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Arthur Robson
Myrna Holtz Wooders

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Citation of this paper:
RESEARCH REPORT 9409

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by

Arthur Robson
and
Myrna Holtz Wooders

May 1994

Department of Economics
Social Science Centre
University of Western Ontario
London, Ontario, CANADA
N6A 5C2
On the Evolution of the Distribution of Income*

Arthur Robson
Department of Economics
University of Western Ontario

Myrna Holtz Wooders
Department of Economics
University of Toronto

November, 1993; revised May, 1994

Abstract

This paper presents a social selection argument based on population growth to bolster marginal productivity theory. Consider an economy with a single output produced according to constant returns to scale from a number of different types of labor. Suppose each type of labor is reproduced according to a constant returns to scale technology from that labor itself and from the amount of the output devoted to it. Output is distributed across the input production processes according to some arbitrary norm. Any norm which fails to induce convergence to maximal balanced growth is "growth dominated" in the sense that the population it induces can eventually be overwhelmed. It is shown that on the maximal balanced growth path, the norm must divide output over the types of labor according to their marginal products. An example investigates the effects of exogenous social mobility. For this example it is shown that one type may be paid less than another, but it may also maximize the balanced growth rate to pay the poorer type more than its marginal product and the richer type less. This provides a novel rationale for redistributive taxation.

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*This paper benefited from the comments of participants in the 1994 NBER Decentralization Conference at the University of Toronto. Both authors also thank the Social Sciences and Humanities Research Council of Canada for its support.
1. Introduction

Marginal productivity pricing is the cornerstone of the neoclassical theory of the distribution of income. Earlier writers intimated that such pricing had normative content. J. B. Clark [9], for example, is well known for advancing the position that factors "should" be paid according to their marginal products. Marginal productivity theory, based on profit-maximizing behavior by a competitive firm, is now taken to be purely positive. Most recently, the theory has found expression in the marginal productivity pricing of individuals (as in Ostroy [26]) and in the view that, in a competitive economy, persons receive their marginal contributions to groups (see Wooders [36], for example).

Offered here is a complementary theory of the distribution of income. This theory is anchored in dynamic biological and social efficiency. It is postulated that a society which does not grow at the maximal rate will eventually be swamped by faster-growing societies, that is, it will be "growth dominated." Thus societies which are actually observed will tend to be those encouraging maximal growth.

The central model here considers, in particular, an economy with a single output produced according to a constant returns to scale production function from a number of types of labor. Each type of labor is reproduced, in turn, according to a constant returns to scale technology from that type of labor itself and from the amount of the output given to it. Suppose output is distributed across all these reproduction processes according to some arbitrary norm. Such a norm is taken to be a cultural characteristic, enforced by some mechanism sufficient to induce individual compliance. If there exists another norm which eventually induces arbitrarily larger levels of every type of labor, the first norm is said to be growth dominated. It follows from a turnpike theorem that any norm which fails to induce convergence to the maximal balanced growth path, the "von Neumann ray," is growth dominated. It is also shown that the von Neumann ray entails the distribution of income in accordance with marginal productivity. This property is dubbed the "platinum rule," since it represents a generalization of the golden rule of neoclassical growth theory. Altogether, any norm failing to induce the income distribution derived from marginal productivity pricing in the long run is antithetical to social survival.

To illustrate the ideas here, consider an example from biology— an "economy" with two species, ants and aphids.¹ The ants collect plants which they consume themselves

¹Hölldobler and Wilson [18], for example, write of ants that "colony design is effectively a problem in economics," maximizing "energy" rather than wealth.
and which they also give to the aphids for their consumption. The aphids feed on the phloem sap of plants and excrete a complex mixture of nutrients. The mixture, called "honeydew," contains some of the nutrients in the sap and also some nutrients added by the aphids in the process of consumption.\(^2\) The honeydew is highly beneficial to aphid-keeping ants. The ants render a hygienic service to their symbionts by constantly removing the sugary material. The ants also provide the aphids with protection from the weather by building shelters and seem to protect the aphids from predators. Altogether the aphids develop larger, more stable populations when tended by ants.

Consider the ants and the aphids as two types of labor. Together they produce energy for sustaining and reproducing the population. The main result of the current paper is that, if the maximal balanced co-evolutionary growth rate for the two species is attained, then each species receives energy in accordance with its marginal product.\(^3\)

The prototypical human example for the present model is the caste system that formerly prevailed in India. The nearly complete lack of social mobility mandated under this system renders it an especially clear case. The central form of the present model applies literally to an economy organized into various strictly hereditary castes whose members are each permitted to perform only certain economic roles.

Although the present model is not Marxist, it is worth noting that Marx believed that classes in modern societies have their origins in castes. Further, he seems to have had in mind a very similar biological analogy to that inspiring the present model. Marx [22, pp. 154-155] states:

Manufacture, in fact, produces the skill of the specialized labourer, by reproducing, and systematically driving to an extreme within the workshop, the naturally developed differentiation of trades, which it found ready to hand in society at large. On the other hand, the conversion of subdivided work into the life-calling of an individual, corresponds to the tendency shown by earlier societies, to make trades hereditary; either to petrify them into castes, or

\(^2\)The honeydew can also be collected and eaten by man. Hölldobler and Wilson [18] write that the "manna" given to the Israelities was almost certainly honeydew, and some Middle Eastern peoples still gather the material, called "man" in Arabic.

\(^3\)To predict that maximal balanced growth would actually occur, it would be necessary to consider the "incentives" of individual ants and aphids. If these conflicted with maximal growth, the majority view among modern biologists is that the interests of the individual would prevail. That is, for non-human species, individuals and not groups should be the "units of selection." For human societies, which are the main concern here, group selection is more plausible, since human societies utilize enforcement mechanisms. These mechanisms may make it incentive compatible for individuals to adhere to even arbitrary norms. (Mayr [23, especially Ch. 5], expresses the view that group selection may be more important for human beings than for other species.) Roughly speaking, that is, it is more reasonable to invoke group selection to choose among various Pareto-ranked Nash equilibria than it would be to invoke it to "resolve" the one-shot prisoner's dilemma. Indeed, group selection in this sense is a central theme of the present paper.
whenever particular historical conditions give rise to individual variability incompatible with a caste system, to ossify them into guilds. Castes and guilds arise from the action of the same natural law that regulates the differentiation of plants and animals into species and subspecies, except that, when a certain degree of development has been reached, the hereditary nature of castes and the exclusiveness of guilds are decreed as a social law.

