}{e}_i-D^{*'}_{i-1}(E_{i-1})\underset{\sim}{t}_i(E_i))]'$$

$$\times[E'_{i-1}\underset{\sim}{b}_i-E'_{i-1}\underset{\sim}{b}_i+c_{(i)}(\underset{\sim}{e}_i-D^{*'}_{i-1}(E_{i-1})\underset{\sim}{t}_i(E_i))]$$

$$=c^2_{(i)}(\underset{\sim}{e}_i-D^{*'}_{i-1}(E_{i-1})\underset{\sim}{t}_i(E_i))'(\underset{\sim}{e}_i-D^{*'}_{i-1}(E_{i-1})\underset{\sim}{t}_i(E_i))$$

$$=c^2_{(i)}s^2_{(i)}(E_i) \qquad (5.1.17)$$

which together with (5.1.16) proves that

$${}_T\theta_i\,{}_T[E]_i = {}_T[{}_T\theta_iE_i] \quad .$$

Also by using (5.1.15), (5.1.16) and (5.1.17) we get

$$\underset{\sim}{d}^*_i({}_T\theta_iE_i)$$

$$=s^{-1}_{(i)}({}_T\theta_iE_i)(\underset{\sim}{e}^*_i-D^{*'}_{i-1}({}_T\theta_{i-1}E_{i-1})\underset{\sim}{t}_i({}_T\theta_iE_i))$$

$$=c^{-1}_{(i)}s^{-1}_{(i)}(E_i)(E'_{i-1}\underset{\sim}{b}_i+c_{(i)}\underset{\sim}{e}_i-D^{*'}_{i-1}(E_{i-1}){}_T[E]'_{i-1}\underset{\sim}{b}_i-c_{(i)}D^{*'}_{i-1}(E_{i-1})$$

$$\times\ \underset{\sim}{t}_i(E_i))$$

$$=c^{-1}_{(i)}s^{-1}_{(i)}(E_i)[E'_{i-1}\underset{\sim}{b}_i-E'_{i-1}\underset{\sim}{b}_i + c_{(i)}(\underset{\sim}{e}_i-D^{*'}_{i-1}(E_{i-1})\underset{\sim}{t}_i(E_i))]$$

$$=s^{-1}_{(i)}(E_i)(\underset{\sim}{e}_i-D^{*'}_{i-1}(E_{i-1})\underset{\sim}{t}_i(E_i))$$

$$=\underset{\sim}{d}^{*}_i(E_i),$$

so

$$D^{*}_i({}_T\theta_i E_i) = D^{*}_i(E_i) \ . \qquad (5.1.19)$$

Then (5.1.18), by knowing that (5.1.19) holds, proves that $[E]_T$ is a transformation variable for the location-progression group G_T (5.1.3).

This transformation variable $[E]_T$ may be now thought as the first stage of the transformation variable for whole positive affine group G. For this group, the variable $[E]_T$ did not consider the orthogonal projections of coordinate vectors into the linear subspace $L(\underset{\sim}{1}, \underset{\sim}{e}_1, \ldots, \underset{\sim}{e}_p)$.

Denote

$$D^{*}_p(E) = D^{*}(E) \ .$$

Then from (1.3.4) we have

$$E = \begin{pmatrix} \underset{\sim}{1}' \\ \underline{E} \end{pmatrix} = \begin{pmatrix} 1 & \underset{\sim}{0}' \\ \underset{\sim}{m}(E) & T(E) \end{pmatrix} \begin{pmatrix} \underset{\sim}{1}' \\ \underline{D}^{*}(E) \end{pmatrix} ,$$

or

$$E = [E]_T D^{*}(E) \ . \qquad (5.1.20)$$

By (5.1.10) and (5.1.11) $D^{*}(E)$ is an orthogonal set. Consider p orthogonal projections of the coordinate vectors (1, 0, ... 0), ..., (0,..0, 1, 0...), ... into the linear sspace $L(\underset{\sim}{1}, \underset{\sim}{e}_1, \ldots, \underset{\sim}{e}_p)$, ... getting p orthogonal projections $\underset{\sim}{e}_1^o, \ldots, \underset{\sim}{e}_p^o$. The vectors $\underset{\sim}{e}_1^o, \ldots, \underset{\sim}{e}_p^o$ are chosen in such a way that $L(\underset{\sim}{1}, \underset{\sim}{e}_1^o, \ldots, \underset{\sim}{e}_p^o)$ and $L(\underset{\sim}{1}, \underset{\sim}{e}_1, \ldots, \underset{\sim}{e}_p)$ have the same orientation.

Let

$$E^o = \begin{bmatrix} \underset{\sim}{1}' \\ \underline{E}^o \end{bmatrix} \text{ and } D(E) = \begin{bmatrix} \underset{\sim}{1}' \\ \underline{D}(E) \end{bmatrix} = \begin{bmatrix} 1' \\ \underline{D}^*(E^o) \end{bmatrix} = D^*(E^o) \tag{5.1.21}$$

It is to be noted that the vectors in $\underline{D}^*(E)$ and $\underline{D}(E) = \underline{D}^*(E^o)$ are orthogonal sets, have the same orientation and are related by an orthogonal rotation. Let $0(E)$ be a p × p rotation matrix which carries $\underline{D}(E)$ into $\underline{D}^*(E)$, so that

$$\underline{D}^*(E) = 0(E)\underline{D}(E) \ .$$

Therefore

$$D^*(E) = \begin{bmatrix} 1 & \underset{\sim}{0}' \\ \underset{\sim}{0} & 0(E) \end{bmatrix} \begin{bmatrix} \underset{\sim}{1}' \\ \underline{D}(E) \end{bmatrix} = [E]_0 D(E) \tag{5.1.22}$$

Lemma 5.1.2

$$[E]=[E]_T[E]_0=\begin{bmatrix} 1 & \underset{\sim}{0}' \\ \underset{\sim}{m}(E) & T(E) \end{bmatrix} \begin{bmatrix} 1 & \underset{\sim}{0}' \\ \underset{\sim}{0} & 0(E) \end{bmatrix} = \begin{bmatrix} 1 & \underset{\sim}{0}' \\ \underset{\sim}{m}(E) & C(E) \end{bmatrix} \tag{5.1.23}$$

is a transformation variable for the structural model (5.1.1).

Proof:

From (5.1.20) and (5.1.22) we have

$$E = [E]_T[E]_0 D(E)$$

$$= \begin{pmatrix} 1 & \underset{\sim}{0}' \\ \underset{\sim}{m}(E) & T(E) \end{pmatrix} \begin{pmatrix} 1 & \underset{\sim}{0}' \\ \underset{\sim}{0} & 0(E) \end{pmatrix} D(E)$$

$$= \begin{pmatrix} 1 & \underset{\sim}{0}' \\ \underset{\sim}{m}(E) & C(E) \end{pmatrix} D(E)$$

$$= [E]D(E) \qquad (5.1.24)$$

By the construction $[E] \in G$. Since G is unitary, $[E]$ is a unique element in G. By definition $D(E)$ is a fixed reference point on the orbit GE of E and depends wholly on the orbit GE. From (5.1.24) we see that the unique $[E]$ transforms $D(E)$ into E, a unique point on GE and hence from (1.3.4) $[E]$ is a transformation variable for the structural model (5.1.1) which was to be proved.

We will investigate the affine multivariate model with the error variable having multivariate normal distribution. Then following Fraser (1968) and Fraser and Haq (1969) the structural distribution of $\underset{\sim}{\mu}$ and Σ, given the set of responses, is given by

$$g(\underset{\sim}{\mu}, \Sigma/X)d\underset{\sim}{\mu}d\Sigma$$

$$=2^{-p}n^{\frac{p}{2}}(2\pi)^{-\frac{np}{2}} \prod_{j=1}^{p} A_{n-j}\exp\{-\tfrac{1}{2}(\underset{\sim}{m}(X)-\underset{\sim}{\mu})'n\Sigma^{-1}(\underset{\sim}{m}(X)-\underset{\sim}{\mu})\}$$

$$\times\exp\{-\tfrac{1}{2}\mathrm{tr}\ \Sigma^{-1}S(X)\}|S(X)|^{\frac{n-1}{2}}\ |\Sigma|^{-\frac{n+p+1}{2}}\ d\underset{\sim}{\mu}d\Sigma, \qquad (5.1.25)$$

where

$$S(X) = T(X)T'(X). \qquad (5.1.26)$$

Remark. Since [X] is a member of the group G, $\underset{\sim}{m}(X)$ and T(X) are defined for the responses by the same formulas as $\underset{\sim}{m}(E)$ and T(E) for the error variables.

5.2 β-expectation Tolerance Region for This Model.

Theorem 5.2.1 Let the error variable $\underset{\sim}{e}$ have the normal distribution with $\underset{\sim}{0}$ mean and variance-covariance matrix I, i.e.

$$f(\underset{\sim}{e})d\underset{\sim}{e} = (2\pi)^{-\frac{p}{2}}\exp\{-\frac{1}{2}\sum_{j=1}^{p} e_j^2\} \prod_{j=1}^{p} de_j \ .$$

Then for central 100β per cent of normal distribution being sampled the ellipsoidal region

$$Q=\{\underset{\sim}{y}\Big/\frac{n}{n+1}(\underset{\sim}{y}-\underset{\sim}{m}(X))'\left[\frac{S(X)}{n-p}\right]^{-1}(\underset{\sim}{y}-\underset{\sim}{m}(X))\le pF_{p;n-p;1-\beta}\} \qquad (5.2.1)$$

is the β-expectation tolerance region, where S(X) is defined by (5.1.26), $\underset{\sim}{m}(X)$ and T(X) are defined as in (5.1.4) and $F_{p;n-p;1-\beta}$ is the point exceeded with probability 1-β when using the F-distribution with p and n-p degrees of freedom.

Proof:

Since the error variable $\underset{\sim}{e}$ have the multivariate standard normal variable, the distribution of the realized errors for the affine multivariate model (5.1.1) is

$$\prod_{i=1}^{n} f(e_{1i}, \ldots, e_{pi}) de_{1i} \ldots de_{pi} = (2\pi)^{-\frac{np}{2}}$$

$$\times \exp\{-\tfrac{1}{2} \sum_{j=1}^{n} \sum_{i=1}^{n} e_{ij}^{2}\} \prod_{j=1}^{p} \prod_{i=1}^{n} de_{ij} \ .$$

The structural distribution for $\underset{\sim}{\mu}$ and Σ is given by (5.1.25). For the future response variable $\underset{\sim}{Y}$, the distribution is

$$p(\underset{\sim}{y}/\underset{\sim}{\mu},\Sigma) d\underset{\sim}{y} = (2\pi)^{-\frac{p}{2}} |\Sigma|^{-\frac{1}{2}} \exp\{-\tfrac{1}{2}(\underset{\sim}{y}-\underset{\sim}{\mu})'\Sigma^{-1}(\underset{\sim}{y}-\underset{\sim}{\mu})\} d\underset{\sim}{y} \ .$$

Therefore the joint distribution of $\underset{\sim}{y}$, $\underset{\sim}{\mu}$ and Σ is

$$p(\underset{\sim}{y}/\underset{\sim}{\mu},\Sigma) g(\underset{\sim}{\mu},\Sigma/X) d\underset{\sim}{\mu} d\Sigma d\underset{\sim}{y}$$

$$= 2^{-p} (2\pi)^{-\frac{(n+1)p}{2}} n^{\frac{p}{2}} \prod_{j=1}^{p} A_{n-j} \exp\{-\tfrac{1}{2} \mathrm{tr} \Sigma^{-1} S(X)\} |S(X)|^{\frac{n-1}{2}} |\Sigma|^{-\frac{n+p+2}{2}}$$

$$\times \exp\{-\tfrac{1}{2}[(\underset{\sim}{m}(X)-\underset{\sim}{\mu})' n\Sigma^{-1}(\underset{\sim}{m}(X)-\underset{\sim}{\mu}) + (\underset{\sim}{y}-\underset{\sim}{\mu})'\Sigma^{-1}(\underset{\sim}{y}-\underset{\sim}{\mu})]\} d\underset{\sim}{\mu} d\Sigma d\underset{\sim}{y} \ .$$

The expression in the exponent for $\underset{\sim}{\mu}$ can be rearranged as follows:

$$(\underset{\sim}{m}(X)-\underset{\sim}{\mu})' n\Sigma^{-1}(\underset{\sim}{m}(X)-\underset{\sim}{\mu}) + (\underset{\sim}{y}-\underset{\sim}{\mu})'\Sigma^{-1}(\underset{\sim}{y}-\underset{\sim}{\mu})$$

$$= [\underset{\sim}{\mu} - (n+1)^{-1}(n\underset{\sim}{m}(X)+\underset{\sim}{y})]'(n+1)\Sigma^{-1}[\underset{\sim}{\mu} - (n+1)^{-1}(n\underset{\sim}{m}(X)+\underset{\sim}{y})]$$

$$+ (\underset{\sim}{y}-\underset{\sim}{m}(X))' n(n+1)^{-1}\Sigma^{-1}(\underset{\sim}{y}-\underset{\sim}{m}(X)) \ .$$

Then by (1.4.5) the prediction distribution for $\underset{\sim}{Y}$ is

$$h(\underset{\sim}{y}/X)d\underset{\sim}{y}=2^{-p}(2\pi)^{-\frac{(n+1)p}{2}}n^{\frac{p}{2}}\prod_{j=1}^{p}A_{n-j}\int_{\Sigma}\exp\{-\tfrac{1}{2}\mathrm{tr}\Sigma^{-1}S(X)\}$$

$$\times\int_{\underset{\sim}{\mu}}\exp\{-\tfrac{1}{2}[\underset{\sim}{\mu}-(n+1)^{-1}(n\underset{\sim}{m}(X)+\underset{\sim}{y})]'(n+1)\Sigma^{-1}[\underset{\sim}{\mu}-(n+1)^{-1}(n\underset{\sim}{m}(X)+\underset{\sim}{y})]\}d\mu$$

$$\times\ \exp\{-\tfrac{1}{2}(\underset{\sim}{y}-\underset{\sim}{m}(X))'n(n+1)^{-1}\Sigma^{-1}(\underset{\sim}{y}-\underset{\sim}{m}(X))\}$$

$$\times\ |S(X)|^{\frac{n-1}{2}}|\Sigma|^{-\frac{n+p+2}{2}}d\Sigma d\underset{\sim}{y}$$

$$=2^{-p}(2\pi)^{-\frac{np}{2}}[n(n+1)^{-1}]^{\frac{p}{2}}\prod_{j=1}^{p}A_{n-j}\int_{\Sigma}|S(X)|^{\frac{n-1}{2}}|\Sigma|^{-\frac{n+p+1}{2}}$$

$$\times\exp\{-\tfrac{1}{2}\mathrm{tr}\Sigma^{-1}[S(X)+n(n+1)^{-1}(\underset{\sim}{y}-\underset{\sim}{m}(X))(\underset{\sim}{y}-\underset{\sim}{m}(X))']\}d\Sigma d\underset{\sim}{y}\ .$$

Using the integration relationship

$$\int_{\Sigma}\exp\{-\tfrac{1}{2}\mathrm{tr}\Sigma^{-1}R(X)\}|\Sigma|^{-\frac{n+p+1}{2}}d\Sigma=\frac{2^{p}(2\pi)^{\frac{np}{2}}}{\prod_{j=1}^{n}A_{n-(j-1)}}|R(X)|^{-\frac{n}{2}}$$

(for references see Fraser (1968) page 242), we get

$h(\underset{\sim}{y}/X)d\underset{\sim}{y}$

$$=\left(\frac{n}{n+1}\right)^{\frac{p}{2}}\frac{\prod_{j=1}^{p}A_{n-j}}{2^{p}(2\pi)^{\frac{np}{2}}}\ \frac{2^{p}(2\pi)^{\frac{np}{2}}}{\prod_{j=1}^{n}A_{n-(j-1)}}$$

$$\times\ \frac{|S(X)|^{\frac{n-1}{2}}}{|S(X)+n(n+1)^{-1}(\underset{\sim}{y}-\underset{\sim}{m}(X))(\underset{\sim}{y}-\underset{\sim}{m}(X))'|^{\frac{n}{2}}}d\underset{\sim}{y}$$

$$= \left(\frac{n}{n+1}\right)^{\frac{p}{2}} \frac{A_{n-p}}{A_n}$$

$$\times \frac{|S(X)|^{\frac{n-1}{2}}}{|S(X)|^{\frac{n}{2}}\left|1+n(n+1)^{-1}(\underset{\sim}{y}-\underset{\sim}{m}(X))'S^{-1}(X)(\underset{\sim}{y}-\underset{\sim}{m}(X))\right|^{\frac{n}{2}}}\, d\underset{\sim}{y}$$

$$= \left(\frac{n}{n+1}\right)^{\frac{p}{2}} \frac{2\pi^{\frac{n-p}{2}}\Gamma\left(\frac{n}{2}\right)|S(X)|^{-\frac{1}{2}}}{\Gamma\left(\frac{n-p}{2}\right)2\pi^{\frac{n}{2}}}$$

$$\times \left|1+n(n+1)^{-1}(\underset{\sim}{y}-\underset{\sim}{m}(X))'S^{-1}(X)(\underset{\sim}{y}-\underset{\sim}{m}(X))\right|^{-\frac{n}{2}} dy \ .$$

Therefore the prediction distribution for $\underset{\sim}{Y}$ is

$h(\underset{\sim}{y}/X)d\underset{\sim}{y}$

$$= \left(\frac{n}{n+1}\right)^{\frac{p}{2}} \frac{\Gamma\left(\frac{n}{2}\right)|S(X)|^{-\frac{1}{2}}}{\pi^{\frac{p}{2}}\Gamma\left(\frac{n-p}{n}\right)}$$

$$\times \left|1+\frac{n}{n+1}(\underset{\sim}{y}-\underset{\sim}{m}(X))'S^{-1}(X)(\underset{\sim}{y}-\underset{\sim}{m}(X))\right|^{-\frac{n}{2}} d\underset{\sim}{y} \ . \qquad (5.2.2)$$

Now if we let

$$\underset{\sim}{Z} = \left[\frac{n(n-p)}{n+1}\right]^{\frac{1}{2}}(\underset{\sim}{Y}-\underset{\sim}{m}(X)) \ , \qquad (5.2.3)$$

we get

$$h(\underset{\sim}{z}/X)d\underset{\sim}{z} = \frac{\Gamma\left(\frac{n}{2}\right)|S(X)|^{-\frac{1}{2}}}{[\pi(n-p)]^{\frac{p}{2}}\Gamma\left(\frac{n-p}{2}\right)} \left|1 + \frac{z'S^{-1}(X)z}{n-p}\right|^{-\frac{n}{2}} d\underset{\sim}{z} \ . \qquad (5.2.4)$$

That is we have that $\underset{\sim}{Z}$ from (5.2.3) is a multivariate t-variable with n-p degrees of freedom and quadratic form S(X). By Lemma 4.2.1 this means that

$$\frac{\underset{\sim}{Z}'S^{-1}(X)\underset{\sim}{Z}}{n-p} = \frac{n}{n+1}(\underset{\sim}{Y}-\underset{\sim}{m}(X))'S^{-1}(X)(\underset{\sim}{Y}-\underset{\sim}{m}(X)) = \frac{p}{n-p}\,F_{p;n-p}\ .$$

Then by (1.4.6) the region Q defined at (5.2.1) is the β-expectation tolerance region, which was to be proved.

CHAPTER 6

THE GENERALIZED MULTIVARIATE MODEL

6.1 Introduction. In this chapter we will investigate the generalized multivariate model. Such a model is a generalization of the model we have investigated in the previous chapter. For the generalized multivariate model we consider a system which does not deal only with one set of p response variables, but with q such sets. The general levels of each set of response variables are considered to be different, but the linear distortion by which every set of response variables is affected by the error variables is the same for every set of response variables. The β-expectation tolerance region for this model is then constructed, assuming multivariate normal distribution of the error variables.

