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Fabry-perot Resonators With Grating Elements

Fah-chap Choo

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FABRY-PEROT RESONATORS WITH GRATING ELEMENTS

by

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Submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy

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ABSTRACT

A laser cavity, essentially a Fabry-Perot resonator, consists of two highly reflecting mirrors. The study of such a cavity involves determination of the transverse normal mode field distribution, diffraction losses and phase shifts.

When one of the reflecting mirrors is replaced by a diffraction grating, the resonator reveals the following significant properties:

(i) wavelength selectivity;
(ii) polarization selectivity;
(iii) possible discrimination against some transverse normal modes.

In this thesis, theoretical investigations based on the geometrical theory of diffraction are carried out for different types of grating resonators. The following are the main results:

(i) A new formulation of the infinite strip resonator problem is established in which the conditions of large strip size and large strip separation, normally imposed in the Fox and Li formulation, are relaxed. Hence problems of strip grating resonators involving strip sizes comparable to the wavelength can be solved.
(ii) Calculations of intensity distributions of echelle gratings with deep grooves, illuminated by a plane wave are carried out. The agreement between theoretical and experimental results lends support to the use of the geometrical theory of diffraction in the study of grating resonators.

(iii) Diffraction losses, phase shifts and transverse mode patterns are obtained for various types of grating resonators.
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# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>ABSTRACT</td>
<td>iii</td>
</tr>
<tr>
<td>ACKNOWLEDGEMENTS</td>
<td>v</td>
</tr>
<tr>
<td>TABLE OF CONTENTS</td>
<td>vii</td>
</tr>
<tr>
<td>LIST OF TABLES</td>
<td>x</td>
</tr>
<tr>
<td>LIST OF FIGURES</td>
<td>xi</td>
</tr>
<tr>
<td>Chapter I. Introduction</td>
<td>1</td>
</tr>
<tr>
<td>Chapter II. The Geometrical Theory of Diffraction.</td>
<td>6</td>
</tr>
<tr>
<td>2.1 Introduction</td>
<td>6</td>
</tr>
<tr>
<td>2.2 Sommerfeld half-plane problem</td>
<td>8</td>
</tr>
<tr>
<td>2.3 Diffraction by a wedge</td>
<td>15</td>
</tr>
<tr>
<td>2.4 Asymptotic expansion of Green's Theorem</td>
<td>17</td>
</tr>
<tr>
<td>Chapter III. A New Formulation of the Infinite Strip Resonator.</td>
<td>19</td>
</tr>
<tr>
<td>3.1 Introduction</td>
<td>19</td>
</tr>
<tr>
<td>3.2 Round trip integral equation</td>
<td>23</td>
</tr>
<tr>
<td>3.3 Diffraction by slit</td>
<td>25</td>
</tr>
<tr>
<td>3.4 A new formulation of an infinite strip resonator problem</td>
<td>28</td>
</tr>
<tr>
<td>3.5 Interaction kernel from the interaction between edges</td>
<td>35</td>
</tr>
<tr>
<td>3.5a Lowest interaction term for P-polarization</td>
<td>36</td>
</tr>
</tbody>
</table>
Chapter IV

The Intensity Distribution of Echelette Gratings

4.1 Introduction
4.2 The grating equation
4.3 The intensity distribution
4.3.1 Singly diffracted rays
4.3.2 Doubly diffracted ray d-correction
4.3.3 A-correction
4.3.4 B-correction
4.3.5 Triply diffracted ray
4.3.6 N = 1 of 23½° grating
4.4 Discussion

Chapter V

Fabry-Perot Resonators with Grating Elements

5.1 Introduction
5.2 15° grating resonator
5.2.1 Formulation
5.2.2 TEM₁-mode - possible transverse mode discrimination
5.3 Parallel-strip-grating resonator
5.3.1 Formulation - non-interacting term
5.3.2 Numerical results
5.3.3 The leakage behind the parallel-strip-grating
5.3.4 Estimate of the eigenvalue for a 23½° echelette grating resonator
5.3.5 Interaction terms

3.5b Lowest interaction term for S-Polarization
3.6 Numerical Examples

Page 39
Page 40
Page 43
Page 43
Page 47
Page 49
Page 49
Page 56
Page 60
Page 63
Page 67
Page 71
Page 73
Page 79
Page 79
Page 84
Page 84
Page 91
Page 92
Page 92
Page 95
Page 97
Page 99
Page 100
5.4 Generalized strip grating resonator. . . . . . . . . . . . 104

5.4.1 Formulation . . . . . . . . . . . . . . . . . . . . . . . 104

5.4.2 Parallel-strip-grating resonators. 105

5.4.3 Three parallel strip resonators. . 109

5.4.4 Parallel-strip-grating with step width $b \neq m/2\lambda$ . . . . . . . . . . . . . . . . . . 113

5.4.5 Parallel-strip-grating resonator with step width $b = 0$. . . . . . . . 116

Chapter VI Conclusions. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 120

REFERENCES . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 123

Appendix I . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 126

VITA . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 129
LIST OF TABLES

<table>
<thead>
<tr>
<th>TABLE</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>Comparison of diffraction losses and phase shifts</td>
<td>42</td>
</tr>
<tr>
<td>II</td>
<td>Characteristics of a set of gratings</td>
<td>52</td>
</tr>
<tr>
<td>III</td>
<td>Experimental results of Brannen &amp; Rumbold</td>
<td>53</td>
</tr>
<tr>
<td>IV</td>
<td>Intensity distribution when only the singly diffracted ray is considered</td>
<td>54</td>
</tr>
<tr>
<td>V</td>
<td>Intensity distribution when the singly diffracted ray and the d-correction are considered</td>
<td>59</td>
</tr>
<tr>
<td>VI</td>
<td>Intensity distribution when the singly diffracted ray, the d-correction and the a-correction are considered</td>
<td>62</td>
</tr>
<tr>
<td>VII</td>
<td>Strength of the line sources for the b-correction</td>
<td>66</td>
</tr>
<tr>
<td>VIII</td>
<td>Intensity distribution when the singly diffracted ray, the d-, the a-, and the b-corrections are considered</td>
<td>68</td>
</tr>
<tr>
<td>IX</td>
<td>Intensity distribution when the singly diffracted ray, the triply diffracted ray, the d-, the a-, and the b-corrections are considered</td>
<td>70</td>
</tr>
<tr>
<td>Xa</td>
<td>Comparison of results of the intensity distribution (P-polarization)</td>
<td>74</td>
</tr>
<tr>
<td>Xb</td>
<td>Comparison of results of the intensity distribution (S-polarization)</td>
<td>75</td>
</tr>
</tbody>
</table>
LIST OF ILLUSTRATIONS

<table>
<thead>
<tr>
<th>FIGURE</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>Half-plane problem</td>
<td>9</td>
</tr>
<tr>
<td>2.2</td>
<td>Reciprocal relation</td>
<td>14</td>
</tr>
<tr>
<td>2.3</td>
<td>Diffraction by a wedge</td>
<td>16</td>
</tr>
<tr>
<td>2.4</td>
<td>Diffraction of a block</td>
<td>18</td>
</tr>
<tr>
<td>3.1</td>
<td>The transmission line analogue</td>
<td>20</td>
</tr>
<tr>
<td>3.2</td>
<td>Infinite strip resonator</td>
<td>24</td>
</tr>
<tr>
<td>3.3</td>
<td>Diffraction of a slit</td>
<td>26</td>
</tr>
<tr>
<td>3.4</td>
<td>Formulation of the infinite strip resonator problem</td>
<td>30</td>
</tr>
<tr>
<td>3.5</td>
<td>Doubly diffracted ray</td>
<td>37</td>
</tr>
<tr>
<td>4.1</td>
<td>An echelette grating of 90° triangular grooves</td>
<td>44</td>
</tr>
<tr>
<td>4.2</td>
<td>Primary and secondary diffracted waves</td>
<td>57</td>
</tr>
<tr>
<td>4.3</td>
<td>The image of a facet</td>
<td>72</td>
</tr>
<tr>
<td>4.4</td>
<td>Reflected rays</td>
<td>77</td>
</tr>
<tr>
<td>5.1</td>
<td>Echelette grating resonator</td>
<td>85</td>
</tr>
<tr>
<td>5.2</td>
<td>Dominant mode for the 15° grating resonator</td>
<td>88</td>
</tr>
<tr>
<td>5.3</td>
<td>Intensity and phase distributions and phase difference</td>
<td>90</td>
</tr>
<tr>
<td>5.4</td>
<td>Parallel-strip-grating resonator</td>
<td>93</td>
</tr>
<tr>
<td>5.5</td>
<td>Dominant mode for parallel-strip-grating resonator</td>
<td>96</td>
</tr>
</tbody>
</table>
FIGURE
5.6 The leakage behind the parallel-strip-grating .. 98
5.7 Generalized strip grating resonator .......... 105
5.8 Dominant mode for parallel-strip-grating
    resonator .................................. 107
5.9 Three strip resonator ....................... 109
5.10 Dominant mode for three strip resonator ..... 110
5.11 TEM₀ Mode for three strip resonator ......... 111
5.12a Transverse mode for parallel-strip-grating
    & resonator when b = 0.6λ .................. 113
5.12b
5.13 Strip-grating resonator with b = 0 ........... 116
5.14a Dominant mode for strip-grating resonator .... 117
5.14b TEM₀ Mode for strip-grating resonator ...... 118
CHAPTER I
INTRODUCTION

One of the central problems in the generation of coherent radiation by stimulated emission is the distribution of the electromagnetic field and the losses due to diffraction. It was not at all obvious that even a steady state field distribution was possible. The first such analysis of the laser cavity properties was carried out by Fox and Li\(^1\) who assumed an initial field distribution on one mirror of the cavity and calculated the distribution on the other mirror by means of the Kirchoff diffraction formula.* By iterating this procedure and requiring that the distribution of the field be reproduced, a normal mode was defined. The diffraction losses and phase shifts were obtained by comparing the results of the successive iterations which differed from each other by a complex constant after the normal mode was established.

Different diffraction losses and phase shifts for various normal modes make it possible for only one particular mode,

*Another approach favoured by Russian workers is that due to L. A. Vainshtein who extended his theory on open ended waveguides to the solution of resonator problems.\(^{53, 54}\)
normally the dominant one, to be excited in a particular laser cavity. The main interest in the study of a laser cavity is therefore to obtain information as regards the transverse normal mode distribution, diffraction loss and phase shift for relatively low loss modes.

A wide variety of resonators with different mirror configurations has been studied since Fox and Li's paper\textsuperscript{1} was published. They include confocal\textsuperscript{2}, generalized confocal\textsuperscript{3,4}, semi-confocal\textsuperscript{3}, tilted\textsuperscript{5}, flat-roof\textsuperscript{6,7}, multi-reflector\textsuperscript{8}, ring\textsuperscript{9,10}, resonators and resonators with coupling holes\textsuperscript{11,12}. A resonator using a reflection echelle grating\textsuperscript{13} as one of the end mirror was first suggested by Brannen\textsuperscript{14} in 1965. The present work is a theoretical investigation of such a cavity.

Since a reflection echelle grating is wavelength selective*, radiation with wavelengths other than the design wavelength of the grating will simply wander off in different directions and thus will never achieve a state of resonant action in the cavity. This is a very attractive and unique feature of a grating resonator, and it has been used for wavelength discrimination in many far-infrared lasers. This feature arises from the collective interference of a periodic set of identical objects, regardless of the type and the shape of the objects. The result of the collective interference

*The possible operating wavelength $\lambda$ is determined by the grating equation: $N\lambda = d(\sin\theta_{\text{inc}} + \sin\theta_{\text{dифф}})$ where $d$ is the grating constant, $N$ the diffracted order, $\theta_{\text{inc}}$ and $\theta_{\text{dифф}}$ are the incident and diffracted angles with respect to the normal of the grating.
depends only on the angle of incidence, the angle of observation and the spacing of the objects. This effect has been distinguished from those effects \(15\) which depend upon the groove shape (the groove effect) and the finite size of the grating (the end effect). The difficulty in solving the diffraction problem of an echelette grating lies in the problem of fitting the exact boundary conditions on the grating. If, however, the grating is of infinite extent and the grooves are shallow, the solution can then be approximated by a summation of solutions of individual sections (or facets) of the boundary surface. In other words, the contribution from any section of the surface is independent of the contributions from all the other sections, or, the "interaction" between them is neglected. Under this assumption, calculations have been done \(13,16,17,18\) for the intensity distribution of an echelette grating. The calculations are simple and, in many cases, they serve the purpose of practical applications. However, these calculations are not adequate for a grating with deeper grooves. Moreover, they are based on Kirchoff's method in which there is no distinction between the two major polarizations when the electric field vector is parallel to (p-polarization) or perpendicular to (s-polarization) the groove lines. More rigorous approaches \(19,20,21\) to the problem make use of an appropriate series expansion of the reflected field and either (i) the variational principle to minimize the error while fitting the boundary condition with a finite terminated series; or (ii) a Fourier expansion of the boundary surface and the solving of an infinite
set of algebraic equations. For both cases, further assumptions have to be made about the series expansion of the reflected field before the methods become practical. Moreover, the results only apply to an infinite grating rather than a finite one and the procedure in the numerical calculations is usually too tedious to be suitable for studying a grating resonator.

A method which overcomes the above-mentioned difficulties to a certain extent and which is suitable for studying a resonator has been developed in the course of this research. This method is based on Keller's geometrical (ray) theory of diffraction. This theory simply recognizes the fact that all the diffraction phenomena of edges, corners, and vertices are localized effects and can be described by diffracted "rays" the coefficients of which are determined by "canonical problems" such as Sommerfeld's half-plane problem. Thus, a diffraction problem is reduced to the problem of studying rays and the consequent summation of them. Using the ray concept, when an echelette grating is illuminated by some incident waves, diffracted rays result from the saw tooth wedges of the grating. These rays could be the "singly diffracted ray", coming directly from the wedge of individual facets, or "doubly diffracted ray", arising when the singly diffracted ray hits another wedge, or "triply diffracted ray", arising when the doubly diffracted ray hits another wedge and so on. These individual rays can be calculated under certain assumptions and can be summed to give the total diffracted
field. This knowledge of the total diffracted field of an 
ecchelette grating is essential to the study of the intensity 
distribution of a grating and the study of the steady mode in 
a resonator.

A brief description of the geometrical theory of diffrac-
tion and all the results needed in the later chapters are 
given in Chapter II. In Chapter III, a new formulation\textsuperscript{25} of 
an infinite strip resonator is presented, in which no restric-
tion is imposed on the size of the strip mirrors or the 
separation between the mirrors. The removal of such restrictions 
are needed when we consider an infinite strip resonator of 
small Fresnel number or any strip grating resonator in which 
the strip size is comparable to the wavelength. In Chapter IV, 
the study of the intensity distribution of a set of gratings, 
studied experimentally by Brannen and Rumbold\textsuperscript{26}, is pursued. 
The theoretical results are compared with the experimental 
one. Chapter V contains studies of grating resonators in 
which two major types of grating are considered, namely, the 
ecchelette grating and the strip grating. The results for 
various types of grating resonator are given. A summary and 
suggestions for further studies along this line are given in 
Chapter VI.
CHAPTER II

THE GEOMETRICAL THEORY OF DIFFRACTION

2.1 Introduction

Geometrical optics is the high frequency limit of wave optics*. Under the assumption that the wavelength of a wave motion is small compared to any change in the property of the medium, the wave motion can be fully described by the concept of ray, which is the trajectory of the normals of the surfaces of the constant phase i.e. the wave fronts. However, geometrical optics fails to describe the phenomenon of diffraction which smoothes out the discontinuity predicted by geometrical optics at the shadow boundary of an obstacle. Therefore, the concept of a diffracted ray is introduced by extending Fermat's principle to a wider class of trajectories which includes not only the direct ray, but also reflected, refracted and diffracted rays. The total field at any point in space due to scattering by an obstacle

is then the resultant sum of all possible types of rays.

A diffracted ray, just as any other type of ray, can be described by amplitude and phase functions along the trajectory. The ray concept in diffraction is useful because the values of their functions are determined by the immediate neighbourhood at the point of diffraction. Thus, they can be obtained from simpler problems in which only the localized configuration or physical properties are involved. For instance, the solution for the diffraction problem of a slit \(27,28\) can be obtained from the solution of the simpler problem, of a half-plane by realizing that the diffracted ray generated at the edges of the slit is no different from that generated at the edge of a half-plane. Because of this type of simplification, the geometrical theory of diffraction has been shown to be a flexible and powerful method of dealing with problems in which the geometrical configuration is complicated or the solution difficult to obtain.

Since we are interested only in the two-dimensional problems of echelle grating resonators, the edge or wedge diffraction is our prime concern. The "canonical" problems involved are therefore the diffraction problems of a half-plane or a wedge. The solutions of these problems are well known\(^{24,29,30,31}\). For completeness, we will give a brief account in this chapter of the solutions of these problems and include the necessary results for our purposes.
2.2 Sommerfeld half-plane problem

(a) Plane wave incidence

An incident plane wave (a time factor of $e^{-i\omega t}$ has been assumed)

$$u_0 = e^{-ikr\cos(\varphi - \varphi')}$$  \hspace{1cm} (2.1)

is launched at an incident angle $\varphi'$ with respect to the half-plane. (see Fig. 2.1a) The total resultant field is given by

$$u = U(r, \varphi - \varphi') + U(r, \varphi + \varphi')$$  \hspace{1cm} (2.2)

where - or + sign refers to the P-polarization or S-polarization (also as E- or H-polarization) i.e. electric or magnetic vector parallel to the edge. The function $U(r, \varphi_0)$ is given by:

$$U(r, \varphi_0) = u_0 \left( \frac{1}{2} \int_{-\infty}^{\infty} e^{i\epsilon r^2}  \right)$$  \hspace{1cm} (2.3)

where 

$$\epsilon = \frac{2\mu r}{\kappa} \cos\frac{\varphi_0}{2} \quad ; \quad \varphi_0 = \varphi - \varphi' \quad \text{or} \quad \varphi + \varphi'$$

Away from the immediate vicinity of the reflected and shadowed boundaries (see Fig. 2.1), and for large $kr$, $U(r, \varphi_0)$ can be represented by: (asymptotic expansion)

$$U'(r, \varphi_0) = (1+i) e^{\frac{i\pi}{4\sqrt{\kappa r}}} \cos\frac{\varphi_0}{2} + O((kr)^{-\frac{3}{2}}) \quad \text{for} \quad \epsilon < 0$$  \hspace{1cm} (2.4)

$$= u_0 - (1+i) e^{\frac{i\pi}{4\sqrt{\kappa r}}} \cos\frac{\varphi_0}{2} + O((kr)^{-\frac{3}{2}}) \quad \text{for} \quad \epsilon > 0$$

Using the above asymptotic form of the total field, we obtain the field in the three different regions by neglecting the $O((kr)^{-\frac{3}{2}})$ terms; (see Fig. 2.1)

$$u = u^G + u^D$$  \hspace{1cm} (2.5)
(a) Plane wave incidence

(b) Cylindrical wave incidence

Fig. 2.1 Half-plane problem
where
\[
\psi = \begin{cases} 
\psi_0 (r, \varphi' - \varphi'') & \text{in region I} \\
\psi_0 (r, \varphi - \varphi'') & \text{in region II} \\
0 & \text{in region III}
\end{cases}
\tag{2.6}
\]
and
\[
\psi' = \begin{cases} 
-\psi' & \text{for E-polarization} \\
+\psi' & \text{for H-polarization}
\end{cases}
\tag{2.7}
\]

where
\[
\psi' = \frac{e^{ikr}}{2}\left[ \frac{1}{\cos \frac{\varphi - \varphi'}{2}} + \frac{1}{\cos \frac{\varphi - \varphi'}{2}} \right] e^{ikr}
\tag{2.8}
\]
\(\dagger\) for E-polarization

Eq. 2.8 can be rewritten in the following form:
\[
\psi' = D_{E,H} (\varphi', \varphi, 2\pi) \frac{e^{ikr}}{\sqrt{r}}
\tag{2.8a}
\]

which is the form of the leading term in the asymptotic series expansion solution of the reduced wave equation$^{22,32}$. Therefore the diffraction coefficients $D_{E,H}$ can be identified from Sommerfeld's solution:
\[
D_{E,H} (\varphi', \varphi, 2\pi) = \frac{e^{ikr}}{2i\sqrt{r}} \left[ \frac{1}{\cos \frac{\varphi - \varphi'}{2}} + \frac{1}{\cos \frac{\varphi - \varphi'}{2}} \right]
\tag{2.9}
\]

Note that the $2\pi$ in the diffraction coefficient $D(\varphi', \varphi, 2\pi)$ represents the fact that the half-plane is a special case of a wedge with an external angle of $2\pi$. $\psi'$ in Eq. 2.8 is only the leading term in the asymptotic expansion for $kr$ and it is not valid for small $kr$ or near the reflected or shadowed boundary i.e. $\varphi = \pi - \varphi'$ or $\varphi = \pi + \varphi'$ where $D_{E,H}$ diverge.
In cases where $kr$ is small or $\varphi$ lies in those boundaries
Equation 2.3 has to be used or some other uniform asymptotic
expansion has to be developed.

