Temporary Equilibrium with Bankruptcy

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by

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ABSTRACT

The present paper deals with problems arising in economies with credit markets. If the length of credit contracts does not coincide with the length of an agent's planning period and if agents consider bankruptcy as a choice possible for them, then their demand will typically be undefined or, at least, their demand correspondences will not be convex-valued and continuous. These problems are outlined in some examples, and a flexible credit constraint is introduced to overcome them. With such a borrowing constraint, excess demand correspondences are well-defined and well-behaved. Existence of a temporary equilibrium will be established.

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1. **Introduction**

It is well known that borrowing and lending raise several analytical problems for economic model building both on the level of individual decision-making and for the existence of an equilibrium. While commodities can be exchanged without reference to the particular buyer or seller, loans of equal size cannot be considered equivalent, if granted to different people, since different individuals pose different bankruptcy risks depending on their wealth, income, and behavioral characteristics. Hence, loan markets cannot be analyzed in the same way as commodity markets. The possibility that an agent is unable or unwilling to fulfill his debt obligations has to be recognized.

There are two branches in the economic literature on credit. On the one hand, partial equilibrium approaches deal with the behavior of agents on credit markets, mostly from the point of view of a potential lender. Questions addressed in this part of the literature concern typically the extent to which a loan should be granted, if there is uncertainty as to the amount of the repayment, or equilibrium concepts on a credit market, e.g. Jaffee and Modigliani (1969), Hellwig (1977), Stiglitz and Weiss (1981).

On the other hand, problems of default have been investigated in part of the literature on temporary equilibrium. Here the decision-making of a borrower is focused on. In particular, the question has been raised as to how individual behavior will be affected if a borrower is aware of the bankruptcy rule. As Grandmont (1982) in his survey article has pointed out, individual optimizing behavior may be undefined or demand correspondences are
not convex-valued (p. 893). Though several authors have tried to cope with these problems (compare Grandmont (1970), Green (1974), Grandmont and Laroque (1975), Kurz (1976)), there is no generally agreed upon proposal so far as to how to deal with them.

In addition, one important aspect of credit contracts is disregarded in the literature so far. Since individual planning is considered for two periods only, the assumption of no outstanding debt in the first period rules out that old debt can be redeemed by issuing new securities. Besides rendering impossible any kind of Ponzi-strategy, the coincidence of the length of an agent's planning period with the length of the credit contract avoids difficult problems which arise, "when one tries to consider the case where traders live three periods or more. One of them is (once again) related to the existence of bankruptcy. For instance, assume that individuals live three periods and that only short-term loans are allowed. Consider a middle-aged trader, and assume that he contracted a debt in the previous period. If the rule is that he must repay his old debt before being allowed to contract a new one, this may lead to a discontinuous demand function." (Grandmont and Laroque (1975), p. 221).

It is the purpose of this paper to investigate the question of how equilibrium might be possible in an economy where agents may live an arbitrary number of periods, but where only short-term bonds are allowed. Hence, the length of an agent's planning period will be different in general from the length of a credit contract. To deal with the problems arising in such a context, agents are supposed to face a credit constraint. Though credit constraints have been proposed by several authors already, it still lacks an
existence proof in a fairly general framework. In addition, an economic
motivation for such a borrowing constraint is provided. To focus on the
role of this credit constraint for individual decision-making, this paper
follows as closely as possible the one of Grandmont and Laroque (1975), in
particular with respect to the role of banking.

The paper is organized as follows. Since the problems arising from
bankruptcy are not readily available in the literature, Section 2 outlines
them by some examples. Section 3 gives a motivation for the particular credit
constraint used in this paper, while Section 4 presents the complete description
of the economy and states the existence theorem. The paper concludes with
remarks on alternative bankruptcy rules. All proofs are gathered in an
appendix.

2. Individual Behavior and Bankruptcy

Consider an economy with a finite set of commodity markets \( L \), one bond
market and a money market. All markets open in a sequence of periods. There
are only two types of agents, a finite set of consumers, \( I \), and a financial
institution called bank. Trading on the commodity markets takes place between
consumers only. In addition to purchasing and selling commodities, a consumer
may store value by holding money (at a price of one throughout all periods)
or take credit by selling bonds to the bank at a price \( q \in [0,1] \). The sale
of one bond in period \( t \) requires a repayment of one unit of money at the
beginning of period \( t+1 \). Hence the interest rate on bonds equals \( q_t^{-1} - 1 \).
Since consumers exchange bonds with the bank only, the bank is the sole
demander for private bonds and receives all repayments. This implies that
potential bankruptcy losses will concern the bank exclusively and repercussions
of individual bankruptcies need not be considered among private agents.

A typical consumer \( i \in I \) is characterized by his planning horizon \( T^i \), his sequence of endowments \((\omega^i_t)^{T^i}_{t=1}\), a consumption set \( X^i \) equal to \( L \cdot \mathbb{R}_+^{|\cdot|} \), and a preference relation representable by a utility index \( u^i \) on \( X^i \).

In each period \( t \in T^i \), a consumer chooses an action \( a^i_t = (b^i_t, z^i_t, m^i_t) \) where \( b^i_t \in \mathbb{R}_+^L \) denotes the amount of bonds supplied, \( z^i_t \in \mathbb{R}_+^L \) the commodity transactions, and \( m^i_t \in \mathbb{R}_+ \) the money holdings at the end of period \( t \). Let \( p_t^i \in \mathbb{R}_+^L \) be the commodity price vector in period \( t \), then the choice of actions a consumer may take is restricted by the budget constraint,

\[
(1) \quad m^i_{t-1} + b^i_{t-1} = q_t b^i_t + p_t z^i_t + m^i_t,
\]

and by

\[
(2) \quad b^i_t \leq 0, z^i_t + \omega^i_t \geq 0, m^i_t \geq 0.
\]

Obviously, by (2), commodity sales of a consumer during a period \( t \) cannot exceed the endowment \( \omega^i_t \) of this period. Since consumption in period \( t \) \( c^i_t \in \mathbb{R}_+^L \) is related to commodity trades by \( c^i_t = z^i_t + \omega^i_t \), the budget equation can be written equivalently:

\[
(3) \quad m^i_{t-1} + b^i_{t-1} + p_t^i \omega^i_t = q_t b^i_t + p_t c^i_t + m^i_t.
\]

Equation (3) shows on the right-hand side the net wealth of the consumer which comprises money holdings from the previous period, \( m^i_{t-1} \geq 0 \), plus the market value of its non-financial assets, \( p_t^i \omega^i_t \geq 0 \), minus the liabilities, \( b^i_{t-1} \leq 0 \), incurred during the previous period. If an agent cannot meet his credit contract \( b^i_{t-1} \) without resorting to new borrowing, i.e., if \( m^i_{t-1} + b^i_{t-1} + p_t^i \omega^i_t < 0 \) holds, the agent is declared bankrupt and his remaining wealth \( p_t^i \omega^i_t \) is
transferred to the bank to redeem his debt at least partially.\textsuperscript{3} Clearly, in
such a case of insolvency the bank will no longer buy any bonds from this
consumer, and he will be left with the minimum consumption $c_t^i = 0$ for the
future. Hence, for $m_{t-1}^i + b_{t-1}^i + p_t w_t^i < 0$ the consumer is left with the
bankruptcy action $a_{t}^{-i} = (0, -w_t^i, 0)$ for the rest of his planning period.