6.2 The Model. Consdier a system with qp response variables $x_1^{(1)}, \ldots, x_p^{(1)}, x_1^{(2)}, \ldots, x_p^{(2)}, \ldots, x_1^{(q)}, \ldots, x_p^{(q)}$. Let us suppose that the internal error of this system can be described by qp error variables $e_1^{(1)}, \ldots, e_p^{(1)}$, $e_1^{(2)}, \ldots, e_p^{(2)}, \ldots, e_1^{(q)}, \ldots, e_p^{(q)}$, with a known distribution on R^{qp}. Let $\mu_1^{(1)}, \ldots, \mu_p^{(1)}, \mu_1^{(2)}, \ldots, \mu_p^{(2)}, \ldots, \mu_1^{(q)}, \ldots, \mu_p^{(q)}$ be the general levels for the qp response variables (accordingly). And suppose that the every set of p error variables affects the corresponding set of response levels by linear distortion, which is the same for every i-th pair (i = 1, ..., q) of corresponding sets of error variables and response levels: for the j-th response ($x_j^{(i)}$) let γ_{jk} be the coefficient applied to the k-th error ($e_k^{(i)}$). Realized error variables and the corresponding response variables are then connected by the equations:

$$\begin{aligned} x_1^{(i)} &= \mu_1 + \gamma_{11}e_1^{(i)} + \ldots + \gamma_{1p}e_p^{(i)} \\ &\vdots \qquad\qquad\qquad\qquad\qquad\qquad i = 1, \ldots, q \\ x_p^{(i)} &= \mu_p + \gamma_{p1}e_1^{(i)} + \ldots + \gamma_{pp}e_p^{(i)} . \end{aligned}$$

Consider now n_i performances of the i-th component of the system (i = 1, ..., q) and let $\underset{\sim}{x}_1^{(i)} = (x_{11}^{(i)} \ldots x_{1n_i}^{(i)})'$ be the observations for the first response in the i-th set, ..., and $\underset{\sim}{x}_p^{(i)} = (x_{p1}^{(i)}, \ldots, x_{pn_i}^{(i)})'$ be the observations for the p-th response in the i-th set. Let $n = \sum_{i=1}^{q} n_i$. The system and the n performances can then be described by the <u>Generalized Multivariate Model</u>:

$$\begin{cases} X = \theta E \\ f(E)dE, \end{cases} \qquad (6.2.1)$$

where

$$X = \begin{pmatrix} V \\ \underline{X} \end{pmatrix} \; ; \quad E = \begin{pmatrix} V \\ \underline{E} \end{pmatrix} \; ; \quad \theta = \begin{pmatrix} I & 0 \\ M & \Gamma \end{pmatrix} \; ;$$

and

$$V = \begin{pmatrix} 1\ldots1 & 0\ldots0\ldots0\ldots0 \\ 0\ldots0 & 1\ldots1\ldots0\ldots0 \\ \vdots\quad\vdots & \vdots\quad\vdots\quad\vdots\quad\vdots \\ 0\ldots0 & 0\ldots0\ldots1\ldots1 \end{pmatrix} = \begin{pmatrix} \underset{\sim}{1}'_{n_1} & \underset{\sim}{0}' & \ldots\underset{\sim}{0}' \\ \underset{\sim}{0}' & \underset{\sim}{1}'_{n_2} & \ldots\underset{\sim}{0}' \\ \vdots & \vdots & \\ \underset{\sim}{0}' & \underset{\sim}{0}' & \ldots\underset{\sim}{1}'_{n_q} \end{pmatrix} = \begin{pmatrix} \underset{\sim}{v}'_1 \\ \underset{\sim}{v}'_2 \\ \vdots \\ \underset{\sim}{v}'_q \end{pmatrix}; \tag{6.2.2}$$

$$\underline{X} = \begin{pmatrix} x_{11}^{(1)}\ldots x_{1n_1}^{(1)} & x_{11}^{(2)}\ldots x_{1n_2}^{(2)}\ldots x_{11}^{(q)}\ldots x_{1n_q}^{(q)} \\ x_{21}^{(1)}\ldots x_{2n_1}^{(1)} & x_{21}^{(2)}\ldots x_{2n_2}^{(2)}\ldots x_{21}^{(q)}\ldots x_{2n_q}^{(q)} \\ \vdots\qquad\vdots & \vdots\qquad\vdots\qquad\vdots\qquad\vdots \\ x_{p1}^{(1)}\ldots x_{pn_1}^{(1)} & x_{p1}^{(2)}\ldots x_{pn_2}^{(2)}\ldots x_{p1}^{(q)}\ldots x_{pn_q}^{(q)} \end{pmatrix} = \begin{pmatrix} \underset{\sim}{x}'_1 \\ \underset{\sim}{x}'_2 \\ \vdots \\ \underset{\sim}{x}'_p \end{pmatrix}; \tag{6.2.3}$$

$$\underline{E} = \begin{pmatrix} e_{11}^{(1)}\ldots e_{1n_1}^{(1)} & e_{11}^{(2)}\ldots e_{1n_2}^{(2)}\ldots e_{11}^{(q)}\ldots e_{1n_q}^{(q)} \\ e_{21}^{(1)}\ldots e_{2n_1}^{(1)} & e_{21}^{(2)}\ldots e_{2n_2}^{(2)}\ldots e_{21}^{(q)}\ldots e_{2n_q}^{(q)} \\ \vdots\qquad\vdots & \vdots\qquad\vdots\qquad\vdots\qquad\vdots \\ e_{p1}^{(1)}\ldots e_{pn_1}^{(1)} & e_{p1}^{(2)}\ldots e_{pn_2}^{(2)}\ldots e_{p1}^{(q)}\ldots e_{pn_q}^{(q)} \end{pmatrix} = \begin{pmatrix} \underset{\sim}{e}'_1 \\ \underset{\sim}{e}'_i \\ \vdots \\ \underset{\sim}{e}'_p \end{pmatrix}; \tag{6.2.4}$$

I is a q × q identity matrix; 0 is a q × p null matrix;

$$\mathcal{M} = \begin{pmatrix} \mu_1^{(1)} & \mu_1^{(2)}\ldots\mu_1^{(q)} \\ \mu_2^{(1)} & \mu_2^{(2)}\ldots\mu_2^{(q)} \\ \vdots & \vdots\qquad\vdots \\ \mu_p^{(1)} & \mu_p^{(2)}\ldots\mu_p^{(q)} \end{pmatrix}; \tag{6.2.5}$$

$$\Gamma = \begin{pmatrix} \gamma_{11} & \cdots & \gamma_{1p} \\ \vdots & & \vdots \\ \gamma_{p1} & \cdots & \gamma_{pp} \end{pmatrix}. \tag{6.2.6}$$

and

$$f(E)dE = \prod_{i=1}^{q} \prod_{k=1}^{n_i} f(e_{1k}^{(i)} \ldots e_{pk}^{(i)}) de_{1k}^{(i)} \ldots de_{pk}^{(i)} . \quad (6.2.7)$$

If the error variables are standardized such that their variance-covariance matrix is I, then the variance-covariance matrix for possible response variables is $\Gamma\Gamma' = \Sigma$(say).

The model (6.2.1) can be utilized in analysing observations on individual units through time or space. For example, p characteristics of production process can be investigated in q different situations. Those situations could be q different plants of the same corporations producing the same products (It is known that the general levels of characteristics are slightly different even though the variations remain the same). Or those sutuations could be the q different shifts in the same plant.

The transformation θ from the model (6.2.1) is an element of the positive-affine group on R^{qp}:

$$G=\left\{ g=\begin{bmatrix} 1 & \ldots 0 & 0 & \ldots 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots 1 & 0 & \ldots 0 \\ a_1^{(1)} & \ldots a_1^{(q)} & c_{11} & \ldots c_{1p} \\ \vdots & \vdots & \vdots & \vdots \\ a_p^{(1)} & \ldots a_p^{(q)} & c_{p1} & \ldots c_{pp} \end{bmatrix} = \begin{bmatrix} I & 0 \\ A & C \end{bmatrix} \middle/ \begin{array}{l} -\infty < a_j^{(i)} < \infty \\ i = 1, \ldots, q \\ j = 1, \ldots, p; \\ -\infty < c_{jk} < \infty \\ j,k = 1, \ldots, p; \\ |C| > 0 \end{array} \right\},$$

(6.2.8)

where the group operation is defined as a matrix multiplication rule.

Consider now a transformation g applied to the error matrix E,

$$\tilde{E} = gE \ .$$

Then vectors $\underset{\sim}{e}_1, \ldots, \underset{\sim}{e}_p$ (where $\underset{\sim}{e}_1' = (\underset{\sim}{e}_1'^{(1)} \ \underset{\sim}{e}_1'^{(2)} \cdots \underset{\sim}{e}_1'^{(q)}), \ldots,$ $\underset{\sim}{e}_p' = (\underset{\sim}{e}_p'^{(1)} \ \underset{\sim}{e}_p'^{(2)} \cdots \underset{\sim}{e}_p'^{(q)})$ are carried into vectors $\tilde{\underset{\sim}{e}}_1, \ldots, \tilde{\underset{\sim}{e}}_p$. In fact the transformation g carries vectors $\underset{\sim}{e}_1, \ldots, \underset{\sim}{e}_p$ in R^n into vectors $\tilde{\underset{\sim}{e}}_1, \ldots, \tilde{\underset{\sim}{e}}_p$ in the linear subspace $L(\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_q, \underset{\sim}{e}_1, \ldots, \underset{\sim}{e}_p)$ of R^n. Of course the vectors $\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_q$ (defined in (6.2.2)) are not affected by the transformation, or better say, they are carried into themselves by the transformation, so we do not have any changes in them. The transformation g in G produces arbitrary $\tilde{\underset{\sim}{e}}_1, \ldots, \tilde{\underset{\sim}{e}}_p$ in $L(\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_q, \underset{\sim}{e}_1, \ldots, \underset{\sim}{e}_p)$ except that the orientation of $\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_q, \tilde{\underset{\sim}{e}}_1, \ldots, \tilde{\underset{\sim}{e}}_p$ must be the same as the orientation of $\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_q, \underset{\sim}{e}_1, \ldots, \underset{\sim}{e}_p$. To avoid the degeneracy for this model, let us assume that E is of rank $q + p$ and $n \geq p + q$.

6.3. <u>The Transformation Variable</u>. It is evident that any $g \in G$ can be factored as

$$g = {}_T g \ g_0$$

where

$$T^g = \begin{pmatrix} I & 0 \\ A & T \end{pmatrix} \ ,$$

with T a positive lower-triangular matrix and

$$g_0 = \begin{pmatrix} I & 0 \\ 0 & \mathcal{O} \end{pmatrix},$$

with $\mathcal{O}$ an orthogonal matrix.

Therefore for the same reasons as in the previous chapter we will first construct the transformation variable for ${}_TG$, the group of all ${}_Tg$. This group is known as a location-progression group on R^{qp} and has a form:

$${}_TG = \left\{ {}_Tg \left| \begin{pmatrix} 1 & 0 & \dots 0 & 0 & 0 & \dots 0 \\ 0 & 1 & \dots 0 & 0 & 0 & \dots 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & 0 \\ 0 & 0 & \dots 1 & 0 & 0 & \dots 0 \\ a_1^{(1)} & a_1^{(2)} & \dots a_1^{(q)} & c_{(1)} & 0 & \dots 0 \\ a_2^{(1)} & a_2^{(2)} & \dots a_2^{(q)} & b_{21} & c_{(2)} & \dots 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_p^{(1)} & a_p^{(2)} & \dots a_p^{(q)} & b_{p1} & b_{p2} & \dots c_{(p)} \end{pmatrix} = \begin{pmatrix} I & 0 \\ A & T \end{pmatrix} \right/ \begin{array}{l} -\infty < a_j^{(i)} < \infty \\ i=1,\dots,q \\ j=1,\dots,p; \\ 0 < c_{(j)} < \infty \\ j=1,\dots,p; \\ -\infty < b_{jk} < \infty \\ j,k=1,\dots,p; \\ |T| > 0 \end{array} \right\}$$

(6.3.1)

Lemma 6.3.1.

$$[E]_T = \begin{pmatrix} I & 0 \\ M(E) & T(E) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \dots & 1 & 0 & 0 & \dots & 0 \\ m_1^{(1)}(E) & \dots & m_1^{(q)}(E) & s_{(1)}(E) & 0 & \dots & 0 \\ m_2^{(1)}(E) & \dots & m_2^{(q)}(E) & t_{21}(E) & s_{(2)}(E) & \dots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \\ m_p^{(1)}(E) & \dots & m_p^{(q)}(E) & t_{p1}(E) & t_{p2}(E) & \dots & s_{(p)}(E) \end{pmatrix}$$

(6.3.2)

is a transformation variable for the location-progression group ${}_{T}G$ (6.3.1), where non-zero, non-diagonal elements of the (q + j)-th row of this matrix (j = 1, ..., p) are given by

$$\begin{aligned}\underset{\sim}{t}_j(E)&=(m_j^{(1)}(E)\ldots m_j^{(q)}(E)\ \ t_{j1}(E)\ldots t_{jj-1}(E))'\\ &=N_{j-1}^{-1}D_{j-1}^{*}(E)\underset{\sim}{e}_j \qquad (6.3.3)\end{aligned}$$

and the diagonal elements are given by

$$s_{(j)}^{2}(E)=(\underset{\sim}{e}_j-D_{j-1}^{*'}(E)\underset{\sim}{t}_j(E))'(\underset{\sim}{e}_j-D_{j-1}^{*'}(E)\underset{\sim}{t}_j(E)), \qquad (6.3.4)$$

where

$$N_{j-1}^{-1}=\begin{pmatrix}N^{-1} & 0\\ 0 & I_{(j-1)\times(j-1)}\end{pmatrix};$$

$$N^{-1}=\begin{pmatrix}n_1^{-1} & 0 & \ldots 0\\ 0 & n_2^{-1} & \ldots 0\\ \vdots & \vdots & \vdots\\ 0 & 0 & \ldots n_q^{-1}\end{pmatrix}; \qquad (6.3.5)$$

$$D_{j-1}^{*}(E)=\begin{pmatrix}V\\ \underline{D}_{j-1}^{*}(E)\end{pmatrix}=\begin{pmatrix}\underset{\sim}{v}_1'\\ \vdots\\ \underset{\sim}{v}_q'\\ \underset{\sim}{d}_1^{*'}(E)\\ \vdots\\ \underset{\sim}{d}_{j-1}^{*'}(E)\end{pmatrix} \qquad (6.3.6)$$
;

$\underset{\sim}{d}_k^{*\prime}(E)$ for k = 1, ..., j - 1 is given by the recurrence formula

$$\underset{\sim}{d}_k^{*\prime}(E) = s_{(k)}^{-1}(E)(\underset{\sim}{e}_k - D_{k-1}^{*\prime}(E)\underset{\sim}{t}_k(E))' . \qquad (6.3.7)$$

Proof:

Again we will first prove few facts about inner products involving vectors $\underset{\sim}{e}_j$, $\underset{\sim}{d}_j^{*}$ (j = 1, ..., p) and $\underset{\sim}{v}_i$ (i = 1, ..., q).

i) From the form of the matrix V (6.2.2) we see that

$$\underset{\sim}{v}_i' \underset{\sim}{v}_i = n_i \quad \text{for } i = 1, \ldots, q \qquad (6.3.8)$$

and

$$\underset{\sim}{v}_1' \underset{\sim}{v}_i = 0 \quad \text{for } 1 \neq i,\ i = 1, \ldots, q, \qquad (6.3.9)$$

so for any i (i = 1, ..., q)

$$V\underset{\sim}{v}_i = (0 \ldots 0 \; n_i \; 0 \ldots 0)' . \qquad (6.3.10)$$

ii) From (6.3.3) we see that

$$\begin{pmatrix} m_j^{(1)} \\ \vdots \\ m_j^{(q)} \\ t_{j1} \\ \vdots \\ t_{jj-1} \end{pmatrix} = \begin{pmatrix} n_1^{-1} & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \ldots & n_q^{-1} & 0 & \ldots & 0 \\ 0 & \ldots & 0 & 1 & \ldots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \ldots & 0 & 0 & \ldots & 1 \end{pmatrix} \begin{pmatrix} \underset{\sim}{v}_1' \\ \vdots \\ \underset{\sim}{v}_q' \\ \underset{\sim}{d}_1^{*\prime} \\ \vdots \\ \underset{\sim}{d}_{j-1}^{*\prime} \end{pmatrix} \underset{\sim}{e}_j$$

$$= \begin{pmatrix} n_1^{-1}\underset{\sim}{v}_1' \\ \vdots \\ n_q^{-1}\underset{\sim}{v}_q' \\ \underset{\sim}{d}_1^{*'} \\ \vdots \\ \underset{\sim}{d}_{j-1}^{*'} \end{pmatrix} \underset{\sim}{e}_j = \begin{pmatrix} n_1^{-1}\underset{\sim}{v}_1'\underset{\sim}{e}_j \\ \vdots \\ n_q^{-1}\underset{\sim}{v}_q'\underset{\sim}{e}_j \\ \underset{\sim}{d}_1^{*'}\underset{\sim}{e}_j \\ \vdots \\ \underset{\sim}{d}_{j-1}^{*'}\underset{\sim}{e}_j \end{pmatrix},$$

so by comparing

$$\underset{\sim}{v}_i'\underset{\sim}{e}_j = n_i m_j^{(i)} \quad \text{for } j = 1,\ldots,p \text{ and } i = 1,\ldots,q \qquad (6.3.11)$$

and

$$\underset{\sim}{d}_k^{*'}\underset{\sim}{e}_j = t_{jk} \quad \text{for } k < j,\ j = 1,\ \ldots,\ p. \qquad (6.3.12)$$

(iii) Using (6.3.4) and (6.3.7) we get

$$\underset{\sim}{d}_j^{*'}\underset{\sim}{d}_j^{*} = s_{(j)}^{-1}(\underset{\sim}{e}_j - D_{j-1}^{*'}\underset{\sim}{t}_j)' s_{(j)}^{-1}(\underset{\sim}{e}_j - D_{j-1}^{*'}\underset{\sim}{t}_j) = s_{(j)}^{-2}s_{(j)}^{2} = 1,$$

so

$$\underset{\sim}{d}_j^{*'}\underset{\sim}{d}_j^{*} = 1 \quad \text{for } j = 1,\ \ldots,\ p. \qquad (6.3.13)$$

iv) For the inner product $\underset{\sim}{d}_k^{*'}\underset{\sim}{d}_j^{*}$ $j \neq k$ we will use the principle of the mathematical induction. We should point out that whenever vectors $\underset{\sim}{v}_i$ are involved, the results hold for any i (i = 1, ..., q).

1°) Using (6.3.7), (6.3.10) and (6.3.11) we get

$$\underset{\sim}{d}_1^{*'}\underset{\sim}{v}_i = s_{(1)}^{-1}(\underset{\sim}{e}_1 - D_0^{*'}\underset{\sim}{t}_1)'\underset{\sim}{v}_i = s_{(1)}^{-1}(\underset{\sim}{e}_1'\underset{\sim}{v}_i - \underset{\sim}{t}_1'V\underset{\sim}{v}_i)$$

$$= s_{(1)}^{-1}(n_i m_1^{(i)} - n_i m_1^{(i)}) = 0.$$

2°) Let us assume that up to j = k - 1 $\underset{\sim}{d}_j^{*'}\underset{\sim}{v}_i = 0$, so

$$\underline{D}^*_{k-1}\underset{\sim}{v}_i = \begin{pmatrix} d^{*\prime}_1 \\ \vdots \\ d^{*\prime}_{k-1} \end{pmatrix} v_i = \begin{pmatrix} \underset{\sim}{d}^{*\prime}_1 \underset{\sim}{v}_i \\ \vdots \\ \underset{\sim}{d}^{*\prime}_{k-1} \underset{\sim}{v}_i \end{pmatrix} = \underset{\sim}{0}$$

and

$$D^*_{k-1}\underset{\sim}{v}_i = (0 \ldots 0 \; n_i \; 0 \ldots 0 \; 0 \ldots 0)' \; .$$

Then by using (6.3.7) and (6.3.10)

$$\underset{\sim}{d}^{*\prime}_k \underset{\sim}{v}_i = s^{-1}_{(k)}(\underset{\sim}{e}_k - D^{*\prime}_{k-1}\underset{\sim}{t}_k)'\underset{\sim}{v}_i = s^{-1}_{(k)}(\underset{\sim}{e}'_k\underset{\sim}{v}_i - \underset{\sim}{t}'_k D^*_{k-1}\underset{\sim}{v}_i)$$

$$= s^{-1}_{(k)}(n_i m_k^{(i)} - n_i m_k^{(i)}) = 0 \; ,$$

so

$$\underset{\sim}{d}^{*\prime}_j \underset{\sim}{v}_i = 0 \quad \text{for } j = 1,\ldots,p \text{ and } i = 1,\ldots,q \; . \qquad (6.3.14)$$

