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Generalized Exponential Models with Applications

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A thesis submitted in partial fulfillment of the requirements for the Doctor of Philosophy degree in Statistics and Actuarial Sciences

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GENERALIZED EXPONENTIAL MODELS WITH APPLICATIONS
(Thesis format: Monograph)

by

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Graduate Program in Statistics and Actuarial Science

A thesis submitted in partial fulfillment
of the requirements for the degree of
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GENERALIZED EXPONENTIAL MODELS WITH APPLICATIONS

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Abstract

We introduce a generalized exponential model whose exact moments and normalizing constant are obtained in terms of Meijer's generalized hypergeometric G-function. Actually, several widely utilized statistical distributions such as the gamma, Weibull and half-normal constitute particular cases thereof. The generalized inverse Gaussian distribution, which was popularized in the late seventies by Ole Barndorff-Neilsen, is also extended by incorporating an additional parameter in its density function, the moments of the resulting distribution being expressed in terms of Bessel functions. A number of data sets were then fitted with diverse exponential-type models for comparison purposes. Additionally, it is shown that the inverse Mellin transform technique may be employed to derive a multiple series representation of the density function of linear combinations of chi-square random variables, which are encountered for instance in connection with the distribution of certain quadratic forms and some asymptotic distributional results arising in multivariate analysis. The accuracy of the truncated form of this density function is compared to that obtained from a reparameterized generalized gamma distribution. A methodology whereby regression problems are converted into density estimation problems is also proposed and applied to certain actuarial data sets. A technique for modeling bivariate observations is presented as well.

Keywords: Bivariate density estimation; Density estimation; Exponential-type distribution; Inverse Gaussian distribution; Generalized exponential models; Generalized hypergeometric functions; Goodness-of-fit; Inverse Mellin transform; Moments; Mortality data; Linear combination of chi-square random variables .

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Chapter 1

Introduction

1.1 Introduction

As pointed out in Balakrishnan and Basu (1995), the gamma family of distributions was discussed by Karl Pearson as early as 1895. However, it took another 35 years for the exponential distribution, which is a special case, to appear on its own: While discussing the sampling distribution of the standard deviation, Kondo (1930) referred to the exponential distribution as Pearson's Type X distribution. Applications of the exponential distribution in actuarial, biological and engineering problems were respectively proposed by Steffensen (1930), Teissier (1934) and Weibull (1939).

Both the shape and scale parameters of the gamma distribution can have non-integer values. The gamma distribution has two types of applications. First, applications based on intervals between events; in this form, examples of its use include queuing models, the flow of items through manufacturing and distribution processes, the load on web servers, and the many and varied forms of telecom exchange. The other type of applications takes advantage of the gamma distribution moderately skewed profile; accordingly, this model can be utilized in several disciplines such as climatology where it is a workable model for rainfall and in actuarial mathematics where it has been used for modeling insurance claims, the size of loan defaults, and for determining the probability of ruin and the value at risk.

An extension of the exponential distribution referred to as the Weibull distribution was proposed by Weibull (1951). The exponential distribution is a special case wherein the shape parameter equals one. As explained in Lai *et al.* (2006), the Weibull distribution has many applications in survival analysis and reliability engineering. Several applications in industrial quality control are also discussed in Berrettoni (1964).

A generalized exponential model ($\mathcal{GEM}$) distribution is being introduced in Chapter 2. Its density function is given by

$$f_A(x) = c\, x^{\xi+\delta} e^{-\nu\, x^{\delta}} e^{-\tau\, x^{-\rho}}\, \mathcal{I}_{\mathbb{R}^+}(x), \tag{1.1}$$

where $\mathcal{I}_{\mathcal{B}}(x)$ denotes the indicator function of the set $\mathcal{B}$, $\mathbb{R}^+$ is the set of real positive numbers and c is a normalizing constant. The parameters ν, δ, τ and ρ are assumed to be positive while ξ can be any real number. The extension proposed in this thesis is more general than the generalized inverse Gaussian model introduced by Jørgensen (1982).

Numerous distributions are special cases of the proposed generalized exponential model. For instance, the following distributions arise as special cases of (1.1) wherein $\tau = 0$:

(i) The gamma distribution — denoted $\Gamma(\alpha,\beta)$— with density function

$$f(x) = \frac{x^{\alpha-1}\exp(-x/\beta)}{\beta^{\alpha}\,\Gamma(\theta)}\,\mathcal{I}_{\mathbb{R}^+}(x), \quad \alpha,\beta > 0\,,$$

is obtained by letting $\xi = \alpha - 2$, $\nu = 1/\beta$, and $\delta = 1$.

(ii) The Weibull distribution with density function

$$f(x) = \theta\,\phi\,x^{\phi-1}\exp(-\theta\,x^{\phi})\,\mathcal{I}_{\mathbb{R}^+}(x)\,,$$

is obtained by letting $\xi = -1$, $\nu = \theta$ and $\delta = \phi$.

(iii) The Maxwell distribution with density function

$$f(x) = \frac{4\,x^2\exp(-x^2/\theta^2)}{\theta^3\sqrt{\pi}}\,\mathcal{I}_{\mathbb{R}^+}(x)\,,$$

is obtained by letting $\xi = 0$, $\nu = 1/\theta^2$ and $\delta = 2$.

(iv) The half-normal distribution with density function

$$f(x) = \frac{2\exp(-x^2/(2\theta^2))}{\theta\sqrt{2\pi}}\,\mathcal{I}_{\mathbb{R}^+}(x), \;\; \theta > 0\,,$$

is obtained by letting $\xi = -2$, $\nu = 1/2\theta^2$ and $\delta = 2$.

(v) The exponential distribution with density function

$$f(x) = \frac{\exp(-x/\kappa)}{\kappa}\,\mathcal{I}_{\mathbb{R}^+}(x), \;\; \phi > 0\,,$$

is obtained by letting $\xi = -1$, $\nu = 1/\kappa$ and $\delta = 1$.

(vi) The chi-square distribution with density function

$$f(x) = \frac{x^{\nu/2-1}\exp(-x/2)}{2^{\nu/2}\,\Gamma(\nu/2)}\,\mathcal{I}_{\mathbb{R}^+}(x), \;\; \nu > 0\,,$$

is obtained by letting $\xi = \nu/2 - 2$, $\nu = 1/2$ and $\delta = 1$.

(vii) The Rayleigh distribution with density function

$$f(x) = \frac{x\exp(-x^2/(2a^2))}{a^2}\,\mathcal{I}_{\mathbb{R}^+}(x)\,,$$

is obtained by letting $\xi = -1$, $\nu = 1/(2a^2)$ and $\delta = 2$.

The density function of the inverse Gaussian distribution with real parameters $\mu \in \mathbb{R}$ and $\lambda > 0$ has the following form:

$$f(x) = \sqrt{\frac{\lambda}{2\pi\, x^3}}\, \exp(-\lambda(x-\mu)^2/(2x\mu^2))\, \mathcal{I}_{\mathbb{R}^+}(x)\,. \tag{1.2}$$

This density function is a particular case of the density function given in (1.1) with $\xi = -5/2$, $\nu = \lambda/(2\mu^2)$, $\delta = 1$, $\tau = \lambda/2$ and $\rho = 1$. It should be note that several other parameterizations are possible and that, in this case, $\tau \neq 0$.

Jørgensen (1982) proposed the so-called Generalized Inverse Gaussian ($\mathcal{GIG}$) distribution whose density function is given by

$$f(x) = \frac{(\phi/\theta)^{\lambda/2}}{2\,K_\lambda(\sqrt{\theta\,\phi})}\, x^{\lambda-1} \exp(-(\theta\, x^{-1} + \phi\, x)/2)\, \mathcal{I}_{\mathbb{R}^+}(x)\,, \tag{1.3}$$

where $K_\lambda(\cdot)$ denotes a modified Bessel function of the second kind. The density function given in (1.3) is a special case of the five-parameter exponential distribution with $\xi = \lambda - 2$, $\nu = \phi/2$, $\delta = 1$, $\tau = \theta/2$, and $\rho = 1$.

One can also obtain special cases from the symmetrized form of (1.1), that is,

$$f_S(x) = \frac{f_A(|x|)}{2}\, \mathcal{I}_{\mathbb{R}}(x)\,.$$

For instance, the normal distribution with density function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp(-x^2/(2\,\sigma^2))\, \mathcal{I}_{\mathbb{R}}(x)\,,\ \sigma > 0\,,$$

is obtained by letting $\tau = 0$, $\xi = -2$, $\nu = 1/2\sigma^2$, and $\delta = 2$. The lognormal(μ, σ) distribution is then obtained via the transformation $y = e^x$. Another example is the double-exponential distribution with density function

$$f(x) = \frac{\theta}{2} \exp(-\theta\,|x|)\, \mathcal{I}_{\mathbb{R}}(x),\ \theta > 0\,,$$

which turns out to be a particular case of $f_S(x)$ wherein $\tau = 0$, $\xi = -1$, $\nu = \theta$, and $\delta = 1$.

Moreover, a location parameter m can readily be incorporated in the density functions by replacing x with $x - m$.

The inverse Mellin transform technique will be used to determine the moments and the normalizing constant of the proposed distribution and other sub-class distributions. A brief introduction to this transform and its inverse is hereby provided.

If $f(x)$ is a real piecewise smooth function that is defined and single valued almost everywhere for $x > 0$ and such that $\int_0^\infty x^{k-1}|f(x)|\,\mathrm{d}x$ converges for some real value k, then

$M_f(s) = \int_0^\infty x^{s-1} f(x)\,\mathrm{d}x$ is the Mellin transform of $f(x)$. Whenever $f(x)$ is continuous, the corresponding inverse Mellin transform is

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s} M_f(s)\,\mathrm{d}s \tag{1.4}$$

which, together with $M_f(s)$, constitute a transform pair. The path of integration in the complex plane is called the Bromwich path. Equation (1.4) determines $f(x)$ uniquely if the Mellin transform is an analytic function of the complex variable s for $c_1 \le \Re(s) = c \le c_2$ where c_1 and c_2 are real numbers and $\Re(s)$ denotes the real part of s. In the case of a continuous nonnegative random variable whose density function is $f(x)$, the Mellin transform is its moment of order $(s-1)$ and the inverse Mellin transform yields $f(x)$.

Letting

$$M_f(s) = \frac{\left\{\prod_{j=1}^{m}\Gamma(b_j + B_j s)\right\}\left\{\prod_{i=1}^{n}\Gamma(1 - a_i - A_i s)\right\}}{\left\{\prod_{j=m+1}^{q}\Gamma(1 - b_j - B_j s)\right\}\left\{\prod_{i=n+1}^{p}\Gamma(a_i + A_i s)\right\}} \equiv h(s) \tag{1.5}$$

where an empty product (for example when $n = p$) is interpreted as unity and m, n, p, q are nonnegative integers such that $0 \le n \le p$, $1 \le m \le q$, A_i, $i = 1, \ldots, p$, B_j, $j = 1, \ldots, q$, are positive numbers and a_i, $i = 1, \ldots, p$, b_j, $j = 1, \ldots, q$, are complex numbers such that $-A_i(b_j + \nu) \ne B_j(1 - a_i + \lambda)$ for $\nu, \lambda = 0, 1, 2, \ldots$, $j = 1, \ldots, m$, and $i = 1, \ldots, n$, the H–function can be defined as follows in terms of the inverse Mellin transform of $M_f(s)$:

$$f(x) = \mathcal{H}_{p,q}^{m,n}\left(x \,\middle|\, \begin{matrix}(a_1, A_1), \ldots, (a_p, A_p)\\ (b_1, B_1), \ldots, (b_q, B_q)\end{matrix}\right) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} h(s)\, x^{-s}\mathrm{d}s \tag{1.6}$$

where $h(s)$ is as defined in (1.5) and the Bromwich path $(c - i\infty,\ c + i\infty)$ separates the points $s = -(b_j + \nu)/B_j$, $j = 1, \ldots, m$, $\nu = 0, 1, 2, \ldots$, which are the poles of $\Gamma(b_j + B_j s)$, $j = 1, \ldots, m$, from the points $s = (1 - a_i + \lambda)/A_i$, $i = 1, \ldots, n$, $\lambda = 0, 1, 2, \ldots$, which are the poles of $\Gamma(1 - a_i - A_i s)$, $i = 1, \ldots, n$. Thus, one must have

$$\mathcal{M}ax_{1 \le j \le m}\, \Re\{-b_j/B_j\} < c < \mathcal{M}in_{1 \le i \le n}\, \Re\{(1 - a_i)/A_i\}. \tag{1.7}$$

If, for certain parameter values, an H–function remains positive on the entire domain, then whenever the existence conditions are satisfied, a probability density function can be generated by normalizing it. For example, the Weibull, chi-square, half–normal and F distributions can all be expressed in terms of H-functions. For the main properties of the H–function as well as its applicability to various disciplines, the reader is referred to Mathai and Saxena (1978) and Mathai (1993).

When $A_i = B_j = 1$ for $i = 1, \ldots, p$ and $j = 1, \ldots, q$, the H–function reduces to Meijer's G–function, that is,

$$\mathcal{G}_{p,q}^{m,n}\left(x \,\middle|\, \begin{matrix}a_1, \ldots, a_p\\ b_1, \ldots, b_q\end{matrix}\right) \equiv \mathcal{H}_{p,q}^{m,n}\left(x \,\middle|\, \begin{matrix}(a_1, 1), \ldots, (a_p, 1)\\ (b_1, 1), \ldots, (b_q, 1)\end{matrix}\right). \tag{1.8}$$

Moreover, the G–function satisfies the following identity:

$$\mathcal{G}^{m,n}_{p,q}\left(x\,\middle|\,\begin{matrix}a_1,\dots,a_p\\ b_1,\dots,b_q\end{matrix}\right)=\mathcal{G}^{n,m}_{q,p}\left(\frac{1}{x}\,\middle|\,\begin{matrix}1-b_1,\dots,1-b_q\\ 1-a_1,\dots,1-a_p\end{matrix}\right). \tag{1.9}$$

Chapter 3 introduces an extension of the generalized inverse Gaussian distribution, which was extensively discussed in Jørgensen (1982). A related model is proposed as well. The effects of the parameters on these models are illustrated graphically. These distributions are fitted to several data sets, the goodness of fit being determined by means of the Anderson-Darling and the Cramér-von Mises statistics.

It is explained in Chapter 4 that quadratic forms in central normal vectors whose density function is often approximated in terms of exponential-type densities, can be reduced to linear combinations of chi-square random variables. We are making use of the inverse Mellin transform technique to obtain a multiple series representation of the density function of such linear combinations. The accuracy of the truncated form of density function is compared in several examples to that obtained from the reparameterized generalized gamma distribution, which is a particular case of the generalized exponential model.

The hazard and the mean residual life functions are determined for some of the proposed distributions in Chapter 5. Two actuarial data sets are fitted with the generalized exponential model. This is achieved by introducing a method whereby regression problems of this type can be converted into density estimation problems.

In the final chapter, a technique is proposed for modelling bivariate data. First the data is normalized and shifted to ensure that the variables be uncorrelated and that their support be essentially positive. Then, each variable is fitted individually with some of the proposed models, after which the inverse transformation is applied to the resulting bivariate density. Histograms of the data sets and plots of the final bivariate density estimates are included for comparison purposes. This approach could be extended to multivariate data sets.

The proposed extended and generalized exponential distributions should provide more accurate univariate or multivariate models in connection with the host of applications that rely on exponential-type distributions, which arise in numerous fields of scientific investigations. For convenience the *Mathematica* codes utilized in connection with the main applications presented in this thesis are included in the Appendix.

We conclude this section with a diagram showing the relationships between the distributions that we introduced and several known exponential-type distributions.

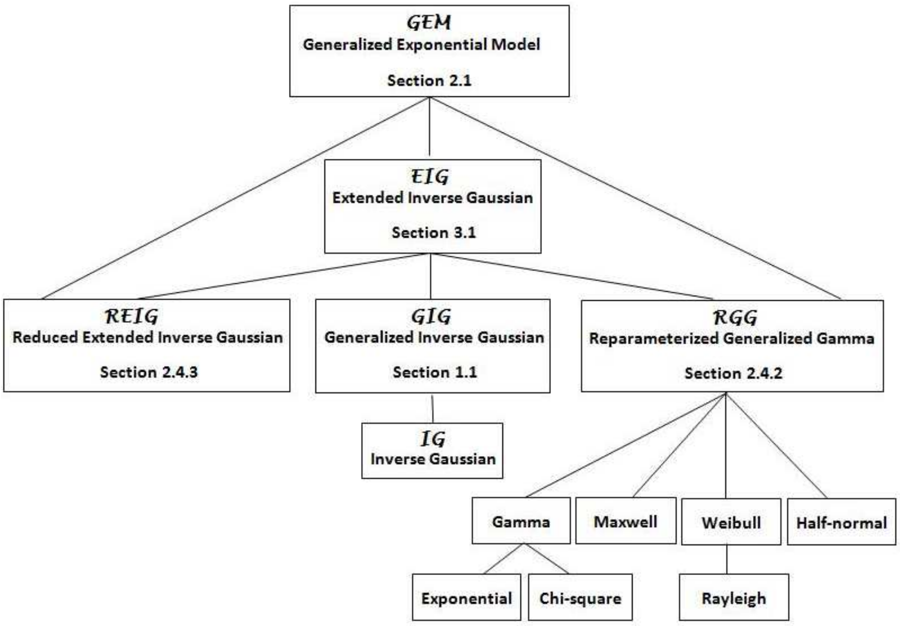

Figure 1.1: Relations between different models introduced in the thesis

Chapter 2

The Generalized Exponential Model

2.1 Introduction

This chapter explores the properties of the proposed probability distribution called the Generalized Exponential Model ($\mathcal{GEM}$) whose associated density function is given by

$$f_A(x) = c\, x^{\xi+\delta} e^{-\nu\, x^{\delta}} e^{-\tau\, x^{-\rho}}\, \mathcal{I}_{\mathbb{R}^+}(x), \tag{2.1}$$

where $\mathcal{I}_{\mathcal{B}}(x)$ denotes the indicator function of the set $\mathcal{B}$, $\mathbb{R}^+$ is the set of real positive numbers and c is a normalizing constant. The parameters ν, δ, τ and ρ, are assumed to be positive while ξ can be any real number. For simplification purposes, at times, δ and ρ will be expressed as fractions, that is, $\delta = a/d$ and $\rho = w/r$ where a, d, w, and r are positive integers.

Section 2.2 shows graphically how the $\mathcal{GEM}$ is affected by its five parameters and Section 2.3 presents the derivation of the h-moment of the $\mathcal{GEM}$ distribution. Some statistical functions such as the mean, certain central moments, the cumulative distribution function, the mode and the moment generating function of the $\mathcal{GEM}$ and some of its sub-class models are given in Section 2.4. In addition, a probability distribution model which approximates the $\mathcal{GEM}$ is introduced in the same section. This distribution is referred to as the proxy distribution and is computationally more convenient than the $\mathcal{GEM}$. Section 2.5 gives an introduction on the parameter estimation methods that are employed in this thesis, while Section 2.6 shows that products and ratios of certain exponential-type distributions can be expressed in terms of the moments of the proposed five-parameter exponential-type distribution, which in turn can be expressed in terms of generalized hypergeometric functions. In section 2.7 three data sets, namely the maximum flood levels, snowfall precipitation and repair time data sets, are fitted to the $\mathcal{GEM}$ and its proxy distribution.

2.2 Parameter Effects

This section illustrates graphically how the generalized exponential model specified by Equation (2.1) is affected by its parameters. Figure 2.1 and 2.2 show that ξ works as a scale and shift parameter. Figure 2.3- 2.6 shows the scale effect of δ, ν and ρ and the shift effect of τ on the proposed distribution.

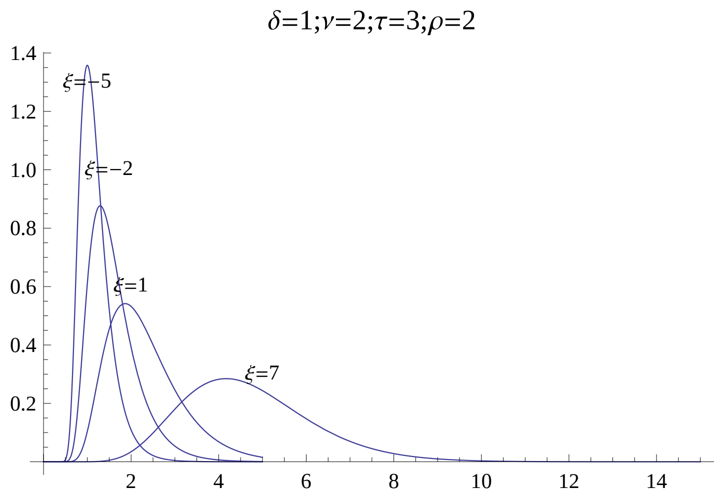

Figure 2.1: *The effect of ξ on the $\mathcal{GEM}$*

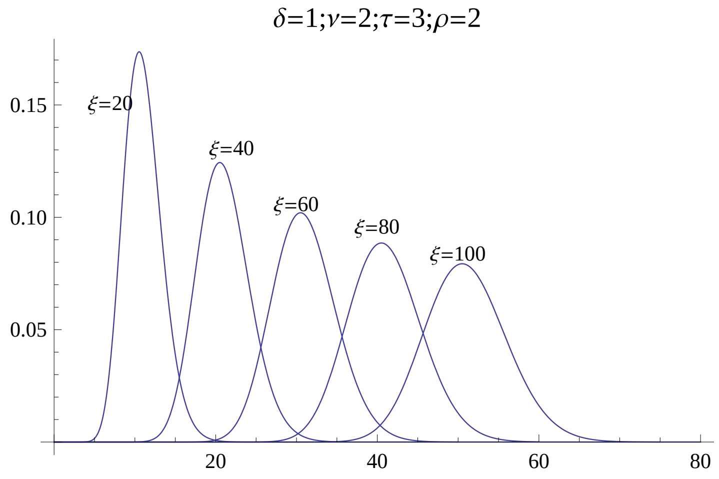

Figure 2.2: *The effect of ξ on the $\mathcal{GEM}$ (Continued)*

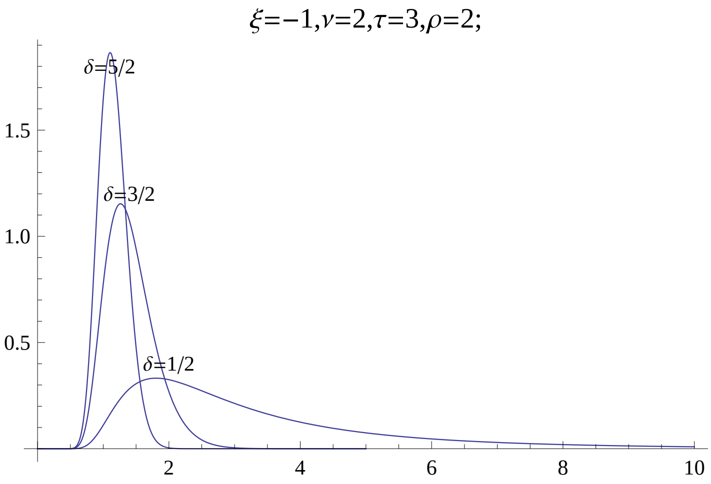

Figure 2.3: *The effect of δ on the $\mathcal{GEM}$*

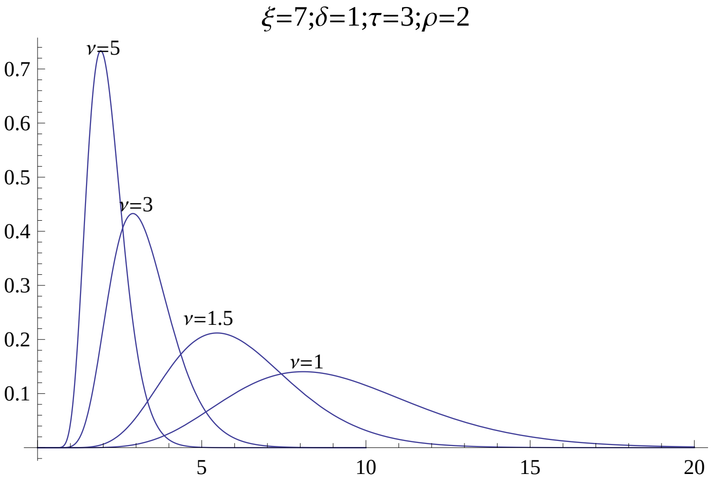

Figure 2.4: *The effect of ν on the $\mathcal{GEM}$*

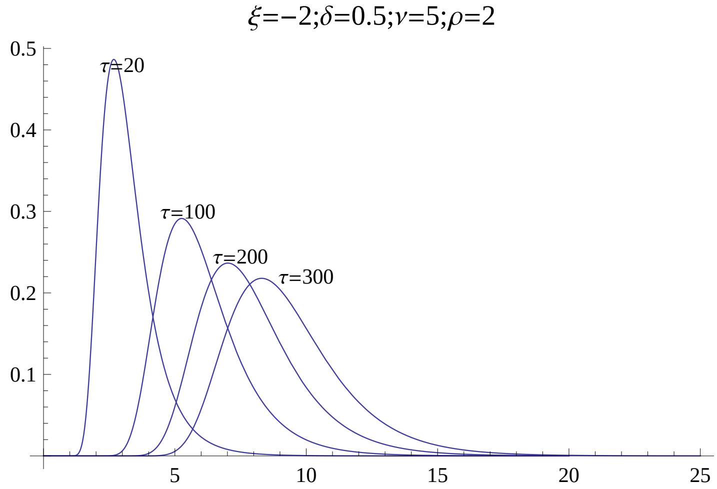

Figure 2.5: *The effect of τ on the $\mathcal{GEM}$*

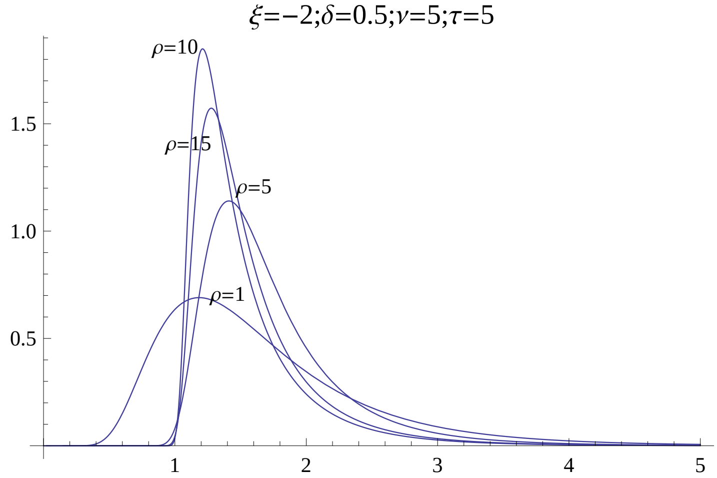

Figure 2.6: *The effect of ρ on the $\mathcal{GEM}$*

2.3 Moments of the Generalized Exponential Model

Consider a random variable X whose density function is given by (2.1). In order to determine its h^{th} moment, one has to evaluate the integral,

$$c\int_0^\infty x^{\xi+\delta+h}\,e^{-\nu x^{\delta}}e^{-\tau x^{-\rho}}\mathrm{d}x\,. \tag{2.2}$$

To this end, we define two random variables such that the density function of their product can be expressed as an integral of the type given in (2.2). By also determining the density function of the product as an inverse Mellin transform, a closed form representation of the integral is then obtained. So, let X_1 and X_2 be independently distributed random variables whose density functions are

$$g_1(x_1) = c_1\, x_1^{\epsilon}\, e^{-\nu\, x_1^{\delta}}\, \mathcal{I}_{\mathbb{R}^+}(x_1)$$

and

$$h_1(x_2) = c_2\, e^{-x_2^{\rho}}\, \mathcal{I}_{\mathbb{R}^+}(x_2)\,,$$

whose $(t-1)^{\text{th}}$ moments are respectively

$$c_1\, \frac{\nu^{-\frac{t+\epsilon}{\delta}}\Gamma\left(\frac{t+\epsilon}{\delta}\right)}{\delta}$$

and

$$c_2\, \frac{\Gamma\left(\frac{t}{\rho}\right)}{\rho}\ ,$$

c_1 and c_2 being normalizing constants. Thus the $(t-1)^{\text{th}}$ moment of $U = X_1X_2$ can be expressed as

$$k(t) = c_1\, c_2\, \frac{\nu^{-\frac{t+\epsilon}{\delta}}\,\Gamma\left(\frac{t+\epsilon}{\delta}\right)\Gamma\left(\frac{t}{\rho}\right)}{\delta\,\rho}\ .$$

The density function of $U = X_1X_2$ obtained by taking the inverse Mellin transform of $k(t)$,that is,

$$\frac{1}{2\pi\, i}\int_C u^{-t}k(t)\,\mathrm{d}t\,,$$

where $i = \sqrt{-1}$ and C is a contour of integration which encompasses the poles of $\Gamma\left(\frac{t}{\rho}\right)$ and $\Gamma\left(\frac{\epsilon}{\delta}+\frac{t}{\delta}\right)$, is

$$\frac{c_1\, c_2\, \nu^{-\frac{\epsilon}{\delta}}}{\delta\,\rho}\, \frac{1}{2\pi\, i}\int_C \left(u\, \nu^{\frac{1}{\delta}}\right)^{-t}\Gamma\left(\frac{t+\epsilon}{\delta}\right)\Gamma\left(\frac{t}{\rho}\right)\, dt \tag{2.3}$$

$$= \frac{c_1\, c_2\, \nu^{-\frac{\epsilon}{\delta}}}{\delta\,\rho}\mathcal{H}^{2,0}_{0,2}\left(u\, \nu^{\frac{1}{\delta}}\,\middle|\, \begin{matrix} \\ \{(0,1/\rho),(\frac{\epsilon}{\delta},\frac{1}{\delta})\}\end{matrix}\right), \tag{2.4}$$

where the H-function is as defined in the Introduction.

When δ and ρ are rational numbers such that $\delta = a/d$ and $\rho = w/r$, where a, d, w, r are positive integers, one can express the integral in (2.3) as a Meijer's G-function by letting $z = t/(a\,w)$ and making use of the Gauss-Legendre multiplication formula:

$$\Gamma(r+q\,s)=(2\pi)^{\frac{1-q}{2}}q^{r+q\,s-\frac{1}{2}}\prod_{k=0}^{q-1}\Gamma\Big(\frac{k+r}{q}+s\Big). \tag{2.5}$$

The density function of U is then

$$\begin{aligned}&c_1\,c_2\,d\,r\,\nu^{-\frac{d\epsilon}{a}}(2\pi)^{-\frac{r\,a}{2}-\frac{d\,w}{2}+1}(d\,w)^{\frac{d\epsilon}{a}-\frac{1}{2}}(r\,a)^{-1/2}\\ &\times\int_{C'}\frac{\left(\frac{u^{a\,w}\nu^{wd}}{(r\,a)^{r\,a}(d\,w)^{d\,w}}\right)^{-z}\prod_{k=0}^{d\,w-1}\Gamma\left(z+\frac{k+\frac{d\epsilon}{a}}{d\,w}\right)\prod_{k=0}^{r\,a-1}\Gamma\left(\frac{k}{r\,a}+z\right)}{2\pi\,i}\,\mathrm{d}z\,,\end{aligned} \tag{2.6}$$

that is,

$$\begin{aligned}&c_1\,c_2\,d\,r\,\nu^{-\frac{d\epsilon}{a}}(2\pi)^{-\frac{r\,a}{2}-\frac{d\,w}{2}+1}(d\,w)^{\frac{d\epsilon}{a}-\frac{1}{2}}(r\,a)^{-1/2}\\ &\quad\times\mathcal{G}^{d\,w+r\,a,0}_{0,d\,w+r\,a}\left(\frac{u^{a\,w}\nu^{w\,d}}{(r\,a)^{r\,a}(d\,w)^{d\,w}}\middle|\ \frac{k+\frac{d\epsilon}{a}}{d\,w},k=0,...,d\,w-1;\frac{k}{r\,a},k=0,...,r\,a-1\ \right)\end{aligned} \tag{2.7}$$

Now, considering the transformation $U=X_1X_2$ and $W=X_1$, it is seen that the density function of U is also given by

$$\begin{aligned}r(u)&=\int_0^\infty\frac{1}{w}\,g_1(w)h_1(u/w)\,\mathrm{d}w\\ &=c_1\,c_2\int_0^\infty\frac{1}{w}\,\nu^\epsilon\,e^{-\nu\,w^\delta}\,e^{-(u/w)^\rho}\,\mathrm{d}w\,,\end{aligned} \tag{2.8}$$

and letting $\epsilon=\xi+\delta+h+1$ and $u=\tau^{1/\rho}$, the integral in (2.8) is seen to coincide with that appearing in Equation (2.2). Thus the h^{th}moment of X is

$$m_X(h)=\frac{c\,\nu^{-\frac{\xi+\delta+h+1}{\delta}}}{\delta\,\rho}\mathcal{H}^{2,0}_{0,2}\left(\tau^{1/\rho}\nu^{1/\delta}\middle|\ \{(0,1/\rho),(\tfrac{\xi+\delta+h+1}{\delta},\tfrac{1}{\delta})\}\ \right) \tag{2.9}$$

or, in light of Equation (2.7),

$$\begin{aligned}&m^{(R)}_X(h)=c\,d\,r\,\nu^{-\frac{d(\xi+h+1)}{a}-1}(2\pi)^{-\frac{r\,a}{2}-\frac{d\,w}{2}+1}(d\,w)^{\frac{d(\xi+h+1)}{a}+\frac{1}{2}}(r\,a)^{-1/2}\\ &\times\mathcal{G}^{d\,w+r\,a,0}_{0,d\,w+r\,a}\left(\frac{\tau^{r\,a}\nu^{wd}}{(r\,a)^{r\,a}(d\,w)^{d\,w}}\middle|\frac{k+\frac{d(\xi+h+1)}{a}+1}{d\,w},k=0,...,d\,w-1;\frac{k}{r\,a},k=0,...,r\,a-1\right)\end{aligned} \tag{2.10}$$

where ρ and δ are rational numbers such that

$$\delta=a/d$$

and

$$\rho = w/r.$$

Now assuming that $\delta = \rho$ and letting $w = t/\delta$,$\epsilon = \xi + \delta + h + 1$ and $u = \tau^{1/\rho}$ in the integral in (2.3), one obtains the h^{th} moment of X as

$$m_X^{(E)}(h) = c\,\frac{\nu^{-\frac{h+\xi+1}{\delta}-1}}{\delta}\mathcal{G}^{2,0}_{0,2}\left(\nu\,\tau\,\middle|\, 0, \frac{\xi+h+1}{\delta}+1\right),$$

which can also be expressed in terms of a Bessel function of the second kind as

$$\frac{2\,\nu^{-\frac{h+\xi+\delta+1}{\delta}}\left(\nu^{\frac{1}{\delta}}\tau^{\frac{1}{\delta}}\right)^{\frac{h+\xi+\delta+1}{2\,\delta}} K_{\frac{h+\xi+\delta+1}{\delta}}\left(2\sqrt{\nu^{\frac{1}{\delta}}\tau^{\frac{1}{\delta}}}\right)}{\delta}.$$

where $K_\lambda(\cdot)$ is a modified Bessel function of the second type that has the following integral representation:

$$K_\lambda(\eta) = \frac{1}{2}\int_0^\infty x^{\lambda-1}e^{\frac{1}{2}\,\eta\,(x+x^{-1})}\mathrm{d}x.$$

Incidentally, $K_\lambda(\cdot)$ is a built-in function in the symbolic computing package *Mathematica*. As explained in Abramowitz and Stegun (1972), the modified Bessel functions of the first and second types, namely $I_\lambda(w)$ and $K_\lambda(w)$, are the two linearly independent solutions of the differential equation $w^2\,\frac{\mathrm{d}^2y}{\mathrm{d}w^2} + w\,\frac{\mathrm{d}y}{\mathrm{d}w} - (w^2 + \lambda^2)\,y = 0$.

Since the null moments are equal to one, the normalizing constant c is seen to be the inverse of the moment expressions $m_X(h)$, $m_X^{(R)}(h)$, and $m_X^{(E)}(h)$, wherein h is set equal to zero and c is omitted. When there are no restrictions on δ and ρ, the normalizing constant in (2.1) is

$$c = \frac{\delta\,\rho\,\nu^{\frac{\xi+1}{\delta}+1}}{\mathcal{H}^{2,0}_{0,2}\left(\tau^{\frac{1}{\rho}}\nu^{\frac{1}{\delta}}\,\middle|\, \{(0, 1/\rho), (\frac{\xi+1}{\delta}+1, \frac{1}{\delta})\}\right)}, \tag{2.11}$$

when δ and ρ are rational numbers, the normalizing constant will be

$$c^{(R)} = \frac{\dfrac{\nu^{\frac{d(\xi+1)}{a}+1}(2\pi)^{\frac{r\,a}{2}+\frac{d\,w}{2}-1}(d\,w)^{-\frac{d(\xi+1)}{a}-\frac{1}{2}}(r\,a)^{1/2}}{d\,r}}{\mathcal{G}^{d\,w+r\,a,0}_{0,d\,w+r\,a}\left(\frac{\tau^{r\,a}\nu^{wd}}{(r\,a)^{r\,a}(d\,w)^{d\,w}}\,\middle|\, \frac{k+\frac{d(\xi+1)}{a}+1}{d\,w}, k=0,\ldots,d\,w-1; \frac{k}{r\,a}, k=0,\ldots,r\,a-1\right)}, \tag{2.12}$$

and when $\delta = \rho$, one has

$$\begin{aligned} c^{(E)} &= \frac{\delta\,\nu^{\frac{\xi+1}{\delta}+1}}{\mathcal{G}^{2,0}_{0,2}\left(\nu\,\tau\,\middle|\, 0, \frac{\xi+1}{\delta}+1\right)} \\ &= \frac{\delta\,\nu^{\frac{\xi+\delta+1}{\delta}}}{2\,(\nu\tau)^{\frac{\xi+\delta+1}{2\,\delta}}K_{\frac{\xi+\delta+1}{\delta}}\left(2\sqrt{\nu\tau}\right)}. \end{aligned} \tag{2.13}$$

2.4 Some Statistical Functions

2.4.1 Generalized Exponential Model ($\mathcal{GEM}$)

Let X be a random variable whose p.d.f is specified by (2.1), then some related statistical functions of the $\mathcal{GEM}$ are obtained.

