PROVING DIRICHLET'S THEOREM ON ARITHMETIC PROGRESSIONS

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Abstract. First proved by German mathematician Dirichlet in 1837, this important theorem states that for coprime integers a, m, there are an infinite number of primes p such that $p = a \pmod{m}$. This is one of many extensions of Euclid's theorem that there are infinitely many prime numbers. In this paper, we will formulate a rather elegant proof of Dirichlet's theorem using ideas from complex analysis and group theory.

Contents

1. Introduction of Dirichlet's Theorem

Dirichlet's Theorem is particularly noteworthy because, despite the complex analysis and group theory required to prove his statement, it can be written in very simple terms.

Theorem 1.1 (Dirichlet's Theorem). *Let $a, m \in \mathbb{Z}^+$ be relatively prime. There exist infinitely many prime numbers p such that $p \equiv a \pmod{m}$.*

The proof that there are infinitely many primes is, as many real analysis students will recall, surprisingly simple. We suppose that $A = \{p_1, p_2, \cdots, p_k\}$ is the ordered set of all prime numbers and let $q = p_1 p_2 \cdots p_k + 1$. If q is prime, then it is missing from our set A and we are done. If q is not prime, then it is divisible by some prime $p_j \in A$. We also know p_j divides $q - 1$ by the definition of q, so it must divide the difference $q - (q - 1) = 1$. But there is no prime which divides 1, so we have a contradiction, which concludes the proof.

However, attempting to prove Dirichlet's stronger statement will require more work, so we proceed by exploring the idea of Dirichlet series.

2. Dirichlet Series

We begin this section with some useful lemmas.

Lemma 2.1. *Let U be an open subset of the complex plane and let (f_n) be a sequence of analytic functions on U that converges uniformly on every compact subset to a function f. Then, f is analytic on U and the derivatives f_n' of the f_n converge uniformly on all compacrt subsets to the derivative f' of f.*

Proof. Let D be a closed disk contained in U with boundary ∂D. By the Cauchy formula, we have

$$f_n(z) = \frac{1}{2\pi i}\int_{\partial D}\frac{f_n(w)}{w-z}dw$$

for all z interior to D. By the uniform convergence of (f_n), we have that

$$f(z) = \frac{1}{2\pi i}\int_{\partial D}\frac{f(w)}{w-z}dw,$$

which shows that f is analytic on D. It follows that f is analytic on U as well. To show that (f_n') uniformly converges to the derivative f', fix $z \in D$ and let (ε_n) be such that $|f_n(z) - f(z)| \leq \varepsilon_n$ for all z, n, with $\varepsilon_n \to 0$. Observe,

$$\begin{aligned}
|f_n'(z) - f'(z)| &= \left|\frac{1}{2\pi i}\int_{\partial D}\frac{f_n(w)}{(w-z)^2}dw - \frac{1}{2\pi i}\int_{\partial D}\frac{f(w)}{(w-z)^2}dw\right| \\
&= \left|\frac{1}{2\pi i}\int_{\partial D}\frac{f_n(w)-f(w)}{(w-z)^2}dw\right| \\
&\leq \frac{1}{2\pi i}\int_{\partial D}\frac{|f_n(w)-f(w)|}{(w-z)^2}dw \\
&\leq \frac{1}{2\pi i}\int_{\partial D}\frac{\varepsilon_n}{(w-z)^2}dw \\
&= \frac{\varepsilon_n}{2\pi i}\int_{\partial D}\frac{1}{(w-z)^2}dw.
\end{aligned}$$

If we let

$$\gamma_n = \frac{\varepsilon_n}{2\pi i}\int_{\partial D}\frac{1}{(w-z)^2}dw,$$

it is clear that $|f_n'(z) - f'(z)| \leq \gamma_n$ for each n, with $\gamma_n \to 0$, thus completing the proof of uniform convergence. □

Lemma 2.2 (Abel's Lemma)**.** *Let (a_n) and (b_n) be two sequences. Put*

$$A_{m,p} = \sum_{n=m}^{n=p} a_n \quad \text{and} \quad S_{m,m'} = \sum_{n=m}^{n=m'} a_n b_n.$$

Then one has:

$$S_{m,m'} = A_{m,m'}b_{m'} + \sum_{n=m}^{n=m'-1} A_{m,n}(b_n - b_{n+1}).$$

Proof. Observe that

$$A_{m,n} - A_{m,n-1} = \sum_{m}^{n} a_k - \sum_{m}^{n-1} a_k = a_n.$$

So, by making this substitution for a_n and expanding the sum we have:

$$\begin{aligned}
S_{m,m'} &= \sum_{m}^{m'} a_n b_n \\
&= \sum_{m}^{m'} (A_{m,n} - A_{m,n-1}) b_n \\
&= \sum_{m}^{m'} (A_{m,n} b_n - A_{m,n-1} b_n) \\
&= (A_{m,m}b_m - A_{m,m-1}b_m) + (A_{m,m+1}b_{m+1} - A_{m,m}b_{m+1}) + \\
&\quad (A_{m,m+2}b_{m+2} - A_{m,m+1}b_{m+2}) + \cdots + \\
&\quad (A_{m,m'-1}b_{m'-1} - A_{m,m'-2}b_{m'-1}) + (A_{m,m'}b_{m'} - A_{m,m'-1}b_{m'})
\end{aligned}$$

Note that $A_{m,m-1} = 0$, and we can regroup these terms to get

$$\begin{aligned}
S_{m,m'} &= A_{m,m}(b_m - b_{m+1}) + A_{m,m+1}(b_{m+1} - b_{m+2}) + \\
&\quad \cdots + A_{m,m'-1}(b_{m'-1} - b_{m'}) + A_{m,m'}b_{m'} \\
&= A_{m,m'}b_{m'} + \sum_{m}^{m'-1} A_{m,n}(b_n - b_{n+1}),
\end{aligned}$$

as required. □

Lemma 2.3. *Let $\alpha, \beta \in \mathbb{R}$ with $0 < \alpha < \beta$, and let $z = x + iy$ with $x, y \in \mathbb{R}$ and $x > 0$. Then,*

$$|e^{-\alpha z} - e^{-\beta z}| \leq \left|\frac{z}{x}\right| (e^{-\alpha x} - e^{-\beta x}).$$