(b) Diffraction of a half-plane illuminated by a line source

If a line source (see Fig. 2.1a)

$$u_o = \sqrt{\frac{\lambda}{2}} e^{i\frac{\pi}{2}} H^\prime_0(kR')$$

(2.10)

is located at $r_0$, the field at any point $r$ in space is given by

$$u(r, \varphi) = U(r, r_0, \varphi - \varphi') \mp U(r, r_0, \varphi + \varphi')$$

(2.11)

where $U(r, r_0, \varphi_0)$ is given by

$$U(r, r_0, \varphi_0) = \frac{e^{-i\varphi_0}}{2\pi R'} \int_{-\infty}^{\infty} e^{-ikR'\chi} d\chi$$

(2.12)

with

$$R' = 2 \sqrt{r_0 r \cos \frac{\varphi + \varphi'}{2}}$$

$$R'' = r_0^2 + r^2 - 2r_0 r \cos \varphi_0$$

To reduce Eq. 2.11 into the form $u = u^G + u^d$, Clemmow obtained the following form for $u^d$:

$$u^d = V_2(-\varphi') \mp V_2(+\varphi')$$

(2.13)

where

$$V_2(\varphi') = \mp \sqrt{\frac{2}{2\pi}} e^{-i\frac{\pi}{2}} \frac{e^{ik}}{\sqrt{R_1 - R_2}} \int_{-\infty}^{\infty} e^{iyR} dy$$

(2.13a)

and

$$V_2(-\varphi') = \mp \sqrt{\frac{2}{2\pi}} e^{-i\frac{\pi}{2}} \frac{e^{-ik}}{\sqrt{R_1 - R_2}} \int_{-\infty}^{\infty} e^{-iyR} dy$$

(2.13b)

with

$$R_1 = r + r_0$$

$$R'' = r^2 + r_0^2 - 2 r_0 r \cos (\varphi + \varphi')$$

$$S'' = r^2 + r_0^2 - 2 r_0 r \cos (\varphi - \varphi')$$
\( u^G \) for the cylindrical wave is given by:

\[
\begin{align*}
\begin{cases}
\int_{r}^{R} \frac{e^{i\eta}}{2} \left[ H_0''(kr) + H_0''(kR) \right] \\
\int_{r}^{R} \frac{e^{-i\eta}}{2} \left[ H_0''(kr) \right] \\
0
\end{cases}
\end{align*}
\]

(2.14)

(c) Reciprocal relations\(^{34,35}\)

(i) The source point \((r_o, \varphi')\) and the observation point \((r, \varphi)\) can be interchanged without affecting the results in section (b). This is because Eq. 12 is symmetrical with respect to \(r, r_0\) and \(\pm \varphi_0\).

(ii) The second reciprocal relation relates the solutions for plane wave incidence and cylindrical wave incidence under the following assumptions: \(Kr\) is large;

\[ r_0/r \text{ is small.} \]

This relation states:

\[
U(r, r_0, \varphi) = \frac{e^{ikr}}{\sqrt{kr}} U(r, \varphi_0)
\]

(2.15)

Physically, it means that a cylindrical line source can be located at the observation point in the plane wave solution and the result is obtained simply by taking \(r\) as \(r_0\), interchanging \(\varphi\) and \(\varphi'\), and multiplying by the asymptotic form of a cylindrical wave \(e^{ikr}/\sqrt{kr}\). To show this, we start from Eq. 12 with the following transformation:

\[
\eta = \sqrt{kr/r_0} \sin \frac{\pi}{2} \]

(2.16)

Under the assumption that \(r_0/r\) is small, the integral becomes:
\[ U(r, r_0, \varphi_0) = \frac{e^{-i\frac{\pi}{4}}}{\sqrt{2\pi}} \sqrt{\frac{\pi}{kr}} \int_{-\infty}^{\infty} \frac{e^{ikr_0 \cos \varphi_0} \cos \frac{1}{2} \varphi_0 - ikr \cos \varphi_0 \sin \frac{1}{2} \varphi_0}{\sqrt{\pi} \sqrt{kr + i}} d\eta \] (2.17)

Since kr is large, the exponential is a rapidly varying function which has a stationary point at \( \eta = 0 \). However, it can be shown by integration by parts that the upper end point contribution is of the order of \( \mathcal{O}\left(\frac{1}{\xi}\right) \) which is significant. Also when \( \cos \frac{1}{2} \varphi_0 < 0 \), the integral does not pass through the stationary point at all and the contribution must be solely from the end point. As a result of these facts and the fact that the Fresnel integral has a constant limit for a large argument, the above integral can be reduced to:

\[ \frac{e^{-i\frac{\pi}{4}}}{\sqrt{2\pi}} \frac{e^{-ikr_0 \cos \varphi_0}}{\sqrt{kr}} \int_{-\infty}^{\infty} \frac{e^{i\frac{\pi}{2} \eta}}{\sqrt{\pi} \sqrt{kr + i}} d\eta \]

\[ = \frac{e^{ikr}}{\sqrt{kr}} U(r_0, \varphi_0) \]

which is the required relation.

Using this result, we obtain the result for the diffraction of a half-plane illuminated by a line source located at \( r_0 \), for any point \( r \) not near the reflected or the shadowed boundary.

\[ U^d(r, \varphi) = \frac{e^{ikr}}{\sqrt{kr}} D^E,H(\varphi', \varphi, 2\pi) \cdot \frac{e^{ikr_0}}{r_0} \] (2.18)
(a) Plane wave incidence

(b) Cylindrical wave incidence

Fig. 2.2 Reciprocal relation
2.3 Diffraction by a wedge\textsuperscript{29,30,31}

The solution of the diffraction problem of a wedge on an external angle $\alpha > \pi + \varphi'$ is given by: (see Fig. 2.3)

\[
\nu = \nu^a + \nu^d \\
= \nu^G + \nu_b(\varphi, \varphi') \mp \nu_b(\varphi, -\varphi') ;
\]  
(2.19)

\[+ \text{ for } P^- \text{Polarization} ;
\]

where

\[\nu_b(\varphi, \varphi') = I(\pi-\varphi+\varphi', \phi) + I(\pi+\varphi-\varphi', \phi) \]  
(2.20)

and

\[I(\delta, \alpha') = -\frac{1}{2\alpha} \sin\left[\frac{\pi}{\alpha}(\pi - |\delta|)\right] \int_0^\infty \frac{H(x) dx}{\sin\left[\frac{\pi}{\alpha}(\pi - |\delta|)\right]} \]

where

\[H(x) = \begin{cases} e^{ik_x x} & \text{for plane wave incidence} \\ H_0^m [k (r_0^2 + x^2) + 2 r_0 x] & \text{for cylindrical wave incidence} \end{cases} \]

For plane wave incidence, there is a similar form for the leading term in an asymptotic expansion as in the case of a half-plane. (Eqs. 2.8a, 2.9):

\[\nu^d = D(\varphi', \varphi, \alpha') \frac{e^{ikr}}{\nu} \]  
(2.21)

where

\[D(\varphi', \varphi, \alpha') = \frac{e^{\frac{i\pi}{\alpha}}}{\sin\frac{\pi}{\alpha} (ikr)^2} \left[ \frac{1}{\cos\frac{\pi}{\alpha} - \cos\frac{\pi}{\alpha} (\varphi'-\varphi)} + \frac{1}{\cos\frac{\pi}{\alpha} - \cos\frac{\pi}{\alpha} (\varphi'-\varphi)} \right] \]  
(2.22)

For cylindrical wave incidence, if we apply the second reciprocal relation as mentioned in the previous section, under the same assumptions, we have:

\[\nu^d = D(\varphi', \varphi, \alpha') \frac{e^{\frac{ik_r}{\nu}}}{\nu} \]  
(2.23)
Fig. 2.3 Diffraction by a wedge
2.4 Asymptotic expansion of Green's theorem

There are cases when the use of the diffraction coefficient (Eq. 2.9) fails to be valid or accurate. For instance, in the consideration of the effect due to the short side of a facet of an echelette grating, since the width of this side is smaller than a wavelength, the representation in term of the diffraction coefficient in the leading term of the asymptotic expansion is no longer good. To overcome this difficulty, we have to resort to other methods.

If the surface field can be obtained by some reasonable approximation, then, by an asymptotic expansion of Green's theorem, the far field can be calculated. The method has been used by Morse\textsuperscript{36} in the calculation of the scattered far field of a rectangular block in grazing incidence.

Following Morse, we consider the scattering of a rectangular block. The scattered field is given by (see Fig. 2.4)

$$u_s(r, \phi) = u_{total}(r, \phi) - u_{inc}(r, \phi)$$

$$= \frac{i}{4} \oint \left[ \frac{1}{2} H_0^{(1)}(kR) - H_0^{(1)}(kR) \right] dS$$

(2.24)

For large $kR$, by asymptotic expansion of $H_0^{(1)}(kR)$, we have, for side $AB$,

$$H_0^{(1)}(kR) = \sqrt{\frac{2}{\pi kR}} e^{ikr - iK \cos \phi - i\frac{\lambda}{4}}$$

(2.25)

$$\frac{\partial H_0^{(1)}(kR)}{\partial n} = - \sqrt{\frac{2}{\pi kR}} iK \sin \phi e^{ikr - iK \cos \phi - i\frac{\lambda}{4}}$$

where

$$R = r - \rho \cos \phi$$
Substitute the above equations back in Eq. 2.24 we obtain

$$u_s^{\text{from side AB}}(r, \omega) = -\frac{i}{\hbar} \int_{AB} A^B \left[ ik \sin \omega u + \frac{2u}{2r} \right] e^{-ikr} \sin \omega \, dp$$

(2.26)

---

**Fig. 2.4** Diffraction of a block
CHAPTER III
A NEW FORMULATION OF THE INFINITE STRIP RESONATOR

3.1 Introduction

In Fox and Li's theory\textsuperscript{1}, in order to search for a steady state transverse field distribution of a wave with a definite phase, bouncing back and forth between the end mirrors, a "transmission line" concept was developed. A sequence of equally spaced black screens (perfectly absorbing) with an aperture of the same size as that of the end mirror of the resonator is set up. Then the waves travelling from aperture to aperture are analogous to the waves travelling back and forth between the mirrors. The field and the phase distributions at one aperture are calculated from the ones at the previous aperture by Kirchhoff's diffraction formula\textsuperscript{34}: (see Fig. 3.1).

\begin{equation}
  u(P) = -\frac{ik}{4\pi} \int u'(P') \frac{e^{ikR}}{R} (1 + \cos \theta) dA
\end{equation}

where \( u'(P') \) is the aperture field, \( k \) is the propagation constant, \( R \) is the distance from a point \( P' \) on one aperture to the point of observation \( P \) at the next aperture and \( \theta \) is the angle between \( R \) and the unit normal to the aperture. It
Fig. 3.1 The transmission line analogue
is assumed that \( u'(p') \) has already encountered many apertures and that it will repeat itself on the next aperture except for a scalar complex constant, i.e.,

\[
\gamma' u'(p') = u(p)
\]  \hspace{1cm} (3.2)

Then, from Eqs. (3.1) and (3.2), we obtain an integral equation of the second kind:

\[
u(p) = \delta \int_A \kappa(p, p') u(p') \, dA
\]  \hspace{1cm} (3.3)

where \( \kappa(p, p') = e^{i k R (r' r') / R} \) and \( \gamma = -i k / 4 \pi \). The Kirchhoff formula and, therefore, the integral equation is based on the assumptions that (i) the linear dimensions of the aperture are large compared to the wavelength, i.e. \( k a \gg 1 \), and (ii) \( R \) is much larger than the linear size of the aperture, i.e. \( R \gg a \). The first assumption is necessary because of the nature of the boundary condition* adopted in Kirchhoff's diffraction formula\(^{34}\). When the size of the apertures decreases, the unperturbed nature of the boundary values is no longer valid, and the multiple scattering of the field has to be included\(^{37,38}\). The second assumption is necessary because the boundary conditions assumed by

*If an incident field is launched from the left of the aperture, the total field on the right is calculated from the Helmholtz integral\(^{39}\) by adopting the values of the incident field and its normal derivative at the aperture as the boundary values. A reasonably large aperture has to be used in order not to disturb the assumed boundary values.
Kirchhoff's formula have been overspecified*. The formula does not hold in the aperture and is a poor approximation near the aperture**.

We will see in the following sections, that conditions (i) and (ii) can be removed*** (under our formulation). In this formulation, the method essentially is based upon the summation of the solutions for perfectly conducting half-planes and the inclusion of the interaction between the two mirrors. The formulation does recover Fox and Li's solution if their assumptions are imposed.

The round trip integral eq. by Kirchhoff's method is derived in Section 2. In Section 3 the solution of the diffraction problem for a slit by the method of Keller's geometrical theory of diffraction is discussed. The new formulation of an infinite parallel strip resonator is presented in Section 4 and 5 and the results are discussed.

*The field and its normal derivative cannot be specified simultaneously on a boundary for an elliptic equation.

**This difficulty has actually been overcome in the boundary wave method 38 which gives consistent results both in the aperture and in the near zone of a black screen with a large aperture.

***The necessity of removing the first condition becomes clear, as has been mentioned in Chapter I, when we consider that the individual facets of a grating form a resonator with the other end mirror.
3.2 Round trip integral equation

For later comparison, we derive in this section a round trip integral equation by the Kirchhoff-Fresnel formula.

The two dimensional analogue of Eq. (3.1) can be written, for small \( \theta \), as (see Fig. 3.2)

\[
U(x) = \gamma \frac{e^{i \frac{kr}{2}}}{\sqrt{\lambda r'}} \int_{-A}^{A} e^{ikr'} U(x') dx'
\]

(3.4)

\[ r' = (x' - x,)^2 + \frac{D^2}{4} \]

To obtain a round trip integral equation we apply the above equation once again to get the field at point \( x \) on mirror 1 due to the distribution \( U(x_1) \) on mirror 2, i.e.

\[
U(x) = \gamma \frac{e^{i \frac{kr}{2}}}{\sqrt{\lambda r'}} \int_{-A}^{A} e^{ikr'} U(x') dx' \int_{-A}^{A} e^{ikr} U(x_1) dx_1
\]

(3.5)

To further simplify Eq. (3.5) we note that for \( D \gg a \)

\[ r' = D + \frac{(x' - x_1)^2}{2D} \]

\[ r = D + \frac{(x - x_1)^2}{2D} \]

and

\[ r' r = D + \frac{(x' - x_1)^2 + (x - x_1)^2}{2D} = 2D + \frac{\left[ x - \left( \frac{x + x_1}{2} \right) \right]^2}{D} \to \frac{(x - x_1)^2}{D} \]

We also introduce the following coordinate transformation:

\[ \eta' = \frac{x'}{A} \quad \eta = \frac{x}{A} \quad \eta_1 = \frac{x_1}{A} \]

Eq. (3.5) then becomes,

\[
U(\eta) = \gamma \frac{e^{2ikD \eta'}}{\sqrt{\lambda r'}} \int_{-\frac{A}{2}}^{\frac{A}{2}} e^{i \eta_1 f} \left( \eta - \frac{\eta_1}{2} \right)^2 \int_{-\frac{A}{2}}^{\frac{A}{2}} e^{2ikf \eta_1} e^{i \eta_1 f} U(\eta_1) \frac{d \eta_1}{A} \frac{d \eta}{A}
\]

(3.6)
Fig. 3.2 Infinite strip resonator
where \( f = \frac{A^3}{X\nu} \); \( f \) is called the Fresnel number.

But the integral
\[
\int e^{2i\pi f(z, z')} dz' = \int e^{\frac{i\nu}{2} \left( F(2 + \nu \eta) + F\left(2 - \eta \right)\right)} \quad (3.7)
\]

where \( F(\omega) \) is the Fresnel integral
\[
F(\omega) = \int e^{i\omega y} dy \quad (3.8)
\]

Combining Eqs. (3.6) and 3.7 we obtain the round trip integral equation for the infinite strip resonator
\[
u(\eta) = \delta \int_{-1}^{+1} k(\eta, \eta') u(\eta') d\eta' \quad (3.9)
\]

where
\[
k(\eta, \eta') = \int e^{i\pi D} e^{\frac{i\nu}{2} \left( F(2 + \nu \eta') + F(2 - \eta')\right)} \quad (3.10)
\]

3.3 Diffraction by a slit\(^{27,28}\)

Diffraction by a slit is considered in terms of the diffraction of two half-planes. The total field at any point in space is the resultant sum of the contributions from both half-planes plus the mutual interaction effects. (see Fig. 3.3)

The singly diffracted ray at point \( P \) is given by:

(Eqs. (2.8) and (2.9))
\[
\begin{align*}
u^{\text{singly}}(P) &= \frac{e^{ik(C_a - \omega \sin^2 \theta) + i\chi}}{2\sqrt{\pi k} k a} \left[ \sec \frac{\chi}{2} (\theta + \theta') \pm \sec \frac{\chi}{2} (\theta - \theta') \right] \\
& \quad - \frac{e^{ik(C_a + \omega \sin^2 \theta') + i\chi}}{2\sqrt{\pi k} k a} \left[ \sec \frac{\chi}{2} (\theta - \theta') \pm \sec \frac{\chi}{2} (\theta + \theta') \right]
\end{align*}
\]
(a) Singly diffracted rays

(b) Doubly diffracted ray

(c) Triply diffracted ray

Fig. 3.3 Diffraction of a slit
The doubly diffracted rays can be obtained by letting the two singly diffracted rays hit the opposite edges. (see Fig. 3.3b). The value of the singly diffracted ray from the lower half-plane at the upper edge is (letting \( r_a = 2a, \vartheta_2 = \pi/2 \)),

\[
U_{\text{singly}}(r_a) = \begin{cases} 
\frac{-e^{ik(r_a + s \sin \theta_1 + i\pi)}}{\sqrt{2 \pi k a}} \cos \left( \frac{s}{2} - \theta_1 \right) & \text{for P-polarization} \\
0 & \text{for S-polarization} 
\end{cases}
\]

The diffracted ray \( U_{\text{singly}}(+a) = 0 \) for S-polarization is due to the fact that the diffraction coefficient vanishes for that polarization when \( \vartheta_1 = \frac{\pi}{2} \) or \( \vartheta_2 = 0 \). Therefore, some other method must be used in this case to determine the doubly diffracted ray from the singly diffracted one. It has been shown that at this angle of incidence, the doubly diffracted ray is proportional to the derivative of the incident field at the edge.\(^{41}\) The proportionality factor is a new diffraction coefficient \( D' \):

\[
U^d = D'(\frac{\pi}{2}, \vartheta) \frac{e^{ikr}}{\sqrt{r}} \frac{\partial U_{\text{inc}}}{\partial \eta} 
\]

where

\[
D'(\frac{\pi}{2}, \vartheta) = \frac{1}{ik} D(\frac{\pi}{2}, \vartheta, 2\eta) = -\frac{e^{-i\pi}}{2\pi \eta} e^{i\pi} \frac{\sin(\frac{2}{2} - \vartheta)}{\cos^2(\frac{\pi}{2} - \vartheta)}
\]

Higher order terms can be obtained by repeating the above processes, i.e. by letting the doubly diffracted ray hit another edge to obtain the triply diffracted ray and so on. The overall interaction effect is obtained by summing
all orders of interaction terms. It is found that even the result in which only rays up to the doubly diffracted rays are included agrees favourably with the exact solution by other methods. The result which is based on the summation of all the interaction terms differs slightly from the one in which only two terms are included.\textsuperscript{27}

3.4 A new formulation of an infinite strip resonator problem\textsuperscript{25}

In the following, a new formulation for an infinite strip resonator problem which removes the two conditions imposed in the Fox and Li formulation, (see section 3.1) is presented. This new formulation allows us to study cases when the Fresnel number of an infinite strip resonator, or the separation of the two mirrors of the resonator is small.

