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The Newsvendor Problem with Pricing

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THE NEWSVENDOR PROBLEM WITH PRICING

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ABSTRACT

In the newsvendor problem with pricing, the seller of an inventory of homogeneous items attempts to maximize expected profit by setting price(s) and inventory level(s) before realizing exact demand for the items. The first step in solving the problem is to model this demand, which in the literature, is most commonly done using additive or multiplicative uncertainty, often without justification for either choice. From here, the problem is solved in a variety of ways.

In this document, a model for demand is derived from two basic quantities: size of customer base and distribution of reservation prices of the population, where a reservation price is the most a customer is willing to pay for an item. This demand model, incorporating price-dependent additive uncertainty, is used as a basis for investigating various scenarios of the newsvendor problem with pricing, including: both discrete inventory levels and respective normal approximations, deterministic and random demand, single and multiple-price strategies, and sales of only primary items or primary and secondary items together.

Where possible, analytical results are obtained from the various models. For example, in a two-price model, an expression of the optimal inventory level of items to be made available for sale at the higher price is derived. If analytical results are not available, numerical examples provide insights. For example, for both deterministic and random sizes of customer base, graphical analysis suggests that for a single-price strategy, the optimal price is the same regardless of size of customer base. In addition, optimal results found using expected profit functions derived here are compared with those optimal results found using other methods found in the literature.

Keywords: demand modeling, newsvendor, pricing, inventory.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

CHAPTER 1

INTRODUCTION

1.1 BACKGROUND

In 2010, the five largest airlines in the United States (Delta, American, U.S. Airways, Continental and United) received a total of $2.7 billion in revenue from baggage fees alone, up from $344 million just three years prior (Bureau of Transportation Statistics, U.S. Department of Transportation). Also in 2010, hotels in the U.S. recorded revenues of over $127 billion (Hotel Operating Statistics Study, 2010, STR Global). These two industries, offering products or services that expire (e.g.; a flight on a plane or an evening in a hotel) incur costs of making those products or services available for sale. The number of items ultimately sold depends on how many items are made available for sale in the first place, and how many customers are willing to buy them. These two issues (uncertain demand and a perishable product) form the basis of the newsvendor problem.

In the basic form of the newsvendor problem, a seller of an inventory of a homogeneous product attempts to maximize expected profit by deciding on an inventory quantity Q, before knowing exactly how many customers will want to buy an item at an exogenous price, P (that is, the seller faces stochastic demand). For each unit of inventory the seller decides to make available for sale, a constant marginal cost of c is incurred ($P > c$), regardless of whether or not the item is ultimately sold. Although the newsvendor problem has been expanded to include the possibility of salvage value for unsold items, here it is assumed that the items are worthless if not sold at full price. Denote the number of customers willing to pay price P for an item as random variable $X(P)$ with corresponding cumulative distribution function $F_{X(P)}(\cdot)$. Given these conditions, the profit-maximizing quantity, Q^*, is determined by the well-known fractile solution:

$$Q^*(P) = F_{X(P)}^{-1}\left(\frac{P-c}{P}\right) \tag{1.1.1}$$

where $F_{X(P)}^{-1}(\cdot)$ denotes the inverse cumulative distribution function of demand. For the case where Q is only allowed to take on discrete values, $Q^*(P)$ is the smallest integer value of Q that is greater than or equal to the right-hand side of (1.1.1).

In the basic newsvendor problem price is fixed at P, and the only decision variable is inventory quantity, Q. In the newsvendor problem with pricing (NPP), it is assumed that the seller also has price-setting ability (as in the case of a monopolist). Now the seller maximizes expected profit over two decision variables: inventory quantity, Q, and selling price, P. Note that setting inventory level Q and selling price P does not imply a guaranteed revenue of PQ as in traditional Economics (where the market determines exactly how many items, $Q(P)$, are sold at a give price P). Rather, since demand is uncertain there is a distribution of possible revenues, and here the seller makes decisions based on an expected outcome.

The first challenge in solving the NPP is modeling demand, where "…the demand function shows, in equation form, the relationship between the quantity sold of a good or service and one or more variables." (Samuelson and Marks, 2010). Denote a deterministic demand function (expected demand) as $\mu(P)$, which often takes on either a linear form:

$$\mu(P) = a - bP, \tag{1.1.2}$$

where *a*, *b* > 0, or an iso-elastic form:

$$\mu(P) = aP^{-b}, \tag{1.1.3}$$

where *a* > 0 and *b* > 1.

If historical data are available, simple analysis can suggest the appropriate form of $\mu(P)$.

See Figure 1.1.1 for an example of a linear demand curve (Equation 1.1.2), and Figure 1.1.2 for an example of an iso-elastic demand curve (Equation 1.1.3).

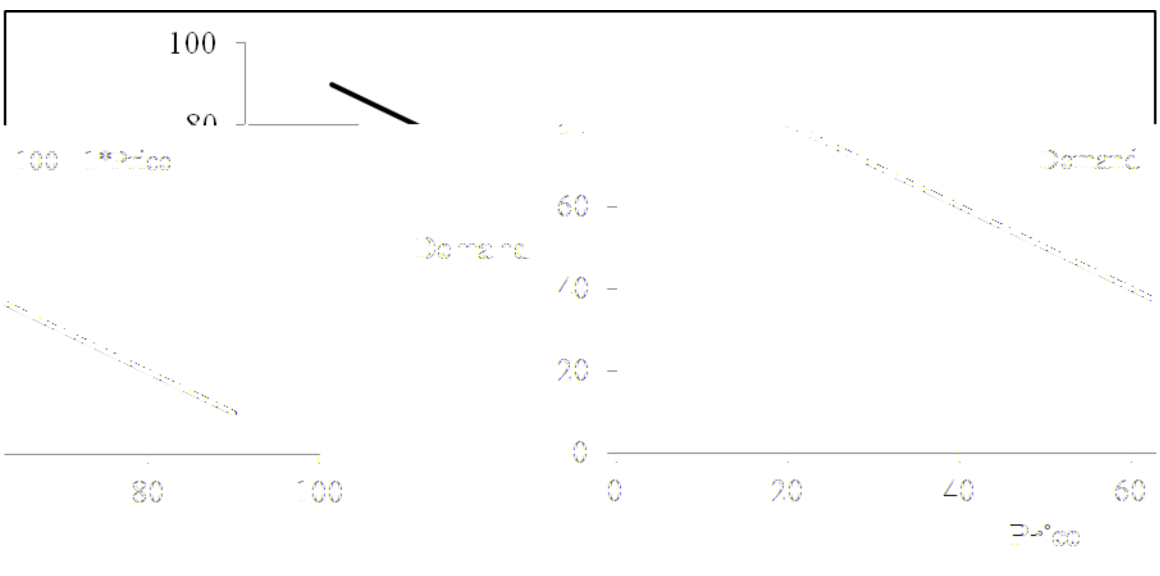

Figure 1.1.1. Linear Demand Curve (Demand = 100 – 1*Price).

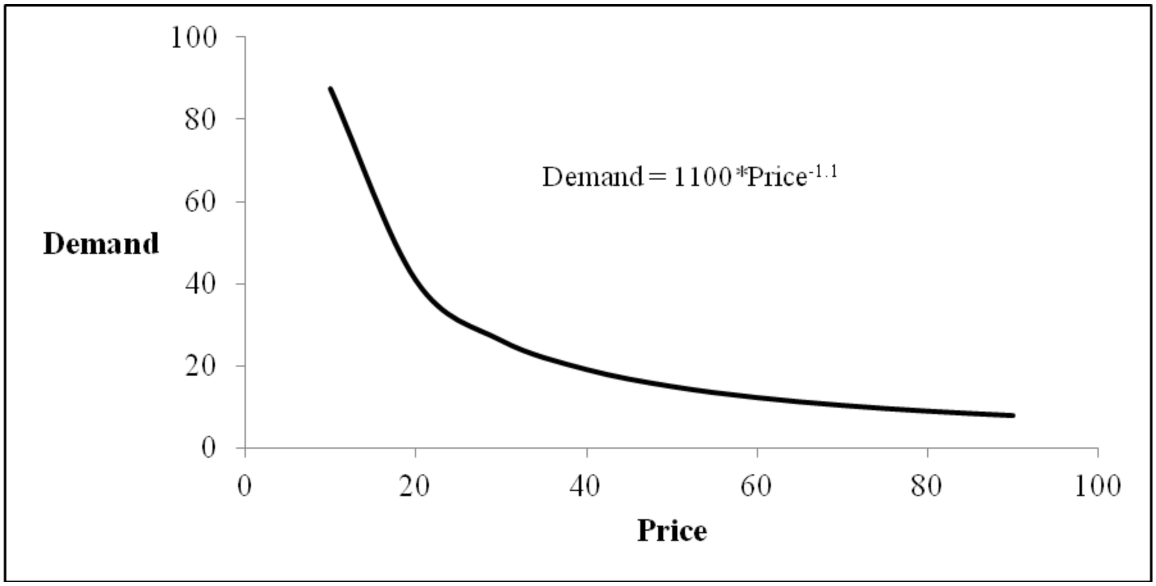

Figure 1.1.2. Iso-Elastic Demand Curve (Demand = 1100*Price$^{-1.1}$).

To these (or other) models of $\mu(P)$, uncertainty is incorporated, usually expressed in one of two forms: multiplicative or additive. To show this uncertainty, denote $X(P)$ as total demand, which is made up of a deterministic component, $\mu(P)$, and a stochastic component, ε. In the case of multiplicative uncertainty:

$$X(P) = \mu(P)\varepsilon\,, \qquad (1.1.4)$$

often where the expected value of ε is one. In the case of additive uncertainty,

$$X(P) = \mu(P) + \varepsilon \qquad (1.1.5)$$

often where the expected value of ε is zero.

Note that in (1.1.4) and (1.1.5), if ε is known the problem becomes deterministic. For any given P, quantity demanded can be determined exactly if the parameters and form of $\mu(P)$ are known. Note also that in most of the literature, the uncertainty term, ε, is typically independent of price, (for an exception, see Young, 1979).

Rarely is the decision to use either of (1.1.4) or (1.1.5) given any convincing justification. An exception is Agrawal and Seshadri (1999), in which a subjective explanation for why additive or multiplicative uncertainty is appropriate in difference situations is provided. In addition, the use of (1.1.5) is appropriate for the case of a linear regression model. Otherwise, the most common justification for choosing any of Equations 1.1.2 – 1.1.5 is for "mathematical tractability." In any case, once a demand model has been chosen, solving the newsvendor problem with pricing begins in a variety of ways, depending on the conditions. Two examples are Karlin and Carr (1962) and Petruzzi and Dada (1999).

1.2 ORGANIZATION OF THE DISSERTATION

The dissertation is organized as follows. Chapter 2 contains a literature reviews on works related to consumer choice models and the newsvendor problem with pricing.

Chapter 3 addresses the issue of modeling demand, with a derivation of a probability-based model and investigation into its properties. The model is compared with existing models from the literature, with respect to forms of uncertainty.

Chapter 4 includes the derivation of a single-price expected profit models using the demand model of Chapter 3. Cases of both deterministic and stochastic customer bases are considered, and examples are provided.

The work in Chapter 5 extends the single-price strategy of Chapter 4 to dual-price and *n*-price strategies. These approaches would be useful for a seller who chooses to make homogeneous products available at different prices. An example is a seller who advertises a few items of inventory available at a low price (to draw customers to the store), and then makes the remaining inventory available at a higher price. Discrete models and their normal approximations are provided. Numerical examples are provided that compare the optimal solutions found using models derived here with those found using models in the literature, that sometimes are derived with incomplete justification.

Chapter 6 contains work that investigates properties of pricing strategies when the seller faces small populations, for both deterministic and stochastic customer bases. Specifically, it is shown whether multiple-price strategies are ever preferable over single-price strategies for four specific cases.

Chapter 7 includes the derivation of an expected profit function for a seller who can realize revenues from sales of both primary and secondary items. A sensitivity analysis is provided using numerical examples. Insight is also provided based on functions derived in the chapter to explain trends in expected profit and variance.

Chapter 8 contains a summary of the dissertation, and comments on future research.

1.3 NOTATION AND ABBREVIATIONS

Shown here are abbreviations and notation used throughout the dissertation. Included for each entry is the page number at which it first appears.

a	Parameter of a function (p.2). Also denotes fraction of leftover customers (p. 120).
b	Parameter of a function (p.2).
c	Cost of making one item available for sale (p. 1).
d	Size of a deterministic customer base (p. 13).
$f_Z(\cdot)$	Probability density function of random variable Z (p.12).
$h_Z(\cdot)$	Probability mass function of random variable Z (p.12).
n	Number of selling prices and corresponding inventory levels (p.8).
r	Arbitrary reservation price (p. 13).
w_i	Defining characteristic i of an item available for sale (p. 12).
$\boldsymbol{w}$	Vector of defining characteristics of an item available for sale (p. 12).
$\boldsymbol{z}$	Possible value of $\boldsymbol{Z}$ (p. 12).

D	Size of random customer base (p. 13).
E	Expectation (p. 25).
$F_Z(\cdot)$	Cumulative distribution function for random variable Z (p.13).
$F_Z^{-1}(\cdot)$	Inverse cumulative distribution function for random variable Z (p.1).
$\overline{F_i}$	Probability a randomly-selected customer is willing to pay price P_i for an item

p.65).

P	Selling price (p. 1).
$Prob(\cdot)$	Probability (p. 13).
Q	Inventory level (p. 1).
R	Revenue (p. 26).
RP	Reservation price (p. 12).
$Sold$	Number of items sold (p. 26).
U	Uniform distribution (p. 19).
X	Number of customers willing to pay price P_1 for an item (p. 1).
Y	Number of customers willing to pay price P_2 for an item (p. 32).
Z_i	Secondary characteristic i of an item available for sale (p. 12).
$\mathbf{Z}$	Vector of secondary characteristics of an item available for sale (p. 12).

ε_Z	Uncertainty in random variable Z (p. 17).
ζ	Price elasticity of demand (p. 20).
μ_Z	Expected value of random variable Z (p. 14).
σ_Z	Standard deviation of random variable Z (p. 14).
σ_Z^2	Variance of random variable Z (p. 14).

Φ	Fraction (p. 130).
Π	Profit (p. 25).

CHAPTER 2

LITERATURE REVIEW

2.1 THE NEWSVENDOR PROBLEM WITH PRICING

The first to approach the newsvendor problem with price as a decision variable is Whitin (1955), who provides examples of the simultaneous determination of profit-maximizing selling price and order quantity. He considers setting a single selling price in both single and multi-period cases, for multiple products and stochastic demand. In one case, he substitutes the economic order quantity into a deterministic demand curve, and uses first order conditions to solve for optimal price. Another example applies a demand distribution that is a function of price, and demonstrates how the optimal quantity is found where expected marginal revenue equals expected marginal loss. Optimal price is then found where, using the optimal quantity, the difference between total expected profits and total expected losses is maximized.

Mills (1959) expands on the problem by more formally addressing the issue of uncertainty, treating it as additive and independent of price. He considers stochastic demand, both single and multi-period cases, single pricing (per period) and a single product. He notes that in general, it is not profit-maximizing to set quantity at expected sales corresponding to optimal price. He also makes explicit that with demand uncertainty, expected revenue is not price times expected demand, but price times expected sales (since realized demand might exceed quantity available for sale). His conclusions relate primarily to how the equilibrium price (the profit-maximizing price under additive demand uncertainty) differs from the profit-maximizing price in the riskless case (if demand were known exactly). First, when marginal cost is constant, the equilibrium price is lower than the riskless price. Second, when marginal cost is rising, and either 1) equilibrium quantity is no more than riskless quantity or 2) riskless quantity is no more than average demand at equilibrium price, then equilibrium price is no more than riskless price. Finally, when marginal cost is falling, and either 1) equilibrium quantity is no less than riskless quantity or 2) riskless quantity is no more than average demand at equilibrium price, then equilibrium price is no more than riskless price.

Karlin and Carr (1962) also include demand uncertainty in their models, both additive and multiplicative. They assume stochastic demand for a single product, both single and multi-period demands, and a single price set for each period. Their findings are that with multiplicative uncertainty the optimal price is always greater than the optimal riskless price, while with additive uncertainty, the optimal price is always less than the optimal riskless price. In either case, the optimal price is determined first and then used in a fractile equation to determine optimal quantity. The authors present these fractile equations in forms that include either the inverse cumulative distribution function (c.d.f.) of the random variable representing uncertainty or the inverse c.d.f. of quantity demanded.

Nevins (1966) uses the works of Mills (1959) and Karlin and Carr (1962) as a basis for his simulations to further investigate the divergence of equilibrium price and quantity and riskless price and quantity. He considers stochastic demand, multiple time periods, a single product and a single equilibrium price for each period. His models are based on multiplicative uncertainty, are multi-period, and include various discount rates, marginal storage costs and variable costs of production. He finds that over the long run, uncertainty alone does not create a divergence between equilibrium price and riskless price. In order for divergence, either a positive discount rate or storage cost is required. In addition, the sensitivity of equilibrium price and quantity to changes in discount rate and storage cost depend on the elasticity of demand. At low elasticities, riskless price and quantity are both good approximations for equilibrium price and quantity, even for fairly large discount rates and storage costs. Similar results are found when elasticity approached unity. However, at higher elasticities, the equilibrium inventory level (the level toward which inventory is expected to move if the optimal price-quantity policy is followed) fall to zero, and this is the necessary condition to note significant divergences in equilibrium price and quantity and riskless price and quantity.

To compare how expected profit differs between simultaneous and sequential equilibrium price-quantity determination (as when firms make centralized and decentralized decisions, respectively), Kunreuther and Richard (1971) use deterministic demand. They consider a single product, multiple periods and a single price. They find that the more inelastic the demand for a product, the more desirable to use the simultaneous procedure. In addition, when using the simultaneous procedure and equilibrium price is greater than unit cost, an upper bound can be calculated, that determines whether or not the firm will order any product.

Another multi-period policy is given by Thomas (1974), who produces a list of price-production pairs for each inventory level in each period. He considers stochastic demand for a single product, and single pricing in each period. For each period, *n*, two inventory levels are calculated, S_n and s_n (analogous to the *S*, *s* model) and the optimal order quantity and selling price are determined depending on the actual inventory level. If the actual inventory level is greater than or equal to s_n, no production order is placed, and the price is set at the point on an optimal price line in the price-inventory plane that corresponds to the actual inventory level. For actual levels below s_n, inventory is ordered up to level S_n, and the price charged is that on the optimal price line corresponding to S_n. The policy is tested for optimality using two sets of examples. The first uses a discrete distribution whose mean (only) depended on price. The second assumes a maximum possible demand and a binomial or multinomial demand distribution with the probability parameter being a function of price. In the first set of examples, the policy is optimal for 15 of 16 cases, and in the second set it is optimal for all 22 cases.

To study the influence of market structure (level of competitiveness) on the newsvendor problem, Young (1979) presents demand as $\xi = \mu(b, P) + \alpha(P)n$, where $\mu(b, P)$ and $\alpha(P)$ are deterministic demand functions (with respect to price) and *n* is a random variable with a known density function, $\varphi(n)$, standard deviation *s*, and expected value of 0. The competitiveness parameter, *b*, only applies to the first deterministic demand function. An increase in *b* means an increase in level of competition. Young provides an optimal policy and shows that if the optimal policy is followed, marginal production cost plus average inventory and shortage costs is:

a. Less than riskless (no demand uncertainty) marginal revenue if $\gamma(P)$ (coefficient of variation) decreases with price,

b. More than riskless marginal revenue if $\gamma(P)$ increases with price, and

c. Equal to riskless marginal revenue if $\gamma(P)$ is independent of price.

In addition, he shows that for any P, an increase in b effects the elasticity of the expected demand curve, increases $\gamma(P)$, and, provided that inventory and shortage costs are significant, increases price and reduces expected sales (even though it increases the elasticity of expected demand). Finally, under certain strict conditions, a change in market competitiveness which either 1) increases expected sales or 2) decreases profitability, will also increase average inventory and shortage costs (the marginal cost of increasing sales exceeds the average cost of sales). He considers both deterministic and stochastic demand, a single product, with single-pricing in a single period.

In Dana (1999), the author investigates Nash equilibrium price dispersion and demand uncertainty. One of the primary goals is to investigate price dispersion both within individual firms, and also between firms, under different competitive environments (monopoly, oligopoly, etc.). Dana provides methods for determining optimal pricing and inventory strategies that relies heavily on methods commonly found in the Economics literature. He considers stochastic demand for a single product, with multiple-pricing in a single period.

In a different approach to the problem, Petruzzi and Dada (1999) maximize profit by solving for optimal price and optimal z, where under 1) additive uncertainty,

$D(P,\varepsilon) = y(P) + \varepsilon$, $z = q - y(P)$ and 2) multiplicative uncertainty, $D(P,\varepsilon) = y(P)\varepsilon$, $z = q/y(P)$. Note that in both cases, the uncertainty, ε, is independent of price. They show that in both cases, determining optimal price is straightforward, but optimal quantity depends on the c.d.f. of ε. In addition, they show that if uncertainty is additive, optimal price is no higher than under riskless conditions. If uncertainty is multiplicative, optimal price is no lower that under riskless conditions. Note that these findings are consistent with Karlin and Carr (1962). Petruzzi and Dad consider a single and multiple periods, stochastic demand for a single product, and a single price for each period.

In Dana and Petruzzi (2001) the issues of consumer utility and competition are addressed. They model demand as a function of both price and inventory level, since consumers' likelihood to visit a firm depends partially on their belief that a product will be available when they arrive. They also consider cases of price being set both by the firm and exogenously. Regardless of who sets the price, a firm that considers the influence of inventory level on demand carries more inventory, provides a higher level of service, attracts more customers and earns higher profits that one that ignores the inventory influence. The authors also show that in the case in which price is determined endogenously, the two-dimensional problem can be reduced to two single-variable optimizations that result in the socially efficient stocking factor.

Using the same approach as Petruzzi and Dada (1999), Zhan and Shen (2005) treat the problem as a nonlinear system of two variables, as opposed to the usual method of reducing it to an optimization over a single variable. They consider stochastic demand for a single product, and setting a single price for a single period. They show analytically how the number of interior solutions ranges from zero to two, depending on the c.d.f. of ε (for the additive uncertainty case) and assuming that P is concave in z and z is concave in P. In addition, the authors provide both iterative and simulation-based algorithms that can be used to determine the optimal conditions.

CHAPTER 3

A DEMAND MODEL INCORPORATING PRICE-DEPENDENT UNCERTAINTY

3.1 INTRODUCTION

In most attempts to solve the newsvendor problem with pricing, the first step is to collect or assume some information on how customer demand is influenced by price, and represent that information in a mathematical form as a demand equation. Often a deterministic demand equation is taken as the basic quantity (assuming some distributional properties), with common forms being those shown in (1.1.2) and (1.1.3). To the deterministic demand equation, uncertainty is (sometimes) introduced, with common forms being those shown in (1.1.4) and (1.1.5). From here, optimal price and inventory quantities are determined.

In this chapter, demand is modeled using via customer behaviour, using information about individual customers and size of customer base as the basic quantities. The result is a justified form of a demand equation, whose properties are studied and compared with those commonly found in the literature.

3.2 GENERAL MODEL OF DEMAND

Consider demand as a choice model where each customer in the population has a choice – buy or not buy one item that is available for sale. Here, individual consumer behaviour is modelled and aggregated into the demand for the product.

The product to be sold will have a number of defining characteristics that will define its customer base. For instance, consider an airline selling seats in different fare classes. The customer base willing to buy advanced purchase fares that require a Saturday night stay-over will often be different to the customer base who will consider buying first class tickets with no restriction, and this group will be different to those who will consider purchasing last minute coach fares. So it can make sense to assume that a product has a set of defining characteristics that can be presented by a vector $\boldsymbol{w} = (w_1, w_2, \ldots, w_r)$.

For each vector of characteristics $\boldsymbol{w}$, let $X(\boldsymbol{w})$ denote the number of customers who would be willing to buy the product at *some* price. Now consider a customer who would like to buy this product. The maximum or "reservation" price the customer is willing to pay will depend on factors intrinsic to the product and intrinsic to the customer. For instance, a customer will be willing to pay more for a first class ticket than a coach class ticket. A customer's personal characteristics will also have an influence—for instance, a student will be willing to pay less in general than a business person. Other factors such as location, gender, time of year, etc., may have an impact. Let these characteristics be denoted by the vector $\boldsymbol{Z} = (Z_1, \ldots, Z_s)$. The vector $\boldsymbol{Z}$ will have a distribution with a density function given by $f_Z(\boldsymbol{z}, \boldsymbol{w})$ among the customers willing to buy the product, where $\boldsymbol{z}$ is a possible value of the random (vector-valued) variable $\boldsymbol{Z}$. Among the substratum of customers who would like to buy the product whose characteristics are described by $\boldsymbol{w}$ and whose personal characteristics are described by $\boldsymbol{Z}$, there will be a distribution for the maximum price a randomly selected customer is willing to pay. Let $RP(\boldsymbol{Z}, \boldsymbol{w})$ denote the random variable that gives the reservation price for a customer drawn from this group.