3°) Now by using (6.3.7), (6.3.12), (6.3.13) and (6.3.14) we get

$$\underset{\sim}{d}^{*\prime}_2 \underset{\sim}{d}^*_1 = s^{-1}_{(2)}(\underset{\sim}{e}_2 - D^{*\prime}_1 \underset{\sim}{t}_2)'\underset{\sim}{d}^*_1 = s^{-1}_{(2)}(\underset{\sim}{e}'_2\underset{\sim}{d}^*_1 - \underset{\sim}{t}'_2 D^*_1 \underset{\sim}{d}_1)$$

$$= s^{-1}_{(2)}\left[\underset{\sim}{d}^{*\prime}_1 \underset{\sim}{e}_2 - \underset{\sim}{t}'_2 \begin{pmatrix} V \\ \underset{\sim}{d}^{*\prime}_1 \end{pmatrix} \underset{\sim}{d}^*_1\right] = s^{-1}_{(2)}\left[t_{21} - \underset{\sim}{t}'_2 \begin{pmatrix} V\underset{\sim}{d}^*_1 \\ \underset{\sim}{d}^{*\prime}_1 \underset{\sim}{d}^*_1 \end{pmatrix}\right]$$

$$= s^{-1}_{(2)}(t_{21} - t_{21}) = 0.$$

4°) Let us now assume that up to $j = k-1$ $\underset{\sim}{d}^{*\prime}_j \underset{\sim}{d}^*_l = 0$, $j \neq l$. Without loss of generality we can assume that $0 < l \leq k - 1$. Then by using (6.3.7), (6.3.12), (6.3.13) and (6.3.14)

$$\underset{\sim}{d}^{*\prime}_k \underset{\sim}{d}^*_l = s^{-1}_{(k)}(\underset{\sim}{e}_k - D^{*\prime}_{k-1}\underset{\sim}{t}_k)'\underset{\sim}{d}^*_l = s^{-1}_{(k)}(\underset{\sim}{e}'_k d^*_l - \underset{\sim}{t}'_k D^*_{k-1}\underset{\sim}{d}^*_l)$$

$$= s^{-1}_{(k)}(t_{kl} - t_{kl}) = 0 \; ,$$

so

$$\underset{\sim}{d}_k^{*}\underset{\sim}{d}_j^{*} = 0 \quad \text{for } j \neq k,\ j = 1, \ldots, p \qquad (6.3.15)$$

v) By using (6.3.7), (6.3.13), (6.3.14) and (6.3.15) we get

$$\underset{\sim}{d}_j^{*'}\underset{\sim}{e}_j = \underset{\sim}{d}_j^{*'}(\underset{\sim}{e}_j - D_{j-1}^{*'}\underset{\sim}{t}_j + D_{j-1}^{*'}\underset{\sim}{t}_j) = \underset{\sim}{d}_j^{*'}(\underset{\sim}{e}_j - D_{j-1}^{*'}\underset{\sim}{t}_j) + \underset{\sim}{d}_j^{*'}D_{j-1}^{*'}\underset{\sim}{t}_j$$

$$= s_{(j)}\underset{\sim}{d}_j^{*'}\underset{\sim}{d}_j^{*} + 0 = s_{(j)} \,,$$

so

$$\underset{\sim}{d}_j^{*'}\underset{\sim}{e}_j = s_{(j)} \quad \text{for } j = 1, \ldots, p \qquad (6.3.16)$$

vi) By using (6.3.7), (6.3.14) and (6.3.15) we get for

$k > j$

$$\underset{\sim}{d}_k^{*'}\underset{\sim}{e}_j = \underset{\sim}{d}_k^{*'}(\underset{\sim}{e}_j - D_{j-1}^{*'}\underset{\sim}{t}_j + D_{j-1}^{*'}\underset{\sim}{t}_j) = \underset{\sim}{d}_k^{*'}(\underset{\sim}{e}_j - D_{j-1}^{*'}\underset{\sim}{t}_j) + \underset{\sim}{d}_k^{*'}D_{j-1}^{*'}\underset{\sim}{t}_j$$

$$= s_{(j)}\underset{\sim}{d}_k^{*'}\underset{\sim}{d}_j^{*} + 0 = 0 \,,$$

so

$$\underset{\sim}{d}_k^{*'}\underset{\sim}{e}_j = 0 \quad \text{for } k > j,\ j = 1, \ldots, p-1 \,. \qquad (6.3.17)$$

Let us now denote

$$E_j = (V\ \underset{\sim}{e}_1 \cdots \underset{\sim}{e}_j)' \qquad (6.3.18)$$

and

$$\underset{T}{[E]}_j = \begin{pmatrix} I & 0 \\ M_j(E) & T_j(E) \end{pmatrix} = \begin{pmatrix} 1 & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \ldots & 1 & 0 & \ldots & 0 \\ m_1^{(1)}(E) & \ldots & m_1^{(q)}(E) & s_{(1)}(E) & \ldots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ m_j^{(1)}(E) & \ldots & m_j^{(q)}(E) & t_{j1}(E) & \ldots & s_{(j)}(E) \end{pmatrix},$$

(6.3.19)

for $j = 1, \ldots, p$. Then we can prove the following relationships between $\underset{T}{[E]}_j$ and E_j:

vii) $$\underset{T}{[E]}'_j = N_j^{-1} D_j^* E'_j \ . \qquad (6.3.20)$$

vii) $$E_j = \underset{T}{[E]}_j D_j^* . \qquad (6.3.21)$$

To prove vii):

$$N_j^{-1} D_j^* E'_j = \begin{pmatrix} N^{-1} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} V \\ \underline{D}_j^* \end{pmatrix} (V'\underline{E}'_j) = \begin{pmatrix} N^{-1}V \\ \underline{D}_j^* \end{pmatrix} (V'\underline{E}'_j)$$

$$= \begin{pmatrix} N^{-1}VV' & N^{-1}V\underline{E}'_j \\ \underline{D}_j^* V' & \underline{D}_j^{**}\underline{E}'_j \end{pmatrix} .$$

Now, by (6.3.10)

$$N^{-1}VV' = N^{-1}N = I \ .$$

By (6.3.14)

$$\underline{D}_j^* V' = \begin{pmatrix} \underset{\sim}{d}_1^{*\prime} \\ \vdots \\ \underset{\sim}{d}_j^{*\prime} \end{pmatrix} (\underset{\sim}{v}_1 \cdots \underset{\sim}{v}_q) = \begin{pmatrix} \underset{\sim}{d}_1^{*\prime} \underset{\sim}{v}_1 \cdots \underset{\sim}{d}_1^{*\prime} \underset{\sim}{v}_q \\ \vdots \qquad\qquad \vdots \\ \underset{\sim}{d}_j^{*\prime} \underset{\sim}{v}_1 \cdots \underset{\sim}{d}_j^{*\prime} \underset{\sim}{v}_q \end{pmatrix} = \begin{pmatrix} 0 \cdots 0 \\ \vdots \quad \vdots \\ 0 \cdots 0 \end{pmatrix} = 0 \quad .$$

By (6.3.11)

$$N^{-1}V\underline{E}'_j = N^{-1} \begin{pmatrix} \underset{\sim}{v}'_1 \\ \vdots \\ \underset{\sim}{v}'_q \end{pmatrix} (e_1 \cdots e_j) = N^{-1} \begin{pmatrix} \underset{\sim}{v}'_1 \underset{\sim}{e}_1 \cdots \underset{\sim}{v}'_1 \underset{\sim}{e}_j \\ \vdots \qquad\qquad \vdots \\ \underset{\sim}{v}'_q \underset{\sim}{e}_1 \cdots \underset{\sim}{v}'_q \underset{\sim}{e}_j \end{pmatrix}$$

$$= \begin{pmatrix} n_1^{-1} & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & n_q^{-1} \end{pmatrix} \begin{pmatrix} n_1 m_1^{(1)} & \cdots & n_1 m_j^{(1)} \\ \vdots & & \vdots \\ n_q m_1^{(q)} & \cdots & n_q m_j^{(q)} \end{pmatrix}$$

$$= \begin{pmatrix} m_1^{(1)} & \cdots & m_j^{(1)} \\ \vdots & & \vdots \\ m_1^{(q)} & \cdots & m_j^{(q)} \end{pmatrix} = M_j' .$$

By (6.3.12), (6.3.16) and (6.3.17)

$$\underline{D}_j^* \underline{E}_j = \begin{pmatrix} \underset{\sim}{d}_1^{*'} \\ \underset{\sim}{d}_2^{*'} \\ \vdots \\ \underset{\sim}{d}_j^{*'} \end{pmatrix} (\underset{\sim}{e}_1 \; \underset{\sim}{e}_2 \cdots \underset{\sim}{e}_j) = \begin{pmatrix} \underset{\sim}{d}_1^{*'} \underset{\sim}{e}_1 & \underset{\sim}{d}_1^{*'} \underset{\sim}{e}_2 \cdots & \underset{\sim}{d}_1^{*'} \underset{\sim}{e}_j \\ \underset{\sim}{d}_2^{*'} \underset{\sim}{e}_1 & \underset{\sim}{d}_2^{*'} \underset{\sim}{e}_2 \cdots & \underset{\sim}{d}_2^{*'} \underset{\sim}{e}_j \\ \vdots & \vdots & \vdots \\ \underset{\sim}{d}_j^{*'} \underset{\sim}{e}_1 & \underset{\sim}{d}_j^{*'} \underset{\sim}{e}_2 \cdots & \underset{\sim}{d}_j^{*'} \underset{\sim}{e}_j \end{pmatrix}$$

$$= \begin{pmatrix} s_{(1)} & t_{21} & \cdots t_{p1} \\ 0 & s_{(2)} & \cdots t_{p2} \\ \vdots & \vdots & \vdots \\ 0 & 0 & \cdots s_{(p)} \end{pmatrix} = T_j' .$$

Therefore

$$N_j^{-1} D_j^* E_j' = \begin{pmatrix} N^{-1}VV' & N^{-1}V\underline{E}_j' \\ \underline{D}_j^* V' & \underline{D}_j^* \underline{E}_j' \end{pmatrix} = \begin{pmatrix} I & M_j' \\ 0 & T_j' \end{pmatrix} = \begin{pmatrix} I & 0 \\ M_j & T_j \end{pmatrix}' = \underset{T}{[E]}_j' ,$$

which was to be proved.

To prove viii):

$$\underset{T}{[E]}_j D_j^* = \begin{pmatrix} I & 0 \\ M_j & T_j \end{pmatrix} \begin{pmatrix} V \\ \underline{D}_j^* \end{pmatrix} = \begin{pmatrix} V \\ M_j V + T_j \underline{D}_j^* \end{pmatrix} .$$

Now, by (6.3.7)

$$M_j V + T_j \underline{D}_j^* = \begin{pmatrix} \mathbf{m}_1' \\ \vdots \\ \mathbf{m}_j' \end{pmatrix} V + \begin{pmatrix} s_{(1)} & \cdots & 0 \\ \vdots & & \vdots \\ t_{j1} & \cdots & s_{(j)} \end{pmatrix} \begin{pmatrix} \mathbf{d}_1^{*'} \\ \vdots \\ \mathbf{d}_j^{*'} \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{m}_1' V \\ \vdots \\ \mathbf{m}_j' V \end{pmatrix} + \begin{pmatrix} s_{(1)} \mathbf{d}_1^{*'} \\ \vdots \\ \sum_{k=1}^{j-1} t_{jk} \mathbf{d}_k^{*'} + s_{(j)} \mathbf{d}_j^{*'} \end{pmatrix} = \begin{pmatrix} \mathbf{e}_1' \\ \vdots \\ \mathbf{e}_j' \end{pmatrix} = \underline{E}_j ,$$

so

$${}_T[E]_j D_j^* = \begin{pmatrix} V \\ M_j V + T_j \underline{D}_j^* \end{pmatrix} = \begin{pmatrix} V \\ \underline{E}_j \end{pmatrix} = E_j .$$

Now we can prove our lemma. For this we will again use the principle of the mathematical induction.

1°) Let us assume p = 1. Then

$${}_T[E]_1 = \begin{pmatrix} 1 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \\ m_1^{(1)}(E) & \cdots & m_1^{(q)}(E) & s_{(1)}(E) \end{pmatrix} = \begin{pmatrix} I & \mathbf{0} \\ \mathbf{m}_1'(E) & s_{(1)}(E) \end{pmatrix},$$

where

$$\mathbf{m}_1(E) = \mathbf{t}_1(E) = N_0^{-1} D_0^*(E) \mathbf{e}_1 = N^{-1} V \mathbf{e}_1$$

and

$$s_{(1)}^2(E) = (\mathbf{e}_1 - D_0^{*'}(E) \mathbf{t}_1(E))'(\mathbf{e}_1 - D_0^{*'}(E) \mathbf{t}_1(E))$$

$$= (\mathbf{e}_1 - V'NV\mathbf{e}_1)'(\mathbf{e}_1 - V'NV\mathbf{e}_1) .$$

The transformation ${}_T\theta \in {}_TG$ in this case (using the notation ${}_T\theta_i$ for i = 1, ..., p) is

$${}_T\theta_1 = \begin{pmatrix} 1 & \cdots 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & \cdots 1 & 0 \\ a_1^{(1)} & \cdots a_1^{(q)} & c_{(1)} \end{pmatrix} = \begin{pmatrix} I & \underset{\sim}{0} \\ \underset{\sim}{a}_1' & c_{(1)} \end{pmatrix} .$$

Then

$${}_T\theta_1\,{}_T[E]_1 = \begin{pmatrix} I & \underset{\sim}{0} \\ \underset{\sim}{a}_1' & c_{(1)} \end{pmatrix} \begin{pmatrix} I & \underset{\sim}{0} \\ \underset{\sim}{m}_1'(E) & s_{(1)}(E) \end{pmatrix} = \begin{pmatrix} I & \underset{\sim}{0} \\ \underset{\sim}{a}_1'I+c_{(1)}\underset{\sim}{m}_1'(E) & c_{(1)}s_{(1)}(E) \end{pmatrix}$$

and

$$E_1^* = \begin{pmatrix} V \\ \underset{\sim}{e}_1^{*\prime} \end{pmatrix} = {}_T\theta_1 E_1 = \begin{pmatrix} I & 0 \\ \underset{\sim}{a}_1' & c_{(1)} \end{pmatrix} \begin{pmatrix} V \\ \underset{\sim}{e}_1' \end{pmatrix} = \begin{pmatrix} V \\ \underset{\sim}{a}_1'V + c_{(1)}\underset{\sim}{e}_1' \end{pmatrix} .$$

Therefore

$$\underset{\sim}{e}_1^{*\prime} = \underset{\sim}{a}_1'V + c_{(1)}\underset{\sim}{e}_1' .$$

Then

$$\underset{\sim}{m}_1({}_T\theta_1E_1) = N^{-1}V\underset{\sim}{e}_1^* = N^{-1}V(V'\underset{\sim}{a}_1+c_{(1)}\underset{\sim}{e}_1) = N^{-1}VV'\underset{\sim}{a}_1 + c_{(1)}N^{-1}V\underset{\sim}{e}_1$$

$$= I\underset{\sim}{a}_1 + c_{(1)}m_{(1)}(E_1) ,$$

from which

$$\underset{\sim}{m}_1'({}_T\theta_1E_1) = \underset{\sim}{a}_1'I + c_{(1)}\underset{\sim}{m}_1'(E).$$

Also

$$s_{(1)}^2({}_T\theta_1E_1) = (\underset{\sim}{e}_1^*-V'N^{-1}V\underset{\sim}{e}_1^*)'(\underset{\sim}{e}_1^*-V'N^{-1}V\underset{\sim}{e}_1^*)$$

$$= [V'\underset{\sim}{a}_1+c_{(1)}\underset{\sim}{e}_1-V'N^{-1}V(V'\underset{\sim}{a}_1+c_{(1)}\underset{\sim}{e}_1)]'$$

$$\times[V'\underset{\sim}{a}_1+c_{(1)}\underset{\sim}{e}_1-V'N^{-1}V(V'\underset{\sim}{a}_1+c_{(1)}\underset{\sim}{e}_1)]$$

$$= [V'\underset{\sim}{a}_1 - V'\underset{\sim}{a}_1 + c_{(1)}(\underset{\sim}{e}_1 - V'N^{-1}V\underset{\sim}{e}_1)]'$$

$$\times[V'\underset{\sim}{a}_1 - V'\underset{\sim}{a}_1 + c_{(1)}(\underset{\sim}{e}_1 - V'N^{-1}V\underset{\sim}{e}_1)]$$

$$= c^2_{(1)}(\underset{\sim}{e}_1 - V'N^{-1}V\underset{\sim}{e}_1)'(\underset{\sim}{e}_1 - V'N^{-1}V\underset{\sim}{e}_1)$$

$$= c^2_{(1)}s^2_{(1)}(E_1) ,$$

so

$$s_{(1)}({}_T\theta_1 E_1) = c_{(1)}s_{(1)}(E_1) .$$

From this we get

$${}_T[{}_T\theta_1 E_1]_1 = \begin{pmatrix} I & 0 \\ \underset{\sim}{m}_1'({}_T\theta_1 E_1) & s_{(1)}({}_T\theta_1 E_1) \end{pmatrix}$$

$$= \begin{pmatrix} I & \underset{\sim}{0} \\ \underset{\sim}{a}_1 I + c_{(1)}\underset{\sim}{m}_1'(E) & c_{(1)}s_{(1)}(E) \end{pmatrix} = {}_T\theta_1 {}_T[E]_1 ,$$

which by (1.3.5) proves that ${}_T[E]_1$ is the transformation variable for the location-progression group ${}_TG$ (6.3.1) for $p = 1$. Also

$$\underset{\sim}{d}_1^*({}_T\theta_1 E_1) = s^{-1}_{(1)}({}_T\theta_1 E_1)[\underset{\sim}{e}_1^* - D_0^{*'}({}_T\theta_1 E_1)\underset{\sim}{t}_1({}_T\theta_1 E_1)]$$

$$= s^{-1}_{(1)}({}_T\theta_1 E_1)[\underset{\sim}{e}_1^* - V'\underset{\sim}{m}_1({}_T\theta_1 E_1)]$$

$$= c^{-1}_{(1)}s^{-1}_{(1)}(E_1)[V'\underset{\sim}{a}_1 + c_{(1)}\underset{\sim}{e}_1 - V'(I\underset{\sim}{a}_1 + c_{(1)}\underset{\sim}{m}_1(E_1))]$$

$$= c^{-1}_{(1)}s^{-1}_{(1)}(E_1)[V'\underset{\sim}{a}_1 - V'\underset{\sim}{a}_1 + c_{(1)}(\underset{\sim}{e}_1 - V'\underset{\sim}{m}_1(E_1))]$$

$$= c^{-1}_{(1)}c_{(1)}s^{-1}_{(1)}(E_1)[\underset{\sim}{e}_1 - V'\underset{\sim}{m}_1(E_1)] = \underset{\sim}{d}_1^*(E_1) ,$$

so

$$D_1^*({}_T\theta_1 E_1) = D_1^*(E_1) .$$

2°) Let us now assume that up to $p = k - 1$

$${}_T\theta_{k-1}\,{}_T[E]_{k-1} = {}_T[{}_T\theta_{k-1}E_{k-1}]_{k-1} \quad \text{and} \quad D_{k-1}^*({}_T\theta_{k-1}E_{k-1}) = D_{k-1}^*(E_{k-1})$$

and let us show that this is true for $p = k$.

