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Regulating the Oligopoly with Unknown Costs

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Abstract

We examine the planner’s problem of regulating the oligopoly with unknown costs. We consider the regulation problem under three different information structures; (1) independently and identically distributed (i.i.d.) case. (2) independently distributed case. (3) correlated case. We provide the full characterization of the optimal mechanism for the arbitrary number of firms in all three cases. We also show that in (1), when the number of the firms increases, the welfare to the planner converges to the first best case where the planner has the complete information of lowest cost. The same result holds in (2) and (2) under additional mild condition. Finally, we compare the welfare to the planner under these three different information structures. The planner becomes better off under correlated case than under i.i.d. case or independent case only when the costs are not too much positively correlated.
1 Introduction

Since the classical papers by Dupuit (1952) and Hotelling (1938), with the development of economics of information, a huge literature on new regulatory economics under asymmetric information has emerged. Baron and Myerson (1982) examine the problem of how to regulate a monopolistic firm whose costs are unknown to the planner. Lewis and Sappington (1988) analyze the regulation problem with unknown demand. Caillaud (1990) extends the Baron and Myerson's model to the case where a monopolistic firm is regulated by the planner, but the competitive fringes are not. Biglaiser and Ma (1993) consider the case where the regulated firm behaves as a Stackelberg leader and the unregulated fringe behaves as a Stackelberg follower. Auriol and Laffont (1992) analyze the problem of regulating the industry when each firm incurs fixed cost. They show that the duplication of fixed costs in a duopoly structure can be justified by three effects: sampling effect, yardstick effect and the increasing marginal cost effect. Wolinsky (1993) also considers the problem of regulating duopoly under asymmetric information using the spatial model for the differentiated product industry.

In this paper, we investigate the planner's problem of regulating oligopoly under different informational structures about the unknown marginal costs. Unlike the monopoly, when there are many firms, many different informational structures about the costs are possible, from the independently and identically distributed (i.i.d.) case, to independently but not identically distributed case to the correlated case. We consider the regulation problem by planner under these three different informational structures.

We assume that each firm has linear cost function without fixed cost, and that the marginal cost for each firm takes two values, high or low. In i.i.d case and independent case, we provide the complete characterization of the optimal mechanism for arbitrary number of the firms. In correlated case, for duopoly, the planner can always extract whole surplus. For oligopoly with more than two firms, under some condition (called Condition 1) on the joint distribution, the planner can still extract the whole surplus.

In all cases, the optimal mechanism shares the following simple property. When at least one firm reports the low cost, the firms with low reported cost together
produce the first best level corresponding to low cost. When all firms report the high cost, in i.i.d. case, all the firms together produces the Baron-Myerson solution. In independent case, only the firm with highest probability for high cost produces the corresponding Baron-Myerson solution and all the other firms produce nothing. In correlated case, all the firms together produce the first best level corresponding to high cost.

The second interest is about the limiting result when the number of firms increases. In all cases, the welfare for the planner in oligopoly take a simple form, a convex combination of two monopoly cases. For i.i.d. case, when the number of the firm increases indefinitely, the planner attains the welfare level as if the planner has the complete information of low cost. The same result holds in independent case and correlated case under mild condition.

The third interest is about the welfare comparison under different informational structures. For duopoly, we make a pairwise comparison of the planner's welfare between i.i.d case and correlated case, and between independent case and correlated case by assuming that the marginal distributions are the same. The fact that the planner can extract the whole surplus under correlated case, but he can not under i.i.d. case and independent case does not mean that the planner always becomes better off under correlated case. The planner becomes better off under correlated case than either i.i.d.case or independent case only when the marginal costs are not too much positively correlated. When the marginal costs are very strongly positively correlated, although the planner should allow positive informational rent, the planner becomes better off under i.i.d. case or independent case than under correlated case.

In the limiting case where the number of the firms increases indefinitely, we can show that with the same marginal distributions, the planner always becomes at least weakly better off under i.i.d. case than under correlated case. Between independent case and correlated case, generally the welfare comparison is ambiguous in the limiting case.

The paper is organized as follows. A model is presented in Section 2. As a benchmark, we analyze the monopoly in Section 3. In Section 4, we begin with duopoly in i.i.d., independent and the correlated cases and generalize the result to the oligopoly. We also provide the limiting result when the number of the firm increases indefinitely. Finally, in Section 5, we provide the welfare comparisons under
three different informational structures.

2 Model

A central planner wants to regulate the oligopoly market with a homogeneous good. $V(q)^1$ is the valuation function of the planner when the total industry supply is $q$. We impose the following standard assumptions on the shape of $V(q)$.

**Assumption 1.** $V(0) = 0$, $V' > 0$, $V'' < 0$, $\lim_{q \to 0} V'(q) = \infty$, and $
\lim_{q \to \infty} V'(q) = 0$.

There are $n$ firms in the industry. We index the firm by $i = 1, 2, \cdots, n$. We assume that each firm has the following identical linear cost function without fixed cost, $c(q) = \theta q$.\(^2\) Let $\theta_i$ be the marginal cost to the firm $i$. We assume that $\theta_i$ is private information to firm $i$. We assume that for all $i = 1, 2, \cdots, n$, $\theta_i$ takes two values, $\theta_h$ and $\theta_l$ with $0 < \theta_l < \theta_h$.

First, we consider the first best solution under complete information. When $\theta$ is known to the planner, due to the linear cost without fixed cost, we can determine the first best level which is independent of the number of the firm. The first best level, $q^*(\theta)$ is defined by $q^*(\theta) \in \text{Argmax}\{V(q) - \theta q\}$. Under Assumption 1, $q^*(\theta)$ is unique and completely determined by the first order condition, $V'(q^*(\theta)) = \theta$. Let $q_i = q^*(\theta_l)$ and $q_h = q^*(\theta_h)$. Note that $q_i > q_h$.

---

\(^1\) $V(q)$ may be interpreted as the gross consumer surplus. Then, the planner tries to maximize the consumer surplus.

\(^2\) Our main interest in this paper is the informational aspect rising from the increased number of the firms. If the cost function is convex, the planner can benefit by spreading the production across the firms, which Auriol and Laffont (1992) call the increasing marginal cost effect. If each firm incurs the fixed cost, the planner will make as few as possible firms produce the quantity. By assuming the linear cost function without fixed cost, we can avoid these problems. We can solely focus on the informational aspect rising from the increased number of the firms.
3 Monopoly

Since we are interested in the effect of the increased number of firms, we briefly consider the monopoly as a benchmark. Suppose there is only one firm who can produce $q$. Let the probability that $\theta = \theta_l$ be $\pi \in (0,1)$. A (direct) mechanism or a contract the planner can use has the following form: $M^1 = \{(q_l, p_l), (q_h, p_h)\}$\(^3\) whose interpretation is obvious: The planner asks the firm to report its marginal cost. If the firm reports $\theta_l$, it will produce $q_l$, and as a reward it will get paid $p_l$, similarly for $\theta_h$.\(^4\)

Any feasible mechanism should satisfy two requirements, incentive constraints and participation constraints. By incentive constraints, any feasible mechanism should not give incentive for one type to mimic the other type. By participation constraints, any feasible mechanism should guarantee the firm at least as much as the reservation utility which is assumed to be zero.

1. Incentive Constraints:

$$p_h - \theta_h q_h \geq p_l - \theta_l q_l \text{ and } p_l - \theta_l q_l \geq p_h - \theta_h q_h.$$ 

2. Participation Constraints:

$$p_j - \theta_j q_j \geq 0, \ j = h, l.$$ 

How the planner maximizes the objective function subject to these two constraints is, by now, well-known in the optimal mechanism design literature. Among the four constraints, the participation constraint for $\theta_h$ and the incentive constraint for $\theta_l$ should be binding in the optimal mechanism. From these, we have

$$p_h = \theta_h q_h, \text{ and } p_l = \theta_h q_h + \theta_l (q_l - q_h) \text{ with } q_l \geq q_h.$$ 

From this, we have

$$E_{\theta \rho}(\theta) = \theta_h q_h + \pi \theta_l (q_l - q_h).$$

\(^3\)The superscript $1$ over $M$ means that we are considering monopoly. When there are $n$ firms, we denote a mechanism by $M^n$.