Then, the probability that a randomly chosen person from the group who wants to buy the product whose characteristics are given by ***w*** will have a reservation price less than or equal to *r* is given by:

$$Prob\big(RP(\mathbf{Z}, \mathbf{w}) \le r\big) = \int \ldots \int \; Prob\big(RP(\mathbf{Z}, \mathbf{w}) \le r \mid \mathbf{Z} = \mathbf{z}\big) f_{\mathbf{Z}}(\mathbf{z}, \mathbf{w}) d\mathbf{z} \,. \qquad (3.2.1)$$

The purpose of the above very general description is to show how one might, in practice, model consumer behaviour. One can certainly conceive of standard statistical/marketing models where one can estimate the distributions for $X(\boldsymbol{w})$, $\boldsymbol{Z} = (Z_1, \ldots, Z_s)$, and $RP(\mathbf{Z}, \boldsymbol{w})$. A modeller will, in general, have a lot of experience and intuition to bring to bear on deciding which factors, for instance, influence a customer's reservation price.

This dissertation considers only the case of a known product and consequently, the ***w*** notation is suppressed. Also note that from (3.2.1), the notation ***Z*** can be suppressed because of integration over this random variable. Therefore, assume that the size of the customer base interested in the product is provided by the random variable *D* (or *d* in the case of a deterministic customer base) and that a randomly selected customer has a reservation price represented by the random variable *RP* whose cumulative distribution function will be denoted by $F_{RP}(\cdot)$.

Given this framework, demand, the number of customers (assumed to be discrete) willing to buy an item at price *P*, is a random variable, and is denoted $X(P)$. For a deterministic customer base, *d*, the random demand follows a binomial distribution with probability mass function, $h_{X(P)}(\cdot)$, given as:

$$h_{X(P)}(x) = Prob(X(P) = x) = \binom{d}{x}(1 - F_{RP}(P))^x (F_{RP}(P))^{d-x},\ x = 0, 1, \ldots d\ . \tag{3.2.2}$$

The cumulative distribution function, $F_{X(P)}(\cdot)$, is:

$$F_{X(P)}(x) \equiv \sum_{y=0}^{x} Prob(X(P) = y), \tag{3.2.3}$$

or

$$F_{X(P)}(x) \equiv \sum_{y=0}^{x} \binom{d}{y}(1 - F_{RP}(P))^y (F_{RP}(P))^{d-y}. \tag{3.2.4}$$

Note that the normal approximation to the binomial distribution can be used, with parameters, mean,

$$\mu_{X(P)} = d(1 - F_{RP}(P)) \tag{3.2.5}$$

and variance,

$$\sigma^2_{X(P)} = d(1 - F_{RP}(P))(F_{RP}(P)), \tag{3.2.6}$$

provided both parameters are greater than five. If such is the case, demand can be approximated as:

$$X(P) \sim N\left(\mu_{X(P)} = d(1 - F_{RP}(P)), \sigma^2_{X(P)} = d(1 - F_{RP}(P))(F_{RP}(P))\right), \tag{3.2.7}$$

with the corresponding probability density function, $f_{X(P)}(\cdot)$, approximated as:

$$f_{X(P)}(x) \approx \frac{1}{\sqrt{2\pi d(1 - F_{RP}(P))(F_{RP}(P))}} \exp\left(-\frac{(x - d(1 - F_{RP}(P)))^2}{2d(1 - F_{RP}(P))(F_{RP}(P))}\right). \tag{3.2.8}$$

The cumulative distribution function, is approximated as:

$$F_{X(P)}(t) \approx \int_{x=o}^{t} f_{X(P)}(x)dx, \tag{3.2.9}$$

or

$$F_{X(P)}(t) \approx \int_{x=0}^{t} \frac{1}{\sqrt{2\pi d(1-F_{RP}(P))(F_{RP}(P))}} \exp\left(-\frac{(x-d(1-F_{RP}(P)))^2}{2d(1-F_{RP}(P))(F_{RP}(P))}\right) dx. \qquad (3.2.10)$$

3.3 UNIFORMLY-DISTRIBUTED RESERVATION PRICES IN THE GENERAL MODEL OF DEMAND

An example of expressing demand from the basic quantities given in Section 3.2 can be found in Dana (1999). In this work, a given example suggests (without obvious justification) that if the reservation prices of customers follow a uniform distribution, the corresponding demand function is linear. Here, for a demand model with the basic quantities of population base (d) and probability of success ($1-F_{RP}(\cdot)$), if the reservation prices of the customers follow a uniform distribution, the expected demand function is linear in P.

Lemma 3.3.1. For a deterministic customer base, d, and a known cumulative distribution function of reservation prices, $F_{RP}(P)$, if reservation prices follow a uniform distribution, the expected demand function is linear in selling price, P.

Proof. First, recall the expected demand equation given by (3.2.5):

$$\mu_{X(P)} = d(1-F_{RP}(P)), \qquad (3.2.5)$$

and consider that if the reservation prices follow a uniform distribution on $[P_1, P_2]$, the cumulative distribution function for reservation price, $F_{RP}(P)$, is given by:

$$F_{RP}(P) = \frac{P - P_1}{P_2 - P_1}\,. \tag{3.3.1}$$

Substituting (3.3.1) into (3.2.5) gives:

$$\mu_{X(P)} = d\left(1 - \frac{P - P_1}{P_2 - P_1}\right), \tag{3.3.2}$$

which can be expanded and rearranged to give:

$$\mu_{X(P)} = d\left(1 + \frac{P_1}{P_2 - P_1}\right) - \left(\frac{d}{P_2 - P_1}\right)P\,. \tag{3.3.3}$$

By inspection, Equation 3.3.3 is linear in P (of the form: $\mu_{X(P)} = b + mP$), with an intercept given by:

$$b = d\left(1 + \frac{P_1}{P_2 - P_1}\right), \tag{3.3.4}$$

and a slope given by:

$$m = -\left(\frac{d}{P_2 - P_1}\right). \tag{3.3.5}$$

Q.E.D.

Lemma 3.3.1 justifies a form of a demand equation often found in the literature (linear and downward sloping) that is typically implemented because of its ease for mathematical manipulation.

3.4 ADDITIVE UNCERTAINTY IN THE GENERAL MODEL OF DEMAND

Approaches to solving the NPP often begin by arbitrarily assuming some form of uncertainty (for example, see Equations 1.1.4 and 1.1.5). Here it is shown that given the basic quantities of demand as described in Section 3.2 (size of customer base and distribution of reservation prices), the appropriate form of the demand equation is that which incorporates additive uncertainty, similar, but not identical to that shown in (1.1.5).

Lemma 3.4.1. Demand from a deterministic customer base of size d, and a cumulative distribution function of reservation prices for the customer base, $F_{RP}(P)$, *can be expressed with price-dependent, additive uncertainty, as*:

$$X(P) = d\left(1 - F_{RP}(P)\right) + \varepsilon_{X(P)}(P). \tag{3.4.1}$$

Proof. First, consider the demand model given by (3.2.7). Demand, the number of people willing to pay P for an item, is random, and can be approximated by the normal distribution:

$$X(P) \sim N\left(\mu_{X(P)}, \sigma^2_{X(P)}\right) \tag{3.4.2}$$

with the parameters given by (3.2.5) and (3.2.6). Substitution into (3.4.2) gives

$$X(P) \sim N\left(\mu_{X(P)} = d\left(1 - F_{RP}(P)\right), \sigma^2_{X(P)} = d\left(1 - F_{RP}(P)\right)\left(F_{RP}(P)\right)\right) \tag{3.4.3}$$

which can be expressed in the form:

$$X(P) = \mu_{X(P)}(P) + \varepsilon_{X(P)}(P) \tag{3.4.4}$$

where $\mu_{X(P)}(P)$ is given by (3.2.5):

$$\mu_{X(P)} = d\left(1 - F_{RP}(P)\right) \tag{3.2.5}$$

and

$$\varepsilon_{X(P)}(P) \sim N\left(\mu_{\varepsilon_{X(P)}(P)} = 0, \sigma^2_{\varepsilon_{X(P)}(P)} = d(1 - F_{RP}(P))(F_{RP}(P))\right). \tag{3.4.5}$$

Thus, the demand model given by (3.4.3) can be expressed as:

$$X(P) = d(1 - F_{RP}(P)) + \varepsilon_{X(P)}(P), \tag{3.4.1}$$

which incorporates additive uncertainty in a form similar to (1.1.5). *Q.E.D.*

Note that the uncertainty term in (3.4.5), $\varepsilon_{X(P)}(P)$, is a function of selling price, P. For reasons demonstrated in subsequent sections and chapters, this result is both interesting and relevant. Also worth mentioning is that it is rare to find an example in the literature of a newsvendor problem with pricing which incorporates a variable form of uncertainty in demand (for an example, see Young, 1979).

As was the case with Lemma 3.3.1, Lemma 3.4.1 provides justification for use of a form of demand (one incorporating additive uncertainty) often found in the literature with little explanation other than "mathematical tractability."

To demonstrate price-dependent additive uncertainty in demand, $\varepsilon_{X(P)}(P)$ in (3.4.5), two examples are presented. In the first example, the size of the customer base is d = 50, and reservation prices follow a normal distribution as $RP \sim N(\mu_{RP} = 50, \sigma_{RP} = 10)$. The demand distribution indicated by (3.4.3) is used to calculate various expected demands and corresponding confidence intervals, which are shown in Figure 3.4.1.

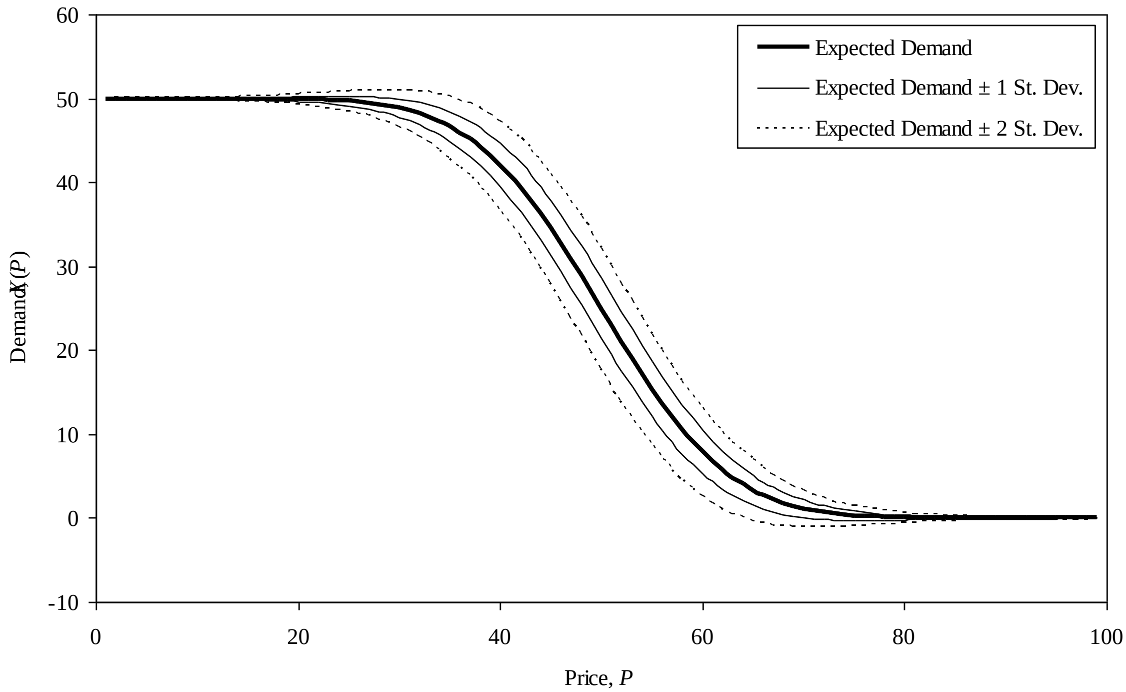

Figure 3.4.1. Expected Demand and Confidence Intervals as Functions of Price for d = 50 and $RP \sim N(\mu_{RP} = 50, \sigma_{RP} = 10)$.

In a second example, the size of the customer base is d = 50, and reservation prices follow a uniform distribution as $RP \sim U[0,100]$. The demand distribution indicated by (3.4.3) is used to calculate various expected demands and corresponding confidence intervals, which are shown in Figure 3.4.2.

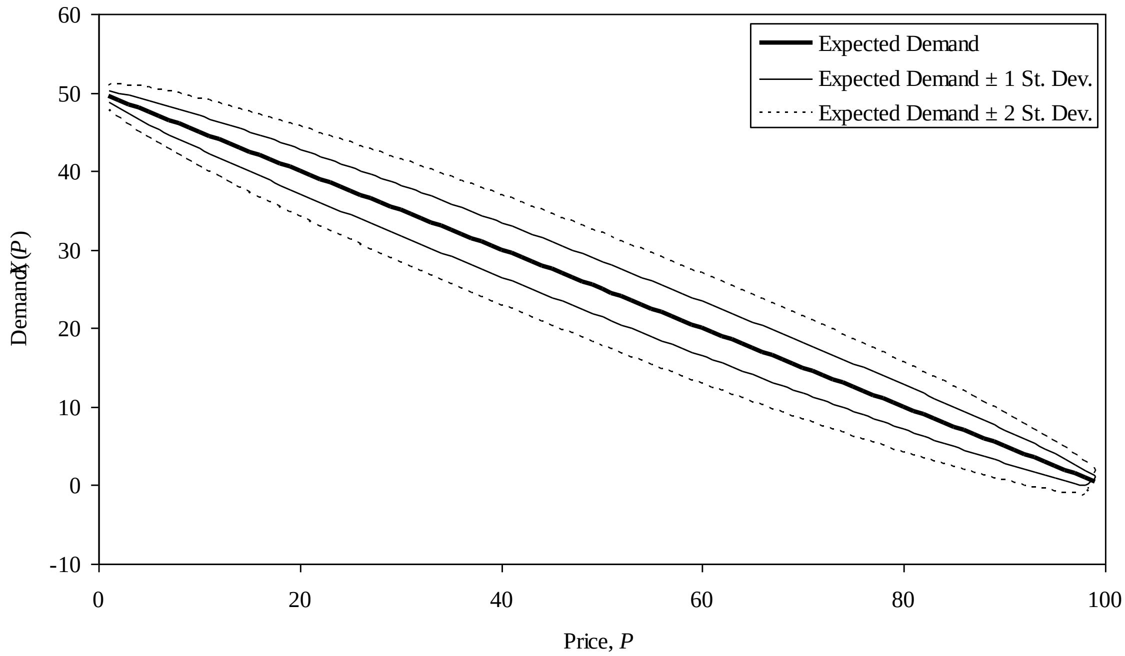

Figure 3.4.2. Expected Demand and Confidence Intervals as Functions of Price for $d = 50$ and $RP \sim U[0, 100]$.

3.5 ELASTICITY IN THE GENERAL MODEL OF DEMAND

Here, the price elasticity of demand, denoted $\zeta(P)$, for the general model is given, where the definition of elasticity is:

$$\zeta(P) \equiv -\frac{P}{\mu(P)}\frac{\partial}{\partial P}\mu(P). \tag{3.5.1}$$

By definition, price elasticity of demand is the percent change in quantity demanded resulting from a percent change in price. The form shown in (3.5.1) arises from applying this definition at a point, rather than over a "large" percent. By convention, the negative sign in (3.5.1) is included to make $\zeta(P)$ positive, as in most cases demand curves are downward-sloping, resulting in a negative partial derivative above.

Lemma 3.5.1. For a demand equation derived from the basic quantities of size of customer base, d, and probability of success, $(1-F_{RP}(P))$, *price elasticity of demand is given by:*

$$\zeta(P)=\frac{P}{(1-F_{RP}(P))}\frac{\partial}{\partial P}F_{RP}(P) \tag{3.5.2}$$

Proof. Begin by substituting (3.2.5) into (3.5.1) to give:

$$\zeta(P)=-\frac{P}{d(1-F_{RP}(P))}\frac{\partial}{\partial P}d(1-F_{RP}(P)). \tag{3.5.3}$$

Cancelling like terms gives:

$$\zeta(P)=-\frac{P}{(1-F_{RP}(P))}\frac{\partial}{\partial P}(1-F_{RP}(P)) \tag{3.5.4}$$

which becomes:

$$\zeta(P)=\frac{P}{(1-F_{RP}(P))}\frac{\partial}{\partial P}F_{RP}(P). \tag{3.5.5}$$

Q.E.D.

Lemma 3.5.1 is useful for business practitioners who use elasticity of demand as a tool for setting price and inventory level. While a revenue-maximizing decision maker sets these decision variables where elasticity of demand is equal to one, there are cases where a decision maker prices in the inelastic region ($0<\zeta(P)<1$), even though an increase in selling price leads to a decrease in units sold, but an increase in revenue.

3.6 VARIANCE IN THE GENERAL MODEL OF DEMAND

Here, it is shown that the variance demand in the general model is greatest at the median value of the reservation prices.

Lemma 3.6.1. For a demand equation derived from size of customer base and distribution of reservation prices, the variance is greatest at the median value of the reservation prices.

Proof. Recall the form of the variance of demand, as given in (3.2.6):

$$\sigma^2_{X(P)}(P) = d\left(1 - F_{RP}(P)\right)\left(F_{RP}(P)\right). \tag{3.2.6}$$

Begin by taking the first derivative of (3.2.6) with respect to *P*:

$$\frac{\partial}{\partial P}\sigma^2_{X(P)}(P) = d\left(\left(1 - F_{RP}(P)\right)\left(\frac{\partial}{\partial P}F_{RP}(P)\right) - \left(F_{RP}(P)\right)\left(\frac{\partial}{\partial P}F_{RP}(P)\right)\right). \tag{3.6.1}$$

Rearranging gives:

$$\frac{\partial}{\partial P}\sigma^2_{X(P)}(P) = d\left(\frac{\partial}{\partial P}F_{RP}(P) - 2\left(F_{RP}(P)\right)\left(\frac{\partial}{\partial P}F_{RP}(P)\right)\right). \tag{3.6.2}$$

Setting (3.6.2) equal to zero and solving for $F_{RP}(P)$ gives:

$$F_{RP}(P) = 1/2 \tag{3.6.3}$$

which, by definition, occurs at the median value of *RP*. To show that this is a maximum, take the second derivative of (3.2.6) with respect to *P* to get:

$$\frac{\partial^2}{\partial P^2}\sigma^2_{X(P)}(P) = d\left(\frac{\partial^2}{\partial P^2}F_{RP}(P) - 2\left(F_{RP}(P)\right)\left(\frac{\partial^2}{\partial P^2}F_{RP}(P)\right) - 2\left(\frac{\partial}{\partial P}F_{RP}(P)\right)^2\right). \tag{3.6.4}$$

Rearranging (3.6.4) and substituting in $F_{RP}(P) = 1/2$ gives:

$$\frac{\partial^2}{\partial P^2}\sigma^2_{X(P)}(P) = -2d\left(\frac{\partial}{\partial P}F_{RP}(P)\right)^2 \tag{3.6.5}$$

which is negative for all distributions of *RP* and allowable values of *d* and *P*. Therefore, the maximum variance in demand is observed at the median value of *RP*. *Q.E.D.*

Graphical representations of this lemma can be seen in Figures 3.4.1 and 3.4.2, where in both figures, the confidence intervals are greatest at the median reservation price of $P = 50$.

Lemma 3.6.1 is useful for a business practitioner who is concerned with the trade-off between risk (variance in demand) and revenue. For example, a risk-averse decision maker can use this lemma to determine an acceptable level of uncertainty in revenues, with the understanding that expected revenue itself is reduced.

CHAPTER 4

THE NEWSVENDOR PROBLEM WITH PRICING: A SINGLE-PRICE MODEL

4.1 INTRODUCTION

This chapter incorporates the demand model with price-dependent uncertainty from Chapter 3 into the newsvendor problem with pricing, where the seller of homogeneous items decides on a single selling price, P, and the corresponding number of items to be made for sale, Q. The seller must decide on P and Q before realizing exact demand, which, from Chapter 3, depends upon 1) the number of customers in the population, denoted d if known exactly, or D if random, and 2) the distribution of reservation prices (how much, at most, individuals are willing to pay for an item) of the customers. The reservation price of each customer, denoted *RP*, is random with a known cumulative distribution function, $F_{RP}(\cdot)$. It is assumed that customers arrive in random order, and each customer's willingness to buy depends only on the selling price of an item relative to that customer's reservation price. For each item the seller makes available for sale, a marginal cost of c is incurred, regardless of whether or not the item is ultimately sold.

4.2 SINGLE-PRICE EXPECTED PROFIT FUNCTIONS

In this section, two expected profit functions are derived: one assuming a deterministic customer base, and one assuming a stochastic customer base. For both functions, normal approximations to the discrete cases are included. The section concludes with comments on joint optimization of selling price and inventory level.

4.2.1 DETERMINISTIC CUSTOMER BASE

Here, the expected profit, $E[\Pi(P,Q)]$, for a seller who sets only one selling price *P*, and the corresponding order quantity *Q*, is derived. The size of the customer base is deterministic and denoted *d*. For each item the seller makes available for sale, a marginal cost of *c* is incurred, regardless of whether or not the item is ultimately sold. Demand is denoted by the random variable $X(P)$, the number of customers with reservation prices at least as high as *P*, with corresponding probability mass function $h_{X(P)}(\cdot)$, or in the continuous case, probability density function $f_{X(P)}(\cdot)$.

Lemma 4.2.1. For a deterministic customer base, the expected profit function for the seller is given by:

$$E[\Pi(P,Q)] = P\left(\sum_{x=0}^{Q} x h_{X(P)}(x) + Q\sum_{x=Q+1}^{d} h_{X(P)}(x)\right) - cQ\,, \tag{4.2.1}$$

which is approximated by:

$$E[\Pi(P,Q)] \approx P\left(\int_{x=0}^{Q} x f_{X(P)}(x)dx + Q\int_{x=Q}^{d} f_{X(P)}(x)dx\right) - cQ\,. \tag{4.2.2}$$

Proof. Consider first, that the revenue realized by the seller depends on the number of items sold. Denoting realized demand as *x* (i.e.; the realized value of $X(P)$), if *x* is less than the number of items made available for sale, the revenue is *Px*. If *x* is greater than the number of items made available for sale, the revenue is *PQ*. Note that in both scenarios, the incurred cost is *cQ*, as the seller pays for each item made available for sale, regardless of whether or not it is sold. From these two possible revenue scenarios, the profit function, $\Pi(P, Q)$, is given as:

$$\Pi(P, Q) = \begin{cases} Px - cQ, & x \le Q \\ PQ - cQ, & x > Q \end{cases}. \tag{4.2.3}$$

In the first revenue scenario, all demand is met ($x \le Q$), and the expected number of items sold is given by:

$$E[Sold \mid x \le Q] = \sum_{x=0}^{Q} x h_{X(P)}(x), \tag{4.2.4}$$

where $h_{X(P)}(\cdot)$ is the probability mass function of demand, as given by (3.2.2):

$$h_{X(P)}(x) = Prob\big(X(P) = x\big) = \binom{d}{x}\big(1 - F_{RP}(P)\big)^{x}\big(F_{RP}(P)\big)^{d-x},\ x = 0, 1, \ldots d\,. \tag{3.2.2}$$

Expected revenue in this case, ($E[R(P, Q) \mid x \le Q]$), is found by multiplying (4.2.4) by P to give:

$$E[R(P, Q) \mid x \le Q] = P\sum_{x=0}^{Q} x h_{X(P)}(x), \tag{4.2.5}$$

In the second revenue scenario, there is unmet demand ($x > Q$), and the expected number of items sold is given by:

$$E[Sold \mid x > Q] = Q \sum_{x=Q+1}^{d} h_{X(P)}(x)\,. \tag{4.2.6}$$

Expected revenue in this case ($E[R(P, Q) \mid x > Q]$, unmet demand) is found by multiplying (4.2.6) by P to give:

$$E[R(P,Q)\,|\,x>Q]=PQ\sum_{x=Q+1}^{d}h_{X(P)}(x)\,. \qquad (4.2.7)$$

Summation of (4.2.5) and (4.2.7) gives the expected total revenue:

$$E[R(P,Q)]=P\sum_{x=0}^{Q}xh_{X(P)}(x)+PQ\sum_{x=Q+1}^{d}h_{X(P)}(x)\,, \qquad (4.2.8)$$

from which the total cost, cQ, is subtracted to give expected total profit:

$$E[\Pi(P,Q)]=P\left(\sum_{x=0}^{Q}xh_{X(P)}(x)+Q\sum_{x=Q+1}^{d}h_{X(P)}(x)\right)-cQ\,. \qquad (4.2.9)$$

As a normal approximation to the (4.2.9), the summations in (4.2.9) are replaced with integrals, the probability mass function is replaced with the probability density function as given by (3.2.8), $f_{X(P)}(\cdot)$:

$$f_{X(P)}(x)\approx\frac{1}{\sqrt{2\pi d\left(1-F_{RP}(P)\right)\left(F_{RP}(P)\right)}}\exp\left(-\frac{\left(x-d\left(1-F_{RP}(P)\right)\right)^{2}}{2d\left(1-F_{RP}(P)\right)\left(F_{RP}(P)\right)}\right). \qquad (3.2.8)$$

and expected profit is approximated by:

$$E[\Pi(P,Q)]\approx P\left(\int_{x=0}^{Q}xf_{X(P)}(x)dx+Q\int_{x=Q}^{d}f_{X(P)}(x)dx\right)-cQ. \qquad (4.2.10)$$

Q.E.D.