If the consumer recognizes the working of this bankruptcy rule, however,
this implies in general an unbounded and non-convex intertemporal choice
set, as the following example shows.

**Example 1:** Deleting $i$, the agent's index, throughout the example, let
$L=1$, $T=2$, $w_t = 1$ for $t=1, 2$ and consider a consumer with $m_o + b_o + p_1 > 0$
who is not bankrupt in the present period $1$. If this consumer knows the
bankruptcy rule outlined above, his consumption choice is constrained by

$m_o + b_o + p_1 = q_1 b_1 + p_1 c_1 + m_1$

and

$max\{0, m_1 + b_1 + p_2\} = p_2 c_2 + m_2,$

where new borrowing in the second period has been ruled out, i.e., $b_2 = 0$ has been
assumed. Eliminating $m_1, b_1$ and $m_2$ one easily derives the set of attainable
consumption bundles:

$\{(c_1, c_2) | p_1 c_1 + p_2 c_2 \leq W_o + p_2\}$ for $p_1 c_1 \leq W_o$

$\{(c_1, c_2) | p_1 c_1 + q_1 p_2 c_2 \leq W_o + q_1 p_2\}$ for $W_o < p_1 c_1 \leq W_o + q_1 p_2$

$\{(c_1, c_2) | c_1 \geq 0, c_2 = 0\}$ for $W_o + q_1 p_2 < p_1 c_1,$

where $W_o = m_o + b_o + p_1$ denotes the initial net wealth of the consumer.
Figure 1 gives a graphical representation of this set.

![Graphical representation of a set](image)

FIGURE 1

Obviously, this set is unbounded and non-convex.

It is evident that for all preferences which allow for total substitution of one period's consumption there will be no well-defined demand and supply schedule, if the agent takes into account the working of the bankruptcy rule. Since the worst that can happen to a consumer, if he goes bankrupt, is his minimal consumption \( c_t^i = 0 \) for the future, but the consumption of a particular period can be extended without limits by increasing indebtedness, it will pay a consumer with such preferences to plan to go bankrupt. Furthermore, it should be noted that introducing an arbitrary constraint on borrowing (e.g. \( K + b \geq 0 \) for some \( K > p_2 \)) will lead to a compact choice set and, consequently, to well-defined demand and supply correspondences, but will not guarantee convex-valuedness of these mappings (compare Figure 1).
This problem of a possibly planned bankruptcy has been discussed in the literature by Grandmont (1970), Green (1973), Grandmont and Laroque (1975). In these papers the following proposals to deal with it have been advanced. (i) If there is a disutility entailed by the extent of bankruptcy, then demand correspondences will be non-empty, but still not convex-valued (Green (1973)). (ii) If an additional constraint \( \sum_{t} m_t + b_t + p_{t+1} w_{t+1} \geq 0 \) is introduced and if price expectations exhibit subjective certainty, then demand correspondences are well-defined and convex-valued (Grandmont and Laroque (1975)). (iii) Finally, if a consumption sequence \( (c_t^i)_{t=1}^{T} \) with \( c_t^i > 0 \) for all \( t \in T \) is always strictly preferred to a sequence \( (\bar{c}_t^i)_{t=1}^{T} \) with \( \bar{c}_t^i = 0 \) for some \( t \in T \) and if there are point expectations for prices, then no bankruptcy will be planned and demand correspondences are non-empty and convex-valued. Grandmont and Laroque advocate the latter proposal in their paper (1975) as a "natural answer" to this problem, since "it seems safe to assume that a consumer would find it unbearable to be at the lower boundary of his consumption set, in any one period of his life", if one interprets "consumption sets as representing quantities of consumption goods which are needed for subsistence" (Grandmont and Laroque (1975), p. 220).

In this paper the latter assumption will be taken to avoid planned bankruptcies. Denote by \( S_t = [0,1] \times \mathbb{R}_+^{L} \) the set of prices in period \( t \) with a typical element \( s_t = (q_t, p_t) \).

**Assumption (A):**

(i) The utility of each private agent \( i \) is a concave, on the interior of \( \mathbb{R}_+^{L} \) strictly increasing function \( u_i : \mathbb{R}_+^{L} \to \mathbb{R} \) with the property:
for any sequence \((c_t^i)_{t=1}^{T_i}\) where \(c_t^i > 0\) holds for all \(t \in T_i\) and any sequence \((\tilde{c}_t^i)_{t=1}^{T_i}\) where \(\tilde{c}_t^i = 0\) for some \(t \in T_i\) holds \(u^i(c_1^i, \ldots, c_T^i) > u^i(\tilde{c}_1^i, \ldots, \tilde{c}_T^i)\) is implied.

(ii) The expectations of agent \(i\) are a continuous function \(\bar{Y}_i: \mathbb{S}_i \rightarrow \mathbb{Q}\), where \(\mathbb{Q}\) is a compact subset of \(\mathbb{X}_\mathbb{S}_i\).

While strict monotonicity and concavity of the utility function as well as the continuity of the expectations \(\bar{Y}\) are standard assumptions in temporary equilibrium theory, the restriction of the range of \(\bar{Y}\) to lie in a compact set is required to generate a sufficiently strong intertemporal substitution effect (compare Grandmont (1983) for a thorough motivation and analysis of this effect). As pointed out above, the assumption that any consumption plan with zero consumption in one future period is dominated by every consumption plan with positive consumption in every period rules out the possibility of an agent deciding to go bankrupt in some future period in exchange for sufficiently large consumption before that event.

Assumption (A), however, is only sufficient to guarantee well-behaved demand correspondences if the initial wealth of the consumer \(w_0 = m_0 + b_0 + p_1\) is positive (as assumed so far). A discontinuity can arise, when an agent becomes bankrupt because his initial wealth is negative, but he is still willing to supply bonds to finance his debt payments and some consumption without defaulting in the future according to his expectations. The following example may illustrate this point.

