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Market Institutions and Core Allocations*

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Abstract

We seek to provide a method of analysis that will, in applications, allow us to explain why we observe the market institutions that we do. We define a market by a set of agents, their types, preferences and beliefs, and a set of resource feasible allocations. Any process used to determine an allocation is interpreted as a set of market institutions. We maintain the assumption that we may model as a noncooperative game any such set of market institutions and we consider only payoffs that are obtainable in Nash equilibria of such games. However, we drop the assumption that the game is exogenously given and immutable. We assume that market institutions are created, and may be circumvented, in order to capture “gains from trade”.

An allocation game is any game whose outcomes correspond to some allocation. Payoffs are obtainable if they are obtainable in a Nash equilibrium of an allocation game. We assume that individuals will circumvent the market’s institutions if there exist alternative institutions that provide them a preferred allocation. The allocations that a given subset of individuals, with given beliefs, may obtain are those that are obtainable in Nash equilibria of allocation games for the subset. An allocation is in the incomplete information core if it is obtainable in a Nash equilibrium of an allocation game, and no subset of individuals can improve upon it.
1. Introduction

The purpose of this paper is to provide an explanation for why we observe the market institutions that we do. The analysis applies to markets with incomplete information—markets where individuals, at the time they contract, have private information that is relevant to the individuals with whom they trade. The study of such markets and the institutions used to facilitate trade in them, has dominated much of applied microeconomic theory for nearly two decades. A standard approach has been to specify, either explicitly or implicitly, a noncooperative game theoretic model that captures the essential features of the particular institutions used in the market being studied and to treat the specified game as exogenously given and immutable. In this paper, we maintain the assumption that we may model as a noncooperative game any given set of institutions employed by a given set of individuals to determine an allocation, and we consider only payoffs that are obtainable in Nash equilibria of such games. However, we drop the assumption that the game is exogenously given and immutable. We take seriously the fundamental economic perspective that individuals act in order to capture gains from trade and assume in this paper that market institutions are created, and may be circumvented, in order to capture such gains.¹

If a given set of institutions leaves gains from trade for some subset of individuals uncaptured, individuals will seek to circumvent the institutions in order that these gains may be captured. We define a core by ruling out allocations relative to which there exist uncaptured gains from trade for some subset of individuals. If we assume that a set of market institutions will not survive if it leaves individuals an incentive to circumvent it, then we should expect to observe only institutions that provide equilibrium allocations in the core. An example of such an institution, in a market with complete information, is the price system which, as is well known, in equilibrium yields allocations in the core. In a market with incomplete information, the Nash equilibria of the noncooperative game that captures the essential features of a market's institutions should provide allocations that are in an incomplete information core. If they do not, then there are individuals "leaving money on the table" contrary our fundamental economic intuitions.

¹The phrase "gains from trade" should be interpreted broadly as referring to any increase in the expected utilities of the individuals involved.
being used in a market, we separate in our analysis the specification of the set of noncooperative games that correspond to these market institutions from the other essential features of a market. These other essential features are captured by the specification of an allocation problem. This is an environment in which there is a finite set, $N$, of individuals and a finite set, $D$, of resource feasible allocations. An individual’s type specifies his private information. For each individual we specify a finite set of possible types, a utility function, and beliefs regarding the types of the other individuals in the market. Each individual’s utility is a function of the allocation and all individuals’ types.

Having specified the allocation problem, we then consider the set of all allocation games. This is the set of games for which every outcome corresponds to an allocation in $D$. A payoff vector is obtainable in this setup if it is obtainable in a Nash equilibrium of an allocation game. In section 2, we show a basic setup that incorporates this approach and provide a definition of an incomplete information core. In section 3, the analysis is specialized for an insurance market model where the structure allows an uncomplicated existence proof, and where the unique core allocation is provided by standard separating contracts.

We take the defining characteristic of a market (as the term is commonly used) to be that the trades that are obtained in equilibrium are determined by the fact that individuals always have outside options; individuals are always free to seek to obtain more favorable trades with other individuals. The incomplete information core defined in this paper captures this idea. Because we study allocation problems in this context, we identify them with markets. In this paper, the word market means allocation problem. Since we assume that any process used to determine an allocation may be modelled as a noncooperative game, the phrase set of market institutions means allocation game. Thus, a market is defined by the agents that participate in it, these agents’ preferences and beliefs, and a set of resource feasible allocations. These defining characteristics are the same no matter what the market institutions. A set of market institutions is defined by the strategic options available to each agent within it.

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2 Beliefs other than these, such as those specific to particular equilibrium in a given noncooperative game, are not specified in the allocation problem.

3 The word trade is taken to mean any economic activity that increases the expected utility of the relevant individuals.
The analysis in section 2 shows that, by the revelation principle, the core may be defined in terms of incentive feasible mechanisms because the set of payoffs obtainable with such mechanisms is equivalent to the set of payoffs obtainable in Nash equilibria of allocation games. As defined here, the core is weak; it only rules out allocations that can, in a sense, be surely improved upon. This weak definition is due to a problem that must be confronted in any definition of an incomplete information core: the types of the individuals in a subset $S$ that circumvents the established market institutions cannot be observed, and in the general setting studied here, we lack a basis for ruling out any beliefs for individuals in such a subset.\footnote{See Boyd, Prescott and Smith [1] and Marimon [5] for models in which sorting constraints impose restrictions on the distribution of types in deviating coalitions.} Thus, the incomplete information core is defined here as follows. First, we consider, using the revelation principle, the set of all payoffs that are obtainable in Nash equilibria of allocation games. We then specify, for each subset of individuals, and each possible specification of beliefs, the payoffs obtainable in Nash equilibria in the set of allocation games defined for that subset given their beliefs. We define the core by taking the set of all payoffs obtainable by $N$ in Nash equilibria of allocation games and throwing out those relative to which some subset $S$ of individuals can improve upon no matter what their beliefs. Thus, the incomplete information core only excludes allocations that could be improved upon by some subset of individuals no matter what beliefs individuals in that subset hold.

