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Real Options Models in Real Estate

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REAL OPTIONS MODELS IN REAL ESTATE

Investigating the utility of realistic real options models in real estate

Integrated Article

by

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Abstract

Our aim in this thesis is to investigate the usefulness of real options analysis, taking case studies of problems in real estate. In the realm of real estate, we consider the following three problems. First, we consider the valuation and usefulness of presale contracts of condominiums, which can be viewed as similar to call options on condominiums. Secondly, we consider the valuation of farm land from the perspective of land developers, who may think of farm land as being similar to call options on subdivision lots. Third, we consider the valuation of opportunities to install solar panels on properties, in which properties may be considered call options on electricity generators. Through consultation with industry professionals, we created models with the aim of being as realistic as possible without losing analytical tractability. In all three problems, we assess the potential value added through the usage of real options modeling techniques over more traditional techniques, using realistic parameter regimes. We utilize a set of sophisticated mathematical and numerical tools to mathematically model problems. We find that for some problems, real options models only add minimal value to more traditional capital budgeting techniques such as the Net Present Value model. In other problems, we find that real options models lead to significantly different sets of conclusions from those predicted by more traditional techniques.

Keywords

Real Options, Real Estate, Presale, Condominiums, Subdivision, Solar Panels, Geometric Brownian Motion, Stochastic Calculus, Monte Carlo Simulation.

Co-Authorship Statement

This thesis was assembled from research articles co-authored with Dr. Matt Davison and, in one case, Henning Rasmussen. Henning Rasmussen was Professor of Applied Mathematics at the University of Western Ontario while Davison is Professor of Applied Mathematics and Statistical & Actuarial Sciences at the University of Western Ontario. Davison is the academic supervisor of Jin Choi.

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Table of Contents

CERTIFICATE OF EXAMINATION ... ii

Abstract ... iii

Co-Authorship Statement ... iv

Acknowledgments ... v

Table of Contents ... vi

List of Tables ... ix

List of Figures ... xi

Chapter 1 ... 1

1 Real Options Overview ... 1

1.1 Motivation ... 1

1.2 Solution Methodologies ... 4

1.3 Focus ... 6

Chapter 2 ... 9

2 Value of Presale Contracts of Condominiums ... 9

2.1 Business Overview ... 10

2.2 Analysis of the Purchaser's Position ... 15

2.2.1 Valuation under Risk Neutral Environment ... 15

2.2.2 Valuation under Risk Adjusted Return ... 24

2.3 Analysis of the Developer's Position ... 26

2.3.1 Model and Analytical Solution ... 26

2.3.2 Risk of Profit and Loss ... 31

2.3.3 Comparison with No Presale ... 35

2.4 Conclusion ... 38

Chapter 3 ... 41

3 Value of Urban Land under Regulatory Uncertainty 41
3.1 Business Practices and Modeling Assumptions 45
3.1.1 Price Process of Developable Lots 46
3.1.2 Permit Approval Time 47
3.1.3 Land Rent 48
3.1.4 Development Costs 50
3.1.5 Discount Rates 50
3.2 Model of Subdivision Project 51
3.3 Base Case Parameter Estimation 53
3.4 Results 56
3.4.1 Land Valuation under Homogeneous Discount Rate 56
3.4.2 Land Valuation under Heterogeneous Discount Rate 59
3.4.3 Valuation Discrepancies between Real Options and Discounted Cash Flows 62
3.4.4 Development Lags and Policy Implications 63
3.1 Conclusion 64
Chapter 4 69
4 Economic Value and Impact of Subsidies on Solar Panel Installations 69
4.1 Model Specification 72
4.2 Solution 76
4.3 Base Case Parameters 77
4.4 Results 78
4.4.1 Net Present Value 79
4.4.2 No subsidies 79
4.4.3 Fixed Installation cost subsidies 81
4.4.4 Declining Installation Cost Subsidies 83

4.5 Conclusions 85
Chapter 5 86
5 Conclusions 86
Bibliography 88
6 Bibliography 88
Appendices 93
A. Analysis of Lai et al. (2004) 93
B. Modeling Assumptions for Chapter 2 94
C. Analytic Solution to Purchaser's Position 99
D. Variance of Purchaser's Profitability 100
E. Solutions to Bellman Equation for Land 102
Curriculum Vitae 104

List of Tables

Table 1-1 Cash flows of a hypothetical oil well. 3

Table 2-1: Parameter descriptions and their base case values. 11

Table 2-2: The construction cost margin of some publicly traded companies which participate in condominium construction. 13

Table 2-3: Risk profile of holding a presale contract for different environments, assuming the purchaser requires a risk free rate of return.. 23

Table 2-4: Risk profile of holding a presale contract for different environments, assuming the purchaser requires $\boldsymbol{\gamma p}$ above the risk free rate of return... 25

Table 2-5: Developer's risk profile for a presale contract in different environments 34

Table 2-6: Average deviation between the total profitability expected from holding presales ($\boldsymbol{Z}$) and profitability expected if without presales ($\boldsymbol{Y}$).. 38

Table 3-1: Parameter descriptions and base case values, placed at beginning of chapter for easy reference.. 44

Table 3-1: Discounted Cash Flow valuation of Land 56

Table 3-3: The value of land and the changes in its values according to different market assumptions.. 57

Table 3-4: The value of land and the changes in its values according to different market assumptions. 60

Table 3-5: Comparison of land values projected by real options model and the discounted cash flow model under different market assumptions.. 62

Table 4-1: Parameter description and their base case values, placed at the beginning of the chapter for easy reference. 71

Table 4-2: Computed values in dollars per kW, of the option to install a solar panel on a roof.. 80

Table 4-3: Computed values in dollars per kW, of the option to install a solar panel on a roof with fixed subsidies of $5,422/kW throughout life of the option.. 82

Table 4-4: Probabilities of early exercise for different rates of declines, obtained using 200,000 Monte Carlo simulations. 83

Table 4-5: Computed values in dollars per kW of the option to install a solar panel on a roof with declining subsidies.. 84

List of Figures

Figure 2-1: Solutions to Purchasers' options.. 18

Figure 2-2: Cumulative probability distribution functions for profit, accounting for down payment.. 21

Figure 2-3: Comparison of the probability density function of purchaser's profitability against the normal distribution of equal mean and variance.. 23

Figure 2-4: Comparison of solution curves between when a developer require risk free rates of returns, and when she requires risk adjusted rates of returns.. 24

Figure 2-5: Solution curves for the developer's real option. 29

Figure 2-6: Comparison of the total profitability expected from holding presales ($\boldsymbol{Z}$), to the profitability expected from build condos without holding a presales ($\boldsymbol{Y}$). 37

Figure 3-1: 95% confidence interval for mean-reverting price process described in equation (3-1) for differing values of $\boldsymbol{\eta}$.. 47

Figure 3-2: Probability distribution for the time taken for a developer to get a subdivision plan approved by the local government.. 48

Figure 3-3: Waiting times between land purchase and commencement of development. 61

Figure 3-4: Histogram of excess profit – i.e. profit above normal profits. 63

Figure 3-5: Waiting times between permit grant and finalization of development. 65

Figure 3-6: Histogram of the number of years taken from purchase of raw land to completion of development. 65

Figure 3-7: Plot of Mean Permit Approval Time vs Q^* and Land Value. 66

Figure 3-8: Plot of Standard Deviation of Permit Approval Time vs $\boldsymbol{Q}*$ and Land Value, after adjusting for ρ. 66

Figure 4-1: Q-Q plot of the natural log of monthly wholesale electricity prices as reported by the Independent Electricity System Operator (IESO).. 73

Figure 4-2: Q-Q plot of monthly global solar module prices, as provided by Solarbuzz........ 74

Figure 4-3: Exercise boundary for option on solar panels, where t=0 and T=5. 81

Figure 0-1: Quarterly New Housing Price Index and Construction Price Index, as provided in the CANSIM database by Statistics Canada (1997 price = 100)... 95

Figure 0-2: Histogram of the of the monthly New Housing Price Index................................ 96

Figure 0-3: Scatter plot of log returns of monthly New Housing Price Index........................ 97

Figure 0-4: The Partial Autocorrelation Function of NHPI log returns................................. 98

Chapter 1

1 Real Options Overview

1.1 Motivation

The seminal work of Black, Scholes and Merton (1973) revolutionized the financial world by providing practitioners with a rational framework in which they could value European stock options. The main hurdle that stood in the way of valuing options up to that point had been the treatment of uncertainty. Without an objective method to model uncertainty, the non-linear payoff structure of options made valuing them difficult. Using the Ito process, and working under certain idealized conditions, Black, Scholes and Merton were able to present a valuation model of options, the validity of which is supported by the ability of traders to make risk-free profits whenever discrepancies between price and model arose.

We can also apply Black, Scholes and Merton's methodology to value non-financial assets, where significant elements of uncertainty underpin the valuation of such assets. Examples include mines, oil wells, dams, factories and even technological research projects. The decision to build one of these assets can be thought of as being similar to exercising an option, for in all cases, the potential economic benefits derived from them are highly uncertain. Modeling such assets is part art, since the modeller needs to determine which economic factors are relevant, and in which way they should be modelled. However, it is also part science, since each model should yield results that are congruent with reality. The art and science of rationally valuing such hard assets using mathematics is the discipline of Real Options.

While the Black-Scholes model and subsequent financial option models have received rapid and widespread acceptance in the financial industry, adoption of real options models have been slow (Block, 2007). There are several reasons for the slow pace of adoption. Firstly, real options generally require models with higher degrees of sophistication. While the assumptions underlying the Black-Scholes model are close enough to reality to make them useful, the same assumptions often do not hold for real assets. For instance, a practical real option model for a certain asset might be a compound option. This adds complexity to the model, which often hinders adoption.

Second, using real options requires technical sophistication that many higher ranking managers lack. While they could hire someone to model an asset, managers might not feel that they themselves are able to fully understand the resulting model. It is understandable that managers would not be comfortable with models they do not fully understand.

Finally, in most cases, mispriced assets do not create arbitrage opportunities to the advantage of savvy investors. Investors can take advantage of arbitrage opportunities in the financial markets because the underlying stocks of call options can be shorted, and the ratio between options and stocks can be adjusted flexibly. At least one of these conditions are usually not satisfied with real assets. With no opportunities to make riskless profits from mispriced assets, the benefit of using real options is more obscured, making it harder to persuade management to use them.

However, the reasons stated above do not preclude real options from being useful in the real world. Real options analysis may not lead to arbitrage strategies, but it does lead to a more accurate valuation of assets through the capital budgeting process, and equally important, it advises on the optimal strategy to extract the maximum financial benefit from the usage of the asset. One would reason that if there are significant benefits to be gained from employing real options models, the models would be in wider use despite the barriers of use. But if real options analysis doesn't yield significant improvements on traditional capital budgeting models, management may not feel compelled to overcome the barriers.

The traditional model of choice for capital budgeting has been the Net Present Value (NPV) model. This method of analysis fixes all variables to constant values. The modeller estimates present and future cash flows, discounting future cash flows to the degree that they reflect the riskiness of the project being modeled. Unlike real options analysis, NPV models do not advise on the optimal time to exercise. Rather, they advise on whether to exercise or not today.

We can take an oil well as an example to illustrate the NPV approach. To model an oil well using NPV, we estimate values of variables such as present and future prices of oil, capital costs, operating costs and taxes. In a spreadsheet, an analyst would lay out expected cash flows over a time horizon. Future cash flows are discounted to incorporate the time value of money, and then added up. This sum is the NPV of the oil well. If the NPV is positive, it is deemed economical to build the oil well today. An example is given in Table 1-1.

Year	0	1	2	3
Oil price (per barrel)	80	90	100	110
Number of barrels produced	0	200	100	50
Revenue	0	18000	10000	5500
Operating Expenditures	0	3000	3000	3000
Capital Expenditures	10000	0	0	0
Undiscounted net cash flow	-10000	15000	7000	2500
Discounted cash flow (discounted @ 10%/year)	-10000	13636	5785	1878

Table 1-1 Cash flows of a hypothetical oil well.

In this example, the value of the opportunity is the sum of the discounted cash flows, which is $11,299. Since this number is positive, the recommendation would be to start building the well today.

The drawback of using the NPV method is that the recommended action may not be the optimal action that maximizes the financial potential of assets. This is especially true in instances where future revenue stream is uncertain. Let us again consider the example of the oil well. If we believe that oil prices going to double in year 2, and increase more slowly thereafter, it would not be optimal to start building the well today. Rather, it would be optimal to wait this year and build the well next year, or perhaps even later. Assuming that we will build the oil well at the optimum point in time, the value of the well (V) can be described mathematically as follows

$$V = \max_t[NPV(t)].$$

In the above equation, t is the time at which the oil company decides to build the well. This type of analysis can be completed rather easily for the types of problems where future revenue is deterministic. However, it becomes more complicated when revenue is uncertain. For instance, while one can try to divine the future price of oil, the reality is that oil prices move in a stochastic fashion that are quite hard to predict. In such cases, the optimal decision that would get the most out of an oil well, would also depend on the price of oil. For example, in an extreme case, if oil was at $1/barrel, and it cost $30/barrel to produce at some time in the future, it would not be optimal to build the oil well. On the other hand, if oil was at $200/barrel, and if oil prices will in all likelihood decline from that level, it would be optimal to build the well at that time.

The land owner need not exercise the option to build an oil well today, but rather wait until there is more relevant information. The NPV method does not allow room for the option of waiting. In contrast, the art and science of analyzing the optimal decision, and the value of the asset assuming the owner exercises at the optimal moment, is the heart of real options modeling.

1.2 Solution Methodologies

Modeling and solving problems using real options is non-trivial. Unlike for financial options, in which the traded contracts clearly identify all the value drivers, one must identify the myriad of factors that affect the valuation of each asset under consideration. Our aim is to create simple models that capture the essence of each asset, only incorporating factors which materially influence the economics of each asset. However, it's often difficult to know the importance of each factor in advance, and it is part of the purpose of this thesis to investigate which factors prove to be material in the real world.

Our models involve the use of stochastic variables. A common model is one in which an asset or other variable follows a Geometric Brownian Motion (GBM). GBM describes a variable whose percentage change in incremental value is normally distributed with mean (μ) and standard deviation (σ). Mathematically, it is expressed using the Wiener process (W) as follows.

$$\frac{dX_t}{X_t} = \mu dt + \sigma dW.$$

For this thesis, it is assumed that the reader understands at least the elements of stochastic calculus. The focus of this thesis will be on extracting business insights rather than on advancing the mathematical theory of real options. While all mathematical statements will be precise and correct, at times technical conditions will not be mentioned when their inclusion would distract the reader with mathematical detail, obscuring the larger picture. An introduction sufficient for our purposes here may be found, for example, in “Paul Wilmott on Quantitative Finance” (Wilmott P. , 2006). The principle of dynamic optimization is used to construct real options models in this thesis. There are a number of books that outline dynamic optimization (see Dixit & Pindyck, 1994). Apart from dynamic optimization, one can also use contingent claims analysis to create real options models. However, contingent claims analysis can only apply if the underlying asset is spanned by existing assets – i.e. a portfolio of existing assets can be assembled to precisely match the economic characteristics of the asset under analysis. As we shall see, that assumption does not apply to the types of assets analyzed in this thesis, and therefore we focus our efforts on utilizing dynamic optimization.

The principle of dynamic optimization is centered on asking the following questions at each decision point - do we exercise the option or wait? Assuming we can extract the most economic value by making optimal decisions, what is the value of the asset given that the owners require a specific rate of return?

For the owner of asset Y to achieve a specific rate of return ρ, the change in the value of the asset must match this rate. This may be mathematically written as follows.

$$\rho Y_t dt = dY_t.$$

The right hand side of this equation is particular to each asset. As a simple example, let’s assume that we can build a mine capable of producing one unit of minerals valued X_t at time t. Once we decide to build the mine and pay M, we get X instantly. X follows GBM, and the mineral rights of the land expires in time T. Since this is a stochastic process, we must use Ito’s lemma to find the function’s differential. Using the lemma, we have

$$dY_t = \left(\mu X_t \frac{\partial Y_t}{\partial X_t} + \frac{\partial Y_t}{\partial t} + \frac{1}{2}\sigma^2 {X_t}^2 \frac{\partial^2 Y_t}{\partial {X_t}^2}\right) dt + \sigma X_t \frac{\partial Y_t}{\partial X_t} dW.$$

Taking expectations over the random variable given by dW, we have

$$E[dY_t] = \left(\mu X_t \frac{\partial Y_t}{\partial X_t} + \frac{\partial Y_t}{\partial t} + \frac{1}{2}\sigma^2 {X_t}^2 \frac{\partial^2 Y_t}{\partial {X_t}^2}\right) dt = \rho Y_t dt.$$

This equation is identical to the Black Scholes equation, except that the discount rate may not equal the risk-free rate. This is because there are no ways to achieve riskless profits in the absence of a replicating portfolio. Instead of representing "The" price of the asset as with the replicating argument, the real options equation represents the expected value of the asset if managed as recommended. However, the equation can be solved using the same methodology used to solve the Black Scholes equation.

In this thesis, we've compensated the holder of the option by allowing for the presence of risk premiums. In the standard Black Scholes equation, the discount rate required by the holder of the option is equal to the risk free rate. Whenever the discount rate is different to the risk free rate, the difference between the discount rate and the risk free rate is the risk premium. Such premiums may be necessary to compensate the holder of assets investigated in this thesis, since it's not possible for the holders to hedge the assets. However, the introduction of the risk premiums is not the only way one can compensate option holders. One can also introduce different utility functions that layer onto the payoff functions to reflect the risk aversion of the option holder. For example, such a utility function can demand that any possible losses are magnified by a multiplicative factor. However, we choose not to employ utility functions in this paper for the following reasons – to keep with industry practices, because utility functions are hard to accurately measure, and because we want payoffs to be linear for a company which may deal with a portfolio of options.

1.3 Focus

There are many areas in business where one can apply the real options paradigm. In this thesis, the focus will be on problems in real estate. Real estate was an attractive subject for several reasons. One, there are relatively few papers regarding the application of real options on real estate. Yet, the real estate sector is a significant component of the economies of nearly every country around the world, and it is one that occupies a significant place in the hearts and minds of ordinary people. It is a sector that is amenable to real options analysis, as real estate prices are unpredictable.

There are many real options problems one can investigate in real estate. In this thesis, we have chosen to focus our attention to the following three problems. In chapter 2, we investigate the value of pre-sale contracts of condominiums. Developers usually sell new condominiums through a presale, whereby purchasers make down payments to enter into contracts that allow them to purchase finished condominiums on a later date at a predetermined price. The presale can be thought of as a call option for the purchaser on the condominium. In this chapter, we value the presale contract from both the purchaser's and the developer's points of view. We also analyze the extent of risk sharing between the purchasers and the developers according to varying levels of down payments.

In chapter 3, we present a real option based model for the valuation of vacant urban land with residential zoning. The model incorporates several factors that were overlooked by earlier papers in the subject, as we strove to reflect business realities more accurately. These factors include the treatment of land as a compound option, the introduction of mean-reversion to subdivision lot prices, incorporation of property taxes, and the introduction of time lags associated with waiting on regulatory permits. The model is tailored to market conditions in London, Ontario, Canada, and we compare results with prevailing market prices and observed developer behaviour. We use the model to present a quantitative discussion of the risks associated with undertaking such land development projects. We present a model that is realistic and computable, and that which incorporates key parameters that drive the value of vacant land. We show that neither taxes nor rents are, in realistic parameter regimes, particularly material. However, it is shown that regulatory uncertainty plays a more important role, particularly as it relates to the mean and variance of permit approval times.

In chapter 4, we model the option to install solar panels on private property. Solar panels can be installed on rooftops or on the ground, and owners are able to either produce electricity for their own use, or sell it into the grid. Making some assumptions about the behaviour of electricity prices and installation costs, we build a model that examines the economic value of owning such an option for the property owner. We also examine the problem from the perspective of governments who wish to foster solar industries within their jurisdictions, by encouraging as much installations as possible. We examine the implications of implementing different subsidy regimes, and we determine the optimal way to implement such subsidies. We also compare and contrast the conclusions derived from applying the NPV and Real Options approaches.

Each problem we've examined is self-contained, and offers various insights that are of specific interest to the problem being considered, even if those insights may not be directly related to the examination of real options analysis. But there is a common thread that binds each of the problems. We've strived to build real options models that are as realistic as possible, leveraging our correspondence with industry professionals. We compare valuations and optimal decisions inferred from using real options models, and compare them against those inferred from using NPV models. As such, we hope this thesis can provide some insights as to the value of real options to real estate decision makers in practice, as distinct from the in-theoretical principle. We summarize our findings and conclusions in chapter 5, where we also discuss possible future work.

Chapter 2

2 Value of Presale Contracts of Condominiums

Developers of large building projects regularly sell units before construction in order to raise capital. With this capital, they are able to secure loans used to build properties. In order to secure bank financing, developers must sell a minimum number of units (Belford, 2008). By selling their units before they are built, both developers and the banks which finance them benefit from lower uncertainty.

Comparatively little analysis of the presale process has appeared in the literature. Chang and Ward (1993) analyzed the implications to developers, viewing presales as forward contracts. Yiu and Wong (2005) investigated the feedback relationship between existing condo prices and presale prices. Yiu et al. (2009) studied the impact of noise traders on volume and price dispersion of presale contracts. Leung et al. (2007) used a linear regression to model the price of a presale contract in terms of a few independent variables such as market price of existing condominiums, interest rate and hidden forward risks.

Lai et al. (2004) laid the foundation of a real option based analysis. This chapter treated the problem from the view both of the purchaser and the developer of a condominium unit in an environment in which both parties demanded a risk free rate of return on their investments. They presented property price data from Shanghai, Hong Kong and Taipei which showed the long term price changes as well as short term price volatilities. From this, they surmised that the price could be modeled using Geometric Brownian Motion.

The present work builds on the foundation of the Lai paper. After correcting a small but important error in their work (see Appendix A), we form a more sophisticated and realistic model.

Chan et al. (2008) have also taken the options approach to the treatment of presale contracts. Our models assume that condo prices move in a lognormal fashion more similar to that of stocks, whereas property prices in Chan et. al.'s model are normally distributed. We also allow uncertainty in the construction cost by treating the cost as a stochastic variable, whereas Chan et. al assumes a deterministic construction cost.

Crucially, whereas Chan et. al. focused only on minimizing the purchaser's payments, we have also taken into account the profit potential through immediate sales upon completion of the property. This can best be illustrated with a hypothetical scenario. Suppose the original purchase price is $100,000 and the down payment is $20,000. Let's also suppose the condo price has 50% chance of doubling to $200,000 and 50% chance of halving to $50,000. The buyer according to Chan et. al. would seek to minimize the cost, so she would be faced with 50% chance of paying $80,000 and 50% chance of paying $30,000. However in our model, the buyer would consider the profit potential: 50% chance of earning $100,000 and a 50% chance of losing $20,000. These are fundamentally different considerations that yield different fair values for the same condominium units.

The outline of the chapter is as follows. In section 2.1 we explain prevalent business practices and justify our assumptions and parameters. In section 2.2 we investigate the problem from the purchaser's point of view, calculating the fair down payment value and the extent of risk sharing. In section 2.3 we investigate the problem from the developer's point of view. We calculate the extent of risk sharing through participation in a presale, and compare the benefits against that of not holding a presale. In section 2.4, we finish the main body of our text with conclusions. We present our analysis of Lai et al.'s model in Appendix A. We justify our reasons for choosing the lognormal process to model condo price movements in Appendix B. In Appendices C and D, we give details of our calculations regarding the value of the purchaser's option and the variance of the purchaser's profitability, respectively.

2.1 Business Overview

This section gives a brief overview of the general practices for buying and selling a condo via a presale contract. We state our assumptions regarding the movement of condo prices and construction costs, and we estimate reasonable values for parameters involved in our model. We note that the parameters selected here are for the purpose of scenario generation and do not represent the fruits of a detailed econometric study.

Symbol	Parameter Description	Base Case
Q_1	Down payment amount	Output of Analysis
Q_2	Final payment amount	\$400,000
$S(t)$	Condominium price	Variable
$R(t)$	Construction cost	Variable
$R(0)$	Initial construction cost	\$350,000
A	Penalty to opt out of contract	\$0
B	Brokerage fee	5%
T	Total time between down payment and final payment	4 years
μ	Expected annual rate of appreciation	4%
r_p	Risk-free interest rate for purchaser	4%
r_d	Interest rate for construction loan	6%
σ_S	Volatility of condo prices	11%
σ_R	Volatility of construction cost	2%
ρ	Correlation between condo price and construction cost	Tunable Parameter
γ_p	Risk Premium required by purchaser	5%
γ_d	Risk Premium required by developer	6.75%

Table 2-1: Parameter descriptions and their base case values, placed at beginning of chapter for easy reference.

A typical presale involves the following steps: The purchaser agrees to pay two pre-specified amounts when she agrees to enter into a contract with the developer. The first amount is the down payment Q_1, paid by the purchaser at the time of the agreement. The down payment is usually not a single lump sum but a series of monthly payments for the first three or four months.

Since the down payments are close together in time, we approximate the series of down payments as one lump sum. When construction is nearly complete, the purchaser makes the final payment Q_2. The construction time T is typically 4 years.

In Canada, the down payment ratio must be at least 20% as of 2008 (i.e. Q_1 is normally 20% of $Q_2 + Q_1$). This ratio is set by the Government of Canada as part of the rules governing mortgage insurance. If a buyer wants to make a down payment less than 20%, she is obligated to purchase mortgage insurance to protect the mortgage lender. Since mortgage insurance is expensive, purchasers generally make the 20% down payment.

Presold condo prices vary widely, with a typical total price being $500,000 for a condo in a big city in 2008. This means a typical Q_1 would be around $100,000 while a typical Q_2 would be about $400,000. We can fix either of Q_1 or Q_2 and vary the other. In this chapter, we set Q_2 to be $400,000 and vary the value of Q_1.

From the purchaser's point of view, the presale contract is similar to a call option on the condominium. The difference is that failing to “exercise" the option may be penalized. We assume condo prices S follow a Geometric Brownian Motion (GBM), motivating this assumption in Appendix B. Using the National Housing Price Index (StatCan, 2009), we estimate the annual appreciation rate μ to be 4% per annum under normal market conditions. This is lower than some of other reported measures. For example, an online resource RealEstate ABC (2004) reports that house prices have been appreciating at a 6% rate on average from 1968 until 2004. The major reason for this divergence is the adjustment StatCan makes for housing improvements in creating their price indices, which other measures don't capture.

The purchaser may or may not be free to walk away from the contract after making the down payment. In some markets, a penalty A must be paid to opt out of the contract. The no penalty situation is modeled by choosing $A = 0$. We assume that purchasers exercise rationally and will only make the final payment if $S(T) > Q_2 - A$. Otherwise, she will cancel the contract. We are aware that purchasers may not be completely rational in their decision to honour the contract. However, as Forsyth and Karp (2008) reveal, the assumption is approximately true. Since $S > 0$, the purchaser is always better off honouring the contract than walking away from the deal if $A = Q_2$. Therefore, when considering legal environments in which the purchaser cannot walk away from the deal, the choice $A = Q_2$ is appropriate. However, unless otherwise stated, this chapter assumes the purchaser can walk away without penalty (i.e. $A = 0$).

Rather than entering into a presale contract, the purchaser can set Q_1 aside to invest in risk free assets earning r_p per year. Since we assume the purchaser makes the down payment using her savings, we take r_p to be the rate of return on savings, rather than interest paid on borrowed money. The purchaser's risk-free rate (yield on GICs) has historically been around 4% annually for a 4 year term.

