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SEQUENTIAL EXPLOITATION OF UNCERTAIN
DEPOSITS OF A DEPLETABLE NATURAL RESOURCE

by
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Sequential Exploitation of Uncertain Deposits
of a Depletable Natural Resource^{*}

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Sequential Exploitation

ABSTRACT

This paper shows that it may well be optimal to exploit "risky" reserves of a natural resource before "less risky" reserves. This is shown precisely in the case of one certain and one uncertain reserve and in the case of distinct exponentially distributed reserves. Hence risk aversion does not imply that risky prospects should be deferred. However, plausible market failures might actually produce such a result. In policy terms this suggests a social motive for early exploitation of reserves such as the Athabasca tar sands. This motive is not necessarily reflected in market behavior.

1. Introduction

How should different deposits of a single depletable natural resource be exploited? When the deposits differ as to the cost of extraction, it is generally optimal to exploit the low cost deposits first. (See Solow and Wan [9], and Heal [6], for discussion of this case.) The present paper addresses this question: When the deposits have differing size probability distributions, how should these deposits be exploited? In particular, in which sequence should they be exploited? (See Kemp [7] for the formulation of the problem with a single uncertain reserve.)

In a simple case, one deposit has a known size and another an uncertain size described by a subjective probability distribution. It is shown that the optimal strategy exhausts the uncertain reserve prior to any exploitation of the certain deposit. Indeed this is true for arbitrary utility and probability distribution functions.

The Arrow-Debreu market structure needed to decentralize the optimum is unrealistically complex. Exactly which markets fail to exist is arguable. However, under plausible assumptions, it is shown that market incentives to exploit the risky reserve first do not exist.

What is the optimal strategy when both deposits are of uncertain extent? General results may not exist. However, with distinct exponential distributions, and iso-elastic utility, it is shown that the high mean, high variance deposit should be exploited first.

The investigation of how the above results might be modified by differing costs of extraction will be deferred to a subsequent paper.

2. One Certain, One Uncertain Deposit

Suppose there are two deposits of a depletable natural resource. The first of these has known size, S_1 , say. However, the size of the second deposit, S_2 , is described by a probability density function, $f(S_2)$, with cumulative distribution function, $F(S_2)$, $S_2 \geq 0$. Suppose that R_1 and R_2 are the rates of extraction from the first and second deposits, respectively, so that the total flow is

$$R = R_1 + R_2 \quad (1)$$

The cumulative total amounts extracted will be denoted by A_1 and A_2 , respectively,

$$\dot{A}_1 = R_1, \quad \dot{A}_2 = R_2 \quad (2)$$

For simplicity, suppose that the resource is consumed directly. (The introduction of production using the resource and investment would not change the qualitative results. See Dasgupta and Heal [2], for discussion of the relationship of such simplifications to more general models.) Also, take the dynamic utility functional to be

$$W = E\left\{ \int_0^{\infty} u(R)e^{-\rho t} dt \right\} \quad u' > 0 \quad u'' < 0 \quad (3)$$

This is a form in which the concavity of $u(\cdot)$ forms the basis of both the attitude to inequality over time and the attitude to risk. This is somewhat restrictive; however, the form confers obvious analytic advantages.

How does uncertainty arise here? Although the basic random variable is the size of the second deposit, a given contingent plan for its exploitation, $R_2(t)$, induces a probability of the second deposit, T , say. In fact, the probability density function of T is given by

$$g(T) = f(A_2)R_2 \quad (4)$$

Hence, for any contingent plan for exploitation of the two deposits, expected utility prior to exhaustion of the second reserve is

$$\begin{aligned} E_T \left\{ \int_0^T u(R_1 + R_2) e^{-\rho t} dt \right\} &= \int_0^\infty \int_0^t u(R_1 + R_2) e^{-\rho s} ds f(A_2) R_2 dt \\ &= \int_0^\infty u(R_1 + R_2) [1 - F(A_2)] e^{-\rho t} dt \end{aligned} \quad (5)$$

The last equality follows using integration by parts. However, it is also intuitively clear, since $1 - F(A_2)$ is the fraction of cases for which exhaustion of the second reserve will not have occurred by time t . (See Dasgupta and Heal [2].)