4.2.2 RANDOM CUSTOMER BASE

Here, the expected profit function, $E[\Pi(P,Q)]$, for a seller who sets only one selling price *P*, and the corresponding order quantity *Q*, is derived. The size of the customer base is random and denoted *D,* with corresponding probability mass function $h_D(\cdot)$, or in the continuous case, probability density function $f_D(\cdot)$. It is assumed that if Q can take on only whole values, the same restriction applies to D. In the case of allowable values of Q including non-integer values, D can take on non-integer values as well. As in Section 4.2.1, demand is given by the random variable $X(P)$, with corresponding probability mass function $h_{X(P)}(\cdot)$, or in the continuous case, probability density function $f_{X(P)}(\cdot)$.

Lemma 4.2.2. For a random-size customer base, the expected profit function for the seller is given by:

$$E[\Pi(P,Q)] = \sum_d \left(P\left(\sum_{x=0}^{Q} x h_{X(P)}(x) + Q \sum_{x=Q+1}^{d} h_{X(P)}(x) \right) h_D(d) \right) - cQ, \tag{4.2.11}$$

and is approximated by:

$$E[\Pi(P,Q)] \approx \int_d P\left(\int_{x=0}^{Q} x f_{X(P)}(x)dx + Q \int_{x=Q}^{d} f_{X(P)}(x)dx \right) f_D(d)dd - cQ. \tag{4.2.12}$$

Proof. Consider first the discrete case where Q and D can only take on whole values. Beginning with (4.2.1), the expected profit function for a deterministic customer base:

$$E[\Pi(P,Q)] = P\left(\sum_{x=0}^{Q} x h_{X(P)}(x) + Q \sum_{x=Q+1}^{d} h_{X(P)}(x) \right) - cQ, \tag{4.2.1}$$

note that the terms inside the brackets both depend on the size of the customer base, d, but the cost term outside the brackets does not. Therefore, to account for all the possible sizes of customer base and their respective probabilities (as given by the probability mass function, $h_D(\cdot)$), the expected revenue over all possible sizes of customer base is calculated. This is achieved by fixing a value of *D*, calculating the corresponding expected revenue, multiplying by the probability of observing the fixed value of *D*, repeating for all allowable values of *D*, and taking the summation of those expected revenues. This expected revenue is expressed as:

$$E[R(P,Q)] = \sum_{d} \left(P\left(\sum_{x=0}^{Q} x h_{X(P)}(x) + Q \sum_{x=Q+1}^{d} h_{X(P)}(x) \right) h_D(d) \right). \qquad (4.2.13)$$

Subtracting the cost gives the expected profit, as in (4.2.11):

$$E[\Pi(P,Q)] = \sum_{d} \left(P\left(\sum_{x=0}^{Q} x h_{X(P)}(x) + Q \sum_{x=Q+1}^{d} h_{X(P)}(x) \right) h_D(d) \right) - cQ. \qquad (4.2.14)$$

Similarly, in the case of *Q* and *D* being allowed to take on non-integer values, the continuous form of expected revenue from (4.2.12) is multiplied by the probability density function $f_D(\cdot)$, and integrated over all allowable values of *D* to give the expected revenue:

$$E[R(P,Q)] \approx \int_{d} \left(P\left(\int_{x=0}^{Q} x f_{X(P)}(x)dx + Q \int_{x=Q}^{d} f_{X(P)}(x)dx \right) f_D(d)dd \right) \qquad (4.2.15)$$

from which total cost, *cQ*, is subtracted to give expected profit:

$$E[\Pi(P,Q)] \approx \int_{d} \left(P\left(\int_{x=0}^{Q} x f_{X(P)}(x)dx + Q \int_{x=Q}^{d} f_{X(P)}(x)dx \right) f_D(d)dd \right) - cQ. \qquad (4.2.16)$$

Q.E.D.

4.2.3 JOINT PRICE AND INVENTORY LEVEL OPTIMIZATION

Here, optimization of the single price-quantity pair model is demonstrated using the expected profit equation given by (4.2.14). The decision variables for the seller are selling price, P, and quantity of items to be made available for sale, Q. The number of customers who are willing to pay price P for an item is a random variable denoted $X(P)$.

First, recall the standard solution (optimal inventory level, Q^*) to the newsvendor problem, as given by (1.1.1):

$$Q^*(P) = F_{X(P)}^{-1}\left(\frac{P-c}{P}\right) \qquad (1.1.1)$$

where $F_{X(P)}^{-1}(\cdot)$ denotes the inverse cumulative distribution function of demand. In the case of discrete demand, the optimal inventory level is the smallest value of Q that is greater than the right-hand side of (1.1.1).

Using (1.1.1) in the expected profit equation given by (4.2.14), or in the corresponding normal approximation form given by (4.2.16), reduces the problem to finding the maximum of a one-dimensional function:

$$\max_P \left(E[\Pi(P, Q^*(P))]\right) \qquad (4.2.17)$$

for which a solution can be found by searching over a grid of possible values.

4.3 EXAMPLES

Here we provide numerical examples to demonstrate the expected profit functions derived in this chapter.

Consider first the case of a deterministic customer base of $d = 100$, where $c = 20$ and $RP \sim U[0,100]$. Using Equation 4.2.1, the expected profits for various sizes of inventory level are calculated and shown in Figure 4.3.1.

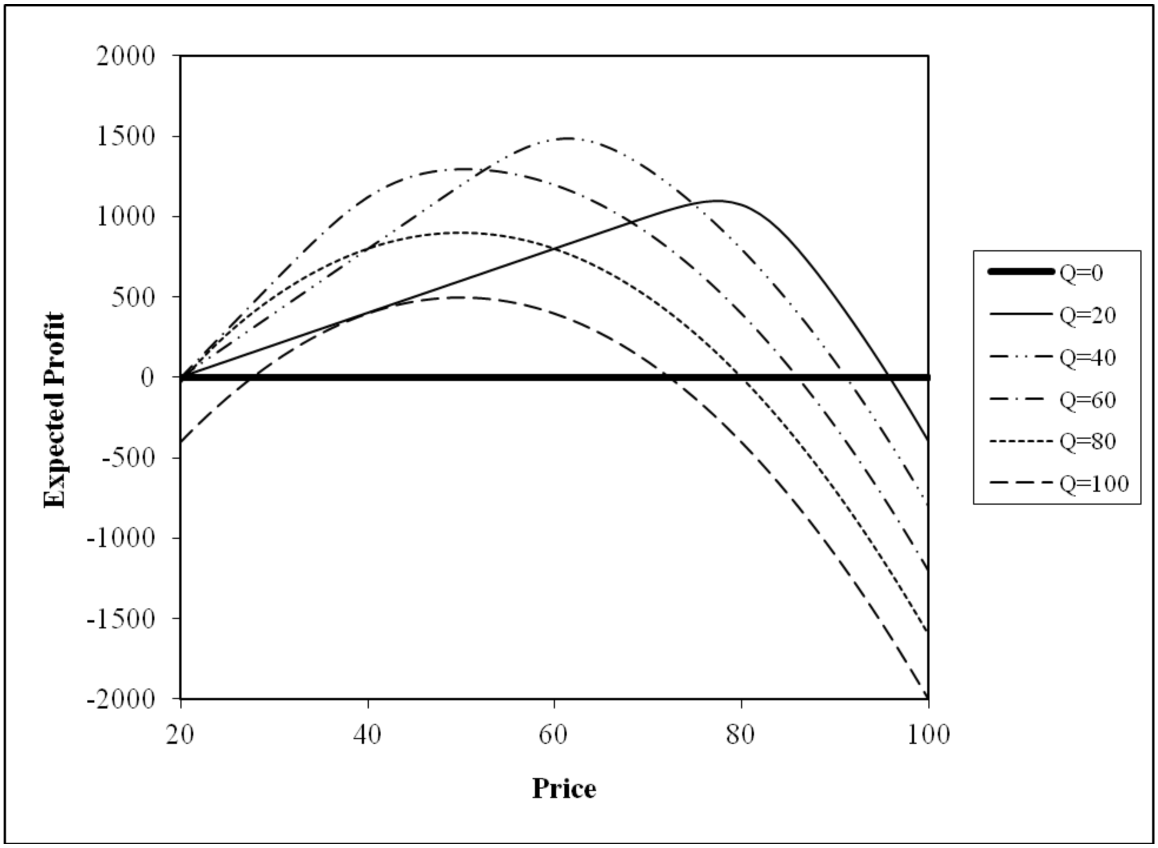

Figure 4.3.1. Expected Profit as a Function of Price for Various *Q* when *d*=100, *c*=20 and *RP*~U[0, 100].

Using the same parameters, expected profit is calculated for $Q = 41, 42$ and 43, and shown in Figure 4.3.2. The maximum expected profit of 1493.1 occurs at $P^* = 59.9$ and $Q^* = 42$.

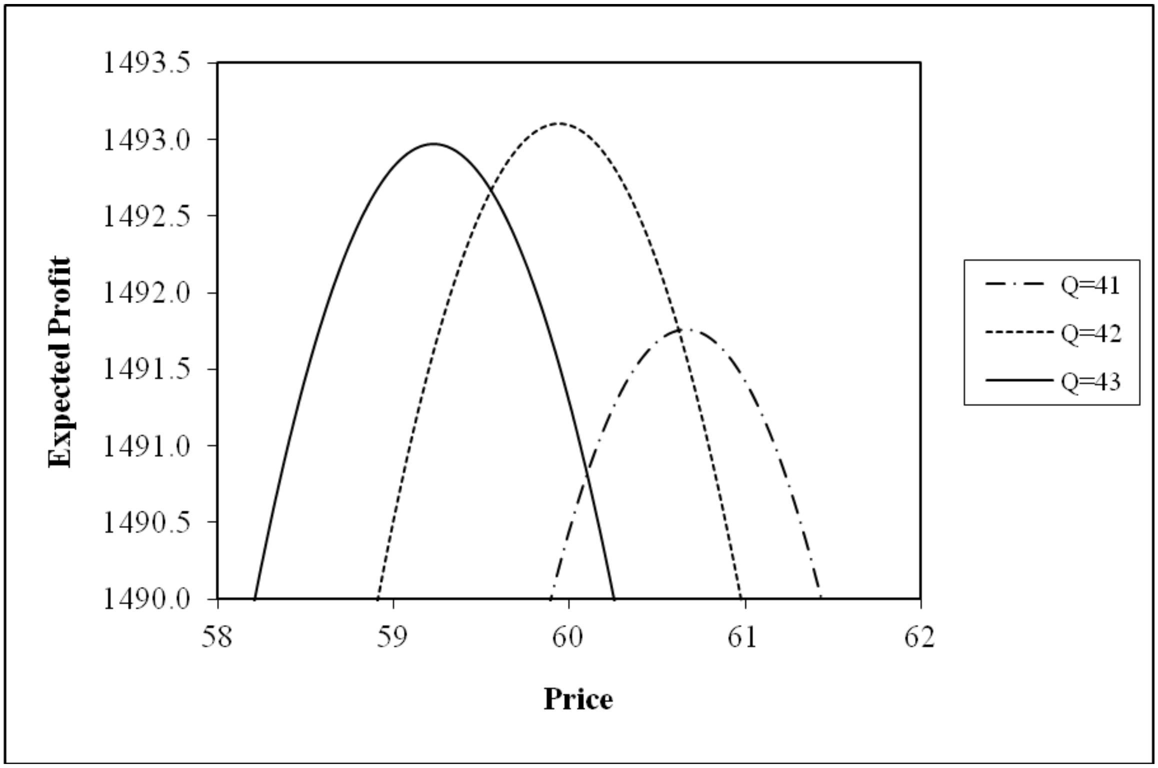

Figure 4.3.2. Maximum Expected Profit when *d*=100, *c*=20 and *RP*~U[0, 100].

Using the same parameters as above, the maximum expected profit is calculated as a function of price for various sizes of customer base, *d*. The results are shown in Figure 4.3.3.

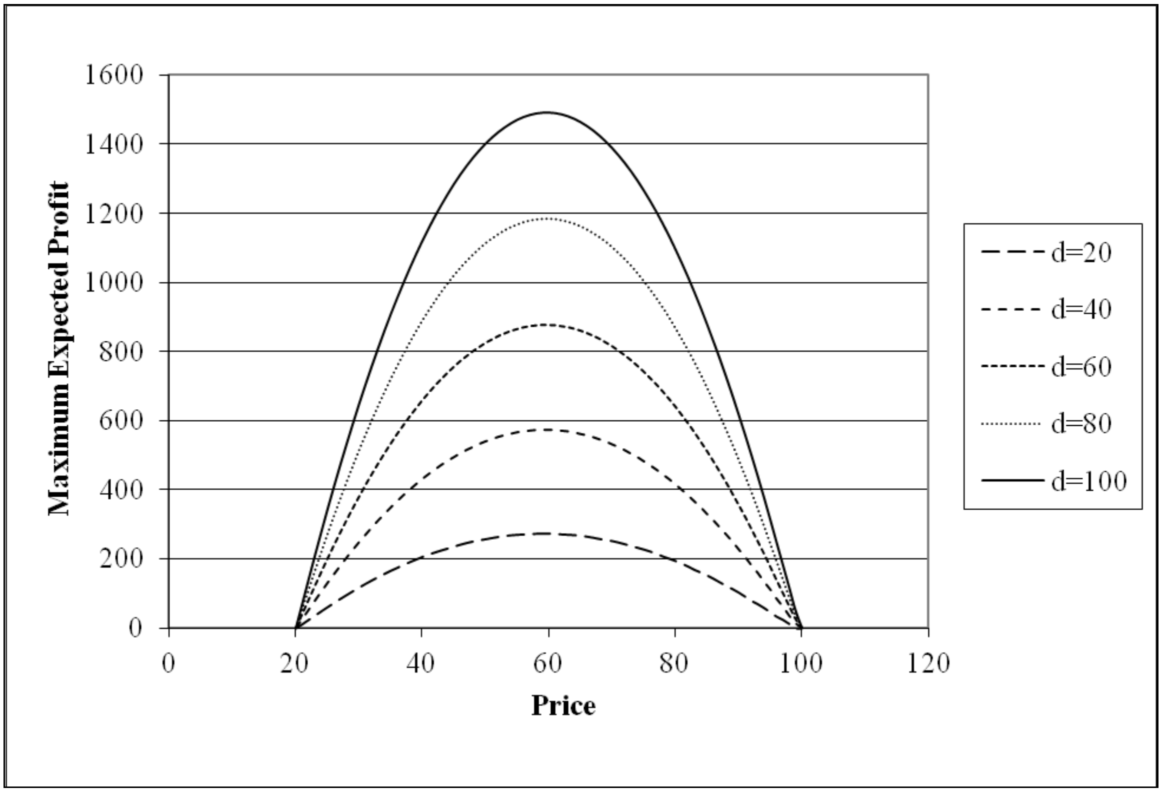

Figure 4.3.3. Maximum Expected Profit as a Function of Price for Various *d* when *c*=20 and *RP*~U[0, 100].

Consider now, the cases where the size of the customer base is random as $D \sim U[0, Dmax]$, where $c = 20$ and $RP \sim U[0, 100]$. Using Equation 4.2.11, the expected profits for various sizes of inventory level are calculated and shown in Figure 4.3.4.

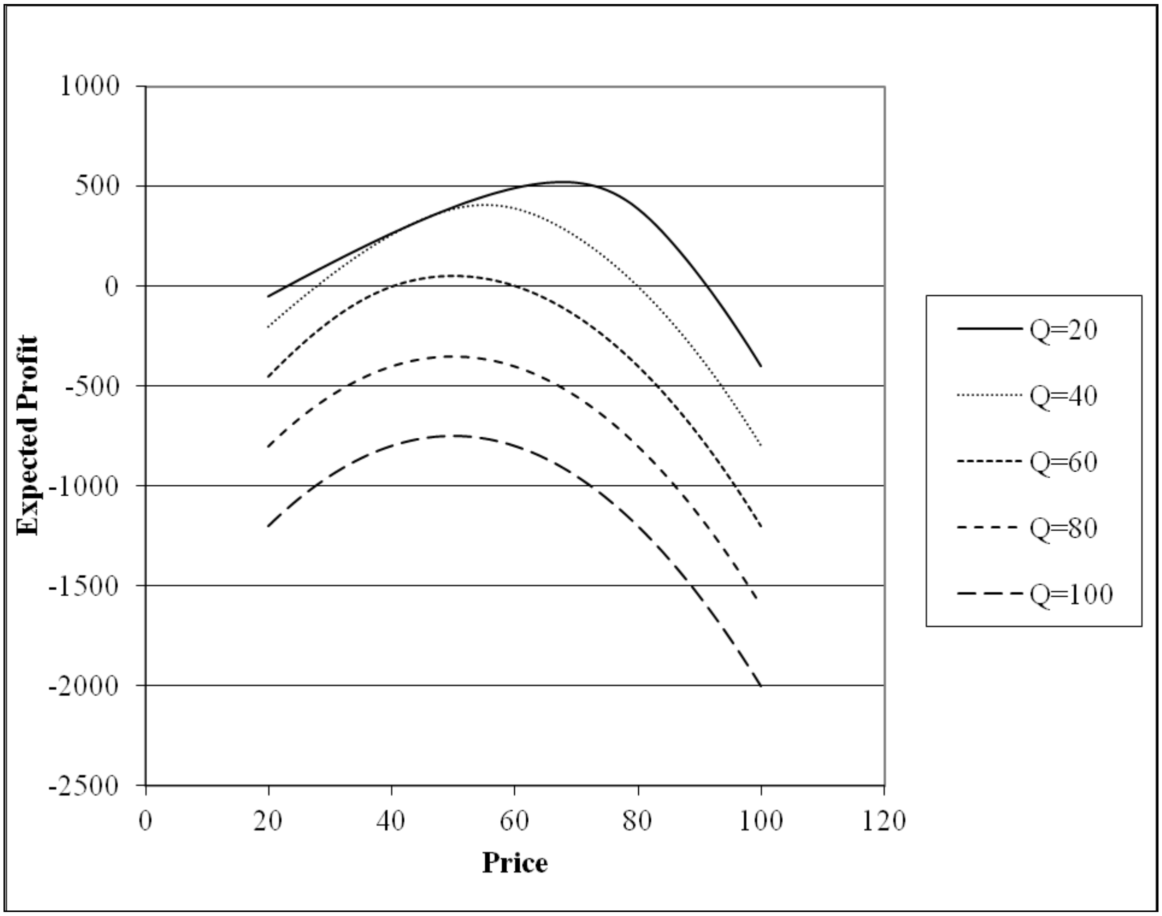

Figure 4.3.4. Expected Profit as a Function of Price for Various *Q* when

***D*~U[0, 100], *c*=20 and *RP*~U[0, 100].**

Using the same parameters, expected profit is calculated for $Q = 22 - 26$, and shown in Figure 4.3.5. The maximum expected profit of 528.4 occurs at $P^* = 65$ and $Q^* = 24$.

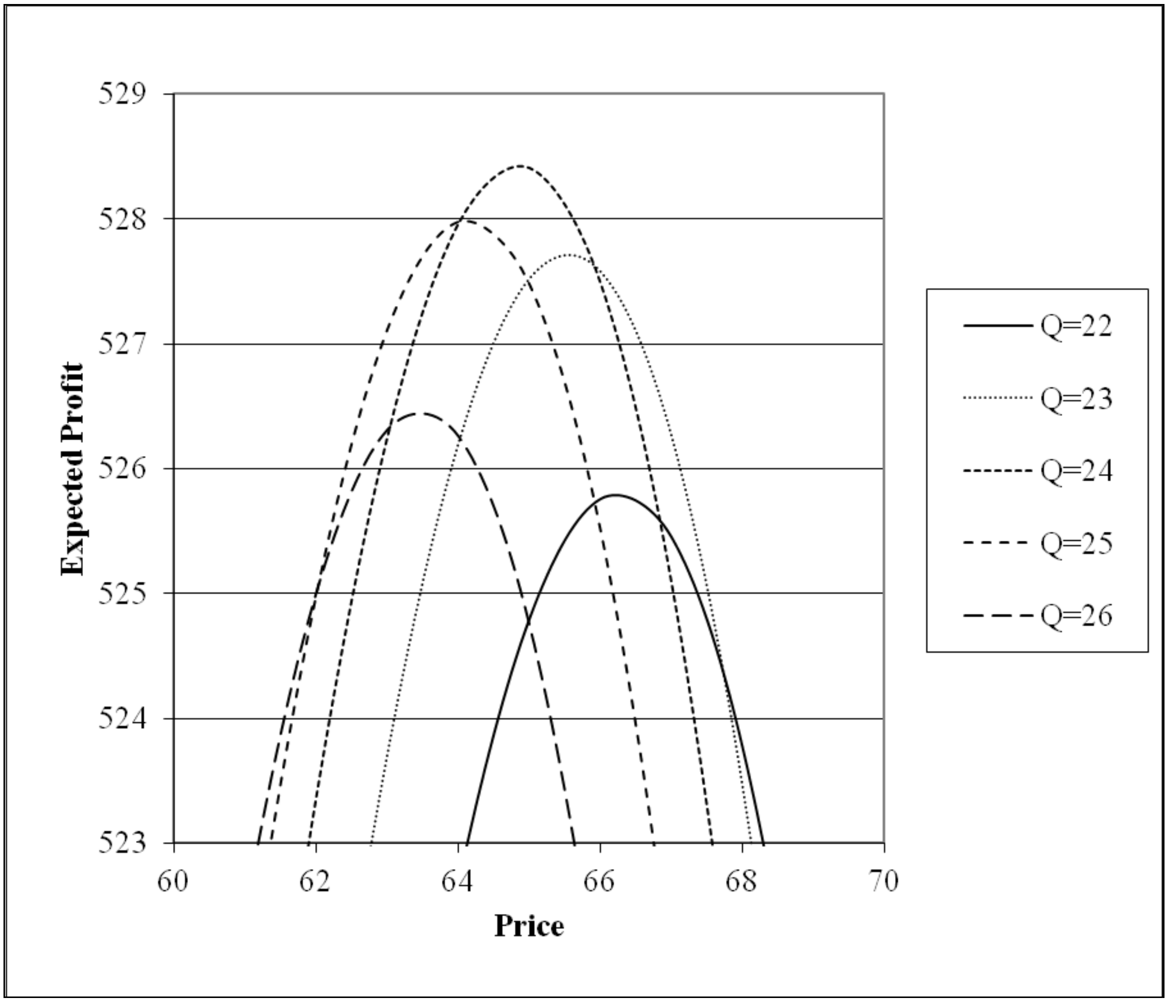

Figure 4.3.5. Maximum Expected Profit when *D*~U[0, 100], *c*=20 and *RP*~U[0, 100].

Using the same parameters as above, the maximum expected profit is calculated as a function of price for various maximum sizes of customer base, $Dmax$, where $D \sim U[0, Dmax]$. The results are shown in Figure 4.3.6.