Marx apparently did not believe that the fundamental reason for the existence of social classes is biological inheritance. Some modern Marxists stress the role of the educational system in perpetuating classes. (See Bowles [6], for example.) A mechanism which seems realistic, and which is at once more obviously analogous to genetic inheritance, is parental inculcation, nurture rather than nature. A parental nurture mechanism has received attention in models of cultural evolution. Cavalli-Sforza and Feldman [8, especially Ch. 2] discuss such “vertical transmission” of preferences toward political parties in the U.S. More recently, Boyd and Richerson [7, p. 8, for example] also mention parental inculcation as a particular form of their theory of “dual” genetic and cultural inheritance.

The key assumption of the central form of the model here is the full heritability of class. Although parental inculcation seems to be the leading candidate, the mechanism underlying this heritability need not be made explicit, but is assumed to operate in a fashion analogous to genetic inheritance.

As an empirical matter, the implied restrictions on social mobility seem immediately plausible for feudal societies, but there is also good evidence that such imperfect mobility characterizes even modern industrial societies. Although the U.S. is generally thought to be characterized by a high degree of mobility, a widely cited study by Blau and Duncan [4, p. 36], for example, includes the following statement, with reference to tables of “mobility ratios”:

These three tables bring the main characteristics of the American occupational structure into high relief. First, occupational inheritance is in all cases greater than expected on the assumption of independence; note the consistently high values on the major diagonal. Second, social mobility is nevertheless pervasive, as revealed by the large number of underlined values off the diagonal. Third, upward mobility (to the left of the diagonal) is more prevalent than downward mobility (to the right), and short-distance movements occur more often than long-distance ones.

4 A relevant observation supporting the independence even of castes from genetic inheritance is that the castes of India share essentially the same gene pool. (See Lumsden and Wilson [21, p. 151], for example.)
In the tables referred to, Blau and Duncan use 17 different occupations, ranging from "self-employed professionals" to "farm laborers." For example, Table 2.5 [4, p. 32] demonstrates the effect of a man's occupation on his son's occupation by presenting ratios of observed frequencies in each case to those expected on the basis of independence. In particular, the frequency with which the sons of self-employed professionals are themselves in this occupational category is 11.7 times greater than that to be expected if there were independence.

Given that type ("class") is inherited, it is natural to assume that the inherent fertilities of these types differ. Such differential inherent fertility need not be a heritable physiological or psychological trait, but rather is taken to reflect the differing costs of raising children who will enter different occupations. For example, it seems plausible that the cost of producing children who will be self-employed professionals typically exceeds that of producing those who will be farm laborers. In a sense, then, the fertility functions here are reduced forms incorporating both biological fertility and the economic cost of children.\(^5\) (The data also suggest the need for the model to accommodate differential fertility. Differential fertility is further discussed in Section 6.) Although the model thus permits non-biological interpretations of inheritance of type and of differential fertility, it remains biological in the key respect that social selection is based on population.\(^6\)

Despite the strong tendency for children to follow in the footsteps of their parents, significant mobility does exist in modern societies. An example is presented here to extend the present model to allow such social mobility, in the presence of differential fertility. This example shows that balanced growth may involve an inverse observed relationship between fertility and income. Moreover, consistent with Blau and Duncan, there may be more upward mobility than downward mobility. Finally, maximal balanced growth may dictate that the poorer type of labor should be paid more than its marginal product while the richer type should be paid less, as is the essence of redistributive taxation.

To summarize the present paper, Section 2 presents an example of a simple norm for distributing income that, in the long run, induces maximal balanced growth. Section 3 presents the general model and the two main results. The first of these is the turnpike theorem showing that norms failing to induce convergence to the von Neumann ray are growth dominated. The second shows that, along the von Neumann ray, any norm must allocate output to each factor in accordance with its marginal productivity, as is dubbed the platinum rule. Section 4 shows by means of an example that direct competitive factor pricing may lead to dynamic instability. Section 5 presents the example which extends the present model to include exogenous social mobility. The issue of the relationship

\(^5\)In a very simplified form, the present formulation thus incorporates the theory of human capital.

\(^6\)A more general form of social selection might be based also on the endowment of natural resources and on the levels of various forms of capital, including weapons.
between fertility and income is further discussed in Section 6. Section 7 discusses the implications of considering the presence of various types of capital as well as labor and relates the present model to neoclassical growth theory. Also discussed in this section is the work of Hansson and Stuart [17], who argue that preferences should be selected to exhibit a lack of impatience in a related model. Section 8 concludes by speculating that the model might bear on the growth implications of serfdom. It considers, in particular, the emancipation of the Russian serfs following the Crimean War. If serfs were paid less than their marginal product under serfdom and if emancipation meant they would be paid this marginal product, the long-run growth rate might, indeed, be expected to rise after emancipation.

2. A Key Example

The flavor of the present approach can be conveyed by a simple example. This illuminates, in particular, the intimate relationship between growth and the distribution of income. Consider an economy with two types of labor. Suppose that the single output is produced by means of a constant returns to scale Cobb-Douglas production function so that

\[ Y^t = (L_1^t)^{1/3}(L_2^t)^{2/3}, \]

where \( Y^t \) is the amount of output, \( L_1^t \) is the quantity of the first type of labor and \( L_2^t \) is the quantity of the second type of labor, all at time \( t \).

Suppose that each type of labor grows according to a biological reproduction function also exhibiting constant returns to scale, where the two inputs are the previous level of that type of labor and the amount of output paid to that type of labor. Constant returns to scale means that doubling the amount of any type of labor, for example, while keeping its per capita income constant, results in a doubling of the number of descendants of that type of that type. Suppose that the second type of labor is possibly more inherently fertile than is the first. As is discussed in the Introduction, this differential inherent fertility need not have a biological basis but is a natural economic consequence of inheritance of type. For the sake of simplicity, take the reproduction functions as Cobb-Douglas, where

\[ L_1^{t+1} = \alpha(L_1^t W_1^t)^{1/2}, \quad \text{and} \quad L_2^{t+1} = k\alpha(L_2^t W_2^t)^{1/2}, \]

where \( \alpha > 0 \) and \( k \) is at least 1. Here \( W_i^t \) is the total amount of output or income distributed to labor of type \( i \), for \( i = 1, 2 \).