Kirchhoff's method in solving the diffraction problem of a plane screen with an aperture is to assume that the screen is black, and the field in the aperture has the value of the incident wave\textsuperscript{34,39}. In solving the infinite strip resonator problem, Fox and Li used a similar approach by assuming that the field on one mirror can be calculated only from the field on the other mirror and that the field outside the strip is identically zero. This gives rise to the transmission line analogue because the wave which travels back and forth between the strips is equivalent to the wave which travels from aperture to aperture.

Since the solution of the diffraction problem of a perfectly conducting strip can be obtained from the solution of
a slit by Babinet's Principle, a transmission analogue can also be set up for an infinite strip resonator. A perfectly conducting screen is adopted because it is physically more realizable than the black one. Also, the exact solution of the half-plane problem which is needed in our formulation is available. However, in the following formulation when Fox and Li's assumptions are imposed, the kernels of the two formulations are identical. This perhaps gives us an explanation why Kirchhoff's diffraction formula has been very successful for cases when the aperture is reasonably large.

A perfectly conducting screen with an aperture of the same size as that of the end mirror is placed in front of a line distribution of line sources at the same distance as the separation of the resonator. The line sources which occupy a width of 2A are located at the position of an end mirror. The field distribution is then calculated at the image position of the line sources with respect to the perfectly conducting screen. (see Fig. 3.4)

The field at any point in the observation plane is the sum of the direct incident field, the diffracted field due to the upper and the lower half-plane and the interaction field between the edges.

First we consider only the direct incident field and the singly diffracted rays. The incident field is given by:

\[ u_0 = \sqrt{\frac{k}{2}} e^{ikz} \mu_e^{(1)}(kR) \]  

(3.14)
Fig. 3.4 Formulation of the infinite strip resonator problem
where
\[ R^2 = (x - x')^2 + (2d)^2 \]

The singly diffracted ray due to the diffraction at a half-plane illuminated by a line source located at \( x' \) (see Fig. 3.4 and 2.1) is given by (Eq. 2.13):

\[ u_{\text{singly}} = v_2(-\theta_0) \pm v_x(\theta_0) \quad \text{for } P\text{-polarization (3.15)} \]

\[ v_2(-\theta_0) = \mp \sqrt{\frac{k}{\pi}} e^{-i\pi/4} e^{ikR} \int_{\sqrt{R_2-R^2}}^{\infty} \frac{e^{iky}}{\sqrt{y^2-R_2}} \, dy \quad \text{for } \cos^2 \left( \frac\theta 2 - \theta_0 \right) > 0 \]

\[ v_2(\theta_0) = \sqrt{\frac{k}{\pi}} e^{-i\pi/4} e^{ikS} \int_{\sqrt{S-\sqrt{R}}}^{\infty} \frac{e^{iky}}{\sqrt{y^2-S}} \, dy \quad \text{for } \cos^2 \left( \frac\theta 2 + \theta_0 \right) > 0 \]

where
\[ R_1 = r + r_0 \]

We first consider the contribution from the lower (negative) half-plane. A similar derivation applied for the upper (positive) half-plane. Since

\[ \pi - 2\theta_1 \leq \theta - \theta_0 \leq \pi \]

where
\[ \theta_1 = \tan^{-1} \frac{2a}{p} \]

therefore
\[ \cos \frac{\theta}{2} (\theta_0 - \theta) > 0 \]

and

\[ u_2^\pm(-\theta_0) = -\sqrt{\frac{k}{\pi}} e^{-i\pi/4} e^{ikR} \int_{\sqrt{R_2-R^2}}^{\infty} \frac{e^{iky}}{\sqrt{y^2-R_2}} \, dy \quad (3.16) \]
For large $D$

$$R = 2D + \frac{(x-x')^2}{4D}$$

$$R_1 = r_0 + r = 2D + \frac{(x'+a)^2}{2D} + \frac{(x+a)^2}{2D}$$

and

$$\sqrt{R - 2yR} = \frac{(x'+x) + 2a}{2\sqrt{D}}$$

The integral becomes

$$V^1_2(-\theta_0) = -\sqrt{\frac{k}{\pi}} \int_{\frac{2kD + 4k(x-x')^2}{x-x'+2a}}^{\infty} e^{\frac{ikRy^*}{\sqrt{y^2+2}}} dy$$

(3.17)

Note that the integrand is a rapidly varying function for large $kR$. It has a stationary point at $y = 0$. Since the integral limits do not include this point, the contribution comes solely from the lower end point at which $y \geq 0$ for large $D/A$. Therefore the integral can now be written as

$$V^1_2(-\theta_0) = -\sqrt{\frac{k}{\pi}} \int_{\frac{2kD + 4k(x-x')^2}{x-x'+2a}}^{\infty} e^{\frac{i\pi y^*}{\sqrt{y^2+2}}} dy$$

(3.18)

after the following substitution:

$$kRy^* = \frac{\pi}{2} \nu^*$$

(3.19)

Then the integral in Eq. 3.25 can be evaluated in terms of the Fresnel integral:
\[
\int_{\frac{2\pi}{k}}^{\infty} e^{iKx} \, dv = \left[ \int_{0}^{\frac{2\pi}{k}} + \int_{\frac{2\pi}{k}}^{\infty} \frac{dK}{k} \right] e^{iKx} \, dv
\]
\[= \frac{1}{2} (1 + i) - \int_{0}^{\frac{2\pi}{k}} e^{iKx} \, dv
\]
\[= \frac{1}{\sqrt{2}} - e^{i\frac{\pi}{4}} \left( 1 + e^{i\frac{\pi}{4}} \right)
\]

Finally \( \nu_{\frac{1}{2}} (-\theta) \) becomes:

\[
\nu_{\frac{1}{2}} (-\theta) = -\frac{1}{\sqrt{2}} e^{\frac{-i\pi}{4}} \int_{0}^{\frac{2\pi}{k}} \frac{e^{ikx + i\frac{\pi^2}{4}}} {e^{\frac{iS}{\sqrt{y^2 + 2}}}} \, dy
\]
\[= \frac{1}{\sqrt{2}} e^{i\frac{\pi}{4}} \left( 1 + e^{i\frac{\pi}{4}} \right)
\]
(3.20)

Now we consider

\[
\nu_{\frac{1}{2}} (\theta) = -\frac{1}{\sqrt{2}} e^{\frac{-i\pi}{4}} \int_{0}^{\frac{2\pi}{k}} \frac{e^{ikS}} {\sqrt{y^2 + 2}} \, dy
\]
(3.21)

\[= \text{ for } \cos \frac{1}{2} (\theta + \theta) > 0
\]

Since

\[
\frac{3\pi}{2} - \theta_{1} \leq \theta \leq \frac{3\pi}{2}
\]
\[
\frac{\pi}{2} \leq \theta_{0} = \frac{\pi}{2} + \theta_{1}
\]
\[
\pi - \theta_{1} \leq \theta_{0} + \theta = \pi + \frac{\theta_{1}}{2}
\]

therefore \( \cos \frac{1}{2} (\theta_{0} + \theta) < 0 \) since \( \theta_{1} = \tan^{-1} \frac{2\pi}{B} < \frac{\pi}{2} \)

From these considerations, we fix the sign for \( \nu_{\frac{1}{2}} (\theta_{0}) \):
\[ \nu_2^-(\theta_0) = \frac{1}{\nu \pi} e^{-i\frac{\pi}{2}} \int_{\frac{1}{2\pi}}^\infty e^{i\frac{S_2 y_2^2}{y_2^2 + 2D^2}} \frac{e^{ikS_2 y_2^2}}{\sqrt{y_2^2 + 2D^2}} dy_2 \]  

(3.22)

Since

\[ R_i = r_0 + r = 2D + \frac{(x-x')^2}{2D} + \frac{(x-A)^2}{2D} \quad \text{also} \quad S = |x-x'| \]

and

\[ \frac{1}{(R_i - 3)^2} = \sqrt{\frac{2D}{|x-x'|}} \quad \text{for} \quad D/A \gg 1 \]

we have

\[ \nu_2^-(\theta_0) = \frac{1}{\nu \pi} e^{-i\frac{\pi}{2}} \int_{\frac{1}{2\pi}}^\infty e^{i\frac{S_2 y_2^2}{y_2^2 + 2D^2}} \frac{e^{ik(S - 3) y_2^2}}{\sqrt{2D(S - 3)^2}} dy_2 \]  

(3.23)

If

\[ D/A \gg 1, \quad \nu_2^-(\theta_0) \rightarrow 0 \]

By similar considerations, the contribution from the upper half-plane is given by

\[ \nu_2^+(\theta_0) = -\frac{1}{\nu \pi} e^{i\frac{\pi}{2}} \int_{\frac{1}{2\pi}}^\infty e^{-i\frac{S_2 y_2^2}{y_2^2 + 2D^2}} \frac{e^{ik(S - 3) y_2^2}}{\sqrt{y_2^2 + 2D^2}} dy_2 \]

(3.24)

\[ \nu_2^+(\theta_0) \rightarrow 0 \quad \text{for} \quad D/A \gg 1 \]

The total field is then given by
\[ u_{\text{total}} = u_0 + u_{\text{sing}} + u_{\text{edge}} + u_{\text{far}} \]

\[ = \frac{\hbar}{\pi} e^{i k D} e^{-i \frac{\pi}{2} (\frac{1}{r_1} - \frac{1}{r_2})} F\left(\frac{1}{r_1} + \frac{1}{r_2}\right) + F\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \]  

where the asymptotic form of the Hankel function has been used for \( u_0 \). The above is the total field due to a unit line source, and therefore can be used as a kernel in the integral Eq. 3.3. If the interaction between the edges can be neglected, as is the case under Fox and Li's assumptions, then the kernel will be

\[ K(\eta, \eta') = \frac{\hbar}{2} e^{i k D} e^{-i \frac{\pi}{2} (\frac{1}{r_1} - \frac{1}{r_2})} F\left(\frac{1}{r_1} + \frac{1}{r_2}\right) + F\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \]  

which is the same as Fox and Li's round trip kernel except for a factor of \( \hbar \). The factor of \( \hbar \) arises because, in the derivation of Fox and Li's round trip kernel, Kirchhoff-Fresnel's formula has been applied twice, while in the derivation of the above kernel, the diffraction formula has been applied essentially once. If we apply the Kirchhoff-Fresnel formula only once, a factor of \( \hbar \) also appears (see Appendix I.)

3.5 Interaction kernel from the interaction between edges

The lowest order of interaction is the doubly diffracted ray which results when the singly diffracted ray from
one half-plane hits the other one. To calculate a doubly diffracted ray from one edge, we have to find the field of the singly diffracted ray at that edge due to the other half-plane. First we consider the case of P-polarization.

3.5a Lowest interaction term for P-polarization

When the singly diffracted ray which originated from the lower half-plane travels across the slit, (see Fig. 3.5)

$$\theta = \pi$$

and

$$\frac{\pi}{2} \leq \theta_0 \leq \pi$$

Therefore

$$3\frac{\pi}{2} \leq \theta + \theta_0 \leq \pi + \frac{\pi}{2} + \theta_1$$

$$\frac{\pi}{2} - \theta_1 \leq \theta - \theta_0 \leq \frac{\pi}{2}$$

$$\theta_1 = \tan^{-1} \frac{2A}{D}$$

and

$$\cos \frac{1}{2}(\theta_0 - \theta) > 0 ; \quad \cos \frac{1}{2}(\theta + \theta_0) < 0$$

We have then

$$u_{\text{sing}y} = V_{\frac{1}{2}}(-\theta_0) - V_{\frac{1}{2}}(\theta_0)$$

$$= -2\sqrt{\frac{k}{\pi}} e^{ikR} \sin \theta \int_{\sqrt{kx^2}}^{\infty} \frac{e^{-iy^2}}{\sqrt{y^2 + \frac{k}{2}}} dy$$  \hspace{1cm} (3.27)

since \( R = S \). For \( D \gg A \), we have

$$\frac{R - R}{R} = \frac{1}{D} \left[ \frac{1}{2D} (A + x')^2 - \frac{1}{2D} (A - x')^2 + 2A \right] = \frac{2A}{D}$$
Fig. 3.5 Doubly diffracted ray
From this:

\[
U_{\text{singly}}^{-(+A)} = -\frac{2e^{i\beta_y}}{\sqrt{2\pi}} D \int_0^\infty \frac{e^{-iR y_2}}{\sqrt{x^2 + y_2^2}} dy
\]  

(3.28)

For \( kR \) is large, by the same argument for this type of integral as we have done before, we have

\[
U_{\text{singly}}^{-(+A)} = -\frac{2e^{i\beta_y}}{\sqrt{2\pi}} D \left[ \frac{e^{ik_\phi}}{k_\phi} - F\left(\frac{2r}{k_\phi} \sqrt{A}\right) \right]
\]

\[
= -\frac{2e^{i\beta_y}}{\sqrt{2\pi}} D e^{ik_\phi} e^{i\beta_x} \frac{e^{i\beta_y}}{i\beta_y} \left[ \frac{e^{ik_\phi}}{k_\phi} - F\left(\frac{2r}{k_\phi} \sqrt{A}\right) \right]
\]

(3.29)

The first term in the above expression cancels with the incident field if \( R \) is large enough. Therefore

\[
U_{\text{singly}}^{-}(+A) = \frac{2e^{i\beta_y}}{\sqrt{2\pi}} D e^{ik_\phi} e^{i\beta_x} \left[ \frac{e^{ik_\phi}}{k_\phi} - F\left(\frac{2r}{k_\phi} \sqrt{A}\right) \right]
\]

(3.30)

Similarly the field at \(-A\) due to the upper half-plane is

\[
U_{\text{singly}}^{+(+A)} = \frac{2e^{i\beta_y}}{\sqrt{2\pi}} D e^{ik_\phi} e^{i\beta_x} \left[ \frac{e^{ik_\phi}}{k_\phi} - F\left(\frac{2r}{k_\phi} \sqrt{A}\right) \right]
\]

(3.31)

From these two edge fields, the field at point \( x \) is given by, according to the geometrical theory of diffraction, (see Fig. 3.5)

\[
U(x) = U_{\text{singly}}^{-}(+A) \frac{e^{ik_\phi}}{\sqrt{2\pi}} D \left( \frac{x}{2}, \beta, 2\pi \right) + U_{\text{singly}}^{+(+A)} \frac{e^{ik_\phi}}{\sqrt{2\pi}} D \left( \frac{x}{2}, \beta_A, 2\pi \right)
\]

(3.32)

where \( r_A \) and \( r_{-A} \) are given by

\[
\frac{r_{-A}}{r_A} = D + \frac{(x + A)^2}{2D}
\]
and
\[ D(\varphi', \varphi, 2\pi) = \frac{e^{i\pi\varphi}}{2\sqrt{2\pi k}} \left[ -\frac{1}{\cos \varphi - \varphi} + \frac{1}{\cos \varphi + \varphi} \right] \]

For large D, \( \theta_a = \theta_d = \pi \)

\[ D(\frac{\pi}{2}, \pi, 2\pi) = -2\sqrt{2} \frac{e^{i\pi}}{2\sqrt{2\pi k}} = -\frac{\sqrt{2}}{\sqrt{\pi k}} \]

Using this, we have
\[
\psi(x) = \sqrt{\frac{2}{\pi^D}} \mathcal{F}(\frac{2}{\pi A}) \left\{ \frac{\text{e}^{i\frac{x}{2\sqrt{2\pi k}}}}{\sqrt{2\pi k}} \left[ \text{e}^{i\frac{\pi x^2}{2\sqrt{2\pi k}}} + \text{e}^{-i\frac{\pi x^2}{2\sqrt{2\pi k}}} \right] \right\}
\]

\[ = -\sqrt{\frac{2}{\pi A \pi}} \sqrt{\frac{2}{\pi^D}} \mathcal{F}(\frac{2}{\pi A}) \left\{ \text{e}^{i\pi x^2/2\sqrt{2\pi k}} + \text{e}^{-i\pi x^2/2\sqrt{2\pi k}} \right\} \]

(3.33)

By taking out a factor of \( \sqrt{\pi} \) for the reason we have mentioned before, the interaction kernel for the integral equation

\[ u(\eta) = \delta \int_{-1}^{+1} [ K(\eta, \eta') + K_{\text{int}}(\eta, \eta') ] u(\eta') d\eta' \quad (3.34) \]

is

\[ K_{\text{int}}(\eta, \eta') = -\sqrt{\frac{2}{\pi A \pi}} \sqrt{\frac{2}{\pi^D}} \left\{ \text{e}^{i\pi (\eta + \eta')^2/2\sqrt{2\pi k}} + \text{e}^{-i\pi (\eta + \eta')^2/2\sqrt{2\pi k}} \right\} \]

(3.35)

3.5b Lowest interaction term for S-polarization

The diffraction coefficient \( D(\varphi', \varphi, 2\pi) \) vanishes when \( \varphi' = \pi/2 \) or \( \varphi = \pi/2 \) for S-polarization. The next non-vanishing term is proportional to the normal derivative of
the incident field at the edge of diffraction \(^4\) (see section 3.3)

\[
U(x) = D' \frac{i}{\sqrt{r}} e^{ikr} \frac{\partial U_{inc}}{\partial n}
\]  

(3.12)

where \(D'\) is given by Eq. 3.13 and

\[
\frac{\partial U_{inc}}{\partial n} \bigg|_{x_r = -A} = -\frac{U_{inc}^{(1)}}{2(2A)^{3/2}} e^{i\frac{\pi}{4}} \frac{e^{i\frac{\pi}{2}}}{\sqrt{2i\pi k}} \frac{1}{2} \left\{ \frac{\sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2} - \frac{\pi}{2}\right)} + \frac{\cos\left(\frac{\pi}{4} + \frac{\theta}{2}\right)}{\sin\left(\frac{\pi}{4} + \frac{\theta}{2}\right)} \right\}
\]  

(3.36)

The doubly diffracted ray due to the lower half-plane is then

\[
\bar{u}(x) = \sqrt{\frac{\pi}{2A}} F\left(\frac{2}{2A}\right) \frac{1}{\cos\theta} \left(\frac{x}{A}\right)^{1/2} \frac{\sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2} - \frac{\pi}{2}\right)} \left[ \frac{\sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2} - \frac{\pi}{2}\right)} + \frac{\cos\left(\frac{\pi}{4} + \frac{\theta}{2}\right)}{\sin\left(\frac{\pi}{4} + \frac{\theta}{2}\right)} \right]^{\frac{1}{2}}
\]  

(3.37)

So

\[
K_{int} \approx 0 \left( \frac{\pi}{2} \left(\frac{A}{2}\right)^{1/2} \frac{1}{\sqrt{D} \cdot 10^{-3}} \right)
\]  

(3.38)

for S-polarization

We can see from Equation 3.38 that the interaction effect due to the doubly diffracted ray is small for S-polarization, and so it can be neglected.