For that

$${}_T\theta_k = \begin{pmatrix} {}_T\theta_{k-1} & \underset{\sim}{0} \\ \underset{\sim}{b}'_k & c_{(k)} \end{pmatrix} ,$$

where

$$\underset{\sim}{b}'_k = (a_k^{(1)} \ \ldots \ a_k^{(q)} \ b_{k1} \ \ldots \ b_{kk-1}) .$$

Also

$${}_T[E]_k = \begin{pmatrix} {}_T[E]_{k-1} & \underset{\sim}{0} \\ \underset{\sim}{t}'_k(E_k) & s_{(k)}(E_k) \end{pmatrix} .$$

Therefore

$${}_T\theta_k\,{}_T[E]_k = \begin{pmatrix} {}_T\theta_{k-1} & \underset{\sim}{0} \\ \underset{\sim}{b}'_k & c_{(k)} \end{pmatrix} \begin{pmatrix} {}_T[E]_{k-1} & \underset{\sim}{0} \\ \underset{\sim}{t}'_k(E_k) & s_{(k)}(E_k) \end{pmatrix}$$

$$= \begin{pmatrix} {}_T\theta_{k-1}\,{}_T[E]_{k-1} & \underset{\sim}{0} \\ \underset{\sim}{b}'_k\,{}_T[E]_{k-1} + c_{(k)}\underset{\sim}{t}'_k(E_k) & c_{(k)}s_{(k)}(E_k) \end{pmatrix} .$$

Also

$$E_k^* = {}_T\theta_k E_k = \begin{pmatrix} {}_T\theta_{k-1} & 0 \\ \underset{\sim}{b}'_k & c_{(k)} \end{pmatrix} \begin{pmatrix} E_{k-1} \\ \underset{\sim}{e}'_k \end{pmatrix} = \begin{pmatrix} {}_T\theta_{k-1}E_{k-1} \\ \underset{\sim}{b}'_k E_{k-1} + c_{(k)}\underset{\sim}{e}'_k \end{pmatrix} ,$$

therefore

$$\underset{\sim}{e}_k^{*'} = \underset{\sim}{b}_k' E_{k-1} + c_{(k)} \underset{\sim}{e}_k' \ .$$

Then by using (6.3.20) we get

$$\begin{aligned}
\underset{\sim}{t}_k({}_{T}\theta_k E_k) &= N_{k-1}^{-1} D_{k-1}^{*}({}_{T}\theta_{k-1} E_{k-1}) \underset{\sim}{e}_k^{*} \\
&= N_{k-1}^{-1} D_{k-1}^{*}(E_{k-1})(E_{k-1}' \underset{\sim}{b}_k + c_{(k)} \underset{\sim}{e}_k) \\
&= N_{k-1}^{-1} D_{k-1}^{*}(E_{k-1}) E_{k-1}' \underset{\sim}{b}_k + c_{(k)} N_{k-1}^{-1} D_{k-1}^{*}(E_{k-1}) \underset{\sim}{e}_k \\
&= {}_{T}[E]_{k-1}' \underset{\sim}{b}_k + c_{(k)} \underset{\sim}{t}_k(E_k) \ ,
\end{aligned}$$

so

$$\underset{\sim}{t}_k'({}_{T}\theta_k E_k) = \underset{\sim}{b}_k' {}_{T}[E]_{k-1} + c_{(k)} \underset{\sim}{t}_k'(E_k) \ .$$

Also by using (6.3.21) we get

$$\begin{aligned}
s_{(k)}^2({}_{T}\theta_k E_k) &= (\underset{\sim}{e}_k^{*} - D_{k-1}^{*'}({}_{T}\theta_{k-1} E_{k-1}) \underset{\sim}{t}_k({}_{T}\theta_k E_k))' \\
&\quad \times (\underset{\sim}{e}_k^{*} - D_{k-1}^{*'}({}_{T}\theta_{k-1} E_{k-1}) \underset{\sim}{t}_k({}_{T}\theta_k E_k))
\end{aligned}$$

$$\begin{aligned}
&= [E_{k-1}' \underset{\sim}{b}_k + c_{(k)} \underset{\sim}{e}_k - D_{k-1}^{*'}(E_{k-1}) {}_{T}[E]_{k-1}' \underset{\sim}{b}_k - c_{(k)} D_{k-1}^{*'}(E_{k-1}) \underset{\sim}{t}_k(E_k)]' \\
&\times [E_{k-1}' \underset{\sim}{b}_k + c_{(k)} \underset{\sim}{e}_k - D_{k-1}^{*'}(E_{k-1}) {}_{T}[E]_{k-1}' \underset{\sim}{b}_k - c_{(k)} D_{k-1}^{*'}(E_{k-1}) \underset{\sim}{t}_k(E_k)] \\
&= [E_{k-1}' \underset{\sim}{b}_k - E_{k-1}' \underset{\sim}{b}_k + c_{(k)} (\underset{\sim}{e}_k - D_{k-1}^{*'}(E_{k-1}) \underset{\sim}{t}_k(E_k)]' \\
&\times [E_{k-1}' \underset{\sim}{b}_k - E_{k-1}' b_k + c_{(k)} (\underset{\sim}{e}_k - D_{k-1}^{*'}(E_{k-1}) \underset{\sim}{t}_k(E_k)] \\
&= c_{(k)}^2 (\underset{\sim}{e}_k - D_{k-1}^{*'}(E_{k-1}) \underset{\sim}{t}_k(E_k))'(e_k - D_{k-1}^{*'}(E_{k-1}) \underset{\sim}{t}_k(E_k)) \\
&= c_{(k)}^2 s_{(k)}^2(E_k) \ ,
\end{aligned}$$

so

$$s_{(k)}({}_{T}\theta_k E_k) = c_{(k)} s_{(k)}(E_k) \ .$$

Then

$$ {}_T\theta_k\,{}_T[E]_k = \begin{pmatrix} {}_T\theta_{k-1}\,{}_T[E]_{k-1} & \underset{\sim}{0} \\ \underset{\sim}{b}_k'\,{}_T[E]_{k-1} + c_{(k)}\underset{\sim}{t}_k'(E_k) & c_{(k)}s_{(k)}(E_k) \end{pmatrix} $$

$$ = \begin{pmatrix} {}_T[{}_T\theta_{k-1}E_{k-1}]_{k-1} & \underset{\sim}{0} \\ \underset{\sim}{t}_k'({}_T\theta_kE_k) & s_{(k)}({}_T\theta_kE_k) \end{pmatrix} $$

$$ = {}_T[{}_T\theta_kE_k]_k \quad , $$

so

$$ {}_T[{}_T\theta_kE_k]_k = {}_T\theta_k\,{}_T[E]_k \quad . \tag{6.3.22} $$

Also

$$ \underset{\sim}{d}_k^*({}_T\theta_kE_k) = s^{-1}_{(k)}({}_T\theta_kE_k)(\underset{\sim}{e}_k^* - D^{*\prime}_{k-1}({}_T\theta_{k-1}E_{k-1})\underset{\sim}{t}_k({}_T\theta_kE_k)), $$

$$ = c^{-1}_{(k)}s^{-1}_{(k)}(E_k) $$

$$ \times(E'_{k-1}\underset{\sim}{b}_k + c_{(k)}\underset{\sim}{e}_k - D^{*\prime}_{k-1}(E_{k-1})\,{}_T[E]'_{k-1}\underset{\sim}{b}_k - c_{(k)}D^{*\prime}_{k-1}(E_{k-1})\underset{\sim}{t}_k(E_k)) $$

$$ = c^{-1}_{(k)}s^{-1}_{(k)}(E_k)[E'_{k-1}\underset{\sim}{b}_k - E'_{k-1}b_k + c_{(k)}(\underset{\sim}{e}_k - D^{*\prime}_{k-1}(E_{k-1})\underset{\sim}{t}_k(E_k))] $$

$$ = s^{-1}_{(k)}(E_k)(\underset{\sim}{e}_k - \underset{\sim}{D}^{*\prime}_{k-1}(E_{k-1})\underset{\sim}{t}_k(E_k)) $$

$$ = \underset{\sim}{d}_k^*(E_k) $$

so

$$ D_k^*({}_T\theta_kE_k) = D_k^*(E_k) \tag{6.3.23} $$

Then (6.3.22), by knowing that (6.3.23) holds, proves that

$[E]_T$ is the transformation variable for the location-progression group ${}_TG$ (6.3.1), which was to be proved.

This transformation variable $[E]_T$ may be now thought as the first stage of the transformation variable for whole positive-affine group G. For this group, the variable $[E]_T$ did not consider the orthogonal projections of coordinate vectors into the linear space $L(\underset{\sim}{v}_1,\ldots,\underset{\sim}{v}_q, \underset{\sim}{e}_1,\ldots,\underset{\sim}{e}_p)$.

Denote

$$D_p^*(E) = D^*(E) .$$

Then from (1.3.4) we have

$$E = \begin{pmatrix} V \\ \underline{E} \end{pmatrix} = \begin{pmatrix} I & 0 \\ M(E) & T(E) \end{pmatrix} \begin{pmatrix} V \\ \underline{D}^*(E) \end{pmatrix},$$

or

$$E = [E]_T D^*(E) . \tag{6.3.24}$$

By (6.3.13) and (6.3.15) $D^*(E)$ is an orthogonal set. Consider p orthogonal projections of the coordinate vectors (1, 0, ..., 0), ..., (0...0, 1, 0...), ... into the linear space $L(\underset{\sim}{v}_1,\ldots,\underset{\sim}{v}_q, \underset{\sim}{e}_1,\ldots,\underset{\sim}{e}_p)$, getting p orthogonal projections $\underset{\sim}{e}_1^o, \ldots, \underset{\sim}{e}_p^o$. The vectors $\underset{\sim}{e}_1^o, \ldots, \underset{\sim}{e}_p^o$ are chosen in such a way that $L(\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_q, \underset{\sim}{e}_1^o, \ldots, \underset{\sim}{e}_p^o)$ and $L(\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_q, \underset{\sim}{e}_1, \ldots, \underset{\sim}{e}_p)$ have the same orientation.

Let

$$E^{o} = \begin{pmatrix} V \\ E^{o} \end{pmatrix} \text{ and } D(E) = \begin{pmatrix} V \\ \underline{D}(E) \end{pmatrix} = \begin{pmatrix} V \\ \underline{D}^{*}(E^{o}) \end{pmatrix} = D^{*}(E^{o}). \qquad (6.3.25)$$

It is to be noted that the vectors in $\underline{D}^{*}(E)$ and $\underline{D}(E)=\underline{D}^{*}(E^{o})$ are orthogonal sets, have the same orientation and are related by an orthogonal rotation. Let $0(E)$ be a $p \times p$ rotation matrix which carried $\underline{D}(E)$ into $\underline{D}^{*}(E)$, so that

$$\underline{D}^{*}(E) = 0(E)\underline{D}(E) \; .$$

Therefore

$$D^{*}(E) = \begin{pmatrix} I & 0 \\ 0 & 0(E) \end{pmatrix} \begin{pmatrix} V \\ \underline{D}(E) \end{pmatrix} = [E]_{o} D(E) \; . \qquad (6.3.26)$$

Then the following theorem holds.

Theorem 6.3.1.

$$[E]=[E]_{T}[E]_{o}=\begin{pmatrix} I & 0 \\ M(E) & T(E) \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & 0(E) \end{pmatrix} = \begin{pmatrix} I & 0 \\ M(E) & C(E) \end{pmatrix}$$
$$(6.3.27)$$

is a transformation variable for the structural model (6.2.1).

Proof:

From (6.3.24) and (6.3.26) we have

$$E = [E]_{T}[E]_{o}D(E)$$

$$= \begin{pmatrix} I & 0 \\ M(E) & T(E) \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & 0(E) \end{pmatrix} D(E) = \begin{pmatrix} I & 0 \\ M(E) & C(E) \end{pmatrix} D(E)$$

$$= [E]D(E). \tag{6.3.28}$$

By the construction $[E] \in G$. Since G is unitary, [E] is a unique element in G. By definition D(E) is a fixed reference point on the orbit GE of E and depends wholly on the orbit GE. From (6.3.28) we see that the unique [E] transforms D(E) into E, a unique point on GE and hence from (1.3.4) [E] is a transformation variable for the structural model (6.2.1) which was to be proved.

6.4 The Generalized Multivariate Model:Distributions.

Before we proceed with the distributions for this model we will investigate the Jacobians for the transformations used in this model.

Consider the invariante differential on the error space. A transformation g applies column-by-column on the matrix E. Its effect on the $\begin{pmatrix} (i) \\ k \end{pmatrix}$-th column $(k = 1, \ldots, n_i; i = 1, \ldots, q)$ is

$$A \begin{pmatrix} v_{1k}^{(i)} \\ \vdots \\ v_{qk}^{(i)} \end{pmatrix} + C \begin{pmatrix} e_{ik}^{(i)} \\ \vdots \\ e_{pk}^{(i)} \end{pmatrix},$$

which has Jacobian $|C|$. Hence

$$J_{pn}(g:E) = |C|^n = |g|^n, \quad J_{pn}(E) = |C(E)|^n = |[E]|^n ,$$

$$dm(E) = \frac{\prod_{i,j,k} de_{jk}^{(i)}}{|C(E)|^n} = \frac{dE}{|[E]|^n} .$$

Now consider the invariant differentials on the group:

$$\begin{pmatrix} I & 0 \\ \tilde{A} & \tilde{C} \end{pmatrix} = \begin{pmatrix} I & 0 \\ A & C \end{pmatrix} \begin{pmatrix} I & 0 \\ A^* & C^* \end{pmatrix} .$$

The left transformation operates column-by-column. For any given column the Jacobian is $|C|$; hence

$$J = |C|^{p+q} , \quad J(g) = |g|^{p+q}$$

$$d\mu(g) = \frac{dg}{|g|^{p+q}} .$$

The right transformation operates row-by-row. For any given row the Jacobian is $|C^*|$; hence

$$J^* = |C^*|^p , \quad J^*(g) = |g|^p$$

$$d\gamma(g) = \frac{dg}{|g|^p} .$$

The modular function is

$$\Delta(g) = \frac{|g|^p}{|g|^{p+q}} = \frac{1}{|g|^q} .$$

The distribution of the transformation variable [E] given the orbit then is

$$f^*([E]/D)d[E] = k(D)f([E]D) \mid [E]|^n \mid [E]|^{-(p+q)} d[E] .$$

The differential can be factored:

$$\frac{d[E]}{|[E]|^{p+q}} = \frac{d[E]_T}{|[E]_T|_\Delta} d[E]_o = \frac{dM(E)\ dT(E)}{|T(E)|^q |T(E)|_\Delta} dO(E) .$$

The distribution of [E] can then be expressed in terms of components M(E), T(E), O(E):

$$f^*([E]/D)d[E] = k(D)f([E]D) \frac{|T(E)|^{n-q}}{|T(E)|_\Delta} dM(E)dT(E)dO(E) . \tag{6.4.1}$$

The structural distribution for θ given X is

$$g(\theta/X)d\theta = k(D)f(\theta^{-1}X)|[X]|^{n-q}\theta^{-(n-q)}d\nu(\theta). \tag{6.4.2}$$

6.5 The Generalized Multivariate Model:Normal Error.

We will consider now that the error variables have the standard normal distribution. Then the generalized multivariate model (6.2.1) in reduced form is:

$$\begin{cases} [X] = \theta[E] , \quad D(X) = D(E) \\ f(E)dE = (2\pi)^{-\frac{np}{2}} \exp\{-\frac{1}{2} \sum_{i=1}^{q} \sum_{j=1}^{p} \sum_{k=1}^{n_i} e_{jk}^{2(i)}\} \prod_{i,j,k} de_{jk}^{(i)} \end{cases} \tag{6.5.1}$$

Let us note, that from the results in Lemma 6.3.1 follows that

$$VV' = N, \quad \underline{D}\underline{D}' = I \text{ and } V\underline{D}' = 0$$

and let us denote

$$N^{\frac{1}{2}} = \begin{pmatrix} \sqrt{n_1} & 0 & \cdots & 0 \\ 0 & \sqrt{n_2} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & \sqrt{n_q} \end{pmatrix} .$$

Then the sum of squares in the exponential in the distribution in (6.5.1) can be expressed in terms of transformation variable

$$\sum_{i=1}^{q} \sum_{j=1}^{n} \sum_{k=1}^{n_i} e_{jk}^{2(i)} = \text{tr } EE' - \text{tr } VV' = \text{tr } [E]DD'[E]' - n$$

$$= \text{tr}\left[\begin{pmatrix} I & 0 \\ M(E) & C(E) \end{pmatrix} \begin{pmatrix} V \\ \underline{D} \end{pmatrix} (V'\underline{D}') \begin{pmatrix} I & M'(E) \\ 0 & C'(E) \end{pmatrix}\right] - n$$

$$= \text{tr}\left[\begin{pmatrix} I & 0 \\ M(E) & C(E) \end{pmatrix} \begin{pmatrix} VV' & V\underline{D}' \\ \underline{D}V' & \underline{DD}' \end{pmatrix} \begin{pmatrix} I & M'(E) \\ 0 & C'(E) \end{pmatrix}\right] - n$$

$$= \text{tr}\left[\begin{pmatrix} I & 0 \\ M(E) & C(E) \end{pmatrix} \begin{pmatrix} N & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & M'(E) \\ 0 & C'(E) \end{pmatrix}\right] - n$$

$$= \text{tr}\left[\begin{pmatrix} I & 0 \\ M(E) & C(E) \end{pmatrix} \begin{pmatrix} N^{\frac{1}{2}} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} N'^{\frac{1}{2}} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & M'(E) \\ 0 & C'(E) \end{pmatrix}\right] - n$$

$$= \text{tr}\left[\begin{pmatrix} N^{\frac{1}{2}} & 0 \\ M(E)N^{\frac{1}{2}} & C(E) \end{pmatrix} \begin{pmatrix} N^{\frac{1}{2}} & 0 \\ M(E)N^{\frac{1}{2}} & C(E) \end{pmatrix}'\right] - n$$

$$= \text{tr}[E][E]' - n ,$$

where

$$[E]=\begin{pmatrix} N^{\frac{1}{2}} & 0 \\ M(E)N^{\frac{1}{2}} & C(E) \end{pmatrix}=\begin{pmatrix} \sqrt{n_1} & \dots 0 & 0 & \dots 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots \sqrt{n_q} & 0 & \dots 0 \\ \sqrt{n_1}m_1^{(1)}(E) & \dots \sqrt{n_q}m_1^{(q)}(E) & c_{11}(E) & \dots c_{1p}(E) \\ \vdots & \vdots & \vdots & \vdots \\ \sqrt{n_1}m_p^{(1)}(E) & \dots \sqrt{n_q}m_p^{(q)}(E) & c_{p1}(E) & \dots c_{pp}(E) \end{pmatrix}$$

The adjusted transformation variable [E] can be factored into triangular and orthogonal components as we saw in the section 6.3:

$$[E] = [E]_T[E]_0 = \begin{pmatrix} N^{\frac{1}{2}} & 0 \\ M(E)N^{\frac{1}{2}} & T(E) \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & 0(E) \end{pmatrix} .$$

The sum of squares in the exponential in the distribution in (6.5.1) can then be further expressed in terms of triangular components:

$$\sum_{i=1}^{q}\sum_{j=1}^{p}\sum_{k=1}^{n_i} e_{jk}^{2(i)} = \mathrm{tr}[E][E]' - n = \mathrm{tr}[E]_T[E]_T' - n$$

$$= \mathrm{tr}\, M(E)N^{\frac{1}{2}}(M(E)N^{\frac{1}{2}})' + \mathrm{tr}\, T(E)T'(E)$$

$$= \sum_{i=1}^{q}\sum_{j=1}^{p} n_i m_j^{2(i)}(E) + \sum_{j>j'=1}^{p} t_{jj'}^2(E) + \sum_{j=1}^{p} s_{(j)}^2(E).$$

The distribution of the transformation variable [E] given the orbit then by (6.4.1) is

$$f([E]/D)d[E] = k(D)(2\pi)^{-\frac{np}{2}} \exp\{-\tfrac{1}{2}(\mathrm{tr}[E][E]' - n\}$$

$$\times \frac{|T(E)|^{n-q}}{|T(E)|_{\Delta}} dM(E)dT(E)d0(E)$$

$$= (2\pi)^{-\frac{np}{2}} \prod_{j=1}^{p} A_{n-q-j+1}$$

$$\times \exp\{-\tfrac{1}{2} \sum_{i=1}^{q} \sum_{j=1}^{p} n_i m_j^{2(i)}(E) - \tfrac{1}{2} \sum_{j>j'=1}^{p} t_{jj'}^{2}(E) - \tfrac{1}{2} \sum_{j=1}^{p} s_{(j)}^{2}(E)\}$$

$$\times s_{(1)}^{n-q-1}(E) \ldots s_{(p)}^{n-q-p}(E) \prod_{i,j} d(\sqrt{n_i} m_j^{(i)}(E)) \prod_{j>j'} dt_{jj'}(E)$$

$$\times \prod_j ds_{(j)}(E) \frac{d0(E)}{\prod_{j=2}^{p} A_j} . \qquad (6.5.2)$$

The structural distribution for θ given X then by (6.4.2) is

$$g(\theta/X)d\theta = (2\pi)^{-\frac{np}{2}} \prod_{j=1}^{p} A_{n-q-j+1} \exp\{-\tfrac{1}{2}\mathrm{tr}(\theta^{-1}XX'\theta'^{-1})-n\}$$

$$\times \frac{\prod_{i=1}^{q} n_i^{\frac{p}{2}}}{\prod_{j=2}^{p} A_j} \frac{|T(X)|^{n-q}}{|\tau|^{n-q}} \frac{dM d\tau d\theta}{|\tau|^{q}|\tau|_{\Delta}} ,$$

where we have factored $\theta \in G$ into the triangular and orthogonal component as members of group G in the section 6.3 as follows:

$$\begin{pmatrix} I & 0 \\ \mathcal{M} & \Gamma \end{pmatrix} = \theta = \theta_T\theta_0 = \begin{pmatrix} I & 0 \\ \mathcal{M} & \tau \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & \mathcal{O} \end{pmatrix} ,$$

so

$$\Gamma = \tau\mathcal{O}.$$

For the purpose of finding the structural distribution for θ expressed in its terms the exponential in the distribution in (6.5.1) can be rearranged as follows:

$$\sum_{i=1}^{q}\sum_{j=1}^{p}\sum_{k=1}^{n_i} e_{jk}^{2(i)} = \text{tr } EE'-n=\text{tr } M(E)N^{\frac{1}{2}}(M(E)N^{\frac{1}{2}})'+\text{tr } C(E)C'(E)$$

$$= \text{tr}\Gamma^{-1}(M(X)-\mathcal{M})N^{\frac{1}{2}}N'^{\frac{1}{2}}(M(X)-\mathcal{M})'\Gamma'^{-1}+\text{tr}\Gamma^{-1}C(X)C'(X)\Gamma'^{-1}$$

$$= \text{tr}(\Gamma\Gamma')^{-1}(M(X)-\mathcal{M})N^{\frac{1}{2}}N'^{\frac{1}{2}} (M(X)-\mathcal{M})'+\text{tr}(\Gamma\Gamma')^{-1}C(X)C'(X)$$

$$= \text{tr } \Sigma^{-1}[(M(X)-\mathcal{M})N^{\frac{1}{2}}N'^{\frac{1}{2}}(M(X)-\mathcal{M})'+S(X)] ,$$

where two inner-product matrices are defined by

$$\Sigma = \Gamma\Gamma' = \tau\mathcal{O}\mathcal{O}'\tau' = \tau\tau' \qquad (6.5.3)$$

$$S(X) = C(X)C'(X) = T(X)O(X)O'(X)T'(X) = T(X)T'(X). \qquad (6.5.4)$$

The structural distribution for θ then is:

$$g(\theta/X)d\theta = (2\pi)^{-\frac{np}{2}} \prod_{j=1}^{p} A_{n-q-j+1}$$

$$\times \exp\{-\frac{1}{2}\text{ tr}\Sigma^{-1}[(M(X)-\mathcal{M})N^{\frac{1}{2}}N'^{\frac{1}{2}}(M(X)-\mathcal{M})'+S(X)]\}$$

$$\times \frac{\prod_{i=1}^{q} n_i^{\frac{p}{2}}}{\prod_{j=2}^{p} A_j} \frac{|S(X)|^{\frac{n-q}{2}}}{|\tau|^{n-q}} \frac{d\mathcal{M}d\tau d\theta}{|\tau|^{q}|\tau|_{\Delta}} .$$

The structural distribution for θ can then be integrated over the rotations $\mathcal{O}$, the rotations in effect being absorbed by the density f. Also the structural distribution for τ induces a structural distribution for Σ, by using Jacobian matrix

$$\frac{\sigma\Sigma}{\sigma\tau} = 2^{p}|\tau|_{\nabla} ,$$

so we get the structural distribution for $\mathcal{M}$ and Σ:

$$g(\mathcal{M},\Sigma/X)d\mathcal{M}d\Sigma = 2^{-p}(2\pi)^{-\frac{np}{2}} |N^{\frac{1}{2}}N'^{\frac{1}{2}}|^{\frac{p}{2}} \prod_{j=1}^{p} A_{n-q-j+1}$$

$$\times \exp\{-\frac{1}{2}\, \mathrm{tr}(\mathrm{M}(X)-\mathcal{M})'N'^{\frac{1}{2}}\Sigma^{-1}N^{\frac{1}{2}}(\mathrm{M}(X)-\mathcal{M})\}$$

$$\times \exp\{-\frac{1}{2}\mathrm{tr}\Sigma^{-1}S(X)\}|S(X)|^{\frac{n-q}{2}}|\Sigma|^{-\frac{n+p+1}{2}} d\mathcal{M}d\Sigma. \qquad (6.5.5)$$

Note: Using the terminology of classical method of inference we can say that our model is investigating q-multivariate normal distributions with mean vectors $\underset{\sim}{\mu}^{(i)}$ (i = 1, ..., q) and the same variance-covariance matrix Σ (Anderson (1958), pg. 212).