What happens after exhaustion of the second reserve? Consumption must take place using only what remains of the first reserve. Define then

$$J(A_1) = \max_{R_1} \int_0^\infty u(R_1) e^{-\rho t} dt$$

$$\text{where } \int_0^\infty R_1 dt = S_1 - A_1 \quad (6)$$

Note at this point that Bellman's equation for $J(\cdot)$ is

$$\max_{R_1} \left\{ u(R_1) + R_1 \frac{dJ}{dA_1}(A_1) \right\} = \rho J(A_1) \quad (7)$$

(See Bellman [1].) Now expected utility subsequent to exhaustion of the second reserve is

$$E_T \{ e^{-\rho T} J(A_1(T)) \} = \int_0^\infty e^{-\rho t} J(A_1) f(A_2) R_2 dt \quad (8)$$

Hence total expected utility before and after exhaustion of the second reserve is given by the sum of (5) and (8), or

$$W = \int_0^\infty e^{-\rho t} \{ u(R_1 + R_2) [1 - F(A_2)] + J(A_1) f(A_2) R_2 \} dt \quad (9)$$

This functional is to be maximized over choice of the rates of extraction,

R_1 and R_2 , where

$$\begin{aligned} \dot{A}_1 &= R_1 \geq 0, A_1(0) = 0 & A_1(\infty) &= S_1 \\ \dot{A}_2 &= R_2 \geq 0, A_2(0) = 0 \end{aligned} \quad (10)$$

Note that there is no terminal condition for A_2 . This is because R_2 is an extraction plan conditional upon non-exhaustion of the second reserve. The "cost" of a higher rate is simply the increased probability of failure of the second reserve. This is included in (9).

It is not hard to show that consistency, as in Strotz [10], obtains in this model. In other words, there will be no revision of the conditional plan at future times, even if this is permitted. This follows because the updated distribution of the size of the second reserve given cumulative extraction $\bar{A}_2$ is

$$\bar{f}(A_2) = \frac{f(A_2)}{1 - F(\bar{A}_2)} \quad A_2 \geq \bar{A}_2 \quad (11)$$

under the assumption that only non-exhaustion can be observed. Since relative probabilities are not affected, no revision is optimal. (See Loury [8], for the single uncertain reserve case.)

Formally the problem of maximizing (9) subject to (10) can be treated by Maximum Principle. Define, then, the appropriate Hamiltonian,

$$\mathcal{H} = e^{-\rho t} \{u(R_1 + R_2)[1 - F(A_2)] + J(A_1)f(A_2)R_2\} + \psi_1 R_1 + \psi_2 R_2 \quad (12)$$

where ψ_1 and ψ_2 are adjoint variables corresponding to A_1 and A_2 , respectively.

The Kuhn-Tucker conditions for maximization of $\mathcal{H}$ are

$$\begin{aligned} \frac{\partial \mathcal{H}}{\partial R_1} &= e^{-\rho t} u'(R_1 + R_2)[1 - F(A_2)] + \psi_1 \leq 0 \\ \frac{\partial \mathcal{H}}{\partial R_1} \cdot R_1 &= 0 \end{aligned} \quad (13)$$

and

$$\begin{aligned} \frac{\partial \mathcal{H}}{\partial R_2} &= e^{-\rho t} \{ u' (R_1 + R_2) [1 - F(A_2)] + J(A_1) f(A_2) \} + \Psi_2 \leq 0 \\ \frac{\partial \mathcal{H}}{\partial R_2} \cdot R_2 &= 0 \end{aligned} \quad (14)$$

The adjoint equations are

$$\dot{\Psi}_1' = \frac{-\partial \mathcal{H}}{\partial A_1} = e^{-\rho t} \frac{dJ}{dA_1} f(A_2) R_2 \quad (15)$$

and

$$\dot{\Psi}_2' = \frac{-\partial \mathcal{H}}{\partial A_2} = e^{-\rho t} \{ u(R_1 + R_2) f(A_2) - J(A_1) f'(A_2) R_2 \} \quad (16)$$

In order to consider the optimal sequence of exploitation of the two reserves, consider

$$\frac{\partial \mathcal{H}}{\partial R_1} - \frac{\partial \mathcal{H}}{\partial R_2} = \Psi_1 - \Psi_2 - e^{-\rho t} J(A_1) f(A_2) \quad (17)$$

from which, using (15) and (16), it follows that

$$\frac{d}{dt} \left\{ \frac{\partial \mathcal{H}}{\partial R_1} - \frac{\partial \mathcal{H}}{\partial R_2} \right\} = e^{-\rho t} f(A_2) \left\{ \rho J - \frac{dJ}{dA_1} \cdot (R_1 + R_2) - u(R_1 + R_2) \right\} \quad (18)$$