In section 3 we show that, for a model of the market for insurance based on Rothschild and Stiglitz [15], the core corresponds to the standard set of separating contracts. Although the incomplete information core defined here is weak, it allows us to rule out any market institutions for which, in every equilibrium, there is pooling or other cross-subsidization. Thus, the analysis makes the sharp prediction for the insurance market that we should observe nothing but separating contracts without cross-subsidization. An empirical test of these predictions is provided by Puelz and Snow [14]. In their study of a market for automobile insurance, they were able to reject any equilibrium with cross-subsidization. This result provides reason for some optimism that the incomplete information core defined here may be a valuable tool for the understanding why we observe the market institutions we do.

The analysis developed here was inspired by the study of sustainable matching
plans in Myerson [13] and Green [3]. In these papers, the market is viewed as a matching process, and the matching plans are interpreted as a direct description of this process. Sustainable matching plans may be interpreted as defining a core for this matching process. In this paper we study markets assuming that market institutions can be anything that can be described as a game, as long as the outcome of the game is an allocation in the allocation set. Myerson [13] and Green [3] use dynamic settings to obtain existence results. Here, we consider the type of static setting that is more common in the study of markets with incomplete information. The existence result in this paper is obtained using the single crossing property that is implied by the assumed preferences of consumers in the insurance model.

Marimon [5], and Boyd, Prescott and Smith [1], and Green [2] study cores for markets that share all of the structure of the model of the market for insurance in Rothschild and Stiglitz [15]. Green [2] applies the matching approach. Marimon [5] interprets the core as the outcome of a process of individuals making proposals and counterproposals in nonmarket settings. Boyd, Prescott and Smith [1] interpret different possibilities for the formation of coalitions and the determination of an allocation in a cooperative setting as representing different organizational forms. None of the definitions of incomplete information cores provided in these papers are based on a model that allows for a range of possible market institutions as rich as the set of those that can be modeled as the noncooperative games allowed in this paper.

2. A Basic Setup

We wish to separate the specification of the noncooperative game that describes the essential features of a set of market institutions from the specification of the fundamental features of the market that define the allocation problem that these institutions solve. An allocation problem is given by

\[ \mathcal{A} = (N, D, (T_i)_{i \in N}, (p_i)_{i \in N}, (u_i)_{i \in N}) \]

where \( N \) is the finite set of players, \( D \) is a finite set of resource feasible allocations, \( T_i \) is the finite set of \( i \)'s possible types, \( p_i(t_{-i}|t_i) \) are beliefs, and, \( u_i(d, t) \) is \( i \)'s von Neumann-Morgenstern payoff with allocation \( d \), given type profile \( t \). We assume that the elements of \( \mathcal{A} \) are common knowledge.
The essential features of the institutions in the market are represented by a noncooperative game with incomplete information. For each such game, we consider its representation as an allocation game for \( A \). The set of all such games is given by the set of all games

\[
\Gamma = \left( N, (C_i)_{i \in N}, (T_i)_{i \in N}, (p_i)_{i \in N}, (\tilde{u}_i)_{i \in N} \right)
\]

such that \( N, (T_i)_{i \in N} \), and \( (p_i)_{i \in N} \) are as in \( A \), \( C_i \) is a finite set of strategies for \( i \), there exists a function \( g: C \rightarrow \Delta(D) \), and, \( \tilde{u}_i(c, t) = \sum_{d \in D} g(d|c)u_i(d, t) \), where \( g(d|c) \) is the probability of \( d \) given \( c \). Thus, each strategy profile in each allocation game determines, possibly through a randomization, an allocation in \( D \).

Allocation problems share most of the structure of what Myerson [11] defines to be Bayesian collective choice problems, and in Myerson's terminology, allocation games are subsumed by the allocation problem.\(^5\) However, allocation games are not Bayesian games; in an allocation game the strategic options available to individuals are limited to those that capture the essential features of the given set of market institutions even though we are implicitly assuming that individuals recognize that alternative market institutions exist, and that they may be able to opt to use them. We are not, however, modelling the process by which a given set of institutions becomes established, or how a given set of individuals is able to affect a change in the market institutions they use. Rather we seek to define an incomplete information core by ruling out any allocation that leaves a subset of individuals with the incentive to seek such alternatives.\(^6\)

A Nash equilibrium of a given allocation game \( \Gamma \) is a mixed strategy profile

\[
\sigma \in \times_{i \in N} \times_{t_i \in T_i} \Delta(C_i)
\]

such that, for each \( i \) in \( N \), and each \( t_i \) in \( T_i \), \( \sigma_i \) maximizes

\[
\sum_{t_{-i} \in T_{-i}} \sum_{c \in C_i} p_i(t_{-i}|t_i)\sigma(c|t)_i \tilde{u}_i(c, t)
\]

where \( \sigma(c|t) = \prod_{i \in N} \sigma_i(c_i|t_i) \). The vector of payoffs \( w = (w_i(t_i)_{t_i \in T_i})_{i \in N} \) is said to be obtainable in a Nash equilibrium of an allocation game for \( A \) if there exists

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\(^6\)For work related to the study of Bayesian collective choice problems see Myerson [7,8,9,10,12] and Holmström and Myerson [4].
an allocation game $\Gamma$ for $A$ and a Nash equilibrium mixed strategy profile $\sigma$ for $\Gamma$
such that, for each $i$ in $N$, and each $t_i$ in $T_i$,

$$u_i(t_i) = \sum_{t_{-i} \in T_{-i}} \sum_{c \in C} p_i(t_{-i}|t_i)\sigma(c|t)\hat{u}_i(c, t).$$