(i) The expectation of X is

$$\mathcal{E}(X) = \frac{\left(\frac{dw}{\nu}\right)^{d/a} \mathcal{G}^{dw+ra,0}_{0,dw+ra}\left(\frac{\tau^{ra}\nu^{wd}}{(ra)^{ra}(dw)^{dw}} \Big|_{\frac{k+\frac{d\left(\frac{a}{d}+\xi+2\right)}{a}}{dw},k=0,\ldots,dw-1,\frac{k}{ra},k=0,\ldots,ra-1} \right)}{\mathcal{G}^{dw+ra,0}_{0,dw+ra}\left(\frac{\tau^{ra}\nu^{dw}}{(ra)^{ra}(dw)^{dw}} \Big|_{\frac{k+\frac{d\left(\frac{a}{d}+\xi+1\right)}{a}}{dw},k=0,\ldots,dw-1,\frac{k}{ra},k=0,\ldots,ra-1} \right)} \tag{2.14}$$

(ii) The variance of X is

$$\begin{aligned} \mathcal{V}ar(X) = {} & \frac{\left(\frac{dw}{\nu}\right)^{\frac{2d}{a}} \mathcal{G}^{dw+ra,0}_{0,dw+ra}\left(\frac{\tau^{ra}\nu^{dw}}{(ra)^{ra}(dw)^{dw}} \Big|_{\frac{k+\frac{d\left(\frac{a}{d}+\xi+3\right)}{a}}{dw},k=0,\ldots,dw-1,\frac{k}{ra},k=0,\ldots,ra-1} \right)}{\mathcal{G}^{dw+ra,0}_{0,dw+ra}\left(\frac{\tau^{ra}\nu^{dw}}{(ra)^{ra}(dw)^{dw}} \Big|_{\frac{k+\frac{d\left(\frac{a}{d}+\xi+1\right)}{a}}{dw},k=0,\ldots,dw-1,\frac{k}{ra},k=0,\ldots,ra-1} \right)} \\ & - \frac{\left(\frac{dw}{\nu}\right)^{\frac{2d}{a}} \mathcal{G}^{dw+ra,0}_{0,dw+ra}\left(\frac{\tau^{ra}\nu^{dw}}{(ra)^{ra}(dw)^{dw}} \Big|_{\frac{k+\frac{d\left(\frac{a}{d}+\xi+2\right)}{a}}{dw},k=0,\ldots,dw-1,\frac{k}{ra},k=0,\ldots,ra-1} \right)^2}{\mathcal{G}^{dw+ra,0}_{0,dw+ra}\left(\frac{\tau^{ra}\nu^{dw}}{(ra)^{ra}(dw)^{dw}} \Big|_{\frac{k+\frac{d\left(\frac{a}{d}+\xi+1\right)}{a}}{dw},k=0,\ldots,dw-1,\frac{k}{ra},k=0,\ldots,ra-1} \right)^2} \end{aligned} \tag{2.15}$$

(iii) The skewness of X is

$$
\begin{aligned}
s(X) \;=\;& \Bigg\{(dw)^{\frac{2d}{a}}\nu^{-\frac{3d}{a}} \\
&\times\Bigg\{3\,\nu^{\frac{d}{a}}\mathcal{G}_{0,dw+ra}^{dw+ra,0}\Bigg(\frac{\tau^{ra}\nu^{dw}}{(ra)^{ra}(dw)^{dw}}\,\Bigg|\,{}_{\frac{k+\frac{d\left(\frac{a}{d}+\xi+2\right)}{a}}{dw},k=0,\ldots,dw-1,\frac{k}{ra},k=0,\ldots,ra-1}\Bigg)^{2} \\
&+\Bigg\{\mathcal{G}_{0,dw+ra}^{dw+ra,0}\Bigg(\frac{\tau^{ra}\nu^{dw}}{(ra)^{ra}(dw)^{dw}}\,\Bigg|\,{}_{\frac{k+\frac{d\left(\frac{a}{d}+\xi+4\right)}{a}}{dw},k=0,\ldots,dw-1,\frac{k}{ra},k=0,\ldots,ra-1}\Bigg) \\
&\times\mathcal{G}_{0,dw+ra}^{dw+ra,0}\Bigg(\frac{\tau^{ra}\nu^{dw}}{(ra)^{ra}(dw)^{dw}}\,\Bigg|\,{}_{\frac{k+\frac{d\left(\frac{a}{d}+\xi+1\right)}{a}}{dw},k=0,\ldots,dw-1,\frac{k}{ra},k=0,\ldots,ra-1}\Bigg) \\
&-3\,\mathcal{G}_{0,dw+ra}^{dw+ra,0}\Bigg(\frac{\tau^{ra}\nu^{dw}}{(ra)^{ra}(dw)^{dw}}\,\Bigg|\,{}_{\frac{k+\frac{d\left(\frac{a}{d}+\xi+2\right)}{a}}{dw},k=0,\ldots,dw-1,\frac{k}{ra},k=0,\ldots,ra-1}\Bigg) \\
&\times(dw)^{\frac{d}{a}}\mathcal{G}_{0,dw+ra}^{dw+ra,0}\Bigg(\frac{\tau^{ra}\nu^{dw}}{(ra)^{ra}(dw)^{dw}}\,\Bigg|\,{}_{\frac{k+\frac{d\left(\frac{a}{d}+\xi+3\right)}{a}}{dw},k=0,\ldots,dw-1,\frac{k}{ra},k=0,\ldots,ra-1}\Bigg)\Bigg)\Bigg\}\Bigg\}\Bigg\} \\
&\div\Bigg\{\mathcal{G}_{0,dw+ra}^{dw+ra,0}\Bigg(\frac{\tau^{ra}\nu^{dw}}{(ra)^{ra}(dw)^{dw}}\,\Bigg|\,{}_{\frac{k+\frac{d\left(\frac{a}{d}+\xi+1\right)}{a}}{dw},k=0,\ldots,dw-1,\frac{k}{ra},k=0,\ldots,ra-1}\Bigg)^{2} \\
&\times\Bigg\{\Bigg\{\Big(\frac{dw}{\nu}\Big)^{\frac{2d}{a}}\Bigg\{\mathcal{G}_{0,dw+ra}^{dw+ra,0}\Bigg(\frac{\tau^{ra}\nu^{dw}}{(ra)^{ra}(dw)^{dw}}\,\Bigg|\,{}_{\frac{k+\frac{d\left(\frac{a}{d}+\xi+3\right)}{a}}{dw},k=0,\ldots,dw-1,\frac{k}{ra},k=0,\ldots,ra-1}\Bigg) \\
&\times\mathcal{G}_{0,dw+ra}^{dw+ra,0}\Bigg(\frac{\tau^{ra}\nu^{dw}}{(ra)^{ra}(dw)^{dw}}\,\Bigg|\,{}_{\frac{k+\frac{d\left(\frac{a}{d}+\xi+1\right)}{a}}{dw},k=0,\ldots,dw-1,\frac{k}{ra},k=0,\ldots,ra-1}\Bigg) \\
&-\mathcal{G}_{0,dw+ra}^{dw+ra,0}\Bigg(\frac{\tau^{ra}\nu^{dw}}{(ra)^{ra}(dw)^{dw}}\,\Bigg|\,{}_{\frac{k+\frac{d\left(\frac{a}{d}+\xi+2\right)}{a}}{dw},k=0,\ldots,dw-1,\frac{k}{ra},k=0,\ldots,ra-1}\Bigg)^{2}\Bigg\}\Bigg\} \\
&\div\mathcal{G}_{0,dw+ra}^{dw+ra,0}\Bigg(\frac{\tau^{ra}\nu^{dw}}{(ra)^{ra}(dw)^{dw}}\,\Bigg|\,{}_{\frac{k+\frac{d\left(\frac{a}{d}+\xi+1\right)}{a}}{dw},k=0,\ldots,dw-1,\frac{k}{ra},k=0,\ldots,ra-1}\Bigg)^{2}\Bigg\}^{3/2}\Bigg\}
\end{aligned}
\tag{2.16}
$$

(iv) The kurtosis of X is

$$
\begin{aligned}
k(X) \;=\;& \Bigg\{\Big(\frac{dw}{\nu}\Big)^{\frac{4d}{a}} \\
&\times\Bigg\{-4\,\mathcal{G}_{0,dw+ra}^{dw+ra,0}\Bigg(\frac{\tau^{ra}\nu^{dw}}{(ra)^{ra}(dw)^{dw}}\,\Bigg|\,{}_{\frac{k+\frac{d\left(\frac{a}{d}+\xi+2\right)}{a}}{dw},k=0,\ldots,dw-1,\frac{k}{ra},k=0,\ldots,ra-1}\Bigg)^{4} \\
&\times 8\,\mathcal{G}_{0,dw+ra}^{dw+ra,0}\Bigg(\frac{\tau^{ra}\nu^{dw}}{(ra)^{ra}(dw)^{dw}}\,\Bigg|\,{}_{\frac{k+\frac{d\left(\frac{a}{d}+\xi+3\right)}{a}}{dw},k=0,\ldots,dw-1,\frac{k}{ra},k=0,\ldots,ra-1}\Bigg) \\
&\times\mathcal{G}_{0,dw+ra}^{dw+ra,0}\Bigg(\frac{\tau^{ra}\nu^{dw}}{(ra)^{ra}(dw)^{dw}}\,\Bigg|\,{}_{\frac{k+\frac{d\left(\frac{a}{d}+\xi+1\right)}{a}}{dw},k=0,\ldots,dw-1,\frac{k}{ra},k=0,\ldots,ra-1}\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&\times \mathcal{G}^{dw+ra,0}_{0,dw+ra}\left(\frac{\tau^{ra}\nu^{dw}}{(ra)^{ra}(dw)^{dw}}\,\middle|\,_{\frac{k+\frac{d\left(\frac{a}{d}+\xi+2\right)}{a}}{dw},k=0,\dots,dw-1,\frac{k}{ra},k=0,\dots,ra-1}\right)^{2}\\
&+ \mathcal{G}^{dw+ra,0}_{0,dw+ra}\left(\frac{\tau^{ra}\nu^{dw}}{(ra)^{ra}(dw)^{dw}}\,\middle|\,_{\frac{k+\frac{d\left(\frac{a}{d}+\xi+5\right)}{a}}{dw},k=0,\dots,dw-1,\frac{k}{ra},k=0,\dots,ra-1}\right)\\
&\times \mathcal{G}^{dw+ra,0}_{0,dw+ra}\left(\frac{\tau^{ra}\nu^{dw}}{(ra)^{ra}(dw)^{dw}}\,\middle|\,_{\frac{k+\frac{d\left(\frac{a}{d}+\xi+1\right)}{a}}{dw},k=0,\dots,dw-1,\frac{k}{ra},k=0,\dots,ra-1}\right)^{3}\\
&- \Bigg\{\mathcal{G}^{dw+ra,0}_{0,dw+ra}\left(\frac{\tau^{ra}\nu^{dw}}{(ra)^{ra}(dw)^{dw}}\,\middle|\,_{\frac{k+\frac{d\left(\frac{a}{d}+\xi+3\right)}{a}}{dw},k=0,\dots,dw-1,\frac{k}{ra},k=0,\dots,ra-1}\right)^{2}\\
&+ 4\,\mathcal{G}^{dw+ra,0}_{0,dw+ra}\left(\frac{\tau^{ra}\nu^{dw}}{(ra)^{ra}(dw)^{dw}}\,\middle|\,_{\frac{k+\frac{d\left(\frac{a}{d}+\xi+2\right)}{a}}{dw},k=0,\dots,dw-1,\frac{k}{ra},k=0,\dots,ra-1}\right)\\
&\times \mathcal{G}^{dw+ra,0}_{0,dw+ra}\left(\frac{\tau^{ra}\nu^{dw}}{(ra)^{ra}(dw)^{dw}}\,\middle|\,_{\frac{k+\frac{d\left(\frac{a}{d}+\xi+4\right)}{a}}{dw},k=0,\dots,dw-1,\frac{k}{ra},k=0,\dots,ra-1}\right)\Bigg\}\\
&\times \mathcal{G}^{dw+ra,0}_{0,dw+ra}\left(\frac{\tau^{ra}\nu^{dw}}{(ra)^{ra}(dw)^{dw}}\,\middle|\,_{\frac{k+\frac{d\left(\frac{a}{d}+\xi+1\right)}{a}}{dw},k=0,\dots,dw-1,\frac{k}{ra},k=0,\dots,ra-1}\right)^{2}\\
&+ \mathcal{G}^{dw+ra,0}_{0,dw+ra}\left(\frac{\tau^{ra}\nu^{dw}}{(ra)^{ra}(dw)^{dw}}\,\middle|\,_{\frac{k+\frac{d\left(\frac{a}{d}+\xi+5\right)}{a}}{dw},k=0,\dots,dw-1,\frac{k}{ra},k=0,\dots,ra-1}\right)\\
&\times \mathcal{G}^{dw+ra,0}_{0,dw+ra}\left(\frac{\tau^{ra}\nu^{dw}}{(ra)^{ra}(dw)^{dw}}\,\middle|\,_{\frac{k+\frac{d\left(\frac{a}{d}+\xi+1\right)}{a}}{dw},k=0,\dots,dw-1,\frac{k}{ra},k=0,\dots,ra-1}\right)^{3}\Bigg\}\Bigg\}\\
&\times \mathcal{G}^{dw+ra,0}_{0,dw+ra}\left(\frac{\tau^{ra}\nu^{dw}}{(ra)^{ra}(dw)^{dw}}\,\middle|\,_{\frac{k+\frac{d\left(\frac{a}{d}+\xi+1\right)}{a}}{dw},k=0,\dots,dw-1,\frac{k}{ra},k=0,\dots,ra-1}\right)^{4}
\end{aligned}
$$

(v) The mode of $f(x)$ satisfies the following equation:

$$x = \rho\,\tau\,x^{-\rho} - \delta\,\nu\,x^{\delta} + \delta + \xi.$$

It is obtained by equating the derivative of the probability density function of X given in (2.1) to zero.

2.4.2 Reparameterized Generalized Gamma ($\mathcal{RGG}$) Model

The $\mathcal{RGG}$ model is a reduced form of the $\mathcal{GEM}$ model, which is obtained by omitting $e^{-\tau x^{-\rho}}$ (or equivalently by letting $\tau = 0$) in the density function (2.1), which yields

$$f(x) = \frac{\delta\nu^{\frac{\delta+\xi}{\delta}}}{\Gamma(\frac{\delta+\xi}{\delta})}\,x^{\xi+\delta-1}e^{-\nu x^{\delta}}\,\mathcal{I}_{\mathbb{R}^{+}}(x), \tag{2.17}$$

where $\delta+\xi > 0$. This density function is in fact a Reparameterized Generalized Gamma ($\mathcal{RGG}$) density function, which is obtained by letting $\beta = \delta,\ \theta = \nu^{-1/\beta}$ and $k = \frac{\delta+\xi+1}{\delta}$ in the generalized gamma density,

$$g_1(x) = \frac{\beta}{\theta^{k\beta}\,\Gamma(k)}\,x^{k\beta-1}e^{-(\frac{x}{\theta})^{\beta}}\,\mathcal{I}_{\mathfrak{R}^{+}}(x)\,. \tag{2.18}$$

For specific distributional results in connection with the generalized gamma distribution, the reader is referred to Johnson *et al.* (1994).

Let X be an $\mathcal{RGG}$ random variable. Then,
(i) The k^{th} moment of X is

$$\mathcal{E}(X^k) = \frac{\nu^{-\frac{k}{\delta}}\Gamma(\frac{k+\delta+\xi}{\delta})}{\Gamma(\frac{\delta+\xi}{\delta})}; \tag{2.19}$$

(ii) The expectation of X is

$$\mathcal{E}(X) = \frac{\nu^{-1/\delta}\Gamma(\frac{\delta+\xi+1}{\delta})}{\Gamma(\frac{\delta+\xi}{\delta})}; \tag{2.20}$$

(iii) The variance of X is

$$\mathrm{Var}(X) = \frac{\nu^{-2/\delta}\Gamma(\frac{\delta+\xi+2}{\delta})}{\Gamma(\frac{\delta+\xi}{\delta})} - \frac{\nu^{-2/\delta}\Gamma(\frac{\delta+\xi+1}{\delta})^2}{\Gamma(\frac{\delta+\xi}{\delta})^2}; \tag{2.21}$$

2.4.3 Reduced Extended Inverse Gaussian ($\mathcal{REIG}$) Model

The $\mathcal{REIG}$ model is a reduced form of $\mathcal{GEM}$ model which is obtained by omitting $e^{-\nu x^{\delta}}$ (or equivalently by letting $\nu = 0$) in the density function (2.1). It is also a reduced form of the extended inverse Gaussian distribution which is defined in Section 3.1, hence the name. Thus, the density function of $\mathcal{REIG}$ model is given by

$$f(x) = \frac{\rho\,\tau^{-\frac{\xi+\rho+1}{\rho}}}{\Gamma(-\frac{\xi+\rho+1}{\rho})}\, x^{\xi+\rho}\, e^{-\tau x^{-\rho}}\, \mathcal{I}_{\mathbb{R}^{+}}(x),\ \xi+\rho+1<0, \tag{2.26}$$

Let X be an $\mathcal{REIG}$ random variable. Then,
(i) The k^{th} moment of X is

$$\frac{\tau^{k/\rho}\,\Gamma(-\frac{k+\xi+\rho+1}{\rho})}{\Gamma(-\frac{\xi+\rho+1}{\rho})}; \tag{2.27}$$

(ii) The expectation of X is

$$\mathcal{E}(X) = \frac{\tau^{\frac{1}{\rho}}\,\Gamma(-\frac{\xi+\rho+2}{\rho})}{\Gamma(-\frac{\xi+\rho+1}{\rho})}; \tag{2.28}$$

(iii) The variance of X is

$$\mathrm{Var}(X) = \frac{\tau^{2/\rho}\,\Gamma(-\frac{\xi+\rho+3}{\rho})}{\Gamma(-\frac{\xi+\rho+1}{\rho})} - \frac{\tau^{2/\rho}\Gamma(-\frac{\xi+\rho+2}{\rho})^2}{\Gamma(-\frac{\xi+\rho+1}{\rho})^2}; \tag{2.29}$$

(iv) The skewness of X is

$$s(X) = \frac{2\Gamma\left(-\frac{\xi+\rho+2}{\rho}\right)^3 - 3\Gamma\left(-\frac{\xi+\rho+1}{\rho}\right)\Gamma\left(-\frac{\xi+\rho+3}{\rho}\right)\Gamma\left(-\frac{\xi+\rho+2}{\rho}\right) + \Gamma\left(-\frac{\xi+\rho+1}{\rho}\right)^2\Gamma\left(-\frac{\xi+\rho+4}{\rho}\right)}{\left(\Gamma(-\frac{\xi+\rho+1}{\rho})\Gamma(-\frac{\xi+\rho+3}{\rho}) - \Gamma(-\frac{\xi+\rho+2}{\rho})^2\right)^{3/2}}; \tag{2.30}$$

(v) The kurtosis of X is

$$\begin{aligned} k(X) \;=\; & \frac{6\tau^{4/\rho}\Gamma\left(-\frac{\xi+\rho+2}{\rho}\right)^2\Gamma\left(-\frac{\xi+\rho+3}{\rho}\right)}{\Gamma\left(-\frac{\xi+\rho+1}{\rho}\right)^3} - \frac{4\tau^{4/\rho}\Gamma\left(-\frac{\xi+\rho+2}{\rho}\right)\Gamma\left(-\frac{\xi+\rho+4}{\rho}\right)}{\Gamma\left(-\frac{\xi+\rho+1}{\rho}\right)^2} \\ & -\left(\frac{\tau^{2/\rho}\Gamma\left(-\frac{\xi+\rho+3}{\rho}\right)}{\Gamma\left(-\frac{\xi+\rho+1}{\rho}\right)} - \frac{\tau^{2/\rho}\Gamma\left(-\frac{\xi+\rho+2}{\rho}\right)^2}{\Gamma\left(-\frac{\xi+\rho+1}{\rho}\right)^2}\right)^2 \\ & -\frac{3\tau^{4/\rho}\Gamma\left(-\frac{\xi+\rho+2}{\rho}\right)^4}{\Gamma\left(-\frac{\xi+\rho+1}{\rho}\right)^4} + \frac{\tau^{4/\rho}\Gamma\left(-\frac{\xi+\rho+5}{\rho}\right)}{\Gamma\left(-\frac{\xi+\rho+1}{\rho}\right)}; \end{aligned} \tag{2.31}$$

(vi) The mode of $\mathcal{REIG}$ model is

$$x = e^{-\frac{i\pi}{\rho}}\,\rho^{\frac{1}{\rho}}\,(\xi+\rho)^{-1/\rho}\,\tau^{\frac{1}{\rho}}\ ;$$

(vii) Its cumulative distribution function (CDF) is

$$F_R(y) = \frac{\Gamma\left(-\frac{\xi+\rho+1}{\rho}, y^{-\rho}\tau\right)}{\Gamma\left(-\frac{\xi+\rho+1}{\rho}\right)}\ ;$$

where $\Gamma(\alpha,\beta)$ denotes the incomplete gamma function.

(viii) Its Moment Generating function (MGF) is

$$\begin{aligned} M_X(s) \;=\; & \frac{(2\pi)^{1-\frac{(w+r)}{2}}(-s)^{-(\xi+\frac{w}{r}+1)}r^{\frac{1}{2}}w^{\xi+\frac{w}{r}+\frac{3}{2}}}{\tau^{\frac{\xi+\frac{w}{r}+1}{\rho}}\Gamma\left(-\frac{\xi+\frac{w}{r}+1}{\rho}\right)} \\ & \times \mathcal{G}^{d\,w+r\,v,0}_{0,d\,w+r\,v}\left(\left(\frac{\tau}{r}\right)^r\left(\frac{-s}{w}\right)^w \Bigg|_{\frac{k+\frac{w}{r}+\xi+1}{w},\,k=0,\dots,w-1;\,\frac{k}{r},\,k=0,\dots,r}\right). \end{aligned} \tag{2.32}$$

We now derive the moment generating function of the $\mathcal{REIG}$. In order to determine the MGF of $\mathcal{REIG}$ model, one has to evaluate the integral,

$$\frac{\rho\,\tau^{-\frac{\xi+\rho+1}{\rho}}}{\Gamma\left(-\frac{\xi+\rho+1}{\rho}\right)}\int_0^\infty x^{\xi+\rho}e^{sx-\tau x^{-\rho}}\,\mathrm{d}x. \tag{2.33}$$

Let X_1 and X_2 be independently distributed random variables whose density functions are

$$g_1(x_1) = c_1\,x_1^{\epsilon}e^{sx_1}\,\mathcal{I}_{(0,\infty)}(x_1)$$

and

$$g_2(x_2) = c_2\,e^{-x_2^{\rho}}\,\mathcal{I}_{(0,\infty)}(x_2)\,.$$

Let $U = X_1X_2$ and $Y = X_1$; then the density function of U is given by

$$\begin{aligned} r(u) &= \int_0^\infty \frac{1}{y}\,g_1(y)h_1(u/y)\,\mathrm{d}y \\ &= c_1\,c_2\int_0^\infty \frac{1}{y}y^{\epsilon}\,e^{-sy}\,e^{-(u/y)^{\rho}}\,\mathrm{d}y. \end{aligned} \tag{2.34}$$

The integral in (2.34) is seen to coincide with that appearing in Equation (2.33) when $u = \tau^{\frac{1}{\rho}}$ and $\epsilon = \xi+\rho+1$. This indicates that the integral in Equation (2.34) is equivalent to the integral in Equation (2.33) and $c_1c_2 = \frac{\rho\,\tau^{-\frac{\xi+\rho+1}{\rho}}}{\Gamma\left(-\frac{\xi+\rho+1}{\rho}\right)}$.

The $(t-1)^{\text{th}}$ moments of $U = X_1 X_2$ is

$$\begin{aligned} E(U) &= E(X_1)\,E(X_2) \\ &= c_1\,c_2\,\frac{(-s)^{-t-\epsilon}}{\rho}\Gamma\left(\frac{t}{\rho}\right)\Gamma(t+\epsilon) \equiv k(t),\ s<0 \end{aligned} \tag{2.35}$$

The density function of $U = X_1 X_2$ can be obtained by taking the inverse Mellin transform of $k(t)$, that is,

$$\begin{aligned} r(u) &= \frac{c_1\,c_2}{2\,\rho\pi\,i}(-s)^{-(\xi+\rho+1)}\int_c(-u\,s)^{-t}\Gamma\left(\frac{t}{\rho}\right)\Gamma(t+\xi+\rho+1)\,\mathrm{d}t \\ &= \frac{c_1\,c_2}{2\,\rho}(-s)^{-(\xi+\rho+1)}\mathcal{H}^{2,0}_{0,2}\left(-u\,s\,\middle|_{(0,\frac{1}{\rho}),\,(\xi+\rho+1)}\right) \end{aligned} \tag{2.36}$$

where the H-function is as defined in Section 1.1.

Using Gauss-Legendre multiplication formula presented by Equation (2.5) and letting $\rho = w/r$ and $z = t/w$ the density function of U is then

$$\begin{aligned} r(u) &= \frac{c_1 c_2}{2\pi i}(2\pi)^{\frac{1-r}{2}+\frac{1-w}{2}}(-s)^{-(\xi+\frac{w}{r}+1)}\,r^{\frac{1}{2}}w^{\xi+\frac{w}{r}+\frac{1}{2}} \\ &\quad\times\int_c(r^{-r}w^{-w}s^{w}(-u)^{w})^{-z}\prod_{k=0}^{w-1}\Gamma\left(\frac{k+\xi+\frac{w}{r}+1}{w}+z\right)\prod_{k=0}^{r-1}\Gamma\left(\frac{k}{r}+z\right)\mathrm{d}z \\ &= \frac{(2\pi)^{1-\frac{(w+r)}{2}}(-s)^{-(\xi+\frac{w}{r}+1)}r^{\frac{-1}{2}}w^{\xi+\frac{w}{r}+\frac{3}{2}}}{\tau^{\frac{\xi+\frac{w}{r}+1}{\rho}}\Gamma\left(-\frac{\xi+\frac{w}{r}+1}{\rho}\right)} \\ &\quad\times\mathcal{G}^{w+r,0}_{0,w+r}\left(\left(\frac{\tau}{r}\right)^{r}\left(-\frac{s}{w}\right)^{w}\,\middle|_{\frac{k+\frac{w}{r}+\xi+1}{w},\,k=0,\ldots,w-1;\,\frac{k}{r},\,k=0,\ldots,r}\right). \end{aligned} \tag{2.37}$$

2.4.4 The Proxy Distribution

Parameter estimation is essential in order to fit a probability distribution to a data set. The proposed probability distribution, as given in Equation (2.1), has five parameters. It is challenging to estimate these parameters at once since the normalizing constant, given in Equation (2.11), is expressed in terms of an H-function, which is difficult to evaluate; besides, it is not available in *Mathematica*. Equation (2.12) gives the normalizing constant expressed in terms of the G-function which is available in *Mathematica*. However, in the latter case, the proposed distribution will have seven parameters, which means that estimating the parameters will be time consuming taking in consideration that the G-function takes time to be evaluated. Therefore, we propose a proxy distribution that approximates the density function given in Equation (2.1), which is obtained by replacing the exponential term $e^{-\nu x^{\delta}}$ by a truncated Taylor series expansion around a point m, assumed to be in the vicinity of the mean or the median of the distribution. For instance, the three term expansion is given by

$$
\begin{aligned}
f_A(x) \;=\;& c_p\, x^{\delta+\xi} e^{-\tau x^{-\rho}} \Big(-\frac{1}{6} \nu^3 e^{-m\nu} \left(x^\delta - m\right)^3 + \frac{1}{2} \nu^2 e^{-m\nu} \left(x^\delta - m\right) \\
& -\nu e^{-m\nu} \left(x^\delta - m\right) + e^{-m\nu} \Big)\, \mathcal{I}_{\mathbb{R}^+}(x), \qquad (2.38)
\end{aligned}
$$

where c_p is the normalizing constant of the proxy distribution.

It has been noted that expansions around the mean converge faster. Of course, in practice one could use the sample mean. The normalizing constant can be found by integrating the resulting function. For the density given in Equation (2.38) the normalizing constant c_p is such that

$$
\begin{aligned}
1/c_p \;=\;& \frac{e^{-m\nu}}{\rho} \Bigg(\frac{1}{6} m^3 \nu^3 \tau^{\frac{\delta+\xi+1}{\rho}} \Gamma\left(-\frac{\delta+\xi+1}{\rho}\right) + \frac{1}{2} m^2 \nu^2 \tau^{\frac{\delta+\xi+1}{\rho}} \Gamma\left(-\frac{\delta+\xi+1}{\rho}\right) \\
& + m\nu \tau^{\frac{\delta+\xi+1}{\rho}} \Gamma\left(-\frac{\delta+\xi+1}{\rho}\right) + \tau^{\frac{\delta+\xi+1}{\rho}} \Gamma\left(-\frac{\delta+\xi+1}{\rho}\right) \\
& - \frac{1}{2} m^2 \nu^3 \tau^{\frac{\delta+\xi+1}{\rho}+\frac{\delta}{\rho}} \Gamma\left(-\frac{2\delta+\xi+1}{\rho}\right) - m\nu^2 \tau^{\frac{\delta+\xi+1}{\rho}+\frac{\delta}{\rho}} \Gamma\left(-\frac{2\delta+\xi+1}{\rho}\right) \\
& - \nu \tau^{\frac{\delta+\xi+1}{\rho}+\frac{\delta}{\rho}} \Gamma\left(-\frac{2\delta+\xi+1}{\rho}\right) + \frac{1}{2} \nu^2 \tau^{\frac{\delta+\xi+1}{\rho}+\frac{2\delta}{\rho}} \Gamma\left(-\frac{3\delta+\xi+1}{\rho}\right) \\
& + \frac{1}{2} m\nu^3 \tau^{\frac{\delta+\xi+1}{\rho}+\frac{2\delta}{\rho}} \Gamma\left(-\frac{3\delta+\xi+1}{\rho}\right) - \frac{1}{6} \nu^3 \tau^{\frac{\delta+\xi+1}{\rho}+\frac{3\delta}{\rho}} \Gamma\left(-\frac{4\delta+\xi+1}{\rho}\right) \Bigg),
\end{aligned}
\qquad (2.39)
$$

provided that $4\delta + \xi < -1$. Figure 2.7 and Figure 2.8 show plots of the original $\mathcal{GEM}$ density superimposed on the proxy model (red line) expanded with 3 and 4 terms, respectively, around 2/3 when $\xi = -10$, $\delta = 0.5$, $\nu = 2$, $\tau = 3$ and $\rho = 2$. Note that the value of the original normalizing constant is 167.986, the normalizing constant for the proxy distribution when it is expanded for 3 terms being 170.959. For the proxy distribution expanded with 4 terms c_p equals 167.354. It can be seen from Figure 2.8 that the proxy distribution is nearly identical to the original $\mathcal{GEM}$ model when it is expanded only with 4 terms.

We determined that the generalized normalizing constant of the proxy model is

$$
c_p = \frac{e^{-m\nu}}{\rho} \sum_{i=0}^{n} \frac{(-1)^{n-i} \nu^{n-i} \tau^{\frac{\delta(n-i+1)+\xi+1}{\rho}} \Gamma\left(-\frac{(n-i+1)\delta+\xi+1}{\rho}\right)}{(n-i)!}, \qquad (2.40)
$$

which is determined by finding a general pattern for the normalizing constant starting with expansions of the proxy distribution with 3, 4, 5 and 6 terms. Similarly, the general form of the h^{th} moment of the proxy distribution is determined by looking at its h^{th} moments for various number of terms in the expansion, and then by investigating the general form. We determined that the general form of the h^{th} moment of the proxy distribution is

$$
\frac{\sum_{i=0}^{n} \frac{(-1)^{n-i} \nu^{n-i} \tau^{\frac{h+\delta(n-i+1)+\xi+1}{\rho}} \Gamma\left(-\frac{h+(n-i+1)\delta+\xi+1}{\rho}\right)}{(n-i)!}}{\sum_{i=0}^{n} \frac{(-1)^{n-i} \nu^{n-i} \tau^{\frac{\delta(n-i+1)+\xi+1}{\rho}} \Gamma\left(-\frac{(n-i+1)\delta+\xi+1}{\rho}\right)}{(n-i)!}}. \qquad (2.41)
$$

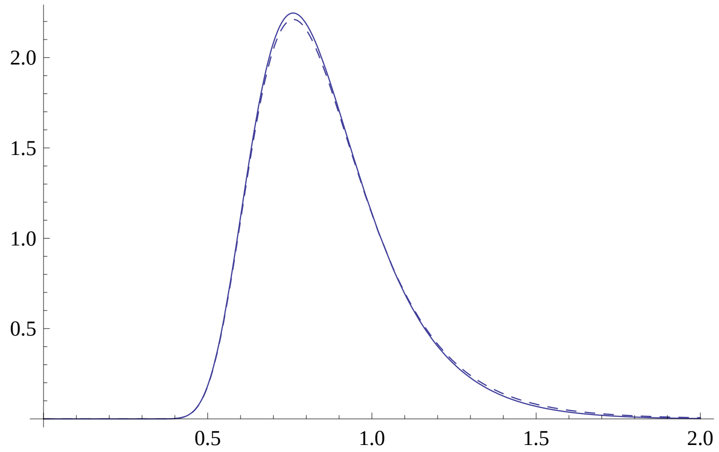

Figure 2.7: *Original $\mathcal{GEM}$ superimposed on the proxy model (red line) expanded with 3 terms around 2/3*

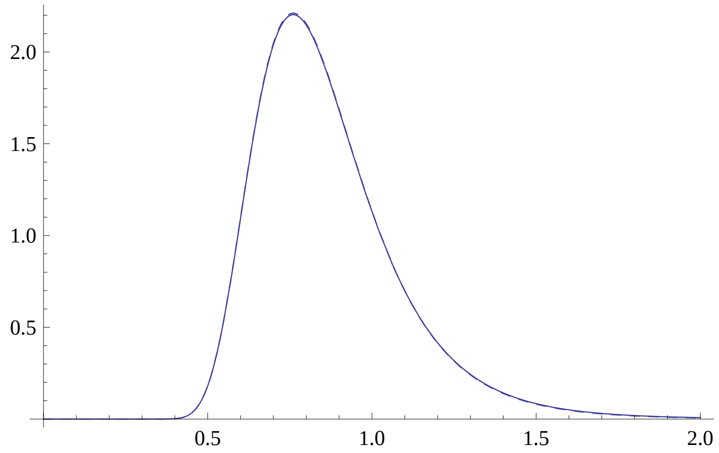

Figure 2.8: *Original $\mathcal{GEM}$ superimposed on the proxy model (red line) expanded with 4 terms around 2/3*

Both the h^{th} moment and the normalizing constant of the proxy distribution can be expressed in terms of seven parameters when δ is replaced by v/d and ρ, by w/r. To test the accuracy of the h^{th} moment of the proxy distribution and that of $\mathcal{GEM}$ model which is specified by Equation (2.10), the first 4 moments are calculated using both formulas. Assuming that $n = 4$ and $m = 2/3$, and that the parameters are $\xi = -10$, $\delta = 0.5$, $\nu = 2$, $\tau = 3$ and $\rho = 2$, the first four moments using Equation (2.10) are $m_1 = 0.86692$, $m_2 = 0.80134$, $m_3 = 0.796764$, and $m_4 = 0.861906$ while making use of Equation (2.41), they are $\mu_1 = 0.869954$, $\mu_2 = 0.810614$, $\mu_3 = 0.823303$, and $\mu_4 = 0.955122$. However, if $n = 6$ the first four moments are $\mu_1 = 0.86715$, $\mu_2 = 0.802284$, $\mu_3 = 0.801055$, and $\mu_4 = 0.89503$. For additional accuracy in the higher moments, one would have to include more terms in the expansion.

Noting that the proxy density function can be written as

$$f(x) = \frac{1}{c_p} x^{\xi+\delta} \, e^{-\tau x^{-\rho}} e^{-m\nu} \sum_{i=0}^{n} \frac{-1^i \nu^i}{i!} \sum_{j=0}^{i} \binom{i}{j} (-m)^{i-j} \, x^{\delta j}, \tag{2.42}$$

where c_p is given in Equation (2.40), the distribution function of the proxy distribution is seen to be

$$F(x) \;=\; \frac{e^{-m\nu}}{c_p} \sum_{i=0}^{n} \frac{-1^i \, \nu^i}{i!} \sum_{j=0}^{i} \binom{i}{j} (-m)^{i-j} \, \frac{y^{1+\delta+j\delta+\xi}}{\rho} E\left[\frac{1+\delta+j\delta+\xi+\rho}{\rho}, y^{-\rho}\tau\right]$$

where

$$E[n, z] \equiv \int_1^{\infty} \frac{e^{-zt}}{t^n} \mathrm{d}t.$$

Similarly, the survival function associated with the proxy distribution is

$$\begin{aligned} S_X(x) \;=\; & \frac{e^{-m\nu}}{c_p} \sum_{i=0}^{n} \frac{-1^i \, \nu^i}{i!} \sum_{j=0}^{i} \binom{i}{j} (-m)^{i-j} \\ & \times \frac{\tau^{\frac{j\delta+\delta+\xi+1}{\rho}} \left(\Gamma\left(-\frac{j\delta+\delta+\xi+1}{\rho}\right) - \Gamma\left(-\frac{j\delta+\delta+\xi+1}{\rho}, y^{-\rho}\tau\right)\right)}{\rho}. \end{aligned} \tag{2.43}$$

The density function of the proxy distribution given by Equation (2.42) is seen to be a mixture of $\mathcal{REIG}$ densities as defined in Section 2.4.3. Accordingly, the moment generating function of the proxy distribution can be expressed in terms of that of the $\mathcal{REIG}$ distribution.

2.5 Parameter Estimation

2.5.1 Maximum Likelihood Estimation

Let $X_1, X_2, \ldots, X_n$ be a data set that is assumed to be the realization of a random variable X that has the probability distribution function $f(x|\boldsymbol{\theta})$, where the p-dimensional unknown vector of parameters $\boldsymbol{\theta} \in \Omega_{\boldsymbol{\theta},}$ the parametric space. The maximum likelihood estimate of $\boldsymbol{\theta}$ is the value $\hat{\boldsymbol{\theta}}$ that maximizes the likelihood or equivalently the loglikelihood, that is, $\ell(\boldsymbol{\theta}) = Log(\prod f(x|\boldsymbol{\theta}))$. Thus, the maximum likelihood estimate $\hat{\boldsymbol{\theta}}$ (MLE) satisfies $\ell(\hat{\boldsymbol{\theta}}) > \ell(\boldsymbol{\theta})$, for all $\boldsymbol{\theta} \in \Omega_{\boldsymbol{\theta}.}$ The approximate covariance matrix associated with the MLE's is $Cov(\hat{\boldsymbol{\theta}}) = \mathbb{I}(\boldsymbol{\theta})^{-1}$ where $\mathbb{I}$ is the (Fisher) information, $\mathbb{I}(\boldsymbol{\theta}) = \mathcal{E}[J(\boldsymbol{\theta})]$, and $J(\boldsymbol{\theta}_{ij}) = -\frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}_i \partial \boldsymbol{\theta}_j}$ is the ij^{th} element of the observed information matrix. The observed and expected information matrices are of dimension $p \times p$ when $\boldsymbol{\theta}$ is p-dimensional.

Asymptotic likelihood theory deals with statistical inference based on likelihood functions under the assumption that the sample size approaches infinity. Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution specified by the density function $f(x|\boldsymbol{\theta})$ and suppose that the true value for the parameter is some constant(written $\boldsymbol{\theta}_0$ when used in the null hypothesis). As the sample size approaches infinity the maximum likelihood estimator has the following properties:

(i) It is normally distributed
(ii) It is unbiased
(iii) It has the smallest variance among all estimators that are asymptotically normal.
In particular the following results hold:

$$\sqrt{n}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}) \xrightarrow{d} \mathcal{N}_p(\mathbf{0}, \mathbb{I}(\boldsymbol{\theta})^{-1}), \tag{2.44}$$

$$\sqrt{n}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}) \xrightarrow{d} \mathcal{N}_p(\mathbf{0}, J(\hat{\boldsymbol{\theta}})^{-1}) . \tag{2.45}$$

The $\mathcal{RGG}$ Model

For this model, which was defined in Section 2.4.2, given the observations $x_1, \ldots, x_n$, the loglikelihood is

$$\begin{aligned} \ell(\xi, \nu, \delta) &= \sum_{i=1}^{n} \log f(x) \\ &= \frac{n(\delta + \xi)\log(\nu)}{\delta} + n\log(\delta) - n\log\left(\Gamma\left(\frac{\delta+\xi}{\delta}\right)\right) - (\delta + \xi - 1)\sum_{i=1}^{n}\log(x_i) - \nu\sum_{i=1}^{n} x_i^{\delta} \end{aligned} \tag{2.46}$$

where $f(x)$ is given in (2.17). On equating the partial derivatives of (2.47) with respect to ξ, ν and δ to zero, one can obtain the maximum likelihood estimates of ξ, τ and ρ by solving the following equations:

$$\frac{n\Big(\log(\nu) - \psi^{(0)}(\frac{\xi}{\delta} + 1)\Big)}{\delta} + \sum_{i=1}^{n} \log(x_i) = 0 , \tag{2.47}$$

$$\frac{n\left(\xi\,\psi^{(0)}(\frac{\xi}{\delta} + 1) - \xi\log(\nu) + \delta\right)}{\delta^2} + \sum_{i=1}^{n}\log(x_i) - \nu\sum_{i=1}^{n} x_i^{\delta}\log(x_i) = 0 , \tag{2.48}$$

$$\frac{n(\frac{\xi}{\delta} + 1)}{\nu} - \sum_{i=1}^{n} x_i^{\delta} = 0 , \tag{2.49}$$

where $\psi^{(n)}(z) = \frac{d^n \Gamma'(z)}{\Gamma(z)}$ is the polygamma function.