Bellman's equation, (7), shows precisely that

$$\frac{d}{dt} \left\{ \frac{\partial \mathcal{H}}{\partial R_1} - \frac{\partial \mathcal{H}}{\partial R_2} \right\} \geq 0 \quad (19)$$

and equality holds, if and only if $R_1 + R_2$ is the appropriate rate of extraction at time t for a policy starting at time t and using only the remainder of the first certain reserve for all subsequent time. It is not hard to show that $R_1 + R_2$ should be greater than this rate, since the uncertain reserve is still available in addition to the first reserve. It follows then that simultaneous exploitation of the two reserves is never optimal, for then

$$\frac{\partial \mathcal{H}}{\partial R_1} = \frac{\partial \mathcal{H}}{\partial R_2} = 0 \quad (20)$$

on some interval, but

$$\frac{d}{dt} \left\{ \frac{\partial \mathcal{H}}{\partial R_1} - \frac{\partial \mathcal{H}}{\partial R_2} \right\} > 0 \quad (21)$$

which is a contradiction. Also, a switch from a period of exploitation of the first certain reserve to the second uncertain reserve cannot occur on the conditional plan. For suppose such a switch occurs at $t^* > 0$. Then

$$\frac{\partial \mathcal{H}}{\partial R_1} = 0 \text{ on some interval before } t^* \quad (22)$$

but

$$\frac{\partial \mathcal{H}}{\partial R_2} = 0 \text{ on some interval after } t^* \quad (23)$$

Hence, from (21),

$$\frac{d}{dt} \left\{ \frac{\partial \mathcal{H}}{\partial R_1} \right\} > 0 \text{ on an interval after } t^* . \quad (24)$$

For optimality along the conditional path, $R_1 + R_2$ must be continuous across even such a switch-point (for otherwise, a small "swapping" variation would improve expected utility). Since the variable Ψ_1 is continuous also,

$$\frac{\partial \mathcal{H}}{\partial R_1}$$

must be continuous across t^* . Hence (22) and (24) establish a contradiction.

Hence, if the conditional path ever involves exploitation of the certain reserve, the uncertain reserve can never be exhausted. This is clearly not optimal, and so no such exploitation of the certain reserve should occur prior to exhaustion of the uncertain reserve.

Theorem 1. When one deposit of a depletable natural resource has known size, but another is of uncertain size, the optimal strategy involves an initial phase of exploitation solely of the uncertain deposit until it is exhausted. Only then should the certain reserve be exploited. This is true regardless of the

form of the utility and probability density functions. (Utility was assumed to be concave, but not necessarily to satisfy Inada-type conditions.)

This result can be made intuitively clear by means of the following argument. Suppose that there is an optimal conditional path,

$$R(t) = R_1(t) + R_2(t), \quad R_1(t) > 0 \quad \text{some } t \quad (25)$$

For any particular realization of the uncertain second reserve, $\bar{S}_2$, say, the time of exhaustion is determined by

$$A_2(T) = \bar{S}_2 \quad (26)$$

At this time, a deterministic plan can be effected based upon the remaining stock of the first certain reserve. Now consider a new conditional path

$$R'_1(t) = 0, \quad R'_2(t) = R_1(t) + R_2(t) \quad (27)$$

Now an earlier exhaustion date, T' , for the uncertain reserve is determined by

$$A'_2(T') = \bar{S}_2 \quad (28)$$

At T' , the amount left is the complete certain stock, S_1 . This was, in fact, the total amount left at T' under the old conditional path, because total cumulative consumption at T' is the same. An option available at T' for the new path is to duplicate precisely the realized old path. This is not generally optimal, however. In particular, it is easy to show that

$$R(T) > \bar{R}_1(T) \quad (29)$$

where $\bar{R}_1(T)$ is the optimal initial consumption along the deterministic plan for consuming the remaining stock of the certain reserve. Such a discrete jump in consumption means that the realized old path can certainly be improved upon as a deterministic plan. Since this argument is valid for each realization of S_2 , it is valid for arbitrary probability distributions. Clearly this result

is related to the result that "the value of information is positive" (see DeGroot [3]).

3. Market Operation

What problems arise in decentralizing the optimum of the previous section? Even with futures markets at time zero, there may be a certain externality arising from consumers' actions.

