1971

The Distribution Of The Standard Deviation For Samples From Rectangular And Exponential Populations

Winston Ashton Richards

Follow this and additional works at: https://ir.lib.uwo.ca/digitizedtheses

Recommended Citation
Richards, Winston Ashton, "The Distribution Of The Standard Deviation For Samples From Rectangular And Exponential Populations" (1971). Digitized Theses. 469.
https://ir.lib.uwo.ca/digitizedtheses/469

This Dissertation is brought to you for free and open access by the Digitized Special Collections at Scholarship@Western. It has been accepted for inclusion in Digitized Theses by an authorized administrator of Scholarship@Western. For more information, please contact tadam@uwo.ca, wlswadmin@uwo.ca.
THE DISTRIBUTION OF THE STANDARD DEVIATION
FOR SAMPLES FROM PECTANGULAR AND
EXPONENTIAL POPULATIONS

by

Winston Ashton Richards

Department of Mathematics

Submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy

Faculty of Graduate Studies
The University of Western Ontario
London, Canada.
May 1971
ABSTRACT

Let \( x_1, \ldots, x_n \) be \( n \) independent, identically distributed, random variables drawn from a continuous distribution \( F(x) \), \( a \leq x \leq b \). Let

\[
S^2 = n^{-1} \sum_{i=1}^{n} (x_i - \bar{x})^2
\]

where \( \bar{x} = n^{-1} \sum_{i=1}^{n} x_i \).

In this thesis we let

\[ dF = dx, \ 0 \leq x \leq 1 \]

and we obtain an expression for \( P(S \leq s) \) for \( 0 < s \leq (2n)^{-\frac{1}{2}} \)

for \( \frac{(n^2-4)^{\frac{1}{2}}}{2n} \leq s \leq \frac{1}{2} \)

when \( n \) is even

for \( \frac{(5n-9)^{\frac{1}{2}}}{2n} \leq s \leq \frac{(n^2-1)^{\frac{1}{2}}}{2n} \)

when \( n \) is odd.

For the case when \( 0 < s \leq (2n)^{-\frac{1}{2}} \) the distribution and frequency functions of \( S \) are expressed as a multiple integral over an \((n-2)\)-dimensional unit cube. For the other stated ranges \( P(S \leq s) \) is expressed as an \((n-1)\)-dimensional integral with trigonometric arguments. The integrals have been evaluated numerically for \( n = 2, 3, 4, 5, 6 \) and 7 respectively for the range when \( 0 < s \leq (2n)^{-\frac{1}{2}} \).
Next we consider the negative exponential distribution given by

\[ dF = e^{-x} \, dx, \quad 0 \leq x < \infty \]

and we obtain

\[ P(S \leq s), \quad 0 \leq s < \infty, \]

that is, we obtain the distribution of the standard deviation \( S \) as multiple integrals over an \((n-2)\)-dimensional unit cube.
ACKNOWLEDGEMENTS

I am deeply indebted to my teacher, Professor Mir M. Ali, for his constant encouragement and continuous interest in my work. Without his kind supervision, keen interest, and valuable advice completion of this work would have been most difficult.

Acknowledgements are also due to, Professor J. D. Talman, who during the sabbatical of Professor Ali in 1968, gave generously of his time and advice; to Eddy Smet who later helped me check the tedious calculations of Chapter 3; and to my wife for her supreme typing effort.

I wish to thank the Province of Ontario for financial assistance.

Winston Ashton Richards
October 1970
London, Ontario CANADA.
# Table of Contents

## Abstract

iii

## Acknowledgments

v

### Chapter 1. Summary and Introduction

1-1

General Outline of the Thesis

1-5

### Chapter 2. The Problem and Its Geometrical Interpretation

2.1 Statement of the Problem

2-1

2.1.1 The "S" Statistic From the Rectangular Distribution

2-1

2.1.2 The "S" Statistic From the Exponential Distribution

2-1

2.2 Geometrical Interpretation of The "S" Statistic

2-3

### Chapter 3. Method of "Sections" for Obtaining the Distribution of the Standard Deviation of Sets of 2, 3, and 4 Variates Respectively, Drawn at Random from a Rectangular Population

3.1 Introduction

3-1

3.2 Method of "Sections"

3-2

3.2.1 Sample Size n=2

3-6

3.2.2 Sample Size n=3

3-8

3.2.2.1 Lemma 3.2.2.1

3-9

3.2.2.2 Lemma 3.2.2.2

3-14

3.2.3 Sample Size n=4

3-35
3.2.3.1  Lemma 3.2.3.1.  
3.2.3.2  Lemma 3.2.3.2.  
3.2.3.3  Lemma 3.2.3.3.  

Chapter 4.  CUMULATIVE DISTRIBUTION OF THE SAMPLE
STANDARD DEVIATION FOR SAMPLES OF SIZE
n DRAWN FROM A RECTANGULAR POPULATION
FOR 0 ≤ s ≤ (2n)^{-1/2}, WHEN n=2k FOR
(\frac{n^2-4}{2n})^{1/2} ≤ s ≤ \frac{n}{2} AND WHEN n=2k+1 FOR
(\frac{5n-9}{2n})^{1/2} ≤ s ≤ \frac{n^2-1}{2n}^{1/2}.

4.1  Introduction.  
4.2  Lemma 4.2.  
4.3  The Distribution For S ≤ (2n)^{-1/2}.  
4.4  Lemma 4.4.  
4.5  Lemma 4.5.  
4.6  The Expression For F(s) as an
(n-2)-integral over an (n-2)-
unit cube.  
4.7  Evaluation of (4.6.ii) For
n=2, 3, 4 respectively.  
4.8  For S Large.  
4.9  Case n=2.  

Chapter 5.  ANOTHER APPROACH TO THE DERIVATION OF
THE CUMULATIVE DISTRIBUTION OF THE
SAMPLE STANDARD DEVIATION S FOR
SAMPLES OF SIZE n DRAWN FROM A REC-
TANGULAR POPULATION WHEN S ≤ (2n)^{-1/2}.  

vii
5.1 Introduction. 5-1

5.2 A Second Approach To Finding 5-2
The Distribution of $S$. 5-2

5.3 Lemma 5.3. 5-6

5.4 Lemma 5.4. 5-7

5.5 $F(r_0)$ as an (n-2)ple Integral 5-10
Over An n-cube.

5.6 Evaluation of $F(r_0)$ for n=5. 5-11

---

Chapter 6. 6-1
SPHERICAL APPROXIMATION.

6.1 Introduction. 6-1

6.2 Spherical Approximation and 6-1
Exact Upper and Lower Bounds
For $P(S \leq s)$, $0 \leq s \leq (2n)^{-\frac{1}{2}}$.

Chapter 7. 7-1
THE DISTRIBUTION OF $S$ FOR SETS OF
N VARIATES WHEN THE UNDERLYING POPU-
LATION IS EXPONENTIAL.

7.1 Introduction. 7-1

7.2 Statement of the Problem. 7-2

7.3 Lemma 7.3. 7-8

7.4 The Cumulative Distribution 7-12
And Distribution Functions For
$S$, When the Underlying Popula-
tion is Exponential.

7.5 Method of "Sections". 7-17

7.6 Spherical Approximation and 7-17
Exact Lower and Upper Bounds For

viii
P(S ≤ s), 0 ≤ s < ∞, The
Underlying Distribution
Being Exponential.

Chapter 8. NUMERICAL RESULTS.
8.1 Introduction.
8.2 Methods Adopted For Numerical
Integration.
8.3 Tables and Graphs.
Tables and Graphs of the S
Statistic From Uniform and
Exponential Populations.
Tables and Graphs For \( \chi = \frac{ns^2}{\sigma^2} \)
Where \( ns^2/\sigma^2 \) is From The Uniform
And Exponential Populations.
8.4 Some Comments On The Results.

Chapter 9. SOME FUTURE PROBLEMS.
REFERENCES
EPILOGUE
VITA
CHAPTER 1

SUMMARY AND INTRODUCTION

In this thesis we deal with the distribution of the sample standard deviation $S$ when the underlying population is continuous rectangular and when the underlying population is negative exponential.

When the sets of $n$ variates are drawn from a continuous rectangular population we obtain an expression for the tail ends of the distribution of $S$, the sample statistic.

In the case when the underlying population is negative exponential we obtain the distribution of the standard deviation $S$ over its entire range.

The motivation for these problems came from the paper of Hotelling (1961) [18] "Statistical Tests Under Nonstandard Conditions" in which he commented, "Even the poor consolation provided by limit theorems in the central part of some of the distributions of $t$ is further weakened when we pass to more complicated statistics such as sample variances and correlation coefficients. Here the moments of the statistic in samples from non-normal populations betray the huge errors that arise when non-normality is neglected."

Rider (1929) [27] derived the distribution of $S$ for $n=2$, the underlying population being continuous rectangular, "in the hope that they may offer some clue to the forms of the distribution for larger values of $n$."
Then Rietz (1931) [28] gave the distribution of $S$ for the case when the sampling distribution was continuous rectangular. The sample size was $n=3$. Rietz then conjectured, "Since the distribution of $S$ from two items is a straight line, and from three items is a parabola for the major portion of the range, it is natural to surmise that the distribution of $S$ from $n$ items may be a polynomial of degree $n-1$ for an interval of the range of $S$ near zero." He further added, "attempts to prove or disprove this conjecture seem to lead to difficult problems of $n$ dimensional geometry."

In Chapter 3 the problem of obtaining the distribution of $S$ when the underlying population is uniform is stated as a problem in geometry. In Chapter 4, where the distribution of the $S$ statistic is found by analytical methods, we also solve the related geometric problem, namely, to find the common content of an $n$-dimensional cylinder (the radius of the cylinder being limited) having its axis the same as the diagonal of an $n$-dimensional unit cube and bounded by that cube. Rietz (1931) [28] had already given the result for $n=3$.

The geometrical interpretation of the distribution of $S$ for samples arising from a continuous rectangular population and for the samples from a negative exponential distribution is exhibited in Chapter 2 and Chapter 7. By employing a method of "sections", explained in Abbott's [1] book Flatland page 77 and expanded on by Coxeter [8] Chapter (XIII) "Sections and Projections", we derive the common contents of 2, 3, and 4 dimensional cylinders having their axes the same
as the diagonals of the 2, 3, and 4 dimensional cubes respectively and bounded by these cubes by cutting the solids by a plane orthogonal to the diagonal of the cube and axis of the cylinder. The common contents of these individual sections are then found. Since the density of the negative exponential "S" is constant on a plane orthogonal to the equiangular line, the results obtained for the contents of the various sections can be employed to find the cumulative distribution of the negative exponential S.

There have been many studies and many approaches to the study of the distribution of the sample variances, significant among them is the approach taken by Hotelling (1961)[18] who "attempted to throw some light on the distribution of the sample variances from the family of populations specified by the Pearson Type VII frequency curves by means of inequalities determined by maximum and minimum values."

Dunlap (1931) [10] by empirical methods determined the distribution of the standard deviation for a sample of size 10 drawn from a discrete uniform population and fitted a Gram - Charlier curve to the distribution.

Le Roux (1931) [24], by using the moments of the sampled population showed that the sampling distribution of the variance could be "adequately represented by certain Pearson Curves." In his introduction he recalled the related works of Student (1908) and Church (1925).

The present work is primarily of a theoretical nature but it may also have some practical value. Le Roux (1931 [24]
states, "The Type III sampling distribution of the variance, appropriate for a normal population, has formed the basis of a number of statistical tests of considerable practical importance. It is not the object of this paper to discuss whether or not similar tests can be developed for use with non-normal material, but it is clear that an essential preliminary to further research is a fuller knowledge of the form of these basic sampling curves of variance."
General Outline of The Thesis.

A summary of the contents of the chapters of this thesis now follows:

Chapter 2. In this Chapter both problems, namely, the cumulative distribution of the $S$ statistic from a uniform population and the cumulative distribution of the $S$ statistic from an exponential population are formulated. A geometric interpretation of the $S$ statistic is given. Finally, the problem is stated in geometrical terms.

Chapter 3. The method of "sections" is used to solve the problem of the $S$ statistic from a continuous rectangular population for $n=2, 3$ completely and for the left tail for the case when $n=4$. The exploration of $2, 3$, and $4$ dimensions is instructive, it gives insight into the problem and emphasises the difficulties involved in going from $4$ to higher dimensions.

Chapter 4. We derive the cumulative distribution of the sample standard deviation $P(S \leq s)$

for $0 < s \leq \frac{(2n)^{-\frac{1}{2}}}{2n}$

for $\frac{(n^2-4)^{\frac{1}{2}}}{2n} \leq s \leq \frac{1}{2}$

when the sample size is even

and for $\frac{(5n-9)^{\frac{1}{2}}}{2n} \leq s \leq \frac{(n^2-1)^{\frac{1}{2}}}{2n}$

when the sample size is odd.

The samples are drawn from a continuous rectangular population.
The distribution of the $S$ statistic is found as a multiple integral over an $(n-2)$-dimensional cube for the case when $0 \leq s \leq (2n)^{-\frac{3}{2}}$. When $n$ is even or odd for large $S$ we find the $P(S \leq s)$ by considering the parallel problem in geometry. We find the content enclosed by the farthest vertices of the $n$-dimensional unit cube $0 \leq x_i \leq 1$, $i=1, \ldots, n$ and the exterior of the cylinder given by

$$\sum_{i=1}^{n} (x_i - \bar{x})^2 \geq r_0^2.$$

The right tail of the distribution of $S$ is then expressed as an $(n-1)$-fold integral in terms of trigonometric arguments. Rietz's conjecture is simultaneously verified by the results obtained in this Chapter.

Chapter 5. Chapter 5 gives another approach to the cumulative distribution of the continuous rectangular $S$ statistic by using a complete order statistic approach. An exact evaluation for $n=5$ when $0 \leq s \leq 10^{-\frac{3}{2}}$ is obtained.

Chapter 6. We consider an approximation of the $(n-2)$-fold integral of Chapter 5 by a double integral by replacing the cubic domain of the integral of Chapter 4 by a spherical domain having the same volume. We obtain exact upper and lower bounds for the left tail probabilities of $S$. 
Chapter 7. We apply all the techniques and results obtained in the previous chapters in the derivation of the $S$ statistic from the negative exponential population. A spherical approximation for the integral is also found. Also the results of Chapter 3 are used to derive the distribution of the $S$ statistic for $n = 2, 3, 4$.

Chapter 8. Chapter 8 contains the details of the computations and methods used accompanied by tables and various graphs. The sampling distribution of the statistic $ns^2/\sigma^2$, appropriate for a normal population, is compared with its sampling distribution, appropriate for a uniform population and its sampling distribution appropriate for a negative exponential population. $\sigma^2$ is taken to be the $\sigma^2$ of the uniform population in the first comparison and in the second comparison it is taken to be the $\sigma^2$ of the negative exponential.
CHAPTER 2
THE PROBLEM AND ITS GEOMETRICAL INTERPRETATION

2.1 Statement of The Problem.

2.1.1 The "S" Statistic From The Rectangular Distribution. Let \( x_1, x_2, \ldots, x_n \) be \( n \) independent identically distributed random variables from the rectangular distribution:

\[
(2.1.1.i) \quad dF = dx, \quad 0 \leq x \leq 1.
\]

(2.1.1.ii) Let

\[
S^2 = n^{-1} \sum_{i=1}^{n} (x_i - \bar{x})^2
\]

where \( \bar{x} = n^{-1} \sum_{i=1}^{n} x_i \).

In this thesis we will find the cumulative distribution

\[
P(S \leq s) \text{ for } 0 < s < (2n)^{-\frac{1}{2}}
\]

\[
\text{for } \frac{(n^2-4)^{\frac{1}{2}}}{2n} < s \leq \frac{1}{2}
\]

when \( n \) is even and

\[
\text{for } \frac{(5n-9)^{\frac{1}{2}}}{2n} < s \leq \frac{(n^2-1)^{\frac{1}{2}}}{2n}
\]

when \( n \) is odd,

for sets of \( n \) variates drawn from the continuous rectangular population given in (2.1.1.i).

2.1.2 The "S" Statistic from The Exponential Distribution. The second problem will now be stated:

Let \( x_1, x_2, \ldots, x_n \) be \( n \) independent identically distributed random variables from the exponential distribution

\[
(2.1.2.i) \quad dF = e^{-x} \, dx \quad 0 \leq x < \infty.
\]
(2.1.2.ii) Let \[ s^2 = n^{-1} \sum_{i=1}^{n} (x_i - \bar{x})^2 \]

where \[ \bar{x} = n^{-1} \sum_{i=1}^{n} x_i. \]

When the random sample is from the population (2.1.2.i) we will find the cumulative distribution function \( F_1(s) \) of the standard deviation \( S \) for sets of \( n \) variates.
2.2 Geometrical Interpretation of The "S" Statistic.

Fig. 2.2A
Assume the values $x_1, \ldots, x_n$ of any given sample to be the coordinates of a point $P$ in $n$-dimensional Euclidean hyper-space. Let $OD$ be the equiangular line whose direction cosines are $n^{-\frac{1}{2}}$, $n^{-\frac{1}{2}}$, \ldots, $n^{-\frac{1}{2}}$. Let $PM$ be the perpendicular from $P$ the sample point to $M$ on $OD$. Then the length of $OM$ is

$$n^{-\frac{1}{2}}x_1 + n^{-\frac{1}{2}}x_2 + \ldots + n^{-\frac{1}{2}}x_n = n^{-\frac{1}{2}}x.$$

The length of $OP$ is $(Ex_1^2)^{\frac{1}{2}}$. Thus the length of $PM$ is $(E x^2 - n\bar{x})^{\frac{1}{2}} = n^{\frac{1}{2}}S = r$. The region for which $PM$ equal to $r$ is a hyper-cylinder whose axis is the equiangular line and whose cross section is an $(n-1)$-dimensional hyper-sphere. The density of $r$ is then found by determining the probability mass between the cylinders determined by $r$ and $r+dr$.

The cumulative distribution function of $S$ is then

$$F_1(s) = P(S \leq s) = P(n^{-\frac{1}{2}}r \leq s) = P(r \leq n^{\frac{1}{2}}s) = F(n^{\frac{1}{2}}s).$$

The inequality $r \leq n^{\frac{1}{2}}s$ has for its solution set all points inside the cylinder obtained by rotating $PM$ about the equiangular line $OD$.

Consider a sample of $n$ values $(x_1, x_2, \ldots, x_n)$ from the rectangular population: $dF = dx$, $0 \leq x \leq 1$, $0$ otherwise. In the $n$-space the density will be $1$ everywhere inside the hyper-cube and $0$ outside it. The unit-vector will be along the diagonal of the hyper-cube, which is that part of the equiangular line inside the hyper-cube. Now if $P$ is
the sample point \((x_1, x_2, \ldots, x_n)\) and \(PM\) the perpendicular to the diagonal of the cube, then as was previously shown

\[PM = n^{\frac{1}{2}}S = (\sum_{i=1}^{n} x_i^2 - nx^2)^{\frac{1}{2}} = r.\]

Let \(T^*\) represent the set of all points inside the cylinder obtained by rotating \(PM\) about the axis \(OD\). Let \(C^*\) be the set of points inside the hyper-cube, \(0 \leq x_i \leq 1, i=1, 2, \ldots, n\). Then \(Pr(r \leq n^{\frac{1}{2}}S) = [T^* \cap C^*]\), where \([T^* \cap C^*]\) denotes the common content of \(T^* \cap C^*\).

\[\therefore F_1(s) = F(n^{\frac{1}{2}}s) = [T^* \cap C^*],\]

where \([T^* \cap C^*]\) geometrically is the common content of the unit-cube and the cylinder having as its axis of symmetry the diagonal of the unit-cube.

When \(r \leq 2^{-\frac{1}{2}}\), that is, when \(S \leq (2n)^{-\frac{1}{2}}\) the hyper-cylinder contains only the vertices \((0, 0, \ldots, 0)\) and \((1, 1, \ldots, 1)\) of the cube. The configuration of \(T^* \cap C^*\) changes for larger values \(r\).

Since our primary concern will be with the tail ends of the distribution we will confine ourselves to small values and large values of \(S\).

The geometric nature of the problem as stated enables us to proceed to the solution of the problem for small \(S\) by finding the common content of the hyper-cylinder and the \(n\)-cube.
Notation: To avoid confusion, unless otherwise stated, we use, in the following chapters, \( r \) and \( S \) for random variables and \( r_0 \) and \( s \) for fixed numbers.
CHAPTER 3

METHOD OF "SECTIONS" FOR OBTAINING THE DISTRIBUTION OF
THE STANDARD DEVIATION OF SETS OF 2, 3, AND 4
VARIATES RESPECTIVELY, DRAWN AT RANDOM
FROM A RECTANGULAR POPULATION.

3.1 Introduction.

Paul Rider (1929) [27] found the distribution of the
standard deviation $S$ for samples of size two drawn from a
rectangular population. Later H. L. Rietz (1931) [28] ob-
tained the distribution of $S$ for samples of size three.
In the case when the sample size $n = 2$ the derivation was
not difficult. For the case $n = 3$ the method employed in
the derivation was not readily extendable for larger sample
sizes.

We will therefore, independently derive by another
method, which we call the "method of sections", the same dis-
tributions of $S$ for $n = 2$ and 3. This "method of sec-
tions" we will also use to derive a partial result for the
distribution of $S$ when $0 < s \leq (2n)^{-\frac{1}{2}}$. This result for
$n = 4$ will be used as a check on the results obtained in
Chapter 4 for any sample size.
The method of "sections" which will be discussed in 3.2 will, in a natural way, permit the investigation to continue for larger n's and at the same time expose the complexities involved in the solution of the present problem.

In order to generate a feeling for the method of "sections" we quote directly from Coxeter [8]:

"Inhabitants of Flatland (the two dimensional world imagined by Abbott 1), desiring to get an idea of solid figures, would have two general methods available to them: section and projection. According to the first method, they would imagine the solid figure gradually penetrating their two-dimensional world, and consider its successive sections; (Hinton 1, pp. 106-108.) e.g., the sections of a cube, beginning with a vertex, would be equilateral triangles of increasing size, then alternate-sided hexagons ("truncated" triangles), and finally equilateral triangles of decreasing size, ending with a single point - the opposite vertex. According to the second method, they would study the shadow of the solid figure in various positions; e.g., a cube in one position appears as a square, in another as a hexagon."

3.2 Method of "Sections."

Let $T$ be a cylinder of radius $r$, having its axis which lies along the positive extension of the equiangular
line, coincide with the diagonal \( OD \) of the hyper-cube
\[ 0 \leq x_i \leq 1, \ i=1,\ldots,n \] which also lies along the positive extension of the equiangular line (See Fig. 2.2A). We will consider only that part of \( T \) bounded by the hyper-cube
\[ 0 \leq x_i \leq 1, \ i=1,\ldots,n. \] Further let us cut the \( n \)-cube with a plane orthogonal to the equiangular line. Call this plane \( \pi_n \).

If the plane \( \pi_n \) lies between the origin \( (0^n) \) and the plane which passes through the vertices \( A_1, A_2,\ldots,A_n \) whose coordinates are permutations of \( (1, 0^{n-1}) \) then the sections of the hyper-cube we would obtain if we were at liberty to move the plane \( \pi_n \) starting from the origin and moving forward would be the point \( (0^n) \). Then as we moved along the equiangular line the sections would be \( (n-1) \)-dimensional regular simplexes of increasing sizes with the one of maximum size having vertices \( A_1, A_2,\ldots,A_n \). As we moved beyond the vertices \( A_1, A_2,\ldots,A_n \) and approached the centre of the cube we would experience various configurations of \( (n-1) \)-dimensional truncated regular simplexes. As we moved away from the centre of the cube the sections produced by the cutting plane would be identical to those that preceded the centre except that they would be inverted. Eventually we would arrive at the plane formed by the points whose coordinates are the permutations of \( (0, 1^{n-1}) \) and as we continued past that plane to the vertex we would again experience \( (n-1) \)-dimensional regular simplexes of decreasing sizes until we arrived at the vertex \( (1^n) \) where the intersection would be a point.
Consider now the cross section of the cylinder and the cutting plane $\pi_n$. The cross section would be a sphere of (n-1)-dimensions of radius $r_0$. Since $r_0$ is fixed the size of the sphere remains constant. Now let us examine the cross section of the common part of the cylinder and the n-cube, as we move our cutting plane from the origin $(0^n)$ to the plane passing through the vertices $A_1', A_2', \ldots, A_n'$. The first configuration we would find is a regular (n-1)-dimensional simplex completely inscribed in an (n-1)-dimensional sphere. The second configuration would be that of the sphere intersecting the 1-dimensional edges of the simplex. In the third configuration the sphere would intersect the two-dimensional faces. This pattern will continue until the sphere intersects only the (n-2)-dimensional faces. Finally, the sphere would be completely inscribed in the regular simplex. When the plane moves beyond the vertices $A_1', A_2', \ldots, A_n'$ on its way to the centre of the cube, if $r_0$ is sufficiently small, the (n-1)-dimensional sphere will lie completely in the truncated simplices. When $r_0$ is not sufficiently small and $n > 4$ a geometric description becomes extremely complicated and so will not be presented here. Beyond the centre the configurations here mentioned will occur in reverse order.

Therefore, in order to find the common content of the cylinder and the cube we integrate along the equiangular line from the vertex $(0^n)$ to the centre of the cube $(\frac{1}{2^n})$ over different limits for which the configurations of the cross sections are the same. We then multiply the result by 2.
The development of the cases for \( n=2, 3, 4 \) respectively will clarify our explanation.
3.2.1 Sample size \( n = 2 \).

When \( n = 2 \), the hyper-cube \( 0 \leq x_i \leq 1, \ i = 1, \ldots, n \) degenerates into a square and the cylinder

\[
\sum_{i=1}^{n} (x_i - \bar{x})^2 = r_0^2 \quad \text{where} \quad r_0^2 = \sqrt{n}s^2
\]

becomes two parallel lines equidistant from the equiangular line. (This will later be referred to as the flat cylinder.)

The hyper-plane \( \pi_n \):

\[
\sum_{i=1}^{n} x_i = \sqrt{n}y \quad \text{where} \quad \sqrt{n}y = nx
\]

becomes the line \( \pi_2 \):

\[
\sum_{i=1}^{2} x_i = \sqrt{2}y \quad \text{where} \quad \sqrt{2}y = 2\bar{x}
\]

Our objective, to find the common content \([T^* \cap C^*]\), will be accomplished if we observe that as the orthogonal line

\[
\sum_{i=1}^{2} x_i = \sqrt{2}y \quad \text{moves along the equiangular axis} \ y \quad \text{the sec-}
\]


tions of common content between the flat cylinder and the cube are lines \( P_1 P_2 \) of lengths \( 2y \) when \( 0 \leq y \leq r_0 \), and when \( r_0 \leq y \leq \frac{\sqrt{2}}{2} \) they are of lengths \( 2r_0 \).

The common content

\[
(3.2.1.i) \\
[T^* \cap C^*] = 2 \cdot \left( \int_0^{r_0} 2y \, dy + \int_{r_0}^{\sqrt{2}/2} 2r_0 dy \right) \\
= 2 \left( y^2 \bigg|_0^{r_0} + 2r_0 y \bigg|_{r_0}^{\sqrt{2}/2} \right) \\
= 2 \left( r_0^2 + r_0 \sqrt{2} - 2r_0^2 \right) \\
= 2\sqrt{2} r_0 - 2r_0^2 = P(r \leq r_0) 
\]

where \( 0 \leq r_0 \leq \frac{\sqrt{2}}{2} \) and \( r_0 = \frac{1}{2}s \).

Therefore

\[
P(r \leq r_0) = F(r_0) = F\left(\frac{1}{2}s\right) 
\]

\[
(3.2.1.iii) \\
F_1(s) = 4s - 4s^2, \quad 0 \leq s \leq \frac{1}{2}. 
\]
3.2.2. Sample Size \( n = 3 \).

When \( n = 3 \), the hyper-cube, hyper-cylinder and hyper-plane reduce respectively to

the cube \( C: 0 \leq x_i \leq 1, \ i = 1, 2, 3; \)

the cylinder \( T: \sum_{i=1}^{3} (x_i - \bar{x})^2 = r_0^2 \) where \( r_0^2 = 3s^2 \);

and the plane \( \pi_3: \sum_{i=1}^{3} x_i = \sqrt{3}y \) where \( \sqrt{3}y = 3\bar{x} \).

The cylinder \( T \) has for its axis of symmetry the equiangular line which passes through the vertices \((0^3)\) and \((1^3)\) of the cube.

Let \( \pi_3 \) intersect the equiangular line at an arbitrary distance \( y \) from the origin \( 0 \). First we consider the case \( 0 \leq y \leq 1/\sqrt{3} \) then the case where \( 1/\sqrt{3} \leq y \leq \sqrt{3}/2 \).

Let \( r_0 \) be the radius of the cross section of the cylinder and \( d \) be the distance from the centre of the tri-
angular cross section to the centre of a 1-dimensional face of the triangle with centre at E.

The coordinates of E are \( \left( \frac{\sqrt{3}y}{3}, \frac{\sqrt{3}y}{3}, \frac{\sqrt{3}y}{3} \right) \). The coordinates of the centre of a 1-dimensional face are \( \left( \frac{\sqrt{3}y}{2}, \frac{\sqrt{3}y}{2}, 0 \right) \).

\[
d^2 = 2 \left( \frac{\sqrt{3}y}{2} - \frac{\sqrt{3}y}{3} \right)^2 + \left( \frac{\sqrt{3}y}{3} \right)^2
\]

\[
d = \frac{1}{\sqrt{2}} \text{ when } 0 \leq y \leq \frac{1}{\sqrt{3}}.
\]

**LEMMA 3.2.2.1.**

Let \( r_0 \) be the radius of a circle C which has common centre with an equilateral triangle T and let d be the distance from the centre of that triangle to a 1-dimensional face. Let \([T \cap C]\) be the area common to the triangle and the circle. Then

\[
[T \cap C] = \pi r_0^2 - 3r_0^2 \cos^{-1} \left( \frac{d}{r_0} \right) + 3d(r_0^2 - d^2)^{\frac{1}{2}}
\]

for \( d \leq r_0 \leq 2d \)

\[
3\sqrt{3}d^2 \text{ for } r_0 \geq 2d.
\]

**Proof:**

**Case I:** The circle is inside the triangle if \( r_0 \leq d \). The common part is a circle and its interior. The area is therefore

\[
[T \cap C] = \pi r_0^2.
\]
Case II: The circle and triangle intersect if
\[ d < r_0 < 2d. \]

To find the area of the common part we first find the areas of the segments \( B_1, B_2, B_3 \) and subtract their sum from the area \( r_0^2 \) of the circle.

Refer the circle \( x^2 + y^2 = r_0^2 \) and the equilateral triangle to the cartesian coordinate system as shown in the diagram. The area of the shaded segment \( B_2 \) is

\[
g(r_0, d) = \int_{d}^{r_0} \int_{-(r_0^2 - x^2)^{\frac{1}{2}}}^{(r_0^2 - x^2)^{\frac{1}{2}}} dy \, dx
\]
\[ r_0 = \int_{d}^{r_0} 2(r_0^2 - x^2)^{\frac{1}{2}} \, dx \, d \]

To evaluate the integral let \( x = r_0 \sin \theta \), then \( dx = r_0 \cos \theta \, d\theta \). When \( x = d \), \( \theta = \sin^{-1} \frac{d}{r_0} \); when \( x = r_0 \), \( \theta = \pi/2 \).

\[
g(r_0, d) = 2r_0^2 \int_{\sin^{-1} \frac{d}{r_0}}^{\pi/2} \cos^2 \theta \, d\theta
\]

\[= 2r_0^2 (\theta/2 + \frac{1}{2} \sin 2\theta) \bigg|_{\sin^{-1} \frac{d}{r_0}}^{\pi/2}
\]

\[= r_0^2 \cos^{-1} \frac{d}{r_0} - d(r_0^2 - d^2)^{\frac{1}{2}}.\]

Since the segments \( B_1, B_2, \) and \( B_3 \) are congruent and the area of the circle is \( \pi r_0^2 \), the area of the circle inside the triangle

\[\{T \cap C\} = \pi r_0^2 - 3r_0^2 \cos^{-1} \frac{d}{r_0} + 3d(r_0^2 - d^2)^{\frac{1}{2}}\]

Case III: The triangle is inside the circle if \( r_0 > 2d \). The common part is the triangle, therefore,

\[\{T \cap C\} = 3\sqrt{3}d^2.\]

Q.E.D.
Let \( \pi_3 \) intersect the equiangular line at an arbitrary distance \( y \) from the origin \( 0 \) when \( \frac{1}{\sqrt{3}} \leq y \leq \frac{\sqrt{3}}{2} \).

\[
d = d_1 = \frac{\sqrt{3}y}{\sqrt{6}}.
\]

The coordinates of the centre of the 1-dimensional face farthest from the centre are \( \left( \frac{\sqrt{3}y - 1}{2}, \frac{\sqrt{3}y - 1}{2}, 1 \right) \).

The coordinates of \( E \) are

\[
\left( \frac{\sqrt{3}y}{3}, \frac{\sqrt{3}y}{3}, \frac{\sqrt{3}y}{3} \right).
\]

\[
\therefore \quad d_2 = 2 \left( \frac{\sqrt{3}y - \sqrt{3}y - 1}{2} \right)^2 + \left( \frac{\sqrt{3}y - 1}{2} \right)^2
\]

\[
= 2 \left( \frac{3 - \sqrt{3}y}{3} \right)^2 + 4 \left( \frac{3 - \sqrt{3}y}{3} \right)^2
\]

\[
= \frac{6 (3 - \sqrt{3}y)^2}{36}
\]

and

\[
d_2 = \frac{1 (3 - \sqrt{3}y)}{\sqrt{6}}
\]

also

\[
d_1 + d_2 = \frac{\sqrt{3}}{\sqrt{2}}.
\]

The coordinates of one of the vertices of the hexagonal cross section (also viewed as the vertices of the figure formed by two intersecting equilateral triangles) are

\( (0, \sqrt{3}y-1, 1) \).

The coordinates of \( E \) are

\[
\left( \frac{\sqrt{3}y}{3}, \frac{\sqrt{3}y}{3}, \frac{\sqrt{3}y}{3} \right).
\]

\[
\rho^2 = \left( \frac{\sqrt{3}y}{3} - 0 \right)^2 + \left( \frac{\sqrt{3}y}{3} - \sqrt{3}y + 1 \right)^2 + \left( \frac{\sqrt{3}y}{3} - 1 \right)^2
\]

\[
\rho = \sqrt{2} (1 - \sqrt{3}y + y^2)^{\frac{1}{2}}.
\]
**Lemma 3.2.2.2.**

Let $r_0$ be the radius of a circle $C$ which has common centre with two intersecting equilateral triangles $T_1$ and $T_2$. Let $d_1$ be the distance from a 1-dimensional face in triangle $T_1$ and let $d_2$ be the distance from a 1-dimensional face in triangle $T_2$, let $T$ be the area common to both triangles $T_1$ and $T_2$. Then the area common to both triangles and the circle is represented by $[T \cap C]$ where

$$[T \cap C] = \pi r_0^2 - 3r_0^2 \cos^{-1} \frac{d_1}{r_0} + 3d_1 (r_0^2 - d_1^2)^{\frac{1}{2}}$$

for $0 \leq r_0 \leq d_1$

$$\pi r_0^2 - 3r_0^2 \cos^{-1} \frac{d_1}{r_0} + 3d_1 (r_0^2 - d_1^2)^{\frac{1}{2}} - 3r_0^2 \cos^{-1} \frac{d_2}{r_0} + 3d_2 (r_0^2 - d_2^2)^{\frac{1}{2}}$$

for $d_1 \leq r_0 \leq d_2$

$$3\sqrt{3}d_1^2 - \sqrt{3}(3d_1 - \sqrt{3/2})^2$$

for $d_2 \leq r_0 \leq \rho$

$$r^0 \geq \rho.$$

Note: In Lemma 3.2.2.2 $d_1 + d_2 = (3/2)^{\frac{1}{2}}$. 
Proof:

Case I: The circle is inside the truncated equilateral triangle if \( r_0 \leq d_1 \). The common part is a circle and its interior. The area is therefore \( [T \cap C] = \pi r_0^2 \).

Case II:

If \( d_1 \leq r_0 \leq d_2 \) consider the \( \triangle A_1 A_2 A_3 \). By case II of Lemma 3.2.2.1, the shaded segments have area

\[
3r_0^2 \cos^{-1} \frac{d_1}{r_0} - 3d_1(r_0^2 - d_1^2)^{\frac{1}{2}}.
\]

Therefore, the common content of the truncated equilateral \( \triangle B_1 B_2 B_3 B_4 B_5 B_6 \) (i.e., the intersection of the two equilateral triangles of the above diagram) and the circle \( C \) of radius \( r_0 \) is

\[
[T \cap C] = r_0^2 - 3r_0^2 \cos^{-1} \frac{d_1}{r_0} + 3d_1(r_0^2 - d_1^2)^{\frac{1}{2}}.
\]
Case III:

If \( d_2 \leq r_0 \leq \rho \) consider the \( \triangle A_1A_2A_3 \). By Case II of Lemma 3.2.2.1., the segments \( B_1B_2, B_3B_4, B_5B_6 \), have total area

\[
3r_0^2 \cos^{-1} \frac{d_2}{r_0} - 3d_2(r_0^2 - d_2^2)^{\frac{1}{2}}.
\]

Consider again the \( \triangle A_1A_2A_3 \). From Case II of Lemma 3.2.2.1., the segments \( B_1B_6, B_4B_5, B_2B_3 \), have total area

\[
3r_0^2 \cos^{-1} \frac{d_1}{r_0} - 3d_2(r_0^2 - d_1^2)^{\frac{1}{2}}.
\]

Therefore, the common content of the truncated equilateral triangle \( B_1B_2B_3B_4B_5B_6 \) and the circle \( C \) of radius \( r_0 \) is

\[
[T \cap C] = r_0^2 - 3r_0^2 \cos^{-1} \frac{d_1}{r_0} + 3d_1(r_0^2 - d_1^2)^{\frac{1}{2}}
\]

\[
- 3r_0^2 \cos^{-1} \frac{d_2}{r_0} + 3d_2(r_0^2 - d_2^2)^{\frac{1}{2}}.
\]

Case IV:

The hexagon is inside the circle if \( r_0 \geq \rho \). The common part is the hexagon, therefore

\[
[T \cap C] = 3\sqrt{3}d_1^2 - \sqrt{3}(3d_1 - \sqrt{3/2})^2.
\]
Case III:
When $0 \leq y \leq 1/\sqrt{3}$ the plane $\pi_3$ will intersect the cube $0 \leq x_i \leq 1$, $i = 1, 2, 3$ in an equilateral triangle which will have as coordinates of its vertices $(\sqrt{3}y, 0, 0)$, $(0, \sqrt{3}y, 0)$, $(0, 0, \sqrt{3}y)$. Evidently the limiting equilateral triangle occurs when $y = 1/\sqrt{3}$, and the vertices of that triangle are $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$. $\pi_3$ will also intersect the cylinder $T$ in the circle

$$\sum_{i=1}^{3} (x_i - \sqrt{3}y)^2 = r^2_0.$$

Both the circle and the triangle lie in the same plane and have the same centre. Therefore as $\pi_3$ moves along orthogonal to the axis of symmetry we will have superimposed on each other, triangles of increasing sizes and circles of fixed radii. For $r_0$ sufficiently small the triangles will be completely inside the circles, then the triangles will intersect the circles, finally the triangles will contain the circles.

When $1/\sqrt{3} \leq y \leq \sqrt{3}/2$ the intersection is a truncated equilateral triangle and for the same $r_0$, the circle is inscribed.

To get a more comprehensive picture and to complete the analysis we will employ a graphical technique to study the relationship between $r_0$ and $d$. Substitute $(1/\sqrt{2})y$ for $d$ first in Lemma (3.2.2.1); setting $r_0 = d = (1/\sqrt{2})y$ then

$$(3.2.2.i) \quad 2r^2_0 = y^2.$$ 

Set $r_0 = 2d$ then

$$(3.2.2.ii) \quad 2r^2_0 = 4y^2.$$
For convenience we graph $2r_0^2$ against $y$ where $0 \leq y \leq 1/\sqrt{3}$.

Set $d_1 = (1/\sqrt{2})y$ in Lemma (3.2.2.2) and recall that $d = d_1$ and $d_2 = (\sqrt{3}/\sqrt{2}) - d_1$. Let $r_0 = d_1 = (1/\sqrt{2})y$ then (3.2.2.iii)

$$2r_0^2 = y^2.$$

Set $r_0 = d_2 = (\sqrt{3}/\sqrt{2}) - (1/\sqrt{2})y$ then (3.2.2.iv)

$$2r_0^2 = (\sqrt{3} - y)^2.$$

Let $r_0 = \rho = \sqrt{2}(1 - \sqrt{3}y + y^2)^{1/2}$ then (3.2.2.v)

$$2r_0^2 = 4(1 - \sqrt{3}y + y^2).$$

Again we graph $2r_0^2$ against $y$ where $1/\sqrt{3} \leq y \leq \sqrt{3}/2$.

The graphs of these functions of $y$ will now follow.

In Lemma (3.2.2.1) common contents for various configurations were found. Now label on the graph the common content found in Case I as $K_1$, in Case II as $K_2$ and in Case III as $K_3$.

For the common contents found in Case I, Case II, Case III, and Case IV of Lemma (3.2.2.2) let us label them respectively $K_1^*, K_2^*, K_3^*$ and $K_4^*$. 
Now from (3.2.2.i) \( 2r_0^2 = y^2 \) implies \( y = \sqrt{2}r_0 \)

and (3.2.2.ii) \( 2r_0^2 = 4y^2 \) implies \( y = \frac{\sqrt{2}r_0}{2} \)

when \( 0 \leq y \leq 1/\sqrt{3} \).

In (3.2.iii) we get \( y = \sqrt{2}r_0 \);

(3.2.2.iv) and from \( 2r_0^2 = (\sqrt{3} - y)^2 \) follows

\( y = \sqrt{3} - \sqrt{2}r_0 \)

(3.2.2.v) also from \( 2r_0^2 = 4(1 - \sqrt{3}y + y^2) \) follows

\( y = \frac{\sqrt{3}}{2} - \frac{\sqrt{2}a}{2} - \frac{1}{4} \cdot \frac{r_0^2}{r_0} \).

When \( 1/\sqrt{3} \leq y \leq \sqrt{3}/2 \).

We can now determine from the graph the limits for the regions over which the common content is the same. It can also be seen that the value for \( r_0 \) also plays a part in determining the different common contents.