Proof. First observe that

$$z \int_{\alpha}^{\beta} e^{-zt} dt = z\left(-\frac{1}{z}e^{-\beta z} + \frac{1}{z}e^{-\alpha z}\right) = e^{-\alpha z} - e^{-\beta z}.$$

Thus,

$$\begin{aligned}|e^{-\alpha z}-e^{-\beta z}| &= \left|z\int_{\alpha}^{\beta} e^{-zt}dt\right| \\ &\leq |z|\int_{\alpha}^{\beta} e^{-xt}dt \\ &= \frac{|z|}{x}(e^{-\alpha x}-e^{-\beta x}),\end{aligned}$$

completing the proof. □

We will now introduce Dirichlet series and use the previous lemmas to deduce some important results.

Definition 2.4. Let (λ_n) be an increasing sequence of real numbers tending to $+\infty$. A **Dirichlet Series** with exponents (λ_n) is a series with the form

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n z},$$

where $a_n \in \mathbb{C}, z \in \mathbb{C}$.

Remark. It is important to note that we will assume that $\lambda_n \geq 0$ for all n, although this is not strictly required as we can always supress a finite number of terms to achieve this property.

Proposition 2.5. *If the series $f(z) = \sum a_n e^{-\lambda_n z}$ converges for $z = z_0$, it comverges uniformly on the domain $\Re(z - z_0) > 0$.*

Proof. Without loss of generality we can assume that $f(z) = \sum a_n e^{-\lambda_n z}$ converges at $z_0 = 0$. So, $f(0) = \sum a_n$ is a convergent series, and we must now show that there is uniform convergence in the domain $D = \{z \in \mathbb{C} : \Re(z) > 0\}$.

Fix a point $z \in D$, and observe that we must have that $\frac{|z|}{\Re(z)} \leq k$ for some $k \in \mathbb{R}^+$. Now fix $\varepsilon > 0$ and choose $N \in \mathbb{N}$ so that if $m, m' > N$, then

$$\left|\sum_{n=m}^{n=m'} a_n\right| < \varepsilon,$$

or equivalently (using notation from Lemma 2.2),

$$|A_{m,m'}| < \varepsilon.$$

Let (b_n) be the sequence with entries given by $b_n = e^{-\lambda_n z}$. Then

$$S_{m,m'} = \sum_{n=m}^{n=m'} a_n b_n = \sum_{n=m}^{n=m'} a_n e^{-\lambda_n z},$$

and thus to show uniform convergence, we will prove that $|S_{m,m'}|$ is bounded.

We apply Lemma 2.2 to get

$$S_{m,m'} = \sum_{m}^{m'-1} A_{m,n}(e^{-\lambda_n z} - e^{-\lambda_{n+1} z}) + A_{m,m'} e^{-\lambda_{m'} z},$$

and putting $z = x + iy$ and applying lemma 2.3 we get:

$$\begin{aligned}
|S_{m,m'}| &= \left| \sum_{m}^{m'-1} A_{m,n}(e^{-\lambda_n z} - e^{-\lambda_{n+1} z}) + A_{m,m'} e^{-\lambda_{m'} z} \right| \\
&\le \left| \sum_{m}^{m'-1} A_{m,n}(e^{-\lambda_n z} - e^{-\lambda_{n+1} z}) \right| + \left| A_{m,m'} e^{-\lambda_{m'} z} \right| \\
&\le \sum_{m}^{m'-1} \left| A_{m,n}(e^{-\lambda_n z} - e^{-\lambda_{n+1} z}) \right| + \left| A_{m,m'} \right| e^{-\lambda_{m'} z} \\
&\le \sum_{m}^{m'-1} \left| A_{m,n} \right| \left| e^{-\lambda_n z} - e^{-\lambda_{n+1} z} \right| + \varepsilon e^{-\lambda_{m'} z} \\
&\le \sum_{m}^{m'-1} \varepsilon \left| \frac{z}{x} \right| (e^{-\lambda_n x} - e^{\lambda_{n+1} x}) + \varepsilon \\
&= \varepsilon \left(\left| \frac{z}{x} \right| \sum_{m}^{m'-1} (e^{-\lambda_n x} - e^{\lambda_{n+1} x}) + 1 \right) \\
&\le \varepsilon \left(k \sum_{m}^{m'-1} (e^{-\lambda_n x} - e^{\lambda_{n+1} x}) + 1 \right).
\end{aligned}$$

Observe that $\sum_{m}^{m'-1}(e^{-\lambda_n x} - e^{-\lambda_{n+1} x})$ is a telescoping series, and thus we conclude that

$$|S_{m,m'}| \le \varepsilon[k(e^{-\lambda_m x} - e^{-\lambda_{m'} x}) + 1] \le \varepsilon(k+1),$$

which completes the proof. □

Corollary 2.6. *If f converges for $z = z_0$, then f is analytic on the domain $\Re(z - z_0) > 0$.*

Proof. From Proposition 2.5 we know that f converges uniformly on the domain $\Re(z - z_0) > 0$. Now define a sequence of functions (f_j), where $f_j(z) = \sum_{n=1}^{n=j} a_n e^{-\lambda_n z}$. It is obvious that f_j converges uniformly to f, and thus f is analytic by Lemma 2.1. □

Proposition 2.7. *Let $f = \sum a_n e^{-\lambda_n z}$ be a Dirichlet series with $a_n \ge 0$ real for each n. Suppose that f converges for $\Re(z) > \rho$, with $\rho \in \mathbb{R}$, and that f can be extended analytically to a function analytic in a neighborhood of the point $z = \rho$. Then there exists a number $\varepsilon > 0$ such that f converges for $\Re(z) > \rho - \varepsilon$.*

Proof. Without loss of generality we can assume that $\rho = 0$. By Corollary 2.6 we have that f is analytic for $\Re(z) > 0$. Additionally, we know that f can be extended analytically in a neighborhood of 0, and thus we get that f is analytic on a disk $|z| \leq \varepsilon$ for some $\varepsilon > 0$. In particular, f converges at $z = -\varepsilon$, and by Proposition 2.5, it converges for $\Re(z) > -\varepsilon$ also, completing the proof. □

In the case where $\lambda_n = \ln n$, the corresponding Dirichlet series is known as an **ordinary Dirichlet series** and is given by

$$F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

.

