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IDENTIFICATION IN EMPIRICAL MODELS OF AUCTIONS*

by

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and

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Identification in Empirical Models of Auctions

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In this paper, we examine the question of identification in empirical models of auctions. In particular, we prove that the probability density function of the equilibrium winning bid at a sealed-bid auction within the independent private values paradigm is uniquely defined.

1. Introduction

In empirical applications of game theory to auctions, the distribution of unobserved (or latent) characteristics is assumed to be common knowledge to the players of the game. For example, in the independent private values model of an auction the distribution of valuations is known to all bidders. Moreover, each bidder knows that his opponent knows the distribution of valuations, and his opponents know that he knows, etc.. Based upon their knowledge of the distribution of latent characteristics, players are assumed to choose strategies which maximize their expected pay-offs. That is, given their realization from the valuation distribution, players choose bids to maximize the expected return from winning the auction. By appealing to a particular concept of equilibrium (e.g., Bayesian-Nash), the equilibrium of the game can be characterized.

The goal of some recent empirical research has been to determine if the predictions of game theory are consistent with observed data. One proposed research strategy (see, for example, Paarsch [1989, 1992]) involves noting that the equilibrium strategies of players depend upon the distribution of latent characteristics. This implies that the equilibrium strategies of players are random variables. If the distribution of latent characteristics comes from a particular class of distributions, then rational behaviour within that class of distributions will impose certain restrictions upon the data generating process of the equilibrium strategies which can be tested. Paarsch (1989, 1992) as well as Laffont, Ossard, and Vuong (1991) have attempted to
apply this structural econometric framework to the interpretation of field data from actual auctions using the econometric methods developed in Paarsch (1992), Donald and Paarsch (1992, 1993) as well as Laffont, Ossard, and Vuong (1991). Implicit in the above mentioned empirical applications is the assumption that the econometrician uses the same family of distributions for the latent random variable as the bidders, but that unlike the bidders in the model the econometrician does not know the structural parameters of the distribution of latent characteristics, and must therefore estimate them. A natural question to ask is whether more than one latent distribution exists which would be consistent with the observed data. Put another way “Are empirical models of auction identified?”

In this paper, we investigate the issue of identification in empirical models of auctions. The paper is in three more parts. In section 2, we outline a simple empirical model of an auction within the independent private values paradigm. In section 3, we prove the uniqueness of the probability density function of the equilibrium winning bid, while in section 4 we conclude.

2. An Auction Model within the Independent Private Values Paradigm

In this section, we develop a simple model of a procurement auction. To illustrate the particular class of estimation problems in which we are interested, we model a sealed-bid auction as a non-coöperative game.\(^1\) We consider auctions at which a known number of bidders \(n\) compete to perform a single task for a government agency, with the lowest bidder winning the auction. We assume that the heterogeneity across agents in the cost of performing the task can be described by a continuous random variable \(c\) which has the probability density function \(f(c)\) and the cumulative distribution function \(F(c)\). Each player is assumed to know his own cost, but not those of his opponents. The costs of players are assumed to be independent draws from \(F(c)\), and \(F(c)\) is assumed to be common knowledge. We assume that bidders are risk neutral with respect to winning the auction, or if they are risk averse, that the von Neumann-Morgenstern utility function falls within the constant relative risk aversion class. Thus,

\[
U(y) = \eta y^{1/\eta} \quad \eta \geq 1.
\]

\(^1\) A reader who is unfamiliar with the auction literature will find the surveys by Milgrom (1985, 1987) and McAfee and McMillan (1987) helpful.
The \( i \)th bidder is assumed to choose a bid \( b_i \) to maximize his expected profit (utility). Finally, we focus upon symmetric Bayesian-Nash equilibria.

To construct the equilibrium, suppose that the \( m = n - 1 \) opponents of player \( i \) are using a common bidding rule \( \beta(c) \) which is increasing and differentiable in \( c \). Since costs are modelled as independent draws from a common distribution, the probability of player \( i \) winning with bid \( b_i \) equals the probability that each of his opponents bids higher because each has a higher cost

\[
[1 - F(\beta^{-1}(b_i))]^m.
\]

Here \( \beta^{-1}(b_i) \) denotes the inverse of the bid function. Given that his cost \( c_i \) is determined before the bidding, player \( i \)'s choice of \( b_i \) has only two effects upon his expected utility

\[ U(b_i - c_i) \cdot [1 - F(\beta^{-1}(b_i))]^m. \]

The lower is \( b_i \), the higher is his probability of winning the auction \([1 - F(\beta^{-1}(b_i))]^m\), but the lower is his pay-off when he wins \( U(b_i - c_i) \). Maximizing behaviour implies that the optimal bid solves the first-order condition

\[
U'(b_i - c_i) \cdot [1 - F(\beta^{-1}(b_i))]^m - mU(b_i - c_i) \cdot f(\beta^{-1}(b_i)) \cdot [1 - F(\beta^{-1}(b_i))]^{m-1} \cdot \frac{d\beta^{-1}(b_i)}{db_i} = 0. \tag{2.1}
\]