6.6 β-expectation Tolerance Region. Before proceeding with the main result in this chapter let us state two Lemmas.

Lemma 6.6.1. (Anderson (1958) pg. 319). If the distribution of Z (p × n*) is h(ZZ')dZ, then the distribution of U = ZZ' is

$$f(U)dU = \frac{\pi^{\frac{1}{2}p[n^*-\frac{1}{2}(p-1)]}}{\prod_{j=1}^{p}\Gamma(\frac{n^*-j+1}{2})} |U|^{\frac{1}{2}(n^*-p-1)} h(U)dU . \qquad (6.6.1)$$

Lemma 6.6.2. If the distribution of Y(p × n*) is

$$h(Y)dY = \frac{|H|^{\frac{p}{2}} \prod_{j=1}^{p} A_{n-q-j+1}}{\prod_{j=1}^{p} A_{n+n^*-q-j+1}}$$

$$\times \frac{|S(X)|^{\frac{n-q}{2}}}{|S(X)+(Y-M(X)V^*)H(Y-M(X)V^*)'|^{\frac{n+n^*-q}{2}}} dY , \qquad (6.6.2)$$

where S(X) and H are symetric non-singular matrices, then the distribution of

$$U = (I + U_1)^{-1}U_1, \qquad (6.6.3)$$

where

$$U_1 = ZZ',$$

with

$$Z = T(Y-M(X)V^*)K,$$

where T is such that

$$T'T = S^{-1}(X)$$

and K is such that

$$KK' = H$$

is

$$f(U)dU = B_p^{-1}\left(\frac{n^*}{2}, \frac{n-q}{2}\right)|U|^{\frac{1}{2}(n^*-p-1)}|I-U|^{\frac{1}{2}(n-q-p-1)}dU, \tag{6.6.4}$$

which is generalized Beta distribution with

$$B_p(a,b) = \frac{\Gamma_p(a)\Gamma_p(b)}{\Gamma_p(a+b)},$$

where

$$\Gamma_p(a) = \pi^{\frac{1}{4}p(p-1)} \prod_{j=1}^{p} \Gamma(a - \frac{j+1}{2}) .$$

(For references to generalized Beta distribution see Olkin (1959)).

Proof:

In the distribution (6.6.2) let us first make the transformation

$$Z_1 = (Y - M(X)V^*)K,$$

where K is such that KK' = H and $|K| = |H|^{\frac{1}{2}}$, which exists since H is symetric and non-singular. Then

$$(Y-M(X)V^*)H(Y-M(X)V^*)' = (Y-M(X)V^*)KK'(Y-M(X)V^*)' = Z_1Z_1'$$

and

$$J(Y \to Z_1) = |K|^{-p} = |H|^{-\frac{p}{2}} ,$$

(For the references on the Jacobians of matrix transformations see Deemer and Olkin (1951) and Olkin (1953)),

so we get

$$h(Z_1)dZ_1 = \frac{|H|^{\frac{p}{2}} \prod\limits_{j=1}^{p} 2\pi^{\frac{n-q-j+1}{2}} \Gamma\left(\frac{n+n^*-q-j+1}{2}\right)}{\prod\limits_{j=1}^{p} 2\pi^{\frac{n+n^*-q-j+1}{2}} \Gamma\left(\frac{n-q-j+1}{2}\right)}$$

$$\times \frac{|S(X)|^{\frac{n-q}{2}} |H|^{-\frac{p}{2}}}{|S(X)+Z_1Z_1'|^{\frac{n+n^*-q}{2}}} dZ_1$$

$$= \pi^{-\frac{n^*p}{2}} \frac{\prod\limits_{j=1}^{p} \Gamma\left(\frac{n+n^*-q-j+1}{2}\right)}{\prod\limits_{j=1}^{p} \Gamma\left(\frac{n-q-j+1}{2}\right)} \frac{|S(X)|^{\frac{n-q}{2}}}{|S(X)+Z_1Z_1'|^{\frac{n+n^*-q}{2}}} dZ_1 .$$

Now by Lemma 1.5.1

$$|S(X)+Z_1Z_1'| = |I|\,|S(X)+Z_1 I^{-1} Z_1'| = \begin{vmatrix} I & -Z_1 \\ Z_1' & S(X) \end{vmatrix}$$

$$= |S(X)|\,|I+Z_1'S^{-1}(X)Z_1| ,$$

so that

$$h(Z_1)dZ_1 = \pi^{-\frac{n^*p}{2}} \frac{\prod\limits_{j=1}^{p} \Gamma\left(\frac{n+n^*-q-j+1}{2}\right)}{\prod\limits_{j=1}^{p} \Gamma\left(\frac{n-q-j+1}{2}\right)}$$

$$\frac{|S(X)|^{\frac{n-q}{2}}}{|S(X)|^{\frac{n+n^*-q}{2}} \; |I+Z_1'S^{-1}(X)Z_1|^{\frac{n+n^*-q}{2}}} \, dZ_1 \; .$$

Let now

$$Z = TZ_1 \; ,$$

where T is such that $T'T = S^{-1}(X)$ and $|T| = |S(X)|^{-\frac{1}{2}}$, which exists since S(X) is symetric and non-singular. Then

$$Z_1'S^{-1}(X)Z_1 = Z_1'T'TZ_1 = Z'Z$$

and

$$J(Z_1 \rightarrow Z) = |T|^{-n^*} = |S|^{\frac{n^*}{2}} \; ,$$

so we get

$$h(Z)dZ = \pi^{-\frac{n^*p}{2}} \frac{\prod_{j=1}^{p} \Gamma\left(\frac{n+n^*-q-j+1}{2}\right)}{\prod_{j=1}^{p} \Gamma\left(\frac{n-q-j+1}{2}\right)} \frac{|S(X)|^{\frac{n-q}{2}} \, |S(X)|^{\frac{n^*}{2}}}{|S(X)|^{\frac{n+n^*-q}{2}} \; |I+Z'Z|^{\frac{n+n^*-q}{2}}} dZ$$

$$= \pi^{-\frac{n^*p}{2}} \frac{\prod_{j=1}^{p} \Gamma\left(\frac{n+n^*-q-j+1}{2}\right)}{\prod_{j=1}^{p} \Gamma\left(\frac{n-q-j+1}{2}\right)} |I + ZZ'|^{-\frac{n+n^*-q}{2}} \; dZ \; ,$$

by using the fact that $|I + Z'Z| = |I+ZZ'|$. Note that

$$Z = TZ_1 = T(Y - M(X)V^*)K,$$

with T such that $T'T = S^{-1}(X)$ and K such that $KK' = H$.

Then by Lemma 6.6.1 the distribution of

$$U_1 = ZZ'$$

is

$$f(U_1)dU_1 = \frac{\pi^{\frac{1}{2}p[n^*-\frac{1}{2}(p-1)]}}{\prod_{j=1}^{p} \Gamma\left(\frac{n^*-j+1}{2}\right)} |U_1|^{\frac{1}{2}(n^*-p-1)} h(U_1)dU_1$$

$$= \frac{\pi^{\frac{1}{2}p[n^*-\frac{1}{2}(p-1)]}}{\prod_{j=1}^{p} \Gamma\left(\frac{n^*-j+1}{2}\right)} |U_1|^{\frac{1}{2}(n^*-p+1)} \pi^{-\frac{n^*p}{2}}$$

$$\times \frac{\prod_{j=1}^{p} \Gamma\left(\frac{n+n^*-q-j+1}{2}\right)}{\prod_{j=1}^{p} \Gamma\left(\frac{n-q-j+1}{2}\right)} |I+U_1|^{-\frac{n+n^*-q}{2}} dU_1$$

$$= \frac{\pi^{-\frac{1}{4}p(p-1)} \prod_{j=1}^{p} \Gamma\left(\frac{n+n^*-q-j+1}{2}\right)}{\prod_{j=1}^{p} \Gamma\left(\frac{n^*-j+1}{2}\right) \prod_{j=1}^{p} \Gamma\left(\frac{n-q-j+1}{2}\right)} |U_1|^{\frac{1}{2}(n^*-p-1)} |I+U_1|^{-\frac{n+n^*-q}{2}} dU_1 .$$

Let us now investigate $B_p\left(\frac{n^*}{2}, \frac{n-q}{2}\right)$ as defined in Olkin (1959):

$$B_p\left(\frac{n^*}{2}, \frac{n-q}{2}\right) = \frac{\Gamma_p\left(\frac{n^*}{2}\right)\Gamma_p\left(\frac{n-q}{2}\right)}{\Gamma_p\left(\frac{n+n^*-q}{2}\right)} ,$$

where

$$\Gamma_p\left(\frac{n^*}{2}\right) = \pi^{\frac{1}{4}p(p-1)} \prod_{j=1}^{p} \Gamma\left(\frac{n^*-j+1}{2}\right) .$$

Similarly

$$\Gamma_p\left(\frac{n-q}{2}\right) = \pi^{\frac{1}{4}p(p-1)} \prod_{j=1}^{p} \Gamma\left(\frac{n-q-j+1}{2}\right)$$

and

$$\Gamma_p\left(\frac{n+n^*-q}{2}\right) = \pi^{\frac{1}{4}p(p-1)} \prod_{j=1}^{p} \Gamma\left(\frac{n+n^*-q-j+1}{2}\right) ,$$

so that

$$B_p\left(\frac{n^*}{2}, \frac{n-q}{2}\right) = \frac{\pi^{\frac{1}{4}p(p-1)} \prod_{j=1}^{p} \Gamma\left(\frac{n^*-j+1}{2}\right) \pi^{\frac{1}{4}p(p-1)} \prod_{j=1}^{p} \Gamma\left(\frac{n-q-j+1}{2}\right)}{\pi^{\frac{1}{4}p(p-1)} \prod_{j=1}^{p} \Gamma\left(\frac{n+n^*-q-j+1}{2}\right)}$$

which implies that

$$\frac{\pi^{-\frac{1}{4}p(p-1)} \prod_{j=1}^{p} \Gamma\left(\frac{n+n^*-q-j+1}{2}\right)}{\prod_{j=1}^{p} \Gamma\left(\frac{n^*-j+1}{2}\right) \prod_{j=1}^{p} \Gamma\left(\frac{n-q-j+1}{2}\right)} = B_p^{-1}\left(\frac{n^*}{2}, \frac{n-q}{2}\right) .$$

Using this result we get

$$f(U_1)dU_1 = B_p^{-1}\left(\frac{n^*}{2}, \frac{n-q}{2}\right) |U_1|^{\frac{1}{2}(n^*-p-1)} |I+U_1|^{-\frac{1}{2}(n+n^*-q)} dU_1 .$$

Now, if we let

$$U = (I + U_1)^{-1}U_1,$$

then

$$|U| = \frac{|U_1|}{|I+U_1|} ,$$

$$|I + U_1| = |I - U|^{-1}$$

and

$$J(U_1 \rightarrow U) = |I - U|^{-(p+1)} ,$$

so

$$|U_1|^{\frac{1}{2}(n^*-p-1)} |I+U_1|^{-\frac{1}{2}(n+n^*-q)} dU_1 = \left[\frac{|U_1|}{|I+U_1|}\right]^{\frac{1}{2}(n^*-p-1)}$$

$$\times |I+U_1|^{-\frac{1}{2}(n-q+p+1)} dU_1$$

$$= |U|^{\frac{1}{2}(n^*-p-1)} |I-U|^{\frac{1}{2}(n-q+p+1)} |I-U|^{-(p+1)} dU$$

$$= |U|^{\frac{1}{2}(n^*-p-1)} |I-U|^{\frac{1}{2}(n-q-p-1)} dU \text{ .}$$

From this we see that

$$f(U)dU = B_p^{-1}\left(\frac{n^*}{2}, \frac{n-q}{2}\right)|U|^{\frac{1}{2}(n^*-p-1)} |I-U|^{\frac{1}{2}(n-q-p-1)} dU \text{ for } 0<U<I,$$

which was to be proved.

Theorem 6.6.1. Let the independent error variables $\underset{\sim}{e}^{(i)}$ $(i = 1,\ldots,q)$ of the structural model (6.2.1) have normal distribution with $\underset{\sim}{0}$ mean and variance-covariance matrix I, i.e.

$$f(\underset{\sim}{e}^{(i)})d\underset{\sim}{e}^{(i)} = (2\pi)^{-\frac{p}{2}} \exp\{-\frac{1}{2}\sum_{j=1}^{p} e_j^{2(i)}\} \prod_{j=1}^{p} de_j^{(i)}$$

$$i = 1, \ldots, q.$$

Then for central 100β per-cent of normal distribution being sampled, the region

$$Q = \{U/U < U_\beta\} \qquad (6.6.5)$$

is the β-expectation tolerance region, where U is defined as in (6.6.3) and U_β is the point exceeded with probability $1-\beta$ when using the generalized Beta distribution with $\frac{n^*}{2}$ and $\frac{n-q}{2}$ degrees of freedom (i.e. U_β is the point such that

$$B_p^{-1}\left(\frac{n^*}{2}, \frac{n-q}{2}\right)\int_0^{U_\beta} |U|^{\frac{1}{2}(n^*-p-1)} |I-U|^{\frac{1}{2}(n-q-p-1)} dU=\beta.)$$

Proof:

Since the error variable $\underset{\sim}{e}^{(i)}$ for $i = 1, \ldots, q$ have standard multivariate normal distributions, the distribution of the realized errors in the generalized multivariate model (6.2.1) is

$$\prod_{i=1}^{q} \prod_{k=1}^{n_i} f(e_{1k}^{(i)} \ldots e_{pk}^{(i)}) de_{1k}^{(i)} \ldots de_{pk}^{(i)}$$

$$= (2\pi)^{-\frac{np}{2}} \exp\{-\frac{1}{2} \sum_{i=1}^{q} \sum_{j=1}^{p} \sum_{k=1}^{n_i} e_{jk}^{2(i)}\} \prod_{i,j,k} de_{jk}^{(i)} .$$

Then by (6.5.5) the structural distribution for M and Σ is

$$g(M,\Sigma/X)dMd\Sigma$$

$$= 2^{-p}(2\pi)^{-\frac{np}{2}} |VV'|^{\frac{p}{2}} \prod_{j=1}^{p} A_{n-q-j+1}|S(X)|^{\frac{n-q}{2}} |\Sigma|^{-\frac{n+p+1}{2}}$$

$$\times \exp\{-\frac{1}{2}tr\Sigma^{-1}(M(X)-M)VV'(M(X)-M)' - \frac{1}{2}tr\Sigma^{-1}S(X)\}dMd\Sigma.$$

For the n* future responses $\underline{Y}$, the distribution is

$$p(\underline{Y}/M,\Sigma)d\underline{Y} = (2\pi)^{-\frac{n^*p}{2}}|\Sigma|^{-\frac{n^*}{2}}\exp\{-\tfrac{1}{2}\mathrm{tr}(\underline{Y}-MV^*)\Sigma^{-1}(\underline{Y}-MV^*)'\}d\underline{Y} .$$

Therefore the joint distribution of $\underline{Y}$, M and Σ is

$$p(\underline{Y}/M,\Sigma)g(M,\Sigma/X)dMd\Sigma d\underline{Y}$$

$$= 2^{-p}(2\pi)^{-\frac{(n+n^*)p}{2}}|VV'|^{\frac{p}{2}}\prod_{j=1}^{p}A_{n-q-j+1}|S(X)|^{\frac{n-q}{2}}|\Sigma|^{-\frac{n+n^*+p+1}{2}}$$

$$\times \exp\{-\tfrac{1}{2}\mathrm{tr}\Sigma^{-1}[(M(X)-M)VV'(M(X)-M)' + (\underline{Y}-MV^*)(\underline{Y}-MV^*)']\}$$

$$\times \exp\{-\tfrac{1}{2}\mathrm{tr}\Sigma^{-1}S(X)\}dMd\Sigma d\underline{Y}.$$

The matrix expression in the bracket in the exponential can be rearranged following Lemma 1.5.3:

$$(M(X)-M)VV'(M(X)-M)' + (\underline{Y}-MV^*)(\underline{Y}-MV^*)'$$

$$= (M-F)(VV'+V^*V^{*\prime})(M-F)' + (\underline{Y}-M(X)V^*)H(\underline{Y}-M(X)V^*)' ,$$

where

$$F = (M(X)VV'+\underline{Y}V^{*\prime})\ (VV' + V^*V^{*\prime})^{-1}$$

and

$$H = (I - V^{*\prime}(VV' + V^*V^{*\prime})^{-1}V^*) . \qquad (6.6.6)$$

Then

$$p(\underline{Y}/M,\Sigma)g(M,\Sigma/X)dMd\Sigma d\underline{Y}$$

$$= 2^{-p}(2\pi)^{-\frac{(n+n^*)p}{2}}|VV'|^{\frac{p}{2}}\prod_{j=1}^{p}A_{n-q-j+1}|S(X)|^{\frac{n-q}{2}}|\Sigma|^{-\frac{n+n^*+p+1}{2}}$$

$$\times\exp\{-\tfrac{1}{2}\mathrm{tr}\Sigma^{-1}[(M-F)(VV'+V^*V^{*\prime})(M-F)'+(\underline{Y}-M(X)V^*)H(\underline{Y}-M(X)V^*)']\}$$

$$\times\exp\{-\tfrac{1}{2}\mathrm{tr}\Sigma^{-1}S(X)\}dMd\Sigma d\underline{Y}.$$

Then by (1.4.5) the prediction distribution for $\underline{Y}$ is

$h(\underline{Y}/X)d\underline{Y}$

$$= 2^{-p}(2\pi)^{-\frac{(n+n^*)p}{2}}|VV'|^{\frac{p}{2}}\prod_{j=1}^{p}A_{n-q-j+1}|S(X)|^{\frac{n-q}{2}}\int_{\Sigma}|\Sigma|^{-\frac{n+n^*+p+1}{2}}$$

$$\times\ [\int_{M}\exp\{-\tfrac{1}{2}\mathrm{tr}\Sigma^{-1}(M-F)(VV'+V^*V^{*\prime})(M-F)'\}dM]$$

$$\times\ \exp\{-\tfrac{1}{2}\mathrm{tr}\Sigma^{-1}[(\underline{Y}-M(X)V^*)H(\underline{Y}-M(X)V^*)'+S(X)]\}d\Sigma d\underline{Y}$$

$$= \frac{|VV'|^{\frac{p}{2}}\prod_{j=1}^{p}A_{n-q-j+1}|S(X)|^{\frac{n-q}{2}}}{|VV'+V^*V^{*\prime}|^{\frac{p}{2}}2^{p}(2\pi)^{\frac{(n+n^*-q)p}{2}}}\int_{\Sigma}|\Sigma|^{-\frac{n+n^*-q+p+1}{2}}$$

$$\times\exp\{-\tfrac{1}{2}\mathrm{tr}\Sigma^{-1}[(\underline{Y}-M(X)V^*)H(\underline{Y}-M(X)V^*)' + S(X)]\}d\Sigma d\underline{Y}\ .$$