The developer posts the down payments as collateral against their bank loans. After developers post adequate collateral, they receive bank loans with interest rate r_d. This rate varies from developer to developer, but their annual reports show it to be in the 5-7% range (NVR, 2008, Hovnanian, 2008, KB Home, 2008, M/I Homes, 2008).

Company	NVR	Hovnanian	KB Home	M/I Homes
2003	75.3%	74.5%	77.5%	75.0%
2004	74.4%	74.5%	76.0%	74.6%
2005	72.2%	73.6%	73.2%	74.9%
2006	77.9%	76.9%	77.3%	75.3%
2007	83.7%	84.9%	86.7%	81.9%
Average	76.7%	76.9%	78.1%	76.3%

Table 2-2: The construction cost margin of some publicly traded companies which participate in condominium construction. Data retrieved from the respective annual reports.

Once the project is financed, the developer builds the condominium. The construction cost R includes materials and labour cost, and varies over the period. We assume R also follows GBM with expected appreciation μ. We estimate the initial construction cost based on the following calculations. Table 2-2 depicts the % of gross revenue reported by major publicly traded developers. Average construction cost per unit revenue is 77%. This implies construction after time T costs $\$500{,}0000 \times 0.77 = \$385{,}000$. Since we expect costs to increase at rate μ, we discount the value by $\mu = 4\%$ over $T = 4$ years to get roughly $350,000.

If the price of the condominium decreases to the point where the purchaser walks away, the developer must find another purchaser for the newly finished unit. To sell a condo, the developers usually work through a broker, who charges a percentage B of the value of the condominium. According to SmartMoney (Todorova, 2007), the standard quoted brokerage fee is at around 6%, but actual brokerage fees range from around 4.5 to 5.5%. We use 5% throughout this chapter.

The volatility of condominium prices are denoted by σ_S, and the volatility of construction costs is denoted by σ_R. Throughout this chapter, we use de Jong and Driessen's (2008) estimate for the total volatility of a residential property of 11% per annum. From the composite construction cost price index, we see that the average standard deviation of the index is around 1.75% per year, which we round to 2%. We believe that purchasers and developers require a rate of return above that of the risk free rate to compensate for the risk of losses. Unlike for stocks, condos cannot be traded during the time that the contract is in effect, so uncertainty cannot be hedged away by either the purchasers or the developer. To compensate, they both require annual risk premiums which we denote by γ_p for the purchaser and γ_d for the developer. Shilling (2003) studied the risk premium for real estate investments. However, he dealt with investments in existing properties, so his results (6-6.75%) are not directly applicable here. Leung et al. (2007) studied the risk premium on presale contracts in Hong Kong, estimating them at around 5% for the purchaser.

We calculate the developer's risk premium using their reported gross margins. The cost of sales include expenses related to land acquisition, construction and development specific financing cost. Since financing costs are included in cost of sales, the return beyond the risk-neutral rate is $P = G/(1 - G)$ where G is the gross profit margin. This is the total return over T years. The annual rate is therefore $(1 + P)^{1/T} - 1$. Using $T = 4$ and $G = 0.23$, we take 6.75% to be the rate of annual risk premium.

2.2 Analysis of the Purchaser's Position

We present calculations showing the value of the presale contract for the purchaser in this section. In contrast to the assumptions required to derive the Black-Scholes equation for stock options, delta hedging is not possible for purchasers since condos cannot be continuously purchased and sold. An appropriate discount rate must be chosen to compensate the purchaser for her risk. We first derive the value of the contract where the purchaser is risk-neutral, and show the risk profile that arises from investing in the contract. We then introduce a risk premium and show the changes in valuation and risk profile.

2.2.1 Valuation under Risk Neutral Environment

The value of the presale of condominium contracts to the purchaser can be calculated using the principles of dynamic optimization as expounded by Dixit and Pindyck (1994). At each decision period, the decision maker chooses the option that maximizes his expected profit over the whole time period. The equation that expresses these dynamics is called the Bellman equation. Assuming the purchaser is risk-neutral, the expected profit potential for choosing to invest in the presale contract is expressed in the following Bellman equation.

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + \mu S \frac{\partial C}{\partial S} = r_p C. \qquad \textbf{(2-1)}$$

where C is the price of the presale contract on a condo, with values of S evolving in a GBM

$$dS = \mu S dt + \sigma_S S dW.$$

where W is a Wiener process. However, it is worth reiterating that since there is no early exercise, only the initial and final condo prices are relevant. The assumption of GBM does not need to hold for us to calculate the solutions. It is only necessary that the final condo prices be log normally distributed. Equation (2-1) requires final and boundary conditions. Under the assumption of rational exercise, the final condition at t = T can be written in the forms

$$C = \max(S - Q_2, -A) = \max(S - (Q_2 - A), 0) - A. \qquad \textbf{(2-2)}$$

If S ever falls to zero, it remains there until final time T. Therefore, the boundary condition at S = 0 is $C = \max(-Q_2, -A) = A$ discounted to present value. The value is discounted since the payment of A occurs in the future. At $S \to \infty$, there is no chance that the value of S will fall sufficiently for the purchaser to walk away. Therefore, the boundary condition at $S \to \infty$ is $C = (S - Q_2)e^{r_p(t-T)}$, discounted to present value. In summary, the boundary conditions are

$$C = -Ae^{r_p(t-T)} \qquad \text{at } S = 0,$$

$$C \approx S - Q_2 e^{r_p(t-T)} \quad \text{as } S \to \infty.$$

The solution for C can now be written in the form

$$C(S,t) = SN(d_1)e^{(\mu - r_p)(t-T)} - [Q_2 N(d_2) + A(1 - N(d_2))]e^{r_p(t-T)}, \qquad \textbf{(2-3)}$$

where

$$d_1 = \frac{\ln\left(\frac{S}{Q_2 - A}\right) + \left(\mu + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}},$$

$$d_2 = d_1 - \sigma\sqrt{T-t}.$$

The derivation of the formula can be found in Appendix B. Assuming the purchaser behaves rationally, she will only make the down payment Q_1 for the presale if $Q_1 \leq C(S, 0)$. Otherwise, the purchaser is expected to earn less than the risk-free rate of return and is better off not purchasing the condo.

Some solution curves of $C(S, 0)$ are plotted in Figure 2-1. The solution curves are plotted where there is no penalty to opt out, and when the purchaser cannot opt out of the contract (The maximum penalty case). Contract values under boom and bust conditions are also investigated, with μ taking on the rate of appreciation a purchaser sees as likely for the next T years (in this case, 4 years). The dotted lines mark the prevailing down payment rate.

We can draw a few conclusions from looking at the solution curves. First, the flexibility of the purchaser to opt out of the contract does not materially affect the purchaser's valuation of the contract at the prevailing down payment rate. The two solution curves merge very quickly for reasonable initial condominium prices of above $400,000. Assuming purchasers pay at or near $C(S, 0)$, we can expect the down payments to be similar regardless of whether the purchaser can opt out or not.

Secondly, there is a substantial difference in contract valuations across boom and bust conditions. The purchaser would pay the same down payment for a cheaper condominium in a boom period than for a more expensive condominium in a bust period. This agrees with our intuition that a down payment that secures the right to purchase an asset is worth more in a fast rising market.

We cannot delta hedge and earn risk free returns by trading condominiums. Buying a contract for less than its value can skew the odds of abnormal returns in the purchaser's favour, but it does not guarantee it. Let us now look at the probabilities of buying or selling a presale contract. The concept is related to the Value at Risk (VaR) used in financial institutions. While VaR is usually calculated over days and weeks, we calculate our potential profit and loss over a period of $T = 4$ years.

How much a purchaser gains or loses from a presale is primarily determined by the final outcome of the condominium price at the time of the final payment. The profit is calculated by subtracting $Q_1 e^{rT}$ (the value of the down payment at time T) and the final payment from the final price of the condominium.

Since S is assumed to follow GBM, and is the only random variable for purchasers, we can analytically calculate the probabilities of profit and loss for the purchaser of the condominium. The probability of gains excluding sunk costs (i.e. down payment), is calculated as shown below. The following equation shows the cumulative probability that gains made are under an arbitrary value p.

$$P(\text{gains} < p) = P(-A < S(T) - Q_2 < p) + P(S(T) - Q_2 < -A < p)$$

$$= P\left(\frac{-A+Q_2}{S(0)} < \frac{S(T)}{S(0)} < \frac{p+Q_2}{S(0)}\right) + P\left(\frac{S(T)}{S(0)} < \frac{-A+Q_2}{S(0)} < \frac{p+Q_2}{S(0)}\right).$$

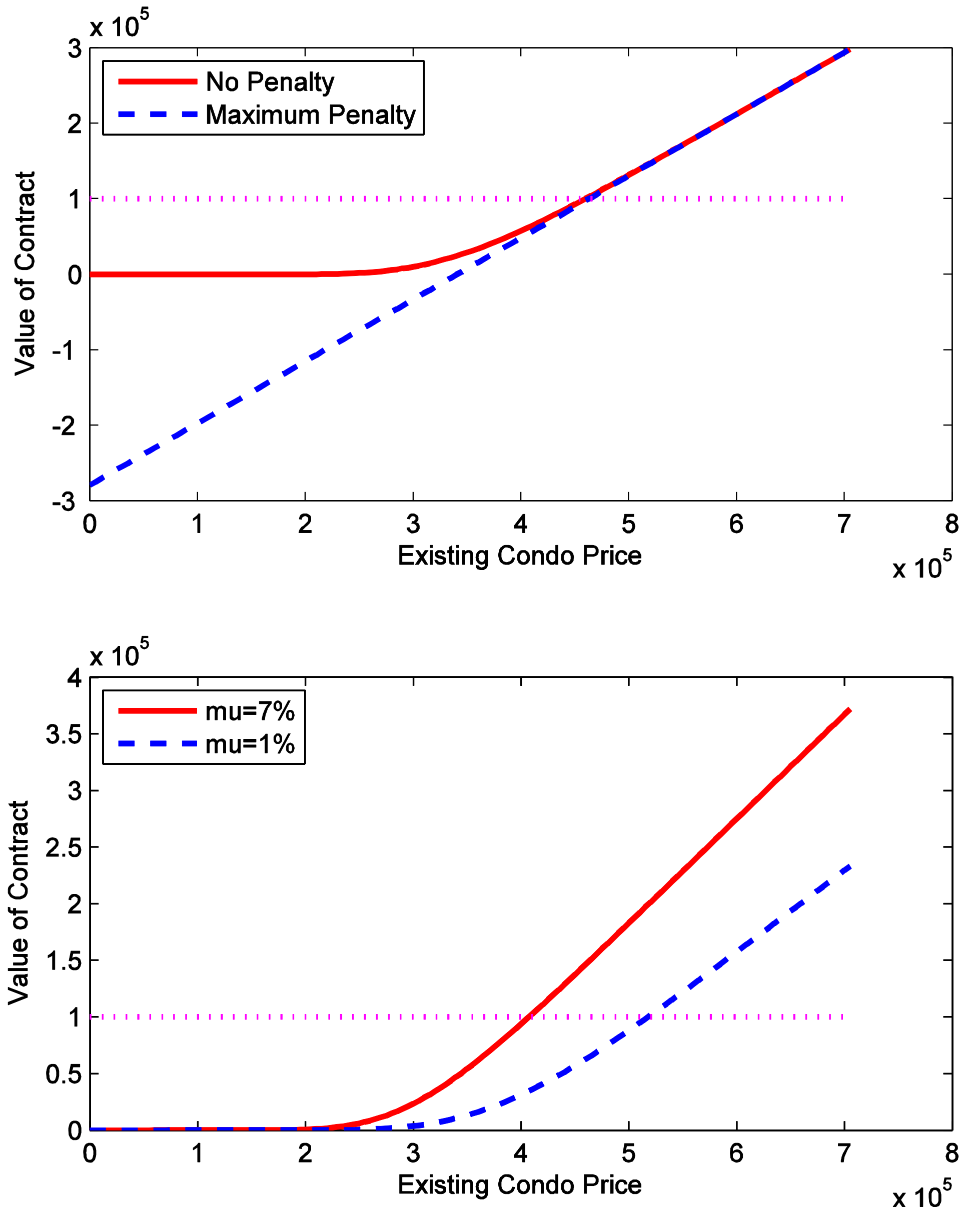


Figure 2-1: Top: No penalty and maximum penalty solutions. For a realistic range of condo prices, there is little difference between the value of contracts containing opt-out provisions and those which do not. Realistic prices occur near the intersection between the solid/dashed lines and the dotted line. For lower prices, this contract provision is much more important. Bottom: Boom and bust environment solutions. Purchasers' expectations about the rate of appreciation have a large effect on the perceived value of the presale contracts.

The probability of the profit falling below $-A$ is 0. Therefore, we only consider $p > -A$ in the next calculations. Since $\ln(S(T)/S(0)\,)$ has a mean of $\left(\mu - \frac{\sigma_S^2}{2}\right)T$ and a standard deviation of $\sigma_S\sqrt{T}$, the probability is

$$P(\text{gains} < p) = \Phi_{\left(\mu-\frac{\sigma_S^2}{2}\right)T,\sigma_S^2 T}\left(\ln\left(\frac{p+Q_2}{S(0)}\right)\right) - \Phi_{\left(\mu-\frac{\sigma_S^2}{2}\right)T,\sigma_S^2 T}\left(\ln\left(\frac{Q_2-A}{S(0)}\right)\right) + \Phi_{\left(\mu-\frac{\sigma_S^2}{2}\right)T,\sigma_S^2 T}\left(\ln\left(\frac{Q_2-A}{S(0)}\right)\right)$$

$$= \Phi_{\left(\mu-\frac{\sigma_S^2}{2}\right)T,\sigma_S^2 T}\left(\ln\left(\frac{p+Q_2}{S(0)}\right)\right).$$

where Φ_{μ,σ^2} denotes the cumulative normal distribution function with mean μ and variance σ^2. To calculate the total profit and loss distribution, we subtract the down payment from the gains calculated above. It follows that $P(\text{profit} < p) = P(\text{gains} - Q_1 e^{r_p T} < p)$.

The graph depicting profit and loss distribution for select $S(0)$ values are given in Figure 2-2. As a reminder, if we were to have a down payment to total payment ratio of 20%, the existing condo value would have to be around $S(0)\ =\ \$435{,}000$ according to Figure 2-1.

The steep drop in probability in the curves in the top graph of Figure 2-2 may be attributed to the purchaser's ability to opt out of the contract without a penalty – i.e. she can at most lose her down payment. The magnitude of the drop corresponds to the probability that the purchaser will opt out. Increasing $S(0)$ makes it more likely the purchaser will honor the contract. This is not surprising, since the purchaser would have made a bigger down payment for higher $S(0)$.

The curves for lower $S(0)$ in Figure 2-2 rise faster than for higher $S(0)$ with increasing profits. Since the cumulative distribution curve reaches a higher point earlier in comparison to curves for higher $S(0)$, the probability of achieving large profitability is lower. When a purchaser makes a smaller down payment, the purchaser is risking less money up front, so even though the percentage rate of return on capital might be higher, the potential for large profitability in absolute dollar terms is smaller.

Finally, we investigate the median and worst case profitability scenarios, as well as the standard deviation of profitability. The formula for the variance is calculated in Appendix D. The results for various environments are tabulated in Table 2-3.

Table 2-3 shows that changing risk free rates for the purchaser do not affect the profitability. This is because during calculation of the down payment as determined by $C(S,T)$, the down payment is discounted by the purchaser's risk free rate. To make a fair time T comparison, we appreciate the down payment with the same risk free rate. These two influences combine to negate the effects of the change in purchaser's risk free rate.

Although the purchaser on average breaks even on her investments, we see that even in the median case, she records a loss, since the probability distribution of profitability is skewed to the right. Figure 2-3 depicts this skewness under the maximum penalty environment. The skewness to the right can be observed in all other environments we have examined in Table 2-3.

The influences of two factors explain the median profitability for different $S(0)$ values. The first is that as $S(0)$ gets larger, the swings in the condo prices also become larger. We have seen that the probability distribution of profitability is skewed to the right. The median profitability declines to counteract the increased potential for profitability, keeping the average profitability at zero. The other factor is the premium paid for the right to walk out of the contract. The probability that the purchaser would benefit by reneging increases with lower $S(0)$, so the purchaser is made to pay more for the right to walk out. This results in a higher Q_1 and negatively impacts the median profitability for lower $S(0)$. These two factors oppose one another. In contrast, if the purchaser is unable to walk out, we see a monotone decrease in the median profitability with increasing $S(0)$, since the median profitability is affected by the first factor but not the second.

Given that a down payment of $100,000 would be appropriate for a condo roughly valued at $435,000, the chance of opting out for the purchaser is at 15.9% in our standard environment, higher than what our intuition would suggest. The introduction of the risk premium changes this picture as we shall see in the next section.

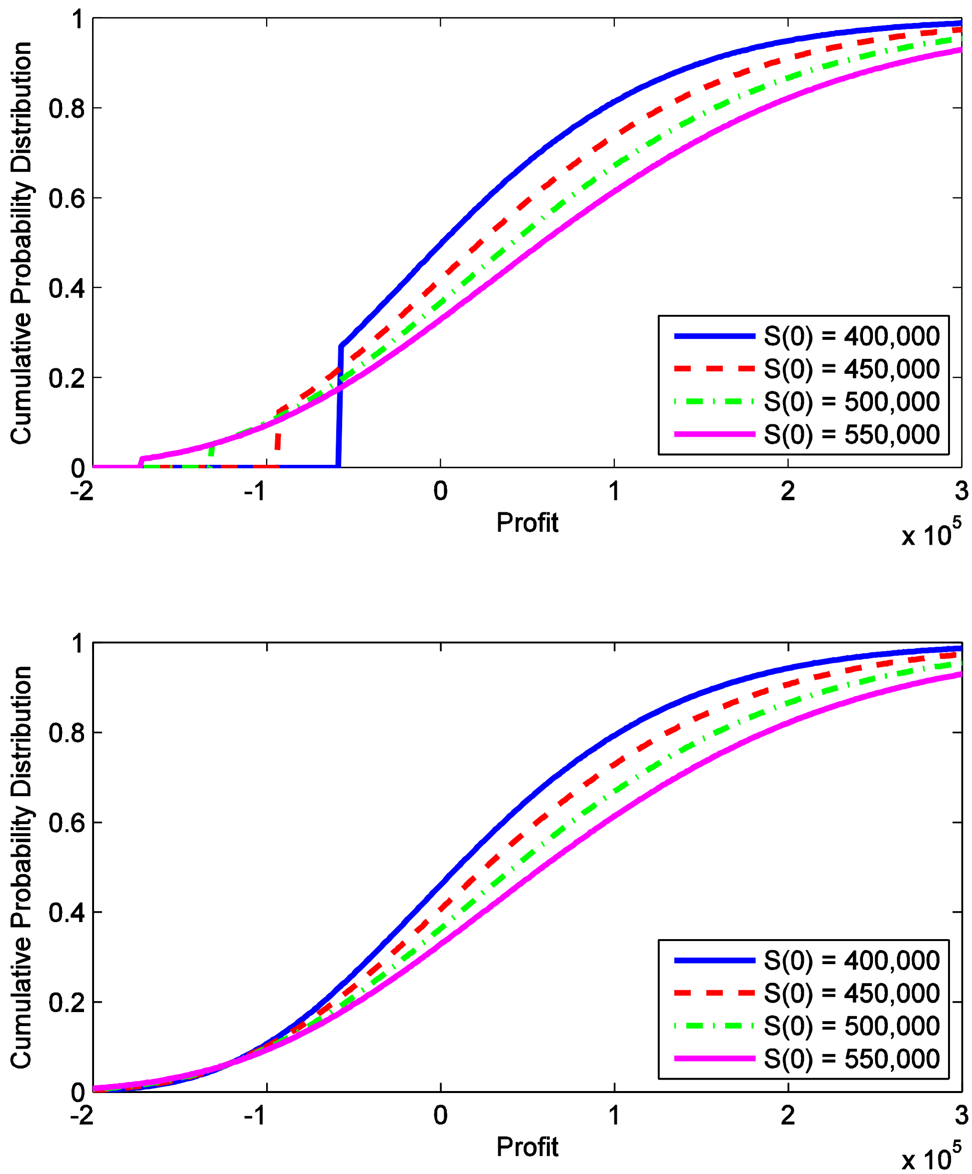


Figure 2-2: Cumulative probability distribution functions for profit, accounting for down payment. Top graph displays the no penalty case, and the bottom graph displays the maximum penalty case. The discontinuity in the top graph shows the limited risk borne by a purchaser when she can opt out - there is no chance that she can make a loss greater than a set threshold.

Environment	S(0)	Median	Worst 5%	Worst 1%	Default %	Std. Dev.
Standard Values	400,000	-24,181	-82,362	-82,362	26.9%	44,666
	450,000	-17,587	-133,040	-133,040	12.5%	55,249
	500,000	-15,781	-188,510	-188,510	5.1%	63,864
	550,000	-16,016	-207,300	-246,010	2.0%	71,311
Max Penalty	400,000	-11,223	-150,340	-194,760	0%	52,266
	450,000	-12,626	-169,130	-219,110	0%	58,799
	500,000	-14,029	-187,920	-243,450	0%	65,332
	550,000	-15,432	-206,720	-267,800	0%	71,865
$\mu = 7\%$	400,000	-17,516	-134,110	-134,110	12.3%	55,438
	450,000	-15,731	-196,390	-196,900	4.5%	65,018
	500,000	-16,246	-212,310	-261,199	1.5%	73,243
	550,000	-17,517	-233,190	-302,060	0.5%	80,901
$\mu = 1\%$	400,000	-44,343	-44,500	-44,500	47.1%	33,122
	450,000	-24,367	-81,533	-81,533	27.2%	44,570
	500,000	-18,096	-126,060	-126,060	13.9%	53,993
	550,000	-15,959	-174,720	-174,720	6.4%	61,901
$r_p = 7\%$	400,000	-24,181	-82,362	-82,362	26.9%	44,666
	450,000	-17,587	-133,040	-133,040	12.5%	55,249
	500,000	-15,781	-188,510	-188,510	5.1%	63,864
	550,000	-16,016	-207,300	-246,010	2.0%	71,311

$r_p = 1\%$	400,000	-24,181	-82,362	-82,362	26.9%	44,666
	450,000	-17,587	-133,040	-133,040	12.5%	55,249
	500,000	-15,781	-188,510	-188,510	5.1%	63,864
	550,000	-16,016	-207,300	-246,010	2.0%	71,311

Table 2-3: Risk profile of holding a presale contract for different environments, assuming the purchaser requires a risk free rate of return.

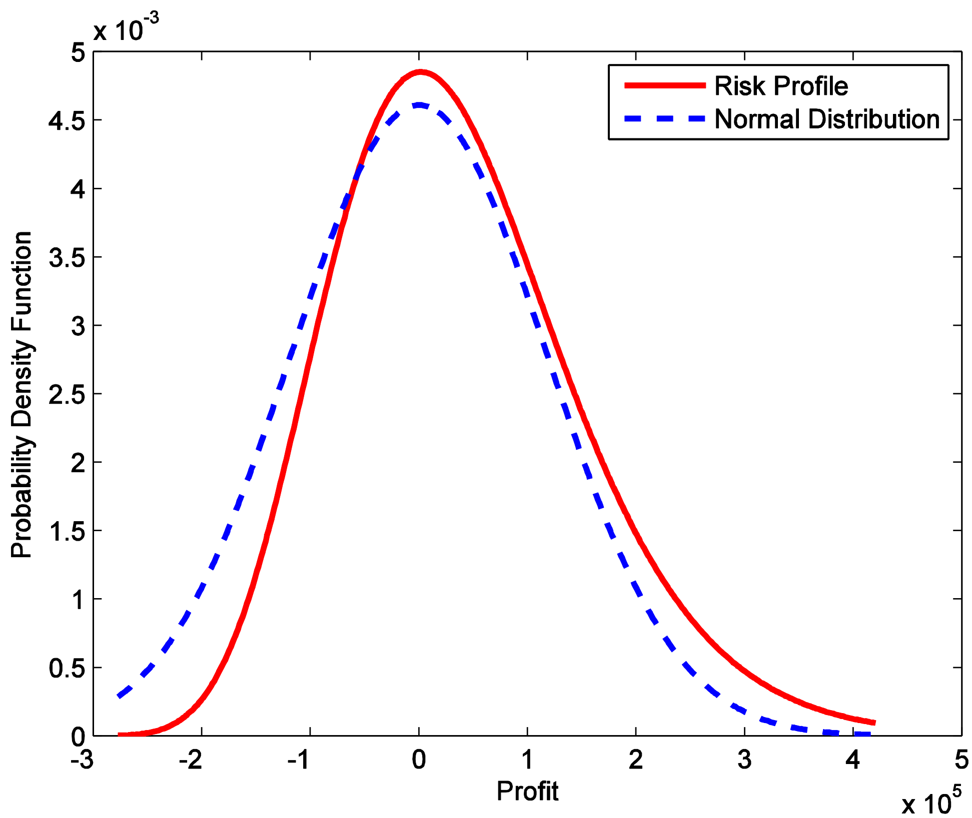


Figure 2-3: Comparison of the probability density function of purchaser's profitability against the normal distribution of equal mean and variance. The profitability curve chosen represents the environment where S(0) = 500,000 and where the purchaser is unable to opt out of the contract. The purchaser's profitability profile is skewed to the right, which gives the purchaser a greater chance of earning outsized profits.

2.2.2 Valuation under Risk Adjusted Return

In the previous subsection, we calculated the value of the contract when the purchaser demands only the minimum rate of return. But since the contract's risk cannot be hedged away, we believe that it is reasonable for the purchaser to demand a risk premium γ_p, in which case the Bellman equation must be modified to give

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + \mu S \frac{\partial C}{\partial S} = (r_p + \gamma_p)C. \tag{2-4}$$

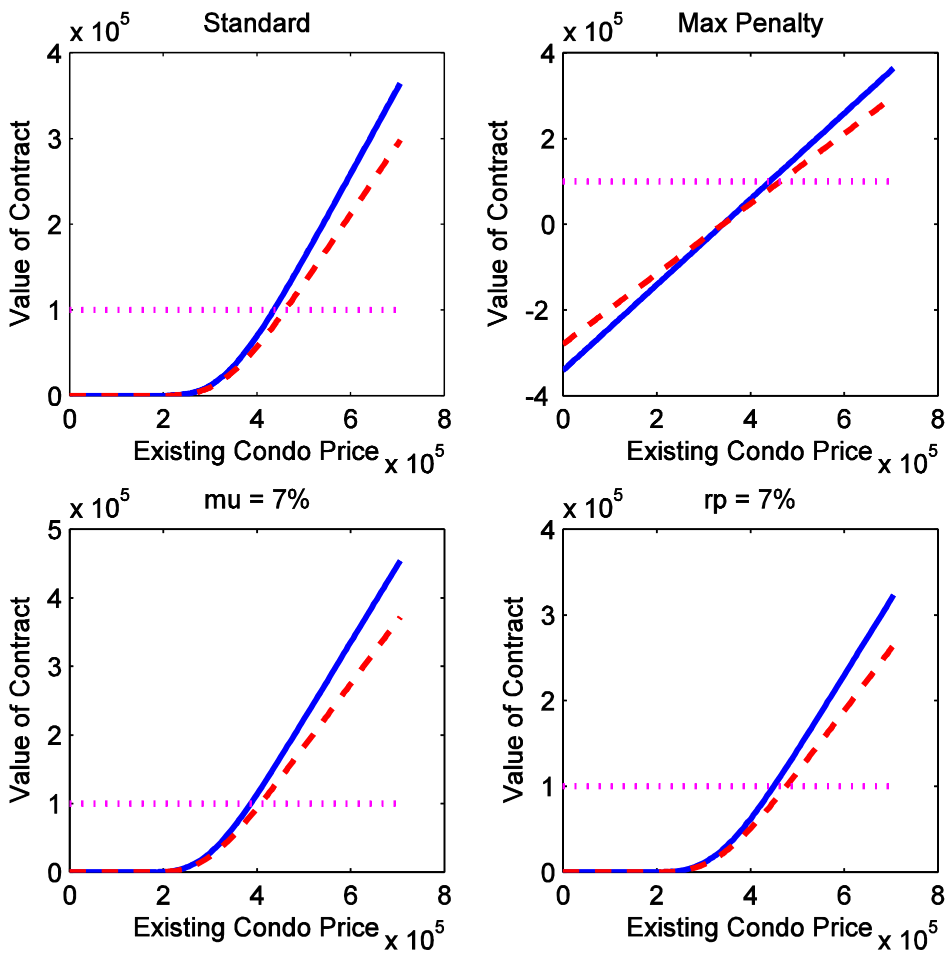


Figure 2-4: Comparison of solution curves between when a purchaser require risk free rates of returns, and when she requires risk adjusted rates of returns. Solid curves denote the risk free curves and the dashed curves denote the risk adjustedf curves. The differences between the curves are relatively small compared to the scale of overall condo prices, but big enough push median profitability into the positive under most circumstances.