**Example 2:** Let \(L = 1, T = 2, \omega_t = 1\) for \(t = 1, 2\) hold (omitting again the index of the consumer \(i\)) and assume \(u(c_1, c_2) = c_1 \cdot c_2\). Clearly, this utility obeys Assumption (A). Hence, maximizing it subject to the budget constraints of Example 1 will lead to well-defined and well-behaved demand functions as
long as the consumer is not bankrupt, i.e., as $m_o + b_o + p_1 \geq 0$ holds. One can easily derive the following demand schedules for period 1, where $W_o = m_o + b_o + p_1$ is written to shorten notation.

$$b_1 = \begin{cases} \min \left\{ 0, \frac{1}{2q_1} (W_o - q_1 p_2) \right\} & \text{for } W_o \geq 0 \\ 0 & \text{for } W_o < 0 \end{cases}$$

$$c_1 = \begin{cases} \frac{1}{2p_1} \max \left\{ \min \{2, W_o, W_o + p_2\}, W_o + q_1 p_2 \right\} & \text{for } W_o \geq 0 \\ 0 & \text{for } W_o < 0 \end{cases}$$

$$m_1 = \begin{cases} \max \{0, W_o - p_2\} & \text{for } W_o \geq 0 \\ 0 & \text{for } W_o < 0 \end{cases}$$

Obviously these functions are discontinuous for all $(b_o, m_o, p_1)$ where $W_o = 0$ holds.

The discontinuity of the demand functions in Example 2 are due to the strict bankruptcy rule $m_o^i + b_o^i + p_1^i w_1^i < 0$ which forbids an agent to raise new debt to meet his previous debt obligations. Thus, a consumer may very well consider his future income sufficiently high to cover old and new debt, but the resulting demand plan is not feasible because of the bankruptcy law. It is easy to check that there will be no discontinuity of this kind, if the bond supply of an agent were rationed below by a credit constraint which decreases continuously to zero as the net wealth of the agent $m_o^i + b_o^i + p_1^i w_1^i$ approaches zero. This would imply that consumption and borrowing would decline continuously with a shrinking budget set. The following section will provide a reasoning for such a flexible credit constraint.
3. **Credit Rationing**

So far the credit relationship has been treated from the point of view of a borrower only. This section turns to the lender's problem: how much credit to grant a particular borrower. Clearly, the answer to this question will depend on the prevailing bond price (interest rate) and the bankruptcy rule. To simplify the argument it will be assumed without loss of generality that the lender (in this paper the bank only) is risk-neutral.

Knowing the bankruptcy rule which states that an agent goes bankrupt, if his net wealth $\bar{W}_t = m_{t-1} + b_{t-1} + p_t w_t$ becomes negative, the bank knows that it will receive the agent's remaining wealth $\bar{W}_t = m_{t-1} + p_t w_t$ in this case, since it is the role creditor. Thus, the profit the bank will earn from buying $b_t$ bonds ($b_t \geq 0$) at the price $q_t$ is given by

$$
(4) \quad \pi_t(b_t, \bar{W}_{t+1}) = \min\{b_t, \bar{W}_{t+1}\} - q_t b_t.
$$

Observe that bonds have to be repaid in the next period. By (4) it obviously pays the bank to grant a customer credit up to his wealth $\bar{W}_{t+1}$ at the beginning of the following period, since this is the maximal amount the borrower can redeem when the repayment of the debt is due. Hence, the expected wealth of a borrower constitutes a constraint on borrowing, if a lender's expectations are subjectively certain. Furthermore, forming his expectations a lender can rely on information only which is currently available. In particular, the actual net wealth of a borrower is an important element of information. Consequently, one can expect that the credit line which a lender will impose on a particular borrower varies with the current net wealth $\bar{W}_t$ of the respective borrower.

This observation that a lender will set a credit limit to a borrower which depends on the borrower's actual net wealth carries over to the case where
a lender has subjectively uncertain expectations concerning the borrower's future wealth. In this case as well the bank extends its loans to a borrower only up to the point where its expected profit is maximized. Again this credit constraint will change with varying actual net wealth of the borrower, if the bank's expectations are conditional on this variable. The following example will make the latter point more precise.

**Example 3:** Suppose the bank expects the wealth \( \bar{W} \) at the beginning of the following period to be distributed uniformly on the interval \((\alpha W, \beta W)\), \(\alpha > 0\), \(\beta > 0\). Then the probability function can be described by the following distribution function \( F(\bar{W}, W) = \frac{\bar{W} - \alpha W}{(\beta - \alpha) W} \). For a given purchase of bonds \( b \), a given bond price \( q \), and actual net wealth of the borrower \( W \), the expected profit function can be written:

\[
E\pi(b, q, W) = \int_{\alpha W}^{\beta W} \pi(b, \bar{W}) dF(\bar{W}, W)
\]

\[
= - \frac{b^2}{2(\beta - \alpha) W} + \frac{\beta}{\beta - \alpha} q b - \frac{\alpha^2 W}{2(\beta - \alpha)}.
\]

One can easily check that \( E\pi(b, q, W) \) is a concave function of \( b \) which attains a maximum at \( b^* = (\beta - q(\beta - \alpha)) W \). Clearly, for all \( b < b^* \) the bank can raise its expected profit by extending its credit to that borrower. Hence, \( (\beta - q(\beta - \alpha)) W \) constitutes the credit limit the bank will enforce upon the borrower for a certain level of his net wealth \( W \).

The preceding argument shows that a creditor will rationally restrict borrowing of an individual agent. Furthermore, this credit constraint will depend on the net wealth of the borrower. Hence, an agent \( i \) who wants to borrow faces a constraint \( \rho_i^1(W_i^1) \), his credit limit, besides his budget constraint, namely
(5) \[ b_t^i + \rho_t^i(W_t^i) \geq 0. \]

In the case of Example 3, \( \rho_t^i(W_t^i) = (\beta - q_t(\beta-\alpha))u_t^i \) forms the credit constraint of the borrower. Obviously, \( \rho_t^i \) is a continuous, increasing function of the agent's net wealth \( W_t^i \), and \( \rho_t^i(0) = 0 \) holds true for the example.

The following section provides an existence proof for an equilibrium in an economy where private agents maximize their utility subject to a credit limit (5) in addition to their budget constraint. Sufficient assumptions with respect to the credit constraints will simply be: (i) continuity, (ii) monotonicity, (iii) concavity, and (iv) no credit in the case of bankruptcy, i.e. for \( W_t^i \leq 0 \). The following assumption states these conditions formally.

**Assumption (B):**

The credit constraint of each private agent \( i \in I \) is a continuous, strictly increasing, and concave function \( \rho_t^i : \mathbb{R}_+ \to \mathbb{R}_+ \) with the property \( \rho_t^i(0) = 0 \) for all periods \( t \in T_i \).