Proposition 2.8. *If the a_n are bounded, then F absolutely converges for $\Re(s) > 1$.*

Proof. Assume the a_n are bounded above by A. Then,

$$F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \leq A \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

But for $\Re(s) > 1$, we have that $|s| > 1$, for which the series $\sum 1/n^s$ converges absolutely. So $F(s)$ converges absolutely as well. □

Proposition 2.9. *If the partial sums $A_{p,q} = \sum_p^q a_n$ are bounded, then F converges for $R(s) > 0$.*

Proof. Suppose $|A_{p,q}| \leq B$ for any $p, q \in \mathbb{N}^+$. Fix $s \in \mathbb{C}$ such that $\Re(s) > 0$, and let $b_n = 1/n^s$. We have

$$S_{m,m'} = \sum_{m}^{m'} a_n b_n = \sum_{m}^{m'} \frac{a_n}{n^s}.$$

Applying Lemma 2.2 yields

$$\begin{aligned}
|S_{m,m'}| &= \left| A_{m,m'} b_{m'} + \sum_{m}^{m'-1} A_{m,n}(b_n - b_{n+1}) \right| \\
&= \left| A_{m,m'} \frac{1}{m'^s} + \sum_{m}^{m'-1} A_{m,n} \left(\frac{1}{n^s} - \frac{1}{(n+1)^s} \right) \right| \\
&\leq B \left| \frac{1}{m'^s} \right| + B \sum_{m}^{m'-1} \left| \frac{1}{n^s} - \frac{1}{(n+1)^s} \right| \\
&= B \left(\left| \frac{1}{m'^s} \right| + \sum_{m}^{m'-1} \left| \frac{1}{n^s} - \frac{1}{(n+1)^s} \right| \right)
\end{aligned}$$

We can suppose that s is real (since $F(s)$ converges if and only if $F(\Re(s))$ converges). Thus,

$$\begin{aligned} |S_{m,m'}| &\leq B\left(\frac{1}{m'^s} + \sum_{m}^{m'-1}\left(\frac{1}{n^s} - \frac{1}{(n+1)^s}\right)\right) \\ &= B\left(\frac{1}{m'^s} + \left(\frac{1}{m^s} - \frac{1}{m'^s}\right)\right) \\ &= \frac{B}{m^s}, \end{aligned}$$

which completes the proof that F converges uniformly for $\Re(s) > 0$. □

3. The Zeta Function

Made famous by mathematician Bernhard Riemann, the zeta function is in fact a Dirichlet series, as we will see soon.

Definition 3.1. We say a function $f : \mathbb{N} \to \mathbb{C}$ is **multiplicative** if $f(1) = 1$ and

$$f(mn) = f(m)f(n)$$

whenever m, n are relatively prime. We say f is **strictly multiplicative** if $f(1) = 1$ and

$$f(mn) = f(m)f(n)$$

for any two positive integers m, n.

For the remainder of this paper, we write $\mathbb{P}$ to be the set of all prime numbers.

Lemma 3.2. *Let $g : \mathbb{N} \to \mathbb{C}$ be a bounded, multiplicative function. Then, the Dirichlet series $\sum_{n=1}^{\infty} g(n)/n^s$ converges absolutely for $\Re(s) > 1$, and*

$$\sum_{n=1}^{\infty} \frac{g(n)}{n^s} = \prod_{p\in\mathbb{P}}\left(\sum_{m=0}^{\infty} \frac{g(p^m)}{p^{ms}}\right).$$

when $\Re(s) > 1$.

Proof. Since g is bounded, Proposition 2.8 implies that $\sum g(n)/n^s$ is absolutely convergent for $\Re(s) > 1$.

Now, let $S = \{p_1, p_2, \cdots, p_k\} \subseteq \mathbb{P}$ be a finite collection of primes, and let $N(S)$ be the set of positive integers whose prime factors belong to S. By induction, we will first prove that

$$\sum_{n\in N(S)} \frac{g(n)}{n^s} = \prod_{p\in S}\left(\sum_{m=0}^{\infty} \frac{g(p^m)}{p^{ms}}\right).$$

Suppose S has one element, call it p_1. Then

$$N(S) = \{1, p_1, p_1^2, \cdots\} = \{p_1^a : a \in \mathbb{N}\},$$

and we have:

$$\prod_{p\in S}\left(\sum_{m=0}^{\infty}\frac{g(p^m)}{p^{ms}}\right)=\sum_{m=0}^{\infty}\frac{g(p_1^m)}{p_1^{ms}}$$
$$=1+\frac{g(p_1)}{p_1^s}+\frac{g(p_1^2)}{(p_1^2)^s}+\cdots$$
$$=\sum_{n\in N(S)}\frac{g(n)}{n^s}.$$

Now, assume that for all finite collections of primes S with cardinality $1, 2, \ldots, k-1$, we have that

$$\sum_{n\in N(S)}\frac{g(n)}{n^s}=\prod_{p\in S}\left(\sum_{m=0}^{\infty}\frac{g(p^m)}{p^{ms}}\right).$$

Suppose that S has k elements (i.e. $S=\{p_1,p_2,\cdots,p_k\}$) and let $T=S\setminus\{p_k\}$. Then,

$$N(T)=\{p_1^{a_1}p_2^{a_2}\cdots p_{k-1}^{a_{k-1}}:a_1,a_2,\cdots,a_{k-1}\in\mathbb{N}\},$$

and

$$N(S)=\{p_1^{a_1}p_2^{a_2}\cdots p_k^{a_k}:a_1,a_2,\cdots,a_k\in\mathbb{N}\}$$
$$=\{qp_k^a:a\in\mathbb{N},q\in N(T)\}.$$