Suppose the distribution of income is determined by a percentage norm as:

\[ W_1^t = \theta Y^t \quad \text{and} \quad W_2^t = (1 - \theta) Y^t, \]
for some \( \theta \in [0, 1] \). It follows readily that
\[
Y^{t+1}/Y^t = \alpha k^{2/3} \theta^{1/6} (1 - \theta)^{1/3},
\]
a constant. The above formulation also implies that
\[
L_2^{t+1}/L_1^{t+1} = k((1 - \theta)/\theta)^{1/2} (L_2^t/L_1^t)^{1/2}.
\]
It follows that, starting from any initial population with positive levels of factors of each type, the ratio \( L_2^t/L_1^t \) converges to \( k^2((1 - \theta)/\theta) \) as \( t \) grows large, so growth is eventually "balanced." In this sense the factor ratio is globally stable for any value of \( \theta \). Further, since \( Y^t/L_1^t \) converges to \( k^{4/3}((1 - \theta)/\theta)^{2/3} \) and \( Y^t/L_2^t \) converges to \( k^{-2/3}(\theta/(1 - \theta))^{1/3} \), as \( t \) grows large, the factors ultimately share the constant growth rate of output.

The growth rate of output is clearly maximized at \( \theta = 1/3 \), illustrating the main theoretical result of this paper: Maximizing the balanced growth rate entails marginal productivity pricing of the two types of labor.\(^7\) Indeed, in this example, the "platinum rule" norm turns out to maximize the growth rate of output in every short run as well.

It is of key interest to examine the behavior of the factor rewards. First note that the present model induces inequality to the extent that the two types of labor have differing inherent fertility, as reflected in the parameter \( k \). The per capita factor rewards are determined as
\[
w_1 = \theta Y^t/L_1^t \to k^{4/3} \theta^{1/3} (1 - \theta)^{2/3} \quad \text{and} \quad w_2 = (1 - \theta) Y^t/L_2^t \to k^{-2/3} \theta^{1/3} (1 - \theta)^{2/3},
\]
so that \( w_1/w_2 \to k^2 \geq 1 \), given \( k \geq 1 \). This implies an inverse relationship between inherent fertility and per capita income, a conclusion with a rather Malthusian ring to it. However, while there may be asymmetries between inherent fertility, as expressed in the reproduction functions, the definition of balanced growth requires that actual fertilities cannot differ across the two types of labor.

The effect of income distribution on growth can now be demonstrated. In the present example, the choice of \( \theta = 1/3 \) not only maximizes the balanced growth rate but also maximizes both per capita factor rewards \( w_1 \) and \( w_2 \). Suppose, for example, that in an attempt to favor the low-wage type 2 workers, \( \theta \) is chosen to be strictly less than 1/3. This will mean that the low-wage type will always be paid more than its marginal product and the high-wage type will always be paid less. However, the long-run ratio of type 2's to type 1's, as measured by \( k^2((1 - \theta)/\theta) \), will rise, and both \( w_1 \) and \( w_2 \) will fall. Moreover, there will be no effect on the ratio of the high wage to the low wage, given as \( w_1/w_2 = k^2 \). Thus, in any reasonable sense, the long-run distribution of income is worse than it was before. It is only in the short-run that choosing a share value

\(^7\)The maximal balanced growth path, as defined in the next section, is able to be generated in this way for some \( \theta \), so that the procedure of optimizing over \( \theta \in [0, 1] \) is valid.
greater than that dictated by marginal productivity will raise the per capita income of the corresponding type. Note that the above argument would apply equally well to redistribution in favor of the richer class. This observation is the basis of the discussion of Russian serfdom in Section 8.

3. The Platinum Rule

The first task of the present section is to present the general model. It is assumed that there are $K$ types of labor given at times $t = 0, 1, 2, \ldots$ by the vector $N^t = (N^t_1, \ldots, N^t_K) \in R^K_+$. There is a single output represented at time $t$ by

$$Y^t = F(N^t_1, \ldots, N^t_K), \text{ where } F : R^K_+ \to R_+.$$  (3.1)

It is assumed that

$$F \in C^0(R^K_+), \text{ and } F \mid_{R^K_+} \in C^1(R^K_+);$$  (3.2)

that is, $F$ is continuous on the non-negative orthant and continuously differentiable on the interior of this orthant. Production of output monotonically increases with each input in the sense that:

$$F_k(N) > 0 \text{ for all } N \in R^K_+ \text{ and for } k = 1, \ldots, K.$$  (3.3)

It simplifies matters to suppose that all inputs are essential as factors of production for the single output, ensuring that maximal balanced growth occurs at an interior point. Hence–

$$F(N) = 0 \text{ for all } N \in R^K_+ \text{ such that } N_k = 0 \text{ for some } k.$$  (3.4)

The production function satisfies constant returns to scale. Finally, it is concave and strictly quasi-concave except where this is precluded by (3.4). Equivalently:

$$F(kN) = kF(N) \text{ for all } N \in R^K_+ \text{ and } k \geq 0.$$  (3.5)

If $N^{(1)} \in R^K_+ \text{ and } N^{(2)} \in R^K_+ \sim \{0\}$ are not proportional, and $t \in (0, 1)$, then $F(tN^{(1)} + (1 - t)N^{(2)}) > tF(N^{(1)}) + (1 - t)F(N^{(2)})$.

Output is distributed as an input over the reproduction processes. The amount of the output devoted to the reproduction of each input $k = 1, \ldots, K$ at time $t = 0, 1, 2, \ldots$ is

$$W^t_k \geq 0, k = 1, \ldots, K, \text{ where } \sum_{k=1}^K W^t_k \leq F(N^t).$$  (3.6)

---

8 The present results extend in a straightforward way to the case of a number of outputs.

9 Cobb-Douglas production functions, in particular, satisfy conditions (3.2), (3.3), (3.4) and (3.5).
The production of each input takes place from the previous level of the input itself and from this allocation of the output. Hence

\[ N_{k+1} = \varphi_k(N_k, W_k), \]  
where \( \varphi_k : R^2_+ \to R_+ \), \( k = 1, \ldots, K \), \( t = 0, 1, \ldots \). \hfill (3.7)

Reproduction of each type of labor is assumed to satisfy similar properties to those of the production of output. The production function \( \varphi_k \) for the \( k \)th factor satisfies

\[ \varphi_k \in C^0(R^2_+), \quad \varphi_k \mid_{R_+^2} \in C^1(R^2_+), \quad \text{for } k = 1, \ldots, K. \] \hfill (3.8)

The production of each input is monotonic in the corresponding inputs; more specifically, for \( k = 1, \ldots, K \),

\[ \varphi_{kN}(N, W_k) > 0, \quad \text{and} \quad \varphi_{kW}(N_k, W) > 0, \quad \text{for all } N_k \text{ and } W_k > 0. \] \hfill (3.9)