\(^4\)By the revelation principle, without loss of generality, we need only consider the direct mechanism.
The planner's objective function is

\[
W^1 = E_\theta \{ V(q(\theta)) - p(\theta) \} = E_\theta V(q(\theta)) - E_\theta p(\theta) \\
= \pi V(q_l) + (1 - \pi)V(q_h) - \theta_h q_h - \pi \theta_l (q_l - q_h).
\]

By differentiating \( W^1 \) with respect to \( q_l \) and \( q_h \) and setting them to zero, we have

\[
V'(q_l) = \theta_l, \quad \text{and} \quad V'(q_h) = \frac{\theta_h - \pi \theta_l}{1 - \pi} > \theta_h. \tag{1}
\]

From (1), \( \forall \pi \in (0, 1), q_l = q_l^* \), the first best level. From (2), \( \forall \pi \in (0, 1), q_h < q_h^* \). In order to emphasize the dependence of \( q_h \) on \( \pi \), we write down the solution to (2) as \( q(\pi) \). \( q(\pi) \) is decreasing in \( \pi \).

For the later comparison, we introduce some notations. Let \( C^1(\pi) \) be the expected cost to the planner for implementing the optimal mechanism in monopoly when the prior probability for \( \theta = \theta_i \) is \( \pi \). Then, \( C^1(\pi) = \theta_h q(\pi) + \pi \theta_l (q_l^* - q(\pi)) \). \( C^1(\pi) \) is decreasing in \( \pi \). Let \( W^1(\pi) \) be the expected welfare to the planner from the optimal mechanism. \( W^1(\pi) = \pi V(q_l^*) + (1 - \pi)V(q(\pi)) - C^1(\pi) \). \( W^1(\pi) \) is increasing and convex in \( \pi \).

4 Oligopoly

Although we eventually consider the general oligopoly, we begin with the duopoly and generalize the result to the oligopoly. We assume that there are two firms, 1 and 2. Let \( \theta_i \) denote the marginal cost to firm \( i = 1, 2 \). Even in duopoly, there are three possible cases depending upon how \( \theta_1 \) and \( \theta_2 \) are related with each other: (1) independently and identically distributed (i.i.d.), (2) independently but not identically distributed, (3) correlated. In increasing order, it becomes more general. We consider three cases separately for duopoly, and we extend the result to oligopoly. Then, we examine the limiting behavior when \( n \) goes to infinity.

Finally, we provide the welfare comparison. We examine how the different information structure affects the optimal mechanism and the welfare of the planner.
4.1 Independently and Identically Distributed Case

Suppose \( \theta_i \) is i.i.d. with \( \pi = \text{Prob}(\theta_i = \theta_i) \). In duopoly, the mechanism that the planner can use takes the following form:

\[
M^2 = \{ q^i(\theta_1, \theta_2), p^i(\theta_1, \theta_2) \}_{i=1,2}.
\]

In general, the pair of quantity and payment for, say, firm 1 may depends upon not only the report by the firm 1 but also the report by firm 2, and vice versa.

For simplifying the notation, the superscript always refers to the firm and the first and second subscripts refer to the reported type of firm 1 and firm 2, respectively. For example, \( q^i_{1l} \) is the amount assigned to the firm 1 when both firms report \( \theta_i \). The single subscript means that we take an expectation with respect to the other subscript. For example, \( q^i_1 = \pi q^i_{1l} + (1 - \pi)q^i_{1h} \).

Like the monopoly, any feasible mechanism must satisfy both the incentive and the participation constraints.

1. Incentive Constraints:

\[
\forall i = 1, 2, \; p^i_i - \theta_i q^i_i \geq p^i_h - \theta_i q^i_h, \; \text{and} \; p^i_h - \theta_h q^i_h \geq p^i_i - \theta_h q^i_i.
\]

2. Participation Constraints:

\[
\forall i = 1, 2 \; \text{and} \; j = l, h, \; p^j_j - \theta_j q^j_j \geq 0.
\]

There are 8 constraints, 4 for each firm, 2 for incentive constraints and 2 for participation constraints. Again, like the monopoly, we can easily show that for each firm, the participation constraint for \( \theta_h \) and the incentive constraint for \( \theta_i \) should be binding at the optimal mechanism. We, therefore, have:

\[
\forall i = 1, 2, \; p^i_h = \theta_h q^i_h, \; \text{and} \; p^i_i = \theta_h q^i_h + \theta_i(q^i_i - q^i_h) \; \text{with} \; q^i_i \geq q^i_h.
\]

The planner's objective function is

\[
W^2 = E_{\theta_1, \theta_2} V(q^1(\theta_1, \theta_2) + q^2(\theta_1, \theta_2)) - E_{\theta_1, \theta_2} \{ p^1(\theta_1, \theta_2) + p^2(\theta_1, \theta_2) \}.
\]

Note that \( E_{\theta_1, \theta_2} p^1(\theta_1, \theta_2) = E_{\theta_1} \{ E_{\theta_2} p^1(\theta_2, \theta_2) \} = E_{\theta_1} p^1(\theta_1) \). From the incentive and participation constraints, \( p^i_h = \theta_h q^i_h \) and \( p^i_i = \theta_h q^i_h + \theta_i(q^i_i - q^i_h) \). So
\[ E_{\theta_1, \theta_2} p^1(\theta_1, \theta_2) = \theta_h q^1_h + \pi \theta_l (q^1_l - q^1_h). \] Similarly, \[ E_{\theta_1, \theta_2} p^2(\theta_1, \theta_2) = \theta_h q^2_h + \pi \theta_l (q^2_l - q^2_h). \] Therefore, \[ E_{\theta_1, \theta_2} \{ p^1(\theta_1, \theta_2) + p^2(\theta_1, \theta_2) \} = \theta_h (q^1_h + q^2_h) + \pi \theta_l ((q^1_l + q^2_l) - (q^1_h + q^2_h)). \]

The planner’s objective function is:

\[
W^2 = E_{\theta_1, \theta_2} V(q^1(\theta_1, \theta_2) + q^2(\theta_1, \theta_2)) - E_{\theta_1, \theta_2} \{ p^1(\theta_1, \theta_2) + p^2(\theta_1, \theta_2) \} \\
= \pi^2 V(q^1_{ih} + q^2_{ih}) + \pi (1 - \pi) V(q^1_{ih} + q^2_{ih}) + (1 - \pi) \pi V(q^1_{ih} + q^2_{ih}) \\
+ (1 - \pi)^2 V(q^1_{hh} + q^2_{hh}) - \theta_h (q^1_h + q^2_h) - \pi \theta_l ((q^1_l + q^2_l) - (q^1_h + q^2_h)).
\]

By differentiating \( W^2 \) with respect to \( q^i_{jk}, i = 1, 2, j = h, l \) and \( k = h, l, \) and setting them to zero, we have following 8 equations:

\[
\frac{\partial W^2}{\partial q^1_{ih}} = \pi^2 V'(q^1_{ih} + q^2_{ih}) - \pi^2 \theta_l = 0, \tag{3}
\]

\[
\frac{\partial W^2}{\partial q^2_{ih}} = \pi^2 V'(q^1_{ih} + q^2_{ih}) - \pi^2 \theta_l = 0, \tag{4}
\]

\[
\frac{\partial W^2}{\partial q^1_{hh}} = \pi (1 - \pi) V'(q^1_{ih} + q^2_{ih}) - \pi (1 - \pi) \theta_l = 0, \tag{5}
\]

\[
\frac{\partial W^2}{\partial q^2_{hh}} = \pi (1 - \pi) V'(q^1_{ih} + q^2_{ih}) - \pi \theta_h + \pi^2 \theta_l = 0, \tag{6}
\]

\[
\frac{\partial W^2}{\partial q^1_{hl}} = \pi (1 - \pi) V'(q^1_{ih} + q^2_{ih}) - \pi \theta_h + \pi^2 \theta_l = 0, \tag{7}
\]

\[
\frac{\partial W^2}{\partial q^2_{hl}} = \pi (1 - \pi) V'(q^1_{ih} + q^2_{ih}) - \pi (1 - \pi) \theta_l = 0, \tag{8}
\]

\[
\frac{\partial W^2}{\partial q^1_{hh}} = (1 - \pi)^2 V'(q^1_{ih} + q^2_{ih}) - (1 - \pi) \theta_h + \pi (1 - \pi) \theta_l = 0, \tag{9}
\]

\[
\frac{\partial W^2}{\partial q^2_{hh}} = (1 - \pi)^2 V'(q^1_{ih} + q^2_{ih}) - (1 - \pi) \theta_h + \pi (1 - \pi) \theta_l = 0. \tag{10}
\]