Using the integration relationship

$$\int_{\Sigma}\exp\{-\tfrac{1}{2}\mathrm{tr}\Sigma^{-1}R(X)\}|\Sigma|^{-\frac{n+p-r+1}{2}}d\Sigma = \frac{2^{p}(2\pi)^{\frac{p(n-r)}{2}}}{\prod_{j=1}^{p}A_{n-(r+j-1)}|R(X)|^{\frac{n-r}{2}}}$$

(for references see Fraser and Haq (1970) pg. 106)

we get

$$h(\underline{Y}/X)d\underline{Y} = \frac{|VV'|^{\frac{p}{2}}}{|VV'+V^*V^{*\prime}|^{\frac{p}{2}}}\frac{\prod_{j=1}^{p}A_{n-q-j+1}}{\prod_{j=1}^{p}A_{n+n^*-q-j+1}}$$

$$\frac{|S(X)|^{\frac{n-q}{2}}}{|S(X)+(\underline{Y}-M(X)V^*)H(\underline{Y}-M(X)V^*)'|^{\frac{n+n^*-q}{2}}}\, d\underline{Y} \; .$$

Applying Lemma 1.5.2 to (6.6.6) we see that

$$\frac{|VV'|^{\frac{p}{2}}}{|VV'+V^*V^{*\prime}|^{\frac{p}{2}}} = |H|^{\frac{p}{2}} \; ,$$

so

$$h(\underline{Y}/X)d\underline{Y} = \frac{|H|^{\frac{p}{2}} \prod_{j=1}^{p} A_{n-q-j+1}}{\prod_{j=1}^{p} A_{n+n^*-q-j+1}}$$

$$\times \frac{|S(X)|^{\frac{n-q}{2}}}{|S(X)+(\underline{Y}-M(X)V^*)H(\underline{Y}-M(X)V^*)'|^{\frac{n+n^*-q}{2}}}\, d\underline{Y} \qquad (6.6.7)$$

Now from (6.3.10) we see that VV' = N and V*V*' = N*, so

$$VV'+V^*V^{*\prime} = \begin{pmatrix} n_1+n_1^* & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & n_q+n_q^* \end{pmatrix}$$

which implies that

$$(VV'+V^*V^{*\prime})^{-1} = \begin{pmatrix} (n_1+n_1^*)^{-1} & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & (n_q+n_q^*)^{-1} \end{pmatrix} \; .$$

Therefore

$$V^{*\prime}(VV'+V^*V^{*\prime})^{-1}V^*=\begin{pmatrix}\underset{\sim}{1}_{n_1^*}&\cdots&\underset{\sim}{0}\\ \vdots&&\\ \vdots&&\\ \underset{\sim}{0}&\cdots&\underset{\sim}{1}_{n_q^*}\end{pmatrix}\begin{pmatrix}(n_1+n_1^*)^{-1}&\cdots&0\\ \vdots&&\vdots\\ 0&\cdots&(n_q+n_q^*)^{-1}\end{pmatrix}\begin{pmatrix}\underset{\sim}{1}'_{n_1^*}&\cdots&\underset{\sim}{0}'\\ \vdots&&\vdots\\ \underset{\sim}{0}'&\cdots&\underset{\sim}{1}'_{n_q^*}\end{pmatrix}$$

$$=\begin{pmatrix}H_1^*&\cdots&0\\ \vdots&&\vdots\\ 0&\cdots&H_q^*\end{pmatrix},$$

where

$$H_i^*=\begin{pmatrix}(n_i+n_i^*)^{-1}&\cdots&(n_i+n_i^*)^{-1}\\ \vdots&&\vdots\\ (n_i+n_i^*)^{-1}&\cdots&(n_i+n_i^*)^{-1}\end{pmatrix}_{(n_i^*\times n_i^*)}\qquad i=1,\ \ldots,\ q$$

Then

$$H=I-V^{*\prime}(VV'+V^*V^{*\prime})^{-1}V^*=\begin{pmatrix}I-H_1^*&\cdots&0\\ \vdots&&\vdots\\ 0&\cdots&I-H_q^*\end{pmatrix}=\begin{pmatrix}H_1&\cdots&0\\ \vdots&&\vdots\\ 0&\cdots&H_q\end{pmatrix},$$

where

$$H_i=\begin{pmatrix}1-(n_i+n_i^*)^{-1}&-(n_i+n_i^*)^{-1}&\cdots&-(n_i+n_i^*)^{-1}\\ -(n_i+n_i^*)^{-1}&1-(n_i+n_i^*)^{-1}&\cdots&-(n_i+n_i^*)^{-1}\\ \vdots&\vdots&&\vdots\\ -(n_i+n_i^*)^{-1}&-(n_i+n_i^*)^{-1}&\cdots&1-(n_i+n_i^*)^{-1}\end{pmatrix},\quad i=1,\ldots,q \tag{6.6.8}$$

which shows that H is symetric and non-singular. S(X) is symetric and non-singular by the definition, so (6.6.7) fulfills the assumptions of Lemma 6.6.2, so U defined by (6.6.3) follows generalized Beta distribution and Q defined at (6.6.5) is the β-expectation tolerance region, which was to be proved.

CHAPTER 7

PAIRWISE DIFFERENCE OF THE SAMPLES FROM q MULTIVARIATE NORMAL DISTRIBUTIONS

7.1 Introduction. In Chapter 3 we have investigated the construction of β-expectation tolerance region for the variable $Z = X_1 - X_2$, where the variables X_1 and X_2 were normally distributed with the different means and the same variance. The multivariate analogue of this problem is to find the β-expectation tolerance region for the variable $\underset{\sim}{Z} = \underset{\sim}{X}^{(1)} - \underset{\sim}{X}^{(2)}$, where $\underset{\sim}{X}^{(1)}$ is $N(\underset{\sim}{\mu}_1, \Sigma)$ and $\underset{\sim}{X}^{(2)}$ is $N_2(\underset{\sim}{\mu}_2, \Sigma)$. But this is only special case of more complex problem of finding the β-expectation tolerance region for q - 1 variables $\underset{\sim}{Z}^{(i)} = \underset{\sim}{X}^{(i)} - \underset{\sim}{X}^{(q)}$ $(i=1,\ldots,q-1)$, where $\underset{\sim}{X}^{(i)}$'s are distributed as $N_i(\underset{\sim}{\mu}_i, \Sigma)$ and $\underset{\sim}{X}^{(q)}$ is distributed as $N_q(\underset{\sim}{\mu}_q, \Sigma)$.

In the previous chapter we have derived the prediction distribution for the future responses $Y^{(1)}, \ldots, Y^{(q)}$ for response variables $\underset{\sim}{X}^{(1)}, \ldots, \underset{\sim}{X}^{(q)}$. So to find the β-expectation tolerance region for variables $\underset{\sim}{Z}^{(1)},\ldots,\underset{\sim}{Z}^{(q-1)}$ it is enough to find the prediction distribution of the following linear combination of the future response variables:

$$(Y^{(1)} - M(X)V^{(1)} - (Y^{(q)} - M(X)V^{(q)}) \dots \qquad (7.1.1)$$
$$\dots Y^{(q-1)} - M(X)V^{(q-1)} - (Y^{(q)} - M(X)V^{(q)})) ,$$

where $Y^{(i)}$ is a $p \times n_i^*$ matrix of n_i^* future responses for variable $\underset{\sim}{X}^{(i)} = (X_1^{(i)} X_2^{(i)} \dots X_p^{(i)})'$, $V^{(i)}$ is a $q \times n_i^*$ matrix, having 1's in the i-th row and 0's as other elements ($i = 1, \dots, q$) and $M(X)$ is $p \times q$ matrix of $m_j^{(i)}(X)$ ($i = 1,\dots,q$; $j = 1,\dots,p$).

7.2 The Distribution of Linear Combination (7.1.1) of Future Response Variables.

In the previous chapter we have obtained the distribution of future response variables $\underline{Y} = (Y^{(1)}\ Y^{(2)}\ \dots\ Y^{(q)})$. We will now investigate the distribution of

$$Z_{q-1} = (Z^{(1)}\ Z^{(2)}\ \dots\ Z^{(q-1)}), \qquad (7.2.1)$$

where

$$Z^{(i)} = \begin{pmatrix} \underset{\sim}{z}_1^{'(i)} \\ \underset{\sim}{z}_2^{'(i)} \\ \vdots \\ \underset{\sim}{z}_p^{'(i)} \end{pmatrix} = \begin{pmatrix} \underset{\sim}{y}_i^{'(i)} - m_1^{(i)}(X)\underset{\sim}{1}' - (\underset{\sim}{y}_1^{'(q)} - m_1^{(q)}(X)\underset{\sim}{1}') \\ \underset{\sim}{y}_2^{'(i)} - m_2^{(i)}(X)\underset{\sim}{1}' - (\underset{\sim}{y}_2^{'(q)} - m_2^{(q)}(X)\underset{\sim}{1}') \\ \cdot \qquad \vdots \\ \underset{\sim}{y}_p^{'(i)} - m_p^{(i)}(X)\underset{\sim}{1}' - (\underset{\sim}{y}_p^{'(q)} - m_p^{(q)}(X)\underset{\sim}{1}') \end{pmatrix} \text{for } i=1,\dots,q-1 \qquad (7.2.2)$$

which is the linear combination (7.1.1) of the future response variables $Y^{(1)}, Y^{(2)}, \dots, Y^{(q)}$. From (7.2.2) we see that this combination is possible only if we have the same number of future response variables for $i = 1,\dots,q$, or the vectors $\underset{\sim}{y}_j^{(i)}$ are of the same dimension, say n_d^* for all i and j. It means that we have to assume $n_1^* = n_2^* = \dots = n_q^* = n_d^*$. Therefore n* from Chapter 6 is qn_d^*, i.e. $n^* = qn_d^*$. Under these assumptions the following lemma holds.

<u>Lemma 7.2.1</u>. If the distribution of $\underline{Y} = (Y^{(1)} \dots Y^{(q)})$, where $Y^{'(i)} = (\underset{\sim}{y}_1^{(i)}\ \underset{\sim}{y}_2^{(i)} \dots \underset{\sim}{y}_p^{(i)})$ for $i = 1, \dots, q$ is

$$h(\underline{Y})d\underline{Y} = \frac{|H|^{\frac{p}{2}} \prod_{j=1}^{p} A_{n-q-j+1} |S(X)|^{\frac{n-q}{2}}}{\prod_{j=1}^{p} A_{n+(n_d^*-1)q-j+1}} \tag{7.2.3}$$

$$\times |S(X)+(\underline{Y}-M(X)V^*)H(\underline{Y}-M(X)V^*)'|^{-\frac{n+q(n_d^*-1)}{2}} \quad d\underline{Y} \ ,$$

then the distribution of Z_{q-1} defined by (7.2.1) is

$$h(Z_{q-1})dZ_{q-1} = \frac{|\mathcal{H}_{q-1}|^{-\frac{p}{2}} \prod_{j=1}^{p} A_{n-q-j+1} |S(X)|^{\frac{n-q}{2}}}{\prod_{j=1}^{p} A_{n+(n_d^*-1)(q-1)-j}} \tag{7.2.4}$$

$$\times |S(X)+Z_{q-1}\mathcal{H}_{q-1}^{-1}Z'_{q-1}|^{-\frac{n+(n_d^*-1)(q-1)-1}{2}} \quad dZ_{q-1} \ ,$$

where

$$\mathcal{H}_{q-1} = \begin{pmatrix} H_1^{-1}+H_q^{-1} & H_q^{-1} & \dots & H_q^{-1} \\ H_q^{-1} & H_2^{-1}+H_q^{-1} & \dots & H_q^{-1} \\ \vdots & \vdots & & \vdots \\ H_q^{-1} & H_q^{-1} & \dots & H_{q-1}^{-1}+H_q^{-1} \end{pmatrix} ;$$

$H_1, H_2, \ldots, H_q$ are $n_d^* \times n_d^*$ matrices defined in (6.6.8) and

$$H = \begin{pmatrix} H_1 & \ldots & 0 \\ \vdots & & \vdots \\ 0 & \ldots & H_q \end{pmatrix} .$$

Proof:

Let

$$\underset{\sim}{y}_j^{'(q)} - m_j^{(q)}(X)\underset{\sim}{1}' = \underset{\sim}{z}_j^{'(q)} \qquad j = 1, \ldots, p,$$

then from (7.2.2) we see that

$$\underset{\sim}{y}_j^{'(i)} - m_j^{(i)}(X)\underset{\sim}{1}' = \underset{\sim}{z}_j^{'(i)} + \underset{\sim}{z}_j^{'(q)} \qquad i=1,\ldots,q-1;\ j=1,\ldots,p.$$

Hence

$$(\underline{Y}-M(X)V^*) = \begin{pmatrix} \underset{\sim}{y}_1^{'(1)}-m_1^{(1)}(X)\underset{\sim}{1}' & \ldots & \underset{\sim}{y}_1^{'(q)}-m_1^{(q)}(X)\underset{\sim}{1}' \\ \vdots & & \vdots \\ \underset{\sim}{y}_p^{'(1)}-m_p^{(1)}(X)\underset{\sim}{1}' & \ldots & \underset{\sim}{y}_p^{'(q)}-m_p^{(q)}(X)\underset{\sim}{1}' \end{pmatrix}$$

$$= \begin{pmatrix} \underset{\sim}{z}_1^{'(1)}+\underset{\sim}{z}_1^{'(q)} & \ldots & \underset{\sim}{z}_1^{'(q-1)}+\underset{\sim}{z}_1^{'(q)} & \underset{\sim}{z}_1^{'(q)} \\ \vdots & & \vdots & \vdots \\ \underset{\sim}{z}_p^{'(1)}+\underset{\sim}{z}_p^{'(q)} & \ldots & \underset{\sim}{z}_p^{'(q-1)}+\underset{\sim}{z}_p^{'(q)} & \underset{\sim}{z}_p^{'(q)} \end{pmatrix}$$

$$= (Z^{(1)}+Z^{(q)} \ \ldots \ Z^{(q-1)}+Z^{(q)} \quad Z^{(q)}) = Z_s ,$$

where

$$Z^{'(i)} = (\underset{\sim}{z}_1^{(i)} \ \ldots \ \underset{\sim}{z}_p^{(i)}) \quad \text{for } i = 1, \ldots, q. \qquad (7.2.5)$$

Therefore

$$|S(X)+(\underline{Y}-M(X)V^*)H(Y-M(X)V^*)'|=|S(X)+Z_sHZ_s^{'}| \ . \qquad (7.2.6)$$

In Chapter 6 it has been shown that H is symetric and that there exists K such that

$$\begin{pmatrix} H_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & H_q \end{pmatrix} = H = KK' = \begin{pmatrix} K_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & K_q \end{pmatrix} \begin{pmatrix} K_1' & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & K_q' \end{pmatrix} . \qquad (7.2.7)$$

Then by using Lemma 1.5.1

$$|S(X)+Z_sHZ_s'| = |I||S(X)+Z_sKI^{-1}K'Z_s'| = \begin{vmatrix} I & K'Z_s' \\ -Z_sK & S(X) \end{vmatrix}$$

$$= |S(X)||I+K'Z_s'S^{-1}(X)Z_sK| = |S(X)||R_1| . \qquad (7.2.8)$$

Let us now investigate $|R_1|$ from (7.2.8):

$$|R_1| = |I+K'Z_s'S^{-1}(X)Z_sK|$$

$$= \left| \begin{pmatrix} I \cdots 0 & 0 \\ \vdots \quad \vdots & \vdots \\ 0 \cdots I & 0 \\ 0 \cdots 0 & I \end{pmatrix} + \begin{pmatrix} K_1' \cdots 0 & 0 \\ \vdots \quad \vdots & \vdots \\ 0 \cdots K_{q-1}' & 0 \\ 0 \cdots 0 & K_q' \end{pmatrix} \begin{pmatrix} Z'^{(1)}+Z'^{(q)} \\ \vdots \\ Z'^{(q-1)}+Z'^{(q)} \\ Z'^{(q)} \end{pmatrix} \right.$$

$$\left. \times\ S^{-1}(X)(Z^{(1)}+Z^{(q)} \ldots Z^{(q-1)}+Z^{(q)} Z^{(q)}) \begin{pmatrix} K_1 \cdots 0 & 0 \\ \vdots \quad \vdots & \vdots \\ 0 \cdots K_{q-1} & 0 \\ 0 \cdots 0 & K_q \end{pmatrix} \right|$$

$$= \left| \begin{pmatrix} I \cdots 0 & 0 \\ \vdots \quad \vdots & \vdots \\ 0 \cdots I & 0 \\ 0 \cdots 0 & I \end{pmatrix} + \begin{pmatrix} K_1'(Z'^{(1)}+Z'^{(q)}) \\ \vdots \\ K_{q-1}'(Z'^{(q-1)}+Z'^{(q)}) \\ K_q'Z'^{(q)} \end{pmatrix} \right.$$

$$\left. \times\ S^{-1}(X)((Z^{(1)}+Z^{(q)})K_1 \ldots (Z^{(q-1)}+Z^{(q)})K_{q-1}Z^{(q)}K_q) \right|$$

$$= \left| \begin{matrix} I+K_1'(Z'^{(1)}+Z'^{(q)})S^{-1}(X)(Z^{(1)}+Z^{(q)})K_1 & \ldots \\ \vdots & \\ K'_{q-1}(Z'^{(q-1)}+Z'^{(q)})S^{-1}(X)(Z^{(1)}+Z^{(q)})K_1 & \ldots \\ K_q'Z'^{(q)}S^{-1}(X)(Z^{(1)}+Z^{(q)})K_1 & \ldots \end{matrix} \right.$$

$$\left. \begin{matrix} \ldots & K_1'(Z'^{(1)}+Z'^{(q)})S^{-1}(X)Z^{(q)}K_q \\ & \vdots \\ \ldots & K'_{q-1}(Z'^{(q-1)}+Z'^{(q)})S^{-1}(X)Z^{(q)}K_q \\ \ldots & I+K_q'Z'^{(q)}S^{-1}(X)Z^{(q)}K_q \end{matrix} \right| .$$

The value of the determinant does not change after making elementary operations, so let us multiply the last row of this determinant by $K_i'K_q'^{-1}$ from left and subtract it from the ith row (i = 1, ..., q-1). Then we get

$$|R_1| = \begin{vmatrix} I+K_1'Z'^{(1)}S^{-1}(X)(Z^{(1)}+Z^{(q)})K_1 & \cdots & K_1'Z'^{(1)}S^{-1}(X)Z^{(q)}K_q-K_1'K_q'^{-1} \\ \vdots & & \vdots \\ K_{q-1}'Z'^{(q-1)}S^{-1}(X)(Z^{(1)}+Z^{(q)})K_1 & \cdots & K_{q-1}'Z'^{(q-1)}S^{-1}(X)Z^{(q)}K_q-K_{q-1}'K_q'^{-1} \\ K_q'Z'^{(q)}S^{-1}(X)(Z^{(1)}+Z^{(q)})K_1 & \cdots & K_q'Z'^{(q)}S^{-1}(X)Z^{(q)} \end{vmatrix}.$$