It seems reasonable to suppose that any input itself and the output are essential as factors of production for the next period’s production of that input. That is,

\[ \varphi_k(0, W_k) = \varphi_k(N_k, 0) = 0, \quad \text{for all } N_k \text{ and } W_k \geq 0. \] \hfill (3.10)

It is assumed that the reproduction function satisfies constant returns to scale. That is, a scaling up of the amount of any type of labor, with the per capita output to that factor held constant, is assumed to result in the same scaling up of offspring. Reproduction functions are also assumed to be concave and strictly quasi-concave except where this is precluded by (3.10). Equivalently: \hfill (3.11)

\[ \varphi_k(kN_k, kW_k) = k \varphi_k(N_k, W_k), \quad \text{for all } N_k, W_k \text{ and } k \geq 0. \]

If \((N(1), W(1)) \in R^2_+ \) and \((N(2), W(2)) \in R^2_+ \sim \{0\} \) are not proportional, then

\[ t \varphi_k(tN(1) + (1-t)N(2), tW(1) + (1-t)W(2)) > t \varphi_k(N(1), W(1)) + (1-t) \varphi_k(N(2), W(2)), \quad \text{for all } t \in (0, 1). \]

Different types of labor are permitted to have different reproduction functions. Along any balanced growth path, however, whether maximal or not, realized fertility must be equal across types, as, indeed, these are the defining conditions of balance. Greater inherent fertility of one type as compared to another, as reflected in a reproduction function which is uniformly greater, then must entail a lower per capita income. Thus these two properties of the example of Section 2 are quite general.

\[^{10}\text{It is natural here to identify the per capita production function for offspring with an individual one-period utility or "felicity" function. This is an increasing concave function of per capita income. Robson [30], takes an analogous approach to the evolution of expected (and non-expected) utility.}\]

\[^{11}\text{Cobb-Douglas production functions, again, satisfy conditions (3.8), (3.9), (3.10) and (3.11).}\]
It may also be illuminating to consider the case where all the reproduction functions are identical. In this case, it is immediate that, on any balanced growth path, per capita incomes are equated across all types. That is, the complete immobility of the classes does not prevent the attainment of the usual (short-run) efficiency condition. It is irrelevant for determining relative incomes that one factor might, in any sense, be more important than another in the production of output. Instead, the relative incomes of the types are determined, in the long run, by supply conditions.

The following definitions are required:

**Definition 1.** Define the technology set for the above model as
\[ T = \{(N^t, N^{t+1}) \mid N^t_{k+} = \varphi_k(N^t_k, W^k_t), \text{for } W^k_t \geq 0, \sum_{k=1}^K W^k_t \leq F(N^t), k = 1, ..., K\}. \]

**Definition 2.** A maximal balanced growth path (or von Neumann ray) and maximal growth rate are a vector \(v \in \Delta^{K-1}\) and a real number \(\gamma > 0\), respectively, such that \((v, \gamma v) \in T\) and \(\gamma \geq \lambda(x, y) = \max\{\lambda \mid y \geq \lambda x\}\) for all \((x, y) \in T, x \neq 0\).

Such a ray and an associated growth rate exist and are unique. In addition, there exist shadow prices at which "profits" are uniquely maximized on the von Neumann ray. In the following, \(\Delta^{K-1}\) represents the unit simplex in \(R^K\) and \(\text{Int}(\Delta^{K-1})\) represents its relative interior.

**Theorem 1.** Given the above model, there exists a unique von Neumann ray represented by a unique set of factor proportions \(v \in \text{Int}(\Delta^{K-1})\) and a maximal balanced growth rate \(\gamma > 0\). There is also a set of shadow prices \(p \in \text{Int}(\Delta^{K-1})\) such that \(p \cdot (y - \gamma x) < 0\), for all \((x, y) \in T \sim \{0\}\) whenever \((x, y)\) is not proportional to \((v, \gamma v)\).

**Proof.** See the Appendix.

The following definition introduces the criterion for social inefficiency.

**Definition 3.** Consider a path given by \(N^t\) where \((N^t, N^{t+1}) \in T\), for \(t = 0, 1, 2, \ldots\). This path is growth dominated (in every component) if and only if there exists another path \(\tilde{N}^t\), where \(\tilde{N}^0 = N^0\) and \((\tilde{N}^t, \tilde{N}^{t+1}) \in T\) for \(t = 0, 1, 2, \ldots\), such that \(N^t_k / \tilde{N}^t_k \to 0\), as \(t \to \infty\), for \(k = 1, \ldots, K\).

The next result is a "turnpike theorem" adapted to the present context.

**Theorem 2.** Consider the model given above. Suppose that a norm has the property that the associated feasible path \(N^t\) does not induce proportions converging to the von Neumann ray. More precisely, suppose that, if
\[ x^t_k = N^t_k / \sum_{k=1}^K N^t_k, \text{ for } k = 1, \ldots, K, \text{ then } x^t \not\rightarrow v^t \text{ as } t \to \infty. \]
Suppose also that \( N^0 \in R^\mathbb{K}_{++} \). It then follows that such a norm is growth dominated.\(^{12}\)

**Proof.** See the Appendix.

Finally, the "platinum rule" is defined and shown to hold on the von Neumann ray.

**Definition 4.** The **platinum rule** is the norm under which output is distributed over the reproduction of the inputs in accordance with the marginal productivities of these inputs.

**Theorem 3.** On the von Neumann ray, income is distributed across the types of inputs in accordance with the platinum rule. That is,

\[
W^t_k = F_k(N^t_k)n^t_k, \quad k = 1, \ldots, K; \quad t = 0, 1, 2, \ldots
\]

**Proof.** See the Appendix.

In the case where all the reproduction functions are identical, this characterization of the maximal growth rate reduces simply to the condition that all marginal products are equal.

Thus, there is a sense in which any norm not inducing the distribution of income derived from marginal productivity is antithetical to social survival: Any society in which inputs are not rewarded according to their marginal productivities in the long run can be outperformed by another.

**4. The Possible Instability of Competitive Pricing**

The previous section showed that failure to generate marginal productivity pricing in the long run implies that a norm is growth dominated. A natural hypothesis is then that marginal productivity pricing would have desirable properties if applied in the short run. As the following example shows, this need not be true.