Equations (3) and (4) give the same equation, \( V'(q^1_{ih} + q^2_{ih}) = \theta_l. \) In other words, \( q^1_{ih} + q^2_{ih} = q^*_i. \) When both firms report low cost, the planner does not care who supplies how much, whenever the total supply is \( q^*_i, \) the first best. This is due to the linear cost without fixed cost. Similarly, Equations (9) and (10) give the same equation, \( (1 - \pi) V'(q^1_{ih} + q^2_{ih}) = \theta_h - \pi \theta_l. \) Namely, \( q^1_{ih} + q^2_{ih} = q(\pi). \) When both firms report high cost, whenever the total supply is \( q(\pi), \) the planner does not care, either. Equations (5) and (6) contradict with each other. Equation (6) gives \( V'(q^1_{ih} + q^2_{ih}) = \theta_l, \) i.e.,
\( q_{1h} + q_{2l}^* = q_l^* \). Equation (7) gives \((1 - \pi)\theta'_{1h}(q_{1h}^1 + q_{2h}^2) = \theta_h - \pi\theta_i\), i.e., \( q_{1h} + q_{2l}^2 = q_l^* \). Since \( q_l^* > q_l(\pi) \), both Equations (5) and (6) cannot hold. This means we must have a corner solution. Suppose Equation (6) holds, i.e., \( q_{1h}^1 + q_{2l}^2 = q_l(\pi) \). Then, \( \partial W^2/\partial q_{1h}^1 > 0 \). So \( W^2 \) increases as \( q_{1h}^1 \) increases. Hence, it cannot be optimal. Suppose Equation (5) holds. Then, \( q_{1h}^1 + q_{2h}^2 = q_l^* \). Then, \( \partial W^2/\partial q_{1h}^2 < 0 \). So \( W^2 \) increases as \( q_{2h}^2 \) decreases. However, if \( q_{2h}^2 = 0 \), the planner cannot decrease \( q_{2h}^2 \) any further. Put together, Equations (5) and (6) determine \( q_{1h}^1 = q_l^* \) and \( q_{2h}^2 = 0 \). In other words, when the firm 1 reports the low cost and the firm 2 reports high cost, the firm 1 produce \( q_l^* \) and the firm 2 produce nothing. Equations (7) and (8) give the opposite condition when the firm 1 reports high cost and the firm 2 reports low cost. Then, the firm 2 produces \( q_l^* \) and the firm 1 produce nothing.

In summary, we have \( q_{1l}^1 + q_{2l}^2 = q_l^* \), \( q_{1l}^1 = q_{2l}^2 = q_l^* \), \( q_{1h}^1 = q_{2h}^2 = 0 \), and \( q_{1h}^1 + q_{2h}^2 = q_l(\pi) \). Due to the linear cost without fixed cost, the optimal mechanism itself is not unique. Any mechanism satisfying above conditions with suitable payment scheme is optimal in the planner’s viewpoint. Since each firm is ex-ante identical, it may be reasonable to impose the symmetry condition on the class of optimal mechanisms. In other words, when two firms report the same cost, the planner will let each firm produce half amount, i.e., \( q_{1l}^1 = q_{2l}^2 = q_l^*/2 \) and \( q_{1h}^1 = q_{2h}^2 = q_l(\pi)/2 \). Then, the unique symmetric optimal mechanism exists.

We now compare the welfare to the planner in monopoly and duopoly. Let \( C^2(\pi) \) be the expected cost for implementing any optimal mechanism in duopoly:

\[
C^2(\pi) = E_{\theta_1, \theta_2} \{ p^1(\theta_1, \theta_2) + p^2(\theta_1, \theta_2) \} = \theta_h(q_{1h}^1 + q_{2h}^2) + \pi \theta_i((q_{1l}^1 + q_{2l}^2) - (q_{1h}^1 + q_{2h}^2)) \\
= (1 - \pi)\{\theta_h q_l(\pi) + \pi \theta_i(q_l^* - q_l(\pi))\} + \pi \theta_i q_l^*.
\]

Note that \( C^1(\pi) = \theta_h q_l(\pi) + \pi \theta_i(q_l^* - q_l(\pi)) \). In particular, \( C^1(1) = \theta_i q_l^* \). So \( C^2(\pi) \) is a convex combination of \( C^1(\pi) \) and \( C^1(1) \), \( C^2(\pi) = \pi C^1(1) + (1 - \pi)C^1(\pi) \). Hence the cost reduction to the planner from monopoly to duopoly is \( C^1(\pi) - C^2(\pi) = \pi \{ C^1(\pi) - C^1(1) \} > 0 \).

For benefit to the planner, in monopoly the planner has the first best \( V(q_l^*) \) with probability \( \pi \) and the second best \( V(q_l(\pi)) \) with probability \( (1 - \pi) \). In duopoly, the planner attains the first best \( V(q_l^*) \) with probability \( 1 - (1 - \pi)^2 \) and gets the second best \( V(q_l(\pi)) \) with probability \( (1 - \pi)^2 \). Hence, the benefit also increases
from monopoly to duopoly. Let $W^2(\pi)$ the expected welfare to the planner from any optimal mechanism in duopoly. Then, $W^2(\pi)$ is also a convex combination of $W^1(\pi)$ and $W^1(1)$, $W^2(\pi) = \pi W^1(1) + (1-\pi) W^1(\pi)$. So the welfare increases from monopoly to duopoly is $W^2(\pi) - W^1(\pi) = \pi \{W^1(1) - W^1(\pi)\} > 0$.

We summarize the result in duopoly in Proposition 1.

**Proposition 1.** In duopoly under i.i.d case, in any optimal mechanism, if at least one firm reports low cost, the firms with low reported cost together produce $q^*_i$ and the firm with high reported cost produces nothing. When both firms report high cost, the firms together produce $q(\pi)$. Furthermore, $C^2(\pi) = \pi C^1(1) + (1 - \pi) C^1(\pi)$ and $W^2(\pi) = \pi W^1(1) + (1 - \pi) W^1(\pi)$.

We can extend the analysis to oligopoly under i.i.d. case. Clearly whenever the number of the firm increases, the planner becomes better off. As seen in duopoly, the worst case for the planner is that all the firms report the high cost. Except for this case, the planner obtains the first best for the low cost. However, when the number of firms increases, this probability becomes smaller and the competition among the firms becomes more severe, therefore the incentive constraints becomes weaker. We may anticipate that in planner's viewpoint, when the number of the firm increases indefinitely, the situation becomes arbitrarily close to the complete information case with low cost. As we will see in oligopoly, this intuition is correct. Let $C^n(\pi)$ and $W^n(\pi)$ be the expected cost to the planner for implementing the optimal mechanism and the expected welfare to the planner from the optimal mechanism, respectively when there are $n$ firms. In Appendix, we prove the following generalization of duopoly result.

**Proposition 2.** In $n$-firm oligopoly under i.i.d. case, in any optimal mechanism, if at least one firm reports the low cost, the firms with low reported cost together produce $q^*_i$ and the firms with high reported cost produce nothing. When all the firms report high cost, the firms together produce $q(\pi)$. Furthermore, $C^n(\pi) = (1 - (1 - \pi)^{n-1}) C^1(1) + (1 - \pi)^{n-1} C^1(\pi)$ and $W^n(\pi) = (1 - (1 - \pi)^{n-1}) W^1(1) + (1 - \pi)^{n-1} W^1(\pi)$.

Proof: See Appendix.

We can easily examine what happens if the number of firms increases to infinity.
As \( n \) goes to infinity, \( \forall \pi \in (0,1], C^n(\pi) \) and \( W^n(\pi) \) converge to \( C^1(1) \) and \( W^1(1) \), respectively. Namely, the planner’s welfare converges to what he could obtain as if he has the complete information about the firm’s marginal cost, \( \theta = \theta_i \).