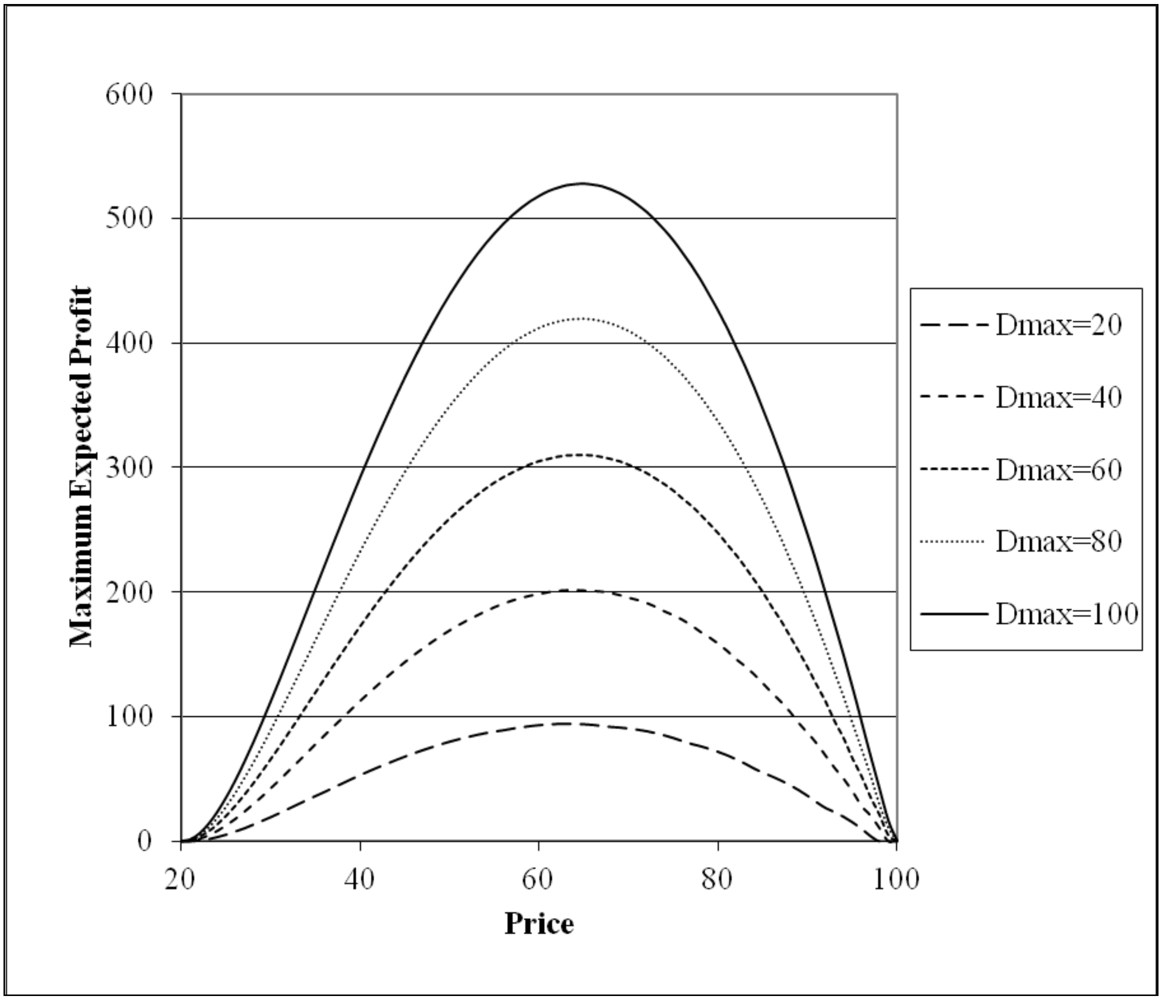

Figure 4.3.6. Maximum Expected Profit as a Function of Price for Various *Dmax* when *D*~U[0, *Dmax*], *c*=20 and *RP*~U[0, 100].

CHAPTER 5

THE NEWSVENDOR PROBLEM WITH PRICING: MULTIPLE-PRICE MODELS

5.1 INTRODUCTION

Now that we have considered a single-price model, we turn our attention to the possibility that the seller of an inventory of homogeneous items may wish to set multiple selling prices and restrict the number of items to be made available for sale at each price. An example in practice is an electronics retailer who, in an attempt to increase customers visiting his store, advertizes that a small number of items are to be sold at a deeply-discounted price, and after all of those are sold the remainder of the inventory will be sold at a regular price.

This chapter incorporates the demand model with price-dependent uncertainty from Chapter 3 into the newsvendor problem with pricing (NPP). The general form of the NPP used in this chapter is as follows. The seller decides on selling prices, $P_1, P_2, ..., P_n$, where $P_1 < P_2 < ... < P_n$, and the corresponding number of items to be made for sale at each of those price levels, $Q_1, Q_2, ..., Q_n$. The condition of strictly increasing prices may be appropriate for a seller of "urgent" items such as airplane tickets, as opposed to non-urgent items such as newspapers that are easily replaced with an alternative when they expire. The seller must decide on these prices and inventory levels before realizing exact demand, which, from Chapter 3, is dependent upon 1) the number of customers in the population, denoted *d* if known exactly, or *D* if random, and 2) the distribution of reservation prices (how much, at most, individuals are willing to pay for an item) of the customers. The reservation price of each customer, denoted *RP*, is random with a known cumulative distribution function, $F_{RP}(\cdot)$. It is assumed that customers arrive in random order, and each customer's willingness to buy depends only on the selling price of an item relative to that customer's reservation price. For each item the seller makes available for sale, a marginal cost of c is incurred, regardless of whether or not the item is ultimately sold.

Section 5.2 considers the case of a seller who chooses to set two selling prices with corresponding inventory levels. First, an expected profit function for the case where both selling prices and inventory levels are discrete values is derived. From this, a normal approximation is derived, which is used to determine the optimal number of items to be made available for sale at the higher price. This reduces the number of decision variables to be optimized to three. The section finishes with a numerical example that is solved using the model presented here. The solution is compared with one found using a model from the literature.

Section 5.3 considers the general case of a seller who chooses n selling prices and corresponding inventory levels. To find the optimal values of the decision variables, a dynamic programming solution is presented, and demonstrated with an example.

5.2 DUAL-PRICE MODELS

In this section, the seller sets two selling prices (P_1 and P_2 , $P_1 < P_2$) and corresponding inventory levels (Q_1 and Q_2). Two distributions are of importance, and are derived first. First, the number of customers willing to pay price P_1 for an item, denoted $X(P_1)$. Second, the number of customers leftover (once all of the items at price P_1 have been sold) who are willing to pay price P_2 for an item is denoted $Y \mid X(P_1, Q_1, P_2)$. Using these distributions, an expected profit function is derived. Following this, normal approximations of $X(P_1)$, $Y \mid X(P_1, Q_1, P_2)$ and the expected profit function are presented. The normal approximation of the expected profit function is used to derive a calculation for Q_2^*, the optimal number of items to be made available for sale at price P_2. The section finishes with a numerical example.

5.2.1 DISTRIBUTION OF $X(P_1)$ IN THE DISCRETE MODEL

In the case of the seller selecting two selling prices, P_1 and P_2, and two corresponding inventory levels, Q_1 and Q_2, customers purchase low-price items first, since the items are homogeneous and no customer will pay the high price for an item if the same item is available at the low price. In this section, the probability mass function of demand for low-price items, $X(P_1)$, denoted $h_{X(P_1)}(\cdot)$, is derived.

Lemma 5.2.1. For $X(P_1)$, *the number of customers with reservation prices at least as high as* P_1, *the probability mass function is that of a random variable that follows a binomial distribution with parameters d and* $1 - F_{RP}(P_1)$:

$$h_{X(P_1)}(x) = Prob\big(X(P_1) = x\big) = \binom{d}{x}\big(1 - F_{RP}(P_1)\big)^x \big(F_{RP}(P_1)\big)^{d-x}. \tag{5.2.1}$$

Proof. The number of customers available to possibly purchase a low-price item is the entire population *d*, and can be thought of as the number of independent yes/no experiments (each customer either wants to buy one item or does not). The probability that a customer has a reservation price of at least P_1 is $1 - F_{RP}(P_1)$, and can be thought of as the probability of a success. Given these two conditions, the number of customers who are willing to buy an item at price P_1 is a random variable, $X(P_1)$, that follows a binomial distribution with parameters *d* and $1 - F_{RP}(P_1)$, as in (5.2.1). Note that this result is the same as in the single price-quantity pair model. *Q.E.D.*

5.2.2 DISTRIBUTION OF $Y(P_1, Q_1, P_2) \mid X(P_1)$ IN THE DISCRETE MODEL

Once all of the low-price items have been sold, it is possible that there are still some leftover customers who were willing to pay P_1 for an item, but were unable to purchase one since they were sold out. Of these leftover customers, some have reservation prices high enough that they are willing to pay P_2 for an item. Denote $Y(P_1, Q_1, P_2) \mid X(P_1)$ as the number of leftover customers willing to buy a high-price item. Here, the probability mass function for the number of leftover customers willing to purchase an item at the high price, denoted $h_{Y(P_1, Q_1, P_2)|X(P_1)}(\cdot)$, is derived.

Lemma 5.2.2. For $Y(P_1, Q_1, P_2) \mid X(P_1)$, the number of leftover customers with reservation prices at least as high as P_2 after all of the low-price items have been sold, *the probability mass function is that of a random variable that follows a binomial distribution with parameters* $\left(X(P_1)-Q_1\right)^+$ *and* $\left(1-F_{RP}(P_2)\right)/\left(1-F_{RP}(P_1)\right)$*, as given by*:

$$h_{Y(P_1,Q_1,P_2)|X(P_1)}(y\,|\,x)=\binom{x-Q_1}{y}\left(\frac{1-F_{RP}(P_2)}{1-F_{RP}(P_1)}\right)^{y}\left(1-\frac{1-F_{RP}(P_2)}{1-F_{RP}(P_1)}\right)^{x-Q_1-y}, \qquad (5.2.2)$$

where x denotes a realized value of $X(P_1)$, $Q_1 \le x \le d$ *and* $0 \le y \le x-Q_1$.

Proof. Since the items being sold are homogeneous, all low-price items must be sold before any high-price items are sold. Of the original *d* customers, $X(P_1)$ are willing to pay P_1 for an item, and $\min\left(X(P_1),Q_1\right)$ purchase one. This reduces the size of the customer base to which high-price items can be sold to $\left(X(P_1)-Q_1\right)^+$ customers. Of these $\left(X(P_1)-Q_1\right)^+$ customers, the number with reservation prices at least as high as P_2 is a random variable denoted $Y(P_1,Q_1,P_2)\,|\,X(P_1)$.

The probability that one of these $\left(X(P_1)-Q_1\right)^+$ customers will be willing to buy a high-price item is found using the definition of conditional probability:

$$Prob\left(RP \ge P_2 \,|\, RP \ge P_1\right)=\frac{Prob\left(\left(RP \ge P_1\right)\cap\left(RP \ge P_2\right)\right)}{Prob\left(RP \ge P_1\right)}. \qquad (5.2.3)$$

Since anyone with a reservation price of at least P_2 also has a reservation price of at least P_1, $Prob\left(\left(RP \ge P_1\right)\cap\left(RP \ge P_2\right)\right)$ is simply $Prob\left(RP \ge P_2\right)$. Therefore, (5.2.3) reduces to:

$$Prob\left(RP \ge P_2 \,|\, RP \ge P_1\right)=\frac{Prob\left(RP \ge P_2\right)}{Prob\left(RP \ge P_1\right)} \qquad (5.2.4)$$

or

$$Prob\left(RP \geq P_2 \mid RP \geq P_1\right) = \frac{1 - F_{RP}(P_2)}{1 - F_{RP}(P_1)}. \quad (5.2.5)$$

Similar to $X(P_1)$ in Lemma 5.2.1, $Y(P_1, Q_1, P_2) \mid X(P_1)$ is a random variable that follows a binomial distribution. The parameters of $Y(P_1, Q_1, P_2) \mid X(P_1)$ are $\left(X(P_1) - Q_1\right)^+$, the number of independent yes/no experiments (the size of the leftover customer base) and $\left(1 - F_{RP}(P_2)\right)/\left(1 - F_{RP}(P_1)\right)$, the probability of a success (the probability one of the leftover $\left(X(P_1) - Q_1\right)^+$ customers will have a reservation price at least as high as P_2) as given by (5.2.5). *Q.E.D.*

5.2.3 THE DUAL-PRICE DISCRETE MODEL

In this section, a new model for a seller who is going to maximize expected profit by selling an inventory of homogeneous items at two difference prices: P_1 (low price) and P_2 (high price), where $0 < P_1 < P_2$, is presented. Note that because the items being sold are identical, all low-price items have to be sold before any high-price items can be sold (no one will pay the high price for an item if the same item is available at the low price). Two inventory levels are chosen by the seller: Q_1 (quantity of items to be made available for sale at the low price) and Q_2 (quantity of items to be made available for sale at the high price). The seller knows the size of the customer base, *d*. Denote the number of customers willing to purchase a low-price item (i.e.; those with a reservation price at least as high as P_1) as $X(P_1)$. Denote the number of leftover customers willing to purchase a high-price item (i.e.; those with a reservation price at least as high as P_2, but who couldn't buy a low-price item because they were sold out) as $Y(P_1, Q_1, P_2) \mid X(P_1)$ For any random variable used here, $Z(\cdot)$, the corresponding probability function is denoted $h_Z(\cdot)$ or $f_Z(\cdot)$ (for discrete and continuous cases, respectively) and the corresponding cumulative distribution function is denoted $F_Z(\cdot)$.

Lemma 5.2.3. Expected total profit, $E[\Pi_{Total}(P_1, Q_1, P_2, Q_2)]$, *for the dual-price discrete model is given by:*

$$\begin{aligned} E[\Pi_{Total}(P_1, Q_1, P_2, Q_2)] &= P_1 \sum_{x=0}^{Q_1} x h_{X(P_1)}(x) \\ &+ \sum_{x=Q_1+1}^{Q_1+Q_2} \left(P_1 Q_1 + P_2 \sum_{y=1}^{x-Q_1} y h_{Y(P_1, Q_1, P_2)|X(P_1)}(y \mid x) \right) h_{X(P_1)}(x) + P_1 Q_1 \sum_{x=Q_1+Q_2}^{d} h_{X(P_1)}(x) \\ &+ P_2 \sum_{x=Q_1+Q_2+1}^{d} \left(\sum_{y=1}^{Q_2} y h_{Y|X(P_1, Q_1, P_2)}(y \mid x) + Q_2 \sum_{y=Q_2+1}^{x-Q_1} h_{Y|X(P_1, Q_1, P_2)}(y \mid x) \right) h_{X(P_1)}(x) \\ &- c(Q_1 + Q_2). \end{aligned} \tag{5.2.6}$$

Proof. The proof of Lemma 5.2.3 begins with the derivation of expected revenue terms covering three sets of conditions of $X(P_1)$:

1. There are enough customers with reservation prices at least as high as P_1 such that the low-price items may sell-out, but with no possibility of unmet demand for low-price items $(X(P_1) \le Q_1)$.
2. There are enough customers with reservation prices at least as high as P_1 such that the low-price items will sell-out and high-price items may sell-out, but with no possibility of unmet demand for high-price items $(Q_1 + 1 \le X(P_1) \le Q_1 + Q_2)$.

3. There are enough customers with reservation prices at least as high as P_1 such that the low-price items will sell-out, high-price items may sell-out, and there is the possibility of unmet demand for high-price items $(Q_1 + Q_2 + 1 \le X(P_1) \le d)$.

Refer to Figure 5.2.1:

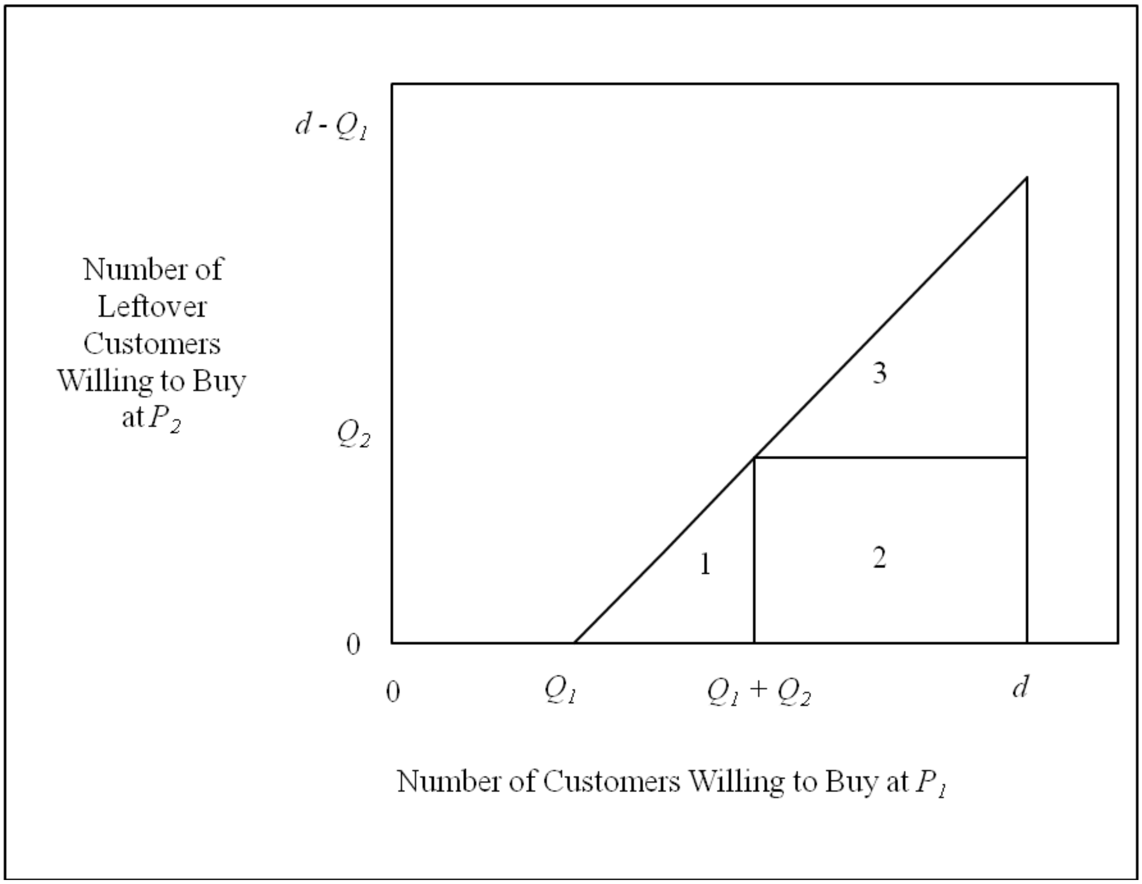

Figure 5.2.1. Diagram of Regions in Expected Profit Function.

Case 1: Expected Revenue for $X(P_1) \le Q_1$.

Here, $X(P_1)$, the number of customers with reservation prices at least as high as P_1, can take on values of $[0, Q_1]$. Since there is no possibility of unmet demand for low-price items, there is no possibility of selling high-price items (recall that all low-price items must be sold before any high-price items can be sold). Therefore, for $X(P_1) \le Q_1$, $Y(P_1, Q_1, P_2) \mid X(P_1)$ and the expected revenue is independent of $Y(P_2)$. The result is an expected revenue term, $E[R_1(P_1, Q_1)]$, that is simply the low-price multiplied by the expected quantity of low-price items sold for $X(P_1) \le Q_1$:

$$E[R_1(P_1, Q_1)] = P_1 \sum_{x=0}^{Q_1} x h_{X(P_1)}(x). \tag{5.2.7}$$

Case 2: Expected Revenue for $Q_1 + 1 \le X(P_1) \le Q_1 + Q_2$.

Here, $X(P_1)$, the number of customers with reservation prices at least as high as P_1, can take on values on $[Q_1 + 1, Q_1 + Q_2]$. As such, all of the low price items will sell out, some of the high-price items may be sold, but there is no possibility of unmet demand for high-price items. The result is an expected total revenue term, $E[R_2(P_1, Q_1, P_2, Q_2)]$, that is made up of the expected revenue from a sell-out of low-price items, $E[R_{2,1}(P_1, Q_1, Q_2)]$:

$$E[R_{2,1}(P_1, Q_1, Q_2)] = P_1 Q_1 \sum_{x=Q_1+1}^{Q_1+Q_2} h_{X(P_1)}(x) \tag{5.2.8}$$

and expected revenue from high-price items, $E[R_{2,2}(P_1, Q_1, P_2, Q_2)]$:

$$E[R_{2,2}(P_1, Q_1, P_2, Q_2)] = P_2 \sum_{x=Q_1+1}^{Q_1+Q_2} \left(\sum_{y=1}^{x-Q_1} y h_{Y(P_1, Q_1, P_2) \mid X(P_1)}(y \mid x) \right) h_{X(P_1)}(x). \tag{5.2.9}$$

Note that in (5.2.9), $Y(P_1, Q_1, P_2) \mid X(P_1)$ can only take on values up to and including $x - Q_1$, since that is the size of the remaining customer base after all of the low-price items have been sold. For a diagrammatic representation, refer to the region labelled “1” in Figure 5.2.1.

Adding (5.2.8) and (5.2.9) and rearranging gives $E[R_2(P_1, Q_1, P_2, Q_2)]$, an expression for the expected revenue for $Q_1 + 1 \le X(P_1) \le Q_1 + Q_2$:

$$E[R_2(P_1, Q_1, P_2, Q_2)] = \sum_{x=Q_1+1}^{Q_1+Q_2} \left(P_1 Q_1 + P_2 \sum_{y=1}^{x-Q_1} y h_{Y(P_1, Q_1, P_2)|X(P_1)}(y \mid x) \right) h_{X(P_1)}(x). \qquad (5.2.10)$$

Case 3: Expected Revenue for $Q_1 + Q_2 + 1 \le X(P_1) \le d$.

Here, $X(P_1)$, the number of customers with reservation prices at least as high as P_1, can take on values $[Q_1 + Q_2 + 1, d]$. As such, all of the low price items will sell out, some of the high-price items may be sold, and there is the possibility of unmet demand for high-price items. The result is an expected revenue term, $E[R_3(P_1, Q_1, P_2, Q_2)]$, that is made up from the expected revenue from a sell-out of low-price items, $E[R_{3,1}(P_1, Q_1, Q_2)]$:

$$E[R_{3,1}(P_1, Q_1, Q_2)] = P_1 Q_1 \sum_{x=Q_1+Q_2+1}^{d} h_{X(P_1)}(x) \qquad (5.2.11)$$

and the expected revenue from high-price items, $E[R_{3,2}(P_1, Q_1, P_2, Q_2)]$:

$$E[R_{3,2}(P_1, Q_1, P_2, Q_2)] =$$

$$\sum_{x=Q_1+Q_2+1}^{d} \left(P_2 \sum_{y=1}^{Q_2} y h_{Y(P_1, Q_1, P_2)|X(P_1)}(y \mid x) + P_2 Q_2 \sum_{y=Q_2+1}^{x-Q_1} h_{Y(P_1, Q_1, P_2)|X(P_1)}(y \mid x) \right) h_{X(P_1)}(x). \qquad (5.2.12)$$

Adding (5.2.11) and (5.2.12) and rearranging gives $E[R_3(P_1, Q_1, P_2, Q_2)]$, an expression for the expected revenue for $Q_1 + Q_2 + 1 \le X(P_1) \le d$:

$$E[R_3(P_1, Q_1, P_2, Q_2)] = P_1 Q_1 \sum_{x=Q_1+Q_2+1}^{d} h_{X(P_1)}(x)$$
$$+ P_2 \sum_{x=Q_1+Q_2+1}^{d} \left(\sum_{y=1}^{Q_2} y h_{Y|X(P_1, Q_1, P_2)}(y \mid x) + Q_2 \sum_{y=Q_2+1}^{x-Q_1} h_{Y|X(P_1, Q_1, P_2)}(y \mid x) \right) h_{X(P_1)}(x). \quad (5.2.13)$$

For a diagrammatic representation of the revenues from high-price items, refer to Figure 5.2.1; the regions labelled "2" (no unmet demand of high-price items, the second term inside the brackets of (5.2.13)) and "3" (unmet demand of high-price items, the third term inside the brackets of (5.2.13)).