3.6 Numerical Examples

The above formulation has been used to calculate the eigenvalues and phase shifts for a strip resonator of Fresnel number less than 0.6. Table I lists the results of calculations for the dominant TEM\(_0\) mode of infinite strip resonators using Fox and Li's round trip kernel and our proposed kernel \(K + K_{int}\). From the table, the correction to
Fox and Li's results for P-polarization can be as much as 3.5% for the diffraction loss and 14% for phase shift. For S-polarization, however, Fox and Li's results stand as they are because of the negligible interaction term. As we can see from the Table, the results for power loss from $K + K_{int}$ approach those from Fox and Li's kernel when the Fresnel number is less than 0.05. The reason for this perhaps is twofold: a) a higher order interaction has been neglected which is greater than 7% of the first order in these regions; b) the power loss is asymptotically approaching 100% and the information of interaction is buried under the computer round-off in this type of numerical procedure. For comparison with calculations using a single trip eigenvalue, the second last column in Table I shows the power loss per transit calculated from the single trip eigenvalue $\gamma$, where $\gamma^2$ is equal to our round trip eigenvalue $\gamma$. Since for small Fresnel numbers, the energy diffracted back to the other mirror approaches zero, both power losses per round trip and per transit approach 100%.
TABLE I

Comparison of Diffraction losses and phase shifts per transit for parallel strip resonators using Fox and Li's Kernel $K$ and the present kernel $K' = K + K_{int}$ for TEM$_0$ mode. ($D = 115\lambda$)

| $a/\lambda$ | Fresnel Number $f$ | Eigenvalue per round trip $|\gamma|$ | Power loss per round trip (%) $1 - |\gamma|^2$ | Power loss per transit (%) $1 - |\gamma_s|^2 = 1 - |\gamma|$ | Phase shift per transit (degrees) |
|------------|-------------------|---------------------|------------------|----------------------|---------------------|
| 8.03       | 0.56              | 0.8503 0.8505       | 27.70 27.66      | 14.97 14.95          | 12.42 12.30         |
| 6.78       | 0.4               | 0.7822 0.7854       | 38.83 38.32      | 21.78 21.46          | 15.42 15.18         |
| 4.80       | 0.2               | 0.6098 0.5918       | 62.82 64.98      | 39.03 40.82          | 24.33 23.60         |
| 3.39       | 0.1               | 0.3707 0.3504       | 86.26 87.72      | 62.93 64.96          | 33.45 34.13         |
| 2.40       | 0.05              | 0.1960 0.1876       | 96.16 96.48      | 80.40 81.24          | 39.06 41.19         |
| 1.52       | 0.02              | 0.0800 0.0762       | 99.36 99.42      | 92.00 92.38          | 42.60 46.48         |
| 1.07       | 0.01              | 0.045 0.045         | 99.8 99.8        | 95.5 95.5            | 43.80 50.00         |
CHAPTER IV
THE INTENSITY DISTRIBUTION OF
ECHOLETTE GRATINGS

4.1 Introduction

Before we proceed to investigate the behavior of a Fabry-Perot resonator with an echelette grating as one of the end mirrors, we will first study, by the method of the geometrical theory of diffraction, the intensity distribution of an echelette grating when a uniform plane wave is incident upon it. In this analysis, the 90° triangular grooves of an echelette grating are replaced by a set of overlapping 90° wedges, with the assumption that the diffraction of an edge or a wedge is a localized effect. (See Fig. 4.1) The intensity distribution is then calculated by the summation of all the singly and doubly diffracted rays. There are three possible types of doubly diffracted rays (see Fig. 4.2) which are termed a-, b-, and d-corrections, for reasons which will be made clear below. The d-correction is the doubly diffracted ray which results when any singly diffracted ray hits the nearest neighbouring wedge and is of the order of $(kd)^{-\frac{1}{2}}$. The a-correction is the doubly diffracted ray which results when the singly diffracted ray grazes along the long side of
Fig. 4.1 An Echelette grating of 90° triangular grooves
a facet, is reflected back from the step and is then diffracted at the same wedge where the singly diffracted ray originated. This doubly diffracted ray is of the order of \((2ka)^{-b}\) where \(a\) is the width of the long side of the facet. The b-correction is the contribution from the short side of the facet of width \(b\). Since this width \(b\) is normally less than a wavelength, for those gratings in which we are interested, this type of correction has to be treated differently from those of \(d\) - and a-corrections. The method used in this case is that of Morse's expansions of Green's Theorem. (see section 2.5 in Chapter II) The surface field or current along the side facet is first calculated and then the far field follows from Green's Theorem.

The only diffracted ray of higher order considered is the triply diffracted ray which results when the doubly diffracted ray from the d-correction hits another neighbouring wedge.

When calculating the intensity distribution by the summation of all the different types of diffracted rays, we start first with the singly diffracted rays. Then we include, one at a time, the d-correction, the a-correction, the b-correction and the triply diffracted rays. In this way, we can study the effect of adding each individual type of diffracted ray. All the calculations are made for a chosen set of gratings and are tabulated and compared with the experimental results obtained by Brannen and Rumbold for the same set of gratings. Excellent agreement is obtained already by
including only the d- and a-corrections in the calculation of gratings with shallow grooves \( (b < \frac{1}{4}\lambda) \). With further consideration of the b-correction, excellent agreement is also obtained for gratings with deeper grooves \( (\frac{1}{2}\lambda < b < \lambda) \). This confirms the applicability of the method. The results indicate that the types of interaction that we have considered account for the major part of the intensity distribution as far as a particular spectral order is concerned.

Since, however, a canonical diffraction coefficient is a continuous function of the incident and the diffracted angles, (or rather, the total diffracted field is continuous) the total contribution from the summation of only the singly diffracted rays plus the nearest neighbour interaction excludes the possibility of the existence of rapid intensity variation of a spectral order as a function of the incident angle (grating anomaly*). This phenomenon has been shown to be the result of multiple scattering\textsuperscript{37}, i.e. the overall effect of all orders of interaction. Nevertheless, in the study of a

*Wood's grating anomalies have been studied extensively since they were discovered by Wood\textsuperscript{43} in 1902. (An excellent review with an extensive bibliography has been given by Hessel and Oliner\textsuperscript{44}. It was thought that the P-anomalies (which have the electric vector parallel to the grooves) which were not predicted by Rayleigh's theory\textsuperscript{45} were the result of resonant interactions\textsuperscript{44,46,37} among the deep grooves. It has been shown\textsuperscript{47} recently that the P-anomaly could also be predicted from a shallow groove grating with its refractive index taken into account.
grating resonator, this type of phenomenon is not our prime concern since we are interested in the steady state of a cavity. Any spectral order which shows a rapid intensity variation at the Rayleigh wavelength or of the resonance type should be avoided. Moreover, since we are interested only in a low loss mode of a cavity, we require a propagating diffracted order of high intensity concentration in the direction of incidence. Any other propagating spectral order is a lossy channel and therefore should be avoided.

4.2 The Grating Equation

If a uniform plane wave

\[ u_\phi = e^{-ikr \cos(\phi - \omega)} \] (4.1)

is incident upon an échelette grating (see Fig. 4.1), the far field approximation of the total diffracted field can be written as:

\[ u_\phi = f(\phi', \phi) \frac{e^{ikr \sin \theta}}{\sqrt{r}} \sum_{n=1}^{N_G} e^{-ikr[(n-(\frac{N_G}{2}+1))d \sin \theta' \sin \theta]} \] (4.2)

where \( N_G \) is the total number of facets (for convenience, we choose \( N_G \) to be an even number); \( \theta \) and \( \theta' \) are the incident and diffracted angles with respect to the grating normal; \( \phi' \) and \( \phi \) are the incident and diffracted angles with respect to the reference axis which is normal to the long side of the facets; \( \Psi \) and \( d \) are the grating angle and the grating constant.

The summation can be simplified as
\[
\sum_{n=1}^{N_G} e^{-i k \left[ n - \left( \frac{N_G}{2} + 1 \right) \right] d (\sin \theta' + \sin \theta)}
\]

\[
= \sum_{j=-\frac{N_G}{2}}^{\frac{N_G}{2} - 1} e^{-i k d (\sin \theta' + \sin \theta)}
\]

\[
= e^{i \frac{k d}{2} (\sin \theta' + \sin \theta)}
\]

\[
= B e^{i \frac{k d}{2} (\sin \theta' + \sin \theta)}
\]

where \( B \) is called "Bragg factor" which gives the interference pattern for the \( N_G \) scatterers. The maxima occur at

\[
k d (\sin \theta' + \sin \theta) = 2 N R
\]

i.e. the grating equation:

\[
N \lambda = d (\sin \theta' + \sin \theta)
\]  \hspace{1cm} (4.4)

where \( N \) is the spectral order (any integer which satisfies Eq. 4.4 for \( \sin \theta' = 1 \) and \( \sin \theta < 1 \)). The actual intensity distribution \( J(\phi) \) is obtained by considering the total time average energy at an element of cylindrical surface of unit length far away from the grating:

\[
\text{Energy flux} = \int \overline{E_x H_y} \, dS = \int_{\phi} \overline{E_x H_y} \, r \, d\phi
\]  \hspace{1cm} (4.5)

Let \( u \) represent the only component of \( E \) or \( H \), depending on whether it is \( P \)-polarization (electric field vector parallel to the grooves) or \( S \)-polarization (electric field vector perpendicular to the grooves). The diffracted wave is almost plane, in the far field, and
\[ \mathbf{E} \times \mathbf{H} = \frac{1}{2} \sqrt{\frac{\varepsilon_0}{\mu_0}} \mathbf{u}_z \hat{\mathbf{n}} \quad \text{P-polarization} \]  \hspace{1cm} (4.6a)

\[ \mathbf{E} \times \mathbf{H} = \frac{1}{2} \sqrt{\frac{\mu_0}{\varepsilon_0}} \mathbf{u}_z \hat{\mathbf{n}} \quad \text{S-polarization} \]  \hspace{1cm} (4.6b)

where \( \varepsilon_0 \) and \( \mu_0 \) are the free space permittivity and permeability. \( \mathbf{n} \) is the unit normal along the Poynting vector \( \mathbf{E} \times \mathbf{H} \).

By the use of Eqs. (4.6) and (4.5) can be written as:

\[ \text{Energy flux} = \oint_{\Omega} \mathbf{E} \times \mathbf{H} \cdot \hat{\mathbf{n}} \, d\Omega = \oint_{\Omega} \mathbf{D}(\varphi) \cdot \hat{\mathbf{n}} \, d\varphi \]  \hspace{1cm} (4.5a)

where \( \mathbf{D}(\varphi) \) is the intensity distribution which is given by

\[ \mathbf{D}(\varphi) = \frac{1}{2} \left( \sqrt{\frac{\varepsilon_0}{\mu_0}} \right)^2 f(\varphi', \varphi) \mathbf{u}_z \mathbf{E} \]  \hspace{1cm} (4.7)

\[ + \text{ for P-polarization.} \]

4.3 The Intensity Distribution

4.3.1 Singly diffracted rays

\( \mathbf{D}(\varphi) \) can be calculated if \( f(\varphi', \varphi) \) is known. Away from the shadow boundaries; if only the singly diffracted ray is considered, \( f(\varphi', \varphi) \) can be given by the diffraction coefficient of a 90° wedge:

\[ f(\varphi', \varphi) = D(\varphi', \varphi, \frac{\pi}{2}) = \frac{\varepsilon_0}{\sqrt{6\pi K}} \left[ \frac{1}{k} \frac{1}{\omega_0} \frac{1}{\sqrt{\omega_0^2 (\varphi' - \varphi) (\varphi' - \varphi)}} \right] \]  \hspace{1cm} (4.8)
To verify the assumption that diffraction of an edge or a wedge is a localized effect, we also include the results calculated from the diffraction coefficient of an edge for comparison, i.e.

\[
f (\varphi', \varphi) = D (\varphi', \varphi, 2 \pi) = \frac{i N}{2 / \lambda k} \left[ \frac{1}{\cos \varphi' \cos \varphi} \pm \frac{1}{\cos \varphi'} \right] \tag{4.9}
\]

\pm \text{ for P-polarization}

\text{S-polarization}

The energy in a particular spectral order is defined by Eq. (4.5a) by choosing \( \varphi_0 \) and \( \varphi_2 \) to be the locations of the first minimum on each side of the peak for that order. In order to compare with the experimental results, we first normalize the energy by the factor of

\[
\frac{1}{2} \int \frac{C_i}{E x H} dS = \frac{1}{2} \sqrt{ \frac{E_x}{\lambda_0} } 2 C_1 \quad \text{for P-polarization}
\]

\[-C_i \quad \text{S-polarization}
\]

which is proportional to the energy per unit length for a plane wave falling upon a plate of size \( 2C_1 \). That is, the energy contained in the \( N \)th order is

\[
U_N = \int_{\varphi_0}^{\varphi_2} \frac{E_x}{E} d\varphi / \left( \frac{E_x}{\lambda_0} \right) \frac{1}{2} C_1 \quad \text{for P-polarization}
\]

\(-C_i \quad \text{S-polarization}
\]

Then the normalized percentage energy in a particular spectral order \( N \) is given by

\[
\bar{U}_N = \frac{U_N}{\sum U_N} \times 100 \%
\]

\((4.12)\)
A set of echelette gratings with different facet angles which has been studied experimentally is chosen for theoretical calculations. The grating constant $d$ is fixed at $1.254\lambda$ such that only two spectral orders can exist, namely $N = 1$ and 0. The incident angle is also fixed i.e. at $23\frac{1}{2}^\circ$ with respect to the grating normal, such that the diffracted angles for these two orders are

$$\phi = \psi - 23\frac{1}{2}^\circ \quad \text{for} \quad N = 1$$
$$\phi = \psi + 23\frac{1}{2}^\circ \quad \text{for} \quad N = 0$$

The experiment was conducted at a wavelength of 4.31 mm. For our calculations we choose a wavelength of a round number of 4 mm. In Table II, the facet angles, the steps $b$, the width of the long side of the facet $a$ and the half-size of the gratings are given.

The results of Eq. 4.7 with Eq. 4.9 and with Eq. 4.8 are given in Table IV. The experimental results of Brannen and Rumbold are tabulated in Table III.

In Table IV, the result for $U_1$ is not obtained for $23\frac{1}{2}^\circ$ grating because the diffracted ray for $N = 1$ is right on the reflected boundary and the diffraction coefficients (Eq. 4.8 and Eq. 4.9) diverge. This is similar to the case of the grazing (normal) incidence in the diffraction problem of a rectangular block, and will be treated later. The results with an asterisk for $U_1$ in the case of $30^\circ$ grating are not obtained from Eqs. 4.8 and 4.9 because these equations do not hold for cases when the diffracted ray is near the reflected boundary. Instead, these values are calculated from
TABLE II  Characteristics of a set of gratings

\[ d = 1.254\lambda, \lambda = 0.4 \text{ cm, Number of facets} = 14 \text{ or } 28 \]

Incident angle = 23\(\frac{1}{2}\)° w.r.t. grating normal

<table>
<thead>
<tr>
<th>Facet angle (degrees)</th>
<th>b (in (\lambda))</th>
<th>a (in (\lambda))</th>
<th>(C_1) (in cm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.2177</td>
<td>1.2349</td>
<td>3.458</td>
</tr>
<tr>
<td>15</td>
<td>0.3245</td>
<td>1.2112</td>
<td>3.392</td>
</tr>
<tr>
<td>23(\frac{1}{2})</td>
<td>0.5000</td>
<td>1.1500</td>
<td>3.220</td>
</tr>
<tr>
<td>30</td>
<td>0.6270</td>
<td>1.0860</td>
<td>3.041</td>
</tr>
<tr>
<td>35</td>
<td>0.7193</td>
<td>1.0273</td>
<td>2.876</td>
</tr>
<tr>
<td>40</td>
<td>0.8061</td>
<td>0.9606</td>
<td>2.691</td>
</tr>
<tr>
<td>45</td>
<td>0.8867</td>
<td>0.8867</td>
<td>2.483</td>
</tr>
<tr>
<td>\text{NG = 14}</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>\text{NG = 28}</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
TABLE III  Experimental Results of Brannen & Rumbold\textsuperscript{26}

Grating size: 68 mm square

<table>
<thead>
<tr>
<th>Facet Angle (degrees)</th>
<th>P-polarization</th>
<th>S-polarization</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$U_0$ (in percent)</td>
<td>$U_1$</td>
</tr>
<tr>
<td>10</td>
<td>84</td>
<td>16</td>
</tr>
<tr>
<td>15</td>
<td>67</td>
<td>33</td>
</tr>
<tr>
<td>23$\frac{1}{2}$</td>
<td>34</td>
<td>66</td>
</tr>
<tr>
<td>30</td>
<td>11</td>
<td>89</td>
</tr>
<tr>
<td>35</td>
<td>4</td>
<td>96</td>
</tr>
<tr>
<td>40</td>
<td>1.5</td>
<td>98.5</td>
</tr>
<tr>
<td>45</td>
<td>2</td>
<td>98</td>
</tr>
</tbody>
</table>
TABLE IV  Numerical results of the intensity
distribution when only the singly
diffracted ray is considered. \((N_G = 14)\)

a. With the use of half-plane diffraction coefficient.

<table>
<thead>
<tr>
<th>Facet Angle (degrees)</th>
<th>P-polarization</th>
<th>S-polarization</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(U_0)</td>
<td>(U_1)</td>
</tr>
<tr>
<td>10</td>
<td>17.4</td>
<td>3.97</td>
</tr>
<tr>
<td>15</td>
<td>9.1</td>
<td>12.28</td>
</tr>
<tr>
<td>23(^1/2)</td>
<td>5.1</td>
<td></td>
</tr>
<tr>
<td>30</td>
<td>4.0</td>
<td></td>
</tr>
<tr>
<td>35</td>
<td>3.5</td>
<td>16.13</td>
</tr>
<tr>
<td>40</td>
<td>3.3</td>
<td>9.63</td>
</tr>
<tr>
<td>45</td>
<td>3.2</td>
<td>7.07</td>
</tr>
</tbody>
</table>

b. With the use of the wedge diffraction coefficient.

<table>
<thead>
<tr>
<th>Facet Angle (degrees)</th>
<th>P-polarization</th>
<th>S-polarization</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(U_0)</td>
<td>(U_1)</td>
</tr>
<tr>
<td>10</td>
<td>21.5</td>
<td>4.37</td>
</tr>
<tr>
<td>15</td>
<td>11.8</td>
<td>14.32</td>
</tr>
<tr>
<td>23(^1/2)</td>
<td>7.1</td>
<td></td>
</tr>
<tr>
<td>30</td>
<td>5.9</td>
<td>46.27*</td>
</tr>
<tr>
<td>35</td>
<td>5.6</td>
<td>20.18</td>
</tr>
<tr>
<td>40</td>
<td>5.6</td>
<td>12.57</td>
</tr>
<tr>
<td>45</td>
<td>5.9</td>
<td>9.66</td>
</tr>
</tbody>
</table>

*Equation (4.13) is used.
the $I$-function given by Oberhettinger$^{31}$:

\[
f(\phi, \phi') \frac{e^{i\pi r}}{r} = I(\pi - \phi + \phi', \frac{\pi}{2}) + I(\pi + \phi - \phi', \frac{\pi}{2})
\]

\[
\pm I(\pi - \phi - \phi', \frac{\pi}{2}) \pm I(\pi + \phi + \phi', \frac{\pi}{2})
\]

(4.13)

where

\[
J(\delta, \alpha) \sim \frac{e^{i\pi r}}{\sqrt{kr}} \frac{1}{\cot \left(\frac{\delta}{2}\right)} + O(\frac{1}{kr})
\]

for \((kr)^{\frac{1}{2}} \sin \frac{1}{2} \delta \gg 1\)

From Table IV, we can see that the calculated results for the gratings of shallow grooves (i.e. 10° and 15° gratings) are already in good agreement with the experimental results. The good agreement for the results using the half-plane diffraction coefficient (Eq. 4.9) in these cases shows that when the side step is small the singly diffracted rays calculated from either half-plane solution or wedge solution give almost the same results. For gratings with large groove steps, however, the results are no longer in agreement with the experimental ones. Nevertheless, the results from the wedge diffraction coefficient are better, (Table IVb) especially in the case of S-polarization, than those from the half-plane diffraction coefficient. This shows that the side step for the wedge already plays a role in the improvement. We will see, in the next section, that the doubly diffracted rays from the $d$-correction provide a big step towards better results for gratings with deeper grooves.
4.3.2 Doubly diffracted ray - d-correction

The field strength along the singly diffracted ray after hitting the wedge \( n \) is given by: (see Fig. 4.2a)

\[
    u_s = u_n \frac{e^{ikr}}{\sqrt{r}} \mathcal{D}(\varphi', \varphi, \frac{\pi}{2})
\]

(4.14)

where \( r \) is the distance from the \( n \)th wedge and \( u_n \) is the strength and the phase of the incident wave at the \( n \)th wedge. To calculate the doubly diffracted ray result from hitting the next wedge, we simply consider

\[
    u_s(d) = u_n \frac{e^{ikd}}{\sqrt{d}} \mathcal{D}(\varphi', \varphi', \frac{\pi}{2})
\]

(4.15)

being the strength of the plane wave at the \((n-1)\)th wedge and therefore:

\[
    u_{\text{double}} = u_s(d) \frac{e^{ikr}}{\sqrt{r}} \mathcal{D}(\varphi - \varphi', \varphi', \frac{\pi}{2})
\]

(4.16)

The alternate way to describe the situation is that there is a line source located at the \( n \)th wedge with a strength of

\[
    u_s = u_n \frac{\mathcal{D}(\varphi', \varphi, \frac{\pi}{2})}{\sqrt{r}}
\]

(4.17)

Then the same result is obtained by the use of the reciprocal theorem.