### 4.2 Independently distributed Case.

In this subsection, we continue to analyze the independently distributed case. Again, we begin the analysis with duopoly. We assume \( \theta_1 \) and \( \theta_2 \) are independently distributed with \( \pi_i = \text{Prob}[\theta_i = \theta_i], i = 1, 2 \). Since we assume independence only, generally \( \pi_1 \neq \pi_1 \). Without loss of generality, we assume \( \pi_1 > \pi_2 \). In other words, it is more likely that the firm 1 will realize low cost. We maintain the same notation here as in the i.i.d. case. Most part of the analysis is the same as the i.i.d. case except that when taking the expectation, each firm should use the other firm’s distribution. So we have the following relations:

\[
q^1_i = \pi_2 q^1_{il} + (1 - \pi_2) q^1_{ih}, \text{ and } q^1_h = \pi_2 q^1_{hl} + (1 - \pi_2) q^1_{hh}. \\
q^2_i = \pi_1 q^2_{il} + (1 - \pi_1) q^2_{ih}, \text{ and } q^2_h = \pi_1 q^2_{hl} + (1 - \pi_1) q^2_{hh}.
\]

As in the i.i.d. case, we have 8 constraints, 4 for each firm, 2 for incentive constraints and 2 for participation constraints, and among these constraints, at optimal mechanism the participation constraint for \( \theta_h \) and the incentive constraint for \( \theta_i \) should be binding for each firm. We, therefore, have:

\[
\forall i = 1, 2, \ p^i = \theta_h q^i, \text{ and } p^i = \theta_h q^i + \theta_i (q^i - q^i) \text{ with } q^i \geq q^i.
\]

The expected cost to the planner is given by

\[
E_{\theta_1, \theta_2} \{p^1(\theta_1, \theta_2) + p^2(\theta_1, \theta_2)\} = \theta_h (q^1_h + q^2_h) + \theta_l (\pi_1 q^1_l + \pi_2 q^2_l) - (\pi_1 q^1_h + \pi_2 q^2_h).
\]

The planner’s objective function is:

\[
W^2 = \pi_1 \pi_2 V(q^1_{il} + q^2_{il}) + \pi_1 (1 - \pi_2) V(q^1_{ih} + q^2_{ih}) + (1 - \pi_1) \pi_2 V(q^1_{hl} + q^2_{hl}) \\
+ (1 - \pi_1)(1 - \pi_2) V(q^1_{hh} + q^2_{hh}) - \theta_h (q^1_h + q^2_h) - \theta_l \{\pi_1 q^1_l + \pi_2 q^2_l\} - (\pi_1 q^1_h + \pi_2 q^2_h).
\]

By differentiating \( W^2 \) with respect to \( q^i_{jk}, i = 1, 2, j = h, l \) and \( k = h, l \), and setting them to zero, we have following 8 equations:
\[
\frac{\partial W^2}{\partial q^1_{li}} = \pi_1 \pi_2 V'(q^1_{li} + q^2_{li}) - \pi_1 \pi_2 \theta_l = 0, \tag{11}
\]
\[
\frac{\partial W^2}{\partial q^2_{li}} = \pi_1 \pi_2 V'(q^1_{li} + q^2_{li}) - \pi_1 \pi_2 \theta_l = 0, \tag{12}
\]
\[
\frac{\partial W^2}{\partial q^1_{lh}} = \pi_1 (1 - \pi_2) V'(q^1_{lh} + q^2_{lh}) - \pi_1 (1 - \pi_2) \theta_l = 0, \tag{13}
\]
\[
\frac{\partial W^2}{\partial q^2_{lh}} = \pi_1 (1 - \pi_2) V'(q^1_{lh} + q^2_{lh}) - \pi_1 \theta_h + \pi_1 \pi_2 \theta_l = 0, \tag{14}
\]
\[
\frac{\partial W^2}{\partial q^1_{hl}} = (1 - \pi_1) \pi_2 V'(q^1_{hl} + q^2_{hl}) - \pi_2 \theta_h + \pi_1 \pi_2 \theta_l = 0, \tag{15}
\]
\[
\frac{\partial W^2}{\partial q^2_{hl}} = (1 - \pi_1) \pi_2 V'(q^1_{hl} + q^2_{hl}) - (1 - \pi_1) \pi_2 \theta_l = 0, \tag{16}
\]
\[
\frac{\partial W^2}{\partial q^1_{hh}} = (1 - \pi_1)(1 - \pi_2) V'(q^1_{hh} + q^2_{hh}) - (1 - \pi_2) \theta_h + \pi_1 (1 - \pi_2) \theta_l = 0, \tag{17}
\]
\[
\frac{\partial W^2}{\partial q^2_{hh}} = (1 - \pi_1)(1 - \pi_2) V'(q^1_{hh} + q^2_{hh}) - (1 - \pi_1) \theta_h + (1 - \pi_1) \pi_2 \theta_l = 0. \tag{18}
\]

Equations (11) through (16) give the same consideration as in the i.i.d. case. By Equations (11) and (12), when both firms report low cost, \(q^1_{li} + q^2_{li} = q^*_l\). By Equations (13) through (16), when one firm reports low cost and the other firm reports high cost, the firm with low reported cost produces \(q^*_l\) and the firm with high reported cost produces nothing. Equations (17) and (18), however, are not the same as (9) and (10). Equation (17) gives \(q^1_{hh} + q^2_{hh} = q(\pi_1)\), and Equation (18) gives \(q^1_{hh} + q^2_{hh} = q(\pi_2)\). Since \(q(\pi)\) is decreasing in \(\pi\) and \(\pi_1 > \pi_2\), both Equations (17) and (18) cannot hold. We must have another corner solution. Since \(q(\pi_2) > q(\pi_1)\), the correct condition is that at optimum, \(\partial W^2/\partial q^1_{hh} < 0\) and \(\partial W^2/\partial q^2_{hh} = 0\). In other words, \(q^1_{hh} = 0\) and \(q^2_{hh} = q(\pi_2)\). The intuition is as follows. Note that whenever the planner allows the firm \(i\) with high cost to produce positive quantity, i.e., \(q^i_h > 0\), the planner should allow positive informational rent to the firm \(i\) with low cost in order to induce the truth-telling. Since \(\pi_1 > \pi_2\), ex-ante it is more likely that firm 1 will be a low cost firm. In the planner’s viewpoint, it is more important to extract the whole surplus from the firm 1. Then, the most efficient way to induce truth-telling is that the planner does not allow any positive production when the firm 1 reports high cost.
At any optimal mechanism, the informational rent for firm 1 with low cost should be zero.

In summary, we have $q_{l1}^* + q_{l2}^* = q_{l1}^2$, $q_{h1} = q_{h2} = q_{h1} = q_{h2} = 0$, and $q_{h1}^1 = q_{h2}^2 = q(\pi_2)$. Still, the optimal mechanism itself is not unique. However, there is only one case for the indeterminacy where both firms report the low cost. In i.i.d. case, we may resolve this indeterminacy based upon the symmetry. If we assume independence only, however, then the firms are not ex-ante identical. So it may not be reasonable to impose the symmetry condition in this case.

The expected cost, $C^2(\pi_1, \pi_2)$ and the expected welfare, $W^2(\pi_1, \pi_2)$ for the planner from any optimal mechanism have the following forms in independent case.

\[
C^2(\pi_1, \pi_2) = \theta_h(q_{h1} + q_{h2}) + \theta_I\{((\pi_1 q_1^1 + \pi_2 q_2^2) - (\pi_1 q_{h1}^1 + \pi_2 q_{h2}^2))
= (1 - \pi_1)\{\theta_h q(\pi_2) + \pi_2 \theta_I(q_1^* - q(\pi_2))\} + \pi_1 \theta_I q_1^*.
\]

Again, $C^2(\pi_1, \pi_2)$ is a convex combination of $C^1(\pi_2)$ and $C^1(1)$, $C^2(\pi_1, \pi_2) = \pi_1 C^1(1) + (1 - \pi_1) C^1(\pi_2)$. Note that this representation depends upon the assumption that $\pi_1 > \pi_2$. If $\pi_2 < \pi_1$, the role of $\pi_1$ and $\pi_2$ should be reversed. If $\pi_1 = \pi_2 = \pi$, then $C^2(\pi_1, \pi_2) = C^2(\pi)$ in i.i.d. case. $W^2(\pi_1, \pi_2)$ is also a convex combination of $W^1(\pi_2)$ and $W^1(1)$, $W^2(\pi_1, \pi_2) = \pi_1 W^1(1) + (1 - \pi_1) W^1(\pi_2)$.

We summarize the result in Proposition 3.

**Proposition 3.** In duopoly under independent case, in any optimal mechanism, if at least one firm reports low cost, the firms with low reported cost together produce $q_1^*$ and the firm with high reported cost produces nothing. When both firms report high cost, the firm 2 produces $q(\pi_2)$ and the firm 1 produces nothing. Furthermore, $C^2(\pi_1, \pi_2) = \pi_1 C^1(1) + (1 - \pi_1) C^1(\pi_2)$ and $W^2(\pi_1, \pi_2) = \pi_1 W^1(1) + (1 - \pi_1) W^1(\pi_2)$.