To calculate expected total revenue, $E[R_{Total}(P_1, Q_1, P_2, Q_2)]$, add (5.2.7), (5.2.10) and (5.2.13) to give:

$$E[R_{Total}(P_1, Q_1, P_2, Q_2)] = P_1 \sum_{x=0}^{Q_1} x h_{X(P_1)}(x)$$
$$+ \sum_{x=Q_1+1}^{Q_1+Q_2} \left(P_1 Q_1 + P_2 \sum_{y=1}^{x-Q_1} y h_{Y(P_1, Q_1, P_2)|X(P_1)}(y \mid x) \right) h_{X(P_1)}(x) + P_1 Q_1 \sum_{x=Q_1+Q_2+1}^{d} h_{X(P_1)}(x) \quad (5.2.14)$$
$$+ P_2 \sum_{x=Q_1+Q_2+1}^{d} \left(\sum_{y=1}^{Q_2} y h_{Y(P_1, Q_1, P_2)|X(P_1)}(y \mid x) + Q_2 \sum_{y=Q_2+1}^{x-Q_1} h_{Y(P_1, Q_1, P_2)|X(P_1)}(y \mid x) \right) h_{X(P_1)}(x).$$

Finally, as expected total profit, $E[\Pi_{Total}(P_1, Q_1, P_2, Q_2)]$, is equal to expected total revenue (Equation 4.7.9) less costs, subtract $c(Q_1 + Q_2)$ from (5.2.14) to get (5.2.6). *Q.E.D.*

5.2.4 DISTRIBUTION OF $X(P_1)$ IN THE NORMAL APPROXIMATION MODEL

In this section, the distribution of the number of customer willing to pay price P_1 for an item is derived. Recall from Chapter 3, $X(P_1)$ is approximated by a normal random variable with mean

$$\mu_{X(P_1)} = d\left(1 - F_{RP}(P_1)\right) \tag{5.2.15}$$

and variance

$$\sigma^2_{X(P_1)} = d\left(1 - F_{RP}(P_1)\right)\left(F_{RP}(P_1)\right), \tag{5.2.16}$$

assuming both parameters are greater than five. Using (5.2.15) and (5.2.16), the probability density function for $X(P_1)$ is approximated by:

$$f_{X(P_1)}(x) \approx \frac{1}{\sqrt{2\pi d\left(1 - F_{RP}(P_1)\right)\left(F_{RP}(P_1)\right)}} \exp\left(-\frac{\left(x - d\left(1 - F_{RP}(P_1)\right)\right)^2}{2d\left(1 - F_{RP}(P_1)\right)\left(F_{RP}(P_1)\right)}\right). \tag{5.2.17}$$

5.2.5 DISTRIBUTION OF $Y(P_1, Q_1, P_2) \mid X(P_1)$ IN THE NORMAL APPROXIMATION MODEL

For $Y(P_1, Q_1, P_2) \mid X(P_1)$, the probability density function is approximated by that of a normal random variable with mean

$$\mu_{Y(P_1, Q_1, P_2)\mid X(P_1)} = \left(x - Q_1\right)\left(\frac{1 - F_{RP}(P_2)}{1 - F_{RP}(P_1)}\right) \tag{5.2.18}$$

and variance

$$\sigma^2_{Y(P_1, Q_1, P_2)\mid X(P_1)} = \left(x - Q_1\right)\left(\frac{1 - F_{RP}(P_2)}{1 - F_{RP}(P_1)}\right)\left(1 - \frac{1 - F_{RP}(P_2)}{1 - F_{RP}(P_1)}\right), \tag{5.2.19}$$

assuming both parameters are greater than five. If the assumptions are not met, the exact model is used. Using (5.2.18) and (5.2.19), the corresponding probability density function for $Y(P_1, Q_1, P_2) \mid X(P_1)$ is approximated by:

$$f_{Y(P_1, Q_1, P_2)|X(P_1)}(y \mid x) \approx \frac{1}{\sqrt{2\pi\left(x - Q_1\right)\left(\frac{1 - F_{RP}(P_2)}{1 - F_{RP}(P_1)}\right)\left(1 - \frac{1 - F_{RP}(P_2)}{1 - F_{RP}(P_1)}\right)}} * \\ \exp\left(-\frac{\left(y - \left(x - Q_1\right)\left(\frac{1 - F_{RP}(P_2)}{1 - F_{RP}(P_1)}\right)\right)^2}{2\left(x - Q_1\right)\left(\frac{1 - F_{RP}(P_2)}{1 - F_{RP}(P_1)}\right)\left(1 - \frac{1 - F_{RP}(P_2)}{1 - F_{RP}(P_1)}\right)}\right). \quad (5.2.20)$$

5.2.6 THE DUAL-PRICE NORMAL APPROXIMATION MODEL

The model from section 5.2.3 requires that inventory levels take on only discrete values. To allow for non-discrete values of inventory (e.g.; when the units of measurement are for weights or volumes), summations are replaced with integrals and the normal approximation to the binomial distribution is used. A random variable that follows a binomial distribution with parameters *d* (number of independent yes/no experiments) and *Prob* (probability of success) can be approximated by a normally distributed random variable with a mean of *dProb* and a variance of *dProb*(1-*Prob*), provided these parameters are both greater than five.

The expected profit approximation model, analogous to (5.2.6) is given by:

$$E[\Pi_{Total}(P_1, Q_1, P_2, Q_2)] = P_1 \int_{x=0}^{Q_1} x f_{X(P_1)}(x)dx$$

$$+ \int_{x=Q_1}^{Q_1+Q_2} \left(P_1 Q_1 + P_2 \int_{y=0}^{x-Q_1} y f_{Y(P_1, Q_1, P_2)|X(P_1)}(y \mid x)dy \right) f_{X(P_1)}(x)dx + P_1 Q_1 \int_{x=Q_1+Q_2}^{\infty} f_{X(P_1)}(x)dx$$

(5.2.21)

$$+ \int_{x=Q_1+Q_2}^{\infty} \left(P_2 \left(\int_{y=0}^{Q_2} y f_{Y(P_1, Q_1, P_2)|X(P_1)}(y \mid x)dy + Q_2 \int_{y=Q_2}^{x-Q_1} f_{Y(P_1, Q_1, P_2)|X(P_1)}(y \mid x)dy \right) \right) f_{X(P_1)}(x)dx$$

$$-c(Q_1 + Q_2).$$

5.2.7 SOLUTION FOR Q_2^* IN THE DUAL-PRICE MODEL

In this section, a solution for Q_2^*, the optimal number of items to be made available for sale at a given high price, P_2, is provided. The result can be used to reduce the number of decision variables required to optimize expected profit from four to three.

Lemma 5.2.4. For the seller who is setting two prices and two corresponding inventory levels, the optimal number of items to be made available for sale at price P_2 *is given by:*

$$Q_2^* = F^{-1}_{Y(P_1, Q_1, P_2)|X(P_1)}\left(\frac{P_2 - c}{P_2} \right). \tag{5.2.22}$$

Proof. Taking the partial derivative of the expected profit function with respect to Q_2, setting equal to zero and solving for Q_2, gives Q_2^*. Using the Leibniz integral rule, the partial derivative of (5.2.21) is:

$$\frac{\partial}{\partial Q_2} E[\Pi_{Total}(P_1, Q_1, P_2, Q_2)] = P_2 \int_{y=0}^{Q_1} y f_{Y(P_1, Q_1, P_2)|X(P_1)}(y \mid Q_1 + Q_2) dy$$

$$+ P_2 \left(\int_{x=Q_1+Q_2}^{\infty} Q_2 f_{Y(P_1, Q_1, P_2)|X(P_1)}(Q_2 \mid x) dx - \left(\int_{y=0}^{Q_1} y f_{Y(P_1, Q_1, P_2)|X(P_1)}(y \mid Q_1 + Q_2) dy \right) \right)$$

$$+ P_2 \left(\int_{x=Q_1+Q_2}^{\infty} \int_{y=Q_2}^{x-Q_1} f_{Y(P_1, Q_1, P_2)|X(P_1)}(y \mid x) dy dx \right) \quad (5.2.23)$$

$$+ P_2 Q_2 \left(\int_{x=Q_1+Q_2}^{\infty} \left(- f_{Y(P_1, Q_1, P_2)|X(P_1)}(Q_2 \mid x) \right) dx - \left(\int_{y=Q_2}^{Q_2} f_{Y(P_1, Q_1, P_2)|X(P_1)}(y \mid Q_1 + Q_2) dy \right) \right) - c,$$

which reduces to:

$$\frac{\partial}{\partial Q_2} E[\Pi_{Total}(P_1, Q_1, P_2, Q_2)] = P_2 \left(\int_{x=Q_1+Q_2}^{\infty} \int_{y=Q_2}^{x-Q_1} f_{Y(P_1, Q_1, P_2)|X(P_1)}(y \mid x) dy dx \right) - c. \quad (5.2.24)$$

Setting (5.2.24) equal to zero and rearranging gives:

$$\int_{x=Q_1+Q_2^*}^{\infty} \int_{y=Q_2^*}^{x-Q_1} f_{Y(P_1, Q_1, P_2)|X(P_1)}(y \mid x) dy dx = \frac{c}{P_2} \quad (5.2.25)$$

Note that the left-hand side of (5.2.25) is simply the probability that $Y(P_1, Q_1, P_2) \mid X(P_1)$ is greater than Q_2^*. Therefore, (5.2.25) can be rewritten as:

$$1 - F_{Y(P_1, Q_1, P_2)|X(P_1)}(Q_2^*) = \frac{c}{P_2}. \quad (5.2.26)$$

Rearranging (5.2.26) gives:

$$F_{Y(P_1, Q_1, P_2)|X(P_1)}(Q_2^*) = \frac{P_2 - c}{P_2} \quad (5.2.27)$$

or

$$Q_2^* = F^{-1}_{Y(P_1,\,Q_1,\,P_2)|X(P_1)}\left(\frac{P_2 - c}{P_2}\right). \tag{5.2.28}$$

To confirm that using (5.2.28) results in a maximum expected profit, begin by taking the second derivative of (5.2.21):

$$\frac{\partial^2}{\partial {Q_2}^2} E[\Pi_{Total}(P_1, Q_1, P_2, Q_2)] =$$

$$P_2\left(\int_{x=Q_1+Q_2}^{\infty} \left(- f_{Y(P_1,\,Q_1,\,P_2)|X(P_1)}(Q_2 \mid x)\right)dy - \int_{y=Q_2}^{Q_2} f_{Y(P_1,\,Q_1,\,P_2)|X(P_1)}(y \mid Q_1 + Q_2)dy \right), \tag{5.2.29}$$

which simplifies to:

$$\frac{\partial^2}{\partial {Q_2}^2} E[\Pi_{Total}(P_1, Q_1, P_2, Q_2)] = -P_2 \int_{x=Q_1+Q_2}^{\infty} f_{Y(P_1,\,Q_1,\,P_2)|X(P_1)}(Q_2 \mid x)dy. \tag{5.2.30}$$

Since (5.2.30) is negative for all allowable values of Q_2, (5.2.28) does occur at the maximum of (5.2.21). *Q.E.D.*

5.2.8 NUMERICAL EXAMPLE WITH A DUAL-PRICE SOLUTION

This section discusses a dual price-quantity NPP example from the literature (Dana, 1999), and shows how the approach and solution from the literature are different from what is derived above.

The example is as follows. A seller of homogeneous items decides on selling prices and corresponding inventory levels before knowing the exact size of the customer base, and how much each customer is willing to pay for an item. There are two possible sizes of customer base, $d_1 = 100$ and $d_2 = 400$, which are equally likely. Denote the probability of the high-demand state being realized as $Prob_{d2}$, which is 50%. The maximum amount each customer is willing to pay for an item (reservation price) is random, and follows a uniform distribution as RP ~ U[0, P_{max} = 100]. In addition, the seller incurs a unit cost of c = 20 for each item made available for sale, regardless or whether or not the item ultimately sold.

Dana's solution (presented in full in Appendix A) is based upon the well-known monopolist's pricing strategy of setting marginal revenue equal to marginal cost to find the optimal quantity, and then using that quantity in the original demand function to find optimal price. Such is the procedure to find the optimal low price P_1^*, and corresponding quantity Q_1^*, with the demand curve found using the low-demand state d_1. Next, a residual demand curve is found (since the potential customer base for high-price items has been reduced by the sale of low-price items) and the process is repeated to find P_2^* and Q_2^*. For the example given above, Dana's solution is $P_1^* = 60$, $Q_1^* = 40$, $P_2^* = 70$ and $Q_2^* = 90$.

To address this example with an expected profit equation derived above, (5.2.6) is used as follows. When the number of customers is 100, $D = d_1 = 100$ is used with the corresponding expected profit occurring 50% of the time. When the number of customers is 400, $D = d_2 = 400$ is used with the corresponding expected profit occurring 50% of the time. Using the definition of expected value, the overall expected profit is given by:

$$E[\Pi_{Overall}] = 0.5E[\Pi_{Total} \mid D = d_1] + 0.5E[\Pi_{Total} \mid D = d_2]. \qquad (5.2.31)$$

where $E[\Pi_{Total} \mid D]$ for each case of D is found using (5.2.6).

For the example given above, the optimal two price-quantity pair that maximizes (5.2.31) is found to be $P_1^* = 60$, $Q_1^* = 42$, $P_2^* = 70$ and $Q_2^* = 87$ (found using a brute force search). Note the optimal prices found using Dana's approach and Equation 4.7.1 are the same, however the optimal quantities are different. The difference in maximum expected profit found using the two approaches is small (2821 using Dana's model and 2824 using the exact model derived here). For this specific example, this suggests that the difference in underlying approaches (Dana assumes the number of items purchased at a given price is deterministic, whereas we assume it to be probabilistic) may not be relevant.

For further comparison, consider the optimal single-price solution found by a search over a grid of values using Equation 4.2.14: $P^* = 65$, $Q^* = 135$, and $E[\Pi^*(P^*, Q^*)] = 2766$. In this case the two-price strategy of Section 5.2.3 (maximum expected profit of 2824) provides a solution that is superior to the single-price strategy of Section 4.2.2.

5.2.9 THE BANDWAGON EFFECT

Consider a seller with an inventory of homogeneous items who sets selling price(s) and corresponding inventory level(s) before realizing exact demand. Specifically, consider the case of a seller who can choose a certain number of items, Q_1, to be sold at a first price, P_1, and a certain number of items, Q_2, to be sold at a second price, P_2. The seller incurs a unit cost of c for each item made available for sale, regardless of whether or not the item is ultimately sold. Assume that once all of the items at price P_1 have been sold, the seller makes it known to the public that Q_1 items have already been sold and that only Q_2 items are remaining. This increases demand such that the new distribution of reservation prices, RP_2, is stochastically larger than the original distribution of reservation prices, RP_1 (the Bandwagon Effect). While the seller does not know the reservation prices of individual customers, it is assumed that he knows both distributions, $F_{RP1}(\cdot)$ and $F_{RP2}(\cdot)$. For other examples of the Bandwagon Effect, see DeSerpa, 1996 and Becker, 1991.

Here we derive the distributions for the number of customers willing to pay the two possible selling prices, and the expected profit function.

5.2.10 DISTRIBUTION OF $X(P_1)$

Consider first, the number of customers willing to pay P_1 for one of the initial Q_1 items. As in the basic model, denote this number of customers as random variable $X(P_1)$, which follows a binomial distribution with probability mass function:

$$h_{X(P_1)}(x) = Prob\big(X(P_1) = x\big) = \binom{d}{x}\big(1 - F_{RP}(P_1)\big)^x \big(F_{RP}(P_1)\big)^{d-x}. \qquad (5.2.32)$$

The proof is provided in Lemma 5.2.1.

5.2.11 DISTRIBUTION OF $Y(P_1, Q_1, P_2) \,|\, X(P_1)$

Once all of the low-price items have been sold, the seller makes it known to the public that Q_1 items have already been sold and that only Q_2 items are remaining. The result is a shift in reservation prices to RP_2. It is possible that there are still some leftover customers who were willing to pay P_1 for an item, but were unable to purchase one since they were sold out. Of these leftover customers, some have reservation prices high enough that they are willing to pay P_2 for an item. Denote $Y(P_1, Q_1, P_2) \mid X(P_1)$ as the number of leftover customers willing to buy a high-price item. Here, the probability mass function for the number of leftover customers willing to purchase an item at the high price, denoted $h_{Y(P_1, Q_1, P_2)|X(P_1)}(\cdot)$, is derived.

Lemma 5.2.5. For $Y(P_1, Q_1, P_2) \mid X(P_1)$, *the number of leftover customers with reservation prices at least as high as* P_2 *after all of the low-price items have been sold, the probability mass function is that of a random variable that follows a binomial distribution with parameters* $(X(P_1) - Q_1)^+$ *and* $(1 - F_{RP2}(P_2))/(1 - F_{RP1}(P_1))$, *as given by*:

$$h_{Y(P_1, Q_1, P_2)|X(P_1)}(y \mid x) = \binom{x - Q_1}{y} \left(\frac{1 - F_{RP2}(P_2)}{1 - F_{RP1}(P_1)} \right)^y \left(1 - \frac{1 - F_{RP2}(P_2)}{1 - F_{RP1}(P_1)} \right)^{x - Q_1 - y}, \tag{5.2.33}$$

where x denotes a realized value of $X(P_1)$, $Q_1 \le x \le d$ *and* $0 \le y \le x - Q_1$.

Proof. Since the items being sold are homogeneous, all low-price items must be sold before any high-price items are sold. Of the original *d* customers, $X(P_1)$ are willing to pay P_1 for an item, and $\min(X(P_1), Q_1)$ purchase one. This reduces the size of the customer base to which high-price items can be sold to $(X(P_1) - Q_1)^+$ customers. Of these $(X(P_1) - Q_1)^+$ customers, the number with reservation prices at least as high as P_2 is a random variable denoted $Y(P_1, Q_1, P_2) \mid X(P_1)$.

The probability that one of these $\left(X(P_1)-Q_1\right)^+$ customers will be willing to buy a high-price item is found using the definition of conditional probability:

$$Prob\left(RP_2 \geq P_2 \mid RP_1 \geq P_1\right) = \frac{Prob\left(\left(RP_1 \geq P_1\right) \cap \left(RP_2 \geq P_2\right)\right)}{Prob\left(RP_1 \geq P_1\right)}. \tag{5.2.34}$$

Since anyone with a reservation price of at least P_2 also has a reservation price of at least P_1, $Prob\left(RP_1 \geq P_1 \cap RP_2 \geq P_2\right)$ is simply $Prob\left(RP_2 \geq P_2\right)$. Therefore, (5.2.3) reduces to:

$$Prob\left(RP_2 \geq P_2 \mid RP_1 \geq P_1\right) = \frac{Prob\left(RP_2 \geq P_2\right)}{Prob\left(RP_1 \geq P_1\right)} \tag{5.2.35}$$

or

$$Prob\left(RP_2 \geq P_2 \mid RP_1 \geq P_1\right) = \frac{1-F_{RP2}(P_2)}{1-F_{RP1}(P_1)}. \tag{5.2.36}$$

Similar to $X(P_1)$ in Lemma 5.2.1, $Y(P_1, Q_1, P_2) \mid X(P_1)$ is a random variable that follows a binomial distribution. The parameters of $Y(P_1, Q_1, P_2) \mid X(P_1)$ are $\left(X(P_1)-Q_1\right)^+$, the number of independent yes/no experiments (the size of the leftover customer base) and $\left(1-F_{RP2}(P_2)\right)/\left(1-F_{RP1}(P_1)\right)$, the probability of a success (the probability one of the leftover $\left(X(P_1)-Q_1\right)^+$ customers will have a reservation price at least as high as P_2) as given by (5.2.33). *Q.E.D.*

5.2.12 THE EXPECTED PROFIT FUNCTION FOR THE BANDWAGON EFFECT

Here we present the expected profit function for the seller.

Lemma 5.2.6. Expected total profit, $E[\Pi_{Total}(P_1, Q_1, P_2, Q_2)]$, *for the dual-price with bandwagon effect model is given by:*

$$E[\Pi_{Total}(P_1, Q_1, P_2, Q_2)] = P_1 \sum_{x=0}^{Q_1} x h_{X(P_1)}(x)$$

$$+ \sum_{x=Q_1+1}^{Q_1+Q_2} \left(P_1 Q_1 + P_2 \sum_{y=1}^{x-Q_1} y h_{Y(P_1, Q_1, P_2)|X(P_1)}(y \mid x) \right) h_{X(P_1)}(x) + P_1 Q_1 \sum_{x=Q_1+Q_2}^{d} h_{X(P_1)}(x)$$

$$+ P_2 \sum_{x=Q_1+Q_2+1}^{d} \sum_{y=1}^{Q_2} y h_{Y|X(P_1, Q_1, P_2)}(y \mid x) h_{X(P_1)}(x) \qquad (5.2.37)$$

$$+ P_2 Q_2 \sum_{x=Q_1+Q_2+1}^{d} \sum_{y=Q_2+1}^{x-Q_1} h_{Y|X(P_1, Q_1, P_2)}(y \mid x) h_{X(P_1)}(x)$$

$$- c(Q_1 + Q_2).$$

The proof of Lemma 5.2.6 is the same as the proof for the expected total profit function for the basic dual-price model, provided in Lemma 5.2.3. The only difference is that the probability mass function for the number of leftover customers willing to buy an item at price P_2 is given by (5.2.33) instead of (5.2.2). *Q.E.D.*

5.2.13 BANDWAGON EFFECT EXAMPLE

Consider the case of a deterministic customer base of size, $d = 25$, a unit cost to the seller of $c = 5$, and an initial reservation price distribution of $RP_1 \sim U[0, 25]$. Assume that once half of the items are sold (i.e.; $Q_1 / (Q_1 + Q_2) \geq 50\%$), the bandwagon effect changes the distribution of reservation prices such that $RP_2 = a(Q_1, Q_2) RP_1$, where $a(Q_1, Q_2) = 1.25$.

The optimal conditions are found using a brute force search with the expected profit functions indicated below. If the seller chooses to set only one selling price and inventory level, the optimal conditions of $P^* = 15$, and $Q^* = 11$ result in an expected profit of $E[\Pi^*(P^*, Q^*)] = 86.7$ (found using Equation 4.2.1). If the seller chooses to set two selling prices and corresponding inventory levels, the optimal conditions of $P_1^* = 13$, $Q_1^* = 6$, $P_2^* = 19$ and $Q_2^* = 6$ resulting in an expected profit of $E[\Pi^*(P_1^*, Q_1^*, P_2^*, Q_2^*)] = 102.5$ (found using Equation 5.2.37), an increase of approximately 18%.

5.3 *n*-PRICE MODEL

In this section we consider the general case of a seller who sets *n* selling prices and corresponding inventory levels. First, we provide a dynamic programming approach which is used to maximize expected profit. The approach reduces the search space of the problem, giving a more efficient algorithm for finding the optimal solution than a traditional dynamic programming approach. The section is concluded with a numerical example.

5.3.1 EXPECTED PROFIT

Here we present a step-by-step procedure that can be used to maximize the total expected profit for the seller. The following information is required:

1. The number of selling prices that will be set, n. This is also the number of corresponding inventory levels that will be set, with at least one item made available for sale at each of the n prices.
2. The size of the customer base, d. Note that $d \geq n$.
3. The selling prices that the seller can choose from (for example, set by the market), $Price_1$, $Price_2$, …, $Price_{Pnum}$ (strictly increasing). Note that $Pnum \geq n$.
4. The cost incurred by the seller to make one unit available for sale, c.
5. The distribution of reservation prices of the customers, such that for each allowable selling price, the probability that a customer will be willing to pay a selling price can be determined: $Prob_{Price1}$, $Prob_{Price2}$, …, $Prob_{PricePNum}$.

Step 1 – Determine the prices that can possibly be charged at each stage, and the allowable sets of those prices.

Since each stage corresponds to one price level and there are n price levels, there are n stages. As the stage number increases, the prices being charged must strictly increase:

$$P_1 < P_2 < ... < P_n. \tag{5.3.1}$$

In the first stage (the stage with the lowest selling price), the lowest possible selling price is the lowest selling price that the seller can choose from, $Price_1$. In the second stage, the lowest possible selling price is the lowest selling price that is higher than the lowest possible selling price in the first stage. Therefore, the lowest possible selling price in the second stage is $Price_2$. In general, the lowest selling price that can possibly be charged in the m^{th} stage is $Price_m$.

In the final stage (the stage with the highest price, stage n), the highest possible selling price is the highest selling price that the seller can choose from, $Price_{Pnum}$. Therefore, the range of possible selling prices in the final stage is:

$$Price_n \le P_n \le Price_{PNum} . \tag{5.3.2}$$

In the penultimate stage, the highest possible selling price is the highest selling price that is lower than the highest possible selling price in the final stage, $Price_{PNum-1}$. Therefore, the range of possible selling prices in the penultimate stage is:

$$Price_{n-1} \le P_{n-1} \le Price_{PNum-1} . \tag{5.3.3}$$

This procedure can be repeated for all stages, and the range of possible selling prices in stage m is observed to be:

$$Price_m \le P_m \le Price_{PNum+m-n} . \tag{5.3.4}$$

Step 2 – Determine the inventory levels that can possibly be selected at each stage, and the allowable sets of those inventory levels.