Suppose that there are two types of labor and that the production function has constant elasticity of substitution, \( \sigma \in (0, 1) \), so that, in the notation of (3.1),

\[
Y^t = F(N^t_1, N^t_2) = [(N^t_1)^\rho + (N^t_2)^\rho]^{1/\rho},
\]

where \( \rho = (\sigma - 1)/\sigma \in (-\infty, 0) \), for \( t = 0, 1, 2, \ldots \)

\(^{12}\)A norm which induces sufficiently slow convergence to the von Neumann ray may also be growth dominated. On the other hand, a norm which converges to the von Neumann ray in one period cannot be growth dominated in any component.
It is readily verified that this satisfies (3.2), (3.3), (3.4) and (3.5). In the notation of (3.7), take the reproduction functions to be Cobb-Douglas:

\[ N_1^{t+1} = \alpha(N_1^tW_1^t)^{1/2} \quad \text{and} \quad N_2^{t+1} = \alpha(N_2^tW_2^t)^{1/2}, \alpha > 0, t = 0, 1, 2, ..., \]

satisfying (3.8), (3.9), (3.10) and (3.11). Suppose the total payments to each factor are determined by the contemporaneous marginal products, so that, for \( t = 0, 1, 2, ... \),

\[
W_1^t = F_1(N^t)N_1^t = \left\{ (N_1^t)^{\rho} + (N_2^t)^{\rho} \right\}^{(1-\rho)/\rho} \cdot (N_1^t)^{\rho} \\
\text{and} \quad W_2^t = F_2(N^t)N_2^t = \left\{ (N_2^t)^{\rho} + (N_2^t)^{\rho} \right\}^{(1-\rho)/\rho} \cdot (N_2^t)^{\rho}.
\]

Defining the factor ratio at time \( t = 0, 1, 2, ... \), as \( r^t = N_2^t/N_1^t \), it follows that

\[ r^t = (r^{t-1})^{(1+\rho)/2}, \text{ so that } \ln r^t = \left( \frac{1+\rho}{2} \right)^t \cdot \ln r^0. \]

If \( \rho > -3 \), or, equivalently, if \( \sigma \in (1/4, 1) \) this difference equation yields \( r^t \to 1 \), for all \( r^0 \in (0, \infty) \). If \( \rho \leq -3 \), or, equivalently, if \( \sigma \in (0, 1/4] \), then the equation generates undamped or explosive oscillations. It is readily checked that the maximal balanced growth path is \( v = (1/2, 1/2) \). If \( \sigma \in (1/4, 1) \), the competitive pricing norm therefore induces convergence to the maximal balanced growth path. If, however, \( \sigma \in (0, 1/4] \), Theorem 2 implies that the competitive pricing norm, which induces a path for which \( r^t \neq 1 \), is growth dominated.

It should finally be noted that the possible instability here is, to some extent, an artifact of the difference equation formulation. Discrete time allows the competitive pricing norm to induce overshooting from one period to the next. In continuous time such overshooting is less likely. That is, as the maximal balanced growth path is continuously approached, competitive pricing will tend to slow down the rate of convergence and thus to dampen oscillations.

5. Social Mobility and Optimal Redistribution

Consider now the introduction of exogenous social mobility into the model so that the offspring of each class can belong to another class.\(^{13}\) The following example shows that the existence of such social mobility, together with differential fertility, can account for the simultaneous existence of inequality and optimal redistribution. This redistribution is in the sense that the poorer and larger of the two classes should receive more than its marginal product, whereas the richer and smaller class should receive less. Furthermore,

\[^{13}\text{It would be desirable to eventually consider endogenous social mobility, with some degree of responsiveness to income differentials, for example.}\]
the type with greater inherent fertility may also now produce more offspring in balanced growth, thus generating a negative correlation between actual fertility and income.\footnote{A paper with a similar view of exogenous social mobility, but with exogenous differential fertility, is due to Lam \cite{20}. His model does not include an aggregate production function and so cannot address the effect of income distribution on growth in the same way as we do here. His focus is rather the effect of differential fertility on measured income inequality.}

Suppose then that the production function is \( Y^t = (L_1^t L_2^t)^{1/2} \), and that the evolution of each of the two types is described by

\[
L_1^{t+1} = (1 - \lambda) \alpha (L_1^t W_1^t)^{1/2} + \lambda k \alpha (L_2^t W_2^t)^{1/2}, \quad t = 0, 1, 2, 
\]

\[
L_2^{t+1} = (1 - \lambda) k \alpha (L_2^t W_2^t)^{1/2} + \alpha (L_1^t W_1^t)^{1/2}, \quad t = 0, 1, 2, 
\]

Suppose the distribution of income is given by \( W_1^t = \theta Y^t \) and \( W_2^t = (1 - \theta) Y^t \). The hypothesis that type 2 is more inherently fertile than type 1 is incorporated by taking \( k > 1 \). The fraction of the offspring of each type which is of the other type is \( \lambda \in [0, 1] \).

If \( \lambda = 0 \), as in the example of Section 2, it is straightforward to show that, since \( w_1/w_2 = k^2 > 1 \) the long-run per capita reward to type 1 exceeds that to type 2, for all \( \theta \). Choice of \( \theta = 1/2 \) also maximizes the balanced growth rate, in which case the per capita factor rewards equal the marginal products.

If \( \lambda \in (0, 1/2) \) and \( \theta \in (0, 1] \) the difference equation describing the evolution of the factor ratio is given by

\[
r^{t+1} = \frac{(1 - \lambda) k (1 - \theta)^{1/2} r^t + \lambda \theta^{1/2}}{(1 - \lambda) \theta^{1/2} + \lambda k (1 - \theta)^{1/2} (r^t)^{1/2}}, \quad r^t = \frac{L_2^t}{L_1^t}, \quad t = 0, 1, 2, 
\]

It can be shown that for all \( r^0 \in (0, \infty) \), \( r^t \to r(\theta, \lambda) \) as \( t \to \infty \), where \( r(\theta, \lambda) \), the balanced growth rate, is the (unique) positive solution for \( r = r(\theta, \lambda) \) of the equation\footnote{This follows directly for this example. Solow and Samuelson \cite{35} show that a more general system must converge to a uniquely determined balanced growth path. The "essentiality" assumptions imposed here, but not in Solow and Samuelson, do not alter this property.}

\[
\lambda k (1 - \theta)^{1/2} r^{3/2} + (1 - \lambda) \theta^{1/2} r = (1 - \lambda) k (1 - \theta)^{1/2} r^{1/2} + \lambda \theta^{1/2}.
\]

It follows that

\[
r(\theta, 0) = \frac{k^2 (1 - \theta)}{\theta} \quad \text{and} \quad \frac{\partial r(\theta, 0)}{\partial \lambda} = \frac{2 (\theta^2 - k^4 (1 - \theta)^2)}{\theta^2}, \quad \text{so that} \quad \frac{\partial r(1/2, 0)}{\partial \lambda} < 0.
\]