We can also extend the analysis to more than two firms. Suppose there are $n$ firms. $\theta_i$'s are independently distributed with $\pi_i = \text{Prob}[\theta_i = \theta_1]$. We assume that all $\pi_i$'s are distinct. In particular, if necessary, by relabelling the firms, we assume $\pi_1 > \pi_2 > \cdots > \pi_n$. Let $\pi = (\pi_1, \pi_2, \cdots, \pi_n)$. Let $C^n(\pi)$ and $W^n(\pi)$ be the expected cost and expected welfare to the planner from the optimal mechanism when there are $n$ firms. We prove the following generalization of duopoly result to the oligopoly with $n$ firms.
Proposition 4. In \( n \)-firm oligopoly under independent case, in any optimal mechanism, if at least one firm reports the low cost, the firms with low reported cost together produce \( q^* \) and the firms with high reported cost produce nothing. When all the firms report high cost, the \( n \)-th firm produces \( q(\pi_n) \) and all the other firms produce nothing. Furthermore, \( C^n(\pi) = (1 - \prod_{i=1}^{n-1} (1-\pi_i)) C^1(1) + \prod_{i=1}^{n-1} (1-\pi_i) C^1(\pi_n) \) and \( W^n(\pi) = (1 - \prod_{i=1}^{n-1} (1-\pi_i)) W^1(1) + \prod_{i=1}^{n-1} (1-\pi_i) W^1(\pi_n) \).

Proof: See Appendix.

Unlike the i.i.d. case, \( C^n \) and \( W^n \) does not necessarily converge to \( C^1(1) \) and \( W^1(1) \). It is true if and only if \( \prod_{i=1}^n (1-\pi_i) \) converges to zero when \( n \) goes to infinity. A necessary and sufficient condition for this is that \( \sum \pi_n = \infty \). In independent case, therefore, \( C^n \) and \( W^n \) converge to \( C^1(1) \) and \( W^1(1) \) respectively if and only if \( \sum \pi_n = \infty \).

4.3 Correlated Case.

In this subsection, we consider the correlated case. We first focus on the duopoly case and generalize the result to oligopoly.

Unlike i.i.d. or independent case, in correlated case, it does not suffice to specify the marginal distributions. We have to specify the joint distribution, \( \pi = \{\pi(\theta_1, \theta_2)\} \). From the joint distribution, we can derive the marginal and conditional distributions. Let \( \pi(\theta_i) \) and \( \pi^i(\theta_{-i}|\theta_i) \) be the marginal distribution for \( \theta_i \) and conditional distribution of \( \theta_{-i} \) given \( \theta_i \), respectively. We assume \( \pi(\theta_i) > 0 \) for \( i = 1, 2 \) and \( \theta_i = \theta_1, \theta_2 \).

Let \( \Pi^1 \) be the matrix of the conditional distribution of \( \theta_2 \) given \( \theta_1 \):

\[
\Pi^1 = \begin{pmatrix}
\pi^1(l|l) & \pi^1(h|l) \\
\pi^1(l|h) & \pi^1(h|h)
\end{pmatrix}
\]

Note that assuming \( \theta_1 \) and \( \theta_2 \) are correlated is equivalent to assuming that \( \Pi^1 \) has rank 2. In other words, \( \Pi^1 \) is non-singular.

In correlated case, when taking the expectation, each firm should use the conditional distribution given its type. Again, we have incentive constraints and participation constraints.
Incentive constraints:

$$\forall i = 1, 2, E_{\theta_{-i}|\theta_i=\theta_i}(p^i(\theta_{-i}, \theta_i) - \theta_i q^i(\theta_{-i}, \theta_i)) \geq E_{\theta_{-i}|\theta_i=\theta_i}(p^i(\theta_{-i}, \theta_h) - \theta_i q^i(\theta_{-i}, \theta_h)),$$

and

$$E_{\theta_{-i}|\theta_i=\theta_h}(p^i(\theta_{-i}, \theta_h) - \theta_h q^i(\theta_{-i}, \theta_h)) \geq E_{\theta_{-i}|\theta_i=\theta_h}(p^i(\theta_{-i}, \theta_i) - \theta_h q^i(\theta_{-i}, \theta_i)).$$

Participation constraints:

$$\forall i = 1, 2, E_{\theta_{-i}|\theta_i=\theta_i}(p^i(\theta_{-i}, \theta_i) - \theta_i q^i(\theta_{-i}, \theta_i)) \geq 0 \text{ and }$$

$$E_{\theta_{-i}|\theta_i=\theta_h}(p^i(\theta_{-i}, \theta_h) - \theta_h q^i(\theta_{-i}, \theta_h)) \geq 0.$$  

In i.i.d. or independent case, the fact that the firm uses the same distribution regardless of its type simplifies the analysis. In correlated case, the firm uses the different conditional distribution depending upon its type. Seemingly, this will complicate the analysis under correlated case. On the contrary, the correlated case admits simpler characterization due to the result by Cremer and Mclean (1985). In our context, their result guarantees that under correlated case, the planner can extract whole surplus from the firms using the correlation. Furthermore, the planner can obtain the first best outcome as if he has the complete information about $\theta_1$ and $\theta_2$. In the sequel, we will construct a mechanism which has the above properties. Moreover, in that mechanism, each type of the firm has truth-telling as the dominant strategy regardless of the type of the other firm.\(^5\)

Under complete information about $\theta_1$ and $\theta_2$, the first best outcome is such that $q^*_h + q^*_l = q^*_l$, $q^*_h = q^*_l = q^*_l$, $q^*_h = q^*_l = 0$, and $q^*_h + q^*_h = q^*_h$. Moreover, the planner extracts the whole surplus from each firm. Hence, the expect welfare to the planner under complete information is such that $W^2_c = (1 - \pi(h, h))W^1(1) + \pi(h, h)W^1(0)$. Without loss of generality, set $q^*_h = q^*_l$, $q^*_h = q^*_h$ and $q^*_h = q^*_h = 0$. In other words, when both firms have the identical costs, the planner will let the firm 1 produce everything and the firm 2 produce nothing.

Suppose the planner has the incomplete information. We will construct a suitable payment schemes which will implement the first best outcome, satisfy the incentive and participation constraints and attain $W^2_c$.

\(^5\)Cremer and Mclean (1985) called such a mechanism ex post dominant strategy mechanism.
For firm 2, set $p_{hl}^2 = p_{lh}^2 = p_{hh}^2 = 0$ and $p_{hl}^2 = q_l^*$. For firm 1, set $p_{hl}^1 = p_{hi}^1 + \theta_i q_l^*$ and $p_{hh}^1 = p_{hh}^1 + \theta_i (q_l^* - q_h^*)$, where $p_{hl}^1$ and $p_{hh}^1$ satisfy the following equations:

\begin{align}
\pi^1(l|l)p_{hl}^1 + \pi^1(h|l)p_{hh}^1 &= \theta_i q_h^* \pi^1(h|l) \\
\pi^1(l|h)p_{hl}^1 + \pi^1(h|h)p_{hh}^1 &= \theta_h q_h^* \pi^1(h|h)
\end{align}

These equations can be written as follows using $\Pi^1$:

$$
\Pi^1 \begin{pmatrix} p_{hl}^1 \\ p_{hh}^1 \end{pmatrix} = \begin{pmatrix} \theta_i q_h^* \pi^1(h|l) \\ \theta_h q_h^* \pi^1(h|h) \end{pmatrix}
$$

Since $\Pi^1$ is non-singular, $p_{hi}^1$ and $p_{hh}^1$ are uniquely determined, thereby $p_{hl}^1$ and $p_{hh}^1$ are also uniquely determined. Notice that Equations (19) and (20) are exactly the zero profit conditions for each type of firm 1, respectively. It can be easily shown that both incentive and participation constraints are satisfied. Actually, it can be shown that much stronger form of incentive constraints is satisfied:

For all $i = 1, 2, \theta_i, \theta_i'$ and $\theta_{-i}$, $p^i(\theta_{-i}, \theta_i) - \theta_i q^i(\theta_{-i}, \theta_i) \geq p^i(\theta_{-i}, \theta_i') - \theta_i q^i(\theta_{-i}, \theta_i')$.