To illustrate the working of the credit constraint (5), this section concludes by reconsidering Example 2. Introducing a borrowing constraint which satisfies Assumption (B) will solve the discontinuity problem.

**Example 4:** Let \( L = 1, T = 2, \omega_t = 1 \) for \( t = 1,2 \) hold as in Example 2 and consider an agent who maximizes \( u(c_1, c_2) = c_1 \cdot c_2 \) subject to \( m_t + b_t + p_t = q_1 b_1 + p_1 c_1 + m_1 \), \( b_1 + \rho_1(m_t + b_t + p_t) \geq 0 \), and \( \max\{0, m_t + b_t + p_t\} \geq p_2 c_2 \) by choosing \( (b_1, c_1, m_1, c_2) \in \mathbb{R}_+ \times \mathbb{R}_+^3 \), if he is not bankrupt, i.e., if \( m_t + b_t + p_t \geq 0 \) holds. In case of bankruptcy his choice is restricted to the vector \( (0,0,0,0,0) \).

Obviously, the only difference to Example 2 is the credit limit. Now denote \( m_t + b_t + p_t \) by \( W_t \), then the following demand and supply functions are easily derived.
\[ b_1 = \begin{cases} \min\{0, \max\left\{ \frac{1}{2q_1} (W_o - q_1 p_2), -\rho_1 (W_o) \right\} \} & \text{for } W_o \geq 0 \\ 0 & \text{otherwise} \end{cases} \]

\[ c_1 = \begin{cases} \frac{1}{2p_1} \max\{\min\{2W_o, W_o + p_2\}, \min\{W_o + q_1 p_2, 2(W_o + q_1 \rho_1(W_o))\}\} & \text{for } W_o \geq 0 \\ 0 & \text{otherwise} \end{cases} \]

\[ m_1 = \begin{cases} \max\{0, \frac{1}{2}(W_o - p_2)\} & \text{for } W_o \geq 0 \\ 0 & \text{otherwise} \end{cases} \]

It is not difficult to check that these functions are continuous for all prices, since the choice set of the agent shrinks continuously to the origin with declining \( W_o \) by virtue of the credit constraint.

4. **Existence of an Equilibrium**

It is the purpose of this section to show that existence of an equilibrium can be guaranteed, if consumer's choice is subject to a credit constraint as developed in the previous sections. This holds true, even if a consumer's planning horizon extends beyond the maturity date of a bond. In the first subsection some lemmas on the choice behavior of private agents are proved, while subsection 4.2 gives a short description of the activity of the (central) bank. Finally, the existence theorem is stated.

4.1 **Consumers**

According to the analysis in the previous sections, a consumer \( i \in I \) can choose in each period \( t \) an action \( a^i_t = (b^i_t, z^i_t, m^i_t) \) which is an element of the set \( A^i_t = \{ a^i_t \in \Re^3 \mid b^i_t \leq 0, z^i_t + \omega^i_t \geq 0, m^i_t \geq 0 \} \). Recall that a market signal in period \( t \), \( s_t = (q_t, p_t) \) consists of a bond price and a commodity price
vector and is an element of the set $S_t = [0,1] \times R^+_t$. Now denote by $D^i_t$ the subset of $A^i_{t-1} \times S_t$ such that the agent remains solvent, i.e.

$$D^i_t = \{(a^i_{t-1}, s_t) \in A^i_{t-1} \times S_t | m^i_{t-1} + b^i_{t-1} + p_t \omega^i_t \geq 0 \},$$

then the correspondence $\tilde{A}^i_t : D^i_t \rightarrow A^i_t$, where

$$\tilde{A}^i_t(a^i_{t-1}, s_t) = \{a^i_t \in A^i_t | m^i_t + b^i_{t-1} = q_t b^i_t + p_t z^i_t + m^i_t,$$

$$b^i_t + p_t (m^i_{t-1} + b^i_{t-1} + p_t \omega_t) \geq 0 \},$$

is obviously well defined. One can easily check that for $(a^i_{t-1}, s_t) \notin D^i_t$ the set $\tilde{A}^i_t(a^i_{t-1}, s_t)$ is empty. In this latter case the agent is bankrupt and there remains no action which is consistent with his budget, since by the credit constraint additional borrowing is excluded. As explained before, the only choice of a bankrupt agent is a vector $(0, -\omega^i_t, 0)$ where his real wealth is sold and transferred to the lending bank to redeem his debt at least partially. Thus, the choice correspondence $A^i_t : A^i_{t-1} \times S_t \rightarrow A^i_t$ can be defined as follows:

$$A^i_t(a^i_{t-1}, s_t) = \begin{cases} \tilde{A}^i_t(a^i_{t-1}, s_t) & \text{for } (a^i_{t-1}, s_t) \in D^i_t \\ \{(0, -\omega^i_t, 0)\} & \text{otherwise} \end{cases}$$

The following lemma which is proved in the appendix states the properties of the choice correspondence.

**Lemma 1:** The choice correspondence $A^i_t$ is non-empty, convex- and compact-valued and continuous for all $(a^i_{t-1}, s_t) \in A^i_{t-1} \times S_t$. 
The problem of the consumer can now be described as maximizing his utility \( u^i \) subject to \( a^i_t \in A^i_t(a^i_{t-1}, s^i_t) \) for all \( t \in T^i \). Of course \( b^i_{T_i} = 0 \) should hold for the last period. Since the action in the first period \( a^i_1 \) where the existence of an equilibrium shall be established is the only relevant component of the solution to this problem, it is convenient to work with the expected utility index. Define, therefore, the expected utility

\[
v^i: A^i_1 \times S_1 \to \mathbb{R}
\]

as

\[
v^i(a^i_1, s^i_1) = \max \{ u^i(z^i_1 + w^i_1, \ldots, z^i_T + w^i_T) | a^i_T \in A^i_T(a^i_{T-1}, v^i_t(s^i_t)) \text{ for all } t \geq 2 \},
\]

where \( v^i_t(s^i_t) \) denotes the projection of the expectation function \( v^i_t(s^i_t) \) on the components referring to period \( t \). The properties of this expected utility index are again gathered in a lemma. Let

\[
C = \{(a^i_1, s^i_1) \in A^i_1 \times S_1 | m^i_1 + b^i_1 + v^i_2(a^i_1)w^i_2 > 0, z^i_1 \gg -w^i_1 \}
\]

denote the set of actions and market signals of the first period where the agent expects to have a strictly positive net wealth in the second period and is left with a strictly positive consumption vector in period 1.

**Lemma 2:**

(i) \( v^i(a^i_1, s^i_1) \) is a continuous function on \( A^i_1 \times S_1 \).

(ii) \( v^i(a^i_1, s^i_1) \) is a strictly increasing and concave function of \( a^i_1 \) on \( C \).