When the above doubly diffracted rays of the neighbouring wedges are included, the intensity distribution is given by

\[
    \mathcal{J}(\omega) = \frac{1}{2} \left| \frac{\mathcal{D}}{\mu_0} \right|^2 \int f(\omega, \omega) e^{i(kd + \omega t)} \mathcal{F}(\varphi, \omega) \mathcal{F}(\varphi, \omega) d\omega
\]

\[
    \pm \text{ for } p- \text{polarization}
\]

(4.18)
Fig. 4.2 Primary and secondary diffracted fields
where $f^{(d)}(\varphi', \varphi)$ is given by:

$$
\begin{align*}
&f^{(d)}(\varphi', \varphi) = \sum_{n=0}^{N_d-1} \int e^{-i k \left[ (n-1) + \frac{M_d + 1}{2} \right] d \sin \theta} \cdot D(\varphi, \varphi) D(\theta, \varphi') \\
&\quad \cdot e^{-i k \left[ (n+1) - \frac{M_d + 1}{2} \right] d \sin \theta} \cdot D(\varphi', \varphi) D(\theta, \varphi)
\end{align*}
$$

(4.19)

where

$$
D(\varphi', \varphi) = \frac{1}{\sqrt{2\pi k}} e^{-i \frac{\pi}{2} \varphi} D(\varphi', \varphi, \frac{\pi}{2}) \quad \text{and}
$$

$$
\phi = \frac{\pi}{2} + \psi, \quad \phi' = \frac{\pi}{2} - \psi.
$$

The diffraction coefficients $D(\varphi, \varphi)$ and $D(\varphi', \varphi)$ in Equation 4.19 diverge when the doubly diffracted ray is along or near the reflected boundary, i.e., when $\phi' - \phi$ or $\phi - \phi' = \pi$. These diffraction coefficients should be replaced by the $v_B$ function (Eq. 2.20) whenever these situations arise. (e.g., for the $35^\circ$ grating where $\phi' - \phi$). That is,

$$
D(\varphi', \varphi) \quad \text{is replaced by}
$$

$$
\sqrt{\rho k d} e^{-i \frac{\pi}{2} \varphi} \int \{ V_B(\varphi', \varphi) - V_B(-\varphi', \varphi) \}
$$

(4.20)

where

$$
V_B(\varphi', \varphi) \quad \text{is given by Eq. (2.20)}
$$
TABLE V  Numerical results of the intensity distribution when the singly diffracted ray and the d-correction are considered. (Eqs. 4.18, 4.8, and 4.19).

\[ \text{Table V} \quad \text{Numerical results of the intensity distribution when the singly diffracted ray and the d-correction are considered. (Eqs. 4.18, 4.8, and 4.19).} \]

a. Number of facet \( N_G = 14 \).

<table>
<thead>
<tr>
<th>Facet Angle (degrees)</th>
<th>( U_0 )</th>
<th>( U_1 )</th>
<th>( U_0 ) (in %)</th>
<th>( U_1 )</th>
<th>( U_0 ) (in %)</th>
<th>( U_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>22.3</td>
<td>4.23</td>
<td>80.06</td>
<td>15.94</td>
<td>11.1</td>
<td>10.11</td>
</tr>
<tr>
<td>15</td>
<td>13.0</td>
<td>13.87</td>
<td>48.38</td>
<td>51.62</td>
<td>4.3</td>
<td>23.88</td>
</tr>
<tr>
<td>23(\frac{1}{2})</td>
<td>10.9</td>
<td></td>
<td>1.2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>30</td>
<td>10.0</td>
<td>54.94</td>
<td>15.4</td>
<td>84.6</td>
<td>4.0</td>
<td>34.81</td>
</tr>
<tr>
<td>35</td>
<td>7.6</td>
<td>24.20</td>
<td>23.9</td>
<td>76.1</td>
<td>5.2</td>
<td>9.48</td>
</tr>
<tr>
<td>40</td>
<td>5.5</td>
<td>18.97</td>
<td>22.48</td>
<td>77.52</td>
<td>11.2</td>
<td>3.61</td>
</tr>
<tr>
<td>45</td>
<td>4.1</td>
<td>24.89</td>
<td>14.14</td>
<td>85.85</td>
<td>8.8</td>
<td>2.21</td>
</tr>
</tbody>
</table>

b. Number of facet \( N_G = 28 \).

<table>
<thead>
<tr>
<th>Facet Angle (degrees)</th>
<th>( U_0 )</th>
<th>( U_1 )</th>
<th>( U_0 ) (in %)</th>
<th>( U_1 )</th>
<th>( U_0 ) (in %)</th>
<th>( U_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>22.73</td>
<td>4.29</td>
<td>84.12</td>
<td>15.88</td>
<td>11.18</td>
<td>10.34</td>
</tr>
<tr>
<td>15</td>
<td>13.39</td>
<td>13.93</td>
<td>49.0</td>
<td>51.0</td>
<td>4.29</td>
<td>24.2</td>
</tr>
<tr>
<td>23(\frac{1}{2})</td>
<td>11.28</td>
<td></td>
<td>1.18</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>30</td>
<td>10.0</td>
<td>52.53</td>
<td>16.0</td>
<td>84.0</td>
<td>5.61</td>
<td>32.44</td>
</tr>
<tr>
<td>35</td>
<td>5.5</td>
<td>24.84</td>
<td>18.13</td>
<td>81.87</td>
<td>2.2</td>
<td>9.48</td>
</tr>
<tr>
<td>40</td>
<td>5.7</td>
<td>19.75</td>
<td>22.4</td>
<td>77.6</td>
<td>11.85</td>
<td>3.58</td>
</tr>
<tr>
<td>45</td>
<td>4.27</td>
<td>26.36</td>
<td>13.94</td>
<td>86.06</td>
<td>9.44</td>
<td>2.25</td>
</tr>
</tbody>
</table>
We first use Eq. 4.19 to obtain results for 14 facet and 28 facet gratings, as tabulated in Table V. The results for the two gratings are not significantly different except that the width of the peak of the 14 facet grating is approximately double of that of the 28 facet ones.

The results in Table V show considerable improvement over those in Table IV (calculated from only singly diffracted rays). The improvement is more significant for gratings with deeper grooves (35°, 40°, and 45° gratings).

4.3.3 a-correction

The a-correction is a doubly diffracted ray which results when a singly diffracted ray grazes the long side of the facet, reflected back from the side step and diffracted from the same edge where the singly diffracted rays originated. (see Fig. 4.2c)

First we treat the case of S-polarization. The singly diffracted ray which results from the nth wedge and grazes the long side of surface is given by

\[ U_{\text{singly}} = \frac{u_n}{\sqrt{r_{1}}} \cdot \frac{ikr}{i} D(\varphi', o, \frac{3\pi}{2}) \]

where \( u_n \) is the incident field at the nth wedge and \( r_{1} \) is some point on the long side of the facet. The doubly diffracted ray which results from this singly diffracted ray is given by

\[ U_{\text{doubly}} = \frac{u_n}{\sqrt{2a}} \cdot \frac{2ika}{\sqrt{2a}} \cdot D(0, \varphi, \frac{3\pi}{2}) \cdot D(\varphi', o, \frac{3\pi}{2}) \cdot \frac{ikr}{\sqrt{r_{2}}} \]

(4.22)
This is the leading term in the Zitron and Karp's expansion formula\textsuperscript{48,36} for the plane wave field in the vicinity of the closest scatterer. The factor of $\frac{1}{2}$ is due to the grazing incidence of the singly diffracted ray, or to the coalescing of the reflected and incident field at this angle of incidence.

For the case of P-polarization, the diffraction coefficient along the grazing incidence vanishes. The next non-vanishing term in Zitron and Karp's expansion is proportional to the product of the $\varphi$-derivative of the diffraction coefficient and the normal derivative of the singly diffracted ray:

$$\tilde{u}^{(a)}_{\text{doubly}} = \frac{\varphi}{2} \frac{e^{2ik\varphi}}{ik(2a)^{3/2}} D_{\varphi}(\alpha', \alpha, \frac{3\pi}{2}) D_{\varphi}(0, \varphi, \frac{3\pi}{2}) e^{ikr} \quad (4.23)$$

When the doubly diffracted ray of the $a$-correction is included, Eq. 4.18 can be rewritten as

$$s(\varphi) = \frac{1}{2} \sqrt{E} \frac{2}{\hbar} \left[ f_{(a)}(\varphi', \varphi) + f_{(a)}^{(2)}(\varphi', \varphi) \right] B + \frac{i(\varphi - \varphi')}{\sqrt{2}} f_{(a)}^{(1)}(\varphi', \varphi) \quad (4.24)$$

where $f_{(a)}^{(1)}(\varphi', \varphi)$ is given by

$$f_{(a)}^{(1)}(\varphi', \varphi) = \frac{1}{2} \frac{e^{2ik\varphi}}{\sqrt{2a}} D_{\varphi}(0, \varphi, \frac{3\pi}{2}) D_{\varphi}(\varphi, 0, \frac{3\pi}{2}) \quad (4.25)$$

for S-polarization;

$$f_{(a)}^{(2)}(\varphi', \varphi) = \frac{1}{2} \frac{e^{2ik\varphi}}{ik(2a)^{3/2}} D_{\varphi}(\varphi', 0, \frac{3\pi}{2}) D_{\varphi}(0, \varphi, \frac{3\pi}{2}) \quad (4.26)$$

for P-polarization.
TABLE VI  Numerical results of intensity distribution
when the singly diffracted ray, the d-correction
and the a-correction are considered.  (Eqs. 4.24,
4.8, 4.19, 4.25 and 4.26)

\[ N_G = 28 \]

<table>
<thead>
<tr>
<th>Facet Angle (degrees)</th>
<th>( U_0 )</th>
<th>( U_1 )</th>
<th>( \bar{U}_0 ) (in %)</th>
<th>( \bar{U}_1 )</th>
<th>( U_0 )</th>
<th>( U_1 )</th>
<th>( \bar{U}_0 ) (in %)</th>
<th>( \bar{U}_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>21.05</td>
<td>3.93</td>
<td>84.27</td>
<td>15.73</td>
<td>9.45</td>
<td>9.95</td>
<td>48.71</td>
<td>51.29</td>
</tr>
<tr>
<td>15</td>
<td>12.54</td>
<td>12.65</td>
<td>49.78</td>
<td>50.22</td>
<td>3.31</td>
<td>22.97</td>
<td>12.60</td>
<td>87.40</td>
</tr>
<tr>
<td>23( \frac{1}{2} )</td>
<td>10.91</td>
<td></td>
<td>0.90</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>30</td>
<td>14.51</td>
<td>50.25</td>
<td>22.41</td>
<td>77.59</td>
<td>0.12</td>
<td>30.20</td>
<td>0.40</td>
<td>99.60</td>
</tr>
<tr>
<td>35</td>
<td>0.26</td>
<td>23.37</td>
<td>1.10</td>
<td>98.90</td>
<td>8.82</td>
<td>8.89</td>
<td>49.80</td>
<td>50.20</td>
</tr>
<tr>
<td>40</td>
<td>0.94</td>
<td>19.16</td>
<td>4.68</td>
<td>95.32</td>
<td>12.80</td>
<td>3.69</td>
<td>77.62</td>
<td>22.38</td>
</tr>
<tr>
<td>45</td>
<td>5.20</td>
<td>26.98</td>
<td>16.16</td>
<td>83.84</td>
<td>14.94</td>
<td>2.12</td>
<td>87.57</td>
<td>12.43</td>
</tr>
</tbody>
</table>
The numerical results of the intensity distribution from Eq. (4.24) are tabulated in Table VI. They do not show an overall consistent improvement. Perhaps this is because no consideration is given to the b-correction which is the same order of magnitude as the a-correction.

4.3.4 b-correction

The singly diffracted ray from the nth wedge can also graze the side facet, be reflected back from the long side of the facet and be diffracted again from the same wedge where it originated. The doubly diffracted ray, resulting in this manner cannot be treated as in the case of the a-correction because the widths of the side facets of the set of gratings we have chosen are less than a wavelength. The leading term in the asymptotic expansion (Eq. 2.8a) is not good enough in this situation. However, the expansion of Green's theorem (see section 2.4) can be applied if the surface field $u$ or $\frac{\partial u}{\partial t}$ is known along the side facet. The expansion states that, for far field,

$$u(r) = \frac{e^{i kr - ik r_0}}{2 \sqrt{2 \pi kr}} \int_0^b \left[ k u \sin \phi - i \frac{\partial u}{\partial t} \right] e^{-i k \rho \cos \phi} d\rho$$

(4.27)

The $u$ or $\frac{\partial u}{\partial t}$ can be calculated from Oberhettinger's I-function by letting $t \rightarrow t_0$ in $u_d$ or $\frac{\partial u}{\partial t}$ where $u_d$ is given by

$$u_d = I\left(2\nu - 2\xi_t, r f_0\right) + I\left(2\xi_t - 1\right) I\left(f_0\right) - I\left(2\nu - f_0\right)$$

(4.28)

$t$ for S-polarization
with \( \delta_0 = \pi - \phi \), \( \delta = \delta_0 - \phi \).

For S-polarization,
\[
\left. u_d \right|_{d = \delta_0} = 2 I(2\pi - \delta_0) + 2 I(\delta_0) .
\]
(4.29)

For P-polarization \( u_d \left|_{d = \delta_0} = 0 \right. \),
\[
\frac{\partial u_d}{\partial n} \left|_{d = \delta_0} = 2 \frac{1}{\rho} \frac{\partial I^p}{\partial \delta} \left|_{d = \delta_0} \right. - 2 \frac{1}{\rho} \frac{\partial I^s}{\partial \delta} \left|_{d = \delta_0} \right. \right.
\]
(4.30)

The required I-function which is valid for small \( k\rho \) is given by
\[
I^s(\delta, \alpha) = \frac{1}{2} \left( \sum_{n = -\infty}^\infty \frac{n}{\sin \alpha} J_n(k\rho) \sin \frac{n(\alpha - \delta)}{\alpha} \right.
\]
\[
- \frac{1}{2} \exp \left( \frac{\pi}{\alpha} \right) \frac{\pi}{\alpha} \sin \frac{n(\alpha - \delta)}{\alpha} \right)
\]
(4.31)

where \( \alpha \) in our case is \( \frac{3\pi}{2} \).*

For small \( k\rho \) (\( k\rho < 6 \)) \( J_n(k\rho) \) and \( J_{2n}(k\rho) \) are rapidly convergent functions as the order \( n \) increases, so we could terminate the series for a reasonable \( n \) (\( n < 15 \)) to obtain

*The right hand side of eq.(4.31) is not defined for the case of \( \alpha = \frac{3\pi}{2} \). However, when the diffracted field is expressed in terms of the I-functions \( I(\delta, \alpha) \), they are always in pairs[31] with respect to the following arguments:
\[
\delta = \pi + \phi
\]
where \( \phi \) is an angle which is defined by the incident and diffracted angles. (see e.g. eq.(4.13) or ref. 31). The factors \( \sin \alpha \) or \( \sin \left( \frac{n\pi}{\alpha} \right) \) in the denominator of eq.(4.31) are cancelled out by the factor \( \sin \left( \frac{n\pi}{\alpha} \right) \) or \( \sin \frac{n\pi}{\alpha} \) respectively when the pairs are combined, i.e.
\[
\left| \sin \pi \left( \alpha - \pi + \phi \right) + \sin \pi \left( \alpha - \pi - \phi \right) \right| = 2 \sin \pi \alpha \cos \pi \phi
\]
\[
\sin \frac{n\pi}{\alpha} \left( \pi - \phi \right) + \sin \frac{n\pi}{\alpha} \left( \pi + \phi \right) = 2 \sin \frac{n\pi}{\alpha} \cos \frac{n\pi}{\alpha} \phi
\]
and hence finally we may allow \( \alpha \to \frac{3\pi}{2} \).
a reasonable \( I(\varphi, \alpha) \). Using this,

\[
U|_{\varphi \leq \phi_0} = -\frac{2}{3} J_0(k \rho) - \frac{4}{3} \sum_{n=1}^{\infty} \left\{ 3 \cos n \phi_0 \ J_n(k \rho) - 2 e^{-\frac{2n}{3} \frac{\rho}{\sin \frac{n}{3} (k \cdot \alpha)}} J_{2n}(k \rho) \right\}
\]

(4.32)

\[
\frac{\partial U}{\partial n}|_{\varphi \leq \phi_0} = -\frac{4}{3} \sum_{n=1}^{\infty} \left\{ \frac{n}{3} \cos n \phi_0 \ J_n(k \rho) + \frac{2n}{3} e^{-\frac{2n}{3} \frac{\rho}{\sin \frac{n}{3} (k \cdot \alpha)}} J_{2n}(k \rho) \right\}
\]

(4.33)

By applying Eqs. (4.27), and (4.32) or (4.33), the b-correction is given by

\[
U^{(b)}(r) = U_n C^{(b)} e^{i kr \cdot \varphi}
\]

(4.34)

where \( U_n \) is the incident field at the nth wedge and \( C^{(b)} \) can be physically interpreted as the strength of the line source located at the nth wedge and is given by

\[
C^{(b)} = \begin{cases} 
\frac{1}{2j b} \int_{0}^{b} k \sin \varphi e^{-ik \rho \cos \varphi} d \rho & \text{for S-polarization} \\
-\frac{1}{2j b} \int_{0}^{b} \frac{d n}{dn} e^{-ik \rho \cos \varphi} d \rho & \text{for P-polarization}
\end{cases}
\]

(4.35)

(4.36)

where \( U_n \) or \( \frac{d n}{dn} \) is given by Eqs. (4.32) or (4.33).

Table VII gives the results of \( C \) for P- and S-polarization.