The revelation principle allows us to characterize all of the payoffs that are
obtainable in Nash equilibria in the set of allocation games for $A$. An incentive
feasible mechanism for $A$ is any function $\mu: T \to \Delta(D)$ such that

$$\sum_{t_{-i} \in T_{-i}} \sum_{d \in D} p_i(t_{-i}|t_i)\mu(d|t)u_i(d, t)$$

$$\geq \sum_{t_{-i} \in T_{-i}} \sum_{d \in D} p_i(t_{-i}|t_i)\mu(d|t_{-i}, s_i)u_i(d, t),$$

$$\forall i \in N, \ \forall t_i \in T_i, \ \forall s_i \in T_i.$$

The vector of payoffs $u$ is said to be obtainable with an incentive feasible
mechanism for $A$ if there exists an incentive feasible mechanism $\mu$ for $A$ such that,
for each $i$ in $N$, and each $t_i$ in $T_i$,

$$u_i(t_i) = \sum_{t_{-i} \in T_{-i}} \sum_{d \in D} p_i(t_{-i}|t_i)\mu(d|t)u_i(d, t).$$

An incentive feasible mechanism may be viewed as no more than a mathematical
artifice, or it may be viewed as representing a game in which individuals confidentially
communicate with a mediator.

The revelation principle: The set of payoffs obtainable with incentive feasible
mechanisms for $A$ is exactly equal to the set of all the payoffs obtainable in Nash
equilibria in allocation games for $A$.

To see the revelation principle result, let $\sigma$ be a Nash equilibrium of a given
allocation game $\Gamma$ for $A$. Then for each $i$ in $N$, and each $t_i$ in $T_i$, $\sigma_i$ maximizes

$$\sum_{t_{-i} \in T_{-i}} \sum_{c \in C} \sum_{d \in D} p_i(t_{-i}|t_i)\sigma(c|t)g(d|c)u_i(d, t).$$

Let $\mu(d|t) = \sum_{c \in C} \sigma(c|t)g(d|c)$. Conversely, any incentive feasible mechanism for $A$
may be interpreted as defining a game in which individuals use Nash equilibrium
strategies.
The Core

We want to define the allocations that are obtainable by individuals in subsets of $N$, and the payoffs individuals obtain with these allocations. For each $S \subseteq N$, $D^S$ is a set of resource feasible allocations for $S$. For each $S \subseteq N$, let $T^S = \times_{i \in S} T_i$, $T_i = \times_{j \in S \setminus \{i\}} (T_j)$. For each $S \subseteq N$, $i \in S$, $d^S$ in $D^S$ and each $t^S$ in $T^S$, $u_i(d^S, t^S)$ denotes individual $i$'s payoff with allocation $d^S$ when the type profile for $S$ is given by $t^S$. We assume that the payoffs to individuals in $S$ with any allocation $d^S$ in $D^S$ do not depend on the types of the individuals in $N \setminus S$.

The allocation problem faced by any given $S \subseteq N$ will depend on the beliefs held by the individuals in $S$. Let $Q^S = \times_{i \in S} \times_{t_i \in T_i} \Delta(T_i^S)$. For each $S \subseteq N$ and $q \in Q^S$, an allocation problem is given by

$$A^S(q) = (S, D^S, (T_i)_{i \in S}, (q_i)_{i \in S}, (u_i)_{i \in S})$$

where $D^S$ is a set of allocations for $S$, and, $q_i(\cdot | t_i) \in \Delta(T_i^S)$ are beliefs. Note that, for each $S \subseteq N$, and each $q \in Q^S$, a unique allocation problem is defined. In each such allocation problem $A^S(q)$, we assume that, for each $j$ in $S$, the set of $j$'s possible types is $T_j$ where this set differs from the set of $j$'s possible types in the allocation problem $A$ only in that, for each $t_j$ in $T_j$, $j$'s beliefs are given by $q_j(\cdot | t_j)$ rather than by $p_j(\cdot | t_j)$.

The set of all allocation games for $A^S(q)$ is given by the set of all games

$$\Gamma = (S, (C_i)_{i \in S}, (T_i)_{i \in S}, (q_i)_{i \in S}, (\hat{u}_i)_{i \in S})$$

such that $S, (T_i)_{i \in S}$, and $(q_i)_{i \in S}$ are as in $A^S(q)$, $C_i$ is a set of strategies for $i$, there exists a function $g: C^S \rightarrow \Delta(D^S)$, where $C^S = \times_{i \in S} C_i$, and, $\hat{u}_i(c, t) = \sum_{d \in D^S} g(d | c) u_i(d, t)$.

Again, the revelation principle allows us to restrict our attention to incentive feasible mechanisms. An incentive feasible mechanism for $A^S(q)$ is any function $\eta: T^S \rightarrow \Delta(D^S)$ such that

$$\sum_{t_i \in T_i} \sum_{d \in D^S} q_i(t_i | t_i) \eta(d | t_i) u_i(d, t) \geq \sum_{t_i \in T_i} \sum_{d \in D} q_i(t_i | t_i) \eta(d | t_i, s_i) u_i(d, t),$$