For each of the n stages, at least one unit of inventory is made available for sale. Therefore, the minimum inventory level for any stage m is one.

If the minimum inventory level (one unit) is chosen at every stage other than the last, there can be up to $d - n + 1$ units of inventory made available for sale at the final stage. No more than this amount would be chosen, since doing so would result in the total number of units being made available for sale exceeding the size of the customer base. Therefore, the range of units to be made available for sale in the final stage is:

$$1 \le Q_n \le d - n + 1. \qquad (5.3.5)$$

Note that this range applies to each stage. Therefore, for any stage, m:

$$1 \le Q_m \le d - n + 1, \qquad (5.3.6)$$

providing

$$\sum_{i=1}^{n} Q_i \le d\ . \qquad (5.3.7)$$

Step 3 – For each stage and allowable combination of prices and inventory levels, determine the probabilities that a given number of customers is willing to pay the price being charged at that stage. The probability that at stage m there are exactly i customers willing to pay price P_m is denoted $r_m(i)$.

Step 3a – Calculate the probability distribution for the first stage.

The number of customers willing to pay the first-stage price P_1, is found using the probability mass function of the binomial distribution, where the population size is the size of the customer base d (since at the first stage, the entire customer base is available to possibly purchase an item), and the probability of a success (the probability of a customer having a reservation price at least as high as P_1) is $Prob_1$:

$$r_1(i \mid P_1) = \binom{d}{i} (Prob_1)^i (1\text{-}Prob_1)^{d-i}, \tag{5.3.8}$$

Equation 4.13.8 is used to calculate probabilities for i = 1, 2, …, d, and $Prob_1$ = $Prob_{Price1}$, $Prob_{Price2}$, …, $Prob_{Price(Pnum+1-n)}$. (Note that $r_1(i \mid P_1)$ are not calculated for i = 0. This is because there will be no use for such probabilities in the expected profit calculations later.)

Step 3b – Calculate the probability distribution for the second stage.

For the second stage, calculating the probability that exactly j customers (from a population size of $Population_2$) are remaining and willing to pay P_2 begins with the binomial probability mass function:

$$r_2(j) = \binom{Population_2}{j} (Prob_2)^j (1\text{-}Prob_2)^{Population_2 - j}, \tag{5.3.9}$$

where $Prob_2$ is the probability of a success, which in this case, is the probability that a customer has a reservation price at least as high as P_2, given that he had a reservation price at least as high as P_1. From the definition of conditional probability, this is given as:

$$Prob_2(P_1, P_2) = \frac{Prob_{P2}}{Prob_{P1}}. \tag{5.3.10}$$

The size of the population in the second stage, $Population_2$, depends on how many customers are "leftover" from the first stage. If there were more than Q_1 customers willing to pay P_1 (i.e.; there was unmet demand), there are "leftover" customers who now make up the population of customers who might have reservation prices high enough to be willing to buy an item at P_2. This means that the size of $Population_2$ can take on values between zero and $d - Q_1$. However, to allow for j customers remaining and possibly willing to pay P_2, the size of $Population_2$ must be at least j.

For each necessary size of $Population_2$, the respective probability of such an occurrence can is given by one of the previously-calculated values of $r_1(i \mid P_1)$. For example, the probability of having exactly j leftover customers in the second stage ($Population_2 = j$) is the probability that at the first stage, there were $Q_1 + j$ customers willing to pay P_1. This was previously calculated as $r_1(Q_1 + j \mid P_1)$. Therefore, the probability that exactly j customers are willing to pay P_2 at the second stage, given that exactly j customers are "leftover" from the first stage is found my multiplying (9) by $r_1(Q_1 + j \mid P_1)$. To account for all possible sizes of $Population_2$ that allow for j successes in the second stage, all allowable forms of (5.3.9) are multiplied by their respective values of $r_1(i \mid P_1)$, and the summation of the products is taken to give:

$$r_2(j \mid Q_1, P_1, P_2) = r_1(Q_1 + j \mid P_1)\binom{j}{j}\left(Prob_2(P_1, P_2)\right)^j \left(1\text{-}Prob_2(P_1, P_2)\right)^{j-j}$$

$$+ r_1(Q_1 + j + 1 \mid P_1)\binom{j+1}{j}\left(Prob_2(P_1, P_2)\right)^j \left(1\text{-}Prob_2(P_1, P_2)\right)^{j+1-j}$$

$$+ r_1(Q_1 + j + 2 \mid P_1)\binom{j+2}{j}\left(Prob_2(P_1, P_2)\right)^j \left(1\text{-}Prob_2(P_1, P_2)\right)^{j+2-j} \qquad (5.3.11)$$

$$+ \ldots + r_1(D \mid P_1)\binom{d-Q_1}{j}\left(Prob_2(P_1, P_2)\right)^j \left(1\text{-}Prob_2(P_1, P_2)\right)^{d-Q_1-j}.$$

which can be expressed as:

$$r_2(j \mid Q_1, P_1, P_2) =$$

$$\sum_{index_1=0}^{D-Q_1-j} r_1(Q_1 + j + index_1 \mid P_1)\binom{j+index_1}{j}\left(Prob_2(P_1, P_2)\right)^j \left(Prob_2(P_1, P_2)\right)^{index_1}. \qquad (5.3.12)$$

Step 3c – Calculate the probability distribution for the third stage.

The procedure for calculating the probabilities for the third stage is similar to that for the second stage. For the third stage, calculating the probability that exactly k customers (from a population size of $Population_3$) are remaining and willing to pay P_3 begins with the binomial probability mass function:

$$r_3(k) = \binom{Population_3}{k}\left(Prob_3\right)^k \left(1\text{-}Prob_3\right)^{Population_3-k}, \qquad (5.3.13)$$

where $Prob_3$ is the probability of a success, which in this case, is the probability that a customer has a reservation price at least as high as P_3, given that he had a reservation price at least as high as P_2. From the definition of conditional probability, this is given as:

$$Prob_3(P_2, P_3) = \frac{Prob_{P3}}{Prob_{P2}}. \tag{5.3.14}$$

The size of the population in the second stage, $Population_3$, depends on how many customers are "leftover" from the second stage. If there were more than Q_2 customers remaining after the first stage and willing to pay P_2 (i.e.; there was unmet demand), there are "leftover" customers who now make up the population of customers who might have reservation prices high enough to be willing to buy an item at P_3. This means that the size of $Population_3$ can take on values between zero and $d - (Q_1 + Q_2)$. However, to allow for k customers remaining and possibly willing to pay P_3, the size of $Population_3$ must be at least k.

For each necessary size of $Population_3$, the respective probability of such an occurrence can is given by one of the previously-calculated values of $r_2(j \mid Q_1, P_1, P_2)$. For example, the probability of having exactly k leftover customers in the second stage ($Population_3 = k$) is the probability that at the second stage, there were $Q_2 + k$ "leftover" customers willing to pay P_2. This was previously calculated as $r_2(Q_2 + k \mid Q_1, P_1, P_2)$. Therefore, the probability that exactly k customers are willing to pay P_3 at the third stage, given that exactly k customers are "leftover" from the second stage is found my multiplying (5.3.12) by $r_2(Q_2 + k \mid Q_1, P_1, P_2)$. To account for all possible sizes of $Population_3$ that allow for k successes in the third stage, all allowable forms of (5.3.12) are multiplied by their respective values of $r_2(j \mid Q_1, P_1, P_2)$, and the summation of the products is taken to give:

$$r_3(k \mid Q_1, Q_2, P_1, P_2, P_3) =$$

$$\begin{aligned} & r_2(Q_2 + k \mid Q_1, P_1, P_2)\binom{k}{k}\left(Prob_3(P_2, P_3)\right)^k\left(1\text{-}Prob_3(P_2, P_3)\right)^{k-k} \\ & + r_2(Q_2 + k + 1 \mid Q_1, P_1, P_2)\binom{k+1}{k}\left(Prob_3(P_2, P_3)\right)^k\left(1\text{-}Prob_3(P_2, P_3)\right)^{k+1-k} \\ & + r_2(Q_2 + k + 2 \mid Q_1, P_1, P_2)\binom{k+2}{k}\left(Prob_3(P_2, P_3)\right)^k\left(1\text{-}Prob_3(P_2, P_3)\right)^{k+2-k} + \ldots \\ & + r_2(\mathrm{D} - Q_1 \mid Q_1, P_1, P_2)\binom{d-(Q_1+Q_2)}{k}\left(Prob_3(P_2, P_3)\right)^k\left(1\text{-}Prob_3(P_2, P_3)\right)^{d-(Q_1+Q_2)-k} \end{aligned} \tag{5.3.15}$$

which can be expressed as:

$$\begin{aligned} r_3(k \mid Q_1, Q_2, P_1, P_2, P_3) = & \sum_{index_2=0}^{d-(Q_1+Q_2)-k} r_2(Q_2 + k + index_2 \mid Q_1, P_1, P_2) \\ & * \binom{k + index_2}{k}\left(Prob_3(P_2, P_3)\right)^k\left(1\text{-}Prob_3(P_2, P_3)\right)^{index_2}. \end{aligned} \tag{5.3.16}$$

Step 3d – Calculate the probability distribution for the m^{th} stage.

From inspection of (5.3.10), (5.3.12), (5.3.13) and (5.3.16), a pattern is seen that leads to the general form of the probability that at the m^{th} stage ($2 \le m \le n$), exactly t customers will be remaining and willing to pay P_m for an item:

$$r_m(t \mid Q_1, Q_2, ..., Q_{m-1}, P_1, P_2, ..., P_m) =$$

$$\sum_{index_{m-1}=0}^{d-Q_{Sum,\,m-1}-t} r_{m-1}(Q_{m-1} + t + index_{m-1} \mid Q_1, Q_2, ..., Q_{m-2}, P_1, P_2, ..., P_{m-1}) \quad (5.3.17)$$

$$* \binom{t + index_{m-1}}{t} \left(Prob_m(P_{m-1}, P_m)\right)^t \left(1 - Prob_m(P_{m-1}, P_m)\right)^{index_{m-1}}$$

where

$$Prob_m(P_{m-1}, P_m) = \frac{Prob_{Pm}}{Prob_{P(m-1)}} \quad (5.3.18)$$

and

$$Q_{Sum,m-1} = \sum_{v=1}^{m-1} Q_v \ . \quad (5.3.19)$$

Step 4 – Use dynamic programming to determine the maximum expected profit, and the corresponding inventory levels and selling prices for each stage.

Step 4a – For every possible state of the final stage, i.e.; every allowable combination of inventory levels and selling prices $(Q_1, Q_2, ..., Q_n, P_1, P_2, ..., P_n)$, calculate the expected profit from the final-stage items. Doing so required the probabilities that various numbers of customers are "leftover" and willing to pay P_n. These probabilities are given from (16) with $m = n$, and are used in the expected profit function as:

$$\mathrm{E}\left[\Pi_n(Q_1, Q_2, ..., Q_n, P_1, P_2, ..., P_n)\right] =$$

$$P_n \sum_{u=1}^{D-Q_{Sum,n-1}} \left(r_n(u \mid Q_1, Q_2, ..., Q_n, P_1, P_2, ..., P_n) * \min\left[u, Q_n\right]\right) - cQ_n \ . \quad (5.3.20)$$

Step 4b – For every possible state of the penultimate stage, i.e.; every allowable combination of inventory levels and selling prices $(Q_1, Q_2, ..., Q_{n-1}, P_1, P_2, ..., P_{n-1})$, determine the maximum expected profit and respective inventory level and selling price from the sale of final-stage items (as found in Step 4b). These quantities are denoted $E\left[\Pi_n^*(Q_1, Q_2, ..., Q_{n-1}, P_1, P_2, ..., P_{n-1})\right]$, $Q_n^*(Q_1, Q_2, ..., Q_{n-1}, P_1, P_2, ..., P_{n-1})$ and $P_n^*(Q_1, Q_2, ..., Q_{n-1}, P_1, P_2, ..., P_{n-1})$, respectively.

Step 4c – For every possible state of the penultimate stage, i.e.; every allowable combination of inventory levels and selling prices $(Q_1, Q_2, ..., Q_{n-1}, P_1, P_2, ..., P_{n-1})$, calculate the expected profit from the penultimate-stage items. The form of the expected profit function is similar to (5.3.20):

$$\mathrm{E}\left[\Pi_{n-1}(Q_1, Q_2, ..., Q_{n-1}, P_1, P_2, ..., P_{n-1})\right] = P_{n-1} \sum_{v=1}^{d-Q_{Sum,\, n-2}} \left(r_{n-1}(v \mid Q_1, Q_2, ..., Q_{n-1}, P_1, P_2, ..., P_{n-1}) * \min\left[v, Q_{n-1}\right]\right) - cQ_{n-1}. \tag{5.3.21}$$

Step 4d – For every possible state of the penultimate stage, i.e.; every allowable combination of inventory levels and selling prices $(Q_1, Q_2, ..., Q_{n-1}, P_1, P_2, ..., P_{n-1})$, determine the expected profit from the sale of penultimate-stage items (as found in Step 4b) plus the maximum expected profit for the given state from sales of the final-stage items (as found in Step 4a). This is expressed as:

$$\mathrm{E}\left[\Pi_{n-1}^*(Q_1, Q_2, ..., Q_{n-1}, P_1, P_2, ..., P_{n-1})\right] = \mathrm{E}\left[\Pi_{n-1}(Q_1, Q_2, ..., Q_{n-1}, P_1, P_2, ..., P_{n-1})\right] + \mathrm{E}\left[\Pi_n^*(Q_1, Q_2, ..., Q_{n-1}, P_1, P_2, ..., P_{n-1})\right]. \tag{5.3.22}$$

Step 4e – Continue with this backward induction procedure until the first stage is reached. At that point, for each combination of allowable Q_1 and P_1, the corresponding optimal inventory levels, $Q_2^*, Q_3^*, \ldots, Q_n^*$, selling prices, $P_2^*, P_3^*, \ldots, P_n^*$ and expected profit, $\mathrm{E}\left[\Pi_2^*\right]$, (notation suppressed) will be known. By inspection, the combination of Q_1^* and P_1^* are chosen by that which gives the highest value of $\mathrm{E}\left[\Pi_1^*\right]$.

5.3.2 NUMERICAL EXAMPLE OF *n*-PRICE MODEL

To demonstrate the steps outlined in Section 5.3.1, the following example is presented:

1. The number of selling prices is $n = 3$. Thus, the solution will include three optimal price-inventory level pairs ($P_1^*, Q_1^*, P_2^*, Q_2^*, P_3^*, Q_3^*$).
2. The size of the customer base is $d = 4$.
3. The selling prices that the seller can choose from are:
    - $Price_1 = 6$,
    - $Price_2 = 8$,
    - $Price_3 = 10$ and
    - $Price_4 = 12$.
4. The reservation prices of the customers follow a normal distribution with mean of 9 and a standard deviation of 2. Therefore, the probabilities that a customer will have a reservation price at least as high as each of the four given selling prices are:
    - $Prob_{Price1} = 0.993$,
    - $Prob_{Price2} = 0.691$,
    - $Prob_{Price3} = 0.309$ and

- $Prob_{Price4}$ = 0.067.

Step 1 – Determine the prices that can possibly be charged at each stage, and the allowable sets of those prices.

Using (5.3.4) and the parameters given in the example, the selling prices that can possibly be charged at each stage are:

P_1 = 6 or 8,

P_2 = 8 or 10 and

P_3 = 10 or 12.

From (5.3.1), the allowable sets of prices (P_1, P_2, P_3) are (6, 8, 10), (6, 8, 12), (6, 10, 12) and (8, 10, 12).

Step 2 – Determine the inventory levels that can possibly be selected at each stage, and the allowable sets of those inventory levels.

Using (5.3.6) and the parameters given in the example, the inventory levels that can possibly be chosen at each stage are:

Q_1 = 1 or 2,

Q_2 = 1 or 2 and

Q_3 = 1 or 2.

From (5.3.7), the allowable sets of inventory levels (Q_1, Q_2, Q_3) are (1, 1, 1), (1, 1, 2), (1, 2, 1) and (2, 1, 1).

Step 3a – Calculate the probabilities for the first stage.

Using (5.3.8) and the parameters given in the example, the values of $r_1(i \mid P_1)$ are calculated and shown in Table 5.3.1:

Table 5.3.1. Values of $r_1(i \mid P_1)$.

P_1	i	$r_1(i \mid P_1)$
6	1	0.0011
6	2	0.0233
6	3	0.2172
6	4	0.7584
8	1	0.0812
8	2	0.2731
8	3	0.4080
8	4	0.2286

Step 3b – Calculate the probabilities for the second stage.

Using (5.3.12) and the parameters given in the example, the values of $r_2(j \mid Q_1, P_1, P_2)$ are calculated and shown in Table 5.3.2.

Step 3c – Calculate the probabilities for the third stage.

Using (5.3.16) and the parameters given in the example, the values of $r_3(k \mid Q_1, Q_2, P_1, P_2, P_3)$ are calculated and shown in Table 5.3.3.

Table 5.3.2. Values of $r_2(j \mid Q_1, P_1, P_2)$.

Q_1	P_1	P_2	j	$index_1$ 0	1	2	$r_2(j \mid Q_1, P_1, P_2)$
1	6	8	1	0.0173	0.0834	0.1131	0.2138
1	6	8	2	0.1192	0.3236		0.4428
1	6	8	3	0.3085			0.3085
1	6	10	1	0.0077	0.0961	0.3370	0.4409
1	6	10	2	0.0237	0.1665		0.1902
1	6	10	3	0.0274			0.0274
1	8	10	1	0.1219	0.2016	0.0938	0.4173
1	8	10	2	0.0812	0.0756		0.1569
1	8	10	3	0.0203			0.0203
2	6	8	1	0.1609	0.2911		0.4520
2	6	8	2	0.4164			0.4164
2	6	10	1	0.0718	0.3357		0.4075
2	6	10	2	0.0829			0.0829
2	8	10	1	0.1821	0.1130		0.2950
2	8	10	2	0.0455			0.0455

Table 5.3.3. Values of $r_3(k \mid Q_1, Q_2, P_1, P_2, P_3)$.

Q_1	Q_2	P_1	P_2	P_3	k	$index_2$ 0	1	$r_3(k \mid Q_1, Q_2, P_1, P_2, P_3)$
1	1	6	8	10	1	0.1976	0.1525	0.3501
1	1	6	8	12	1	0.0428	0.0539	0.0966
1	1	6	10	12	1	0.0412	0.0093	0.0505
1	1	6	8	10	2	0.0614		0.0614
1	1	6	8	12	2	0.0029		0.0029
1	1	6	10	12	2	0.0013		0.0013
1	1	8	10	12	1	0.0340	0.0069	0.0409
1	1	8	10	12	2	0.0010		0.0010
1	2	6	8	10	1	0.1377		0.1377
1	2	6	8	12	1	0.0298		0.0298
1	2	6	10	12	1	0.0059		0.0059
1	2	8	10	12	1	0.0044		0.0044
2	1	6	8	10	1	0.1858		0.1858
2	1	6	8	12	1	0.0402		0.0402
2	1	6	10	12	1	0.0180		0.0180
2	1	8	10	12	1	0.0099		0.0099

Step 4 – Use dynamic programming to determine the maximum expected profit, and the corresponding inventory levels and selling prices for each stage.

Step 4a – For every possible state of the final stage, i.e.; every allowable combination of inventory levels and selling prices $(Q_1, Q_2, ..., Q_n, P_1, P_2, ..., P_n)$, calculate the expected profit from the final-stage items.

Using (5.3.20) and the parameters given in the example, the values of $E[\Pi_3(Q_1, Q_2, Q_3, P_1, P_2, P_3)]$ are calculated and shown in Table 5.3.4.

Table 5.3.4. Values of $E[\Pi_3(Q_1, Q_2, Q_3, P_1, P_2, P_3)]$.

Q_1	Q_2	Q_3	P_1	P_2	P_3	$E[\Pi_3(Q_1, Q_2, Q_3, P_1, P_2, P_3)]$
1	1	1	6	8	10	3.1147
1	1	1	6	8	12	0.1942
1	1	1	6	10	12	-0.3787
1	1	1	8	10	12	-0.4983
1	1	2	6	8	10	2.7290
1	1	2	6	8	12	-0.7712
1	1	2	6	10	12	-1.3633
1	1	2	8	10	12	-1.4869
1	2	1	6	8	10	0.3766
1	2	1	6	8	12	-0.6423
1	2	1	6	10	12	-0.9287
1	2	1	8	10	12	-0.9472
2	1	1	6	8	10	0.8578
2	1	1	6	8	12	-0.5172
2	1	1	6	10	12	-0.7846
2	1	1	8	10	12	-0.8817

Step 4b – For every possible state of the penultimate stage, i.e.; every allowable combination of inventory levels and selling prices $(Q_1, Q_2, ..., Q_{n-1}, P_1, P_2, ..., P_{n-1})$, determine the maximum expected profit and respective inventory level and selling price from the sale of final-stage items (as found in Step 4b).

From inspection of Table 5.3.4, the values of $E\left[\Pi_3^*(Q_1, Q_2, P_1, P_2)\right]$ are determined and shown in Table 5.3.5:

Table 5.3.5. Values of $E\left[\Pi_3^*(Q_1, Q_2, P_1, P_2)\right]$.

Q_1	Q_2	P_1	P_2	$Q_3^*(Q_1, Q_2, P_1, P_2)$	$P_3^*(Q_1, Q_2, P_1, P_2)$	$E\left[\Pi_3^*(Q_1, Q_2, P_1, P_2)\right]$
1	1	6	8	1	10	3.1147
1	1	6	10	1	12	-0.3787
1	1	8	10	1	12	-0.4983
1	2	6	8	1	10	0.3766
1	2	6	10	1	12	-0.9287
1	2	8	10	1	12	-0.9472
2	1	6	8	1	10	0.8578
2	1	6	10	1	12	-0.7846
2	1	8	10	1	12	-0.8817

Step 4c – For every possible state of the penultimate stage, i.e.; every allowable combination of inventory levels and selling prices $(Q_1, Q_2, ..., Q_{n-1}, P_1, P_2, ..., P_{n-1})$, calculate the expected profit from the penultimate-stage items.

Using (5.3.21) and the parameters given in the example, the values of $E\left[\Pi_2(Q_1, Q_2, P_1, P_2)\right]$ are calculated and shown in Table 5.3.6:

Table 5.3.6. Values of $E\left[\Pi_2(Q_1, Q_2, P_1, P_2)\right]$.

Q_1	Q_2	P_1	P_2	$E\left[\Pi_2(Q_1, Q_2, P_1, P_2)\right]$
1	1	P1	P2	6.7206
1	1	P1	P3	5.5850
1	1	P2	P3	4.9451
1	2	P1	P2	11.7311
1	2	P1	P3	6.7612
1	2	P2	P3	5.7167
2	1	P1	P2	5.9472
2	1	P1	P3	3.9038
2	1	P2	P3	2.4055

Step 4d – For every possible state of the penultimate stage, i.e.; every allowable combination of inventory levels and selling prices $(Q_1, Q_2, ..., Q_{n-1}, P_1, P_2, ..., P_{n-1})$, determine the expected profit from the sale of penultimate-stage items (as found in Step 4b) plus the maximum expected profit for the given state from sales of the final-stage items (as found in Step 4a).

Using (5.3.22) and the parameters given in the example, the values of $E\left[\Pi_2^*(Q_1, Q_2, P_1, P_2)\right]$ are calculated and shown in Table 5.3.7:

Table 5.3.7. Values of $E\left[\Pi_2^*(Q_1, Q_2, P_1, P_2)\right]$.

Q_1	Q_2	P_1	P_2	$Q_3^*(Q_1, Q_2, P_1, P_2)$	$P_3^*(Q_1, Q_2, P_1, P_2)$	$E\left[\Pi_2^*(Q_1, Q_2, P_1, P_2)\right]$
1	1	6	8	1	10	9.8354
1	1	6	10	1	12	5.2062
1	1	8	10	1	12	4.4468
1	2	6	8	1	10	12.1077
1	2	6	10	1	12	5.8324
1	2	8	10	1	12	4.7695
2	1	6	8	1	10	5.4300
2	1	6	10	1	12	3.1192
2	1	8	10	1	12	1.5238

Step 4e – Continue with this backward induction procedure until the first stage is reached. At that point, for each combination of allowable Q_1 and P_1, the corresponding optimal inventory levels, $Q_2^*, Q_3^*, \ldots, Q_n^*$, selling prices, $P_2^*, P_3^*, \ldots, P_n^*$ and expected profit, $\mathrm{E}\left[\Pi_2^*\right]$, (notation suppressed) will be known.