When \( 0 \leq r_0 \leq (\sqrt{3}/2)/2 \), we find the common content of the cylinder and the cube as we integrate along the y-axis to be

(3.2.2.vi) \[ \int_{(\sqrt{2}/2)r_0}^{r_0} \int_{1/\sqrt{3}}^{\sqrt{2}r_0} \int_{\sqrt{3}/2}^{\sqrt{2}r_0} \left[ T^* \cap C^* \right] dy \]

When \( (\sqrt{3}/2)/2 \leq r_0 \leq 1/\sqrt{2} \)

(3.2.2.vii)
(3.2.2.vii) \[
[T* \cap C*] = 2 \left\{ \begin{array}{c}
\frac{1}{\sqrt{3}} \int_{0}^{\sqrt{3}/2} K_3 \, dy + \int_{\sqrt{3}/2}^{\sqrt{3} - \sqrt{2} r_0} K_2 \, dy + \int_{\sqrt{3} - \sqrt{2} r_0}^{\sqrt{3}/2} K_* \, dy + \int_{\sqrt{3}/2}^{\sqrt{3}/2} K_3 \, dy \\
\end{array} \right.
\]

Finally when \( 1/\sqrt{2} \leq r_0 \leq \sqrt{2}/3 \)

(3.2.2.viii) \[
[T* \cap C*] = 2 \left\{ \begin{array}{c}
\frac{1}{\sqrt{3}} \int_{0}^{\sqrt{3}/2} K_3 \, dy + \int_{\sqrt{3}/2}^{3/2 - (r_0^2/2 - 1/4)^{1/2}} K_2 \, dy + \int_{3/2 - (r_0^2/2 - 1/4)^{1/2}}^{3/2} K_* \, dy + \int_{3/2}^{\sqrt{3}/2} K_3 \, dy \\
\end{array} \right.
\]

Making use of the fact that

(3.2.2.ix) \[
\int \left\{ -3r^2 \cos^{-1} \frac{x}{r} + 3x(r^2 - x^2)^{1/4} \right\} \, dx = -3r^2 x \cos^{-1} \frac{x}{r} + 3r^2 (r^2 - x^2)^{1/4} - (r^2 - x^2)^{3/2}
\]

and setting \( d = y/\sqrt{2} \) in \( K_1, K_2, K_3 \)

and \( d_2 = (\sqrt{3}/2) - y/\sqrt{2} \) in \( K_1, K_2, K_3 \) and \( K_* \)

we will now evaluate the integrals which give the common content \( [T* \cap C*] \). Let us now rewrite (3.2.2.vi).

(3.2.2.x) \[
[T* \cap C*] = 2 \left\{ \begin{array}{c}
\frac{3\sqrt{2}}{2} \int_{0}^{\pi} y^2 \, dy + \left[ \pi r_0^2 - 3r^2 \cos^{-1} \frac{y}{r_0 \sqrt{2}} \right] \\
\end{array} \right. 
\]

\[+
3 \cdot \frac{y}{\sqrt{2}} \left\{ r_0^2 \left( \frac{y}{\sqrt{2}} \right)^{1/2} \right\} dy \]
(3.2.2.x) cont.

\[
\begin{aligned}
&= 2\left[ \frac{3\sqrt{3}}{2} \int_0^{(\sqrt{2}/2)r_0} y^2 dy + \int_{(\sqrt{2}/2)r_0}^{\sqrt{3}/2} \pi r_0^2 dy + \sqrt{2}\int_{(\sqrt{2}/2)r_0}^{\sqrt{2}/2} [-3r_0 \cos^{-1} \left( \frac{y}{r_0} \right) + 3 \frac{y}{\sqrt{2}} r_0^2] dy + \int_{(\sqrt{2}/2)r_0}^{\sqrt{3}/2} r_0^2 dy \right] \\
&= 2\left[ \frac{\sqrt{3}}{2} \cdot y^3 \int_0^{(\sqrt{2}/2)r_0} \right. \\
&\quad \left. + \pi r_0^2 y \int_{(\sqrt{2}/2)r_0}^{\sqrt{3}/2} + \sqrt{2}\left[ -3r_0 \cos^{-1} \left( \frac{y}{r_0} \right) \right. \right. \\
&\quad \left. \left. + 3r_0^2 \left\{ r_0^2 - \left( \frac{y}{\sqrt{2}} \right)^2 \right\} - \left\{ r_0^2 - \left( \frac{y}{\sqrt{2}} \right)^2 \right\} \right\} \int_{(\sqrt{2}/2)r_0}^{\sqrt{3}/2} \right] \\
&= 2\left[ \frac{\sqrt{6}r_0^3}{8} - 0 + \pi r_0^2 \left( \frac{\sqrt{3}}{2} - \frac{\sqrt{2}r_0}{2} \right) + 0 - \sqrt{2}\{-\pi r_0^3/2 + 3\sqrt{3}r_0^3/2 - 3\sqrt{3}r_0^3/8\} \right] \\
\end{aligned}
\]

(3.2.2.xi) \( = \sqrt{3}\pi r_0^2 - 2\sqrt{6}r_0^3 \) where \( 0 \leq r_0 \leq (\sqrt{3}/2)/2. \)

The evaluation of (3.2.2.vii) now follows

(3.2.2.xii) \[ [T^* \cap C^*] = 2\left( \int_0^{(\sqrt{2}/2)r_0} \frac{3\sqrt{3}y^2 dy}{2} + \int_{(\sqrt{2}/2)r_0}^{\sqrt{3}/2} \pi r_0^2 - 3r_0^2 \cos^{-1} \left( \frac{y}{r_0} \right) + 3 \frac{y}{\sqrt{2}} r_0^2 dy \right) - \left( \frac{y}{\sqrt{2}} \right)^2 \left( \frac{3}{2} \right) dy \]
(3.2.2.xii) cont.

\[
\left(\sqrt{3} - \sqrt{2}r_0 \right)
+ \int_{1/\sqrt{3}}^{\sqrt{3}/2} \left[ \pi r_0^2 - 3r_0^2 \cos^{-1} \frac{y}{r_0 \sqrt{2}} + 3 \frac{y}{\sqrt{2}} \left( r_0^2 - \left(\frac{y}{\sqrt{2}}\right)^2 \right) \right] dy
+ \int_{\sqrt{3}/2}^{\sqrt{3}/2} \left[ \pi r_0^2 - 3r_0^2 \cos^{-1} \frac{y}{r_0 \sqrt{2}} + 3 \frac{y}{\sqrt{2}} \left( r_0^2 - \left(\frac{y}{\sqrt{2}}\right)^2 \right) - 3r_0^2 \cos^{-1} \frac{\sqrt{3} - y}{\sqrt{2} r_0} \right] dy
+ 3 \left( \frac{\sqrt{3} - y}{\sqrt{2}} \right) \left( r_0^2 - \left(\frac{\sqrt{3} - y}{\sqrt{2}}\right)^2 \right) dy
\]

(3.2.2.xiii) \( \frac{(\sqrt{2}/2)r_0}{2} \)

\[
2 \left( \int_{0}^{\sqrt{3}/2} \frac{3\sqrt{3}y^2}{2} \ dy + \int_{\sqrt{3}/2}^{\sqrt{3}/2} \pi r_0^2 \ dy + \int_{\sqrt{3}/2}^{\sqrt{3}/2} \left[ -3r_0^2 \cos^{-1} \frac{y}{r_0 \sqrt{2}} \right] \right)
+ \frac{3\sqrt{3}/2}{\sqrt{3} - \sqrt{2}r_0} \left( r_0^2 - \left(\frac{\sqrt{3} - y}{\sqrt{2}}\right)^2 \right) dy
+ 3 \left( \frac{\sqrt{3} - y}{\sqrt{2}} \right) \left( r_0^2 - \left(\frac{\sqrt{3} - y}{\sqrt{2}}\right)^2 \right) dy.
\]

Let us break up (3.2.2.xiii) into \( I_1', I_2', I_3', I_4' \), where

\[
I_1 = \int_{0}^{(\sqrt{2}/2)r_0} \frac{3\sqrt{3}y^2}{2} \ dy = \frac{\sqrt{6}r^3}{8}.
\]

\[
I_2 = \int_{(\sqrt{2}/2)r_0}^{\sqrt{3}/2} \pi r_0^2 \ dy = \pi r_0^2 \left( \frac{\sqrt{3}}{2} - \frac{\sqrt{2}r_0}{2} \right).
\]
\[ I_3 = \frac{\sqrt{3}}{2} \left| -3r_0^2 \cos^{-1} \frac{\sqrt{3}}{2\sqrt{2}r_0} + 3r_0^2 \left\{ r_0^2 - \left(\frac{\sqrt{3}}{2}\right)^2 \right\} \right|_0^{\sqrt{3}/2} \]

\[ = \frac{\sqrt{3}}{2} \left[ -3r_0^2 \cos^{-1} \frac{\sqrt{3}}{2\sqrt{2}r_0} + 3r_0^2 \left\{ r_0^2 - \left(\frac{\sqrt{3}}{2}\right)^2 \right\} \right] \]

\[ = \frac{\sqrt{3}}{2} \left[ -3\sqrt{3}r_0^2 \cos^{-1} \frac{\sqrt{3}}{2\sqrt{2}r_0} + 3r_0^2 \left\{ r_0^2 - \left(\frac{\sqrt{3}}{2\sqrt{2}}\right)^2 \right\} - \left\{ r_0^2 - \left(\frac{\sqrt{3}}{2}\right)^2 \right\} \right] \]

\[ = \frac{\sqrt{3}}{2} \left[ -3\sqrt{3}r_0^2 \cos^{-1} \frac{\sqrt{3}}{2\sqrt{2}r_0} + 3r_0^2 \left\{ r_0^2 - \left(\frac{\sqrt{3}}{2\sqrt{2}}\right)^2 \right\} - \left\{ r_0^2 - \left(\frac{\sqrt{3}}{2}\right)^2 \right\} \right] \]

\[ I_4 = \frac{\sqrt{3}}{2} \left| -3r_0^2 \cos^{-1} \frac{\sqrt{3} - y}{\sqrt{2}r_0} + 3(\sqrt{3} - y) \left\{ r_0^2 - \left(\frac{\sqrt{3} - y}{\sqrt{2}}\right)^2 \right\} \right|_{\sqrt{3} - \sqrt{2}r_0}^{\sqrt{3}/2} \]

\[ = \frac{\sqrt{3}}{2} \left[ -3r_0^2 \cos^{-1} \frac{\sqrt{3} - y}{\sqrt{2}r_0} + 3(\sqrt{3} - y) \left\{ r_0^2 - \left(\frac{\sqrt{3} - y}{\sqrt{2}}\right)^2 \right\} \right] \]

\[ = -\sqrt{2} \left[ -3r_0^2 (\sqrt{3} - y) \cos^{-1} \frac{\sqrt{3} - y}{\sqrt{2}r_0} + 3r_0^2 \left\{ r_0^2 - \left(\frac{\sqrt{3} - y}{\sqrt{2}}\right)^2 \right\} \right] \]

\[ = -\sqrt{2} \left[ -3\sqrt{3}r_0^2 (\sqrt{3} - y) \cos^{-1} \frac{\sqrt{3}}{2\sqrt{2}r_0} + 3r_0^2 \left\{ r_0^2 - \left(\frac{\sqrt{3}}{2\sqrt{2}}\right)^2 \right\} - \left\{ r_0^2 - \left(\frac{\sqrt{3}}{2}\right)^2 \right\} \right] \]

\[ I_4 = \frac{\sqrt{3}}{2} \left| -3\sqrt{3}r_0^2 \cos^{-1} \frac{\sqrt{3}}{2\sqrt{2}r_0} + 3r_0^2 \left\{ r_0^2 - \left(\frac{\sqrt{3}}{2\sqrt{2}}\right)^2 \right\} - \left\{ r_0^2 - \left(\frac{\sqrt{3}}{2}\right)^2 \right\} \right| \]
Taking the sum \( 2(I_1 + I_2 + I_3 + I_4) \) gives

\[
(3.2.2.xiv)
\]

\[
[T^\ast \setminus C^\ast] = 2\left[\frac{\sqrt{6}r_0^3}{8} + \frac{\sqrt{3}r_0^2}{2} - \frac{\sqrt{2}r_0^3}{2} + \frac{\sqrt{2}r_0^2}{2} - \frac{3\sqrt{6}r_0^3}{8} + \frac{3\sqrt{6}r_0^3}{8}\right]
\]

\[
= \sqrt{3}r_0^2 - 2\sqrt{6}r_0^3;
\]

\[
\sqrt{3}/2 \leq r_0 \leq 1/\sqrt{2}.
\]

And finally the evaluation of (3.2.2.viii)

\[
(3.2.2.xv)
\]

\[
[T^\ast \setminus C^\ast] = 2\left\{ \frac{3\sqrt{3}y^2}{2} dy + \left[ \frac{\pi r_0^2 - 3r_0^2\cos^{-1}\frac{y}{r_0}}{r_0\sqrt{2}} + \frac{3y}{r_0}\frac{r_0^2 - (\frac{y}{\sqrt{2}})^2}{\sqrt{2}} \right] dy \right\}
\]

\[
\int_{0}^{(\sqrt{3}/2)r_0} \int_{(-1/\sqrt{3})}^{1/\sqrt{3}} \]

\[
\sqrt{3} - \sqrt{2}r_0 \]

\[
+ \left[ \frac{\pi r_0^2 - 3r_0^2\cos^{-1}\frac{y}{r_0}}{r_0\sqrt{2}} + \frac{3y}{r_0}\frac{r_0^2 - (\frac{y}{\sqrt{2}})^2}{\sqrt{2}} \right] dy \]

\[
\int_{(\sqrt{3}/2)}^{1/\sqrt{3}} \]

\[
[(r_0^2/2) - (1/4)]^{1/2} \]

\[
+ \left[ \frac{\pi r_0^2 - 3r_0^2\cos^{-1}\frac{y}{r_0}}{r_0\sqrt{2}} + \frac{3y}{r_0}\frac{r_0^2 - (\frac{y}{\sqrt{2}})^2}{\sqrt{2}} \right] dy \]

\[
\sqrt{3} - \sqrt{2}r_0 \]

\[
- 3r_0^2\cos^{-1}\left(\frac{\sqrt{3} - y}{\sqrt{2}r_0}\right) + 3\left(\frac{\sqrt{3} - y}{\sqrt{2}r_0}\right)\left\{r_0^2 - \left(\frac{\sqrt{3} - y}{\sqrt{2}}\right)^2\right\}^{1/2} \right] dy \]

\[
\int_{(\sqrt{3}/2)}^{(r_0^2/2 - 1/4)} \]

\[
\left[\frac{3\sqrt{3}y^2}{2} - \sqrt{3}\left(\frac{3y}{\sqrt{2}} - \frac{\sqrt{3}}{\sqrt{2}}\right)^2 \right] dy \]
\[ 2 \int_{0}^{(\sqrt{2}/2)r_0} \frac{3\sqrt{3}y^2}{2} dy + \int_{(\sqrt{3}/2) - \left( r_0^2/2 - 1/4 \right)^{1/2}}^{r_0} \left[ \pi r_0^2 - 3r_0^2 \cos^{-1} \frac{y}{\sqrt{2}} + 3 \frac{y}{\sqrt{2}} \left( r_0^2 - \left( \frac{y}{\sqrt{2}} \right)^2 \right)^{1/2} \right] dy \\
+ \int_{\sqrt{3}/2 - (r_0^2/2 - 1/4)^{1/2}}^{(\sqrt{3}/2) - \left( r_0^2/2 - 1/4 \right)^{1/2}} \left[ -3r_0^2 \cos^{-1} \left( \frac{\sqrt{3}}{\sqrt{2}} - \frac{y}{\sqrt{2}} \right) + 3 \frac{\sqrt{3}}{\sqrt{2}} \left( r_0^2 - \left( \frac{\sqrt{3}}{\sqrt{2}} - \frac{y}{\sqrt{2}} \right)^2 \right)^{1/2} \right] dy \]

Let

\[ I_1 = \int_{0}^{(\sqrt{2}/2)r_0} \frac{3\sqrt{3}y^2}{2} dy = \frac{\sqrt{6}r_0^3}{8} \]

\[ I_2 = \int_{(\sqrt{3}/2) - \left( r_0^2/2 - 1/4 \right)^{1/2}}^{r_0} \pi r_0^2 dy \]

\[ = \pi r_0^2 \left( (\sqrt{3}/2) - \left( r_0^2/2 - 1/4 \right)^{1/2} - (\sqrt{2}/2)r_0 \right) \]

\[ I_5 = \int_{\sqrt{3}/2}^{\sqrt{3}/2 - (r_0^2/2 - 1/4)^{1/2}} \left[ \frac{3\sqrt{3}y^2}{2} - \sqrt{3} \left( \frac{3y}{\sqrt{2}} - \frac{\sqrt{3}}{\sqrt{2}} \right)^2 \right] dy \]
\[ I_3 = \int_{\sqrt{3}/2}^{(\sqrt{3}/2) - [(r_0^2/2) - 1/4]} \left[ -3r_0^2 \cos^{-1} \frac{y}{r_0 \sqrt{2}} + 3 \frac{y}{r_0 \sqrt{2}} \left( r_0^2 - \left( \frac{y}{\sqrt{2}} \right)^2 \right)^{1/2} \right] dy \]

\[ = \sqrt{2} \left[ -3r_0^2 \frac{\cos^{-1} \frac{y}{r_0 \sqrt{2}}}{r_0 \sqrt{2}} + 3r_0 \left( r_0^2 - \left( \frac{y}{\sqrt{2}} \right)^2 \right)^{1/2} \right]_{(\sqrt{2}/2)r_0}^{(\sqrt{3}/2) - [(r_0^2/2) - 1/4]} \]

\[ = \sqrt{3} [3/2 - 6(y - \sqrt{3}/2)] dy \]

\[ = \left[ \frac{\sqrt{3}}{2} \left( \frac{r_0^2}{2} - \frac{1}{4} \right)^{3/2} - 2 \frac{r_0^2 - 1}{4} \right] \]

\[ = \frac{\sqrt{3}}{2} \left[ (r_0^2/2 - 1/4)^{1/2} (2 - r_0^2) \right] \]

\[ = \frac{\sqrt{3}}{2} (r_0^2/2 - 1/4)^{1/2} - \frac{\sqrt{3}r_0^2}{2} (r_0^2/2 - 1/4)^{1/2} . \]

At the upper limit
\[ r_0^2 - (\frac{y}{\sqrt{2}})^2 = \frac{r_0^2}{8} + \frac{\sqrt{3}}{2} (r_0^2/2 - 1/4)^{1/2} - \frac{1}{2} (r_0^2/2 - 1/4) \]
\[
\begin{align*}
\mathbf{I}_3 &= -3r_0^2 \left[ \frac{\sqrt{3}}{2} - (r_0^2/2 - 1/4)^{1/2} \right] \cos^{-1}(\sqrt{3}/2) - \left[ (r_0^2/2 - 1/4)^{1/2} \right] r_0^{1/2} \\
&\quad + 3\sqrt{2}r_0^2 \left[ \frac{1}{2\sqrt{2}} + (3r_0^2 - 3)^{1/2} \right] \sqrt{2} \left[ \frac{1}{2\sqrt{2}} + (3r_0^2 - 3)^{1/2} \right]^3 \\
&\quad + \frac{\sqrt{2}\pi r_0^3}{2} - 3\sqrt{6}r_0^3 + 3\sqrt{6}r_0^3.
\end{align*}
\]

It follows that
\[
\mathbf{I}_4 = \left\{ \begin{array}{l}
-3r_0^2 \cos^{-1}(\sqrt{3}/2 - \sqrt{2}/2) \\
\frac{1}{2\sqrt{2}} + (\sqrt{3}/2 - \sqrt{2}/2)^{1/2}
\end{array} \right\}
\]

\[
\begin{align*}
\mathbf{I}_4 &= \left[ 3\sqrt{2}r_0^2 \cos^{-1}(\sqrt{3}/2 - \sqrt{2}/2) + 3r_0^2 \frac{(\sqrt{3}/2) - (\sqrt{3}/2)^{1/2}}{\sqrt{2}} \right. \\
&\quad \left. - \frac{(\sqrt{3}/2)^{1/2}}{ \sqrt{2}} \right] \\
&\quad \left[ 3\sqrt{2}r_0^2 \cos^{-1}(\sqrt{3}/2) - \left[ (r_0^2/2 - 1/4)^{1/2} \right] r_0^{1/2} \\
&\quad - \left[ (r_0^2 - (\sqrt{3}/2)^{1/2})^{1/2} \right] \right]
\end{align*}
\]

At the upper limit
\[
\begin{align*}
\frac{r_0^2 - (\sqrt{3}/2)^{1/2}}{\sqrt{2}} &= \frac{r_0^2}{8} + \frac{\sqrt{3}(r_0^2/2 - 1/4)^{1/2}}{2} + \frac{1}{2}(r_0^2/2 - 1/4) \\
&= \left[ \frac{(3r_0^2 - 3)^{1/2}}{2\sqrt{2}} \right]^2.
\end{align*}
\]

\[
\begin{align*}
\left[ r_0^2 - (\sqrt{3}/2)^{1/2} \right]^{1/2} &= \frac{1}{2\sqrt{2}} - \frac{(3r_0^2 - 3)^{1/2}}{8}.
\end{align*}
\]

The sign is chosen to be positive for \( 1/\sqrt{2} \leq r_0 \leq \sqrt{2}/3 \).

It follows that
\[ I_4 = 3r_0^2 \left[ \sqrt{3} + \frac{r_0^2}{2} - \frac{1}{4} \right] \cos^{-1} \left( \frac{\sqrt{3}/2}{r_0^2} \right) + \left[ \frac{r_0^2}{2} - \frac{1}{4} \right]^{\frac{1}{2}} \]

\[-3\sqrt{2}r_0^2 \left[ \frac{1}{2\sqrt{2}} - \frac{3r_0^2}{4} - \frac{3}{8} \right] + \sqrt{2} \left[ \frac{1}{2\sqrt{2}} - \frac{3r_0^2}{4} - \frac{3}{8} \right]^3.\]

Taking the sum \(2(I_1 + I_2 + I_3 + I_4 + I_5)\) gives

\[2 \left[ \frac{\sqrt{6}r_0^3}{8} + \frac{3\pi r_0^2}{2} \right] - \pi r_0^2 \left( \frac{r_0^2}{2} - \frac{1}{4} \right)^{\frac{1}{2}} - \frac{\pi \sqrt{2}r_0^3}{2}
\]

\[+ \sqrt{3} \left( \frac{r_0^2}{2} - \frac{1}{4} \right)^{\frac{1}{2}} - \frac{\sqrt{3}r_0^2}{2} \left( \frac{r_0^2}{2} - \frac{1}{4} \right)^{\frac{1}{2}}\]

\[-3r_0^2 \left( \frac{\sqrt{3}}{2} - \left( \frac{r_0^2}{2} - \frac{1}{4} \right)^{\frac{1}{2}} \right) \cos^{-1} \left( \frac{\sqrt{3}/2}{r_0^2} \right) + \left[ \frac{r_0^2}{2} - \frac{1}{4} \right]^{\frac{1}{2}} \]

\[+ 3\sqrt{2}r_0^2 \left[ \frac{1}{2\sqrt{2}} - \frac{3r_0^2}{4} - \frac{3}{8} \right] - \sqrt{2} \left[ \frac{1}{2\sqrt{2}} - \frac{3r_0^2}{4} - \frac{3}{8} \right]^3\]

\[+ \sqrt{2} \pi r_0^3 \left( \frac{3\sqrt{6}r_0^3}{8} \right) + \frac{3\sqrt{6}r_0^3}{8}\]

\[+ 3r_0^2 \left( \frac{\sqrt{3}}{2} + \left( \frac{r_0^2}{2} - \frac{1}{4} \right)^{\frac{1}{2}} \right) \cos^{-1} \left( \frac{\sqrt{3}/2}{r_0^2} \right) + \left[ \frac{r_0^2}{2} - \frac{1}{4} \right]^{\frac{1}{2}} \]

\[-3\sqrt{2}r_0^2 \left[ \frac{1}{2\sqrt{2}} - \frac{3r_0^2}{4} - \frac{3}{8} \right] + \sqrt{2} \left[ \frac{1}{2\sqrt{2}} - \frac{3r_0^2}{4} - \frac{3}{8} \right]^3\]

Let \(\theta_1 = \cos^{-1} \left( \frac{\sqrt{3}/2}{r_0^2} \right) + \left[ \frac{r_0^2}{2} - \frac{1}{4} \right]^{\frac{1}{2}}\);

then \(\cos \theta_1 = \left( \frac{\sqrt{3}/2}{r_0^2} \right) + \left[ \frac{r_0^2}{2} - \frac{1}{4} \right]^{\frac{1}{2}}\)

and \(\sin \theta_1 = \frac{1}{r_0^2} \left[ \frac{1}{2\sqrt{2}} - \frac{3r_0^2}{4} - \frac{3}{8} \right].\)

Let \(\theta_2 = \cos^{-1} \left( \frac{\sqrt{3}/2}{r_0^2} \right) - \left[ \frac{r_0^2}{2} - \frac{1}{4} \right]^{\frac{1}{2}};\)

then \(\cos \theta_2 = \left( \frac{\sqrt{3}/2}{r_0^2} \right) - \left[ \frac{r_0^2}{2} - \frac{1}{4} \right]^{\frac{1}{2}}\)

and \(\sin \theta_2 = \frac{1}{r_0^2} \left[ \frac{1}{2\sqrt{2}} + \frac{3r_0^2}{4} - \frac{3}{8} \right].\)
Note: \[ \cos \theta_1 > \cos \theta_2 \]

\[ \therefore \quad \theta_1 < \theta_2. \]

\[ \cos(\theta_1 + \theta_2) = \frac{1}{r_0^2} \left[ \frac{3}{8} - \frac{r_0^2}{4} - \frac{1}{8} \right] - \frac{1}{8} + \frac{3r_0^2}{4} - \frac{3}{8} \]

\[ = \frac{1}{2} \]

\[ \therefore \quad \theta_1 + \theta_2 = \pi/3 \]

\[ \cos(\theta_1 - \theta_2) = \frac{1}{r_0^2} \left[ \frac{3}{8} - \frac{r_0^2}{4} - \frac{1}{8} \right] + \frac{1}{8} - \frac{3r_0^2}{4} - \frac{3}{8} \]

\[ = \frac{1}{r_0^2} - 1 \]

Since \[ \theta_1 - \theta_2 < 0 \]

\[ \theta_1 - \theta_2 = -\cos^{-1}(1/r_0^2 - 1). \]
Rewriting (3.2.2.xvi), after collecting like items, we get

\[
[T* \cap C^*] = 2 \left[ \frac{\sqrt{3} r_0^3}{8} - \frac{12 \sqrt{6} r_0^3}{8} + \frac{3 \sqrt{6} r_0^3}{8} - \frac{r_0^2 (r_0^2/2 - 1/4)^{1/2}}{8} 
- 3 r_0^2 \{ \frac{\sqrt{3}}{2} - (r_0^2/2 - 1/4)^{1/2} \} \cos^{-1} \left( \frac{\sqrt{3}/2}{r_0^{1/2}} \right) - \left[ \frac{(r_0^2/2 - 1/4)^{1/2}}{r_0^{1/2}} \right] \right]
+ 3 r_0^2 \{ \frac{\sqrt{3}}{2} + (r_0^2/2 - 1/4)^{1/2} \} \cos^{-1} \left( \frac{\sqrt{3}/2}{r_0^{1/2}} \right) + \left[ \frac{(r_0^2/2 - 1/4)^{1/2}}{r_0^{1/2}} \right] 
+ \sqrt{3} (r_0^2/2 - 1/4)^{1/2} - \frac{\sqrt{3}}{2} r_0^2 (r_0^2/2 - 1/4)^{1/2}
+ 3 \sqrt{2} r_0^2 \{ \frac{1 + \sqrt{3}(r_0^2/2 - 1/4)^{1/2}}{2 \sqrt{2}} \} - 3 \sqrt{2} r_0^2 \{ \frac{1 + \sqrt{3}(r_0^2/2 - 1/4)^{1/2}}{2 \sqrt{2}} \}
- \sqrt{2} \{ \frac{1 + \sqrt{3}(r_0^2/2 - 1/4)^{1/2}}{2 \sqrt{2}} \}^3 + \sqrt{2} \{ \frac{1 - \sqrt{3}(r_0^2/2 - 1/4)^{1/2}}{2 \sqrt{2}} \}^3
+ \frac{\sqrt{2} \pi r_0^3}{2} - \frac{\sqrt{2} \pi r_0^3}{2} + \frac{\sqrt{3} \pi r_0^3}{2}].
\]

Simplifying this result gives (3.2.2.xvii)

\[
[T* \cap C^*] = \sqrt{3} \pi r_0^2 - 2 \sqrt{6} r_0^3 - 3 \sqrt{3} r_0^2 \cos^{-1}(1/r_0^2 - 1) + 4 \sqrt{6} r_0^2 (r_0^2 - 1)^{1/2} + (6)^{1/2} (r_0^2 - 1)^{1/2}
\]

where \(1/\sqrt{2} \leq r_0 \leq \sqrt{2}/3\) and \(r_0 = 3^ {1/2}\) s.

After substituting in (3.2.2.xi), (3.2.2.xiv) and (3.2.2.xvii) we have

\[
P(r \leq r_0) = F(r_0) = F(3^{1/2}s)
\]

(3.2.2.xviii)

\[
= F_1(s) = 3 \sqrt{3} \pi s^2 - 18 \sqrt{2} s^3 \quad 0 \leq s \leq 6^{1/2}
\]

(3.2.2.xix)

\[
= F_1(s) = 3 \sqrt{3} \pi s^2 - 18 \sqrt{2} s^3
- 9 \sqrt{3} s^2 \cos^{-1}(1/3s^2 - 1) + 12 \sqrt{6} s^2 (3s^2 - 1)^{1/2} + (6)^{1/2} (3s^2 - 1)^{1/2}
6^{1/2} \leq s \leq \sqrt{2}/3.
\]
Which is the cumulative distribution of the standard deviation of sets of 3 variates drawn from a rectangular population.

When we take the derivative

\[ F'(s) = (6\sqrt{3}\pi s - 54\sqrt{2}s^2)ds \quad 0 \leq s \leq 1/\sqrt{6} \]

and

\[ F'(s) = (6\sqrt{3}\pi s - 54\sqrt{2}s^2 - 18\sqrt{3}s \cos^{-1}(1/3s^2-1) + 36\sqrt{6}s(3s^2-\frac{1}{2})^{\frac{1}{2}})ds \]

\[ 1/\sqrt{6} \leq s \leq \sqrt{2}/3. \]

Thus the frequency function or distribution function of \( S \) after further simplification is

(3.2.2.xx)

\[ f(s)ds = (6\sqrt{3}\pi s - 54\sqrt{2}s^2)ds \quad 0 \leq s \leq 1/\sqrt{6} \]

and

\[ f(s)ds = 18\left(\frac{\sqrt{3}\pi}{3} - 3\sqrt{2}s - \sqrt{3}\cos^{-1}(1/3s^2-1) + 2(18s^2-3)^{\frac{1}{2}}\right)ds \]

\[ 1/\sqrt{6} \leq s \leq \sqrt{2}/3. \]

which is identical with the result obtained by Rietz [28].

Let us now examine the result (3.2.2.xviii).

\[ [T^* \cap C^*] = \sqrt{3}\pi r_0^2 - 2\sqrt{6}r_0^3 - 3\sqrt{3}r_0^2 \cos^{-1}(1/r_0^2 - 1) \]

\[ + 4\sqrt{6}r_0^2(r_0^2 - \frac{1}{2}) + \sqrt{6}(r_0^2 - \frac{1}{2})^{\frac{1}{2}} \]

where \( 1/\sqrt{2} \leq r_0 \leq \sqrt{2}/3. \)

When \( r_0 \leq 1/\sqrt{2} \) the last three terms become meaningless in terms of volume and the first two terms are the same as that obtained in (3.2.2.xi) and (3.2.2.xiv) namely,
(3.2.2.xxi) \[ [T^* \backslash C^*] = \sqrt{3} \pi r_0^2 - 2\sqrt{6} r_0^3 \]

\[ 0 \leq r_0 \leq 1/\sqrt{2}. \]
3.2.3 Sample Size \( n = 4 \).

Fig. [3.2.3.1]
When \( n = 4 \) the hyper-cube, hyper-cylinder and hyper-plane reduce respectively to

the hyper-cube \( C: \quad 0 \leq x_i \leq 1, \quad i=1, 2, 3, 4 \)

the hyper-cylinder \( T: \quad \sum_{i=1}^{4} (x_i - \bar{x})^2 = r_0^2 \quad \text{where} \quad r_0^2 = 4s^2 \)

and the hyper-plane \( \pi_4: \quad \sum_{i=1}^{4} x_i = 2y \quad \text{where} \quad 2y = 4\bar{x} \).

The axis of symmetry of the cylinder \( T \) is the equi-angular line joining the vertices \( (0^4) \) and \( (1^4) \) of the 4-cube.

When \( 0 \leq y \leq \frac{1}{4} \) the plane \( \pi_4 \) intersects the 4-cube in the regular tetrahedron whose coordinates are 
\( (2y, 0, 0, 0), (0, 2y, 0, 0), (0, 0, 2y, 0) \) and 
\( (0, 0, 0, 2y) \). \( \pi_4 \) also intersects the hyper-cylinder in the sphere \( \sum (x_i - \frac{y}{2})^2 = r_0^2 \). Hence, both sphere and regular tetrahedron have the same centre. Let \( d \) be the distance from the centre of the tetrahedron to the centre of a 2-dimensional face. Referring to our figure [3.2.3.1] the coordinates of the centre of the 2-dimensional face \( P \) are
\[ (\frac{2y}{3}, \frac{2y}{3}, \frac{2y}{3}, 0). \]
The coordinates of \( C \) are
\[ (\frac{1y}{2}, \frac{1y}{2}, \frac{1y}{2}, \frac{1y}{2}) \]
hence,
\[ d^2 = y^2 \left[3\left(\frac{2}{3} - \frac{1}{2}\right)^2 + \frac{y^2}{2}\right] \]
and
\[ (3.2.3.i) \quad d = \frac{1y}{\sqrt{3}} \quad 0 \leq y \leq \frac{1}{4}. \]
Let \( d_1 \) be the distance from \( C \) the centre, to the
Fig. [3.2.3.2]
vertex A whose coordinates are \((0, 0, 0, 2y)\).

Then
\[
d_1^2 = y^2 \left[ \frac{3}{2} \left( \frac{1}{2} \right)^2 + \left( \frac{3}{2} \right)^2 \right]
\]
and
\[
(3.2.3.ii) \quad d_1 = \sqrt{3}y = 3d.
\]

Let \(\rho\) be the distance from C to the centre of the edge E whose coordinates are \((y, 0, 0, y)\).

Then
\[
\rho^2 = y^2 \cdot 4 \left( \frac{1}{2} \right)^2 = y^2
\]
and
\[
(3.2.3.iii) \quad \rho = y = \sqrt{3}d.
\]

When \(\frac{1}{2} < y \leq 1\), the plane \(z = 2y\) intersects the 4-cube in the truncated regular tetrahedron of figure [3.2.3.2]. Again
\[
(3.2.iv) \quad d = \frac{1}{\sqrt{3}}y \quad \frac{1}{2} < y \leq 1
\]

\[\rho = \sqrt{3}d \quad \frac{1}{2} < y \leq 1.\]

The coordinates of the centre of the face \(P_2\) are
\[
\left( \frac{2y-1}{3}, \frac{2y-1}{3}, \frac{2y-1}{3}, 1 \right)
\]
and the coordinates of C are
\[
\left( \frac{1}{2}y, \frac{1}{2}y, \frac{1}{2}y, \frac{1}{2}y \right)
\]
hence,
\[
d_1^2 = 3 \left( \frac{2y-1}{3} - \frac{1}{2}y \right)^2 + \left( 1 - \frac{1}{2}y \right)^2
\]
and
\[
(3.2.3.v) \quad d_1 = \frac{1}{\sqrt{3}}(2 - y) \quad \frac{1}{2} < y \leq 1.
\]

It follows that
\[
d + d_1 = \frac{2}{\sqrt{3}}
\]
therefore
\[
(3.2.3.vi) \quad d_1 = \frac{2}{\sqrt{3}} - d.
\]
The coordinates of the centre of the edge $E_1$ are
\[
\left( \frac{2y-1}{2}, \frac{2y-1}{2}, 0, 1 \right) \quad \text{and those of } C \text{ are}
\left( \frac{1y}{2}, \frac{1y}{2}, \frac{1y}{2}, \frac{1y}{2} \right)
\]
hence, \( s_1^2 \) is
\[
2 \left( \frac{2y-1}{2} - \frac{1y}{2} \right)^2 + \left( \frac{1y}{2} \right)^2 + \left( 1 - \frac{1y}{2} \right)^2
\]

(3.2.3.vii) \( s_1 = \frac{1}{\sqrt{2}} \left( 2y^2 - 4y + 3 \right)^{\frac{1}{2}} \quad \frac{1}{2} < y \leq 1. \)

The coordinates of the vertex $P$ are
\[
\left( 0, 2y-1, 0, 1 \right) \quad \text{and those of the}
\]
centre $C$ are
\[
\left( \frac{1y}{2}, \frac{1y}{2}, \frac{1y}{2}, \frac{1y}{2} \right)
\]
hence, \( s_2^2 \) is
\[
2 \left( \frac{1y}{2} \right)^2 + \left( 2y-1 - \frac{1y}{2} \right)^2 + \left( 1 - \frac{1y}{2} \right)^2
\]
\[
= \frac{1y}{2}^2 + \left( 3y - 1 \right)^2 + \left( 1 - \frac{1y}{2} \right)^2
\]
therefore, \( s_2 = \left( 3y^2 - 4y + 2 \right)^{\frac{1}{2}} \quad \frac{1}{2} < y \leq 1. \)

Now that we have the distances $d, d_1, \rho$ as a function of $y$ when $0 \leq y \leq \frac{1}{2}$ and $d, d_1, \rho, s_1, s_2$ also a function of $y$ when $\frac{1}{2} < y \leq 1$, we will set $r_0^2$ the squared radius of the sphere equal to each individual distance squared and examine the graph of $3r_0^2$ as a function of $y$. We now graph the functions.
Graph of The Domain of $F(r_0)$. 

$y = 2 + \sqrt{(3r_0^2 - 2)/3}$

$y = 2 - \sqrt{3r_0}$

$y = 1 - \sqrt{r_0^2 - \frac{1}{3}}$
Fig. [3.2.3.3]
Fig. [3.2.3.4]
The information which we receive from the graph can be put in the form of a table:

<table>
<thead>
<tr>
<th>I</th>
<th>II</th>
<th>III</th>
<th>IV</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 ≤ y ≤ 2⁻¹</td>
<td>2⁻¹ ≤ y ≤ √3⁻¹</td>
<td>√3⁻¹ ≤ y ≤ 3/4</td>
<td>3/4 ≤ y ≤ 1</td>
</tr>
<tr>
<td>f₁</td>
<td>f₁</td>
<td>f₁</td>
<td>f₁</td>
</tr>
<tr>
<td>e₁</td>
<td>e₁</td>
<td>f₂</td>
<td>f₂</td>
</tr>
<tr>
<td>v₁</td>
<td>f₂</td>
<td>e₁</td>
<td>e₂</td>
</tr>
<tr>
<td></td>
<td>e₂</td>
<td>e₂</td>
<td>e₁</td>
</tr>
<tr>
<td></td>
<td>v₂</td>
<td>v₂</td>
<td>v₂</td>
</tr>
</tbody>
</table>

Table 3.2.3.1

In the case when 0 ≤ y ≤ 2⁻¹, the sections made by the plane π₄ with the 4-dimensional cube, are regular tetrahedrons and f₁, e₁, and v₁ are labels given to the faces, edges and vertices of the regular tetrahedron as shown in Fig [3.2.3.3]. The order of f₁, e₁, v₁ indicates the manner in which the sphere of increasing radius and having common centre with a regular tetrahedron, expands first intersecting the faces then the edges and finally circumscribes the regular tetrahedron.

When 2⁻¹ ≤ y ≤ 1, the sections made by the plane with the 4-dimensional cube are truncated regular tetrahedrons. The manner in which the expanding sphere penetrates the faces, edges, etc. of the truncated regular tetrahedron is given in
the order shown in the table. It is to be noted that the order of intersecting the various parts of the regular tetrahedron by the sphere changes for different ranges of $y$. Fig.[3.2.3.4] gives a labeled figure of the various parts of the truncated regular tetrahedron.

We now prove two lemmas which will be used in the finding of the cumulative distribution of $S$, when the sample size is $4$, $S \leq 8^{-\frac{1}{2}}$, and the underlying cumulative distribution is rectangular.
LEMMA 3.2.3.1.

Let \( r_0 \) be the radius of a sphere \( S \) which has common centre with a regular tetrahedron \( T \) and let \( d \) be the distance from the centre of that regular tetrahedron to a 2-dimensional face. Let \([S \setminus T]\) be the content common to the tetrahedron and the sphere. Then

\[
C_1 = \frac{4\pi r_0^3}{3} \quad \text{if} \quad r_0 \leq d
\]

\[
C_2 = -\frac{4\pi r_0^3}{3} + 4\pi r_0^2d - \frac{4\pi d^3}{3} \quad \text{if} \quad d \leq r_0 \leq \sqrt{3}d
\]

\[
[S \setminus T] = C_3 = \frac{4\sqrt{2}d^2(r_0^2 - 3d^2)^{1/2}}{3} + 2r_0^3\left(\cos^{-1}\frac{r_0^2+3d^2}{3r_0^2-3d^2} - \cos^{-1}\frac{2\sqrt{2}d(r_0^2-3d^2)^{1/2}}{3r_0^2-3d^2} - \frac{\pi}{6}\right) + 2(3r_0^2d-3d^3)(\cos^{-1}\frac{r_0^2-5d^2}{r_0^2-d2} - \frac{\pi}{6}) \quad \sqrt{3}d \leq r_0 \leq 3d
\]

\[
C_4 = 8\sqrt{3}d^3 \quad \text{if} \quad r_0 \geq 3d.
\]

Proof:

Case I: The sphere is inside the regular tetrahedron if \( r_0 \leq d \). The common part is a sphere and its interior. The common content \( C_1 \) is therefore

\[
[S \setminus T] = \frac{4\pi r_0^3}{3}.
\]

Case II: The sphere intersects the faces of the regular tetrahedron if \( d \leq r_0 \leq \sqrt{3}d \).
To find the common content \([S \cap T]\) we refer the regular tetrahedron \(ABCD\) and the sphere to a rectangular coordinate system as shown in Fig.[3.2.3.1]. "Sections" are then made parallel to the base of the regular tetrahedron and the common areas of the equilateral triangles and circles are found by making use of Lemma 3.2.2.1. Integrating over the regions for which the common contents are the same finally gives the value for the content \([S \cap T]\).