We have:

$$\prod_{p\in S}\left(\sum_{m=0}^{\infty}\frac{g(p^m)}{p^{ms}}\right)=\left[\prod_{p\in T}\left(\sum_{m=0}^{\infty}\frac{g(p^m)}{p^{ms}}\right)\right]\left(\sum_{m=0}^{\infty}\frac{g(p_k^m)}{p_k^{ms}}\right)$$
$$=\left(\sum_{n\in N(T)}\frac{g(n)}{n^s}\right)\left(\sum_{m=0}^{\infty}\frac{g(p_k^m)}{p_k^{ms}}\right)$$
$$=\left(\sum_{n\in N(T)}\frac{g(n)}{n^s}\right)\left(1+\frac{g(p_k)}{p_k^s}+\frac{g(p_k^2)}{p_k^{2s}}+\cdots\right)$$
$$=\sum_{n\in N(T)}\frac{g(n)}{n^s}+\sum_{n\in N(T)}\frac{g(np_k)}{(np_k)^s}+\sum_{n\in N(T)}\frac{g(np_k^2)}{(np_k^2)^s}+\cdots$$
$$=\sum_{n\in N(S)}\frac{g(n)}{n^s}.$$

Now that this identity has been proven, we let S_j denote the set of the first j primes. It is obvious that as $j\to\infty$, both $S_j\to\mathbb{P}$ and $N(S_j)\to\mathbb{N}$.

Thus

$$\sum_{n=1}^{\infty} \frac{g(n)}{n^s} = \lim_{j\to\infty}\left[\sum_{n\in N(S_j)} \frac{g(n)}{n^s}\right] = \lim_{j\to\infty}\left[\prod_{p\in S_j}\left(\sum_{m=0}^{\infty} \frac{g(p^m)}{p^{ms}}\right)\right] = \prod_{p\in\mathbb{P}}\left(\sum_{m=0}^{\infty} \frac{g(p^m)}{p^{ms}}\right),$$

as required. □

Lemma 3.3. *If g is bounded and strictly multiplicative, one has*

$$\sum_{n=1}^{\infty} \frac{g(n)}{n^s} = \prod_{p\in\mathbb{P}} \frac{1}{1-\frac{g(p)}{p^s}}.$$

Proof. We apply the previous lemma and see that a strictly multiplicative function yields a geometric series:

$$\sum_{n=1}^{\infty} \frac{g(n)}{n^s} = \prod_{p\in\mathbb{P}}\left(\sum_{m=0}^{\infty} \frac{g(p^m)}{p^{ms}}\right) = \prod_{p\in\mathbb{P}}\left(\sum_{m=0}^{\infty} \frac{g(p)^m}{p^{ms}}\right) = \prod_{p\in\mathbb{P}}\left(\sum_{m=0}^{\infty} \left(\frac{g(p)}{p^s}\right)^m\right) = \prod_{p\in\mathbb{P}} \frac{1}{1-\frac{g(p)}{p^s}}.$$

□

We now introduce the zeta function by letting $g = 1$:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p\in\mathbb{P}} \frac{1}{1-\frac{1}{p^s}}.$$

Proposition 3.4. *The zeta function is analytic and non-zero in the open half plane $\Re(s) > 1$.*

Proof. Let s_0 be a point in the half plane $\Re(s) > 1$. Then by Proposition 2.8, ζ absolutely converges at $s = s_0$, and furthermore, Corollary 2.6 implies that ζ is analytic on the domain $\Re(s) > s_0$. Since s_0 was chosen arbitrarily, we conclude that ζ is analytic on all half planes $\Re(s) > s_0$ with $s_0 > 1$. This is equivalent to stating ζ is analytic on the domain $\Re(s) > 1$. The fact that the zeta function is nonzero is clear. □

Proposition 3.5. *One has:*

$$\zeta(s) = \frac{1}{s-1} + \sigma(s),$$

where σ is analytic for $\Re(s) > 0$.

Proof. First observe that

$$\frac{1}{s-1} = \int_1^\infty t^{-s} dt = \sum_{n=1}^\infty \int_n^{n+1} t^{-s} dt.$$

We write

$$\begin{aligned}\zeta(s) &= \sum_{n=1}^\infty \frac{1}{n^s} + \frac{1}{s-1} - \sum_{n=1}^\infty \int_n^{n+1} t^{-s} dt \\ &= \frac{1}{s-1} + \sum_{n=1}^\infty \left(\frac{1}{n^s} - \int_n^{n+1} t^{-s} dt \right) \\ &= \frac{1}{s-1} + \sum_{n=1}^\infty \int_n^{n+1} (n^{-s} - t^{-s}) dt\end{aligned}$$

Now set:

$$\sigma_n(s) = \int_n^{n+1} (n^{-s} - t^{-s}) dt$$

and

$$\sigma(s) = \sum_{n=1}^\infty \sigma_n(s).$$

We must first show that the σ_n are analytic; it is sufficient to show the existence of the first derivative σ_n'. Since each σ_n is continuous for $\Re(s) > 0$, we differentiate under the integral sign:

$$\begin{aligned}\sigma_n'(s) &= \int_n^{n+1} \frac{\partial}{\partial s} (n^{-s} - t^{-s})\, dt \\ &= \int_n^{n+1} s \left(t^{-(s+1)} - n^{-(s+1)} \right) dt \\ &= s \left(\int_n^{n+1} t^{-(s+1)}\, dt - n^{-(s+1)} \int_n^{n+1} dt \right) \\ &= s \left(-\frac{(n+1)^{-s}}{s} + \frac{n^{-s}}{s} - n^{-(s+1)} \right) \\ &= n^{-s} - (n+1)^{-s} - s n^{-(s+1)}.\end{aligned}$$

Since the derivative is defined for all s with $\Re(s) > 0$, we conclude that the σ_n are analytic on this domain.