The proof is contained in the appendix.

Clearly, it is equivalent to the original optimization problem of the consumer, to maximize the expected utility index \( v^i \) subject to \( a^i_1 \in A^i_1(a^i_0, s^i_1) \). The excess demand correspondence of the consumer \( i \) in period 1 can be defined now as the mapping

\[
\alpha^i: A^i_0 \times S_1 \to A^i_1,
\]
\[ \alpha^i(a^i_o, s^i_1) = \arg \max \{ v^i(a^i_1, s^i_1) | a^i_1 \in A^i_1(s^i_0, s^i_1) \}. \]

The following lemma states the properties of the excess demand correspondence \( \alpha^i \). It is proved in the appendix.

**Lemma 3:**

(i) The correspondence \( \alpha^i \) is non-empty, compact- and convex-valued and u.h.c. on \( A^i_0 \times S^i_1 \).

(ii) For all \( a^i_1 \in \alpha^i(a^i_o, s^i_1) \) \( m^i_1 + b^i_1 \) is bounded below.

(iii) For all \( s^i_1 \in S^i_1 \) where \( q^i_1 = 0 \) holds: \( a^i_1 \in \alpha^i(a^i_o, s^i_1) \) implies \( b^i_1 = 0 \).

(iv) For all \( s^i_1 \in S^i_1 \) where \( q^i_1 = 1 \) holds: if \( (a^i_o, s^i_1) \in D^i_1 \), then \( a^i_1 \in \alpha^i(a^i_o, s^i_1) \) and \( 0 \leq b^i_1 \leq m^i_1 + b^i_1 + p^i_1(m^i_0 + b^i_1 + p^i_1 \omega^i_1) \) imply \( (b^i_1 - \delta, z^i_1, m^i_1 + \delta) \in \alpha^i(a^i_o, s^i_1) \).

(v) Let \( a^i_1 \) be a sequence where \( \Vert a^i_1 \Vert \to \infty \) and let \( T^i > 2 \) and \( \omega^i \gg 0 \) hold, then for all \( a^i_1 \in \alpha^i(a^i_o, s^i_1) \) \( \Vert a^i_1 \Vert \to \infty \) follows.

(vi) Let \( a^i_1 \) be a sequence where \( p^i_{1\omega} \to 0 \) for some \( \omega \in L \) and let \( m^i_1 + b^i_1 > 0 \), then for all \( a^i_1 \in \alpha^i(a^i_o, s^i_1) \) \( \Vert a^i_1 \Vert \to \infty \) follows.

Most of the statements of Lemma 3 are easy to understand. Boundedness of \( m^i_1 + b^i_1 \) follows, since otherwise the consumer will be bankrupt in the second period (observe that \( v^i_2(s^i_1) \cdot \omega^i_2 \) is bounded by Assumption (A)). A consumption plan where \( c^i_2 = 0 \), however, can be dominated quite easily (Assumption (A)). Statements (iii) and (iv) are obvious, since no bonds will be supplied at a bond price of zero, while the consumer is indifferent between any action which leaves his consumption and \( m^i_1 + b^i_1 \) unchanged, if the bond price equals the money
price. Given that the expected utility function is strictly increasing in its arguments, statements (v) and (vi) are standard boundary results. Obviously these properties carry over to the aggregate private excess demand, if it is guaranteed that there are consumers in the economy with the required properties.

4.2 The Bank

In this paper operations of the bank will be restricted to buying bonds and to supplying money. The bank decides how many bonds $B^i_1 \geq 0$ to buy from the private sector within the frame given by the credit constraints $\rho^i_q$ which the bank assigns to the agents individually. The associated money supply $M^i_1 \geq 0$ then follows by the budget constraint of the bank.\(^6\)

Denote by $M^o_o \geq 0$ the stock of money at the beginning of the first period and by $B^o_o \geq 0$ the stock of outstanding debt. Since money and bond markets had to be cleared in the previous period, $M^o_o = \sum_{i \in I} m^i_o$ and $B^o_o = -\sum_{i \in I} b^i_o$ must hold. As explained above, all outstanding debt $B^o_o$ is due at the beginning of the first period, but the redemption of the debt may fall short of the face value $B^o_o$, since some agents may default. The repayment $R^1_1$ the bank will receive, hence, is a function of the prevailing market signal $S^1_1$. Define this function $R^1_1: S^1_1 \rightarrow R^+_+$ by

\[
R^1_1(s^1_1) = \sum_{i \in I} \min\{-b^i_o, m^i_o + p^1_1 a^i_1\}.
\]

Clearly, $R^1_1(s^1_1) \leq B^o_o$ must be true. The change of the money stock $(M^1_1 - M^o_o)$ which takes place in period 1 by way of the operations of the bank, therefore, must be equal to the net change of outstanding debt $(B^1_1 - B^o_o)$ minus the profit.
of the bank \([(1-q_i)B_1 - (B_0 - R_1(s_1))\)]. The profit consists of interest earnings \((1-q_i)B_1\) minus the default losses \((B_0 - R_1(s_1))\). Hence, one can write the budget constraint of the bank as

\[(7) \quad M_1 = M_0 + q_i B_1 - R_1(s_1)\].

Since the bank is inactive in all commodity markets, its action \(a^b_i\) is a vector of the form \((B_1, 0, -M_1)\). The choice of this action is subject to the budget constraint (7). It will be assumed in this paper\(^7\) that the bank fixes its bond demand at a level \(\beta_1 \geq 0\). The money supply, then, is determined by \(\mu_1(s_1, \beta_1) = M_0 + q_i \beta_1 - R_1(s_1)\). Thus, the excess demand correspondence of the bank can be written as \(\alpha^b_1: S_1 \times \mathbb{R}_+ \rightarrow \mathbb{R}^{L+2}\),

\[\alpha^b_1(s_1, \beta_1) = \{\beta_1, 0, -\mu_1(s_1, \beta_1)\}\].

In this case, the correspondence is a continuous function on \(s_1\) for all \(\beta_1 \geq 0\). Furthermore, \(\alpha^b_1\) is bounded below by \((0, 0, -(M_0 + \beta_1))\).

4.3 Existence

Denote the aggregate excess demand correspondence of the economy as

\(\alpha_1: S_1 \times \mathbb{R}_+ \rightarrow \mathbb{R}^{L+2}\),

\[\alpha_1(s_1, \beta_1) = \sum_{i \in I} \alpha^i_1(a^i_0, s_1) + \alpha^b_1(s_1, \beta_1)\].