For any $S \subseteq N$, if the sets $S$ and $N \setminus S$ are orthogonal, then the set of allocations that are in $D^S$ cannot be restricted by any actions of the members of $N \setminus S$. In the definition of $A^S(q)$ we do not assume that $D^S$ satisfies orthogonality. However, in many economic applications this assumption is natural. For a discussion of the orthogonality assumption, see Myerson [12].
\[ \forall i \in S, \quad \forall t_i \in T_i, \quad \forall s_i \in T_i, \]

and, for every \( i \) in \( S \),
\[ \sum_{t_{-i} \in T_{-i}} \sum_{d \in D^S} q_i(t_{-i}|t_i) \eta(d|t) u_i(d, t) \geq w_i, \quad \forall t_i \text{ in } T_i. \]

Let \( \eta \) be an incentive feasible mechanism for \( A^S(q) \). Let \( y = (y_i(t_i)_{t_i \in T_i})_{i \in S} \) denote the payoffs obtained by the individuals in \( S \) with mechanism \( \eta \). That is, for each \( i \in S \), and each \( t_i \) in \( T_i \),
\[ y_i(t_i) = \sum_{t_{-i} \in T_{-i}} \sum_{d \in D^S} q_i(t_{-i}|t_i) \eta(d|t) u_i(d, t). \]

Let \( \mu \) be a given incentive feasible mechanism for \( A \) with payoff vector \( w \). For any given \( S \subseteq N \), let \( |S| = \sum_{i \in S} |T_i| \). An alternative for \( S \) is an upper-hemicontinuous correspondence
\[ F: Q^S \rightarrow \mathbb{R}^{\|S\|} \]
such that for each \( q \in Q^S \), \( y \in F(q) \) is obtainable with an incentive feasible mechanism for \( A^S(q) \).

We say that \( \mu \) can be improved upon if for some \( S \subseteq N \), there exists an alternative \( F \) for \( S \) such that for every \( q \) in \( Q^S \), and every \( y \) in \( F(q) \), \( y_j(t_j) > w_j(t_j) \), for some \( t_j \) in \( T_j \), for some \( j \) in \( S \). (Recall that incentive feasibility requires that for every \( i \) in \( S \), \( y_i(t_i) \geq w_i(t_i) \), for every \( t_i \) in \( T_i \).)

The incomplete information core of the allocation game \( A \), denoted \( C(A) \), is given by the set of payoff vectors \( w \) obtained by incentive feasible mechanisms for \( A \) such that none of these mechanisms can be improved upon.

**Proposition 2.1** *The core may be empty.*

To see this consider a standard example with three individuals and complete information. Let \( N = \{1, 2, 3\} \), \( D = \{(x_1, x_2, x_3)|x_1 + x_2 + x_3 = 1\} \), and \( u_i = x_i \). If for any \( S = \{i, j\} \), \( D^S = \{(x_i, x_j)|x_i + x_j = 1\} \), then any allocation in \( D \) can be improved upon.

3. The Core for an Insurance Market

Rothschild and Stiglitz [15] introduced a canonical example of a market with incomplete information. In this section we apply the method of analysis introduced in
section 2 above to a model of the market for insurance based on this example. With
an equal treatment assumption, we are able to apply the analysis directly to pairs of
insurance contracts that are incentive feasible. We show that the only pair of con-
tracts in the incomplete information core is the standard set of separating contracts.
At the end of this section we extend the analysis to the case with \( n \) types.

**The allocation problem**

The set of individuals is given by \( N = J \cup I \) where \( J \) is a finite set of consumers
and \( I \) is a finite set of insurers. Each consumer \( j \) in \( J \) has wealth \( w \) if he does not
have an accident and \( w - d \) if he does. There are two types of consumers; a high risk
type for which the probability of an accident is \( p_h \), and a low risk type for which this
probability is \( p_l \), where \( p_h > p_l \). For each consumer \( j \) in \( J \), the set of possible types
is \( T_j = \{ h, l \} \). It is common knowledge that each consumer’s type is drawn from a
distribution where the probability of being a high risk type is \( p \), where \( 0 < p < 1 \). The
beliefs of consumers and insurers are derived from this commonly known distribution.

An insurance contract is described by a vector \( \alpha = (\alpha^1, \alpha^2) \), where \( \alpha^1 \) is the
premium and \( \alpha^2 \) is the insured individual’s net benefit in the event of an accident.
Insurers are risk neutral and face zero costs. We assume that insurers have enough
capital to always honor the contracts they sell. When a consumer with accident
probability \( p_{t_j} \) buys insurance contract \( \alpha \) from insurer \( i \), the insurer’s expected profit
on the contract is

\[
\pi(\alpha, t_j) = (1 - p_{t_j})\alpha^1 - p_{t_j}\alpha^2.
\]

(3.1)

Individuals share a common utility of wealth function \( v(\cdot) \). With insurance
contract \( \alpha \), the expected utility for a consumer \( j \) in \( J \) of type \( t_j = h, l \) is given by

\[
u_j(\alpha, t_j) = (1 - p_{t_j})v(w - \alpha^1) + p_{t_j}v(w - d + \alpha^2) + k_{t_j},
\]

(3.2)

where, for \( t_j = h, l \), \( k_{t_j} \) is a constant that normalizes the utility functions so that the
expected utility of being uninsured is zero. Individuals prefer more wealth to less,
and are risk averse: \( v' > 0 \) and \( v'' < 0 \).

An allocation specifies, for each consumer, the contract he purchases and the
insurer from whom he purchases it. We may take the allocation set \( D \) to be a set
of mappings from \( J \) into \( \mathcal{F} \times I \), where \( \mathcal{F} \) is a subset of \( \mathbb{R}^2_+ \). For each \( j \) in \( J \), and
each \( d \) in \( D \), let \( \alpha(j, d) \) denote the contract that consumer \( j \) obtains in allocation \( d \). An allocation set satisfies equal treatment if, for every \( d \) in \( D \), \( \alpha(j, d) = \alpha(j', d) \), for every \( j \) and \( j' \) in \( J \) such that \( t_j = t_{j'} \). We assume that \( D \) satisfies this condition. This implies that in every allocation, there are at most two types of contracts traded. The utility that a consumer of a given type obtains with a given allocation is just the utility he obtains with the contract he obtains in that allocation.