Symmetry among bidders implies

\[ b_i = \beta(c_i). \tag{2.2} \]

Substituting (2.2) into (2.1), recalling that \( d\beta^{-1}(b_i)/db_i = 1/\beta'(c_i) \), and requiring (2.1) to hold for all feasible \( c_i \)'s, yields the following differential equation for \( \beta \):

\[
\beta'(c)[1 - F(c)]^m - m\eta\beta(c)f(c)[1 - F(c)]^{m-1} = -m\eta(cf(c)[1 - F(c)]^{m-1}. \tag{2.3}
\]

Integrating (2.3), and imposing the boundary condition \( \beta(\infty) = \infty \), yields

\[
\beta(c) = c + \int_c^\infty \frac{[1 - F(\xi)]^{m\eta} \, d\xi}{[1 - F(c)]^{m\eta}}. \tag{2.4}
\]

\[ \text{In fact, simply imposing } \beta(\infty) = \infty \text{ is insufficient to guarantee a unique solution since adding any constant } a \text{ to that solution is also a solution. In this case, } a = 0 \text{ is the appropriate constant.} \]
Because the bidding rules are functions of the random variable $c$, the bids are also random variables and their densities are related to $f(c)$. The density of $\beta(c)$, for example, is

$$\frac{f(\beta^{-1}(b))}{\beta'(\beta^{-1}(b))}$$

where

$$\beta'(c) = \frac{m\eta f(c) \int_0^\infty [1 - F(\xi)]^{m\eta} d\xi}{[1 - F(c)]^{m\eta+1}}$$

is the Jacobian for the transformation of $c$ to $\beta(c)$.

The winning bid is a simple function of $c$. Thus, its density is related to $f(c)$. The density of the lowest cost $z = c_{(1:n)}$ is

$$n[1 - F(z)]^m f(z),$$

so the density of the winning bid $w = \beta(z)$, denoted $h(w)$, is

$$h(w) = \frac{n[1 - F(\beta^{-1}(w))]}{\beta'(\beta^{-1}(w))} = \frac{n[1 - F(\beta^{-1}(w))]^{m\eta+n}}{m\eta \int_{\beta^{-1}(w)}^\infty [1 - F(\xi)]^{m\eta} d\xi}.$$

Suppose that the random variable $c$ comes from some distribution which can be uniquely characterized by the $(p \times 1)$ parameter vector $\theta$. Thus,

$$F(c) = F(c; \theta). \quad (2.5)$$

The parameter vector $\theta$ will embed itself in the density of $w$

$$h(w; \theta, \eta) = \frac{n[1 - F(\beta^{-1}(w; \theta); \theta)]^{m\eta+n}}{m\eta \int_{\beta^{-1}(w; \theta)}^\infty [1 - F(\xi; \theta)]^{m\eta} d\xi}. \quad (2.6)$$

If the econometrician knows the family from which (2.5) is drawn, then for a random sample of observation he can derive the likelihood function of the sample data and then, using the methods of Donald and Paarsch (1992, 1993), estimate the parameters of the latent process.

We shall assume that $F(c; \theta^0)$ is the true distributions of costs, and that $\eta^0$ is the true value of the risk aversion parameter $\eta$. Note that the derived distribution of the winning bid has support upon

$$\left[ \int_0^\infty [1 - F(\xi; \theta^0)]^{m\eta^0}, \infty \right) = [\mathcal{Z}_0, \infty).$$
Suppose, however, that the econometrician must guess at (2.5). Is there a class of distributions \( \mathcal{G} \) in which members \( G \) other than \( F \) solve

\[
h(w; \theta) = \frac{n[1 - G(B^{-1}(w))]^{mn+n}}{mn \int_{B^{-1}(w)}[1 - G(\xi)]^{mn} \, d\xi}
\]  
(2.7)

where

\[
B(c) = c + \frac{\int_{c}^{\infty} [1 - G(\xi)]^{mn} \, d\xi}{[1 - G(c)]^{mn}}
\]  
(2.8)

3. Conditions for Identification

In showing that the parameters vector \((\theta^0, \eta^0)\) is identifiable, we shall proceed in two steps. To begin we shall show that \((F, \eta^0)\) is the only distribution–risk aversion parameter pair that give rise to the true distribution for the winning bid. The identifiability of \(\theta^0\) will then follow from standard results concerning uniqueness of the parameters defining \(F\).