The elements of the information matrix for the $\mathcal{RGG}$ model are given by

$$\frac{\partial^2 \ell}{\partial \xi^2} = -\frac{n\psi^{(1)}\left(\frac{\delta+\xi}{\delta}\right)}{\delta^2} \tag{2.50}$$

$$\frac{\partial^2 \ell}{\partial \xi\, \partial \delta} = \frac{n\left(-\delta \log(\nu) + \delta\psi^{(0)}\left(\frac{\delta+\xi}{\delta}\right) + \xi\psi^{(1)}\left(\frac{\delta+\xi}{\delta}\right)\right)}{\delta^3} \tag{2.51}$$

$$\frac{\partial^2 \ell}{\partial \xi\, \partial \nu} = \frac{n}{\delta \nu} \tag{2.52}$$

$$\frac{\partial^2 \ell}{\partial \delta^2} = (-n\left(\delta\left(2\xi\psi^{(0)}\left(\frac{\delta+\xi}{\delta}\right) + \delta - 2\xi\log(\nu)\right) + \xi^2\psi^{(1)}\left(\frac{\delta+\xi}{\delta}\right)\right))/\delta^4 - \nu \textstyle\sum_{i=1}^n x_i^\delta \log^2(x_i) \tag{2.53}$$

$$\frac{\partial^2 \ell}{\partial \delta\, \partial \nu} = -\sum_{i=1}^{n} x_i^\delta \log(x_i) - \frac{n\xi}{\delta^2 \nu} \tag{2.54}$$

$$\frac{\partial^2 \ell}{\partial \nu^2} = -\frac{n(\delta+\xi)}{\delta \nu^2}. \tag{2.55}$$

The $\mathcal{REIG}$ Model

For this model, which was defined in Section 2.4.3, given the observations $x_1, \ldots, x_n$, the loglikelihood is

$$\begin{aligned} \ell(\xi, \tau, \rho) &= n\log(\rho) - \frac{n(\xi+\rho+1)\log(\tau)}{\rho} - n\log\left(\Gamma\left(-\frac{\xi+\rho+1}{\rho}\right)\right) \\ &\quad +(\xi+\rho)\sum_{i=1}^{n}\log(x_i) - \tau\sum_{i=1}^{n} x_i^{-\rho} \end{aligned} \tag{2.56}$$

where $f(x;\xi,\tau,\rho)$ is given in (2.26). On equating the partial derivatives of (2.57) with respect to ξ, τ and ρ to zero, one can obtain the maximum likelihood estimates of ξ, τ and ρ by solving the following equations:

$$\frac{n\Big(-\log(\tau) + \psi^{(0)}(-\frac{\xi+\rho+1}{\rho})\Big)}{\rho} + \sum_{i=1}^{n}\log(x_i) = 0\,, \tag{2.57}$$

$$\frac{n\left(-(\xi+1)\psi^{(0)}\left(-\frac{\xi+\rho+1}{\rho}\right)+(\xi+1)\log(\tau)+\rho\right)}{\rho^2} + \textstyle\sum_{i=1}^n \log(x_i) - \tau\sum_{i=1}^n x_i^{-\rho}\log(x_i) = 0\,, \tag{2.58}$$

$$-\frac{n(\xi+\rho+1)}{\rho\tau} - \sum_{i=1}^{n} x_i^{-\rho} = 0\,. \tag{2.59}$$

The elements of the information matrix for the $\mathcal{REIG}$ model are given by

$$\frac{\partial^2 \ell}{\partial \xi^2} = -\frac{n\psi^{(1)}\left(-\frac{\xi+\rho+1}{\rho}\right)}{\rho^2} \tag{2.60}$$

$$\frac{\partial^2 \ell}{\partial \xi\, \partial \tau} = -\frac{n}{\rho\tau} \tag{2.61}$$

$$\frac{\partial^2 \ell}{\partial \xi\, \partial \rho} = \frac{n\left(-\rho\psi^{(0)}\left(-\frac{\xi+\rho+1}{\rho}\right) + (\xi+1)\psi^{(1)}\left(-\frac{\xi+\rho+1}{\rho}\right) + \rho\log(\tau)\right)}{\rho^3} \tag{2.62}$$

$$\frac{\partial^2 \ell}{\partial \tau^2} = \frac{n(\xi+\rho+1)}{\rho\tau^2} \tag{2.63}$$

$$\frac{\partial^2 \ell}{\partial \tau\, \partial \rho} = \frac{n(\xi+1)}{\rho^2\tau} - \sum_{i=1}^{n} x_i^{-\rho}\left(-\log\left(x_i\right)\right) \tag{2.64}$$

$$\frac{\partial^2 \ell}{\partial \rho^2} = -\frac{1}{\rho^4}\Big\{n\Big(\rho(2(\xi+1)\log(\tau)+\rho) + (\xi+1)^2\psi^{(1)}\left(-\frac{\xi+\rho+1}{\rho}\right) \tag{2.65}$$

$$-\ 2(\xi+1)\rho\psi^{(0)}(-\frac{\xi+\rho+1}{\rho}) + \rho^4\tau\left(\sum_{i=1}^{n} x_i^{-\rho}\log^2\left(x_i\right)\right)\Big)\Big\}. \tag{2.66}$$

2.5.2 The Method of Moments

The method of moments is a method of estimation of population parameters whereby the sample moments are equated to unobservable population moments. The parameters are thus determined by solving the resulting equation system. That is, if X is a random variable with a probability density function $f(x, \boldsymbol{\theta})$ where $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_p)$, then the k^{th} population moment is

$$\mu_k^{'} = E(X^k), \quad k = 1, 2, \ldots, p \ ,$$

where population moments are functions in $\boldsymbol{\theta}$, that is, $\mu_k^{'} = \mu_k^{'}(\boldsymbol{\theta})$. Suppose that $X_1, X_2, \ldots\ldots, X_n$ is a random sample generated from that distribution, then the k^{th} sample moment is

$$m_k^{'} = \frac{1}{n}\sum_{i=1}^{n} x_i^k, \quad k = 1, 2, \ldots, p\,.$$

By equating the population moments with the corresponding sample moments, one has

$$m_k^{'} = \mu_k^{'}(\theta),\ k = 1, 2, \ldots, p.$$

Solving these equations we obtain the estimates $\tilde{\boldsymbol{\theta}} = (\tilde{\theta}_1, \tilde{\theta}_2, \ldots, \tilde{\theta}_p)$ which are called the method-of-moments estimates for $\boldsymbol{\theta}$.

The method of moments in general yields estimators that are consistent but are not as efficient as the maximum likelihood estimators. They are often used because they usually involve simple computations, unlike the maximum likelihood approach, which can become cumbersome.

2.6 Related Distributional Results

It is shown in this section that the product and ratios of certain exponential-type random variables can be expressed in an integral form corresponding to (2.2) and thus in terms of H-functions as explained in Section 2.3 .

(i) Let $X_i \sim \Gamma(\theta_i, \phi_i)$ with p.d.f. $f_i(x_i) = x_i^{\theta_i-1}\exp(-x_i/\phi_i)/({\phi_i}^{\theta_i}\Gamma(\theta_i))\,\mathcal{I}_{\mathbb{R}^+}(x_i)$, $\theta_i,\ \phi_i > 0, i = 1,2.$ Letting $z_1 = x_1x_2$ and $z_2 = x_2$, the absolute value of the Jacobian of the inverse transformation is $\frac{1}{z_2}$. Thus, the joint p.d.f of Z_1 and Z_2 is $f_1(\frac{z_1}{z_2})f_2(z_2)\frac{1}{z_2}$ and the marginal p.d.f of $Z_1 = X_1X_2$ is

$$\begin{aligned} g_1(z_1) &= \int_0^\infty f_1\Big(\frac{z_1}{z_2}\Big)f_2(z_2)\frac{1}{z_2}\mathrm{d}z_2 \\ &= \frac{1}{\Gamma(\theta_1)\Gamma(\theta_2)\phi_1^{\theta_1}\phi_2^{\theta_2}}z_1^{\theta_1-1}\int_0^\infty z_2^{\theta_2-\theta_1-1}e^{-\frac{1}{\phi_2}z_2}e^{-\frac{z_1}{\phi_1}z_2^{-1}}\mathrm{d}z_2\,, \theta_2-\theta_1>0\,, \end{aligned}$$

which corresponds to the integral (2.2) with $c^{-1} = \Gamma(\theta_1)\Gamma(\theta_2)\phi_1^{\theta_1}\phi_2^{\theta_2}z_1^{1-\theta_1}$, $\xi = \theta_2 - \theta_1 - 2,\ \nu = \frac{1}{\phi_2},\ \delta = 1,\ \tau = \frac{z_1}{\phi_1}$, and $\rho = 1$. This result also holds for chi-square and exponential distributions, which are particular cases of the gamma distribution.

(ii) Let $X_i \sim$ Weibull(θ_i, ϕ_i) with p.d.f. $f_i(x_i) = \theta_i\,\phi_i\,x_i^{\phi_i-1}\exp(-\theta_i\,x_i^{\phi_i})\,\mathcal{I}_{\mathbb{R}^+}(x_i)$, $i = 1,2$. Now, letting $z_1 = x_1x_2$ and $z_2 = x_2$, the absolute value of the Jacobian of the inverse transformation is $\frac{1}{z_2}$, and the joint p.d.f of Z_1 and Z_2 is $f_1(\frac{z_1}{z_2})f_2(z_2)\frac{1}{z_2}$, so that the marginal p.d.f of $Z_1 = X_1X_2$ is

$$\begin{aligned} g_2(z_1) &= \int_0^\infty f_1\Big(\frac{z_1}{z_2}\Big)f_2(z_2)\frac{1}{z_2}\mathrm{d}z_2 \\ &= \phi_1\,\phi_2\,\theta_1\,\theta_2\,z_1^{\phi_1-1}\int_0^\infty z_2^{\phi_2-\phi_1-1}e^{-\theta_2z_2^{\phi_2}}e^{-\theta_1\,z_1^{\phi_1}\,z_2^{-\phi_1}}\mathrm{d}z_2\,, \end{aligned}$$

which is also in the form of the integral in (2.2) with $c = \phi_1\,\phi_2\,\theta_1\,\theta_2\,z_1^{\phi_1-1}$, $\xi = -\phi_1 - 1,\ \delta = \phi_2,\ \nu = \theta_2\,, \tau = \theta_1\,z_1^{\phi_1}$, and $\rho = \phi_1$. This result also applies to the Rayleigh(a) and exponential(ϕ) distributions as they are particular cases of the Weibull distribution with $\phi = 2,\ \theta = 1/(2a^2)$ and $\phi = 1,\ \theta = 1/\kappa$, respectively.

(iii) If one lets $X_i,\ i = 1,2$, be distributed as in (1.1) with common parameters$(\xi_1, \delta_1, \nu_1, \tau_1, \rho_1)$, then, letting $z_1 = \frac{x_1}{x_2}$ and $z_2 = x_2$, the absolute value of the Jacobian of the inverse transformation is z_2 and the p.d.f of $Z_1 = \frac{X_1}{X_2}$ is

$$\begin{aligned} g_3(z_1) &= \int_0^\infty f_1(z_1z_2)f_2(z_2)z_2\mathrm{d}z_2 \\ &= c_1\,c_2\,z_1{}^{\xi_1+\delta_1}\int_0^\infty z_2^{2\xi_1+2\delta_1+1}e^{-(\nu_1z_1^{\delta_1}+\nu_1)z_2^{\delta_1}}e^{-(\tau_1z_1^{-\rho_1}+\tau_1)z_2^{\rho_1}}\mathrm{d}z_2\,, \end{aligned}$$

which corresponds to the integral (2.2) with $\xi = 2\xi_1 + \delta_1 + 1,\ \nu = \nu_1z_1^{\delta_1} + \nu_1,\ \delta = \delta_1,\ \tau = \tau_1z_1^{-\rho_1} + \tau_1$, and $\rho = \rho_1$.

Thus, in light of the representations of the moments of the five-parameter exponential distribution provided in (2.9) and (2.10), the density functions $g_i(z_1)$, $i = 1, 2, 3$, respectively obtained in (i), (ii) and (iii) are seen to be expressible in terms of generalized hypergeometric functions.

2.7 Illustrative Examples

In order to assess the fit of a distribution with respect to a given data set, one may make use of the following goodness-of-fit statistics:

(i) The Anderson-Darling statistic denoted by A_0^2 and given by

$$A_0^2 = -n - \frac{1}{n}\sum_{i=1}^{n}(2\,i - 1)\log\left(z_i(1 - z_{n+1-i})\right)$$

where $z_i = \mathrm{cdf}(x_i)$, the x_i's, $i = 1, \ldots, n$, being the *ordered* observations;

(ii) The Cramér-von Mises statistic, which is

$$W_0^2 = \sum_{i=1}^{n}\Big(z_i - \frac{2\,i-1}{2\,n}\Big)^2 + \frac{1}{12\,n}\,.$$

The smaller these statistics are, the better the fit.

The five-parameter generalized exponential model defined in Section 2.1 is applied to the Flood, Snowfall precipitation and Repair data sets where the parameters are estimated using the maximum likelihood approach and the method of moments. Parameter estimates have been obtained using the original density function with its normalizing constant (exact or determined by numerical integration) and the proxy density function.

The flood data set, given in Table 2.1, corresponds to maximum flood levels (in millions cubic feet per second) for the Susquehanna River at Harrisburg, Pennsylvania over 20 four-year periods, *cf.* Dumonceaux and Antle (1973). The Buffalo snowfall data set, given in Table 2.4 (and available for instance from the S-PLUS data library) comprises a record of the annual snowfall precipitations in centimeters over 63 consecutive years in the city of Buffalo. The repair time data set given by Jørgensen (1982) and presented in Table 2.5 represents the active time in hours for an airborne communication transceiver.

2.7.1 Maximum Likelihood Estimates

The maximum likelihood method is being employed in this section to estimate the model parameters. We fit for the three data sets to the distribution specified by (2.1), as well as some particular cases thereof. We made use of the symbolic computing package *Mathematica* in which the G-function is a built-in function, in conjunction with the command *NMaximize*

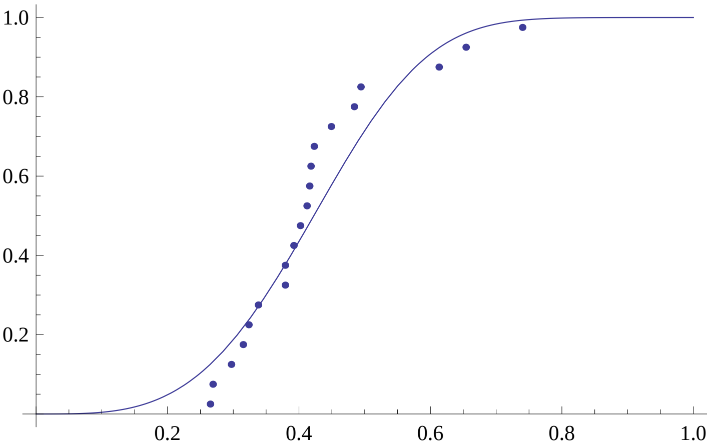

Figure 2.9: *The empirical CDF and the fitted Weibull CDF for the flood data set.*

applied to the likelihood functions to estimate the parameters, assuming the Weibull and inverse Gaussian models as defined in Chapter 1, and then assuming that $v = d = w = r = 1$, which corresponds to the generalized inverse Gaussian distribution whose p.d.f. is given in (2.2), and finally that $\delta = v/d = 5/2$ and $\rho = w/r = 5/2$ in the most general model specified by (2.1). It can be seen from the results presented in Tables 2.2, 2.4 and 2.6 that the $\mathcal{GEM}$ distribution provides a better fit than that obtained from the other models for all the three data sets. It has been found that the parameter estimates of the lognormal model were found to be $\mu = -0.898$ and $\sigma = 0.269$ for the flood data, $\mu = 4.337$ and $\sigma = 0.327$ for the snow data, and $\mu = 0.65839$ and $\sigma = 1.10179$ for the repair data.

Table 2.1: Maximum Flood Levels

.654	.613	.402	.379	.269
.740	.416	.338	.315	.449
.297	.423	.379	.3235	.418
.412	.494	.392	.484	.265

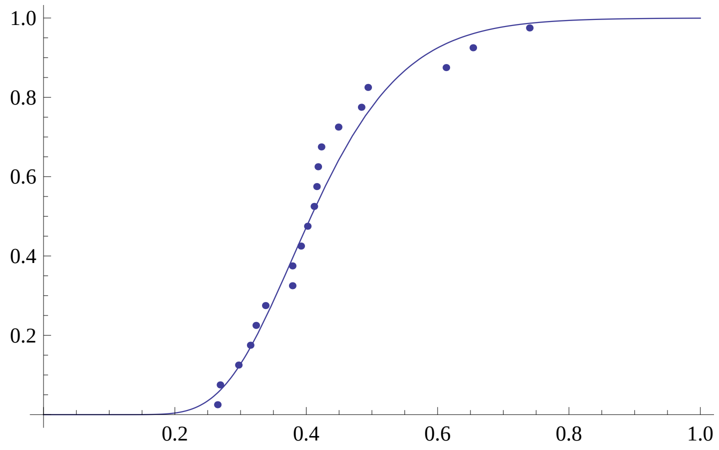

Figure 2.10: *The empirical CDF and the fitted lognormal CDF for the flood data set.*

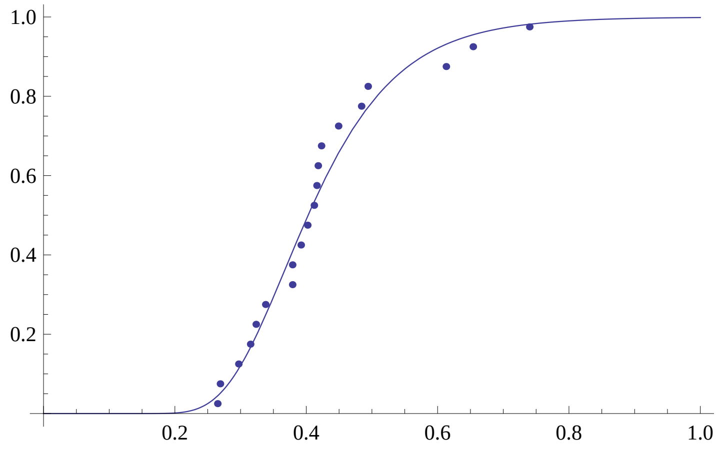

Figure 2.11: *The empirical CDF and the fitted generalized inverse Gaussian CDF for the Flood data set.*

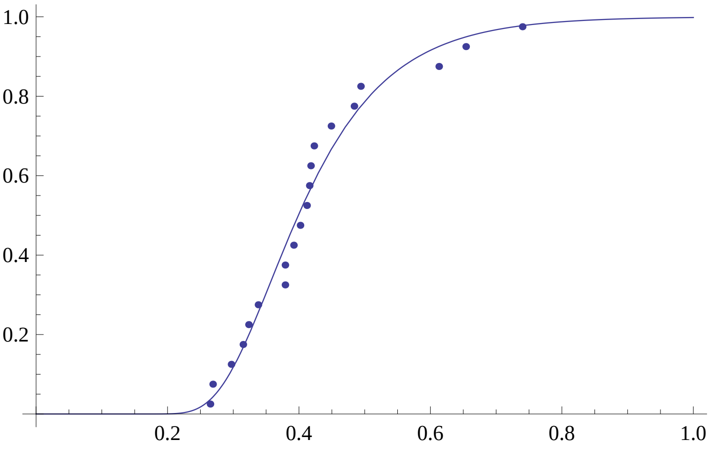

Figure 2.12: *The empirical CDF and the fitted five parameter $\mathcal{GEM}$ CDF for the Flood data set.*

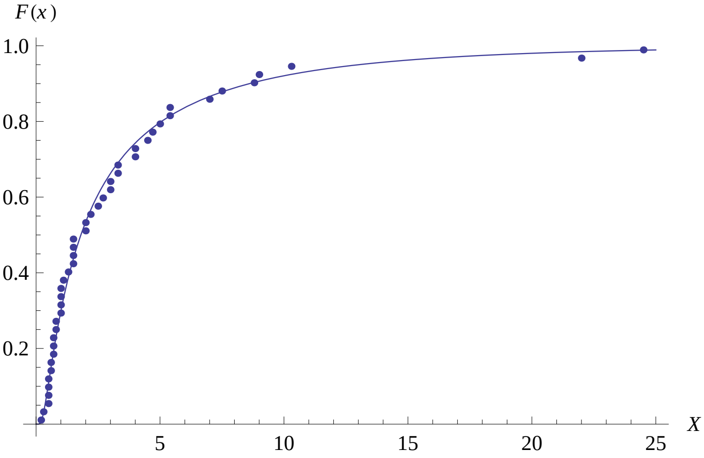

Figure 2.13: *The empirical CDF and the fitted inverse Gaussian CDF using the method of moments for the Flood data set*

Table 2.2: Estimates of the Parameters and Goodness-of-Fit Statistics for Maximum Flood Levels

MLE's	$\hat{\xi}$	$\hat{\nu}$	$\hat{\tau}$	A_0^2	W_0^2
Inverse Gaussian	−2.50	15.745	2.819	7.17061	1.56141
Weibull($\delta = 3.526$)	−1	14.450	0	0.8213	0.13998
Lognormal	–	–	–	0.34701	0.05396
$\delta = \rho = 1$	−16.572	0.001	5.736	0.28605	0.04488
$\delta = \rho = 5/2$	−8.698	1.608	0.199	0.26880	0.04371

Table 2.3: The Bufffalo Snow Data Set

25	39.8	39.9	40.1	46.7	49.1	49.6	51.2	51.6
53.5	54.7	55.5	55.9	58	60.3	63.6	65.4	66.1
69.3	70.9	71.4	71.5	71.8	72.9	74.4	76.2	77.8
78.2	78.4	79	79.3	79.6	80.7	82.4	82.4	83
83.6	83.6	84.8	85.5	87.4	88.7	89.6	89.8	89.9
90.9	97.	98.3	101.4	102.4	103.9	104.5	105.2	110.
110.5	110.5	113.7	114.5	115.6	120.5	120.7	124.7	126.4

2.7.2 Method of Moment Estimates

The method of moment was applied to the Flood data and the results are included in Table 2.7. It is seen that the goodness-of-fit results are similar to those obtained in Table 2.2 by making use of the maximum likelihood approach.

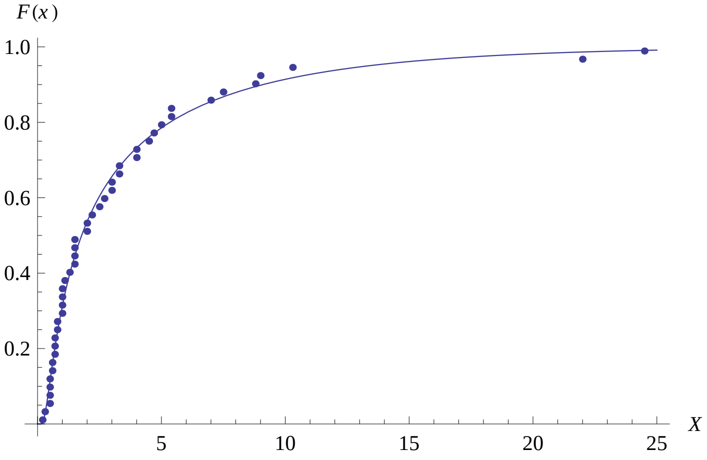

Figure 2.14: *The empirical CDF and the fitted five parameter $\mathcal{GEM}$ CDF using the method of moments for the Flood data set*

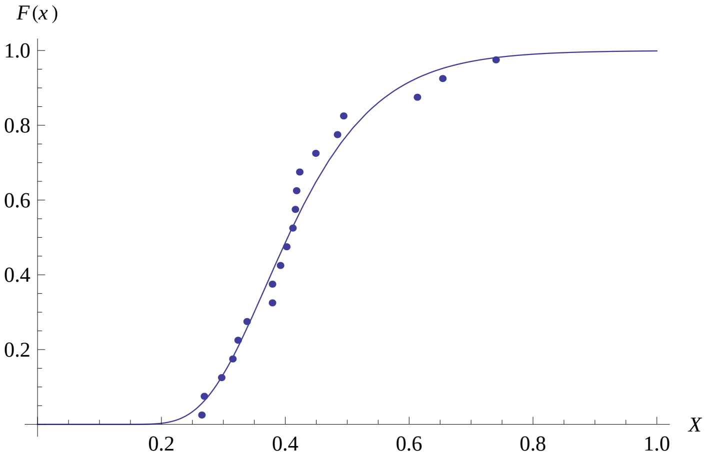

Figure 2.15: *The empirical CDF and the fitted inverse Gaussian CDF using the method of moments for the Flood Data set*

Table 2.4: Estimates of the Parameters and Goodness-of-Fit Statistics for the Snowfall Data Set

MLE's	$\hat{\xi}$	$\hat{\nu}$	$\hat{\tau}$	A_0^2	W_0^2
Inverse Gaussian	-2.5	0.055	353.261	0.86761	0.15043
Weibull($\delta = 3.8338$)	-1	3.3748×10^{-8}	0	0.29638	0.04543
Lognormal	—	—	—	0.77525	0.12835
$\delta = \rho = 1$	8.354	0.129	1.003	0.48211	0.07684
$\delta = 3, \rho = 8/3$	0.427	2.267×10^{-6}	0.0001	0.27864	0.04105

Table 2.5: The Repair Time Data Set

.2	.3	.5	.5	.5	.5	.6	.6	.7	.7
.7	.8	.8	1	1	1	1	1.1	1.3	1.5
1.5	1.5	1.5	2	2	2.2	2.5	2.7	3	3
3.3	3.3	4	4	4.5	4.7	5	5.4	5.4	7
7.5	8.8	9	10.3	22	24.5				

2.7.3 Estimates Using a Proxy Distribution

The Proxy distribution described in Section 2.3.4 is utilized in order to be able to estimate the five parameters at once. The exponential term $e^{-\nu x^{\delta}}$ is expanded with 7 terms in order to closely approximate the original $\mathcal{GEM}$. Table 2.8 shows the parameter estimates and the goodness-of-fit statistics obtained for the three data sets.

2.7.4 Determining the Normalizing Constant

In order to avoid fixing the parameters, δ and ρ and estimating the other parameters, we obtained an approximation to the normalizing constant by using numerical integration. This method has the advantage of estimating all the parameters at once without needing to specify any other quantities such as the number of terms in the expansion used to obtain the proxy density. Tables 2.9 and 2.10 respectively give the estimates of the parameters and the goodness-of-fit attained for the different data sets. It can be observed that the fit measures determined from any of the three proposed approaches are comparable.

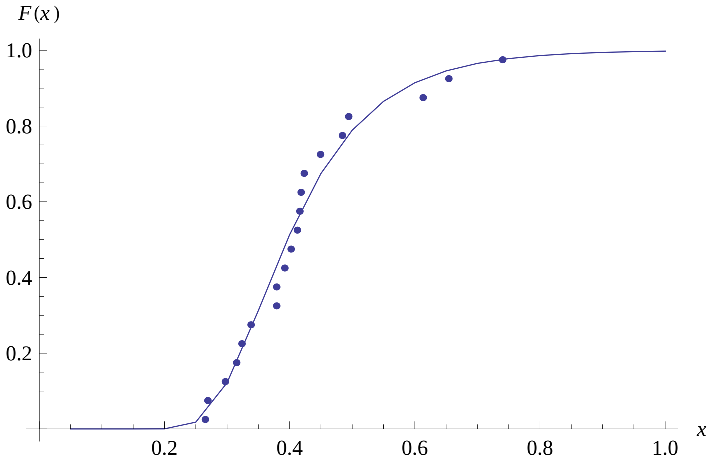

Figure 2.16: *The empirical CDF and the fitted five parameter $\mathcal{GEM}$ using the method of moments for the Flood Data set*

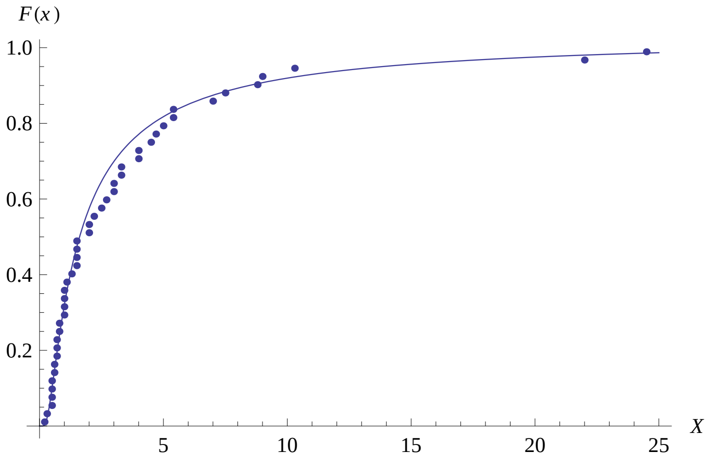

Figure 2.17: *The empirical CDF and the fitted proxy $\mathcal{GIG}$ CDF using the maximum likelihood method for the Flood data set*

Table 2.6: Estimates of the Parameters and Goodness-of-Fit Statistics for the Repair Data Set

MLE's	$\hat{\xi}$	$\hat{\nu}$	$\hat{\tau}$	A_0^2	W_0^2
Inverse Gaussian	-5/2	0.33284	2.25	6.44415	0.58717
Weibull($\delta = 0.89858$)	-1	0.33375	0	0.88782	0.12046
Lognormal	—	—	—	0.33951	0.05468
$\delta = \rho = 1$	−2.442	0.070	0.789	0.21718	0.03186
$\delta = \rho = 3/2$	−2.814	0.013	0.306	0.20844	0.02502

Table 2.7: Estimates of the Parameters and Goodness-of-Fit Statistics for the Maximum Flood data using the Moment Method

	$\hat{\xi}$	$\hat{\nu}$	$\hat{\tau}$	A_0^2	W_0^2
Inverse Gaussian	-2.5	14.1892	2.5404	0.337483	0.05588
Weibull	-1	15.8368	2.2252	0.347327	0.05749
Lognormal	—	—	—	0.335996	0.05536
$\delta = \rho = 1$	−10.274	5.871	4.216	0.29652	0.04897
$\delta = \rho = 8/3$	−8.760	1.265	0.148	0.264028	0.04436

2.7.5 Model Comparison Based on Likelihood Criteria

Comparison between the $\mathcal{GEM}$ and some other models based on likelihood criteria, namely Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC) for the the three data sets is discussed in this section. As mentioned by Andrews (1999), the AIC criterion was introduced by Akaike (1969,1974) and the BIC was introduced by Schwarz (1978). More specifically AIC and BIC are defined as follows:

$$AIC = -2\ln(L) + 2k$$

$$BIC = -2\ln(L) + k\ln(n)$$

where L is the likelihood function, k is the number of parameters and n is the sample size.

Both the AIC and the BIC take into account the number of parameters; however a larger penalty for the number of parameters results from making use of the BIC when the sample size is greater than 7. Accordingly, we only present the results for the BIC in Table 2.13.

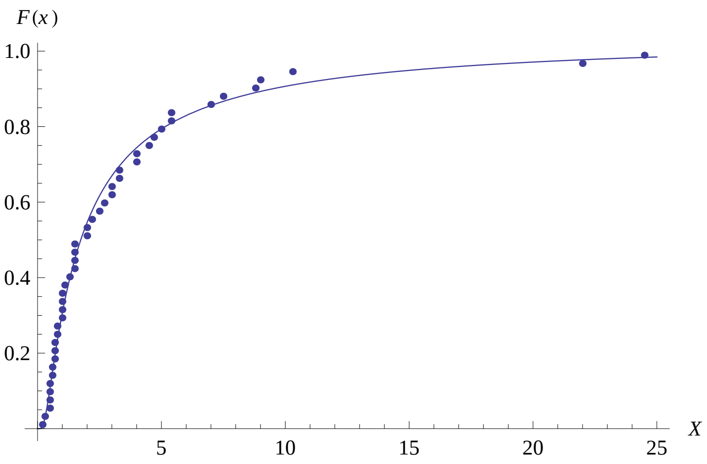

Figure 2.18: *The empirical CDF and the fitted proxy $\mathcal{GEM}$ CDF using the maximum likelihood method for the Flood data set*

Table 2.8: Estimates of Parameters and Goodness-of-Fit Statistics for the Three Data Sets Using a Proxy Distribution with 7 Terms

Data Set	Model	$\hat{\xi}$	δ	$\hat{\nu}$	$\hat{\tau}$	ρ	A_0^2	W_0^2
Flood	$\mathcal{GIG}$	-16	1	0.27707	5.55738	1	0.28452	0.04576
	$\mathcal{GEM}$	-9	1	0.01201	0.32961	2.29095	0.26508	0.04306
Repair	$\mathcal{GIG}$	-3	1	0.000001	1.174	1	0.35384	0.05995
	$\mathcal{GEM}$	-1.806	0.101	2.976	1.363	0.845	0.22607	0.03290

Table 2.9: Estimates of Parameters for Various Data Sets Using NIntegrate for Determining the Normalizing Constant

	$\hat{\xi}$	v	d	$\hat{\nu}$	$\hat{\tau}$	w	r
Flood	-7.0509	5	1	5	1.6636	5	2
Snow	3.4075	11	6-	0.00102963	0.143345	4	8
Repair	-2.383	1	1	0.0760	0.7129	2	2

Table 2.10: Goodness-of-Fit Statistics for Various Data Sets Using NIntegrate for Determining the Normalizing Constant

	A_0^2	W_0^2	D	DS
Flood	0.2568	0.0444	0.12603	0.00155
Snow	0.343113	0.0522957	0.0721831	0.000754621
Repair	0.2183	0.0303	0.07327	0.00047

Table 2.11: Estimates of the Parameters for Various Data Sets Using NIntegrate for Determining the Normalizing Constant (5 Parameters)

	$\hat{\xi}$	$\hat{\delta}$	$\hat{\nu}$	$\hat{\tau}$	$\hat{\rho}$
Flood	-7.089	1.530	2.098	0.141	2.701
Snow	0.102	3.159	1×10^{-6}	2.586	3.490
Repair	-3.046	1.55	0.008	0.611	1.156

Table 2.12: Goodness-of-Fit Statistics for Various Data Sets Using NIntegrate for Determining the Normalizing Constant (5 Parameters)

	A_0^2	W_0^2	D	DS
Flood	0.26665	0.04358	0.12710	0.0015209
Snow	0.27397	0.04077	0.05809	0.0005710
Repair	0.20747	0.02760	0.06521	0.0004050

Table 2.13: Loglikelihood Function and BIC for Various Data Sets

	Flood		Snowfall		Repair	
	LogL	BIC	LogL	BIC	LogL	BIC
IG	15.8488	-25.7061	-292.574	593.434	-119.798	247.253
Weibull	13.264	-20.5365	-287.959	584.204	-104.47	216.597
Lognormal	52.5784	-99.1653	-176.398	361.081	-15.474	38.6052
$\mathcal{GIG}$	16.1382	-23.2892	-289.908	592.245	-99.0509	209.588
$\mathcal{GEM}$	16.3393	-23.6915	-288.069	588.567	-99.0216	209.529
Proxy	16.3107	-20.6385	—	—	—	—
Bessel	16.3283	-20.6737	-290.367	597.306	-98.9936	213.302

Chapter 3

An Extended Inverse Gaussian Model

3.1 Introduction

We are proposing an extension of the Generalized Inverse Gaussian ($\mathcal{GIG}$) distribution specified by the following density function:

$$f(x) = \frac{(\phi/\theta)^{\lambda/2}}{2\,K_\lambda(\sqrt{\theta\,\phi})}\, x^{\lambda-1} \exp(-(\theta\, x^{-1} + \phi\, x)/2)\, \mathcal{I}_{\mathbb{R}^+}(x)\,, \tag{3.1}$$

where $\lambda \in \Re$ and ϕ and θ are positive numbers, which will be referred to as the Extended Inverse Gaussian ($\mathcal{EIG}$) distribution. Its density function is given by

$$f_E(x) = \frac{\delta\, (\nu/\tau)^{\frac{\delta+\xi+1}{2\delta}}\, x^{\delta+\xi}\, e^{-\tau x^{-\delta}-\nu x^{\delta}}}{2\,K_{\frac{\delta+\xi+1}{\delta}}\left(2\sqrt{\nu\tau}\right)}\, \mathcal{I}_{\Re^+}(x) \tag{3.2}$$

where $\xi \in \Re$, $\nu > 0$, $\tau > 0$ and $\delta > 0$ and $K_\alpha(\cdot)$ denotes a Bessel function of the second type, which is defined in Section 2.3. By introducing a single additional parameter, we aim to obtain a more flexible modeling distribution while keeping the resulting model relatively parsimonious. A location parameter could also be introduced in (3.2) for modeling purposes. Note that the $\mathcal{GIG}$ density function can be obtained from (3.2) by making the following substitutions: $\delta = 1,\ \tau = \theta/2,\ \nu = \phi/2$ and $\xi = \lambda - 2$. This distribution can also be obtained as a special case of the $\mathcal{GEM}$ given by Equation (2.1) when $\delta = \rho \neq 1$.

A reduced model called the Reduced Extended Inverse Gaussian ($\mathcal{REIG}$) distribution, is obtained by omitting $e^{-\nu\, x^{\delta}}$ (or equivalently letting $\nu = 0$) in the density function (4), which gives

$$f_R(x) = \frac{\delta\, \tau^{-\frac{\xi+\delta+1}{\delta}}}{\Gamma(-\frac{\xi+\delta+1}{\delta})}\, x^{\xi+\delta}\, e^{-\tau\, x^{-\delta}}\, \mathcal{I}_{\Re^+}(x)\,,\ \ \xi \in \Re,\ \nu > 0,\ \tau > 0,\ \ \delta > 0, \tag{3.3}$$

provided that $1 + \delta + \xi < 0$. This model is also a reduced form of the $\mathcal{GEM}$ where the exponent term that contains τ and ρ is excluded

Another reduced version of the $\mathcal{EIG}$ model is obtained by omitting $e^{-\tau x^{-\delta}}$ (or equivalently by letting $\tau = 0$) in the density function (3.2), which yields

$$g(x) = \frac{\delta\, \nu^{\frac{\delta+\xi+1}{\delta}}}{\Gamma(\frac{\delta+\xi+1}{\delta})}\, x^{\xi+\delta} e^{-\nu x^{\delta}}\, \mathcal{I}_{\mathfrak{R}^+}(x), \tag{3.4}$$

where $\delta+\xi > 0$. This density function is in fact a Reparameterized Generalized Gamma ($\mathcal{RGG}$) density function, which is obtained by letting $\beta = \delta$, $\theta = \nu^{-1/\beta}$ and $k = \frac{\delta+\xi+1}{\delta}$ in the standard generalized gamma density given by

$$g_1(x) = \frac{\beta}{\theta^{k\beta}\,\Gamma(k)}\, x^{k\beta-1} e^{-(\frac{x}{\theta})^{\beta}}\, \mathcal{I}_{\mathfrak{R}^+}(x)\,. \tag{3.5}$$

The $\mathcal{RGG}$ model is also a form of the $\mathcal{GEM}$ model where the exponent term that contains τ and ρ is excluded. For specific distributional results in connection with the generalized gamma distribution, the reader is referred to Johnson *et al.* (1994).

3.2 Parameter Effects

This section illustrates graphically how the extended generalized inverse Gaussian model and its reduced version are affected by their parameters.

3.2.1 The Extended Inverse Gaussian ($\mathcal{EIG}$) Model

Figures 3.1-3.3 indicate that the parameters ξ, δ and ν somewhat affect the shape of the $\mathcal{EIG}$ model while ξ and τ have a noticeable shifting effect on the distribution. Moreover, the parameters ν and τ in the density expression (3.2) are clearly scale parameters.