Now, from Lemma 2.1, it is clear that the convergence $\sum \sigma_n \to \sigma$ will complete the proof. Put $f(t) = n^{-s} - t^{-s}$ and note that

$$|\sigma_n(s)| \leq \int_n^{n+1} |f(t)|\, dt \leq \sup_{n \leq t \leq n+1} |f(t)|.$$

Since $f(n) = 0$, we get that for any $t \in [n, n+1]$,

$$\begin{aligned} |f(t)| &= |f(t) - f(n)| \\ &= \left| \int_n^t f'(z) dz \right| \\ &\leq \int_n^t |f'(z)| dz \\ &\leq \int_n^{n+1} |f'(z)| dz \\ &\leq \sup_{n \leq z \leq n+1} |f'(z)| \\ &= \sup_{n \leq z \leq n+1} \left| \frac{s}{z^{s+1}} \right| \\ &= \frac{|s|}{|n^{s+1}|} \\ &\leq \frac{|s|}{n^{\Re(s)+1}} \end{aligned}$$

So, we have shown that $|\sigma_n(s)| \leq \frac{|s|}{n^{x+1}}$ where $x = \Re(s)$. It is clear that the series $\sum \sigma_n$ converges normally for $\Re(s) \geq \varepsilon$, for all $\varepsilon > 0$. □

Corollary 3.6. *The zeta function has a simple pole at $s = 1$ (that is, $(s-1)\zeta(s)$ is analytic in a neighborhood of $s = 1$).*

Proof. This immediately follows from the proposition above. □

4. Asymptotic Equivalence

Definition 4.1. Let f, g be non-zero complex-valued functions and let $c \in \mathbb{C} \cup \{-\infty, \infty\}$. We say that f and g are **aysmptotically equivalent** as z tends to c provided:

$$\lim_{z \to c} \frac{f(z)}{g(z)} = 1.$$

We denote asymptotic equivalence by $f \sim_c g$. This relation is one of the keys to the final proof, and has some important properties which we must prove.

Proposition 4.2. *Let $\mathcal{F}_c$ be the set of all complex-valued with a nonzero limit at $c \in \mathbb{C} \cup \{-\infty, \infty\}$. Then, $\sim_c$ is an equivalence relation on $\mathcal{F}_c$.*

Proof. Pick any $c \in \mathbb{C} \cup \{-\infty, \infty\}$ and let $f, g, h \in \mathcal{F}_c$. It is clear that $\sim_c$ is reflexive:

$$\lim_{z \to c} \frac{f(z)}{f(z)} = \lim_{z \to c} 1 = 1.$$

To show that $\sim_c$ is symmetric, suppose that $f \sim_c g$. Then we have

$$\lim_{z \to c} \frac{g(z)}{f(z)} = \frac{1}{\lim_{z \to c} \frac{f(z)}{g(z)}} = 1.$$

Lastly, to show that $\sim_c$ is transitive, suppose that $f \sim_c g$ and $g \sim_c h$. Then,

$$\lim_{z \to c} \frac{f(z)}{h(z)} = \frac{\lim_{z \to c} \frac{f(z)}{g(z)}}{\lim_{z \to c} \frac{h(z)}{g(z)}} = \frac{1}{1} = 1,$$

which completes the proof. □

Proposition 4.3. *Suppose $f, g \in \mathcal{F}_c$ such that*

$$\lim_{z \to c} f(z) = \lim_{z \to c} g(z) = \pm\infty.$$

Then if $\lim_{z \to c}(f(z) - g(z)) = \gamma \in \mathbb{C}$, we have $f \sim_c g$.

Proof. This is clear:

$$\lim_{z \to c} \frac{f(z)}{g(z)} = \lim_{z \to c} \frac{f(z) - g(z) + g(z)}{g(z)} = \lim_{z \to c} \frac{f(z) - g(z)}{g(z)} + 1 = 1.$$

□

Lemma 4.4. *We have*

$$\frac{1}{s-1} \sim_1 \zeta(s).$$

Proof. Using the formula from Proposition 3.5, we have:

$$\lim_{s \to 1} \frac{\frac{1}{s-1}}{\zeta(s)} = \lim_{s \to 1} \frac{1}{1 + (s-1)\sigma(s)} = 1.$$

□

Proposition 4.5. *We have*

$$\ln \frac{1}{s-1} \sim_1 \ln \zeta(s).$$

Proof. Using Proposition 4.3, it suffices to show that

$$\lim_{s \to 1} \left(\ln \frac{1}{s-1} - \ln \zeta(s) \right) = \gamma, \qquad \gamma \in \mathbb{C}.$$

We can assume that s tends to 1 along the real axis (from the right, of course) and we proceed by contradiction. If the above equality does not hold, then there are two possible cases (since the limit clearly must exist).

If $\lim_{s\to 1}(\ln \frac{1}{s-1} - \ln \zeta(s)) = \infty$, then we must have

$$\lim_{s\to 1} e^{\ln \frac{1}{s-1} - \ln \zeta(s)} = \lim_{s\to 1} \frac{\frac{1}{s-1}}{\zeta(s)} = \infty.$$

This contradicts the previous lemma. If $\lim_{s\to 1}(\ln \frac{1}{s-1} - \ln \zeta(s)) = -\infty$, then we must have

$$\lim_{s\to 1} e^{\ln \frac{1}{s-1} - \ln \zeta(s)} = \lim_{s\to 1} \frac{\frac{1}{s-1}}{\zeta(s)} = 0.$$

Once again, this contradicts the previous lemma, so we have completed the proof. □

Lemma 4.6. *The sum*

$$\sum_{p\in\mathbb{P},k\geq 2} \frac{1}{kp^{ks}}$$

remains bounded for $\Re(s) > 1$.