It is immediate that if \( \theta \) is close enough to 1/2, then \( r(\theta, \lambda) > 1 \) and there is more upward mobility than downward mobility. It also follows that if \( \lambda > 0 \) is small enough
and \( \theta \) is close enough to \( 1/2 \) then \( r(\theta, \lambda) < r(\theta, 0) \). In this case, the ratio of the actual fertility of type 2 at time \( t \), say \( F_2^{t+1} \), to that of type 1 at time \( t \), say \( F_1^t \), is

\[
\frac{F_2^{t+1}}{F_1^{t+1}} = \frac{k(L_2^tW_2^t)^{1/2}/L_2^t}{(L_1^tW_1^t)^{1/2}/L_1^t} = k(1 - \theta)^{1/2} \theta^{-1/2} r(\theta, \lambda)^{-1/2} > 1, \ t = 0, 1, 2, \ldots .
\]

Hence type 2's greater inherent fertility is then reflected in greater actual fertility.

Since the balanced growth rate is given by

\[
\gamma = \gamma(\theta, \lambda) = (1 - \lambda) \alpha \theta^{1/2} r^{1/4} + \lambda k \alpha (1 - \theta)^{1/2} r^{3/4},
\]

it follows that

\[
\frac{\partial \gamma(\theta, 0)}{\partial \lambda} = \alpha \left( \frac{\theta^2 - 2k^2 \theta (1 - \theta) + k^4 (1 - \theta)^2}{2k^3/2 \theta^{3/4} (1 - \theta)^{3/4}} \right).
\]

Since

\[
\frac{\partial^2 \gamma(1/2, 0)}{\partial \theta^2} < 0 \quad \text{and} \quad \frac{\partial^2 \gamma(1/2, 0)}{\partial \theta \partial \lambda} < 0,
\]

the value of \( \theta \) which maximizes the balanced growth rate is a differentiable decreasing function of \( \lambda \) near \( \lambda = 0 \), with the optimal value \( \theta = 1/2 \) at \( \lambda = 0 \).\(^{16}\)

Altogether, if \( \lambda > 0 \) is small enough, the following results hold. First:

\[
MP_1 = (1/2)Y^t/L_1^t > w_1 = \theta Y^t/L_1^t
\]

\[
> (1 - \theta)Y^t/L_2^t = w_2 > (1/2)Y^t/L_2^t = MP_2,
\]

since \( L_2^t/L_1^t \) and \( w_1/w_2 \) then remain close to \( k^2 > 1 \), the value they both assume at \( \lambda = 0 \). In addition, it follows that there is a greater prevalence of upward mobility than of downward mobility, consistent with the evidence due to Blau and Duncan [4] discussed in the Introduction to the present paper. Finally, type 2 must have more offspring per capita than type 1, thus generating a negative correlation between actual fertility and income. (This issue is further discussed in the next section.)

Such a divergence between income and marginal product is reminiscent of redistributive taxation. Here, such redistribution promotes rapid social growth. The intuition for this is as follows. In the absence of mobility, the more fertile of the two types is more numerous and has a lower marginal product and hence wage. Mobility for a given factor tends to increase the return to that factor above its marginal product, perhaps

\(^{16}\)It can be shown that results similar to Theorems 3.1 and 3.2 hold for a version of the model extended to include exogenous mobility. Thus the maximal balanced growth path again has a "turnpike" property. Further, since the maximal balanced growth path must be of the form given here for some \( \theta \), the procedure of maximizing \( \gamma \) over choice of \( \theta \in [0, 1] \) is valid.
because this mobility reduces the supply of the first factor and increases that of the other. When mobility is small and symmetrical across the factors, this effect has more force for the poorer and more numerous of the two types, so that it is this type which should be paid more than its marginal product.

6. Fertility and Income

An inverse relationship between income or wealth and actual fertility is often to be seen in the census data of modern economies, although the effect is not always strong or uniform over all income classes.¹⁷ (See Becker [2, p. 102], for example. Daly and Wilson [10, p. 334] claim there was a positive correlation between father's income and fertility for the U.S. in 1970.) Indeed, Becker [2] has developed a theory under which low fertility is a consequence of increasing wealth or income. The demand for "quality" of children is taken to increase sufficiently rapidly with wealth or income. The last section showed how the present model, in which low fertility is the fundamental cause of high income, can yield this observed inverse relationship in a way that nevertheless does not contradict an underlying positive relationship between income and fertility, ceteris paribus.

The relationship between fertility and income or wealth was discussed extensively by R. A. Fisher, who favored the present direction of cause and effect. For example, Fisher [14, p. 248] relates a description due to Galton [16] of the hereditary peerages received by English judges. Galton claimed that a disproportionate number of these lines had died out, although it did not seem that such men were themselves unusually infertile. Galton points out, however, that such men tended to marry heiresses whose wealth derived at least partially from having an unusually low number of siblings. He suggests this might, in turn, be due to a lower than usual level of heritable fertility on the part of the heiresses' parents. Fisher amplifies the argument that low fertility is the underlying cause of high levels of income and wealth, rather than the reverse. It is noted by Fisher that it is not necessarily the case that such heritable infertility derive simply from the physiological capacity to bear children. Presumably heritable psychological tendencies to marry late or to enjoy children less would be expressed in a similar fashion. Indeed, as noted in the Introduction to the present paper, it is unnecessary here to postulate such heritable fertility at all. It is enough to suppose that the costs of children vary across types in the appropriate fashion.

¹⁷Note that a flat relationship is consistent with the central model of the previous several sections.
7. Neoclassical Growth and the Golden Rule

Suppose now that some of the factors of production are various types of capital. Note, first, that the usual technology for augmenting a capital stock satisfies the constant returns to scale assumption made in this paper. Indeed, the only reason that attention has been limited so far to the case without capital is clarity of interpretation. When various types of capital are present, "income" as usually defined may include a return from the ownership of capital, or may be reduced by new investment. This definition of income derives from private ownership of capital and competitive pricing of all factors. The platinum rule then requires that the amount reinvested in each input, whether a type of labor or of capital, should be the amount imputed as its return under marginal product pricing. This might now be reinterpreted as characterizing the distribution of consumption rather than income. On the other hand, under the platinum rule, ownership of capital is essentially irrelevant since the returns to each type of capital are reinvested in that good.