We shall first be concerned with the identifiability of the pair \((F, \eta^0)\); i.e., we consider the question of whether \((F, \eta^0)\) is the only pair that gives rise to the true distribution of the winning bid. Stated another way, is there another \((G, \eta)\), such that either \(G\) is different from \(F\) (in a sense to be defined below), \(\eta \neq \eta^0\), or both differ, that gives rise to the same probability law for the winning bid? The main regularity assumption on the class of distributions (of which \(F\) is a member) is contained in Assumption 1.

**Assumption 1.**

The class of distributions \( \mathcal{G} \) contains distributions \(G(c)\) defined upon \([0, \infty)\) that are monotonically increasing, continuously differentiable with continuous density \(g(c)\) such that \(g(c) > 0\) on \((0, \infty)\).

Note that this assumption imposes no restrictions upon the value of the probability density function at its lower bound. Our results also hold when the support of \(g(c)\) is compact. A large number of families of probability density functions satisfies Assumption 1. Identification of \(F\) will be relative to the class \( \mathcal{G} \), with \(F\) assumed to be a member of \( \mathcal{G} \). We shall assume that \( \eta \) belongs to some set of real numbers \( \mathcal{E} = [1, \Delta] \), for some large real number \( \Delta \). The sense in which we say that \( G \in \mathcal{G} \) differs from \( F \), written \( G \neq F \), is given by the following definition:
**Definition 1.**

We say that $G \neq F$ if there exists an open interval $A \subset (0, \infty)$ for which either $F(c) > G(c)$ or $G(c) > F(c)$ for all $c \in A$.

The sense in which $(F, \eta^0)$ is identified is given in the final definition. Note that this definition requires that there exist some event (concerning the winning bid) for which the probability under $(F, \eta^0)$ differs from the probability under $(G, \eta)$ for any $(G, \eta) \neq (F, \eta^0)$. This is the standard notion of identification as in Wald (1949).

**Definition 2.**

$(F, \eta^0)$ is identifiably unique relative to $\mathcal{G} \times \mathcal{E}$, if for any $G \in \mathcal{G}$ and $\eta \in \mathcal{E}$ such that $G \neq F$, $\eta \neq \eta^0$, or both,

$$\Pr[w \in A \mid F, \eta^0] \neq \Pr[w \in A \mid G, \eta]$$

for any Lebesgue measurable set $A$, where $\Pr[\cdot \mid F, \eta^0]$ denotes the probability calculated under $(F, \eta^0)$.

Note that it will be sufficient to show that the density of $w$ under $(F, \eta^0)$ differs from that under $(G, \eta)$ over some open set in $[\mathfrak{S}_0, \infty)$.

**Theorem 1.**

Given Assumption 1, $(F, \eta^0)$ is identifiably unique in $\mathcal{G} \times \mathcal{E}$.

**Proof:** Note that

$$\Pr[w < k \mid F, \eta^0] = 1 - [1 - F(\beta^{-1}(k))]^n$$

and

$$\Pr[w < k \mid G, \eta] = 1 - [1 - G(B^{-1}(k))]^n$$

where $\beta^{-1}$ depends upon the pair $(F, \eta^0)$ and $B^{-1}$ depends upon the pair $(G, \eta)$. The result proceeds by showing that if $(F, \eta^0) \neq (G, \eta)$, then there exists a $k$ for which these differ.

First note that if $(F, \eta^0) \neq (G, \eta)$, and if

$$\mathfrak{S}_0 = \int_0^\infty [1 - F(\xi)]^{m\eta} d\xi \neq \int_0^\infty [1 - G(\xi)]^{m\eta} d\xi,$$

6
then it is obvious that one can find a $k$ such that $\Pr[w < k \mid F, \eta^0] \neq \Pr[w < k \mid G, \eta]$. Suppose that

$$\Xi_0 > \int_0^\infty [1 - G(\xi)]^{mn} \, d\xi$$

then let $k = \Xi_0$ and by Assumption 1,

$$\Pr[w < k \mid G, \eta] > \Pr[w < k \mid F, \eta^0] = 0$$

while if

$$\Xi_0 < \int_0^\infty [1 - G(\xi)]^{mn} \, d\xi$$

then, letting $k = \int_0^\infty [1 - G(\xi)]^{mn} \, d\xi$, we have by Assumption 1,

$$0 = \Pr[w < k \mid G, \eta] < \Pr[w < k \mid F, \eta^0].$$

If, on the other hand, $(F, \eta^0) \neq (G, \eta)$ and

$$\Xi_0 = \int_0^\infty [1 - F(\xi)]^{mn^0} \, d\xi = \int_0^\infty [1 - G(\xi)]^{mn} \, d\xi, \quad (3.1)$$

then there are two cases to consider. First, if $\eta \neq \eta^0$, then the densities at the lower bounds ($\Xi_0$) under $(F, \eta^0)$ and $(G, \eta)$ are such that

$$h(\Xi_0 | F, \eta^0) = \frac{n}{mn^0, \Xi_0} \neq \frac{n}{mn, \Xi_0} = h(\Xi_0 | G, \eta). \quad (3.2)$$