We now show that this mechanism attains $W_c^2$. The expected cost for implementing this mechanism $C$ is such that

$$
C = \pi(l, l)p_{hl}^1 + \pi(l, h)p_{hl}^1 + \pi(h, h)p_{hh}^1 + \pi(h, l)(p_{hl}^1 + p_{hh}^1)
$$

From Equation (19), $\pi(l, l)p_{hl}^1 + \pi(l, h)p_{hl}^1 = \theta_i q_l^* (\pi(l, l) + \pi(l, h))$. From Equation (20), $\pi(h, l)p_{hl}^1 + \pi(h, h)p_{hh}^1 = \theta_h q_h^* \pi(h, h)$. Note that $p_{hl}^2 = \theta_l q_l^*$. Putting these together, we have $C = (1 - \pi(h, h))\theta_l q_l^* + \pi(h, h)\theta_h q_h^*$. Also, the planner enjoys $q_l^*$ with probability $(1 - \pi(h, h))$ and $q_h^*$ with probability $\pi(h, h)$. Hence, the expected welfare to the planner is $W_c^2$.

We summarize the result in Proposition 5.

**Proposition 5.** In duopoly under correlated case, in any optimal mechanism, if at least one firm reports low cost, the firms with low reported cost together produce $q_l^*$ and the firm with high reported cost produces nothing. When both firms report high cost, two firms together produces $q_h^*$. Both types of each firm have zero profit. Furthermore, the planner attains the first best outcome as if he has the complete information.
The first best result in duopoly can be generalized to oligopoly with some qualifications. When there are more than two firms, even for two types of each firm, it does not suffice for the first best outcome result under incomplete information that \( \theta_i \)'s are correlated. \( \theta_i \)'s are correlated in some specific way:

**Condition 1.** For all \( i = 1, 2, \ldots, n, \pi^i(\theta_{-i}|\theta_i = \theta_i) \neq \pi^i(\theta_{-i}|\theta_i = \theta_h) \) for some \( \theta_{-i}. \)

Under Condition 1, Proposition 5 can be generalized to oligopoly.

**Proposition 6.** Under Condition 1, in \( n \)-firm oligopoly under correlated case, in any optimal mechanism, if at least one firm reports low cost, the firms with low reported cost together produce \( q^*_i \) and the firm with high reported cost produces nothing. When all firms report high cost, all firms together produces \( q^*_h \). Both types of each firm have zero profit. Furthermore, the planner attains the first best outcome.


Under Condition 1, the expected welfare to the planner \( W^{n}_{\text{cor}} \) is given by

\[
W^{n}_{\text{cor}} = (1 - \pi(h, h, \ldots, h))W^1(1) + \pi(h, h, \ldots, h)W^1(0),
\]

where \( \pi(h, h, \ldots, h) \) is the probability that \( \theta_i = \theta_h \) for all \( i = 1, 2, \ldots, n. \) \( W^{n}_{\text{cor}} \) does not always converge to \( W^1(1) \) when \( n \) increases. \( W^{n}_{\text{cor}} \) converges to \( W^1(1) \) if and only if \( \pi(h, h, \ldots, h) \) goes to 0 as \( n \) increases. Condition 1 does not guarantee that \( \pi(h, h, \ldots, h) \) goes to 0. Besides Condition 1, therefore, extra assumption about the joint probability distribution is needed for the planner to attain \( W^1(1). \)

## 5 Welfare Comparison

In this section, we compare the planner's welfare under different information structures. We begin with duopoly. First, we compare the i.i.d. case and the correlated case. In order to make a legitimate comparison, we assume that marginal distributions are the same under both information structures, i.e., \( \pi^1(l) = \pi^2(l) = \pi \in (0, 1). \)

\(^6\)This is a special case of Assumption 4 in Cremer and Mclean (1985).
Then, there is only one degree of freedom in the joint distribution. For some \( \epsilon \in [\max\{0, 2\pi - 1\}, \pi] \), \( \pi(l, l) = \epsilon, \pi(l, h) = \pi(h, l) = \pi - \epsilon \) and \( \pi(h, h) = 1 + \epsilon - 2\pi \).

When \( \epsilon = \pi \), \( \theta_1 \) and \( \theta_2 \) are perfectly positively correlated. If \( \epsilon = \pi^2 \), then \( \theta_1 \) and \( \theta_2 \) are i.i.d.

![Diagram](image)

**Figure 1**

By Proposition 5, when \( \epsilon \neq \pi^2 \), the expected welfare to the planner under correlated case, \( W_{cor}^2 \) is given by \( W_{cor}^2(\epsilon) = (2\pi - \epsilon)W^1(1) + (1 + \epsilon - 2\pi)W^1(0) \). When \( \epsilon = \pi^2 \), however, the expected welfare to the planner is not \( W_{cor}^2(\pi^2) = (1 - (1 - \pi^2)W^1(1) + (1 - \pi^2)W^1(0) \). When \( \epsilon = \pi^2 \), it reduces to i.i.d. case and \( \Pi_1 \) becomes singular. The planner, then, cannot extract whole surplus from both types of each firm. In this case, the expected welfare to the planner is, by Proposition 1, \( W^2(\pi)_{iid} = \pi W^1(1) + (1 - \pi)W^1(\pi) \). Note that \( W_{cor}^2(\pi^2) > W^2(\pi)_{iid} \) because \( W_{cor}^2(\pi^2) \) is the expected welfare to the planner which can be attained under complete information. This shows that the expected welfare to the planner as a function of \( \epsilon \) is discontinuous at \( \epsilon = \pi^2 \).

Since \( W_{cor}^2(\epsilon) \) is linear in \( \epsilon \) with negative coefficient \( (W^1(0) - W^1(1)) \), \( W_{cor}^2(\epsilon) \) is decreasing in \( \epsilon \). The larger \( \epsilon \) is, the more positively correlated \( \theta_1 \) and \( \theta_2 \) are and the planner becomes worse off.

We now compare the welfare to the planner under i.i.d. case and correlated case.
$W_{\text{cor}}^2(\epsilon) > W_{\text{id}}^2$ if and only if $\epsilon < \epsilon^*$ where

$$
\epsilon^* = 1 - (1 - \pi)\frac{(W^1(\pi) - W^1(0))}{(W^1(1) - W^1(0))}.
$$

Clearly $\epsilon^* < \pi$. Since $W_{\text{cor}}^2(\pi^2) > W_{\text{id}}^2$, $\epsilon^* > \pi^2$. Hence, the planner becomes better off under correlated case than under i.i.d. case if and only if $\epsilon < \epsilon^*$ with $\epsilon \neq \pi^2$. If $\epsilon > \epsilon^*$, the planner becomes better off under i.i.d. case than under correlated case. The intuition is follows. Unless $\epsilon = \pi^2$, the planner can extract the whole surplus by using correlation. In i.i.d. case, the planner cannot extract the whole surplus. The planner should allow some positive informational rent for low type of at least one firm. With $\epsilon > \epsilon^*$, although the planner can extract the whole surplus, $\theta_1$ and $\theta_2$ are too much positively correlated. With the same marginal distribution, the probability that both firms realize the high cost becomes higher and the probability that at least one firm realizes the low cost becomes lower under correlated case than under i.i.d. case. These do harm to the planner. With large $\epsilon$, this harmful effect outweighs the benefit from the full extraction of the surplus. Therefore, when $\epsilon$ is larger than $\epsilon^*$, the planner becomes better off under i.i.d. case although some positive informational rent should be allowed.

We now compare the independent case and the correlated case. Again, we assume that marginal distributions are the same under both information structures, i.e., $\pi^1(l) = \pi_1$ and $\pi^2(l) = \pi_2$ with $\pi_1 > \pi_2$.

\begin{tabular}{ccc}
\hline
$\theta_1$ & 1 & h \\
\hline
1 & $\pi_1 \pi_2$ & $\pi_1 (1 - \pi_2)$ \\
(1-$\pi_1$)$\pi_2$ & $1 - \pi_1$ & 1- $\pi_1$ \\
\hline
$\pi_2$ & 1-$\pi_2$ & \\
\hline
\end{tabular}

\begin{tabular}{ccc}
\hline
$\theta_1$ & 1 & h \\
\hline
1 & $\pi_1$ & \\
$\pi_2 - \epsilon$ & $1+\epsilon-\pi_1-\pi_2$ & 1- $\pi_1$ \\
\hline
\end{tabular}

**Figure 2**
Like the i.i.d. case, there is only one degree of freedom in the joint distribution. For some \( \epsilon \in [\max\{0, \pi_1 + \pi_2 - 1\}, \pi_2] \), \( \pi(l, l) = \epsilon, \pi(l, h) = \pi_1 - \epsilon, \pi(h, l) = \pi_2 - \epsilon \) and \( \pi(h, h) = 1 + \epsilon - \pi_1 - \pi_2 \). Notice that when \( \epsilon = \pi_1 \pi_2 \), \( \theta_1 \) and \( \theta_2 \) are independent.