In the definition of an allocation problem, we assume that the number of individuals in the market is fixed. Therefore, we cannot maintain the assumption made the original model of Rothschild and Stiglitz that there is free entry into the insurance market. However, we maintain the spirit of the free entry assumption by restricting the allocations to those in which insurers have zero expected profits.\(^8\)

The equal treatment of consumers and zero profit for insurers assumptions imply that without loss of generality, we may assume that, in every allocation, every insurer sells the same contracts.\(^9\) This completes the definition of the allocation problem for the insurance market, \( \mathcal{A} = (N, D, (T_i)_{i \in N}, (p_i)_{i \in N}, (u_i)_{i \in N}) \).

### Allocation games and incentive feasible contracts

For the market described by the allocation problem above, consider the representation of any given set of market institutions as an allocation game

\[
\Gamma = (N, (C_i)_{i \in N}, (T_i)_{i \in N}, (p_i)_{i \in N}, (\hat{u}_i)_{i \in N}),
\]

where every pure strategy profile \( c \) determines a probability distribution over allocations in \( D \). That is, there exists a function \( g : C \rightarrow \Delta(D) \). Let \( \alpha_0 \) be the null contract with a zero premium and a zero net benefit. An allocation game \( \Gamma \) is voluntary if every consumer \( j \) in \( J \) has a strategic option \( c_j \) in \( C_j \) that ensures he will obtains the contract \( \alpha_0 \). This may be interpreted as meaning that individuals have the option of not participating in the market. We restrict out analysis to voluntary allocation games.\(^{10}\)

---

\(^8\)We may motivate this approach by dropping the zero profits assumption and considering any allocation in which some insurers sell no contracts while those that do earn positive profits. Using arguments developed in this section of the paper, it is straightforward to show that no such allocation is in the core; the insurer who is inactive in any such proposed allocation could offer a contract that earns nonnegative expected profits, and is strictly preferred by at least one type.

\(^9\)If \( \alpha_0 \neq \alpha_i \) in some allocation in which not all firms sold both contracts, then each contract allows an insurer to break even, and selling both contracts would allow an insurer to break even.

\(^{10}\)The restriction to voluntary allocation games is natural in this model, even though, for the sake of generality, such a restriction was not made in the basic setup in section 2.
By the revelation principle, the set of payoffs obtainable in Nash equilibria in allocations games is equal to the set of payoffs obtainable with incentive feasible mechanisms. From the perspective of an individual, an insurance contract is a lottery, and a mechanism corresponds to a compound lottery. With any given mechanism, all individuals of a given type face the same compound lottery. Therefore, if the set of contracts \( \mathcal{F} \) is large enough, every mechanism is equivalent to another mechanism that, for some \( \alpha_h \) and \( \alpha_l \) in \( \mathcal{F} \), gives every consumer of type \( h \) the contract \( \alpha_h \), and gives every consumer of type \( l \) the contract \( \alpha_l \). In what follows, we apply the analysis directly to pairs of insurance contracts.

For any \( p \) in \([0,1]\), and \( \alpha_h, \alpha_l \) in \( \mathcal{F} \), let

\[
\pi(\alpha_h, \alpha_l, p) = p \left[ (1 - p_h)\alpha_h^1 - p_h\alpha_h^2 \right] + (1 - p) \left[ (1 - p_l)\alpha_l^1 - p_l\alpha_l^2 \right].
\]

A pair of contracts \((\alpha_h, \alpha_l)\) is incentive feasible if

\begin{align*}
(3.3) & \quad u_j(\alpha_{t_j}, t_j) \geq 0, \quad t_j \in \{h, l\}, \quad j \in J, \\
(3.4) & \quad u_j(\alpha_{t_j}, t_j) \geq u_j(\alpha_{t_j'}, t_j), \quad t_j, t_j' \in \{h, l\}, \quad j \in J, \quad \text{and,} \\
(3.5) & \quad \pi(\alpha_h, \alpha_l, p) = 0.
\end{align*}

The set of inequalities in (3.3) require that the pair of contracts \((\alpha_h, \alpha_l)\) provide each type of individual an expected utility at least as great as is obtainable by not participating in the market. Neither type of individual has an incentive to purchase the contract intended for the other type if \((\alpha_h, \alpha_l)\) satisfy (3.4). The equality in (3.5) ensures that insurers in \( I \) earn zero profits. For simplicity, in the discussion that follows, we consider the set of all incentive feasible pairs of contracts rather than just a finite subset.

**The core**

There are three kinds of contracts that can be traded: pooling contracts, where \( \alpha_h = \alpha_l \), \( \pi(\alpha_h, h) < 0 \), \( \pi(\alpha_l, l) > 0 \), and, \( \pi(\alpha_h, \alpha_l, p) = 0 \), separating contracts with cross-subsidization, where \( \alpha_h \neq \alpha_l \), \( \pi(\alpha_h, h) < 0 \) and \( \pi(\alpha_l, l) > 0 \), or \( \pi(\alpha_h, h) > 0 \) and \( \pi(\alpha_l, l) < 0 \), and, \( \pi(\alpha_h, \alpha_l, p) = 0 \), and, separating contracts without cross-subsidization, where \( \alpha_h \neq \alpha_l \) and \( \pi(\alpha_h, h) = \pi(\alpha_l, l) = 0 \).