Consider first the operation of a price-taking firm owning the certain first reserve. If $q(t)$ is the supply price at $t = 0$ for delivery at time t , clearly

$$q(t) = P_0, \quad (30)$$

a constant. Hence the firm's profit is

$$P_0 S_1 \quad (31)$$

which is independent of the rate of extraction, $R_1(t)$

Consider the more complex case of a "price-taking" firm owning the uncertain second reserve. Clearly prices for delivery at time t are no longer adequate, since the probability of fulfillment depends upon the cumulative total, $A_2(t)$, extracted by time t . Define, then,

$$P(A_2, t) \quad (32)$$

to be the supply price at $t = 0$ for delivery at time t , given cumulative extraction, A_2 , with probability of fulfillment $1 - F(A_2)$. (It is then the function $P(A_2, t)$ which is parametric to the price-taking firm.) Profits are

$$\int_0^{\infty} P(A_2, t) R_2 dt = \int_0^{\infty} P(A_2, t) dA_2$$

Clearly, in order for the firm to be indifferent among alternative extraction paths over time, it is necessary that

$$P(A_2, t) = P(A_2) \quad (33)$$

This then is the supply price at $t = 0$ for delivery at any time with probability $1 - F(A_2)$. In fact, if the firm owning the second reserve is risk-neutral,

$$P(A_2) = P_0(1 - F(A_2)) \quad (34)$$

where P_0 is the price at $t = 0$ for certain delivery at any time, as for the firm owning the first certain reserve. For suppose (34) does not hold, so that there is an A_2^* such that

$$\frac{P(A_2^*)}{1 - F(A_2^*)} \geq \frac{P(A_2)}{1 - F(A_2)} \quad (35)$$

for all A_2 and with strict inequality on some interval. (As follows from continuity.) Hence

$$\int_0^{\infty} P(A_2) dA_2 < \frac{P(A_2^*)}{1 - F(A_2^*)} \int_0^{\infty} (1 - F(A_2)) dA_2 = \frac{P(A_2^*)}{1 - F(A_2^*)} E(S_2) \quad (36)$$

The last term is the expected profit which the firm can obtain in the following way. At time zero, extract the entire uncertain reserve with mean outcome $E(S_2)$. Now sell $E(S_2)/(1 - F(A_2^*))$ tickets at price $P(A_2^*)$ in a lottery to deliver units of resource with probability $1 - F(A_2^*)$. With no storage, as has already been implicitly assumed, all consumption must take place at time zero. Clearly demand considerations must render the firm at least indifferent to such a strategy, and hence (34) must hold.

Now consider the actions of a price-taking consumer in these markets. Optimality can be obtained, but only by making two strong assumptions.

- (1) The consumer is permitted to make contracts not simply for delivery at a specified time, but also contingent upon exhaustion or non-exhaustion of the second uncertain reserve.

- (2) The consumer accounts for the effect his consumption has upon cumulative extraction paths. The cumulative extraction of the uncertain reserve determines the probability of exhaustion over time, and cumulative extraction of the first reserve determines the amount which will be available if the uncertain reserve becomes exhausted. (Although an analogous assumption was made for firms, it is more reasonable that information is available about their extraction paths than that it is available about consumption.)

In order merely to suggest an example of how markets might fail in this situation, suppose,

- (1') The consumer can make contracts only for delivery at specified dates, regardless of exhaustion or non-exhaustion of the second reserve.
- (2') The consumer neglects the impact of his consumption upon aggregate extraction paths. In particular, the probability of exhaustion of the uncertain reserve (implying non-delivery) is parametric at each point in time.

Redefine, then

$$\Pi(t) = 1 - F(A_2(t)) \quad (37)$$

to be the now parametric probability of fulfillment of a contract made for delivery at time t , by the firm owning the uncertain reserve. The consumer's expected utility is

$$\int_0^{\infty} \{ \Pi(t)u(R_1 + R_2) + (1 - \Pi(t))u(R_1) \} e^{-\rho t} dt \quad (38)$$

which is to be maximized for some level of expenditure,

$$Y = P_0 \int_0^{\infty} \{ \Pi(t)R_2 + R_1 \} dt \quad (39)$$

It is easily established that R_1 and R_2 are determined by (39) and

$$u'(R_1)e^{-\rho t} = \lambda P_0 \text{ and } R_2 = 0 \quad (40)$$

for some Lagrange multiplier λ . Hence only contracts for the certain reserve will be entered into by consumers. This result is clearly due to the risk-neutrality of the firms and the risk aversion of consumers.

Theorem 2. When consumers can make contracts for delivery at specified times only, and take the probability of exhaustion of the second reserve as parametric, no contracts for delivery of the uncertain reserve will be entered into. Thus, far from exploiting this reserve first as is optimal, plausible market failures will lead to it never being exploited at all.

Note: If firms are allowed to be also risk-averse, the price for uncertain delivery, $P(A_2)$, will fall below $P_0(1 - F(A_2))$. Then consumers will be prepared to enter into contracts. Generally, however, at each point in time, both reserves will be exploited (if the uncertain one has not run out) which is, again, non-optimal.