Under the risk adjusted scenario, we do not need to adjust our solution methodology. Although we expect the value of the contract to change, we do not expect the standard deviation of the profits to change. This is because in calculating the profits $\max(S(T) - Q_2, -A) - Q_1 e^{r_p T}$, we only change the fixed amount Q_1. Figure 2-4 compares risk free and risk adjusted solutions.

Environment	S(0)	Median	Worst 5%	Worst 1%	Default %	Std. Dev.
Standard Values	400,000	-9,251	-67,432	-67,432	26.9%	40,855
	450,000	6,529	-108,920	-108,920	12.5%	51,777
	500,000	18,389	-154,340	-154,340	5.1%	63,864
	550,000	28,579	-162,700	-201,420	2.0%	71,311
Max Penalty	400,000	1,358	-137,760	-182,180	0%	52,266
	450,000	10,591	-145,910	-195,890	0%	58,799
	500,000	19,824	-154,070	-209,600	0%	65,332
	550,000	29,057	-162,230	-223,310	0%	71,865
$\mu = 7\%$	400,000	6,795	-109,800	-109,800	12.3%	62,024
	450,000	19,962	-156,430	-161,210	4.5%	65,018
	500,000	31,245	-164,820	-214,500	1.5%	73,243
	550,000	41,910	-173,760	-242,630	0.5%	80,901
$\mu = 1\%$	400,000	-30,098	-36,434	-36,434	47.1%	33,112
	450,000	-9,587	-66,754	-66,754	27.2%	44,457
	500,000	4,755	-103,210	-103,210	13.9%	53,993
	550,000	15,712	-143,050	-143,050	6.4%	61,901

Table 2-4: Risk profile of holding a presale contract for different environments, assuming the purchaser requires γ_p above the risk free rate of return. The median profitability is generally positive with the addition of the risk premium.

The top left graph of Figure 2-4 reveals that if the developer wishes to demand a $100,000 down payment, the present value of the condominium developed should be $460,000. This is slightly under the total of $500,000 the purchaser is required to pay for both the down payment and the final payment. By comparison, the present value should be $435,000 for the risk-neutral purchaser.

Risk adjusted contracts should always be less valuable than risk free contracts. However, this does not hold in the maximum penalty case shown in the top right graph, when the value of the contract is less than zero. In such a case, the developer is expected to pay the purchaser to enter into the contract, since the purchaser will be forced to pay Q_2 when the condo is finished. We consider this scenario to be unrealistic.

The new risk profile for the purchaser is given in Table 2-4. Unlike in Table 2-3, the median profitability for the purchaser is usually positive and monotonically increasing for $S(0)$. Higher $S(0)$ leads to higher risk premium in absolute dollars demanded by the purchaser. Thus the purchaser who takes a bigger risk by buying more expensive condos benefits more.

2.3 Analysis of the Developer's Position

In this section, we present calculations showing the value of the presale contract for the developer. We assume that the developer requires a risk premium and show the risk profile that arises from selling the contract.

2.3.1 Model and Analytical Solution

In modeling the value of the presale contract from the developer's perspective, we have to take into account construction costs. The profit for a developer of a condo is the revenue resulting from the project, less the cost associated with the construction of the condo.

The condo price is assumed to follow GBM as before. Since we are also taking into account construction costs which we assume follow GBM with some correlation, we have two equations as follows.

$$dS = \mu S dt + \sigma_S S dW_S,$$

$$dR = \mu R dt + \sigma_R R dW_R,$$

where $E[dW_S dW_R] = \rho dt.$

The Bellman equation is formed on the same principle as was for the purchaser - the expected gain from holding the presale contract is the same as the gain from earning a desired rate of return. For the developer, this rate is the interest rate of the bank loan plus the risk premium.

Developers borrow money from banks to finance construction. Immediately after borrowing the money, the developer has two alternatives. One is to repay the money to the bank. Interest not paid is interest earned, and repaying the loan “earns" the developer money at the rate of the loan. If the developer chooses to build, she must expect a return greater than the rate at which the loan is negotiated. Otherwise, the developer would be assuming uncompensated risks. Therefore, the developer requires a risk premium to increase the chance of profitability. The Bellman equation which takes these factors into account is as follows.

$$(r_d+\gamma_d)V=\frac{\partial V}{\partial t}+\frac{1}{2}\sigma_S^2S^2\frac{\partial^2 V}{\partial S^2}+\frac{1}{2}\sigma_R^2R^2\frac{\partial^2 V}{\partial R^2}+\rho\sigma_S\sigma_R SR\frac{\partial^2 V}{\partial S\partial R}+\mu S\frac{\partial V}{\partial S}+\mu R\frac{\partial V}{\partial R}. \qquad \textbf{(2-5)}$$

This is almost identical to the multidimensional version of the Black-Scholes equation. There are two differences. We allow μ to differ from r_d, allowing rate of condo price appreciation to diverge from risk free rates. We also see the presence of the risk premium γ_d.

We can solve equation (2-5) analytically. The solution of the multidimensional Black Scholes in terms of the payoff function is given by Wilmott (2006). We follow a similar solution but with $\mu \neq r_d$ and including γ_d.

$$\begin{gathered}V(S,R,t)=e^{-(r_d+\gamma_d)(T-t)}\big(2\pi\sigma_S\sigma_R(T-t)\big)^{-1}\\ \times(1-\rho^2)^{-\frac{1}{2}}\int_0^\infty\int_0^\infty\frac{V(S',R',T)}{S'R'}e^{-\frac{1}{2}\alpha^T\Sigma^{-1}\alpha}dS'dR',\\ \alpha=\begin{pmatrix}1/(\sigma_S(T-t)^{\frac{1}{2}}\left(\ln\left(\frac{S}{S'}\right)+\left(\mu-\frac{\sigma_S^2}{2}\right)(T-t)\right)\\ 1/(\sigma_R(T-t)^{\frac{1}{2}}\left(\ln\left(\frac{R}{R'}\right)+\left(\mu-\frac{\sigma_R^2}{2}\right)(T-t)\right)\end{pmatrix},\ \Sigma=\begin{pmatrix}1&\rho\\ \rho&1\end{pmatrix}.\end{gathered} \qquad \textbf{(2-6)}$$

At $t=T$, the developer will receive Q_2-R if $S+A>Q_2$. If $S+A<Q_2$, the developer will receive A and in addition, she will sell the condo for $S(1-B)$where B is the brokerage fee. Thus the final condition at $t = T$ can be written as

$$V(S,R,T) = W(S,T) - R = \begin{cases} Q_2 - R & \text{if } S + A > \ Q_2 \\ A + S(1-B) - R & \text{if } S + A < \ Q_2 \end{cases}. \qquad \textbf{(2-7)}$$

We have introduced $W(S,T)$ to emphasize that $V(S,R,T)$ can be broken into separate components involving S and R. In terms of W and R, the integral in equation (2-6) becomes

$$\int_0^\infty \int_0^\infty \frac{V(S',R',T)}{S'R'} e^{-\frac{1}{2}\alpha^T\Sigma^{-1}\alpha} dS'dR' = \int_0^\infty \int_0^\infty \frac{W(S',T) - R'}{S'R'} e^{-\frac{1}{2}\alpha^T\Sigma^{-1}\alpha} dS'dR'$$

$$= \int_0^\infty \frac{W(S',T)}{S'} \int_0^\infty \frac{1}{R'} e^{-\frac{1}{2}\alpha^T\Sigma^{-1}\alpha} dR'\, dS' - \int_0^\infty \int_0^\infty \frac{1}{S'} e^{-\frac{1}{2}\alpha^T\Sigma^{-1}\alpha} dS'dR'.$$

The above equation consists of integrations over bivariate lognormal probability distributions. Solving the inner integrals yields

$$\sigma_R[2\pi(T-t)(1-\rho^2)]^{1/2} \int_0^\infty \frac{W(S',T)}{S'} e^{-\frac{\alpha_1^2}{2}} dS' - \sigma_S[2\pi(T-t)(1-\rho^2)]^{1/2} \int_0^\infty e^{-\frac{\alpha_2^2}{2}} dR'.$$

The remaining integrals can be integrated using the same technique employed for calculating the Purchaser's Position in Appendix C, yielding

$$\int_0^\infty \frac{W(S',T)}{S'} e^{-\frac{\alpha_1^2}{2}} dS' = \left[(1-B)\big(1-N(d_1)\big)Se^{\mu(T-t)} + Q_2N(d_2))\right]\sigma_S\sqrt{2\pi(T-t)}$$

$$\int_0^\infty e^{-\frac{\alpha_2^2}{2}} dR' = -Re^{\mu(T-t)}\sigma_R\sqrt{2\pi(T-t).}$$

Where

$$d_1 = \frac{\ln\left(\frac{S}{Q_2 - A}\right) + \left(r_d + \gamma_d + \frac{1}{2}\sigma_S^2\right)(T-t)}{\sigma_S\sqrt{T-t}}, \quad d_2 = d_1 - \sigma_S\sqrt{T-t}.$$

Substitute the results back into equation (2-6) to obtain

$$V(S,R,t) = \left[(1-B)\big(1-N(d_1)\big)S - R\right]e^{(\mu - r_d - \gamma_d)(T-t)} + [Q_2N(d_2) + A(1 - N(d_2))]e^{-(r_d - \gamma_d)(T-t)}. \qquad \textbf{(2-8)}$$

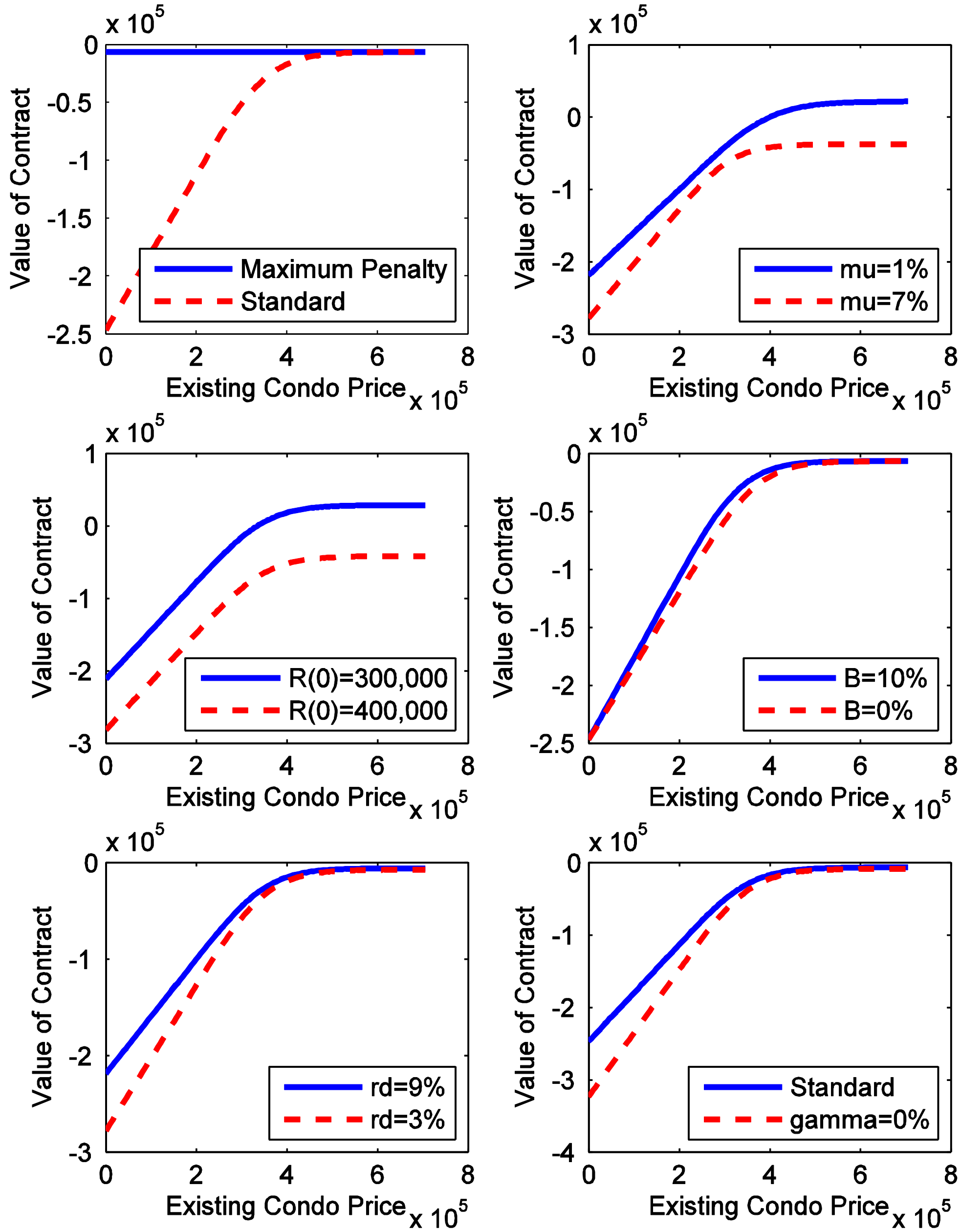


Figure 2-5: Solution curves for the developer's real option. The curves represent the fair developer's presale contract price in different environments. Near the realistic price of around $460,000, variations in the value of expected price appreciation(top right) and initial construction cost(mid left) significantly impacts the price of presale contracts. Variations in other factors do not appear to impact the prices as significantly.

It might surprise the reader that the correlation coefficient ρ does not feature in equation (2-8). In other words, $V(S,R,t)$ does not depend on whether condo prices and construction costs are correlated. This is because the expected revenues and construction costs are separated by an additive operand, as seen from the relation $E[W(S,t) - R] = E[W(S,t)] - E[R]$.

Solution curves under select environments are graphed in Figure 2-5. Unless otherwise stated, standard values presented in Table 2-1 are used. We remind the reader that the solution curves are equal to the expected returns on fulfilling the contracts, without taking into account the down payment. Also, note that since a presale contract is analogous to a call option, the payoff to the developer appears similar to having sold a covered call option. The fair down payment varies for each environment. We can make a few observations about the results.

The solutions incorporating no penalty and maximum penalty may seem very different at first glance. However, we can see that for realistic values of existing condo prices, the value to the developer is quite similar. As noted in section 2.2.2, the condo should be worth \$460,000 today if the developer were to charge a \$100,000 down payment. For the solution curves at the \$460,000 point in the x axis and beyond, we can see that there is actually very little difference between the two solution curves in the top left graph of Figure 2-5. There is actually very little value in placing a clause in the contract that binds the purchaser to honour it, unless we are negotiating for a condominium the present value of which is much lower than the purchaser's total payment.

Contract values materially differ depending on the expected rate of condo value appreciation. The contract is worth less to the developer when condo values are expected to appreciate more, and vice versa. In effect, the developer has sold a call, capping his revenue on the condo price. No matter how far the condo price appreciates, the developer can at most earn $Q_1 + Q_2$, whereas she is still responsible for paying the total construction cost. The cost, which rises with drift μ, hurts the developer's expected profit so the developer expects to earn more money when μ is lower. To compensate, the developer charges a higher down payment for a higher μ.

If the quoted initial construction cost is lower, then the developer is expected to make more profit. This conclusion is reflected in the middle left graph of the figure, where the gap between the two solution curves equal the difference between initial construction costs.

There is very little difference between solution curves of different B values for $S(0) > \$450{,}000$, as can be seen in the middle right graph. If the condo price is greater than $450,000 at the time the contract is signed, there is only a small chance (12.5% in the standard environment) that the condo price will depreciate so much that the purchaser will walk away. Since the developer only pays a brokerage fee if the purchaser forfeits the condo unit, the impact of brokerage fees to the developer is small for initial condo value over $450,000. However, brokerage fees do make an impact when the contract is being negotiated for smaller initial condo values, since then the purchaser is much more likely to default. At very small $S(0)$, the impact of the brokerage fee is hard to observe. Brokerage fees are proportional to the final condo price, and so the expected fee is lower for smaller $S(0)$.

Finally, the effects of changing r_d and γ_d are similar, as can be expected from looking at equation (2-5). Increase in either variable shifts the curves towards zero, reflecting the higher discounting involved. The solution curves seem to converge to the same level for higher initial condo prices for increasing r_d or γ_d. However, this happens only because the regions where contract values start levelling off are close to 0. A closer inspection of the graph reveals that two curves do not meet.

2.3.2 Risk of Profit and Loss

In this section, we investigate the risk of taking on a condo construction project with a presale. The profit that the condo developer takes at time T is

$$V(S,R,T) + C(S,0)\, e^{r_d T}.$$

where the first term $V(S,R,T)$ comes from equation (2-7), and the second term is the down payment plus interest accrued. We are assuming here that the purchaser pays the fair down payment calculated in section 2.2.2.

In constructing the profit and loss profiles for the developer, we would like to know the realistic range of values for ρ. The correlation between the NHPI and the CPI is 0.57. Letting N be the variable representing the path of the NHPI,

$$\frac{\sigma_{NR}}{\sigma_N \sigma_R} = 0.57$$

Note that S has two components - the systematic component and the idiosyncratic component($S = NI$). The idiosyncratic component incorporate mainly the property's location, so we expect it to be uncorrelated to R. The covariance of S and R is therefore

$$\sigma_{NR} = E\left[\left(\ln\left(\frac{S}{S(0)}\right) - \mu + \frac{1}{2}\sigma_S^2\right)\left(\ln\left(\frac{R}{R(0)}\right) - \mu + \frac{1}{2}\sigma_R^2\right)\right]$$
$$= E\left[\left(\ln\left(\frac{N}{N(0)}\right) + \ln\left(\frac{I}{I(0)}\right) - (\mu_N + \mu_I) + \frac{1}{2}(\sigma_N^2 + \sigma_I^2)\right)\left(\ln\left(\frac{R}{R(0)}\right) - \mu + \frac{1}{2}\sigma_R^2\right)\right]$$
$$= \sigma_{NR} + \sigma_{IR}.$$

As a reminder, de Jong and Driessen's (2008) estimation is that $\sigma_{IR} = 0.11$ while $\sigma_{IR} = 0.06$. Therefore, the correlation between S and R can be expected to be

$$\frac{\sigma_{SR}}{\sigma_S \sigma_R} = \frac{\sigma_{NR}}{\frac{11}{6}\sigma_N \sigma_R} = 0.31.$$

Therefore, it would be useful to look at the risk profiles assuming $\rho = 0.3$, and the higher and lower ρs relative to the benchmark.

The profit and loss profiles under different environments are tabulated in Table 2-5. We have used the Monte Carlo method to obtain our results. For each line of results, 100,000 pairs of price paths for condo price and construction costs were generated. Some conclusions we can draw from the table are as follows.

It appears that a higher ρ translates into less risk for the developer. We can understand this because big losses accrue when condo price is low and construction cost is high. With a high correlation, the chance of construction cost increasing while revenue decreasing is small, and this leads to stable profits. The lower risk appears to come at the cost of a slightly smaller median profit. The benefit of a higher correlation is non-existent for the maximum penalty case where revenue, but not cost, is fixed.

If r_d is small, we see a smaller profit, and vice versa. This is to compensate for the different rate of return required for the developer. The down payment is calculated using the purchaser's risk free rate r_p. At time T, the value of money is adjusted using r_d. Therefore, unlike for the purchaser, the effects do not cancel out and we have different risk profiles for different r_d values.

Environment	S(0)	ρ	Median	Worst 5%	Worst 1%	Std. Dev.
Standard Values	450,000	0	104,760	45,173	-3,060	27,378
		0.3	103,850	53,447	6,836	24,960
		0.6	103,080	60.774	19,901	22,076
	500,000	0	155,760	120,530	77,619	21,083
		0.3	155,140	122,020	89,817	19,602
		0.6	154,980	124,080	100,560	17,996
Max penalty	450,000	0	103,240	75,371	63,257	16,471
		0.3	103,050	75,334	63,081	16,446
		0.6	103,070	75,275	63,154	16,430
	500,000	0	155,150	127,300	115,160	16,422
		0.3	155,190	127,200	115,190	16,426
		0.6	155,250	127,460	115,460	16,409
$\mu = 7\%$	450,000	0	110,090	73,677	37,912	22,035
		0.3	110,510	75,537	49,903	20,733
		0.6	110,170	76,805	57,285	19,336
	500,000	0	169,350	136,200	117,590	19,710
		0.3	169,240	136,880	121,200	19,027
		0.6	169,130	137,140	122,760	18,587
$\mu = 1\%$	450,000	0	101,430	8,285	-35,504	37,308
		0.3	99,897	15,528	-25,148	34,138

		0.6	98,517	23,050	-13,775	30,620
	500,000	0	144,870	81,341	33,058	27,562
		0.3	143,930	88,349	42,018	25,239
		0.6	143,930	95,680	54,489	22,419
R(0)=300,000	450,000	0	163,690	104,890	56,049	26,127
		0.3	162,910	113,320	67,123	23,432
		0.6	162,210	119,380	76.240	21,289
	500,000	0	214,470	182,480	138,180	19,241
		0.3	214,070	184,630	147,230	17,971
		0.6	213,830	186,040	157,430	16,475
$r_d = 9\%$	450,000	0	120,050	61,963	12,001	27,199
		0.3	118,740	68,623	24,098	24,724
		0.6	118,090	75,381	35,191	22,052
	500,000	0	177,707	141,730	100,470	20,927
		0.3	176,530	143,490	110,970	19,529
		0.6	176,290	145,350	121,350	18,026
$r_d = 3\%$	450,000	0	91,459	32,364	-16,999	27,461
		0.3	90,536	40,267	-4,718	24,761
		0.6	89,697	46,261	6,529	22,169
	500,000	0	136,830	101,330	58,660	21,131
		0.3	136,420	103,420	70,614	19,562
		0.6	136,140	105,100	81,370	18,057

Table 2-5: Developer’s risk profile for a presale contract in different environments

Additionally, we can draw the following conclusions from examining the table. The developer is likely to earn more profit if the condo market appreciates faster, or if construction cost is initially cheaper, or if the condo starts out with a higher initial value.

2.3.3 Comparison with No Presale

In this section, we compare the profit and loss profiles of developers who chose to hold a presale, against that of those who chose not to hold a presale. For the purchaser, the risk profile for buying a finished property is trivial - there is no risk involved since she is buying the finished property. The total value $Z(S,t)$ for the developer with a presale is

$$Z(S,R,t) = Q_1 + V(S,R,t).$$

In other words, this is the sum of the down payment and the expected profit after construction. Supposing that the purchaser and the developer agree at $Q_1 = C(S,0)$. Then, the value for the developer is

$$Z(S,R,t) = SN(d_1)\left(e^{(\mu-r_p-\gamma_p)(T-t)} - e^{(\mu-r_d-\gamma_d)(T-t)}\right) + \left[\left(1-B+BN(d_1)\right)S - R\right]e^{(\mu-r_d-\gamma_d)(T-t)}$$

$$\times\left[Q_2N(d_2) + A(1-N(d_2))\right]\left(e^{(\mu-r_p-\gamma_p)(T-t)} - e^{(\mu-r_d-\gamma_d)(T-t)}\right).$$

Note that should $r_d + \gamma_d = r_p + \gamma_p = r + \gamma$, the formula simplifies considerably.

$$Z(S,R,t) = \left[\left(1-B+BN(d_1)\right)S - R\right]e^{(\mu-r-\gamma)(T-t)}.$$

This must be compared with the situation where the developer simply holds on to the condos and sell them for S at $t = T$. In this case we have a situation similar to that of holding a European call option. The value of the opportunity $Y(S,t)$ has a payoff of

$$Y(S,R,T) = S(T) - R(\mathrm{T}).$$

Thus

$$Y(S,R,t) = E[S(T) - R(T)|S(t),R(t)]e^{-(r_d+\gamma_Y)(T-t)} = (S(t) - R(t))e^{(\mu-r_d-\gamma_Y)(T-t)}.$$

where γ_Y is the risk premium for when the developer does not hold a presale. This number is expected to be higher than γ_d because there is more risk associated with not holding a presale. Thus we see that a developer at some price S at $t = 0$ will have a presale for a given value of Q_2 provided

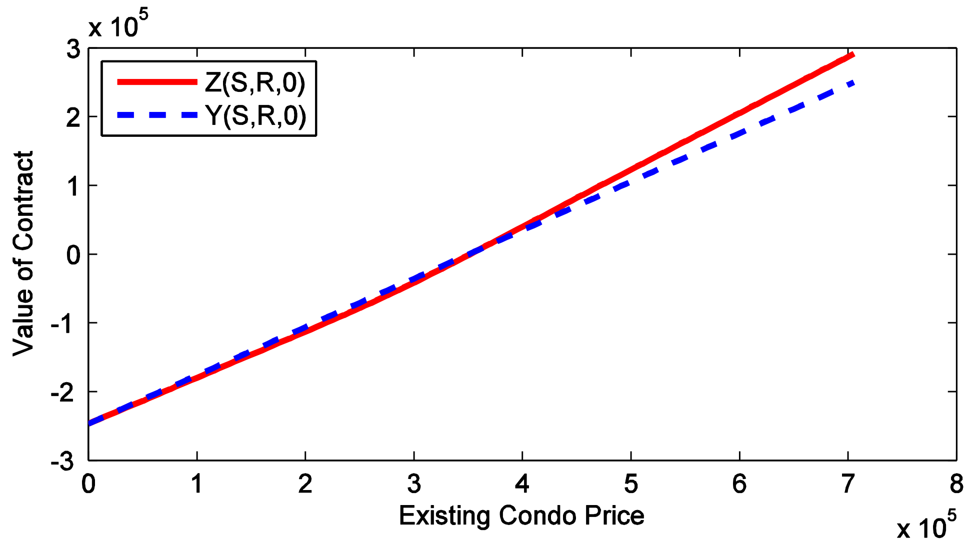
$$Z(S, 0) > Y(S, 0).$$

If we were to incorporate fees from holding the presale, we can assume that the fees would be similar whether the developer holds a presale or holds sales after construction. In that case, we subtract similar values from the left and right hand side of the above equation, leaving the inequality unchanged. Figure 2-6 compares Z and Y curves.