Cut the \(z\) axis with a plane \(\pi_z\) parallel to the base of the tetrahedron at an arbitrary distance \(z\) from the base of the tetrahedron. The plane \(\pi_z\) will intersect the regular tetrahedron in an equilateral triangle and the sphere in a circle. The radius \(R\) of the circle at \(z\) which lies in this plane will be given by

\[
(3.2.3.1.i) \quad R = (r_0^2 - z^2)^{\frac{1}{2}}.
\]

On examining the vertical cross section made by the plane \(\pi_{xz}\) and the horizontal cross section in the plane \(\pi_z\), illustrated in figures [3.2.3.2] and [3.2.3.3] respectively, we find from similar triangles that
VERTICAL CROSS SECTION OF THE
SPHERE AND REGULAR TETRAHEDRON

CASE II.

Fig. [3.2.3.2]*
HORIZONTAL CROSS SECTION OF THE SPHERE AND REGULAR TETRAHEDRON

CASE II.

Fig. [3.2.3.3]*
\[
\frac{3d-z}{D} = \frac{4d}{\sqrt{2d}} = \frac{4}{\sqrt{2}}
\]

and hence

\[(3.2.3.1.ii) \quad D = \frac{\sqrt{2}}{4} (3d - z)\]

where \(D\) is the distance from the centre of the triangle of figure [3.2.3.3] to a 1-dimensional edge.

We will now compute the common contents for the various configurations of triangles and circles as the cutting plane \(\pi_z\) moves along the \(Z\) axis.

From Lemma (3.2.2.1) when \(R \leq D\) we get Case I,

hence \([T \cap C] = R^2\). It follows from 3.2.3.1.i and 3.2.3.1.ii that \([T \cap C] = (r_0^2 - z^2)\) when \((r_0^2 - z^2)^{\frac{1}{2}} \leq \frac{\sqrt{2}}{4} (3d - z)\).

When \(D \leq R \leq 2D\) Case II occurs.

Hence \([T \cap C] = \pi R^2 - 3R^2 \cos^{-1}_D + 3D(R^2 - D^2)^{\frac{1}{2}}\). Again from (3.2.3.1.i) and (3.2.3.1.ii)

\[
[T \cap C] = \pi (r_0^2 - z^2) - 3(r_0^2 - z^2) \cos^{-1} \frac{\sqrt{2}}{4} (3d - z) - \frac{1}{2} (3d - z)^{\frac{1}{2}}
\]

\[
+ \frac{3}{8} (3d - z) \cdot (8r_0^2 - 9z^2 + 6dz - 9d^2)^{\frac{1}{2}}
\]

where \(\frac{1}{2\sqrt{2}} (3d - z) < (r_0^2 - z^2)^{\frac{1}{2}} < \frac{1}{\sqrt{2}} (3d - z)\).

When \(R \geq 2D\) Case III occurs.

\[
[T \cap C] = 3\sqrt{3}D^2. \quad \text{From (3.2.3.1.i) and (3.2.3.1.ii)}
\]

\[
[T \cap C] = 3\sqrt{3}(3d - z) \quad \text{where} \quad (r_0^2 - z^2)^{\frac{1}{2}} \geq \frac{1}{2} (3d - z).
\]

It should be noted that Case III only occurs in a degenerate form in case \(C_2\) of this Lemma that is when the sphere is
tangent to the edge of the regular tetrahedron. The section through the points of tangency is an equilateral triangle with zero thickness.

Now choosing the appropriate limits for \( z \) and integrating over the regions for which the common content is the same we get

\[
C_2 = \left( \frac{d}{3} \right) - \frac{1}{3} \left( \frac{8r_0^2 - 8d^2}{2} \right)^{\frac{1}{2}} + \frac{r_0}{\pi} \int_{-d}^{0} \pi (r_0^2 - z^2) \, dz + \frac{r_0}{\pi} \int_{d/3}^{1/3} \pi (r_0^2 - z^2) \, dz
\]

\[
+ \frac{1}{3} \left( \frac{8r_0^2 - 8d^2}{2} \right)^{\frac{1}{2}} + \frac{1}{3} \left( \frac{8r_0^2 - 8d^2}{2} \right)^{\frac{1}{2}}
\]

\[
= \left( \frac{d}{3} \right) + \frac{1}{3} \left( \frac{8r_0^2 - 8d^2}{2} \right)^{\frac{1}{2}}
\]

\[
- \frac{3}{\pi} \left( \frac{r_0^2 - d^2}{2} \right) \cos^{-1} \left( \frac{3d - z}{\frac{8}{8}} \right) \frac{3d - z}{\frac{8}{8}} + \frac{3}{8} (3d - z) \cdot z^{\frac{1}{2}} \, dz
\]

\[
- \left( \frac{d}{3} \right) - \frac{1}{3} \left( \frac{8r_0^2 - 8d^2}{2} \right)^{\frac{1}{2}}
\]

where \( z = \frac{8r_0^2 - 9z^2 + 6dz - 9d^2}{8} \).

Rewriting this expression we get

\[
C_2 = \frac{r_0}{\pi} \int_{-d}^{0} \pi (r_0^2 - z^2) \, dz
\]

\[
= \left( \frac{d}{3} \right) + \frac{1}{3} \left( \frac{8r_0^2 - 8d^2}{2} \right)^{\frac{1}{2}}
\]

\[
+ \frac{1}{3} \left( \frac{8r_0^2 - 8d^2}{2} \right)^{\frac{1}{2}}
\]

\[
- \frac{3}{\pi} \left( \frac{r_0^2 - d^2}{2} \right) \cos^{-1} \left( \frac{3d - z}{\frac{8}{8}} \right) \frac{3d - z}{\frac{8}{8}} + \frac{3}{8} (3d - z) \cdot z^{\frac{1}{2}} \, dz.
\]

\[
= \left( \frac{d}{3} \right) - \frac{1}{3} \left( \frac{8r_0^2 - 8d^2}{2} \right)^{\frac{1}{2}}
\]

Call the first and second integrals of the preceding
expression \( I_1 \) and \( I_2 \) respectively. Integrating out \( z \) we get

\[ I_1 = \pi r^2 z - \frac{\pi z^3}{3} \left\{ r_0 = \pi r_0^3 + \pi r_0^2 d - \frac{\pi r_0^3}{3} - \frac{\pi d^3}{3} \right\} - d \]

And

\[ I_2 = -(3r_0^2 z - 3z^3) \cos^{-1} \frac{3d-z}{r_0^2} - \frac{r_0^2 z^2}{9} + \frac{d(z+d) z}{6} \]

\[ + \frac{z^{3/2}}{72} + \frac{3z-d}{6} \cdot d \cdot z^{1/2} + (3r_0^2 d - 3d^3) \sin^{-1} \frac{3z-d}{(8r_0^2 - 8z^2)} \]

\[ - r_0^3 \sin^{-1} \frac{6(d-3r)(z-r_0) - 2(r_0-3d)^2}{6(z-r_0)(8r_0^2 - 8d^2)} \]

\[ + \sin^{-1} \frac{6(d+3r)(z+r_0) - 2(r_0+3d)^2}{6(z+r_0)(8r_0^2 - 8d^2)} \]

\[ = - 2\pi r_0^3 - \pi d^3 + 3r_0^2 d) \]

It follows that

(3.2.3.1.v)

\[ [S \cap T] = I_1 + I_2 = - \frac{4\pi r_0^3}{3} + 4\pi r_0^2 d - \frac{4\pi d^3}{3} \]

\[ d \leq r_0 \leq \sqrt{3}d. \]

The result in (3.2.3.1.v) is easily checked by writing

\[ [S \cap T] = \frac{4\pi r_0^2}{3} - 4\left( \frac{\pi}{3} \right)(r_0 - d)^2 (2r_0 + d) \text{ using the formula for a spherical segment as noted on page 3-59.} \]
VERTICAL CROSS SECTION OF THE
SPHERE AND REGULAR TETRAHEDRON
CASE III.

Fig. [3.2.3.4]*
Case III: When $\sqrt{3d} \leq r_0 < 3d$ all the cases of Lemma 3.2.2.1 occur. Figure [3.2.3.4] should clarify this case. Again choosing the appropriate limits for $z$ we get

$$[T \cap C] = \int_{-d}^{d} \left( \frac{1}{3} \pi (r_0^2 - z^2) - 3(r_0^2 - z^2) \cos^{-1} \frac{3d - z}{8(r_0^2 - 8z^2)} \right) dz$$

$$+ \frac{3}{8} (3d - z) \cdot z^{3/2} dz$$

$$+ \int_{d}^{d + \frac{1}{3} (8r_0^2 - 8d^2)^{1/2}} \pi (r_0^2 - z^2) - 3(r_0^2 - z^2) \cos^{-1} \frac{3d - z}{8(r_0^2 - 8z^2)} + \frac{3}{8} (3d - z) \cdot z^{3/2} dz$$

$$= \int_{d}^{d + \frac{1}{3} (8r_0^2 - 8d^2)^{1/2}} \pi (r_0^2 - z^2) dz + \int_{d}^{d - \frac{1}{3} (8r_0^2 - 8d^2)^{1/2}} \pi (r_0^2 - z^2) dz$$

Rearranging the terms we get

$$[T \cap C] = \left[ \int_{d}^{d + \frac{1}{3} (8r_0^2 - 8d^2)^{1/2}} \pi (r_0^2 - z^2) dz - \int_{d}^{d - \frac{1}{3} (8r_0^2 - 8d^2)^{1/2}} \pi (r_0^2 - z^2) dz \right]$$

$$+ \left[ \frac{1}{3} \pi (r_0^2 - z^2) - 3(r_0^2 - z^2) \cos^{-1} \frac{3d - z}{8(r_0^2 - 8z^2)} + \frac{3}{8} (3d - z) \cdot z^{3/2} \right]$$

$$+ \left[ -3(r_0^2 - z^2) \cos^{-1} \frac{3d - z}{8(r_0^2 - 8z^2)} + \frac{3}{8} (3d - z) \cdot z^{3/2} \right]$$

$$+ \left[ -3(r_0^2 - z^2) \cos^{-1} \frac{3d - z}{8(r_0^2 - 8z^2)} + \frac{3}{8} (3d - z) \cdot z^{3/2} \right]$$
\[
\begin{align*}
I_1 &= \int_0^r \pi (r_0^2 z^2) \, dz = \pi r_0^3 + \pi r_0^2 d - \pi r_0^2 d - \frac{\pi r_0^3}{3} - \frac{\pi d^3}{3} \\
&= d + \frac{(2r_0^2 - 2d^2)^{\frac{3}{2}}}{3} \\
I_2 &= \int_0^r \frac{\pi (r_0^2 z^2) \, dz}{d} = \pi r_0^2 z - \pi z^3 \frac{3}{3} \\
&= d - \frac{(2r_0^2 - 2d^2)^{\frac{3}{2}}}{3} \\
I_3 &= \int_0^r 3 (r_0^2 - z^2) \, dz = 3d - z \frac{3d - z}{8} + \frac{3 (3d - z) \cdot z^{\frac{1}{2}}}{72} \\
&= \frac{r_0^2 z^{\frac{3}{2}}}{9} + \frac{d(z+d)}{6} z^{\frac{3}{2}} + \frac{z^{3/2}}{72} \\
&= \frac{(3z - d) z^{\frac{1}{2}}}{8} + (3r_0^2 d - d^3) \sin^{-1} \frac{3z - d}{(8r_0^2 - 8d^2)} \\
&= r_0^3 \left[ \sin^{-1} \frac{6(d - 3r)(z - r) - 2(r - 3d)^2}{6(z - r)(8r_0^2 - 8d^2)} + \sin^{-1} \frac{6(d + 3r)(z + r) - 2(r + 3d)^2}{6(z + r)(8r_0^2 - 8d^2)} \right]
\end{align*}
\]
\[ r_t^3 = d^2 (8r_0^2 - 24d^2) \]

\[ + r_t^3 \left( \frac{16r_0^2 + 24dr - 24d^2}{-6(d+r)(8r_0^2 - 8d^2)} \right) \]

\[ + 2(3r_0^2 - d^3) \sin^{-1} \left( \frac{4d}{(8r_0^2 - 8d^2)} \right) \]

\[ d + \left( \frac{2r_0^2 - 2d^2}{3} \right)^{\frac{1}{2}} \]

\[ I_4 = \int_{d - \left( \frac{2r_0^2 - 2d^2}{3} \right)^{\frac{1}{2}}}^{d + \left( \frac{2r_0^2 - 2d^2}{3} \right)^{\frac{1}{2}}} \text{(Integrand of } I_3) \, dz \]

\[ = \frac{2\pi}{3} \left( \frac{2r_0^2 - 2d^2}{3} \right)^{\frac{1}{2}} \left( \frac{7r_0^2 - d^2}{3} \right) + \frac{1}{8} \left( \frac{2r_0^2 - 6d^2}{3} \right) \left( (A+B)^{\frac{1}{3}} + (A+B)^{\frac{1}{3}} \right) \]

\[ - \frac{d}{2} \left( \frac{2r_0^2 - 2d^2}{3} \right)^{\frac{1}{2}} \left( (A-B)^{\frac{1}{3}} + (A+B)^{\frac{1}{3}} \right) \]

\[ -(3r_0^2 - d^3) \sin^{-1} \left( \frac{2d}{(8r_0^2 - 8d^2)} + \left( \frac{6r_0^2 - 18d^2}{8r_0^2 - 8d^2} \right) \right) - \sin^{-1} \left( \frac{2d}{(8r_0^2 - 8d^2)} \right) \]

\[ + \frac{6r_0^2 - 18d^2}{8r_0^2 - 8d^2} \]

\[ - \left( \frac{6r_0^2 - 18d^2}{8r_0^2 - 8d^2} \right) \]

\[ + r_t^3 \left( \frac{16r_0^2 + 12dr - 12d^2 + 6(d-3r)(2r_0^2 - 2d^2)}{6[(d-r) + (2r_0^2 - 2d^2)](8r_0^2 - 8d^2)^{\frac{1}{2}}} \right) \]

\[ + \sin^{-1} \left( \frac{16r_0^2 + 12rd - 12d^2 + 6(d+3r)(2r_0^2 - 2d^2)}{6[(d+r) + (2r_0^2 - 2d^2)](8r_0^2 - 8d^2)^{\frac{1}{2}}} \right) \]
\[-r_0^3 \sin^{-1} \left( \frac{16r_0^2 - 12rd - 12d^2}{(d-3r_0)(2r_0^2 - 2d^2)} \right) \frac{b_2}{6}\left( \frac{d-r}{2} \right) \frac{b_2}{(8r_0^2 - 8d^2)} \]

\[+ \sin^{-1} \left( \frac{16r_0^2 + 12rd - 12d^2}{(d+r)(2r_0^2 - 2d^2)} \right) \frac{b_2}{(8r_0^2 - 8d^2)} \]

Where \( A = 2r_0^2 + 6d^2 \), and \( B = 12d[(2/3)r_0^2 - 2d^2] \)

\[d + (2r_0^2 - 2d^2)^{1/2}\]

and \( I_5 = \int \frac{3\sqrt{3}}{8} (3d - z)^2 dz \]

\[d - (2r_0^2 - 2d^2)^{1/2}\]

\[-\frac{3\sqrt{3}}{8} \frac{(3d-z)^3}{3} \left| \begin{array}{c} \frac{r_0^2}{3} \\ \frac{r_0^2}{3} \end{array} \right| \]

\[= \frac{\sqrt{3}}{2} \left( 5d^2 + \frac{r_0^2}{3} \right) \left( 2r_0^2 - 2d^2 \right)^{1/2} \]

It follows that

\([Tf \setminus C] = I_1 + I_2 + I_3 + I_4 + I_5 \).

After combining the \( \sin^{-1} \) functions and the \( \cos^{-1} \) functions and simplifying we get

\([T \cap C] = 4\sqrt{2}d^2 \left( r_0^2 - 3d^2 \right)^{1/2} \]

\[+ 2r_0^3 \left( \cos^{-1} \frac{r_0^2 + 3d^2}{3r_0^2 - 3d^2} - \cos^{-1} \frac{2\sqrt{2}r_0 \left( r_0^2 - 3d^2 \right)^{1/2}}{3r_0^2 - 3d^2} - \frac{\pi}{6} \right) \]

\[+ 2(3r_0^2 d - d^3) \left( \cos^{-1} \frac{r_0^2 - 5d^2}{r_0^2 - d^2} - \frac{\pi}{3} \right) \]
\[ \sqrt{3}d \leq r_0 \leq 3d. \]

**Case IV:** The sphere circumscribes the regular tetrahedron when \( r_0 > 3d \). The common part is the regular tetrahedron and its interior. The common content is therefore

\[ [S \cap T] = 8\sqrt{3}d^3. \]

Q.E.D.
3.2.3.2 **Lemma 3.2.3.2.**

Let \( r_0 \) be the radius of a sphere \( S \) which has common centre with a truncated regular tetrahedron \( T \) and let \( d \) be the distance from the centre of that truncated regular tetrahedron to the nearest 2-dimensional face. Let \([S \setminus T]\) be the content common to the truncated regular tetrahedron and the sphere. Then

\[
C^*_1 = \frac{4\pi r_0^3}{3} \quad r_0 \leq d
\]

\[
C^*_2 = -\frac{4\pi r_0^3}{3} + \pi r_0^2 d - \frac{4\pi d^3}{3} \quad d \leq r_0 \leq \rho = \sqrt{3}d
\]

\[\text{[S \setminus T]} = \]

\[
C^*_3 = 4\sqrt{2}d^2 (r_0^2 - 3d^2)^{\frac{1}{2}}
\]

\[
+2r_0^3 \left( \cos^{-1} \left( \frac{r_0^2 + 3d^2}{3r_0^2 - 3d^2} \right) - \cos^{-1} \left( \frac{2\sqrt{2}r_0 (r_0^2 - 3d^2)^{\frac{1}{2}}}{3r_0^2 - 3d^2} \right) - \frac{\pi}{6} \right)
\]

\[
+2(3r_0^2d - d^3) \left( \cos^{-1} \left( \frac{r_0^2 - 5d^2}{r_0^2 - d^2} \right) - \frac{\pi}{6} \right)
\]

\[
\sqrt{3}d \leq r_0 \leq 3d.
\]

No proof will be given for this lemma since the results for \( C^*_1, C^*_2, C^*_3 \) are identical with those for \( C_1, C_2, C_3 \) of Lemma 3.2.3.1. From our graph however, it is to be noted that the lemma is only complete when \( 2^{-1} \leq y \leq 3/4 \)
and \( 0 \leq r_0 \leq \frac{1}{\sqrt{2}} \).
3.2.3.3  **Lemma 3.2.3.3.**

Let $r_0$ be the radius of a sphere $S$ which has common centre with a truncated regular tetrahedron $T$ and let $d$ be the distance from the centre of that truncated regular tetrahedron to the nearest 2-dimensional face, let $d_1$ be the distance to the farthest 2-dimensional face. Let $d + d_1 = \frac{2}{\sqrt{3}}$. Let $[S \cap T]$ be the content common to the truncated regular tetrahedron and the sphere, then

$$C^{**} = \frac{4\pi r_0^3}{3}$$

$$r_0 \leq d$$

$$[S \cap T] = C^{**} = -\frac{4\pi r_0^3}{3} + \frac{\pi r_0^2 d}{3} - \frac{4\pi d^3}{3}$$

$$d \leq r_0 \leq \rho = \sqrt{3}d$$

$$C^{**} = -\frac{4\pi r_0^3}{3} - \frac{\pi r_0^2 d}{3} + \frac{4\pi d^3}{3}$$

$$-\left(\frac{8\pi r_0^3}{3} - \frac{\pi r_0^2 d_1}{3} + \frac{4\pi d_1^3}{3}\right)$$

$$\sqrt{3}d \leq r_0 \leq d_1.$$  

**Proof:** The results for **Case I** and **Case II** follow from Lemma 3.2.3.1.

**Case III:** In Case II we have

$$C^{**} = -\frac{4\pi r_0^3}{3} + \frac{\pi r_0^2 d}{3} - \frac{4\pi d^3}{3}$$

$$= \frac{4\pi r_0^3}{3} - \frac{8\pi r_0^3}{3} + \frac{\pi r_0^2 d}{3} - \frac{4\pi d^3}{3}$$

$$= \frac{4\pi r_0^3}{3} - \left(\frac{8\pi r_0^3}{3} - \frac{\pi r_0^2 d}{3} + \frac{4\pi d^3}{3}\right).$$

In this last form we see that $[S \cap T]$ in Case II is really the volume of the sphere minus the total volume of the four spherical segments of the sphere when the planes forming the base of these segments are the nearest faces of the truncated
regular tetrahedron, which are at a distance \( d \) from the common centre of the sphere and truncated tetrahedron. It follows therefore, that when the sphere \( S \) bulges through the farthest faces that

\[
[S \cap T] = \frac{4}{3} \pi r_0^3 - \left( \frac{8}{3} \pi r_0^3 - \pi r_0^2 d + 4 \pi d^3 \right)
- \left( \frac{8}{3} \pi r_0^3 - \pi r_0^2 d_1 + 4 \pi d^3 \right)
\]

where \( d_1 \) is the distance to the individual farthest faces.

Collecting terms and simplifying our results we get

\[
[S \cap T] = -\frac{4}{3} \pi r_0^3 + \pi r_0^2 d - 4 \pi d^3 - \frac{8}{3} \pi r_0^3 + \pi r_0^2 d_1 - 4 \pi d^3.
\]

Q.E.D.

From the graph it will be noted that the above lemma is only complete when \( 0 \leq r_0 \leq 1/\sqrt{2} \) and \( \frac{3}{4} < y < 1 \).

We also note that \( C_1, C_2, C_3 \) in Lemma 3.2.3.1 are respectively equal to \( C^*_1, C^*_2, C^*_3 \) in Lemma 3.2.3.2. Also \( C^*_1 \) and \( C^*_2 \) of Lemma 3.2.3.2 are respectively equal to \( C^{**}_1 \) and \( C^{**}_2 \) of Lemma 3.2.3.3.

We now determine from the graph (3.2.3.1A) the regions over which the common content is the same. When \( 0 \leq r_0 \leq \frac{1}{\sqrt{3}} \)

we find the common content

\[
(3.2.3.ix)
\]

\[
[T^* \cap C^*] = 2 \int_0^{\frac{1}{\sqrt{3}}} C_4 \, dy + \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}r_0} C_3 \, dy + \int_{\sqrt{3}r_0}^1 C_2 \, dy + \int_0^{\frac{1}{\sqrt{3}}} C_1 \, dy.
\]
Again when \( \frac{d}{\sqrt{3}} \leq r_0 \leq \frac{1}{\sqrt{2}} \) we find the common content

\[
\frac{1r}{\sqrt{3}} \int_0^{r_0} r_0 \int_{r_0}^{2-\sqrt{3}r_0} \frac{1}{2-\sqrt{3}r_0} \int C_2 dy + C_3 dy + C_{**} dy \]

\[
[T*(\cap C^*)] = 2[ \int_0^{1r/\sqrt{3}} C_4 \frac{dy}{8} + \int_0^{r_0} C_3 \frac{dy}{r_0} + \int_{2-\sqrt{3}r_0}^{2-\sqrt{3}r_0} C_{**} \frac{dy}{2-\sqrt{3}r_0} ]
\]

Writing out (3.2.3.ix) explicitly after substituting
\[
\frac{1y}{\sqrt{3}}
\]

for \( d \) gives

\[
(3.2.3.xi) \quad \frac{1r}{\sqrt{3}} \int_0^{r_0} \int_0^{r_0} \int_0^{r_0} \int_0^{r_0} \int_0^{r_0} \int_0^{r_0}
\]

\[
[T*(\cap C^*)] = 2 \left[ \frac{8}{3} \int_0^{r_0} y^3 \frac{dy}{3} \right.
\]

\[
+ \frac{\pi}{3} \int_0^{r_0} \left( -r_0^3 - 2\sqrt{3}r_0^2 y + \frac{2}{3}\sqrt{3} y^3 \right) \frac{dy}{\sqrt{3}}
\]

\[
+ \frac{4\sqrt{2}}{3} \int_0^{r_0} \left( r_0^2 - y^2 \right) \frac{dy}{\sqrt{3}}
\]

\[
- \int_0^{r_0} \left( \cos^{-1} \frac{2\sqrt{2}r_0}{\sqrt{3}} \left( r_0^2 - y^2 \right)^{1/2} \right) \frac{dy}{\sqrt{3}}
\]

\[
\left. \int_0^{r_0} \left( \cos^{-1} \frac{r_0^2 + y^2}{3r_0^2 - y^2} \right) \frac{dy}{\sqrt{3}} \right]
\]

\[
+ \int_0^{r_0} \left( \cos^{-1} \frac{2(\sqrt{3}r_0^2 y - \frac{1}{3\sqrt{3}} y^3)}{3r_0^2 - 5y^2} \right) \frac{dy}{\sqrt{3}}
\]
\[
\begin{align*}
&\sqrt{3}r_0 \\
&+ \int_{r_0}^{\frac{1}{\sqrt{3}}} \left( -\frac{4\pi r_0^3}{3} + \frac{4\pi r_0^2 y}{\sqrt{3}} - \frac{4 y^3}{9\sqrt{3}} \right) dy \\
&\int_{\sqrt{3}r_0}^{r_0} \frac{1}{4\pi r_0^3} dy \bigg] \\
&= 2 \left[ \frac{4\pi r_0^3}{3} - 3\sqrt{2}r_0^4 \cos^{-1} \frac{1}{\sqrt{3}} \right] = \frac{8\pi r_0^3}{3} - 6\sqrt{2}r_0^4 \cos^{-1} \frac{1}{\sqrt{3}} \\
&0 \leq r_0 \leq \frac{1}{\sqrt{3}}.
\end{align*}
\]

We now write (3.2.3.xii) explicitly as

\[
(3.2.3.xii) \quad \frac{1r}{\sqrt{3}} \\
[T^* \cap C^*] = 2 \left[ \frac{8}{3} \int_{0}^{\frac{1}{\sqrt{3}}} y^3 dy \\
+ \frac{\pi}{3} \int_{0}^{\frac{1}{\sqrt{3}}} \left( -r_0^3 - 2\sqrt{3}r_0^2 y + \frac{2 y^3}{3\sqrt{3}} \right) dy \\
+ \frac{4\sqrt{2}}{3} \int_{0}^{\frac{1}{\sqrt{3}}} (r_0^2 - y^2)^{\frac{1}{2}} dy \\
- 2r_0^3 \int_{0}^{1r} \cos^{-1} \frac{2\sqrt{2}r_0 (r_0^2 - y^2)^{\frac{1}{2}}}{(3r_0^2 - y^2)} dy \\
+ \int_{0}^{1r} \cos^{-1} \frac{r_0^2 + y^2}{3r_0^2 - y^2} dy \\
\right]
\]
\[
\begin{align*}
&\int_{r_0}^{\sqrt{3}r_0} \left( 2(\sqrt{3}r_0^2y - \frac{1}{3}y^3) \cos^{-1} \frac{3r_0^2 - 5y^2}{3r_0^2 - y^2} \right) dy \\
&+ \int_{\sqrt{3}r_0}^{2-\sqrt{3}r_0} \left( \frac{4\pi r_0^3}{3} + \frac{4\pi r_0^2y - 4\pi y^3}{9\sqrt{3}} \right) dy \\
&+ \int_{\frac{1}{2}}^{1} \left( \frac{4\pi r_0^3}{3} + \frac{4\pi r_0^2y - 4\pi y^3}{9\sqrt{3}} \right) dy \\
&+ \int_{2-\sqrt{3}r_0}^{\frac{1}{\sqrt{3}}} \left( \frac{4\pi r_0^3}{3} + \frac{4\pi r_0^2y - 4\pi y^3}{9\sqrt{3}} \right) dy \\
&+ \int_{\frac{1}{\sqrt{3}}}^{\frac{1}{\sqrt{2}}} \left( \frac{8\pi r_0^3}{3} + \frac{4\pi r_0^2y - 4\pi y^3}{9\sqrt{3}} \right) dy \\
\end{align*}
\]

where \( \frac{1}{\sqrt{3}} \leq r_0 \leq \frac{1}{\sqrt{2}} \).

After examining the integrals for \([T^* \cap C^*]\) when \(0 \leq r_0 \leq \frac{1}{\sqrt{3}}\) and when \(\frac{1}{\sqrt{3}} \leq r_0 \leq \frac{1}{\sqrt{2}}\) we find that their first six integrals are identical. We will now show that the remaining integrals for both expressions of \([T^* \cap C^*]\) over different domains, are identical and hence, the expression for \([T^* \cap C^*]\) is the same when \(0 \leq r_0 \leq \frac{1}{\sqrt{3}}\) as that found when \(\frac{1}{\sqrt{3}} \leq r_0 \leq \frac{1}{\sqrt{2}}\).

Consider the last three expressions of (3.2.3.xii) and let

\[
L_1 = \int_{r_0}^{2-\sqrt{3}r_0} \left( -\frac{4\pi r_0^3}{3} + \frac{4\pi r_0^2y - 4\pi y^3}{9\sqrt{3}} \right) dy
\]
\[
\frac{1}{2-\sqrt{3}r_0} \left( -4\pi r^3 + 4\pi r^2 y - \frac{4\pi y^3}{9\sqrt{3}} \right) dy
\]

\[
\frac{1}{2-\sqrt{3}r_0} \left( -8\pi r^3 + 4\pi r^2 (2-y) - \frac{4\pi (2-y)^3}{9\sqrt{3}} \right) dy.
\]

Combining the first two integrals and making the transformation \( z = (2-y) \) where \( dz = -dy \) in the third integral gives

\[
\frac{1}{r_0} \left( -4\pi r^3 + \frac{4\pi r^2 y}{3} - \frac{4\pi y^3}{9\sqrt{3}} \right) dy
\]

\[
-\left( -4\pi r^3 dz \right) \left( -\frac{4\pi r^3}{3} + \frac{4\pi r^2 z}{\sqrt{3}} - \frac{4\pi z^3}{9\sqrt{3}} \right) dz.
\]

Further simplification, interchanging the limits of the last integral then combining all the integrals, gives

\[
\frac{1}{r_0} \left( -4\pi r^3 + \frac{4\pi r^2 y}{3} - \frac{4\pi y^3}{9\sqrt{3}} \right) dy
\]

\[
+ \frac{1}{\sqrt{3}r_0} \left( 4\pi r^3 \right) dy.
\]

It follows that \( l_1 \) consisting of the last three integrals of \( \{T^* \land C^*\} \) for \( \frac{1}{\sqrt{3}} \leq r_0 \leq \frac{1}{\sqrt{2}} \) is also identical with
the last two expressions on (3.2.3.xi) for \([T^* \cap C^*]\) when
\[0 \leq r_0 \leq \frac{1}{\sqrt{3}}.\]

It follows that
\[
[T^* \cap C^*] = \frac{8\pi r_0^3}{3} - 6\sqrt{2}r_0^4 \cos^{-1} \frac{1}{\sqrt{3}} \quad 0 \leq r_0 \leq \frac{1}{\sqrt{2}}
\]
\[= F'(r_0')\]

Hence, since \(r_0 = 2s\)

\[
P(S \leq s) = F_1(s) = \frac{64\pi s^3}{3} - 96\sqrt{2} s^4 \cos^{-1} \frac{1}{\sqrt{3}}
\]
\[0 \leq s \leq 8^{-1/2}.
\]
CHAPTER 4

CUMULATIVE DISTRIBUTION OF THE SAMPLE STANDARD DEVIATION

FOR SAMPLES OF SIZE \( n \) DRAWN FROM A RECTANGULAR

POPULATION FOR \( 0 < s \leq (2n)^{-\frac{1}{2}} \), WHEN \( n=2k \)

FOR \( \frac{(n^2-4)^{\frac{1}{2}}}{2n} \leq s \leq \frac{1}{2} \) AND WHEN \( n=2k+1 \)

FOR \( \frac{(5n-9)^{\frac{1}{2}}}{2n} \leq s \leq \frac{(n^2-1)^{\frac{1}{2}}}{2n} \).

4.1 Introduction.

In this Chapter we find the cumulative distribution \( P(S \leq s) \) for \( 0 < s \leq (2n)^{-\frac{1}{2}} \),

for \( \frac{(n^2-4)^{\frac{1}{2}}}{2n} \leq s \leq \frac{1}{2} \)

when the sample size is even,

and for \( \frac{(5n-9)^{\frac{1}{2}}}{2n} \leq s \leq \frac{(n^2-1)^{\frac{1}{2}}}{2n} \)

when the sample size is odd,

for sets of \( n \) variates drawn from the continuous rectangular population. Once we derive this result we find in (4.6.iii) the conjecture of Rietz (1931) [28], "that the distribution of \( S \) from \( n \) items may be a polynomial of degree \( n-1 \) for an interval of the range of \( S \) near zero."

In section 4.3 this statistical problem is related to the problem in geometry and so we simultaneously obtain the results for the common content of an \( n \)-dimensional unit cube and an \( n \)-dimensional cylinder having as its axis a diagonal of the \( n \)-dimensional cube and also bounded by the \( n \)-dimensional cube. We obtain results for the case when the radius of the \( n \)-dimensional cylinder is \( \leq 1/\sqrt{n} \). For even and odd
dimensions we find corresponding contents when

\[
\left( \frac{n^2-4}{4n} \right)^{\frac{1}{2}} \leq r_0 \leq \frac{\sqrt{n}}{2}
\]

and when

\[
\left( \frac{5n-9}{4n} \right)^{\frac{1}{2}} \leq r_0 \leq \left( \frac{n^2-1}{4n} \right)^{\frac{1}{2}}
\]

From the cases for \( n=2, 3, 4 \) one realises that though it is possible to arrive at the cumulative distribution of the statistic \( S \) for larger values of \( n \) by using the method of Chapter 3, it would be unwise to do so because of the complex nature of the geometry involved. In this chapter there, fore, we introduce the more powerful analytical technique of transformations.

We will first make the partial order transformation:

1. \( x(1) \geq 0 \)
2. \( x(n) \leq 1 \)
3. \( 0 < x(1) \leq x_i \leq x(n) < 1 \)

\( x_i \) for \( i=2, \ldots, n-1 \) unordered

on the variables \( x(i) \) for \( i=1, \ldots, n \).

This will divide the domain of the cumulative distribution function into \( n(n-1) \) mutually exclusive sets. This order transformation will be followed by a version of Helmert's Transformation in which the rows and columns will be somewhat rearranged. Helmert's Transformation will be followed by a cylindrical transformation. This will give the cumulative distribution in the form of a multiple iterated integral. However, its form will be reduced to an \( (n-2) \)-dimensional integral over a unit cube which can be evaluated by numerical techniques. We now prove the following lemma.
4.2 **LEMMA 4.2.**

The shortest distance from the equiangular line to an (n-2)-dimensional bounding face which does not contain the points \((0,\ldots,0)\) and \((1,\ldots,1)\) is \(2^{-k}\).

**Proof:**

Let \(x_1 = 0, x_2 = 1, 0 \leq x_i \leq 1, i=3,\ldots,n\) be an (n-2)-dimensional bounding face of an n-dimensional cube \(0 \leq x_i \leq 1\) \(i=1,\ldots,n\). Choose an arbitrary point: \(P_1 : (0, 1, y_3,\ldots,y_n)\) in this (n-2)-dimensional face. The projection of this point onto the equiangular line will be

\[
P_2 : \left( \frac{1}{n} \sum_{i=3}^{n} y_i, \ldots, \frac{1}{n} \sum_{i=3}^{n} y_i \right)
\]

The squared distance between the points \(P_1\) and \(P_2\) is

\[
d^2 = \left( \frac{1}{n} \sum_{i=3}^{n} y_i \right)^2 + \left( \frac{1}{n} \sum_{i=3}^{n} y_i - 1 \right)^2 + \sum_{j=3}^{n} \left( \frac{1}{n} y_i - y_j \right)^2.
\]

Let \(s = \sum_{i=3}^{n} y_i\) then

\[
f = d^2 = \left( \frac{1+s}{n} \right)^2 + \left( \frac{1+s-1}{n} \right)^2 + \sum_{j=3}^{n} \left( \frac{1+s-y_j}{n} \right)^2.
\]

We will now find the values of \(y_i, i=3,\ldots,n\) which minimizes \(d^2\). First we find the critical point of \(f\).

\[
\frac{\partial f}{\partial y_j} = \frac{2}{n} \frac{(1+s)}{n} + \frac{2}{n} \frac{(1+s-1)}{n} + \frac{2}{n} \sum_{j=3}^{n} \frac{(1+s-y_j)}{n} - \frac{2}{n} \frac{(1+s-y_j)}{n} - 2 \frac{(1+s-y_j)}{n}
\]

\[
= 2 \frac{(1+s+1+s-n) + \sum_{j=3}^{n} (1+s-ny_j) - n-ns+n^2y_j}{n^2}
\]

\[
(4.2.ii) = 2 \frac{[-ns-n+n^2y_j]}{n^2}.
\]
Setting $\frac{\partial f}{\partial y_j} = 0$, we get $y_j = \frac{1+s}{n}$ by symmetry we see that

$$y_i = \frac{1+s}{n}$$

therefore, $y_i = y_j$ for $i=3,\ldots,n$.

Therefore,

$$ny_j = 1 + (n-2)y_j$$

$$ny_j - (n-2)y_j = 1$$

then

$$y_j = \frac{1}{2}, \quad j=3,\ldots,n$$

which is the critical point of $f$.

To determine whether this point minimises or maximises $d^2f$ we examine the determinant of the matrix of second order partial derivatives of $f$.

Since

$$\frac{\partial^2 f}{\partial y_j^2} = \frac{2}{n^2} \left[-ns-n+n^2y_j\right]$$

then

$$\frac{\partial^2 f}{\partial y_j^2} = \frac{2}{n} (n-1)$$

and

$$\frac{\partial^2 f}{\partial y_i \partial y_j} = -\frac{2}{n}$$

Hence, the matrix of partial derivatives is the $(n-2) \times (n-2)$ matrix given by

$$A = \begin{bmatrix}
\frac{2(n-1)}{n} & -\frac{2}{n} & \cdots & -\frac{2}{n} \\
-\frac{2}{n} & \frac{2(n-1)}{n} & \cdots & -\frac{2}{n} \\
\vdots & \ddots & \ddots & \ddots \\
-\frac{2}{n} & \cdots & \cdots & \frac{2(n-1)}{n}
\end{bmatrix}. $$
Now since the quadratic form $X^\top A X$
\[
= \frac{2}{n} \left[ (n-1) \sum_{i \neq j} x_i x_j \right] + \frac{1}{n} \left[ (n-2) \sum x_i^2 + 2 \sum x_i x_j \right] \\
= \frac{2}{n} \left[ (n-2) \sum x_i^2 + \left( \sum x_i \right)^2 \right],
\]
it is readily seen that the matrix $A$ is positive definite. Therefore, the critical point $(\frac{1}{2}, \ldots, \frac{1}{2})$ of $f$ corresponds to the minimum of $f$.

When we substitute $\frac{1}{2}$ for $y_i$, $i=3, \ldots, n$ in $d^2$ in (4.2.i) we get
\[
d^2 = \frac{1}{2}
\]
and hence
\[
d = \frac{1}{\sqrt{2}}
\]
is the minimum distance from the equiangular line to an (n-2)-dimensional bounding plane.

Q.E.D.

The coordinates of $P_1:(0, 1, \frac{1}{2}, \ldots, \frac{1}{2})$ also happen to be the centre of gravity of the (n-2)-dimensional face $x_1 = 0, \quad x_2 = 1, \quad 0 \leq x_i \leq 1, \quad i=3, \ldots, n.$
4.3 The Distribution For $s \leq (2n)^{-\frac{1}{2}}$.

Let $x_1, x_2, \ldots, x_n$ be n-independent identically distributed random variables from the continuous rectangular distribution

$$dF = dx, \quad 0 \leq x \leq 1.$$ 

The joint distribution of the $n$ variables is

$$dF = \prod_{i=1}^{n} dx_i, \quad 0 \leq x_i \leq 1, \quad i = 1, \ldots, n.$$ 

Therefore the sampling cumulative distribution of $S$ is

$$P(S \leq s) = F(s) = \left\{ \prod_{i=1}^{n} dx_i \right\}_{D}$$

where

$$D = \begin{cases} 
(a) & n^{-1} \sum_{i=1}^{n} (x_i - \bar{x})^2 \leq s^2 \\
(b) & 0 \leq x_i \leq 1, \quad i = 1, \ldots, n.
\end{cases}$$

Let $r_0^2 = ns^2$, then

$$F(r_0) = \left\{ \prod_{i=1}^{n} dx_i \right\}_{D}$$

where

$$D = \begin{cases} 
(a) & \sum_{i=1}^{n} (x_i - \bar{x})^2 \leq r_0^2 \\
(b) & 0 \leq x_i \leq 1, \quad i = 1, \ldots, n.
\end{cases}$$

The form of (4.3.ii) is geometrically the common content of a cylinder and an n-dimensional unit cube. $r_0$ is therefore chosen to be $\leq 2^{-\frac{1}{2}}$ which is the distance from the axis of the cylinder (or diagonal of the cube) to an $(n-2)$-dimensional bounding face.
To evaluate the integral (4.3.ii) we will partially order the x's such that

(4.3.iii)
(a) \(0 \leq x_1\)
(b) \(x_n \leq 1\)
(c) \(0 < x_1 \leq x_i \leq x_n < 1\)

where \(x_1\) denotes the smallest, \(x_n\) the largest and 
\(x_2, \ldots, x_{n-1}\) are unordered.

The integral (4.3.ii) may be rewritten as

(4.3.iv)
\[
F(r_0) = n(n-1) \int_D \prod_{i=2}^{n-1} dx_i dx_n
\]

where \(D = (a) \sum_{i=1}^{n} (x_i - \bar{x})^2 \leq r_0^2\)

(b) \(0 \leq x_1', x_n' \leq 1\)
\(0 < x_1 \leq x_i \leq x_n < 1, x_i \) unordered.

To further facilitate evaluation of this integral \(F(r_0)\) an orthogonal transformation is made. The order notation on the variables can be dropped and replaced by regular subscripts.

When we use a prime to denote a column vector, we may transform orthogonally to new variables of integration by setting

(4.3.v)
\[ y' = Ax', \]

where we require that \(A\) be the \(n \times n\) Helmert's transformation matrix with a rearrangement of rows and columns.
Upon the application of the transformation inverse we get

(4.3.vii) \[ x' = A^T y'. \]

Where \( A^T \) is the transpose of the matrix A. Since A is orthogonal the Jacobian of the transformation is 1.