In order to show that a pair of contracts is not in the core, we need to show that
there exists a subset of individuals that, for any beliefs they may hold, can obtain higher expected payoffs with another pair of contracts that are incentive feasible for them given their beliefs. We will consider subsets of \( N \) that contain both consumers and insurers. Let \( J' \) and \( I' \) denote nonempty subset of \( J \) and \( I \), respectively. Let \( q_i \) denote the proportion of high risk types that insurers believe are in \( J' \).\(^{11} \) The incentive feasibility of a pair of contracts traded by the individuals in \( J' \cup I' \) depends only on the beliefs of the insurers. The beliefs of consumers may therefore be suppressed in this analysis.

For each \( j \) in \( J \), and for each \( t_j \) in \( \{h, l\} \), let \( w_{t_j} \) denote the payoff that individual \( j \) of type \( t_j \) obtains from the contracts \((\alpha_h, \alpha_l)\) traded in \( \mathcal{A} \). We assume that insurers in \( I' \) always have the option of specifying a null contract \( \beta_{t_j} \) for either type \( t_j \) in \( \{h, l\} \). With such a contract, individuals of type \( t_j \) remain with the status quo set of institutions, where the contracts \((\alpha_h, \alpha_l)\) are traded.

A pair of contracts \((\beta_h, \beta_l)\) is incentive feasible for \( S \) given \( q_i \) if

\[
(3.6) \quad u_j(\beta_{t_j}, t_j) \geq w_{t_j}, \quad t_j \in \{h, l\}, \quad j \in J',
\]

\[
(3.7) \quad u_j(\beta_{t_j}, t_j) \geq u_j(\beta_{t_j'}, t_j), \quad t_j, t_j' \in \{h, l\}, \quad j \in J', \quad \text{and},
\]

\[
(3.8) \quad \pi(\beta_h, \beta_l, q_i) \geq 0.
\]

The set of inequalities in (3.6) require that the pair of contracts \((\beta_h, \beta_l)\) give each type of individual a payoff at least as large as the payoff obtained with the contracts \((\alpha_h, \alpha_l)\) traded in \( \mathcal{A} \). No individual has an incentive to misrepresent his type if \((\beta_h, \beta_l)\) satisfy (3.7). Insurers in \( I' \) earn nonnegative profits if (3.8) is satisfied.

**Proposition 3.1** No pooling contract is in the core.

Before proving this proposition, we first introduce a weaker definition of the core that is useful in models in which preferences satisfy the single crossing property. We will show that pooling contracts and separating contracts with cross subsidization are not in this weak core.

Let \( \mathcal{A} \) be an allocation problem, \( \omega \) a vector of payoffs obtainable with an incentive feasible mechanism for \( \mathcal{A} \), and \( S \subset N \). A strong alternative for \( S \) is an upper-

\(^{11}\)All insurers in \( I' \) share the same belief \( q_i \) because we assume that there is only one type for insurers.
hemicontinuous correspondence \( G : Q^S \rightarrow \mathbb{R}^{\|S\|} \) such that for each \( q \in Q^S \), \( y \in G(q) \) is obtainable with a mechanism \( \eta \) that is incentive feasible for \( A^S(q) \), for all \( q \in Q^S \).

The weak incomplete information core of the allocation game \( A \), denoted \( W(A) \), is given by the set of payoff vectors \( w \) obtained by incentive feasible mechanisms for \( A \) such that none of these mechanisms can be improved upon with a strong alternative.

This definition gives us the following obvious result.

**Proposition 3.2** \( C(A) \subseteq W(A) \).

To prove proposition 3.1, we will show that no pooling contract is in the weak incomplete information core. It is sufficient to show that, for any pooling contract \( \alpha \), and for any \( q_i \) in \([0, 1]\), there exists an incentive feasible pair of contracts \((\beta_h, \beta_l)\) that is preferred by at least one type.

With any pooling contract \( \alpha \), low risk individuals are willing to substitute the net benefit in the event of an accident for a lower premium at a higher rate than are high risk individuals. That is,

\[
\frac{(1 - p_l)v'(w - \alpha^1)}{p_lv'(w - d + \alpha^2)} < \frac{(1 - p_h)v'(w - \alpha^1)}{p_hv'(w - d + \alpha^2)}.
\]

This implies that there exists a contract \( \beta_l \) such that \( \beta_l^1 < \alpha^1 \), \( \beta_l^2 < \alpha^2 \),

\[
(3.10) \quad u_j(\beta_l, l) > u_j(\alpha, l), \quad j \in J,
\]

\[
(3.11) \quad u_j(\beta_l, h) < u_j(\alpha, h), \quad j \in J, \text{ and,}
\]

\[
(3.12) \quad \pi(\beta_l, l) > 0.
\]

Equations (3.10) – (3.12) hold for every \( q_i \) in \([0, 1]\). That is, for any beliefs an insurer may hold, the pair of contracts \((\beta_h, \beta_l)\), defined above, are incentive feasible for any subset \( J' \cup I' \), and this pair improves upon the pooling contract \((\alpha_h, \alpha_l)\); low risk individuals strictly prefer \( \beta_l \) to \( \alpha \).

**Proposition 3.3** No separating contract with cross-subsidization is in the core.

Let \((\alpha_h, \alpha_l)\) be a pair of separating contracts with cross-subsidization. Incentive feasibility requires that \( u_j(\alpha_h, h) \geq u_j(\alpha_l, h) \). As with the pair of pooling contracts
considered above, with contract \( \alpha_i \), low risk individuals are willing to substitute the net benefit in the event of an accident for a lower premium at a higher rate than are high risk individuals. That is, (3.9) holds with \( \alpha_i \), and there exists a contract \( \beta_i \) such that (3.10) – (3.12) hold, and the pair \((\bar{\beta}_i, \beta_i)\) is incentive feasible, for any \( q_i \) in \([0, 1]\).