Let us now multiply the last column of this determinant by $K_q^{-1}K_i$ from right and subtract it from ith column $(i = 1, \ldots, q-1)$. Then

$$|R_1| = \begin{vmatrix} I+K_1'K_q'^{-1}K_q^{-1}K_1+K_1'Z'^{(1)}S^{-1}(X)Z^{(1)}K_1 & \cdots & -K_1'K_q'^{-1}+K_1'Z'^{(1)}S^{-1}(X)Z^{(q)}K_q \\ \vdots & & \vdots \\ K_{q-1}'K_q'^{-1}K_q^{-1}K_1+K_{q-1}'Z'^{(q-1)}S^{-1}(X)Z^{(1)}K_1 & \cdots & -K_{q-1}'K_q'^{-1}+K_{q-1}'Z'^{(q-1)}S^{-1}(X)Z^{(q)}K_q \\ -K_q^{-1}K_1+K_q'Z'^{(q)}S^{-1}(X)Z^{(1)}K_1 & \cdots & I+K_q'Z'^{(q)}S^{-1}(X)Z^{(q)}K_q \end{vmatrix}$$

$$= \left| \begin{pmatrix} I+K_1'H_q^{-1}K_1 & \dots & -K_1'K_q'^{-1} \\ \vdots & & \vdots \\ K_{q-1}'H_q^{-1}K_1 & \dots & -K_{q-1}'K_q'^{-1} \\ -K_q^{-1}K_1 & \dots & I \end{pmatrix} \right.$$

$$\left. + \begin{pmatrix} K_1'Z'^{(1)}S^{-1}(X)Z^{(1)}K_1 & \dots K_1'Z'^{(1)}S^{-1}(X)Z^{(q)}K_q \\ \vdots & \vdots \\ K_{q-1}'Z'^{(q-1)}S^{-1}(X)Z^{(1)}K_1 \dots & K_{q-1}'Z'^{(q-1)}S^{-1}(X)Z^{(q)}K_q \\ K_q'Z'^{(q)}S^{-1}(X)Z^{(1)}K_1 \quad \dots & K_q'Z'^{(q)}S^{-1}(X)Z^{(q)}K_q \end{pmatrix} \right|$$

$$= |\mathcal{K}_q + K'Z'S^{-1}(X)ZK| \ , \tag{7.2.9}$$

where

$$\mathcal{K}_q = \begin{pmatrix} I+K_1'H_q^{-1}K_1 \dots & K_1'H_q^{-1}K_1 & -K_1'K_q'^{-1} \\ \vdots & \vdots & \vdots \\ K_{q-1}'H_q^{-1}K_1 \dots & I+K_{q-1}'H_q^{-1}K_{q-1} & -K_{q-1}'K_q'^{-1} \\ -K_q^{-1}K_1 \quad \dots & -K_q^{-1}K_{q-1} & I \end{pmatrix} \tag{7.2.10}$$

and

$$Z = (Z^{(1)} \dots Z^{(q-1)} \quad Z^{(q)}) . \tag{7.2.11}$$

In Chapter 6 we have also seen that there exists a p × p matrix T, such that

$$S(X) = TT' \ . \tag{7.2.12}$$

Then we can further simplify (7.2.9) using Lemma 1.5.1:

$$|R_1| = |\mathcal{K}_q + K'Z'T'^{-1}I^{-1}T^{-1}ZK| = \begin{vmatrix} I & T^{-1}ZK \\ -K'Z'T'^{-1} & \mathcal{K}_q \end{vmatrix}$$

$$= \begin{vmatrix} I & T^{-1}Z_{q-1}K^{(q-1)} & T^{-1}Z^{(q)}K_q \\ -K'^{(q-1)}Z'_{q-1}T'^{-1} & \mathcal{K}_{q-1} & -K^{(q)} \\ -K'_qZ'^{(q)}T'^{-1} & -K'^{(q)} & I \end{vmatrix} , \tag{7.2.13}$$

where

$$K^{(q-1)} = \begin{pmatrix} K_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & K_{q-1} \end{pmatrix} , \tag{7.2.14}$$

$$\mathcal{K}_{q-1} = \begin{pmatrix} I+K'_1H_q^{-1}K_1 & \cdots & K'_1H_q^{-1}K_{q-1} \\ \vdots & & \vdots \\ K'_{q-1}H_q^{-1}K_1 & \cdots & I+K'_{q-1}H_q^{-1}K_{q-1} \end{pmatrix} \tag{7.2.15}$$

and

$$K'^{(q)} = (K_q^{-1}K_1 \; \ldots \; K_q^{-1}K_{q-1}) \; . \tag{7.2.16}$$

Now let

$$L = \begin{pmatrix} I & T^{-1}Z_{q-1}K^{(q-1)} \\ -K'^{(q-1)}Z'_{q-1}T'^{-1} & \mathcal{K}_{q-1} \end{pmatrix} . \tag{7.2.17}$$

From this, using Lemma 1.5.1, we get

$$|L| = \begin{vmatrix} I & T^{-1}Z_{q-1}K^{(q-1)} \\ -K'^{(q-1)}Z'_{q-1}T'^{-1} & \mathcal{K}_{q-1} \end{vmatrix}$$

$$= |I|\,|\mathcal{K}_{q-1}+K'^{(q-1)}Z'_{q-1}T'^{-1}I^{-1}T^{-1}Z_{q-1}K^{(q-1)}|$$

$$= |\mathcal{K}_{q-1}+K'^{(q-1)}Z'_{q-1}S^{-1}(X)Z_{q-1}K^{(q-1)}|$$

$$= |S(X)|^{-1}\begin{vmatrix} \mathcal{K}_{q-1} & -K'^{(q-1)}Z'_{q-1} \\ Z_{q-1}K^{(q-1)} & S(X) \end{vmatrix}$$

$$= |S(X)|^{-1}|\mathcal{K}_{q-1}|\,|S(X)+Z_{q-1}K^{(q-1)}\mathcal{K}_{q-1}^{-1}K'^{(q-1)}Z'_{q-1}|. \tag{7.2.18}$$

Let us now investigate $\mathcal{K}_{q-1}$. In (7.2.15) for $i = 1,\ldots,q-1$

$$I+K'_iH_q^{-1}K_i = K'_iK_i'^{-1}K_i^{-1}K_i+K'_iH_q^{-1}K_i = K'_i(K_iK'_i)^{-1}K_i+K_iH_q^{-1}K_i$$

$$= K'_iH_i^{-1}K_i+K'_iH_q^{-1}K_i = K'_i(H_i^{-1}+H_q^{-1})K_i \ ,$$

so (7.2.15) becomes

$$\mathcal{K}_{q-1} = \begin{pmatrix} K'_1(H_1^{-1}+H_q^{-1})K_1 \ldots & K'_1H_q^{-1}K_{q-1} \\ \vdots & \vdots \\ K'_{q-1}H_q^{-1}K_1 & \ldots K'_{q-1}(H'_{q-1}+H_q^{-1})K_{q-1} \end{pmatrix}$$

$$= \begin{pmatrix} K'_1 \ldots & 0 \\ \vdots & \vdots \\ 0 & \ldots K'_{q-1} \end{pmatrix} \begin{pmatrix} H_1^{-1}+H_q^{-1} \ldots & H_q^{-1} \\ \vdots & \vdots \\ H_q^{-1} & \ldots H_{q-1}^{-1}+H_q^{-1} \end{pmatrix} \begin{pmatrix} K_1 \ldots & 0 \\ \vdots & \vdots \\ 0 & \ldots K_{q-1} \end{pmatrix}$$

$$= K^{'(q-1)}\mathcal{H}_{q-1}K^{(q-1)} \qquad (7.2.19)$$

using (7.2.14) and letting

$$\mathcal{H}_{q-1} = \begin{pmatrix} H_1^{-1}+H_q^{-1} & H_q^{-1} & \cdots & H_q^{-1} \\ H_q^{-1} & H_q^{-1}+H_q^{-1}\cdots & & H_q^{-1} \\ \vdots & \vdots & & \vdots \\ H_q^{-1} & H_q^{-1} & \cdots H_{q-1}^{-1}+H_q^{-1} & \end{pmatrix} . \qquad (7.2.20)$$

Then from (7.2.19) we see that

$$\mathcal{K}_{q-1}^{-1} = K^{-1(q-1)}\mathcal{H}_{q-1}^{-1}K^{'-1(q-1)} \qquad (7.2.21)$$

and

$$|\mathcal{K}_{q-1}|=|K^{'(q-1)}||\mathcal{H}_{q-1}||K^{(q-1)}|=|\mathcal{H}_{q-1}|\prod_{i=1}^{q-1}|K_i|^2=|\mathcal{H}_{q-1}|\prod_{i=1}^{q-1}|H_i| .$$

$$(7.2.22)$$

Using (7.2.21) and (7.2.22) in (7.2.18) we get

$$|L| = |S(X)|^{-1}|\mathcal{H}_{q-1}|$$

$$\times\left(\prod_{i=1}^{q-1}|H_i|\right)|S(X)+Z_{q-1}K^{(q-1)}K^{-1(q-1)}\mathcal{H}_{q\ 1}^{-1}K^{'-1(q-1)}K^{'(q-1)}Z'_{q-1}|$$

$$= |S(X)|^{-1}|\mathcal{H}_{q-1}|\left(\prod_{i=1}^{q-1}|H_i|\right)|S(X)+Z_{q-1}\mathcal{H}_{q-1}^{-1}Z'_{q-1}|$$

$$= |S(X)|^{-1}|\mathcal{H}_{q-1}||\mathcal{Z}|\prod_{i=1}^{q-1}|H_i| , \qquad (7.2.23)$$

where

$$\mathcal{Z} = S(X)+Z_{q-1}\mathcal{H}_{q-1}^{-1}Z'_{q-1} . \qquad (7.2.24)$$

For L^{-1} let us use Lemma 1.5.2. Since

$$L = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}$$

as we can see from (7.2.17),

$$L^{-1} = \begin{pmatrix} L^{11} & L^{12} \\ L^{21} & L^{22} \end{pmatrix} ,$$

where

$$L^{11} = (L_{11}-L_{12}L_{22}^{-1}L_{21})^{-1}$$

$$L^{12} = -(L_{11}-L_{12}L_{22}^{-1}L_{21})^{-1}L_{12}L_{22}^{-1}$$

$$L^{21} = -L_{22}^{-1}L_{21}(L_{11}-L_{12}L_{22}^{-1}L_{21})^{-1}$$

$$L^{22} = L_{22}^{-1}+L_{22}^{-1}L_{21}(L_{11}-L_{12}L_{22}^{-1}L_{21})^{-1}L_{12}L_{22}^{-1} .$$

Substituting L_{11}, L_{12}, L_{21} and L_{22} from (7.2.17) and using (7.2.12), (7.2.21) and (7.2.24) we get

$$\begin{aligned} L^{11} &= (I+T^{-1}Z_{q-1}K^{(q-1)}\mathcal{K}_{q-1}^{-1}K'^{(q-1)}Z'_{q-1}T'^{-1})^{-1} \\ &= (I+T^{-1}Z_{q-1}K^{(q-1)}K^{-1(q-1)}\mathcal{H}_{q-1}^{-1}K'^{-1(q-1)}K'^{(q-1)}Z'_{q-1}T'^{-1})^{-1} \\ &= (T^{-1}TT'T'^{-1}+T^{-1}Z_{q-1}\mathcal{H}_{q-1}^{-1}Z'_{q-1}T'^{-1})^{-1} \\ &= (T^{-1}[S(X)+Z_{q-1}\mathcal{H}_{q-1}^{-1}Z'_{q-1}]T'^{-1})^{-1} = T'\mathcal{Z}^{-1}T . \qquad (7.2.25) \end{aligned}$$

$$\begin{aligned} L^{12} &= -T'\mathcal{Z}^{-1}TT^{-1}Z_{q-1}K^{(q-1)}\mathcal{K}_{q-1}^{-1} \\ &= -T'\mathcal{Z}^{-1}Z_{q-1}K^{(q-1)}K^{-1(q-1)}\mathcal{H}_{q-1}^{-1}K'^{-1(q-1)} \end{aligned}$$

$$= -T'\mathcal{Z}^{-1}Z_{q-1}\mathcal{H}_{q-1}^{-1}K^{'-1(q-1)} \quad . \tag{7.2.26}$$

$$L^{21} = \mathcal{K}_{q-1}^{-1}K^{'(q-1)}Z'_{q-1}T^{'-1}T'\mathcal{Z}^{-1}T = K^{-1(q-1)}\mathcal{H}_{q-1}^{-1}Z'_{q-1}\mathcal{Z}^{-1}T \quad . \tag{7.2.27}$$

$$\begin{aligned} L^{22} = {} & K^{-1(q-1)}\mathcal{H}_{q-1}^{-1}K^{'-1(q-1)} \\ & - K^{-1(q-1)}\mathcal{H}_{q-1}^{-1}Z'_{q-1}\mathcal{Z}^{-1}Z_{q-1}\mathcal{H}_{q-1}^{-1}K^{'-1(q-1)} \end{aligned} \tag{7.2.28}$$

Now substituting (7.2.17) into (7.2.15) and using Lemma 1.5.1 again, we get

$$|R_1| = \begin{vmatrix} L & \begin{pmatrix} T^{-1}Z^{(q)}K_q \\ -K^{(q)} \end{pmatrix} \\ -(K_q'Z^{'(q)}T^{'-1} \quad K^{'(q)}) & I \end{vmatrix}$$

$$= |L|\left| I+(K_q'Z^{'(q)}T^{'-1} \quad K^{'(q)})L^{-1}\begin{pmatrix} T^{-1}Z^{(q)}K_q \\ -K^{(q)} \end{pmatrix}\right|$$

$$= |L|\left| I+(K_q'Z^{'(q)}T^{'-1} \quad K^{'(q)})\begin{pmatrix} L^{11} & L^{12} \\ L^{21} & L^{22} \end{pmatrix}\begin{pmatrix} T^{-1}Z^{(q)}K_q \\ -K^{(q)} \end{pmatrix}\right|$$

$$\begin{aligned} = |L|\Big| & I+(K_q'Z^{'(q)}T^{'-1}L^{11}+K^{'(q)}L^{21} \\ & \qquad K_q'Z^{'(q)}T^{'-1}L^{12}+K^{'(q)}L^{22})\begin{pmatrix} T^{-1}Z^{(q)}K_q \\ -K^{(q)} \end{pmatrix}\Big| \end{aligned}$$

$$\begin{aligned} = |L||I & +K_q'Z^{'(q)}T^{'-1}L^{11}T^{-1}Z^{(q)}K_q+K^{'(q)}L^{21}T^{-1}Z^{(q)}K_q \\ & -K_q'Z^{'(q)}T^{'-1}L^{12}K^{(q)}-K^{'(q)}L^{22}K^{(q)}| \end{aligned} \tag{7.2.29}$$

Substituting for L^{11}, L^{12}, L^{21}, L^{22} from (7.2.25) up to (7.2.28) we then get in (7.2.29)

$$
\begin{aligned}
|R_1| = {} & |L|\,|I+K'_qZ^{\prime(q)}T^{\prime -1}T'Z^{-1}TT^{-1}Z^{(q)}K_q \\
& +K^{\prime(q)}K^{\prime -1}(q-1)\mathcal{H}^{-1}_{q-1}Z'_{q-1}T^{\prime -1}T'Z^{-1}TT^{-1}Z^{(q)}K_q \\
& +K'_qZ^{\prime(q)}T^{\prime -1}T'Z^{-1}TT^{-1}Z_{q-1}\mathcal{H}^{-1}_{q-1}K^{-1}(q-1)K^{(q)} \\
& +K^{\prime(q)}K^{\prime -1}(q)\mathcal{H}^{-1}_{q-1}Z'_{q-1}T^{\prime -1}T'Z^{-1}TT^{-1}Z_{q-1}\mathcal{H}^{-1}_{q-1}K^{-1}(q-1)K^{(q)} \\
& -K^{\prime(q)}K^{-1}(q-1)\mathcal{H}^{-1}_{q-1}K^{\prime -1}(q-1)K^{(q)}|
\end{aligned}
$$

$$
\begin{aligned}
= {} & |L|\,|I-K^{\prime(q)}K^{-1}(q-1)\mathcal{H}^{-1}_{q-1}K^{\prime -1}(q-1)K^{(q)} \\
& +(K'_qZ^{\prime(q)}+K^{\prime(q)}K^{\prime -1}(q-1)\mathcal{H}^{-1}_{q-1}Z'_{q-1}) \\
& \qquad \times Z^{-1}(Z^{(q)}K_q+Z_{q-1}\mathcal{H}^{-1}_{q-1}K^{-1}(q-1)K^{(q)}|
\end{aligned}
$$

$$
\begin{aligned}
= {} & |L|\,|L_{q-1}+(K'_qZ^{\prime(q)}K^{\prime(q)}K^{\prime -1}(q-1)\mathcal{H}^{-1}_{q-1}Z'_{q-1} \\
& \qquad \times Z^{-1}(Z^{(q)}K_q+Z_{q-1}\mathcal{H}^{-1}_{q-1}K^{-1}(q-1)K^{(q)})| \;, \qquad (7.2.30)
\end{aligned}
$$

by letting

$$L_{q-1} = I-K^{\prime(q)}K^{-1}(q-1)\mathcal{H}^{-1}_{q-1}K^{\prime -1}(q-1)K^{(q)} \; . \qquad (7.2.31)$$

Let us now calculate $|L_{q-1}|$ using Lemma 1.5.1, (7.2.14), (7.2.16) and (7.2.20):

$$|L_{q-1}| = |I-K^{\prime(q)}K^{-1}(q-1)\mathcal{H}^{-1}_{q-1}K^{\prime -1}(q-1)K^{(q)}|$$

$$= |\mathcal{H}_{q-1}|^{-1}\begin{vmatrix} I & K^{\prime -1}(q-1)K^{(q)} \\ K^{\prime(q)}K^{-1}(q-1) & \mathcal{H}_{q-1} \end{vmatrix}$$

$$= |\mathcal{H}_{q-1}|^{-1}|I||K^{\prime -1(q-1)}K^{(q)}K^{\prime (q)}K^{-1(q-1)}|$$

$$= |\mathcal{H}_{q-1}|^{-1}$$

$$\times \left| \mathcal{H}_{q-1} - \begin{pmatrix} K_1^{\prime -1} & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & K_{q-1}^{\prime -1} \end{pmatrix} \begin{pmatrix} K_1^{\prime}K_q^{\prime -1} \\ \vdots \\ K_{q-1}^{\prime}K_q^{\prime -1} \end{pmatrix} (K_q^{-1}K_1 \ldots K_q^{-1}K_{q-1}) \begin{pmatrix} K_1^{-1} & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & K_{q-1}^{-1} \end{pmatrix} \right|$$

$$= |\mathcal{H}_{q-1}|^{-1} \left| \begin{pmatrix} H_1^{-1}+H_q^{-1} & \cdots & H_q^{-1} \\ \vdots & & \vdots \\ H_q^{-1} & \cdots & H_{q-1}^{-1}+H_q^{-1} \end{pmatrix} - \begin{pmatrix} (K_qK_q^{\prime})^{-1} & \cdots & (K_qK_q^{\prime})^{-1} \\ \vdots & & \vdots \\ (K_qK_q^{\prime})^{-1} & \cdots & (K_qK_q^{\prime})^{-1} \end{pmatrix} \right|$$

$$= |\mathcal{H}_{q-1}|^{-1} \left| \begin{pmatrix} H_1^{-1}+H_q^{-1} & \cdots & H_q^{-1} \\ \vdots & & \vdots \\ H_q^{-1} & \cdots & H_{q-1}^{-1}+H_q^{-1} \end{pmatrix} - \begin{pmatrix} H_q^{-1} & \cdots & H_q^{-1} \\ \vdots & & \vdots \\ H_q^{-1} & \cdots & H_q^{-1} \end{pmatrix} \right|$$

$$= |\mathcal{H}_{q-1}|^{-1} \begin{vmatrix} H_1^{-1} & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & H_{q-1}^{-1} \end{vmatrix}$$

$$= |\mathcal{H}_{q-1}|^{-1} \prod_{i=1}^{q-1} |H_i|^{-1} \; . \qquad (7.2.32)$$

Now let

$$W_1 = Z^{(q)}K_q + Z_{q-1}\mathcal{H}_{q-1}^{-1}K^{-1(q-1)}K^{(q)} \; , \qquad (7.2.33)$$

then

$$W_1^{\prime} = K^{\prime (q)}K^{\prime -1(q-1)}\mathcal{H}_{q-1}^{-1}Z_{q-1}^{\prime} + K_q^{\prime}Z^{\prime (q)} \; ,$$

since $\mathcal{H}_{q-1}$ is symetric, so $\mathcal{H}_{q-1}^{-1} = \mathcal{H}_{q-1}^{\prime -1}$. Then (7.2.30) becomes

$$|R_1| = |L||L_{q-1}+W_1^{\prime}Z^{-1}W_1| \; . \qquad (7.2.34)$$

From (7.2.24) we see that Z is a symetric matrix, so there exists a matrix Z_1 such that

$$Z^{-1} = Z_1'Z \tag{7.2.35}$$

and

$$|Z_1| = |Z|^{-\frac{1}{2}} . \tag{7.2.36}$$

Substituting (7.2.35) into (7.2.34) and using Lemma 1.5.1 we get

$$|R_1| = |L||L_{q-1}+W_1'Z^{-1}W_1| = |L||L_{q-1}+W_1'Z_1'Z_1W_1|$$

$$= |L|\begin{vmatrix} L_{q-1} & W_1'Z_1' \\ -Z_1W_1 & I \end{vmatrix} = |L||L_{q-1}||I+Z_1W_1L_{q-1}^{-1}W_1'Z_1'| . \tag{7.2.37}$$

Let now

$$Z_1W_1 = W , \tag{7.2.38}$$

so that

$$|R_1| = |L||L_{q-1}||I+WL_{q-1}^{-1}W'| . \tag{7.2.39}$$

Combining (7.2.6), (7.2.9), (7.2.23), (7.2.32)   (7.2.39)

we get

$$|S(X)+(\underline{Y}-M(X)V*)H(\underline{Y}-M(X)V*)'|$$

$$= |S(X)||S(X)|^{-1}|H_{q-1}||Z|\left(\prod_{i=1}^{q-1}|H_i|\right)$$

$$\times|H_{q-1}|^{-1}\left(\prod_{i=1}^{q-1}|H_i|^{-1}\right)|I+WL_{q-1}^{-1}W'|$$

$$= |Z||I+WL_{q-1}^{-1}W'| ,$$

so using (7.2.24) we get

$$|S(X)+(\underline{Y}-M(X)V^*)H(\underline{Y}-M(X)V^*)'|$$
$$= |S(X)+Z_{q-1}H^{-1}_{q-1}Z'_{q-1}||I+WL^{-1}_{q-1}W'| \ . \qquad (7.2.40)$$