When the b-correction is included in the calculation, Eq. 4.18 can be rewritten as

\[
\mathcal{J}(\varphi) = \frac{j}{2} \left( \frac{E_0}{\mu_0} \right)^{1/2} \left[ f(\varphi, \phi) + f^{(a)}(\varphi, \phi) + f^{(b)}(\varphi, \phi) \right] B^i(k \rho + \varphi)
\]

\[
+ \frac{e}{\sqrt{2 \pi \rho_0}} \left| f^{(d)}(\varphi, \phi) \right|^2
\]

(4.37)

\[\text{for P-}\text{S-polarization}\]
# TABLE VII  Strength of the line sources for b-correction

<table>
<thead>
<tr>
<th>Facet Angle (degrees)</th>
<th>N=0</th>
<th>N=1</th>
<th>S-polarization</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>ReC</td>
<td>ImC</td>
<td>ReC</td>
</tr>
<tr>
<td>10</td>
<td>+0.4700</td>
<td>-0.0037</td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>+0.2539</td>
<td>+0.2586</td>
<td></td>
</tr>
<tr>
<td>23½</td>
<td>-0.1154</td>
<td>-0.1336</td>
<td>-0.0924</td>
</tr>
<tr>
<td>30</td>
<td>-0.1167</td>
<td>+0.0949</td>
<td>-0.1162</td>
</tr>
<tr>
<td>35</td>
<td>-0.1341</td>
<td>+0.1747</td>
<td>-0.2504</td>
</tr>
<tr>
<td>40</td>
<td>-0.1944</td>
<td>+0.2555</td>
<td>-0.2995</td>
</tr>
<tr>
<td>45</td>
<td>-0.2561</td>
<td>+0.2949</td>
<td>-0.2242</td>
</tr>
</tbody>
</table>
where \( f^{(b)}(\psi, \psi') \) is given by
\[
f^{(b)}(\psi, \psi') = C
\]
with \( C \) given in Table VII for both P- and S-polarization.

The results for the intensity distribution from Eq. (4.37) are tabulated in Table VIII. There are good agreements for all cases in S-polarization with the exception of the 45° grating. For P-polarization, however, the \( b \)-correction seems to have little effect on the previous results.

4.3.5 Triply diffracted ray

A triply diffracted ray results when a doubly diffracted ray hits another wedge. Since we have three types of doubly diffracted rays, there will be many types of triply diffracted rays. However, to test the significance of including such a diffracted ray in the calculation of intensity distribution we will consider only one type of triply diffracted ray which is similar to the doubly diffracted ray in all respects except that it involves three wedges. Consider a line source of strength
\[
\mathcal{U}_{nr2} \frac{e}{\sqrt{6\pi}} D(\psi, \phi')
\]
located at \((n + 2)\)th wedge, where
\[
D(\psi, \phi') = \sqrt{6\pi} e^{-i\pi/2} D(\psi', \phi', \frac{5\pi}{2})
\]
TABLE VIII  Numerical results of the intensity distribution
when the singly diffracted ray, the d-, the a-
and the b-corrections are considered. (Eqs. 4.37,
4.8, 4.9 and 4.38).

\[ N_G = 28 \]

<table>
<thead>
<tr>
<th>Facet Angle (degrees)</th>
<th>( U_0 )</th>
<th>( U_1 )</th>
<th>( \overline{U}_0 ) (in % )</th>
<th>( \overline{U}_1 )</th>
<th>( U_0 )</th>
<th>( U_1 )</th>
<th>( \overline{U}_0 ) (in % )</th>
<th>( \overline{U}_1 )</th>
</tr>
</thead>
<tbody>
<tr>
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<td>82.09</td>
<td>17.91</td>
<td>8.41</td>
<td>9.95</td>
<td>45.81</td>
<td>54.19</td>
</tr>
<tr>
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<td>12.65</td>
<td>47.31</td>
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<td>3.51</td>
<td>22.97</td>
<td>13.26</td>
<td>86.74</td>
</tr>
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<td></td>
</tr>
<tr>
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<td>14.95</td>
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<td>22.48</td>
<td>77.52</td>
<td>2.53</td>
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</tr>
<tr>
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<td>0.46</td>
<td>25.34</td>
<td>1.80</td>
<td>98.20</td>
<td>2.57</td>
<td>8.58</td>
<td>23.73</td>
<td>76.27</td>
</tr>
<tr>
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<td>93.59</td>
<td>4.24</td>
<td>3.27</td>
<td>56.46</td>
<td>43.54</td>
</tr>
<tr>
<td>45</td>
<td>4.25</td>
<td>28.15</td>
<td>13.12</td>
<td>86.88</td>
<td>7.36</td>
<td>0.26</td>
<td>96.59</td>
<td>3.41</td>
</tr>
</tbody>
</table>
Since the doubly diffracted ray, which is used as the incident ray for the triply diffracted ray, lies along the shadowed boundary, the field at the nth wedge is given by

$$u_n = \frac{U_{nt} e^{ik}}{2} \frac{e^{ikd}}{\sqrt{k}d} D(\psi', \phi') \frac{e^{ikd}}{\sqrt{k}d}$$

and the triply diffracted ray,

$$u_{triply} = \frac{e^{ikr}}{\sqrt{r}} D(\psi, \phi, \frac{r^2}{2}) u_n$$

There is a similar triply diffracted ray which results from n-2, n-1 and nth wedges. To include these rays in Eq. 3.18 we have

$$\mathcal{J} = \frac{1}{2} \left( \int \frac{e^{ikd}}{\sqrt{kd}} \sum_{n=2}^{n} \left[ f(\psi'; \psi) + f^{(A)}(\psi'; \psi) + f^{(B)}(\psi'; \psi) \right] \cdot \frac{e^{ikd}}{\sqrt{kd}} \right) \cdot \left\{ f^{(d)}(\psi', \psi) + f^{(f)}(\psi', \psi) \right\}^2$$

where $f^{(d)}(\psi', \psi)$ is given by

$$f^{(d)}(\psi', \psi) = \frac{e^{ikd}}{2^{\frac{n-2}{2}}} \sum_{n=3}^{n} \frac{e^{ikd}}{\sqrt{kd}} \cdot \rho(n-2) - \frac{\sqrt{(n-2) + 1}}{\rho(n-2) + 1} \cdot \frac{e^{i\theta}}{\sqrt{kd}}$$

$$\int e^{-ik([n-2] - (\frac{\sqrt{n} + 1}{2})]} D(\phi', \phi') D(\theta, \phi')$$

$$+ e^{-ik([n+1] - (\frac{\sqrt{n} + 1}{2})]} D(\phi', \phi') D(\theta, \phi')$$

$$+ e^{-ik([n+2] - (\frac{\sqrt{n} + 1}{2})]} D(\phi', \phi') D(\theta, \phi')$$

$$\frac{e^{ikd}}{\sqrt{kd}} \sum_{n=3}^{n} \frac{e^{ikd}}{\sqrt{kd}}$$

$$\int e^{-ik([n-2] - (\frac{\sqrt{n} + 1}{2})]} D(\phi', \phi') D(\theta, \phi')$$

$$+ e^{-ik([n+1] - (\frac{\sqrt{n} + 1}{2})]} D(\phi', \phi') D(\theta, \phi')$$

$$+ e^{-ik([n+2] - (\frac{\sqrt{n} + 1}{2})]} D(\phi', \phi') D(\theta, \phi')$$

$$\frac{e^{ikd}}{\sqrt{kd}} \sum_{n=3}^{n} \frac{e^{ikd}}{\sqrt{kd}}$$

$$\int e^{-ik([n-2] - (\frac{\sqrt{n} + 1}{2})]} D(\phi', \phi') D(\theta, \phi')$$

$$+ e^{-ik([n+1] - (\frac{\sqrt{n} + 1}{2})]} D(\phi', \phi') D(\theta, \phi')$$

$$+ e^{-ik([n+2] - (\frac{\sqrt{n} + 1}{2})]} D(\phi', \phi') D(\theta, \phi')$$
TABLE IX  Numerical results of the intensity distribution
when the singly diffracted ray, the triply diffracted
ray, the d-, the a-, and the b-corrections are
considered.  (Eqs. 4.41, 4.8, 4.19, 4.38 and 4.42).

\[ N_G = 28 \]

<table>
<thead>
<tr>
<th>Facet Angle (degrees)</th>
<th>( U_0 )</th>
<th>( U_1 )</th>
<th>( \overline{U}_0 ) (in %)</th>
<th>( \overline{U}_1 )</th>
<th>( U_0 )</th>
<th>( U_1 )</th>
<th>( \overline{U}_0 ) (in %)</th>
<th>( \overline{U}_1 )</th>
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<tbody>
<tr>
<td>10</td>
<td>18.14</td>
<td>3.90</td>
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<td>9.85</td>
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<td>53.42</td>
</tr>
<tr>
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<td>11.56</td>
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<td>52.09</td>
<td>3.58</td>
<td>22.81</td>
<td>13.57</td>
<td>86.43</td>
</tr>
<tr>
<td>23( \frac{1}{2} )</td>
<td>11.97</td>
<td>0.86</td>
<td></td>
<td></td>
<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>30</td>
<td>13.77</td>
<td>52.18</td>
<td>20.88</td>
<td>79.12</td>
<td>2.13</td>
<td>30.62</td>
<td>6.50</td>
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</tr>
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<td>8.94</td>
<td>25.75</td>
<td>74.25</td>
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<td>89.67</td>
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<td>7.72</td>
<td>0.68</td>
<td>91.90</td>
<td>8.10</td>
</tr>
</tbody>
</table>
The results, in which the triply diffracted ray (Eq. (4.40)) is included, are listed in Table IX. There are no significant changes from the previous results (Table VIII) except for the case of the 45° grating in S-polarization. Hence, we are justified in ignoring diffracted rays of a higher order.

4.3.6 $N = 1$ of 23½° grating

In order to calculate the intensity of the first diffracted order ($N = 1$) for the 23½° grating, we have to apply Green's theorem since the incident ray, the diffracted ray and one side of facet are parallel to each other. However, the surface field along the facet which is required in the application of Green's theorem is not known until we have solved the boundary value problem of a grating. To avoid this difficulty, we assume that each facet can be approximated as a block of dimension $2a \times 2b$ if we consider that the neighbouring facets serve as mirrors. (see Fig. 4.3) However, this representation is not exact since the neighbouring facets which serve as mirrors are not of infinite extent. Hence, the diffracted field of a facet is assumed to be some portion $g$ of the diffracted field of a block.

The diffracted far field in $N = 1$ direction will be the reflected forward field of the scattering of a block which is given by Morse\(^{36}\):

For P-polarization:
Fig. 4.3 The image of a facet
\[
U_d = \frac{e^{i(kr - \frac{1}{2} \pi)}}{\sqrt{2 \pi kr}} \left[ 2kA + \frac{4e^{i\frac{\pi}{8}}}{\sqrt{2k}b} - \frac{19}{18} \frac{e^{-i\frac{\pi}{8}}}{\sqrt{2k}b} + O(ka^2) \right] 
\]

(4.43a)

For S-polarization:

\[
U_d = \frac{e^{i(kr - \frac{1}{2} \pi)}}{\sqrt{3 \pi kr}} \left[ -2kA + \frac{4e^{i\frac{\pi}{8}}}{9} + \frac{4}{\sqrt{kA}} - \frac{2}{\sqrt{2\pi k}b} \right] + O(\frac{1}{kA}) 
\]

(4.43b)

By assuming \( g = \frac{1}{4} \) which physically may be interpreted as meaning that although the facet is imaged as a block, the total contribution is from the actual size of the facet.

\( U_1 \) for 23\(^\circ\) grating is calculated to be 32.01 for

P-polarization and 24.04 for S-polarization. Together with the results for \( U_0 \) from Table IX, the intensity distributions for 23\(^\circ\) gratings are:

<table>
<thead>
<tr>
<th>P-polarization</th>
<th>S-polarization</th>
</tr>
</thead>
<tbody>
<tr>
<td>( U_0 )</td>
<td>( U_0 )</td>
</tr>
<tr>
<td>27.22</td>
<td>3.45</td>
</tr>
<tr>
<td>( U_1 )</td>
<td>( U_1 )</td>
</tr>
<tr>
<td>72.78</td>
<td>96.55</td>
</tr>
</tbody>
</table>

Comparing the experimental (Table III) with the theoretical results we note that \( g \) is less than \( \frac{1}{4} \) for P-polarization and greater than \( \frac{1}{4} \) for S-polarization.

4.4 Discussion

We have seen from the above calculations, that the results for S-polarization when all four types of rays are included, give a better fit with the experimental results than those of P-polarization (See Table X). Except for the 23\(^\circ\) and the 45\(^\circ\) gratings, the results for S-polarization agree excellently with the experimental ones. Poor agreement
<table>
<thead>
<tr>
<th>Facet Angle (degrees)</th>
<th>Experimental results</th>
<th>Singly diffr. ray</th>
<th>Singly + doubly* diffr. rays</th>
<th>Singly &amp; doubly* + a-correction</th>
<th>Singly + doubly* + a- + b-correction</th>
<th>Singly + doubly* + triply + a- and b-correction</th>
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<tbody>
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<td>15.73</td>
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<td>17.70</td>
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<td>54.82</td>
<td>51.00</td>
<td>50.22</td>
<td>52.69</td>
<td>52.09</td>
</tr>
<tr>
<td>23½</td>
<td>66</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
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<td>88.69</td>
<td>84.00</td>
<td>77.59</td>
<td>77.52</td>
<td>79.12</td>
</tr>
<tr>
<td>35</td>
<td>96</td>
<td>78.28</td>
<td>81.87</td>
<td>98.90</td>
<td>98.20</td>
<td>96.12</td>
</tr>
<tr>
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<td>98.5</td>
<td>69.12</td>
<td>77.60</td>
<td>95.32</td>
<td>93.59</td>
<td>89.67</td>
</tr>
<tr>
<td>45</td>
<td>98</td>
<td>62.08</td>
<td>86.06</td>
<td>83.84</td>
<td>86.88</td>
<td>91.91</td>
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</tbody>
</table>

* The doubly diffracted ray indicates only the d-correction.
<table>
<thead>
<tr>
<th>Facet Angle (degrees)</th>
<th>Experimental results</th>
<th>Singly differ. ray</th>
<th>Singly + doubly* differ. rays</th>
<th>Singly &amp; doubly* + a-correction</th>
<th>Singly + doubly* + a- + b-correction</th>
<th>Singly + doubly* + triply + a-and b-correction</th>
</tr>
</thead>
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<td>87.40</td>
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<td>86.43</td>
</tr>
<tr>
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</tr>
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<td>22.38</td>
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<td>44.19</td>
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<tr>
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<td>74.00</td>
<td>19.25</td>
<td>12.43</td>
<td>3.41</td>
<td>8.10</td>
</tr>
</tbody>
</table>

* The doubly diffracted ray indicates only the d-correction.
in both P- and S-polarization can perhaps be attributed to
the fact that the widths of both sides of a facet are in
the order of a wavelength. When this region is reached,
the "corner" (the concave part of the grating) has to be
taken into consideration. The result for the $23\frac{1}{2}^\circ$ grating
has been explained in the previous section. It is essen-
tially equivalent to the inadequacy of determining the
g-factor accurately. However, the fact that $g > \frac{1}{4}$ for
S-polarization and less than $\frac{1}{4}$ for P-polarization provides
some insight to the "poor" results in the case of
P-polarization. Physically, a smaller g-factor indicates
a stronger imaging process. It should be recalled that the
g-factor is that fraction of an area of a complete block
that gives the correct amount of imaging, because the "mirrors"
are not infinite. For mirrors of infinite extent, the
imaging process is complete and the g-factor is equal to 1.
The reflection is essentially an imaging process, it is more
susceptible to reflection in P-polarization than in S-
polarization. We see that, for P-polarization, the percentage
energy in the $N = 1$ order is less in the theoretical results
than in the experimental ones, for facet angles greater than
$23\frac{1}{2}^\circ$. This may be the consequence of neglecting the type of
reflected field shown in Fig. 4.4a. Also the reason for the
smaller value for $N = 0$ in the $15^\circ$ grating is that the ref-
lected field shown in Fig. 4.4b has not been accounted for.
In fact, if we put a line source of strength $1/\sqrt{6\pi kd}$ at each
wedge for $N = 0$ for $15^\circ$ grating, and for $N = 1$ for $30^\circ$, $35^\circ$,
Fig. 4.4 Reflected rays
40°, and 45° grating, then we have the following results:

<table>
<thead>
<tr>
<th>Facet Angle</th>
<th>$U_0$</th>
<th>$U_1$</th>
<th>$\bar{U}_0$</th>
<th>$\bar{U}_1$</th>
<th>Experimental results for $\bar{U}_1$ (in %)</th>
</tr>
</thead>
<tbody>
<tr>
<td>15°</td>
<td>19.24</td>
<td>12.57</td>
<td>61.43</td>
<td>38.57</td>
<td>33</td>
</tr>
<tr>
<td>30°</td>
<td>13.77</td>
<td>55.76</td>
<td>19.80</td>
<td>80.20</td>
<td>89</td>
</tr>
<tr>
<td>35°</td>
<td>1.05</td>
<td>38.44</td>
<td>2.66</td>
<td>97.34</td>
<td>96</td>
</tr>
<tr>
<td>40°</td>
<td>2.56</td>
<td>35.00</td>
<td>6.82</td>
<td>93.18</td>
<td>98.5</td>
</tr>
<tr>
<td>45°</td>
<td>2.67</td>
<td>38.21</td>
<td>6.53</td>
<td>93.47</td>
<td>98</td>
</tr>
</tbody>
</table>
CHAPTER V

FABRY-PEROT RESONATORS WITH GRATING ELEMENTS

5.1 Introduction

A Fabry-Perot resonator requires two highly reflecting mirrors. If one of the mirrors is replaced by an echelle grating, the grating must have a diffracted order which has a highly concentrated intensity in the direction of incidence. To determine the diffracted order in the direction of incidence, we set

\[ \sin \theta = \sin \theta' \]

in the grating equation (Eq. 4.4) and this particular order is denoted \( N_B \), the so-called "blazed" diffracted order:

\[ \frac{N_B \lambda}{2d} = \sin \theta = \sin \theta' \]

(5.1)

Any other possible diffracted order \( N \) has to satisfy the following criterion for a particular incident angle:

\[ N\lambda = \frac{N_B \lambda}{2} + d \sin \theta \]

With \( |\sin \theta| \leq 1 \), we deduce from the above equation

\[ |N - \frac{N_B}{2}| \leq \frac{d}{\lambda} \]

(5.2)

From Eq. 5.2, for a particular \( N_B \), the larger \( d \), the larger
number of possible \( N \), i.e. the larger number of possible diffracted orders. In order to keep the total number of diffracted orders to be two, we can only choose \( N_B = 1 \), and
\[
\frac{\lambda}{2} < d < \frac{3\lambda}{2}
\]
This has been the case in those gratings, having \( d = 1.254\lambda \), which we have chosen for the study of intensity distributions in the previous chapter.

To form a grating resonator (resonator with a plane mirror and a grating), we arrange the grating so that the direction of the blazed order coincides with the optical axis of the resonator. To study such a cavity, as we have mentioned in Chapter I and III, we set up an integral equation by requiring the field on the plane mirror to reproduce itself (up to a complex constant) after one round trip. The kernel of the integral equation is constructed by calculating the response from a grating illuminated by a unit incident field (plane wave or a line source) and setting the response to be the kernel. The integral equation thus obtained is then solved by an iteration method by first assuming an appropriate initial field. The iterative process has the physical meaning of simulating a round trip travel for the field, starting from the plane mirror and ending on itself. The solution of the equation is obtained when the result from the iterative process reaches a certain "steady state". The steady state is defined as the state when the solution from one iteration is repeating itself in the next
in all respects except a complex constant factor. The complex constant $f$ is the eigenvalue of the integral equation. Its absolute value square has the physical meaning of the fraction of energy remained after one round trip and its argument represents the phase shift (other than the geometrical one) introduced in a round trip. The absolute value of the eigenvalue can be obtained by comparing the absolute value of the field at any point across the plane mirror with the results of two successive iterations. In a similar manner, the phase shift can be obtained by comparing the phase (at any point across the mirror) with the two successive phase distributions. In the following, however, $|X|^2$ will be calculated from the ratio of the two successive integrations of intensity distributions across the plane mirror.