Follow this orthogonal transformation by the cylindrical transformation:
\( y_1 = y_1 \)
\( y_2 = \rho \cos \theta_1 \)
\( y_3 = \rho \sin \theta_1 \cos \theta_2 \)
\[ \ldots \]
\( y_i = \rho \sin \theta_1 \sin \theta_2 \ldots \sin \theta_{i-2} \cos \theta_{i-1} \)
\[ \ldots \]
\( y_{n-1} = \rho \sin \theta_1 \sin \theta_2 \ldots \sin \theta_{n-3} \cos \theta_{n-2} \)
\( y_n = \rho \sin \theta_1 \sin \theta_2 \ldots \sin \theta_{n-3} \sin \theta_{n-2} \)

where \( 0 < \theta_i < \pi, \quad i=1, \ldots, n-3, \quad 0 < \theta_{n-2} < 2\pi, \quad 0 < \rho < \infty. \)

This 1 to 1 transformation has the Jacobian
\[
|J| = \rho^{n-2} \sin^{n-3} \theta_1 \sin^{n-4} \theta_2 \ldots \sin \theta_{n-3}.
\]

The integral (4.3.iv) becomes

\[
(4.3.ix) \quad F(r_0) = n(n-1) \int_D \rho^{n-2} \sin^{n-3} \theta_1 \sin^{n-4} \theta_2 \ldots \sin \theta_{n-3} \, dy_1 \, d\rho \, d\theta_1 \ldots d\theta_{n-2}
\]

where \( D \) is given by

\[
(a) \quad \rho^2 < r_0^2
\]
(b) \[ \frac{n - i + 1}{(n - i + 1)(n - i)^{1/2}} \rho \sin \theta_1 \sin \theta_2 \cdots \cos \theta_i \]
+ \[ \frac{1}{(n - i)(n - i - 1)^{1/2}} \rho \sin \theta_1 \sin \theta_2 \cdots \sin \theta_i \cos \theta_{i+1} \]
+ \[ \cdots \quad \cdots \quad \cdots \]
+ \[ \frac{1}{6^{1/2}} \rho \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{n-3} \cos \theta_{n-2} \]
+ \[ \frac{1}{2^{1/2}} \rho \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{n-3} \sin \theta_{n-2} \geq 0 \]
\quad i = 1, \ldots, n - 1

(c) \[ y_{1/2} - \frac{n^{1/2}}{n(n - 1)^{1/2}} \rho \cos \theta_1 \]
- \[ \frac{n^{1/2}}{(n - 1)(n - 2)^{1/2}} \rho \sin \theta_1 \cos \theta_2 \]
- \[ \cdots \quad \cdots \quad \cdots \]
- \[ \frac{n^{1/2}}{(n - 1)(n - 2)^{1/2}} \rho \sin \theta_1 \cdots \sin \theta_{i-2} \cos \theta_{i-1} \]
- \[ \frac{n}{6} \rho \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{n-3} \sin \theta_{n-2} \]
- \[ \frac{n}{2} \rho \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{n-3} \cos \theta_{n-2} \]

(d) \[ \frac{n^{1/2}}{(n(n - 1))^{1/2}} \rho \cos \theta_1 - \frac{n - 2}{(n - 1)(n - 2)^{1/2}} \rho \sin \theta_1 \cos \theta_2 \geq 0 \]
\[ \vdots \]
\[ \vdots \]
\[ \frac{n^{1/2}}{n(n - 1)^{1/2}} \rho \cos \theta_1 + \frac{1}{(n - 1)(n - 2)^{1/2}} \rho \sin \theta_1 \cos \theta_2 \cdots \]
- \[ \frac{n - i}{(n - i + 1)(n - i)^{1/2}} \rho \sin \theta_1 \cdots \cos \theta_i \geq 0 \]
(d) cont.

\[ \frac{n}{\sqrt{n(n-1)}} \rho \cos \theta_1 + \frac{1}{\sqrt{(n-1)(n-2)}} \rho \sin \theta_1 \cos \theta_2 + \ldots + \frac{1}{\sqrt{n}} \rho \sin \theta_1 \ldots \cos \theta_{n-2} \]

\[ - \frac{1}{\sqrt{2^n}} \rho \sin \theta_1 \ldots \sin \theta_{n-2} \geq 0. \]

(e)

\[ y_1 \geq (n-1)^{\frac{1}{2}} \rho \cos \theta_1. \]

The domain \( D \) of (4.3.ix) was arrived at thus: From the transformation (4.3.iii) we get

\[ \sum_{i=1}^{n} x_i^2 - nx_i^2 \leq r_0^2 \]

which transformed into

\[ \sum_{i=2}^{n} y_i^2 \leq r_0^2 \]

under transformation (4.3.vii.). Under the transformation (4.3.viii)

\[ \sum_{i=2}^{n} y_i^2 \leq r_0^2 \]

transformed into \( \rho^2 \leq r_0^2. \)

The order transformation (4.3.iii) is equivalent to

(b) \( x(n) - x_i \geq 0, \quad i = 1, \ldots, n-1 \)

(c) \( x(n) \leq 1. \)

(d) \( x_i - x(1) \geq 0, \quad i = 2, \ldots, n-1 \)

(e) \( x(1) \geq 0. \)

Under the transformation (4.3.vii) (b), (c), (d) and (e) became respectively

(b) \[ \frac{n - i + 1}{\sqrt{(n - i + 1)(n - i)}} y_{i+1} + \frac{1}{\sqrt{(n - i)(n - i - 1)}} y_{i+2} \]

\[ \ldots + \frac{1}{\sqrt{6}} y_{n-1} + \frac{1}{\sqrt{2}} \frac{y_n}{2} \geq 0, \quad i = 1, \ldots, n-1. \]
(c) \[
\frac{1}{n^2} y_1 + \frac{1}{[n(n-1)]^{\frac{1}{2}}} y_2 + \frac{1}{[(n-1)(n-2)]^{\frac{1}{2}}} y_3 \\
\cdots + \frac{1}{6^{\frac{1}{2}}} y_{n-1} + \frac{1}{2^{\frac{1}{2}}} y_n \leq 1, \ i = 1, \ldots, n - 1.
\]

(d) \[
\frac{n}{[n(n-1)]^{\frac{1}{2}}} y_2 - \frac{n-2}{[(n-1)(n-2)]^{\frac{1}{2}}} y_3 \geq 0 \\
\vdots \\
\frac{n}{[n(n-1)]^{\frac{1}{2}}} y_2 + \frac{1}{[(n-1)(n-2)]^{\frac{1}{2}}} y_3 \cdots - \frac{n-i}{[(n-i+1)(n-i)]^{\frac{1}{2}}} y_{i+1} \\
\vdots \\
\frac{n}{[n(n-1)]^{\frac{1}{2}}} y_2 + \frac{1}{[(n-1)(n-2)]^{\frac{1}{2}}} y_3 \cdots + \frac{1}{6^{\frac{1}{2}}} y_{n-1} - \frac{1}{2^{\frac{1}{2}}} y_n \geq 0.
\]

(e) \[
\frac{1}{n^2} y_1 - \frac{n-1}{[n(n-1)]^{\frac{1}{2}}} y_2 \geq 0.
\]

After the application of transformation (4.3.viii) (b), (c), (d) and (e) transform into (b), (c), (d) and (e) of the domain D of (4.3.ix).

Returning to (4.3.ix)

\[
F(r_0) = n(n-1) \int_D \rho^{n-2} \sin^{n-3} \theta_1 \sin^{n-4} \theta_2 \cdots \sin^{n-3} \theta_{n-3} \\
dy_1 \ d\rho \ d\theta_1 \cdots d\theta_{n-2}
\]

We first integrate out \( y_1 \) whose upper limit of integration is

\[
y^* = n^{\frac{1}{2}} \left( \frac{n}{[n(n-1)]^{\frac{1}{2}}} \rho \cos \theta_1 - \frac{n}{[(n-1)(n-2)]^{\frac{1}{2}}} \rho \sin \theta_1 \cos \theta_2 \\
\cdots \frac{n}{(2)^{\frac{1}{2}}} \rho \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{n-3} \sin \theta_{n-2}, \right.
\]

then \( \rho \) whose upper limit is \( r_0 \).

(4.3.ix) now becomes (4.3.x)
\( F(r_0) = \left( \frac{\rho}{(n-1)} \right)^3 r_0^{n-1} \left\{ \begin{array}{l} \sin^{-3}\theta_1 \sin^{-4}\theta_2 \ldots \sin^{-3}\theta_{n-3} \cos\theta_1 \cos\theta_2 \ldots \cos\theta_{n-2} \\
^{-2} \sin\theta_1 \cos\theta_2 \ldots \sin\theta_{n-2} \end{array} \right\} \)

where the domain \( D_1 \) is given by:

(b) \[ \frac{n-i+1}{(n-i+1)(n-i)} \rho \sin\theta_1 \sin\theta_2 \ldots \cos\theta_i \]

+ \[ \frac{1}{(n-i)(n-i-1)} \rho \sin\theta_1 \sin\theta_2 \ldots \sin\theta_i \cos\theta_{i+1} \]

+ \[ \ldots \]

+ \[ \frac{1}{n-3} \rho \sin\theta_1 \sin\theta_2 \ldots \sin\theta_{n-3} \cos\theta_{n-2} \]

+ \[ \frac{1}{2} \rho \sin\theta_1 \sin\theta_2 \ldots \sin\theta_{n-3} \sin\theta_{n-2} \geq 0 \]

i = 1, \ldots, n - 1.

(d) \[ \frac{n}{n(n-1)} \rho \cos\theta_1 \]

\[ \vdots \]

\[ \frac{n}{n(n-1)} \rho \cos\theta_1 + \frac{1}{(n-1)(n-2)} \rho \sin\theta_1 \cos\theta_2 \geq 0 \]

\[ \vdots \]

\[ \frac{n}{n(n-1)} \rho \cos\theta_1 + \frac{1}{(n-1)(n-2)} \rho \sin\theta_1 \cos\theta_2 \]

\[ \vdots \]

\[ \frac{n}{n(n-1)} \rho \cos\theta_1 + \frac{1}{(n-1)(n-2)} \rho \sin\theta_1 \cos\theta_2 \]

\[ + \frac{1}{n-1} \rho \sin\theta_1 \ldots \cos\theta_{n-2} \]
(d) cont.

\[-\frac{1}{2} \frac{\rho \sin \theta_1 \ldots \sin \theta_{n-2}}{1} > 0.\]

We will prove in Lemmas 4.4 and 4.5 that

\[
\left. \begin{array}{c}
\sin^{n-3} \theta_1 \sin^{n-4} \theta_2 \ldots \sin \theta_{n-3} \prod_{i=1}^{n-2} \sin \theta_i \\
\end{array} \right|_{D_1} \frac{2\pi^{\frac{1}{2}} (n-1)}{n(n-1) \Gamma \left\{ \frac{1}{2} (n-1) \right\}}.
\]

\[
\left. \begin{array}{c}
\{ \frac{\cos \theta_1 + \ldots + \frac{1}{2} \sin \theta_1 \ldots \sin \theta_{n-2}}{\frac{1}{2}} \\
\end{array} \right|_{D_1} [n(n-1)] \frac{2}{\sin^{n-3} \theta_1 \sin^{n-4} \theta_2 \ldots \sin \theta_{n-3} \prod_{i=1}^{n-2} \sin \theta_i}.
\]

\[
= \frac{1}{2^n} \left. \begin{array}{c}
\frac{n-1}{n-1} \prod_{i=2}^{n-1} \frac{\sin \theta_i}{u_i} \\
\end{array} \right| \frac{1}{2^n} \left[ (1 + \sum_{i=2}^{n-1} u_i^2) - \frac{1}{n} (1 + \sum_{i=2}^{n-1} u_i^2) \right] \frac{1}{\ln}.
\]

0 < u_i < 1

i=2, \ldots, n-1.

Therefore,

\[(4.3.xi) \]

\[
F(r_0) = \frac{2\pi^{\frac{1}{2}} (n-1) \frac{1}{2} r_0 n-1}{(n-1) \Gamma \left\{ \frac{1}{2} (n-1) \right\}} - (n-1) r_0^n \left. \begin{array}{c}
\left( \prod_{i=2}^{n-1} \frac{1}{u_i} \\
\end{array} \right| \frac{1}{2^n} \left[ (1 + \sum_{i=2}^{n-1} u_i^2) - \frac{1}{n} (1 + \sum_{i=2}^{n-1} u_i^2) \right] \frac{1}{\ln}.
\]

0 < u_i < 1

i=2, \ldots, n-1.
4.4 **LEMMA 4.4.**

(4.4.i)

\[
J_{n-2} = \begin{vmatrix}
\sin^{n-3}\theta_1 & \sin^{n-4}\theta_2 & \ldots & \sin\theta_{n-3} & \prod_{i=1}^{n-2} d\theta_i \\
\end{vmatrix}_{D_{1}}
\]

\[
= \frac{2\pi \frac{k}{2}(n-1)}{n(n-1) \Gamma\left(\frac{k}{2}(n-1)\right)}
\]

where \( D_{1} \) is

(b) \[
\frac{n-i+1}{k} \rho \sin^{i}_{1} \sin_{2} \ldots \cos_{1} \frac{1}{(n-i+1)(n-i)} \]

+ \[
\frac{1}{k} \rho \sin^{i}_{1} \sin_{2} \ldots \sin_{1} \cos_{i+1} \frac{1}{(n-i)(n-i-1)} \]

+ \[
\ldots \quad \ldots \quad \ldots \]

+ \[
\frac{1}{6k} \rho \sin^{i}_{1} \sin_{2} \ldots \sin_{1} \cos_{i} \]

+ \[
\frac{1}{2k} \rho \sin^{i}_{1} \sin_{2} \ldots \sin_{1} \sin_{n-3} \cos_{i} \]

+ \[
\frac{1}{2k} \rho \sin^{i}_{1} \sin_{2} \ldots \sin_{1} \sin_{n-3} \sin_{n-2} \geq 0
\]

i=1, \ldots , n-1.

(d) \[
\frac{n}{k} \rho \cos_{1} - \frac{n-2}{k} \rho \sin^{1}_{1} \cos_{2} \geq 0
\]

\[
\frac{n}{k} \rho \cos_{1} + \frac{1}{k} \rho \sin^{1}_{1} \cos_{2} \ldots
\]

\[
\frac{n}{k} \rho \cos_{1} + \frac{1}{k} \rho \sin^{1}_{1} \cos_{2} \ldots
\]

\[
\frac{n}{k} \rho \cos_{1} - \frac{n-i}{k} \rho \sin^{1}_{1} \cos_{i} \geq 0
\]

\[
\frac{n}{k} \rho \cos_{1} + \frac{1}{k} \rho \sin^{1}_{1} \cos_{2} \ldots
\]

\[
\frac{n}{k} \rho \cos_{1} + \frac{1}{k} \rho \sin^{1}_{1} \cos_{2} \ldots
\]

+ \[
\frac{1}{6k} \rho \sin^{1}_{1} \cos_{n-2}
\]
(d) cont.

\[- \frac{1}{2^\frac{n}{2}} \rho \sin \theta_1 \cdots \sin \theta_{n-2} \geq 0.\]

**Proof:**

Let \(x_1, \ldots, x_n\) be \(n\)-independent identically distributed random variables from the normal population

\[(4.4.\text{ii}) \quad dG = \frac{1}{2^\frac{n}{2}} e^{-\frac{1}{2}x^2} \, dx \quad - \infty < x < \infty.\]

Then the cumulative distribution of \(s^2\) given by

\[(4.4.\text{iii}) \quad G = \frac{1}{2^\frac{n}{2}(n)} \left| \frac{-\frac{1}{2} \sum x_i^2}{\prod d\xi_i} \right| D_2\]

where

\[D_2 = \begin{cases} \sum (x_i - \bar{x})^2 \leq ns^2 & (a) \\ -\infty < x_i < +\infty & (b) i = 1, \ldots, n; \end{cases}\]

is well known to be \(\chi^2\) with \((n-1)\)-degrees of freedom.

Setting \(r = \sqrt{n}s\) we rewrite this result given in Kendall and Stuart [19] page 256 (11.4.1) as

\[(4.4.\text{iv}) \quad dG = \frac{1}{2^\frac{n}{2}(n-1)} \frac{1}{\Gamma\left(\frac{1}{2}(n-1)\right)} \, e^{-\frac{1}{2}r^2} \, (r^2)^{\frac{1}{2}(n-3)} \, d(r^2).\]

We will show that \(G\) can also be factored into two parts.
\[(4.4.v)\]
\[G = \left\{ \frac{n(n-1)}{2(n-1)} \left[ \int_{0}^{r} e^{-\frac{1}{2} \rho^2} \cdot (\rho^2)^{\frac{1}{2}(n-3)} d(\rho^2) \right] \cdot J_{n-2} \right\} \]

where \( J_{n-2} \) is the expression for the integral (4.4.i).

By using results (4.4.iv) and (4.4.v) we will evaluate \( J_{n-2} \).

First let us make the partial order transformation

\[(4.4.vi)\]
\[x_{(1)} > -\infty; \quad x_{(n)} < +\infty\]

on the \( x_i \)'s of (4.4.ii). \( x_i \), \( i=2, \ldots, n-1 \) are unordered.

Follow this by the composite transformation whose inverse is

\[x_{(1)} = \frac{1}{n} y_1 - \frac{n-1}{n} \frac{\rho \cos \theta_1}{\sqrt{n(n-1)}}\]

\[x_2 = \frac{1}{n} y_1 + \frac{1}{n} \frac{\rho \cos \theta_1}{\sqrt{n(n-1)}} - \frac{n-2}{(n-1)(n-2)} \frac{\rho \sin \theta_1 \cos \theta_2}{\sqrt{n(n-1)}}\]

\[\vdots\]

\[x_i = \frac{1}{n} y_1 + \frac{1}{n} \frac{\rho \cos \theta_1}{\sqrt{n(n-1)}} + \frac{1}{(n-1)(n-2)} \frac{\rho \sin \theta_1 \cos \theta_2}{\sqrt{n(n-1)}}\]

\[+ \ldots - \frac{(n-i)}{(n-i+1)(n-i)} \frac{\rho \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{i-1} \cos \theta_i}{\sqrt{n(n-1)}}\]

\[x_{(n)} = \frac{1}{n} y_1 + \frac{1}{n} \frac{\rho \cos \theta_1}{\sqrt{n(n-1)}} + \frac{1}{(n-1)(n-2)} \frac{\rho \cos \theta_1 \cos \theta_2}{\sqrt{n(n-1)}}\]

\[+ \ldots + \frac{1}{n} \frac{\rho \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{n-3} \cos \theta_{n-2}}{6}\]

\[+ \frac{1}{n} \frac{\rho \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{n-3} \sin \theta_{n-2}}{2}\]

where \( 0 < \theta_i < \pi, \quad i=1, \ldots, n-3, \quad 0 < \theta_{n-2} < 2\pi, \quad 0 < \rho < \infty \).
This transformation is really the composite of (4.3.vii) and (4.3.viii) whose Jacobians are \( |J_1| = 1 \) and 
\[ |J_2| = \rho^{n-2} \sin^{n-3} \theta_1 \sin^{n-4} \theta_2 \cdots \sin \theta_{n-3} \]
respectively. The Jacobian of the composite transformation is therefore
\[ |J| = |J_1| \cdot |J_2| = \rho^{n-2} \sin^{n-3} \theta_1 \sin^{n-4} \theta_2 \cdots \sin \theta_{n-3}. \]
The integral (4.4.iii) becomes
\[(4.4.vii)\]
\[
G(r) = \frac{n(n-1)}{(2\pi)^{\frac{1}{2}}(n)} \int_0^\infty \int_0^{2\pi} e^{-\frac{1}{2}r^2} \rho^{n-2} \rho^2 \rho^{n-2} d\rho \cdot d\theta_1 \cdot \sin^{n-3} \theta_1 \sin^{n-4} \theta_2 \cdots \sin \theta_{n-3} \prod_{i=1}^{n-2} d\theta_i.
\]
where \( D_1^* \) is the same as \( D_1 \) in (4.4.i)
\[(4.4.viii)\]
\[
G(r) = \frac{n(n-1)}{2(2\pi)^{\frac{1}{2}}(n-1)} \int_0^\infty \exp\left(-\frac{1}{2}\rho^2\right) \rho^{n-3} d\rho \cdot J_{n-2}.
\]
Differentiating with respect to \( r \) we get
\[
G'(r) = \frac{n(n-1)}{2(2\pi)^{\frac{1}{2}}(n-1)} e^{-\frac{1}{2}r^2} r^{n-3} d\rho \cdot J_{n-2}
\]
But from (4.4.iv)
\[
G'(r) = \frac{1}{2^{n-3}(n-1)} \Gamma\left\{\frac{1}{2}(n-1)\right\} e^{-\frac{1}{2}r^2} r^{n-3} d\rho.
\]
After equating both results for \( G'(r) \) it follows that
\[(4.4.ix)\]
\[
J_{n-2} = \frac{2 \cdot \pi^{\frac{1}{2}(n-1)}}{n(n-1) \Gamma\left\{\frac{1}{2}(n-1)\right\}}.
\]
We were able to find that \( D_1^* = D_1 \) in the following
manner. The initial transformation (4.4.vi) took $D_2$ of

(4.4.iii) into

(4.4.x) (a) $\sum_{i=2}^{n} y_i^2 \leq r^2$ 

(b) $x_{(n)} - x_i \geq 0$, $i=1,\ldots,n$.

(c) $x_{(n)} < \infty$

(d) $x_i - x_{(1)} \geq 0$

(e) $x_{(1)} > -\infty$.

Under the composite, transformation (4.4.x) became

(4.4.xi)

(a) $\rho^2 \leq r^2$

(b) $\frac{n-i+1}{\frac{\rho}{k_i}} \frac{\rho \sin \theta_1 \sin \theta_2 \cdots \cos \theta_i}{[(n-i+1)(n-i)]}

+ \frac{1}{\frac{\rho}{k_i}} \frac{\rho \sin \theta_1 \sin \theta_2 \cdots \sin \theta_i \cos \theta_{i+1}}{[(n-i)(n-i-1)]}$

+ $\ldots$ $\ldots$ $\ldots$

+ $\frac{1}{\frac{\rho}{k_2}} \frac{\rho \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{n-3} \cos \theta_{n-2}}{6}$

+ $\frac{1}{\frac{\rho}{k_2}} \frac{\rho \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{n-3} \sin \theta_{n-2}}{2}$ $\geq 0$

$\quad i=1,\ldots,n-1$.

(c) $y_1 < \infty$

(d) $\frac{n}{\frac{\rho}{k_1}} \frac{\rho \cos \theta_1}{[n(n-1)]}$

$\cdot$

$\cdot$

$\cdot$

$\frac{n}{\frac{\rho}{k_1}} \frac{\rho \cos \theta_1}{[n(n-1)]}$

$\frac{n-2}{\frac{\rho}{k_2}} \frac{\rho \sin \theta_1 \cos \theta_2}{[(n-1)(n-2)]}$ $\geq 0$

$\cdot$

$\cdot$

$\cdot$

$\frac{1}{\frac{\rho}{k_2}} \frac{\rho \sin \theta_1 \cos \theta_2 \cdots}{[(n-1)(n-2)]}$

$\frac{n-i}{\frac{\rho}{k_2}} \frac{\rho \sin \theta_1 \cdots \cos \theta_i}{[(n-i+1)(n-i)]} \geq 0$
(d) cont.

\[ \frac{n \rho \cos \theta_1}{\sqrt{n(n-1)}} + \frac{1}{\sqrt{(n-1)(n-2)}} \rho \sin \theta_1 \cos \theta_2 \ldots \]

\[ + \frac{1}{\sqrt{6}} \rho \sin \theta_1 \cos \theta_{n-2} \]

\[ - \frac{1}{\sqrt{2}} \rho \sin \theta_1 \sin \theta_{n-2} \geq 0 \]

(e) \( y_1 > -\infty \)

and finally we were able to write \( G(r) \) as (4.4.vii).

Q.E.D.
4.5  **LEMMA 4.5.**

(4.5.i) \[ I_{n-2} = \frac{1}{D_1 [n(n-1)]} \left\{ \frac{n}{2} \cos^2 \theta_1 \cdots + \frac{1}{2} \sin \theta_2 \cdots \sin \theta_{n-2} \right\} \]

\[ \sin^{n-3} \theta_1 \sin^{n-4} \theta_2 \cdots \sin^{n-3} \theta_{n-3} \prod_{i=1}^{n-2} d\theta_i \]

where \( D_1 \) is the same as in Lemma 4.4.

(4.5.ii) \[ = \frac{1}{\frac{n}{2}} \left\{ \frac{n}{2} \prod_{i=2}^{n-1} du_i \right\} \frac{1}{\frac{n}{2}} \left\{ \frac{1}{n} (1 + \sum_{i=2}^{n-1} u_i^2) - \frac{1}{n} (1 + \sum_{i=2}^{n-1} u_i^2) \right\} \]

\[ 0 < u_i < 1 \]

\[ i=2, \ldots, n-1. \]

**Proof:**

We will first show that the integral

(4.5.iii) \[ -\frac{1}{2(n-2)} \frac{1}{\Gamma(\frac{n}{2})} \left| \begin{array}{cc} x(n) & \frac{n}{2} \prod_{i=2}^{n-1} dx_i dx(n) \\ \frac{1}{n} (\sum_{i=2}^{n-1} x_i^2 - \frac{1}{n} (\sum_{i=2}^{n-1} x_i^2)) \end{array} \right| \]

is equal to the integral of (4.5.i) and secondly that it also equals (4.5.ii). Consider the integral

(4.5.iv) \[ K_{n-2} = (n-1) \left| \begin{array}{cc} x(n) & \frac{n}{2} \prod_{i=2}^{n-1} dx_i dx(n) \\ j D_3 \end{array} \right| \]

where \( D_3 = 0 < x_i < x(n) < \infty, \quad i=2, \ldots, n-1. \)

\( D_3 \) is equivalent also to
(a) \( x_{(n)} - x_i \geq 0 \) \( i = 2, \ldots, n - 1 \).
(b) \( x_i \geq 0 \) \( i = 2, \ldots, n - 1 \).
(c) \( x(n) < \infty \).

(4.5.v) Make the transformation whose inverse is

\[
\begin{align*}
x_2 &= \frac{n}{[n(n - 1)]^{1/2}} \rho \cos \theta_1 - \frac{n - 2}{[(n - 1)(n - 2)]^{1/2}} \rho \sin \theta_1 \cos \theta_2 \\
x_3 &= \frac{n}{[n(n - 1)]^{1/2}} \rho \cos \theta_1 + \frac{1}{[(n - 1)(n - 2)]^{1/2}} \rho \sin \theta_1 \cos \theta_2 \\
&\quad - \frac{n - 3}{[(n - 3)(n - 4)]^{1/2}} \rho \sin \theta_1 \sin \theta_2 \cos \theta_3 \\
&\quad \vdots \\
x(n) &= \frac{n}{[n(n - 1)]^{1/2}} \rho \cos \theta_1 + \cdots \\
&\quad + \frac{1}{2^{1/2}} \rho \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{n-2}.
\end{align*}
\]

where

\[
0 < \theta_i < \pi, \quad i = 1, \ldots, n - 3
\]

\[
0 < \theta_{n-2} < 2\pi.
\]

We see that (4.5.v) is the composite of two transformations. The first is

(4.5.vi) \( x' = By' \)

where \( x' \) and \( y' \) are \((n - 1 \times 1)\) column vectors given by

\[
\begin{bmatrix}
x_2 \\
x_3 \\
\vdots \\
x(n)
\end{bmatrix}, \quad \begin{bmatrix}
y_2 \\
y_3 \\
\vdots \\
y_n
\end{bmatrix}
\]

and
B is an \((n-1 \times n-1)\) square matrix (which is not orthogonal) and is given by

\[
(4.5.vii) \quad B = \begin{bmatrix}
\frac{n}{\sqrt{\frac{1}{2}}} & \frac{-(n-2)}{\sqrt{\frac{1}{2}}} & 0 & \cdots & 0 \\
\frac{n}{\sqrt{\frac{1}{2}}} & \frac{1}{\sqrt{\frac{1}{2}}} & \frac{-(n-3)}{\sqrt{\frac{1}{2}}} & \cdots & \frac{-1}{\sqrt{\frac{1}{2}}} \\
\frac{n}{\sqrt{\frac{1}{2}}} & \frac{1}{\sqrt{\frac{1}{2}}} & \frac{1}{\sqrt{\frac{1}{2}}} & \cdots & \frac{1}{\sqrt{\frac{1}{2}}} \\
\frac{n}{\sqrt{\frac{1}{2}}} & \frac{1}{\sqrt{\frac{1}{2}}} & \frac{1}{\sqrt{\frac{1}{2}}} & \cdots & \frac{1}{\sqrt{\frac{1}{2}}}
\end{bmatrix}
\]

Now the Jacobian of the transformation

\[
|J|^2 = |B^TB| = |\text{Diag } [n, 1, \ldots, 1]| = n.
\]

(4.5.viii) Therefore \(|J| = \sqrt{n}\).

\(x' = By'\) can be written as

\[
(4.5.ix) \quad z' = B'y'
\]

\[
\begin{bmatrix}
x_2 \\ x_3 \\ \vdots \\ x_{(n)}
\end{bmatrix} = \begin{bmatrix}
\frac{n - \sqrt{n}}{\sqrt{\frac{1}{2}}} y_2 \\
\frac{n - \sqrt{n}}{\sqrt{\frac{1}{2}}} y_2 \\
\vdots \\
\frac{n - \sqrt{n}}{\sqrt{\frac{1}{2}}} y_2
\end{bmatrix}
\]

where \(z' = \)
and

\[
\bar{B} = \begin{bmatrix}
\frac{1}{(n-1)^{\frac{1}{2}}} & \frac{-(n-2)}{[(n-1)(n-2)]^{\frac{1}{2}}} & 0 & 0 \cdots 0 & 0 \\
\frac{1}{(n-1)^{\frac{1}{2}}} & \frac{1}{[(n-1)(n-2)]^{\frac{1}{2}}} & \frac{-(n-3)}{[(n-3)(n-4)]^{\frac{1}{2}}} & 0 \cdots 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots \\
\frac{1}{(n-1)^{\frac{1}{2}}} & \frac{1}{[(n-1)(n-2)]^{\frac{1}{2}}} & \frac{1}{[(n-3)(n-4)]^{\frac{1}{2}}} & \cdots & \frac{-1}{2^{\frac{1}{2}}} \\
\frac{1}{(n-1)^{\frac{1}{2}}} & \frac{1}{[(n-1)(n-2)]^{\frac{1}{2}}} & \frac{1}{[(n-3)(n-4)]^{\frac{1}{2}}} & \cdots & \frac{1}{2^{\frac{1}{2}}}
\end{bmatrix}
\]

From which we get

\[(4.5.x)\]

\[
zz^\prime = \sum_{i=2}^{n} (x_i - \frac{n - \sqrt{n}}{n(n-1)} y_2)^2 = \frac{n}{n(n-1)} y_{\bar{B}}^\prime \bar{B} y^\prime = \sum_{i=2}^{n} y_i^2;
\]

where \(\bar{B}^\prime \bar{B} = I\).

From (4.5.vi) we also get

\[(4.5.xi)\]

\[
\sum_{i=2}^{n} x_i = \sum_{i=2}^{n} y_i = \frac{n}{n(n-1)} y_2.
\]

Simplifying (4.5.x) and making use of (4.5.xi) we get

\[(4.5.xii)\]

\[
\frac{1}{n} \sum_{i=2}^{n} x_i^2 - \frac{1}{n} \left( \sum_{i=2}^{n} x_i \right)^2 = \sum_{i=2}^{n} y_i^2.
\]

The second transformation is the spherical transformation

\[(4.5.xiii)\]

\[
Y_2 = \rho \cos \theta_1 \\
Y_3 = \rho \sin \theta_1 \cos \theta_2 \\
\vdots \\
Y_n = \rho \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{n-2}
\]
where
\[ 0 < \theta_i < \pi, \quad i = 1, 2, \ldots, n-3, \]
\[ 0 < \theta_{n-2} < 2\pi, \quad \text{and} \quad 0 < \rho < \infty. \]

The Jacobian of this transformation is
\[ |J| = \rho^{n-2} \sin^{n-3}\theta_1 \ldots \sin\theta_{n-3}. \]

The Jacobian of the composite transformation (4.5.vi) is therefore
\[ \sqrt{n} \rho^{n-2} \sin^{n-3}\theta_1 \ldots \sin\theta_{n-3}. \]

From (4.5.xii) \[ \sum_{i=2}^{n} y_i^2 = \rho^2 \] therefore, (4.5.xii) is now rewritten
\[ (4.5.xiv) \quad \sum_{i=2}^{n} x_i^2 - \frac{1}{n} \left( \sum_{i=2}^{n} x_i \right)^2 = \rho^2. \]

(4.5.iv) becomes (4.5.xv)
\[ (4.5.xv) \]
\[ K_{n-2} = (n-1) \left| \begin{array}{c}
\frac{n}{2} \rho \cos \theta_1 + \ldots + \frac{1}{2} \rho \sin \theta_1 \ldots \sin \theta_{n-2} \\
\frac{1}{n(n-1)} \frac{D_3}{2} \rho^2 \sin^{n-3} \theta_1 \sin^{n-4} \theta_2 \ldots \sin \theta_{n-3} \rho \sin \theta_i \, d\rho \prod_{i=2}^{n} d\theta_i,
\end{array} \right| \]

\[ D_3 \] being
\[ (a) \quad \frac{n - i + 1}{[n - i + 1](n - i)]^{1/2}} \rho \sin \theta_1 \sin \theta_2 \ldots \cos \theta_i \]
\[ + \frac{1}{[n - \frac{i}{2}(n - i - 1)]^{1/2}} \rho \sin \theta_1 \sin \theta_2 \ldots \sin \theta_i \cos \theta_{i+1} \]
\[ \ldots \quad \ldots \quad \ldots \]
\[ + \frac{1}{6^{1/2}} \rho \sin \theta_1 \sin \theta_2 \ldots \sin \theta_{n-3} \cos \theta_{n-2} \]
\[ + \frac{1}{6^{1/2}} \rho \sin \theta_1 \sin \theta_2 \ldots \sin \theta_{n-2} \geq 0; \]
\[ i = 2, \ldots, n-1. \]
(b) \[ \frac{n\rho \cos\theta_1}{(n-1)n} - \frac{n-2\rho \sin\theta_1 \cos\theta_2}{(n-1)(n-2)} \geq 0 \]

\[ \frac{n \rho \cos\theta_1}{(n-1)n} + \frac{1}{(n-1)(n-2)} \rho \sin\theta_1 \cos\theta_2 \]

\[ -\frac{n-3}{(n-3)(n-4)} \rho \sin\theta_1 \sin\theta_2 \cos\theta_3 \geq 0 \]

\[ \vdots \]

\[ \frac{n \rho \cos\theta_1}{(n-1)n} + \ldots \]

\[ + \frac{1}{2^\frac{n}{2}} \rho \sin\theta_1 \sin\theta_2 \ldots \sin\theta_{n-2} \geq 0 \]

(c) \( 0 < \rho < \infty \).

Integrating out \( \rho \), \( K_{n-2} \) reduces to (4.5.xvi).

(4.5.xvi)

\[ K_{n-2} = (n-1)^{\frac{1}{2}} 2^{\frac{n}{2}} (n-2) \Gamma(\frac{n}{2}) \int_{D_4[n(n-1)]} \left( \frac{n}{2} \cos\theta_1 + \ldots + \rho \sin\theta_1 \ldots \sin\theta_{n-2} \right) \sin^{n-3}\theta_1 \sin^{n-4}\theta_2 \ldots \sin^{n-3}\theta_{n-3} \prod_{i=1}^{n-2} \frac{d\theta_i}{2} \]

where \( D_4 = D_1 \). That is

(4.5.xvii) \( K_{n-2} = (n-1)^{\frac{1}{2}} 2^{\frac{n}{2}} (n-2) \Gamma(\frac{n}{2}) \cdot I_{n-2} \).

Consider again (4.5.iv)

\[ \frac{K_{n-2}}{n-1} = \left| \int (n) e^{-\frac{1}{2} \left[ \frac{1}{n} \sum_{i=2}^{n} \left( \frac{1}{2} x_i \right)^2 \right]} \prod_{i=2}^{n-1} dx_i dx(n) \right| \]

\[ 0 < x_i < x(n) < \infty \]

Let \( x_i = u_i x(n) \) for \( i = 2, \ldots, n-1 \); \( x_n = x(n) \).
This transformation has Jacobian $|J| = x_n^{n-2}$ therefore

$$K_{n-2} = \int_{x_1^{n-1}}^{x_1^{n-2}} \left[ \frac{x^2}{2} : \frac{n-1}{(1 + \sum_{i=2}^{n-1} u_i^2) - \frac{1}{n}(1 + \sum_{i=2}^{n-1} u_i^2)^2} \right] dx_n \prod_{i=2}^{n-1} du_i$$

for $0 < u_i < 1$

$i=2, \ldots, n-1$.

By letting

$$\frac{x^2}{2} : \frac{n-1}{(1 + \sum_{i=2}^{n-1} u_i^2) - \frac{1}{n}(1 + \sum_{i=2}^{n-1} u_i^2)^2} = u,$$

it follows that

$$x_n dx_n = \frac{du}{[(1 + \sum_{i=2}^{n-1} u_i^2) - \frac{1}{n}(1 + \sum_{i=2}^{n-1} u_i^2)^2]^{\frac{n}{2}}}.$$

By integrating out the $x_n$, it follows that

$$K_{n-2} = 2^{\frac{1}{2}}(n-2) \Gamma(\frac{n}{2}) \int_{0}^{1} \prod_{i=2}^{n-1} \frac{du_i}{[(1 + \sum_{i=2}^{n-1} u_i^2) - \frac{1}{n}(1 + \sum_{i=2}^{n-1} u_i^2)^2]^{\frac{n}{2}}}$$

for $0 < u_i < 1$

$i=2, \ldots, n-1$.

From (4.5.xvii) $K_{n-2} = \frac{1}{n-1} 2^{\frac{1}{2}}(n-2) \Gamma(\frac{n}{2}) \cdot I_{n-2}$. Hence, upon equating both values for $K_{n-2}$ we get

(4.5.xviii)

$$I_{n-2} = \frac{1}{n-1} 2^{\frac{1}{2}}(n-2) \Gamma(\frac{n}{2}) \cdot I_{n-2}$$

for $0 < u_i < 1$

$i=2, \ldots, n-1$.

Q.E.D.

4.6 Now using Lemma 4.5 and 4.4 we are able to express

(4.3.x) as: $F(s)$ as an $(n-2)$-integral over an $(n-2)$-unit cube.
\[ (4.6.i) \]

\[
F(r_0) = 2\pi^{\frac{1}{2}}(n-1)^{-\frac{1}{2}}r_0^{n-1} \frac{r_0^n}{(n-1)} \Gamma\left\{\frac{1}{2}(n-1)\right\} - (n-1)r_0^n \left\{ \begin{array}{c}
\prod_{i=2}^{n-1} du_i \\
\int_{u_i^2}^{u_i^2} \left[ \frac{1}{2(n-1)} \frac{n-1}{i=2} \frac{1}{(1+\sum u_i^2) \left(1+\sum u_i^2\right)^2} \right]^{\frac{1}{2}} \end{array} \right.
\]

\[
0 < u_i^2 < 1 \quad i=2, \ldots, n-1.
\]

Since \( r_0^2 = ns^2 \) then

\[ (4.6.ii) \]

\[
F(s) = 2\pi^{\frac{1}{2}}(n-1)^{-\frac{1}{2}}n^{\frac{1}{2}} \left(\frac{1}{2} \right) \left(\begin{array}{c}
\prod_{i=2}^{n-1} du_i \\
\int_{u_i^2}^{u_i^2} \left[ \frac{1}{2(n-1)} \frac{n-1}{i=2} \frac{1}{(1+\sum u_i^2) \left(1+\sum u_i^2\right)^2} \right]^{\frac{1}{2}} \end{array} \right. 
\]

\[
0 < u_i^2 < 1 \quad i=2, \ldots, n-1 
\]

\[ 0 \leq s \leq (2n)^{-\frac{1}{2}}. \]

We have thus expressed \( F(s) \) as an \((n-2)\)-integral over an \((n-2)\)-unit cube. It follows also that

\[ (4.6.iii) \]

\[
f(s) = 2\pi^{\frac{1}{2}}(n-1)^{-\frac{1}{2}}n^{\frac{1}{2}} \left(\frac{1}{2} \right) \left(\begin{array}{c}
\prod_{i=2}^{n-1} du_i \\
\int_{u_i^2}^{u_i^2} \left[ \frac{1}{2(n-1)} \frac{n-1}{i=2} \frac{1}{(1+\sum u_i^2) \left(1+\sum u_i^2\right)^2} \right]^{\frac{1}{2}} \end{array} \right. 
\]

\[
0 < u_i^2 < 1 
\]

\[ i=2, \ldots, n-1 
\]

\[ 0 \leq s \leq (2n)^{-\frac{1}{2}}. \]

We observe that \( f(s) \) is a polynomial of order \( n-1 \) in \( s \).

This proves the conjecture of Rietz (1931) [28].
Here we will evaluate (4.6.ii) for \( n = 2, 3, 4 \) respectively.

When \( n = 2 \). Since two variables have been already integrated out, the cumulative distribution function can be immediately written as

\[
(4.7.i) \quad F(s) = 4s - 4s^2 \quad 0 \leq s \leq \frac{1}{2}.
\]

When \( n = 3 \). We will first evaluate the integral obtained by setting \( n = 3 \) in (4.6.ii).

\[
(4.7.ii) \quad \int_0^1 \frac{du_2}{[(1+u_2^2) - \frac{1}{3}(1+u_2^2)]^{3/2}} = \int_0^{\frac{3\sqrt{3}}{2\sqrt{2}}} \frac{du_2}{(1-u_2+u_2^2)^{3/2}}
\]

\[
= \frac{3\sqrt{3}}{2\sqrt{2}} \left\{ \frac{4u_2^2 - 2}{3(1-u_2+u_2^2)} \right\} \quad \mid_{0}^{\frac{1}{2}} = \sqrt{6}.
\]

It follows that for \( n = 3 \) the cumulative distribution function is

\[
(4.7.iii) \quad F_1(s) = 3\sqrt{3}\pi s^2 - 18\sqrt{2}s^3 \quad 0 \leq s \leq 6^{-\frac{1}{2}}.
\]

Both (4.7.i) and (4.7.ii) check with (3.2.1.ii) and (3.2.2.xviii).

When \( F'(s) \) is taken for \( n = 2 \) and \( n = 3 \), the results also check with that given by Rider [27] and Rietz [28] respectively for the identical ranges.