By the budget constraints (1) and (7), Walras' Law holds true. Now, an equilibrium of this economy relative to a bank policy \(\beta_1 \geq 0\) can be defined as follows:
**Definition:** An equilibrium relative to a bank policy $\beta_1$ is defined as a market signal $a_1^*$ and an allocation $(a_1^{i*})_{i \in I}$, $a_1^{b*}$ such that

(i) $\sum_{i \in I} a_1^{i*} + a_1^{b*} = 0$,

(ii) $a_1^{i*} \in \alpha_1^i(a_o^i, s_1^*)$ for all $i \in I$,

(iii) $a_1^{b*} \in \alpha_1^b(s_1^*, \beta_1)$

hold.

Before the existence theorem can be stated, a final assumption has to be made. Denote by $I^* = \{ i \in I | m_o^i + b_o^i > 0 \}$ the set of private agents with strictly positive financial wealth. It is obvious that these agents will not be bankrupt regardless of the prevailing market signal.

**Assumption (C):**

(i) The set $I^*$ is non-empty.

(ii) There is an agent $i \in I^*$ with $T^i \geq 2$ and $\omega^i > 0$.

By Assumption (C), the existence of agents is guaranteed who display the properties required for the application of Lemma 3. Assumption (C.ii) could be weakened considerably at the cost of notational complexity, since it suffices that there is at least one consumer for each commodity $l \in L$ who is endowed with a strictly positive amount of this commodity and who plans for at least two periods ahead. For such a consumer can benefit from a price increase of the respective commodity by increasing his demand for money.

**Theorem:** Given Assumptions (A), (B), (C), there exists an equilibrium for every bank policy $0 \leq \beta_1 \leq \sum_{i \in I^*} \rho_1^i(m_o^i + b_o^i)$.

**Proof:** See the appendix.
This theorem demonstrates that there is an equilibrium in an economy where private agents may go bankrupt and where the planning horizon of agents extends beyond the maturity date of credit contracts. It seems quite natural that the bank policy parameter $\beta_1$ is restricted to values smaller than the total amount of credit which the bank is willing to supply to the private sector regardless of the market signal. For otherwise there may remain an excess demand on the bond market even for an interest rate of zero ($q_1 = 1$), simply because consumers cannot borrow in excess of their credit constraints which are set by the bank.

5. **Final Remarks**

The paper will be concluded by some remarks on bankruptcy rules. According to the bankruptcy rule of this paper an agent defaults, if his repayment obligations exceed his actual wealth. This means he cannot repay an old debt by borrowing anew. Though this rule is used in most of the literature, it may be considered unduly strong. In particular, the difficulties with discontinuous demand correspondences are consequences of this rule.

Considering a similar multi-period planning problem of a consumer, Grandmont (1983), therefore, assumed a consumer to default only if his repayment obligations were not covered by the present value of his future endowments. Since the evaluation of the future endowments as well as the discounting is performed with the consumer's own expectations, the decision whether to declare himself bankrupt or not is completely subjective. Furthermore, this rule implies that a bankrupt agent still obtains credit up to the level of the present value of his future wealth (as stated by himself). Clearly, such a bankruptcy rule avoids the kind of problems described in
Section 2 of this paper, but does not correspond to any actual default rule.

Of course, some consideration of future income in the bankruptcy rule seems desirable, but the present value of this future income should be evaluated with price and interest expectations of the creditor. In this case, however, the same problems as with the bankruptcy rule of this paper arise, because the expectations with respect to future prices and interest rates may differ between debtor and creditor.

Here again, a constraint on borrowing can solve the problem. The basic property of such a constraint is its dependence on the observed net wealth, since with declining net wealth borrowing will shrink as well and, hence, consumption will approach the lower boundary of the consumer's consumption set. If the choice of an agent's action is subject to such a flexible credit constraint, the existence of an equilibrium can be established even in an economy where bankruptcy can occur and agents take into account this possibility.
FOOTNOTES

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1Compare, for example, Grandmont and Laroque (1975, p. 220), Kurz (1976) and Milne (1980). Milne (1980) proves existence of an equilibrium for an economy consisting of two agents only.

2It should be pointed out that firms could be introduced into such a model without any problem except complicating the exposition (compare Eichberger (1984)).

3This is the bankruptcy concept mostly used in the literature. Compare the last section of this paper for a short discussion of potential alternatives.

4Observe that the assumption of point expectations is essential. It is easy to construct examples where even for a Cobb-Douglas utility function convex-valued correspondences for demand cannot be guaranteed, if expectations are uncertain. Clearly, a plan which implies bankruptcy with probability one is ruled out, but an action might be chosen which allows for bankruptcy with a positive probability.

5One can think of changing the last period's choice set by imposing a credit constraint \( \rho^i_T(\cdot) = 0 \) to guarantee \( h^i_T = 0 \).

6Since the bank is inactive on commodity markets, its choice variables are interdependent by the budget constraint.
7. Here only the case where the bank chooses a bond demand will be considered. As in Grandmont and Laroque (1975) one could as well consider a policy of the bank trying to peg the interest rate. There are, however, problems with respect to the ability of the bank to control $M_1$, the money supply, directly, since it has to cover losses from bankruptcies of private agents.
REFERENCES


APPENDIX

A.1 Proof of Lemma 1: Let's drop the individual index $i$.

1. Obviously, the statement of the lemma is true for all $(a_{t-1}, s_t) \notin D_t$, since $A_t$ is a constant function there.

2. For $(a_{t-1}, s_t) \in D_t$ and $W = m_{t-1} + b_{t-1} + p_t w_t > 0$, again it is easy to check that the statement of the lemma holds.

3. Now suppose that $(a_{t-1}, s_t) \in D_t$ and $W = 0$ holds, then $\rho(0) = 0$ implies $0 \geq b_t \geq 0$ and consequently $p_t(z_t + \omega_t) + m_t = 0$ must hold, i.e.

$$A_t(a_{t-1}, s_t) = \{(0, -\omega_t, 0)\}$$

in this case. Thus $A_t(a_{t-1}, s_t)$ is trivially non-empty, compact- and convex-valued. Again, continuity is obvious for every sequence $(a_{t-1}^\nu, s_t^\nu)$ which stays in $(A_{t-1} \times S_t) \cap D_t$ for large $\nu$. If $(a_{t-1}^\nu, s_t^\nu) \in D_t$ for large $\nu$ holds, however, then for every sequence $a_t^\nu \in A_t(a_{t-1}^\nu, s_t^\nu)$ \[0 \geq b_t^\nu \geq -\rho(W^\nu)\] holds and $b_t^\nu \to 0$ must be true. Hence, $p_t(z_t^\nu + \omega_t^\nu) + m_t^\nu = W^\nu - q_t^\nu b_t^\nu \to 0$ implies $m_t^\nu \to 0$ and $z_t^\nu \to -\omega_t$. Thus, $a_t^\nu \to (0, -\omega_t, 0)$ must hold and $A_t$ is u.h.c. in this case. Obviously, $A_t$ is $A$-h.c. as well in this case, since $(0, -\omega_t, W^\nu)$ converges to $(0, -\omega_t, 0)$.