Consider a pair \((\alpha_h, \alpha_l)\) separating contracts with cross-subsidization where \( \pi(\alpha_h, h) > 0 \) and \( \pi(\alpha_l, l) < 0 \). Such a pair is not in the core. To see this, let \( \alpha_h^* \) be the contract that maximizes \( u_j(\alpha, h) \) subject to \( [(1 - p_l)\alpha_l - p_l\alpha_l^+] = 0 \). Then the pair \((\beta_h, \beta_l)\) where \( \beta_h = \alpha_h^* \) is incentive feasible for all \( q_i \) in \([0, 1]\) and is strictly preferred by high risk types.

The standard set of separating contracts is given by the pair \((\alpha_h^*, \alpha_l^*)\) such that \( \alpha_h^* \) maximizes \( u_j(\alpha, h) \) subject to \( \pi(\alpha, h) = 0 \), and \( \alpha_l^* \) maximizes \( u_j(\alpha, l) \) subject to \( u_j(\alpha_h^*, h) \geq u_j(\alpha, h) \), and \( \pi(\alpha, l) = 0 \).

**Proposition 3.4** The pair of contracts \((\alpha_h^*, \alpha_l^*)\) is in the core.

In order to show that the pair \((\alpha_h^*, \alpha_l^*)\) is in the core, it is sufficient to demonstrate that, for all \( J' \cup I' \subseteq J \cup I \), there are beliefs \( q_i \) in \([0, 1]\) and a sequence \( \{q_i^n\} \) converging to \( q_i \) such that the payoffs obtainable with incentive feasible contracts given \( q_i^n \) converge to \((u_i(\alpha_h^*, h), u_i(\alpha_l^*, l))\). There are two types of contracts to consider; pooling contracts, and separating contracts with cross-subsidization. Let \( \{q_i^n\} \) be an increasing sequence in \([0, 1]\) that converges to 1. We will argue that as \( q_i^n \) approaches 1, the only feasible pair of contracts is \((\beta_h, \beta_l) = (\alpha_h^*, \alpha_l^*)\). To see that pooling contracts can be ruled out for \( q_i^n \) close enough to 1, recall that the assumptions on \( v(\cdot) \) imply that

\[
-(1 - p_l)v'(w) < \frac{-(1 - p_h)}{p_h}.
\]

Therefore, for some \( \varepsilon > 0 \), for all \( q_i^n \in (1 - \varepsilon, 1) \) no pooling contract \( \beta \) could satisfy \( u_j(\beta, l) \geq u_j(\alpha_l^*, l) \) and \( \pi(\beta, \beta, q_i^n) \geq 0 \).

Miyazaki [6] shows that, for any \( q_i^n \in (0, 1) \), there exists a pair of separating contracts \((\beta_h, \beta_l)\) with cross-subsidization that earn nonnegative expected profit, are at least weakly preferred by high and low risk types to \((\alpha_h^*, \alpha_l^*)\). Such a pair of contracts is a solution to the following programming problem.
\[(3.14) \quad \max_{\beta_h, \beta_l} u_j(\beta_l, l) \]

subject to

\[u_j(\beta_h, h) \geq u_j(\alpha_h, h),\]
\[u_j(\beta_h, h) \geq u_j(\beta_l, h), \text{ and},\]
\[q_i \left[ (1 - p_h)\beta_h^1 - p_h\beta_h^2 \right] + (1 - q_i) \left[ (1 - p_l)\beta_l^1 - p_l\beta_l^2 \right] \geq 0.\]

As \(q_i^n\) approaches 1, the solution to this program approaches \((\alpha_h^*, \alpha_l^*)\). That is, the optimal separating contract converges to the separating contracts without cross-subsidization. (See Miyazaki [6].) Thus, the standard set of separating contracts cannot be improved upon for all \(q_i\) in \([0, 1]\) and are therefore in the core.

To see that no other pair of contracts \((\alpha_h, \alpha_l)\) that are separating without cross-subsidization can be in the core, notice that the pair \((\beta_h, \beta_l) = (\alpha_h^*, \alpha_l^*)\) is incentive feasible for any \(J' \cup I'\) with any \(q_i\) in \([0, 1]\), and this pair of contracts is strictly preferred by at least one type. If follows that the only set of contracts in the core is \((\alpha_h^*, \alpha_l^*)\).

**Interim efficiency**

A desirable property of an allocation is that it be *interim efficient*. (See Holmström and Myerson [4].) A pair of contracts \((\alpha_h, \alpha_l)\) is interim efficient if it is incentive feasible and there does not exist another set of incentive feasible contracts \((\beta_h, \beta_l)\) that at least one type strictly prefers, where the incentive feasibility of both pairs of these contracts is evaluated using the same beliefs, \(p\), for insurers. It may be the case that interim efficient contracts can be improved upon.

**Proposition 3.5** There exists an \(\varepsilon > 0\) such that if \(\varepsilon > p > 0\), then the set of core and interim efficient allocations are disjoint.

To show the proposition, for any given \(p\), let \(\alpha^* (p)\) be the pooling contract that earns zero expected profits and maximizes the expected utility of low risk individuals. That is, \(\alpha^*\) maximizes \(u_j(\alpha, l)\) subject to \(\pi(\alpha, \alpha, p) = 0\). By the strict concavity of \(v\), it follows immediately that for \(p\) close enough to zero \((\alpha^*, \alpha^*)\) interim dominates \((\alpha_h^*, \alpha_l^*)\).
The $n$ type case

Extending the above analysis to markets in which there are more than two types is straightforward. No equilibrium with pooling or other cross-subsidization is in the core. The unique core allocation is the standard set of separating contracts in which the highest risk type gets full insurance and each other type gets the maximum amount of insurance at actuarially fair rates subject to the constraint that the next highest risk type does not prefer this contract to its own. Here we present an outline of the argument that demonstrates this.