4. Two Uncertain Reserves

Suppose now that the sizes of the two reserves are given by independent subjective probability distributions. Suppose, in fact, that the probability density functions are $f_1(S_1)$ and $f_2(S_2)$ with cumulative distribution functions $F_1(S_1)$ and $F_2(S_2)$. For any contingent plan it can be readily shown that expected utility prior to exhaustion of either reserve is given by

$$\int_0^{\infty} e^{-\rho t} u(R_1 + R_2)(1 - F_1(A_1))(1 - F_2(A_2))dt \quad (41)$$

where A_1 and A_2 are, as before, the cumulative amounts extracted from each reserve. (This follows because $(1 - F_1(A_1))(1 - F_2(A_2))$ is the proportion of cases in which neither reserve has failed at time t .) To consider the optimal strategy subsequent to exhaustion of the second reserve, define

$$J_1(A_1) = \max_{R_1} \int_0^{\infty} u(R_1)(1 - F_1(A_1))e^{-\rho t} dt$$

$$A_1(0) = A_1 \quad (42)$$

and similarly,

$$J_2(A_2) = \max_{R_2} \int_0^{\infty} u(R_2)(1 - F_2(A_2))e^{-\rho t} dt$$

$$A_2(0) = A_2 \quad (43)$$

as the expected utility for an optimal path using only the remaining first and second reserve, respectively. (It has been implicitly assumed that $u(0) = 0$, which is a harmless normalization, as long as $u(0)$ is finite. If $u(0) = -\infty$, the problem either will not be well-defined, or will lead to a trivial result. See Kemp [7].) If, indeed, the second reserve fails first, the expected utility subsequent to this failure is

$$\int_0^{\infty} J_1(A_1)f_2(A_2)R_2e^{-\rho t} dt \quad (44)$$

and, similarly, if the first reserve fails first, the expected utility subsequent to failure is

$$\int_0^{\infty} J_2(A_2)f_1(A_1)R_1e^{-\rho t} dt \quad (45)$$

Hence, total expected utility is the sum of (41), (44), and (45), or

$$W = \int_0^{\infty} e^{-\rho t} \{u(R_1 + R_2)(1 - F_1(A_1))(1 - F_2(A_2)) + J_1(A_1)f_2(A_2)R_2 + J_2(A_2)f_1(A_1)R_1\} dt \quad (46)$$

which is to be maximized subject to

$$\dot{A}_1 = R_1, \quad A_1(0) = 0$$

$$\dot{A}_2 = R_2, \quad A_2(0) = 0 \quad (47)$$

The appropriate Hamiltonian thus becomes

$$\begin{aligned} \mathcal{H} = e^{-\rho t} \{ & u(R_1 + R_2)(1 - F_1(A_1))(1 - F_2(A_2)) + J_1(A_1)f_2(A_2)R_2 \\ & + J_2(A_2)f_1(A_1)R_1 \} + \Psi_1 R_1 + \Psi_2 R_2 \end{aligned} \quad (48)$$

where Ψ_1 and Ψ_2 are adjoint variables corresponding to A_1 and A_2 , respectively.

The adjoint equations are

$$\begin{aligned} \Psi_1' = \{ & u(R_1 + R_2)f_1(A_1)(1 - F_2(A_2)) - J_2(A_2)f_1'(A_1)R_1 \\ & - J_1'(A_1)f_2(A_2)R_2 \} e^{-\rho t} \end{aligned} \quad (49)$$

and

$$\begin{aligned} \Psi_2' = \{ & u(R_1 + R_2)f_2(A_2)(1 - F_1(A_1)) - J_1(A_1)f_2'(A_2)R_2 \\ & - J_2'(A_2)f_1(A_1)R_1 \} e^{-\rho t} \end{aligned} \quad (50)$$

Now, for an optimum

$$\begin{aligned} \frac{\partial \mathcal{H}}{\partial R_1} = \{ & u'(R_1 + R_2)(1 - F_1(A_1))(1 - F_2(A_2)) + J_2(A_2)f_1(A_1) \} e^{-\rho t} \\ & + \Psi_1 \leq 0 \end{aligned} \quad (51)$$

$$\frac{\partial \mathcal{H}}{\partial R_1} \cdot R_1 = 0$$

and

$$\begin{aligned} \frac{\partial \mathcal{H}}{\partial R_2} = \{ & u'(R_1 + R_2)(1 - F_1(A_1))(1 - F_2(A_2)) + J_1(A_1)f_2(A_2) \} e^{-\rho t} \\ & + \Psi_2 \leq 0 \end{aligned} \quad (52)$$