Since the density functions in both cases are continuous, there must exist an $\epsilon > 0$ such that over the interval $[\Xi_0, \Xi_0 + \epsilon)$ one density is larger than the other (depending upon whether $\eta^0 > \eta$ or $\eta^0 < \eta$). This implies that

$$\Pr[w < \Xi_0 + \epsilon | F, \eta^0] \neq \Pr[w < \Xi_0 + \epsilon | G, \eta]. \quad (3.3)$$

The second possibility assuming that (3.1) holds is for $\eta^0 = \eta$ and $F \neq G$. Under these conditions there exist disjoint intervals $A_1$ and $A_2$ such that for $c \in A_1$, $F < G$ and

$$[1 - F(c)]^{mn^0} > [1 - G(c)]^{mn} \quad (3.4)$$

and for $c \in A_2$, $F > G$

$$[1 - F(c)]^{mn^0} < [1 - G(c)]^{mn}. \quad (3.5)$$
Without loss of generality, suppose that $A_1$ occurs before $A_2$ as $c$ increases. Let

$$c_l = \inf\{c : F(c) < G(c)\}.$$  

Note that there exists a $\delta$ such that on $(c_l, c_l + \delta)$ $F(c) < G(c)$. This implies that $\left[1 - F(c)\right]^{\eta^0} > \left[1 - G(c)\right]^{\eta}$. Also define

$$c_u = \inf\{c : c > c_l + \delta, F(c) = G(c)\}$$

where $c_u$ exists because of (3.1). By construction $F(c_u) = G(c_u)$ and over $(\mathcal{G}_0, c_u)$, $F(c) \leq G(c)$. Hence,

$$\left[1 - F(c)\right]^{\eta^0} \geq \left[1 - G(c)\right]^{\eta}.$$  

Also by construction, for all $c \in (c_l, c_u)$, $F(c) < G(c)$. Thus,

$$\left[1 - F(c)\right]^{\eta^0} > \left[1 - G(c)\right]^{\eta}.$$  

Let

$$k = k_u = \beta(c_u)$$

so that $\beta^{-1}(k_u) = c_u$. Note, however, that

$$B^{-1}(k_u) < c_u.$$  

To see why this is, note that by construction

$$\int_{c_u}^{\infty} \left[1 - F(c)\right]^{\eta^0} d\xi < \int_{c_u}^{\infty} \left[1 - G(c)\right]^{\eta} d\xi,$$

so that

$$k_u = \beta(c_u) < B(c_u)$$

since we also have that $F(c_u) = G(c_u)$. Hence, we have that

$$\Pr[w < k_u | F, \eta^0] = 1 - \left[1 - F(c_u)\right]^n$$

$$= 1 - \left[1 - G(c_u)\right]^n > 1 - \left[1 - G(B^{-1}(k_u))\right]^n = \Pr[w < k_u | G, \eta]$$

and the result follows.
A consequence of this result is that if we have proposed a family of distributions \( F(c; \theta) = F_\theta \), then if the parameter \( \theta^0 \) is identifiably unique in the sense of generating a unique probability model for \( c \), then it is also identifiably unique in the sense of generating a unique probability model for the winning bid.

**Corollary 1.**

If for any \( \theta \in \Theta \), such that \( \theta \neq \theta^0 \), we have \( F_\theta \neq F \), then \( \theta^0 \) is identifiably unique in the model for the winning bid.

In these results, we have not imposed any regularity conditions beyond those contained in Assumption 1. There may be assumptions on \( F \) that are needed for there to be an equilibrium in the auction model, but these do not need to be imposed to identify the distribution of costs given the distribution of winning bids. Although the result of Theorem 1 implies the identifiability of the distribution of costs and the risk aversion parameter given the winning bid, it is not necessarily the case that the bidding rule is unique. Consider the following example with exponentially distributed costs. Suppose

\[
[1 - F(c)] = \exp(-\theta c)
\]

and \( \eta > 1 \) so agents are risk averse, then

\[
\beta(c) = c + \frac{1}{\eta \theta m}.
\]

When

\[
[1 - G(c)] = \exp(-\eta \theta c),
\]

but agents are risk neutral (\( \eta = 1 \)), then

\[
\beta(c) = c + \frac{1}{\eta \theta m}
\]

too.

4. Conclusions

In this note, we have proven that the distribution of the equilibrium winning bid for agents bidding at a sealed-bid auction within the independent private values paradigm is uniquely defined, thus identified. This result carries over to the distribution of any one of the bids at sealed-bid auctions within this paradigm. The presence of a parametric class of risk aversion does not change this result.
B. Bibliography