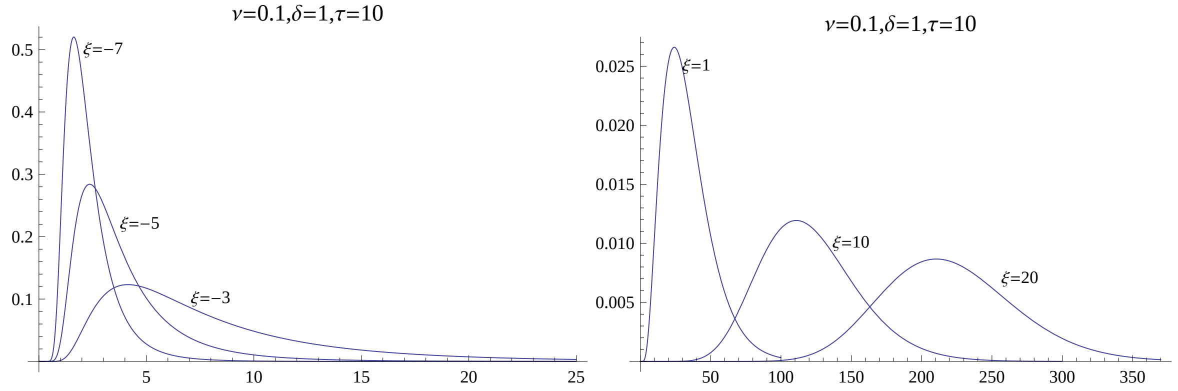

Figure 3.1: *Effect of ξ on the $\mathcal{EIG}$ distribution.*

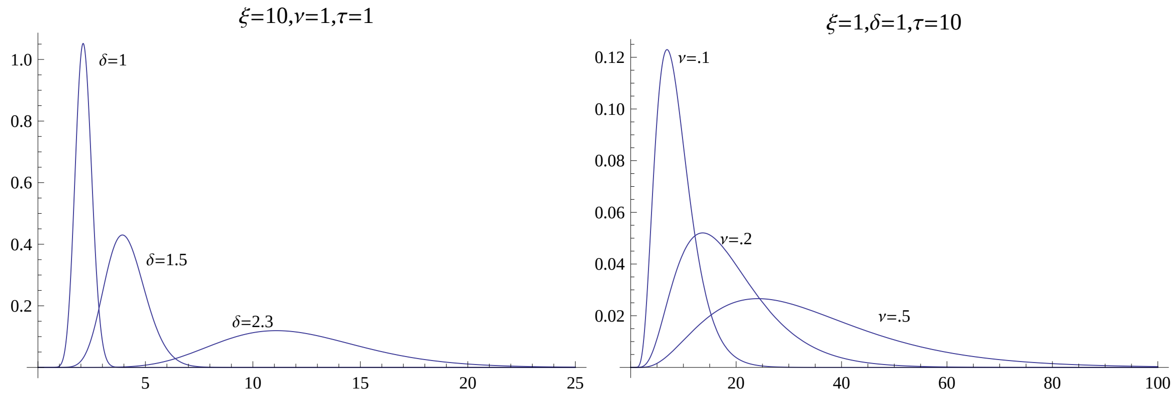

Figure 3.2: *Effects of δ (left panel) and ν (right panel) on the $\mathcal{EIG}$ model.*

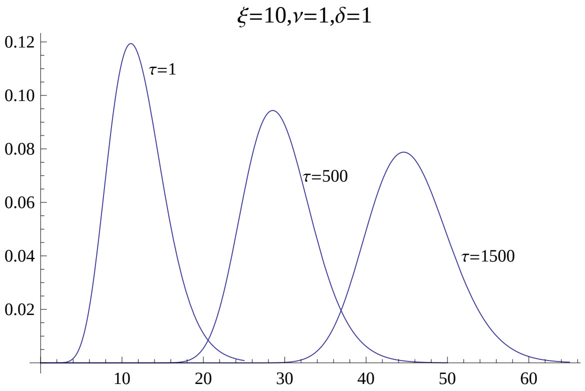

Figure 3.3: *Effect of τ on the $\mathcal{EIG}$ distribution.*

3.2.2 The Reduced Extended Inverse Gaussian ($\mathcal{REIG}$) Model

Figure 3.4 and 3.5 suggest that the parameter δ acts somewhat as a shifting parameter while ξ affects the shape of the $\mathcal{REIG}$ distribution. The scale parameter τ acts as a shifting parameter as it did for the $\mathcal{EIG}$ model.

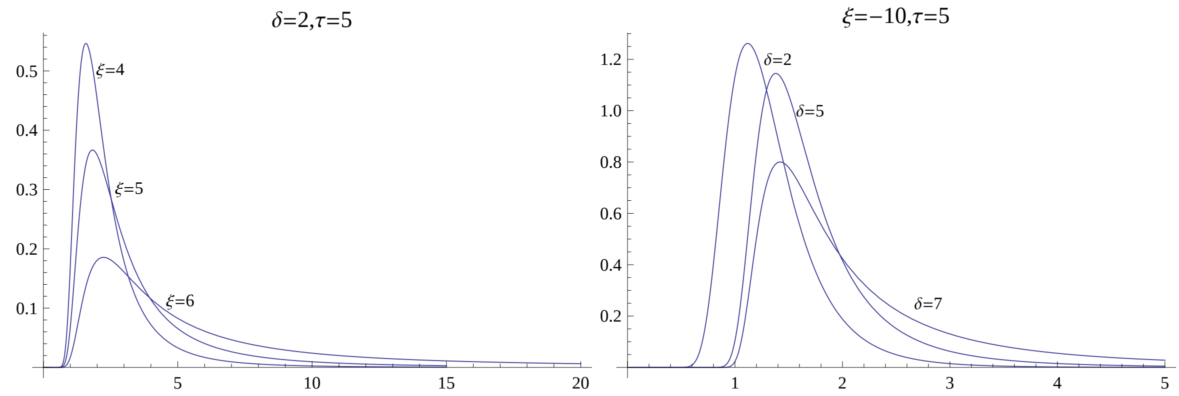

Figure 3.4: *Effects of* ξ *(left panel) and* δ *(right panel) on the* $\mathcal{REIG}$ *model.*

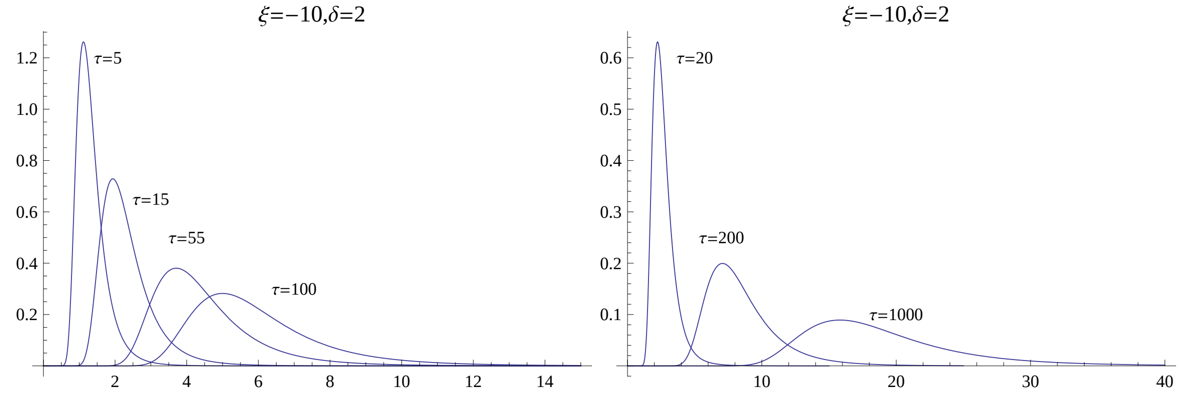

Figure 3.5: *Effect of* τ *on the* $\mathcal{REIG}$ *distribution.*

3.3 Certain Statistical Functions

Some statistical functions are provided in this section for the $\mathcal{EIG}$ model. Let X be an $\mathcal{EIG}$ random variable. Then,

(i) its h^{th} moment is

$$E(X^h) = \frac{\nu^{-\frac{h}{\delta}}(\nu\tau)^{\frac{h}{2\delta}} K_{\frac{h+\delta+\xi+1}{\delta}}\left(2\sqrt{\nu\tau}\right)}{K_{\frac{\delta+\xi+1}{\delta}}\left(2\sqrt{\nu\tau}\right)}\,; \tag{3.6}$$

(ii) its expectation, $E(X)$, is as given above for $h = 1$;

(iii) its variance, $E(X^2) - (E(X))^2$, can be directly obtained from (3.6);

(iv) its skewness is given by $(E(X^3) - 3E(X^2)\mu + 2\mu^3)/\sigma^3$;

(v) its kurtosis is given by $(E(X^4) - 4E(X^3)\mu + 6E(X^2)\mu^2 - 3\mu^4)/\sigma^4 - 3$;

(vi) its mode is

$$2^{-1/\delta}\left(\frac{\delta+\xi+\sqrt{4\delta^2\nu\tau+\delta^2+2\delta\xi+\xi^2}}{\delta\,\nu}\right)^{\frac{1}{\delta}}\,;$$

(vii) its survival function is

$$S_X(x) = \frac{x^{\delta+\xi+1}\left(\frac{\nu}{\tau}\right)^{\frac{\delta+\xi+1}{2\delta}} K_{\frac{\delta+\xi+1}{\delta}}\left(\nu x^{\delta},\, \tau x^{-\delta}\right)}{2\, K_{\frac{\delta+\xi+1}{\delta}}\left(2\sqrt{\nu\tau}\right)}, \tag{3.7}$$

where $K_\nu\,(x,\, y)$ is the incomplete Bessel function and, as explained in Harrisa (2008), has the following integral representation :

$$K_\nu\,(x,\, y) = \int_1^\infty t^{\nu-1} e^{-x t - \frac{y}{t}} \mathrm{d}t. \tag{3.8}$$

To derive the survival function of the $\mathcal{EIG}$ distribution, the integral

$$\int_x^\infty \frac{\delta\ (\nu/\tau)^{\frac{\delta+\xi+1}{2\delta}}}{2\, K_{\frac{\delta+\xi+1}{\delta}}\left(2\sqrt{\nu\tau}\right)} x^{\delta+\xi}\ e^{-\tau x^{-\delta} - \nu x^{\delta}},$$

has to be evaluated. First, we let $Y = X^\delta$. Noting that the derivative of the inverse transformation is $\frac{1}{\delta}\, y^{\frac{1}{\delta}-1}$, we have

$$S_Y(y) = \int_y^\infty \frac{(\nu/\tau)^{\frac{\delta+\xi+1}{2\delta}}}{2\, K_{\frac{\delta+\xi+1}{\delta}}\left(2\sqrt{\nu\tau}\right)} y^{\frac{\xi+1}{\delta}}\ e^{-\tau y^{-1}\nu y}.$$

Since this last integral would be identical to that given by Equation (3.8) if its lower bound were 1, a second change of variables, namely $W = Y/y$ is utilized. Accordingly, the survival function of Y can be expressed in terms of an incomplete Bessel function as follows

$$S_Y(y) = \frac{y^{\frac{\delta+\xi+1}{\delta}} (\nu/\tau)^{\frac{\delta+\xi+1}{2\delta}}}{2\, K_{\frac{\delta+\xi+1}{\delta}}\left(2\sqrt{\nu\tau}\right)} K_{\frac{\delta+\xi+1}{\delta}}\left(\nu y,\, \frac{\tau}{y}\right).$$

Since $S_X(x) = F_Y(x^\delta)\, J$ where J is the Jacobian, then

$$S_X(x) = \frac{x^{\delta+\xi+1}\left(\frac{\nu}{\tau}\right)^{\frac{\delta+\xi+1}{2\delta}} K_{\frac{\delta+\xi+1}{\delta}}\left(\nu x^{\delta},\, \tau x^{-\delta}\right)}{2\, K_{\frac{\delta+\xi+1}{\delta}}\left(2\sqrt{\nu\tau}\right)}.$$

3.4 The Observed Information Matrix

The elements of the information matrix for $\mathcal{EIG}$ are given by

$$\begin{aligned}
\frac{\partial^2 L}{\partial \xi^2} &= \frac{n\mathrm{K}^{(1,0)}\left(\frac{\delta+\xi+1}{\delta}, 2\sqrt{\nu\tau}\right)^2}{\delta^2 \mathrm{K}_{\frac{\delta+\xi+1}{\delta}}\left(2\sqrt{\nu\tau}\right)^2} - \frac{n\mathrm{K}^{(2,0)}\left(\frac{\delta+\xi+1}{\delta}, 2\sqrt{\nu\tau}\right)}{\delta^2 \mathrm{K}_{\frac{\delta+\xi,+1}{\delta}}\left(2\sqrt{\nu\tau}\right)}, \\
\frac{\partial^2 L}{\partial \xi\, \partial \delta} &= \frac{n\left(\frac{1}{\delta} - \frac{\delta+\xi+1}{\delta^2}\right)\mathrm{K}^{(1,0)}\left(\frac{\delta+\xi+1}{\delta}, 2\sqrt{\nu\tau}\right)^2}{\delta \mathrm{K}_{\frac{\delta+\xi+1}{\delta}}\left(2\sqrt{\nu\tau}\right)^2} + \frac{n\mathrm{K}^{(1,0)}\left(\frac{\delta+\xi+1}{\delta}, 2\sqrt{\nu\tau}\right)}{\delta^2 \mathrm{K}_{\frac{\delta+\xi+1}{\delta}}\left(2\sqrt{\nu\tau}\right)} \\
&\quad - \frac{n\left(\frac{1}{\delta} - \frac{\delta+\xi+1}{\delta^2}\right)\mathrm{K}^{(2,0)}\left(\frac{\delta+\xi+1}{\delta}, 2\sqrt{\nu\tau}\right)}{\delta \mathrm{K}_{\frac{\delta+\xi+1}{\delta}}\left(2\sqrt{\nu\tau}\right)} - \frac{n\log(\nu)}{\delta} + \frac{n\log(\nu\tau)}{2\delta^2}, \\
\frac{\partial^2 L}{\partial \xi\, \partial \nu} &= \frac{n\tau\left(-\mathrm{K}_{\frac{\delta+\xi+1}{\delta}-1}\left(2\sqrt{\nu\tau}\right) - \mathrm{K}_{\frac{\delta+\xi+1}{\delta}+1}\left(2\sqrt{\nu\tau}\right)\right)\mathrm{K}^{(1,0)}\left(\frac{\delta+\xi+1}{\delta}, 2\sqrt{\nu\tau}\right)}{2\delta\sqrt{\nu\tau}\mathrm{K}_{\frac{\delta+\xi+1}{\delta}}\left(2\sqrt{\nu\tau}\right)^2} \\
&\quad - \frac{n\tau\mathrm{K}^{(1,1)}\left(\frac{\delta+\xi+1}{\delta}, 2\sqrt{\nu\tau}\right)}{\delta\sqrt{\nu\tau}K_{\frac{\delta+\xi+1}{\delta}}\left(2\sqrt{\nu\tau}\right)} + \frac{n}{2\delta\nu}, \\
\frac{\partial^2 L}{\partial \xi\, \partial \tau} &= \frac{n\nu\left(-\mathrm{K}_{\frac{\delta+\xi+1}{\delta}-1}\left(2\sqrt{\nu\tau}\right) - \mathrm{K}_{\frac{\delta+\xi+1}{\delta}+1}\left(2\sqrt{\nu\tau}\right)\right)\mathrm{K}^{(1,0)}\left(\frac{\delta+\xi+1}{\delta}, 2\sqrt{\nu\tau}\right)}{2\delta\sqrt{\nu\tau}\mathrm{K}_{\frac{\delta+\xi+1}{\delta}}\left(2\sqrt{\nu\tau}\right)^2} \\
&\quad - \frac{n\nu\mathrm{K}^{(1,1)}\left(\frac{\delta+\xi+1}{\delta}, 2\sqrt{\nu\tau}\right)}{\delta\sqrt{\nu\tau}\mathrm{K}_{\frac{\delta+\xi+1}{\delta}}\left(2\sqrt{\nu\tau}\right)} - \frac{n}{2\delta\tau}, \\
\frac{\partial^2 L}{\partial \delta^2} &= -\nu\frac{n\log(\nu\tau) - 2n\log(\nu) - n}{\delta^2} + \frac{n\left(\frac{1}{\delta} - \frac{\delta+\xi+1}{\delta^2}\right)^2\mathrm{K}^{(1,0)}\left(\frac{\delta+\xi+1}{\delta}, 2\sqrt{\nu\tau}\right)^2}{K_{\frac{\delta+\xi+1}{\delta}}\left(2\sqrt{\nu\tau}\right)^2} \\
&\quad - \frac{n\left(\frac{1}{\delta} - \frac{\delta+\xi+1}{\delta^2}\right)^2\mathrm{K}^{(2,0)}\left(\frac{\delta+\xi+1}{\delta}, 2\sqrt{\nu\tau}\right)}{K_{\frac{\delta+\xi+1}{\delta}}\left(2\sqrt{\nu\tau}\right)} - \frac{n(\delta+\xi+1)\log(\nu\tau)}{\delta^3} \\
&\quad - \frac{n\left(\frac{2(\delta+\xi+1)}{\delta^3} - \frac{2}{\delta^2}\right)\mathrm{K}^{(1,0)}\left(\frac{\delta+\xi+1}{\delta}, 2\sqrt{\nu\tau}\right)}{K_{\frac{\delta+\xi+1}{\delta}}\left(2\sqrt{\nu\tau}\right)} + \frac{2n(\delta+\xi+1)\log(\nu)}{\delta^3} \\
&\quad + \sum_{i=1}^{n} x_i^{\delta}\log^2(x_i) - \tau\sum_{i=1}^{n} x_i^{-\delta}\log^2(x_i)\,, \\
\frac{\partial^2 L}{\partial \delta\, \partial \nu} &= \frac{n\tau\left(\frac{1}{\delta} - \frac{\delta+\xi+1}{\delta^2}\right)\left(-K_{\frac{\delta+\xi+1}{\delta}-1}\left(2\sqrt{\nu\tau}\right) - K_{\frac{\delta+\xi+1}{\delta}+1}\left(2\sqrt{\nu\tau}\right)\right)\mathrm{K}^{(1,0)}\left(\frac{\delta+\xi+1}{\delta}, 2\sqrt{\nu\tau}\right)}{2\sqrt{\nu\tau}K_{\frac{\delta+\xi+1}{\delta}}\left(2\sqrt{\nu\tau}\right)^2} \\
&\quad - \frac{n\tau\left(\frac{1}{\delta} - \frac{\delta+\xi+1}{\delta^2}\right)\mathrm{K}^{(1,1)}\left(\frac{\delta+\xi+1}{\delta}, 2\sqrt{\nu\tau}\right)}{\sqrt{\nu\tau}K_{\frac{\delta+\xi+1}{\delta}}\left(2\sqrt{\nu\tau}\right)} - \sum_{i=1}^{n} x_i^{\delta}\log(x_i) - \frac{n(\delta+\xi+1)}{2\delta^2\nu} + \frac{n}{2\delta\nu},
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial^2 L}{\partial\delta\,\partial\tau} ={}& \frac{n\nu\left(\frac{1}{\delta}-\frac{\delta+\xi+1}{\delta^2}\right)\left(-K_{\frac{\delta+\xi+1}{\delta}-1}\left(2\sqrt{\nu\tau}\right)-K_{\frac{\delta+\xi+1}{\delta}+1}\left(2\sqrt{\nu\tau}\right)\right)\mathrm{K}^{(1,0)}\left(\frac{\delta+\xi+1}{\delta},2\sqrt{\nu\tau}\right)}{2\sqrt{\nu\tau}K_{\frac{\delta+\xi+1}{\delta}}\left(2\sqrt{\nu\tau}\right)^2} \\
&-\frac{n\nu\left(\frac{1}{\delta}-\frac{\delta+\xi+1}{\delta^2}\right)\mathrm{K}^{(1,1)}\left(\frac{\delta+\xi+1}{\delta},2\sqrt{\nu\tau}\right)}{\sqrt{\nu\tau}K_{\frac{\delta+\xi+1}{\delta}}\left(2\sqrt{\nu\tau}\right)}-\sum_{i=1}^{n}x_i^{-\delta}\left(-\log\left(x_i\right)\right)+\frac{n(\delta+\xi+1)}{2\delta^2\tau}-\frac{n}{2\delta\tau}\,, \\
\frac{\partial^2 L}{\partial\nu^2} ={}& -\frac{n(\delta+\xi+1)}{2\delta\nu^2}+\frac{n\tau^2\left(-K_{\frac{\delta+\xi+1}{\delta}-1}\left(2\sqrt{\nu\tau}\right)-K_{\frac{\delta+\xi+1}{\delta}+1}\left(2\sqrt{\nu\tau}\right)\right)}{4(\nu\tau)^{3/2}K_{\frac{\delta+\xi+1}{\delta}}\left(2\sqrt{\nu\tau}\right)} \\
&-\frac{n\tau\left(-\frac{\tau\left(-K_{\frac{\delta+\xi+1}{\delta}}\left(2\sqrt{\nu\tau}\right)-K_{\frac{\delta+\xi+1}{\delta}-2}\left(2\sqrt{\nu\tau}\right)\right)}{2\sqrt{\nu\tau}}-\frac{\tau\left(-K_{\frac{\delta+\xi+1}{\delta}}\left(2\sqrt{\nu\tau}\right)-K_{\frac{\delta+\xi+1}{\delta}+2}\left(2\sqrt{\nu\tau}\right)\right)}{2\sqrt{\nu\tau}}\right)}{2\sqrt{\nu\tau}K_{\frac{\delta+\xi+1}{\delta}}\left(2\sqrt{\nu\tau}\right)} \\
&+\frac{n\tau\left(-K_{\frac{\delta+\xi+1}{\delta}-1}\left(2\sqrt{\nu\tau}\right)-K_{\frac{\delta+\xi+1}{\delta}+1}\left(2\sqrt{\nu\tau}\right)\right)^2}{4\nu K_{\frac{\delta+\xi+1}{\delta}}\left(2\sqrt{\nu\tau}\right)^2}\,, \\
\frac{\partial^2 L}{\partial\nu\,\partial\tau} ={}& \frac{n\left(-K_{\frac{\delta+\xi+1}{\delta}-1}\left(2\sqrt{\nu\tau}\right)-K_{\frac{\delta+\xi+1}{\delta}+1}\left(2\sqrt{\nu\tau}\right)\right)^2}{4K_{\frac{\delta+\xi+1}{\delta}}\left(2\sqrt{\nu\tau}\right)^2} \\
&-\frac{n\left(-K_{\frac{\delta+\xi+1}{\delta}-1}\left(2\sqrt{\nu\tau}\right)-K_{\frac{\delta+\xi+1}{\delta}+1}\left(2\sqrt{\nu\tau}\right)\right)}{2\sqrt{\nu\tau}K_{\frac{\delta+\xi+1}{\delta}}\left(2\sqrt{\nu\tau}\right)} \\
&-\frac{n\tau\left(-\frac{\nu\left(-K_{\frac{\delta+\xi+1}{\delta}}\left(2\sqrt{\nu\tau}\right)-K_{\frac{\delta+\xi+1}{\delta}-2}\left(2\sqrt{\nu\tau}\right)\right)}{2\sqrt{\nu\tau}}-\frac{\nu\left(-K_{\frac{\delta+\xi+1}{\delta}}\left(2\sqrt{\nu\tau}\right)-K_{\frac{\delta+\xi+1}{\delta}+2}\left(2\sqrt{\nu\tau}\right)\right)}{2\sqrt{\nu\tau}}\right)}{2\sqrt{\nu\tau}K_{\frac{\delta+\xi+1}{\delta}}\left(2\sqrt{\nu\tau}\right)} \\
&+\frac{n\nu\tau\left(-K_{\frac{\delta+\xi+1}{\delta}-1}\left(2\sqrt{\nu\tau}\right)-K_{\frac{\delta+\xi+1}{\delta}+1}\left(2\sqrt{\nu\tau}\right)\right)}{4(\nu\tau)^{3/2}K_{\frac{\delta+\xi+1}{\delta}}\left(2\sqrt{\nu\tau}\right)}\,, \\
\frac{\partial^2 L}{\partial\tau^2} ={}& \frac{n(\delta+\xi+1)}{2\delta\tau^2}+\frac{n\nu^2\left(-K_{\frac{\delta+\xi+1}{\delta}-1}\left(2\sqrt{\nu\tau}\right)-K_{\frac{\delta+\xi+1}{\delta}+1}\left(2\sqrt{\nu\tau}\right)\right)}{4(\nu\tau)^{3/2}K_{\frac{\delta+\xi+1}{\delta}}\left(2\sqrt{\nu\tau}\right)} \\
&-\frac{n\nu\left(-\frac{\nu\left(-K_{\frac{\delta+\xi+1}{\delta}}\left(2\sqrt{\nu\tau}\right)-K_{\frac{\delta+\xi+1}{\delta}-2}\left(2\sqrt{\nu\tau}\right)\right)}{2\sqrt{\nu\tau}}-\frac{\nu\left(-K_{\frac{\delta+\xi+1}{\delta}}\left(2\sqrt{\nu\tau}\right)-K_{\frac{\delta+\xi+1}{\delta}+2}\left(2\sqrt{\nu\tau}\right)\right)}{2\sqrt{\nu\tau}}\right)}{2\sqrt{\nu\tau}K_{\frac{\delta+\xi+1}{\delta}}\left(2\sqrt{\nu\tau}\right)} \\
&+\frac{n\nu\left(-K_{\frac{\delta+\xi+1}{\delta}-1}\left(2\sqrt{\nu\tau}\right)-K_{\frac{\delta+\xi+1}{\delta}+1}\left(2\sqrt{\nu\tau}\right)\right)^2}{4\tau K_{\frac{\delta+\xi+1}{\delta}}\left(2\sqrt{\nu\tau}\right)^2}\,,
\end{aligned}
$$

where

$$\begin{aligned} K^{(0,n)}(\nu,z) &= \frac{\partial^n K}{\partial z^n}(\nu,z) \\ &= \left(-\frac{1}{2}\right)^n \sum_{k=0}^{n} \binom{n}{k} K(2k-n+\nu,z) \\ &\qquad \text{for } n \in \mathbb{Z} \text{ and } n \geq 0\,, \end{aligned} \tag{3.9}$$

$$\begin{aligned} \mathrm{K}^{(m,0)}(\nu,z) &= \frac{\partial^m K}{\partial \nu^m} \\ &= \frac{1}{2}\pi \sum_{k=0}^{\infty} \frac{2^{-2k} z^{2k} \frac{\partial^m}{\partial \nu^m} \left(\csc(\pi\nu)\left(\frac{2^{\nu} z^{-\nu}}{\Gamma(k-\nu+1)} - \frac{2^{-\nu} z^{\nu}}{\Gamma(k+\nu+1)}\right)\right)}{k!}, \\ &\qquad m \in \mathbb{Z}\ .m \geq 0,\ \nu \in \mathbb{Z}. \end{aligned} \tag{3.10}$$

3.5 Proposed Maximization Methodology

Parameter estimation for a multi-parameter density function such as the $\mathcal{EIG}$ model can be challenging. The command *NMaximize*, available in *Mathematica*, attempts to find a global maximum subject to certain constraints. In the context of maximum likelihood estimation, this command requires setting an interval for each parameter to determine a region within which *Mathematica* seeks the global maximum. Since the region where the global maximum lies has yet to be determined, it is helpful to have a methodology that can specify the appropriate parameter intervals to be used in conjunction with the command *NMaximize*. Such an iterative methodology is proposed in this section. Accordingly, the parameter intervals are initially chosen to be very wide, with such intervals containing at the very least five points. Then, the log-likelihood function is evaluated for all the possible combinations of the points specified within these intervals. Next, we consider potential candidates for the maximum value and their corresponding regions. New intervals are chosen based only on the m highest values of the log-likelihood so that narrower intervals with finer grids can be set, where m is the fourth root (four being for instance the number of parameters to be estimated) of the number of the log-likelihood values that were evaluated at the previous step . These steps are repeated until a global maximum can be identified. The *Mathematica* code used for applying this methodology is given below. (A complete numerical example is included in the Appendix.)

```
vt2 = Table[Evaluate[LogLikelihood[ξ, δ, ν, τ]], {ξ, −10, 10, .5}, {δ, .1, 10, .5},
{ν, .1, 10, .5}, {τ, .1, 10, .5}];
vtc = Table[{ξ, δ, ν, τ}, {ξ, −10, 10, .5}, {δ, .1, 10, .5}, {ν, .1, 10, .5}, {τ, .1, 10, .5}];
Max[vt2]
ps = Position[vt2, Max[vt2]]
vm = vtc[[ps[[1, 1]], ps[[1, 2]], ps[[1, 3]], ps[[1, 4]]]]
Evaluate[LogLikelihood[vm[[1]], vm[[2]], vm[[3]], vm[[4]]]]
```

$$\text{rt} = \text{Floor}\left[\left(\prod_{j=1}^{\text{Length[Dimensions[vt2]]}} \text{Dimensions[vt2][[}j\text{]]}\right)\wedge\left(\frac{1}{\text{Length[Dimensions[vt2]]}}\right)\right]$$

```
tb = Table[−Sort[Flatten[−vt2]][[j]], {j, 1, rt}]
psv = Flatten[Table[Position[vt2, tb[[j]]], {j, 1, rt}], 1]
Table[vtc[[psv[[j, 1]], psv[[j, 2]], psv[[j, 3]], psv[[j, 4]]]], {j, 1, rt}]
```

More specifically, this methodology was applied to the flood data, which is modeled in the next section. The resulting maximum value of the loglikelihood turned out to be larger than that corresponding to the best model in Table 3.2.

3.6 Numerical Examples

The following density functions, all related to the $\mathcal{EIG}$ model, will be considered. The gamma density function which is given by

$$f(x) = \frac{x^{\theta-1} e^{-x/\phi}}{\phi^{\theta}\,\Gamma(\theta)}\,\mathcal{I}_{\Re^+}(x), \quad \theta > 0,\ \phi > 0\,, \tag{3.11}$$

is clearly a particular case of the $\mathcal{EIG}$ density as specified by (3.2) with $\xi = \theta - 2$, $\delta = 1$, $\nu = 1/\phi$ and $\tau = 0$. On letting $\xi = -5/2$, $\delta = 1$, $\nu = \lambda/(2\mu^2)$ and $\tau = \lambda/2$ in (3.2), the inverse Gaussian distribution with parameters $\mu \in \Re$ and $\lambda > 0$ whose density is given in Chapter 1, is also seen to be a special case of the $\mathcal{EIG}$ distribution. The reparameterized generalized gamma $\mathcal{RGG}$ density as given in (3.4) can be obtained from the $\mathcal{EIG}$ model by letting $\tau = 0$ in (3.2). The $\mathcal{EIG}$ density reduces to the Weibull density function,

$$f(x) = \theta\,\phi\,x^{\phi-1} e^{-\theta\,x^{\phi}}\,\mathcal{I}_{\Re^+}(x)\,,\ \ \theta > 0,\ \phi > 0\,, \tag{3.12}$$

with the substitutions, $\delta = \phi$, $\tau = 0$ $\nu = \theta$ and $\xi = -1$ in (3.2). The relationship between the $\mathcal{GIG}$ density, as given in (3.1), and the $\mathcal{EIG}$ density function is specified in Section 3.1. Finally, the $\mathcal{REIG}$ model as defined by the density (3.3) is obtained by letting $\nu = 0$ in (3.2). Two data sets were fitted with each one of these models as well as the lognormal distribution, and the resulting parameter estimates and goodness-of-fit statistics were tabulated. Several of the fitted cumulative distribution functions are graphically displayed along the empirical cumulative distribution functions for comparison purposes.

3.6.1 Maximum Flood Level Data

Consider the data set presented in Table 3.1. This data which was studied by Dumonceaux and Antle (1973), consists of maximum flood levels (in millions cubic of feet per second) of the Susquehanna River at Harrisburg, Pennsylvania, observed over 20 four-year periods.

Table 3.1: Maximum Flood Level Data

.654	.613	.402	.379	.269	.740	.416	.338	.315	.449
.297	.423	.379	.3235	.418	.412	.494	.392	.484	.265

This data was fitted to several distributions including those specified by (3.2) and (3.3). We made use of the symbolic computing package *Mathematica* in conjunction with the command *NMaximize* applied to the loglikelihoods to estimate the parameters. This command always attempts to find a global maximum subject to certain constraints. In this case, such constraints are specified by inequalities that certain functions of the parameters should satisfy and intervals within which the parameters can vary. The determination of such intervals was guided by the parameter estimates obtained for the reduced models. The results are presented in Table 3.2. For comparison purposes, the lognormal model whose parameters estimates were found to be $\hat{\mu} = -0.8978$ and $\sigma = 0.2692$, was also considered. It can be seen that the proposed $\mathcal{EIG}$ model and its reduced version provide a better fit than that resulting from the other models. Figures 3.6 and 3.7 show the cumulative distribution functions of the lognormal, $\mathcal{RGG}$, $\mathcal{REIG}$ and $\mathcal{EIG}$ models superimposed on the empirical cumulative distribution function. Admittedly, the $\mathcal{EIG}$ and $\mathcal{REIG}$ models fit the data nearly equally well in this case. However, it should be noted that the sample size is minute and that only scant data is available in the tails of the distribution, which apparently precludes taking full advantage of the additional parameter in this instance.

Table 3.2: Parameter Estimates and A_0^2 & W_0^2 for the Flood Data

	$\hat{\xi}$	$\hat{\delta}$	$\hat{\nu}$	$\hat{\tau}$	A_0^2	W_0^2
Weibull	-1	3.5260	14.450	0	.8213	0.1400
Gamma	11	1	30.769	0	0.4433	0.0712
Inverse Gaussian	-2.5	1	15.745	2.8195	0.3514	0.0558
Lognormal					0.3470	0.0540
$\mathcal{RGG}$	339.113	0.0364	9600	0	0.3390	0.0560
$\mathcal{GIG}$	-16.567	1	0.005	5.7343	0.2861	0.0449
$\mathcal{REIG}$	-10	2.3	0	0.3108	0.2567	0.0436
$\mathcal{EIG}$	-9.95	2.24	0.09	0.34	0.2551	0.0437

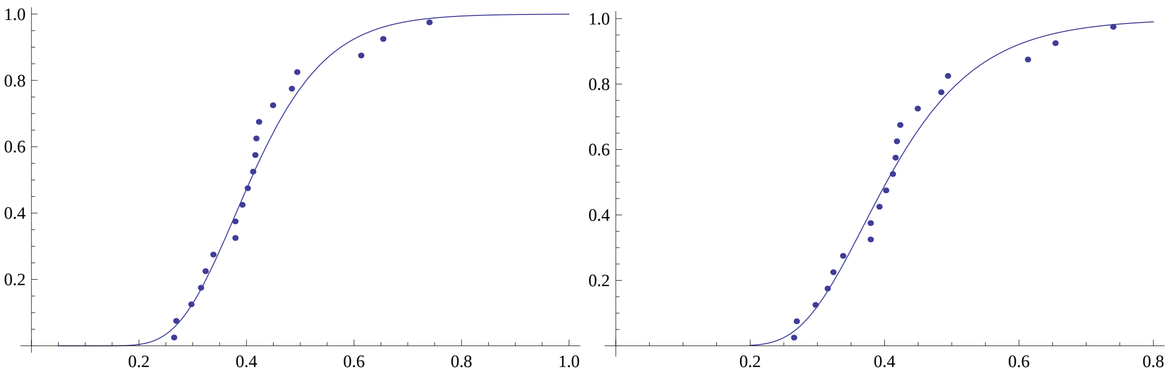

Figure 3.6: *CDF (solid line) and empirical CDF (dots) for the flood data set. Left panel: Lognormal; Right panel:* $\mathcal{GIG}$.

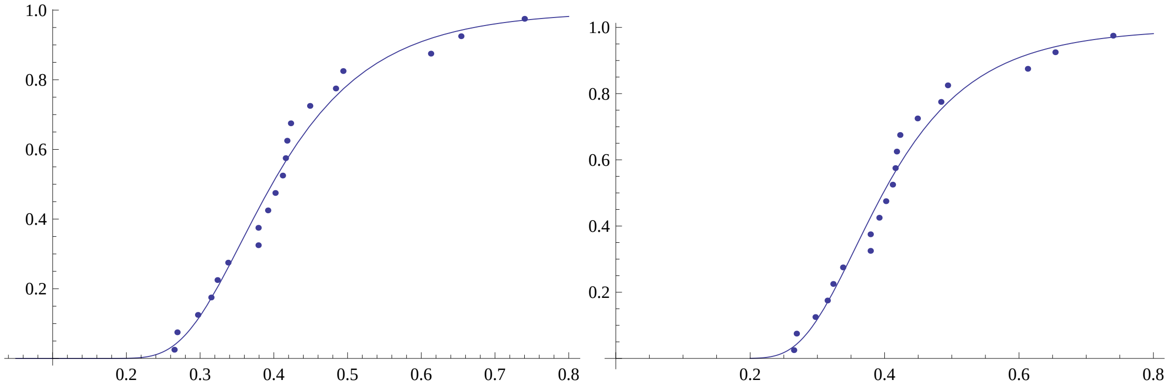

Figure 3.7: *CDF (solid line) and empirical CDF (dots) for the flood data set. Left panel:* $\mathcal{EIG}$*; Right panel:* $\mathcal{REIG}$.

3.6.2 Snowfall Precipitations in Buffalo

The same models are now fitted to the Buffalo snowfall data set, as given in Table 3.3 (available for instance from the S-PLUS data library). This set comprises a record of the annual snowfall precipitations in centimeters over 63 consecutive years in the city of Buffalo. It can be seen from Table 3.4 that the $\mathcal{EIG}$ distribution provides the best fit. In this case, the goodness-of-fit measures indicate that a close fit can also be obtained by making use of the $\mathcal{RGG}$ distribution. This is corroborated by the graphs of the cumulative distribution functions superimposed on the empirical cumulative distribution function (Figures 3.8 and 3.9). Again, the lognormal was considered as an alternative model. In this case, referring to Table 4.3, the $\mathcal{EIG}$ model clearly produces a superior fit as compared to the $\mathcal{REIG}$ model. Note that the parameter estimates of the lognormal model were found to be $\mu = -4.3368$ and $\sigma = .3270$.

Table 3.3: The Snowfall Precipitation Data

25	39.8	39.9	40.1	46.7	49.1	49.6	51.2	51.6	53.5	54.7
55.5	55.9	58	60.3	63.6	65.4	66.1	69.3	70.9	71.4	71.5
71.8	72.9	74.4	76.2	77.8	78.2	78.4	79	79.3	79.6	80.7
82.4	82.4	83	83.6	83.6	84.8	85.5	87.4	88.7	89.6	89.8
89.9	90.9	97.	98.3	101.4	102.4	103.9	104.5	105.2	110	110.5
110.5	113.7	114.5	115.6	120.5	120.7	124.7	126.4			

Table 3.4: Parameter Estimates and A_0^2 & W_0^2 for the Snowfall Data

	$\hat{\xi}$	$\hat{\delta}$	$\hat{\nu}$	$\hat{\tau}$	A_0^2	W_0^2
Inverse Gaussian	−2.5	1	0.0548	353.261	0.8676	0.1504
Lognormal					0.7752	0.1284
$\mathcal{REIG}$	−53.66	0.1671	0	650.2	0.7417	0.0886
Gamma	8	1	0.124536	0	0.4840	0.0792
$\mathcal{GIG}$	7.97	1	0.1219	0.0025	0.4291	0.0532
Weibull	−1	3.8338	3.37×10^{-8}	0	0.2964	0.0454
$\mathcal{RGG}$	1.2889	3.629	9.31×10^{-8}	0	0.2817	0.0428
$\mathcal{EIG}$	-0.0557	3.144	1.03×10^{-6}	1.743	0.2625	0.0403

3.6.3 Breaking Stress Data

The same models are now fitted to a data set, which is presented in Table 3.5 and was obtained from Nicholas and Padgett (2006) of carbon fibres (in Gba). In addition confidence intervals for the estimated parameters have been found using the observed information matrix. In addition to the Anderson-Darling and the Cramér-von Mises statistics, the AIC (Akaike Information Criterion), BIC (Bayesian Information Criterion) and HQIC (Hannan-Quinn Information Criterion) have been used to assess the fit of a given distribution with the Breaking Stress data set in order to compare our models The parameter estimates and Anderson-Darling and Cramér-von Mises statistics are given in Table 3.6. The AIC, BIC and HQIC statistics are tabulated in Table 3.7. According to these five of goodness-of-fit criteria, the best fit is obtained with $\mathcal{EIG}$ model.