A.2 Proof of Lemma 2: Let's drop the index $i$ again.

(i) To see that $v(a_1, s_1)$ is continuous on $A_1 \times S_1$, define $v_t(a_1, a_2, \ldots, a_t, s_t)$

$$= \max\{u(z_1 + \omega_1, \ldots, z_t + \omega_t) \mid a_t \in A_t(a_{t-1}, s_t)\}$$

for all $t > 1$. By the Bellman principle of optimality $v_t(a_1, a_2, \ldots, a_t, s_t)$

$$= \max\{v_{t+1}(a_1, \ldots, a_{t+1}, s_{t+1}) \mid a_{t+1} \in A_{t+1}(a_{t}, s_{t+1})\}$$

must be true.

Now $v_{T-1}(a_1, \ldots, a_{T-1}, s_1)$ is a well-defined and continuous function, since $u$ is a continuous function maximized on a non-empty, compact-valued and continuous correspondence $A_{T-1}$ by the maximum theorem.
Let $\mathcal{V}_{t+1}$ be a continuous function, then $\mathcal{V}_t$ is a well-defined and continuous function as well by the maximum theorem and Lemma 1. Thus, by induction $\mathcal{V}_t$ is a continuous function for all $t$. Obviously, $\mathcal{V}(a^*_1, s^*_1) = \mathcal{V}_1(a^*_1, s^*_1)$.

(ii) Let $a^*_2, \ldots, a^*_T$ be an optimal sequence of actions maximizing $\mathcal{V}(z_1 + w_1, \ldots, z_T + w_T)$ subject to $a^*_t \in \mathcal{A}_t(a^*_{t-1}, \mathcal{V}_t(s_i))$ for all $t \geq 2$ and a given $a^*_1$. First observe that $a^*_2, \ldots, a^*_T$ cannot be a bankruptcy plan, since by $(a^*_1, s^*_1) \in \mathcal{C}$ a plan with strictly positive consumption stream is possible, which dominates each bankruptcy plan by assumption A.

1. Monotonicity. Pick any $a^*_1' \geq a^*_1$. If there is a $t \in \mathcal{I}$ where $z^*_t > z^*_t$ holds, then it follows by assumption A immediately that $\mathcal{V}(a^*_1', s^*_1) > \mathcal{V}(a^*_1, s^*_1)$ holds. If $b^*_1 > b_1$ or $m^*_1 > m_1$ holds, then $w^*_2 = m^*_1 + b^*_1 + \eta^*_2$ holds. Hence, it is possible to increase consumption in the second period without changing consumption in any other future period and this implies $\mathcal{V}(a^*_1', s^*_1) > \mathcal{V}(a^*_1, s^*_1)$, since $\mathcal{V}$ is strictly increasing in its arguments.

2. Concavity. It is easy to check that for any $a^*_t \in \mathcal{A}_t(a^*_{t-1}, \mathcal{V}_t(s_i))$, any $a^*_t' \in \mathcal{A}_t(a^*_{t-1}', \mathcal{V}_t(s_i))$, and any $\lambda \in [0, 1]$ $a^*_t + (1-\lambda)a^*_t' \in \mathcal{A}_t(\lambda a^*_{t-1} + (1-\lambda)a^*_{t-1}')$, $\mathcal{V}_t(s_i)$ holds (observe that by assumption B, the functions $p^*_t$ are concave). Let $a^*_2, \ldots, a^*_T$ be an optimal plan for some $a^*_1$ and $a^*_2, \ldots, a^*_T$ the optimal plan for $a^*_1'$. Now clearly $a^*_1 = \lambda a^*_1 + (1-\lambda)a^*_1'$ is a feasible plan for $a^*_1 = \lambda a^*_1 + (1-\lambda)a^*_1'$. Hence, $\mathcal{V}(a^*_1, s^*_1)$ $\geq \mathcal{V}(z^*_1 + w_1, z^*_2 + w_2, \ldots, z^*_T + w_T)$ must hold and by concavity of $\mathcal{U}$, $\mathcal{U}(z^*_1 + w_1, \ldots, z^*_T + w_T) \geq \lambda \mathcal{U}(z^*_1 + w_1, \ldots, z^*_T + w_T) + (1-\lambda)\mathcal{U}(z^*_1 + w_1, \ldots, z^*_T + w_T)$.
must be true. Since \( v(a_i, s_i) = u(z_i + \omega_i, z_i^*, \ldots, z_i^*) \) and \\
v(a_i', s_i) = u(z_i' + \omega_i, z_i'^*, \ldots, z_i'^*) \) holds, \( v \) is concave in \( a_i \).

A.3 Proof of Lemma 3: Let's drop the index \( i \)

(i) For \((a_o, s_1)\) such that \( m_o + b_o + p_1 \omega_1 \leq 0 \) holds true, the statement is trivial. Otherwise, there is always a plan with positive consumption in every period. Hence a planned bankruptcy is always suboptimal according to assumption A. The statement then follows immediately from the maximum theorem (Green and Heller (1981), 49) and Lemma 1.

(ii) From \( m_1 + b_1 + \gamma_2 (s_1) \omega_2 \geq 0 \) holds (a planned bankruptcy is never optimal). By assumption A(ii), \( \gamma_2 (s_1) \) is bounded, hence \( m_1 + b_1 \) is bounded below.

(iii) This is obvious, since \( v(a_1, s_1) \) is strictly increasing in \( b_1 \). Hence there is no bond supply for \( q_1 = 0 \).

(iv) For \( q_1 = 1 \) the price of money and bond is equal. Since for future choices only \( m_1 + b_1 \) matters, every feasible plan which leaves \( m_1 + b_1 \) unchanged is equivalent.

(v) Suppose the statement to be false. Then there exists a converging subsequence (retain the notation) \( a_1^v \to a_1^0 = (b_1^0, z_1^0, m_1^0) \). Since by assumption A(ii) \( \gamma(s_1^v) \) is contained in a compact set, there is again a converging subsequence \( \gamma(s_1^v) \to \gamma^0 \). Now define \( \bar{v}(a_1) = \max \{ u(z_1 + \omega_1, \ldots, z_T + \omega_T) | a_t \in A_t(a_{t-1}, \gamma^0), t \geq 2 \} \) where \( \gamma^0 \) is the projection of \( \gamma^0 \) on its \( t^{th} \) components. It is easy to check by repeating the proof of Lemma 2 that all the statements of Lemma 2 hold for \( \bar{v}(a_1) \) as well.