There is little difference between the two curves. The difference is more pronounced if $\gamma_Y = \gamma_d$. As long as the purchasers make down payments which equal the value of the contracts, the developer can be expected to earn approximately the same amount through a presale, as she would through selling finished units. For higher γ_Y, it makes more sense to hold a presale. This is because having a higher γY means the developer is more risk averse, and therefore is willing to forgo more profit in favour of earnings stability. Holding $\gamma_Y = \gamma_d$, the two curves stay very close to each other even if we alter other environment variables. Table 2-6 shows the average deviation between the two curves for a variety of environment factors.

The effects of altering γ_p and γ_d are the same as altering r_p and r_d, so we do not investigate them separately. The shapes of the graphs remain mostly consistent through different environments. When r_p and r_d are closer, the two solutions curves are closer together for higher existing condo values. The two curves are closer together for lower existing condo values when B is lower.

The average deviation assuming all standard values is close to \$10,000, which is not large compared to the total amount of purchases involved. For reasonable condo prices above \$450,000, the total benefit developers expect from holding presales exceeds the amount that developers expect from selling finished units. The source of this advantage resides in the different $r_d + \gamma_d$ and $r_p + \gamma_p$ values. If they are equal, the curves are nearly equal to each other for reasonable condo prices. With lower $r_p + \gamma_p$, the purchaser is paying slightly more in down payment, since she does not require as high a return on her savings as the developer.



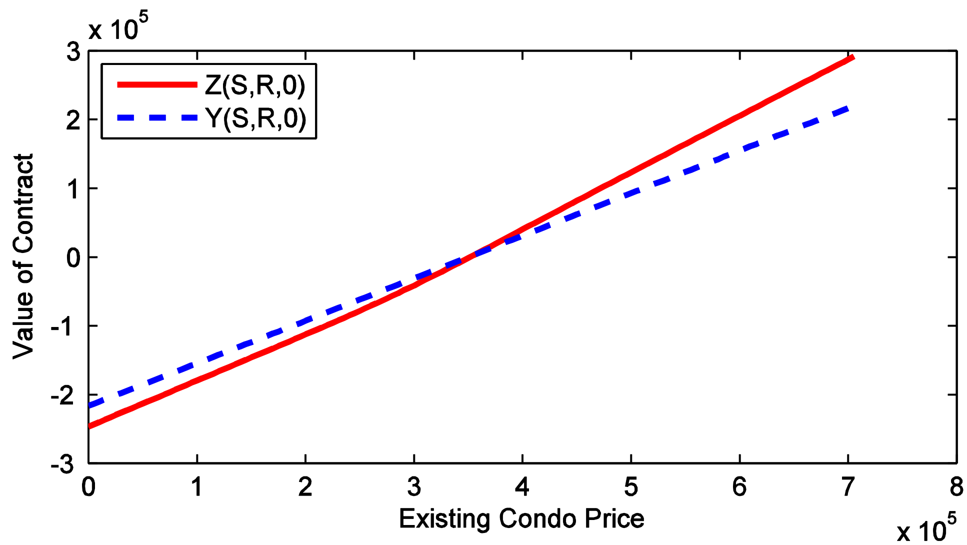


Figure 2-6: Comparison of the total profitability expected from holding presales (Z), to the profitability expected from build condos without holding a presales (Y). Top: $\gamma_Y = \gamma_d$ - If risk premiums are held to be the same, the profitabilities are roughly equal with holding presales having a slight advantage. Bottom: $\gamma_Y = 10\%$ - holding presales are more attractive if the developer requires a bigger risk premium for not holding presales.

Environment	Average Deviation
Standard values	$11,664
Max penalty	$19,449
R(0)=$300,000	$11,664
R(0)=$400,000	$11,664
$\mu = 7\%$	$16,624
$\mu = 1\%$	$9,677
$B = 0\%$	$10,082
$B = 10\%$	$14,041
$r_d = 9\%$	$18,166
$r_d = 3\%$	$4,878
$r_p = 7\%$	$4,327
$r_p = 1\%$	$20,483

Table 2-6: Average deviation between the total profitability expected from holding presales (Z) and profitability expected if without presales (Y). Results shown are $\sum_{i=1}^{N} |Z_i - Y_i|/N$, with each Z_i and Y_i calculated with $S(0)$ ranging over 700 to 700,000. The average deviations are often noticeable but not material in the context of the overall sums of money usually involved ($450,000~$500,000).

2.4 Conclusion

This chapter computes values of presale contracts and discusses the consequences of entering into presale agreements. The purchaser can expect, in the median case, to make a gain on her investment above the risk free rate of return, provided the presale contract is priced to include a risk premium. In return, the purchaser bears some of the risk that the developer would otherwise assume. According to our model, the developer can expect at least as much profit from holding a presale, as she would by not holding it. It appears that the reduction in risk comes at no cost to the expected profitability for the developer. Given the lower risk to reward ratio expected by holding a presale, we are not surprised to find that most development projects are sold through presales.

Developers are not required to sell all available units through a presale before starting construction. The assumption is that developers will be able to sell the rest of the units through a presale during construction, or sell them after they have been finished. However, it remains that the more units they presell, the less risky the project. With the credit crisis making many lenders anxious, the lenders have increased the percentage of units that needs to be presold from 60% to 70% (Belford, 2008).

The profitability and the risk profile for both the purchaser and the developer are sensitive to the expected rate of appreciation of the condominium. Both developers and purchasers suffer from reduced profitability and increased uncertainty when μ is lower. Consequently, the current trend in housing prices in the US significantly impacts the decisions made by both developers and purchasers.

During 2008-9, the US had experienced a severe recession led by the housing market. The seasonally adjusted Case-Shiller Home Price Index dropped an average of 1.7% per month from October 2007 to September 2008. The decline in property prices leads purchasers to expect an even further decline in prices. To compensate, purchasers will demand a lower final condo price for the same down payment, than they would have in better times.

The low μ does not guarantee that developers would not be willing to build new units, even if the developer is forced to sell condos of higher value at a discount. The construction cost can determine whether the developer can expect to make a profit or a loss. Up until the summer of 2008, the cost for building materials was at an all-time high. Since then however, we have seen a dramatic reduction in the cost of materials. Labour is becoming cheaper, and we are able to discern from the annual reports of developers that land prices are dropping. Without knowing the extent to which costs are dropping, we are uncertain about the profitability of the developers going forward. However, it remains a possibility that profitability for developers will improve in the near future, even given the low μ.

Throughout this chapter, we have modeled the movement in condo prices using Geometric Brownian Motion. In reality, the historical probability distribution of returns on New Housing Price Index and Construction Price Index exhibit heavier tails. We leave the analysis involving distributions with heavier tails as possible future work. Intuitively, we would expect that the risk profile increases for both the purchaser and the developer, and the presale contract value to increase in order to compensate for the increased volatility.

Chapter 3

3 Value of Urban Land under Regulatory Uncertainty

Real option price models have been widely used in the academic literature to model the value of land. In this view, land is seen as being similar to a perpetual financial call option where the underlying asset is the constructed building and the strike price is the development cost. The common practice in the industry is to use the Net Present Value paradigm to calculate the value of land. However, there are many complications specific to the real estate market that may lend to the adaptation of the real options paradigm.

These different paradigms can lead to different land valuations, the usage of which can affect the developers' purchasing decisions. Given the same market price, a developer who places a higher valuation on a tract of land may purchase and develop land, while a developer who values it less may hold out. Accurate land values are important, as purchasing land based on a misleadingly high valuation can lead to suboptimal profitability for the developer, whereas avoiding a purchase based on a misleadingly low valuation may deprive the developer of a profitable opportunity. In addition to enabling us to more accurately value land, the real options model also allows us to analyze the development lags in the time between the purchase of raw land and the development of lots.

In early real options literature, Titman (1985) valued underutilized urban land. He modeled different payoffs according to the different heights of buildings that could be constructed, and reasoned that it is rational to leave land underutilized in the hopes of realizing bigger payoffs in the future. Grenadier (1996) introduced a game-theoretic approach to model the behaviour of multiple land owners. He gave different valuations of land, differentiating between developers who are first to develop and those who are last to develop. He used his model to explain why we observe bursts of construction activity rather than developments at a steady rate.

Capozza and Helsley (1990) valued the option to turn agricultural land into urban land. In their model, an irreversible decision can be made to construct a building on agricultural land. The building earns rent which they assumed would follow Arithmetic Brownian Motion (ABM). They explain that the irreversibility of their decisions causes developers to postpone development, which in turn reduces city size. Capozza and Li (2001) use their theory to explain the positive relationship observed between interest rates and rate of land development.

Bar-Ilan and Strange (1996) present a more sophisticated model in a similar vein to Capozza and Helsley. Rents are modeled to follow a Geometric Brownian Motion (GBM), and decisions to develop agricultural land are made reversible. Their model incorporated fixed development time lags to explain the phenomena in which distant land is sometimes developed prior to nearby land.

Leishman et al. ((2000)) conducted empirical research on builder behaviour to test the validity of the real options based approach, but didn't provide conclusive evidence either in support or in opposition to the real option valuation approach. They stopped short of modeling the relationship between house and land prices, instead relying on builders' projections versus observed prices to test the influence of uncertainty on land prices.

Buttimer et al. (2008) sought to model business realities more closely and used real options to determine what effects holding presales had on the risk and return characteristics of subdivision developers. While we agree with the intuitive concepts presented in their paper, we do not agree with their mathematical formulation. For instance, their formula for the payoff to the homebuilder is the following.

$$[P(T) - \left(X(T) - V(T_0, P(T_0); T_C)\right)]^+.$$

Here P is the completed lot price, X is the strike and V is the presale option value. As an option, the price that builders pay for V should always be positive, and paying a higher price for the option should cut into the builders' profits. But according to the above formulation, a higher V translates into higher payoffs for the builders. Moreover, since V is a sunk cost, it should reside outside the square brackets.

In general, discussions with developers lead us to believe that models in previous literature overlook several significant factors. For one, building on land is not a one step process. For residential land, subdivision developers face two decision points. They must decide when to apply for a permit and commit to developing land, and after the permit has been granted, decide when to fully develop land into lots so that they are ready for builders to start building. Builders purchase these lots and construct houses which they in turn sell to home buyers. Land is therefore better modeled as a compound option.

Previous literature has assumed real estate prices follow either ABM or GBM. However, an examination of price series such as the New Housing Price Index (StatCan, 2009) or the Macromarkets LLC repeat-sales index Lincoln Institute of Land Policy reveals that price movements are cyclical around a growing mean. The usage of ABM or GBM can overstate the uncertainty surrounding future real estate prices, particularly over long periods of time. The Martingale property of ABM and GBM fails to explain the behaviour of some developers who choose to sit on the sidelines during periods when they believe the market is overheated.

Only limited effort has been made to understand regulatory risk. There are many ways in which regulation can significantly impact the value of real estate properties. Riddiough (1997) modeled land values taking into account one form of regulatory risk in which governments have the ability to confiscate property. Sunding et al. (2004) discusses the impact of environmental laws which increases cost of construction and limits development scale, negatively impacting housing prices. In some instances, the government has been known to grant development permits, only to block development later through rezoning (e.g. Pacific National Investments Ltd. V. City of Victoria, 2000).

Extreme forms of regulatory risks, such as confiscation of property and rezoning, occur only rarely and most residential developers are not worried about their impacts. A more common worry is compliance with environmental laws, but this can be incorporated into overall development costs. However, permitting risk arising due to extensive waits to obtain development permits, is a regular concern which can’t easily be incorporated into models presented in previous literature. Developers told us that it takes a mean of 3 years and in extreme cases, several decades, for development permits to be approved by the city of London, Ontario. This introduces great uncertainty surrounding lot market conditions if and when permits are approved.

In this chapter, we seek to value land by modeling the business of subdivision lot developers, who are the major purchasers of residentially zoned raw urban land. By taking a real options approach, our work is in a similar vein to Bar-Ilan & Strange and Buttimer, Clark, & Ott. We present a more sophisticated model that treats land as a compound option, introduces mean reversion in lot price evolution, and incorporates permit application lag. On the other hand, we also limit our model by stopping our analysis at the sale of completed lots. Earlier work expected builders to value options as though they were retaining the finished properties for rental purposes. However, our conversations with developers revealed that this is not the norm in Ontario. Since we assume that developers do not retain possession of lots, we can assume that the impacts of homebuilder behaviour are all subsumed in the market price of lots.

Our analysis focuses on the regulatory environment in London, Ontario, Canada. We utilize public data available from published papers and government statistics, but our analysis also derives from conversations with local urban planners, land appraisers, bankers and subdivision developers.

Variable	Description	Value
S_0	Current Lot Price	$415,000/acre
μ	Appreciation Rate	3%/year
$\rho = \rho_1 = \rho_2$	Discount Rate	13%/year
σ	Standard Deviation of Lot Price	25%/year
η	Mean Reversion	0.7
R	Land Rent less Property Taxes	-$700/acre/year
I_1	Phase 1 Costs	$85,000/acre
I_2	Phase 2 & 3 Costs	$25,000/acre

Table 3-1: Parameter descriptions and base case values, placed at beginning of chapter for easy reference. Base case values derived using information obtained from Sifton Properties and other sources.

This chapter is organized as follows. We provide an overview of business practices and modeling assumptions in the next section. We then devote a section to formulating and solving our real options model. In the following section, we estimate realistic values for our model parameters. The next section contains our results under different market conditions and we compare our results against prevailing land prices. We also examine optimal decision points and development time lags, as well as the risks associated with entering into development. We finish with a conclusion.

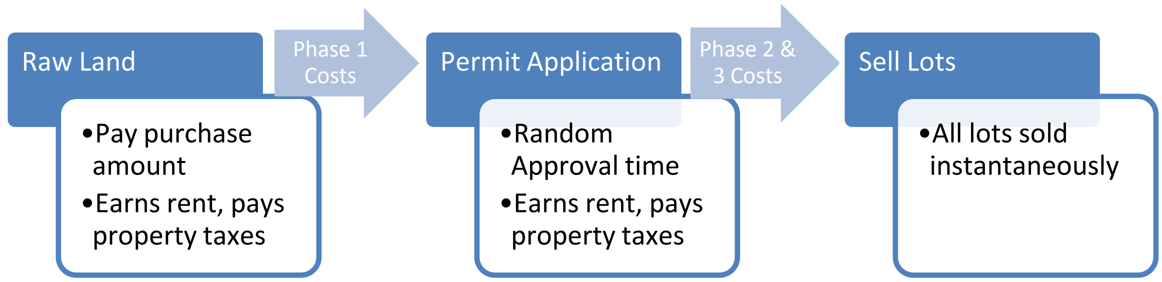
3.1 Business Practices and Modeling Assumptions

Subdivision developers buy land with the intention of developing them into serviced subdivision lots, and then selling them to builders. Sevelka (2004) gives a detailed overview of the business process and risks involved in subdivision development. In summary, when developers hold raw land, they pay property taxes and may receive rent from farmers or others.

Provided the land is zoned residential, they may apply to develop land into subdivisions. This process may take several years, during which some development occurs. After receiving a permit, they have the option to complete development as planned. Land can be viewed as a compound option on subdivision lots with two exercise points – the first exercise occurring at the time of permit application, and the second exercise occurring at the time of completion of development. The permit application stage is referred to in the industry as Phase 1. Post-approval, there are two more stages of development until lots are completed. However, these Phases incorporate no decision points so it is convenient to consider them as a single development cycle which we denote by Phase 2&3. Developers state that permits, once granted, don't expire. We therefore assume that both component options are perpetual.

We assume that lot development and lot sales are made instantaneously. We think that these simplifying assumptions are reasonable for the following reasons. Permit application typically takes long enough that Phase 1 development is usually completed before permit approval, and Phase 2 development occurs at the same time that lots are being marketed.

Developers may sell lots all at once or in piecemeal at an agreed price depending on the project. In the case that they are sold piecemeal, we may regard our model's lot prices as the Net Present Value of the sold lots contained within an acre. We do not attempt to model cash flow structures for the sale of lots because each development project is handled differently in this regard. The following graph illustrates typical developer cash flows.



3.1.1 Price Process of Developable Lots

Most previous literature that utilized real options didn't distinguish between land and developable lots. Lots have access to services such as sewers, electricity and storm water management, unlike raw land. Lots are the end product produced by subdivision developers, and we seek to model their price process. In previous literature, real estate assets were assumed to follow GBM. However, discussions with developers revealed their belief that housing prices follow a cyclical, or mean reverting process. Papers by Hwang and Quigley (2004) as well as Capozza et al. (2002) confirm this view. To capture this behaviour, we assume the following pricing process for lots

$$dS = \left(\mu + \eta\left(1 - \frac{S}{\bar{S}}\right)\right) S dt + \sigma S dW, \text{where } \ \bar{S} = S_0 e^{\mu t}. \qquad \textbf{(3-1)}$$

Under this price process, lot prices are drawn towards $\bar{S}$ in the long run. The strength of the mean reversion is determined by η. Unfortunately, analytic expressions for the probability distribution of this process are unavailable. However if the condition $\frac{2(\mu+\eta)}{\sigma^2} > 1$ is met, the distribution of S is stationary and asymptotically converges to a gamma distribution (see Karlin & Taylor, 1981).

$$\frac{S}{\bar{S}} \sim \text{Gamma}(a, b), \text{where } a = \frac{2(\mu+\eta)}{\sigma^2} - 1, b = \frac{\sigma^2}{2\eta}.$$

The 95% confidence interval for the process is shown in Figure 3-1. The confidence interval to the left of the dotted vertical line is estimated using Monte Carlo simulations. The confidence interval to the right of the line is estimated using the gamma distribution.

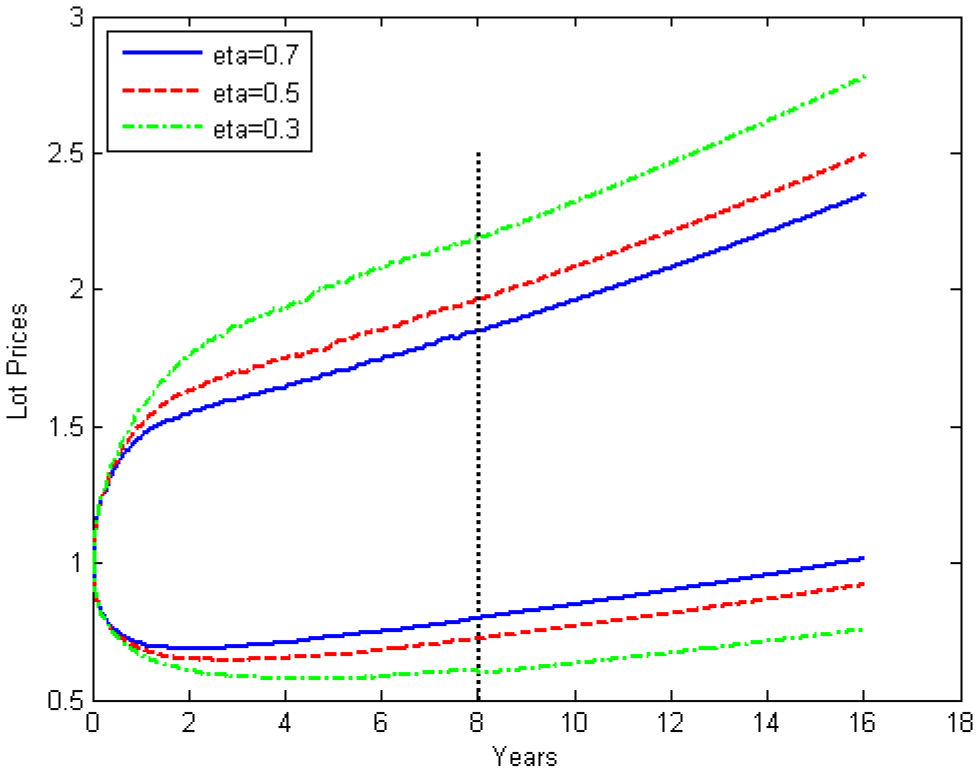


Figure 3-1: 95% confidence interval for mean-reverting price process described in equation (3-1) for differing values of η. Other values used are $\mu = 3\%/yr$, $\sigma = 25\%/yr$ and $S_0 = 1$. The intervals to the left of the dotted line were calculated using Monte Carlo simulations, and the intervals to the right are calculated using the gamma distribution. The gamma distribution does not accurately describe the intervals for small time periods.

3.1.2 Permit Approval Time

Obtaining a development permit is often a lengthy political process of uncertain duration. We model the permit approval time using the inverted gamma distribution to satisfy these characteristics. In London, Ontario, the mean length of time is around 3 years according to developers. We don't possess a similarly clear estimate for the standard deviation of permit approval times. Upon examining the probability distributions using several numbers, we found that using a standard deviation of 1 year produced a satisfactory distribution that included rare instances of several decades long permit approval times. Unless otherwise stated, we use the inverted gamma distribution with mean of 3 years and a standard deviation of 1 year.

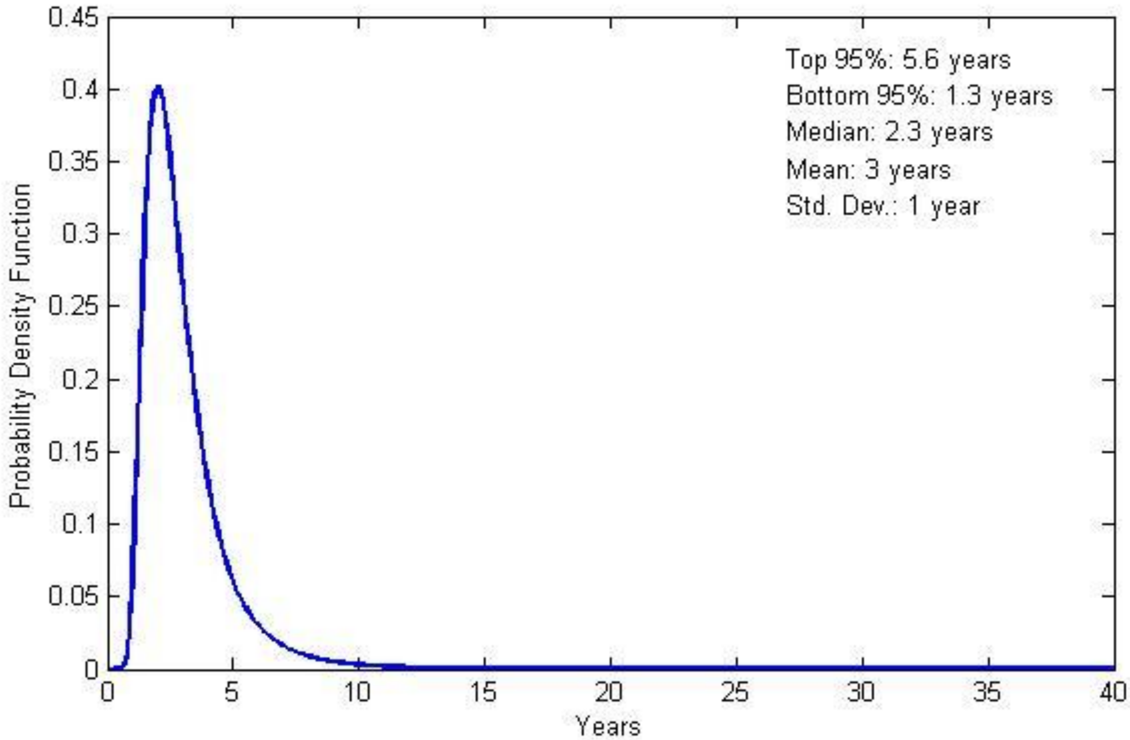


Figure 3-2: Probability distribution for the time taken for a developer to get a subdivision plan approved by the local government. The distribution follows an inverted gamma distribution. Parameters for this particular distribution were calibrated to yield a mean of 3 years and a standard deviation of 1 year.

Figure 3-2 displays the probability distribution histogram. The figure points to the small possibility of a very lengthy permit approval waiting time. Our conversations with land appraisers and bankers revealed that such lengthy wait times do occasionally occur, the evidence of which supports our choice of distribution.

3.1.3 Land Rent

Before land is serviced into lots, we assume it earns rent from farming. It is possible to have other uses for land before development – for instance, as a parking lot. However, examples of such instances are relatively rare and we have decided such scenarios are out of scope for this thesis. Subdivision developers purchase large tracts of land on the outskirts of developed areas. To make use of the land as parking space, for instance, is not feasible since the space is usually too large compared to the demand.

The owner of land pays property taxes and in this chapter, we define rent as being the net post-tax rent per acre. Property taxes are levied as a proportion of the value of land each year. The property tax rate in the City of London in 2010 is 1.5% (City of London, 2010). For farm land, the tax rate is a quarter of this at 0.375%. In our model, we assume that the land qualifies for farm tax rate in the base case. In other words, if a developer purchases raw land for $200,000/acre, the property tax is $750 in the first year. The estimation of taxes is problematic as it is an input variable which depends on the output variable. To solve this dilemma, we choose to use the Discounted Cash Flow estimate of the value of land to derive the amount paid in property taxes. Although the DCF method gives a different estimate to that obtained by the Real Options approach, the resulting small errors in the estimate of property taxes are immaterial, as we will see.

We make the simplifying assumption that rent and lot prices both grow at rate μ. The assumption is necessary to simplify the solutions of our model, as we shall see later. However, the assumption is not without merit. Rent can be expected to rise with inflation. But also, rent can be expected to rise as cities expand, making land closer and closer to the built-up urban areas. The decreased distance between the land and the built-up urban areas opens up the land to more competing uses of land (e.g. parking). Thus, rent goes up faster than the rate of inflation. Similarly, we assume that property taxes also appreciate at rate μ. This is justifiable given that land appreciates in value, and taxes, as a proportion of land values, increase at the same rate.

To estimate the annual rate of increase in farm rent, we analyzed data on US cropland rent. Cropland rent has increased at a rate of 2.54%/year during the years from 2000 to 2010, according to the United States Agricultural Department (2009). We do not have time series data for lot prices, but we know that housing prices have increased by 3.91%/year during the same period according to the Case-Shiller Index Standard and Poor's, which doesn’t adjust for the increase in the quality of homes. Adjusting for the quality of homes is expected to yield a lower rate of appreciation. Even if we don’t account for the quality of homes, the differences in rates are statistically insignificant according to the two-sample t-statistic.

3.1.4 Development Costs

Since land is modeled as a compound option, we separate development costs according to the times in which costs are incurred. Phase 1 costs (I_1) are chiefly comprised of roughly 70% of the servicing costs, permit application fees and subdivision design fees. Phase 2 & 3 costs (I_2) mainly include the remaining 30% of servicing costs and marketing costs. Servicing costs are by far the biggest costs involved overall and include installation of sewers, storm water management and electricity. We make the simplifying assumption that these prices increase at rate μ, the average rate of appreciation of lot prices, in order to simplify our solutions. We have reason to believe this assumption in the long run. If costs always increase at a slower pace, it is beneficial for the land owner to leave land undeveloped indefinitely. If costs increase at a faster pace, every developer will go out of business eventually.

There is uncertainty surrounding future servicing costs. As an example, servicing costs in London, Ontario in 2010 were higher than in 2008, despite falling housing prices. This increase was the result of a diversion of resources towards infrastructure projects resulting from the Government of Canada's economic stimulus plan. However, servicing costs do not fluctuate as much as housing prices, varying at most 10% over a few years. Therefore, we assume that servicing costs increase deterministically in our model, and we leave modeling uncertain servicing costs to possible future work.