In section 5.2 the resonator with an optical axis which is not in the direction of grazing incidence with respect to the grating will be discussed. We intend to study the case where the singly diffracted ray is dominant. For this purpose, a 15° grating has been chosen because the result of the intensity distribution from the singly diffracted ray alone has already shown a good agreement with the experimental one (see previous chapter).

In section 5.3, the case of grazing incidence will be discussed. In the previous chapter, we mentioned that we could not determine adequately the intensity distribution of the diffracted order $N = 1$ for this angle of incidence. However, if the step width $b$ is sufficiently small when
compared to the wavelength and the width of the long side of the facet, its contribution to the resonant action in the grating resonator is negligible. Assuming this, we can replace the grating by a set of parallel strips which coincide with the long side of the facets. This set of parallel strips will be referred to as the parallel-strip-grating. The parallel-strip-grating resonator is studied by considering the contribution from all strips.

When replacing an echelette grating by parallel-strip-grating, we always have to keep in mind that the latter is a lossy structure because there is leakage behind the grating. Justification for making such a replacement depends entirely on how much loss occurs through leakage of the grating. When the loss is large, the solution (transverse mode with certain diffraction loss and phase shift) obtained is that of the parallel-strip-grating resonator rather than that of the echelette grating resonator, since the steady state mode pattern is actually determined by the total diffraction loss or vice versa. Estimating such a loss is equivalent to calculating the intensity distribution by using the diffraction coefficient of a half-plane. This was done in the previous chapter. It will be shown that the loss is not dependant upon the polarization of the field, because it is actually the sum of the energies of all possible diffracted orders (other than the blazed one) for both P- and S-polarizations.

The only interaction effect between the facets, the
d-correction, is considered in the case of 23½° grating resonator. The interaction between edges, because of the small width of the strip, is considered. These effects are comparatively small, as far as the resonant mode is concerned because they are of the order of $\sqrt{D}$, while D is usually large for a laser resonator. As for the other cases, the degree of importance of including certain interaction correction is similar to that already discussed in the previous chapter. Therefore, the intensity distribution of a particular grating which is being used as an end mirror in a resonator, should be the guide line for setting up the resonator integral equation.

In section 5.4, the kernel for a generalized grating resonator will be given. The grating in such a resonator consists of an arbitrary number of strips which are arranged in a reasonably arbitrary manner such that the interactions among the strips are small. Within the scope of the given kernel, a few cases of different types of resonators are studied.

5.2 15° grating resonator

5.2.1 Formulation

The optical axis of a grating resonator is the line which joins the centre of the plane mirror and the centre of the grating, i.e. the vertex of the $N_g/2 + 1$ th wedge; which also lies perpendicularly to the plane mirror and in the direction of the blazed diffracted order of the grating. For
the 15° grating resonator, the optical axis has an angle of
23.5° with respect to the grating normal. (see Fig. 5.1)
The separation D of the resonator is the length of the opti-
cal axis between the plane mirror and the grating. The
width of the plane mirror is 2C₀ and the width of the grating
projected onto the line parallel to the plane mirror is 2C₁.
The Fresnel number of the resonator is given by f:
\[ f = \frac{C₀C₁}{λD} \]  
(5.3)

To set up an integral equation for the resonator, we
first have to find the response from the grating when it is
illuminated by the field distribution \( u(x') \) on the plane
mirror. With the assumptions \( D >> C₀ >> λ \) we can apply the
Kirchoff-Fresnel formula to calculate the "incident" field
\( u_n(x_1) \) at each vertex of the wedges of the grating:
\[ u_n = \frac{e^{-i\frac{R}{2}}}{iNd} e^{ikR} u(x') \]  
(5.4)
where
\[ R^2 = (x' - a_n)^2 + D_n^2 \]
and \( a_n \) is the projection of the nth wedge onto the line which
passes through the centre of the grating and which is
parallel to the plane mirror. \( D_n \) is the perpendicular dis-
tance (with respect to the plane mirror) from the wedge to
the plane mirror. For large \( D \), and \( D/C₀ \gg 1 \)
\[ R = D_n + \frac{(C₀-x')^2}{2D_n} \]
where
\[ D_n = D - \left[ n - \left( \frac{n_0}{2} + 1 \right) \right] d \sin \theta_{inc} \]

From these and the coordinate transformation
\[ \eta' = \frac{x'}{c_0} \]

Eq. 5.4 becomes
\[ u_n = \sqrt{f_0} e^{-i \frac{\pi}{2}} \int_{-1}^{+1} e^{i k D_n + i n f_n \left( \frac{c_0}{c_0} - \eta' \right)^2} u(\eta') d\eta' \]

where
\[ f_0 = \frac{c_0}{\lambda D}, \quad f_n = \frac{c_0}{\lambda D_n} \]

To obtain the diffracted far field from each wedge, we apply Eq. 2.8
\[ U_D = \frac{e^{ik r_n}}{\sqrt{r_n}} D(\phi', \varphi, \frac{3\pi}{2}) u_n \]  \( (5.6) \)

The total diffracted field at point x on the plane mirror will then be
\[ U_{total} = \sum_n \frac{e^{ik r_n}}{\sqrt{r_n}} D(\phi', \varphi, \frac{3\pi}{2}) u_n \]  \( (5.7) \)

Rewriting it, we obtain
\[ U_{total}(x) = \int_{-1}^{+1} u(\eta') \sum_n \frac{e^{ik r_n}}{\sqrt{r_n}} D(\phi', \varphi, \frac{3\pi}{2}) e^{i k D_n + i n f_n \left( \frac{c_0}{c_0} - \eta' \right)^2} d\eta' \]

The resonator integral equation can then be set up by letting
\[ u(x) \] reproduce itself up to a complex constant \( \gamma \). Therefore
we have

\[ u(\eta) = \gamma \int_{-1}^{+1} K(\eta, \eta') u(\eta') d\eta' \quad (5.8) \]

where \( K(\eta, \eta') \) is given by

\[ K(\eta, \eta') = \sum_n \frac{e^{i k r_n}}{\sqrt{r_n}} D(\eta', \varphi, \frac{\pi}{2}) e^{i k D_n + i n f_n (\frac{\pi}{2} - \eta')} \quad (5.9) \]

where \( r_n = r_{k, l} - \left( n - \left( \frac{N}{L} + 1 \right) \right) D \cos (\varphi + \frac{\pi}{2} - \psi) \)

\[ r_{k, l} = D \cos (\varphi + \theta_{inc} - \psi) \]

5.2.2 Numerical results - Dominant mode

For the dominant mode in the resonator, a uniform field distribution across the plane mirror is chosen to be the initial field for the integral equation. We can set \( \text{Re} \ u(\eta') = 1.0 \) and \( \text{Im} \ u(\eta') = 0.0 \). Simpson's rule is applied for the integration and the net of the integration is chosen to be 51 points spreading across \(-1 \leq \eta' \leq +1\). The results of the intensity distribution across the plane mirror of the first three iterations are shown in Fig. 5.2. The result of the first iteration is actually the intensity distribution of the \( N = 1 \) order for the plane wave incidence. We can see from the third iteration that the two side peaks which appear in the first iteration have already been smoothed
Fig. 5.2 Dominant mode for the 15° grating resonator
out by the resonator action. The transverse mode seems to settle down quite readily after 10 iterations. The eigenvalue $\gamma^i$ fluctuates within 0.001 after 30 iterations.

Fig. 5.3 shows the results (intensity distribution, phase distribution, and the phase difference from previous iterations) at the 41th iteration. Phase differences across the plane mirror from iteration to iteration are not uniform, and have a minimum around the centre of the plane mirror. The minimum is of the order of $\pm 0.01$ radians for an approximate angular range of 0.02 radians, i.e. approximately a half-width of the peak of the dominant mode. The phase difference plot has two maxima near the edge of the plane mirror. They are of the order of $\pm 0.3$ radians with the smaller one appearing at the lower end and the larger one at the upper end of the mirror. This indicates that there is a complicated structure of waves travelling at different speeds with different angles in the range subtended by the plane mirror. Near the centre of the plane mirror (optical axis), where the phase shift introduced from iteration to iteration is practically negligible, we can assume that the waves which have the maximum constructive interference near the optical axis are actually travelling with the velocity of light.

The above discussion is true for both P- and S-polarizations. The difference between these two polarizations is the diffraction loss per round trip. For S-polarization, the eigenvalue $|\gamma|^i$ is 0.754 while for P-polarization it is
Fig. 7.1 Relative Intensity & Phase Distributions and Phase Difference for the 15th Grating Resonator
0.465. Therefore, the diffraction loss per round trip for S-polarization is 24.6% and for P-polarization, it is 53.5%. These values are larger than those for \( N = 0 \) order for the grating only because there is additional diffraction loss produced because of the finite size of the mirrors. This diffraction loss, obtained by subtracting from the above figures the diffraction loss for \( N = 0 \) for the grating alone, is 9% for the S-polarization and 5.5% for the P-polarization. The diffraction loss per transit for a parallel strip resonator of Fresnel number of unity is about 8% (Fox and Li's figure). Perhaps this can be explained by the fact that there is a stronger imaging process (see section 4.4) for P-polarization than for S-polarization and a resonator or the resonance action in a resonator is essentially a repeating imaging process between the mirrors with respect to each other.

5.2.2 \( \text{TEM}_1 \) - mode - Possible transverse mode discrimination

In order to induce a steady state solution other than the dominant mode a sine wave with zero at the centre of the strip as an initial field distribution is used. Iterating the integral equation as before, we found that after 40 iterations, the result of every iteration is that of the dominant mode. From this we can draw two possible conclusions: (i) there is discrimination against the \( \text{TEM}_1 \) mode for the possible physical reason of the high intensity concentration at the centre of the plane mirror due to the multiple diffraction action of the grating: (ii) the iterative procedure
fails for this case as it does for normal modes higher than TEM₁ mode in a plane-plane resonator. If the first case holds true, the use of a grating resonator is of great advantage because it discriminates not only with respect to wavelengths and polarizations, but also with respect to different transverse modes.

5.3 Parallel-strip-grating resonator

5.3.1 Formulation - Non-interacting term

To find the non-interacting kernel for the integral equation of a parallel-strip-grating resonator, we first have to calculate the field at any point x on one of the strips n, i.e. (see Fig. 5.4)

$$u_n(x) = \frac{e^{-ikx}}{\sqrt{\lambda D}} \int_{-\infty}^{\infty} e^{ikR} u(x') dx'$$

(5.10)

where

$$R = (x' - x_n)^2 + D_n^2$$

The field $\bar{U}(x)$ due to the nth strip at point x on the plane mirror is thus:

$$\bar{U}(x) = \frac{e^{-ikx}}{\sqrt{\lambda D}} \int_{n\text{th strip}} e^{ikr} u_n(x) dx,$$

(5.11)

where

$$r^2 = (x - x_n)^2 + D_n^2$$

$$D_n = D - nb$$

$$n = -N_{b/2}, -N_{b/2} - 1, \ldots, N_{b/2} - 1$$
Fig. 5.4 Parallel-strip-grating resonator
Combine the two equations (Eq. 5.10 and Eq. 5.11)

\[ \tilde{u}(x) = \frac{e^{-ikR}}{\lambda D} \int \int e^{ikR + ikR'} u(x', x) \, dx \, dx' \]  

(5.12)

and make use of the assumptions \( \frac{D_0}{C_0} \gg 1 \), \( \frac{D_0}{C_1} \gg 1 \),

\[ R + r = 2D - 2n b + \frac{n b^2}{D} + \frac{(v' - v)^2}{4D} + \frac{1}{D} \left[ x_1 - \frac{x_1' x_2}{2} \right] \]

and

\[ \tilde{u}(x) = \frac{e^{-ikD - 2ikn b + ikn b^2}}{\sqrt{\lambda D}} e^{ikn b^2} \int \int e^{ik(n b^2) - C_0} \frac{e^{ik(n b^2) - C_1}}{\sqrt{\lambda D}} d x \, d x' \]

(5.13)

By summing the contributions from all the strips and requiring that the field reproduces itself except for a complex constant \( \gamma \), we obtain the following integral equation:

\[ u(\eta) = \gamma \int_{-1}^{+1} K_0(\eta, \eta') u(\eta') d \eta' \]

(5.14)

where

\[ K_0(\eta, \eta') = \frac{\sqrt{2}}{2} \frac{1}{\sqrt{10}} e^{2ikD + \frac{i\pi}{2} f(\eta - \eta') \frac{\eta_0}{\eta_1} - \frac{1}{2} \eta_1^2} e^{ikn b + \pi \eta_1^2} \left[ F(\eta_{01}) - F(\eta_{10}) \right] \]

(5.15)

with \( f_0 = b \frac{b}{D} \), \( \eta_0 = b \frac{\sqrt{2}}{C_0} = \frac{1}{\sqrt{2}} \),

\[ F(\eta) = \int_0^\eta e^{i\pi f_0} d \eta \]

\[ \eta_{01} = 2\sqrt{f_0} (\eta_0 n a - \frac{1}{2} \eta_1 (\eta' + \eta)) \]
5.3.2 Numerical results

The above equation is used to study a 28 strip grating resonator with the following parameters:

\[ f_0 = 1.137, \ \lambda = 0.4\text{cm}, \ \sigma = 6.44\text{cm}, \ D = 228\lambda, \ a = 1.15\lambda, \]

\[ b = 0.5\lambda, \ d = 1.254\lambda, \ \psi = 23^\circ \]

The transverse dominant mode settles down after 10 iterations. The mode pattern is shown in Fig. 5.5. The eigenvalue and the phase shift are listed in the following table along with those of the infinite strip resonator of Fresnel number 1:

<table>
<thead>
<tr>
<th>Solution for</th>
<th>No. of iteration</th>
<th>Round trip eigenvalue</th>
<th>Phase shift per round trip</th>
</tr>
</thead>
<tbody>
<tr>
<td>28 strip resonator</td>
<td>35</td>
<td>0.716</td>
<td>0.14° (lagging)</td>
</tr>
<tr>
<td>[ f_0 = 1.137 ]</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Plane-plane resonator</td>
<td>22</td>
<td>0.846</td>
<td>15.6° (leading)</td>
</tr>
<tr>
<td>[ f_0 = 1.0 ]</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The phase shift for the grating resonator shows a small degree of lagging (relative to the geometrical phase shift) which leads to the possibility of designing a resonator in which the mode has an effective phase velocity less than the speed of light. This property is useful in a Fabry-Perot radiator. The diffraction loss per round trip for the grating resonator is about 13% higher than that of a plane-plane resonator of equivalent Fresnel number. This is attributed to the lossy nature of the grating structure.

The above results are identical for the two polarizations. In order to estimate the correct diffraction losses
Fig. 5.5 Dominant mode for parallel-strip-grating resonator
for P- and S-polarizations in an echelette grating resonator
with similar parameters, we have to calculate the leakage
behind the parallel-strip-grating. In other words, we have
to utilize the results from the intensity distribution cal-
culation of \( N = 0 \) by using the half-plane diffraction
coefficient.

5.3.3 The leakage behind the parallel-strip-grating

To show that the loss behind the grating does not
depend on the polarization we note that summing up the edge
fields \( A \) or \( B \) (see Fig. 5.6) will give the interference
maxima in the same direction. However, the strength of the
equivalent line sources at edge \( A \) and \( B \) depends on the dif-
fractive coefficient \( D(\phi', \phi, 2\pi) \). The far field due to edge
\( A \) or edge \( B \) is given by:

\[
U_A = \frac{e^{ikr_A}}{\sqrt{r_A}} D(0, \phi_A, 2\pi)
\]

\[
U_B = \frac{e^{ikr_B}}{\sqrt{r_B}} D(0, \phi_B, 2\pi)
\]

(5.16)

where

\[
D(0, \phi, 2\pi) = \frac{e^{i\frac{\psi}{2}}}{2 \sin k} \left( \frac{1}{\omega_s \sin \frac{\phi}{2}} - \frac{1}{\omega_p \sin \frac{\phi}{2}} \right)
\]

+ for P- polarization

\[
- \text{ for S- polarization}
\]

For the far field, \( r_A \) and \( r_B \) are approximately the same and
so do \( \phi_A \) and \( 2\pi - \phi_B \). In order to show that the loss behind
the grating is the same for P- and S-polarizations, we need
Fig. 5.6 The leakage behind the parallel-strip-grating
to prove
\[ u_A^P = u_B^S \quad \text{or} \quad u_A^S = u_B^P \quad (5.17) \]

Therefore
\[ u_A^P + u_B^P = u_A^S + u_B^S \quad (5.17a) \]

To do this we need only to show that
\[ D^S(\varphi_A, 0, 2\pi) = D^P(\varphi_B, 0, 2\pi) \]
or
\[ D^P(\varphi_A, 0, 2\pi) = D^S(\varphi_B, 0, 2\pi) \quad (5.18) \]

This is actually the case because \( \varphi_A = 2\pi - \varphi_B \).

The loss in front of the grating in the direction of \( N = 0 \) is equal to the loss behind the grating. We have calculated \( U^o \) for 23\(^{\circ}\) grating using the half-plane diffraction coefficient to be 10.37 and 1.61 units for P- and S-polarization respectively. Therefore, the total energy loss in front of and behind the parallel-strip-grating is 2 \times (10.37 + 1.61) while for the echelette grating, it is only 10.37 units for P-polarization and 1.61 units for S-polarization.

5.3.4 Estimate of the eigenvalue for a 23\(^{\circ}\) echelette grating resonator

The percentage loss for the P- and S-polarizations of the total loss for the parallel-strip-grating is 86.6% and 13.4% respectively. If we assume that the additional loss of 13% in the eigenvalue (see section 5.3.2) is the total loss due to the leakage, we should return 93.3% of the 13% for the
S-polarization and 6.7% of the 13% for the P-polarization after allowing \( N = 0 \) for the echelette grating. This is equivalent to increasing the eigenvalue \( |\gamma|^2 \) obtained from the integral equation by 0.0087 for P-polarization and by 0.1213 for S-polarization. In this manner, the eigenvalue for S-polarization is 0.837, which is now comparable to the value for the equivalent plane-plane resonator. For P-polarization, the eigenvalue turns out to be 0.7247.

The above argument is only an estimation of the eigenvalue from a physical point of view. In an actual case, the transverse mode pattern will be influenced by the actual loss due to the leakage of the grating. It is suspected that the dip near the centre of the mirror in the transverse mode pattern plot (see Fig. 5.5) for the dominant mode is produced by the leakage. This suspicion is substantiated by further studies of two more cases with different \( b/a \) as shown in later sections. The deeper dip corresponds to the case of larger \( b/a \), i.e. larger leakage.

5.3.5 Interaction terms

The interaction between the facets in a resonator can be calculated in a manner similar to that of the previous chapter with the exception that the uniform incident plane wave is replaced by the field, as calculated by the Kirchhoff-Fresnel formula from the field distribution of the plane mirror. We consider only the double diffracted ray in the \( d \)-correction and the edge interaction as discussed in
Chapter III. The resultant kernels from the d-correction \( K_2 \) and from the edge interaction \( K_3 \) are given by:

\[
K_2^p(\eta, \eta') = B_2 \sum_{n=1}^{N_{\eta}} e^{i k D_n} \left[ e^{i \pi \eta_n} \left( \frac{C_{\eta}}{C_{\eta}} - \eta^2 \right) + e^{i \pi \eta_n} \left( \frac{C_{\eta}}{C_{\eta}} - \eta'^2 \right) + e^{i \pi \eta_n} \left( \frac{C_{\eta}}{C_{\eta}} - \eta^2 \right) \right]
\]

\[
B_2 = \frac{\sqrt{A}}{2 \sqrt{\pi \lambda}} \sqrt{D} \left\{ \frac{1}{\eta n - \lambda} \right\} e^{i \pi n}
\]

\[
D(\theta_n) = 2 \sqrt{\pi \kappa} D(\theta, \theta_n, \lambda)
\]

\[
K_2^s(\eta, \eta') \sim O \left( \left( \frac{A}{\lambda} \right)^{\frac{1}{4}} \sqrt{D} \right)
\]

\[
K_3^{pors}(\eta, \eta') = B_3 \sum_{n=1}^{N_{\eta}} e^{i k D_n} \left[ e^{i \pi \eta_n} \left( \frac{C_{\eta}}{C_{\eta}} - \eta^2 \right) + e^{i \pi \eta_n} \left( \frac{C_{\eta}}{C_{\eta}} - \eta'^2 \right) + e^{i \pi \eta_n} \left( \frac{C_{\eta}}{C_{\eta}} - \eta^2 \right) \right]
\]

\[
B_3 = \frac{\sqrt{A}}{2 \sqrt{\pi \lambda}} \sqrt{D} \cdot \frac{1}{\eta n - \lambda} e^{i \pi n}
\]

\[
D^{pors}(\nu, \nu') = \sqrt{\pi \kappa} D(\nu, \nu, \frac{\pi n}{2})
\]

The above expressions for the kernel include a factor of \( \frac{1}{\sqrt{D}} \) which is small for resonators of large separation. We therefore anticipate a small effect from these contributions for \( D/\lambda >> 1 \). By including only the d-correction in the kernel along with the non-interacting term, the resultant eigenvalue
(after following the procedure discussed in section 5.3.2) does not differ by more than 0.0002 and the phase shift is the same as before when no interaction is considered. By also including the edge interaction, the eigenvalue differs by not more than 0.03 and the phase difference from iteration to iteration is not uniform across the plane mirror. The non-uniformity is slight. In other words, we can consider this as a small perturbation.