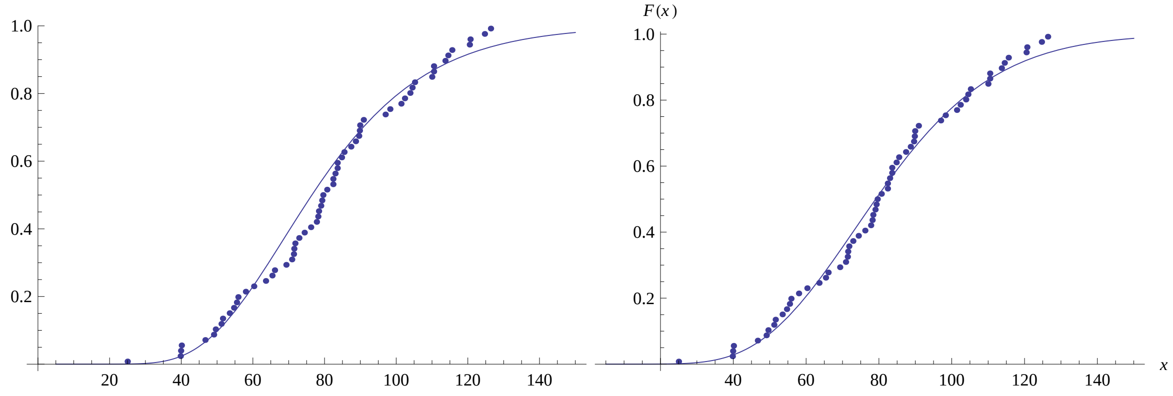

Figure 3.8: *CDF (solid line) and empirical CDF (dots) for the snowfall data set. Left panel: Lognormal; Right panel: $\mathcal{GIG}$.*

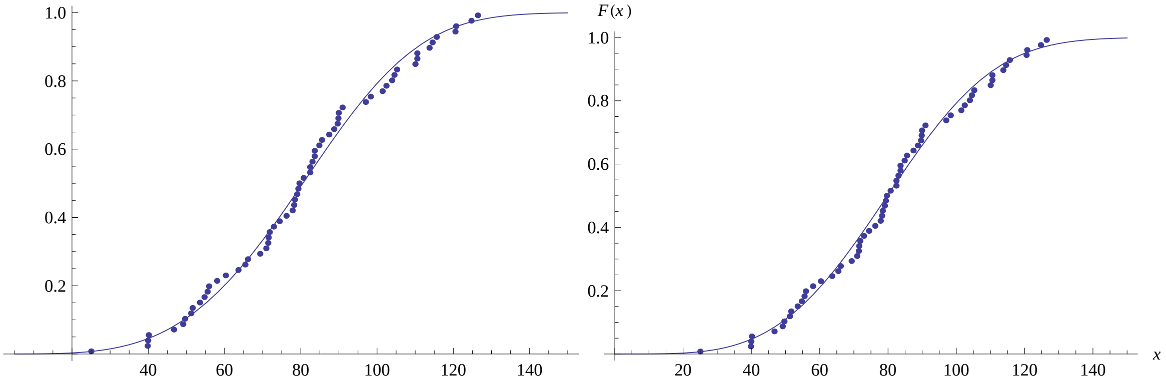

Figure 3.9: *CDF (solid line) and empirical CDF (dots) for the snowfall data set. Left panel: $\mathcal{RGG}$; Right panel: $\mathcal{EIG}$.*

Table 3.5: The Breaking Stress Data

3.7	2.74	2.73	2.5	3.6	3.11	3.27	2.87	1.47	3.11
4.42	2.41	3.19	3.22	1.69	3.28	3.09	1.87	3.15	4.9
3.75	2.43	2.95	2.97	3.39	2.96	2.53	2.67	2.93	3.22
3.39	2.81	4.2	3.33	2.55	3.31	3.31	2.85	2.56	3.56
3.15	2.35	2.55	2.59	2.38	2.81	2.77	2.17	2.83	1.92
1.41	3.68	2.97	1.36	.98	2.76	4.91	3.68	1.84	1.59
3.19	1.57	.81	5.56	1.73	1.59	2	1.22	1.12	1.71
2.17	1.17	5.08	2.48	1.18	3.51	2.17	1.69	1.25	4.38
1.84	.39	3.68	2.48	.85	1.61	2.79	4.7	2.03	1.8
1.57	1.08	2.03	1.61	2.12	1.89	2.88	2.82	2.05	3.65

Table 3.6: Parameter Estimates and A_0^2 & W_0^2 for the Breaking Stress Data

	$\hat{\xi}$	$\hat{\delta}$	$\hat{\nu}$	$\hat{\tau}$	A_0^2	W_0^2
$\mathcal{REIG}$	−111.967	0.0453	0	2543.7	1.51096	0.2831
$\mathcal{RGG}$	−0.1875	2.3428	0.1141	0	0.4091	0.0692
$\mathcal{EIG}$	−0.1875	2.3428	0.1142	0.0040	0.0692	0.0692

Table 3.7: Parameter Estimates and A_0^2 & W_0^2 for the Breaking Stress Data

	AIC	*BIC*	HQIC
$\mathcal{REIG}$	303.526	311.341	306.688
$\mathcal{RGG}$	288.692	296.507	291.855
$\mathcal{EIG}$	288.687	296.502	291.85

Chapter 4

The Distribution of Weighted Sums of Chi-square Random Variables

4.1 Introduction

The distribution of linear combinations of chi-square random variables and that of quadratic forms in normal vectors which can be expressed as weighted sums of chi-square random variables, have already received much attention in the statistical literature. Box (1954) considered a linear combination of chi-square variables having even degrees of freedom. Some representations of the density function of linear combinations of chi-square variables were derived by Mathai and Saxena (1978).

It is pointed out in Szatrowski (1979) that the null distribution of certain likelihood ratio tests on mean vectors and covariance matrices arising in multivariate analysis can be approximated by linear combinations of chi-square random variables, see for instance Willks (1946), Votaw (1948)and Gleser and Olkin (1966, 1969).

Various representations of the distribution function of a quadratic form are available, and several procedures have been proposed for computing percentage points and preparing tables. Gurland (1948, 1953, 1956), Pachares (1955), Ruben (1960, 1962), Shah and Khatri (1961), and Kotz *et al.* (1967a,b) among others, have given representations of the distribution function of quadratic forms in terms of MacLaurin series and the distribution function of chi-square variables. Gurland (1956) and Shah (1963) considered respectively central and noncentral indefinite quadratic forms, but as pointed by Shah (1963), the expansions obtained are not practical. Various representations of the exact density and distribution functions of indefinite quadratic forms have been given by Imhof (1961), Davis (1973) and Rice (1980).

As pointed out in Mathai and Provost (1992), a wide array of statistics can be expressed in terms of quadratic forms in normal random vectors. For example, one may consider the lagged regression residuals developed by De Gooijer and MacNeill (1999) and discussed in Provost *et al.* (2005), or certain change point test statistics derived by MacNeill (1978).

Hillier (2001) considered ratios of quadratic forms in normal random variables and expressed their density functions in terms of top-order zonal polynomials involving difference quotients of the characteristic roots of the matrix in the numerator quadratic form. The sample serial correlation coefficient as defined in Anderson (1990) and discussed in Provost and Rudiuk (1995) as well as the sample innovation cross-correlation function for an ARMA time series whose asymptotic distribution was derived by McLeod (1979) have such a structure.

Abadir and Larsson (1996, 2001) derived the exact finite-sample joint moment generating functions of the three quadratic forms constituting the sufficient statistics of a discrete multivariate Gaussian autoregressive process of order one. Phillips (1978) considered the comparative performance of two-well known approximation techniques in the case of the coefficient estimator in the first order noncircular autoregression model, and Jeong (1985) developed a new approximation of the critical point of the Durbin-Watson statistic for testing for autoregressive disturbances in the linear regression model with a lagged dependent variable.

Monte Carlo simulations, whereby artificial data are generated and sampling distributions and moments then are estimated, can be easily implemented on an extensive array of models. These simulations may, however, result in some limitations such as sampling variations and simulation inadequacies, and their results may be specific to the set of parameter values assumed in the simulations. Hendry and Harrison (1974), Dempster *et al.* (1977), Hendry (1979), and Hendry and Mizon (1980) among others, attempted to cope with these issues.

On the other hand, the analytical approach derives results which hold over the entire parameter space but may find some limitations in terms of simplifications on the model which are imposed to render the problem tractable. The analytical approach has been applied to various statistics involving quadratic forms. Examples include certain heteroscedastic models studied by Taylor (1977, 1978), the first-order autoregressive process considered by Sawa (1978) and Phillips (1977, 1978), the regression models analyzed by Dwivedi and Srivastava (1979), a linear model with unknown covariance structure studied by Yamamoto (1979), as well as the Bayesian analysis of simultaneous equations models carried out by Zellner (1971) and Dreze (1976).

A representation of the density function of a linear combination of independently distributed chi-square random variables is obtained by means of inverse Mellin transform technique in Section 4.2. The connection between a linear combination of independently distributed chi-square random variables and central quadratic forms is explained in Section 4.3. This representation of the density is utilized to calculate the distribution function of certain weighted sums of chi-square random variables at certain percentiles of the distribution in Section 4.4. The results are compared with those obtained by making use of the $\mathcal{RGG}$ model.

4.2 Derivation of the Density Function

Let $S = \sum_{j=1}^{k} m_j X_j$ where $m_j > 0$ and X_j are independently distributed chi-square random variables having r_j degrees of freedom each, $j = 1, \ldots, k$. We will determine the density function of S by applying the inverse Mellin transform technique as described in Chapter 1. First, we obtain a representation of $E(S^{-h})$:

$$E(S^{-h}) = \int_0^\infty \cdots \int_0^\infty s^{-h} \prod_{j=1}^{k} \frac{x_j^{r_j/2-1} e^{-x_j/2}}{2^{r_j/2}\,\Gamma(r_j/2)} \mathrm{d}x_j\,, \tag{4.1}$$

where $s = \sum_{j=1}^{k} m_j x_j$. But since

$$s^{-h} = \{\Gamma(h)\}^{-1} \int_0^\infty t^{h-1} e^{-st} \mathrm{d}t \quad \text{for } \Re(h > 0),\ s > 0 \tag{4.2}$$

where $\Re(.)$ denotes the real part of (.), we have

$$\begin{aligned} E(S^{-h}) &= \{\Gamma(h)\}^{-1} \int_0^\infty \cdots \int_0^\infty \int_0^\infty t^{h-1} e^{-t(m_1 x_1 + \cdots + m_k x_k)} \left(\prod_{j=1}^{k} \frac{x_j^{r_j/2-1} e^{-x_j/2}}{2^{r_j/2}\,\Gamma(r_j/2)} \mathrm{d}x_j \right) \mathrm{d}t \\ &= \{\Gamma(h)\}^{-1} \int_0^\infty t^{h-1} \prod_{j=1}^{k} \int_0^\infty \left\{ \frac{x_j^{r_j/2-1} e^{-x_j(m_j t+1/2)}}{2^{r_j/2}\,\Gamma(r_j/2)} \mathrm{d}x_j \right\} \mathrm{d}t. \end{aligned} \tag{4.3}$$

Noting that for $i = 1, \ldots, k$,

$$\int_0^\infty \{\Gamma(\frac{r_j}{2})\}^{-1} (x_j^{r_j/2-1} e^{-x_j(m_j t+1/2)})\, \mathrm{d}x_j = (m_j t + 1/2)^{-r_j/2} \tag{4.4}$$

(4.3) becomes

$$\begin{aligned} E(S^{-h}) &= \{\Gamma(h)\}^{-1} \int_0^\infty t^{h-1} \left\{ \prod_{j=1}^{k} \frac{(m_j t + 1/2)^{-r_j/2}}{2^{r_j/2}} \right\} \mathrm{d}t \\ &= \{\Gamma(h)\}^{-1} \int_0^\infty t^{h-1} \left\{ \prod_{j=1}^{k} (2m_j t + 1)^{-r_j/2} \right\} \mathrm{d}t. \end{aligned} \tag{4.5}$$

Now letting $u = 1/(1+t)$, that is, $t = (1-u)/u$, one has $|\mathrm{d}t/\mathrm{d}u| = 1/u^2$, and (4.5) becomes

$$\begin{aligned} E(S^{-h}) &= \{\Gamma(h)\}^{-1} \prod_{j=1}^{k} (2m_j t)^{-r_j/2} \times \\ &\quad \int_0^1 u^{\rho-h-1} (1-u)^{h-1} \prod_{j=1}^{k} \left\{ 1 - u \frac{2m_j - 1}{2m_j} \right\}^{-r_j/2} \mathrm{d}u \end{aligned} \tag{4.6}$$

where $\rho = (r_1 + r_2 + \cdots + r_k)/2$. Let $\rho - h = q$ and $\gamma_j = (2m_j - 1)/(2m_j)$; then

$$\begin{aligned} E(S^{-h}) &= \left\{ \prod_{j=1}^{k} \frac{(2m_j)^{-r_j/2}\Gamma(\rho)}{\Gamma(\rho)} \right\} \\ &\quad \times \left[\frac{\Gamma(\rho)}{\Gamma(q)\Gamma(p-q)} \int_0^1 u^{q-1} (1-u)^{\rho-q-1} \prod_{j=1}^{k} (1-\gamma_j u)^{-r_j/2}\, \mathrm{d}u \right] \\ &= \left\{ \prod_{j=1}^{k} \frac{(2m_j)^{-r_j/2}\Gamma(\rho)}{\Gamma(\rho)} \right\} F_D\left(q; \frac{r_1}{2}, \ldots, \frac{r_k}{2}; \gamma_1, \ldots, \gamma_k \right), \end{aligned} \tag{4.7}$$

provided $\Re(q) = \Re(\rho - h) > 0$, $\Re(p - q) = \Re(h) > 0$, and $|\gamma_j| = |(2m_j - 1)/(2m_j)| < 1$, that is, $m_j > 1/4$ for $j = 1, \ldots, k$, where $F_D(.)$ denotes Lauricella's hypergeometric function of k variables (see Mathai and Saxena 1978, p. 162). We may now express (4.7) as an infinite series:

$$\begin{aligned} E(S^{-h}) &= \left\{ \prod_{j=1}^{k} \frac{(2m_j)^{-r_j/2}\Gamma(\rho)}{\Gamma(\rho)} \right\} \\ &\quad \times \sum_{\nu=0}^{\infty} \sum_{\nu_1+\cdots+\nu_k=\nu} \frac{(q)_\nu}{(\rho)_\nu} \left\{ \left(\frac{r_1}{2}\right)_{\nu_1} \cdots \left(\frac{r_k}{2}\right)_{\nu_k} \right\} \left(\frac{\gamma_1^{\nu_1} \cdots \gamma_k^{\nu_k}}{\nu_1! \ldots \nu_k!} \right) \equiv \gamma, \end{aligned} \tag{4.8}$$

where $(\theta)_\nu = \Gamma(\theta + \nu)/\Gamma(\theta)$,

$$\sum_{\nu_1+\ldots+\nu_k=\nu} f(\nu_1, \ldots, \nu_k) = \sum_{\nu_1=0}^{\nu} \sum_{\nu_2=0}^{\nu-\nu_1} \cdots \sum_{\nu_{k-1}=0}^{\nu-\nu_1-\cdots-\nu_{k-2}} f(\nu_1, \cdots, \nu_{k-1}, \nu - \nu_1 - \cdots - \nu_{k-1})$$

this series being convergent for $|\gamma_j| < 1$, that is, for $m_j > 1/4, j = 1, \ldots, k$. If the condition $m_j > 1/4$ is not satisfied for $j = 1, \ldots, k$, then we multiply m_j by a scalar quantity B chosen such that $(Bm_j) > 1/4$ for all j.

Let $T = S^{-k}$; then $E(T^h) = E\{(S^{-kh})\}$. We can therefore obtain the hth moment of T upon substituting hk for h in (4.8). Let $b = \rho - hk$ where $0 < \Re(hk) < \rho$ and $|\gamma_j| = |(2m_j-1)/2m_j| < 1$. Noting that

$$\Gamma(\rho)(\rho)_\nu = \Gamma(\rho + \nu) \tag{4.9}$$

and that

$$\begin{aligned} \Gamma(b)(b)_\nu &= \Gamma(b + \nu) = \Gamma(\rho - hk + \nu) = \Gamma(k(\frac{\rho}{k} + \frac{\nu}{k} - h)) \\ &= (2\pi)^{(1-k)/2} k^{\rho+\nu-hk-1/2} \prod_{i=0}^{k-1} \Gamma(\frac{\rho + \nu + 1}{k} - h), \end{aligned} \tag{4.10}$$

by the Gauss-Legendre multiplication formula:

$$\Gamma(mz) = (2\pi)^{(1-m)/2} m^{mz-1/2} \prod_{j=0}^{m-1} \Gamma(z + \frac{j}{m}), \tag{4.11}$$

one has

$$E(T^h) = \sum_{\nu=0}^{\infty} \sum_{\nu_1+\ldots+\nu_k=\nu} C_{\nu_1\ldots\nu_k} \prod_{l=1}^{k} \Gamma(\frac{\rho + \nu + l - 1}{k} - h) \Big(\frac{1}{k}\Big)^{hk} \tag{4.12}$$

for $0 < \Re(h) < \rho/k$ and $m_j > 1/4,\ j = 1 \ldots, k$, where

$$C_{\nu_1\ldots\nu_k} = \frac{(2\pi)^{(1-k)/2} k^{\rho+\nu-1/2} (\frac{r_1}{2})_{\nu_1} \cdots (\frac{r_k}{2})_{\nu_k} (\gamma_1^{\nu_1} \cdots \gamma_k^{\nu_k})}{\prod_{i=1}^{k} \{(2m_i)^{r_i/2}\} \Gamma(\rho + \nu)(\nu_1! \ldots \nu_k!)}. \tag{4.13}$$

From the uniqueness of the inverse Mellin transform, it is seen that the moment expression in (4.12) uniquely determines $g(T)$, the density function of T, where $T = 1/S^k$. Hence

$$g(t) = (2\pi)^{-1} \int_{c-i\infty}^{c+i\infty} E(T^h) t^{-(h+1)} \mathrm{d}h, \tag{4.14}$$

where $i = (-1)^{1/2}$. Since the infinite series in Equation (4.2.12) is uniformly convergent within its radius of convergence $(m_j > 1/4)$, $j = 1, \ldots, k$, the density of T may be written as follows:

$$\begin{aligned} g(t) &= \sum_{\nu=0}^{\infty} \sum_{\nu_1+\ldots+\nu_k=\nu} t^{-1} C_{\nu_1 \ldots \nu_k} \\ &\quad \times \left\{ (2\pi)^{-1} \int_{c-i\infty}^{c+i\infty} \prod_{\ell=1}^{k} \Gamma(1 - a_\ell - h)(\theta t)^{-h} \mathrm{d}h \right\}, \\ &= \sum_{\nu=0}^{\infty} \sum_{\nu_1+\cdots+\nu_k=\nu} t^{-1} C_{\nu_1 \ldots \nu_k} \mathcal{G}_{k,0}^{0,k}\left(\theta t \,\middle|\, \begin{matrix} a_1, \ldots, a_k \\ 0, \ldots, 0 \end{matrix} \right) \end{aligned} \tag{4.15}$$

where $1 - a_j = (\rho + \nu + l - 1)/k$, $\theta = k^k$, $C_{\nu_1 \ldots \nu_k}$is given in (4.13) and $\mathcal{G}_{k,0}^{0,k}$ is the Meijer's G-function which is defined for $0 < |\theta T| < 1$. Since $|dt/ds| = ks^{-(k+1)}$ and $0 < t < 1/\theta$ implies that $\theta^{1/k} < s < \infty$, the density of R is therefore

$$\phi(s) = ks^{-(k+1)} g(1/s^k) \tag{4.16}$$

Using (4.16) and the following identity:

$$\mathcal{G}_{p,q}^{m,n}\left(x \,\middle|\, \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} \right) = \mathcal{G}_{q,p}^{n,m}\left(\frac{1}{x} \,\middle|\, \begin{matrix} 1 - b_1, \ldots, 1 - b_q \\ 1 - a_1, \ldots, 1 - a_p \end{matrix} \right), \tag{4.17}$$

we have

$$\phi(s) = ks^{-(k+1)} \sum_{\nu=0}^{\infty} \sum_{\nu_1+\ldots+\nu_k=\nu} s^k C_{\nu_1 \ldots \nu_k} \mathcal{G}_{0,k}^{k,0}\left(\frac{s^k}{\theta} \,\middle|\, \begin{matrix} 0, \ldots\ldots, 0 \\ 1 - a_1, \ldots, 1 - a_k \end{matrix} \right) \tag{4.18}$$

which becomes

$$\phi(s) = ks^{-1} \sum_{\nu=0}^{\infty} \sum_{\nu_1+\ldots+\nu_k=\nu} C_{\nu_1 \ldots \nu_k} \frac{(2\pi)^{\frac{k-1}{2}} (\frac{s^k}{\theta})^{\frac{\rho+\nu}{k}} e^{-k(\frac{s^k}{\theta})^{\frac{1}{k}}}}{\sqrt{k}} \tag{4.19}$$

using the identity

$$\mathcal{G}_{0,k}^{k,0}\left(z \,\middle|\, \begin{matrix} 0, \ldots\ldots, 0 \\ b, b + \frac{1}{m} \ldots, b + \frac{m-1}{m} \end{matrix} \right) = \frac{(2\pi)^{\frac{m-1}{2}} z^b e^{-mz^{\frac{1}{m}}}}{\sqrt{m}} \tag{4.20}$$

The last identity was obtained from the *Mathematica* website, functions.wolfram.com, with reference number 07.34.03.1081.01.

The density of s can be simplified to

$$\begin{aligned} \phi(s) &= \sum_{\nu=0}^{\infty} \sum_{\nu_1+\ldots+\nu_k=\nu} \frac{k^{\nu+\rho} \theta^{-\frac{\nu+\rho}{k}} (\frac{r_1}{2})_{\nu_1} \ldots (\frac{r_k}{2})_{\nu_k} (\gamma_1^{\nu_1} \ldots \gamma_k^{\nu_k}) s^{\nu+\rho-1} e^{-s(k\theta^{-1/k})}}{\left\{ \prod_{i=1}^{k} (2m_i)^{r_i/2} \right\} \Gamma(\rho + \nu)(\nu_1! \ldots \nu_k!)} \quad (4.21) \\ &= \sum_{\nu=0}^{\infty} \sum_{\nu_1+\ldots+\nu_k=\nu} \frac{(\frac{r_1}{2})_{\nu_1} \ldots (\frac{r_k}{2})_{\nu_k} (\gamma_1^{\nu_1} \ldots \gamma_k^{\nu_k})}{\left\{ \prod_{i=1}^{k} (2m_i)^{r_i/2} \right\} \Gamma(\rho + \nu)(\nu_1! \ldots \nu_k!)} s^{\nu+\rho-1} e^{-s} \quad (4.22) \end{aligned}$$

since $\theta = k^k$. This representation which is a mixture of gamma densities with $\beta = 1$ is suitable when $\frac{Var(S)}{E(S)} = 1$. Let Y be a linear combination of chi-square random variables such that $\frac{Var(S)}{E(S)} = \beta \neq 1$, then let $Y = \beta S$, where $\beta = Var(\sum_{i=1}^{k} m_j X_j)/E(\sum_{i=1}^{k} m_j X_j) = \sum_i^k 2r_i m_i^2 / \sum_i^k r_i m_i$, one has the following representation of the density of Y:

$$\phi(y) = \sum_{\nu=0}^{\infty} \sum_{\nu_1+\ldots+\nu_k=\nu} \frac{(\frac{r_1}{2})_{\nu_1} \cdots (\frac{r_k}{2})_{\nu_k} (\gamma_1^{\nu_1} \cdots \gamma_k^{\nu_k})\, y^{\nu+\rho-1} e^{-y/\beta}}{\left\{\prod_{i=1}^{k} (2m_i)^{r_i/2}\right\} \Gamma(\rho+\nu)(\nu_1! \ldots \nu_k!)\beta} \frac{\sum_i^k r_i m_i}{\sum_i^k 2r_i m_i^2}. \tag{4.23}$$

4.3 Connection to Central Quadratic Forms

We now show that central quadratic forms can be expressed as linear combinations of central chi-square random variables.

Let $\mathbf{X} \sim \mathcal{N}_p(\mathbf{0}, \Sigma)$ where Σ is a positive definite covariance matrix. On letting $\mathbf{Z} \sim \mathcal{N}_p(\mathbf{0}, I)$, where I is a $p \times p$ identity matrix, one has $\mathbf{X} = \Sigma^{\frac{1}{2}} \mathbf{Z}$ where $\Sigma^{\frac{1}{2}}$ denotes the symmetric square root of Σ. Then, the quadratic form $Q = \mathbf{X}' A \mathbf{X}$ where A is a $p \times p$ real symmetric matrix and $\mathbf{X}'$ denotes the transpose of $\mathbf{X}$ can be expressed as follows:

$$\begin{aligned} Q &= \mathbf{Z}' \Sigma^{\frac{1}{2}} A \Sigma^{\frac{1}{2}} \mathbf{Z} \\ &= \mathbf{Z}' P P' \Sigma^{\frac{1}{2}} A \Sigma^{\frac{1}{2}} P P' \mathbf{Z} \\ &= \mathbf{Z}' \Lambda \mathbf{Z} \end{aligned} \tag{4.24}$$

where P is an orthogonal matrix that diagonalizes $\Sigma^{\frac{1}{2}} A \Sigma^{\frac{1}{2}}$, that is, $P' \Sigma^{\frac{1}{2}} A \Sigma^{\frac{1}{2}} P = \mathrm{diag}(\lambda_1, \ldots, \lambda_p) = \Lambda$, $\lambda_1, \ldots, \lambda_p$ being the eigenvalues of $\Sigma^{\frac{1}{2}} A \Sigma^{\frac{1}{2}}$ (or equivalently those of $A\Sigma$) in *decreasing* order.

As shown in Mathai and Provost (1992), the s^{th} cumulant of $\mathbf{X}' A \mathbf{X}$ where $\mathbf{X} \sim \mathcal{N}_p(\boldsymbol{\mu}, \Sigma)$ is

$$k(s) = 2^{s-1}(s-1)!\mathrm{tr}(A\,\Sigma)^s \ , \tag{4.25}$$

tr(·) denoting the trace of (·). It should be noted that $\mathrm{tr}(A\Sigma)^s = \sum_{j=1}^{p} \lambda_j^s$ where the λ_j's, $j = 1, \ldots, p$, are the eigenvalues of $A\Sigma$. The moments of a random variable can be obtained from its cumulants by means of a recursive relationship that is derived for instance in Smith (1995). Accordingly, the h^{th} moment of $\mathbf{X}' A \mathbf{X}$ is given by

$$\mu(h) = \sum_{i=0}^{h-1} \frac{(h-1)!}{(h-1-i)!\, i!}\, k(h-i)\, \mu(i)\,, \tag{4.26}$$

where $k(s)$ is as specified by Equation (4.24).

Alternatively, we can obtain the h^{th}moment of a linear combination of chi-square random variables specified in the previous section by making use of the following code in *Mathematica*:

$$\left(\sum_{j=1}^{k} m_j X_j\right)^h /. \left\{X_{\mathrm{j\_.}}^{\mathrm{i\_.}} :\to \frac{2^i \Gamma\left(i + \frac{r_j}{2}\right)}{\Gamma\left(\frac{r_j}{2}\right)}\right\}$$

where k is number of variables in the linear combination, r_j is number of degrees of freedom of the j^{th} chi-square random variable in the linear combination and

$$E(X_j^i) = \frac{2^i \Gamma\left(i + \frac{r_j}{2}\right)}{\Gamma\left(\frac{r_j}{2}\right)}.$$

4.4 Numerical Examples

We will make use of the representation of the density function of a linear combination of independent chi-square random variables which is given in (4.23), and investigate its accuracy. The percentiles have been determined by simulating one million linear combination of chi-square random variables involving 2, 4 and 5 terms. Since the density given in (4.23) is a mixture of the gamma densities, the Reparameterized Generalized Gamma distribution ($\mathcal{RGG}$) is being used to fit the percentiles obtained by the simulation. The sum of the squared differences between the sample and population moments are minimized using *Mathematica* for determining the parameters. Tables 4.2-4.4 give the CDF obtained from the density function given in (4.23) at different percentiles and the CDF resulting from the $\mathcal{RGG}$ model. Let ν^* denote the number of terms used in the sum over ν in (4.23), that is, ν^* is the truncation point. Upon truncation, it is indicated to normalize the resulting function in order to obtain a *bona fide* density function. The most accurate CDF values that were achieved are indicated in bold face numbers. Table 4.1 presents the degrees of freedom and the k coefficients of the chi-square variables in the linear combinations being considered. Tables 4.2-4.4 indicate that in order to attain a reasonable level of accuracy, one should increase the values of ν^* as the number of variables in the linear combination increases and that in general the approximated distribution functions are in close agreement.

Table 4.1: Parameter Values of the Three Linear Combinations

k	r_1	r_2	r_3	r_4	r_5	m_1	m_2	m_3	m_4	m_5
2	4	2	—	—	—	1/3	2/3	—	—	—
4	4	2	3	1	—	3/10	2/10	3/10	2/10	—
5	4	2	2	4	2	1/10	2/10	4/10	2/10	1/10

Figures 4.1-4.3 present the empirical CDF and the fitted $\mathcal{RGG}$ CDF for $k = 2$, $k = 4$ and $k = 5$ based on the Tables 4.2-4.4 values.

Table 4.2: CDF Approximations for a Linear Combination of Two Variables, (k=2), Using the $\mathcal{RGG}$ Model and the Truncated Density (4.23)

CDF	Rank	Percentile	ν^*=5	ν^*=10	ν^*=15	$\mathcal{RGG}$
0.01	10 000	0.369131	0.0100614	0.0100614	**0.0100614**	0.00908585
0.05	50 000	0.693639	0.0499273	0.0499273	**0.0499273**	0.0485347
0.10	100 000	0.941485	0.100584	0.100585	**0.100585**	0.0996755
0.15	150 000	1.14003	0.150754	0.150754	0.150754	**0.150464**
0.20	200 000	1.31794	0.20068	0.200681	**0.200681**	0.200951
0.25	250 000	1.48594	0.250489	0.250492	**0.250492**	0.251213
0.30	300 000	1.65104	0.30067	0.300677	**0.300677**	0.301728
0.35	350 000	1.81543	0.350784	0.350798	**0.350798**	0.352056
0.40	400 000	1.98083	0.400491	0.400517	**0.400517**	0.401866
0.45	450 000	2.15279	0.450678	0.450724	**0.450724**	0.452061
0.50	550 000	2.33188	0.500711	0.500792	**0.500792**	0.502027
0.55	600 000	2.52242	0.550932	0.551069	**0.551069**	0.552126
0.60	660 000	2.7246	0.600447	0.600675	**0.600675**	0.601499
0.65	700 000	2.94815	0.650431	0.650808	**0.650808**	0.65136
0.70	750 000	3.19737	0.700203	0.700825	**0.700825**	0.701089
0.75	800 000	3.48134	0.749494	0.750527	0.750527	**0.750515**
0.80	850 000	3.81988	0.798645	0.800399	0.800399	**0.80015**
0.85	900 000	4.24535	0.847299	0.850396	0.850395	**0.849989**
0.90	950 000	4.82435	0.894204	0.900054	**0.900051**	0.899619
0.95	990 000	5.78943	0.936957	0.949954	**0.949939**	0.949675
0.99	990 000	8.00217	0.952753	0.990375	**0.990193**	0.990216

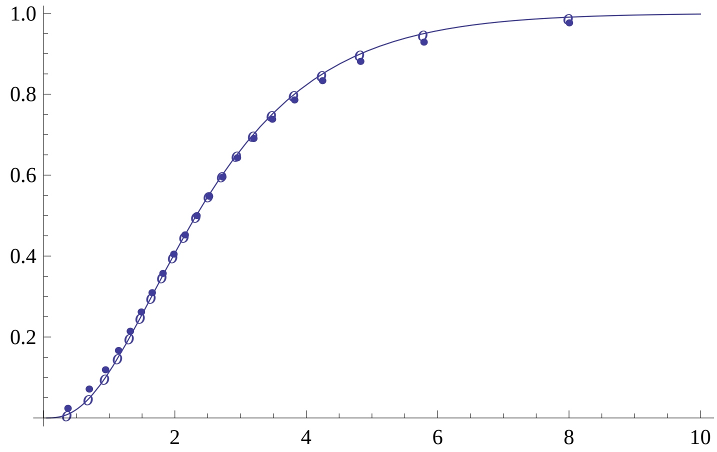

Figure 4.1: *The empirical CDF (dots), the CDF resulting from Equation (4.22) (circles) and the fitted $\mathcal{RGG}$ CDF (solid line) k=2*

Table 4.3: CDF Approximations for a Linear Combination of Four Variables, (k=4), Using the $\mathcal{RGG}$ Model and the Truncated Density (4.23)

CDF	Rank	Percentile	ν^*=5	ν^*=10	$\mathcal{RGG}$
0.01	10 000	0.995105	0.00987755	**0.00987755**	0.0097909
0.05	50 000	1.59237	0.0498953	**0.0498953**	0.0498032
0.10	100 000	1.99972	0.0999317	**0.0999318**	0.0999005
0.15	150 000	2.31116	0.149547	0.149547	**0.149574**
0.20	200 000	2.58651	0.199936	0.199937	**0.200011**
0.25	250 000	2.83862	0.249995	**0.249997**	0.250103
0.30	300 000	3.07916	**0.300027**	0.300031	0.300156
0.35	350 000	3.31547	**0.350282**	0.350288	0.35042
0.40	400 000	3.54929	**0.400155**	0.400166	0.400294
0.45	450 000	3.78622	**0.450038**	0.450055	0.450172
0.50	550 000	4.03145	**0.500243**	0.500269	0.500367
0.55	600 000	4.28532	**0.550049**	0.550088	0.550163
0.60	660 000	4.55482	0.599951	**0.60001**	0.600058
0.65	700 000	4.84807	**0.650311**	0.650398	0.650418
0.70	750 000	5.16931	**0.700425**	0.700553	0.700547
0.75	800 000	5.52905	**0.750146**	0.750333	0.750305
0.80	850 000	5.94958	**0.799908**	0.800186	0.800141
0.85	900 000	6.4669	0.849693	0.850118	**0.850064**
0.90	950 000	7.15227	0.898963	**0.899638**	0.899589
0.95	990 000	8.27585	0.948909	0.950109	**0.950083**
0.99	990 000	10.6338	0.987401	**0.989849**	**0.989849**

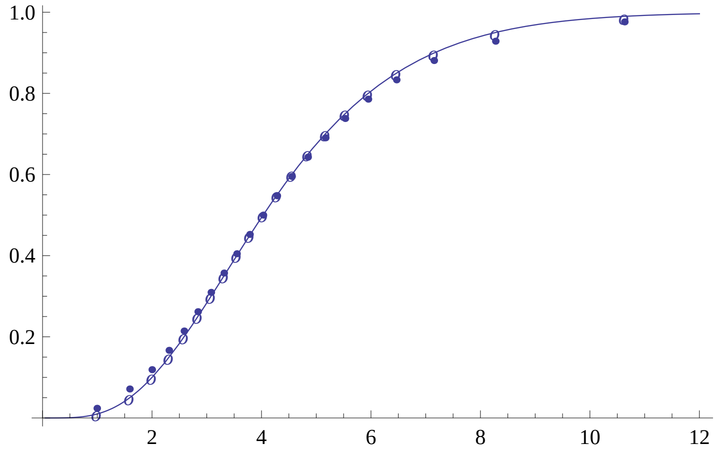

Figure 4.2: *The empirical CDF (dots), the CDF resulting from Equation (4.22) (circles) and the fitted* $\mathcal{RGG}$ *CDF (solid line) for k=4*

Table 4.4: CDF Approximations for a Linear Combination of Five Variables, (k=5), Using the $\mathcal{RGG}$ Model and the Truncated Density (4.23)

CDF	Rank	Percentile	$\nu^*=35$	$\nu^*=40$	$\nu^*=45$	$\mathcal{RGG}$
0.01	10 000	1.6614	0.0100821	0.0100813	**0.0100784**	0.00922397
0.05	50 000	2.36962	0.0497643	**0.0497637**	0.0497665	0.0487915
0.10	100 000	2.83474	**0.0994873**	0.0994868	0.0994873	0.0991038
0.15	150 000	3.18827	0.14951	0.14951	0.149506	**0.149741**
0.20	200 000	3.49272	0.199566	0.199565	0.199566	**0.200302**
0.25	250 000	3.77031	0.249431	0.249431	0.249435	**0.250536**
0.30	300 000	4.03468	0.299432	0.299432	**0.299438**	0.300772
0.35	350 000	4.29261	0.349488	0.349487	**0.349491**	0.350939
0.40	400 000	4.54692	0.399115	0.399115	0.399113	**0.400567**
0.45	450 000	4.80695	0.449286	0.449286	0.449284	**0.450643**
0.50	550 000	5.07294	0.499234	0.499234	0.499237	**0.500418**
0.55	600 000	5.35159	0.549388	0.549389	0.549393	**0.550336**
0.60	660 000	5.64557	0.5993	0.599299	0.5993	**0.599968**
0.65	700 000	5.9661	0.649711	0.649711	0.649708	**0.650074**
0.70	750 000	6.31386	0.699337	0.699337	0.699337	**0.699397**
0.75	800 000	6.7102	0.749344	0.749344	**0.749348**	0.749123
0.80	850 000	7.17469	**0.799336**	0.799336	0.799335	0.798889
0.85	900 000	7.74978	**0.849428**	0.849428	0.849426	0.84885
0.90	950 000	8.53224	**0.899849**	0.899848	0.899849	0.899289
0.95	990 000	9.8016	0.949875	0.949881	**0.949879**	0.949559
0.99	990 000	12.6141	0.973384	**0.990161**	0.989994	0.990033

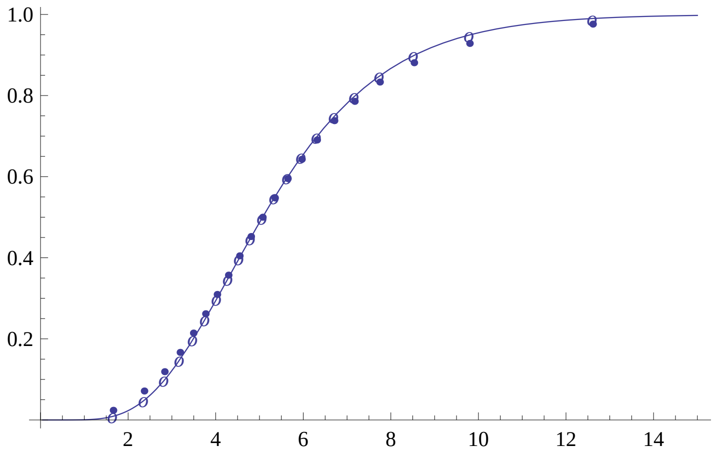

Figure 4.3: *The empirical CDF (dots), the CDF resulting from Equation (4.22) (circles) and the fitted $\mathcal{RGG}$ CDF (solid line) for k=5*

Chapter 5

Actuarial Examples

5.1 Some Actuarial Functions

Closed form representations of the hazard function and mean residual life function for the $\mathcal{RGG}$, $\mathcal{REIG}$, $\mathcal{EIG}$ and proxy distributions are presented in this section.

The hazard function is defined as $Z(x) = f(x)/S(x)$ where $S(x)$ is the survival function and the mean residual life function is defined as

$$K(x) = \frac{\int_x^\infty (y - x) f(y) \mathrm{d}y}{S(x)}.$$

(i) The hazard function for $\mathcal{RGG}$ distribution (Section 2.4.2) is

$$Z(x) = \frac{\delta\, e^{\nu\left(-x^{\delta}\right)} x^{\delta+\xi-1} \nu^{\frac{\delta+\xi}{\delta}}}{\Gamma\left(\frac{\delta+\xi}{\delta}, x^{\delta}\nu\right)}.$$

(ii) The hazard function for $\mathcal{REIG}$ distribution (Section 2.4.3) is

$$Z(x) = \frac{\rho\, x^{\xi+\rho}\, e^{\tau\left(-x^{-\rho}\right)} \tau^{-\frac{\xi+\rho+1}{\rho}}}{\Gamma\left(-\frac{\xi+\rho+1}{\rho}\right) - \Gamma\left(-\frac{\xi+\rho+1}{\rho}, x^{-\rho}\tau\right)},$$

provided that $1 + \delta + j\,\delta + \xi < 0$.

(iii) The hazard function for $\mathcal{EIG}$ distribution (Section 3.1) is

$$\frac{e^{-\tau x^{-\delta} - \nu x^{\delta}}}{x\, K_{\frac{\delta+\xi+1}{\delta}}\left(\nu x^{\delta},\ \tau x^{-\delta}\right)}. \tag{5.1}$$

(iv) Using the survival function of the proxy distribution as defined in Equation (2.44), its hazard function is

$$\frac{x^{\xi+\delta}\, e^{-\tau x^{-\rho}} \sum_{i=0}^{n} \frac{-1^i \nu^i}{i!} \sum_{j=0}^{i} \binom{i}{j} (-m)^{i-j}\, x^{\delta j}}{\sum_{i=0}^{n} \frac{-1^i\, \nu^i}{i!} \sum_{j=0}^{i} \binom{i}{j} (-m)^{i-j} \tau^{\frac{j\delta+\delta+\xi+1}{\rho}} \left(\Gamma\left(-\frac{j\delta+\delta+\xi+1}{\rho}\right) - \Gamma\left(-\frac{j\delta+\delta+\xi+1}{\rho}, x^{-\rho}\tau\right)\right)}. \tag{5.2}$$

(v) The mean residual life function for $\mathcal{RGG}$ distribution is

$$K(x) = \frac{\nu^{-1/\delta}\Gamma\left(\frac{\delta+\xi+1}{\delta}, x^{\delta}\nu\right)}{\Gamma\left(\frac{\delta+\xi}{\delta}, x^{\delta}\nu\right)} - x.$$

(vi) The mean residual life function for $\mathcal{REIG}$ distribution is

$$K(x) = \frac{\tau^{\frac{1}{\rho}}\left(\Gamma\left(-\frac{\xi+\rho+2}{\rho}\right) - \Gamma\left(-\frac{\xi+\rho+2}{\rho}, x^{-\rho}\tau\right)\right) + x\Gamma\left(-\frac{\xi+\rho+1}{\rho}, x^{-\rho}\tau\right) - x\Gamma\left(-\frac{\xi+\rho+1}{\rho}\right)}{\Gamma\left(-\frac{\xi+\rho+1}{\rho}\right) - x^{\xi+1}\left(x^{-\rho}\right)^{\frac{\xi+1}{\rho}}\Gamma\left(-\frac{\xi+\rho+1}{\rho}, x^{-\rho}\tau\right)}.$$

(vii) The mean residual life function for $\mathcal{EIG}$ distribution is

$$\frac{x\,K_{\frac{\delta+\xi+2}{\delta}}\left(\nu x^{\delta},\ \tau x^{-\delta}\right)}{K_{\frac{\delta+\xi+1}{\delta}}\left(\nu x^{\delta},\ \tau x^{-\delta}\right)} - x. \tag{5.3}$$

(viii) The Mean residual life function of the proxy distribution is

$$\begin{aligned}
&\Bigg(\sum_{i=0}^{n}\frac{-1^{i}\nu^{i}}{i!}\sum_{j=0}^{i}\binom{i}{j}(-m)^{i-j}\,\tau^{\frac{j\delta+\delta+\xi+1}{\rho}}\bigg(-\tau^{\frac{1}{\rho}}\Gamma\left(-\frac{j\delta+\delta+\xi+2}{\rho}, x^{-\rho}\tau\right)\\
&+x\Gamma\left(-\frac{j\delta+\delta+\xi+1}{\rho}, x^{-\rho}\tau\right) - x\Gamma\left(-\frac{j\delta+\delta+\xi+1}{\rho}\right) + \tau^{\frac{1}{\rho}}\Gamma\left(-\frac{j\delta+\delta+\xi+2}{\rho}\right)\bigg)\Bigg)\\
&\Bigg/\left(\sum_{i=0}^{n}\frac{-1^{i}\,\nu^{i}}{i!}\sum_{j=0}^{i}\binom{i}{j}(-m)^{i-j}\tau^{\frac{j\delta+\delta+\xi+1}{\rho}}\left\{\Gamma\left(-\frac{j\delta+\delta+\xi+1}{\rho}\right) - \Gamma\left(-\frac{j\delta+\delta+\xi+1}{\rho}, x^{-\rho}\tau\right)\right\}\right),
\end{aligned} \tag{5.4}$$

provided that $2+\delta+j\,\delta+\xi < 0$. A methodology for converting problems into density estimation problem is proposed and applied to actuarial data sets in the next sections.