When \( n = 4 \). We again evaluate the integral obtained by making this substitution for \( n \) into (4.6.ii). Using \( x \) and \( y \) for \( u_2 \) and \( u_3 \) we have
\[
\begin{align*}
(4.7.\text{iv}) & \quad \frac{\frac{\mathrm{dy}}{\mathrm{dx}}}{1} \left[ (1 + x^2 + y^2) - \frac{1}{4}(1 + x + y)^2 \right]^{2} \\
& = 16 \left[ \frac{\frac{\mathrm{dy}}{\mathrm{dx}}}{1} \left[ (3 - 2y + 3y^2) - 2x(1 + y) + 3x^2 \right]^{2} \right] \\
& = \left[ \frac{3 - (1 + y) + 3x}{(1 - y + y^2)(3 - 2y + 3y^2) - 2x(1 + y) + 3x^2} \right]^{2} \\
& + \frac{3}{(1 - y + y^2)} \tan^{-1} \left( \frac{3x - (1 + y)}{\sqrt{8}(1 - y + y^2)} \right) \left[ \frac{3}{8(1 - y + y^2)} \right]^{3/2} \\
& = \frac{(2 - y)\mathrm{dy}}{(1 - y + y^2)(4 - 4y + 3y^2)} \\
& + \left[ \frac{3}{\sqrt{8}(1 - y + y^2)} \tan^{-1} \left( \frac{2 - y}{\sqrt{8}(1 - y + y^2)} \right) \right]^{3/2} \\
& + \left[ \frac{(1 + y)\mathrm{dy}}{(1 - y + y^2)(3 - 2y + 3y^2)} \right]^{3/2} \\
& - \left[ \frac{3}{\sqrt{8}(1 - y + y^2)} \tan^{-1} \left( \frac{1}{\sqrt{8}(1 - y + y^2)} \right) \right]^{3/2} \\
& \end{align*}
\]
Let $I(1)$ be the first integral of the above expression. The second, $I(2)$, the third and $I(4)$ the fourth.

Then

$$I(2) = \int_0^1 \frac{3}{\sqrt{8}(1 - y + y^2)} \tan^{-1} \left( \frac{2 - y}{\sqrt{2}} \right) dy - \frac{3}{\sqrt{8}(1 - y + y^2)} \frac{(2 - y)dy}{\sqrt{2}} \cdot$$

Let $u = \frac{\tan^{-1} (2 - y) \sqrt{2}}{2y - 1}$; $du = \frac{-\sqrt{2}y}{(4 - 4y + 3y^2)} dy$; $v = \frac{2y - 1}{\sqrt{2}(1 - y + y^2)}$

Integrating by parts we get

$$I(2) = \left\{ \tan^{-1} \frac{2 - y}{\sqrt{2}} \right\} \left\{ \frac{2y - 1}{2(1 - y + y^2)} \right\} \left[ \frac{2y - 1}{\sqrt{2}(1 - y + y^2)} \right]_0^1$$

$$+ \int_0^1 \frac{(2y - 1)y}{(1 - y + y^2)(4 - 4y + 3y^2)} dy.$$

Combining $I(1)$ and $I(2)$ gives

$$I(1) + I(2) = \int_0^1 \frac{2dy}{4 - 4y + 3y^2}$$

$$+ \left\{ \tan^{-1} \frac{2 - y}{\sqrt{2}} \right\} \left\{ \frac{2y - 1}{2(1 - y + y^2)} \right\} \left[ \frac{2y - 1}{\sqrt{2}(1 - y + y^2)} \right]_0^1$$

$$= \frac{2\tan^{-1} \frac{3y - 2}{\sqrt{8}}}{\sqrt{8}} \left[ \frac{1}{\sqrt{2}} \left\{ \tan^{-1} \frac{1}{2\sqrt{2}} + \tan^{-1} \frac{1}{\sqrt{2}} \right\} \right]_0^1$$
\[
I(4) = \left[ \frac{\tan^{-1} \frac{1 + y}{\sqrt{2}}}{8(1 - y + y^2)^{3/2}} \right]_0^1 - \frac{3}{\sqrt{8}} \left[ \tan^{-1} \frac{1 + y}{\sqrt{2}} \right]_{1 - y + y^2}^{8(1 - y + y^2)^{3/2}}
\]

Let \( u = \tan^{-1} \frac{1 + y}{\sqrt{2}} \)
\( dv = \frac{3dy}{\sqrt{8}(1 - y + y^2)^{3/2}} \)
\( du = \frac{\sqrt{2}(1 - y)}{(3 - 2y + 3y^2)(1 - y + y^2)^{1/2}} \)
\( v = \frac{2y - 1}{\sqrt{2}(1 - y + y^2)^{1/2}} \)

Integrating by parts we find

\[
I(4) = \left[ \frac{\tan^{-1} \frac{1 + y}{\sqrt{2}}}{8(1 - y + y^2)^{3/2}} \right]_0^1
\]

\[
- \int_0^1 \frac{(1 - y)(2y - 1)}{(3 - 2y + 3y^2)(1 - y + y^2)^{1/2}} dy.
\]

We combine \( I(3) \) and \( I(4) \) to give

\[
I(3) + I(4) = \left[ \frac{2}{(3 - 2y + 3y^2)} + \frac{1}{\sqrt{2}} \{\tan^{-1} \frac{1}{\sqrt{2}} + \tan^{-1} \frac{1}{2\sqrt{2}} \} \right]_0^1
\]

\[
= \frac{1\tan^{-1} \frac{3y - 1}{\sqrt{8}}}{\sqrt{2}} + \frac{1\cos^{-1} \frac{1}{\sqrt{3}}}{\sqrt{2}}
\]

\[
= \frac{1\cos^{-1} \frac{1}{\sqrt{2}}}{\sqrt{3}} + \frac{1\cos^{-1} \frac{1}{\sqrt{3}}}{\sqrt{2}}
\]

\[
= \sqrt{2}\cos^{-1} \frac{1}{\sqrt{3}}.
\]
Therefore

\[(4.7.v)\quad I_{(1)} + I_{(2)} + I_{(3)} + I_{(4)} = 2\sqrt{2} \cos^{-1} \frac{1}{\sqrt{3}}\]

which is the value of \((4.7.iv)\).

When \(n = 4\), from \((4.6.ii)\)

\[(4.7.vi)\]

\[
F(s) = \frac{2\pi^{3/2} \cdot 2 \cdot (2s)^3}{3 \cdot \Gamma(3/2)} \left[ \sum_{i=2}^{3} \frac{du \cdot du_3}{[(1+ \sum_{i=2}^{3} u_i^2) - 1(1+ \sum_{i=2}^{3} u_i^2)]^2} \right] \quad 0 < u_i < 1 \quad i = 2, 3.
\]

And applying the result of \((4.5.v)\) it follows that

\[(4.7.vii)\]

\[
F(s) = \frac{64\pi s^3}{3} - 96\sqrt{2}s^4 \cos^{-1} \frac{1}{\sqrt{3}} \quad 0 \leq s \leq 8^{-\frac{1}{2}}.
\]

This result coincides with that obtained by the method of sections in Chapter 3 for \(n = 4\).

4.8 \underline{For s Large.}

We will now find the function for the cumulative distribution of \(S\) when \(s\) is large. We first find the content \(K(r_0)\) enclosed by the bounding planes of the farthest vertices of the cube \(0 \leq x_i \leq 1, \ i = 1, \ldots, n\) and the exterior of the cylinder given by \(\Sigma(x_i - \bar{x})^2 \geq r_0^2\). Then it follows that the common content of the hyper-cylinder and hyper-cube for large \(r\) will be \([T^* \cap C^*] = 1 - K(r_0)\)

\[
(n^2-4)^{\frac{1}{2}} \leq r_0 \leq \sqrt{n}/2 \quad \text{and} \quad F(s) = 1 - K(\sqrt{ns}) \left(\frac{n^2-4}{2n}\right)^{\frac{1}{2}} \leq s \leq \frac{1}{2}, \quad (4n)^{\frac{1}{2}}
\]
when \( n = 2k \).

When \( n = 2k + 1 \) the range for \( r_0 \) and for \( s \) is respectively

\[
\frac{(5n-9)^{\frac{1}{2}}}{(4n)^{\frac{1}{2}}} < r_0 < \frac{(n^2-1)^{\frac{1}{2}}}{(4n)^{\frac{1}{2}}}; \quad \frac{(5n-9)^{\frac{1}{2}}}{2n} < s < \frac{(n^2-1)^{\frac{1}{2}}}{2n}
\]
The upper and lower bounds for \( r_0 \), when \( n=2k \), were found in the following manner. The farthest vertices of the cube \( 0 \leq x_i \leq 1, \ i=1, \ldots, n \) have coordinates given by \( \binom{n}{k} \) arrangements of \( k \) ones and \( k \) zeros. The distance from one of these vertices, for example \( (1^k, 0^k) \) to its projection on the equiangular line is \( \left( \frac{k}{2k}, \ldots, \frac{k}{2k} \right) \) is \( d=\frac{\sqrt{n}}{2} \).

The axis of the cube \( \text{OP} \) with coordinates \( (0^n) \) and \( (1^n) \) lies on the positive extension of the equiangular line. Therefore, when \( r_0 \), the radius of the \( n \)-dimensional cylinder with axis the same as that of the \( n \)-dimensional cube is \( \geq \frac{\sqrt{n}}{2} \), the \( n \)-dimensional cylinder circumscribes the \( n \)-dimensional cube. Hence, \( r_0 \) is taken \( \leq \frac{n}{2} \).

The second farthest vertices of the \( n \)-dimensional cube from the equiangular line have coordinates consisting of \( \binom{n}{k-1} \) arrangements of \( k-1 \) ones and \( k+1 \) zeros. A typical vertex \( (1^{k-1}, 0^{k+1}) \) whose projection on the equiangular line is \( \left( \frac{k-1}{n}, \ldots, \frac{k-1}{n} \right) \) is a distance \( d=\left( \frac{n^2-4}{\sqrt{(4n)}} \right)^{\frac{1}{2}} \) from the equiangular line. Hence, for \( n=2k \) \( r_0 \) is chosen such that \( \frac{(n^2-4)^{\frac{1}{2}}}{\sqrt{(4n)}} \leq r_0 \leq \frac{\sqrt{n}}{2} \).

When \( n=2k+1 \) the farthest vertices of the cube have coordinates consisting of \( \binom{n}{k+1} \) arrangements of \( k+1 \) ones and \( k \) zeros or \( k+1 \) zeros and \( k \) ones. The distance to the equiangular line is
\[
d=\left( \frac{n^2-1}{\sqrt{(4n)}} \right)^{\frac{1}{2}}.\]
Since for \( n=2k+1 \) there are \( 2\binom{n}{k+1} \) vertices arranged symmetrically on both sides of the plane passing through the centre \( (\frac{1}{2}, \ldots, \frac{1}{2}) \) of the \( n \)-dimensional cube and orthogonal to its diagonal \( OP \), we consider the distance from the centre of gravity of the one-dimensional edges. A typical edge is given by \( 0 \leq x_1 \leq 1, \ 0, \ldots, 0,1 \). Its centre of gravity is \( (\frac{1}{2}, 0, \ldots, 0, 1) \). Its projection is \( (\frac{3}{2n}, \ldots, \frac{3}{2n}) \).

The distance from the centre of gravity to the projection is 
\[
\frac{(5n^2-9n)^{\frac{1}{2}}}{2n}
\]

Therefore, for the case \( n=2k+1 \), \( r_0 \) is chosen so that 
\[
\frac{(5n^2-9n)^{\frac{1}{2}}}{2n} < r_0 < \frac{(n^2-1)^{\frac{1}{2}}}{4n}
\]

It is clear that (4.8.ii) is the value of the content of the \( n \)-dimensional cube minus the content outside the \( n \)-dimensional cylinder but inside the \( n \)-dimensional unit cube.

When \( \frac{n^2-4}{\sqrt{4n}} < r_0 < \frac{\sqrt{n}}{2} \) we are actually considering the content in the neighborhoods of \( \binom{n}{k} \) arrangements of \( k \) ones and \( k \) zeros. Because of symmetry we consider the neighborhood of \( (0^k, 1^k) \) and find an expression for the integral of \( I \) over the stated domain.
When \( n = 2k \) the farthest vertices have the coordinates given by \( \binom{n}{k} \) arrangements of \( k \) ones and \( k \) zeros.

In this case
\[
(4.8.i) \quad P(r \leq r_0) = F(r_0) = \left[ \prod_{i=1}^{n} dx_i \right] \sum_{j}^{n} (x_i - \bar{x})^2 \leq r_0^2
\]
\[
0 \leq x_i \leq 1, \quad i=1,...,n
\]

\[
(4.8.ii) = 1 - \left[ \prod_{i=1}^{n} dx_i \right] \sum_{j}^{n} (x_i - \bar{x})^2 \geq r_0^2
\]

\[
0 \leq x_i \leq 1, \quad i=1,...,n
\]

\[
(4.8.iii) \quad \left[ \prod_{i=1}^{n} dx_i \right] = \binom{n}{k} \left[ \prod_{i=1}^{n} dx_i \right] \sum_{j}^{n} (x_i - \bar{x})^2 \leq r_0^2
\]
\[
0 \leq x_i \leq 1, \quad i=1,...,n, \quad g_i(r_0) \leq x_i \leq 1, \quad i=1,...,k
\]
\[
g_i(r_0) \leq x_i \geq 0, \quad i=k+1,...,n
\]

Let
\[
I = \left[ \prod_{i=1}^{n} dx_i \right] \sum_{j}^{n} (x_i - \bar{x})^2 \leq r_0^2
\]
\[
g_i(r_0) \leq x_i \leq 1, \quad i=1,...,k
\]
\[
g_i(r_0) \geq x_i \geq 0, \quad i=k+1,...,n
\]

\[
(4.8.iv) \quad \text{Make the transformation}
\]
\[
y_i = 1 - x_i \quad i=1,...,k
\]
\[
y_i = x_i \quad i=k+1,...,n
\]

The Jacobian \( |J| = 1 \). Follow this transformation by the spherical transformation (4.8.v).
(4.8.v)

\[ y_1 = \rho \cos \theta \]
\[ y_2 = \rho \sin \theta \]
\[ \vdots \]
\[ y_{n-1} = \rho \sin \theta s_2 s_3 \ldots s_{n-2} c_{n-1} \]
\[ y_n = \rho \sin \theta s_2 s_3 \ldots s_{n-2} s_{n-1} \]

where \( \theta_i = \cos \theta_i \) \( i=1, \ldots, n-1; \)
\( s_i = \sin \theta_i \) \( i=1, \ldots, n-1 \)
\( 0 < \theta_i < \pi \) \( i=1, \ldots, n-2; \)
\( 0 \leq \theta_{n-1} \leq 2\pi \)

and \( 0 < \rho < \infty \).

The Jacobian \( |J| = \rho^{n-1} s_1 s_2 \ldots s_{n-2} \).

The new expression for \( I \) is now

(4.8.vi)

\[
\begin{vmatrix}
\pi/2 \\
\vdots \\
\pi/2
\end{vmatrix}
\begin{vmatrix}
\rho^* \\
\vdots \\
0
\end{vmatrix}
\begin{vmatrix}
\rho^{n-1} s_1 s_2 \ldots s_{n-2} d\rho \prod_{i=1}^{n-1} d\theta_i \\
\vdots \\
0 \\
0
\end{vmatrix}
\]

where \( \rho^* =\)

\[
\psi_2(\theta_1, \ldots, \theta_{n-1}) - \frac{\psi_2(\theta_1, \ldots, \theta_{n-1}) - 4(n-1-2\psi_1(\theta_1, \ldots, \theta_{n-1}))}{n} \cdot \{n-r^2\}^{1/2}
\]

We will now show how we arrived at \( \rho^* \).

Since \( \Sigma (x_i - \overline{x})^2 = \Sigma x_i^2 - nx^2 \) from the transformation (4.8.iv)

\[
\Sigma x_i^2 = \frac{k}{k} (1 - y_i)^2 + \frac{n}{n} y_i^2 \\
= k - 2 \Sigma y_i + \frac{k}{n} y_i^2 + \frac{n}{n} y_i^2 \\
= \frac{n}{n} y_i^2 - 2 \Sigma y_i + k
\]
and 
\[ n \bar{x}^2 = (1/n) (\sum x_i)^2 \]
\[ = (1/n) \left( \sum_{i=1}^{k} (1 - y_i) + \sum_{i=k+1}^{n} y_i \right)^2 \]
\[ = (1/n) \{ k - (\sum_{i=1}^{k} y_i - \sum_{i=k+1}^{n} y_i) \}^2 \]
\[ = (1/n) \{ \sum_{i=1}^{n} y_i^2 + 2 \sum_{1 \leq i < j \leq k} y_i y_j - 2(\sum_{i=1}^{k} y_i)(\sum_{i=k+1}^{n} y_i) \]
\[ + 2 \sum_{k+1 \leq i < j \leq n} y_i y_j - 2k(\sum_{i=1}^{k} y_i - \sum_{i=k+1}^{n} y_i) + k^2 \} \}

Therefore \[ \sum_{i=1}^{n} x_i^2 - n \bar{x}^2 = \]
\[ \sum_{i=1}^{n} y_i^2 - 2 \sum_{i=1}^{k} y_i^2 + k - (1/n)(\sum_{i=1}^{n} y_i^2 + 2 \sum_{1 \leq i < j \leq k} y_i y_j - 2(\sum_{i=1}^{k} y_i)(\sum_{i=k+1}^{n} y_i) \]
\[ + 2 \sum_{k+1 \leq i < j \leq n} y_i y_j - 2k(\sum_{i=1}^{k} y_i - \sum_{i=k+1}^{n} y_i) + k^2 \} \]

Since \[ n = 2k \] this expression reduces to
\[ \frac{n-1}{n} \sum_{i=1}^{n} y_i^2 - 2\sum_{1 \leq i < j \leq k} y_i y_j - (\sum_{i=1}^{k} y_i)(\sum_{i=k+1}^{n} y_i) + \sum_{k+1 \leq i < j \leq n} y_i y_j \]
\[ - \sum_{i=1}^{n} y_i + n \cdot \frac{n}{4} \]

It follows that \[ \sum (x_i - \bar{x})^2 = r^2 \] is now

(4.8.vii)
\[ \frac{n-1}{n} \sum_{i=1}^{n} y_i^2 - 2\sum_{1 \leq i < j \leq k} y_i y_j - (\sum_{i=1}^{k} y_i)(\sum_{i=k+1}^{n} y_i) + \sum_{k+1 \leq i < j \leq n} y_i y_j \]
\[ - \sum_{i=1}^{n} y_i + n \cdot \frac{n}{4} = r^2. \]

Applying the transformation (4.8.iv) to this result
gives

\[ \frac{n-1}{n} \rho^2 - \frac{2 \psi_1(\theta_1, \ldots, \theta_{n-1})}{n} \rho^2 - \psi_2(\theta_1, \ldots, \theta_{n-1}) \rho + \frac{n}{4} = r_0^2. \]

Simplifying (4.8.viii)

\[ \frac{n-1-2 \psi_1(\theta_1, \ldots, \theta_{n-1})}{n} \rho^2 - \psi_2(\theta_1, \ldots, \theta_{n-1}) \rho + \frac{(n-r_0^2)}{4} = 0 \]

where \( \psi_1(\theta_1, \ldots, \theta_{n-1}) \) is obtained by substituting the appropriate terms from the transformation (4.8.v) for \( y_i \) and \( y_j \) in

\[ \sum_{1 \leq i < j \leq k} \{ y_i y_j - (\sum_{i=1}^{n-k} y_i)(\sum_{i=k+1}^{n} y_i) + \sum_{k+1 \leq i < j \leq n} y_i y_j \} \]

\[ i=1, \ldots, n-1; \quad j=2, \ldots, n. \]

Likewise, \( \psi_2(\theta_1, \ldots, \theta_{n-1}) \) is obtained by making similar substitutions for \( y_i \), \( i=1, \ldots, n \) in \( \sum y_i \).

In (4.8.viii) we now solve for \( \rho \). We make use of the root \( \rho^* \) where

\[ \psi_2(\theta_1, \ldots, \theta_{n-1}) - [\psi_2(\theta_1, \ldots, \theta_{n-1}) - 4(n-1-2 \psi_1(\theta_1, \ldots, \theta_{n-1})) \{n-r_0^2\}]^{\frac{1}{2}} \]

\[ (2/n) (n-1-2 \psi_1(\theta_1, \ldots, \theta_{n-1})) \]

Integrating out the \( \rho \) in (4.8.iii) gives

\[ (4.8.ix) \]

\[ \left( \begin{array}{c} \pi/2 \\ \pi/2 \\ \vdots \\ 0 \\ 0 \end{array} \right) \]

\[ \frac{1}{n} \left( \begin{array}{c} \psi_2(\theta_1, \ldots, \theta_{n-1}) - [\psi_2(\theta_1, \ldots, \theta_{n-1}) - 4(n-1-2 \psi_1(\theta_1, \ldots, \theta_{n-1})) \{n-r_0^2\}]^{\frac{1}{2}} \right) \]

\[ \frac{1}{n} \left( \begin{array}{c} (2/n) (n-1-2 \psi_1(\theta_1, \ldots, \theta_{n-1})) \end{array} \right) \]

\[ \{s_1, s_2, \ldots, s_{n-2} \} \]

\[ \prod_{i=1}^{n} \frac{d \theta_i}{\theta_i}. \]
Making use of this result and \((4.8.\text{ii})\) and \((4.8.\text{iii})\) we see that

\[
\mathbf{F}(r_0) = 1 - \frac{1}{n} \sum_{k=1}^{n} \left( \frac{\psi_2 - \frac{4(n-1-2\psi_1)(n-r_0^2)}{n}}{(2/n)(n-1-2\psi_1)} \right)^n
\]

\[
\cdot \prod_{i=1}^{n-1} s_i^{n-2} s_{n-2} \ldots s_2 s_1 d\theta_i
\]

\[
[(n^2-4)/(4n)]^{\frac{1}{2}} \leq r_0 \leq \sqrt{n}/2.
\]

\((4.8.\text{x})\) Hence

\[
\mathbf{I}(s) = 1 - \frac{1}{n} \sum_{k=1}^{n} \left( \frac{\psi_2 - \frac{4(n-1-2\psi_1)(n-r_0^2)}{n}}{(2/n)(n-1-2\psi_1)} \right)^n
\]

\[
\cdot \prod_{i=1}^{n-1} s_i^{n-2} s_{n-2} \ldots s_2 s_1 d\theta_i
\]

\[
(n^2-4)/(2n) \leq s \leq \frac{1}{2}
\]

when \(n = 2k\).

When \(n = 2k + 1\) the number of vertices that are farthest and at the same distance from the equiangular line are \(2 \binom{n}{k}\) and the procedure for the derivation of \(\mathbf{F}(s)\) is the same. We will therefore consider the integral

\[
I = \int \cdots \int_{\sum (x_i - \bar{x})^2 \geq r_0^2} \prod_{i=1}^{n} dx_i
\]

\[
h_i(r_0) \leq x_i \leq 1, \quad i=1, \ldots, k+1
\]

\[
h_i(r_0) \geq x_i \geq 0, \quad i=k+2, \ldots, n
\]

when \(n = 2k+1\).

After making the transformation

\((4.8.\text{xi})\)

\[
y_i = 1 - x_i \quad i=1, \ldots, k+1
\]

\[
y_i = x_i \quad i=k+2, \ldots, n
\]
where Jacobian \( |J| = 1 \) which is followed by the transformation (4.8.v) we get

\[
\begin{aligned}
I = \left| \begin{array}{ccc}
\pi/2 & \pi/2 & \rho^* \\
0 & 0 & 0 \\
\end{array} \right|
= \rho^{n-1} s_1^{n-2} \ldots s_{n-2} d\rho \Pi d\theta_i
\end{aligned}
\]

where

\[
\rho^* = \frac{\psi_2^2 - \frac{4(n-1-2\psi_1)(n^2 - r_0^2)}{4n}}{(n-1-2\psi_1)(2/n)}
\]

\( \rho^* \) is obtained from

(4.8.xii)

\[
\begin{aligned}
(n-1)\frac{\Sigma y_i^2}{n} = & \frac{2}{n} \sum_{1 \leq i < j \leq k+1} y_i y_j - \left( \frac{\Sigma y_i}{n} \right)^2 \sum_{i=k+2}^{k+1} y_i + \sum_{k+2 \leq i < j \leq n} y_i y_j \\
+ \frac{2}{n} \sum_{i=1}^{k+1} y_i^2 + \frac{n-1}{n} \left( \sum_{i=1}^{k+1} y_i - \frac{\Sigma y_i}{n} \right) + \frac{n^2 - 1}{4n} = r_0^2
\end{aligned}
\]

after the transformation (4.8.v) is made. (4.8.xii) is obtained from \( \Sigma x_i^2 - nx^2 = r_0^2 \) after making substitutions from the transformation (4.8.xi).

\( \psi_1 \) is obtained from

(4.8.xiii)

\[
\begin{aligned}
\left\{ \sum_{1 \leq i < j \leq k+1} y_i y_j - \left( \sum y_i \right) \left( \sum y_i \right) + \sum_{k+2 \leq i < j \leq n} y_i y_j \right\}
= \left\{ \sum_{i=1}^{k+1} y_i \right\} - \frac{\Sigma y_i}{n} \left( \sum_{i=k+2}^{k+1} y_i \right)
\end{aligned}
\]

and \( \psi_2 \) from

(4.8.xiv)

\[
\begin{aligned}
2 \sum_{i=1}^{k+1} \left\{ \frac{\Sigma y_i - \sum y_i}{n} \right\} \quad \text{or} \quad \sum_{i=1}^{k+1} \left\{ \frac{\Sigma y_i + (n+1) y_i}{n} \right\}
\end{aligned}
\]

after the appropriate substitutions are made from (4.8.v).

Finally, when \( n = 2k + 1 \)

(4.8.xv)
(4.8.xv) \[
F(r_0) = 1 - 2\left(\frac{n}{n}\right) \sum_{k=0}^{\pi/2} \sum_{j=0}^{\pi/2} \left[ \psi_2 - \frac{\psi^2 - 4(n-1-2\psi_1)(n^2-1-r^2)}{4n} \right]^n
\]

\[
\left(\frac{\psi_2 - \psi^2 - 4(n-1-2\psi_1)(n^2-1-r^2)}{4n} \right)^n
\]

\[
\left(\frac{s_1^{-2} \cdots s_{n-2}^{-2}}{s_1^{-n} \cdots s_{n-2}^{-n}} \prod_{i=1}^{n-1} \sin \theta_i \right)
\]

\[
\frac{(5n-9)^{\frac{1}{2}}}{2n} < r_0 \leq \frac{(n^2-1)^{\frac{1}{2}}}{2n}.
\]

Hence

(4.8.xvi) \[
F_1(s) = 1 - 2\left(\frac{n}{n}\right) \sum_{k=0}^{\pi/2} \sum_{j=0}^{\pi/2} \left[ \psi_2 - \frac{\psi^2 - 4(n-1-2\psi_1)(n^2-1 ns^2)}{4n} \right]^n
\]

\[
\left(\frac{\psi_2 - \psi^2 - 4(n-1-2\psi_1)(n^2-1 ns^2)}{4n} \right)^n
\]

\[
\left(\frac{s_1^{-2} \cdots s_{n-2}^{-2}}{s_1^{-n} \cdots s_{n-2}^{-n}} \prod_{i=1}^{n-1} \sin \theta_i \right)
\]

\[
\frac{(5n-9)^{\frac{1}{2}}}{2n} < s < \frac{(n^2-1)^{\frac{1}{2}}}{2n}
\]

when \( n = 2k+1 \).

### 4.9 Case \( n = 2 \)

When \( n=2 \), (4.8.vii) becomes

\[
\frac{1}{2}(y_1^2 + y_2^2) + y_1 y_2 - y_1 - y_2 + \frac{1}{2} = r_0^2
\]

it follows that

(4.9.i) \[
\frac{1}{2}(y_1 + y_2)^2 - (y_1 + y_2) - (r_0^2 - \frac{1}{2}) = 0.
\]

Making use of transformation (4.8.v) for the case \( n=2 \), we rewrite (4.9.i) as

\[
\frac{1}{2}(c_1 + s_1)^2 \rho^2 - \rho (c_1 + s_1) - (r_0^2 - \frac{1}{2}) = 0.
\]

We find

\[
\rho^* = \frac{(c_1 + s_1)^2 - [(c_1 + s_1)^2 + 2(c_1 + s_1)^2 (r_0^2 - \frac{1}{2})]^\frac{1}{2}}{2(c_1 + s_1)^2}
\]
\[
\frac{1-\sqrt{2}r_0}{s_1+c_1}
\]

From (4.8.i)

\[
I = \int_0^{\pi/2} \frac{1-\sqrt{2}r_0}{s_1+c_1} \rho \, d\rho \, d\theta_1
\]

\[
= \frac{(1-\sqrt{2}r_0)^2}{2} \int_0^{\pi/2} \frac{1}{(s_1+c_1)^2} \, d\theta_1.
\]

Now

\[
(s_1+c_1)^2 = 1 + 2s_1c_1 = 1 + \sin2\theta_1
\]

\[
I = \frac{(1-\sqrt{2}r_0)^2}{4} \int_0^{\pi/2} \frac{1}{1 + \sin2\theta_1} \, d2\theta_1
\]

\[
= \frac{(1-\sqrt{2}r_0)^2}{4} \left\{ -\tan \left( \frac{\pi}{4} - \theta \right) \right\} \int_0^{\pi/2} 0
\]

\[
I = \frac{(1-\sqrt{2}r_0)^2}{2}.
\]

From (4.8.ii) and (4.8.iii)

\[
P(r < r_0) = 1 - 2I = 1 - (1-\sqrt{2}r_0)^2 = 1 - (1-2\sqrt{2}r_0 + 2r_0^2) = 2\sqrt{2}r_0 - 2r_0^2
\]

which shows that \(F(r_0)\) is in agreement with (3.2.1.ii).
CHAPTER 5

ANOTHER APPROACH TO THE DERIVATION OF THE CUMULATIVE DISTRIBUTION OF THE SAMPLE STANDARD DEVIATION $S$ FOR SAMPLES OF SIZE $n$ DRAWN FROM A RECTANGULAR POPULATION WHEN $S \leq (2n)^{-\frac{1}{2}}$.

5.1 Introduction.

In this Chapter we employ another approach to the derivation of the cumulative distribution of the sample standard deviation $S$, for samples of size $n$, when $S \leq (2n)^{-\frac{1}{2}}$. This approach gives us the cumulative distribution in the form of an iterated integral which we can evaluate directly when the sample size $n \leq 5$.

In this approach we use the order transformation:

$$x(1) \leq x(2) \leq \cdots \leq x(n)$$

on the variable $x_i$, $i=1, \ldots, n$ where $x(1)$ is the smallest, $x(2)$ is the second smallest, etc., $x(n)$ being the largest.

This transformation will divide the domain of the cumulative distribution function into $n!$ mutually exclusive sets.

The sequence of transformations will follow as in 4.3. The result will be an iterated integral, which though
complicated, can be explicitly integrated out for \( n = 2, 3, 4, 5 \).

A Second Approach To Finding The Distribution of \( S \).

5.2 We will begin directly with the expression (4.3.ii) of (4.3) Chapter 4.

\[
(5.2.i) \quad F(r_0) = \int_D^{n} \prod_{i=1}^{n} dx_i \\
(a) \quad \sum_{i=1}^{n} \left(x_i - \bar{x}\right)^2 \leq r_0^2 \\
\text{where } D = \\
(b) \quad 0 \leq x_i \leq 1, \quad i = 1, \ldots, n.
\]

Make the order transformation

\[
(5.2.ii) \quad 0 \leq x(1) \leq x(2) \leq \ldots \leq x(n) \leq 1.
\]

Then the integral (5.2.i) can be rewritten as

\[
(5.2.iii) \quad F(r_0) = n! \int_D^{n} \prod_{i=1}^{n} dx(i) \\
(a) \quad \sum_{i=1}^{n} \left(x(i) - \bar{x}\right)^2 \leq r_0^2 \\
\text{where } D = \\
(b) \quad 0 \leq x(1) \leq x(2) \leq \ldots \leq x(n) \leq 1.
\]

Follow the transformation (5.2.ii) by the orthogonal Helmert's transformation (4.3.v) then the cylindrical transformation (4.3.viii). Then the integral (5.2.iii) becomes

\[
(5.2.iv) \quad F(r_0) = n! \int_D^{n-2} \rho^{n-2} \sin^{n-3} \theta_1 \sin^{n-4} \theta_2 \sin \theta_3 \ldots \sin^n \theta_i \prod_{i=1}^{n-2} d\rho \prod_{i=1}^{n-3} d\theta_i
\]

where \( D \) is given by (5.2.v)
(5.2.v)
(a) \( \rho^2 \leq r_0^2 \)
(b) \( y_1 \geq (n - 1)^{\frac{1}{2}} \rho \cos \theta_1 \)
(c) \( y_1 \leq n^{\frac{1}{2}} - \left\{ \frac{n}{n(n-1)} \right\}^{\frac{1}{2}} \rho \cos \theta_1 - \left\{ \frac{n}{(n-1)(n-2)} \right\}^{\frac{1}{2}} \rho \sin \theta_1 \cos \theta_2 \\
- \left\{ \frac{n}{(n-1+i)(n-1+i+1)} \right\}^{\frac{1}{2}} \rho \sin \theta_1 \cos \theta_2 \ldots \cos \theta_{i-2} \cos \theta_{i-1} \\
\ldots \quad \ldots \quad \ldots \\
- \left\{ \frac{n}{6} \right\}^{\frac{1}{2}} \rho \sin \theta_1 \sin \theta_2 \ldots \sin \theta_{n-3} \cos \theta_{n-2} \\
- \left\{ \frac{n}{2} \right\}^{\frac{1}{2}} \rho \sin \theta_1 \sin \theta_2 \ldots \sin \theta_{n-3} \sin \theta_{n-2} \)
(d) \( 0 \leq \theta_i \leq \cot\left\{ \frac{n-i-1}{n-i+1} \right\}^{\frac{1}{2}} \cos \theta_{i+1} \)
\( i = 1, \ldots, n-3 \)
(e) \( 0 \leq \theta_{n-2} \leq \cot^{-1} \frac{1}{\sqrt{3}} \)
(f) \( \sin \theta_{n-2} > 0 \) implies \( 0 < \theta_{n-2} \leq \cot^{-1} \frac{1}{\sqrt{3}} - \frac{\pi}{2} \).

The domain \( D \) was arrived at in the following manner.

From the transformation (5.2.ii) we get
\[
\sum_{i=2}^{n} x_i^2 - nx_i^2 \leq r_0^2
\]
which transformed into \( \sum_{i=2}^{n} y_i^2 \leq r_0^2 \) under the transformation (4.3.v). Then applying the transformation (4.3.viii) took \( \sum_{i=2}^{n} y_i^2 \leq r_0^2 \) into \( \rho^2 \leq r_0^2 \).

The order transformation (5.2.ii) is equivalent to

(5.2.vi)
(a) \( x(i) \geq 0 \) \( i = 1, 2, \ldots, n \)
(b) \( x(i) \geq 0 \) \( i = 1, 2, \ldots, n \)
(c) \( x(i) - x(i-1) \geq 0 \) \( i = 2, \ldots, n \)
(d) \( x(n) \leq 1 \).

Upon the application of the transformation (4.3.vii)
(b), (c), and (d) were taken into (5.2.vii)
(5.2.vii)

(b) \( \frac{1}{n^2} y_1 - \left( \frac{n - 1}{n} \right)^{\frac{1}{2}} y_2 \geq 0 \)

(c) \( \left( \frac{n - i + 2}{n - i + 1} \right)^{\frac{1}{2}} y_i - \left( \frac{n - i}{n - i + 1} \right)^{\frac{1}{2}} y_{i+1} \geq 0; i = 2, \ldots, n - 1 \)

(d) \( y_n \geq 0 \)

(e) \( \frac{1}{n^2} y_1 + \frac{1}{[n(n-1)]^{\frac{1}{2}}} y_3 + \frac{1}{[(n-1)(n-2)]^{\frac{1}{2}}} y_3 \)

\[ \ldots + \frac{1}{[(n-i+3)(n-i+2)]^{\frac{1}{2}}} y_{i-1} + \frac{1}{[(n-i+2)(n-i+1)]^{\frac{1}{2}}} y_i \]

\[ + \frac{1}{6^{\frac{1}{2}}} y_{n-1} + \frac{1}{2^{\frac{1}{2}}} y_n \leq 1. \]

After application of transformation (4.3.viii) (b), (c), (d), (e) transform into (b), (c), (d), (e) and (f) of the domain D of (5.2.iv). A more detailed discussion now follows.

From (d) of (5.2.vi) after applying transformation (4.3.viii) \( y_n \geq 0 \), becomes \( \rho \sin^\theta_1 \sin^\theta_2 \cdots \sin^\theta_{n-2} \geq 0 \)

but \( 0 < \theta_i < \pi, \ i = 1, \ldots, n - 3 \). Therefore \( \sin^\theta_i \geq 0 \), \( i = 1, \ldots, n - 2 \), hence \( \sin^\theta_{n-2} \geq 0 \) and \( 0 < \theta_{n-2} < \pi \).

In what follows we will find \( \theta_{n-2} \) further restricted.

From (c) of (5.2.vi)

\( \left( \frac{n - i + 2}{n - i + 1} \right)^{\frac{1}{2}} y_i - \left( \frac{n - i}{n - i + 1} \right)^{\frac{1}{2}} y_{i+1} \geq 0; \ i = 2, \ldots, n - 1 \)

transforms (4.3.viii) into

\( \left( \frac{n - i + 2}{n - i + 1} \right)^{\frac{1}{2}} \rho \sin^\theta_1 \sin^\theta_2 \cdots \sin^\theta_{i-2} \cos^\theta_{i-1} \)

\[ \geq \left( \frac{n - i}{n - i + 1} \right)^{\frac{1}{2}} \rho \sin^\theta_1 \sin^\theta_2 \cdots \sin^\theta_{i-1} \cos^\theta_i \]

but \( \sin^\theta_{i-1} \geq 0 \), therefore
\[
\left(\frac{n-i+2}{n-i+1}\right)^{\frac{1}{2}} \cot \theta_{i-1} \geq \left(\frac{n-i}{n-i+1}\right)^{\frac{1}{2}} \cot \theta_i
\]
and
\[
\cot \theta_{i-1} \geq \left(\frac{n-i}{n-i+2}\right)^{\frac{1}{2}} \cos \theta_i.
\]

It follows that
\[
0 < \theta_{i-1} \leq \cot^{-1} \left\{ \left(\frac{n-i}{n-i+2}\right)^{\frac{1}{2}} \cos \theta_i \right\}.
\]

If we let \( i = i + 1 \)
\[
0 \leq \theta_i \leq \cot^{-1} \left\{ \left(\frac{n-i-1}{n-i+1}\right)^{\frac{1}{2}} \cos \theta_{i+1} \right\}
\]
\[
i = 1, \ldots, n - 3.
\]

And when \( i = n - 2 \)
\[
0 \leq \theta_i \leq \cot^{-1} \frac{1}{\sqrt{3}} \leq \frac{\pi}{2}
\]

Since the axis of the cylinder \( \rho^2 = r^2 \) lies along \( y_1 \) and the maximum value of \( y_1 \) is \( \sqrt{n} \) the range of \( y_1 \) will be taken as \( (n-1)^{\frac{1}{2}} \rho \cos \theta_1 \leq y_1 \leq \sqrt{n} \). Where \( \frac{\sqrt{n}}{2} \) is the distance from the origin to midway along the diagonal (axis of symmetry) of the n-cube. Making use of symmetry and integrating over the \( y_1 \) and the \( \rho \) we get

(5.2.viii)
\[
F(r_0) = \frac{1}{\sqrt{n}} \frac{\Gamma(n+1)r^{n-1}}{(n-1)} \int_{D^*} \sin^{n-3}\theta_1 \sin^{n-4}\theta_2 \cdots \sin\theta_{n-3} \prod_{i=1}^{n-2} \theta_i \, d\theta_i
\]
\[
-2(n-1)^{\frac{1}{2}} \frac{\Gamma(n)}{r^n} \int_{D^*} \cos\theta_1 \sin^{n-3}\theta_1 \cdots \sin\theta_{n-3} \prod_{i=1}^{n} \theta_i \, d\theta_i
\]

where \( D^* = \)

\( a \)
\[
0 \leq \theta_i \leq \cot^{-1} \left\{ \left(\frac{n-i-1}{n-i+1}\right)^{\frac{1}{2}} \cos \theta_{i+1} \right\}
\]
\[
i = 1, \ldots, n - 3
\]
\( b \)
\[
0 \leq \theta_{n-2} \leq \cot^{-1} \frac{1}{\sqrt{3}}
\]
(c) \( \sin \theta_{n-2} > 0, \ 0 < \theta_{n-2} < \frac{\pi}{2} \)

5.3 **Lemma 5.3.**

\[
\int \sin^{n-3} \theta_1 \sin^{n-4} \theta_2 \cdots \sin^{n-3} \theta_{n-2} \prod_{i=1}^{n-2} d\theta_i
\]

\[
D^* = \frac{2\pi^{\frac{1}{2}(n-1)}}{\Gamma(n+1) \Gamma\left(\frac{1}{2}(n-1)\right)}
\]

where \( D^* = \)

(a) \( 0 < \theta_i \leq \cot^{-1}\left\{\frac{n-i-1}{n-i+1}\right\} \frac{1}{2} \cos \theta_{i+1} \)

\( i = 1, 2, \ldots, n - 3 \)

(b) \( 0 < \theta_{n-2} \leq \cot^{-1}\left\{\frac{1}{\sqrt{3}} \leq \frac{\pi}{2}\right\} \)

**Proof:**

The technique used to prove this lemma is identical to that employed in the proof of Lemma 4.4., with the exception that rather than use the partial order transformation (4.4.vi) we completely order the variables \( x_1, x_2, \ldots x_n \).
5.4 **LEMMA 5.4.**

(5.4.i) \[ \frac{J^*_{n-2}}{n-2} = \frac{1}{ \mathcal{D}^* } \cos \theta_1 \sin^{n-3} \theta_1 \cdots \sin^{n-3} \theta_{n-3} \prod_{i=1}^{n} d \theta_i \]

where \( \mathcal{D}^* \) is the same as in Lemma 5.3;

(5.4.ii) \[ \frac{(n-1)^{\frac{1}{2}}}{2(n-1)^{1/2}} \left( \prod_{i=2}^{n-1} \left[ (1 + \sum_{i=2}^{n} u_i^2) - \frac{1}{n} \left( 1 + \sum_{i=2}^{n} u_i^2 \right)^2 \right] \right)^{\frac{1}{2}} \]

\[ 0 < u_i < 1 \quad i = 2, \ldots, n - 1. \]

First we will show that

(5.4.iii) \[ \frac{1}{n(n-1)^{\frac{1}{2}} \Gamma \left( \frac{n}{2} \right) } \left( \prod_{i=2}^{n} \left( \Sigma x(i) \right)^e \right) \]

\[ \int_{D_1}^{\infty} \left( \Sigma x(i) \right)^e \prod_{i=2}^{n} dx(i) \]

where \( D_1 = 0 \leq x_2 \leq x_3 \cdots \leq x_n < \infty \)

is equal to the integral of (5.4.i) i.e., \( J_{n-2}^* \), then we will show that it is also equal to (5.4.ii).

Consider the integral

(5.4.iv) \[ K_{n-2}^* = \int_{D_1}^{\infty} \left( \Sigma x(i) \right)^e \prod_{i=2}^{n} dx(i) \]

where \( D_1 = 0 \leq x(i) \leq \cdots \leq x(n) < \infty \).