However, as the separation is decreased, there are some significant changes when the interactions are considered. The following is an example of a 14 facet parallel-strip-grating resonator with Fresnel number 0.56:

\[ f_o = 0.56, \lambda = 0.4\text{cm}, C_o = 3.22\text{cm}, D = 115.25\lambda, a = 1.15\lambda, \]
\[ b = 0.5\lambda, d = 1.254\lambda, \gamma = 23.5\degree \]

The transverse dominant mode pattern is the same, for all cases, as the one for the plane-plane resonator of same Fresnel number (see Fig. 5.5). The eigenvalue and phase shift are listed in the following table:

<table>
<thead>
<tr>
<th>Solution from iteration</th>
<th>No. of iteration</th>
<th>Round trip eigenvalue</th>
<th>Phase shift per round trip</th>
</tr>
</thead>
<tbody>
<tr>
<td>plane-plane resonator</td>
<td>15</td>
<td>0.723</td>
<td></td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>0.655</td>
<td></td>
</tr>
<tr>
<td>( K_o )</td>
<td>more than Average</td>
<td>0.653</td>
<td>Average</td>
</tr>
<tr>
<td>( K_o + K_3 )</td>
<td>55</td>
<td>0.653</td>
<td>0.655</td>
</tr>
<tr>
<td>( K_o + K_2 + K_3 )</td>
<td>more than Average</td>
<td>0.615</td>
<td>Average</td>
</tr>
</tbody>
</table>
From the above table, we can see that when the interactions are considered, the eigenvalues and phase shifts are given in the average sense because the transverse mode does not repeat itself every round trip but in 3 round trips for the case of $K_0 + K_3$ and in 7 round trips for the case of $K_0 + K_2 + K_3$. The eigenvalue $|\psi|^2$ is different for each round trip and so is the phase shift. In the case of $K_0 + K_2 + K_3$ the phase distribution is no longer uniform across the plane mirror. It changes from round trip to round trip indicating that the wave fronts are changing shape periodically.

These phenomena also occur when $K_0$ is applied alone for cases when $b \neq m \sqrt{2}$, where $m$ is an integer. This will be discussed in a later section.

5.4 Generalized strip grating resonator

5.4.1 Formulation

As we have seen from previous sections, the interaction effect from neighbouring facets or neighbouring edges may be small in a grating resonator. The resonator properties can be studied by summing up contributions from all facets. For this reason, we will consider the "generalized" strip grating resonator, in which the grating consists of an arbitrary number of strips of arbitrary size arranged in an arbitrary manner such that the interaction among the strips is small. Proceeding in a similar manner as in the formulation of the
parallel-strip-grating resonator, we obtain the kernel for the generalized strip grating resonator: (see Fig. 5.7)

\[
K(\eta, \eta') = \sum_{n=0}^{N-1} \frac{2ik(D_n - \sin \alpha_n a_n - \frac{1}{2} D_n \sin^2 \alpha_n) + ik \sin \alpha_n C_0 (\eta + \eta')}{\sin \alpha_n}
\]

\[
e^{-\frac{\pi}{2} [2g_n (\eta' + \eta)^2 - (\eta + \eta')^2]}
\]

\[
\int \frac{d \eta'}{C_0 + \frac{D_n}{C_0} \sin \alpha_n}
\]

where

\[
\tilde{\eta}' = \frac{x'}{C_0} + \frac{D_n}{C_0} \sin \alpha_n;
\]

\[
g_n = 1 - \frac{D_n}{p}
\]

Note that the kernel also provides the possibility of replacing the plane mirror by a concave mirror with a radius of curvature \(p\). In fact the case of the plane mirror is a special case when \(p \to \infty\). The parallel-strip-grating resonator is naturally a special case of a generalized resonator when \(\alpha_n = 0; b_n = \text{constant}, \text{ and } a_n = \text{constant} \). In the following section we present two more cases of parallel-strip-grating resonators with different \(b/a\).

5.4.2 Parallel-strip-grating resonators
Fig. 5.7 Generalized strip grating resonator
To study the leakage as a function of the widths of the facet, we include the numerical results from two cases:

(both of \( N_g = 14 \))

(i) \( b/a = 0.1925 \)

\[ f_o = 1.998, \lambda = 0.4\text{cm}, C_o = 7.274\text{cm}, D = 165.5\lambda, a = 2.598\lambda, \]

\[ b = 0.5\lambda, d = 2.6457\lambda, \psi = 11^\circ, N_B = 1 \]

Number of possible diffracted order is 6: \( N = 3,2,1,0,-1,-2. \)

(ii) \( b/a = 1.4434 \)

\[ f_o = 1.998, \lambda = 0.4\text{cm}, C_o = 7.274\text{cm}, D = 1.65.5\lambda, a = 2.598\lambda, \]

\[ b = 1.5\lambda, d = 3.0\lambda, \psi = 30^\circ, N_B = 3 \]

Number of possible diffracted order is 6: \( N = 4,3,2,1,0,-1. \)

The transverse mode pattern of the above two cases are shown in Fig. 5.8. The eigenvalue and the phase shift are listed in the following table:

<table>
<thead>
<tr>
<th>Solution for</th>
<th>No. of iteration</th>
<th>Round trip eigenvalue</th>
<th>Round trip Phase shift</th>
</tr>
</thead>
<tbody>
<tr>
<td>Plane-plane resonator</td>
<td>15</td>
<td>0.9373</td>
<td>3.87°</td>
</tr>
<tr>
<td>case (i)</td>
<td>15</td>
<td>0.8970</td>
<td>3.90°</td>
</tr>
<tr>
<td>case (ii)</td>
<td>40</td>
<td>0.3705</td>
<td>3.5°</td>
</tr>
</tbody>
</table>

We can see that ripples appear on both sides of the peak of the dominant mode for the plane-plane resonator which become more pronounced in the case (i) of a parallel-strip-grating resonator of the same Fresnel number. The two ripples are asymmetrical with respect to the centre of the plane mirror because the grating is not positioned
Fig. 5.8 Dominant mode for parallel-strip-grating resonator
symmetrically with respect to the optical axis of the resonator. In case (ii) the mode pattern changes quite drastically with a higher field near the edges of the plane mirror and therefore a higher loss is shown in the eigenvalue.

5.4.3 Three parallel strip resonator

The three parallel strip resonator has an arrangement as shown in Fig. 5.9. This type of resonator is similar to the resonator with rimmed mirrors\textsuperscript{51}. Three cases with steps $b = 0.3\lambda$, $0.5\lambda$, and $0.6\lambda$ are considered. The remaining parameters are listed below:

$$f = 2.0, \lambda = 0.01\text{cm}, a_0 = -C_1 = -1.4142\text{cm}, a_1 = -0.9051\text{cm},$$

$$a_2 = 0.9051, a_3 = C_1 = 1.4142, D_1 = 1000\lambda, C_0 = 1.4142$$

The transverse dominant mode and $TEM_1$ mode for the three cases are plotted in Fig. 5.10 and 5.11. The eigenvalues and phase shifts are listed in the following table: (obtained after 40 iterations)

<table>
<thead>
<tr>
<th>Solution for</th>
<th>Dominant mode</th>
<th>TEM$1$ mode</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Round trip eigenvalue</td>
<td>Round trip eigenvalue</td>
</tr>
<tr>
<td>Plane-plane resonator</td>
<td>0.9351</td>
<td>0.7737</td>
</tr>
<tr>
<td>The case with $b=0.5\lambda$</td>
<td>0.9352</td>
<td>0.7736</td>
</tr>
<tr>
<td>The case with $b=0.6\lambda$</td>
<td>0.9568</td>
<td>0.8367</td>
</tr>
<tr>
<td>The case with $b=0.3\lambda$</td>
<td>0.7569</td>
<td>0.4747</td>
</tr>
</tbody>
</table>
Fig. 5.9 Three strip resonator
Fig. 5.10 Dominant mode for three strip resonator
Fig. 5.11 TEM₁ mode for the three strip resonator
From the above table, we can see that the results for the case with \( b = \frac{1}{2} \lambda \) do not differ significantly from those of a plane-plane resonator with the same Fresnel number. In addition to the fact that the diffraction loss per round trip is smaller with a larger step (compare the results for the case of \( b = 0.6 \lambda \) and \( b = 0.3 \lambda \)), there is an indication that the interference of the three strips plays a greater role than the leakage through the steps. Note that the increase of diffraction loss for the TEM\(_1\) mode in the case when \( b = 0.3 \lambda \) is much larger than that for the dominant mode. (30% and 18% respectively). Therefore, this type of resonator can be used to discriminate against the TEM\(_1\) mode.

5.4.4 Parallel-strip-grating resonator with step width \( b \neq m\frac{1}{2} \lambda \)

To study a grating resonator with step width not equal to integral number of half-wavelength, we consider the following example:

\[
N_G = 4
\]

\[
f = 2.0, \lambda = 0.01cm, C_0 = C_1 = 2.236cm, D = 25000 \lambda,
\]

\[
a = 111.8 \lambda, b = 0.6 \lambda
\]

A total of 135 iterations are completed. The transverse mode patterns shown in Fig. 5.12 are repeated in sequence with eigenvalues \( |W| \) range from 0.4258 to 0.7600. The phase distribution is different in each iteration and also repeats itself in sequence. The phase difference from iteration to iteration is not uniform across the plane mirror. The reason
*39th or 129th iteration
**40th or 130th iteration
***41st or 131st iteration

Fig. 5.12a Transverse mode for parallel-strip-grating resonator when $b = 0.6\lambda$
*42nd or 132nd iteration
**43rd or 133rd iteration
***44th or 134th iteration

Fig. 5.12b Transverse mode for parallel-strip-grating resonator when $b = 0.6\lambda$
for such a behavior is perhaps due to the step being $0.1\lambda$ off the $\frac{1}{2}\lambda$ mark. Five round trips are required to make up a $\frac{1}{2}\lambda$.

5.4.5 Parallel-strip-grating resonator with step width $b = 0.52$

The type of grating resonator considered here is one with a grating which consists of strips with a width and a separation $d$ (centre to centre of the neighbouring strips) lying in the same plane (see Fig. 5.13). Two cases with the same Fresnel number $f$ and different $N_G$, $a$ and $d$ are considered:

Case (i) $N_G = 5$, $f = 2.0$, $\lambda = 0.01\text{cm}$

$a = 52.56\lambda$, $d = 57.56\lambda$, $c_o = c_i = 1.4142\text{cm}$, $D = 100\text{cm}$

Case (ii) $N_G = 10$, $f = 2.0$, $\lambda = 0.01\text{cm}$

$a = 26.29\lambda$, $d = 31.29\lambda$, $c_o = c_i = 1.4142\text{cm}$, $D = 100\text{cm}$

The transverse dominant mode and TEM$_1$ mode for the two cases are plotted in Fig. 5.14. The eigenvalues and phase shifts are listed in the following table: (obtained after 30 iterations)

<table>
<thead>
<tr>
<th>Solution for Plane-plane resonator</th>
<th>Dominant mode eigenvalue</th>
<th>Round trip phase shift (deg)</th>
<th>TEM$_1$ mode eigenvalue</th>
<th>Round trip phase shift (deg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case (i)</td>
<td>0.9351</td>
<td>8.8</td>
<td>0.7737</td>
<td>34.6</td>
</tr>
<tr>
<td>Case (ii)</td>
<td>0.7815</td>
<td>9.1</td>
<td>0.6583</td>
<td>33.5</td>
</tr>
<tr>
<td></td>
<td>0.6399</td>
<td>8.8</td>
<td>0.5335</td>
<td>34.0</td>
</tr>
</tbody>
</table>

From the above table, we can see that resonators of this type are quite lossy. For a smaller strip width $a$, there is a larger diffraction loss. Also, the dominant mode seems to suffer more severe losses than the TEM$_1$ mode.
Fig. 5.13 Strip-grating resonator
Fig. 14a Dominant mode for strip-grating resonator
Fig. 14b TEM_1 mode for strip-grating resonator
CHAPTER VI

CONCLUSIONS

In the present work, the geometrical theory of diffraction or ray optics approach was used to study: (i) the intensity distribution of an echelette grating when it is illuminated by a uniform plane wave; (ii) a Fabry-Perot resonator with a grating, either an echelette grating or a strip grating, as an end mirror.

The present use of ray optics approach was motivated by its physical plausibility, mathematical simplicity and its success in many other problems. Since our prime objective was to study the normal modes in a Fabry-Perot resonator, the first part of the work was centred on the intensity distribution of the propagating diffracted orders, rather than the diffraction problem of an echelette grating as a whole. This suggests that further work could be done along this line especially the details of the Wood's anomalies.

The geometrical theory of diffraction was used for the first time here to formulate the infinite strip resonator problem. This new formulation was motivated by the needs of removing the conditions of large mirror size and large mirror separation imposed by the Fox and Li formulation. It was
proved that when these conditions were imposed in the new formulation, Fox and Li's integral equation was obtained. This method is required in the study of infinite strip resonators of small Fresnel numbers and in the study of the generalized grating resonator which contains strips whose widths are comparable to the wavelength. The results of a set of infinite strip resonators with Fresnel numbers ranging 0.56 to 0.01 were given.

The intensity distribution of an echelette grating was calculated from the sum of four basic types of diffracted rays; namely, the singly diffracted ray, the doubly diffracted rays of $d-$, $a-$, and $b$-corrections. The first type is the primary diffracted waves, while the others are secondary diffracted waves because of the interaction between scattering wedges. To test the method, we compared our calculated results of the intensity distribution of a chosen set of echelette gratings with the experimental results made available by Brannen and Rumbold. The agreement was excellent for all cases in S-polarization except for the case of $23\frac{1}{2}^\circ$ grating, in which the incident field grazes along the side facet. The agreement for cases in P-polarization was fair. A possible reason for this different degree of agreement for P- and S-polarization was explained. Further work may be done to improve on these results.

The Fabry-Perot resonator with an echelette grating is in common use now in Far-infrared lasers. The present work established the existence of a normal mode in such a cavity
from the theoretical point of view. The eigenvalue of the integral equation of such a resonator is closely related to the percentage intensity of the particular diffracted order which we have chosen for resonant action and to the diffraction loss due to the finite size of the plane mirror. Therefore, the accuracy of the eigenvalue will depend on how accurately we can determine the intensity distribution of the grating alone. The case which we have studied, indicated that TEM\textsubscript{1} mode in such a cavity is not obtainable by the same iteration procedure as that used to obtain the TEM\textsubscript{1} mode for a plane-plane resonator. This shows that either the iterative method fails to obtain even the lowest odd symmetric mode or that physically there is discrimination against this particular mode. Other types of gratings in the resonant cavity were also studied, in particular a lossy type of diffraction grating - parallel strip grating. This led to the study of the generalized strip grating resonator where the grating consists of strips arranged in an arbitrary manner.

In summary, the grating resonator problem has been solved and polarization and intensity diffraction of gratings with deep grooves analyzed by using the geometrical theory of diffraction.
REFERENCES


APPENDIX I

DIFFRACTION OF AN APERTURE BY

THE KIRCHHOFF-FRESNEL FORMULA

According to the Kirchhoff-Fresnel Formula\(^3\), the field at \( P \) due to the diffraction of the source at \( P' \) by the aperture \( A \) (see Fig. I), is given by:

\[
 u(P) = -\frac{i}{2\lambda} \frac{e^{ik_R}}{r_0} \iint_A \frac{e^{ik\theta}}{r} (1 + \cos \theta) \, dA
\]  \tag{I.1}

For \( D/A \gg 1 \), \( \cos \theta \approx 1 \), we have

\[
 u(P) = \frac{e^{-i\theta/2}}{\lambda} \iint_A \frac{e^{i(k_0 + k)R}}{r_0 R} \, dA
\]  \tag{I.2}

Since

\[
 r_0 = D + \frac{(x-x')^2}{2D} + \frac{(y-y')^2}{2D} \tag{I.3}
\]

\[
 R = D + \frac{(x-x)^2}{2D} + \frac{(y-y)^2}{2D}
\]

we have,

\[
 u(P) = \frac{e^{-i\theta/2}}{\lambda} \frac{e^{2ikD}}{D^2} \iint_A e^{\frac{ik}{2D} \left[ (x-x)^2 + (y-y')^2 + (x-x')^2 + (y-y)^2 \right]} \, dx, dy,
\]  \tag{I.4}
Fig. I Diffraction of an aperture
\[
\frac{e^{-i\hbar^2}}{\sqrt{D}} \int e^{i\frac{\hbar}{2D} \left[(x-x')^2 + (y-y')^2\right]} \frac{e^{-i\hbar^2}}{\sqrt{D}} \int e^{i\frac{\hbar}{2D} \left[(y-y')^2 + (y-y')^2\right]} \, dx, dy.
\]

(I.5)

Note that the D in Eqs. (I.4) and (I.5) are not the same. The former is a 3-dimensional length while the latter is a 2-dimensional length, or the projection of the 3-dimensional D on the x- or y-axis. Consider the distribution on the x-axis, from Eq. (I.5):

\[
\mu_x(x) = \frac{e^{-i\hbar^2}}{\sqrt{\lambda D}} e^{i\frac{\hbar}{2D} (x-x')^2} \int e^{i\frac{\hbar}{2D} \left[(x-x')^2 + (y-y')^2\right]} \, dx, \, dy.
\]

(I.6)

where

\[
\gamma' = \frac{x'}{\lambda}, \quad \gamma = \frac{x}{\lambda}, \quad \gamma' = \frac{x'}{\lambda}
\]

Eq. (I.6) can be rewritten as:

\[
\mu_x(x) = \sqrt{\lambda} \sqrt{\frac{2}{\hbar}} e^{-i\hbar^2} e^{i\frac{\hbar}{2D} \left(x-x'\right)^2} F \left(2; \gamma' \left(1 + \gamma' \gamma \right) \right) + F \left(2; \gamma' \left(1 - \gamma' \gamma \right) \right)
\]

(I.8)

For an arbitrary distribution \( \mu(x') \) on plane 1, we have on plane 2:
\[ u_{\lambda}(x) = \sqrt{\lambda} \left( e^{i\pi \varphi} \right)^{\frac{1}{2}} e^{2ik\varphi} \int_{-\varphi}^{\varphi} u(\lambda') e^{\frac{i}{2} \pi(\varphi - \varphi')^2} \cdot \right] \]

\[ F \left( \sqrt{F(2+\varphi' \varphi)} \right) \]

\[ + \left( \sqrt{F(2-\varphi' \varphi)} \right)^{\frac{3}{2}} \]

(1.9)

Note that the above expression does contain a factor of \( \sqrt{\lambda} \) in this consideration.