$$\frac{\partial \mathcal{H}}{\partial R_2} \cdot R_2 = 0$$

In any case,

$$\frac{\partial \mathcal{H}}{\partial R_1} - \frac{\partial \mathcal{H}}{\partial R_2} = \{ J_2(A_2)f_1(A_1) - J_1(A_1)f_2(A_2) \} e^{-\rho t} + \Psi_1 - \Psi_2 \quad (53)$$

from which, using (49) and (50), it follows that

$$\begin{aligned}
\frac{d}{dt} \left\{ \frac{\partial H}{\partial R_1} - \frac{\partial H}{\partial R_2} \right\} &= e^{-\rho t} f_1(A_1) \{ J_2'(A_2)(R_1 + R_2) + u(R_1 + R_2) \cdot (1 - F_2(A_2)) \\
&\quad - \rho J_2(A_2) \} - e^{-\rho t} f_2(A_2) \{ J_1'(A_1)(R_1 + R_2) + u(R_1 + R_2)(1 - F_1(A_1)) \\
&\quad - \rho J_1(A_1) \}
\end{aligned} \tag{54}$$

From Bellman's equation for $J_2(A_2)$ and $J_1(A_1)$, respectively, it follows that each term in braces above is non-positive. In fact, since the conditional path, $R_1 + R_2$ must involve greater consumption than would be appropriate if only either the first or second reserve remained, each term in braces is negative. However, these observations cannot now rule out simultaneous exploitation or determine the optimal sequence of exploitation. It seems plausible that arbitrarily complex patterns of exploitation might arise with general distributions. To gain further insight into the choice between two uncertain reserves, consider then particular probability density functions. The simplest choices are exponential distributions,

$$f_i(s_i) = \alpha_i e^{-\alpha_i s_i} \quad s_i \geq 0, \quad i=1,2 \tag{55}$$

because optimal paths will be constant over time. (See Gilbert [5] for the single reserve case.) This facilitates the global comparisons which are apparently needed. Assume, in fact,

$$\alpha_1 \geq \alpha_2 \tag{56}$$

so that the first distribution has a lower mean and a lower variance than the second. (For an exponential distribution with "failure rate" α , the mean and variance are both $1/\alpha$. See Feller [4].) Now, it is easy to see that

$$J_i(A_i) = e^{-\alpha_i A_i} J_i(0) \quad i=1,2 \tag{57}$$

and that because of this separability, and the "Principle of Optimality" (see Bellman [1]),

$$R_i(t) = R_i(0) = \bar{R}_i \quad i=1,2 \quad (58)$$

that is, the rates of extraction are constant when only a single reserve remains. Now, from (46),

$$W = \int_0^{\infty} e^{-\rho t} e^{-\alpha_1 A_1 t} e^{-\alpha_2 A_2 t} \{u(R_1 + R_2) + \alpha_1 R_1 J_2(0) + \alpha_2 R_2 J_1(0)\} dt \quad (59)$$

so that a similar argument implies

$$R_i(t) = R_i(0) = R_i^*, \quad i=1,2 \quad (60)$$

--the rates of extraction are constant also when both reserves remain. Now

$$W = \text{Max}_{R_1, R_2} \frac{u(R_1 + R_2) + \alpha_1 R_1 J_2(0) + \alpha_2 R_2 J_1(0)}{\rho + \alpha_1 R_1 + \alpha_2 R_2} \quad (61)$$

It is clear from (61), that, if $\alpha_1 = \alpha_2$, the sequence of exploitation is immaterial, as only $R_1 + R_2$ then matters. Assume henceforth that $\alpha_1 > \alpha_2$.

Suppose now that simultaneous exploitation can be optimal. (This is shown to lead to a contradiction.) Then the right-hand side of (54) must be zero, since (50) and (51) must hold with equality on some interval. Hence

$$\begin{aligned} (\alpha_1 - \alpha_2)u(R_1^* + R_2^*) &= \alpha_1(\rho + \alpha_2(R_1^* + R_2^*))J_2(0) \\ &\quad - \alpha_2(\rho + \alpha_1(R_1^* + R_2^*))J_1(0) \end{aligned} \quad (62)$$

and (61) can be simplified to

$$W_s = \frac{\alpha_1 J_2(0) - \alpha_2 J_1(0)}{\alpha_1 - \alpha_2} \quad (63)$$

which is thus the optimum expected utility if simultaneous exploitation is optimal. If instead, the first reserve should be exploited first, define

$$W_1 = \text{Max}_{R_1} \frac{u(R_1) + \alpha_1 R_1 J_2(0)}{\rho + \alpha_1 R_1} \quad (64)$$

and similarly, if the second reserve is optimally exploited first,

$$W_2 = \max_{R_2} \frac{u(R_2) + \alpha_2 R_2 J_1(0)}{\rho + \alpha_2 R_2} \quad (65)$$