5.2 Canadian Quinquennial Mortality Rates

Canadian quinquennial mortality rates (times 1000) in 2006 for the age group 15 and over where were obtained from Statistics Canada's website, are included in Table 5.1. These rates are fitted using the five-parameter $\mathcal{GEM}$. A plot of this data set is presented in Figure 5.1. Since fitting this data set is a regression problem, we need to convert it into a density estimation problem. First, the 'Interpolation' *Mathematica* command (with third degree splines) is applied to the data set. The resulting plot appears in Figure 5.2.

The second step in transforming a regression problem into a density fitting problem is to obtain a density function from the Interpolation function. This is done by differentiating numerically the interpolating function of the Canadian quinquennial mortality rates data set

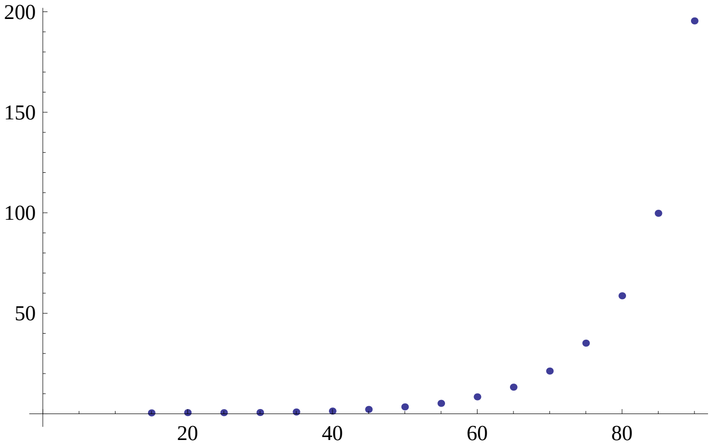

Figure 5.1: *Plot of Canadian quinquennial mortality rates (times 1000) in 2006 for ages 15 and over*

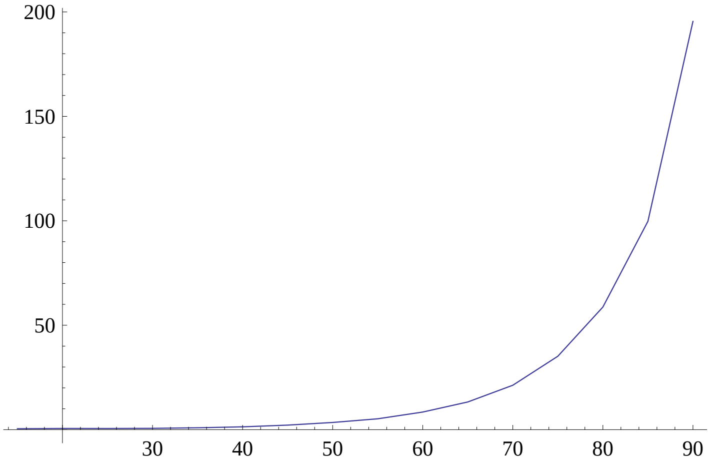

Figure 5.2: *Interpolation function (third degree splines) for the Canadian quinquennial mortality rates data set*

Table 5.1: Canadian Quinquennial Mortality Rates (times 1000) in 2006 for Ages 15 and Over

Age	Mortality Rate	Age	Mortality Rate
15-19	0.453594	55-59	5.2008
20-24	0.58258	60-64	8.43232
25-29	0.560751	65-69	13.2571
30-34	0.63396	70-74	21.2655
35-39	0.899783	75-79	35.1512
40-44	1.35417	80-84	58.704
45-49	2.17811	85-89	99.7542
50-54	3.44684	90-95	195.448

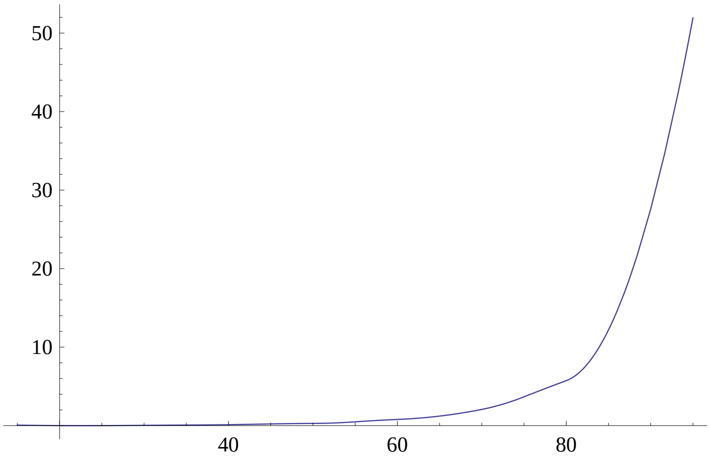

Figure 5.3: *Derivative of the interpolating function for the Canadian quinquennial mortality rates*

and then reflecting the resulting function using the transformation $95 - x$ to obtain a right skewed distribution. As a final step the resulting function is normalized. Figure 5.3 shows the interpolation function after numerical differentiation and Figure 5.4 shows the density function obtained after applying numerical differentiation, reflection and normalization.

The probability density corresponding to the Canadian quinquennial mortality rates data set is fitted from its moments with different distributions and then the reverse steps are applied to model the mortality rates. Goodness-of-fit is assessed by evaluating the average squared differences (ASD) between the original and fitted mortality rates. Table 5.2 shows ASD's for the various fitted models.

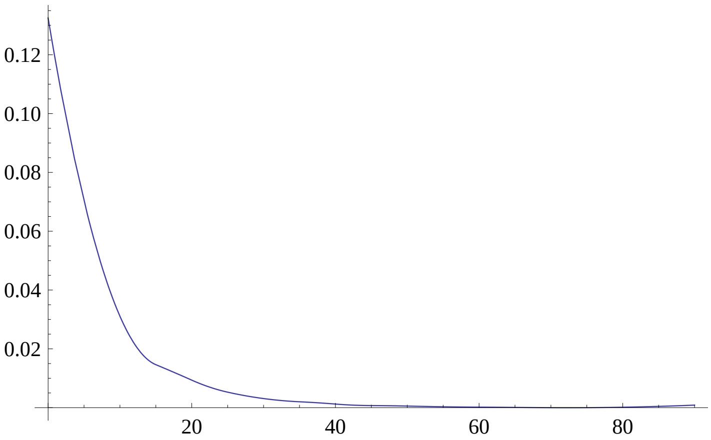

Figure 5.4: *Density function corresponding to Canadian quinquennial mortality rates obtained after applying numerical differentiation, reflection and normalization*

Table 5.2: Parameter Estimates and ASD's for the Canadian Quinquennial Mortality Rates

	$\hat{\zeta}$	$\hat{\nu}$	$\hat{\tau}$	ASD
Exponential $\delta = 1$	-1	0.1216	0	40.526
Inverse Gaussian	-2.5	0.0394	2.665	7.25
Lognormal (1.641,.966)	–	–	–	6.575
Bessel Distribution $\hat{\delta} = 0.553$	$-.533$	0.7729	0.6208	3.703
RGG Distribution $\hat{\delta} = 0.482$	1.047	1.329	0	3.438
GEM $\hat{\delta} = 3/4, \hat{\rho} = 2/3$	-1.906	.1832	2.704	1.742

5.3 American Yearly Mortality Rates for Females

American Yearly Mortality Rates for Females in 2006 for the age group 8 and over where were obtained from www.mortality.org website, are included in Table 5.3. Table 5.4 provides parameter estimates and goodness-of-fit results for the following models: exponential, Weibull, lognormal, $\mathcal{RGG}$, $\mathcal{EIG}$ and $\mathcal{GEM}$. It is seen from that table that the $\mathcal{GEM}$ provides the best fit. The improvement in the fit with this model is not as noticeable as in the previous example wherein the mortality rates were multiplied by 1000, which makes these rates more distinct.

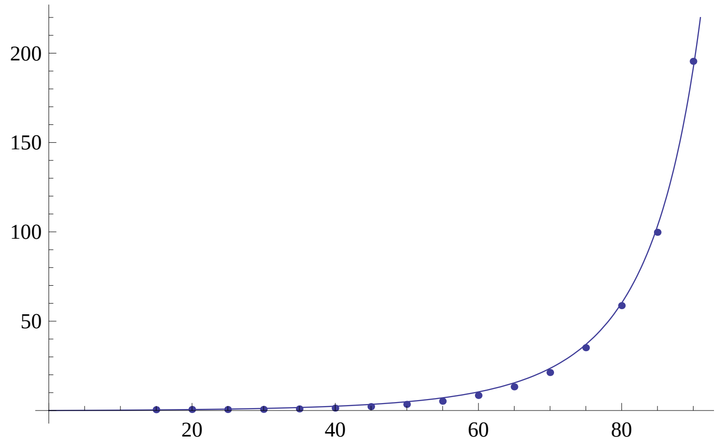

Figure 5.5: *Original and fitted mortality rates using the* $\mathcal{RGG}$ *model*

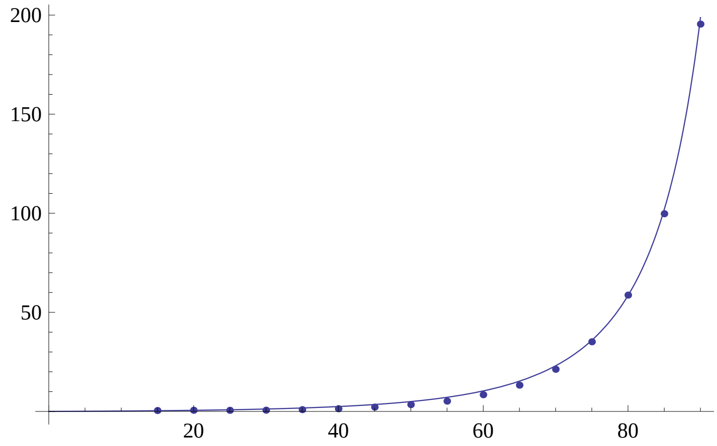

Figure 5.6: *Original and fitted mortality rates using the* $\mathcal{GEM}$

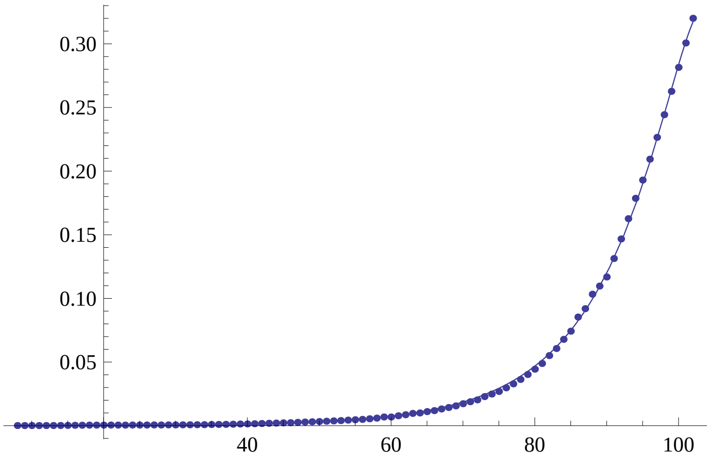

Figure 5.7: *Fitted function superimposed on the mortality rates*

Table 5.3: Mortality Rates for Females for ages 8-102

.00011	.00011	.00012	.00011	.00012	.00014	.00017	.00022
.00033	.00037	.00045	.00048	.00045	.00052	.00048	.00047
.00053	.00052	.00054	.00060	.00056	.00063	.00062	.00069
.00072	.00077	.00081	.00092	.00098	.00104	.00117	.00133
.00139	.00154	.00172	.00191	.00207	.00224	.00235	.00263
.00286	.00307	.00327	.00351	.00378	.00401	.00438	.00468
.00496	.00544	.00595	.00685	.00683	.00780	.00861	.00966
.01003	.01107	.01185	.01315	.01428	.01556	.01727	.01882
.02030	.02286	.02489	.02691	.02976	.03288	.03635	.04023
.04441	.04888	.05509	.06065	.06783	.07427	.08536	.09193
.10330	.10971	.11686	.13134	.14673	.16264	.17865	.19297
.20939	.22653	.24433	.26269	.28151	.30068	.32007	

Table 5.4: Parameter Estimates and ASD's for the Female Mortality Rates

	$\hat{\zeta}$	$\hat{\nu}$	$\hat{\tau}$	ASD
Exponential $\delta = 1$	-1	0.072998	0	0.000124943
REIG Distribution $\hat{\rho} = 0.806$	-3.49	0	11.5	0.000108267
Weibull(1.167)	-1	0.0454	0	0.0000340835
Lognormal (2.357,0.7376)	–	–	–	0.0000140229
RGG Distribution $\hat{\delta} = 0.48202$	2.83462	2.09553	0	2.05652×10^{-6}
$\mathcal{EIG}\,\hat{\delta} = 0.725$	0.300	0.475	0.800	1.38445×10^{-6}
$\mathcal{GEM}\,\hat{\delta} =$ [illegible]/$\hat{\rho} =$ [illegible]	0.300	0.4750	0.800	1.38078×10^{-6}

Chapter 6

Fitting Continuous Distributions to Bivariate Data

6.1 Introduction

Suppose that U and W are jointly distributed random variables with density $h(u, w)$; then in order to remove the correlation between the variables, the original data is normalized by using the transformation

$$V^{-\frac{1}{2}} \begin{pmatrix} u - \overline{u} \\ w - \overline{w} \end{pmatrix},$$

where $V^{-\frac{1}{2}}$ is the inverse of the symmetric square root of the estimated covariance matrix, which is denoted by V.

Since we are modeling the transformed variables with distributions defined on the positive half-line, the following constants are added respectively to each coordinate so that the support of the resulting distribution lies in the first quadrant. These constants are determined as follows:
γ_1=Absolute value of [Min[U]]+(Max[U]-Min[U])/2
γ_2=Absolute value of [Min[W]]+(Max[W]-Min[W])/2.
Thus,

$$\begin{pmatrix} x \\ y \end{pmatrix} = V^{-\frac{1}{2}} \begin{pmatrix} u - \overline{u} \\ w - \overline{w} \end{pmatrix} + \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}, \tag{6.1}$$

where

$$V^{-\frac{1}{2}} = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix}. \tag{6.2}$$

Then the transformed data is fitted to the model specified by $f(x, y) = f_1(x) f_2(y)$. This is done by separately fitting the x_i's and y_i's with appropriate models. In the final step, the distribution of the original data is obtained by applying the inverse transformation, that is,

$$\begin{pmatrix} u \\ w \end{pmatrix} = V^{\frac{1}{2}} \begin{pmatrix} x - \gamma_1 \\ y - \gamma_2 \end{pmatrix} + \begin{pmatrix} \overline{u} \\ \overline{w} \end{pmatrix}. \tag{6.3}$$

Since the Jacobian of the inverse of the inverse transformation is the determinant of $V^{-\frac{1}{2}}$, the probability density function of the original data is taken to be

$$h(u,w) = |V^{-\frac{1}{2}}|\, f(v_{11}(u-\overline{u}) + v_{12}(w-\overline{w}) + \gamma_1, v_{12}(u-\overline{u}) + v_{22}(w-\overline{w}) + \gamma_2). \tag{6.4}$$

6.2 Applications

6.2.1 Bivariate Flood Data

This data which was previously analyzed by Yue (2001), consists of flood peaks (variable u) and volumes (variable w), as observed in the Madawask basin, Quebec, Canada from 1990 to 1995. A bivariate histogram of this data set is shown in Figure 6.1. The bivariate density obtained from the $\mathcal{EIG}$ model (Figure 6.8) reflects more accurately the features of the original data set than that based on the Inverse Gaussian distribution (Figure 6.9).

Table 6.1: The Flood Data Set

292, 12057	208, 10853	289, 10299	146, 10818	183, 7748	279, 9763	260, 11127
279, 10659	137, 8327	311, 13593	309, 12882	261, 9957	162, 5236	202, 9581
306, 12740	405, 11174	183, 4780	219, 14890	210, 6334	200, 9177	289, 7133
239, 6865	294, 8918	371, 8704	245, 6907	189, 4189	229, 8637	240, 8409
331, 13602	206, 8788	157, 5002	184, 5167	275, 10128	286, 12035	230, 10828
233, 8923	351, 11401	156, 6620	168, 3826	343, 8192	214, 6414	303, 8900
300, 9406	143, 7235	232, 8177	182, 7684	121, 3306	186, 8026	173, 4892
292, 8692	416, 11272	246, 8640	248, 6989	297, 9352	371, 12825	442, 13608
260, 8949	236, 12577	334, 11437	310, 9266	383, 14559	151, 5057	197, 9645
, 283, 7241	390, 13543	405, 15003	176, 6460	181, 7502	233, 5650	187, 7350
216, 9506	196, 6728	424, 13315	255, 8041	257, 10174	232, 14769	286, 8711

The following models were used: the inverse Gaussian model ($\mathcal{IG}$), $\mathcal{GIG}$, $\mathcal{REIG}$, $\mathcal{RGG}$, $\mathcal{EIG}$. Their parameter estimates are included in Table 6.2. The Anderson-Darling and Cramér-von Mises the goodness-of-fit statistics are presented in Table 6.3.

6.2.2 Old Faithful Geyser Data

The Old Faithful Geyser data, presented in Table 6.4, is readily available in R as 'faithful {datasets}'. The first variable is the waiting time between eruptions and the second one represents the duration of the eruptions. The bivariate histogram of this data set appears in Figure 6.10. It is seen that the histogram and the density estimates exhibit similar features. The goodness-of-fit results are tabulated in Table 6.5.

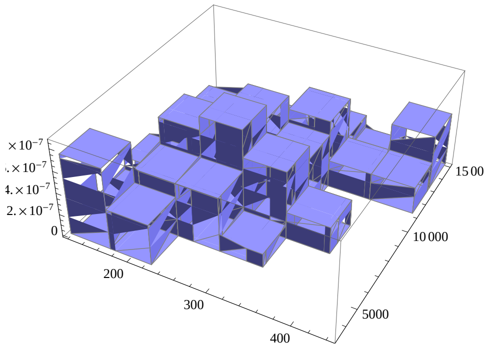

Figure 6.1: *Bivariate histogram of the flood data.*

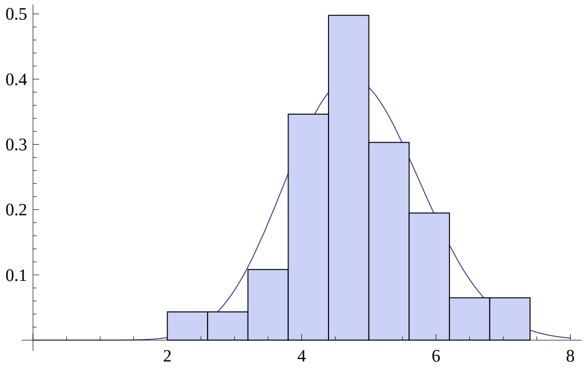

Figure 6.2: $\mathcal{EIG}$ *univariate density estimate for the waiting time in the bivariate flood data*

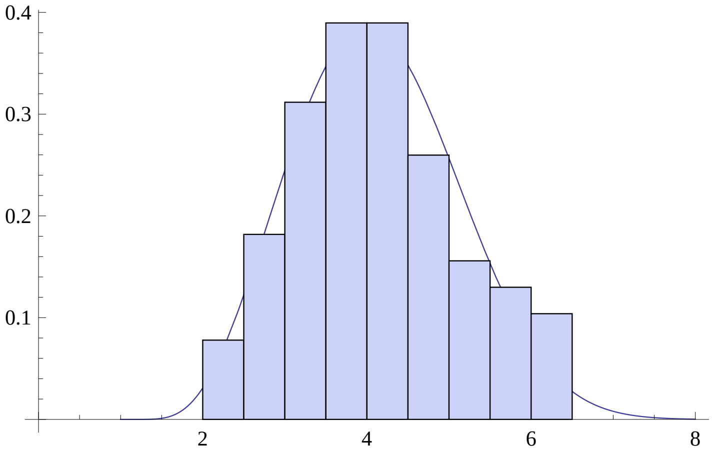

Figure 6.3: $\mathcal{RGG}$ *univariate density estimate for the waiting time variable in the bivariate flood data*

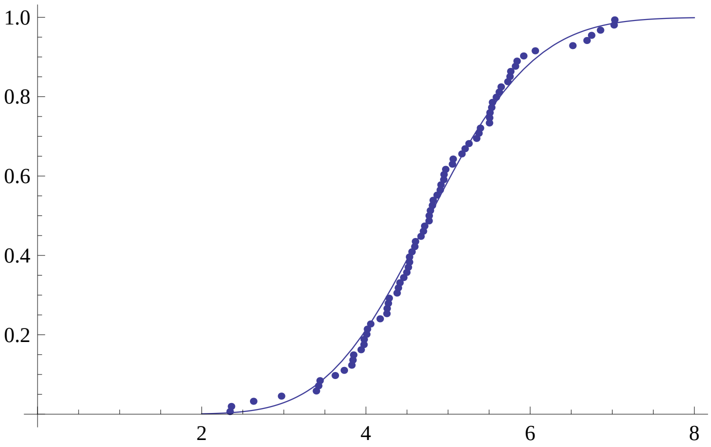

Figure 6.4: *The empirical CDF and the fitted $\mathcal{EIG}$ CDF for the flood peaks variable in the bivariate flood data*

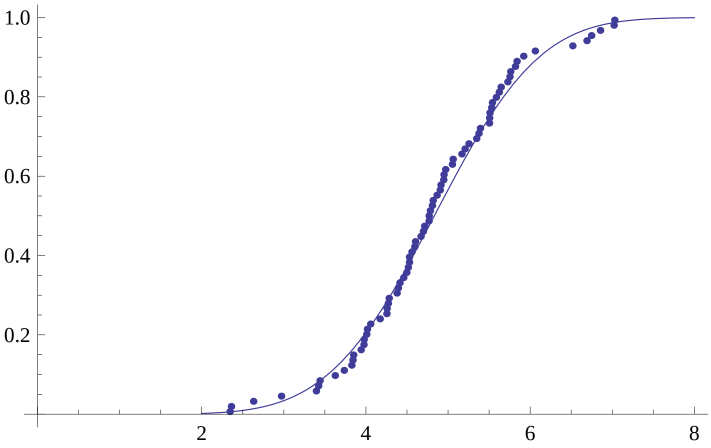

Figure 6.5: *The empirical CDF and the fitted $\mathcal{RGG}$ CDF for the flood peaks variable in the bivariate flood data*

Table 6.2: Parameter Estimates for the Bivariate Flood Data

	$\hat{\xi}_1$	$\hat{\delta}_1$	$\hat{\nu}_1$	$\hat{\tau}_1$	$\hat{\xi}_2$	$\hat{\delta}_2$	$\hat{\nu}_2$	$\hat{\tau}_2$
$\mathcal{REIG}$	-16.1971	1.0141	—	55	-129.3242	0.1550	—	1050
IG	-5/2	1	2.0556	47.3732	-5/2	1	1.76551	30
$\mathcal{GIG}$	19.6891	1	4.51802	0.0010	8.60958	1	3	7
$\mathcal{RGG}$	4.44	3.324	.0107	—	3.543	2.7225	.04232	—
$\mathcal{EIG}$	7.000	2.3685	0.0997	0.0011	-0.4420	3.3413	0.0100	16.2171

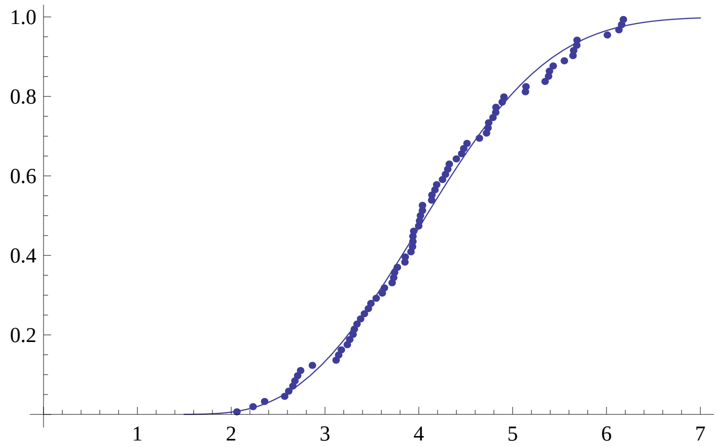

Figure 6.6: *The empirical CDF and the fitted $\mathcal{EIG}$ CDF for the volume variable in the bivariate flood data*

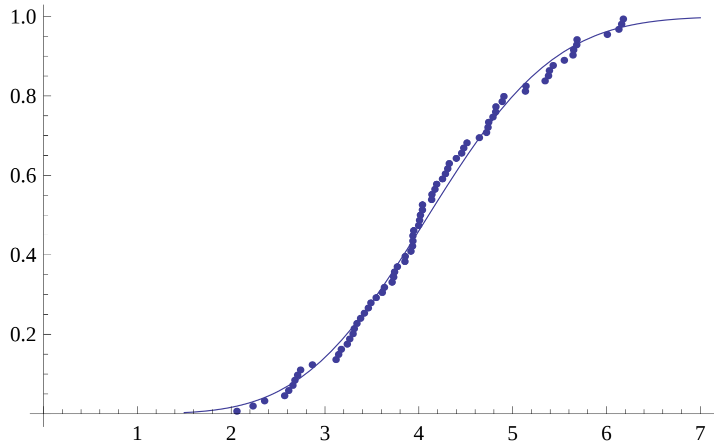

Figure 6.7: *The empirical CDF and the fitted $\mathcal{EIG}$ CDF for the flood peaks variable in the bivariate flood data*

Table 6.3: A_0^2 & W_0^2 for the Bivariate Flood Data

	$A1_0^2$	$W1_0^2$	$A2_0^2$	$W2_0^2$
$\mathcal{REIG}$	0.849674	0.113627	0.651955	0.0951932
$\mathcal{IG}$	0.841639	0.113539	0.413713	0.0585011
$\mathcal{GIG}$	0.511099	0.0628183	0.306132	0.0399535
$\mathcal{RGG}$	0.393214	0.0536596	0.268807	0.039934
$\mathcal{EIG}$	0.335722	0.0360489	0.247784	0.033126

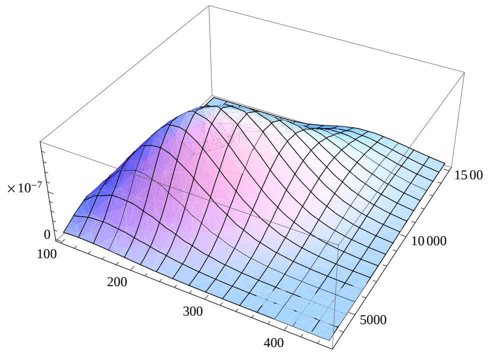

Figure 6.8: *$\mathcal{IG}$ bivariate density estimate for the bivariate flood data*

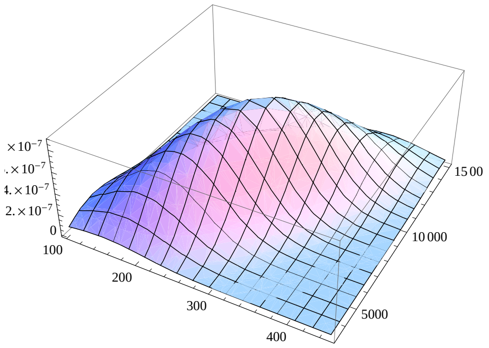

Figure 6.9: *$\mathcal{EIG}$ bivariate density estimate for the bivariate flood data*

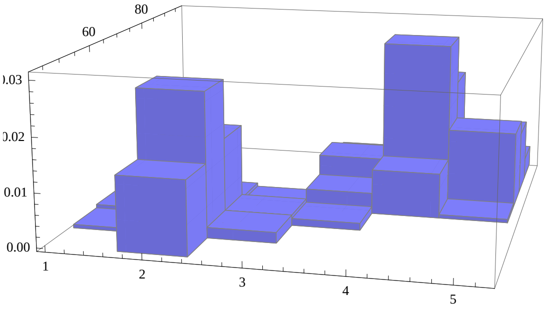

Figure 6.10: *Bivariate histogram for the Old Faithful data*

Table 6.4: The Old Faithful Data Set

1.800, 54	3.333, 74	2.283, 62	4.533, 85	2.883, 55	4.700, 88	3.600, 85
1.950, 51	4.350, 85	1.833, 54	3.917, 84	4.200 , 78	1.750, 47	4.700, 83
2.167, 52	1.750, 62	4.800, 84	1.600, 52	4.250, 79	1.800, 51	1.750, 47
3.450, 78	3.067, 69	4.533, 74	3.600, 83	1.967, 55	4.083, 76	3.850, 78
4.433, 79	4.300, 73	4.467, 77	3.367, 66	4.033, 80	3.833, 74	2.017, 52
1.867, 48	4.833, 80	1.833, 59	4.783, 90	4.350, 80	1.883, 58	4.567, 84
1.750, 58	4.533, 73	3.317, 83	3.833, 64	2.100, 53	4.633, 82	2.000, 59
4.800 , 75	4.716, 90	1.833, 54	4.833, 80	1.733 , 54	4.883, 83	3.717, 71
1.667, 64	4.567, 77	4.317, 81	2.233, 59	4.500, 84	1.750, 48	4.800, 82
1.817, 60	4.400, 92	4.167, 78	4.700, 78	2.067, 65	4.700, 73	4.033, 82
1.967, 56	4.500, 79	4.000, 71	1.983, 62	5.067, 76	2.017, 60	4.567, 78
3.883, 76	3.600, 83	4.133, 75	4.333, 82	4.100, 70	2.633, 65	4.067 , 73
4.933, 88	3.950, 76	4.517, 80	2.167, 48	4.000, 86	2.200, 60	4.333, 90
1.867, 50	4.817, 78	1.833, 63	4.300, 72	4.667, 84	3.750, 75	1.867, 51
4.900, 82	2.483, 62	4.367, 88	2.100, 49	4.500, 83	4.050, 81	1.867, 47
4.700, 84	1.783, 52	4.850, 86	3.683, 81	4.733, 75	2.300, 59	4.900, 89
4.417, 79	1.700, 59	4.633, 81	2.317, 50	4.600, 85	1.817, 59	4.417, 87
2.617, 53	4.067, 69	4.250, 77	1.967, 56	4.600, 88	3.767, 81	1.917, 45
4.500, 82	2.267, 55	4.650, 90	1.867, 45	4.167, 83	2.800, 56	4.333, 89
1.833, 46	4.383, 82	1.883, 51	4.933, 86	2.033, 53	3.733, 79	4.233, 81
2.233, 60	4.533, 82	4.817, 77	4.333, 76	1.983, 59	4.633, 80	2.017, 49
5.100, 96	1.800, 53	5.033, 77	4.000, 77	2.400, 65	4.600, 81	3.567 , 71
4.000, 70	4.500, 81	4.083, 93	1.800, 53	3.967, 89	2.200, 45	4.150, 86
2.000, 58	3.833, 78	3.500, 66	4.583, 76	2.367, 63	5.000, 88	1.933, 52
4.617, 93	1.917, 49	2.083, 57	4.583, 77	3.333, 68	4.167, 81	4.333, 81
4.500, 73	2.417, 50	4.000, 85	4.167, 74	1.883, 55	4.583, 77	4.250, 83
3.767, 83	2.033, 51	4.433, 78	4.083, 84	1.833, 46	4.417, 83	2.183, 55
4.800, 81	1.833, 57	4.800, 76	4.100, 84	3.966, 77	4.233, 81	3.500, 87
4.366, 77	2.250, 51	4.667, 78	2.100, 60	4.350, 82	4.133, 91	1.867, 53
4.600, 78	1.783, 46	4.367, 77	3.850, 84	1.933, 49	4.500, 83	2.383, 71
4.700, 80	1.867, 49	3.833, 75	3.417, 64	4.233, 76	2.400, 53	4.800, 94
2.000, 55	4.150, 76	1.867, 50	4.267, 82	1.750, 54	4.483, 75	4.000, 78
4.117, 79	4.083, 78	4.267, 78	3.917, 70	4.550, 79	4.083, 70	2.417, 54
4.183, 86	2.217, 50	4.450, 90	1.883, 54	1.850, 54	4.283, 77	3.950, 79
2.333, 64	4.150, 75	2.350, 47	4.933, 86	2.900, 63	4.583, 85	3.833, 82
2.083, 57	4.367, 82	2.133, 67	4.350, 74	2.200, 54	4.450, 83	3.567, 73
4.500, 73	4.150, 88	3.817, 80	3.917, 71	4.450, 83	2.000, 56	4.283, 79
4.767, 78	4.533, 84	1.850, 58	4.250, 83	1.983, 43	2.250, 60	4.750, 75
4.117, 81	2.150, 46	4.417, 90	1.817, 46	4.467, 74		

Table 6.5: A_0^2 & W_0^2 for the Old Faithful Data

	$\mathrm{A1}_0^2$	$\mathrm{W1}_0^2$	$\mathrm{A2}_0^2$	$\mathrm{W2}_0^2$	$\mathrm{A3}_0^2$	$\mathrm{W3}_0^4$	$\mathrm{A4}_0^2$	$\mathrm{W4}_0^2$
$\mathcal{RGG}$	1.2641	0.2049	0.2947	0.0439	0.2947	0.0439	0.2708	0.0415
$\mathcal{EIG}$	0.3713	0.0644	0.2493	0.0374	0.2079	0.0259	0.2539	0.0370

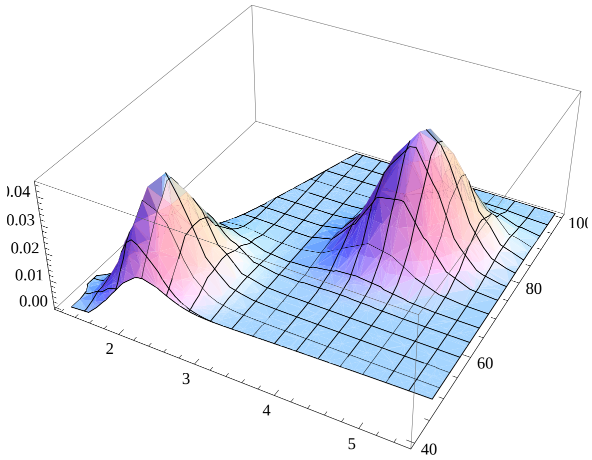

Figure 6.11: $\mathcal{RGG}$ *bivariate density estimate for the Old Faithful data*

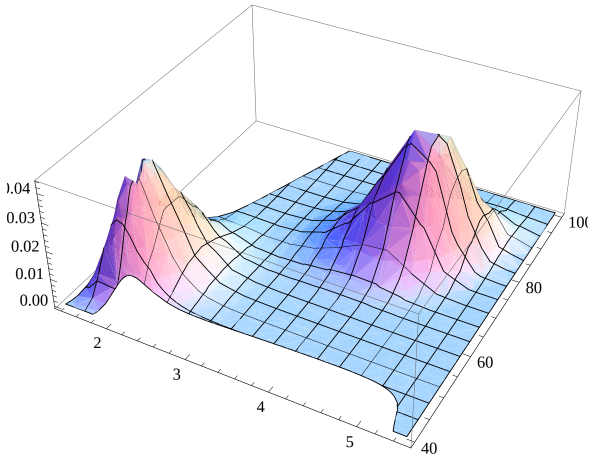

Figure 6.12: $\mathcal{EIG}$ *bivariate density estimate for the Old Faithful data*

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Parameter Estimates of the GEM (2.1) Using the MLE Method for the Flood Data Set (Table 2.1) (Sections 2.5.1 & 2.7.1)

```
f[x_?NumberQ, ξ_?NumberQ, v_?IntegerQ, d_?IntegerQ,
  ν_?NumberQ, τ_?NumberQ, w_?IntegerQ, r_?IntegerQ] :=
 f[x?NumberQ, ξ?NumberQ, v?IntegerQ, d?IntegerQ,
  ν?NumberQ, τ?NumberQ, w?IntegerQ, r?IntegerQ] =
```

$$\left(\left(\left((\nu)^{d\,(\xi+(v/d)+1)/v}\,(2\,\pi)^{((r\,v)/2)+((d\,w)/2)-1}\,(d\,w)^{-((\xi+(v/d)+1)/(v/d))+0.5}\,(r\,v)^{0.5}\right)\Big/(d\,r)\right)\Big/\text{MeijerG}\Big[\{\{\},\{\}\},\{\text{Join}[\text{Table}[(k+d\,(\xi+(v/d)+1)/v)/(d\,w),\{k,0,d\,w-1\}],\text{Table}[k/(r\,v),\{k,0,r\,v-1\}]],\{\}\},(\tau)^{r\,v}\,(\nu)^{w\,d}\,(r\,v)^{-r\,v}\,(d\,w)^{-d\,w}\Big]\right)\,x^{\xi+(v/d)}\,e^{-\nu\,x^{(v/d)}}\,e^{-\tau\,x^{-(w/r)}}$$

```
Lok3[ξ_?NumberQ, v_?IntegerQ, d_?IntegerQ, ν_?NumberQ,
  τ_?NumberQ, w_?IntegerQ, r_?IntegerQ] :=
```

$$n\,\text{Log}\Big[\left(\left(\left((\nu)^{d\,(\xi+(v/d)+1)/v}\,(2\,\pi)^{((r\,v)/2)+((d\,w)/2)-1}\,(d\,w)^{-((\xi+(v/d)+1)/(v/d))+0.5}\,(r\,v)^{0.5}\right)\Big/(d\,r)\right)\Big/\text{MeijerG}\Big[\{\{\},\{\}\},\{\text{Join}[\text{Table}[(k+d\,(\xi+(v/d)+1)/v)/(d\,w),\{k,0,d\,w-1\}],\text{Table}[k/(r\,v),\{k,0,r\,v-1\}]],\{\}\},(\tau)^{r\,v}\,(\nu)^{w\,d}\,(r\,v)^{-r\,v}\,(d\,w)^{-d\,w}\Big]\right)\Big]$$

$$+\sum_{i=1}^{n}(\xi+(v/d))\,\text{Log}[\text{data}[[i]]]-\nu\sum_{i=1}^{n}\text{data}[[i]]^{v/d}$$

$$-\tau\sum_{i=1}^{n}\text{data}[[i]]^{-w/r}$$

```
Logt3[ξ_, v_, d_, ν_, τ_, w_, r_] :=
 If[Lok3[ξ, v, d, ν, τ, w, r] ∈ Reals, Lok3[ξ, v, d, ν, τ, w, r], -10^10]
data = Sort[{.654, .613, .402, .379, .269,
    .740, .416, .338, .315, .449, .297, .423, .379,
    .3235, .418, .412, .494, .392, .484, .265}];
n = Length[data];
x_{i_} := data[[i]]
cdf1[y_] :=
  cdf1[y] = NIntegrate[ f[x, ξ, v, d, ν, τ, w, r], {x, 0, y}];
z_{i_} := cdf1[x_i]



v = 5; d = 2; w = 5; r = 2;
ClearAll[ξ, ν, τ]
vt =
 NMaximize[{Evaluate[Logt3[ξ, v, d, ν, τ, w, r]], -10 < ξ < -7,
   1.5 < ν < 2.3, .01 < τ < 3}, {ξ, ν, τ}, MaxIterations → 1000]

ξ = vt[[2, 1, 2]]
ν = vt[[2, 2, 2]]
τ = vt[[2, 3, 2]]
ClearAll[A0, W0]
```

$$A0 = -n - \frac{1}{n}\sum_{i=1}^{n}((2\,i-1)\,\mathrm{Log}[z_i\ (1-z_{n+1-i})])$$

$$W0 = \frac{1}{12\,n} + \sum_{i=1}^{n}\left(z_i - \frac{2\,i-1}{2\,n}\right)^2$$

```
Plot[f[x, ξ, v, d, ν, τ, w, r], {x, 0, 1.3}, PlotRange → All]
p10 = ListPlot[Table[{y, cdf1[y]}, {y, 0.05, 1.2, 0.05}],
   AxesLabel → {x, F[x]}, PlotStyle → RGBColor[1, 0, 0]];
t = Table[{x_i, (i / n) - 1 / (2 n)}, {i, 1, Length[data]}];
lp1 = ListPlot[t];
Show[p10, lp1]
```