Proof. Observe:

$$\begin{aligned}
\sum_{p\in\mathbb{P},k\geq 2} \frac{1}{kp^{ks}} &= \sum_{p,k\geq 2} \frac{1}{kp^{ks}} \\
&\leq \sum_{p,k\geq 2} \frac{1}{p^{ks}} \\
&= \sum_{p} \frac{1}{p^s(p^s-1)} \\
&\leq \sum_{p} \frac{1}{p(p-1)} \\
&\leq \sum_{n=2}^{\infty} \frac{1}{n(n-1)} \\
&= 1.
\end{aligned}$$

□

Proposition 4.7. *We have*

$$\sum_{p\in\mathbb{P}} \frac{1}{p^s} \sim_1 \ln \frac{1}{s-1}$$

Proof. One has

$$\begin{aligned}
\ln\zeta(s) &= \ln\left(\prod_{p\in\mathbb{P}} \frac{1}{1-\frac{1}{p^s}}\right) \\
&= \sum_{p\in\mathbb{P}}\left(\ln\frac{1}{1-\frac{1}{p^s}}\right) \\
&= \sum_{p\in\mathbb{P}}\Big(-\ln(1-p^{-s})\Big) \\
&= \sum_{p\in\mathbb{P}}\left(\sum_{k=1}^{\infty}\frac{p^{-ks}}{k}\right) \\
&= \sum_{p,k\geq 1}\frac{1}{kp^{ks}} \\
&= \sum_{p\in\mathbb{P}}\frac{1}{p^s} + \sum_{p\in\mathbb{P},k\geq 2}\frac{1}{kp^{ks}}
\end{aligned}$$

By the previous lemma we know the the second term is bounded as $s \to 1$, and thus by Proposition 4.3 it is clear that $\ln\zeta(s) \sim_1 \sum_p 1/p^s$. Since $\sim_1$ is transitive, applying Proposition 4.5 completes the proof. □

5. Dirichlet Characters and L-Functions

Let G be a group.

Definition 5.1. A **character** of G is a homomorphism of G into the multiplicative group $\mathbb{C}^* = (\mathbb{C}\setminus\{0\},\cdot)$ of complex numbers. The set of all characters of G form a group, denoted $\hat{G}$, called the **dual** of G.

Proposition 5.2. *Let $n = card(G)$ and $\chi \in \hat{G}$. Then,*

$$\sum_{x\in G}\chi(x) = \begin{cases} n, & \chi = 1 \\ 0, & \chi \neq 1. \end{cases}$$

Proof. In the case where $\chi = 1$, we have

$$\sum_{x\in G} 1(x) = 1 + 1 + \cdots = \text{card}(G) = n.$$

Now suppose that $\chi \neq 1$ and choose $y \in G$ such that $\chi(y) \neq 1$. By the properties of χ we have:

$$\chi(y)\sum_{x\in G}\chi(x) = \sum_{x\in G}\chi(y)\chi(x) = \sum_{x\in G}\chi(xy) = \sum_{x\in G}\chi(x).$$

So, $(\chi(y)-1)\sum_x \chi(x) = 0$, and since $\chi(y) \neq 0$, it immediately follows that $\sum_x \chi(x) = 0$. □

Corollary 5.3. *Let $x \in G$. Then*

$$\sum_{\chi \in \hat{G}} \chi(x) = \begin{cases} n, & x = 1 \\ 0, & x \neq 1 \end{cases}$$

Proof. Applying the previous proposition to $\hat{G}$ completes the proof. □

Let m be a positive integer. The multiplicative group of integers modulo m, which we will denote with $G(m)$, is the set of all positive integers $\leq m$ which are coprime to m.

Definition 5.4. The **Euler totient function** is a map $\phi : \mathbb{N} \to \mathbb{N}$ which counts the positive integers up to an integer m which are coprime to m.

Remark. It is clear that $|G(m)| = \phi(m)$.

Definition 5.5. For a positive integer m, we say the map $\chi : \mathbb{Z} \to \mathbb{C}$ is a **Dirichlet character of modulus** m if for all $a, b \in \mathbb{Z}$ we have:

(1) $\chi(ab) = \chi(a)\chi(b)$;

(2) $\chi(a) \begin{cases} = 0, & \text{if } (a, m) > 1, \\ \neq 0 & \text{if } (a, m) = 1; \end{cases}$

(3) $\chi(a + m) = \chi(a)$.

Remark. Each element of $\widehat{G(m)}$ can be extended to some Dirichlet character of modulus m.

We will often refer to the **unit character** in the remaining sections. This is the map $1 : \mathbb{Z} \to \mathbb{Z}$ given by

$$1(a) = \begin{cases} 0, & \text{if } (a, m) > 1, \\ 1 & \text{if } (a, m) = 1. \end{cases}$$

Once again, fix a positive integer m and let χ be a Dirichlet charcter mod m. The corresponding L function is a Dirichlet series given by

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

Observe that the nth term of this sum is nonzero if and only if n is coprime to m.

Proposition 5.6. *For $\chi = 1$, we have*

$$L(s, 1) = \zeta(s) H(s),$$

with

$$H(s) = \prod_{p|m} (1 - p^{-s}).$$

In particular, $L(s, 1)$ extends analytically for $\Re(s) > 0$ and has a simple pole at $s = 1$.

Proof. We have

$$\begin{aligned} L(s,1) &= \sum_{n=1}^{\infty} \frac{1(n)}{n^s} \\ &= \prod_{p\in\mathbb{P}} \frac{1}{1-\frac{1(p)}{p^s}} \\ &= \prod_{p\nmid m} \frac{1}{1-p^{-s}} \\ &= \left(\prod_{p\in\mathbb{P}} \frac{1}{1-p^{-s}}\right)\left(\prod_{p\mid m} 1-p^{-s}\right) \\ &= \zeta(s)H(s). \end{aligned}$$

The remainder of the proof is clear, since ζ extends analytically for $\Re(s) > 0$ and has a simple pole at $s = 1$. □

Proposition 5.7. *For $\chi \neq 1$, the series $L(s,\chi)$ converges absolutely for $\Re(s) > 1$ and one has:*

$$L(s,\chi) = \prod_{p\in\mathbb{P}} \frac{1}{1-\frac{\chi(p)}{p^s}}$$

Proof. Since χ is strictly multiplicative, this follows directly from Lemma 3.3 □

Proposition 5.8. *For $\chi \neq 1$, the series $L(s,\chi)$ converges for $\Re(s) > 0$.*