Thus simultaneous exploitation is not, in fact, optimal if W_s is less than either W_1 or W_2 . (The optimal sequence is determined by the comparison of W_1 and W_2 .) It is convenient, although possibly not essential, to assume isoelastic utility. That is

$$u(C) = C^\gamma / \gamma \quad 0 < \gamma < 1 \quad (66)$$

(Recall that $u(0) = 0$.) Then

$$J_i(0) = \max_{R_i} \frac{R_i^\gamma / \gamma}{\rho + \alpha_i R_i} = \left(\frac{1-\gamma}{\rho} \right)^{1-\gamma} \alpha_i^{-\gamma} \quad (67)$$

so that, from (63),

$$W_s = \frac{\alpha_1^{1+\gamma} - \alpha_2^{1+\gamma}}{\alpha_1 - \alpha_2} \frac{1}{\alpha_1^\gamma \alpha_2^\gamma} \left(\frac{1-\gamma}{\rho} \right)^{1-\gamma} \quad (68)$$

and, from (64),

$$W_1 = \max_{R_1} \frac{\frac{R_1^\gamma}{\gamma} + \frac{\alpha_1}{\alpha_2^\gamma} \left(\frac{1-\gamma}{\rho} \right)^{1-\gamma} R_1}{\rho + \alpha_1 R_1} \quad (69)$$

It can be shown now that

$$W_1 > W_s \quad (70)$$

but the proof of this is relegated to Appendix 1. Hence simultaneous exploitation is never optimal.

The remaining question is thus: Which of the two reserves is it optimal to exploit first? It is shown in Appendix 2 that

$$\frac{dW_1}{d\alpha_1} < \frac{dW_2}{d\alpha_1} < 0 \quad \text{at} \quad \alpha_1 = \alpha_2 = \alpha \quad (71)$$

say. Hence it is optimal to exploit the second reserve (with higher mean and variance) first in some neighborhood of α . However, suppose that for some $\alpha_1 > \alpha_2$, it is optimal to exploit the first reserve first. Then by continuity, there would have to be another $\alpha_1 > \alpha_2$, such that $W_1 = W_2$. Hence, either reserve may be exploited at each time. By switching back and forth between the reserves any convex combination of the appropriate extraction rates may be approximated. By continuity, then, simultaneous exploitation must be optimal, contradicting the previous result. Hence it is always optimal to exploit the high-mean, high-variance reserve first until it is exhausted.

Theorem 3. When there are two uncertain reserves, with distinct exponential distributions, it is optimal to exploit the high mean, high variance reserve until it is exhausted before beginning exploitation of the other reserve.

5. Conclusions

In bold terms, the main result of this paper is that it may well be optimal to exploit "riskier" reserves first and to defer exploitation of the "less risky" reserves until the first reserves are exhausted. This was established precisely in the case of one certain and one uncertain reserve, and in the case of distinct exponentially distributed reserves. The result is striking in that it might have seemed that risk aversion would entail deferment of risky prospects. Indeed plausible market failures might actually lead to postponement of consumption of the riskier reserve. In terms of policy, then, this suggests a social motive for early exploitation of, for example, the Athabasca oil sands of Alberta, while substantial conventional reserves remain. However, this social motive is not necessarily reflected in market behavior.

Appendix 1. Proof that simultaneous exploitation is never optimal. From (68) and (69), it follows that

$$W_1 - W_s = \max_{R_1} \left\{ \frac{\frac{R_1^\gamma}{\gamma} + \frac{\alpha_1}{\alpha_2^\gamma} \left(\frac{1-\gamma}{\rho\gamma}\right)^{1-\gamma} R_1 - (\rho + \alpha_1 R_1) \frac{\alpha_1^{1+\gamma} - \alpha_2^{1+\gamma}}{\alpha_1 - \alpha_2} \alpha_1^{-\gamma} \alpha_2^{-\gamma} \left(\frac{1-\gamma}{\rho\gamma}\right)^{1-\gamma}}{\rho + \alpha_1 R_1} \right\} \quad (72)$$