3.1.5 Discount Rates

In a typical real options analysis, the same discount rate is applied throughout the lifetime of each project. However as we shall see in the results section, the application of uniform discount rates yield results that are incongruent with observed developer behaviour. To remain flexible between the choice of homogeneous or heterogeneous discount rates, we provide two symbols for the two discount rates - ρ_1 and ρ_2. ρ_1 is the discount rate when raw land is held, and ρ_2 is the discount rate when the development project is initiated. To use a uniform discount rate throughout, we need only set $\rho_1 = \rho_2$. We discuss the use of different discount rates in the results section.

3.2 Model of Subdivision Project

We use the principles of dynamic programming as expounded by Dixit & Pindyck (among other authors), to value land. We seek the optimal decision point that maximises expected profits, the discounted value of which is the value of the development opportunity. The principles of dynamic programming allow us to arrive at Bellman equations, which describe the relationships between underlying lot prices and land prices. In this section, we describe the process by which we arrive at the Bellman equations.

We hypothesize that there is a lot price S^*, at or above which it is optimal to apply for a development permit. We first derive the equation that governs the price of raw land $L(S(t), t)$ when lot prices are below this value (i.e. $S(t) \leq S^*$). We note that $L(S(t), t)$ depends on the finished lot price S and time t. Since S is stochastic, we use Ito's lemma in order to describe the evolution of L according to S and t.

$$dL = \left(\frac{\partial L}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 L}{\partial S^2} + \left(\mu + \eta\left(1 - \frac{S}{\bar{S}}\right)\right) S \frac{\partial L}{\partial S}\right) dt + \sigma \frac{\partial L}{\partial S} dW. \tag{3-2}$$

Here, W is the standard Wiener process. On average, the developer requires that her investment appreciates at the discount rate. Having bought land, there are two ways that the developer earns the discount rate while land remains undeveloped – through appreciation of the value of land, and through rents collected. In mathematical terms, we have the following

$$E\left[\frac{dL}{dt}\right] + R_t = \rho_1 L. \tag{3-3}$$

Substituting (3-2) into (3-3) yields the following Bellman equation, which governs the price of raw land before the permit application is sent.

$$\frac{\partial L}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 L}{\partial S^2} + \left(\mu + \eta\left(1 - \frac{S}{\bar{S}}\right)\right) S \frac{\partial L}{\partial S} + Re^{\mu t} = \rho_1 L. \tag{3-4}$$

When the permit is applied for, the developer is made to wait τ years while the application is processed. During the time that she waits, she continues to collect rent and completes Phase 1 development. When the project is approved, she receives P, the value of the permit-approved land. We assume the developer makes the decision to apply for a permit when lot prices reach S^*, a rational expected value maximizer. Mathematically, this is expressed as follows.

$$L(S^*(t),t) = E\left[P(S^*(t+\tau),t+\tau)e^{-\rho_2\tau} - I_1 e^{\mu t} + \frac{Re^{\mu t}}{\rho_2 - \mu}\left(1 - e^{-(\rho_2-\mu)\tau}\right)\middle| S^*(t)\right]. \tag{3-5}$$

In order to solve our equation, we need another condition. We observe that for prices under S^*, the value of land follows from solving equation (3-4). At S^*, by definition, the developer is indifferent between applying for a development permit and holding land hence (3-5). In addition, it turns out that optimal solutions of Bellman equations may also be proved to have an additional degree of smoothness at this boundary (see Dixit and Pindyck). The mathematical statement of this smoothness is:

$$\frac{\partial L(S^*(t),t)}{\partial S^*(t)} = E\left[\frac{\partial P\left(S^*(t+\tau)\right)}{\partial S^*(t)} \middle| S^*(t)\right]. \tag{3-6}$$

Between equations (3-5), (3-6) and (3-7), we have sufficient information to compute the solutions for $L(S(t),t)$, provided we know the value of permit approved land $P(S(t),t)$. We therefore turn our attention to finding $P(S(t),t)$. The price point over which it is optimal to exercise $P(S(t),t)$ is denoted by $S^\dagger$. For $S(t) \le S^\dagger$, the same process used to derive (3-4) can be used. We state the derived equation below.

$$\frac{\partial P}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 P}{\partial S^2} + \left(\mu + \eta\left(1 - \frac{S}{\bar{S}}\right)\right) S\frac{\partial P}{\partial S} - \rho_2 P + Re^{\mu t} = 0. \tag{3-7}$$

When $S(t) \ge S^\dagger$, the developer pays the Phase 2 & 3 costs and collects the subdivision sales revenue. This is expressed mathematically as follows.

$$P\left(S^\dagger(t),t\right) = S^\dagger(t) - I_2 e^{\mu t}. \tag{3-8}$$

As was the case for $L(S(t),t)$, we require the transition between pricing regions to be smooth for $P\left(S^\dagger(t),t\right)$.

$$\frac{\partial P\left(S^\dagger(t),t\right)}{\partial S^\dagger(t)} = 1. \tag{3-9}$$

In order to solve the equations (3-4)-(3-9), it is convenient to make the following transformations.

$$Q = S/\bar{S}, \quad \tilde{L} = L/\bar{S} - \frac{R}{S_0(\rho_1-\mu)}, \quad \tilde{P} = P/\bar{S} - \frac{R}{S_0(\rho_2-\mu)}. \tag{3-10}$$

Further details of the solution process are given in Appendix E. The solutions are stated as follows.

$$L(S,t) = \bar{S}\left(\tilde{L}\left(\frac{S}{\bar{S}}\right) + \frac{R}{\rho_1-\mu}\right). \tag{3-11}$$

$$\tilde{L}(Q) = \begin{cases} AQ^{\theta_1}H\left(\frac{2\eta}{\sigma^2}Q;\theta_1,b_1\right) & \text{if } Q \le Q^* \\ E\left[\tilde{P}\big(Q(t+\tau)\big)e^{-(\rho_2-\mu)\tau} - \frac{I_1}{S_0} + \frac{R}{S_0}\left(\frac{1}{\rho_2-\mu} - \frac{1}{\rho_1-\mu}\right)\middle|Q(t)\right] & \text{if } Q \ge Q^* \end{cases} \tag{3-12}$$

$$\tilde{P}(Q) = \begin{cases} AQ^{\theta_2}H\left(\frac{2\eta}{\sigma^2}Q;\theta_2,b_2\right) & \text{if } Q \le Q^\dagger \\ Q - \frac{I_2}{S_0} - \frac{R}{S_0(\rho_2-\mu)} & \text{if } Q \ge Q^\dagger \end{cases} \tag{3-13}$$

$$\theta_n = \frac{1}{2} - \frac{\mu+\eta}{\sigma^2} + \sqrt{\left(\frac{\mu+\eta}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2(\rho_n-\mu)}{\sigma^2}} \tag{3-14}$$

$$b_n = 2\theta_n + 2\left(\frac{\mu+\eta}{\sigma^2}\right). \tag{3-15}$$

Here H is the confluent hypergeometric function (see Hassani, 1999). The function's analytical form is known, but it must be computed to within a finite tolerance as it is an infinite series. The expectation procedure present in the bottom of (3-12) can't be resolved analytically because we don't have the probability distribution of $\tilde{P}\big(Q(t+\tau)\big)$ for small τ. To evaluate the expectation, we opted to use the Monte Carlo simulation method.

3.3 Base Case Parameter Estimation

In this section, we assign realistic numerical values to our parameters and compare the results of our calculations to some commonly observed market values. If the observed market values are close to the values predicted using our model, we have evidence to support our claim that our model incorporates key drivers of the land development business. During this process, among other sources, we have relied on the information shared by Sifton Properties, one of the largest developers in London, Ontario.

Sifton informed us that they expected lot prices to appreciate by 3% a year on average. This number is close to the historical rate of increase of farm rents as well as house prices as mentioned in the previous section on modeling assumptions. Sifton also stated that the market price of finished lots is approximately $415,000/acre at the time of this writing.

Sevelka states that one developer they asked anticipated Internal Rate of Returns (IRR) in the region of 12% to 15% excluding profit. Analyzing the minimum cash flow expected by Sifton reveals an IRR closer to the lower end of the range. For the purposes of this chapter, we assume a discount rate of 13%. To begin, we assume a homogenous discount rate – i.e. let $\rho_1 = \rho_2$.

Estimating the volatility of lot prices is problematic. There is a dearth of information regarding past lot sale prices. But even if we had access to such data, we would have to adjust the data to account for 'hedonic' differences in lot characteristics such as lot size, desirability of neighbourhood, and proximity to amenities. Instead of seeking out and using historical data, we infer the volatility using two sources – the Capital Asset Pricing Model (CAPM) and from published papers.

CAPM hypothesizes that risk premiums are directly proportional to risks as measured by volatilities. At the time of this writing on July 2010, the yield on a 3-5 year Canadian Treasury is at roughly 2%, and has been so since the end of 2008. The historical non-inflation adjusted rate of return on stocks is roughly 9%/year according to Damodaran. The MVX index TMX, which measures volatility of Canadian stocks, has averaged around 17%/year from January to July of 2010. We therefore expect the Canadian stock market to yield a historical 9% a year, and we assume 17% volatility for Canadian stocks. Assuming a discount rate of 13%, the volatility of subdivision projects according to CAPM is 17(13-2)/(9-2)% = 26.7%.

We compare this figure with what we can infer from other sources. According to Davis and Heathcote (2007), land prices are twice as volatile as house prices, and individual house prices are seen to fluctuate with a standard deviation of 11% according to de Jon et al. (2008). Using these data, we infer that land prices fluctuate on the order of 22%. We compromise between the two estimates of volatility and assume a rate of 25% in annualized units.

It is very difficult to estimate the strength of mean-reversion, given the paucity of data we have to work with. We give our best guess based on our observation of sample simulated price paths based on different values of η. When we set η to 0.3 and 0.5, the simulated prices were frequently seen to move into unrealistic territories before falling back into normal levels. When we set η to 0.7, the resulting simulated prices didn't suffer from such extreme episodes as often, even though they still occurred from time to time. Therefore, we choose to work with 0.7 as a base case scenario.

Sifton has provided us with the following estimates of parameter values. Farm rent is minimal at $100/acre. Soft costs committed with the decision to develop are typically around $25,000/acre and servicing costs are estimated at around $85,000 per acre. Phase 1 costs are therefore $25,000+0.7x$85,000=$85,000/acre and Phase 2 & 3 costs are 0.3x$85,000=$25,000/acre. The mean permit approval time is roughly 3 years and, and Sifton believes a standard deviation of 1 year is reasonable.

To estimate property taxes, we need to examine the Discounted Cash Flow (DCF) valuation of land. The components of the DCF calculations are outlined in Table 3-2.

Using $210,000/acre as an estimate of land prices, we can estimate property taxes to be 0.375% of this amount, or $800/acre. Note that this amount is much greater than the $100/acre received in rents, resulting in a net rent of -$700/acre. A summary of the variables and their base case values are given in Table 3-1.

Time (years)	Cash Flow	Description
0	-$85,000/acre	Phase 1 Costs
1~3	$100/acre -0.375% of output	Land Rent less Property Taxes

3	-$25,000/acre	Phase 2 & 3 Costs
3	+$454,000/acre	Expected Sales (Assuming 3% appreciation per year)
3	$210,000/acre	Discounted Total (Using 13% discount rate)

Table 3-2: Discounted Cash Flow valuation of Land

3.4 Results

3.4.1 Land Valuation under Homogeneous Discount Rate

Using our model, we calculated the value of land, the hurdle values and their sensitivities to changes in parameter values. The results are summarized in Table 3-3.

The base case value of land is $219,000/acre. This corresponds quite closely with the going rate of raw land in London, which we are told to be $225,000/acre. We can compare this figure against one obtained using the Net Present Value approach with the following cash flows. According to the cash flow chart, the fair value of raw land is roughly $204,000/acre, which is roughly 7% lower than the computed Real Options value.

Changes in expected housing price appreciation (μ) significantly impact the value of land, suggesting that industry outlook heavily influences the price of land. Similarly, changes in discount rate (ρ) are also seen to significantly impact land values, suggesting that the risk appetite of developers also play a big part in affecting land values. Changes in housing price volatility (σ) appear to have less significant impact, suggesting that near term uncertainty plays a smaller role in determining land values. This is understandable given the small option premium we observe from our results. Option premium for a financial option can be calculated by subtracting the option's intrinsic value from its price. In applying the option valuation methodology to land prices, we can substitute the intrinsic value with NPV, and the option price with land price. In the deterministic calculation of NPV, σ plays no role, but it does influence the real options-calculated value. The option premium is valued at $15,000 compared to a NPV of $204,000, so by analogy to a deep in the money option, it is perhaps not surprising that changes in σ don't influence the value of land very heavily.

Variable	Change	Value of Land	Q^*	$Q^\dagger$	Std. Profits
Base Case	-	$219,000	0.52	1.08	$80,000
μ	+1%/year	+$15,000	0.5	1.12	$80,000
ρ	+1%/year	-$10,000	0.52	1.05	$79,000
Σ	+5%/year	+$3,000	0.46	1.13	$91,000
η	-0.2	-$2,000	0.47	1.04	$89,000
R	-$2400/acre/year	-$7,000	0.49	1.06	$81,000
I_1	+$10,000	-$6,000	0.56	1.08	$79,000
I_2	+$10,000	-$10,000	0.55	1.08	$80,000
Mean Permit Approval Time	+1 year	-$29,000	0.35	1.08	$77,000
Std. Approval Time unadjusted	+1 year	+$6,000	0.58	1.08	$90,000
Std. Approval Time ρ adjusted	+1 year	-$9,000	0.65	1.04	$91,000

Table 3-3: The value of land and the changes in its values according to different market assumptions. Here, we assume $\rho = \rho_1 = \rho_2$. Q^* is the optimal point at which the developer holding raw land should apply for the permit to develop. $Q^\dagger$ is the optimal relative price at which the developer holding a permit-granted land should complete development and sell the lots. Std. Profits denote the standard deviation of cash flows obtained through 50,000 Monte Carlo simulations.

Changes in strength of mean reversion (η) seem to have little impact on the value of land. Various η values determines the likelihood of extreme housing values in the future, but leaves the expected value of houses unchanged. A change in η affects the value of land in a similar way with σ, except the effects are even less pronounced given that it takes a few years for the influence of η to manifest in housing prices. Increases in development costs negatively affect the value of land, with I_2 having less impact than I_1. This occurs because I_2 is paid later, and so is more heavily discounted.

Increases in permit approval times negatively impact the value of land in a significant fashion, which helps us understand why developers raise the issue of permit uncertainty as one of their primary concerns. Property taxes are seen to negatively impact the value of land in a fairly significant way.

Perhaps counter-intuitively, increases in uncertainties surrounding approval times increase the value of land. The increased uncertainty increases the value of the option premium, increasing the value of land overall. However, this only holds because the discount rate remains the same despite the increased uncertainty. It would be reasonable for a developer to demand higher discount rates to compensate for higher uncertainty. We can determine the magnitude of change in discount rates by applying the CAPM. Having already established that the risk free rate is 2%, we can use the std. (standard deviation) of profits as the proxy for risk and use the following formula. \$80,000 refers to the std. of profits in the base case.

$$\rho = 2\% + 11\% \frac{\sigma_{\text{profits}}}{\$80{,}000}. \tag{3-16}$$

If we account for the change in the discount rate, an increase in uncertainties surrounding approval times causes land values to fall.

The uncertainty surrounding realized profits are significant, as evidenced by the large standard deviations of profits. Changes in certain parameters are seen to increase the risks surrounding development. These parameters are volatility, degree of mean reversion and uncertainty surrounding permit approval times. In particular, we note that an increase in one year to the standard deviation of permit approval times has similar effects on profit uncertainties as increasing lot standard deviation by 5%/year.

The optimal point of exercise to begin Phases 2&3 ($Q^{\dagger}$) is seen to be consistently above 1, meaning that it is optimal to build lots when lot prices are above the expected lot price. This may be surprising given the very high discount rate ρ, which penalizes waiting. However, the disadvantage of waiting due to discounting is more than compensated by the advantage of waiting for a boom in the real estate market. Because of the mean-reverting characteristic of lot prices, the prices are expected to oscillate around the long term mean. Since the volatility is high, the prices are expected to deviate greatly from this mean in the short run. The combination of strong mean reversion and high volatility gives a high probability that the prices will deviate above the mean significantly in the near future, and the optimal decision is frequently to wait for this deviation to occur.

Q^* is seen to be consistently extremely small. In all but the worst market conditions, our equations suggest that the best decision is to never keep raw land in inventory with the hopes of developing in some future date. The reason the values are so low lies with the high discount rate. However, one might think that the discount rate should not be so high for raw land in the inventory. Since lot prices are mean reverting, the value of the option to develop is stable in the long run. Once development starts, short term volatilities dominate the riskiness of the project, but such concerns are of less importance for raw land held for the long term.

3.4.2 Land Valuation under Heterogeneous Discount Rate

We assume two different discount rates according to the different stages of development. We hold the discount rate for a project already in development at $\rho_2 = 13\%$. To determine the discount rate in effect while holding raw land, we assume that the purchases are financed through long term loans. As of the summer of 2010, the interest rates on corporate bonds in Canada rated A or BBB with maturities of 10 years were approximately 5%. Therefore, we assume $\rho_1 = 5\%$ as a base case. We have documented some of the results in Table 3-4.

We see a big change in the Q^* values in the inhomogeneous discount rate model. Since Q^* is above 1 in the base case scenario, it is not optimal to develop the raw land immediately after purchasing it. However, Q^* of the base case is very close to the starting value of land of 1.00, so the developer is expected to start developing very soon after purchase on average. This can be seen by examining the duration of waiting times between land purchase and commencement of development, as shown in Figure 3-3.

Variable	Change	Value of Land	Q^*	$Q^\dagger$	Std. Profits
Base Case	-	$220,000/acre	1.01	1.08	$80,000

ρ_1	+1%/year	-$2,000	0.90	1.08	$80,000
ρ_2	+1%/year	-$11,000	1.00	1.05	$79,000
μ	+1%/year	+$16,000	1.13	1.12	$82,000
η	-0.2	No change	1.12	1.04	$91,000
R	-$2100/acre/year	-$8,000	0.92	1.06	$79,000
σ	+5%/year	+$2,000	1.07	1.13	$90,000
I_1	+$10,000	-$9,000	1.04	1.08	$80,000
I_2	+$10,000	-$8,000	1.03	1.08	$81,000
Mean Permit Approval Time	+1 year	-$30,000	0.88	1.08	$76,000
Std. Approval Time unadjusted	+1 year	+$6,000	1.09	1.08	$91,000
Std. Approval Time ρ_2 adjusted	+1 year	-$9,000	1.11	1.04	$90,000

Table 3-4: The value of land and the changes in its values according to different market assumptions, to within ±$700/acre with 95% confidence. We assume different discount rates for the different phases of development (ρ_1=5%, ρ_2=13%). Q^* is the optimal point at which the developer holding raw land should apply for the permit to develop. $Q^\dagger$ is the optimal point at which the developer holding a permit-granted land should complete development and sell the lots. Std. Profits denote the standard deviation of cash flows obtained through 50,000 Monte Carlo simulations.

The wait time is less than 1 year in over 80% of the simulations, as can be seen by the results shown in Figure 3-3. During this time, the value of the Permitted Option $P(S,t)$ is discounted by a small rate. However in extreme cases, the developer may be left holding land in inventory for a significant period of time – over a decade in some instances. Anecdotal evidence points to the occurrence of such long waits, lending credence to our usage of different discount rates.

Q^* values are very sensitive to changes in input variables. Increases in ρ_1 and R make it more expensive for developers to hold undeveloped land, so they are willing to develop despite low housing prices relative to the long term norm. Conversely, if prices are expected to increase at a higher μ, they will wait until housing prices are above the trend line. If housing prices are volatile, developers are also willing to wait longer before developing, since there is an increased chance that housing prices will swing up significantly. If development costs are expected to be high, they are also willing to wait until housing prices are high, to mitigate the risk of committing capital.

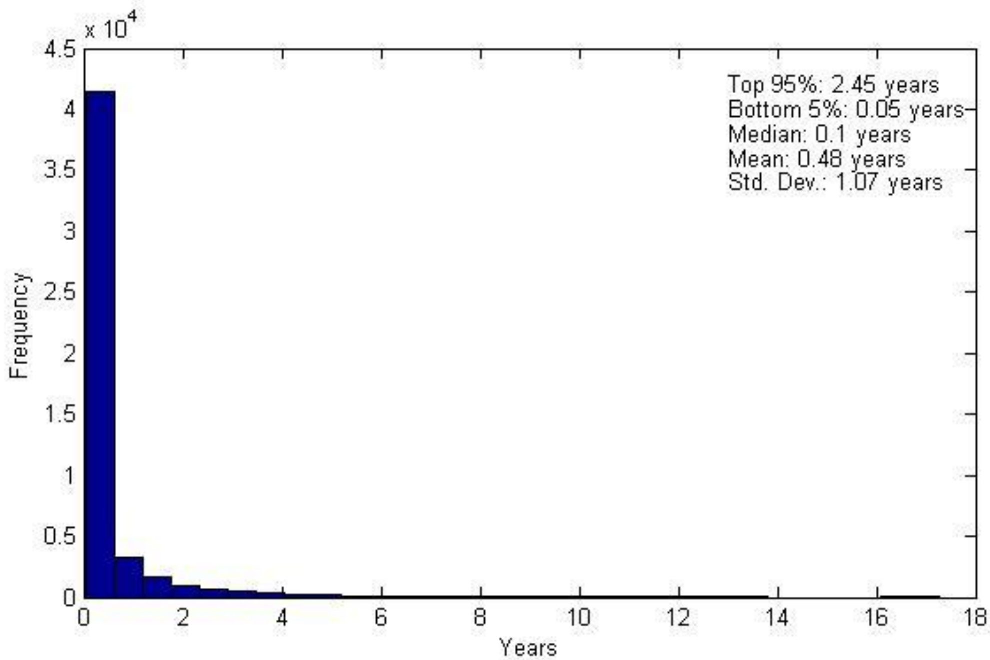


Figure 3-3: Waiting times between land purchase and commencement of development. Histogram is result of 50,000 Monte Carlo simulations under the Base Case scenario.

Although changes in mean reversion do not affect land values significantly, it seems to have a big impact on Q^*. If housing prices are less attracted to the long term trend line, big upward deviations to the upside are made more probable. In such circumstances, developers will wait for the possibility of "hitting the jackpot" in a heated housing market.

If developers believe that the permit approval process will take a long time, they will be willing to develop despite low housing prices. A lengthier process means a lengthier projection completion time, which means that capital is tied up for a lengthier period of time, and the discount rate is applied for longer. Also, the developer is expected to pay more property taxes during their wait. Higher uncertainty surrounding permit approval process has an opposite effect, in that the developer will wait until high housing prices. Higher uncertainty makes developers hesitant about committing capital.

3.4.3 Valuation Discrepancies between Real Options and Discounted Cash Flows

We are interested in determining the scenarios under which the real options valuation differs significantly from the discounted cash flow (DCF) approach to valuation. Table 3-5 shows the differences in valuations under changes in variables. Note that not every listed variable is used for the calculation of DCF values. In such cases, the DCF values are the same as the base case.

Variable	Change	Real Options (/acre)	DCF (/acre)	Difference
Base Case	-	\$220,000	\$210,000	\$10,000
ρ_1	+1%/year	\$218,000	\$210,000	\$8,000
ρ_2	+1%/year	\$209,000	\$203,000	\$6,000
μ	+1%/year	\$236,000	\$220,000	\$16,000
η	-0.2	\$220,000	\$210,000	\$10,000
R	-\$2400/acre/year	\$212,000	\$205,000	\$7,000
σ	+5%/year	\$222,000	\$210,000	\$10,000
I_1	+\$10,000	\$211,000	\$200,000	\$11,000
I_2	+\$10,000	\$212,000	\$203,000	\$9,000
Mean Permit Approval Time	+1 year	\$190,000	\$184,000	\$6,000

Table 3-5: Comparison of land values projected by real options model and the discounted cash flow model under different market assumptions. The DCF tracks the real options valuation well under most circumstances, but diverges significantly with higher μ.

The changes in the differences between the real options and DCF valuations are minor, for the majority of variables examined in Table 3-5. The changes in the first discount rate, mean reversion, servicing costs and volatility only change the differences by a maximum of \$2,000/acre. However, the option value of land increases with a rosy view of the market (increased μ), and the option value suffers under higher discount rate, higher real estate taxes or longer expected permit approval time.

Figure 3-4 shows the histogram of overall profitability for the developer, assuming the developer pays the real options price for land. To have earned $0 signifies that the developer realized normal profits – i.e. yields of ρ_1 and ρ_2. In real terms, the risks are quite significant. There is a real possibility that the developer will earn $200,000 less per acre than she had initially hoped for when she bought land. When we compare this figure to the price of land we had just computed (Approx. $222,000 per acre) and development costs (Approx $85,000 and $25,000 per acre), we can understand why developers require such high discount rates to compensate for their risks.

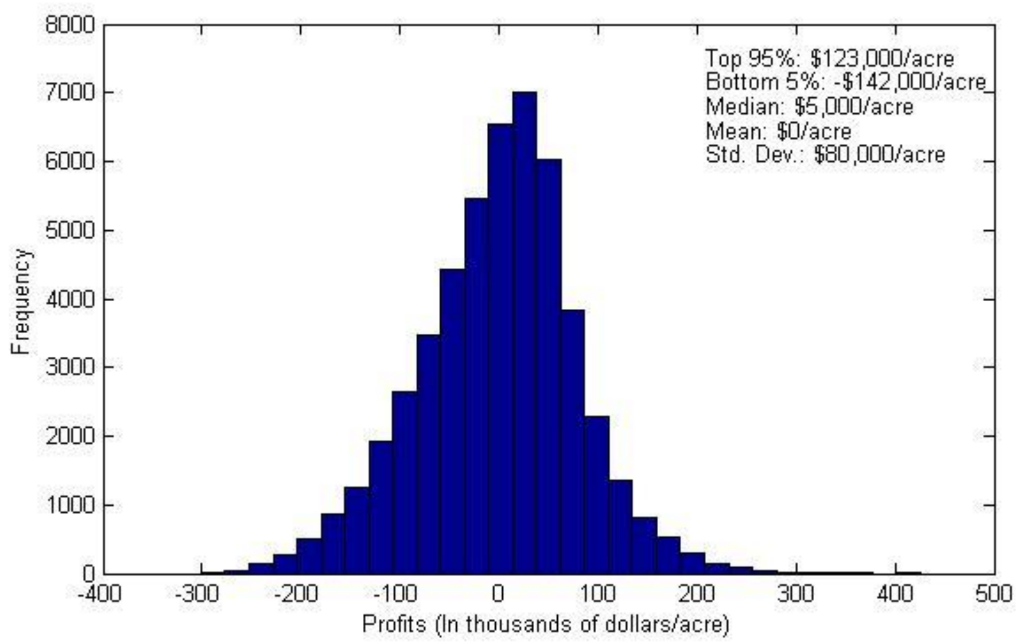


Figure 3-4: Histogram of excess profit – i.e. profit above normal profits. Histogram is a result of 50,000 Monte Carlo simulations under the Base Case scenario.