For the parameters given in the example, the dynamic programming process described above gives the following the expected profits for each allowable combination of Q_1 and P_1 for the first stage:

Table 5.3.8. Values of $E\left[\Pi_1(Q_1, Q_2^*, Q_3^*, P_1, P_2^*, P_3^*)\right]$.

Q_1	P_1	Q_2^*	P_2^*	Q_3^*	P_3^*	$E\left[\Pi_1(Q_1, Q_2^*, Q_3^*, P_1, P_2^*, P_3^*)\right]$
1	6	2	8	1	10	17.1075
1	8	2	10	1	12	11.6970
2	6	1	8	1	10	16.7982
2	8	1	10	1	12	14.7288

The optimal solution is determined by inspection of Table 5.3.8, and shown in Table 5.3.9:

Table 5.3.9. Values of $E\left[\Pi_1^*(Q_1^*, Q_2^*, Q_3^*, P_1^*, P_2^*, P_3^*)\right]$.

Q_1^*	P_1^*	Q_2^*	P_2^*	Q_3^*	P_3^*	$E\left[\Pi_1^*(Q_1^*, Q_2^*, Q_3^*, P_1^*, P_2^*, P_3^*)\right]$
1	6	2	8	1	10	17.1075

CHAPTER 6

THE NEWSVENDOR PROBLEM WITH PRICING:

OPTIMALITY RESULTS FOR SMALL POPULATION SIZES

6.1 INTRODUCTION

In the previous chapter, the example from Dana (1999) with two possible sizes of customer base had an optimal solution of two prices and corresponding inventory levels. In this chapter we investigate the circumstances under which it is optimal for the seller to set a single selling price and corresponding inventory level, as opposed to setting two selling prices and corresponding inventory levels. Two general cases are considered: a deterministic size of customer base and a stochastic size of customer base. For each case, two examples are provided. While certain proofs and derivations are shown in full in this chapter, longer, more tedious derivations of the expected profit functions used in the four examples are presented in Appendix B.

6.2 OPTIMALITY RESULTS FROM SMALL POPULATION SIZES

This section includes the derivation of expected profit functions for both the deterministic and stochastic cases of size of customer base. For each case two examples are provided. New notation includes the denoting of the probability that a randomly-selected customer has a reservation price at least as high as a given selling price P_i as $\overline{F_i}$.

6.2.1 GENERAL FORM OF THE EXPECTED PROFIT FUNCTION FOR A DETERMINISTIC CUSTOMER BASE

In this section, the general form of an expected profit function for a seller who sets *n* inventory levels and corresponding selling prices with a deterministic customer base of size *d* is derived. Commonly used notation in this section includes the following. The total number of items made available for sale is Q_{Total}. Revenue, *R*, is random and can take on values denoted rev_i, where $i = 1, 2, ..., Q_{Total} + 1$.

Theorem 6.2.1. The expected profit function for a seller facing a deterministic customer base is given by

$$E[\Pi(Q_1, Q_2, ..., Q_n, P_1, P_2, ..., P_n)] = \sum_{i=1}^{QTotal+1} rev_i h_R(rev_i) - cQ_{Total}. \tag{6.2.1}$$

Proof. Note that the right hand side of (6.2.1) is made up of an expected revenue calculation and a total cost calculation. The proof of Theorem 6.2.1 requires that the components of the expected revenue calculation (possible values of revenue and their respective probabilities) be determined.

Begin with a deterministic profit function, revenue less total cost, for a seller who sets *n* inventory levels and corresponding selling prices:

$$\Pi(Q_1, Q_2, ..., Q_n, P_1, P_2, ..., P_n) = R(Q_1, Q_2, ..., Q_n, P_1, P_2, ..., P_n) - cQ_{Total}. \tag{6.2.2}$$

Given that the reservation prices of the customers are random, the number of items actually sold, and therefore revenue, *R* (notation suppressed for ease of reading), is also random. To determine the distribution of *R*, consider first the allowable values of *R*.

Lemma 6.2.1. For a seller who sets n inventory levels and prices, the possible revenues are $0, 1P_1, 2P_1, ..., Q_1P_1, Q_1P_1 + 1P_2, Q_1P_1 + 2P_2, ..., Q_1P_1 + Q_2P_2, ..., \sum_{i=1}^{n} Q_iP_i$.

Proof. Since the items being sold are homogeneous, all items available at price P_1 must be sold before any items available for sale at any of the higher prices will be sold. As the number of items that can be sold at price P_1 is $0, 1, 2, ..., Q_1$, the possible revenues from the sale of these items are $0, 1P_1, 2P_1, ..., Q_1P_1$. After all of the items at price P_1 have been sold, items at price P_2 may be sold. Since the number of items that can be sold at price P_2 is $0, 1, 2, ..., Q_2$, the possible revenues from the sale of these items are $0, 1P_2, 2P_2, ..., Q_2P_2$. The possible revenues that can be realized from the sale of items at prices P_1 and P_2 are $0, 1P_1, 2P_1, ..., Q_1P_1, Q_1P_1 + 1P_2, Q_1P_1 + 2P_2, ..., Q_1P_1 + Q_2P_2$. By inspection, the possible revenues if there are *n* inventory levels and corresponding prices are $0, 1P_1, 2P_1, ..., Q_1P_1, Q_1P_1 + 1P_2, Q_1P_1 + 2P_2, ..., Q_1P_1 + Q_2P_2, ..., \sum_{i=1}^{n} Q_iP_i$. *Q.E.D.*

Since the reservation prices of the customers are random, demand for items is random, meaning the number of items sold is random, between zero and $min(Q_{Total}, d)$. Assuming that the seller knows *d* with certainty, no more than one item for each customer will be made available for sale, since no more than d items can be sold. Therefore, inventory levels will be chosen such that $Q_{Total} \leq d$, which means $min(Q_{Total}, d) = Q_{Total}$. Denote the possible values of revenue that can be realized by the seller, that is, the possible values of R, as rev_i, where $i = 1, 2, ..., Q_{Total} + 1$, $Q_{Total} \leq d$, and $rev_1 < rev_2 < ... < rev_{QTotal+1}$.

Now that the allowable values of R are known, the remaining information needed to know the distribution of R are the probabilities of realizing each allowable value of R. The probability of realizing a given possible revenue, rev_i, depends on both item availability and on willingness for a customer to pay for an item. Given that customers have random reservation prices, their willingness to pay given selling prices is also random. The distribution of this willingness to pay is given by a joint probability mass function, which indicates the joint probabilities of customers' reservation prices falling within certain ranges of prices.

Lemma 6.2.2. The joint probability mass function indicating the probabilities of the reservation prices of customers falling within given ranges of prices, denoted " $range_i$*," i=1, 2, …, d, is given by:*

$$h_{RP1, RP2, ..., RPd}(range_1, range_2, ..., range_d) = \prod_{i=1}^{d} Prob(RP_i \in range_i). \qquad (6.2.3)$$

Proof. Consider first a customer with a reservation price so low, such that no revenue can be realized, that is, the customer's reservation price is lower than P_1. Denote this range of prices as $Range_1$, where the probability of a customer's reservation price falling in this range is:

$$Prob(RP < P_1) = Prob(RP \in Range_1) = F_{RP}(P_1) = 1 - \overline{F_1} \,. \tag{6.2.4}$$

Next, consider a customer's reservation price that is such that a revenue of P_1 can be realized from an item priced on $[P_1, P_2)$, that is, the customer's reservation price is $P_1 \le RP < P_2$. Denote this range of prices as $Range_2$, where the probability of a customer's reservation price falling in this range is:

$$\begin{aligned} Prob(P_1 \le RP < P_2) &= Prob(RP \in Range_2) \\ &= 1 - F_{RP}(P_1) - \left(1 - F_{RP}(P_2)\right) = \overline{F_1} - \overline{F_2} \,. \end{aligned} \tag{6.2.5}$$

Next, consider a customer's reservation price that is such that a revenue of P_2 can be realized from an item priced on $[P_2, P_3)$, that is, the customer's reservation price is $P_2 \le RP < P_3$. Denote this range of prices as $Range_3$, where the probability of a reservation price falling in this range is:

$$\begin{aligned} Prob(P_2 \le RP < P_3) &= Prob(RP \in Range_3) \\ &= 1 - F_{RP}(P_2) - \left(1 - F_{RP}(P_3)\right) = \overline{F_2} - \overline{F_3} \,. \end{aligned} \tag{6.2.6}$$

From inspection of (6.2.5) and (6.2.6), for a customer's reservation price that is such that a revenue of P_{m-1} can be realized from an item priced on $[P_{m-1}, P_m)$, denoted $Range_m$, the probability of a reservation price falling in this range is:

$$Prob(P_{m-1} \le RP < P_m) = Prob(RP \in Range_m)$$
$$= 1 - F_{RP}(P_{m-1}) - \left(1 - F_{RP}(P_m)\right) = \overline{F_{m-1}} - \overline{F_m},\ m = 2, 3, ..., n. \quad (6.2.7)$$

For the special case of a customer having a reservation price at least as high as the highest selling price $(P_n \le RP)$, denote $[P_n, \infty)$ as $Range_{n+1}$, and the probability of a reservation price falling in this range, is:

$$Prob(P_n \le RP) = Prob(RP \in Range_{n+1}) = 1 - F_{RP}(P_n) = \overline{F_n}. \quad (6.2.8)$$

For the first customer in the population, the probability mass function that provides the probability that this customer has a reservation price within a certain range is given as:

$$h_{RP1}(range_1) = Prob(RP_1 \in range_1),\ range_1 = Range_1,\ Range_2, ..., Range_{n+1}. \quad (6.2.9)$$

For the second customer in the population, the probability mass function that provides the probability that this customer has a reservation price within a certain range is given as:

$$h_{RP2}(range_2) = Prob(RP_2 \in range_2),\ range_2 = Range_1,\ Range_2, ..., Range_{n+1}. \quad (6.2.10)$$

In general, for the m^{th} customer in the population, the probability mass function that provides the probability that this customer has a reservation price within a certain range is given as:

$$h_{RPm}(range_m) = Prob(RP_m \in range_m),\ range_m = Range_1,\ Range_2, ..., Range_{n+1}$$
$$\text{and } m = 1, 2, ..., d. \quad (6.2.11)$$

Under the assumption that customers behave independently, that is, one customer's reservation price does not depend on any other customer's reservation price, the joint probability of customers having a given combination of reservation prices in certain ranges is simply the product of the individual probabilities of reservation prices falling in certain ranges. Therefore, the joint probability mass function is:

$$h_{RP1,\,RP2,\,...,\,RPd}(range_1, range_2, ..., range_d)$$
$$= Prob(RP_1 \in range_1) * Prob(RP_2 \in range_2) * ... * Prob(RP_d \in range_d)\,, \qquad (6.2.12)$$

which can be expressed as (6.2.3). *Q.E.D.*

The final step in determining the components of the expected revenue calculation is using the joint probability mass function of customers' reservation prices (Equation 6.2.3) to determine a probability mass function that provides the probabilities of realizing each possible value of *R*. Denote this probability mass function as $h_R(rev_i)$, which is defined as:

$$h_R(rev_i) = Prob(R = rev_i \mid Q_1, Q_2, ..., Q_n, P_1, P_2, ..., P_n)\,. \qquad (6.2.13)$$

The values of $h_R(rev_i)$ are determined by inspection of the joint p.m.f of customers' willingness to pay, $h_{RP1,\,RP2,\,...,\,RPd}(range_1, range_2, ..., range_d)$, and the values of the inventory levels and corresponding prices decided by the seller $(Q_1, Q_2, ..., Q_n, P_1, P_2, ..., P_n)$.

Using the definition of expected value, the expected revenue term on the right hand side of (6.2.1) results when using the probability mass function given in (6.2.13) and the allowable values of *R* as found using Lemma 6.2.1. Expected revenue is given as:

$$E[R(Q_1, Q_2, ..., Q_n, P_1, P_2, ..., P_n)] = \sum_{i=1}^{QTotal+1} rev_i h_R(rev_i) . \quad (6.2.14)$$

The final step in the proof of Theorem 6.2.1 is subtraction of total cost from expected revenue to give expected profit, where total cost is the number of items made available for sale, Q_{Total}, multiplied by the unit cost, *c*. This gives (5.2.1), and completes the proof. *Q.E.D.*

6.2.2 DETERMINISTIC CUSTOMER BASE EXAMPLE 1, $d = 2$,

In this section, the expected profit model derived in Section 6.2.1 is used to determine, for a deterministic customer base of $d = 2$, an optimal pricing strategy. The expected profits from two single-price strategies (items are made available for sale at only one price) are compared with the expected profit a dual-price strategy (two inventory and corresponding price decisions are made) to determine which of these two strategies is optimal. The calculations for this example are provided in Appendix B, with important results summarized and discussed below.

In this example, the seller chooses from a set of inventory and corresponding price combinations where $Q_{Total} \le 2$, since the number of items made available for sale will never be more than the number of customers. These inventory combinations are shown in Table 6.2.1. Note that Q_i is the number of units of inventory to be made available for sale at price P_i, $i = 1, 2$.

Table 6.2.1. Inventory Combinations for Example 1

		Q_1		
		0	1	2
Q_2	0	Case 1	Case 2	Case 3
	1	Case 4	Case 5	
	2	Case 6		

While Case 1 is included in Table 6.2.1 for completeness, the decision to make no items available for sale results in a trivial expected profit of zero, and is not considered for the remainder of the example.

The objective of this example is to determine if one of the strategies (single-price or dual-price) is always superior to the other for a customer base of size $d = 2$.

Theorem 6.2.2. For a seller facing a deterministic customer base of size $d = 2$*, a single-price strategy is optimal.*

Proof. The proof of Theorem 6.2.2 is achieved in two steps. First, it is shown that a dual-price strategy is always sub-optimal unless a strict condition is met. Second, it is shown that if this condition is met, there exists at least one, single-price strategy that gives an expected profit that is the same as the expected profit for the dual-price strategy. The first step is the proof of Lemma 6.2.3.

Lemma 6.2.3. For a seller facing a deterministic customer base of size $d = 2$*, a dual-price strategy is always sub-optimal unless* $P_2\overline{F_2} = P_1\overline{F_1}$.

Proof. The proof of Lemma 6.2.3 requires a comparison of the expected profit functions of the cases shown in Table 6.2.1. These expected profit functions, derived using the method outlined in Section 6.2.1, are as follows:

$$E[\Pi_{Case2}] = P_1\overline{F_1}\left(2 - \overline{F_1}\right) - c\,. \tag{6.2.15}$$

$$E[\Pi_{Case3}] = 2P_1\overline{F_1} - 2c\,. \tag{6.2.16}$$

$$E[\Pi_{Case4}] = P_2\overline{F_2}\left(2 - \overline{F_2}\right) - c\,. \tag{6.2.17}$$

$$E[\Pi_{Case5}] = P_1\overline{F_1}\left(2 - \overline{F_1}\right) + P_2\overline{F_1}\,\overline{F_2} - 2c\,. \tag{6.2.18}$$

$$E[\Pi_{Case6}] = 2P_2\overline{F_2} - 2c\,. \tag{6.2.19}$$

For the dual-price strategy (Case 5) to be optimal, it must simultaneously give a higher expected profit than all of the available single-inventory and price strategies (Cases 2, 3, 4 and 6). Shown below are comparisons of the expected profit function for Case 5 with the expected profit functions of two of the single-price strategies.

For Case 5 ($Q_1 = 1, Q_2 = 1$) to be superior to Case 3 ($Q_1 = 2, Q_2 = 0$),

$$E[\Pi_{Case5}] \geq E[\Pi_{Case3}]\,. \tag{6.2.20}$$

Substitution of (6.2.16) and (6.2.18) into (6.2.20) gives:

$$P_1\overline{F_1}\left(2 - \overline{F_1}\right) + P_2\overline{F_1}\,\overline{F_2} - 2c \geq 2P_1\overline{F_1} - 2c\,. \tag{6.2.21}$$

Rearranging (6.2.21) gives:

$$P_2\overline{F_2} \geq P_1\overline{F_1}\,. \tag{6.2.22}$$

For Case 5 ($Q_1 = 1, Q_2 = 1$) to be superior to Case 6 ($Q_1 = 0, Q_2 = 2$),

$$E[\Pi_{Case5}] \geq E[\Pi_{Case6}]\,. \tag{6.2.23}$$

Substitution of (6.2.18) and (6.2.19) into (6.2.23) gives:

$$P_1\overline{F_1}\left(2-\overline{F_1}\right) + P_2\overline{F_1}\,\overline{F_2} - 2c \geq 2P_2\overline{F_2} - 2c\,. \tag{6.2.24}$$

Rearranging (6.2.24) gives:

$$P_2\overline{F_2} \leq P_1\overline{F_1}\,. \tag{6.2.25}$$

The non-trivial condition that satisfies both (6.2.22) and (6.2.25) is

$$P_2\overline{F_2} = P_1\overline{F_1}\,. \tag{6.2.26}$$

Because (6.2.26) must be true for the expected profit from the dual-price model (Case 5, where $Q_1 = 1, Q_2 = 1$) to be at least as high as the expected profits from two of the single-price models (Case 3, where $Q_1 = 2, Q_2 = 0$, and Case 6, where $Q_1 = 0, Q_2 = 2$), the dual-price strategy is always sub-optimal unless $P_2\overline{F_2} = P_1\overline{F_1}$. *Q.E.D.*

The second step in the proof of Theorem 6.2.2 is to show that under the optimality condition found in Lemma 6.2.3, there exists a single-price strategy that has at least the same expected profit as the dual-price strategy. This is shown in the proof of Lemma 6.2.4.

Lemma 6.2.4. For a seller facing a deterministic customer base of size $d = 2$*, if* $P_2 \overline{F_2} = P_1 \overline{F_1}$*, there exists a single-price strategy with an expected profit at least as high as the expected profit of the dual-price strategy.*

Proof. The proof of Lemma 6.2.4 begins with the substitution of (6.2.26) into the expected profit functions for Cases 2 – 6:

$$E[\Pi_{Case2}] = P_1 \overline{F_1}\left(2 - \overline{F_1}\right) - c, \quad (6.2.27)$$

$$E[\Pi_{Case3}] = 2P_1 \overline{F_1} - 2c, \quad (6.2.28)$$

$$E[\Pi_{Case4}] = P_1 \overline{F_1}\left(2 - \overline{F_2}\right) - c, \quad (6.2.29)$$

$$E[\Pi_{Case5}] = 2P_1 \overline{F_1} - 2c, \text{ and} \quad (6.2.30)$$

$$E[\Pi_{Case6}] = 2P_1 \overline{F_1} - 2c. \quad (6.2.31)$$

By inspection, the right hand sides of (6.2.28), (6.2.30) and (6.2.31) are equivalent. Therefore, under the condition that $P_2 \overline{F_2} = P_1 \overline{F_1}$, there are two single-price strategies (Case 3, where $Q_1 = 2, Q_2 = 0$, and Case 6, where $Q_1 = 0, Q_2 = 2$), with expected profits at least as high as the expected profit of the dual-price strategy (Case 5, where $Q_1 = 1, Q_2 = 1$). *Q.E.D.*

The non-trivial condition under which a dual-price strategy (Case 5, where $Q_1 = 1, Q_2 = 1$) simultaneously gives at least the same expected profit as two of the single-price strategies (Case 3, where $Q_1 = 2, Q_2 = 0$, and Case 6, where $Q_1 = 0, Q_2 = 2$), is $P_2 \overline{F_2} = P_1 \overline{F_1}$. Under this condition, these two single-price strategies give the same expected profit as the dual-price strategy. Therefore, on the assumption that simplicity is preferred, *ceteris paribus*, a single-price strategy is optimal. *Q.E.D.*

6.2.3 DETERMINISTIC CUSTOMER BASE EXAMPLE 2, *d* = 3

In this section, the expected profit model derived in Section 6.2.1 is used to determine, for a deterministic customer base of 3, an optimal pricing strategy. Specifically, the expected profits from single-price strategies (items are made available for sale at only one price) are compared with the expected profit of dual-price strategies (two inventory and corresponding price decisions are made) to determine which of these strategies is optimal. The calculations for this example are provided in Appendix B, with important results summarized and discussed below.

In this example, the seller chooses from a set of inventory and corresponding price combinations where $Q_{Total} \leq 3$, since the seller will never make more items available for sale than the number of customers. These inventory combinations are shown and named in Table 6.2.2. Note that Q_i is the number of units of inventory to be made available for sale at price P_i, i = 1, 2.

Table 6.2.2. Inventory Combinations for Example 2.

		Q_1			
		0	1	2	3
Q_2	0	Case 1	Case 2	Case 3	Case 4
	1	Case 5	Case 6	Case 7	
	2	Case 8	Case 9		
	3	Case 10			

While Case 1 is included in Table 6.2.2 for completeness, the decision to make no items available for sale results in a trivial expected profit of zero, and is not considered for the remainder of the example.

The objective of this example is to determine if one of the strategies (single-price or dual-price) is optimal for a customer base of size $d = 3$.

Theorem 6.2.3. For a seller facing a deterministic customer base of size $d = 3$*, a single-price strategy is optimal.*

Proof. The proof of Theorem 6.2.3 is achieved in two steps. First, it is shown that a dual-price strategy is always sub-optimal unless a strict condition is met. Second, it is shown that if this condition is met, there exists at least one, single-price strategy that gives an expected profit that is the same as the expected profit for the dual-price strategy. The first step is the proof of Lemma 6.2.5.

Lemma 6.2.5. For a seller facing a deterministic customer base of size $d = 3$*, a dual-price strategy is always sub-optimal unless* $P_2 \overline{F_2} = P_1 \overline{F_1}$.

Proof. The proof of Lemma 6.2.5 requires comparison of the expected profit functions of the cases shown in Table 6.2.2. These expected profit functions, derived using the method outlined in Section 6.2.1, are as follows:

$$E[\Pi_{Case2}] = P_1\overline{F_1}\left(3 - 3\overline{F_1} + \overline{F_1}^2\right) - c. \tag{6.2.32}$$

$$E[\Pi_{Case3}] = P_1\overline{F_1}\left(3 - \overline{F_1}^2\right) - 2c. \tag{6.2.33}$$

$$E[\Pi_{Case4}] = 3P_1\overline{F_1} - 3c. \tag{6.2.34}$$

$$E[\Pi_{Case5}] = P_2\overline{F_2}\left(3 - 3\overline{F_2} + \overline{F_2}^2\right) - c. \tag{6.2.35}$$

$$E[\Pi_{Case6}] = P_1\overline{F_1}\left(3 - 3\overline{F_1} + \overline{F_1}^2\right) + P_2\overline{F_2}\left(3\overline{F_1} - \overline{F_1}^2 - \overline{F_1}\,\overline{F_2}\right) - 2c. \tag{6.2.36}$$

$$E[\Pi_{Case7}] = P_1\overline{F_1}\left(3 - \overline{F_1}^2\right) + P_2\overline{F_1}^2\overline{F_2} - 3c. \tag{6.2.37}$$

$$E[\Pi_{Case8}] = P_2\overline{F_2}\left(3 - \overline{F_2}^2\right) - 2c. \tag{6.2.38}$$

$$E[\Pi_{Case9}] = P_1\overline{F_1}\left(3 - 3\overline{F_1} + \overline{F_1}^2\right) + P_2\overline{F_2}\left(3\overline{F_1} - \overline{F_1}^2\right) - 3c. \tag{6.2.39}$$

$$E[\Pi_{Case10}] = 3P_2\overline{F_2} - 3c. \tag{6.2.40}$$

For a dual-price strategy (Case 6, 7 or 9) to be optimal, it must simultaneously give a higher expected profit than all of the available single-price strategies (Cases 2, 3, 4, 5, 8 and 10). Shown below are comparisons of the expected profit from the dual-price strategies with the expected profit functions of selected single-price strategies.

The first comparisons are between the dual-price strategy of Case 6 and the single-price strategies of Cases 3 and 8.