The present results bring together two branches of neoclassical growth theory. On the one hand, a turnpike theorem is used here to establish that paths which fail to converge to the maximal balanced growth path are growth dominated. On the other, the present platinum rule, which holds along such a von Neumann ray, generalizes the golden rule to allow for a number of types of endogenously growing labor. (See Phelps [27], or Solow [34] for statements of the golden rule.) Such a generalization to a number of types is, indeed, important in order to obtain a general theory of the distribution of income. Further, the endogeneity of the growth of labor permits the use of the appealing social selection criterion of growth dominance introduced here.

The present model offers the platinum rule as a theory of income distribution. It is of interest then to note that an interpretation of the original golden rule in this spirit was offered by Joan Robinson [29]. Her interpretation was criticized by Samuelson [33] on the basis, for example, of the observation that the golden rule remained well-defined in circumstances that competitive prices were not. Samuelson argued that the golden rule should be taken as a productive efficiency condition rather than as a theory of income distribution. Whatever is true with non-differentiable functions, the present approach coincides with the marginal productivity theory of income distribution when

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18The usual investment relationship is of the form \( K^{t+1} = (1 - \delta)K^t + I^t \), for some \( \delta \in (0, 1) \), where \( K^t \) is the stock of capital and \( I^t \) is investment, both at time \( t \). Although this trivially exhibits constant returns to scale in \( K^t \) and \( I^t \), it is a minor technicality that it does not satisfy the present essentiality conditions. These were imposed on labor since they appear realistic and to avoid various "corner solutions."

19It is difficult to explain the degree of observed inequality of income only by the unequal distribution of the returns from capital. That is, a large degree of actual inequality is due to inequality of labor income. (See Arrow [1], for example.) It is this observation which renders the present model of differentiated labor especially apposite.
the production and reproduction functions are differentiable. More fundamentally, the usual clear distinction between consumption and investment does not hold here, since consumption by labor is identified with investment in labor.

A small body of the growth literature has addressed the extent to which the golden rule can be generalized when a single type of labor grows endogenously. Typically, however, this literature assumes that the growth of labor is driven by either income per capita (see Davis [11] and Samuelson [32], for example) or capital per capita (see Merton [25]), rather than consumption per capita, as in the present model. Under these two alternative criteria the original simple golden rule is lost. (See, however, Davis [11] and Merton [25], for comments that the original golden rule continues to hold if the growth of population is driven by per capita consumption.) From the present biological point of view there seems little doubt that per capita consumption is the appropriate determinant of the growth of population. To employ per capita income in this role, for instance, is to ascribe a double impact to output used to augment the capital stock since it also increases population.

Note how a positive rate of time preference in the usual sense might relate to the present results. In the standard neoclassical growth model, with a single capital good and a single type of exogenously-growing labor, such impatience leads to a lower level of reinvestment in capital than that dictated by the golden rule. In the present model, where the growth rate of labor is endogenous, any such failure to reinvest in each factor the amount dictated by marginal productivity pricing must lower the long-run growth rate. This creates the possibility of growth dominance by a less impatient society so that such impatience is subject to social selection.\footnote{In models allowing for random lifetimes, for example, Rogers [31] and Becker and Mulligan [3] examine the formation of preferences incorporating impatience.}

Selection of preferences to exhibit a lack of impatience is the main result of a related paper by Hansson and Stuart [17].\footnote{To be more precise, they argue that preferences supporting the "golden age" consist of the undiscounted sum of the per capita felicity of per capita consumption levels of the present and future generations to a distant horizon. In the present model, these same intertemporal preferences can be more transparently represented as the product over these generations of the per capita reproduction functions, since this yields the number of descendants of a particular individual.} They examine a model in which there are a number of non-intermarrying "clans." Each clan comprises a single type of labor and can invest in a single type of capital. As in the present paper, labor grows at an endogenous rate determined by per capita consumption. In contrast to the present paper, there is a "carrying capacity" effect which slows down the growth of each clan as the total population of all clans grows. (Such a modification of the present model could presumably be carried out.) Hansson and Stuart argue that the long-run "evolutionarily stable" equilibrium, in which total population is constant, entails the maximization of per capita consumption. That is, the equilibrium is that given by the golden rule.
8. An Application to the Emancipation of Russian Serfs

Consider the emancipation of the serfs in Russia after the Crimean War. In this war, Russia did rather badly against her West European adversaries. One of the arguments advanced in favor of emancipation was that it would stimulate the economy and hence, perhaps, the military development of Russia. Indeed, appeal was made to the doctrines of Adam Smith to back the claim that the creation of an atomistic self-interested economy would foster prosperity. (See Blum [5, especially Chs. 25 and 26].) Although such an assertion has much intuitive appeal, it is not clear that economic theory makes a straightforward prediction to this effect. That is, a transfer of property rights, albeit in human beings, seems to have no special claim to advance prosperity. The issue of the economic comparison between slavery and free labor has, of course, been addressed by economic historians. For example, Fenoaltea [13] advances the argument that slaves, being motivated by pain, would work hard but not carefully, whereas free employees, being motivated by money wages, would have the reverse comparative advantage. The present model contributes a simple one-sided explanation for the ending of serfdom.

Consider then the present model with two types of labor—lords and serfs—and no capital. Suppose that the distribution of income under serfdom involves giving the serfs less than their marginal product so that the lords receive more. Such a norm for the distribution of income is then growth dominated. Indeed, suppose now that emancipation occurs and that competitive markets for both types of labor emerge. If such a competitive pricing norm induces a convergent growth path at all, it induces convergence to maximal balanced growth and so this new norm growth dominates the original norm.

Why might serfs be paid less than their marginal product under serfdom? The arguments of Fogel and Engerman [15] suggest that serfs might be viewed as a form of capital by the lords. It would not then be economically rational for the lords to grossly underinvest in serfs. On the other hand, if lords had positive rates of time preference, it would be optimal to pay serfs less than their marginal product. This is analogous to the “modified golden rule” under which investment falls short of the income derived from capital. (See Solow [34].)

9. Appendix

Proof of Theorem 3.1. The following are readily verified—
(i) $T$ is a closed convex cone in $\mathbb{R}^{2K}$.

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22 The absence of capital is for simplicity. The absence of land from the model is unobjectionable if land were not scarce. Indeed, Domar [12, especially Ch. 12] argues that the non-scarcity of land was a fundamental reason for the existence of serfdom.

23 That is, this property of the example in Section 4 can be shown to hold in general.
(ii) If \((x, y) \in T\), \(x' \geq x\) and \(y' \leq y\), then \((x', y') \in T\).