3.4.4 Development Lags and Policy Implications

Figure 3-5 shows the histogram of the waiting time between the moment a permit is granted, and the moment the developer decides to complete development and sell off the lots. The pattern is similar to that shown in Figure 3-3 where in most cases, development occurs immediately after permit grant. But in a few cases, projects stall for a significant time period. From our conversations with developers and bankers, we have found that this indeed occurs.

Figure 3-6 shows the histogram of the total time taken from purchase of raw land to completion of the development project. The histogram shows that there is significant uncertainty with regards to the duration of development projects.

We have mentioned that Q^* values are very sensitive to changes in the expected permit approval wait times. Figure 3-7 shows the relationship between the mean permit approval wait times and Q^* as well as land value. The results show that developers would be willing to pay more for raw land and hold it longer in their inventories if they knew that permit approvals would be granted expeditiously.

Uncertainty surrounding permit approval times can also affect developer behaviour. The effects of different degrees of uncertainties, ρ adjusted, can be seen in Figure 3-8. We see a fairly steep decline in land values for low standard deviations of permit times, and a levelling off thereafter. This is primarily because changes in ρ are much more sensitive in the low standard deviation regions.

It might be useful to imagine a dramatically different regulatory environment, where the expected permit approval time is low, and where the uncertainties around approval times are also low. If we expect a mean approval time of just 1 year, and a standard deviation of that time of 0.25 annual units, the value of the land is much higher at $305,000/acre, with Q^* of 1.23. The DCF model would yield $276,000/acre in comparison, suggesting that in a favourable regulatory environment, the usefulness of the real options model would increase.

3.1 Conclusion

We have taken a closer look at the underlying business dynamics of residential land development and we presented a real options model that incorporates some of the additional complexities found. We found that the real options method yields a valuation of raw land that is consistent with the going market rate of raw land.

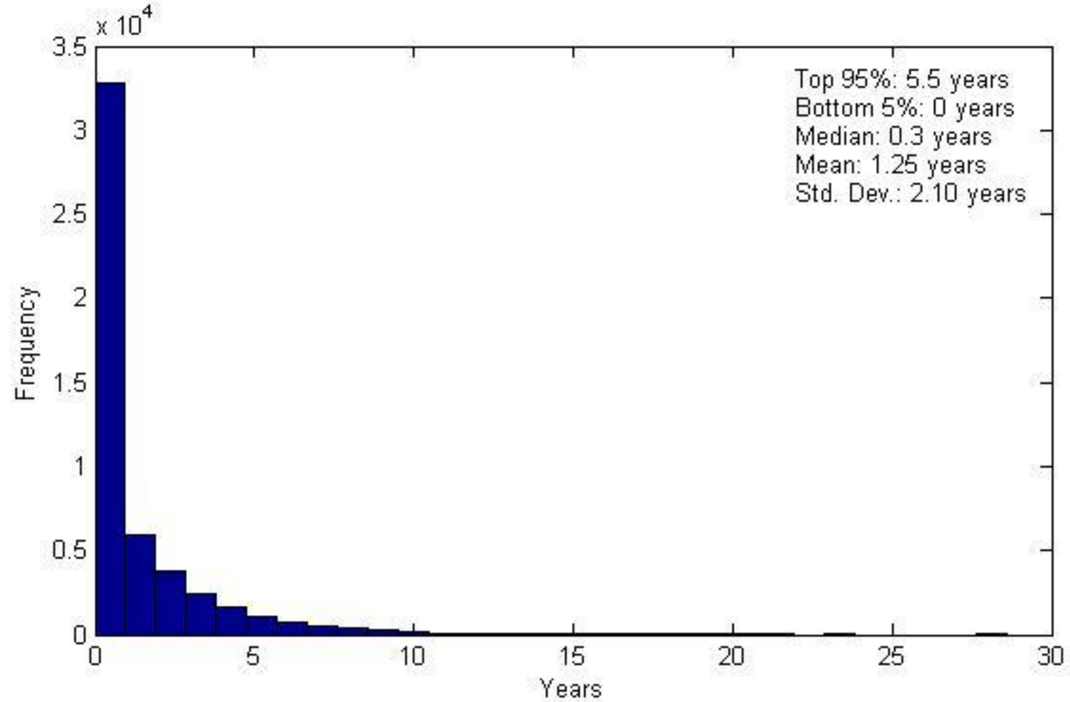


Figure 3-5: Waiting times between permit grant and finalization of development. Histogram is result of 50,000 Monte Carlo simulations under the Base Case scenario.

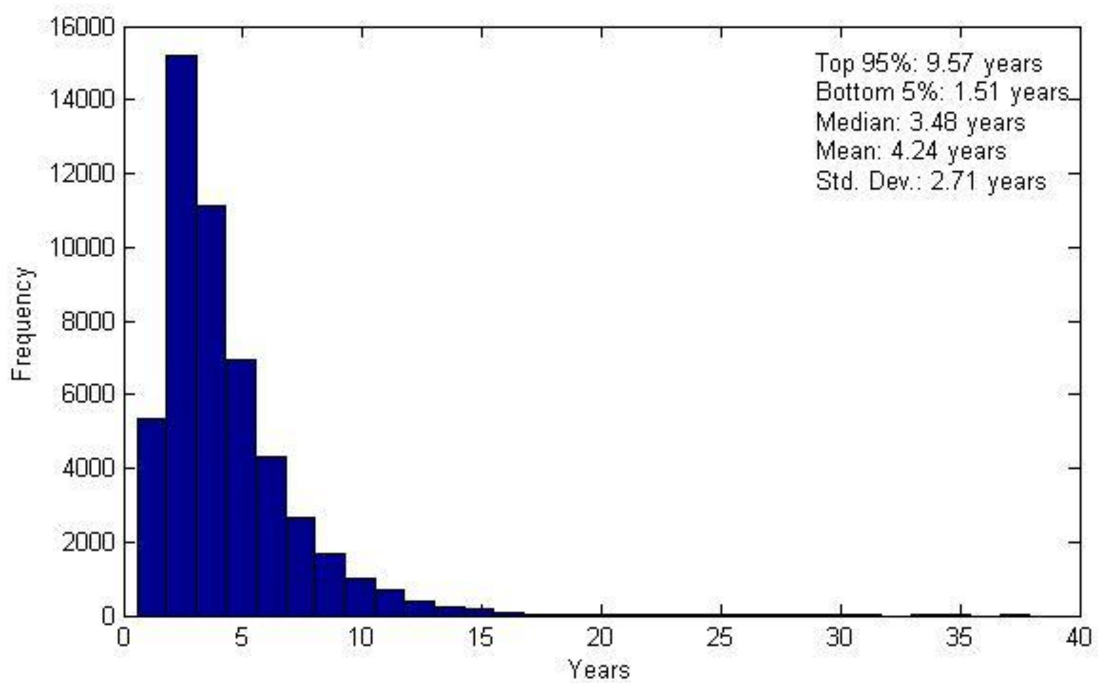


Figure 3-6: Histogram of the number of years taken from purchase of raw land to completion of development. Histogram is result of 50,000 Monte Carlo simulations under the Base Case scenario.

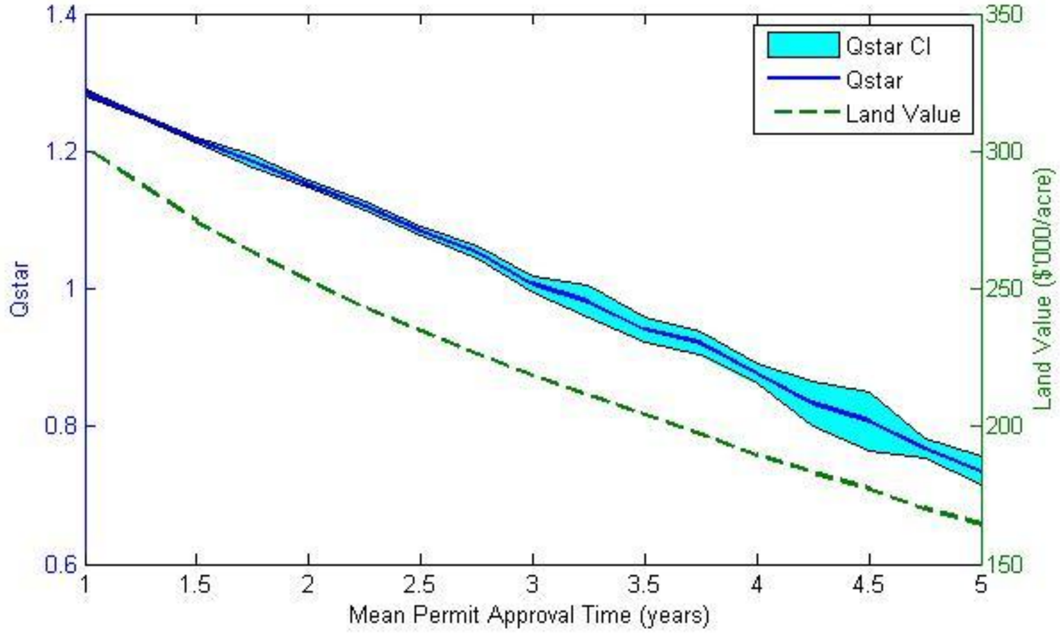


Figure 3-7: Plot of Mean Permit Approval Time vs Q* and Land Value. Land Values and Q* are computed using Monte Carlo simulations with 500,000 runs. The shaded area indicates the 95% confidence interval of Qstar. Standard deviation of approval time is fixed at 1 year.

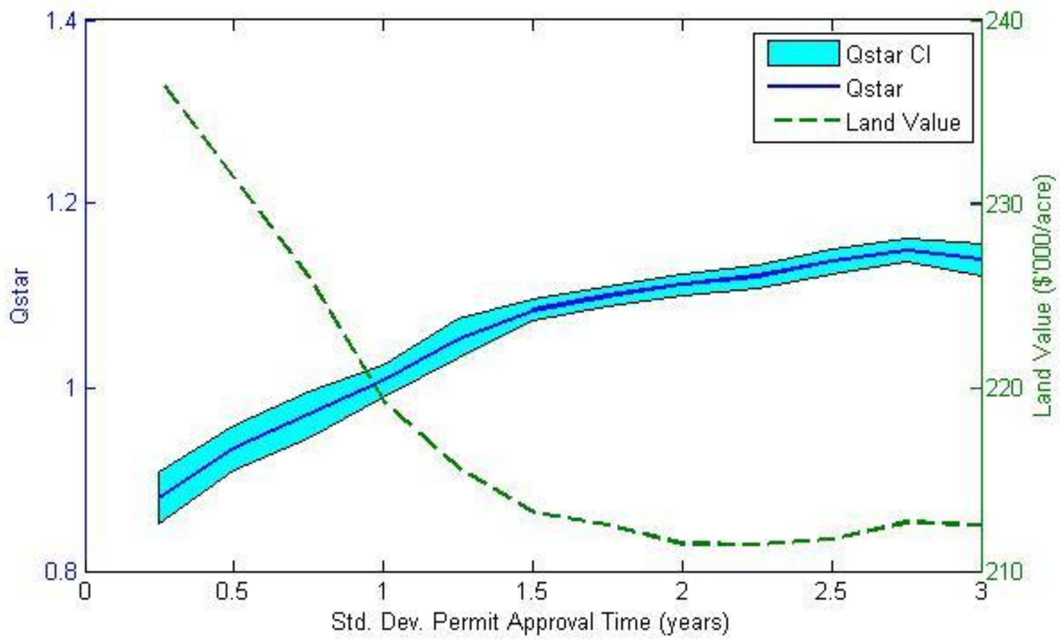


Figure 3-8: Plot of Standard Deviation of Permit Approval Time vs Q^* and Land Value, after adjusting for ρ. Land Values and Q^* are computed using Monte Carlo simulations with 500,000 runs. T. The shaded area indicates the 95% confidence interval of Q^*. Mean approval time is fixed at 3 years.

The Net Present Value approach is also seen to give a value in a reasonable neighborhood, but its deviation from the real options value may still prove significant. Its estimate of $210,000/acre is some 5% less than the $220,000/acre estimated using real options approach under heterogeneous discount rates, and the difference of $10,000/acre is 2% of the current going rate of finished lots, valued at $415,000/acre. The difference in the rates that developers pay for land flows directly through to the operating margin, where 2% may be considered significant. The usage of real options valuation may entice the developer to buy land for which the NPV approach suggests is too expensive. This is especially true in a very favourable regulatory environment, where permits are granted expeditiously and reliably. We have seen that with mean approval time of 1 year and standard deviation of 0.25 annual units, the option premium increases to $29,000.

Rent from farmers, which figures prominently in many past papers, does not appear to have a significant effect on the price of land. On the contrary, property taxes make owning land a drain on cash flows. If a piece of land can't get taxed as farm land, the higher taxes significantly lower the price of land.

The presence of permitting risk significantly influences the development project profitability picture. Increases in the expected number of years to obtain permits significantly lower the developer's profit expectations, which translate into lower land valuations. Increases in the uncertainty surrounding the number years to permit grants, without matching increases in discount rates, has minimal affect on expected profitability while significantly increasing the perceived risk of development projects. With matching increases in discount rates that compensate for higher risks, profitability is affected in a major way. City planners may be interested to see the direct economic impact of their permit approval process, even though this knowledge is just one of many inputs to the complex task of land zoning.

Raw land is a compound option for which we need to apply different discount rates depending on the different stages of development. This is seen to have little bearing on the price of land or the risk characteristics of development projects, but it is an important assumption that explains the behaviour of developers. If there was only a single discount rate and it was assumed to be at the high figure of approximately 13%, we would have closely estimated at the prevailing price of land, but we would not have had an explanation as to why developers chose to hold raw land in their inventories. Conversely, if the single discount rate was too low, we would have expected to see much higher land prices.

In this chapter, we have introduced a model for land development that incorporates some of the business realities that had been overlooked by existing papers. However, our model does not incorporate some significant factors. Whereas we assumed fixed development costs in this chapter, the costs are uncertain in reality, though not as uncertain as lot prices. The model presented in this chapter could be improved by the introduction of such uncertainties.

Another factor that we have not incorporated concerns the use of presales. Each development project is handled differently, but many lots are presold before development begins. The presence of these arrangements is likely to significantly alter the risk characteristics of the development projects. The introduction of such arrangements into the model are expected to significantly add to the complexity of land valuation models as, depending on the details of such contracts, time dependence might not be easily factored away from the model. Such a model would require the solutions of a perpetual option with time dependent parameters.

In conclusion, our model reflects many of the business considerations previously unconsidered by other papers, and in so doing gives fairly accurate estimates of land prices. Our model is also useful for explaining the behaviour of developers and the impact of changes in regulatory risk to their behaviours.

In this chapter on land valuations and in the previous chapter on presale of condominiums, real options analysis offered only marginally different valuations for assets under analysis in comparison to those offered by NPV analysis. The results point to a possible explanation as to why real options analysis have not become widespread in the industry today. However, in the next chapter, we show an example of a problem in which real options analysis draw significantly different conclusions than those drawn by NPV.

Chapter 4

4 Economic Value and Impact of Subsidies on Solar Panel Installations

The rise of solar power in recent years has been nothing short of spectacular. The annual production of solar Photovoltaics (PV) has risen from a mere 2 Megawatts in 1975 to an estimated 40 Gigawatts in 2010 according to the European Photovoltaic Industry Association, and the figures are likely to increase for the foreseeable future. The reasons for such dramatic adoption of solar are numerous. Solar power is environmentally friendly, renewable, and available in most regions.

Although the technology to harvest solar energy has existed for some time, few operational solar power plants have been constructed until recently. The economic case for solar power plants has, again until recently, just not supported their construction. Despite recent advances in technologies in the form of increases in energy conversion efficiencies and lower manufacturing costs, solar power is still not as cost competitive as traditional power plants using coal or natural gas. In order to induce companies and individuals to build solar power plants, financial incentives are necessary.

Some recent trends have led governments to provide such incentives to encourage development of solar industries within their own jurisdictions. Increasing concerns about global warming has led to political pressure to turn to environmentally friendly solutions such as solar power. Higher oil prices have led some countries to be concerned about the high level of dependency they have on oil for their continued economic growth. Oil is a non-renewable resource imported from many countries vulnerable to political unrest. As a result, developed countries are looking to solar power as part of their future energy policy mix. Finally, in the expectation that solar energy will be a key technology for the future, many countries are adopting policies aimed at developing world class solar industries within their own jurisdictions. When solar power becomes competitive without the need for financial incentives, those jurisdictions with large solar industries stand to benefit from increased economic output.

Countries have provided various kinds of financial incentives to develop solar industries. In this chapter, we have focused on a specific form of financial incentives - one provided to encourage installation of solar panels on residential properties. The list of countries that provide such incentives include Germany, Spain, Canada and the US, although not necessarily in the exact form modelled here.

The ability to install solar panels on land or on rooftops adds value to these resources, and their additional value can be calculated using the real options approach. Holding a tract of land or a rooftop can be seen as being similar to holding a financial option. Analogous to being able to buy an underlying stock at a contracted price, the owner of a land or a rooftop may choose to generate solar power income after paying the installation costs. While there have been many articles published on solar power plants from an engineering perspective, only a few articles have analyzed them from a financial perspective.

Kumbaroglu et al. (2008) valued different types of power plants using the real options approach. Their model incorporated expected learning curve cost reductions as well as input cost projections, to evaluate each power sources' financial merits. They used their results to project the likely composition of future power sources under various regulatory scenarios, focusing especially on Turkey. Solar power plants do not appear prominently in Kumbaroğlu et al.'s research, and the idiosyncrasies inherent in solar plants were not factored into their financial consideration in their paper.

Rehman et al. (2007) study the economics of building solar plants in Saudi Arabia. They combine several different statistics including the amount of solar radiation available per region, to project hypothetical power output of solar plants installed in each region. Utilizing information on costs of solar plants, they calculate such measures as Internal Rate of Return, Net Present Value (NPV) and the Cost of Energy.

The NPV approach uses a deterministic forecast of future power prices and installation costs. However, power prices and installation costs do not fluctuate in a predictable fashion over time. As we shall see, the NPV approach may therefore give misleading answers. In this chapter, we take the real options approach to value the options, and to predict the behaviour of property owners.

In this chapter, our aim is to value the option of installing solar panels on residential properties, and also to examine the property owners' inclination to install panels under different forms of government subsidies. In section 4.1, we examine the various economic factors under consideration for solar panels, and build both our economic model and the resulting option valuation model. In section 4.2, we detail our solution methodology. In section 4.4, we estimate realistic values for the parameters we use. In section 5, we analyze the value of the opportunity to the property owner under different subsidy schemes, and the resulting likelihood that rational owners will choose to install panels. In section 6, we end with a conclusion.

Variable	Description	Value
P(0)	Starting Electricity Price	30¢/kwh
μ_P	Appreciation Rate of Electricity Price	0.6%/year
σ_P	Std. Dev. of Electricity Price	72.4%/year
I(0)	Starting Installation Cost	$10,000
μ_I	Appreciation Rate of Installation Cost	-0.5%/year
σ_I	Std. Dev. of Installation Cost	1.4%/year
δ	Rate of Decrease in Power Output	0.5%/year
$\bar{R}$	Expected Annual Power Output per kW	1200kwh
C	Starting Annual Maintenance Costs	$87/year
μ_C	Appreciation Rate of Maintenance Costs	0.6%/year
τ	Life Expectancy of Solar Plant	20 years
P	Discount Rate	4%
T	Option Expiry	Varies
X	Subsidies	Varies

Table 4-1: Parameter description and their base case values, placed at the beginning of the chapter for easy reference.

4.1 Model Specification

Solar panels generate revenue through sales of electricity, and electricity is typically sold to electrical utilities and transmitted to end users. Electricity prices fluctuate stochastically in time. The manner in which they fluctuate can be modeled quite differently depending on the choices of time frames. For example, Knittel and Roberts (2005) provides an empirical investigation into the hourly electricity spot prices. Hourly prices are characterized by heavy seasonality as well as occasional spikes during times of high electricity demand. Hourly prices would be very useful for a power plant which has variable costs and which have the capability to ramp up production quickly.

However, solar panels provide electricity at no marginal cost, and it is therefore beneficial to run them at full capacity at all times. Therefore, we can opt to model wholesale prices of electricity over longer periods of time, which smoothes over short term fluctuations and produces average electricity prices that fluctuate less dramatically.

Modeling monthly electricity prices is relatively straightforward. Figure 4-1 shows the Q-Q plot of the natural log of monthly wholesale electricity prices, obtained from the Independent Electricity System Operator IESO, a regulatory institution in Ontario. The monthly wholesale electricity prices fit the lognormal distribution quite well. We provide the model for the price of annual electricity prices using the Geometric Brownian Motion (GBM) as described in equation (4-1).

$$\frac{dP}{P} = \mu_P dt + \sigma_P dW. \qquad (4\text{-}1)$$

The P in equation (4-1) stands for the wholesale electricity price, and μ_P and σ_P are its annual rate of appreciation and standard deviation respectively. W is a standard Wiener process.

The revenue generated from solar panels is also a function of the amount of sunlight (R) panels receive, as well as the rate of decline (δ) in the efficiency of the panels. We assume that this rate of decline in efficiency is exponential, and we assume that R and P are independently distributed. The expected discounted revenue over the lifetime of the panels is expressed mathematically as follows.

$$\hat{P}(t) = E\left[\sum_{i=1}^{\tau} \frac{P(t+i)R_i}{e^{(r+\delta)i}}\right] = \sum_{i=1}^{\tau} \frac{E[P(t+i)]E[R_i]}{e^{(r+\delta)i}} = \sum_{i=1}^{\tau} P(t)e^{i(\mu_P - r - \delta)}\overline{R_\iota} = P(t)\overline{R_\iota}\frac{a^{\tau+1}-a}{a-1}.$$

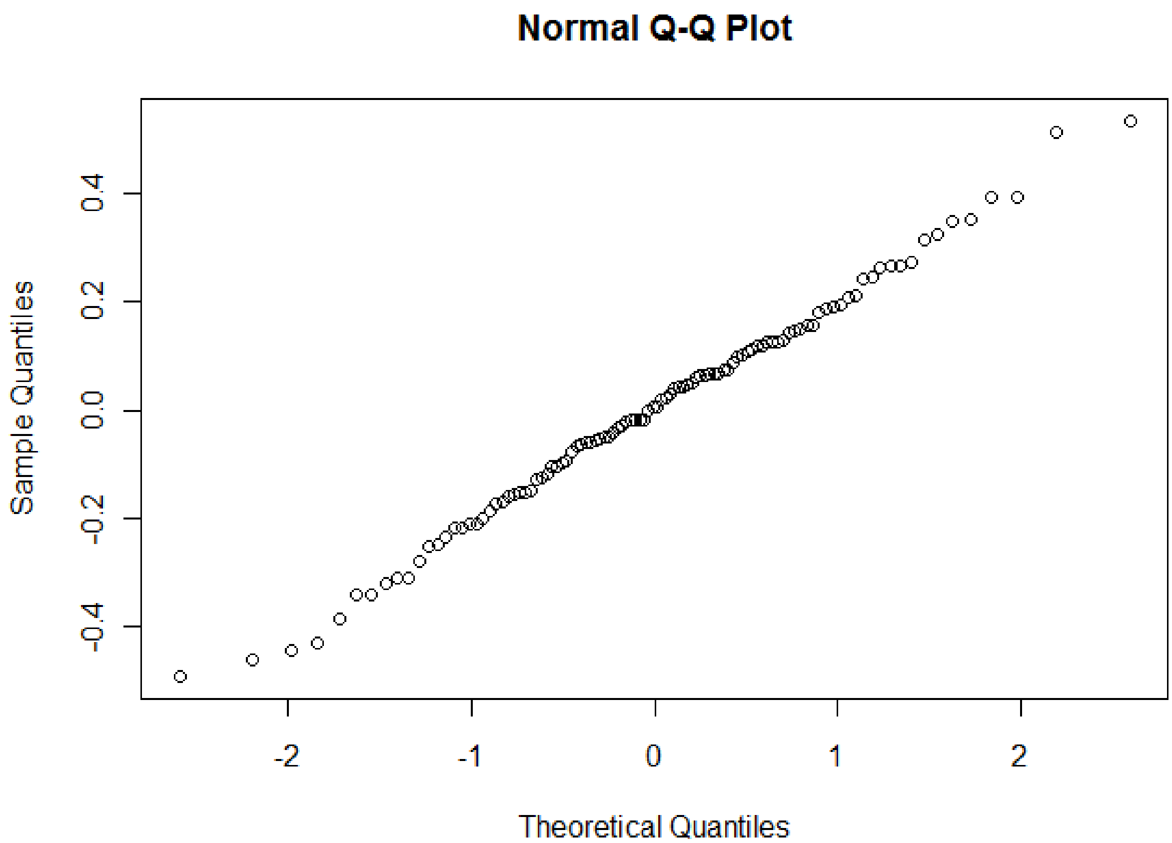


Figure 4-1: Q-Q plot of the natural log of monthly wholesale electricity prices as reported by the Independent Electricity System Operator (IESO).

r is the discount rate required by the owner, and $a = e^{(\mu_P - r - \delta)}$ is introduced to simplify notation. Note that since we only care about the expectation of R_i, we can remain agnostic about the distribution of R_i. Since $\overline{R_\iota}\frac{a^{\tau+1}-a}{a-1}$ is constant, $\hat{P}(t)$ follows the same GBM process that $P(t)$ does.

Installation costs of solar panels have also fluctuated over the years. Increasing economies of scale, learning curves and technological progress are expected to contribute towards lower installation costs in the long run. Figure 4-2 shows the Q-Q plot of the natural log of monthly global solar module prices, which we use as a proxy for overall solar installation costs. The data was provided by Solarbuzz, an international solar energy research and consulting company.

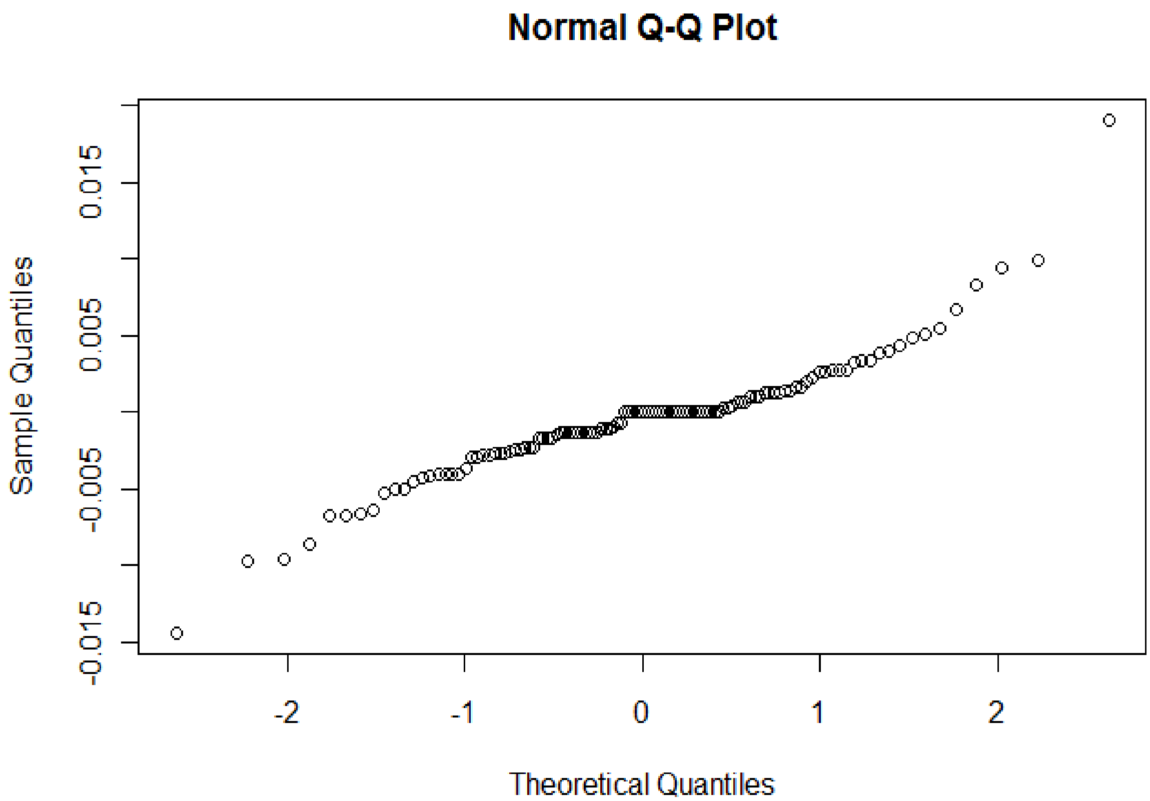


Figure 4-2: Q-Q plot of monthly global solar module prices, as provided by Solarbuzz.