\( D_1 \) is equivalent to

(a) \( x(i) - x(i-1) \geq 0 \quad i = 2, \ldots, n \)

(b) \( x(2) > 0 \)

(c) \( x(n) < \infty \)
Make the transformation (4.5.v) of Lemma (4.5) using the special results obtained from (4.5.xi) and (4.5.xiii). We get

\[(5.4.v) \quad \sum_{i=2}^{n} x(i) = [n(n-1)]^{\frac{1}{2}} \rho \cos \theta_1 \quad \text{and} \quad \sum_{i=2}^{n} x^2(i) - \frac{1}{n} \left( \sum_{i=2}^{n} x(i) \right)^2 = \rho^2.\]

Then

\[(5.4.vi) \quad K^{*}_{n-2} = [n(n-1)]^{\frac{1}{2}} n^{\frac{1}{2}} \int_{D_1} \frac{-\rho^2}{2} e^{-\rho} \cos^{n-3} \theta_1 \cdots \cos \theta_{n-3} \prod_{i=1}^{n-2} d \theta_i \]

where \( D_1 = \begin{cases} (a) & 0 \leq \rho \leq \infty \\ (b) & 0 \leq \theta_i \leq \cot^{-1} \{ \frac{n-i-1}{n-i+1} \} \\ (c) & 0 \leq \theta_{n-2} \leq \cot^{-1} \frac{1}{\sqrt{3}} \leq \frac{1}{2} \pi. \end{cases} \)

Integrating out \( \rho \) the integral (5.4.vii) becomes

\[(5.4.viii) \quad K^{*}_{n-2} = n(n-1)^{\frac{1}{2}} 2^{\frac{1}{2}(n-2)} \Gamma \left( \frac{n}{2} \right) \int_{D^*} \cos^{n-3} \theta_1 \cdots \sin \theta_{n-3} \prod_{i=1}^{n-2} d \theta_i \]

That is

\[(5.4.ix) \quad K^{*}_{n-2} = n(n-1)^{\frac{1}{2}} 2^{\frac{1}{2}(n-2)} \Gamma \left( \frac{n}{2} \right) \cdot J^{*}_{n-2} \quad \text{hence} \]

\[
\frac{K^{*}_{n-2}}{n(n-1)^{\frac{1}{2}} 2^{\frac{1}{2}(n-2)} \Gamma \left( \frac{n}{2} \right)} = J^{*}_{n-2}
\]

thus completing the first part of our proof.
Consider again

\[
K^*_{n-2} = \int_{D_1} \left( \sum_{i=2}^{n} \frac{x(i)}{n} \right)^2 \prod_{i=2}^{n} dx(i)
\]

where \( D_1 = 0 \leq x(2) \leq \ldots \leq x(n) \).

Rewriting the integral

\[
(5.4.x)
K^*_{n-2} = \sum_{i=2}^{n} \int_{D_1} \frac{x(i)}{n} \left( \sum_{i=2}^{n} \frac{x(i)}{n} \right)^2 \prod_{i=2}^{n} dx(i)
\]

Unorder the variables \( x_2, \ldots, x(n) \) and let \( x(i) \) be the maximum \( x(i) \) for the \( i \) integral \( i = 2, \ldots, n \). There are \( (n-1) \) contenders for the maximum position after which there are \((n-2)\) etc., until there is only one. In each case replace \( x(i) \) by \( x(n) \) then factor out the common integral. Then

\[
(5.4.x_i)
K^*_{n-2} = \left( \int_{D_2} x(n) \right)^{1+2+\ldots+n-1} \prod_{i=2}^{n-1} dx_i dx(n) \frac{(n-1)(n)}{2(n-1)!}
\]

where \( D_2 = 0 \leq x_1 \leq x_n \leq \infty, \ i = 2, \ldots, n-1 \).

But \( 1 + 2 + \ldots + n - 1 = \frac{(n-1)(n)}{2} \) therefore

\[
(5.4.x_i)
K^*_{n-2} = \frac{(n-1)n}{2(n-1)!} \int_{D_2} x(n) \prod_{i=2}^{n} dx_i dx(n)
\]
but from Lemma 4.5 we have

\[
\frac{1}{\Gamma^{\frac{1}{2n}}(\frac{1}{2n})} \frac{1}{D_2} \left\{ \begin{array}{c}
-\frac{1}{2n} \sum_{i=2}^{n} \left( \frac{x_i^2}{n} - \frac{1}{n} \sum_{i=2}^{n} x_i \right)^2 \\
\prod_{i=2}^{n-1} dx(i) \ dx(n)
\end{array} \right\}
\]

\[
= \left\{ \begin{array}{c}
\prod_{i=2}^{n-1} du_i \\
0 < u_i < 1 \\
i=2, \ldots, n-1
\end{array} \right\}
\left\{ \begin{array}{c}
\sum_{i=2}^{n-1} \left( \frac{(1 + \sum_{i=2}^{n-1} u_i^2) - \frac{1}{n} \sum_{i=2}^{n-1} u_i^2} {1 + \sum_{i=2}^{n-1} u_i^2} \right) \\
i=2, \ldots, n-1
\end{array} \right\}
\]

(5.4.xii) therefore,

\[
\frac{K^{*}}{\frac{1}{2}(n-2)} \Gamma^{\frac{1}{2n}}(\frac{1}{2n}) = \frac{(n-1)n}{2(n-1)!} \left\{ \begin{array}{c}
\prod_{i=2}^{n-1} du_i \\
0 < u_i < 1 \\
i=2, \ldots, n-1
\end{array} \right\}
\left\{ \begin{array}{c}
\sum_{i=2}^{n-1} \left( \frac{(1 + \sum_{i=2}^{n-1} u_i^2) - \frac{1}{n} \sum_{i=2}^{n-1} u_i^2} {1 + \sum_{i=2}^{n-1} u_i^2} \right) \\
i=2, \ldots, n-1
\end{array} \right\}
\]

(5.4.xiii) and

\[
J^{*}_{n-2} = \frac{K^{*}}{\frac{1}{2}(n-2)} \sqrt{\frac{n}{n-1}} \Gamma^{\frac{1}{2n}}(\frac{1}{2n})
\]

\[
= \frac{(n-1)^{\frac{1}{2}}}{2(n-1)!} \left\{ \begin{array}{c}
\prod_{i=2}^{n-1} du_i \\
0 < u_i < 1 \\
i=2, \ldots, n-1
\end{array} \right\}
\left\{ \begin{array}{c}
\sum_{i=2}^{n-1} \left( \frac{(1 + \sum_{i=2}^{n-1} u_i^2) - \frac{1}{n} \sum_{i=2}^{n-1} u_i^2} {1 + \sum_{i=2}^{n-1} u_i^2} \right) \\
i=2, \ldots, n-1
\end{array} \right\}
\]

which completes the proof of this lemma. Q.E.D.

5.5 \( F(r_0) \) As An (n-2)ple Integral Over An n-cube.

(5.5.i) Employing Lemma 5.3 we can rewrite (5.2.viii) as

\[
F(r_0) = 2\pi \frac{\frac{1}{2}(n-1)}{n} \frac{1}{\Gamma^{\frac{1}{2n}}(\frac{1}{2n})} \frac{1}{r_0^{n-1}} \left\{ \begin{array}{c}
\cos^3 \theta_1 \sin^3 \theta_1 \ldots \sin^3 \theta_{n-3} \prod_{i=1}^{n} d\theta_i \\
D^*
\end{array} \right\}
\]
where $D^* = (a) \ 0 \leq \theta_i \leq \cot^{-1} \left\{ \left( \frac{n - i - 1}{n - i + 1} \right)^{1/2} \cos \theta_{i+1} \right\} \qquad i = 1, \ldots, n - 3$$
(b) \ 0 \leq \theta_{n-2} \leq \cot^{-1} \frac{1}{\sqrt{3}}$.

Applying the Lemmas 5.3 and 5.4 to (5.2.viii) we get

(5.5.ii)

$$F(r_0) = 2\pi^{1/2} (n-1) \frac{1}{n} r_0 \frac{n-1}{(n-1)} \left[ \prod_{i=2}^{n-1} \frac{u_i}{(1 + \sum_{i=2}^{n-1} u_i)^{1/2}} \right]^{1/2} \left[ \prod_{i=2}^{n-1} \frac{u_i}{(1 + \sum_{i=2}^{n-1} u_i)^{1/2}} \right]$$

$$0 < u_i < 1 \quad i = 2, \ldots, n - 1.$$

This result is the same as 4.i) in Chapter 4. Since (5.5.i), (5.4.i) and (5.5.ii) are all equal the assumption of symmetry is validated.

5.6 We will now evaluate $F(r_0)$ for $n = 5$. To do this we must first evaluate the integral obtained by substituting $n = 5$ into the iterated integral of (5.5.i). This gives

(5.6.i)

$$4 \cdot 4! r^4 \int_0^{\pi/3} \cot^{-1} \left( \frac{1}{\sin^2 \theta_1} \right) \cot^{-1} \left( \frac{\sqrt{3/5} \cos \theta_2}{\sin \theta_1} \right)$$

$$= 4 \cdot 4! r^4 \int_0^{\pi/3} \left( \frac{1}{\sin^2 \theta_1} \right) \sin \theta_2 \ d\theta_2 \ d\theta_3$$
\[ = - \frac{4 \cdot 4! r^4}{3} \int_0^{\pi/3} \cot^{-1} \left( \frac{1 \cos \theta_3}{\sqrt{2}} \right) \frac{\cos \theta_2}{(1 + 3/5 \cos^2 \theta_2)^{3/2}} \, d \theta_3 \]

\[ = - \frac{4 \cdot 4! r^4}{3} \int_0^{\pi/3} \frac{\{\cos^2 \theta_3/(2 + \cos^2 \theta_3)\}^{1/2}}{[1 + 3/5\{\cos^2 \theta_3/(2 + \cos^2 \theta_3)\}]^{1/2}} - (5/8)^{1/2} \, d \theta_3 \]

\[ = - \frac{4 \cdot 4!}{3} \int_0^{\pi/3} \frac{\cos \theta_3 \, d \theta_3}{[2 + 8/5 \cos^2 \theta_3]^{1/2}} - (5/8)^{1/4} \, d \theta_3 \]

The first integral can be readily evaluated by writing \( \cos \theta_3 \, d \theta_3 \) as \( d(\sin \theta_3) \). We have

\[ \int_0^{\pi/3} \frac{\sin \theta_3 \, d \theta_3}{[2 + 8/5 \cos^2 \theta_3]^{1/2}} = \int_0^{\pi/3} \frac{\sin \theta_3 \, d \theta_3}{(8/5) [18/8 - \sin^2 \theta_3]^{1/2}} \]

\[ = (5/8)^{1/4} [\sin^{-1} (\sin \theta_3)] \int_0^{\pi/3} \frac{d \theta_3}{\sqrt{(18/8)}} = \sqrt{5/8} \sin^{-1} \sqrt{18/8} \cdot \sqrt{3}/2. \]

Therefore the integral 5.6.i is now equal to

\[ = - \frac{4 \cdot 4!}{3} \left\{ (5/8)^{1/4} \sin^{-1} \frac{1}{\sqrt{3}} - (5/8)^{1/4} \cdot \frac{\pi}{3} \right\} \]

\[ = + 4.4! \cdot (5/8)^{1/4} \left\{ \frac{\pi}{3} - \sin^{-1} \frac{1}{\sqrt{3}} \right\} \]
CHAPTER 6
SPHERICAL APPROXIMATION

6.1 Introduction.

In Chapter 4 we obtained the distribution of $S$ for $0 \leq S \leq (2n)^{-\frac{1}{2}}$ in the form of a $(n-2)$-dimensional multiple integral over an $(n-2)$-cube. Since the computations involved for large $n$ will be excessive, we will approximate this $(n-2)$-dimensional integral by a 2-dimensional integral. The $(n-2)$-cubic domain will be replaced by an $(n-2)$-spherical domain having the same volume. We will also calculate the exact upper and lower bounds for the probability $P(S \leq s)$ when $0 \leq s \leq (2n)^{-\frac{1}{2}}$ by first considering the sphere completely inscribed in the $(n-2)$-unit cube which is the domain of the integral, then considering the sphere completely circumscribed about the $(n-2)$-unit cube which is the domain of the integral.

A comparison of the exact probabilities and the approximate probabilities will follow in a later chapter.

6.2 Spherical Approximation and Exact Upper and Lower Bounds for $P(S \leq s), \ 0 \leq s \leq (2n)^{-\frac{1}{2}}$.

The cumulative distribution of the sample standard deviation $S$ for samples of size $n$ drawn from a rectangular population when $S \leq (2n)^{-\frac{1}{2}}$ is given by (6.2.i).
(6.2.i) 
\[ F(s) = 2\pi^{\frac{1}{2}(n-1)} \frac{\Gamma(\frac{1}{2}(n-1))}{(n-1)} \sum_{j} (n_i^n)^{n-1} - (n-1)(n_i^n)^n \]
\[ \frac{\prod_{i=2}^{n-1} du_i}{(1 + \sum_{i=2}^{n-1} u_i^2)^{\frac{n-1}{2}}} \]
\[ 0 < u_i < 1 \]
\[ i = 2, \ldots, n - 1. \]

Consider the integral part of this expression and let

(6.2.ii) 
\[ I_{n-2} = \int \frac{n-1}{\prod_{i=2}^{n-1} du_i} \]
\[ \frac{(1 + \sum_{i=2}^{n-1} u_i^2) - \frac{1}{n} (1 + \sum_{i=2}^{n-1} u_i^2)^{\frac{2}{n}}}{(1 + \sum_{i=2}^{n-1} u_i^2)^{\frac{n-1}{2}}} \]
\[ 0 < u_i < 1 \]
\[ i = 2, \ldots, n - 1. \]

Make the transformation \( u_i = v_i + \frac{1}{2}, \) \( i = 2, \ldots, n - 1. \)

This transformation shifts the origin to the centre of the cube which is the domain of this integral. The Jacobian is \( |J| = 1, \) and so

(6.2.iii) 
\[ I_{n-2} = \int \frac{n-1}{\prod_{i=2}^{n-1} dv_i} \]
\[ \frac{(\sum_{i=2}^{n-1} v_i^2 + 4) - \frac{1}{n} (\sum_{i=2}^{n-1} v_i^2 + \frac{4}{2})^{\frac{2}{n}}}{(\sum_{i=2}^{n-1} v_i^2 + 4)^{\frac{n-1}{2}}} \]
\[ -\frac{1}{2} < v_i < \frac{1}{2} \]
\[ i = 2, \ldots, n - 1. \]

The domain of this integral which is an \((n-2)\)-dimensional unit cube will now be approximated by an \((n-2)\)-dimensional sphere of unit volume. The equation of the \((n-2)\)-dimensional sphere of radius \( r \) is \( \sum_{i=2}^{n-1} u_i^2 = r^2. \) In this case \( r \) is chosen so that the sphere will have unit volume. The value for this \( r \) is now determined by noting that the volume of an \((n-2)\)-dimensional sphere is
\[ V_{n-2}(r) = \frac{\pi^{\frac{1}{2}}(n-2)}{\Gamma(\frac{3}{2}n)} \cdot r^{n-2}. \]

Setting \( V_{n-2}(r) = 1 \) we get
\[ r = \left[ \frac{\Gamma(\frac{3}{2}n)}{\pi^{1/2}} \right]^{1/(n-2)} = r^*. \]

It follows that

\[
I_{n-2} = \left( \begin{array}{c}
\frac{\prod_{i=2}^{n-1} v_i}{\left( \sum_{i=2}^{n-1} v_i + \frac{2}{n} \sum_{i=2}^{n-1} (v_i + n) \right) \frac{1}{2}} \\
\frac{\sum_{i=2}^{n-1} v_i^2}{\sum_{i=2}^{n-1} v_i} \leq r^* \end{array} \right).
\]

Make the orthogonal transformation

\[
\begin{array}{ccc|c}
 x_2 & (n-2)^{-\frac{1}{2}} & \cdots & (n-2)^{-\frac{1}{2}} \\
 x_3 & a_{3,2} & \cdots & a_{3,n-1} \\
 \vdots & \vdots & \ddots & \vdots \\
 x_{n-1} & a_{n-1,2} & \cdots & a_{n-1,n-1} \\
 x_n \end{array} = A \begin{array}{c}
v_2 \\
v_3 \\
\vdots \\
v_{n-1} \\
v_n \end{array}
\]

where the first row of the matrix \( A_{n-2,n-2} \) is \((n-2)^{-\frac{1}{2}},\ldots,(n-2)^{-\frac{1}{2}}\) and the other rows are chosen to form an orthogonal matrix. Then
\[
x_2 = (n-2)^{-\frac{1}{2}} \sum_{i=2}^{n-1} v_i \quad \text{or} \quad \sum_{i=2}^{n-1} v_i = (n-2)^{\frac{1}{2}} x_2
\]

and
\[
\sum_{i=2}^{n-1} v_i^2 = \sum_{i=2}^{n-1} x_i^2.
\]

Follow this transformation by the spherical transformation
\[ x_2 = \rho \cos \theta_1 \]
\[ x_3 = \rho \sin \theta_1 \cos \theta_2 \]
\[ \vdots \]
\[ x_{n-2} = \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{n-4} \cos \theta_{n-3} \]
\[ x_{n-1} = \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{n-4} \sin \theta_{n-3} \]
\[ 0 < r < \infty, \quad 0 < \theta_i < \pi, \quad i = 1, \ldots, n-4, \quad 0 < \theta_{n-3} < 2\pi. \]

The Jacobian of the composite transformation is
\[ |J| = \rho^{n-3} \sin^{n-4} \theta_1 \sin^{n-5} \theta_2 \cdots \sin \theta_{n-4}. \]

Also
\[ \sum_{i=2}^{n-1} v_i^2 = \sum_{i=2}^{n-1} x_i^2 = \rho^2 \]
and
\[ \sum_{i=2}^{n-1} v_i = (n-2) \frac{\rho^2}{2} x_2 = (n-2) \frac{\rho^2}{2} \cos \theta_1. \]

Then (6.2.iv) becomes
\[ (6.2.v) \]
\[ I_{n-2} = \int_{D} \left( \rho^{n-3} \sin^{n-4} \theta_1 \sin^{n-5} \theta_2 \cdots \sin \theta_{n-4} \right) \frac{\partial^n}{\partial \theta_1 \cdots \partial \theta_{n-4}} \frac{d\theta_1 \cdots d\theta_{n-4}}{(\rho^2 + (n-2) \rho \cos \theta_1 + \frac{n+2}{4} - \frac{1}{n}(n-2) \rho \cos \theta_1 + \frac{n^2}{4})^{\frac{n}{2}}} \]

where \( D = \left\{ 0 \leq \rho \leq \frac{1}{(n-2)} \right\}^{1/(n-2)} = r^* \)
\[ 0 < \theta_i < \pi, \quad i = 1, \ldots, n-4 \]
\[ 0 < \theta_{n-3} < 2\pi. \]

Integrating out \( \theta_2, \ldots, \theta_{n-3} \) we get
\[ (6.2.vi) \]
\[ I_{n-2} \approx 2 \cdot \frac{\pi^{\frac{1}{2}}}{\Gamma\left(\frac{1}{2}(n-3)\right)} \int_{0}^{r^*} \int_{0}^{\pi} \rho^{n-3} \sin^{n-4} \theta_1 dp \, d\theta_1 \left[ 1 - \frac{(n-2) \cos^2 \theta_1 \rho^2 + \frac{1}{2}}{n} \right]^{\frac{1}{2}}. \]
Where
\[ r^* = \frac{\Gamma\left(\frac{1}{2}n\right)}{\sqrt{\pi}}^{1/(n-2)} . \]

(6.2.vii) Therefore
\[
F(s) = 2 \cdot \pi^{\frac{1}{2}(n-1)} n^{\frac{1}{2}} (n_s)^{n-1} \frac{\Gamma\left(\frac{1}{2}(n-1)\right)}{\Gamma\left\{\frac{1}{2}(n-3)\right\}} - 2 \cdot \pi^{\frac{1}{2}(n-3)} (n-1)^{\frac{1}{2}} (n_s)^{n-1} \frac{\Gamma\left(\frac{1}{2}(n-3)\right)}{\Gamma\left\{\frac{1}{2}(n-1)\right\}} \frac{\rho^{n-3} \sin^{n-4} \theta}{\int_0^{\pi} \int_0^{\frac{1}{2}(n-2)} \frac{\cos^2 \theta}{\rho^2 + \frac{1}{2}}} \frac{d\rho \, d\theta}{\int_0^{\frac{1}{2}(n-2)} \frac{\cos^2 \theta}{\rho^2 + \frac{1}{2}} \frac{d\rho \, d\theta}{\int_0^{\frac{1}{2}(n-2)} \frac{\cos^2 \theta}{\rho^2 + \frac{1}{2}}} .
\]

The exact lower and upper bounds for \( F(s) \) will now be computed by finding the upper and lower bounds of the integral \( I_{n-2} \). We will first consider a circumscribed \( (n-2) \)-sphere about the \( (n-2) \)-unit cube, then an inscribed \( (n-2) \)-sphere in the \( (n-2) \)-unit cube. Since for the circumscribed sphere of the unit cube the radius is \( \frac{1}{2}(n - 2) \), the lower bound of \( F(s) \) is

(6.2.viii)
\[
F(s) = 2 \cdot \pi^{\frac{1}{2}(n-1)} n^{\frac{1}{2}} (n_s)^{n-1} \frac{\Gamma\left(\frac{1}{2}(n-1)\right)}{\Gamma\left\{\frac{1}{2}(n-3)\right\}} - 2 \cdot \pi^{\frac{1}{2}(n-3)} (n-1)^{\frac{1}{2}} (n_s)^{n-1} \frac{\Gamma\left(\frac{1}{2}(n-3)\right)}{\Gamma\left\{\frac{1}{2}(n-1)\right\}} \frac{\rho^{n-3} \sin^{n-4} \theta}{\int_0^{\pi} \int_0^{\frac{1}{2}(n-2)} \frac{\cos^2 \theta}{\rho^2 + \frac{1}{2}}} \frac{d\rho \, d\theta}{\int_0^{\frac{1}{2}(n-2)} \frac{\cos^2 \theta}{\rho^2 + \frac{1}{2}}} .
\]

For the inscribed sphere of the unit cube the radius is \( \frac{1}{2} \).

Hence, the upper bound for \( F(s) \) will be

(6.2.viv)
\[
F(s) = 2 \cdot \pi^{\frac{1}{2}(n-1)} n^{\frac{1}{2}} (n_s)^{n-1} \frac{\Gamma\left(\frac{1}{2}(n-1)\right)}{\Gamma\left\{\frac{1}{2}(n-3)\right\}} - 2 \cdot \pi^{\frac{1}{2}(n-3)} (n-1)^{\frac{1}{2}} (n_s)^{n-1} \frac{\Gamma\left(\frac{1}{2}(n-3)\right)}{\Gamma\left\{\frac{1}{2}(n-1)\right\}} \frac{\rho^{n-3} \sin^{n-4} \theta}{\int_0^{\pi} \int_0^{\frac{1}{2}(n-2)} \frac{\cos^2 \theta}{\rho^2 + \frac{1}{2}}} \frac{d\rho \, d\theta}{\int_0^{\frac{1}{2}(n-2)} \frac{\cos^2 \theta}{\rho^2 + \frac{1}{2}}} .
\]
\[
\begin{align*}
&\int_0^\pi \int_0^\pi \frac{\rho^{n-3} \sin^{n-4} \theta_1 \, d\rho \, d\theta_1}{\left[\left(1 - \frac{n-2}{n} \cos^2 \theta_1 \right)^2 + \frac{n}{1}L_1^n\right]^{\frac{1}{2n}}} \\
&\quad \cdot \left[\rho \sin^4 \theta_1 \, d\rho \, d\theta_1\right]^{\frac{1}{2n}}.
\end{align*}
\]
CHAPTER 7

THE DISTRIBUTION OF $S$ FOR SETS OF $n$ VARIATES WHEN THE UNDERLYING POPULATION IS EXPONENTIAL.

7.1 Introduction.

In this chapter we employ the same analytic methods used in Chapter 4 to derive the distribution of $S$ when the samples are drawn from an exponential population. We also use the method of "sections" of Chapter 3 to find in another form the distribution of $S$ for the cases where $n = 2, 3, 4$.

We use the method of "sections" primarily to emphasize the close relationship between the distribution of the $S$ statistic when the underlying population is uniform and the distribution of the $S$ statistic when the underlying population is negative exponential.

In section 7.6 exact lower and upper bounds for $P_n(S \leq s)$ are found and spherical approximations for the integral expression of the cumulative distribution function are given for any $n$ where $n$ is the sample size.
7.2 Statement of The Problem.

Let \( x_1, x_2, \ldots, x_n \) be \( n \) independent identically distributed random variables from the exponential distribution.

\[
dF = e^{-x} dx \quad 0 \leq x < \infty.
\]

The joint distribution of the \( n \) variables is

\[
dF = \prod_{i=1}^{n} e^{-x_i} dx_i \quad 0 \leq x_i < \infty, \quad i=1, \ldots, n.
\]

Therefore, the sampling cumulative distribution of \( S \) is

(7.2.i)

\[
P(S \leq s) = F(s) = \int_{D} e^{-\sum_{i=1}^{n} x_i} \prod_{i=1}^{n} dx_i
\]

where \( D = \)

(a) \( n^{-1} \sum_{i=1}^{n} (x_i - \bar{x})^2 \leq s^2 \)

(b) \( 0 \leq x_i < \infty, \quad i=1, \ldots, n. \)

Let \( r_0^2 = ns^2 \), then

(7.2.ii)

\[
F(r_0) = \int_{D} e^{\sum_{i=1}^{n} (x_i - \bar{x})^2 \leq r_0^2}
\]

where \( D = \)

(a) \( \sum_{i=1}^{n} (x_i - \bar{x})^2 \leq r_0^2 \)

(b) \( 0 \leq x_i < \infty, \quad i=1, \ldots, n. \)

The domain of integration is an infinite cylinder, wedged into the coordinate system, whose axis is the equiangular line. Since the density is constant on the plane
\[
\sum x_i = n\bar{x} = \sqrt{n}y,
\]

we can use the previous results for the various configurations to derive the exact cumulative distribution of \( S \) for \( n = 2, 3, 4 \). First however, we will give the analytic derivation.

Let us now make the partial order transformation

1. \( x_{(1)} \geq 0 \)
2. \( x_{(n)} < \infty \)
3. \( 0 \leq x_{(1)} \leq x_i \leq x_{(n)} < \infty \)

\( x_i, \ i=2, \ldots, n-1 \) unordered,

on the variates \( x_i \) of (7.2.ii) where \( x_{(1)} \) denotes the smallest, \( x_{(n)} \) the largest.

Therefore, (7.2.ii) becomes

\[
(7.2.iii) \quad F(r_0) = n(n-1) \int_{D} e^{-\sum_{i=1}^{n} x_i} \prod_{i=2}^{n-1} dx_i \, dx_{(n)}
\]

where \( D = \)

(a) \( x_{(n)} - x_i \geq 0, \ i=1, \ldots, n-1 \)
(b) \( x_{n} < \infty \)
(c) \( x_i - x_{(1)} \geq 0, \ i=2, \ldots, n-1 \)
(d) \( x_1 \geq 0 \).

Now after applying the orthogonal Helmert's Transformation (4.3.v) then the cylindrical transformation (4.3.viii) the integral (7.2.iii) is

\[
(7.2.iv) \quad F(r_0) = n(n-1) \int_{D} (e^{-\sqrt{n}y_1} dy_1) \rho^{n-2} \sin^3 \theta_1 \ldots \sin^3 \theta_{n-3} \prod_{i=1}^{n-2} d\theta_i
\]
where $D =$

(a) $\rho^2 \leq r^2_0 < \infty$

(b) $\frac{n-i+1}{[(n-i+1)(n-1)]} \rho \sin \theta_1 \sin \theta_2 \ldots \cos \theta_i$

+ $\frac{1}{[(n-i)(n-i-1)]} \rho \sin \theta_1 \sin \theta_2 \ldots \sin \theta_i \cos \theta_{i+1}$

+ \ldots \ldots \ldots

+ $\frac{1}{[\frac{1}{2}]i} \rho \sin \theta_1 \sin \theta_2 \ldots \sin \theta_{n-3} \cos \theta_{n-2}$

$\frac{1}{6}$

+ $\frac{1}{[\frac{1}{2}]i} \rho \sin \theta_1 \sin \theta_2 \ldots \sin \theta_{n-3} \sin \theta_{n-2} \geq 0 \quad i=1, \ldots, n-1$

(c) $\gamma_1 < \infty$

(d) $\gamma_1 \geq (n-1)^{\frac{1}{2}} \rho \cos \theta_1$

(e) $\frac{n}{[n(n-1)\frac{1}{2}]} \rho \cos \theta_1 - \frac{n-2}{[(n-1)(n-2)\frac{1}{2}]} \rho \sin \theta_1 \cos \theta_2 \geq 0$

\ldots

$\frac{n}{[n(n-1)\frac{1}{2}]} \rho \cos \theta_1 + \frac{1}{[(n-1)(n-2)\frac{1}{2}]} \rho \sin \theta_1 \cos \theta_2 \ldots$

$- \frac{n-i}{[(n-i+1)(n-i)\frac{1}{2}]} \rho \sin \theta_1 \ldots \cos \theta_i \geq 0$

$\frac{n}{[n(n-1)\frac{1}{2}]} \rho \cos \theta_1 + \frac{1}{[(n-1)(n-2)\frac{1}{2}]} \rho \sin \theta_1 \cos \theta_2 + \ldots$

$+ \frac{1}{[\frac{1}{2}]i} \rho \sin \theta_1 \ldots \cos \theta_{n-2}$

$\frac{1}{6}$

$- \frac{1}{[\frac{1}{2}]i} \rho \sin \theta_1 \ldots \sin \theta_{n-2} \geq 0.$
Integrating out the $y_1$ variable of (7.2.iv) we get

(7.2.v)

$$F(r_0) = \frac{n(n-1)}{\sqrt{n}} \int_{D^*} e^{-\left[n(n-1)\right]^{\frac{1}{2}} \rho \cos \theta_1} \rho^{n-2} \sin^{n-3} \theta_1 \cdots \sin^{n-3} \theta_{n-3} \prod_{i=1}^{n-2} d\theta_i d\rho$$

where $D^*$ is $D$ of (7.2.iv) with (c) and (d) excluded.

Expand $e^{-\left[n(n-1)\right]^{\frac{1}{2}} \rho \cos \theta_1}$ in a Taylor Series then interchange the integral and summation signs of (7.2.v). Thus we get

(7.2.vi)

$$F(r_0) = \frac{n(n-1)}{\sqrt{n}} \sum_{k=0}^{\infty} \left( \frac{(-1)^k \left[n(n-1)\right]^{\frac{k}{2}} \rho \cos \theta_1}{k!} \right)^k \rho^{n-2} \sin^{n-3} \theta_1 \cdots \sin^{n-3} \theta_{n-3} \prod_{i=1}^{n-2} d\theta_i d\rho.$$

The frequency function of $F(r_0)$ is

(7.2.vii)

$$f(r_0) = \frac{n(n-1)}{\sqrt{n}} r_0^{n-2} \left( \sum_{k=0}^{\infty} \frac{(-1)^k \left[n(n-1)\right]^{\frac{k}{2}} \rho_0 \cos \theta_1}{k!} \right)^k \sin^{n-3} \theta_1 \cdots \sin^{n-3} \theta_{n-3} \prod_{i=1}^{n-2} d\theta_i d\rho.$$

$0 \leq r_0 < \infty.$
where \( D^{**} = \)

\[
\begin{align*}
&n - i + 1 \quad \rho \sin^i \sin_2 \cdots \cos_i \\
&\frac{1}{\sqrt{(n - i) (n - i - 1)}} \\
&\frac{n}{\sqrt{n(n - 1)}} \rho \sin_1 \sin_2 \cdots \sin_{n-3} \cos_{n-2} \\
&\frac{1}{\sqrt{6}} \rho \sin_1 \sin_2 \cdots \sin_{n-3} \sin_{n-2} \leq 0, \quad i = 1, \ldots, n - 1 \\
&n \quad \rho \cos_i - \frac{n - 2}{\sqrt{(n - 1)(n - 2)}} \rho \sin_1 \cos_2 \geq 0 \\
&\frac{n}{\sqrt{n(n - 1)}} \rho \cos_1 + \frac{1}{\sqrt{(n - 1)(n - 2)}} \rho \sin_1 \cos_2 \cdots \\
&\frac{n}{\sqrt{n(n - 1)}} \rho \cos_1 + \frac{1}{\sqrt{(n - 1)(n - 2)}} \rho \sin_1 \cos_2 \cdots \\
&- \frac{1}{\sqrt{(n - i)(n - i + 1)}} \rho \sin_1 \cdots \cos_i \geq 0 \\
&\frac{n}{\sqrt{n(n - 1)}} \rho \cos_1 + \frac{1}{\sqrt{(n - 1)(n - 2)}} \rho \sin_1 \cos_2 \cdots \\
\end{align*}
\]

In section 7.3 Lemma 7.3 we get

\[
\left| \cos^k \sin^{n-3} \sin_1 \cdots \sin_{n-3} \prod_{i=1}^{n-2} d\theta_i \right| \quad D^{**}
\]
\[
\frac{1}{\sqrt{n[n(n-1)]}} \sum_{j=0}^{n-1} \left( \left( 1 + \sum_{i=2}^{n-1} u_i \right)^k - \frac{1}{n} \left( 1 + \sum_{i=2}^{n-1} u_i \right)^2 \right)
\]

Substituting the above result into (7.2.vii) it follows that

\[
f(r_0) = \frac{n(n-1)}{\sqrt{n \pi n}} r_0^{n-2} \sum_{k=0}^{\infty} \frac{(-1)^k r_0^k}{k!} \frac{1}{\pi^{\frac{3}{2}} n^{\frac{1}{2}} n^{-1} \prod_{i=2}^{n-1} u_i^{\frac{1}{2}} n^{-1} \prod_{i=2}^{n-1} \left( 1 + \sum_{i=2}^{n-1} u_i \right)^2 \left( 1 + \sum_{i=2}^{n-1} u_i \right)^2} \]

\[
f(r_0) = (n-1) r_0^{n-2} \prod_{i=2}^{n-1} u_i^{\frac{1}{2}} n^{-1} \prod_{i=2}^{n-1} \left( 1 + \sum_{i=2}^{n-1} u_i \right)^2 \left( 1 + \sum_{i=2}^{n-1} u_i \right)^2
\]

\[
0 \leq u_i \leq 1
\]

\[
i = 2, \ldots, n-1
\]

\[
f(r_0)
\]

\[
\text{can now be rewritten in exponential form:}
\]

\[
f(r_0) = \left( \frac{n-1}{\sqrt{n \pi n}} r_0 \right) r_0^{n-2} \prod_{i=2}^{n-1} u_i^{\frac{1}{2}} n^{-1} \prod_{i=2}^{n-1} \left( 1 + \sum_{i=2}^{n-1} u_i \right)^2 \left( 1 + \sum_{i=2}^{n-1} u_i \right)^2
\]

\[
0 \leq u_i \leq 1
\]

\[
i = 2, \ldots, n-1
\]

\[
0 \leq r_0 < \infty
\]

And

\[
F(r_0) = (n-1) \left( \frac{n-1}{\sqrt{n \pi n}} r_0 \right) r_0^{n-2} \prod_{i=2}^{n-1} u_i^{\frac{1}{2}} n^{-1} \prod_{i=2}^{n-1} \left( 1 + \sum_{i=2}^{n-1} u_i \right)^2 \left( 1 + \sum_{i=2}^{n-1} u_i \right)^2
\]

\[
0 \leq u_i \leq 1
\]

\[
0 \leq r_0 < \infty
\]

\[
i = 2, \ldots, n-1
\]
7.3 **Lemma 7.3.**

\[ J_{n-2}^{**} = \int_{D^{**}} \prod_{i=1}^{n-3} \sin^{n-3} \theta_i \cos^{k} \theta_1 \prod_{i=1}^{n-2} d\theta_i \]

where \( D^{**} \) is (b) and (e) of \( D \) in (7.2.iv).

\[ (7.3.ii) \]

\[ = \frac{1}{\sqrt{n(n-1)}} \frac{1}{\frac{1}{n(n-1)}}^{\frac{1}{2}k} \frac{1}{\frac{1}{n(n-1)}}^{\frac{1}{2}k} \int \left( \frac{1}{(1+ \sum u_i) \prod du_i} \right)_{i=2}^{n-1} \left[ \frac{1}{(1+ \sum u_i^2) - \frac{1}{n}} \prod_{i=2}^{n-1} \frac{1}{(1+ \sum u_i^2) \frac{1}{2}k} \right]_{0}^{u_i < 1} \]

**Proof:**

We will first show that the integral

\[ (7.3.iii) \]

\[ K_{n-2}^* = \frac{-\frac{1}{2}k(n-3+k)}{2} \frac{1}{\frac{1}{2}k} \frac{1}{\frac{1}{2}k} \frac{1}{\Gamma \left[ \frac{1}{2} + (n-i+k) \right]} \]

\[ \int \left( \frac{n}{(1+ \sum x_i) \prod dx_i \prod dx_i} \right)_{i=2}^{n} \left( \frac{n}{(1+ \sum x_i^2) \prod dx_i \prod dx_i} \right)_{i=2}^{n-1} \left( \frac{n}{(1+ \sum x_i^2) - \frac{1}{n}} \prod dx_i \prod dx_i \right)_{i=2}^{n} \]

\[ 0 < x_i < x(n) < \infty \]

\[ i=2, \ldots, n-1 \]

is equal to the integral of (7.3.i) and secondly that it also equals (7.3.ii). Make the transformation (4.5.v) of Lemma 4.5 on \( K_{n-2}^* \), then substituting the results

\[ (4.5.vi) \]

\[ \sum_{i=2}^{n} x_i = [n(n-1)]^{\frac{1}{2}} y_2 = [n(n-1)]^{\frac{1}{2}} \rho \cos \theta_1 \]

and

\[ (4.5.xiv) \]

\[ \sum_{i=2}^{n} x_i^2 - \frac{1}{n} \left( \sum_{i=2}^{n} x_i \right)^2 = \rho^2 \]

into \( K_{n-2}^* \)
and also using the fact that the Jacobian of (4.5.v) is

\[ \sqrt{n^2} \sin^{n-3} \theta_1 \ldots \sin \theta_{n-3} \] we get

(7.3.iv)

\[ K_{n-2} = \frac{-\frac{1}{2} \Gamma \frac{k}{2} \Gamma \left[ \frac{k}{2} (n-1+k) \right]}{n(n-l)} \left[ \rho \cos \theta_1 \right]^k \rho^{n-2} e^{-\frac{\rho^2}{2}} \sin^{n-3} \theta_1 \ldots \sin \theta_{n-3} \]

\[ D \]

\[ \prod_{i=1}^{n-2} d \theta_i \ d \rho \]

where \( D = \)

(a) \[ \frac{n-i+1}{[(n-i)^2]} \rho \sin \theta_1 \sin \theta_2 \ldots \cos \theta_i \]

\[ \frac{1}{[(n-i)(n-i-1)]} \rho \sin \theta_1 \sin \theta_2 \ldots \sin \theta_i \cos \theta_{i+1} \]

\[ \ldots \ldots \ldots \]

\[ + \frac{1}{2} \rho \sin \theta_1 \sin \theta_2 \ldots \sin \theta_{n-3} \cos \theta_{n-2} \]

\[ + \frac{1}{2} \rho \sin \theta_1 \sin \theta_2 \ldots \sin \theta_{n-2} \geq 0 \]

where \( i = 2, \ldots, n-1. \)

(b) \[ \frac{n}{[(n-1)(n-1)]} \rho \cos \theta_1 - \frac{n-2}{[(n-1)(n-2)]} \rho \sin \theta_1 \cos \theta_2 \geq 0 \]

\[ \frac{n}{[(n-1)(n-1)]} \rho \cos \theta_1 + \frac{1}{[(n-1)(n-2)]} \rho \sin \theta_1 \cos \theta_2 \]

\[ - \frac{n-3}{[(n-3)(n-4)]} \rho \sin \theta_1 \sin \theta_2 \cos \theta_3 \geq 0 \]

\[ \ldots \]

\[ \ldots \]
\[
\frac{n}{\sqrt{\frac{1}{2}}} \rho \cos \theta_1 + \ldots + \frac{1}{\sqrt{\frac{1}{2}}} \rho \sin \theta_1 \sin \theta_2 \ldots \sin \theta_{n-2} \geq 0.
\]

(c) \(0 < \rho < \infty\).

Now integrating out \(\rho\) in (7.3.iv) gives

\[
k^* = \frac{\cos \theta^k}{1} \sin^{n-3} \Theta_1 \ldots \sin^{n-3} \Theta_{n-3} \prod_{i=1}^{n-3} d \Theta_i
\]

where \(D_1\) is \(D\) of (7.3.iv) with (c) excluded and is the same as \(D^{**}\) in (7.3.i). Hence the first part of the Lemma 7.3 is proved.

Consider again

\[
k^{*}_{n-2} = \frac{2^{-\frac{n}{2}(n-3+k)} n^{\frac{n}{2}}}{[n(n-1)]^{\frac{1}{2}} \Gamma[\frac{1}{2}(n-1+k)]}
\]

\[
\left( \begin{array}{c}
-\frac{n}{2} \left[ \sum_{i=2}^{n} x_i^2 - \frac{1}{n} \left( \sum_{i=2}^{n} x_i \right)^2 \right] \\
(\sum_{i=2}^{n} x_i k \ e_i \ n-1) \\
\prod_{i=2}^{n-1} dx_i \\
\prod_{i=2}^{n} dx(n)
\end{array} \right)
\]

\(0 < x_i < x(n) < \infty\)

\(i = 2, \ldots, n-1\).

Let \(x_i = u_i x(n)\) \(i = 2, \ldots, n-1; x(n) = x(n)\).