Now, from (7.2.1) we see that

$$J(Y \rightarrow Z) = 1 \ , \qquad (7.2.41)$$

from (7.2.33) we see that

$$J(Z^{(q)} \rightarrow W_1) = |K_q|^{-p} = |H_q|^{-\frac{p}{2}} \qquad (7.2.42)$$

and from (7.2.38) by using (7.2.24) and (7.2.36) we see that

$$J(W_1 \rightarrow W) = |Z_1|^{-n_d^*} = |Z|^{\frac{n_d^*}{2}} = |S(X)+Z_{q-1}H^{-1}_{q-1}Z'_{q-1}|^{\frac{n_d^*}{2}} \ . \qquad (7.2.43)$$

So by substituting (7.2.40) into (7.2.3) and using the Jacobian results (7.2.41), (7.2.42) and (7.2.43) we get

$$h(Z_{q-1}, W)dZ_{q-1}dW = \frac{|H|^{\frac{p}{2}} \prod_{j=1}^{p} A_{n-q-j+1}}{|H_q|^{\frac{p}{2}} \prod_{j=1}^{p} A_{n+(n_d^*-1)q-j+1}}$$
$$\times \frac{|S(X)|^{\frac{n-q}{2}} |S(X)+Z_{q-1}H^{-1}_{q-1}Z'_{q-1}|^{\frac{n_d^*}{2}}}{|S(X)+Z_{q-1}H^{-1}_{q-1}Z'_{q-1}|^{\frac{n+q(n_d^*-1)}{2}} |I+WL^{-1}_{q-1}W'|^{\frac{n+q(n_d^*-1)}{2}}} dZ_{q-1}dW \ . \qquad (7.2.44)$$

Using the integration relationship

$$\int |I+WL^{-1}_{q-1}W'|^{-\frac{n+q(n_d^*-1)}{2}} dW = \frac{|L_{q-1}|^{\frac{p}{2}} \prod_{j=1}^{p} A_{n+(n_d^*-1)q-j+1}}{\prod_{j=1}^{n} A_{n+(n_d^*-1)(q-1)-j}} \qquad (7.2.45)$$

(for references see Fraser and Haq (1970)) and substituting for $|L_{q-1}|$ from (7.2.32) we get

$$h(Z_{q-1})dZ_{q-1}$$

$$=\frac{\left(\prod_{i=1}^{q-1}|H_i|^{\frac{p}{2}}\right)|H_q|^{\frac{p}{2}}\prod_{j=1}^{p}A_{n-q-j+1}\left(\prod_{i=1}^{q-1}|H_i|^{-\frac{p}{2}}\right)|\mathcal{H}_{q-1}|^{-\frac{p}{2}}\prod_{j=1}^{p}A_{n+(n_d^*-1)q-j+1}}{|H_q|^{\frac{p}{2}}\prod_{j=1}^{p}A_{n+(n_d^*-1)q-j+1}\prod_{j=1}^{p}A_{n+(n_d^*-1)(q-1)-j}}$$

$$\times\frac{|S(X)|^{\frac{n-q}{2}}}{|S(X)+Z_{q-1}\mathcal{H}_{q-1}^{-1}Z'_{q-1}|^{\frac{n+(n_d^*-1)(q-1)+1}{2}}}dZ_{q-1}$$

$$=\frac{|\mathcal{H}_{q-1}|^{-\frac{p}{2}}\prod_{j=1}^{p}A_{n-q-j+1}|S(X)|^{\frac{n-q}{2}}}{\prod_{j=1}^{p}A_{n+(n_d^*-1)(q-1)-j}}$$

$$\times|S(X)+Z_{q-1}\mathcal{H}_{q-1}^{-1}Z'_{q-1}|^{-\frac{n+(n_d^*-1)(q-1)+1}{2}}dZ_{q-1}\ ,$$

which was to be proved.

7.3 β-expectation Tolerance Region for This Case.

Theorem 7.3.1 Let the error variables $\underset{\sim}{e}^{(i)}$ (i=1, ..., q) have the multivariate normal distributions with $\underset{\sim}{0}$ mean and variance-covariance matrix I, i.e.

$$f(\underset{\sim}{e}_i)d\underset{\sim}{e}_i = (2\pi)^{-\frac{p}{2}}\exp\{-\frac{1}{2}\sum_{j=1}^{p}e_j^{2(i)}\}\prod_{j=1}^{p}de_j^{(i)}\ .$$

Then for central 100β per cent of the variable $\underset{\sim}{Z} = (\underset{\sim}{X}^{(1)}-\underset{\sim}{X}^{(q)}\ldots\underset{\sim}{X}^{(q-1)}-\underset{\sim}{X}^{(q)})$, where $\underset{\sim}{X}^{(i)}$ is $N(\underset{\sim}{\mu}_i, \Sigma)$ for i = 1, ..., q, being sampled, the region

$$Q = \{U_2/U_2 < U_{2_\beta}\} \qquad (7.3.1)$$

is the β-expectation tolerance region, where

$$U_2 = (I + U_3)^{-1}U_3 , \qquad (7.3.2)$$

with

$$U_3 = TZ_{q-1}H_{q-1}^{-1}Z'_{q-1}T' , \qquad (7.3.3)$$

with T such that

$$TT' = T'^{-1}(X)T^{-1}(X) = S^{-1}(X) \ ,(7.3.4)$$

and Z_{q-1} is defined by (7.2.1) and U_{2_β} is the point exceeded with probability 1-β when using the generalized Beta-distribution with $\frac{n_d^*(q-1)}{2}$ and $\frac{n-q}{2}$ degrees of freedom.

Proof:

By using structural model (6.2.1) and Theorem 6.5.1 we see that the prediction distribution of Y is (6.5.7). Then by Lemma 7.2.1 the prediction distribution of Z_{q-1} is (7.2.4). From (7.2.19) we see that H_{q-1}^{-1} is symetric, S(X) is also symetric, so (7.2.4) fulfils the requirements of Lemma 6.2.2, so the distribution of U_2 is generalized Beta-distribution with $\frac{n_d^*(q-1)}{2}$ and $\frac{n-q}{2}$ degrees of freedom. Therefore by (1.4.6) the region Q defined at (7.3.1) is the β-expectation tolerance region, which was to be proved.

7.4 Special Case: $q=2$, $n_d^* = 1$. If $q = 2$ we are dealing with the variable $\underset{\sim}{Z} = \underset{\sim}{X}^{(1)} - \underset{\sim}{X}^{(2)}$, where $\underset{\sim}{X}^{(1)}$ is $N(\underset{\sim}{\mu}_1, \Sigma)$ and $\underset{\sim}{X}^{(2)}$ is $N(\underset{\sim}{\mu}_2, \Sigma)$. Let us note that in this case $n = \sum_{i=1}^{2} n_i = n_1 + n_2$. Then the following theorem holds:

Theorem 7.4.1. Let the error variables $\underset{\sim}{e}_i$ $(i = 1, 2)$ have the multivariate normal distributions with 0 mean and variance-covariance matrix I, i.e.

$$f(\underset{\sim}{e}_i)d\underset{\sim}{e}_i = (2\pi)^{-\frac{p}{2}} \exp\{-\frac{1}{2} \sum_{j=1}^{p} e_j^{2(i)}\} \prod_{j=1}^{p} de_j^{(i)} .$$

Then for central 100β per cent of the variable $\underset{\sim}{Z} = \underset{\sim}{X}^{(1)} - \underset{\sim}{X}^{(2)}$, where $\underset{\sim}{X}^{(1)}$ is $N(\underset{\sim}{\mu}_1, \Sigma)$ and $\underset{\sim}{X}^{(2)}$ is $N(\underset{\sim}{\mu}_2, \Sigma)$, being sampled, the region

$$Q = \{\underset{\sim}{z} \frac{n_1 n_2}{n_2(n_1+1)+n_1(n_2+1)} \underset{\sim}{z}' \left|\frac{S(X)}{n_1+n_2-p-1}\right|^{-1} \underset{\sim}{z} \leq pF_{p;n_1+n_2-p-1;1-\beta}\} \qquad (7.4.1)$$

is β-expectation tolerance region, where (since $n_d^* = 1$)

$$\underset{\sim}{z} = \begin{bmatrix} z_1^{(1)} \\ \vdots \\ z_p^{(1)} \end{bmatrix} = \begin{bmatrix} y_1^{(1)} - y_1^{(2)} - (\bar{x}_1^{(1)} - \bar{x}_1^{(2)}) \\ \vdots \\ y_p^{(1)} - y_p^{(2)} - (\bar{x}_p^{(1)} - \bar{x}_p^{(2)}) \end{bmatrix}$$

$$= \underset{\sim}{y}^{(1)} - \underset{\sim}{y}^{(2)} - (\bar{\underset{\sim}{x}}^{(1)} - \bar{\underset{\sim}{x}}^{(2)}) , \qquad (7.4.2)$$

$$S(X) = T(X)T'(X), \qquad (7.4.3)$$

and $F_{p;n_1+n_2-p-1;1-\beta}$ is the point exceeded with probability $1-\beta$ when using the

F-distribution with p and n_1+n_2-p-1 degrees of freedom.

Proof:

By the Theorem 7.3.1 the prediction distribution for Z_1 (or for the case when q=1 and $n_d^* = 1$) is

$$h(Z_1/X)dZ_1 = \frac{|H_1|^{-\frac{p}{2}} \prod_{j=1}^{p} A_{n_1+n_2-j-1}}{\prod_{j=1}^{p} A_{n_1+n_2-j}}$$

$$\times \frac{|S(X)|^{\frac{n_1+n_2-2}{2}}}{|S(X)+Z_1 H_1^{-1} Z_1'|^{\frac{n_1+n_2-1}{2}}} dZ_1 . \qquad (7.4.4)$$

By (7.2.1) and (7.2.2)

$$Z_1 = (Z^{(1)}) = \begin{pmatrix} z_1^{(1)} \\ \vdots \\ z_p^{(1)} \end{pmatrix}$$

$$= \begin{pmatrix} y_1^{(1)}-y_1^{(2)}-(m_1^{(1)}(X)-m_1^{(2)}(X)) \\ \vdots \\ y_p^{(1)}-y_p^{(2)}-(m_p^{(1)}(X)-m_p^{(2)}(X)) \end{pmatrix} . \qquad (7.4.5)$$

By (6.3.3) for j = 1, ..., p

$$(m_j^{(1)}(X)\, m_j^{(2)}(X)\, t_{j1}(X) \;\ldots\; t_{jj-1}(X))' = N_{j-1}^{-1} D_{j-1}^*(X) \underset{\sim}{x}_j$$

$$= \begin{pmatrix} N^{-1} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} V \\ \underline{D}_{j-1}^*(X) \end{pmatrix} \underset{\sim}{x}_j = \begin{pmatrix} N^{-1} V \underset{\sim}{x}_j \\ \underline{D}_{j-1}^*(X) \underset{\sim}{x}_j \end{pmatrix} ,$$

from which we get

$$\begin{pmatrix} m_j^{(1)}(X) \\ m_j^{(2)}(X) \end{pmatrix} = \begin{pmatrix} n_1^{-1} & 0 \\ 0 & n_2^{-1} \end{pmatrix} \begin{pmatrix} \underset{\sim}{v}_1' \\ \underset{\sim}{v}_2' \end{pmatrix} \underset{\sim}{x}_j$$

$$= \begin{pmatrix} n_1^{-1} & 0 \\ 0 & n_2^{-1} \end{pmatrix} \begin{pmatrix} \underset{\sim}{1}' & \underset{\sim}{0}' \\ \underset{\sim}{0}' & \underset{\sim}{1}' \end{pmatrix} \begin{pmatrix} \underset{\sim}{x}_j^{(1)} \\ \underset{\sim}{x}_j^{(2)} \end{pmatrix}$$

$$= \begin{pmatrix} n_1^{-1} \underset{\sim}{1}' \underset{\sim}{x}_j^{(1)} \\ n_2^{-1} \underset{\sim}{1}' \underset{\sim}{x}_j^{(2)} \end{pmatrix} = \begin{pmatrix} n_1^{-1} \sum_{k=1}^{n_1} x_{jk}^{(1)} \\ n_2^{-1} \sum_{k=1}^{n_2} x_{jk}^{(2)} \end{pmatrix} = \begin{pmatrix} \bar{x}_j^{(1)} \\ \bar{x}_j^{(2)} \end{pmatrix} , \qquad (7.4.6)$$

so (7.4.5) becomes

$$Z_1 = \begin{pmatrix} z_1^{(1)} \\ \vdots \\ z_p^{(1)} \end{pmatrix} = \underset{\sim}{z} = \begin{pmatrix} y_1^{(1)} - y_1^{(2)} - (\bar{x}_1^{(1)} - \bar{x}_1^{(2)}) \\ \vdots \qquad\qquad \vdots \\ y_p^{(1)} - y_p^{(2)} - (\bar{x}_p^{(1)} - \bar{x}_p^{(2)}) \end{pmatrix} = \underset{\sim}{y}^{(1)} - \underset{\sim}{y}_q^{(2)} - (\bar{\underset{\sim}{x}}^{(1)} - \bar{\underset{\sim}{x}}^{(2)}) . \qquad (7.4.7)$$

From (7.2.20)

$$\mathcal{H}_1 = (H_1^{-1} + H_2^{-1}) ,$$

where

$$H_1^{-1} = \left(\frac{n_1+1}{n_1}\right) \text{ and } H_2^{-1} = \left(\frac{n_2+1}{n_2}\right) ,$$

so

$$\mathcal{H}_1 = \left(\frac{n_1+1}{n_1} + \frac{n_2+1}{n_2}\right) = \left(\frac{(n_1+1)n_2 + n_1(n_2+1)}{n_1 n_2}\right) ,$$

from which

$$|\mathcal{H}_1| = \frac{n_2(n_1+1) + n_1(n_2+1)}{n_1 n_2} . \qquad (7.4.8)$$

Let us now evaluate the constant in (7.4.4):

$$\frac{\prod_{j=1}^{n} A_{n_1+n_2-j-1}}{\prod_{j=1}^{p} A_{n_1+n_2-j}} = \frac{A_{n_1+n_2-p-1}}{A_{n_1+n_2-1}} = \frac{2\pi^{\frac{n_1+n_2-p-1}{2}}}{\Gamma\left(\frac{n_1+n_2-p-1}{2}\right)} \, \frac{\Gamma\left(\frac{n_1+n_2-1}{2}\right)}{2\pi^{\frac{n_1+n_2-1}{2}}}$$

$$= \frac{\Gamma\left(\frac{n_1+n_2-1}{2}\right)}{\pi^{\frac{p}{2}}\Gamma\left(\frac{n_1+n_2-p-1}{2}\right)} \; . \qquad (7.4.9)$$

Using (7.4.7), (7.4.8) and (7.4.9) we get

$$h(\underset{\sim}{z}/X)d\underset{\sim}{z} = \left|\frac{n_1n_2}{(n_1+1)n_2+n_1(n_2+1)}\right|^{\frac{p}{2}} \frac{|S(X)|^{-\frac{1}{2}}\Gamma\left(\frac{n_1+n_2-1}{2}\right)}{\pi^{\frac{p}{2}}\Gamma\left(\frac{n_1+n_2-p-1}{2}\right)}$$

$$\times \left|1+\frac{n_1n_2}{n_2(n_1+1)+n_1(n_2+1)}\underset{\sim}{z}'S^{-1}(X)\underset{\sim}{z}\right|^{-\frac{n_1+n_2-1}{2}} d\underset{\sim}{z} \; . \qquad (7.4.10)$$

That is we have that

$$\sqrt{\frac{n_1n_2}{n_2(n_1+1)+n_1(n_2+1)}}\;\underset{\sim}{z} \qquad (7.4.11)$$

is a multivariate T-variable with n_1+n_2-p-1 degrees of freedom and quadratic form S(X). By the Lemma 4.2.1 this means that

$$\frac{n_1n_2}{n_2(n_1+1)+n_1(n_2+1)}\underset{\sim}{z}'S^{-1}(X)\underset{\sim}{z} = \frac{p}{n_1+n_2-p-1}F_{p;n_1n_2-p-1} \; .$$

Then by (1.4.6) the region Q defined at (7.4.1) is the β-expectation tolerance region, which was to be proved.

APPENDIX

We will prove Lemma 1.5.3 now:

<u>Lemma 1.5.3.</u> The following rearrangement of the matrix expression holds.

$$\begin{aligned}
&(B-A)CC'(B-A)'+(D-AE)(D-AE)' \\
&\quad = [A-(BCC'+DE')(CC'+EE')^{-1}](CC'+EE') \\
&\qquad \times [A-(BCC'+DE')(CC'+EE')^{-1}]' \\
&\qquad + (D-BE)(I-E'(CC'+EE')^{-1}E](D-BE)'.
\end{aligned}$$

Proof:

$$\begin{aligned}
&[A-(BCC'+DE')(CC'+EE')^{-1}](CC'+EE')[A-(BCC'+DE')(CC'+EE')^{-1}]' \\
&\quad + (D-BE)[I-E'(CC'+EE')^{-1}E](D-BE)' \\
&=A(CC'+EE')A'-(BCC'+DE')(CC'+EE')^{-1}(CC'+EE')A' \\
&\quad - A(CC'+EE')(CC'+EE')^{-1}(BCC'+DE')' \\
&\quad + (BCC'+DE')(CC'+EE')^{-1}(CC'+EE')(CC'+EE')^{-1}(BCC'+DE')' \\
&\quad + (D-BE)(D-BE)'-(D-BE)E'(CC'+EE')^{-1}E(D-BE)' \\
&=ACC'A'+AEE'A'-BCC'A'-DE'A'-ACC'B'-AED'+BCC'(CC'+EE')^{-1}CC'B' \\
&\quad + DE'(CC'+EE')^{-1}CC'B'+BCC'(CC'+EE')^{-1}ED'+DE'(CC'+EE')ED'+DD' \\
&\quad - DEB'-BED'+BEE'B'-DE'(CC'+EE')^{-1}ED'+BEE'(CC'+EE')^{-1}ED' \\
&\quad + DE'(CC'+EE')^{-1}EE'B'-BEE'(CC'+EE')^{-1}EE'B' \\
&=BCC'B'-ACC'B'-BCC'A'+ACC'A'+DD'-DE'A'-AED'+AEE'A' \\
&\quad - BCC'B'+BCC'(CC'+EE')^{-1}CC'B'+BEE'B'-BEE'(CC'+EE')^{-1}EE'B'
\end{aligned}$$

$$+ BCC'(CC'+EE')^{-1}ED'+BEE'(CC'+EE')^{-1}ED'-BED'$$

$$+ DE'(CC'+EE')^{-1}CC'B'+DE'(CC+EE')^{-1}EE'B'-DE'B'$$

$$=(B-A)CC'(B-A)'+(D-AE)(D-AE)'$$

$$+ B[EE'-EE'(CC'+EE')^{-1}EE'-CC'+CC'(CC'+EE')^{-1}CC']B'$$

$$+ B[CC'(CC'+EE')^{-1}+EE'(CC'+EE')^{-1}-I]ED'$$

$$+ DE'[(CC'+EE')^{-1}CC'+(CC'+EE')^{-1}EE'-I]B'$$

$$=(B-A)CC'(B-A)'+(D-AE)(D-AE)' ;$$

since

$$B[EE'-EE'(CC'+EE')^{-1}EE'-CC'+CC'(CC'+EE')^{-1}CC']B'$$

$$=B[(CC'+EE')(CC'+EE')^{-1}EE'-EE'(CC'+EE')^{-1}EE'$$

$$- CC'(CC'+EE')^{-1}(CC'+EE')+CC'(CC'+EE')^{-1}CC']B'$$

$$=B[(CC'+EE'-EE')(CC'+EE')^{-1}EE'$$

$$- CC'(CC'+EE')^{-1}(CC'+EE'-CC')]B'$$

$$=B[CC'(CC'+EE')^{-1}EE'-CC'(CC'+EE')^{-1}EE']B'$$

$$=0 ;$$

$$B[CC'(CC'+EE')^{-1}+EE'(CC'+EE')^{-1}-I]ED'$$

$$=B[(CC'+EE')(CC'+EE')^{-1}-I]ED' = B[I-I]ED'$$

$$=0$$

and

$$DE'[(CC'+EE')^{-1}CC'+(CC'+EE')^{-1}EE'-I]B'$$

$$=DE'[(CC'+EE')^{-1}(CC'+EE')-I]B' = DE'[I-I]B'$$

$$=0 .$$

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