and so by rearrangement of the denominator, it follows that

$$\begin{aligned} \text{sgn}(W_1 - W_s) &= \text{sgn} \max_X \left\{ \frac{X^\gamma}{\gamma} - \left[\frac{(1-\gamma)\alpha_1\alpha_2}{\rho\gamma} \right]^{1-\gamma} \left[\frac{\alpha_1^{1+\gamma} - \alpha_2^{1+\gamma}}{\alpha_1 - \alpha_2} \right] X \right. \\ &\quad \left. - \rho \left(\frac{1-\gamma}{\rho\gamma}\right)^{1-\gamma} \alpha_1^{-\gamma} \alpha_2^{-\gamma} \left[\frac{\alpha_1^{1+\gamma} - \alpha_2^{1+\gamma}}{\alpha_1 - \alpha_2} \right] \right\} \end{aligned} \quad (73)$$

although the maximum of (73) obtains at a different value for X than the value for R_1 in (72). Carrying out the maximization, finally,

$$\text{sgn}(W_1 - W_s) = \text{sgn}\{(Z-1) - (Z^{1+\gamma} - 1)^{1-\gamma}(Z^\gamma - 1)\} \quad Z = \frac{\alpha_1}{\alpha_2} > 1, \quad 0 < \gamma < 1. \quad (74)$$

Define then

$$f(\gamma) = (Z^{1+\gamma} - 1)^{1-\gamma} (Z^\gamma - 1) > 0 \quad (75)$$

which has the following properties

$$f(1) = (Z-1), \quad \frac{f'(\gamma)}{f(\gamma)} = \frac{\log Z \cdot Z^\gamma}{(Z^{1+\gamma}-1)(Z^\gamma-1)} [Z(Z^\gamma-1) + \gamma(Z-1)] > 0 \quad (76)$$

so that (70) follows.

Appendix 2. Proof that it is optimal to exploit the high-mean, high-variance reserve first in a neighborhood of $\alpha_1 = \alpha_2 = \alpha$. From (69), and the analogous expression for W_2 ,

$$\left. \frac{dW_1}{d\alpha_1} \right|_{\alpha_1 = \alpha_2 = \alpha} = \frac{\frac{\rho R^*}{\alpha^\gamma} \left(\frac{1-\gamma}{\rho\gamma} \right)^{1-\gamma} - \frac{R^{*1+\gamma}}{\gamma}}{(\rho + \alpha R^*)^2} \quad (77)$$

and

$$\left. \frac{dW_2}{d\alpha_1} \right|_{\alpha_1 = \alpha_2 = \alpha} = - \frac{\frac{\gamma R^*}{\alpha^\gamma} \left(\frac{1-\gamma}{\rho\gamma} \right)^{1-\gamma}}{(\rho + \alpha R^*)} < 0 \quad (78)$$

where $R_1^* = R_2^* = R^*$ say, when $\alpha_1 = \alpha_2 = \alpha$. Now it can be shown that

$$\text{sgn} \left[\frac{dW_2}{d\alpha_1} - \frac{dW_1}{d\alpha_1} \right] = \text{sgn} [W - (1 + \gamma) J(0)] \quad (79)$$

where $J_1(0) = J_2(0)$, and $W_1 = W_2 = W$, when $\alpha_1 = \alpha_2 = \alpha$. Now, from (67) and (69),

$$\begin{aligned} \text{sgn} [W - (1+\gamma) J(0)] &= \text{sgn} \max_R \left\{ \frac{\frac{R^\gamma}{\gamma} + \alpha^{1-\gamma} \left(\frac{1-\gamma}{\rho\gamma} \right)^{1-\gamma} R - (\rho + \alpha R) \frac{1+\gamma}{\alpha^\gamma} \left(\frac{1-\gamma}{\rho\gamma} \right)^{1-\gamma}}{\rho + \alpha R} \right\} \\ &= \text{sgn} \max_X \left\{ \frac{X^\gamma}{\gamma} - \alpha^{1-\gamma} \gamma \left(\frac{1-\gamma}{\rho\gamma} \right)^{1-\gamma} X - \rho \frac{1+\gamma}{\alpha^\gamma} \left(\frac{1-\gamma}{\rho\gamma} \right)^{1-\gamma} \right\} \end{aligned} \quad (80)$$

Carrying out the maximization and upon simplification,

$$\text{sgn} \left[\frac{dW_2}{d\alpha_1} - \frac{dW_1}{d\alpha_1} \right] = \text{sgn} [1 - \gamma^\gamma (1+\gamma)^{1-\gamma}] \quad (81)$$

Now by strict concavity,

$$\gamma \log \gamma + (1-\gamma) \log(1+\gamma) < \log(\gamma^2 + 1 - \gamma^2) = 0 \quad (82)$$

so that (71) follows.

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