This transformation has Jacobian \(|J| = x^{n-2}\) therefore

\[
k^{*}_{n-2} = \frac{2^{-\frac{n}{2}(n-3+k)} n^{\frac{n}{2}}}{[n(n-1)]^{\frac{1}{2}} \Gamma[\frac{1}{2}(n-1+k)]}
\]

\[
\left( \begin{array}{c}
\frac{n-1}{(1 + \sum_{i=2}^{n} u_i)^k} \\
(\sum_{i=2}^{n} u_i)^k \ e_i \ n-1 \\
\prod_{i=2}^{n-1} dx_i \\
\prod_{i=2}^{n} dx(n)
\end{array} \right)
\]

\(0 < u_i < 1\)

\(i = 2, \ldots, n-1\).
\[- \frac{x^2}{n} \left[ \frac{n-1}{\sum_{i=2}^{n-1} u_i^2} - \frac{1}{n} \frac{n-1}{\sum_{i=2}^{n-1} (1+u_i)^2} \right] \]  
\[ e^{x^{n+k-2} \prod_{i=2}^{n-1} du_i} \]

Let \[
\frac{x^2}{2} \left[ \frac{n-1}{\sum_{i=2}^{n-1} u_i^2} - \frac{1}{n} \frac{n-1}{\sum_{i=2}^{n-1} (1+u_i)^2} \right] = u
\]

then

\[ x(n) \frac{dx(n)}{du} = \frac{n-1}{\sum_{i=2}^{n-1} u_i^2} \frac{n-1}{\sum_{i=2}^{n-1} (1+u_i)^2} \]

Then

\[
K_{n-2}^* = \frac{-\frac{1}{2} (n-3+k) \frac{1}{2}}{2} \frac{1}{\Gamma \left[ \frac{1}{2} (n-1+k) \right]} \frac{1}{\sum_{i=2}^{n-1} u_i^2} \left( \frac{1}{\sum_{i=2}^{n-1} (1+u_i)^2} \right)^{\frac{1}{2} (n+k-1)} e^{-u} (2u)^{\frac{1}{2} (n-3+k)}
\]

\[ 0 < u_i < 1 \]

\[ i=2, \ldots, n-1 \]

\[
= \frac{1}{\sqrt{n} \left[ n(n-1) \right]} \frac{-\frac{1}{2} (n-3+k) \frac{1}{2}}{\Gamma \left[ \frac{1}{2} (n-1+k) \right]} \frac{1}{\sum_{i=2}^{n-1} u_i^2} \left( \frac{1}{\sum_{i=2}^{n-1} (1+u_i)^2} \right)^{\frac{1}{2} (n+k-1)} e^{-u} (2u)^{\frac{1}{2} (n-3+k)}
\]

\[ 0 < u_i < 1 \]

\[ i=2, \ldots, n-1 \]

and the Lemma is proved.
7.4 The Cumulative Distribution and Distribution Functions

For \( S \); When The Underlying Population Is Exponential.

From section 7.3 we found that

\[
(7.4.i) \quad F(r_0) = (n-1)! \left[ \prod_{i=2}^{n-1} \frac{\sum_{j=2}^{n-1} t^{n-2} e^{\frac{n-1}{n}} \left( \frac{2}{\sum_{i=2}^{n-1} (1+ \sum_{i=2}^{n-1} u_i^2) - \frac{1}{n} \left( \sum_{i=2}^{n-1} u_i^2 \right)^2 \right) \right]}{n-1} \right]^{\sum_{i=2}^{n-1} u_i} dt
\]

\( 0 < r_0 < \infty \)

Integrating out \( t \)

\[
(7.4.ii) \quad F(r_0) = (n-1)! \left[ \prod_{i=2}^{n-1} \frac{\sum_{j=2}^{n-1} u_i}{n-1} \right]^{\sum_{i=2}^{n-1} u_i} dt
\]

\( 0 < u_i < 1 \)
(7.4.ii) cont.

\[ - \sum_{k=1}^{n-2} r^{n-2-k} k! (n-2)_k \left[ \sum_{i=2}^{n-1} \frac{(1+ \Sigma u_i^2) - \frac{1}{n}(1+ \Sigma u_i)^2}{i=2} \right] \frac{n-1}{\Pi \partial u_i} \]

\[ (1+ \sum_{i=2}^{n-1} u_i) \]

\[ 0 \leq s < \infty. \]
Since \( r_0 = \sqrt{\frac{r}{n}} \), \( P(r < r_0) = F(r_0) = F(n^{\frac{1}{2}}) = F_1(s) \) hence,

\[
F_1(s) = \frac{1}{(n-1)!} \prod_{i=2}^{n-1} du_i \frac{1}{n-1} \frac{n-1}{n-1} \frac{(1 + \sum u_i)}{i=2} \]

\[0 \leq u_i \leq 1\]

\[i=2, \ldots, n-1\]

\[-(n-1)\left[ \begin{array}{c}
\frac{n-1}{(1 + \sum u_i)^{\frac{1}{2}}}
\frac{1}{i=2}
\frac{n-1}{n-1}
\frac{n-1}{n-1}
\frac{(1 + \sum u_i^2) - \frac{1}{n}}{i=2}
\end{array} \right]^{\frac{1}{2}}\]

\[0 \leq u_i \leq 1\]

\[i=2, \ldots, n-1\]

\[\cdot \{ (\sqrt{n}s)^{n-2} \frac{n-1}{(1 + \sum u_i^2) - \frac{1}{n}} \frac{n-1}{i=2} \frac{n-1}{i=2} \frac{n-1}{i=2} \frac{1}{1 + \sum u_i^2} \}

+ \frac{n-2}{k-2} \prod_{k=2}^{n-1} \frac{1}{k!} (n^{\frac{1}{2}}) \frac{n-1}{k+1} \frac{n-1}{k+1} \frac{k+1}{(1 + \sum u_i^2) - \frac{1}{n}} \frac{n-1}{i=2} \frac{n-1}{i=2} \frac{n-1}{i=2} \]

\[n-1 \prod_{i=2}^{n-1} du_i \]
And

(7.4.iv)

\[ f_1(s) = \sqrt{n}(n-1)/(\sqrt{n}s)^{n-2} \left( e^{-\frac{\sum_{i=2}^{n-1} u_i}{n} \sum_{i=2}^{n-1} \frac{(1+\sum_{i=2}^{n-1} u_i^2) - \frac{1}{n} (1+\sum_{i=2}^{n-1} u_i^2)^2}{2}} \right) \]

\[ 0 \leq u_i \leq 1 \]

\[ i=2, \ldots, n-1 \]

\[ 0 \leq s < \infty. \]

When \( n=2 \).

When \( n=2 \), we get the frequency function of \( S \) for a sample of size 2.

\[ f_1(s) = 2e^{-2s} \]

\[ 0 \leq s < \infty \]

and the cumulative distribution function is

\[ F_1(s) = 1-e^{-2s} \]

\[ 0 \leq s < \infty. \]

When \( n=3 \).

\[ F_1(s) = 1-2 \left( \frac{(3/\sqrt{2})(1+u)s}{(u^2-u+1)^{3/2}} \right) \]

\[ 0 \leq u \leq 1 \]

and

\[ f_1(s) = 6s \left[ e^{-\frac{(1+u)^3}{\frac{1}{3} (1+u)^2}} \right] \]

\[ 0 \leq u \leq 1 \]

\[ 0 < s < \infty. \]
Here $F_1(s)$ and $f_1(s)$ are the distribution and cumulative distribution functions for the standard deviation from an exponential population when the sample size $n = 3$. 
7.5 Method of "Sections".

Let \( x_1, \ldots, x_n \) be a random sample from a cumulative distribution having a distribution function

\[
(7.5.i) \quad f(x) = e^{-x} \quad 0 \leq x < \infty.
\]

Any such random sample is represented by a point in the "orthant" of the sample space for which all coordinates are positive. In this part of the space the density which is the product of the \( n \) values of the distribution for independent variates is \( e^{-\sum x_i} \). This density is constant over a regular \((n-1)\)-dimensional simplex defined by the equation

\[
\Sigma x_i = c.
\]

For obvious reasons we take \( c = n\bar{x} \).

In (7.2.i) we are given the cumulative distribution of \( S \) as

\[
(7.5.ii) \quad P(S \leq s) = F(s) = \frac{1}{D} \int_{D} e^{-\sum_{i=1}^{n} x_i} \prod_{i=1}^{n} dx_i
\]

\[
D = (a) \quad n^{-1} \sum_{i=1}^{n} (x_i - \bar{x})^2 \leq s^2
\]

\[
(b) \quad 0 \leq x_i < \infty.
\]

The domain of integration \( D \) is an infinite cylinder of radius \( r_0 = \sqrt{n}s \) wedged into the vertex of the "orthant" \( 0 \leq x_i < \infty, \quad i = 1, \ldots, n; \) having as its axis of symmetry the equiangular line whose direction cosines are \("(n^{-\frac{1}{2}}, \ldots, n^{-\frac{1}{2}})\"\).

This same plane \( \Sigma x_i = n\bar{x} \) over which the density is constant intersects the \( n \)-dimensional cylinder in an \((n-1)\)-
dimensional sphere. Hence, the density is constant over the common contents of the tetrahedron and sphere.

The various configurations and contents have been discussed, and contents for \( n=2, 3, 4 \) have been found in Chapter 3. We now proceed to make use of those results.

The sections of common content of the flat cylinder and flat cube were found to be lines \( P_1P_2 \) of lengths \( 2y \) when \( 0 \leq y \leq r_0 \) and when \( r_0 \leq y \leq \frac{\sqrt{2}}{2} \) they were of lengths \( 2r_0 \).

Since \( y \) ranges from \( 0 \) to \( \infty \), \( r_0 \leq y \leq \frac{\sqrt{2}}{2} \) will now be \( r_0 \leq y \leq \infty \).

If we now multiply each section of common content by the density \( e^{-\Sigma x_1} = \sqrt{n}y \) and integrate over the regions for which the common content is the same we get
(7.5.iii) 
\[
F(r_0) = \int_0^{2y} e^{-2y} \, dy + \int_{r_0}^\infty 2r_0 e^{-2y} \, dy
\]

\[
= \begin{vmatrix}
0 & -\sqrt{2y} \\
\frac{r_0}{2} & -\sqrt{2y} \\
r_0 & 0
\end{vmatrix}
\]

\[
(-\sqrt{2y}-1)e^{r_0} + \sqrt{2} e^{r_0}
\]

\[
= -\sqrt{2r_0} e^{r_0} - \sqrt{2r_0} e^{r_0} + \sqrt{2r_0} e^{r_0} + 1
\]

\[
F(r_0) = 1 - e^{-r_0} \quad 0 \leq r_0 < \infty.
\]

Since 
\[
r_0 = \sqrt{2s}
\]

(7.5.iv) 
\[
P(S \leq s) = F_1(s) = 1 - e^{-2s} \quad 0 \leq s \leq \infty.
\]

This result is the same for the cumulative distribution of \( S \) as that obtained in (7.3.v).

\[n = 3.\]
From Lemma (3.2.2.i) we get the common contents for the various configurations to be

\[ K_1 = \pi r_0^2 \text{ for } r_0 \leq d \]

\[ K_2 = \pi r_0^2 - 3r_0^2 \cos^{-1}\left(\frac{d}{r_0}\right) + 3d(r_0^2 - d^2)^{\frac{1}{2}} \]

\[ \text{for } d \leq r_0 \leq 2d \]

\[ K_3 = 3\sqrt{3}d^2 \]

\[ \text{for } r_0 \geq 2d. \]

When we replace \( d \) by \( \frac{1}{\sqrt{2}} \) the regions for the same common content may be rewritten

\[ K_1 = \pi r_0^2 \]

\[ \sqrt{2}r_0 \leq y < \infty \]

\[ K_2 = \pi r_0^2 - 3r_0^2 \cos^{-1}\left(\frac{y}{\sqrt{2}r_0}\right) + 3r_0\left(r_0^2 - \left(\frac{y}{\sqrt{2}}\right)^2\right)^{\frac{1}{2}} \]

\[ \frac{\sqrt{2}r_0}{2} \leq y \leq \sqrt{2}r_0 \]

\[ K_3 = \frac{3\sqrt{3}}{2}y^2 \]

\[ 0 \leq y \leq \frac{\sqrt{2}r_0}{2}. \]

Observe since \( 0 \leq x_i < \infty \), \( i=1, 2, 3 \) only the results from Lemma (3.2.2.i) apply. It follows that

\[ F(r_0) = \int_0^{(\sqrt{2}/2)r_0} 3\sqrt{3}y^2 e^{-\sqrt{3}y} \, dy \]

\[ + \int_{\sqrt{2}r_0}^{(\sqrt{2}/2)r_0} \left(\pi r_0^2 - 3r_0^2 \cos^{-1}\left(\frac{y}{\sqrt{2}r_0}\right) + 3r_0\left(r_0^2 - \left(\frac{y}{\sqrt{2}}\right)^2\right)^{\frac{1}{2}}\right) e^{-\sqrt{3}y} \, dy \]

\[ + \int_{(\sqrt{2}/2)r_0}^{\infty} \pi r_0^2 e^{-\sqrt{3}y} \, dy. \]
When \( n = 4 \).

From Lemma (3.2.3.1) we get the common contents for the various configurations to be

\[
K_1 = \frac{4\pi r_0^3}{3}
\]

\[
K_2 = -\frac{4\pi r_0^3}{3} + 4\pi r_0^2d - \frac{4\pi d^3}{3} \quad \frac{r_0}{\sqrt{3}} \leq d \leq r_0
\]

\[
K_3 = -\frac{1\pi r_0^3}{3} - \frac{2\pi (3r_0^2d - d^3)}{3} + 4\sqrt{2}d^2 \left( r_0^2 - 3d^2 \right)^{\frac{1}{2}}
\]

\[
- 2r_0^2 \cos^{-1} \left( \frac{2\sqrt{2}r_0 \left( r_0^2 - 3d^2 \right)^{\frac{1}{2}}}{3 \left( r_0^2 - d^2 \right)} \right) + 2r_0^3 \cos^{-1} \left( \frac{3d^2 + r_0^2}{3r_0^2 - 3d^2} \right)
\]

\[
+ 2 \left( 3r_0^2d - d^3 \right) \cos^{-1} \left( \frac{r_0^2 - 5d^2}{r_0^2 - d^2} \right) \quad \frac{r_0}{3} < d < \frac{r_0}{\sqrt{3}}
\]

\[
K_4 = 8\sqrt{3}d^3 \quad d \leq \frac{r_0}{3}
\]

Because \( 0 \leq x_i \leq \infty, \quad i=1, 2, 3, 4 \) only the results of Lemma (3.2.3.1) apply. The more complex configurations of Lemma (3.2.3.2) do not occur. Substituting \( d = \frac{1}{\sqrt{3}} \) into \( K_1, K_2, K_3 \) and \( K_4 \) we can now write the cumulative distribution function \( r_0 \), where \( r_0 = 2s \).

\[
F(r_0) = \begin{cases} 
\frac{1}{\sqrt{3}}r_0 \\
8 \left\{ e^{-2y} y^3 dy \right\}_0^{r_0} \\
+ \frac{\pi}{3} \left\{ e^{-2y} \left( -r_0^3 - 2\sqrt{3}r_0^2y + \frac{2}{3\sqrt{3}}y^3 \right) dy \right\}_{\frac{1}{\sqrt{3}}r_0}^{r_0}
\end{cases}
\]
\[
\begin{align*}
&+ \frac{4\sqrt{2}}{3} \int_{\sqrt{3}r_0}^{r_0} e^{-2y} (r_0^2 - y^2)^{1/4} \, dy \\
&+ \frac{1r}{\sqrt{3}} 0 \\
&- 2r_0 \int_{\sqrt{3}r_0}^{r_0} e^{-2y} \cos \frac{2\sqrt{2}r_0(r_0^2 - y^2)^{1/4}}{(3r_0^2 - y^2)} \, dy \\
&+ \frac{1r}{\sqrt{3}} 0 \\
&+ \int_{\sqrt{3}r_0}^{r_0} e^{-2y} \cos \frac{r_0^2 + y^2}{3r_0^2 - y^2} \, dy \\
&+ \frac{1r}{\sqrt{3}} 0 \\
&- 2y \int_{\sqrt{3}r_0}^{r_0} e^{-2y} 2(\sqrt{3r_0^2 - y^3} \left[ \frac{1}{3} \right] (3r_0^2 - 5y^2) \, dy \\
&+ \frac{1r}{\sqrt{3}} 0 \\
&+ \int_{\sqrt{3}r_0}^{r_0} e^{-2y} (4\pi r_0^3 + \frac{4\pi r_0^2 y - 4\pi y^3}{9\sqrt{3}}) \, dy \\
&+ \frac{1r}{\sqrt{3}} 0 \\
&+ \int_{\sqrt{3}r_0}^{r_0} e^{-2y} \pi r_0^3 \, dy. & 0 \leq r_0 < \infty \\
&+ \int_{\sqrt{3}r_0}^{r_0} \, dy.
\end{align*}
\]
7.6 Spherical Approximations and Exact Lower and Upper Bounds For $P(S < s)$; $0 \leq s < \infty$, the Underlying Distribution Being Exponential.

The cumulative distribution function for the sample standard deviation $s$ for samples of size $n$ drawn from an exponential population is given by

\[(7.6.i)\]

\[
F_{1}(r_{0}) = (n-1)! \left\{ \prod_{i=2}^{n-1} \frac{du_{i}}{n-1} \right\} \frac{1}{n-1} \frac{n-1}{n-1} \\
\frac{[(1+ \Sigma u_{i})]^{n-1}}{[(1+ \Sigma u_{i})]^{n-1}} \\
0 \leq u_{i} \leq 1, i=2, \ldots, n-1
\]

\[-(n-1)\left[ \frac{\sum_{i=2}^{n-1} u_{i}}{n} \right]^{n-1} \frac{e}{[(1+ \Sigma u_{i})^{2} - \frac{1}{n} (1+ \Sigma u_{i})^{2}]^{2}} \]

\[
0 \leq u_{i} \leq 1, i=2, \ldots, n-1
\]

\[
\left\{ (\sqrt{n} s)^{n-2} \frac{[(1+ \Sigma u_{i})^{2} - \frac{1}{n} (1+ \Sigma u_{i})^{2}]^{k}}{i=2} \right\} \frac{n-1}{i=2} \\
\frac{[(1+ \Sigma u_{i})^{2} - \frac{1}{n} (1+ \Sigma u_{i})^{2}]^{k+1}}{i=2} \\
\frac{n-1}{k+1} \\
(1+ \Sigma u_{i})^{k+1}
\]

\[
0 \leq u_{i} \leq 1, i=2
\]

\[
\prod_{i=2}^{n-1} du_{i}
\]
Make the transformation \( u_i = v_i + \frac{1}{2} \), \( i = 2, \ldots, n-1 \). The Jacobian \( |J| = 1 \). The transformation shifts the origin to the centre of the cube which is the domain of this integral, and so

\[
(7.6.ii) \quad F_1(s) = 1 - (n-1) \left( \frac{(n + \sum v_i)\sqrt{ns}}{2} \right) \left( \frac{n-1}{2} \right) \left( \frac{(\sum v_i^2 + \sum v_i + n+2) - \frac{1}{n} (\sum v_i + n)^2}{4} \right) \frac{n-1}{2} \]

\[
-\frac{1}{2} \leq v_i \leq \frac{1}{2}
\]

\[
(n\sqrt{ns})^{n-2} \cdot \left( \frac{n-1}{2} \right) \left( \frac{(\sum v_i^2 + \sum v_i + n+2) - \frac{1}{n} (\sum v_i + n)^2}{4} \right) \frac{n-1}{2}
\]

\[
\sum_{k=1}^{n-2} k! (n-2)^{n-2-k} (\sqrt{ns})^{n-2-k} \frac{n-1}{2} \left( \frac{(\sum v_i^2 + \sum v_i + n+2) - \frac{1}{n} (\sum v_i + n)^2}{4} \right) \frac{n-1}{2}
\]

\[
\prod_{i=2}^{n-1} dv_i.
\]

Following the procedure of Chapter 6 we approximate the domain of the integral by an \((n-2)\)-dimensional sphere of unit volume. The equation of the \((n-2)\)-dimensional sphere or radius \( r \) is

\[
\sum_{i=2}^{n-1} v_i^2 = r^2,
\]

\( r \) being chosen so that the sphere has unit volume. We find
therefore that
\[ r = \frac{\Gamma(\frac{1}{2}n)}{\sqrt{n}} \left( \frac{1}{n-2} \right). \]

Now \( F_1(s) \) is taken over the approximate domain
\[ \sum_{i=2}^{n-1} v_i^2 \leq r^2. \]
We make the orthogonal transformation of (6.2.4) followed by the polar transformation of that section. \( F_1(s) \) can then be written as

(7.6.iii)
\[
\int_0^{r^*} \frac{e^{\frac{n-1}{2}\left(\frac{1}{n}(1-\frac{n-2}{n}\cos^2\theta_1)\rho^2 + \frac{1}{2}\right)}}{\sqrt{n}s} d\theta_1 d\rho.
\]

\[
\frac{2\pi}{\Gamma(\frac{1}{2}(n-3))} \cdot \gamma \left(\sqrt{n}s\right)^n \cdot \frac{\left(1 - \frac{n-2}{n}\cos^2\theta_1\right)^{\frac{1}{2}}}{\sqrt{n}} \cdot \frac{\left[\frac{n}{2} - \frac{n-2}{2}\cos^2\theta_1\right]}{\left[\frac{n}{2} - \frac{n-2}{2}\cos^2\theta_1\right]^\frac{1}{2}}
\]

\[
\sum_{k=1}^{n-2} \frac{\left(\sqrt{n}s\right)^{n-2-k} k!(n-2)}{\Gamma(\frac{1}{2}(n-3))} \cdot \left(\frac{1 - \frac{n-2}{n}\cos^2\theta_1\right)^{\frac{1}{2}}}{\sqrt{n}} \cdot \frac{\left[\frac{n}{2} - \frac{n-2}{2}\cos^2\theta_1\right]}{\left[\frac{n}{2} - \frac{n-2}{2}\cos^2\theta_1\right]^\frac{1}{2}} \cdot \gamma d\theta_1 d\rho.
\]

By using this new approximate expression for \( F_1(s) \) we may find the exact lower and upper bounds for \( F_1(s) \). When \( r^* = \frac{1}{2} (n-2)^{\frac{1}{2}} \) we get an expression for the exact lower bound of \( F_1(s) \). Similarly, when \( r^* = \frac{1}{2} \) we get an expression for the exact upper bound of \( F_1(s) \).
CHAPTER 8
NUMERICAL RESULTS.

8.1 Introduction.

In this chapter we first present tables containing numerical values for the distribution and density functions of the $S$ statistic when the random samples are drawn from a continuous rectangular population. The values of $S$ are taken for $0 < s < (2n)^{-\frac{1}{2}}$ where $n$ is the sample size. When $n=2, 3, 4, 5$ the integrals of the distribution and density functions were evaluated by standard analytical techniques. For $n=6, 7$ the integrals were evaluated by numerical quadrature and by making use of a spherical approximation.

Graphs of the cumulative distribution function and the frequency functions immediately follow their respective tables. For the cases $n=6$ and $7$ we do not graphically present the results of the tables obtained by spherical approximation. It will be noted that for left tail values the approximate results are quite close to the exact results.

Secondly we present tables and graphs of the density functions of the $S$ statistic when random samples are drawn from a negative exponential population for sample sizes 2, 3, 4, 5 and 6.

Thirdly we compare the sampling distribution of the statistic $\frac{ns^2}{\sigma^2}$ appropriate for a normal population with the sampling distribution appropriate for a uniform population when $0 < \frac{ns^2}{\sigma^2} < 6$, and its sampling distribution appropriate for a
negative exponential population \( 0 \leq \frac{s^2}{\sigma^2} < \infty, \quad \sigma^2 \) taken

in the first comparison to be the \( \sigma^2 \) of the uniform population and in the second comparison to be the \( \sigma^2 \) of the negative exponential. Graphs also follow each set of tables.
8.2 Methods Adopted For Numerical Integration.

To compute the multiple integral in (4.6.i) given by

\[(8.2.i)\]

\[
\int \frac{\prod u_i \, du_i}{\left( (1+ \sum u_i^2) - \frac{1}{n} (1+ \sum u_i)^2 \right)^{\frac{1}{2n}}} \\
0 < u_i < 1 \quad i=2, \ldots, n-1
\]

and the multiple integral for the density function of the S statistic for samples from an exponential population in (7.4.iv) given by

\[(8.2.ii)\]

\[
f_1(s) = \sqrt{n} (n-1) (\sqrt{n} s)^{n-2} e^{-\frac{n-1}{n} \left( (1+ \sum u_i^2) - \frac{1}{n} (1+ \sum u_i)^2 \right)^{\frac{1}{2n}}} \\
0 \leq u_i \leq 1 \quad i=2 \ldots n \\quad 0 \leq s < \infty
\]

the Cartesian product of the explicit, one dimensional, 9-point Legendre-Gauss quadrature formula has been used. The computer program, ALGORITHM 32 MULTINT [14] copied from the Communications of the Association For Computing Machinery, (and translated into Fortran IV), transformed the ranges of integration from (0, 1) to (-1, 1).

The degree of precision of the 9-point Legendre-Gauss quadrature for evaluation of a single integral is 17, i.e., it gives exact results for an arbitrary polynomial of degree 17.
or less. For polynomials of degree higher than 17 the truncation error is

\[ E = \frac{2^{19}}{(19)!} \left[ \frac{(9!)^2}{(18)!} \right]^2 f^{(18)}(\eta) \]

where \( \eta \) is a point in the segment \([-1, +1]\).

The 18th derivative of the function is extremely difficult to find and thus it is difficult to get a precise estimate even for single integrals. In repeated quadrature for evaluating multiple integrals it is even more difficult to get an estimate of the error. The approach used under these circumstances was to evaluate these integrals by using a different number of points in the Legendre-Gaussian product formulas. The multiple integrals were evaluated by using 5-point and 9-point formulas. Using 5-point and 9-point formulas gave the same result up to 5 decimal places. It follows therefore, that the truncation error does not affect values up to 5 decimal places as long as the polynomial approximating the function is of degree 17. It follows that the 9-point formula used here for evaluating the multiple integrals gives values accurate up to 5 decimal places. See page 8-35.

Tables for the cumulative distribution of \( ns^2/\sigma^2 \), where \( \chi_0 = ns^2/\sigma^2 \), when the underlying population is continuous rectangular were computed from (4.6.ii) after making the transformation \( \chi > ns^2/s^2 \), where \( \sigma^2 = 1/12 \). Once the multiple integral of

\[ F(\chi_0) = \frac{\frac{1}{2}(n-1) \frac{1}{2} 2^{ \frac{n-1}{2} } }{ (n-1) \Gamma \left( \frac{1}{2}(n-1) \right) } \]

...
(8.2.iii) cont.

\[
(n-1) \left( \sigma^2 \chi_0 \right)^{n/2} \prod_{i=2}^{n-1} \frac{n-1}{n-1} \frac{2^{2i}}{(l+\Sigma u_i)^2} \left[ \left( \frac{1}{n} \Sigma u_i \right) - \frac{1}{n} \left( l+\Sigma u_i \right) \right]^{2} \\
0 < u_i < 1 \quad \quad i=2, \ldots, n-1 \\
0 < \chi_0 < 6,
\]

namely, (8.2.i) had been evaluated, the values for \( F(\chi_0) \) were readily computed. \( \chi_0 \) was chosen so that \( F(\chi_0) \) could be compared with tables for the already existing chi-square distribution.

In (8.2.ii) we make the transformation \( \chi_0 = \frac{n s^2}{\sigma^2} \) and attempt to compare the density

(8.2.iv)

\[
f(\chi_0) = \frac{(n-1)}{2} \prod_{i=2}^{n-1} \frac{n-1}{n-1} \frac{2^{2i}}{(l+\Sigma u_i)^2} \left[ \left( \frac{1}{n} \Sigma u_i \right) - \frac{1}{n} \left( l+\Sigma u_i \right) \right]^{2} \\
0 < u_i < 1 \quad \quad i=2, \ldots, n-1
\]

with the corresponding chi-square density

(8.2.v)

\[
f(\chi_0) = \frac{1}{\Gamma\left(\frac{\chi_0}{2}(n-1)\right)} \left( \frac{n-3}{2} \right)^{\frac{\chi_0}{2}} \chi_0^{\frac{\chi_0}{2}} e^{-\frac{\chi_0}{2}}
\]
To compute the double integrals arising from the use of the spherical approximation in Chapter 6, the Romberg quadrature technique was employed.

"The method is basically iterative in nature. Two dimensional trapezoidal integration of the function is made with 1, 2, 4,... subintervals (of equal length \( \Delta x \)) in the \( x \) direction and the same number of subintervals (of equal length \( \Delta y \)) in the \( y \) direction. From these approximants higher order approximants are calculated recursively using a Richardson-type extrapolation.

In particular let \( T^{(i)}_0 \) be the \( i \)th trapezoidal approximant. The higher order approximants are obtained by means of the recursion formula

\[
T^{(i-1)}_m = \frac{D * T^{(i)}_m - T^{(i-1)}_m}{m-1} \quad \text{where} \quad D = 2^{2m} \quad \text{and} \quad 1 \leq m \leq i. \quad [31]
\]

Convergence was determined by the difference between successive approximations of the integral. The computer program used to evaluate these integrals was taken from an Argonne National Laboratory publication. [31]
8.3 TABLES AND GRAPHS.
SAMPLE SIZE \( n = 2 \).

\( 0 \leq s \leq \frac{1}{2} \)

TABLES OF THE DISTRIBUTION AND DENSITY FUNCTIONS OF

THE \( S \) STATISTIC WHEN THE RANDOM SAMPLES ARE

DRAWN FROM A CONTINUOUS

RECTANGULAR POPULATION.

<table>
<thead>
<tr>
<th>s</th>
<th>( P(S \leq s) )</th>
<th>( f(s) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00000</td>
<td>0.00000</td>
<td>0.00000</td>
</tr>
<tr>
<td>0.01250</td>
<td>0.04937</td>
<td>3.90000</td>
</tr>
<tr>
<td>0.02500</td>
<td>0.09750</td>
<td>3.80000</td>
</tr>
<tr>
<td>0.03750</td>
<td>0.14437</td>
<td>3.70000</td>
</tr>
<tr>
<td>0.05000</td>
<td>0.19000</td>
<td>3.60000</td>
</tr>
<tr>
<td>0.06250</td>
<td>0.23438</td>
<td>3.50000</td>
</tr>
<tr>
<td>0.07500</td>
<td>0.27750</td>
<td>3.40000</td>
</tr>
<tr>
<td>0.08750</td>
<td>0.31937</td>
<td>3.30000</td>
</tr>
<tr>
<td>0.10000</td>
<td>0.36000</td>
<td>3.20000</td>
</tr>
<tr>
<td>0.11250</td>
<td>0.39938</td>
<td>3.10000</td>
</tr>
<tr>
<td>0.12500</td>
<td>0.43750</td>
<td>3.00000</td>
</tr>
<tr>
<td>0.13750</td>
<td>0.47438</td>
<td>2.90000</td>
</tr>
<tr>
<td>0.15000</td>
<td>0.51000</td>
<td>2.80000</td>
</tr>
<tr>
<td>0.16250</td>
<td>0.54438</td>
<td>2.70000</td>
</tr>
<tr>
<td>0.17500</td>
<td>0.57750</td>
<td>2.60000</td>
</tr>
<tr>
<td>0.18750</td>
<td>0.60938</td>
<td>2.50000</td>
</tr>
<tr>
<td>0.20000</td>
<td>0.64000</td>
<td>2.40000</td>
</tr>
<tr>
<td>0.21250</td>
<td>0.66938</td>
<td>2.30000</td>
</tr>
<tr>
<td>0.22500</td>
<td>0.69750</td>
<td>2.20000</td>
</tr>
<tr>
<td>0.23750</td>
<td>0.72438</td>
<td>2.10000</td>
</tr>
<tr>
<td>0.25000</td>
<td>0.75000</td>
<td>2.00000</td>
</tr>
<tr>
<td>0.26250</td>
<td>0.77437</td>
<td>1.90000</td>
</tr>
<tr>
<td>0.27500</td>
<td>0.79750</td>
<td>1.80000</td>
</tr>
<tr>
<td>0.28750</td>
<td>0.81937</td>
<td>1.70000</td>
</tr>
<tr>
<td>0.30000</td>
<td>0.84000</td>
<td>1.60000</td>
</tr>
<tr>
<td>0.31250</td>
<td>0.85938</td>
<td>1.50000</td>
</tr>
<tr>
<td>0.32500</td>
<td>0.87750</td>
<td>1.40000</td>
</tr>
<tr>
<td>0.33750</td>
<td>0.89437</td>
<td>1.30000</td>
</tr>
<tr>
<td>0.35000</td>
<td>0.91000</td>
<td>1.20000</td>
</tr>
<tr>
<td>0.36250</td>
<td>0.92437</td>
<td>1.10000</td>
</tr>
<tr>
<td>0.37500</td>
<td>0.93750</td>
<td>1.00000</td>
</tr>
<tr>
<td>0.38750</td>
<td>0.94937</td>
<td>0.90000</td>
</tr>
<tr>
<td>0.40000</td>
<td>0.96000</td>
<td>0.80000</td>
</tr>
<tr>
<td>0.41250</td>
<td>0.96937</td>
<td>0.70000</td>
</tr>
<tr>
<td>0.42500</td>
<td>0.97750</td>
<td>0.60000</td>
</tr>
<tr>
<td>0.43750</td>
<td>0.98438</td>
<td>0.50000</td>
</tr>
<tr>
<td>0.45000</td>
<td>0.99000</td>
<td>0.40000</td>
</tr>
<tr>
<td>0.46250</td>
<td>0.99437</td>
<td>0.30000</td>
</tr>
<tr>
<td>0.47500</td>
<td>0.99750</td>
<td>0.20000</td>
</tr>
<tr>
<td>0.48750</td>
<td>0.99937</td>
<td>0.10000</td>
</tr>
<tr>
<td>0.50000</td>
<td>1.00000</td>
<td>0.00000</td>
</tr>
</tbody>
</table>
SAMPLE SIZE \( n = 2 \).

\[ P(S \leq s) \quad 0 \leq s \leq \frac{1}{2} \]
SAMPLE SIZE $n = 2$.

$f(s) \quad 0 \leq s \leq \frac{1}{2}$
SAMPLE SIZE  \( n = 3 \).

\( 0 \leq s \leq 6^{-\frac{1}{2}} \)

TABLES OF THE DISTRIBUTION AND DENSITY FUNCTIONS OF THE S STATISTIC WHEN THE RANDOM SAMPLES ARE DRAWN FROM A CONTINUOUS RECTANGULAR POPULATION.

<table>
<thead>
<tr>
<th>s</th>
<th>P(S ( \leq ) s)</th>
<th>f(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00000</td>
<td>0.00000</td>
<td>0.00000</td>
</tr>
<tr>
<td>0.01250</td>
<td>0.00250</td>
<td>0.39622</td>
</tr>
<tr>
<td>0.02500</td>
<td>0.00981</td>
<td>0.76858</td>
</tr>
<tr>
<td>0.03750</td>
<td>0.02162</td>
<td>1.11708</td>
</tr>
<tr>
<td>0.05000</td>
<td>0.03763</td>
<td>1.44171</td>
</tr>
<tr>
<td>0.06250</td>
<td>0.05756</td>
<td>1.74247</td>
</tr>
<tr>
<td>0.07500</td>
<td>0.08110</td>
<td>2.01937</td>
</tr>
<tr>
<td>0.08750</td>
<td>0.10794</td>
<td>2.27241</td>
</tr>
<tr>
<td>0.10000</td>
<td>0.13781</td>
<td>2.50158</td>
</tr>
<tr>
<td>0.11250</td>
<td>0.17038</td>
<td>2.70689</td>
</tr>
<tr>
<td>0.12500</td>
<td>0.20538</td>
<td>2.88833</td>
</tr>
<tr>
<td>0.13750</td>
<td>0.24249</td>
<td>3.04591</td>
</tr>
<tr>
<td>0.15000</td>
<td>0.28143</td>
<td>3.17962</td>
</tr>
<tr>
<td>0.16250</td>
<td>0.32188</td>
<td>3.28946</td>
</tr>
<tr>
<td>0.17500</td>
<td>0.36357</td>
<td>3.37545</td>
</tr>
<tr>
<td>0.18750</td>
<td>0.40617</td>
<td>3.43756</td>
</tr>
<tr>
<td>0.20000</td>
<td>0.44940</td>
<td>3.47581</td>
</tr>
<tr>
<td>0.21250</td>
<td>0.49297</td>
<td>3.49020</td>
</tr>
<tr>
<td>0.22500</td>
<td>0.53656</td>
<td>3.48072</td>
</tr>
<tr>
<td>0.23750</td>
<td>0.57989</td>
<td>3.44738</td>
</tr>
<tr>
<td>0.25000</td>
<td>0.62265</td>
<td>3.39017</td>
</tr>
<tr>
<td>0.26250</td>
<td>0.66454</td>
<td>3.30910</td>
</tr>
<tr>
<td>0.27500</td>
<td>0.70527</td>
<td>3.20417</td>
</tr>
<tr>
<td>0.28750</td>
<td>0.74454</td>
<td>3.07537</td>
</tr>
<tr>
<td>0.30000</td>
<td>0.78206</td>
<td>2.92270</td>
</tr>
<tr>
<td>0.31250</td>
<td>0.81751</td>
<td>2.74616</td>
</tr>
<tr>
<td>0.32500</td>
<td>0.85061</td>
<td>2.54577</td>
</tr>
<tr>
<td>0.33750</td>
<td>0.88106</td>
<td>2.32151</td>
</tr>
<tr>
<td>0.35000</td>
<td>0.90855</td>
<td>2.07338</td>
</tr>
<tr>
<td>0.36250</td>
<td>0.93279</td>
<td>1.80139</td>
</tr>
<tr>
<td>0.37500</td>
<td>0.95349</td>
<td>1.50553</td>
</tr>
<tr>
<td>0.38750</td>
<td>0.97033</td>
<td>1.18581</td>
</tr>
<tr>
<td>0.40000</td>
<td>0.98303</td>
<td>0.84223</td>
</tr>
<tr>
<td>0.40825</td>
<td>0.98900</td>
<td>0.60239</td>
</tr>
</tbody>
</table>
SAMPLE SIZE \( n = 3 \).

\[
f(s) \quad 0 \leq s \leq 6^{-\frac{1}{2}}
\]
SAMPLE SIZE $n = 3$. 

$P(S < s) \ 0 \leq s \leq 6^{-\frac{1}{2}}$
SAMPLE SIZE  \( n = 4 \).

\( 0 \leq s \leq 8^{-\frac{1}{2}} \)

TABLES OF THE DISTRIBUTION AND DENSITY FUNCTIONS OF

THE S STATISTIC WHEN THE RANDOM SAMPLES ARE

DRAWN FROM A CONTINUOUS

RECTANGULAR POPULATION.

<table>
<thead>
<tr>
<th>s</th>
<th>P(S ≤ s)</th>
<th>f(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00000</td>
<td>0.00000</td>
<td>0.00000</td>
</tr>
<tr>
<td>0.01250</td>
<td>0.00013</td>
<td>0.03041</td>
</tr>
<tr>
<td>0.02500</td>
<td>0.00100</td>
<td>0.11757</td>
</tr>
<tr>
<td>0.03750</td>
<td>0.00328</td>
<td>0.25542</td>
</tr>
<tr>
<td>0.05000</td>
<td>0.00757</td>
<td>0.43787</td>
</tr>
<tr>
<td>0.06250</td>
<td>0.01438</td>
<td>0.65884</td>
</tr>
<tr>
<td>0.07500</td>
<td>0.02417</td>
<td>0.91225</td>
</tr>
<tr>
<td>0.08750</td>
<td>0.03730</td>
<td>1.19203</td>
</tr>
<tr>
<td>0.10000</td>
<td>0.05405</td>
<td>1.49209</td>
</tr>
<tr>
<td>0.11250</td>
<td>0.07465</td>
<td>1.80635</td>
</tr>
<tr>
<td>0.12500</td>
<td>0.09924</td>
<td>2.12873</td>
</tr>
<tr>
<td>0.13750</td>
<td>0.12787</td>
<td>2.45316</td>
</tr>
<tr>
<td>0.15000</td>
<td>0.16054</td>
<td>2.77356</td>
</tr>
<tr>
<td>0.16250</td>
<td>0.19715</td>
<td>3.08383</td>
</tr>
<tr>
<td>0.17500</td>
<td>0.23755</td>
<td>3.37792</td>
</tr>
<tr>
<td>0.18750</td>
<td>0.28149</td>
<td>3.64973</td>
</tr>
<tr>
<td>0.20000</td>
<td>0.32865</td>
<td>3.89318</td>
</tr>
<tr>
<td>0.21250</td>
<td>0.37865</td>
<td>4.10220</td>
</tr>
<tr>
<td>0.22500</td>
<td>0.43101</td>
<td>4.27071</td>
</tr>
<tr>
<td>0.23750</td>
<td>0.48519</td>
<td>4.39263</td>
</tr>
<tr>
<td>0.25000</td>
<td>0.54057</td>
<td>4.46187</td>
</tr>
<tr>
<td>0.26250</td>
<td>0.59645</td>
<td>4.47235</td>
</tr>
<tr>
<td>0.27500</td>
<td>0.65206</td>
<td>4.41801</td>
</tr>
<tr>
<td>0.28750</td>
<td>0.70656</td>
<td>4.29276</td>
</tr>
<tr>
<td>0.30000</td>
<td>0.75901</td>
<td>4.09052</td>
</tr>
<tr>
<td>0.31250</td>
<td>0.80842</td>
<td>3.80522</td>
</tr>
<tr>
<td>0.32500</td>
<td>0.85370</td>
<td>3.43076</td>
</tr>
<tr>
<td>0.33750</td>
<td>0.89372</td>
<td>2.96107</td>
</tr>
<tr>
<td>0.35000</td>
<td>0.92723</td>
<td>2.39006</td>
</tr>
<tr>
<td>0.35353</td>
<td>0.93535</td>
<td>2.20963</td>
</tr>
<tr>
<td>0.36250</td>
<td>0.95295</td>
<td>1.71167</td>
</tr>
</tbody>
</table>
SAMPLE SIZE  n = 4.

\[ P(S \leq s) \quad 0 \leq s \leq 8^{-\frac{1}{n}} \]
SAMPLE SIZE $n = 4$.

$f(s) \ 0 \leq s \leq 8^{-\frac{1}{2}}$
SAMPLE SIZE $n = 5$.

$0 \leq s \leq 10^{-4}$

TABLES OF THE DISTRIBUTION AND DENSITY FUNCTIONS OF THE $S$ STATISTIC WHEN THE RANDOM SAMPLES ARE DRAWN FROM A CONTINUOUS RECTANGULAR POPULATION.