Proof. By Proposition 2.9, it is sufficient to show that the partial sums

$$A_{u,v} = \sum_{n=u}^{n=v} \chi(n)$$

are bounded for $u \leq v$. Using the fact that χ is periodic with period m, we apply Proposition 5.2 to get

$$\sum_{u}^{u+m-1} \chi(n) = 0,$$

so we only need to show that the partial sums $A_{u,v}$ with $v - u < m$ are bounded. By the cyclic nature of χ, this is simple. Fix

$$M = \max\{|\chi(n)|, 1 \leq n \leq m\},$$

and we get

$$|A_{u,v}| \leq M \cdot \phi(m).$$

□

With m still a fixed positive integer, we introduce some new notation. If $p \in \mathbb{P}$ does not divide m, then we denote $\overline{p}$ by its image in $G(m)$. Furthermore, we define $f(p)$ to be the order of $\overline{p}$, that is, $f(p)$ is the smallest integer $f > 1$ such that $p^f \equiv 1 \pmod{m}$. Lastly, we put $g(p) = \phi(p)/f(p)$.

Definition 5.9. For a fixed natural number n, the nth **roots of unity** are the solutions to the equation $x^n = 1$, and there are n solutions.

Lemma 5.10. *If $p \nmid m$, we get the identity*

$$\prod_{\chi \in \widehat{G(m)}} (1 - \chi(p)T) = (1 - T^{f(p)})^{g(p)},$$

where $\Re(T) > 0$.

Proof. Let U be the set of the $f(p)$-th roots of unity. We have

$$\prod_{u \in U} (1 - uT) = 1 - T^{f(p)}.$$

The lemma follows from this as well as the fact that for all $u \in U$ there exist $g(p)$ characters χ of $G(m)$ such that $\chi(p) = u$. □

We now define a new function ψ_m as follows:

$$\psi_m(s) = \prod_{\chi} L(s, \chi),$$

where the product extends over all characters of $G(m)$.

Proposition 5.11. *One has*

$$\psi_m(s) = \prod_{p \nmid m} \frac{1}{\left(1 - \frac{1}{p^{f(p)s}}\right)^{g(p)}}.$$

This is a Dirichlet series with positive integral coefficients, converging for $\Re(s) > 1$.

Proof. We apply Proposition 5.7 and Lemma 5.10 with $T = p^{-s}$ to get:

$$\begin{aligned}\psi_m(s) &= \prod_\chi L(s,\chi) \\ &= \prod_\chi\Big(\prod_{p\in\mathbb{P}}\frac{1}{1-\chi(p)p^{-s}}\Big) \\ &= \prod_{p\in\mathbb{P}}\Big(\frac{1}{\prod_\chi(1-\chi(p)p^{-s})}\Big) \\ &= \prod_{p\nmid m}\Big(\frac{1}{\prod_\chi(1-\chi(p)p^{-s})}\Big) \\ &= \prod_{p\nmid m}\Big(\frac{1}{(1-p^{-f(p)s})^{g(p)}}\Big) \\ &= \prod_{p\nmid m}\frac{1}{\Big(1-\frac{1}{p^{f(p)s}}\Big)^{g(p)}},\end{aligned}$$

as required. □

Theorem 5.12. *$L(1,\chi)\neq 0$ for all $\chi\neq 1$.*

Proof. Proceeding by contradiction, suppose $L(1,\chi)=0$ for some $\chi\neq 1$. Then the function ψ_m is analytic at $s=1$. We know that $L(s,1)$ extends analytically for $\Re(s)>0$ (Proposition 5.6) and for $\chi\neq 1$, we know that $L(s,\chi)$ converges for $\Re(s)>0$ (Proposition 5.8), and thus ψ_m is analytic for all s with $\Re(s)>0$. Since ψ_m is a Dirichlet series with positive coefficients, this implies that $\psi_m(s)$ converges for $\Re(s)>0$ as well. However, observe that the pth factor of ψ_m is

$$\frac{1}{(1-p^{-f(p)s})^{g(p)}},$$

which has the MacLauren series expansion

$$\Big(\sum_{n=0}^{\infty}p^{-nf(p)s}\Big)^{g(p)} = (1+p^{-f(p)s}+p^{-2f(p)s}+\cdots)^{g(p)},$$

which dominates the series

$$\sum_{n=0}^{\infty}p^{-n\phi(m)s} = 1+p^{-\phi(m)s}+p^{-2\phi(m)s}+\cdots.$$

So,

$$\psi_m(s) = \prod_{p\nmid m}\frac{1}{(1-p^{-f(p)s})^{g(p)}} \geq \prod_{p\nmid m}\Big(\sum_{n=0}^{\infty}p^{-n\phi(m)s}\Big).$$

But at $s = 1/\phi(m)$, we get that

$$\psi_m(1/\phi(m)) \geq \prod_{p \nmid m} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots\right) \geq \sum_{p \nmid m} \frac{1}{p},$$

which is a divergent series. Hence ψ_m is not analytic for $\Re(s) > 0$, a contradiction which completes the proof. $\square$

6. Dirichlet Density

We now approach the final steps toward proving Dirichlet's theorem.

Definition 6.1. Fix $s \in \mathbb{R}_{>1}$ and let $A \subseteq \mathbb{P}$. We say A has **density** k if the ratio

$$\left(\sum_{p \in A} \frac{1}{p^s}\right) \Big/ \left(\ln \frac{1}{s-1}\right)$$

tends to k as $s \to 1$.

Notice that $k \in [0,1]$ (Proposition 4.7). Fix a positive integer m and let χ be a character of $G(m)$. Put

$$f_\chi(s) = \sum_{p \nmid m} \frac{\chi(p)}{p^s}.$$

Note that f_χ converges for $\Re(s) > 1$.