By Proposition 5, with \( \epsilon \neq \pi_1 \pi_2 \), \( W^2_{\text{cor}}(\epsilon) = (\pi_1 + \pi_2 - \epsilon)W^1(1) + (1 + \epsilon - \pi_1 - \pi_2)W^1(0) \). With \( \epsilon = \pi_1 \pi_2 \), however, the expected welfare to the planner is, by Proposition 3, \( W^2_{\text{ind}} = \pi_1 W^1(1) + (1 - \pi_1)W^1(\pi_2) \). Since \( W^2_{\text{cor}}(\pi_1 \pi_2) > W^2_{\text{ind}} \), \( W^2_{\text{cor}}(\epsilon) \) is discontinuous at \( \epsilon = \pi_1 \pi_2 \).

By comparing \( W^2_{\text{cor}}(\epsilon) \) and \( W^2_{\text{ind}} \), \( W^2_{\text{cor}}(\epsilon) > W^2_{\text{ind}} \) if and only if \( \epsilon < \epsilon^* \) where

\[
\epsilon^* = 1 - (1 - \pi_1) \frac{(W^1(\pi_2) - W^1(0))}{(W^1(1) - W^1(0))}.
\]

Note that \( \pi_1 \pi_2 < \epsilon^* < \pi_1 \). Hence, the planner becomes better off under correlated case than under independent case if and only if \( \epsilon < \epsilon^* \) with \( \epsilon \neq \pi_1 \pi_2 \). The intuition is the same as in the comparison with i.i.d. case and correlated case.

We now consider the welfare to the planner in the limiting case where the number of firms increases indefinitely. In i.i.d. case, by Proposition 2, whenever \( \pi > 0 \), the planner attains \( W^1(1) \) in the limit. In correlated case with the same marginal distribution as i.i.d. case, although the planner may extract the whole surplus by virtue of Condition 1, the planner attains \( W^1(1) \) only when the probability that all the firms realize the high cost goes to zero. Condition 1, however, does not guarantee this. Given \( 0 < \pi < 1 \), we can easily construct a joint distribution which has the same marginal distribution as i.i.d. case, satisfies Condition 1, but the probability that all the firms realize the high cost does not go to zero. So we can conclude that in the limiting case, the planner becomes at least weakly better off under i.i.d. case than under correlated case.

For the independent case and correlated case, it is hard to compare in the limiting case. By Proposition 4, under independent case, the planner attains \( W^1(1) \) if and only if \( \Pi_{i=1}^n (1 - \pi_i) \) converges to zero, which is equivalent to \( \sum \pi_n = \infty \). Think of the following example. Set \( \pi_n = 1/(n + 1)^2 \). Then, \( \sum \pi_n < \infty \). Hence, in the limit, the planner cannot attain \( W^1(1) \) under independent case. It is, however, possible to construct a joint distribution which has the same marginal distribution, satisfies Condition 1, and the probability that all the firms realize the high cost goes to zero. (Actually, it is possible that the probability that all the firms realize the high cost is
zero for all \( n \).) Then, the planner attains \( W^1(1) \) in the limit under correlated case. Of course, we can construct an example using \( \pi_n = 1/(n+1) \) where opposite case holds. In other words, the planner attains \( W^1(1) \) under independent case, but he cannot under correlated case. Hence, generally the welfare comparison between independent case and correlated case is ambiguous in limiting case.
Appendix

In this appendix, we prove Propositions 2 and 4.

Proof of Proposition 2. Suppose there are \( n \) firms. Let \( \theta_i \) be the marginal cost for firm \( i \). We assume \( \theta_i \)'s are i.i.d. with \( \text{Prob}[\theta_i = \theta_i] = \pi \in (0,1) \). Let \( \theta = (\theta_1, \theta_2, \cdots, \theta_n) \). Then, a mechanism takes the following form:

\[
M^n = \{q^i(\theta), p^i(\theta)\}_{i=1,2,\cdots,n}.
\]

Let \( q^i = E_{\theta - i} \{q^i(\theta)|\theta_i = \theta_i\} \) and \( q^i_h = E_{\theta - i} \{q^i(\theta)|\theta_i = \theta_h\} \), where \( \theta - i = (\theta_1, \theta_2, \cdots, \theta_{i-1}, \theta_{i+1}, \cdots, \theta_n) \). Define \( p^j_j, j = l, h \) similarly.

Simplifying the notation, we sometimes write down \( q^i(\theta) \) and \( p^i(\theta) \) by identifying the firms who reports low cost. Let \( C \subset N = \{1,2,\cdots,n\} \) and \( |C| \) be the number of firms in \( C \). Then, \( q^i_C \) is the quantity that firm \( i \) will produce when \( C \) is the set of firms who report low cost. \( p^i_C \) is similarly defined.

Any feasible mechanism must satisfy both incentive and participation constraints. Like the duopoly, for each firm, the participation constraint for \( \theta_h \) and the incentive constraint \( \theta_l \) must be binding at any optimal mechanism:

\[
\forall i = 1,2,\cdots,n, p^i_h = \theta_h q^i_h \quad \text{and} \quad p^i_l = \theta_l q^i_l + \theta_l (q^i_l - q^i_h) \quad \text{with} \quad q^i_l \geq q^i_h.
\]

The planner's objective function is

\[
W^n = E_{\theta} \{V(\sum_{i=1}^{n} q^i(\theta)) - \sum_{i}^{n} p^i(\theta))\}.
\]

From the incentive and participation constraints for each firm, we have:

\[
\forall i, E_{\theta} p^i(\theta) = \theta_h q^i_h + \pi \theta_l (q^i_l - q^i_h).
\]

Therefore, \( E_{\theta} \{\sum_{i=1}^{n} p^i(\theta)\} = \theta_h \sum_{i=1}^{n} q^i_h + \pi \theta_l (\sum_{i=1}^{n} q^i_l - \sum_{i=1}^{n} q^i_h) \). Hence, the planner's objective function is

\[
W^n = E_{\theta} V(\sum_{i=1}^{n} q^i(\theta)) - \theta_h \sum_{i=1}^{n} q^i_l - \pi \theta_l (\sum_{i=1}^{n} q^i_l - \sum_{i=1}^{n} q^i_h).
\]
By differentiating $W^n$ with respect to $q^i(\theta)$ and setting it to zero, we have the following set of equations:

$$\forall i \in C, \frac{\partial W^n}{\partial q_C^i} = \pi |\mathcal{C}|(1 - \pi)^{n-|\mathcal{C}|} V'(\sum_{i=1}^{n} q_C^i) - \pi |\mathcal{C}|(1 - \pi)^n - |\mathcal{C}| \theta_i = 0, \quad (21)$$

$$\forall i \not\in C, \frac{\partial W^n}{\partial q_C^i} = \pi |\mathcal{C}|(1 - \pi)^{n-|\mathcal{C}|} V'(\sum_{i=1}^{n} q_C^i) - \pi |\mathcal{C}|(1 - \pi)^n - |\mathcal{C}| - 1 \theta_i$$

$$+ \pi |\mathcal{C}|(1 - \pi)^n - |\mathcal{C}| - 1 \theta_i = 0. \quad (22)$$

If $C \neq \emptyset$, Equations (27) and (28) give that for all $i \in C$, $\sum_{i=1}^{n} q_C^i = q_1^*$, and for all $i \not\in C$, $\sum_{i=1}^{n} q_C^i = q(\pi)$. Hence, there is a corner solution. Namely, for all $i \not\in C$, $q_C^i = 0$. Therefore, Equation (27) gives $\sum_{i \in C} q_1^i = q_1^*$. If $C = \emptyset$, Equation (29) gives that for all $i$, $\sum_{i=1}^{n} q_0^i = q(\pi)$. Thus, in any optimal mechanism, if at least one firm reports low cost, the firms with low reported cost together produce $q_1^*$. If all firms report high cost, the firms together produce $q(\pi)$.