(iii) If \((0, y) \in T\), then \(y = 0\).

(iv) If \(x \in R^K_{++}\), then there exists \(y \in R^K_{++}\) such that \((x, y) \in T\).

It then follows from Karlin [19, Theorems 9.10.1 and 9.10.2, pp. 338-340], that there exists \(v \in \Delta^{K-1}\) and \(\gamma > 0\) such that \((v, \gamma v) \in T\) and \(\gamma \geq \lambda(x, y) = \max\{\lambda \mid y \geq \lambda x\}\), for all \((x, y) \in T, x \neq 0\). In addition, there exists \(p \in \Delta^{K-1}\) such that \(p \cdot (y - \gamma x) \leq 0\), for all \((x, y) \in T\).

Furthermore, since all types of labor are essential, \(v \in \text{Int}(\Delta^{K-1})\). Also, if \(p_k = 0\), for some \(k\), then "profit" maximization would entail \(w_k = 0\), so that \(y_k = 0 < \gamma v_k\). Hence \(p \in \text{Int}(\Delta^{K-1})\).

Suppose now that there exists \((x, y) \in T \sim \{0\}\), where \((x, y)\) is not proportional to \((v, \gamma v)\), but where \(p \cdot (y - \gamma x) = 0\). Without loss of generality, \(x \in \Delta^{K-1}\). Altogether, then, there exists \(w \in R^K_{++}, \sum_{k=1}^{K} w_k = F(v)\), and \(z \in R^K, \sum_{k=1}^{K} z_k = F(x)\) such that

\[
\sum_{k=1}^{K} p_k \{\varphi_k(v_k, w_k) - \gamma v_k\} = 0 = \sum_{k=1}^{K} p_k \{\varphi_k(x_k, z_k) - \gamma z_k\}
\]

Fix any \(t \in (0, 1)\). Suppose first that \(x = v\), but \(w \neq z\). It is follows from (3.11) that

\[
\sum_{k=1}^{K} p_k \{\varphi_k(v_k, tw_k + (1-t)z_k) - \gamma v_k\} > 0,
\]

a contradiction. Suppose then that \(x \neq v\). It now follows from (3.11) that

\[
\sum_{k=1}^{K} p_k \{\varphi_k(tv_k + (1-t)x_k, tw_k + (1-t)z_k) - \gamma v_k\} \geq 0.
\]

In addition, it follows from (3.5) that

\[
F(tv + (1-t)x) > tF(v) + (1-t)F(x) = t \sum_{k=1}^{K} w_k + (1-t) \sum_{k=1}^{K} z_k.
\]

Take then \(\epsilon > 0\) such that \(\sum_{k=1}^{K} (tw_k + (1-t)z_k + \epsilon) = F(tv + (1-t)x)\) so that

\[
\sum_{k=1}^{K} p_k \{\varphi_k(tv_k + (1-t)x_k, tw_k + (1-t)z_k + \epsilon) - \gamma v_k\} > 0,
\]

again, a contradiction.

The uniqueness of \(v \in \text{Int}(\Delta^{K-1})\) is now immediate. \(\Box\)
Proof of Theorem 3.2. This follows Radner [28].\textsuperscript{24} By hypothesis, there exists $\epsilon > 0$ such that $d(x^t, v^t) > \epsilon$, for all $t \in J$, say, where $J \subset \{0, 1, 2, \ldots\}$ is infinite and $d : (\Delta^{K-1})^2 \to R_+$ is Euclidean distance. It follows from a trivial modification of Radner’s “Lemma” that there exists a $\delta > 0$ such that for any $t \in J$,

$$p \cdot N^{t+1} \leq (\gamma - \delta)p \cdot N^t$$

and, in all periods,

$$p \cdot N^{t+1} \leq \gamma p \cdot N^t.$$  

Hence, since $J$ is infinite,

$$p \cdot N^t / \gamma^t \to 0, \text{ so } N^t_k / \gamma^t \to 0, \text{ as } t \to \infty,$$

given that $p \in Int(\Delta^{K-1})$. It is possible to define a sequence of feasible population vectors as follows, for some $L > 0$:

$$\tilde{N}^0 = N^0, \tilde{N}^1 = Lv, \tilde{N}^{t+1} = \gamma \tilde{N}^t = \gamma^t Lv.$$  

so that

$$N^t_k / \tilde{N}^t_k \to 0, t \to \infty, k = 1, \ldots, K. \Box$$

Proof of Theorem 3.3. It follows from Theorem 1 that the following expression

$$\pi(x, z) = \sum_{k=1}^{K} p_k \{\phi_k(x_k, z_k) - \gamma x_k\}$$

attains a maximum value of $\pi = 0$ at $(v, w)$, say, over $(x, z) \in G$, where

$$G = \{(x, z) | x \in \Delta^{K-1} \text{ and } z \in R^K_+ \text{ and } \sum_{k=1}^{K} z_k \leq F(x)\}.$$

Furthermore, $v \in Int(\Delta^{K-1})$ is unique and it is readily shown that $w \in R^K_+$ is also unique, $\sum_{k=1}^{K} w_k = F(v)$ and, by definition,

$$\phi_k(v_k, w_k) - \gamma v_k = 0, k = 1, \ldots, K.$$

It follows that, if

$$L = \pi(x, z) + \lambda\{F(x) - \sum_{k=1}^{K} z_k\} + \mu\{\sum_{k=1}^{K} x_k\},$$

\textsuperscript{24}McKenzie [24] derives an elegant generalization of Radner’s result.
then, at \((x, z) = (v, w)\),

\[
\frac{\partial L}{\partial x_k} = p_k \frac{\partial \varphi_k}{\partial z_k} - p_k \gamma + \lambda \frac{\partial F}{\partial x_k} + \mu = 0, \ k = 1, ..., K
\]

\[
\frac{\partial L}{\partial z_k} = p_k \frac{\partial \varphi_k}{\partial z_k} - \lambda = 0, \ k = 1, ..., K,
\]

for some \(\lambda > 0\). Using (3.5), (3.11) and Euler's Theorem, it follows that \(\mu = 0\) and that

\[
w_k = F_k(v) v_k.
\]

Along a von Neumann ray given by \(N^t\), where \(N^t_k / \sum_{k=1}^K N^t_k = v_k, \ k = 1, ..., K, \ t = 0, 1, 2, ...,\) it follows that

\[
W^t_k = w_k \cdot \sum_{k=1}^K N^t_k = F_k(N^t) N^t_k,
\]

as required. \(\square\)

References