For each consumer, let the set of types be $T = \{t_1, t_2, \ldots, t_n\}$. Let $p_{t_k}$ be the probability that a type $t_k$ has an accident. Assume $p_{t_1} < p_{t_2} < \cdots < p_{t_n}$. Let $\pi(\alpha, t_k)$ denote the profit an insurer earns on contract $\alpha$ when it is sold to a type $t_k$. That is,

$$\pi(\alpha, t_k) = (1 - p_{t_k})\alpha^1 - p_{t_k}\alpha^2.$$ 

Let $u(\alpha, t_k)$ denote the expected utility for a type $t_k$ with contract $\alpha$. That is,

$$u(\alpha, t_k) = (1 - p_{t_k})v(w - \alpha^1) + p_{t_k}v(w - d + \alpha^2) + k_{t_k}.$$ 

Each consumer is a type $t_k$ with probability $\lambda_{t_k}$. Let $\lambda = (\lambda_{t_1}, \lambda_{t_2}, \ldots, \lambda_{t_n})$. Then, $\lambda \in \Delta(T)$. Let $(\alpha_{t_1}, \alpha_{t_2}, \ldots, \alpha_{t_n})$ denote an n-tuple of contracts traded with the status quo set of institutions.

Given $J' \subseteq J$ and $I' \subseteq I$, for any consumer $j \in J'$ and insurer $i \in I'$, let $q_{t_k}$ denote the probability that insurer $i$ assigns to agent $j$ being a type $t_k$. There is only one type of insurer, so they all hold the same beliefs. Let $q = (q_{t_1}, q_{t_2}, \ldots, q_{t_n})$. Then, $q \in \Delta(T)$.

For each $k = 1, \ldots, n$, let $\tilde{\beta}_{t_k}$ denote the null contract that specifies that consumers of type $t_k$ remain with the status quo set of institutions where the contracts $(\alpha_{t_1}, \ldots, \alpha_{t_n})$ are traded.

To see that no pooling contract $\alpha$ is in the core, recall the argument used in the model with two types based on the relative slopes of the different types’ indifference curves at a pooling contract $\alpha$. Apply this argument in this model to types $t_1$ and $t_2$. Let $\beta_{t_1}$ be a contract that is preferred by $t_1$ to $\alpha$ and not preferred by type $t_2$ to $\alpha$. Such a contract is not preferred by types $t_3, \ldots, t_n$ to $\alpha$. The n-tuple of contracts $(\beta_{t_1}, \tilde{\beta}_{t_2}, \ldots, \tilde{\beta}_{t_n})$ is incentive feasible for every $q \in \Delta(T)$ and strictly preferred by type $t_1$. 

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To see that separating contracts with cross-subsidization are not in the core, let $\beta_{t_1}$ be a contract that is preferred by type $t_1$ to $\alpha_{t_1}$ and not preferred by type $t_k, k \geq 2$, to $\alpha_{t_k}$ and hence not preferred to $\alpha_{t_k}$ by type $t_k, k \geq 2$. The n-tuple of contracts $(\beta_{t_1}, \beta_{t_2}, \ldots, \beta_{t_n})$ is incentive feasible for every $q \in \Delta(T)$ and strictly preferred by type $t_1$.

Now we want to show that the standard set of separating contracts is in the core. This set of contracts is given by $(\alpha_{t_1}^*, \alpha_{t_2}^*, \ldots, \alpha_{t_n}^*)$ where $\alpha_{t_n}^*$ maximizes $u(\alpha, t_n)$ subject to $\pi(\alpha, t_n) = 0$, and, for $k \leq n - 1$, $\alpha_{t_k}^*$ maximizes $u(\alpha, t_k)$ subject to $u(\alpha, t_{k+1}) \leq u(\alpha_{t_{k+1}}^*, t_{k+1})$, and $\pi(\alpha, t_k) = 0$.

Let $\{q^n\}$ be a sequence in $\Delta(T)$ that converges to $(q_{t_1}, q_{t_2}, \ldots, q_{t_n}) = (0, 0, \ldots, 1)$. As this sequence approaches its limit, the payoffs obtainable from a set of incentive feasible contracts converges to the payoffs obtained with the standard set of separating contracts, $(\alpha_{t_1}^*, \alpha_{t_2}^*, \ldots, \alpha_{t_n}^*)$. Thus, with $n$ types, the unique set of contracts in the core is the standard set of separating contracts, $(\alpha_{t_1}^*, \alpha_{t_2}^*, \ldots, \alpha_{t_n}^*)$.

4. Empirical evidence and conclusions

The definition of the incomplete information core provided in this paper is motivated by the belief that individuals act in order to capture gains from trade. Taking this perspective seriously, and recognizing that market institutions are created and may be circumvented by individuals, we should not expect to observe market institutions for which equilibrium outcomes are not in the core. Applying this to the market for insurance, the model developed in this paper yields the prediction that we should only observe the separating contracts without cross-subsidization, $(\alpha_{t_1}^*, \alpha_{t_2}^*)$.

In a study of the market for automobile collision insurance in Georgia, Puelz and Snow [14] were able to reject the hypothesis that high risk types receive contracts that are subsidized by low risk types. That is, they were able to reject the hypothesis that the contracts were pooling or separating with cross-subsidization. This result provides reason to hope that the approach developed in section 2 will prove to be useful in understanding why we use the market institutions that we do. The model developed in section 2 is reasonably general and we hope that it will be applied and tested in a wide variety of markets.
References