While installation costs don't fit the lognormal distribution as closely as electricity prices do, we believe that the data fits closely enough for us to justify the usage of the GBM as an initial model, leaving the usage of alternate distributions to possible future work. Equation (4-2) describes the movement of installation costs.

$$\frac{dI}{I} = \mu_I dt + \sigma_I dZ. \tag{4-2}$$

I stands for installation costs, whereas μ_I and σ_I stand for annual rate of appreciation and standard deviation, respectively. Z is a standard Wiener process.

Once panels are installed, they must be maintained with an associated cost. We model these costs to rise with inflation in the long run, but not to fluctuate stochastically. The resulting maintenance cost model for time t is given by $Ce^{t\mu_C}$, where μ_C is the rate of appreciation. The maintenance cost over the lifetime of the panels is expressed as follows.

$$\hat{C}(t) = \sum_{i=1}^{\tau} \frac{Ce^{\mu_C(i+t)}}{e^{ri}} = Ce^{t\mu_C}\frac{b^{\tau+1}-b}{b-1}.$$

Here, $b = e^{(\mu_c - r)}$ is introduced to simplify notation.

Governments may provide subsidies to encourage solar panel installations. These subsidies may also vary over time. In this chapter, we consider subsidies which help lower the cost of installations, and denote them by $X(t)$. We assume $X(t)$ is deterministic.

When the owner of a resource decides to install the solar plant, the owner pays an up-front installation cost, and receives revenue from selling electricity plus any subsidies, and pays maintenance fees. The value of a solar installation is expressed in equation (4-3).

$$\mathrm{SP}(P, I, t) = \hat{P}(t) - \hat{C}(t) - I(t) + X(t). \tag{4-3}$$

The Bellman equations for the option to build solar plants are shown in equations (4-4). The top equation describes the evolution of the value of the option when the option to build the plant remains unexercised. The bottom equation describes the value of the option when the option is exercised. Equation (4-5) describes the final condition. The equations are similar to that presented for a spread option, except our model must incorporate an increasing strike price.

$$\begin{cases} \dfrac{\partial L}{\partial t} + \dfrac{\sigma_P^2 \hat{P}^2}{2}\dfrac{\partial L^2}{\partial \hat{P}^2} + \mu_P \hat{P}\dfrac{\partial L}{\partial \hat{P}} + \dfrac{\sigma_I^2 I}{2}\dfrac{\partial L^2}{\partial I^2} + \mu_I I \dfrac{\partial L}{\partial I} + \rho\sigma_P\sigma_I \hat{P} I \dfrac{\partial L^2}{\partial I \partial \hat{P}} - rL = 0 & \hat{P}(t) < B(I,t) \\ L\big(\hat{P}, I, t\big) = \max\big(\mathrm{SP}\big(\hat{P}, I, t\big), 0\big) & \hat{P}(t) \geq B(I,t) \end{cases} \tag{4-4}$$

$$L\big(\hat{P}, I, T\big) = \max\big(\mathrm{SP}\big(\hat{P}, I, T\big), 0\big). \tag{4-5}$$

$B(I,t)$ signifies the exercise boundary, in which for $\hat{P} > B(I,t)$, it is optimal to exercise the option – i.e. install the solar panels. If $\hat{P}$ is below the boundary, the owner is better off waiting.

Note that there can only exist one $B(I,t)$ value for each I,t pair. The continuation value of L (top equation of (4-4)) with respect to $\hat{P}$ is a convex function, which slope varies from greater than 0 to less than 1, and is always positive in value. The payoff (bottom equation of (4-4)) is either 0 or have a slope of 1 with respect to $\hat{P}$. Such functions can only intersect once, and that value is $B(I,t)$.

4.2 Solution

Unfortunately, no analytical solution exists that solves such an equation, and we turn to numerical methods. We proceed by following a similar set of principles as used by Kim (1990) to solve the related American option problem for financial options.

First, we break down the time horizon into finite set of intervals. We also divide $\hat{P}$ and I into grids. At each point on the time grid t, we have a choice to make: exercise or wait. If we exercise, we receive $\mathrm{SP}(P,I,t)$. If not, we receive the future option value $e^{-r\Delta t}L\left(\hat{P},I,t+\Delta t\right)$. The decision on whether to exercise or wait, is based on the condition $\hat{P} > B(I,t)$. We start from the last decision period $t = T - \Delta t$, and work our way backwards in time. At time $t = T - \Delta t$, $B(I, T - \Delta t)$ is the value in which the owner is indifferent to exercising or waiting, expressed mathematically as follows.

$$B(I,T-\Delta t) - I(T-\Delta t) - \hat{C}e^{\mu_C (T-\Delta t)} + X(T-\Delta t)$$

$$= e^{-r\Delta t}E\left[\max\left(\hat{P}' - I' - \hat{C}e^{\mu_C T} + X(T),0\right) | B(I,T-\Delta t), I\right].$$

The left hand side (lhs) is the value realized when exercised (the owner develops at that time), and the right hand side (rhs) is the value realized when held (the owner waits to develop). Note that since there is no decision point after $T - \Delta t$, the rhs is just the European option with Δt to expiry. The expectation on the rhs can be expanded to give the following.

$$= e^{-r\Delta t}\int_0^\infty \int_0^\infty \max\left(\hat{P}' - I' - \hat{C}e^{\mu_C T} + X(T),0\right)\psi(\hat{P}', I'; B(I,T-\Delta t), I)\, d\hat{P}' dI',$$

where ψ is the probability density function. Once we have computed $B(I, T-\Delta t)$, we are able to compute the option value $L\left(\hat{P}, I, T-\Delta t\right)$ as follows.

$$L\big(\hat{P}, I, T-\Delta t\big)\begin{cases}\hat{P} - I - \hat{C}e^{\mu_C(T-\Delta t)} + X(T-\Delta t) & \text{if } \hat{P} \geq B(I, T-\Delta t)\\ e^{-r\Delta t}E\big[\max\big(\hat{P}' - I' - \hat{C}e^{\mu_C T} + X(T), 0\big)\,|\hat{P}, I\big] & \text{if } \hat{P} < B(I, T-\Delta t)\end{cases}$$

To compute $B(I,t)$ and $L\big(\hat{P}, I, t\big)$ for earlier times, we compute recursively by applying the following process. Assuming that we know values of $B(I,t)$ and $L\big(\hat{P}, I, t\big)$ at time $t = (n + 1)\Delta t$, we begin by finding the earlier exercise boundary $B(I, n\Delta t)$ from solving the following.

$$B(I, n\Delta t) - I(n\Delta t) - \hat{C}e^{n\Delta t} + X(n\Delta t) = e^{-r\Delta t}E\big[L\big(\hat{P}', I', (n+1)\Delta t\big)|B(I, n\Delta t), I\big].$$

Using the solutions from above, we can calculate the value of our option.

$$L\big(\hat{P}, I, n\Delta t\big)\begin{cases}\hat{P} - I - \hat{C}e^{\mu_C n\Delta t} + X(n\Delta t) & \text{if } \hat{P} \geq B(I, n\Delta t)\\ e^{-r\Delta t}E\big[L\big(\hat{P}, I, (n+1)\Delta t\big)|\hat{P}, I\big] & \text{if } \hat{P} < B(I, n\Delta t)\end{cases}$$

We repeat the above process until we compute $L\big(\hat{P}, I, 0\big)$.

4.3 Base Case Parameters

In order to generate realistic results, we estimate realistic base case values for all the relevant parameters present in our model. We have used historical data to fit base parameters. There is no guarantee the future will reflect past trends, but the past give us a good reference point to work with.

The parameters that affect the economics of solar panels are location-specific. For instance, some regions receive more sunlight than others. The price of electricity differs across regions. Installation costs may vary according to the proximity to established solar panel servicers. In this chapter, we assign base case parameters based on market conditions in Ontario, Canada where solar power is a big focus of a provincial government initiative to "green" the power supply.

Analyzing IESO's monthly wholesale electricity price data suggests values of $\mu_P = 0.6\%/\text{year}$ and $\sigma_P = 72.4\%/\text{year}$. Analyzing solar installation cost data from SolarBuzz yields $\mu_I = -0.5\%/\text{year}$ and $\sigma_I = 1.4\%/\text{year}$.

The expected annual power generation per kilowatt of installed solar units in Ontario is approximately 1,200 kWh/year. The current installation cost is approximately $10,000/kW and they are typically guaranteed to last at least 20 years according to OurPower, an Ontario solar industry association. Information from OurPower also indicates that annual maintenance costs begin at $87/year and is expected to appreciate by 0.6%/year. From conversations with various solar module installers, we understand that solar power output will decline at the rate of 0.5%/year due to the aging of the panels.

It is difficult to determine the expiry of the option. For property owners who wish to install solar panels on a rooftop, the lifetime of the option is limited by the expected remaining life of the house. For ground mounted panels, the option is theoretically perpetual. However, there may be conditions that further limit the lifetime of the option. For example, a property owner might be mindful of the age of their shingles. It might be painful economically and logistically to uninstall and reinstall solar panels on a roof in order to shingle roofs, and owners may wish to avoid such a scenario. For our purposes, we investigate the value of the option over various timeframes. Calculating $L(\hat{P}, I, t)$ can be very computationally demanding, in the order of $O(N_{\hat{P}}^2 N_{\hat{I}}^2 N_t^2)$. Unfortunately, it is necessary to increase $N_{\hat{P}}$ and N_I when we increase N_t, in order to preserve the accuracy of our results. We therefore show computational results up to a maximum of T=5 years with Δt of 1 year.

The discount rate is very hard to determine as it measures the risk appetite of the investor of solar plants. In this chapter, we consider the risk neutral case and match the discount rate to the interest rate of 20 year Canadian treasuries. As of April of 2011, this rate was close to 4%. We leave the analysis of the effects of higher discount rates to future work.

The variables that go into the model, and the base case values discussed in this section, are summarized in Table 4-1.

4.4 Results

In this section, we examine the economics of solar panel installations. We first conduct an NPV analysis, and determine the appropriate policy action implied by the analysis. We then conduct real options analyses under three different government subsidy structures – No subsidies, fixed subsidies and declining rate subsidies. Under each scenario, we examine the economics of solar installations for property owners, and the effects that different subsidy regimes have on influencing owners' behaviours.

4.4.1 Net Present Value

It is straightforward to conduct an NPV analysis on a solar panel installation project. It is calculated using the following formula, which is equivalent to the value of exercise of the real option today.

$$\mathrm{NPV} = \hat{P}(0) - \hat{C}(0) - I(0) + X(0).$$

One may note that $\hat{P}(0)$ is influenced by the value of σ_P, because it is a consideration that affects the average annual rate of increase in electricity prices e^{μ_P}. Under a deterministic method of analysis such as an NPV, σ_P should be set to 0. While this is true, we seek to preserve the annual rate of increase of electricity prices in an NPV analysis. This means preserving the value of e^{μ_P}, and therefore we offset the (deterministic) absence of σ_P with an increase in the value of μ_P. Consequentially, $\hat{P}(0)$ is equal to the cumulative discounted revenue under a deterministic cash flow model.

Under base case parameter values, the NPV of a solar installation project is calculated as follows.

$$\mathrm{NPV} = \$5{,}819 - \$1{,}241 - \$10{,}000 = -\$5422/\mathrm{kW}.$$

Since the NPV is negative, the model predicts that property owners will not install solar panels. However, the model also predicts that if owners are given $5,422/kW or more in subsidies, the project will become economical, and that we will see solar panel installations.

4.4.2 No subsidies

In this section, we show computational results of the real options model using the base case, assuming no subsidies are granted to property owners. The results of the computation for various values of T are given in Table 4-2. Using results for $B(I, t)$, we perform a Monte Carlo simulation to examine the expected probability that owners could end up installing solar panels before expiry (i.e. exercise the option early).

The value of the option increases with almost linearly with T. This phenomenon can be explained by the low probability of an early exercise, or conversely a high probability of a decision by the building owner to wait until the very end to install solar panels. As time passes by, the value of solar panel installations increases, since electricity prices go up on average, while installation costs do the opposite. Therefore, the owner "gains" economic value proportional to the length of time she is able to wait.

T (years)	1	2	3	4	5
$L(\hat{P}, I, t)$ ($/kW)	623	1,272	1,761	2,124	2,377
Prob. of early install (%)	-	0.2	1.1	2.4	3.6

Table 4-2: Computed values in dollars per kW, of the option to install a solar panel on a roof. The probability that the building owner will exercise before the expiry day is also given, and is obtained through 200,000 runs of Monte Carlo simulation. Results for various option expiries T are used.

The exercise boundary $B(I, 0)$ for $T = 5$ is shown in Figure 4-3. The base case $\hat{P}$ for given I are shown as the circle in the figure also. If the ordered pair $\hat{P}, I$ are above the $B(I, 0)$ line, it would indicate that it is optimal to install solar panels at $t = 0$. However, since it is below, it is optimal to wait. In fact, the base case $\hat{P}, I$ are well below the boundary. For early exercise to occur, future $\hat{P}, I$ would have to go above future $B(I, t)$ lines. Given the large gap that must be bridged, it comes as no surprise that early exercises rarely occur.

As our results show, the optimal choice is to hold off installing solar panels as long as possible if owners are not given any extra incentive. If policy makers want to encourage development of the solar industry in their jurisdiction, some form of intervention seems to be needed.

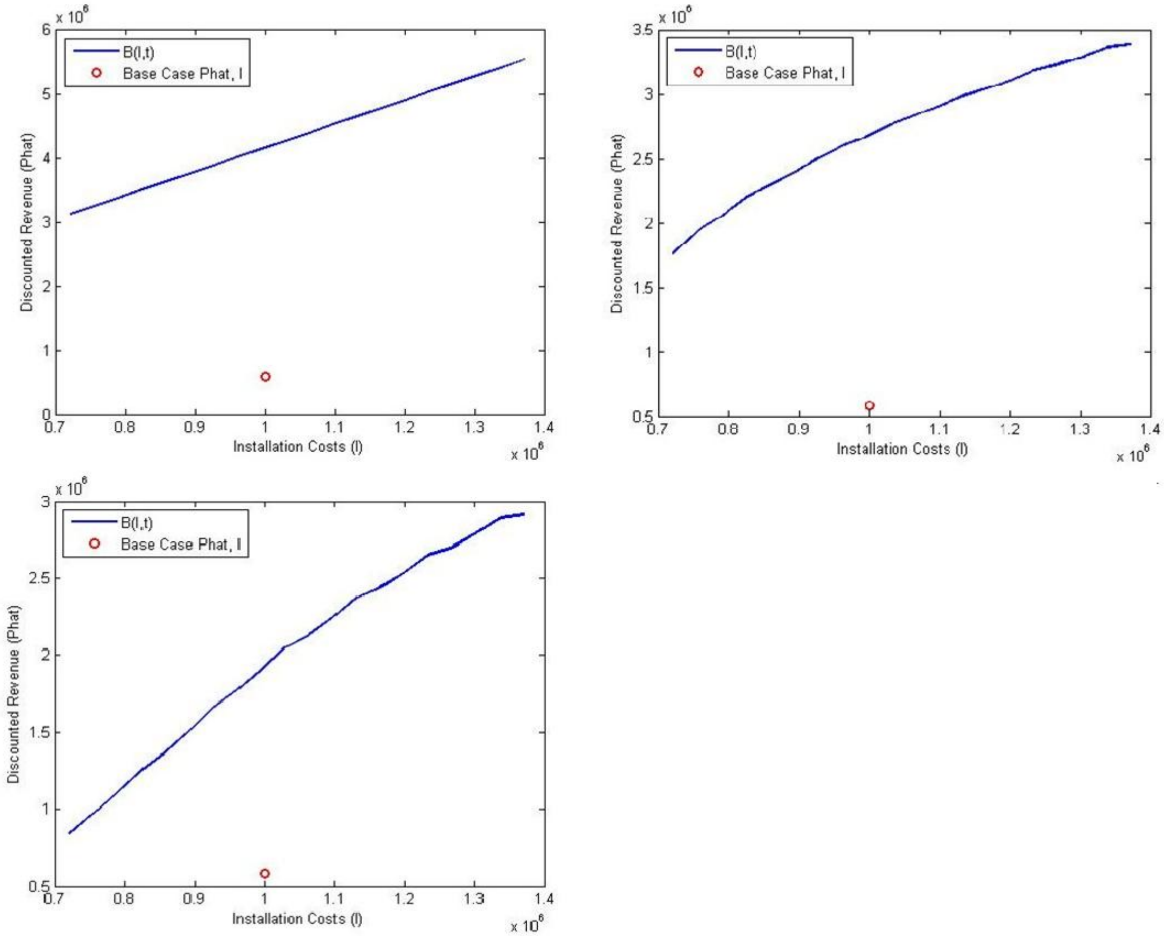


Figure 4-3: Exercise boundaries for t=0 and T=5 are in blue lines. The base case $\widehat{P}$, I coordinates are also shown as red dots. Top left: No subsidies. Top right: Fixed subsidies of \$5,422/kW. Bottom right: Initial subsidies of \$5,422/kW, declining at 10%/year. In all cases, the base case $\widehat{P}$, I are below the exercise boundaries, indicating it's optimal to wait before installing solar panels.

4.4.3 Fixed Installation cost subsidies

Some governments have chosen to give subsidies to those who install solar panels by way of providing tax credits. Examples of such subsidies can be found in the U.S., where people can take advantage of federal tax credits as well as state and county level subsidies if available – e.g. Maryland (2008). According to our NPV analysis, subsidies of \$5,422/kW would be adequate to motivate property owners to install panels. In this section, we examine the effects of providing such amounts to property owners.

Just as important as the existence of these subsidies is the question of whether these subsidies will still be around in future years. In this section, we examine the value of the option under the assumption that the same amount of subsidies will be available until expiry of the option. The resulting values of the option are given in Table 4-3.

T (years)	1	2	3	4	5
$L(\hat{P}, I, t)$ (\$/kW)	1,749	2,344	2,741	3,021	3,204
Prob. of early install (%)	-	2.1	5.9	8.7	10.5

Table 4-3: Computed values in dollars per kW, of the option to install a solar panel on a roof with fixed subsidies of $5,422/kW throughout life of the option. The probability that the building owner will exercise before the expiry day is also given, and is obtained through 200,000 runs of Monte Carlo simulation. Results for various option expiries T are used.

The option to install solar panels is worth more if owners are given the assurance of subsidies, than if no subsidies were offered. We also observe an increased probability that the owner will install the panels early. It is easy to understand this result when we examine the exercise boundaries as shown in Figure 4-3.

We see that the base case $\hat{P}, I$ is still below the boundary line, suggesting that it would not be optimal for property owners to install solar panels at t=0. However, the gap between the base case $\hat{P}, I$ and the boundary line is smaller than seen for the case with no subsidies. This makes it more likely that future $\hat{P}, I$ coordinates would move above the boundary line, causing the owner to install the panels before expiry.

The results suggest that the provision of subsidies encourages property owners to install panels. However, it appears to be a costly policy given that providing $5,422/kwh only modestly increases the chance of early installation. It is also worth contemplating why providing such high level of subsidies only modestly improves the value of the options, which for the 5-year option was in the order of $800. The answer lies in the fact that subsidies are only paid if and when property owners decide to exercise. Since these probabilities are low, the values of options are not as heavily impacted.

The results run contrary to the conclusions obtained from NPV analysis, which states that $5,422/kwh subsidies should be enough to spur property owners to exercise the option today. NPV analysis ignores the fact that since solar installations become more economical over time, it is often more advantageous for property owners to wait. This can be an important insight, as policy makers attempting to encourage solar panel installations may be disappointed by the lack of response generated from their subsidies, should they rely solely on insights provided by the NPV analysis.

4.4.4 Declining Installation Cost Subsidies

Instead of providing fixed subsidies for the foreseeable future, governments may instead choose to provide declining subsidies for the next number of years. One may reason that this policy would encourage owners to act early before subsidies are reduced. In this section, we examine the implications of implementing such a policy.

There are many ways of specifying subsidy decline curves. Governments may choose to implement a straight line decline, reducing an equal dollar amount every year. However, such decline curves are time-constrained; subsidies may reach 0 before the option expires. For our purposes, we model the decline using an exponentially declining curve e^{-at}, where a is the rate of decline. Analysis using different decline curves could be considered in future work.

Rate of decline of subsidies	5%/yr	10%/yr	15%/yr	20%/yr	25%/yr
T = 2years	2.5	2.1	1.7	1.4	1.1
T = 3 years	8.0	7.8	7.0	6.1	5.2
T = 4 years	11.5	11.6	10.8	9.8	8.8
T = 5 years	13.6	14.1	13.6	12.9	12.1

Table 4-4: Probabilities of early exercise for different rates of declines, obtained using 200,000 Monte Carlo simulations. Probabilities are examined for options with different maturities. Sample standard deviations are approximately 0.05%.

In order to find the optimal decline rate that produces the maximum probability of early exercise, we've calculated exercise boundaries for different rates, and ran Monte Carlo simulations to calculate the probabilities of early exercise. The findings are summarized in Table 4-4.

T (years)	1	2	3	4	5
$L(\hat{P}, I, t)$ ($/kW)	1,140	1,812	2,248	2,575	2,822
Prob. of early install (%)	-	2.2	7.8	11.6	14.0

Table 4-5: Computed values in dollars per kW of the option to install a solar panel on a roof with declining subsidies. Subsidies are initially $5,422/kW, and declines at the rate of 10%/year. The probability that the building owner will exercise before the expiry day is also given, and is obtained through 200,000 runs of Monte Carlo simulation. Results for various option expiries T are used.

The findings show that the optimal rate of decline is roughly 10%/year. When the declines are too small, property owners don't have the incentive to act quickly before the subsidies decline. When the declines are too big, the owners may see that subsidies will never suffice to make the projects economical. The values of the options, and the probabilities of early exercise using the decline rate of 10% are shown in Table 4-5.

The results show lower valuations of the option to install solar panels in comparison to the results under fixed subsidies. Given that subsidies decline over the years, this is to be expected. The results also show increases in the probability of early installations, suggesting that the prospect of lower future subsidies act as a positive impetus for property owners to install panels early. If the government aims to encourage as many early installations as possible, reducing the amount of subsidies appears to be an effective strategy. The exercise boundary under declining subsidies is shown in Figure 4-3.

Under declining subsidies, the gap between the boundary line and the base case $\hat{P}, I$ coordinates is smaller than it is under fixed subsidies. This signifies that owners are aware that subsidies are declining, and need a smaller push to get them to install solar panels.

4.5 Conclusions

In our paper, we have valued the option to install solar panels, and investigated the implications of introducing different subsidy regimes aimed at encouraging solar panel installations. At the time of this writing, most residential properties are not big enough to install more than 5kW of generative capacity. For properties where up to 5kW can be installed, if we assume that the option to install panels expire in 5 years, the value of the option is worth roughly $12,000 with no subsidies. This value goes up to roughly $16,000 with fixed subsidies of $27,000. While these numbers are not insignificant, they are but a few percentages of the overall housing prices in Ontario, are typically valued over $300,000. Therefore, while they certainly enhance the value of properties, the introduction of these options is not expected to generate significant “buzz” in the housing market.

From a policy point of view, it is seen that providing subsidies encourages property owners to install solar panels. The NPV analysis shows that subsidies of $5,422/kW should be enough to entice property owners to install panels today. Even though this is over half of the overall installation costs, it is still not enough when viewed from the real options framework. As seen in Figure 4-3, a significant additional amount is required to entice owners to install panels today.

Rather than fixing subsidies, governments who wish to encourage solar panel development can be more effective by introducing subsidies that decline over time. This has the same effect of retailers putting up limited time sales signs on their merchandise, enticing consumers to spend before the bargains end. A high initial subsidy coupled with a moderate decline may give enough incentive for property owners to exercise their options to install panels today.

Chapter 5

5 Conclusions

In this thesis, we've created real options models for different classes of real estate assets, putting special emphasis on bridging the gap between theory and reality by considering input from industry sources. In particular, we have focused on modeling three types of assets – presale contracts of condominiums, raw urban land and properties on which solar panels may be installed. We've chosen these projects both because they are of industries that touch the lives of many, and because, for them, relevant industry contacts were relatively easy to obtain. Real options analysis had not previously been applied extensively to these types of assets, and in each of these types, conversations with industry decision makers has led us to create models which differed materially from those that had been created by other authors. In addition, this thesis had been written with broader usage of mathematical tools than had been generally used so far in the applied real options literature. The usage of these sophisticated tools was necessary to capture some important idiosyncrasies found in assets.

In each of the problems we've analyzed, we compared and contrasted the merits of using a real options approach against more traditional business valuation approaches. We used the models to value the assets under consideration, and to determine profit maximizing decisions. We have seen that in the presale condominium and land development problems, real options do not provide valuations that materially contrast with valuations provided by the NPV approach. The benefits of the real options approach may not be compelling enough for popular adoption.

However, in the case of land development, using the real options approach may explain the behaviour of land developers in ways that the NPV approach can't, and thus real options models may be useful for policy makers. In the solar panel problem we see that the real options method yields very different results, both in terms of value of the opportunities and the expected rational behaviour of property owners, from those inferred from using the NPV approach. The analysis derived from the real options approach may be critical for both property owners and policy makers.

There are many more opportunities to apply real options to the area of real estate. Throughout this chapter, we had assumed that real estate prices move following a GBM process with the exception of chapter 3. However, this is only an approximation, and certainly for short and possibly also for long durations, GBM becomes less representative of real estate price movement. This is because over the short term, real estate prices exhibit momentum. In chapter 3, we introduced mean reversion into our model, which significantly altered the long term behaviour of real estate prices. However, introducing mean reversion did not address the exhibition of short term momentum.

As an appropriate example of a project where a more sophisticated process might be needed, we can take the valuation of subdivision lots. Subdivision lots are an option to build a building. A typical detached residential home takes less than a year to build. Momentum in real estate prices could significantly influence the willingness of builders to purchase lots. Even if you could model short term momentum, land developers probably can't take advantage of such models, as the time horizon to develop land typically outlasts the effects of short term momentum. However, one could model the value of lots as an option to construct buildings, from the perspective of a builder. Such models could benefit from incorporating short term momentum, possibly from utilizing delay equations.

One could also take the solar power project further. In this thesis, we have examined subsidy regimes of the kind that gave tax credits to solar panel purchasers. However, many countries subsidize solar power differently, by guaranteeing purchase of electricity generated from solar panels at a higher price; this is called a feed in tariff. Jurisdictions which provide such feed in tariffs include Germany and Ontario, among others. The presence of feed in tariffs implies the need to create a different model to examine the value of the option to property owners, as well as to analyze the impact of incentives on the willingness of owners to install panels.