Lemma 6.2. *If $\chi = 1$, then*

$$f_\chi(s) \sim_1 \ln \frac{1}{s-1}.$$

Proof. Observe that $f_1(s)$ differs from $\sum_p 1/p^s$ by a finite number of terms:

$$f_1(s) = \sum_{p \nmid m} \frac{1(p)}{p^s} = \sum_{p \in G(m)} \frac{1}{p^s} + \sum_{p > m} \frac{1}{p^s}.$$

So by Proposition 4.3 $f_1(s) \sim_1 \sum_p 1/p^s$ and by Proposition 4.5 and the transitivity of $\sim_1$, we have $f_1(s) \sim_1 \ln \frac{1}{s-1}$, completing the proof. $\square$

Lemma 6.3. *If $\chi \neq 1$, then f_χ remains bounded when $s \to 1$.*

Proof. Let $F_\chi(s) = \sum_{p\in\mathbb{P}}(\sum_{n=2}^{\infty} \frac{\chi(p)^n}{np^{ns}})$ and observe:

$$\begin{aligned}
\ln L(s,\chi) &= \ln\Bigg(\prod_{p\in\mathbb{P}} \frac{1}{1-\chi(p)p^{-s}}\Bigg) \\
&= \sum_{p\in\mathbb{P}} \ln \frac{1}{1-\chi(p)p^{-s}} \\
&= \sum_{p\in\mathbb{P}} \Bigg(\sum_{n=1}^{\infty} \frac{\chi(p)^n}{np^{ns}}\Bigg) \\
&= \sum_{p\in\mathbb{P}} \frac{\chi(p)}{p^s} + \sum_{p\in\mathbb{P}} \Bigg(\sum_{n=2}^{\infty} \frac{\chi(p)^n}{np^{ns}}\Bigg) \\
&= f_\chi(s) + F_\chi(s).
\end{aligned}$$

From Theorem 5.12, we know that $L(s,\chi) \neq 0$ at $s = 1$, so $\ln L(s,\chi)$ must be bounded as $s \to 1$. Furthermore, from Lemma 4.6 it is clear that $F_\chi(s)$ also remains bounded as $s \to 1$. Hence $f_\chi(s)$ must share the same property, completing the proof. □

Theorem 6.4. *Let m be a positive integer and let $a \in \mathbb{Z}$ such that $(a,m) = 1$. Let P_a be the set of primes such that $p \equiv a$ (mod m). Then the set P_a has density $1/\phi(m)$.*

Proof. We begin by defining a function

$$g_a(s) = \sum_{p\in P_a} \frac{1}{p^s}.$$

Put

$$T = \sum_{\chi} \chi(a)^{-1} f_\chi(s),$$

with the sum extending over all characters of $G(m)$, and observe that

$$\begin{aligned}
T &= \sum_{\chi} \chi(a)^{-1} f_\chi(s) \\
&= \sum_{\chi} \chi(a)^{-1} \Bigg(\sum_{p\nmid m} \frac{\chi(p)}{p^s}\Bigg) \\
&= \sum_{p\nmid m} \Bigg(\frac{\sum_\chi \chi(a)^{-1}\chi(p)}{p^s}\Bigg) \\
&= \sum_{p\nmid m} \Bigg(\frac{\sum_\chi \chi(a^{-1}p)}{p^s}\Bigg).
\end{aligned}$$

By Corollary 5.3, we have

$$\sum_{\chi} \chi(a^{-1}p) = \begin{cases} \phi(m) & \text{if } a^{-1}p \equiv 1 \pmod{m}, \\ 0 & \text{otherwise.} \end{cases}$$

However, $a^{-1}p \equiv 1 \pmod{m}$ if and only if $p \equiv a \pmod{m}$, and thus:

$$T = \sum_{\chi} \chi(a)^{-1} f_{\chi}(s) = \sum_{p \nmid m} \left(\frac{\sum_{\chi} \chi(a^{-1}p)}{p^s} \right) = \sum_{p \in P_a} \frac{\phi(m)}{p^s},$$

that is,

$$\begin{aligned} g_a(s) &= \frac{1}{\phi(m)} \sum_{\chi} \chi(a)^{-1} f_{\chi}(s). \\ &= \frac{1}{\phi(m)} \left(1(a)^{-1} f_1(s) + \sum_{\chi \neq 1} \chi(a)^{-1} f_{\chi}(s) \right) \\ &= \frac{1}{\phi(m)} \left(f_1(s) + \sum_{\chi \neq 1} \chi(a)^{-1} f_{\chi}(s) \right). \end{aligned}$$

Lemma 6.3 tells us that the f_{χ} with $\chi \neq 1$ are bounded near $s = 1$, and hence,

$$\lim_{s \to 1} \sum_{\chi \neq 1} \chi(a)^{-1} f_{\chi}(s) = K, \qquad K \in \mathbb{C}.$$

So,

$$\lim_{s \to 1} \Big[\phi(m) g_a(s) - f_1(s) \Big] = K,$$

and by Proposition 4.3, it follows that $\phi(m) g_a(s) \sim_1 f_1(s)$. Furthermore, it follows that $\phi(m) g_a(s) \sim_1 \ln \frac{1}{s-1}$ by Lemma 6.2. So,

$$\lim_{s \to 1} \frac{\phi(m) g_a(s)}{\ln \frac{1}{s-1}} = 1,$$

and equivalently,

$$\lim_{s \to 1} \frac{g_a(s)}{\ln \frac{1}{s-1}} = \lim_{s \to 1} \frac{\sum_{p \in P_a} 1/p^s}{\ln \frac{1}{s-1}} = \frac{1}{\phi(m)},$$

completing the proof. □

Corollary 6.5. *The set P_a is infinite.*

Proof. This is clear. Let $A \subseteq \mathbb{P}$ be a finite set. Then, $\lim_{s \to 1} \sum_{p \in A} 1/p^s$ can be evaluated at $s = 1$. So

$$\lim_{s \to 1} \frac{\sum_{p \in A} 1/p^s}{\ln \frac{1}{s-1}} = 0,$$

and it follows that any set with nonzero density must be infinite. □

This simple corollary is indeed equivalent to Dirichlet's Theorem, and we are finished. The intersection between complex analysis and number theory is much larger and more important than one may think, and is one of the many beautiful aspects of studying mathematics.

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