The expected cost for implementing the optimal mechanism is

$$C^n(\pi) = E_{\theta} \{\sum_{i=1}^{n} p^i(\theta)\} = \theta_h \sum_{i=1}^{n} q_h^i + \pi(\sum_{i=1}^{n} q_1^i - \sum_{i=1}^{n} q_h^i).$$

So we have to calculate $\sum_{i=1}^{n} q_h^i$ and $\sum_{i=1}^{n} q_1^i$. Note that $q_h^i = E_{\theta_i} \{q^i(\theta)|\theta_i = \theta_h\}$. In any optimal mechanism, when firm $i$ reports high cost, it may produce positive quantity only if all the other firm also report high cost whose probability is $(1 - \pi)^{n-1}$. Hence $q_h^i = (1 - \pi)^{n-1} q_0^i$. Therefore, $\sum_{i=1}^{n} q_h^i = (1 - \pi)^{n-1} \sum_{i=1}^{n} q_0^i = (1 - \pi)^{n-1} q(\pi)$ by Equation (29). Note that $q_1^i$ is calculated conditional on that firm $i$ is a low cost firm whose probability is $\pi$. Conditional on that firm $i$ is a low cost firm, the quantity firm $i$ produces $q_C^i$ with probability $\pi |\mathcal{C}| - 1 (1 - \pi)^{n-|\mathcal{C}|}$. Hence when exactly $k$ firms report low cost, $\sum_{i=1}^{n} q_1^i$ is $q_1^*$ with probability $nC_k \pi^{k-1}(1 - \pi)^{n-k}$. Therefore,

$$\sum_{i=1}^{n} q_1^i = \{\sum_{k=1}^{n} nC_k \pi^{k-1}(1 - \pi)^{n-k}\}q_1^*$$
\begin{align*}
&= \pi^{-1}\{\sum_{k=1}^{n} C_{k} \pi^{k}(1-\pi)^{n-k}\} q^{*} \\
&= \pi^{-1}\{1-(1-\pi)^{n}\} q^{*}.
\end{align*}

So we have

\begin{align*}
C^{n}(\pi) &= \theta_{h} \sum_{i=1}^{n} q^{i}_{h} + \pi \theta_{l}(\sum_{i=1}^{n} q^{i}_{l} - \sum_{i=1}^{n} q^{i}_{h}) \\
&= (1-\pi)^{n-1}\theta_{h} q(\pi) + \pi \theta_{l}[\pi^{-1}\{1-(1-\pi)^{n}\} q^{*} - (1-\pi)^{n-1} q(\pi)] \\
&= (1-\pi)^{n-1}\{\theta_{h} q(\pi) + \pi \theta_{l}(q^{*} - q(\pi))\} + (1-(1-\pi)^{n-1})\theta_{l} q^{*}.
\end{align*}

This shows that \( C^{n} \) is a convex combination of \( C^{1}(\pi) \) and \( C^{1}(1) \), \( C^{n}(\pi) = (1-\pi)^{n-1} C^{1}(\pi) + (1-(1-\pi)^{n-1}) C^{1}(1) \).

Note that in any optimal mechanism, \( E_{\theta} V(\sum_{i=1}^{n} q^{i}(\theta)) = (1-\pi)^{n-1}\{\pi V(q^{*}) + (1-\pi) V(q(\pi))\} + (1-(1-\pi)^{n-1}) V(q^{*}) \). Hence the expected welfare for the planner is given by

\begin{align*}
W^{n}(\pi) &= E_{\theta} V(\sum_{i=1}^{n} q^{i}(\theta)) - C^{n}(\pi) \\
&= (1-\pi)^{n-1}\{\pi V(q^{*}) + (1-\pi) V(q(\pi)) - C^{1}(\pi)\} \\
&\quad + (1-(1-\pi)^{n-1})\{V(q^{*}) - C^{1}(1)\} \\
&= (1-\pi)^{n-1} W^{1}(\pi) + (1-(1-\pi)^{n-1}) W^{1}(1).
\end{align*}

Again, \( W^{n}(\pi) \) is given by a convex combination of \( W^{1}(\pi) \) and \( W^{1}(1) \).

**Proof of Proposition 4.** The proof of Proposition 4 is similar to the proof of Proposition 2. We maintain the same notation. Here, we assume \( \theta \)'s are independently distributed with \( \text{Prob}[\theta_{i} = \theta] = \pi_{i} \in (0,1) \). We assume that \( \pi_{1} > \pi_{2} > \cdots > \pi_{n} \). Let \( \pi = (\pi_{1}, \pi_{2}, \ldots, \pi_{n}) \).

In independent case, \( E_{\theta} p^{i}(\theta) = \theta_{h} q^{i}_{h} + \pi_{i} \theta_{l}(q^{i}_{l} - q^{i}_{h}) \). Hence, we have \( E_{\theta}\{\sum_{i=1}^{n} p^{i}(\theta)\} = \theta_{h} \sum_{i=1}^{n} q^{i}_{h} + \theta_{l}(\sum_{i=1}^{n} \pi_{i} q^{i}_{l} - \sum_{i=1}^{n} \pi_{i} q^{i}_{h}) \), and the planner’s objective function is given by

\[ W^{n} = E_{\theta} V(\sum_{i=1}^{n} q^{i}(\theta)) - \theta_{h} \sum_{i=1}^{n} q^{i}_{h} - \theta_{l}(\sum_{i=1}^{n} \pi_{i} q^{i}_{l} - \sum_{i=1}^{n} \pi_{i} q^{i}_{h}) \].

Differentiating \( W^{n} \) with respect to \( q^{i}(\theta) \) and setting to zero gives the same conditions as in i.i.d. case except that \( C = \emptyset \). When \( C = \emptyset \), we have

\[ \forall i, \frac{\partial W^{n}}{\partial q^{i}_{\theta}} = \prod_{i=1}^{n}(1-\pi_{i}) V'(\sum_{i=1}^{n} q^{i}_{\theta}) - \prod_{j \neq i}(1-\pi_{j}) \theta_{h} + \prod_{j \neq i}(1-\pi_{j}) \pi_{i} \theta_{l} = 0. \]
Each equation gives \((1 - \pi_i)V'(\sum_{i=1}^{n} q_i^*) - \theta_i + \pi_i \theta_i = 0\), i.e., \(\sum_{i=1}^{n} q_i^* = q(\pi_i)\). As in the duopoly case, there arises a corner solution when all the firms report high cost. In this case, the planner will let \(n\)-th firm produce \(q(\pi_n)\) and all the other firm produce nothing. The intuition is the same in duopoly case. This is the least costly way to induce truth-telling. So in independent case, in any optimal mechanism, if at least one firm reports low cost, the firms with low reported cost together produce \(q_i^*\) and the firms with high reported cost produce nothing. When all the firms report high cost, the \(n\)-th firm produces \(q(\pi_n)\) and the other firms produce nothing. Notice that this implies that \(q_i^* = 0\) except \(i = n\). Hence \(\sum_{i=1}^{n} q_i^* = \prod_{i=1}^{n-1} (1 - \pi_i)q(\pi_n)\).

Like i.i.d. case, after some algebra we can also show that \(\sum_{i=1}^{n} \pi_i q_i^* = (1 - \prod_{i=1}^{n} (1 - \pi_i)^n)q_i^*\). Therefore, we have

\[
C^n(\pi) = \theta_h \sum_{i=1}^{n} q_i^* + \theta_i \left( \sum_{i=1}^{n} \pi_i q_i^* - \sum_{i=1}^{n} \pi_i q_h^* \right)
\]
\[
= \prod_{i=1}^{n-1} (1 - \pi_i)\theta_h q(\pi_n) + \theta_i \left[ \{1 - \prod_{i=1}^{n} (1 - \pi_i)\} q_i^* - \pi_n \prod_{i=1}^{n-1} (1 - \pi_i)q(\pi_n) \right]
\]
\[
= \prod_{i=1}^{n-1} (1 - \pi_i)\{ \theta_h q(\pi_n) + \pi_n (q_i^* - q(\pi_n)) \} + (1 - \prod_{i=1}^{n-1} (1 - \pi_i))\theta_i q_i^*
\]
\[
= \prod_{i=1}^{n-1} (1 - \pi_i)C^1(\pi_n) + (1 - \prod_{i=1}^{n-1} (1 - \pi_i))C^1(1)
\]

Namely, \(C^n(\pi)\) is given by a convex combination of \(C^1(\pi_n)\) and \(C^1(1)\). Similarly, \(W^n(\pi)\) is given by a convex combination of \(W^1(\pi_n)\) and \(W^1(1)\), \(W^n(\pi) = \prod_{i=1}^{n-1} (1 - \pi_i)W^1(\pi_n) + (1 - \prod_{i=1}^{n-1} (1 - \pi_i))W^1(1)\).
References