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Appendices

A. Analysis of Lai et al. (2004)

This appendix provides a detailed discussion of Lai et al.'s analysis for the purchaser, which is described on page 341 and Appendix A in their paper. We suppose that the purchaser will make a first payment of Q_1 at $t = t_1$. At $t = T$ the purchaser can either make an additional payment Q_2 and thus obtain ownership of the unit or she can pay a penalty of A in order to get out of the contract. It is clear that she will only make the additional payment if $S > Q_2 - A$ where S is the the spot price of a similar unit at the open market at time T.

We suppose that S is given by the stochastic differential equation

$$\boldsymbol{dS = \mu S dt + \sigma S dW}$$

where μ is the expected growth rate of the unit price, σ is the volatility and W is the Wiener process. We will suppose that the purchaser is risk neutral so we can replace μ by the risk free interest rate r_p. Let $C(S,t)$ be the value of the presale option at time t and price S. Hence we can write the Bellman equation in the form

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + \mu S \frac{\partial C}{\partial S} = r_p C. \qquad \textbf{(0-1)}$$

The final condition at $t = T$ can be written in the form

$$C = \max(S - Q_2, -A). \qquad \textbf{(0-2)}$$

Lai et al. claim that they can replace Q_2 by $MSe^{r_p(T-t)}$ and state that Q_2 is "the amount of the last payment and is a function of M (a percentage) and $Se^{r_p(T-t)}$ (the expected spot price at the end of the last period)". It is not clear what they mean by this statement. At the very least M surely must be a function of t. However, their analysis implies that they treat it as a constant. They also replace the penalty A by $\eta Se^{r_p(T-t)}$ where again they state that η is a percentage which they treat as constant. Thus they replace (0-2) by

$$C = \max\left(S - (M-\eta)e^{r_p(T-t)}, 0\right) - \eta S e^{r_p(T-t)}. \qquad \textbf{(0-3)}$$

Since the last term in this condition depends on S, C vanishes at $S = 0$. Thus they get the two boundary conditions

$$\begin{aligned} C &= 0 \text{ at } S = 0, \\ C &\approx S \text{ as } S \to \infty. \end{aligned} \qquad \textbf{(0-4)}$$

Thus they get the solution (their equation A4)

$$C(S,t) = S\left(N\left(d + \sigma\sqrt{T-t}\right) - (M-\eta)N(d) - \eta\right).$$

Where

$$d = \frac{-\ln(M-\eta) - 1/2\sigma^2(T-t)}{\sigma\sqrt{T-t}}.$$

This solution is linear in S, so the second derivative term in (0-1) is identically zero, so σ cannot be relevant in their solution. Despite this, σ does appear in their solution. It is also odd that the risk free interest rate r_p does not appear in their solution.

Our position is that we cannot replace the final payment Q_2 by $MSe^{r_p(T-t)}$ and the penalty A by $\eta S e^{r_p(T-t)}$. Instead we must carry out the calculations using these quantities, Q_2 and A, directly.

B. Modeling Assumptions for Chapter 2

We assume that the price movements of condominiums can be modeled by Geometric Brownian Motion (GBM). As with stock prices, GBM does not completely fit the behaviour of condo price movements, but we choose it for its analytical tractability.

We use the New Housing Price Index (NHPI) and Construction Price Index (CPI) to calibrate our model parameters, which are provided by Statistics Canada (StatCan). These can be found in the Government of Canada's CANSIM database (StatCan, 2009, StatCan, 2009). The NHPI tracks the monthly average housing price across all major metropolitan regions within Canada. The index adjusts for the change of quality in houses.

The CPI is compiled by measuring quarterly changes in building contractors' quoted prices. It excludes land, design, development charges and real estate fees. Development charges are fees payable to the government. It would be reasonable to include the charge as part of our construction cost, but the size of the charge is immaterial relative to the total cost City of Vaughan (e.g. order of $10,000 in the 2009), and so contributes little to the growth of the total construction cost. We therefore do not take it into account.

The graph of the NHPI is shown in Figure 0-1, along with the construction price index. The histogram of the log return of the index is shown in Figure 0-2. We see that the returns exhibit a distribution with heavier tails than the normal distribution. However, we think it is close enough to a normal distribution that we can gain insight into the market by approximating the returns with a normal distribution. The distribution of the CPI behaves similarly to the NHPI, and so we also model it using the lognormal process. This is similar to the treatment of construction cost proposed by Wang and Zhou (2006), who however modeled construction cost as a series of cash flows, whereas we model it as a lump sum.

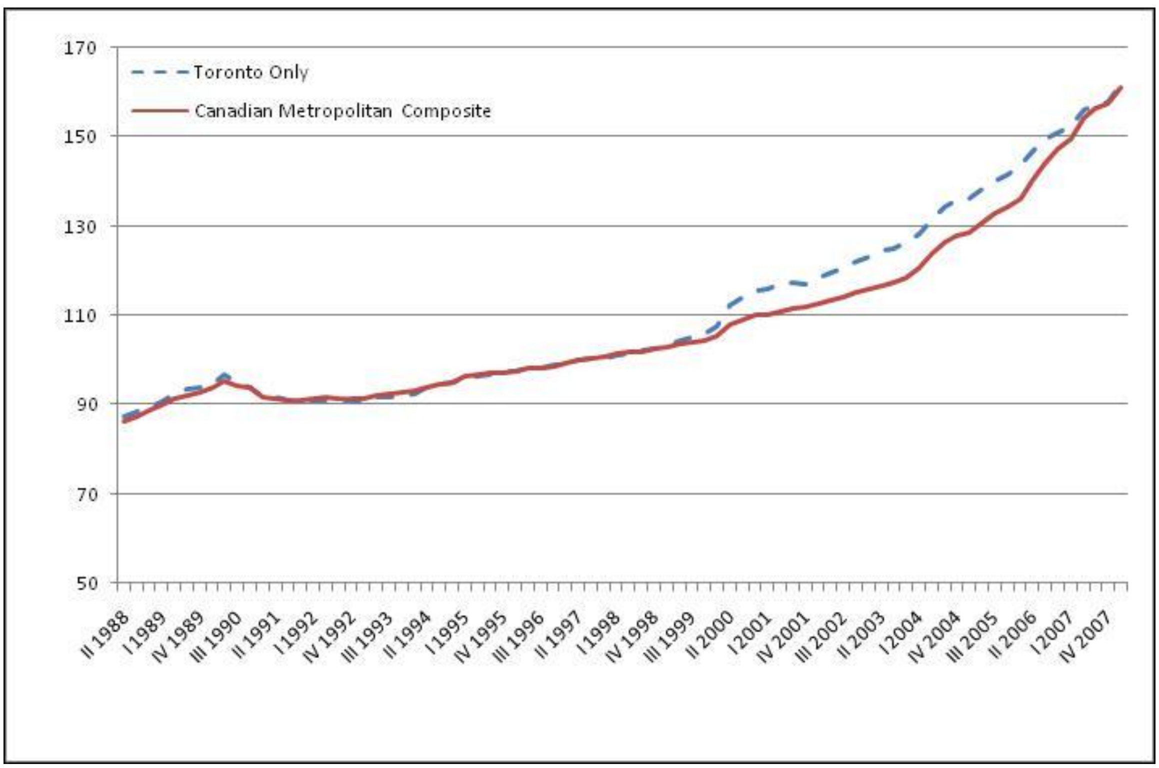

Figure 0-1: Quarterly New Housing Price Index and Construction Price Index, as provided in the CANSIM database by Statistics Canada (1997 price = 100)

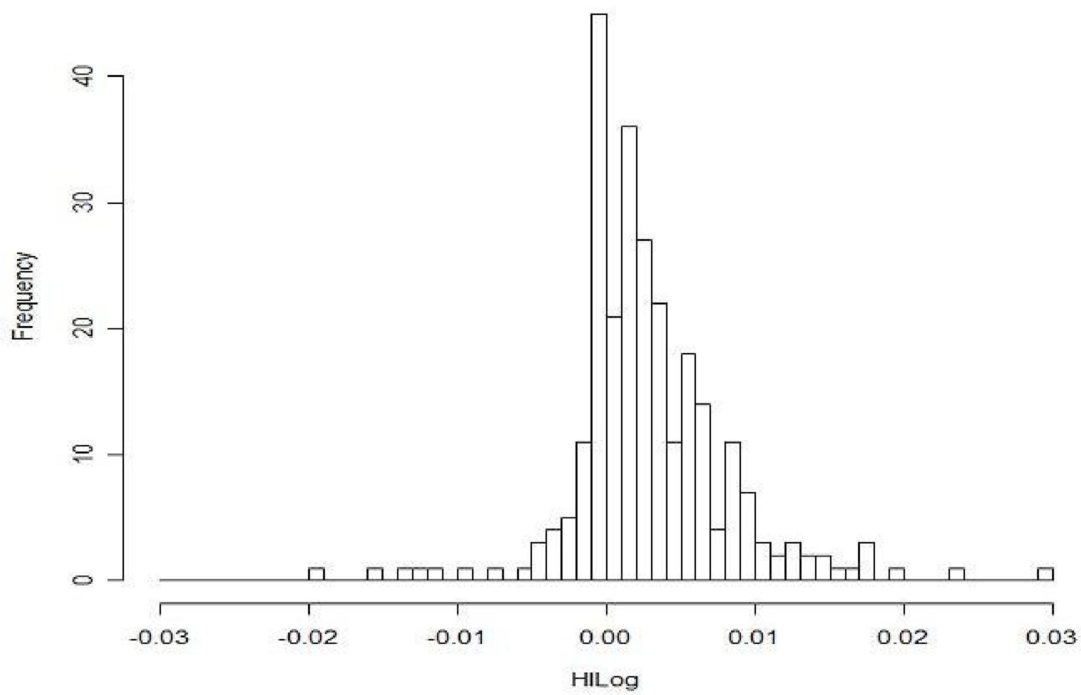

Figure 0-2: Histogram of the of the monthly New Housing Price Index. The histogram shows that, while the normal distribution is not perfect as a model for monthly logarithmic returns, it is reasonable for purposes of our analysis

There is some evidence of autocorrelation in the housing and construction index returns in Fig. 9. However, this would only significantly impact our model if the purchasers and developers could trade the partly constructed condo. In such a scenario, the use of GBM would paint a flawed picture about the profitability for both the purchaser and the developer. However, the underlying cannot be traded while the contract is in effect, and only the initial and the final prices of the condominium matter. The returns on condominium prices four years from the starting period do not seem to bear any significant autocorrelation, and we can treat the condo price as making one geometric Brownian "leap" from period $t = 0$ to $t = T$, with T being four years.

However, the change in expectations of final condo price due to the autoregressive nature of the price series cannot be ignored. Rising prices in the last few months may lead a purchaser to expect higher final prices than she would if prices had been falling. We can keep our assumption of GBM for condo prices while incorporating the change in price expectations by adjusting μ. To find a μ that appropriately reconciles the change in expectations, we need to analyze the autocorrelation in the NHPI data.

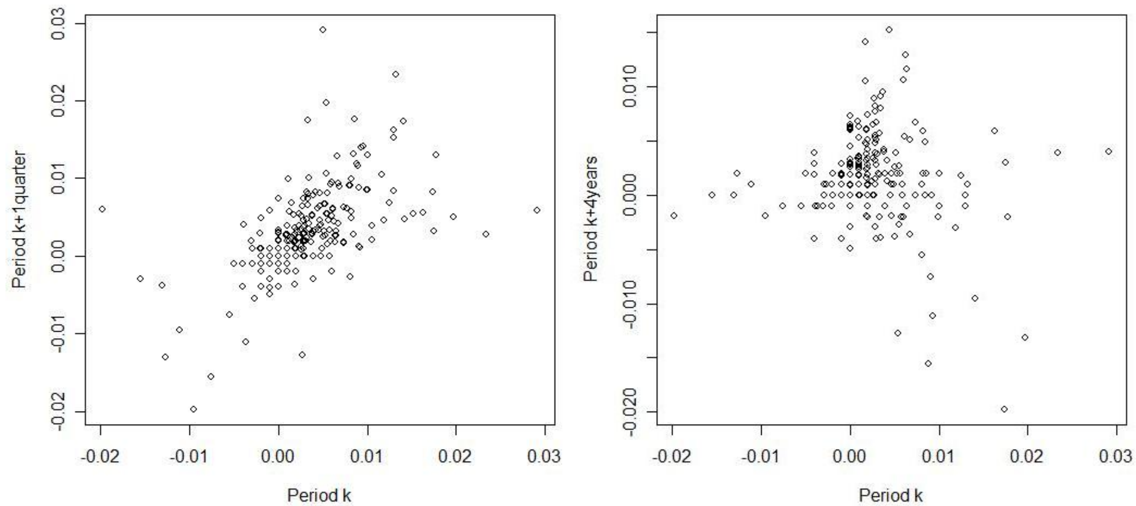

Figure 0-3: Scatter plot of log returns of monthly New Housing Price Index. Plotted returns of period k vs $k + p$, where p is 3 months for the left graph, and 4 years for the right graph. The left graph shows some evidence of pricing momentum in the short term—upward price movement is likely followed by another upward price movement, and vice versa. Over 4 years, the relationship between price movements is weaker. Expectations of condo prices at the time of completion is not affected by the movement in condo prices at the time of the presale agreement since those dates are 4 years apart

The Partial Autocorrelation Function of the log returns of the NHPI is shown in Fig. 10. The graph indicates that it would be a good idea to use either an AR(2) or an AR(5) model. The Akaike Information Criterion (AIC) using the Ordinary Least Squares method supports the use of AR(2), while AIC using the Maximum Likelihood Estimation method supports the use of AR(5). We choose to employ AR(5) because the AIC ranking in support of AR(5) is marginally more decisive. This yields the following

$$E\left[\ln\left(\frac{S(t)}{S(t-1\text{month})}\right)\right] = \lambda_t$$

$$= 4.76 \times 10^{-4} + 5.506\lambda_{t-1} + 2.192\lambda_{t-2} + 1.041\lambda_{t-3} + 0.883\lambda_{t-4} + 0.871\lambda_{t-5}.$$

In order to take the auto regression into account, we can set the expected annual rate of appreciation μ to be $\sum_{i=1}^{48} \lambda_i$ which equals the following

$$\frac{0.118+5.506\lambda_{-1}+2.192\lambda_{-2}+1.041\lambda_{-3}+0.883\lambda_{-4}+0.871\lambda_{t-5}}{4}+\frac{\sigma_S^2}{2}. \quad \textbf{(0-5)}$$

where λ_0 through λ_{-4} are the monthly returns on housing for the previous 5 months. By equating μ to Eq. (0-5), we are able to match our expected appreciation using GBM model over four years to be the same as the expected appreciation implied by the AR(5) model. However, this is not a hard and fast rule. We are free to forecast the value of μ using different methods. Equation (0-5) was merely derived to show that if we choose to, we can incorporate the expectation of the final condo price as implied by AR(5) model in the GBM model.

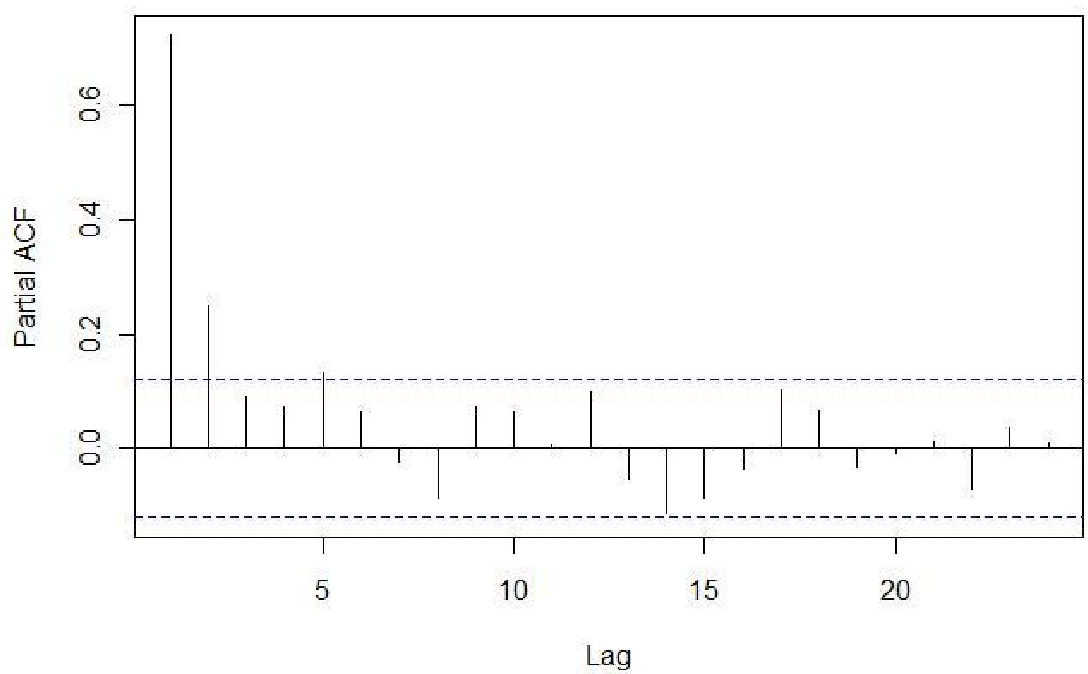

Figure 0-4: The Partial Autocorrelation Function of NHPI log returns. The results confirm the existence of short term pricing momentum in condo prices

Weighing all these factors and considering the significant analytic simplification, we feel that using geometric Brownian motion is adequate to describe the uncertainties associated with the price of a single condominium. By the same logic, we also model the construction cost of condominiums to follow a geometric Brownian motion. The Q-Q plot and the histogram of the returns of CPI are very similar in shape to that of the returns of NHPI.

Rosenthal (1999) finds that construction cost and housing prices are cointegrated. To capture the connection between the two without sacrificing analytic tractability, we instead assume the two time series are correlated, and we denote the correlation as ρ. Any long-term divergence between construction cost and condo price would lead to increasingly greater or smaller profitability for the developer. This leads to our assumption that condo prices and construction costs appreciate at the same rate μ. Our view is validated by comparing the average returns of the NHPI and CPI. However, this does not preclude a divergence of condo and construction prices on a given realization, particularly in the short term.

C. Analytic Solution to Purchaser's Position

The Bellman equation we are trying to solve is identical to the well-known Black Scholes equation, but with the rate of return on an asset allowed not to equal the risk free rate. The fundamental solution of this equation is

$$K(S', t) = e^{-\frac{\left(\ln\frac{S}{S'} + \left(\mu - \frac{1}{2}\sigma^2\right)(T-t)\right)^2}{2\sigma^2(T-t)}} / S'.$$

Given our final condition

$$F(S', T) = \max(S - (Q_2 - A), 0) - A.$$

and defining $B(t)$ to be the following

$$B(t) = e^{-r(T-t)} / \sigma\sqrt{2\pi(T-t)}.$$

Our solution can be obtained by computing the following

$$C(S,t) = B(t)\left[\int_0^\infty \frac{K(S',t)F(S',T)dS'}{S'}\right]$$

$$= B(t)\left[\int_0^\infty \frac{K(S',t)(\max(S-(Q_2-A),0)-A)dS'}{S'}\right]$$

$$= B(t)\left[\int_{Q_2-A}^{\infty}\frac{K(S',t)\big(S'-(Q_2-A)\big)dS'}{S'}\right] - B(t)\left[\int_{-\infty}^{\infty}\frac{K(S',t)AdS'}{S'}\right].$$

Our calculations become much more easy to follow if we use the following easily verifiable relationships.

$$B(t)\int_{\alpha}^{\infty}\frac{K(S',t)S'dS'}{S'} = SN(\delta_1)e^{(\mu-r)(T-t)},$$

$$B(t)\int_{\alpha}^{\infty}\frac{K(S',t)\beta dS'}{S'} = \beta N(\delta_2)e^{-r(T-t)},$$

$$\delta_1 = \frac{\ln\frac{S}{\alpha}+\left(\mu+\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}},$$

$$\delta_2 = \delta_1 - \sigma\sqrt{T-t}.$$

Using the above relationships, the value of a presale contract from the purchaser’s point of view is found to be

$$C(S,t) = SN(d_1)e^{(\mu-r)(T-t)} + (Q_2-A)N(d_2)e^{-r(T-t)} - Ae^{-r(T-t)}.$$

Where

$$d_1 = \frac{\ln\left(\frac{S}{Q_2-A}\right)+\left(\mu+\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}},$$

$$d_2 = d_1 - \sigma\sqrt{T-t}.$$

D. Variance of Purchaser’s Profitability

We calculate the variance of the profitability for the purchaser when she holds the contract without employing any hedging strategies. We do this by calculating the value of the option at time T, which is equal to the payoff of the option.

$$\text{Variance(Profit)}=E[\max(S(T)-Q_2,-A)^2] - E[\max(S(T)-Q_2,-A)]^2$$

$$= E[\max(S(T) - Q_2, -A)^2] - C(S,T)^2.$$

$$E[\max(S(T) - Q_2, -A)^2] = \int_{-\infty}^{\infty} \max\left(S(0)e^{\left(\mu - \frac{\sigma_S^2}{2}\right)T + \sigma_S W(T)} - Q_2, -A\right)^2 P\big(W(T)\big)dW\,.$$

$$= \int_{\alpha}^{\infty}\left(S(0)e^{\left(\mu - \frac{\sigma_S^2}{2}\right)T + \sigma_S W(T)} - Q_2\right)^2 P\big(W(T)\big)dW - A^2\int_{-\infty}^{\alpha} P\big(W(T)\big)dW.$$

Where

$$\alpha = \frac{\ln\left(\frac{Q_2 - A}{S_0}\right) - \left(\mu - \frac{1}{2}\sigma^2\right)T}{\sigma}.$$

To ease notation, denote $S(0) = S_0$, $W(T) = W$ and $\sigma_S = \sigma$. Since W is normally distributed with mean 0 and standard deviation of $\sqrt{T}$,

$$\int_{\alpha}^{\infty}\left(S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma W} - Q_2\right)^2 P(W)dW - A^2\int_{-\infty}^{\alpha} P(W)dW$$

$$= \frac{S_0^2}{\sqrt{2\pi T}}\int_{\alpha}^{\infty} e^{-\frac{1}{2T}\left((W - 2\sigma T)^2 - 4\mu T^2 - 2T^2\sigma^2\right)}dW - \frac{2Q_2 S_0}{\sqrt{2\pi T}}\int_{\alpha}^{\infty} e^{-\frac{1}{2T}\left((W - \sigma T)^2 - 2\mu T^2\right)}dW$$

$$+ \frac{Q_2^2}{\sqrt{2\pi T}}\int_{\alpha}^{\infty} e^{-\frac{W^2}{2T}}dW - \frac{A^2}{\sqrt{2\pi T}}\int_{-\infty}^{\alpha} e^{-\frac{W^2}{2T}}dW$$

$$= \frac{S_0^2}{\sqrt{\pi}} e^{2\mu T + \sigma^2 T}\int_{\frac{\alpha - 2\sigma T}{\sqrt{2T}}}^{\infty} e^{-U^2}dU - \frac{2Q_2 S_0}{\sqrt{\pi}}\int_{\frac{\alpha - \sigma T}{\sqrt{2T}}}^{\infty} e^{-U^2}dW + Q_2^2 N(d_2) - A^2\big(1 - N(d_2)\big)$$

$$= S_0 e^{2\mu T + \sigma^2 T} N(d_3) - 2Q_2 S_0 e^{\mu T} N(d_1) + Q_2^2 N(d_2) - A^2\big(1 - N(d_2)\big).$$

where, as for the purchaser's solutions, we have

$$d_1 = \frac{\ln\left(\frac{S}{Q_2 - A}\right) + \left(\mu + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}},$$

$$d_2 = d_1 - \sigma\sqrt{T}.$$

$$d_3 = d_1 + \sigma\sqrt{T}$$

Substituting our answer for $C(S,T)$ in the full expression for the variance yields

$Variance = S_0^2 e^{2\mu T}\left(N(d_3)e^{\sigma^2 T} - N(d_1)^2\right) + 2AS_0 e^{\mu T} N(d_1)\left(1 - N(d_2)\right) + Q_2^2 N(d_2)\big(1 - N(d_2)\big) + A^2 N(d_2)\left(1 - N(d_2)\right) - 2AQ_2 N(d_2)\left(1 - N(d_2)\right) - 2Q_2 S_0 e^{\mu T} N(d_1)\left(1 - N(d_2)\right).$

E. Solutions to Bellman Equation for Land

Variable Q presented in (3-10) can be interpreted as the degree of deviation from the expected evolution of lot prices. The transformation removes the time-dependence of the original Bellman equations. The new coefficients of $\frac{\partial \tilde{L}}{\partial Q}$ and $\frac{\partial \tilde{P}}{\partial Q}$ terms are time-independent, and since L and P are both perpetual options, $\tilde{L}$ and $\tilde{P}$ are also time-independent. The transformed Bellman equations are as follows.

$$\frac{\sigma^2 Q^2}{2}\frac{\partial^2 \tilde{L}}{\partial Q^2} + \left(\mu + \eta(1-Q)\right)Q\frac{\partial \tilde{L}}{\partial Q} - (\rho_1 - \mu)\tilde{L} = 0. \tag{0-6}$$

$$\tilde{L}(Q^*(t)) = E\left[\tilde{P}(Q^*(t+\tau), t+\tau)e^{-(\rho_2-\mu)\tau} - \frac{I_1}{S_0} + \frac{R}{S_0}\left(\frac{1}{\rho_2-\mu} - \frac{1}{\rho_1-\mu}\right)\middle| Q^*(t)\right],$$

$$\frac{\partial \tilde{L}(Q^*(t))}{\partial Q^*(t)} = E[\frac{\partial \tilde{P}(Q^*(t+\tau))}{\partial Q^*(t)}|Q^*(t)].$$

$$\frac{\sigma^2 Q^2}{2}\frac{\partial^2 \tilde{P}}{\partial Q^2} + \left(\mu + \eta(1-Q)\right)Q\frac{\partial \tilde{P}}{\partial Q} - (\rho_2 - \mu)\tilde{P} = 0. \tag{0-7}$$

$$\tilde{P}\left(Q^\dagger(t)\right) = Q^\dagger(t) - \frac{I_2}{S_0} - \frac{R}{S_0(\rho_2-\mu)},$$

$$\frac{\partial \tilde{P}(Q^\dagger(t))}{\partial Q^\dagger(t)} = 1.$$

Equations (0-6) and (0-7) are similar to perpetual options on underlying assets following an Ornstein-Uhlenbeck process. We can therefore follow a similar logic used to solve the Ornstein-Uhlenbeck equation to solve our equations. The solution to the transformed permit option $\tilde{P}$ is given by equation (3-13)-(3-15). Both A and $Q^\dagger$ must be determined numerically using the two boundary conditions.

The above result can be used to compute the values of the boundary conditions for $\tilde{L}$. Since we don't have the probability distribution function of $Q^*(t+\tau)$ in analytical form, we use Monte

Carlo simulations to take expectations on $Q^*(t+\tau)$. We generate many values of $Q^*(t+\tau)$ given $Q^*(t)$ and compute the average. Equation (0-6) is solved using the same process used to solve equation (0-7). The solution of $\tilde{L}$ is given by equation (3-12).

Curriculum Vitae

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