Furthermore, \( \lim_{v \to \infty} v(a_1^v, s_1^v) = \bar{v}(a_1^0) \) holds obviously. Since \( \| s_1^v \| \to \infty \) holds, there must be at least one \( \lambda \in L \) with \( p^{\lambda}_v \to \infty \). Denote by \( e_\lambda \) the
\(\tilde{a}_1^\nu = (b_1^\nu, z_1^\nu - \frac{1}{\nu} e_{1}, m_1^\nu + 1)\).

It is easy to check that \(\tilde{a}_1^\nu \in \tilde{A}(a_o^\nu, s_1^\nu)\) for large \(\nu\) holds, since \(\omega_{\tilde{\omega}} > 0\) is assumed. Hence \(a_1^\nu \in \alpha(a_o^\nu, s_1^\nu)\) implies \(\nu(a_1^\nu, s_1^\nu) \geq \nu(\tilde{a}_1^\nu, s_1^\nu)\)

and by continuity \(\tilde{\nu}(a_1^\nu) \geq \tilde{\nu}(b_1^\nu, z_1^\nu, m_1^\nu + 1))\) in contradiction to the strict monotonicity of \(\tilde{\nu}\) in \(m_1\).

(vi) Suppose again the statement to be false and \(a_1^\nu \to a_1^0\). Since \(m_o + b_0 > 0\) is assumed, there must be at least one component of \((z_1^0, m_1^0)\) strictly greater than zero. Without loss of generality, let \(m_1^0 > 0\) hold.

For every \(\varepsilon > 0\) the sequence of actions \(\tilde{a}_1^\nu = (b_1^\nu, z_1^\nu + \varepsilon e_{\tilde{\omega}}, m_1^\nu - \varepsilon)\) then satisfies \(\tilde{a}_1^\nu \in \tilde{A}(a_o^\nu, s_1^\nu)\) for large \(\nu\). Hence, \(\nu(a_1^\nu, s_1^\nu) \geq \nu(\tilde{a}_1^\nu, s_1^\nu)\) must hold for large \(\nu\) and \(\tilde{\nu}(a_1^\nu) \geq \tilde{\nu}((b_1^\nu, z_1^\nu + \varepsilon e_{\tilde{\omega}}, m_1^\nu))\) by continuity.

This contradicts, however, the monotonicity of \(\nu\) (Lemma 2).

A.4 Proof of the Theorem: To simplify notation, drop the index 1 referring to the present period. Let \(a = (b, z, m)\) where \(b = \sum_{i \in I} b_i + B\), \(z = \sum_{i \in I} z_i\), \(m = \sum_{i \in I} m_i - M\) and denote by \(\alpha(s, \beta) = \sum_{i \in I} \alpha_i^i(a_o^i, s) + \alpha^b(s, \beta)\) the aggregate excess demand correspondence. It is easy to check that \(\alpha(s, \beta)\) is a compact-, convex-valued, and u.h.c. correspondence (Lemma 3(i)) and that Walras' Law, \((s, 1)a = 0\), holds for every \(a \in \alpha(s, \beta)\). Furthermore, for every \(a \in \alpha(s, \beta)\) \(z\) and \(m + b\) are bounded from below (Lemma 3(ii)). Therefore, \(\|a^\nu\| \to \infty\) follows for every sequence \(a^\nu \in \alpha(s^\nu, \beta)\), if \(p_{\tilde{\omega}} \to \infty\) or \(p_{\tilde{\omega}} \to 0\) holds for some \(\tilde{\omega}\) (Lemma 3(v) and (vi) and assumption C). Finally, for every \(s \in S\) where \(q = 0\) holds \(a \in \alpha(s, \beta)\) implies \(b = \beta\), and for every \(s \in S\) where \(q = 1\) holds \(a \in \alpha(s, \beta)\) implies \(\tilde{a} = (\tilde{b}, \tilde{z}, \tilde{m}) \in \alpha(s, \beta)\) as well where \(\tilde{b} = \beta - \sum_{i \in I} \rho_i(m_o^i + b_o^i) - p_{\tilde{\omega}}\) and \(\tilde{m} = -q \tilde{b} - ps\) holds (Lemma 3(iii) and (iv)).
Now let \((\delta^\gamma)^\infty\) be an increasing sequence of real numbers with \(\delta^1 > 1\) and denote by \(S^\gamma = [0,1] \times \left[\frac{1}{\delta^\gamma}, \delta^\gamma\right]^L\). Since \(\alpha\) is u.h.c. there is a compact set \(A^\gamma\) containing \(\alpha(s,\beta)\) for all \(s \in S^\gamma\). Define the correspondence \(\gamma: A^\gamma \to S^\gamma\) by \(\gamma(a) = \{s \in S^\gamma | (s,1)a \geq (s',1)a\} \) for all \(s \in S^\gamma\). Obviously, \(\gamma\) is compact- and convex-valued, and u.h.c. Hence, the correspondence \(\alpha \times \gamma: S^\gamma \times A^\gamma \to S^\gamma \times A^\gamma\) has a fix point, say \((a^\gamma,s^\gamma)\), which satisfies \(a^\gamma \in \alpha(s^\gamma,\beta)\) and \((s^\gamma,1)a^\gamma \geq (s,1)a^\gamma\) for all \(s \in S^\gamma\).

Clearly, there is \(\bar{s} \in S^1\) with \(\bar{q} < 1\) and by Walras' Law \((s^\gamma,1)a^\gamma = 0\) must hold. This implies \((\bar{s},1)a^\gamma \leq 0\) for all \(\gamma\). Hence, \(z^\gamma\) has to be bounded. Furthermore, \(\bar{q}b^\gamma + m^\gamma\) is bounded from above which implies together with the lower bound on \(b^\gamma + m^\gamma\) that \(b^\gamma\) and \(m^\gamma\) are bounded. Thus, there is a converging subsequence (same notation) \(a^\gamma \to a^0\), and \(s^\gamma \to s^0 \in S\), for otherwise \(\|a^\gamma\|\) would have to diverge. Since \(\alpha\) is u.h.c., \(a^0 \in \alpha(s^0,\beta)\) must hold and one can easily check that \((s^0,1)a^0 \geq (s,1)a^0\) for all \(s \in S\).

Since \(\mathbb{R}_+^L\) is an open set, \(z^0 = 0\) must hold. Now suppose \(b^0 < 0\), then \(q^0 = 0\) must hold contradicting \(b = \beta \geq 0\) in this case. If \(b^0 > 0\) holds, then \(q^0 = 1\) must be true. But in this case there is \(\tilde{a} \in \alpha(s^0,\beta)\) where \(\tilde{b} \leq 0\) holds. Thus, there is an equilibrium allocation for the bond market in this case as well. Finally, \(m^0 = 0\) follows by Walras' Law.