Parameters Estimates of the GEM (2.1) Using the Method of Moments (Sections & 2.7.2)

```
v = 3; d = 1; w = 8; r = 3;
M1 = ((ν)^-(d /v) (d w)^1/(v/d) MeijerG[{{}, {}},
       {Join[Table[(k + d (ξ + (v / d) + 2) / v) / (d w),
          {k, 0, d w - 1}], Table[k / (r v), {k, 0, r v - 1}]], {}} ,
       (τ)^r v (ν)^w d (r v)^-r v (d w)^-d w]  ) / MeijerG[{{}, {}},
     {Join[Table[(k + d (ξ + (v / d) + 1) / v) / (d w), {k, 0, d w - 1}],
       Table[k / (r v), {k, 0, r v - 1}]], {}} ,
     (τ)^r v (ν)^w d (r v)^-r v (d w)^-d w];

M2 = ((ν)^-(2 d /v) (d w)^2/(v/d) MeijerG[{{}, {}},
       {Join[Table[(k + d (ξ + (v / d) + 3) / v) / (d w),
          {k, 0, d w - 1}], Table[k / (r v), {k, 0, r v - 1}]], {}} ,
       (τ)^r v (ν)^w d (r v)^-r v (d w)^-d w]  ) / MeijerG[{{}, {}},
     {Join[Table[(k + d (ξ + (v / d)  + 1) / v) / (d w), {k, 0, d w - 1}],
       Table[k / (r v), {k, 0, r v - 1}]], {}} ,
     (τ)^r v (ν)^w d (r v)^-r v (d w)^-d w];

M3 = ((ν)^-(3 d /v) (d w)^3/(v/d) MeijerG[{{}, {}},
       {Join[Table[(k + d (ξ + (v / d) + 4) / v) / (d w),
          {k, 0, d w - 1}], Table[k / (r v), {k, 0, r v - 1}]], {}} ,
       (τ)^r v (ν)^w d (r v)^-r v (d w)^-d w]  ) / MeijerG[{{}, {}},
     {Join[Table[(k + d (ξ + (v / d) + 1) / v) / (d w), {k, 0, d w - 1}],
       Table[k / (r v), {k, 0, r v - 1}]], {}} ,
```

```
      (τ)^(r v) (ν)^(w d) (r v)^(-r v) (d w)^(-d w)];


m1 = ∑_{i=1}^{n} (x_i / n);

m2 = ∑_{i=1}^{n} (x_i^2 / n);

m3 = ∑_{i=1}^{n} (x_i^3 / n);


f[x_?NumberQ, ξ_?NumberQ, v_?IntegerQ, d_?IntegerQ,
   ν_?NumberQ, τ_?NumberQ, w_?IntegerQ, r_?IntegerQ] :=
  f[x?NumberQ, ξ?NumberQ, v?IntegerQ, d?IntegerQ,
    ν?NumberQ, τ?NumberQ, w?IntegerQ, r?IntegerQ] =
   ((((ν)^(d (ξ+(v/d)+1)/v) (2 π)^(((r v)/2)+((d w)/2)-1) (d w)^(-((ξ+(v/d)+1)/(v/d))+0.5)
          (r v)^(0.5)) / (d r)) / MeijerG[{{}, {}},
       {Join[Table[(k + d (ξ + (v / d) + 1) / v) / (d w),
          {k, 0, d w - 1}], Table[k / (r v), {k, 0, r v - 1}]], {}} ,
       (τ)^(r v) (ν)^(w d) (r v)^(-r v) (d w)^(-d w)]) x^(ξ+(v/d)) ⅇ^(-ν x^(v/d)) ⅇ^(-τ x^(-(w/r)))
data = Sort[{.654, .613, .402, .379, .269, .740,
    .416, .338, .315, .449, .297, .423, .379,
    .3235, .418, .412, .494, .392, .484, .265}];
n = Length[data];
x_i_ := data[[i]]
cdf1[y_] :=
  cdf1[y] = NIntegrate[ f[x, ξ, v, d, ν, τ, w, r], {x, 0, y}];
```

$$z_{i\_} := \text{cdf1}[x_i]$$

```
ClearAll[ξ, ν, τ]
vt = FindRoot[{(m1 - M1)^2 == 0, (m2 - M2)^2 == 0, (m3 - M3)^2 == 0},
   {{ξ, -8.7`}, {ν, 1.6`}, {τ, 0.19`}}];
ξ = vt[[1, 2]]
ν = vt[[2, 2]]
τ = vt[[3, 2]]

ClearAll[A0, W0]
```

$$\text{A0} = -n - \frac{1}{n}\sum_{i=1}^{n}\left((2\,i-1)\,\text{Log}[z_i\,(1-z_{n+1-i})]\right)$$

$$\text{W0} = \frac{1}{12\,n} + \sum_{i=1}^{n}\left(z_i - \frac{2\,i-1}{2\,n}\right)^2$$

```
Plot[f[x, ξ, v, d, ν, τ, w, r], {x, 0, 1.3}, PlotRange → All]
p10 = ListPlot[Table[{y, cdf1[y]}, {y, 0.05, 1.2, 0.05}],
   AxesLabel → {x, F[x]}, PlotStyle → RGBColor[1, 0, 0]];
```

$$\text{t} = \text{Table}[\{x_i,\ (i/n) - 1/(2\,n)\},\ \{i, 1, \text{Length}[\text{data}]\}];$$

```
lp1 = ListPlot[t];
Show[p10, lp1]
```

Parameters Estimates of the GEM (2.1) Using the Proxy Distribution (Sections 2.4.4 & 2.7.3) for the Repair Data Set (Table 2.5)

```
repair = Sort[{.2, .3, .5, .5, .5, .5, .6, .6, .7,
    .7, .7, .8, .8, 1, 1, 1, 1, 1.1, 1.3, 1.5, 1.5, 1.5,
    1.5, 2, 2, 2.2, 2.5, 2.7, 3, 3, 3.3, 3.3, 4, 4, 4.5,
    4.7, 5, 5.4, 5.4, 7, 7.5, 8.8, 9, 10.3, 22, 24.5}];
n = Length[repair]
m = Mean[repair]
Normal[Series[ⅇ^(-ν y), {y, m, 7}]] /. y → x^δ
```

```
46
```

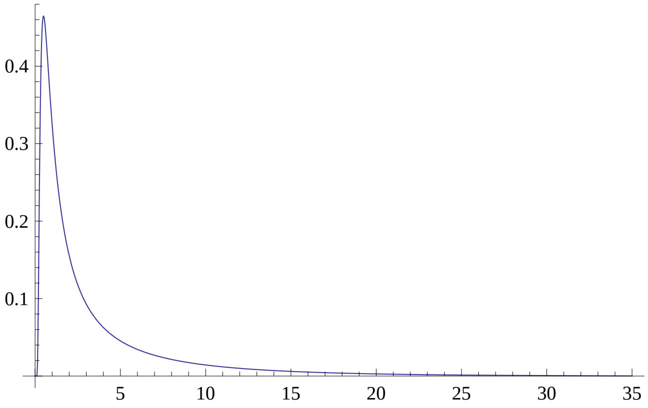
```
3.60652
```

$$\mathrm{e}^{-3.6065217391304345`\,\nu} - \mathrm{e}^{-3.6065217391304345`\,\nu}\left(-3.6065217391304345` + x^{\delta}\right)\nu + \frac{1}{2}\mathrm{e}^{-3.6065217391304345`\,\nu}\left(-3.6065217391304345` + x^{\delta}\right)^{2}\nu^{2} - \frac{1}{6}\mathrm{e}^{-3.6065217391304345`\,\nu}\left(-3.6065217391304345` + x^{\delta}\right)^{3}\nu^{3} + \frac{1}{24}\mathrm{e}^{-3.6065217391304345`\,\nu}\left(-3.6065217391304345` + x^{\delta}\right)^{4}\nu^{4} - \frac{1}{120}\mathrm{e}^{-3.6065217391304345`\,\nu}\left(-3.6065217391304345` + x^{\delta}\right)^{5}\nu^{5} + \frac{1}{720}\mathrm{e}^{-3.6065217391304345`\,\nu}\left(-3.6065217391304345` + x^{\delta}\right)^{6}\nu^{6} - \frac{\mathrm{e}^{-3.6065217391304345`\,\nu}\left(-3.6065217391304345` + x^{\delta}\right)^{7}\nu^{7}}{5040}$$

```
fExp[x_, ξ_, δ_, ν_, τ_, ρ_] := fExp[x, ξ, δ, ν, τ, ρ] = x^(ξ+δ)
  (ⅇ^(-3.6065217391304345` ν) - ⅇ^(-3.6065217391304345` ν) (-3.6065217391304345` + x^δ) ν +
    1/2 ⅇ^(-3.6065217391304345` ν) (-3.6065217391304345` + x^δ)^2 ν^2 -
    1/6 ⅇ^(-3.6065217391304345` ν) (-3.6065217391304345` + x^δ)^3 ν^3 +
    1/24 ⅇ^(-3.6065217391304345` ν) (-3.6065217391304345` + x^δ)^4 ν^4 -
    1/120 ⅇ^(-3.6065217391304345` ν) (-3.6065217391304345` + x^δ)^5 ν^5 +
    1/720 ⅇ^(-3.6065217391304345` ν) (-3.6065217391304345` + x^δ)^6 ν^6 -
    (ⅇ^(-3.6065217391304345` ν) (-3.6065217391304345` + x^δ)^7 ν^7)/5040) ⅇ^(-τ x^-(ρ))
 values of the parameter has been obtained from the paper
ξ = -2.813912569953034`; ν = 0.012865718614719886`;
τ = 0.3061641062613921`; δ = 1.5; ρ = 1.5;
cNum = NIntegrate[ fExp[x, ξ, δ, ν, τ, ρ], {x, 0, 35}]
p1 = Plot[fExp[x, ξ, δ, ν, τ, ρ] / cNum, {x, 0, 35}, PlotRange → All]
```

```
been from has obtained of paper parameter the^2 values

2.24074
```



```
ClearAll[ξ, δ, ν, τ, ρ]
Integrate[ fExp[x, ξ, δ, ν, τ, ρ], {x, 0, 35},
 Assumptions → ρ > 0 && δ > 0 && ν > 0 && τ > 0]
```

$$
\begin{aligned}
&\frac{1}{5040\,\rho}\, e^{-3.6065217391304345`\,\nu}\,\tau^{\frac{1+\delta+\xi}{\rho}}\,\Big(7936.332969693193`\,(0.7650037652698287` + \nu)\\
&\quad (1.5245387297876563` - 0.7800235405908453`\,\nu + \nu^2)\\
&\quad (0.8515177511153613` + 0.6361812144316815`\,\nu + \nu^2)\\
&\quad (0.6394622562309058` + 1.3197668309315294`\,\nu + \nu^2)\ \text{Gamma}\Big[-\frac{1+\delta+\xi}{\rho},\ 35^{-\rho}\,\tau\Big] +\\
&\quad \nu\,\tau^{\delta/\rho}\,\Big(-15403.85302134544`\,(1.100852683662018` - 0.44564354752820917`\,\nu + \nu^2)\\
&\qquad (0.6154910832459223` + 0.7995523082010735`\,\nu + \nu^2)\\
&\qquad (0.4828922098512196` + 1.3097440422204452`\,\nu + \nu^2)\ \text{Gamma}\Big[-\frac{1+2\,\delta+\xi}{\rho},\\
&\qquad 35^{-\rho}\,\tau\Big] + \nu\,\tau^{\delta/\rho}\,\Big(12813.331627158957`\,(0.6046288589850255` + \nu)\\
&\qquad\quad (0.7568223154096397` - 0.13298485970653137`\,\nu + \nu^2)\\
&\qquad\quad (0.4297893731454994` + 0.9147333364659304`\,\nu + \nu^2)\ \text{Gamma}\Big[-\frac{1+3\,\delta+\xi}{\rho},\\
&\qquad\quad 35^{-\rho}\,\tau\Big] + \nu\,\tau^{\delta/\rho}\,\Big(-5921.370854423469`\,(0.4879759526718002` +\\
&\qquad\qquad 0.15003695444105558`\,\nu + \nu^2)\,(0.2907090846887768` +\\
&\qquad\qquad 0.959064914154484`\,\nu + \nu^2)\ \text{Gamma}\Big[-\frac{1+4\,\delta+\xi}{\rho},\ 35^{-\rho}\,\tau\Big] +\\
&\qquad\quad \nu\,\tau^{\delta/\rho}\,\Big(1641.8508698220592`\,(0.4425515090249114` + \nu)\\
&\qquad\qquad (0.28901591702904156` + 0.38927489242174335`\,\nu + \nu^2)\ \text{Gamma}\Big[\\
&\qquad\qquad -\frac{1+5\,\delta+\xi}{\rho},\ 35^{-\rho}\,\tau\Big] + \nu\,\tau^{\delta/\rho}\,\Big(\big(-42.` - 151.47391304347826`\,\nu -\\
&\qquad\qquad\quad 273.14698015122866`\,\nu^2\big)\ \text{Gamma}\Big[-\frac{1+6\,\delta+\xi}{\rho},\ 35^{-\rho}\,\tau\Big] +\\
&\qquad\qquad \nu\,\tau^{\delta/\rho}\,\Big((7.` + 25.24565217391304`\,\nu)\ \text{Gamma}\Big[-\frac{1+7\,\delta+\xi}{\rho},\\
&\qquad\qquad\quad 35^{-\rho}\,\tau\Big] - 1.`\,\nu\,\tau^{\delta/\rho}\,\text{Gamma}\Big[-\frac{1+8\,\delta+\xi}{\rho},\ 35^{-\rho}\,\tau\Big]\Big)\Big)\Big)\Big)\Big)\Big)\Big)
\end{aligned}
$$

$$\mathbf{c}\left[\xi\_,\ \delta\_,\ \nu\_,\ \tau\_,\ \rho\_\right] :=$$

$$\mathbf{c}[\xi,\ \delta,\ \nu,\ \tau,\ \rho] = \frac{1}{5040\,\rho}\, e^{-3.6065217391304345`\,\nu\,\tau^{\frac{1+\delta+\xi}{\rho}}} \Big(7936.332969693193`$$

$$(0.7650037652698287` + \nu)\,\left(1.5245387297876563` - 0.7800235405908453`\,\nu + \nu^2\right)$$

$$\left(0.8515177511153613` + 0.6361812144316815`\,\nu + \nu^2\right)$$

$$\left(0.6394622562309058` + 1.3197668309315294`\,\nu + \nu^2\right)\ \mathbf{Gamma}\left[-\frac{1+\delta+\xi}{\rho},\ 35^{-\rho}\,\tau\right] +$$

$$\nu\,\tau^{\delta/\rho}\,\Big(-15403.85302134544`\,\left(1.100852683662018` - 0.44564354752820917`\,\nu + \nu^2\right)$$

$$\left(0.6154910832459223` + 0.7995523082010735`\,\nu + \nu^2\right)$$

$$\left(0.4828922098512196` + 1.3097440422204452`\,\nu + \nu^2\right)\ \mathbf{Gamma}\left[-\frac{1+2\,\delta+\xi}{\rho},\right.$$

$$\left.35^{-\rho}\,\tau\right] + \nu\,\tau^{\delta/\rho}\,\Big(12813.331627158957`\,(0.6046288589850255` + \nu)$$

$$\left(0.7568223154096397` - 0.13298485970653137`\,\nu + \nu^2\right)$$

$$\left(0.4297893731454994` + 0.9147333364659304`\,\nu + \nu^2\right)\ \mathbf{Gamma}\left[-\frac{1+3\,\delta+\xi}{\rho},\right.$$

$$\left.35^{-\rho}\,\tau\right] + \nu\,\tau^{\delta/\rho}\,\Big(-5921.370854423469`\,\big(0.4879759526718002` +$$

$$0.15003695444105558`\,\nu + \nu^2\big)\,\big(0.2907090846887768` +$$

$$0.959064914154484`\,\nu + \nu^2\big)\ \mathbf{Gamma}\left[-\frac{1+4\,\delta+\xi}{\rho},\ 35^{-\rho}\,\tau\right] +$$

$$\nu\,\tau^{\delta/\rho}\,\Big(1641.8508698220592`\,(0.4425515090249114` + \nu)$$

$$\left(0.28901591702904156` + 0.38927489242174335`\,\nu + \nu^2\right)\ \mathbf{Gamma}\Big[$$

$$-\frac{1+5\,\delta+\xi}{\rho},\ 35^{-\rho}\,\tau\Big] + \nu\,\tau^{\delta/\rho}\,\Big(\big(-42.` - 151.47391304347826`\,\nu -$$

$$273.1469801512286`\,\nu^2\big)\ \mathbf{Gamma}\left[-\frac{1+6\,\delta+\xi}{\rho},\ 35^{-\rho}\,\tau\right] +$$

$$\nu\,\tau^{\delta/\rho}\,\Big((7.` + 25.24565217391304`\,\nu)\ \mathbf{Gamma}\left[-\frac{1+7\,\delta+\xi}{\rho},\right.$$

$$\left.35^{-\rho}\,\tau\right] - 1.`\,\nu\,\tau^{\delta/\rho}\,\mathbf{Gamma}\left[-\frac{1+8\,\delta+\xi}{\rho},\ 35^{-\rho}\,\tau\right]\Big)\Big)\Big)\Big)\Big)\Big)\Big)$$

$$\mathbf{fExp3}\left[\mathrm{x}\_,\ \xi\_,\ \delta\_,\ \nu\_,\ \tau\_,\ \rho\_\right] := \mathbf{fExp3}[\mathrm{x},\ \xi,\ \delta,\ \nu,\ \tau,\ \rho] =$$

$$\mathbf{fExp}[\mathrm{x},\ \xi,\ \delta,\ \nu,\ \tau,\ \rho]\,/\,\mathbf{c}[\xi,\ \delta,\ \nu,\ \tau,\ \rho]$$

```
ClearAll[ξ, δ, ν, τ, ρ]
LogExpan[ξ_, δ_, ν_, τ_, ρ_] := LogExpan[ξ, δ, ν, τ, ρ] =
```

$$\text{Log}\left[\prod_{i=1}^{n} \text{fExp3}[\text{repair}[[i]],\ \xi,\ \delta,\ \nu,\ \tau,\ \rho]\right]$$

```
LogExpan1[ξ_, δ_, ν_, τ_, ρ_] :=
 If[LogExpan[ξ, δ, ν, τ, ρ] ∈ Reals && fExp3[.2, ξ, δ, ν, τ, ρ] ∈ Reals &&
   fExp3[24.5, ξ, δ, ν, τ, ρ] ∈ Reals && fExp3[.2, ξ, δ, ν, τ, ρ] > 0 &&
   fExp3[24.5, ξ, δ, ν, τ, ρ] > 0, LogExpan[ξ, δ, ν, τ, ρ], -10^10]

vt = NMaximize[{Evaluate[LogExpan1[ξ, δ, ν, τ, ρ]],
    -7 < ξ < 3, .1 < δ < 3, .001 < ν < 3, .1 < τ < 10, .1 < ρ < 3},
  {ξ, δ, ν, τ, ρ}, MaxIterations → 1000]

ξ = vt[[2, 1, 2]]
δ = vt[[2, 2, 2]]
ν = vt[[2, 3, 2]]
τ = vt[[2, 4, 2]]
ρ = vt[[2, 5, 2]]
p = Plot[Evaluate[fExp3[x, ξ, δ, ν, τ, ρ]], {x, 0, 35},
  PlotStyle → RGBColor[1, 0, 0], PlotRange → All]
```

$$x_{i\_} := \text{repair}[[i]]$$

```
cdfExpan[y_] :=
  cdfExpan[y] = NIntegrate[ fExp3[x, ξ, δ, ν, τ, ρ], {x, 0, y}];
```

$$z_{i\_} := \text{Chop}[\text{cdfExpan}[x_i]]$$

$$\text{A0} = -n - \frac{1}{n}\sum_{i=1}^{n}((2\,i - 1)\ \text{Log}[z_i\ (1 - z_{n+1-i})])$$

$$\text{W0} = \frac{1}{12\,n} + \sum_{i=1}^{n}\left(z_i - \frac{2\,i - 1}{2\,n}\right)^2$$

Parameters Estimates of the GEM (2.1) for the Flood Data (Table 2.1) by Determining the Normalizing Constant (Section 2.7.4)

```
flood = Sort[{.654, .613, .402, .379, .269,
    .740, .416, .338, .315, .449, .297, .423, .379,
    .3235, .418, .412, .494, .392, .484, .265}];
n = Length[flood];
x_i_ := flood[[i]]

fFlood3[x_?NumberQ, ξ_?NumberQ, δ_?NumberQ,
  ν_?NumberQ, τ_?NumberQ, ρ_?NumberQ] :=
 fFlood3[x, ξ, δ, ν, τ, ρ] = x^(ξ+δ) e^(-ν x^δ)
   e^(-τ x^-ρ) / (Evaluate[NIntegrate[y^(ξ+δ) e^(-ν y^δ) e^(-τ y^-ρ), {y, 0, 10000}]])
Off[NIntegrate::inumr]

LogFlood2[ξ_?NumberQ, δ_?NumberQ,
  ν_?NumberQ, τ_?NumberQ, ρ_?NumberQ] :=
 LogFlood2[ξ, δ, ν, τ, ρ] = Log[Product[fFlood3[flood[[i]], ξ, δ, ν, τ, ρ], {i, 1, n}]]
LogFlood3[ξ_, δ_, ν_, τ_, ρ_] :=
 LogFlood3[ξ, δ, ν, τ, ρ] =  If[LogFlood2[ξ, δ, ν, τ, ρ] ∈ Reals &&
    fFlood3[.265, ξ, δ, ν, τ, ρ] > 0 &&
    fFlood3[.654, ξ, δ, ν, τ, ρ] > 0, LogFlood2[ξ, δ, ν, τ, ρ], -10^10]

cdfFlood[y_] := cdfFlood[y] =
  NIntegrate[ fFlood3[x, ξ, δ, ν, τ, ρ], {x, 0, y}];
z_i_ := cdfFlood[x_i]
```

```
ClearAll[ξ, δ, ν, τ, ρ]
vt = Timing[
  NMaximize[{Evaluate[LogFlood3[ξ, δ, ν, τ, ρ]], -10 < ξ < 3,
    .1 < δ < 3, .1 < ν < 3, .1 < τ < 1, .1 < ρ < 3}, {ξ, δ, ν, τ, ρ},
   MaxIterations → 1000, Method → "SimulatedAnnealing"]]
```

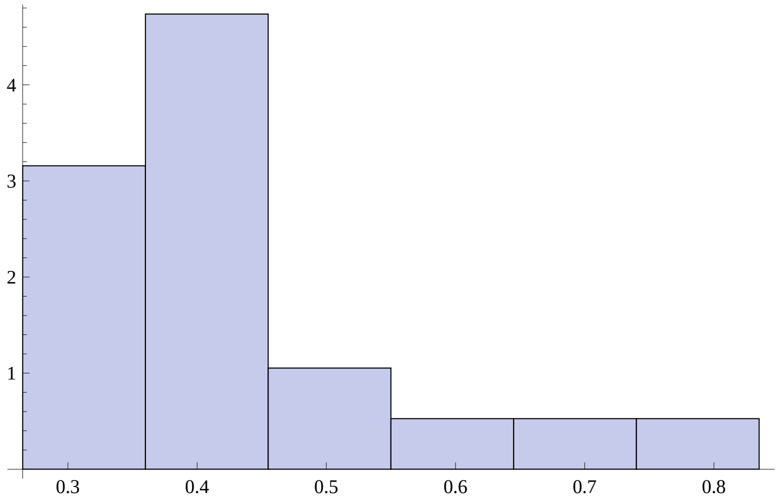
Proposed Maximization Methodology (Section 3.5)

Searching for the maximum on a grid fo the Flood data: An iterative methodology based on previously obtained sequences of maxima

```
fBesselFlood[x_, ξ_, δ_, ν_, τ_] :=
 fBesselFlood[x, ξ, δ, ν, τ] = x^(ξ+δ) ⅇ^(-ν x^δ) ⅇ^(-τ x^-δ) δ
   ν^((ξ+δ+1)/δ) / (2 (ν τ)^((1+δ+ξ)/(2 δ)) BesselK[(1 + δ + ξ)/δ, 2 √(ν τ)])
LgBesselFlood[ξ_, δ_, ν_, τ_] := LgBesselFlood[ξ, δ, ν, τ] =
  Log[∏_(i=1)^n fBesselFlood[data[[i]], ξ, δ, ν, τ]]
Lg1BesselFlood[ξ_, δ_, ν_, τ_] :=
 Lg1BesselFlood[ξ, δ, ν, τ] =
  If[LgBesselFlood[ξ, δ, ν, τ] ∈ Reals &&
    fBesselFlood[.265, ξ, δ, ν, τ] > 0 &&
    fBesselFlood[.654, ξ, δ, ν, τ] > 0,
   LgBesselFlood[ξ, δ, ν, τ], -10^10]

data = Sort[{.654, .613, .402, .379, .269,
    .740, .416, .338, .315, .449, .297, .423, .379,
    .3235, .418, .412, .494, .392, .484, .265}];
n = Length[data];
x_i_ := data[[i]]
cdf1[y_] :=
  cdf1[y] = NIntegrate[ fBesselFlood[x, ξ, δ, ν, τ], {x, 0, y}];
z_i_ := cdf1[x_i]
```

```
Hs = Hist1[data, 5]
```



The loglikelihood corresponding to the best combination in our paper :

Evaluate[LgBesselFlood[-9.95, 2.24, 0.09, 0.34]]

16.274046245042985`

At first, we cast a wide net to determine neighborhoods where maxima could occur :

```
ClearAll[ξ, δ, τ, ν]
vt2 =
  Table[Evaluate[Lg1BesselFlood[ξ, δ, ν, τ]], {ξ, -15, 0, .5},
   {δ, .01, 6.01, .5}, {ν, .01, 6.01, .5}, {τ, .01, 6.01, .5}];
vtc = Table[{ξ, δ, ν, τ}, {ξ, -15, 0, .5}, {δ, .01, 6.01, .5},
   {ν, .01, 6.01, .5}, {τ, .01, 6.01, .5}];
vtmax = Max[vt2]
16.2807
ps = Position[vt2, Max[vt2]]
{{12, 5, 4, 2}}
ps[[1, 3]]
4
```

```
vm = vtc[[ps[[1, 1]], ps[[1, 2]], ps[[1, 3]], ps[[1, 4]]]]
{-9.5, 2.01, 1.51, 0.51}
```

$\{\xi, \delta, \nu, \tau\}$

```
Evaluate[Lg1BesselFlood[vm[[1]], vm[[2]], vm[[3]], vm[[4]]]]
16.2807
```

- **Now consider the [root 4 of 31*13*13*13] largest values of vt2 and the corresponding coordinates. Intervals for the next iteration can be determined from the set of points corresponding to these coordinates.**

```
Dimensions[vt2]
{31, 13, 13, 13}
```

$$\mathtt{rt} = \mathtt{Floor}\left[\sqrt[\mathtt{Length[Dimensions[vt2]]}]{\prod_{\mathtt{j}=1}^{\mathtt{Length[Dimensions[vt2]]}} \mathtt{Dimensions[vt2][[j]]}}\right]$$

```
16
```

```
tb = Table[-Sort[Flatten[-vt2]][[j]], {j, 1, rt}]
{16.2807, 16.2705, 16.2697, 16.2648,
 16.2479, 16.247, 16.2415, 16.239, 16.2344, 16.234,
 16.2332, 16.2331, 16.2124, 16.2096, 16.2063}
```

```
psv = Flatten[Table[Position[vt2, tb[[j]]], {j, 1, rt}], 1]
{{12, 5, 4, 2}, {18, 9, 11, 1}, {11, 5, 2, 2},
 {18, 9, 12, 1}, {12, 5, 5, 2}, {18, 9, 10, 1}, {11, 5, 3, 2},
 {12, 10, 4, 1}, {12, 5, 3, 2}, {18, 9, 13, 1}, {12, 10, 5, 1},
 {13, 5, 6, 2}, {7, 4, 1, 4}, {7, 4, 2, 4}, {8, 4, 3, 4}}
```

```
psv[[4, 3]]
12
```

```
Table[vtc[[psv[[j, 1]], psv[[j, 2]],
  psv[[j, 3]], psv[[j, 4]]]], {j, 1, rt}]
{{-9.5, 2.01, 1.51, 0.51},
 {-6.5, 4.01, 5.01, 0.01}, {-10., 2.01, 0.51, 0.51},
 {-6.5, 4.01, 5.51, 0.01}, {-9.5, 2.01, 2.01, 0.51},
 {-6.5, 4.01, 4.51, 0.01}, {-10., 2.01, 1.01, 0.51},
 {-9.5, 4.51, 1.51, 0.01}, {-9.5, 2.01, 1.01, 0.51},
 {-6.5, 4.01, 6.01, 0.01}, {-9.5, 4.51, 2.01, 0.01},
 {-9., 2.01, 2.51, 0.51}, {-12., 1.51, 0.01, 1.51},
 {-12., 1.51, 0.51, 1.51}, {-11.5, 1.51, 1.01, 1.51}}
```

$\{\xi, \delta, \nu, \tau\}$

It is seen from these points that the following ranges could be investigated

```
ClearAll[ξ, δ, τ, ν]
vt2 = Table[Evaluate[Lg1BesselFlood[ξ, δ, ν, τ]],
   {ξ, -12.5, -6, .25}, {δ, 1.3, 4.3, .25},
   {ν, .01, 7.01, .25}, {τ, .01, 2.01, .25}];
vtc = Table[{ξ, δ, ν, τ}, {ξ, -12.5, -6, .25}, {δ, 1.3,
    4.3, .25}, {ν, .01, 7.01, .25}, {τ, .01, 2.01, .25}];
vtmax = Max[vt2]
16.4928
ps = Position[vt2, Max[vt2]]
{{18, 13, 13, 1}}
ps[[1, 3]]
13
vm = vtc[[ps[[1, 1]], ps[[1, 2]], ps[[1, 3]], ps[[1, 4]]]]
{-8.25, 4.3, 3.01, 0.01}
```

$\{\xi, \delta, \nu, \tau\}$

```
Evaluate[Lg1BesselFlood[vm[[1]], vm[[2]], vm[[3]], vm[[4]]]]
16.4928
```

- **Now consider the [root 4 of the total number of point] largest values of vt2 and the corresponding coordinates. Intervals for the next iteration can be determined from the set of points corresponding to these coordinates.**

```
Dimensions[vt2]
{27, 13, 29, 9}
```

$$\text{rt} = \text{Floor}\left[\sqrt[\text{Length[Dimensions[vt2]]}]{\prod_{\text{j}=1}^{\text{Length[Dimensions[vt2]]}} \text{Dimensions[vt2][[j]]}}\,\right]$$

```
17
tb = Table[-Sort[Flatten[-vt2]][[j]], {j, 1, rt}]
{16.4928, 16.4911, 16.487, 16.4809, 16.4752, 16.4746,
 16.4736, 16.4701, 16.4689, 16.4673, 16.4644, 16.464,
 16.4613, 16.4612, 16.4561, 16.4509, 16.4485}

psv = Flatten[Table[Position[vt2, tb[[j]]], {j, 1, rt}], 1]
{{18, 13, 13, 1}, {18, 13, 12, 1},
 {18, 13, 14, 1}, {18, 13, 11, 1}, {17, 13, 10, 1},
 {18, 13, 15, 1}, {17, 13, 11, 1}, {19, 13, 15, 1},
 {19, 13, 16, 1}, {17, 13, 9, 1}, {19, 13, 14, 1},
 {17, 13, 12, 1}, {19, 13, 17, 1}, {18, 13, 10, 1},
 {18, 13, 16, 1}, {19, 13, 13, 1}, {17, 13, 8, 1}}
psv[[4, 3]]
11
```

```
Table[vtc[[psv[[j, 1]], psv[[j, 2]],
  psv[[j, 3]], psv[[j, 4]]]], {j, 1, rt}]
{{-8.25, 4.3, 3.01, 0.01},
 {-8.25, 4.3, 2.76, 0.01}, {-8.25, 4.3, 3.26, 0.01},
 {-8.25, 4.3, 2.51, 0.01}, {-8.5, 4.3, 2.26, 0.01},
 {-8.25, 4.3, 3.51, 0.01}, {-8.5, 4.3, 2.51, 0.01},
 {-8., 4.3, 3.51, 0.01}, {-8., 4.3, 3.76, 0.01},
 {-8.5, 4.3, 2.01, 0.01}, {-8., 4.3, 3.26, 0.01},
 {-8.5, 4.3, 2.76, 0.01}, {-8., 4.3, 4.01, 0.01},
 {-8.25, 4.3, 2.26, 0.01}, {-8.25, 4.3, 3.76, 0.01},
 {-8., 4.3, 3.01, 0.01}, {-8.5, 4.3, 1.76, 0.01}}
```

**It is seen from these points that
the following ranges could be investigated**

```
ClearAll[ξ, δ, τ, ν]
vt2 = Table[Evaluate[Lg1BesselFlood[ξ, δ, ν, τ]],
   {ξ, -8.7, -7.8, .1}, {δ, 4.1, 4.4, .05},
   {ν, 1.7, 4.1, .2}, {τ, .005, 0.015, .003}];
vtc = Table[{ξ, δ, ν, τ}, {ξ, -8.7, -7.8, .1}, {δ, 4.1,
    4.4, .05}, {ν, 1.7, 4.1, .2}, {τ, .005, 0.015, .003}];
vtmax = Max[vt2]
16.4999
ps = Position[vt2, Max[vt2]]
{{7, 7, 9, 2}}
ps[[1, 3]]
9
vm = vtc[[ps[[1, 1]], ps[[1, 2]], ps[[1, 3]], ps[[1, 4]]]]
{-8.1, 4.4, 3.3, 0.008}
```

$\{\xi, \delta, \nu, \tau\}$

```
Evaluate[Lg1BesselFlood[vm[[1]], vm[[2]], vm[[3]], vm[[4]]]]
16.4999
```

- **Now consider the [root 4 of the total number of point] largest values of vt2 and the corresponding coordinates. Intervals for the next iteration can be determined from the set of points corresponding to these coordinates.**

```
Dimensions[vt2]
{10, 7, 13, 4}
```

$$\mathtt{rt} = \mathtt{Floor}\left[\sqrt[\mathtt{Length[Dimensions[vt2]]}]{\prod_{j=1}^{\mathtt{Length[Dimensions[vt2]]}} \mathtt{Dimensions[vt2][[j]]}}\right]$$

```
7
```

```
tb = Table[-Sort[Flatten[-vt2]][[j]], {j, 1, rt}]
{16.4999, 16.4998, 16.4993, 16.4979, 16.4975, 16.496, 16.4939}
```

```
psv = Flatten[Table[Position[vt2, tb[[j]]], {j, 1, rt}], 1]
{{7, 7, 9, 2}, {8, 7, 10, 2}, {7, 7, 8, 2},
 {8, 7, 9, 2}, {8, 7, 11, 2}, {7, 7, 10, 2}, {7, 7, 7, 2}}
psv[[4, 3]]
9
Table[vtc[[psv[[j, 1]], psv[[j, 2]],
  psv[[j, 3]], psv[[j, 4]]]], {j, 1, rt}]
{{-8.1, 4.4, 3.3, 0.008},
 {-8., 4.4, 3.5, 0.008}, {-8.1, 4.4, 3.1, 0.008},
 {-8., 4.4, 3.3, 0.008}, {-8., 4.4, 3.7, 0.008},
 {-8.1, 4.4, 3.5, 0.008}, {-8.1, 4.4, 2.9, 0.008}}
```

```
It is seen from these points that
 the following ranges could be investigated
```

```
ClearAll[ξ, δ, τ, ν]
vt2 = Table[Evaluate[Lg1BesselFlood[ξ, δ, ν, τ]],
   {ξ, -8.2, -7.9, .05}, {δ, 4.3, 4.6, .05},
   {ν, 3, 4, .1}, {τ, .006, 0.010, .002}];
vtc = Table[{ξ, δ, ν, τ}, {ξ, -8.2, -7.9, .05},
   {δ, 4.3, 4.6, .05}, {ν, 3, 4, .1}, {τ, .006, 0.010, .002}];
vtmax = Max[vt2]
16.5131
ps = Position[vt2, Max[vt2]]
{{1, 7, 6, 1}}
ps[[1, 3]]
6
vm = vtc[[ps[[1, 1]], ps[[1, 2]], ps[[1, 3]], ps[[1, 4]]]]
{-8.2, 4.6, 3.5, 0.006}
```

$\{\xi, \delta, \nu, \tau\}$

```
Evaluate[Lg1BesselFlood[vm[[1]], vm[[2]], vm[[3]], vm[[4]]]]
16.5131
```

- **Now consider the [root 4 of the total number of point] largest values of vt2 and the corresponding coordinates. Intervals for the next iteration can be determined from the set of points corresponding to these coordinates.**

```
Dimensions[vt2]
{7, 7, 11, 3}
```

$$\mathrm{rt} = \mathrm{Floor}\left[\sqrt[\mathrm{Length[Dimensions[vt2]]}]{\prod_{j=1}^{\mathrm{Length[Dimensions[vt2]]}} \mathrm{Dimensions[vt2][[j]]}}\right]$$

```
6
```

```
tb = Table[-Sort[Flatten[-vt2]][[j]], {j, 1, rt}]
{16.5131, 16.513, 16.5121, 16.5119, 16.5117, 16.5113}
```

```
psv = Flatten[Table[Position[vt2, tb[[j]]], {j, 1, rt}], 1]
{{1, 7, 6, 1}, {1, 7, 5, 1}, {1, 7, 7, 1},
 {1, 7, 4, 1}, {2, 7, 7, 1}, {2, 7, 6, 1}}
```

```
psv[[4, 3]]
4
```

```
Table[vtc[[psv[[j, 1]], psv[[j, 2]],
  psv[[j, 3]], psv[[j, 4]]]], {j, 1, rt}]
{{-8.2, 4.6, 3.5, 0.006}, {-8.2, 4.6, 3.4, 0.006},
 {-8.2, 4.6, 3.6, 0.006}, {-8.2, 4.6, 3.3, 0.006},
 {-8.15, 4.6, 3.6, 0.006}, {-8.15, 4.6, 3.5, 0.006}}
```

It is seen from these points that the following ranges could be investigated

```
ClearAll[ξ, δ, τ, ν]
vt2 = Table[Evaluate[Lg1BesselFlood[ξ, δ, ν, τ]],
   {ξ, -8.3, -8.1, .02}, {δ, 4.5, 4.7, .02},
   {ν, 3.2, 3.8, .04}, {τ, .003, 0.007, .001}];
vtc = Table[{ξ, δ, ν, τ}, {ξ, -8.3, -8.1, .02}, {δ, 4.5,
    4.7, .02}, {ν, 3.2, 3.8, .04}, {τ, .003, 0.007, .001}];
```

```
vtmax = Max[vt2]
16.5166
```

The loglikelihood corresponding to the best combination in our paper :

Evaluate[LgBesselFlood[-9.95, 2.24, 0.09, 0.34]]

16.274046245042985` which is lower than 16.5166.

```
ps = Position[vt2, Max[vt2]]
{{8, 11, 13, 3}}
ps[[1, 3]]
13
vm = vtc[[ps[[1, 1]], ps[[1, 2]], ps[[1, 3]], ps[[1, 4]]]]
{-8.16, 4.7, 3.68, 0.005}
```