<table>
<thead>
<tr>
<th>$s$</th>
<th>$P(S \leq s)$</th>
<th>$f(s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00000</td>
<td>0.00000</td>
<td>0.00000</td>
</tr>
<tr>
<td>0.01250</td>
<td>0.00001</td>
<td>0.00208</td>
</tr>
<tr>
<td>0.02500</td>
<td>0.00010</td>
<td>0.01650</td>
</tr>
<tr>
<td>0.03750</td>
<td>0.00050</td>
<td>0.05217</td>
</tr>
<tr>
<td>0.05000</td>
<td>0.00153</td>
<td>0.11889</td>
</tr>
<tr>
<td>0.06250</td>
<td>0.00363</td>
<td>0.22289</td>
</tr>
<tr>
<td>0.07500</td>
<td>0.00728</td>
<td>0.36905</td>
</tr>
<tr>
<td>0.08750</td>
<td>0.01304</td>
<td>0.56048</td>
</tr>
<tr>
<td>0.10000</td>
<td>0.02149</td>
<td>0.79847</td>
</tr>
<tr>
<td>0.11250</td>
<td>0.03320</td>
<td>1.08256</td>
</tr>
<tr>
<td>0.12500</td>
<td>0.04873</td>
<td>1.41046</td>
</tr>
<tr>
<td>0.13750</td>
<td>0.06862</td>
<td>1.77813</td>
</tr>
<tr>
<td>0.15000</td>
<td>0.09333</td>
<td>2.17971</td>
</tr>
<tr>
<td>0.16250</td>
<td>0.12323</td>
<td>2.60757</td>
</tr>
<tr>
<td>0.17500</td>
<td>0.15859</td>
<td>3.05229</td>
</tr>
<tr>
<td>0.18750</td>
<td>0.19956</td>
<td>3.50265</td>
</tr>
<tr>
<td>0.20000</td>
<td>0.24613</td>
<td>3.94565</td>
</tr>
<tr>
<td>0.21250</td>
<td>0.29811</td>
<td>4.36651</td>
</tr>
<tr>
<td>0.22360</td>
<td>0.34851</td>
<td>4.70832</td>
</tr>
<tr>
<td>0.22500</td>
<td>0.35513</td>
<td>4.74864</td>
</tr>
<tr>
<td>0.23750</td>
<td>0.41659</td>
<td>5.07367</td>
</tr>
<tr>
<td>0.25000</td>
<td>0.48165</td>
<td>5.32144</td>
</tr>
<tr>
<td>0.26250</td>
<td>0.54921</td>
<td>5.47005</td>
</tr>
<tr>
<td>0.26726</td>
<td>0.57531</td>
<td>5.49566</td>
</tr>
<tr>
<td>0.27500</td>
<td>0.61789</td>
<td>5.49571</td>
</tr>
<tr>
<td>0.28750</td>
<td>0.68598</td>
<td>5.37291</td>
</tr>
<tr>
<td>0.28867</td>
<td>0.69226</td>
<td>5.35287</td>
</tr>
<tr>
<td>0.30000</td>
<td>0.75148</td>
<td>5.07437</td>
</tr>
<tr>
<td>0.31250</td>
<td>0.81199</td>
<td>4.57095</td>
</tr>
<tr>
<td>0.31622</td>
<td>0.82864</td>
<td>4.37708</td>
</tr>
</tbody>
</table>
SAMPLE SIZE $n = 5$.

$P(S \leq s) \quad 0 \leq s \leq 10^{-\frac{1}{2}}$
SAMPLE SIZE $n = 5$.

$f(s) \quad 0 \leq s \leq 10^{-\frac{1}{2}}$
SAMPLE SIZE  $n = 6$.

$0 \leq s \leq 12^{-\frac{1}{2}}$

TABLES OF THE DISTRIBUTION AND DENSITY FUNCTIONS OF THE S STATISTIC
WHEN THE RANDOM SAMPLES ARE DRAWN FROM A CONTINUOUS
RECTANGULAR POPULATION.

<table>
<thead>
<tr>
<th>s</th>
<th>$P(S&lt;s)$ Obtained By Numerical Quadrature.</th>
<th>$P(S&lt;s)$ Obtained By Spherical Approximation.</th>
<th>$f(s)$ Obtained By Numerical Quadrature.</th>
<th>$f(s)$ Obtained By Spherical Approximation.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00000</td>
<td>0.00000</td>
<td>0.00000</td>
<td>0.00000</td>
<td>0.00000</td>
</tr>
<tr>
<td>0.01250</td>
<td>0.00000</td>
<td>0.00000</td>
<td>0.00001</td>
<td>0.00013</td>
</tr>
<tr>
<td>0.02500</td>
<td>0.00000</td>
<td>0.00008</td>
<td>0.00081</td>
<td>0.00206</td>
</tr>
<tr>
<td>0.03750</td>
<td>0.00000</td>
<td>0.00031</td>
<td>0.00093</td>
<td>0.01001</td>
</tr>
<tr>
<td>0.05000</td>
<td>0.00000</td>
<td>0.00091</td>
<td>0.00793</td>
<td>0.03035</td>
</tr>
<tr>
<td>0.06250</td>
<td>0.00000</td>
<td>0.00218</td>
<td>0.01405</td>
<td>0.07019</td>
</tr>
<tr>
<td>0.07500</td>
<td>0.00221</td>
<td>0.00453</td>
<td>0.14052</td>
<td>0.13869</td>
</tr>
<tr>
<td>0.08750</td>
<td>0.00459</td>
<td>0.00848</td>
<td>0.24818</td>
<td>0.24422</td>
</tr>
<tr>
<td>0.10000</td>
<td>0.00861</td>
<td>0.01462</td>
<td>0.40267</td>
<td>0.39493</td>
</tr>
<tr>
<td>0.11250</td>
<td>0.01489</td>
<td>0.02366</td>
<td>0.51179</td>
<td>0.59786</td>
</tr>
<tr>
<td>0.12500</td>
<td>0.02416</td>
<td>0.03633</td>
<td>0.88186</td>
<td>0.85826</td>
</tr>
<tr>
<td>0.13750</td>
<td>0.03720</td>
<td>0.05339</td>
<td>1.21704</td>
<td>1.17904</td>
</tr>
<tr>
<td>0.15000</td>
<td>0.05486</td>
<td>0.07557</td>
<td>1.61875</td>
<td>1.56064</td>
</tr>
<tr>
<td>0.16250</td>
<td>0.07794</td>
<td>0.10353</td>
<td>2.08504</td>
<td>1.99747</td>
</tr>
<tr>
<td>0.17500</td>
<td>0.10723</td>
<td>0.13780</td>
<td>2.61012</td>
<td>2.48322</td>
</tr>
<tr>
<td>0.18750</td>
<td>0.14340</td>
<td>0.17871</td>
<td>3.18345</td>
<td>3.00428</td>
</tr>
<tr>
<td>0.20000</td>
<td>0.18695</td>
<td>0.22632</td>
<td>3.78945</td>
<td>3.54205</td>
</tr>
<tr>
<td>0.21250</td>
<td>0.23818</td>
<td>0.28034</td>
<td>4.06734</td>
<td>4.07172</td>
</tr>
<tr>
<td>0.22500</td>
<td>0.29705</td>
<td>0.34003</td>
<td>5.00750</td>
<td>4.56167</td>
</tr>
<tr>
<td>0.23750</td>
<td>0.36316</td>
<td>0.40413</td>
<td>5.55700</td>
<td>4.97277</td>
</tr>
<tr>
<td>0.25000</td>
<td>0.43559</td>
<td>0.47072</td>
<td>6.01285</td>
<td>5.25782</td>
</tr>
<tr>
<td>0.26250</td>
<td>0.51288</td>
<td>0.53712</td>
<td>6.32447</td>
<td>5.36084</td>
</tr>
<tr>
<td>0.27500</td>
<td>0.59286</td>
<td>0.59979</td>
<td>6.43247</td>
<td>5.21649</td>
</tr>
<tr>
<td>0.28750</td>
<td>0.67256</td>
<td>0.60531</td>
<td>6.26804</td>
<td>4.74942</td>
</tr>
<tr>
<td>0.28867</td>
<td>0.67987</td>
<td></td>
<td>6.23597</td>
<td>4.68620</td>
</tr>
</tbody>
</table>
SAMPLE SIZE $n = 6$.

$P(S \leq s) \quad 0 \leq s \leq 12^{-t}$
SAMPLE SIZE  $n = 6$.

$f(s) \quad 0 \leq s \leq 12^{-\frac{1}{2}}$
SAMPLE SIZE \( n = 7 \),
\[0 \leq s \leq 14^{-\frac{1}{2}}\]

TABLES OF THE DISTRIBUTION AND DENSITY FUNCTIONS OF THE \( s \) STATISTIC
WHEN THE RANDOM SAMPLES ARE DRAWN FROM A CONTINUOUS
RECTANGULAR POPULATION.

<table>
<thead>
<tr>
<th>( s )</th>
<th>( P(S &lt; s) ) Obtained By Numerical Quadrature.</th>
<th>( P(S &lt; s) ) Obtained By Spherical Approximation.</th>
<th>( f(s) ) Obtained By Numerical Quadrature.</th>
<th>( f(s) ) Obtained By Spherical Approximation.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00000</td>
<td>0.00000</td>
<td>0.00000</td>
<td>0.00000</td>
<td>0.00000</td>
</tr>
<tr>
<td>0.01250</td>
<td>0.00000</td>
<td>0.00000</td>
<td>0.00001</td>
<td>0.00001</td>
</tr>
<tr>
<td>0.02500</td>
<td>0.00000</td>
<td>0.00000</td>
<td>0.00025</td>
<td>0.00026</td>
</tr>
<tr>
<td>0.03750</td>
<td>0.00001</td>
<td>0.00001</td>
<td>0.00185</td>
<td>0.00189</td>
</tr>
<tr>
<td>0.05000</td>
<td>0.00006</td>
<td>0.00007</td>
<td>0.00745</td>
<td>0.00770</td>
</tr>
<tr>
<td>0.06250</td>
<td>0.00023</td>
<td>0.00024</td>
<td>0.02171</td>
<td>0.02266</td>
</tr>
<tr>
<td>0.07500</td>
<td>0.00067</td>
<td>0.00070</td>
<td>0.05149</td>
<td>0.05431</td>
</tr>
<tr>
<td>0.08750</td>
<td>0.00162</td>
<td>0.00171</td>
<td>0.10577</td>
<td>0.11290</td>
</tr>
<tr>
<td>0.10000</td>
<td>0.00346</td>
<td>0.00369</td>
<td>0.19548</td>
<td>0.21136</td>
</tr>
<tr>
<td>0.11250</td>
<td>0.00671</td>
<td>0.00723</td>
<td>0.33292</td>
<td>0.36510</td>
</tr>
<tr>
<td>0.12500</td>
<td>0.01204</td>
<td>0.01312</td>
<td>0.53104</td>
<td>0.59159</td>
</tr>
<tr>
<td>0.13750</td>
<td>0.02029</td>
<td>0.02240</td>
<td>0.80247</td>
<td>0.90975</td>
</tr>
<tr>
<td>0.15000</td>
<td>0.03245</td>
<td>0.03633</td>
<td>1.15833</td>
<td>1.33914</td>
</tr>
<tr>
<td>0.16250</td>
<td>0.04963</td>
<td>0.05642</td>
<td>1.60674</td>
<td>1.89902</td>
</tr>
<tr>
<td>0.17500</td>
<td>0.07302</td>
<td>0.08442</td>
<td>2.15116</td>
<td>2.60711</td>
</tr>
<tr>
<td>0.18750</td>
<td>0.10380</td>
<td>0.12228</td>
<td>2.78848</td>
<td>3.47824</td>
</tr>
<tr>
<td>0.20000</td>
<td>0.14307</td>
<td>0.17210</td>
<td>3.50687</td>
<td>4.52281</td>
</tr>
<tr>
<td>0.21250</td>
<td>0.19172</td>
<td>0.23609</td>
<td>4.28336</td>
<td>5.74499</td>
</tr>
<tr>
<td>0.22500</td>
<td>0.25025</td>
<td>0.31645</td>
<td>5.08122</td>
<td>7.14081</td>
</tr>
<tr>
<td>0.23750</td>
<td>0.31862</td>
<td>0.41528</td>
<td>5.84713</td>
<td>8.69596</td>
</tr>
<tr>
<td>0.25000</td>
<td>0.39599</td>
<td>0.53440</td>
<td>6.50805</td>
<td>10.38353</td>
</tr>
<tr>
<td>0.26726</td>
<td>0.51390</td>
<td>0.73477</td>
<td>7.06548</td>
<td>12.85029</td>
</tr>
</tbody>
</table>
SAMPLE SIZE \( n = 7 \).

\[ P(S\leq s) \quad 0 \leq s \leq 14^{-\frac{1}{2}} \]
SAMPLE SIZE $n = 7$.

$f(s) \ 0 \leq s \leq 14^{-k}$
SAMPLE SIZES  $n = 2, 3, 4.$

TABLES OF DENSITY FUNCTIONS OF THE $S$ STATISTIC
WHEN RANDOM SAMPLES ARE DRAWN FROM A NEGATIVE
EXPONENTIAL POPULATION.

<table>
<thead>
<tr>
<th>s</th>
<th>$f_2(s)$ Obtained By Numerical Quadrature.</th>
<th>$f_3(s)$ Obtained By Numerical Quadrature.</th>
<th>$f_4(s)$ Obtained By Numerical Quadrature.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00000</td>
<td>2.00000</td>
<td>0.00000</td>
<td>0.00000</td>
</tr>
<tr>
<td>0.05000</td>
<td>1.80967</td>
<td>0.45676</td>
<td>0.09722</td>
</tr>
<tr>
<td>0.10000</td>
<td>1.63746</td>
<td>0.76774</td>
<td>0.30173</td>
</tr>
<tr>
<td>0.15000</td>
<td>1.48164</td>
<td>0.96887</td>
<td>0.52826</td>
</tr>
<tr>
<td>0.20000</td>
<td>1.34064</td>
<td>1.08800</td>
<td>0.73287</td>
</tr>
<tr>
<td>0.25000</td>
<td>1.21306</td>
<td>1.14668</td>
<td>0.89620</td>
</tr>
<tr>
<td>0.50000</td>
<td>0.73576</td>
<td>0.99376</td>
<td>1.09898</td>
</tr>
<tr>
<td>0.75000</td>
<td>0.44626</td>
<td>0.66471</td>
<td>0.81254</td>
</tr>
<tr>
<td>1.00000</td>
<td>0.27067</td>
<td>0.40670</td>
<td>0.50585</td>
</tr>
<tr>
<td>1.25000</td>
<td>0.16417</td>
<td>0.23980</td>
<td>0.29278</td>
</tr>
<tr>
<td>1.50000</td>
<td>0.09957</td>
<td>0.13926</td>
<td>0.16394</td>
</tr>
<tr>
<td>1.75000</td>
<td>0.06039</td>
<td>0.08045</td>
<td>0.09045</td>
</tr>
<tr>
<td>2.00000</td>
<td>0.03663</td>
<td>0.04646</td>
<td>0.04962</td>
</tr>
<tr>
<td>2.25000</td>
<td>0.02222</td>
<td>0.02687</td>
<td>0.02718</td>
</tr>
<tr>
<td>2.50000</td>
<td>0.01348</td>
<td>0.01558</td>
<td>0.01491</td>
</tr>
<tr>
<td>2.75000</td>
<td>0.00817</td>
<td>0.00905</td>
<td>0.00819</td>
</tr>
<tr>
<td>3.00000</td>
<td>0.00496</td>
<td>0.00527</td>
<td>0.00451</td>
</tr>
<tr>
<td>3.25000</td>
<td>0.00301</td>
<td>0.00308</td>
<td>0.00249</td>
</tr>
<tr>
<td>3.50000</td>
<td>0.00182</td>
<td>0.00180</td>
<td>0.00138</td>
</tr>
<tr>
<td>3.75000</td>
<td>0.00111</td>
<td>0.00105</td>
<td>0.00076</td>
</tr>
</tbody>
</table>
SAMPLE SIZES n = 5, 6.

TABLES OF DENSITY FUNCTIONS OF THE S STATISTIC WHEN RANDOM SAMPLES ARE DRAWN FROM A NEGATIVE EXPONENTIAL POPULATION.

<table>
<thead>
<tr>
<th>s</th>
<th>( f_5(s) ) Obtained By Numerical Quadrature.</th>
<th>( f_6(s) ) Obtained By Numerical Quadrature.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00000</td>
<td>0.00000</td>
<td>0.00000</td>
</tr>
<tr>
<td>0.05000</td>
<td>0.01957</td>
<td>0.00441</td>
</tr>
<tr>
<td>0.10000</td>
<td>0.11172</td>
<td>0.03969</td>
</tr>
<tr>
<td>0.15000</td>
<td>0.27044</td>
<td>0.13230</td>
</tr>
<tr>
<td>0.20000</td>
<td>0.46226</td>
<td>0.28224</td>
</tr>
<tr>
<td>0.25000</td>
<td>0.65451</td>
<td>0.46305</td>
</tr>
<tr>
<td>0.50000</td>
<td>1.13111</td>
<td>1.12455</td>
</tr>
<tr>
<td>0.75000</td>
<td>0.92592</td>
<td>1.01871</td>
</tr>
<tr>
<td>1.00000</td>
<td>0.58790</td>
<td>0.66150</td>
</tr>
<tr>
<td>1.25000</td>
<td>0.33442</td>
<td>0.37044</td>
</tr>
<tr>
<td>1.50000</td>
<td>0.18060</td>
<td>0.19404</td>
</tr>
<tr>
<td>1.75000</td>
<td>0.09514</td>
<td>0.09702</td>
</tr>
<tr>
<td>2.00000</td>
<td>0.04957</td>
<td>0.04351</td>
</tr>
<tr>
<td>2.25000</td>
<td>0.02573</td>
<td>0.02205</td>
</tr>
<tr>
<td>2.50000</td>
<td>0.01335</td>
<td>0.01323</td>
</tr>
<tr>
<td>2.75000</td>
<td>0.00694</td>
<td>0.00441</td>
</tr>
<tr>
<td>3.00000</td>
<td>0.00362</td>
<td>0.00441</td>
</tr>
<tr>
<td>3.25000</td>
<td>0.00189</td>
<td>0.00000</td>
</tr>
<tr>
<td>3.50000</td>
<td>0.00099</td>
<td>0.00000</td>
</tr>
<tr>
<td>3.75000</td>
<td>0.00052</td>
<td>0.00000</td>
</tr>
</tbody>
</table>
GRAPH OF DENSITY FUNCTIONS OF THE \( S \) STATISTIC

WHEN RANDOM SAMPLES ARE DRAWN FROM A NEGATIVE
EXPONENTIAL POPULATION.
TABLES FOR COMPARISON OF THE CUMULATIVE DISTRIBUTION OF
\( *x^2/s^2 \) WHEN THE UNDERLYING POPULATION IS CONTINUOUS
RECTANGULAR WITH ITS CORRESPONDING CUMULATIVE
DISTRIBUTION WHEN THE UNDERLYING POPULATION
IS NORMAL FOR SAMPLE SIZE \( n = 2 \).

<table>
<thead>
<tr>
<th>( x_0 )</th>
<th>( P_R(x \leq x_0) )</th>
<th>( P_N(x \leq x_0) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00000</td>
<td>0.00000</td>
<td>0.00000</td>
</tr>
<tr>
<td>0.00100</td>
<td>0.02565</td>
<td>0.02500</td>
</tr>
<tr>
<td>0.00400</td>
<td>0.05097</td>
<td>0.05000</td>
</tr>
<tr>
<td>0.02000</td>
<td>0.11214</td>
<td>0.10000</td>
</tr>
<tr>
<td>0.15000</td>
<td>0.29123</td>
<td>0.30000</td>
</tr>
<tr>
<td>0.45000</td>
<td>0.47272</td>
<td>0.50000</td>
</tr>
<tr>
<td>1.07000</td>
<td>0.66626</td>
<td>0.70000</td>
</tr>
<tr>
<td>1.64000</td>
<td>0.77229</td>
<td>0.80000</td>
</tr>
<tr>
<td>2.71000</td>
<td>0.89246</td>
<td>0.90000</td>
</tr>
<tr>
<td>3.84000</td>
<td>0.96000</td>
<td>0.95000</td>
</tr>
<tr>
<td>5.41000</td>
<td>0.99746</td>
<td>0.98000</td>
</tr>
<tr>
<td>6.00000</td>
<td>1.00000</td>
<td>0.99000</td>
</tr>
</tbody>
</table>

FOR SAMPLE SIZE \( n = 3 \).

<table>
<thead>
<tr>
<th>( x_0 )</th>
<th>( P_R(x \leq x_0) )</th>
<th>( P_N(x \leq x_0) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00000</td>
<td>0.00000</td>
<td>0.00000</td>
</tr>
<tr>
<td>0.05100</td>
<td>0.02177</td>
<td>0.02500</td>
</tr>
<tr>
<td>0.10300</td>
<td>0.04281</td>
<td>0.10000</td>
</tr>
<tr>
<td>0.21000</td>
<td>0.08388</td>
<td>0.30000</td>
</tr>
<tr>
<td>0.71000</td>
<td>0.25144</td>
<td>0.50000</td>
</tr>
<tr>
<td>1.39000</td>
<td>0.43716</td>
<td>0.70000</td>
</tr>
<tr>
<td>2.41000</td>
<td>0.65190</td>
<td>0.80000</td>
</tr>
<tr>
<td>3.22000</td>
<td>0.77916</td>
<td>0.90000</td>
</tr>
<tr>
<td>4.60000</td>
<td>0.92317</td>
<td>0.95000</td>
</tr>
<tr>
<td>5.99000</td>
<td>0.98845</td>
<td>0.98000</td>
</tr>
<tr>
<td>6.00000</td>
<td>0.98865</td>
<td>0.99000</td>
</tr>
</tbody>
</table>

\( *x^2/n s^2 = \chi^2 \)
TABLES FOR COMPARISON OF THE CUMULATIVE DISTRIBUTION OF 
\*ns^2/\sigma^2\ WHEN THE UNDERLYING POPULATION IS CONTINUOUS 
RECTANGULAR WITH ITS CORRESPONDING CUMULATIVE 
DISTRIBUTION WHEN THE UNDERLYING POPULATION 
IS NORMAL FOR SAMPLE SIZE \( n = 4 \).

<table>
<thead>
<tr>
<th>( X_0 )</th>
<th>( P_R(X \leq X_0) )</th>
<th>( P_N(X \leq X_0) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00000</td>
<td>0.00000</td>
<td>0.00000</td>
</tr>
<tr>
<td>0.11500</td>
<td>0.00711</td>
<td>0.01000</td>
</tr>
<tr>
<td>0.18500</td>
<td>0.01411</td>
<td>0.02000</td>
</tr>
<tr>
<td>0.35200</td>
<td>0.03511</td>
<td>0.05000</td>
</tr>
<tr>
<td>0.58400</td>
<td>0.07073</td>
<td>0.10000</td>
</tr>
<tr>
<td>1.00500</td>
<td>0.14616</td>
<td>0.20000</td>
</tr>
<tr>
<td>1.42400</td>
<td>0.22826</td>
<td>0.30000</td>
</tr>
<tr>
<td>2.36600</td>
<td>0.41823</td>
<td>0.50000</td>
</tr>
<tr>
<td>3.66500</td>
<td>0.65774</td>
<td>0.70000</td>
</tr>
<tr>
<td>4.64200</td>
<td>0.80240</td>
<td>0.80000</td>
</tr>
<tr>
<td>6.00000</td>
<td>0.93513</td>
<td>0.90000</td>
</tr>
<tr>
<td>6.25100</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

FOR SAMPLE SIZE \( n = 5 \).

<table>
<thead>
<tr>
<th>( X_0 )</th>
<th>( P_R(X \leq X_0) )</th>
<th>( P_N(X \leq X_0) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00000</td>
<td>0.00000</td>
<td>0.00000</td>
</tr>
<tr>
<td>0.29700</td>
<td>0.00571</td>
<td>0.01000</td>
</tr>
<tr>
<td>0.42900</td>
<td>0.01147</td>
<td>0.02000</td>
</tr>
<tr>
<td>0.71100</td>
<td>0.02941</td>
<td>0.05000</td>
</tr>
<tr>
<td>1.06400</td>
<td>0.06121</td>
<td>0.10000</td>
</tr>
<tr>
<td>1.64900</td>
<td>0.13197</td>
<td>0.20000</td>
</tr>
<tr>
<td>2.19500</td>
<td>0.21301</td>
<td>0.30000</td>
</tr>
<tr>
<td>3.35700</td>
<td>0.41172</td>
<td>0.50000</td>
</tr>
<tr>
<td>4.87800</td>
<td>0.67322</td>
<td>0.70000</td>
</tr>
<tr>
<td>5.98900</td>
<td>0.82740</td>
<td>0.80000</td>
</tr>
<tr>
<td>6.00000</td>
<td>0.82868</td>
<td>0.80000</td>
</tr>
</tbody>
</table>

\*_{X}=ns^2/\sigma^2
TABLES FOR COMPARISON OF THE CUMULATIVE DISTRIBUTION OF \( *x^2 / \sigma^2 \) WHEN THE UNDERLYING POPULATION IS CONTINUOUS RECTANGULAR WITH ITS CORRESPONDING CUMULATIVE DISTRIBUTION WHEN THE UNDERLYING POPULATION IS NORMAL FOR SAMPLE SIZE \( n = 6 \).

<table>
<thead>
<tr>
<th>( x_0 )</th>
<th>( P_R(x \leq x_0) )</th>
<th>( P_N(x \leq x_0) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00000</td>
<td>0.00000</td>
<td>0.00000</td>
</tr>
<tr>
<td>0.55400</td>
<td>0.00465</td>
<td>0.01000</td>
</tr>
<tr>
<td>0.75200</td>
<td>0.00953</td>
<td>0.02000</td>
</tr>
<tr>
<td>1.14500</td>
<td>0.02515</td>
<td>0.05000</td>
</tr>
<tr>
<td>1.16000</td>
<td>0.02590</td>
<td>0.10000</td>
</tr>
<tr>
<td>2.34300</td>
<td>0.12196</td>
<td>0.20000</td>
</tr>
<tr>
<td>3.00000</td>
<td>0.20300</td>
<td>0.30000</td>
</tr>
<tr>
<td>4.35100</td>
<td>0.41078</td>
<td>0.50000</td>
</tr>
<tr>
<td>6.00000</td>
<td>0.67990</td>
<td>0.70000</td>
</tr>
</tbody>
</table>

FOR SAMPLE SIZE \( n = 7 \).

<table>
<thead>
<tr>
<th>( x_0 )</th>
<th>( P_R(x \leq x_0) )</th>
<th>( P_N(x \leq x_0) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00000</td>
<td>0.00000</td>
<td>0.00000</td>
</tr>
<tr>
<td>0.87200</td>
<td>0.00385</td>
<td>0.01000</td>
</tr>
<tr>
<td>1.13400</td>
<td>0.00803</td>
<td>0.02000</td>
</tr>
<tr>
<td>1.63500</td>
<td>0.02196</td>
<td>0.05000</td>
</tr>
<tr>
<td>2.20400</td>
<td>0.04881</td>
<td>0.10000</td>
</tr>
<tr>
<td>3.07000</td>
<td>0.11442</td>
<td>0.20000</td>
</tr>
<tr>
<td>3.82800</td>
<td>0.41125</td>
<td>0.50000</td>
</tr>
<tr>
<td>6.00000</td>
<td>0.51395</td>
<td></td>
</tr>
</tbody>
</table>

\( *x = ns^2 / \sigma^2 \)
TABLES FOR COMPARISON OF THE ORDINATES FOR THE FREQUENCY
FUNCTION OF \( *n s^2 / \sigma^2 \), WHEN THE UNDERLYING POPULATION
IS EXPONENTIAL, WITH THE ORDINATES FROM ITS
CORRESPONDING FREQUENCY FUNCTION WHEN THE
UNDERLYING POPULATION IS NORMAL.

<table>
<thead>
<tr>
<th>( \chi_0 )</th>
<th>( f_N(\chi_0) )</th>
<th>( f_E(\chi_0) )</th>
<th>( P_N(\chi \leq \chi_0) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00016</td>
<td>31.83656</td>
<td>55.44211</td>
<td>.01</td>
</tr>
<tr>
<td>0.00063</td>
<td>15.91454</td>
<td>27.23416</td>
<td>.02</td>
</tr>
<tr>
<td>0.00393</td>
<td>6.35128</td>
<td>10.32253</td>
<td>.05</td>
</tr>
<tr>
<td>0.01580</td>
<td>3.14884</td>
<td>4.70928</td>
<td>.10</td>
</tr>
<tr>
<td>0.06420</td>
<td>1.52476</td>
<td>1.95028</td>
<td>.20</td>
</tr>
<tr>
<td>0.14800</td>
<td>0.96303</td>
<td>1.06677</td>
<td>.30</td>
</tr>
<tr>
<td>0.45500</td>
<td>0.47109</td>
<td>0.40382</td>
<td>.50</td>
</tr>
<tr>
<td>1.07400</td>
<td>0.22500</td>
<td>0.15757</td>
<td>.70</td>
</tr>
<tr>
<td>1.64200</td>
<td>0.13698</td>
<td>0.09011</td>
<td>.80</td>
</tr>
<tr>
<td>2.70600</td>
<td>0.06268</td>
<td>0.04198</td>
<td>.90</td>
</tr>
<tr>
<td>3.84100</td>
<td>0.02983</td>
<td>0.02257</td>
<td>.95</td>
</tr>
<tr>
<td>5.41200</td>
<td>0.01146</td>
<td>0.01132</td>
<td>.98</td>
</tr>
<tr>
<td>6.63500</td>
<td>0.00561</td>
<td>0.00719</td>
<td>.99</td>
</tr>
</tbody>
</table>

SAMPLE SIZE \( n = 2 \).

<table>
<thead>
<tr>
<th>( \chi_0 )</th>
<th>( f_N(\chi_0) )</th>
<th>( f_E(\chi_0) )</th>
<th>( P_N(\chi \leq \chi_0) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.02010</td>
<td>0.49500</td>
<td>1.36274</td>
<td>.01</td>
</tr>
<tr>
<td>0.04040</td>
<td>0.49000</td>
<td>1.21041</td>
<td>.02</td>
</tr>
<tr>
<td>0.10300</td>
<td>0.47490</td>
<td>0.95353</td>
<td>.05</td>
</tr>
<tr>
<td>0.21100</td>
<td>0.44994</td>
<td>0.72596</td>
<td>.10</td>
</tr>
<tr>
<td>0.44600</td>
<td>0.40006</td>
<td>0.48402</td>
<td>.20</td>
</tr>
<tr>
<td>0.71300</td>
<td>0.35006</td>
<td>0.34515</td>
<td>.30</td>
</tr>
<tr>
<td>1.38600</td>
<td>0.25004</td>
<td>0.18483</td>
<td>.50</td>
</tr>
<tr>
<td>2.40800</td>
<td>0.15000</td>
<td>0.09343</td>
<td>.70</td>
</tr>
<tr>
<td>3.21900</td>
<td>0.09999</td>
<td>0.06076</td>
<td>.80</td>
</tr>
<tr>
<td>4.60500</td>
<td>0.05000</td>
<td>0.03304</td>
<td>.90</td>
</tr>
<tr>
<td>5.99100</td>
<td>0.02501</td>
<td>0.01986</td>
<td>.95</td>
</tr>
<tr>
<td>7.82400</td>
<td>0.01000</td>
<td>0.01117</td>
<td>.98</td>
</tr>
<tr>
<td>9.21000</td>
<td>0.00500</td>
<td>0.00762</td>
<td>.99</td>
</tr>
</tbody>
</table>

SAMPLE SIZE \( n = 3 \).

\( *x = n s^2 / \sigma^2 \)
TABLES FOR COMPARISON OF THE ORDINATES FOR THE FREQUENCY
FUNCTION OF $x^2 / \sigma^2$, WHEN THE UNDERLYING POPULATION
IS EXPONENTIAL, WITH THE ORDINATES FROM ITS
CORRESPONDING FREQUENCY FUNCTION WHEN THE
UNDERLYING POPULATION IS NORMAL.

<table>
<thead>
<tr>
<th>$\chi_0$</th>
<th>$f_N(\chi_0)$</th>
<th>$f_E(\chi_0)$</th>
<th>$P_N(\chi \leq \chi_0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.11500</td>
<td>0.12773</td>
<td>0.45146</td>
<td>.01</td>
</tr>
<tr>
<td>0.18500</td>
<td>0.15643</td>
<td>0.45735</td>
<td>.02</td>
</tr>
<tr>
<td>0.35200</td>
<td>0.19849</td>
<td>0.42413</td>
<td>.05</td>
</tr>
<tr>
<td>0.58400</td>
<td>0.22767</td>
<td>0.36344</td>
<td>.10</td>
</tr>
<tr>
<td>1.00500</td>
<td>0.24197</td>
<td>0.27386</td>
<td>.20</td>
</tr>
<tr>
<td>1.42400</td>
<td>0.23359</td>
<td>0.21152</td>
<td>.30</td>
</tr>
<tr>
<td>2.36600</td>
<td>0.18800</td>
<td>0.12790</td>
<td>.50</td>
</tr>
<tr>
<td>3.66500</td>
<td>0.12221</td>
<td>0.07213</td>
<td>.70</td>
</tr>
<tr>
<td>4.64200</td>
<td>0.08438</td>
<td>0.04983</td>
<td>.80</td>
</tr>
<tr>
<td>6.25100</td>
<td>0.04380</td>
<td>0.02927</td>
<td>.90</td>
</tr>
<tr>
<td>7.81500</td>
<td>0.02241</td>
<td>0.01863</td>
<td>.95</td>
</tr>
<tr>
<td>9.83700</td>
<td>0.00915</td>
<td>0.01112</td>
<td>.98</td>
</tr>
<tr>
<td>11.34500</td>
<td>0.00462</td>
<td>0.00786</td>
<td>.99</td>
</tr>
</tbody>
</table>

SAMPLE SIZE $n = 4$.

<table>
<thead>
<tr>
<th>$\chi_0$</th>
<th>$f_N(\chi_0)$</th>
<th>$f_E(\chi_0)$</th>
<th>$P_N(\chi \leq \chi_0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.29700</td>
<td>0.06400</td>
<td>0.25900</td>
<td>.01</td>
</tr>
<tr>
<td>0.42900</td>
<td>0.08654</td>
<td>0.27383</td>
<td>.02</td>
</tr>
<tr>
<td>0.71100</td>
<td>0.12457</td>
<td>0.26913</td>
<td>.05</td>
</tr>
<tr>
<td>1.06400</td>
<td>0.15626</td>
<td>0.24217</td>
<td>.10</td>
</tr>
<tr>
<td>1.64900</td>
<td>0.18075</td>
<td>0.19369</td>
<td>.20</td>
</tr>
<tr>
<td>2.19500</td>
<td>0.18312</td>
<td>0.15622</td>
<td>.30</td>
</tr>
<tr>
<td>3.35700</td>
<td>0.15665</td>
<td>0.10129</td>
<td>.50</td>
</tr>
<tr>
<td>4.87800</td>
<td>0.10640</td>
<td>0.06106</td>
<td>.70</td>
</tr>
<tr>
<td>5.98900</td>
<td>0.07495</td>
<td>0.04379</td>
<td>.80</td>
</tr>
<tr>
<td>7.77900</td>
<td>0.03978</td>
<td>0.02698</td>
<td>.90</td>
</tr>
<tr>
<td>9.48800</td>
<td>0.02065</td>
<td>0.01781</td>
<td>.95</td>
</tr>
<tr>
<td>11.66800</td>
<td>0.00854</td>
<td>0.01102</td>
<td>.98</td>
</tr>
<tr>
<td>13.27700</td>
<td>0.00434</td>
<td>0.00797</td>
<td>.99</td>
</tr>
</tbody>
</table>

SAMPLE SIZE $n = 5$.

$*x = ns^2 / \sigma^2$
TABLES FOR COMPARISON OF THE ORDINATES FOR THE FREQUENCY FUNCTION OF $x_n s^2 / \sigma^2$, WHEN THE UNDERLYING POPULATION IS EXPONENTIAL, WITH THE ORDINATES FROM ITS CORRESPONDING FREQUENCY FUNCTION WHEN THE UNDERLYING POPULATION IS NORMAL.

<table>
<thead>
<tr>
<th>$x_0$</th>
<th>$f_N(x_0)$</th>
<th>$f_E(x_0)$</th>
<th>$P_N(x \leq x_0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.55400</td>
<td>0.04157</td>
<td>0.18141</td>
<td>.01</td>
</tr>
<tr>
<td>0.75200</td>
<td>0.05954</td>
<td>0.19527</td>
<td>.02</td>
</tr>
<tr>
<td>1.14500</td>
<td>0.09191</td>
<td>0.19753</td>
<td>.05</td>
</tr>
<tr>
<td>1.16000</td>
<td>0.09302</td>
<td></td>
<td>.10</td>
</tr>
<tr>
<td>2.34300</td>
<td>0.14780</td>
<td></td>
<td>.20</td>
</tr>
<tr>
<td>3.00000</td>
<td>0.15418</td>
<td></td>
<td>.30</td>
</tr>
<tr>
<td>4.35100</td>
<td>0.13704</td>
<td></td>
<td>.50</td>
</tr>
<tr>
<td>6.06400</td>
<td>0.09575</td>
<td></td>
<td>.70</td>
</tr>
<tr>
<td>7.28900</td>
<td>0.06839</td>
<td></td>
<td>.80</td>
</tr>
<tr>
<td>9.23600</td>
<td>0.03685</td>
<td></td>
<td>.90</td>
</tr>
<tr>
<td>11.07000</td>
<td>0.01933</td>
<td>0.01715</td>
<td>.95</td>
</tr>
<tr>
<td>13.38800</td>
<td>0.00807</td>
<td>0.01090</td>
<td>.98</td>
</tr>
<tr>
<td>15.08600</td>
<td>0.00413</td>
<td>0.00800</td>
<td>.99</td>
</tr>
</tbody>
</table>

SAMPLE SIZE $n = 6$.

$x = n s^2 / \sigma^2$
RESULTS OBTAINED FOR EXACT EVALUATION OF THE INTEGRAL $I_{n-2}$ FOR $n=3, 4, 5$ AND RESULTS OBTAINED FOR THE INTEGRAL $I_{n-2}$ FOR $n=3, 4, 5, 6, 7$ BY NUMERICAL QUADRATURE WHEN

$$I_{n-2} = \frac{\int_0^1 \prod_{i=2}^{n-1} u_i \left( \frac{2}{n} \right)^{1/n} \prod_{i=2}^{n-1} \left( 1 + \sum_{i=2}^{n-1} u_i \right) - \frac{1}{n} \left( 1 + \sum_{i=2}^{n-1} u_i \right) \right)}{\left( 1 + \sum_{i=2}^{n-1} u_i \right)}$$

$$0 < u_i < 1, \quad i=2, \ldots, n-1.$$

<table>
<thead>
<tr>
<th>$n$</th>
<th>$I_{n-2}$</th>
<th>Exact value to 6 decimal places</th>
<th>Result obtained 9 point Gaussian Quadrature</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>$I_1$</td>
<td>2.449489</td>
<td>2.449434</td>
</tr>
<tr>
<td>4</td>
<td>$I_2$</td>
<td>2.702042</td>
<td>2.702043</td>
</tr>
<tr>
<td>5</td>
<td>$I_3$</td>
<td>2.730412</td>
<td>2.730417</td>
</tr>
<tr>
<td>6</td>
<td>$I_4$</td>
<td></td>
<td>2.559015</td>
</tr>
<tr>
<td>7</td>
<td>$I_5$</td>
<td></td>
<td>2.25359</td>
</tr>
</tbody>
</table>
8.4 Some Comments on The Results.

A study of the density functions of the $S$ statistic when the population from which the random samples are drawn is continuous rectangular, shows that a significant amount of the probability mass lies between 0 and $(2n)^{-\frac{1}{2}}$ where $n$ is the sample size. As $n$ increases the probability mass to the left of $(2n)^{-\frac{1}{2}}$ decreases.

Comparison of the chi-square distribution with the distribution of $\chi = ns^2/\sigma^2$ when the parent population is continuous rectangular shows that the probability mass at the left tail for critical values of $\chi_0$ is less than that for chi-square.

Comparison of the chi-square frequency function with the distribution of $\chi = ns^2/\sigma^2$ when the underlying population is exponential shows that the graph of the latter is likely to be thicker in the tails than that of chi-square.

The spherical approximation seemed to be very good. Tables using the spherical approximation are only included for $n = 6, 7$ when the distribution of the $S$ statistic from a uniform population is considered. Agreement in the left tail was quite good.
CHAPTER 9
SOME FUTURE PROBLEMS.

In this thesis, for the case when the underlying population is continuous rectangular, we have found for the S statistic,

\[ P(S \leq s) \text{ for } 0 < s \leq (2n)^{-\frac{1}{2}} \]

\[ \text{for } \frac{(n^2-4)^{\frac{1}{2}}}{2n} < s \leq \frac{1}{2} \]

when the sample size \( n \) is even

and for \( \frac{(5n-9)^{\frac{1}{2}}}{2n} < s \leq \frac{(n^2-1)^{\frac{1}{2}}}{2n} \]

when the sample size \( n \) is odd.

We have expressed \( F(s) \) as an \((n-2)\)-ple integral over a cubic domain when \( 0 < s \leq (2n)^{-\frac{1}{2}} \). We have further approximated the \((n-2)\)-ple integral by a double integral by replacing the cubic domain by a spherical domain of the same volume.

When \( n \) is odd for \( \frac{(n^2-4)^{\frac{1}{2}}}{2n} < s \leq \frac{1}{2} \)

and \( n \) is even for \( \frac{(5n-9)^{\frac{1}{2}}}{2n} < s \leq \frac{(n^2-1)^{\frac{1}{2}}}{2n} \)

\( F(s) \) is expressed as an \((n-1)\)-tuple integral.

Finally, by making use of the same techniques we have found the cumulative distribution and frequency function of \( S \) when the underlying population is negative exponential and \( 0 < s < \infty \).

We have further expressed both \( F(s) \) and \( f(s) \) as \((n-2)\)-ple integrals over cubic domains. The \((n-2)\)-ple integral in both \( F(s) \) and \( f(s) \) has been further approximated by
double integrals. In both cases we replaced the cubic domain of each by spherical domains of the same volume.

The problems which arise from this work for future investigation may be summarised thus:

(i) Reduce the (n-2)-ple integrals in (4.6.iii) and (7.4.ii) through further transformations. Also reduction of the integral (4.8.x) and (4.8.xvi) should be considered.

(ii) The spherical approximation should be examined analytically.

(iii) Employ methods developed in this thesis for finding the distribution of the statistic \( S \) in samples from other populations, e.g., the logistic.

(iv) Find the distribution of \( S \) over the whole range for \( n = 4 \) by the "method of sections".

(v) Find the distribution function of \( S \) over the whole range of \( S \) for any \( n \) by the analytical methods developed in this thesis.
REFERENCES


[31] Hillstrom, K. (December, 1966). AN DL53S - DRØMB, 
ØS/360 FORTRAN IV Routine for two dimensional 
Romberg quadrature. Argonne National Laboratory, Argonne, Illinois.
EPILOGUE

After my research had been completed and my results had been submitted, Professor M. M. Ali recommended that I use the thesis of M. M. Ali, (1969), On The "Student"-Fisher Ratio, Dept. of Math., Univ. of Toronto, which he obtained for me, as a model for writing up this present thesis.

The thesis of Lai Kow Chan, (1965), Linear Estimation of Location and Scale Parameters ..., Dept. of Math., Univ. of Western Ont., was used as a secondary model.