$\{\xi, \delta, \nu, \tau\}$

```
Evaluate[Lg1BesselFlood[vm[[1]], vm[[2]], vm[[3]], vm[[4]]]]
16.5166
ξ = -8.16`; δ = 4.7`; ν = 3.68`; τ = 0.005`;
```

$$\mathrm{A0} = -n - \frac{1}{n}\sum_{i=1}^{n}\left((2\,i-1)\ \mathrm{Log}[z_i\ (1-z_{n+1-i})]\right)$$

$$\mathrm{W0} = \frac{1}{12\,n} + \sum_{i=1}^{n}\left(z_i - \frac{2\,i-1}{2\,n}\right)^2$$

```
0.28611
0.0492518
```

- **Now consider the [root 4 of the total number of point] largest values of vt2 and the corresponding coordinates. Intervals for the next iteration can be determined from the set of points corresponding to these coordinates.**

```
Dimensions[vt2]
{11, 11, 16, 5}
```

$$\mathbf{rt} = \mathbf{Floor}\left[\sqrt[\text{Length[Dimensions[vt2]]}]{\prod_{\mathbf{j}=1}^{\text{Length[Dimensions[vt2]]}} \text{Dimensions[vt2][[j]]}}\right]$$

```
9
```

```
tb = Table[-Sort[Flatten[-vt2]][[j]], {j, 1, rt}]
```

```
{16.5166, 16.5166, 16.5165, 16.5165,
 16.5165, 16.5164, 16.5164, 16.5163, 16.5163}
```

It is seen from tb that convergence has more or less been achieved.

```
psv = Flatten[Table[Position[vt2, tb[[j]]], {j, 1, rt}], 1]
```

```
{{8, 11, 13, 3}, {7, 11, 12, 3}, {7, 11, 11, 3},
 {8, 11, 14, 3}, {8, 11, 12, 3}, {7, 11, 13, 3},
 {7, 11, 10, 3}, {8, 11, 15, 3}, {6, 11, 10, 3}}
```

```
psv[[4, 3]]
```

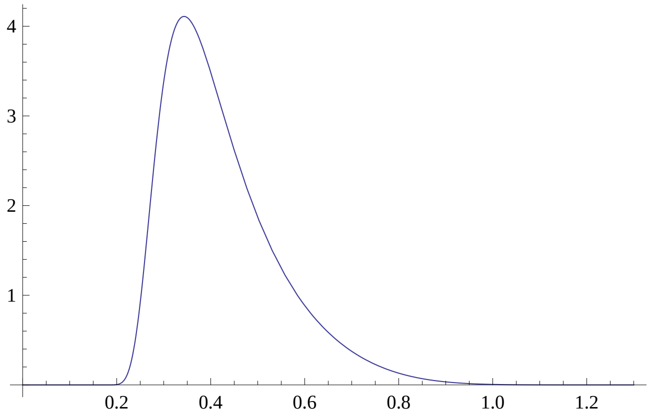
```
14
```

```
Table[vtc[[psv[[j, 1]], psv[[j, 2]],
   psv[[j, 3]], psv[[j, 4]]]], {j, 1, rt}]
```

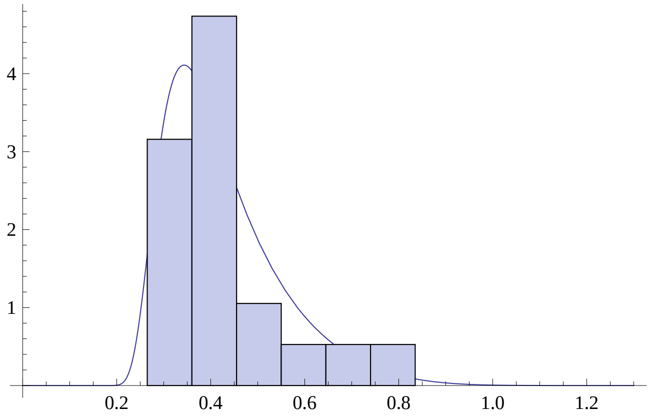
```
{{-8.16, 4.7, 3.68, 0.005},
 {-8.18, 4.7, 3.64, 0.005}, {-8.18, 4.7, 3.6, 0.005},
 {-8.16, 4.7, 3.72, 0.005}, {-8.16, 4.7, 3.64, 0.005},
 {-8.18, 4.7, 3.68, 0.005}, {-8.18, 4.7, 3.56, 0.005},
 {-8.16, 4.7, 3.76, 0.005}, {-8.2, 4.7, 3.56, 0.005}}
```

```
p1 = Plot[Evaluate[
   fBesselFlood[x, vm[[1]], vm[[2]], vm[[3]], vm[[4]]]],
  {x, 0, 1.3}, PlotRange → All]
```

```
General::unfl : Underflow occurred in computation. ≫
```

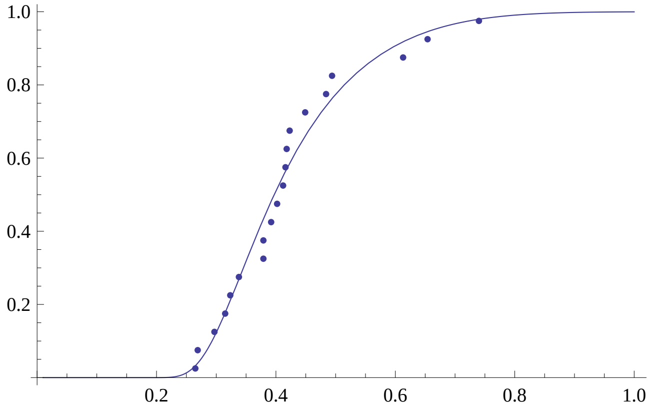


```
Show[p1, Hs]
```



```
cdf1[y_] :=
  cdf1[y] = NIntegrate[Evaluate[fBesselFlood[x, vm[[1]],
      vm[[2]], vm[[3]], vm[[4]]]], {x, 0, y}];
p10 = Plot[cdf1[y], {y, .01, 1}];
t = Table[{xi, (i / n) - 1 / (2 n)}, {i, 1, Length[data]}];
lp1 = ListPlot[t];
Show[p10, lp1]
```

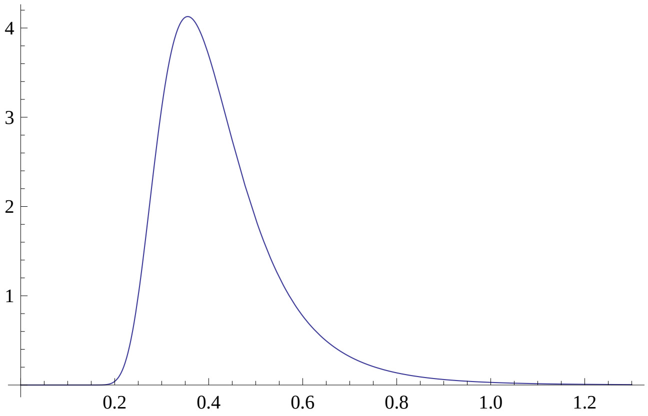
```
NIntegrate::ncvb :
  NIntegrate failed to converge to prescribed accuracy after 9 recursive bisections in x near {x} = {0.00667378}.
      NIntegrate obtained 0.` and 0.` for the integral and error estimates. ≫
```



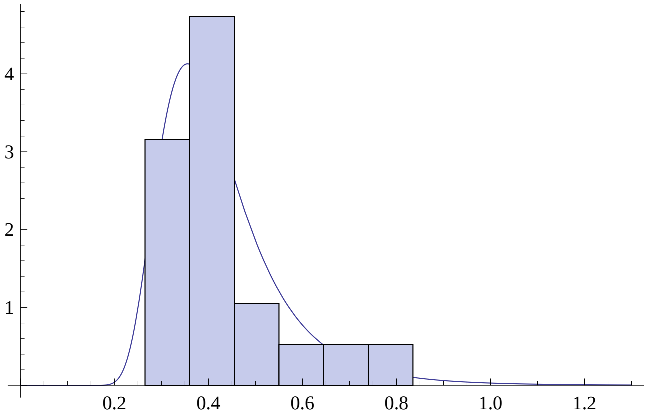
Evaluate[LgBesselFlood[-9.95, 2.24, 0.09, 0.34]]

```
p1 = Plot[Evaluate[fBesselFlood[x, -9.95, 2.24, 0.09, 0.34]],
  {x, 0, 1.3}, PlotRange → All]
```

```
General::unfl : Underflow occurred in computation. ≫
```

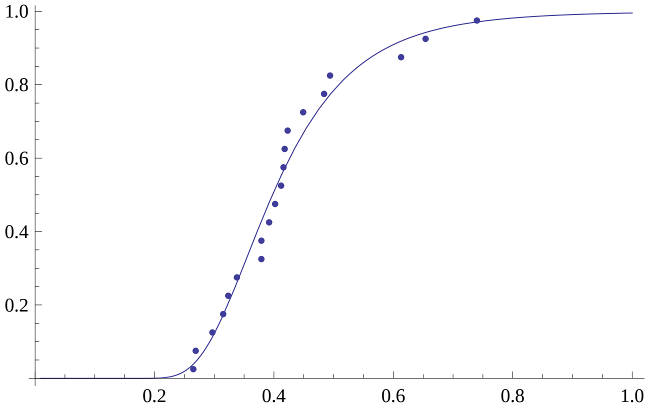


```
Show[p1, Hs]
```



```
cdf2[y_] := cdf2[y] = NIntegrate[Evaluate[
     fBesselFlood[x, -9.95, 2.24, 0.09, 0.34]], {x, 0, y}];
p10 = Plot[cdf2[y], {y, .01, 1}];
t = Table[{x_i, (i / n) - 1 / (2 n)}, {i, 1, Length[data]}];
lp1 = ListPlot[t];
Show[p10, lp1]
```

```
NIntegrate::ncvb :
  NIntegrate failed to converge to prescribed accuracy after 9 recursive bisections in x near {x} = {0.00667378}.
      NIntegrate obtained 0.` and 0.` for the integral and error estimates. ≫
```



Density Approximation (4.23) for a Linear Combination of Five Chi-square Random Variables (Section 4.4)

$$k = 5;$$

$$m_1 = 1/10;\ m_2 = 2/10;\ m_3 = 4/10;\ m_4 = 2/10;\ m_5 = 1/10;$$

$$r_1 = 4;\ r_2 = 2;\ r_3 = 2;\ r_4 = 4;\ r_5 = 2;$$

```
Off[Pattern::"nodef"]
```

$$\text{ss2} = \text{Expand}\left[\left(\sum_{j=1}^{k} m_j\, X_j\right)^2\right] /.\ \left\{X_{j\_.}^{\,i\_.} :\to 2^i\, \text{Gamma}\left[\left(r_j/2\right) + i\right] / \text{Gamma}\left[\left(r_j/2\right)\right]\right\};$$

$$\text{ss1} = \text{Expand}\left[\left(\sum_{j=1}^{k} m_j\, X_j\right)\right] /.\ \left\{X_{j\_.}^{\,i\_.} :\to 2^i\, \text{Gamma}\left[\left(r_j/2\right) + i\right] / \text{Gamma}\left[\left(r_j/2\right)\right]\right\};$$

$$\text{ss2} - (\text{ss1})^2;$$

$$\beta = \left(ss2 - (ss1)^{2}\right) / ss1$$

$$\frac{31}{65}$$

$$c = 1 \Big/ \left(\frac{31}{65}\right);\ p1 = 1\,000\,000;\ n1 = 1;$$

```
r1 = 4; r2 = 2; r3 = 2; r4 = 4; r5 = 2;
m1 = c (1 / 10);  m2 = c (2 / 10); m3 = c (4 / 10);  m4 = c (2 / 10); m5 = c (1 / 10);
cdist1 = ChiSquareDistribution[r1];
cdist2 = ChiSquareDistribution[r2];
cdist3 = ChiSquareDistribution[r3];
cdist4 = ChiSquareDistribution[r4];
cdist5 = ChiSquareDistribution[r5];


ran1 = Table[Random[cdist1], {p1}, {n1}];
ran2 = Table[Random[cdist2], {p1}, {n1}];
ran3 = Table[Random[cdist3], {p1}, {n1}];
ran4 = Table[Random[cdist4], {p1}, {n1}];
ran5 = Table[Random[cdist5], {p1}, {n1}];
data1 = Sort[Flatten[m1 * ran1 + m2 * ran2 + m3 * ran3 + m4 * ran4 + m5 * ran5]];
t1 = Prepend[Append[Table[.05 i, {i, 1, 19}], .99], .01]
t2 = Table[data1[[p1 t1[[i]]]], {i, 21}]
```

$$\mathrm{Prod}[x\_,\ v\_] := \mathrm{Prod}[x,\ v] = \prod_{j=0}^{v-1} (x + j)$$

$$\begin{aligned}
&\mathrm{f40}[r\_] := \mathrm{f40}[r] = \\
&\quad N\Big[\mathrm{Sum}\Big[\Big((\mathrm{Prod}[(r1 / 2),\ \nu1])\ (\mathrm{Prod}[(r2 / 2),\ \nu2])\ (\mathrm{Prod}[(r3 / 2),\ \nu3]) \\
&\qquad\qquad (\mathrm{Prod}[(r4 / 2),\ \nu4])\ (\mathrm{Prod}[(r5 / 2),\ \nu - \nu1 - \nu2 - \nu3 - \nu4]) \\
&\qquad\qquad \left(\gamma1^{\nu1}\right) \left(\gamma2^{\nu2}\right) \left(\gamma3^{\nu3}\right) \left(\gamma4^{\nu4}\right) \left(\gamma5^{\nu-\nu1-\nu2-\nu3-\nu4}\right)\Big) \Big/ \\
&\qquad\quad \Big((2\,m1)^{r1/2}\ (2\,m2)^{r2/2}\ (2\,m3)^{r3/2}\ (2\,m4)^{r4/2}\ (2\,m5)^{r5/2}\ \mathrm{Gamma}[p + \nu] \\
&\qquad\qquad (\nu1\,!\ \nu2\,!\ \nu3\,!\ \nu4\,!\ (\nu - \nu1 - \nu2 - \nu3 - \nu4)\,!)\Big)\ r^{p+\nu-1}\, e^{-r}, \\
&\qquad \{\nu,\ 0,\ 40\},\ \{\nu1,\ 0,\ \nu\},\ \{\nu2,\ 0,\ \nu - \nu1\},\ \{\nu3,\ 0,\ \nu - \nu1 - \nu2\}, \\
&\qquad \{\nu4,\ 0,\ \nu - \nu1 - \nu2 - \nu3\}\Big],\ 50\Big]
\end{aligned}$$

```
CDFf40[y_] := Integrate[f40[r], {r, 0, y}]
k = 5; p = (r1 + r2 + r3 + r4 + r5) / 2;
```

$$\gamma 1 = \frac{(2\,\mathrm{m1}) - 1}{2\,\mathrm{m1}};\ \gamma 2 = \frac{(2\,\mathrm{m2}) - 1}{2\,\mathrm{m2}};$$

$$\gamma 3 = \frac{(2\,\mathrm{m3}) - 1}{2\,\mathrm{m3}};\ \gamma 4 = \frac{(2\,\mathrm{m4}) - 1}{2\,\mathrm{m4}};\ \gamma 5 = \frac{(2\,\mathrm{m5}) - 1}{2\,\mathrm{m5}};$$

```
aprCDF30 = Table[CDFf30[t2[[i]]], {i, 1, 21}]
```

RGG Model (2.18) for a Linear Combination of Five Chi-square Random Variables (Section 4.4)

μ[h_] := Expand$\left[\left(\sum_{j=1}^{k} m_j\, X_j\right)^{h}\right]$ /.

  $\left\{X_{j\_.}{}^{i\_.} :\to 2^{i}\,\text{Gamma}[(r_j/2)+i]\,/\,\text{Gamma}[(r_j/2)]\right\}$

$\beta = \dfrac{31}{65}$;

c = 1 / β;

m_1 = c (1 / 10); m_2 = c (2 / 10);

m_3 = c (4 / 10); m_4 = c (2 / 10); m_5 = c (1 / 10);

r_1 = 4; r_2 = 2; r_3 = 2; r_4 = 4; r_5 = 2;

k = 5;

μ1 = μ[1]

μ2 = μ[2]

μ3 = μ[3]

n = Length[data1];

$x_{i\_}$:= data1[[i]]

fTyp1[x_, ξ_, δ_, ν_] :=

$$\text{fTyp1}[x,\ \xi,\ \delta,\ \nu] = x^{\xi+\delta-1}\, e^{-\nu\, x^{\delta}} \Big/ \frac{\nu^{-\frac{\delta+\xi}{\delta}}\ \text{Gamma}\left[\frac{\delta+\xi}{\delta}\right]}{\delta}$$

$$\text{M}[t\_] := \frac{\nu^{-t/\delta}\ \text{Gamma}\left[\frac{t+\xi+\delta}{\delta}\right]}{\text{Gamma}\left[\frac{\xi+\delta}{\delta}\right]};$$

cdf1[y_] :=

  cdf1[y] = NIntegrate[fTyp1[x, ξ, δ, ν], {x, 0, y}];

$z_{i\_}$:= cdf1[x_i]

```
vt = NMinimize[
   {(μ1 - M[1])^2 + (μ2 - M[2])^2 + (μ3 - M[3])^2, -3 < ξ < 10, .1 < δ < 3,
    .1 < ν < 3, δ + ξ > 0}, {ξ, δ, ν}, MaxIterations → 1000]
ξ = vt[[2, 1, 2]]
δ = vt[[2, 2, 2]]
ν = vt[[2, 3, 2]]
```

$$A0 = -n - \frac{1}{n}\sum_{i=1}^{n}\left((2\,i - 1)\,\mathrm{Log}[z_i\,(1 - z_{n+1-i})]\right)$$

$$W0 = \frac{1}{12\,n} + \sum_{i=1}^{n}\left(z_i - \frac{2\,i - 1}{2\,n}\right)^2$$

```
data1 = t2;
Plot[fTyp1[x, ξ, δ, ν], {x, 0, 10}, PlotRange → All]
p10 = Plot[cdf1[y], {y, 0, 10}];
t = Table[{x_i, (i / n) - 1 / (2 n)}, {i, 1, Length[data1]}];
lp1 = ListPlot[t];
CDFk5 = Sort[{0.01007843017578125`, 0.04976654052734375`, 0.0994873046875`,
     0.149505615234375`, 0.19956588745117188`, 0.2494354248046875`,
     0.2994384765625`, 0.3494911193847656`, 0.3991127014160156`,
     0.4492835998535156`, 0.499237060546875`, 0.5493927001953125`,
     0.5993003845214844`, 0.6497077941894531`, 0.6993370056152344`,
     0.7493476867675781`, 0.7993354797363281`, 0.84942626953125`,
     0.8998489379882812`, 0.9498786926269531`, 0.9899940490722656`}];
y_i_ := CDFk5[[i]]
t3 = Table[{x_i, y_i}, {i, 1, Length[data1]}];
lp3 = ListPlot[t3, PlotStyle → RGBColor[0, 1, 0]];
Show[p10, lp1, lp3]
```

Modeling Canadian Quinquennial Mortality Rates (Section 5.2)

```
Needs["NumericalCalculus`"]
pop = {2 171 546, 2 262 352, 2 236 286, 2 227 271, 2 357 234,
   2 704 985, 2 676 629, 2 368 253, 2 085 833, 1 584 973,
   1 229 157, 1 046 909, 880 880, 640 944, 346 151, 178 569};
death = {985, 1318, 1254, 1412, 2121, 3663, 5830, 8163, 10 848,
   13 365, 16 295, 22 263, 30 964, 37 626, 34 530, 34 901};
age = {15, 20, 25, 30, 35, 40, 45, 50, 55,
   60, 65, 70, 75, 80, 85, 90};
mort := N[(death / pop) * 1000]
Length[death]; Length[pop]; Length[age];

deathp = Table[{age[[i]], mort[[i]]}, {i, 16}];
polate = Interpolation[deathp, InterpolationOrder → 1];
c := Integrate[polate[x], {x, 15, 90}]
(1 / c) Integrate[polate[x], {x, 15, 90}];
p = Plot[polate[x], {x, 15, 90}];
polateDS3[age];
polate6 = Interpolation[deathp,
  InterpolationOrder → 3, Method → "Spline"]
polateDS3[y_] := polateDS3[y] = ND[polate6[x], x, y]
polateDS3[50.5]
polateDS3[2.5]
p1 = Plot[polateDS3[y], {y, 15, 95}, PlotRange → All]
```

```
f[y_] := f[y] = polateDS3[95 - y]
NIntegrate[f[y], {y, 0, 90}]
cons = NIntegrate[f[y], {y, 0, 90}]
NIntegrate[(1 / cons) f[y], {y, 0, 90}]
Plot[f[y], {y, 0, 90}]
p1 = Plot[(1 / cons) f[y], {y, 0, 90}, PlotRange → All]

f1[y_] := f1[y] = (1 / cons) f[y]
mm1 = NIntegrate[y f1[y], {y, 0, 90}]
mm2 = NIntegrate[y^2 f1[y], {y, 0, 90}]
mm3 = NIntegrate[y^3 f1[y], {y, 0, 90}]

fGem[x_?NumberQ, ξ_?NumberQ, v_?IntegerQ, d_?IntegerQ,
   ν_?NumberQ, τ_?NumberQ, w_?IntegerQ, r_?IntegerQ] :=
  fGem[x?NumberQ, ξ?NumberQ, v?IntegerQ, d?IntegerQ,
    ν?NumberQ, τ?NumberQ, w?IntegerQ, r?IntegerQ] =
   ((((ν)^(d (ξ+(v/d)+1)/v) (2 π)^(((r v)/2)+((d w)/2)-1) (d w)^(-((ξ+(v/d)+1)/(v/d))+0.5)
          (r v)^0.5) / (d r)) / MeijerG[{{}, {}},
       {Join[Table[(k + d (ξ + (v / d) + 1) / v) / (d w),
          {k, 0, d w - 1}], Table[k / (r v), {k, 0, r v - 1}]], {}} ,
       (τ)^(r v) (ν)^(w d) (r v)^(-r v) (d w)^(-d w)]) x^(ξ+(v/d)) ⅇ^(-ν x^(v/d)) ⅇ^(-τ x^(-(w/r)))

fGemR[x_] := fGemR[x] = fGem[95 - x, ξ, v, d, ν, τ, w, r]
cdf[y_] := cdf[y] = NIntegrate[fGemR[x], {x, 0, y}]
x_i_ := age[[i]]
z_i_ := cdf[x_i]
n = Length[age];
```

```
M[h_] := M[h] =
   (((ν)^(-((d h) / v)) (d w)^(h / (v / d))) MeijerG[{{}, {}}, {Join[Table[
          (k + d (ξ + (v / d) + h + 1) / v) / (d w), {k, 0, d w - 1}],
         Table[k / (r v), {k, 0, r v - 1}]], {}} ,
       (τ)^(r v) (ν)^(w d) (r v)^(-r v) (d w)^(-d w)]) / (MeijerG[{{}, {}},
      {Join[Table[(k + d (ξ + (v / d) + 1) / v) / (d w),
          {k, 0, d w - 1}], Table[k / (r v), {k, 0, r v - 1}]],
       {}} , (τ)^(r v) (ν)^(w d) (r v)^(-r v) (d w)^(-d w)]);


ClearAll[ξ, v, d, ν, τ, w, r]
v = 3; d = 4; w = 2; r = 3;
vt = NMinimize[
   {Evaluate[(mm1 - M[1])^2 + (mm2 - M[2])^2 + (mm3 - M[3])^2],
    -5 < ξ < 10, .1 < ν < 3, .001 < τ < 3},
   {ξ, ν, τ}, MaxIterations → 1000]
ξ = vt[[2, 1, 2]]
ν = vt[[2, 2, 2]]
τ = vt[[2, 3, 2]]
NIntegrate[
  (fGem[x, ξ, v, d, ν, τ, w, r] - (1 / cons) f[x])^2, {x, 15, 90}]
ClearAll[A0, W0]
```

$$\mathtt{A0} = -n - \frac{1}{n}\sum_{i=1}^{n}\left((2\,i-1)\,\mathrm{Log}[z_i\ (1-z_{n+1-i})]\right)$$

$$\mathtt{W0} = \frac{1}{12\,n} + \sum_{i=1}^{n}\left(z_i - \frac{2\,i-1}{2\,n}\right)^2$$

```
est = Table[{cons cdf[age[[i]]]}, {i, 16}];
Table[{ est[[i]], mort[[i]]}, {i, 16}];
diff = mort - est;
Total[diff^2]
Total[diff]
```

```
p2 = Plot[fGem[x, ξ, v, d, ν, τ, w, r], {x, 0, 90},
   PlotRange → All, PlotStyle → RGBColor[0, 1, 0]];
Show[p1, p2]
p3 = Plot[Evaluate[fGemR[x]],
   {x, 15, 90}, PlotStyle → RGBColor[0, 1, 0]];
p4 = Plot[(1 / cons) polateDS3[y], {y, 15, 95}, PlotRange → All];
Show[p3, p4]
p5 = ListPlot[deathp];
p6 = ListPlot[Table[{y, cons cdf[y]}, {y, 15, 95, 5}],
  PlotStyle → RGBColor[1, 0, 0], PlotRange → All]
Show[
 p5,
 p6]
```

Modeling the Bivariate Flood Data Fitted Using the EIG Model (Sections 3.1 & 6.2.1)

```
flood = {{292, 12 057}, {208, 10 853}, {289, 10 299}, {146, 10 818},
   {183, 7748}, {279, 9763}, {260, 11 127}, {279, 10 659}, {137, 8327},
   {311, 13 593}, {309, 12 882}, {261, 9957}, {162, 5236}, {202, 9581},
   {306, 12 740}, {405, 11 174}, {183, 4780}, {219, 14 890}, {210, 6334},
   {200, 9177}, {289, 7133}, {239, 6865}, {294, 8918}, {371, 8704},
   {245, 6907}, {189, 4189}, {229, 8637}, {240, 8409}, {331, 13 602},
   {206, 8788}, {157, 5002}, {184, 5167}, {275, 10 128}, {286, 12 035},
   {230, 10 828}, {233, 8923}, {351, 11 401}, {156, 6620}, {168, 3826},
   {343, 8192}, {214, 6414}, {303, 8900}, {300, 9406}, {143, 7235},
   {232, 8177}, {182, 7684}, {121, 3306}, {186, 8026}, {173, 4892},
   {292, 8692}, {416, 11 272}, {246, 8640}, {248, 6989}, {297, 9352},
   {371, 12 825}, {442, 13 608}, {260, 8949}, {236, 12 577},
   {334, 11 437}, {310, 9266}, {383, 14 559}, {151, 5057}, {197, 9645},
   {283, 7241}, {390, 13 543}, {405, 15 003}, {176, 6460}, {181, 7502},
   {233, 5650}, {187, 7350}, {216, 9506}, {196, 6728}, {424, 13 315},
   {255, 8041}, {257, 10 174}, {232, 14 769}, {286, 8711}};
n1 = Length[flood]
u1i_ := flood[[i, 1]];
w1i_ := flood[[i, 2]];
U1 = Table[u1i, {i, 1, n1}];
Ubar1 = Mean[U1]
W1 = Table[w1i, {i, 1, n1}];
Wbar1 = Mean[W1] 1
```

$$C1_{11} = \sum_{i=1}^{n1} (u1_i - Ubar1)^2 \Big/ (n1 - 1)$$

$$C1_{22} = \sum_{i=1}^{n1} (w1_i - Wbar1)^2 \Big/ (n1 - 1)$$

$$C1_{12} = \sum_{i=1}^{n1} (u1_{i} - Ubar1)\ (w1_{i} - Wbar1)\ /\ (n1 - 1)$$

$$V1 = \{\{C1_{11},\ C1_{12}\},\ \{C1_{12},\ C1_{22}\}\}\ //\ N$$

```
rV1 = MatrixPower[V1, 1 / 2] // N
rV1.rV1
irV1 = Inverse[rV1]
Det[rV1];
Det[irV1];
floodc = Table[flood[[i]] - {Ubar1, Wbar1}, {i, 1, n1}] // N;
floodn = floodc.irV1;
floodn1 = Table[floodn[[i, 1]], {i, 1, n1}];
floodn2 = Table[floodn[[i, 2]], {i, 1, n1}];
floodnp = Table[{floodn[[i, 1]] + Abs[Min[floodn1]] +
      (Max[floodn1] - Min[floodn1]) / 2, floodn[[i, 2]] +
      Abs[Min[floodn2]] + (Max[floodn2] - Min[floodn2]) / 2}, {i, 1, n1}];
floodnp1 = Table[floodnp[[i, 1]], {i, 1, n1}];
floodnp2 = Table[floodnp[[i, 2]], {i, 1, n1}];
h11 = Histogram[floodnp1, 10, "ProbabilityDensity"];
h12 = Histogram[floodnp2, 10, "ProbabilityDensity"];
```

$$\mathtt{x1}_{\mathtt{i\_}} := \mathtt{x1}_{\mathtt{i}} = \mathtt{floodnp}[[\mathtt{i},\ 1]];$$

$$\mathtt{f11Bessel}\left[\mathtt{x1\_},\ \xi1\_,\ \delta1\_,\ \nu1\_,\ \tau1\_\right] :=$$

$$\mathtt{f11Bessel}[\mathtt{x1},\ \xi1,\ \delta1,\ \nu1,\ \tau1] = \mathtt{x1}^{\xi1+\delta1}\ e^{-\nu1\ \mathtt{x1}^{\delta1}}\ e^{-\tau1\ \mathtt{x1}^{-\delta1}}\ \delta1\ \nu1^{(\xi1+\delta1+1)/\delta1} \Big/ \left(2\ (\nu1\ \tau1)^{\frac{1+\delta1+\xi1}{2\,\delta1}}\ \mathtt{BesselK}\left[\frac{1+\delta1+\xi1}{\delta1},\ 2\sqrt{\nu1\ \tau1}\right]\right)$$

$$\mathtt{Lg11Bessel}\left[\xi1\_,\ \delta1\_,\ \nu1\_,\ \tau1\_\right] :=$$

$$\mathtt{Lg11Bessel}[\xi1,\ \delta1,\ \nu1,\ \tau1] = \mathtt{Log}\left[\prod_{\mathtt{i}=1}^{\mathtt{n1}} \mathtt{f11Bessel}[\mathtt{x1}_{\mathtt{i}},\ \xi1,\ \delta1,\ \nu1,\ \tau1]\right]$$

$$\mathtt{Llg11Bessel}\left[\xi1\_,\ \delta1\_,\ \nu1\_,\ \tau1\_\right] :=$$

$$\mathtt{If}\Big[\mathtt{Lg11Bessel}[\xi1,\ \delta1,\ \nu1,\ \tau1] \in \mathtt{Reals}\ \&\&\ \mathtt{f11Bessel}[2,\ \xi1,\ \delta1,\ \nu1,\ \tau1] \in \mathtt{Reals}\ \&\&\ \mathtt{f11Bessel}[7.5,\ \xi1,\ \delta1,\ \nu1,\ \tau1] \in \mathtt{Reals}\ \&\&\ \mathtt{f11Bessel}[2,\ \xi1,\ \delta1,\ \nu1,\ \tau1] > 0\ \&\&\ \mathtt{f11Bessel}[7.5,\ \xi1,\ \delta1,\ \nu1,\ \tau1] > 0,\ \mathtt{Lg11Bessel}[\xi1,\ \delta1,\ \nu1,\ \tau1],\ -10^{10}\Big]$$

$$\mathtt{M}\left[\mathtt{h\_}\right] = \frac{\nu1^{-\frac{\mathtt{h}}{\delta1}}\ (\nu1\ \tau1)^{\frac{\mathtt{h}}{2\,\delta1}}\ \mathtt{BesselK}\left[\frac{1+\mathtt{h}+\delta1+\xi1}{\delta1},\ 2\sqrt{\nu1\ \tau1}\right]}{\mathtt{BesselK}\left[\frac{1+\delta1+\xi1}{\delta1},\ 2\sqrt{\nu1\ \tau1}\right]};$$

$$\mathtt{m11} = \sum_{\mathtt{i}=1}^{\mathtt{n1}} (\mathtt{x1}_{\mathtt{i}} / \mathtt{n1});$$

$$\mathtt{m12} = \sum_{\mathtt{i}=1}^{\mathtt{n1}} \left(\mathtt{x1}_{\mathtt{i}}^{2} \big/ \mathtt{n1}\right);$$

$$\mathtt{m13} = \sum_{\mathtt{i}=1}^{\mathtt{n1}} \left(\mathtt{x1}_{\mathtt{i}}^{3} \big/ \mathtt{n1}\right);$$

$$\mathtt{m14} = \sum_{\mathtt{i}=1}^{\mathtt{n1}} \left(\mathtt{x1}_{\mathtt{i}}^{4} \big/ \mathtt{n1}\right);$$

```
vt = NMaximize[{Llg11Bessel[ξ1, δ1, ν1, τ1], -3 < ξ1 < 5,
   2 < δ1 < 5, .001 < ν1 < .1, .01 < τ1 < 1}, {ξ1, δ1, ν1, τ1}]
ξ1 = vt[[2, 1, 2]]
δ1 = vt[[2, 2, 2]]
ν1 = vt[[2, 3, 2]]
τ1 = vt[[2, 4, 2]]
cdf11Bessel[y_] := cdf11Bessel[y] =
   NIntegrate[ f11Bessel[x1, ξ1, δ1, ν1, τ1], {x1, 0, y}];
```

$$z_{i\_} := \mathrm{Chop}[\mathrm{cdf11Bessel}[x1_{i}]]$$

$$A0 = -n1 - \frac{1}{n1} \sum_{i=1}^{n1} ((2\,i - 1)\,\mathrm{Log}[z_{i}\,(1 - z_{n1+1-i})])$$

$$W0 = \frac{1}{12\,n1} + \sum_{i=1}^{n1} \left(z_{i} - \frac{2\,i - 1}{2\,n1}\right)^{2}$$

$$\text{y1}_{\text{i\_}} := \text{y1}_{\text{i}} = \text{floodnp}[[\text{i}, 2]];$$

$$\text{f12Bessel}\left[\text{y1\_}, \xi2\_, \delta2\_, \nu2\_, \tau2\_\right] := \text{f12Bessel}[\text{y1}, \xi2, \delta2, \nu2, \tau2] = \text{y1}^{\xi2+\delta2}\, e^{-\nu2\,\text{y1}^{\delta2}}\, e^{-\tau2\,\text{y1}^{-\delta2}}\,\delta2\;\nu2^{(\xi2+\delta2+1)/\delta2} \Big/ \left(2\,(\nu2\,\tau2)^{\frac{1+\delta2+\xi2}{2\,\delta2}}\,\text{BesselK}\left[\frac{1+\delta2+\xi2}{\delta2}, 2\sqrt{\nu2\,\tau2}\right]\right)$$

$$\text{Lg12Bessel}\left[\xi2\_, \delta2\_, \nu2\_, \tau2\_\right] := \text{Lg12Bessel}[\xi2, \delta2, \nu2, \tau2] = \text{Log}\left[\prod_{\text{i}=1}^{\text{n1}} \text{f12Bessel}[\text{y1}_{\text{i}}, \xi2, \delta2, \nu2, \tau2]\right]$$

$$\text{Llg12Bessel}\left[\xi2\_, \delta2\_, \nu2\_, \tau2\_\right] := \text{If}\Big[\text{Lg12Bessel}[\xi2, \delta2, \nu2, \tau2] \in \text{Reals}\ \&\&\ \text{f12Bessel}[2, \xi2, \delta2, \nu2, \tau2] \in \text{Reals}\ \&\&\ \text{f12Bessel}[7, \xi2, \delta2, \nu2, \tau2] \in \text{Reals}\ \&\&\ \text{f12Bessel}[2, \xi2, \delta2, \nu2, \tau2] > 0\ \&\&\ \text{f12Bessel}[7, \xi2, \delta2, \nu2, \tau2] > 0, \text{Lg12Bessel}[\xi2, \delta2, \nu2, \tau2], -10^{10}\Big]$$

$$\text{M2}\left[\text{h\_}\right] = \frac{\nu2^{-\frac{\text{h}}{\delta2}}\,(\nu2\,\tau2)^{\frac{\text{h}}{2\,\delta2}}\,\text{BesselK}\left[\frac{1+\text{h}+\delta2+\xi2}{\delta2}, 2\sqrt{\nu2\,\tau2}\right]}{\text{BesselK}\left[\frac{1+\delta2+\xi2}{\delta2}, 2\sqrt{\nu2\,\tau2}\right]};$$

$$\text{m15} = \sum_{\text{i}=1}^{\text{n1}} (\text{y1}_{\text{i}} / \text{n1});$$

$$\text{m16} = \sum_{\text{i}=1}^{\text{n1}} \left(\text{y1}_{\text{i}}^{2} \big/ \text{n1}\right);$$

$$\text{m17} = \sum_{\text{i}=1}^{\text{n1}} \left(\text{y1}_{\text{i}}^{3} \big/ \text{n1}\right);$$

$$\text{m18} = \sum_{\text{i}=1}^{\text{n1}} \left(\text{y1}_{\text{i}}^{3} \big/ \text{n1}\right);$$

```
vt2 = NMaximize[{Llg12Bessel[ξ2, δ2, ν2, τ2], -1 < ξ2 < 9,
   1 < δ2 < 5, .0001 < ν2 < .3, .01 < τ2 < 3}, {ξ2, δ2, ν2, τ2}]
ξ2 = vt2[[2, 1, 2]]
δ2 = vt2[[2, 2, 2]]
ν2 = vt2[[2, 3, 2]]
τ2 = vt2[[2, 4, 2]]
cdf12Bessel[y_] := cdf12Bessel[y] =
   NIntegrate[ f12Bessel[y1, ξ2, δ2, ν2, τ2], {y1, 0, y}];
```

$$z_{i\_} := \mathrm{Chop}[\mathrm{cdf12Bessel}[y1_{i}]]$$

$$A0 = -n1 - \frac{1}{n1}\sum_{i=1}^{n1}((2\,i - 1)\,\mathrm{Log}[z_i\;(1 - z_{n1+1-i})])$$

$$W0 = \frac{1}{12\,n1} + \sum_{i=1}^{n1}\left(z_i - \frac{2\,i - 1}{2\,n1}\right)^2$$

```
irV111 = irV1[[1, 1]] ;
irV112 = irV1[[1, 2]] ;
irV122 = irV1[[2, 2]] ;
irV211 = irV2[[1, 1]] ;
irV212 = irV2[[1, 2]] ;
irV222 = irV2[[2, 2]] ;
h1[u_, w_] :=
 h1[u, w] = Det[irV1] ff1[irV111 (u - Ubar1)  + irV112 (w - Wbar1) +
     Abs[Min[floodn1]] + (Max[floodn1] - Min[floodn1]) / 2,
    irV112 (u - Ubar1)  + irV122 (w - Wbar1) + Abs[Min[floodn2]] +
     (Max[floodn2] - Min[floodn2]) / 2, α, α1, β, β1]
h2[u_, w_] := h2[u, w] = Det[irV2] ff2[irV211 (u - Ubar2)  +
     irV212 (w - Wbar2) + Abs[Min[faithful2n1]] +
     (Max[faithful2n1] - Min[faithful2n1]) / 2,
    irV212 (u - Ubar2)  + irV222 (w - Wbar2) + Abs[Min[faithful2n2]] +
     (Max[faithful2n2] - Min[faithful2n2]) / 2, α2, α3, β2, β3]
h[u_, w_] := h[u, w] = (98 / 272) h1[u, w] + (174 / 272) h2[u, w]
Plot3D[h[u, w] , {u, 1.2, 5.5}, {w, 40, 100}, PlotRange → All]
Histogram3D[faithful, 7, "ProbabilityDensity"]
```

Curriculum Vitae

Name:	Iman Mabrouk
Post-Secondary Education and Degrees:	*University of Western Ontario, Canada* 2007-2011, Ph.D. 2007-2011, University Teaching and Learning Certificate Student May 2011, GEOIDE/MITACS summer school on Spatial Statistics *Cairo University, Egypt* August 2005, M.Sc in Statistics (Excellent) May 1999, B.Sc in Statistics (Very Good with honors) *Information Technology Institute (ITI), The Cabinet, Egypt* June 2000, Diploma in Software Skills Development (SSD Program)
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Publications:

Provost, B. Serge and Mabrouk, Iman (2011). An extended inverse Gaussian model , International Journal of Statistical Sciences, 11.

Provost, B. Serge and Mabrouk, I. (2010). A generalized exponential-type distribution, Pak. J. Statist., 26 (1), 97-110.

Mabrouk, Iman, (2005). Fuzzy neural network model for Nile river flood forecasting, M.Sc thesis, Cairo University, 2005.

Presentations:

Mabrouk, Iman, (2011). An Extended Inverse Gaussian Model, Statistical Society of Canada Annual Meeting, Wolfville, NS, 2011 (Contributed Talk).

Mabrouk, Iman, (2011). An Extended Inverse Gaussian Model, Southern Ontario Statistics Graduate Students Seminar Days (SOSGSSD), London, ON, 2011 (Contributed Talk).