For Case 6 ($Q_1 = 1, Q_2 = 1$) to be superior to Case 3 ($Q_1 = 2, Q_2 = 0$),

$$E[\Pi_{Case6}] \geq E[\Pi_{Case3}]. \tag{6.2.41}$$

Substitution of (6.2.33) and (6.2.36) into (6.2.41) gives:

$$P_1\overline{F_1}\left(3-3\overline{F_1}+\overline{F_1}^2\right)+P_2\overline{F_2}\left(3\overline{F_1}-\overline{F_1}^2-\overline{F_1}\,\overline{F_2}\right)-2c \geq P_1\overline{F_1}\left(3-\overline{F_1}^2\right)-2c\,. \tag{6.2.42}$$

Rearranging and simplifying (6.2.42) gives:

$$\left(P_2\overline{F_2}-P_1\overline{F_1}\right)\left(3-2\overline{F_1}\right) \geq P_2\overline{F_2}\left(\overline{F_1}-\overline{F_2}\right). \tag{6.2.43}$$

Note that since $\overline{F_1} \geq \overline{F_2}$, the right-hand side of (6.2.43) is non-negative. Since the second term on the left-hand side is strictly positive, the condition which satisfies (6.2.43) is

$$P_2\overline{F_2} \geq P_1\overline{F_1}\ . \tag{6.2.44}$$

For Case 6 ($Q_1 = 1, Q_2 = 1$) to be superior to Case 8 ($Q_1 = 0, Q_2 = 2$),

$$E[\Pi_{Case6}] \geq E[\Pi_{Case8}]\,. \tag{6.2.45}$$

Substitution of (6.2.36) and (6.2.38) into (6.2.45) and gives:

$$P_1\overline{F_1}\left(3-3\overline{F_1}+\overline{F_1}^2\right)+P_2\overline{F_2}\left(3\overline{F_1}-\overline{F_1}^2-\overline{F_1}\,\overline{F_2}\right)-2c \geq P_2\overline{F_2}\left(3-\overline{F_2}^2\right)-2c\,. \tag{6.2.46}$$

Rearranging (6.2.46) gives:

$$\left(P_2\overline{F_2}-P_1\overline{F_1}\right)\left(3\overline{F_1}-\overline{F_1}^2-3\right) \geq P_2\overline{F_2}^2\left(\overline{F_1}-\overline{F_2}\right). \tag{6.2.47}$$

Note that since $\overline{F_1} \geq \overline{F_2}$, the right-hand side of (6.2.47) is non-negative. As the second term on the left-hand side of (6.2.47) is non-positive, the condition which satisfies (6.2.47) is

$$P_2\overline{F_2} \leq P_1\overline{F_1}\ . \tag{6.2.48}$$

The non-trivial condition that satisfies both (6.2.44) and (6.2.48) is

$$P_2\overline{F_2} = P_1\overline{F_1}\ . \tag{6.2.49}$$

Because (6.2.49) must be true for the expected profit from the dual-price model (Case 6, where $Q_1 = 1, Q_2 = 1$) to be at least as high as the expected profits from two of the single-price models (Case 3, where $Q_1 = 2, Q_2 = 0$, and Case 8, where $Q_1 = 0, Q_2 = 2$), this dual-price strategy is always sub-optimal unless $P_2 \overline{F_2} = P_1 \overline{F_1}$.

The next comparisons are between the dual-price strategy of Case 7 and the single-price strategies of Cases 4 and 10.

For Case 7 ($Q_1 = 2, Q_2 = 1$) to be superior to Case 4 ($Q_1 = 3, Q_2 = 0$),

$$E[\Pi_{Case7}] \geq E[\Pi_{Case4}]. \tag{6.2.50}$$

Substitution of (6.2.34) and (6.2.37) into (6.2.50) gives:

$$P_1 \overline{F_1}\left(3 - \overline{F_1}^2\right) + P_2 \overline{F_1}^2 \overline{F_2} - 3c \geq 3P_1 \overline{F_1} - 3c. \tag{6.2.51}$$

Expanding and rearranging (6.2.51) gives:

$$\overline{F_1}^2 \left(P_2 \overline{F_2} - P_1 \overline{F_1}\right) \geq 0. \tag{6.2.52}$$

Since $\overline{F_1}^2$ is non-negative, this requires that

$$P_2 \overline{F_2} \geq P_1 \overline{F_1}. \tag{6.2.53}$$

For Case 7 ($Q_1 = 2, Q_2 = 1$) to be superior to Case 10 ($Q_1 = 0, Q_2 = 3$),

$$E[\Pi_{Case7}] \geq E[\Pi_{Case10}]. \tag{6.2.54}$$

Substitution of (6.2.37) and (6.2.40) into (6.2.54) gives:

$$P_1 \overline{F_1}\left(3 - \overline{F_1}^2\right) + P_2 \overline{F_1}^2 \overline{F_2} - 3c \geq 3P_2 \overline{F_2} - 3c. \tag{6.2.55}$$

Expanding and rearranging (6.2.55) gives:

$$\left(P_2\overline{F_2} - P_1\overline{F_1}\right)\left(\overline{F_1}^2 - 3\right) \geq 0\,. \tag{6.2.56}$$

Since the second term in brackets in (6.2.56) is strictly negative, this requires

$$P_2\overline{F_2} \leq P_1\overline{F_1}\,. \tag{6.2.57}$$

The non-trivial condition that satisfies both (6.2.53) and (6.2.57) is

$$P_2\overline{F_2} = P_1\overline{F_1}\,. \tag{6.2.58}$$

Because (6.2.58) must be true for the expected profit from the dual-price model (Case 7, where $Q_1 = 2, Q_2 = 1$) to be at least as high as the expected profits from two of the single-price models (Case 4, where $Q_1 = 3, Q_2 = 0$, and Case 10, where $Q_1 = 0, Q_2 = 3$), this dual-price strategy is always sub-optimal unless $P_2\overline{F_2} = P_1\overline{F_1}$.

The final comparisons are between the dual-price strategy of Case 9 and the single-price strategies of Cases 4 and 10.

For Case 9 ($Q_1 = 1, Q_2 = 2$) to be superior to Case 4 ($Q_1 = 3, Q_2 = 0$),

$$E[\Pi_{Case9}] \geq E[\Pi_{Case4}]. \tag{6.2.59}$$

Substitution of (6.2.34) and (6.2.39) into (6.2.59) gives:

$$P_1\overline{F_1}\left(3 - 3\overline{F_1} + \overline{F_1}^2\right) + P_2\overline{F_2}\left(3\overline{F_1} - \overline{F_1}^2\right) - 3c \geq 3P_1\overline{F_1} - 3c\,. \tag{6.2.60}$$

Expanding and rearranging (6.2.60) gives:

$$\left(P_2\overline{F_2} - P_1\overline{F_1}\right)\left(3\overline{F_1} - \overline{F_1}^2\right) \geq 0 . \tag{6.2.61}$$

Since the second term is non-negative, this requires that

$$P_2\overline{F_2} \geq P_1\overline{F_1} . \tag{6.2.62}$$

For Case 9 ($Q_1 = 1, Q_2 = 2$) to be superior to Case 10 ($Q_1 = 0, Q_2 = 3$),

$$E[\Pi_{Case9}] \geq E[\Pi_{Case10}] . \tag{6.2.63}$$

Substitution of (6.2.39) and (6.2.40) into (6.2.63) gives:

$$P_1\overline{F_1}\left(3 - 3\overline{F_1} + \overline{F_1}^2\right) + P_2\overline{F_2}\left(3\overline{F_1} - \overline{F_1}^2\right) - 3c \geq 3P_2\overline{F_2} - 3c . \tag{6.2.64}$$

Expanding and rearranging (6.2.64) gives:

$$\left(P_2\overline{F_2} - P_1\overline{F_1}\right)\left(3\overline{F_1} - \overline{F_1}^2 - 3\right) \geq 0 . \tag{6.2.65}$$

Since the second term in brackets in (6.2.65) is strictly negative, this requires that

$$P_2\overline{F_2} \leq P_1\overline{F_1} . \tag{6.2.66}$$

The non-trivial condition that satisfies both (6.2.62) and (6.2.66) is

$$P_2\overline{F_2} = P_1\overline{F_1} . \tag{6.2.67}$$

Because (6.2.67) must be true for the expected profit from the dual-price model (Case 9, where $Q_1 = 1, Q_2 = 2$) to be at least as high as the expected profits from two of the single-price models (Case 4, where $Q_1 = 3, Q_2 = 0$, and Case 10, where $Q_1 = 0, Q_2 = 3$), this dual-price strategy is always sub-optimal unless $P_2\overline{F_2} = P_1\overline{F_1}$.

For each of the three dual-price models (Cases 6, 7 and 9) to have an expected profit at least as high as the expected profit from two of the selected single-price models (Cases 3, 4, 8 and 10), the necessary condition is $P_2\overline{F_2} = P_1\overline{F_1}$, otherwise the dual-price models are sub-optimal. *Q.E.D.*

The second step in the proof of Theorem 6.2.3 is to show that under the optimality condition found in Lemma 6.2.3, at least one, single-price strategy has the same expected profit as the dual-price strategy. This is shown in the proof of Lemma 6.2.6.

Lemma 6.2.6. For a seller facing a deterministic customer base of size $d = 3$*, if* $P_2\overline{F_2} = P_1\overline{F_1}$*, then there exist single-price strategies with expected profits at least as high as the expected profit of each dual-price strategy.*

Proof. The proof of Lemma 6.2.6 begins with the substitution of $P_2\overline{F_2} = P_1\overline{F_1}$ into the expected profit functions for Cases 2 – 10 giving:

$$E[\Pi_{Case2}] = P_1\overline{F_1}\left(3 - 3\overline{F_1} + \overline{F_1}^2\right) - c, \tag{6.2.68}$$

$$E[\Pi_{Case3}] = P_1\overline{F_1}\left(3 - \overline{F_1}^2\right) - 2c, \tag{6.2.69}$$

$$E[\Pi_{Case4}] = 3P_1\overline{F_1} - 3c, \tag{6.2.70}$$

$$E[\Pi_{Case5}] = P_1\overline{F_1}\left(3 - 3\overline{F_2} + \overline{F_2}^2\right) - c, \tag{6.2.71}$$

$$E[\Pi_{Case6}] = P_1\overline{F_1}\left(3 - \overline{F_1}\,\overline{F_2}\right) - 2c, \tag{6.2.72}$$

$$E[\Pi_{Case7}] = 3P_1\overline{F_1} - 3c, \tag{6.2.73}$$

$$E[\Pi_{Case8}] = P_1\overline{F_1}\left(3-\overline{F_2}^{\,2}\right)-2c\,, \tag{6.2.74}$$

$$E[\Pi_{Case9}] = 3P_1\overline{F_1}-3c\text{, and} \tag{6.2.75}$$

$$E[\Pi_{Case10}] = 3P_1\overline{F_1}-3c\,. \tag{6.2.76}$$

Consider first, the expected profit functions for the dual-price strategy of Case 6 and the single-price strategy of Case 8 (compare Equations 46 and 48). Since $\overline{F_2} \le \overline{F_1}$, the expected profit from this single-price strategy will be at least as high as the expected profit of this dual-price strategy.

Second, note that the expected profit functions for Cases 7 and 9 (dual-price strategies) are identical to the expected profit functions for Cases 4 and 10 (single-price strategies).

Therefore, under the condition that $P_2\overline{F_2} = P_1\overline{F_1}$, there exist single-price strategies that give the expected profits at least as high as each of the dual-price strategies. *Q.E.D.*

There exists a necessary condition, $P_2\overline{F_2} = P_1\overline{F_1}$, under which each dual-price strategy simultaneously gives at least the same expected profit as all of the single-price strategies. Under this condition, there exist single-price strategies with expected profits at least as high as the expected profits of each of the dual-price strategies. On the assumption that simplicity is preferred, *ceteris paribus*, a single-price strategy is optimal. *Q.E.D.*

6.2.4 GENERAL FORM OF THE EXPECTED PROFIT FUNCTION FOR A RANDOM CUSTOMER BASE

In this section, the general form of an expected profit function for a seller who sets n inventory levels and corresponding selling prices with a stochastic customer base (two possible sizes) is derived.

Theorem 6.2.4. The expected total profit function for a seller facing a stochastic customer base of possible sizes d_1 *and* d_2 *with probabilities* $Prob_{d1}$, *and* $1-Prob_{d1}$, *respectively, is given by:*

$$E[\Pi_{Total}(Q_1, Q_2, ..., Q_n, .P_1, P_2, ..., P_n)] =$$

$$Prob_{d1} * E[\Pi(Q_1, Q_2, ..., Q_n, P_1, P_2, ..., P_n \mid d_1)] \qquad (6.2.77)$$

$$+\left(1-Prob_{d1}\right) * E[\Pi(Q_1, Q_2, ..., Q_n, P_1, P_2, ..., P_n \mid d_2)].$$

Proof. Theorem 6.2.4 is proved by use of the definition of expected value where there are only two possible outcomes. The two possible values in this theorem are expected profits, each conditional upon the size of its respective customer base. These values are multiplied by their respective probabilities of occurring, and added together to give expected profit. *Q.E.D.*

6.2.5 RANDOM CUSTOMER BASE EXAMPLE 1, $d_1 = 1$, $d_2 = 2$

In this section, the expected profit model derived in Section 6.2.4 is used to determine an optimal pricing strategy for a seller facing a stochastic customer base, where there are two possible sizes of customer base, $d_1 = 1$ with probability $Prob_{d1}$ and $d_2 = 2$ with probability $1-Prob_{d1}$. The expected profits from two single-price strategies (items are made available for sale at only one price) are compared with the expected profit a dual-price strategy (two inventory and corresponding price decisions are made) to determine which of these two strategies is optimal. The calculations for this example are provided in Appendix B, with important results summarized and discussed below.

In this example, the seller chooses from a set of inventory and corresponding price combinations where $Q_{Total} \leq 2$. While it is true that a second item will never be sold if the size of the customer base is one, the possibility of selling the second item if the customer base is two might make it worthwhile to make a second item available for sale, therefore this option is considered. The possible inventory combinations are shown in Table 6.2.3. Note that Q_i is the number of units of inventory to be made available for sale at price P_i, i = 1, 2.

Table 6.2.3. Inventory Combinations for Example 3.

		Q_1		
		0	1	2
Q_2	0	Case 1	Case 2	Case 3
	1	Case 4	Case 5	
	2	Case 6		

While Case 1 is included in Table 6.2.3 for completeness, the decision to make no items available for sale results in a trivial expected profit of zero, and is not considered for the remainder of the example.

The objective of this example is to determine if one of the strategies (single-price or dual-price) is optimal for a seller facing a stochastic customer base with the parameters listed above.

Theorem 6.2.5. If there are two possible sizes of customer base, $d_1 = 1$ *with probability* $Prob_{d1}$*, and* $d_2 = 2$ *with probability* $1 - Prob_{d1}$*, a single- price strategy is optimal.*

The proof of Theorem 6.2.5 is achieved in two steps. First, it is shown that a dual-price strategy is always sub-optimal unless a strict condition is met. Second, it is shown that if this condition is met, there exists at least one, single-price strategy that gives an expected profit that is the same as the expected profit for the dual-price strategy. The first step is the proof of Lemma 6.2.7.

Lemma 6.2.7. If there are two possible sizes of customer base, $d_1 = 1$ *with probability* $Prob_{d1}$*, and* $d_2 = 2$ *with probability* $1 - Prob_{d1}$*, a dual-price strategy is always sub-optimal unless* $P_2\overline{F_2} = P_1\overline{F_1}$ *.*

Proof. The proof of Lemma 6.2.7 requires a comparison of the expected profit functions of the cases shown in Table 6.2.3. These expected profit functions, derived using the method outlined in Section 6.2.4, are as follows:

$$E[\Pi_{Case2}] = P_1\overline{F_1}\left(2 - Prob_{d1}\right) - P_1\overline{F_1}^2\left(1 - Prob_{d1}\right) - c\,. \tag{6.2.78}$$

$$E[\Pi_{Case3}] = P_1\overline{F_1}\left(2 - Prob_{d1}\right) - 2c\,. \tag{6.2.79}$$

$$E[\Pi_{Case4}] = P_2\overline{F_2}(2 - Prob_{d1}) - P_2\overline{F_2}^2(1 - Prob_{d1}) - c. \tag{6.2.80}$$

$$E[\Pi_{Case5}] = P_1\overline{F_1}(2 - Prob_{d1}) - P_1\overline{F_1}^2(1 - Prob_{d1}) + P_2\overline{F_1}\,\overline{F_2}(1 - Prob_{d1}) - 2c. \tag{6.2.81}$$

$$E[\Pi_{Case6}] = P_2\overline{F_2}(2 - Prob_{d1}) - 2c. \tag{6.2.82}$$

For the dual-price strategy (Case 5) to be optimal, it must simultaneously give a higher expected profit than all of the available single-inventory and price strategies (Cases 2, 3, 4 and 6). Shown below are comparisons of the expected profit function for Case 5 with the expected profit functions of two of the single-price strategies.

For Case 5 ($Q_1 = 1, Q_2 = 1$) to be superior to Case 3 ($Q_1 = 2, Q_2 = 0$),

$$E[\Pi_{Case5}] \geq E[\Pi_{Case3}]. \tag{6.2.83}$$

Substitution of (6.2.79) and (6.2.81) into (6.2.83) gives:

$$P_1\overline{F_1}(2 - Prob_{d1}) - P_1\overline{F_1}^2(1 - Prob_{d1}) + P_2\overline{F_1}\,\overline{F_2}(1 - Prob_{d1}) - 2c \geq P_1\overline{F_1}(2 - Prob_{d1}) - 2c. \tag{6.2.84}$$

Rearranging and reducing (6.2.84) gives:

$$\overline{F_1}\left(P_2\overline{F_2} - P_1\overline{F_1}\right)(1 - Prob_{d1}) \geq 0. \tag{6.2.85}$$

Since $\overline{F_1}$ and $1 - Prob_{d1}$ are both non-negative, for (6.2.85) to hold, $P_2\overline{F_2} - P_1\overline{F_1}$ must also be non-negative, or

$$P_2\overline{F_2} \geq P_1\overline{F_1}. \tag{6.2.86}$$

For Case 5 ($Q_1 = 1, Q_2 = 1$) to be superior to Case 6 ($Q_1 = 0, Q_2 = 2$),

$$E[\Pi_{Case5}] \geq E[\Pi_{Case6}]. \qquad (6.2.87)$$

Substitution of (6.2.81) and (6.2.82) into (6.2.87) gives:

$$P_1\overline{F_1}(2-Prob_{d1}) - P_1\overline{F_1}^2(1-Prob_{d1}) + P_2\overline{F_1}\,\overline{F_2}(1-Prob_{d1}) - 2c \geq P_2\overline{F_2}(2-Prob_{d1}) - 2c. \qquad (6.2.88)$$

Rearranging and reducing (6.2.88) gives:

$$\overline{F_1}\left(P_2\overline{F_2} - P_1\overline{F_1}\right)(1-Prob_{d1}) \geq \left(P_2\overline{F_2} - P_1\overline{F_1}\right)(2-Prob_{d1}). \qquad (6.2.89)$$

Since

$$\overline{F_1}(1-Prob_{d1}) < (2-Prob_{d1}), \qquad (6.2.90)$$

for (6.2.89) to be true, $P_2\overline{F_2} - P_1\overline{F_1}$ must be non-positive, or

$$P_2\overline{F_2} \leq P_1\overline{F_1}. \qquad (6.2.91)$$

The non-trivial condition that satisfies both (6.2.86) and (6.2.91) is

$$P_2\overline{F_2} = P_1\overline{F_1}. \qquad (6.2.92)$$

Because (6.2.92) must be true for the expected profit from the dual-price model (Case 5, where $Q_1 = 1, Q_2 = 1$) to be at least as high as the expected profits from two of the single-price models (Case 3, where $Q_1 = 2, Q_2 = 0$, and Case 6, where $Q_1 = 0, Q_2 = 2$), the dual-price strategy is always sub-optimal unless $P_2\overline{F_2} = P_1\overline{F_1}$. *Q.E.D.*

The second step in the proof of Theorem 6.2.5 is to show that under the optimality condition found in Lemma 6.2.3, there exists a single-price strategy that has at least the same expected profit as the dual-price strategy. This is shown in the proof of Lemma 6.2.8.

Lemma 6.2.8. If there are two possible sizes of customer base, $d_1 = 1$ *with probability* $Prob_{d1}$*, and* $d_2 = 2$ *with probability* $1 - Prob_{d1}$*, and if* $P_2\overline{F_2} = P_1\overline{F_1}$ *, there exists a single-price strategy with an expected profit at least as high as the expected profit of the dual-price strategy.*

Proof. The proof of Lemma 6.2.8 begins with the substitution of (6.2.92) into the expected profit functions for Cases 2 – 6:

$$E[\Pi_{Case2}] = P_1\overline{F_1}(2 - Prob_{d1}) - P_1\overline{F_1}^2(1 - Prob_{d1}) - c \,. \tag{6.2.93}$$

$$E[\Pi_{Case3}] = P_1\overline{F_1}(2 - Prob_{d1}) - 2c \,. \tag{6.2.94}$$

$$E[\Pi_{Case4}] = P_1\overline{F_1}(2 - Prob_{d1}) - P_1\overline{F_1}\,\overline{F_2}(1 - Prob_{d1}) - c \,. \tag{6.2.95}$$

$$E[\Pi_{Case5}] = P_1\overline{F_1}(2 - Prob_{d1}) - 2c \,. \tag{6.2.96}$$

$$E[\Pi_{Case6}] = P_1\overline{F_1}(2 - Prob_{d1}) - 2c \,. \tag{6.2.97}$$

By inspection, the right hand sides of Equations 56, 58 and 59 are equivalent. Therefore, under the condition that $P_2\overline{F_2} = P_1\overline{F_1}$, there are two single-price strategies (Case 3, where $Q_1 = 2, Q_2 = 0$, and Case 6, where $Q_1 = 0, Q_2 = 2$), with expected profits at least as high as the expected profit of the dual-price strategy (Case 5, where $Q_1 = 1, Q_2 = 1$). *Q.E.D.*

The non-trivial condition under which a dual-price strategy (Case 5, where $Q_1 = 1, Q_2 = 1$) simultaneously gives at least the same expected profit as two of the single-price strategies (Case 3, where $Q_1 = 2, Q_2 = 0$, and Case 6, where $Q_1 = 0, Q_2 = 2$), is $P_2\overline{F_2} = P_1\overline{F_1}$. Under this condition, these two single-price strategies give the same expected profit as the dual-price strategy. Therefore, on the assumption that simplicity is preferred, *ceteris paribus*, a single-price strategy is optimal. *Q.E.D.*

6.2.6 RANDOM CUSTOMER BASE EXAMPLE 2, $d_1 = 1$, $d_2 = 3$

In this section, the expected profit model derived in Section 6.2.4 is used to determine an optimal pricing strategy for a seller facing a stochastic customer base, where there are two possible sizes of customer base, $d_1 = 1$ with probability $Prob_{d1}$ and $d_2 = 3$ with probability $1 - Prob_{d1}$. The expected profits from two single-price strategies (items are made available for sale at only one price) are compared with the expected profit a dual-price strategy (two inventory and corresponding price decisions are made) to determine which of these two strategies is optimal. The calculations for this example are provided in Appendix B, with important results summarized and discussed below.

In this example, the seller chooses from a set of inventory and corresponding price combinations where $Q_{Total} \le 3$. While it is true that a second or third item will never be sold if the size of the customer base is one, the possibility of selling the additional items if the customer base is three might make it worthwhile to make these items available for sale, therefore these inventory options are considered. The possible inventory combinations are shown in Table 6.2.4. Note that Q_i is the number of units of inventory to be made available for sale at price P_i, $i = 1, 2$.

Table 6.2.4. Inventory Combinations for Example 4.

		Q_1			
		0	1	2	3
Q_2	0	Case 1	Case 2	Case 3	Case 4
	1	Case 5	Case 6	Case 7	
	2	Case 8	Case 9		
	3	Case 10			

While Case 1 is included in Table 6.2.4 for completeness, the decision to make no items available for sale results in a trivial expected profit of zero, and is not considered for the remainder of the example.

The objective of this example is to determine if one of the strategies (single-price or dual-price) is optimal for a seller facing a stochastic customer base with the parameters listed above.

Theorem 6.2.6. If there are two possible sizes of customer base, $d_1 = 1$ *with probability* $Prob_{d1}$*, and* $d_2 = 3$ *with probability* $1 - Prob_{d1}$*, a single- price strategy is optimal.*

The proof of Theorem 6.2.6 is achieved in two steps. First, it is shown that a dual-price strategy is always sub-optimal unless a strict condition is met. Second, it is shown that if this condition is met, there exists at least one, single-price strategy that gives an expected profit that is the same as the expected profit for the dual-price strategy. The first step is the proof of Lemma 6.2.9.

Lemma 6.2.9. If there are two possible sizes of customer base, $d_1 = 1$ with probability $Prob_{d1}$, and $d_2 = 3$ with probability $1 - Prob_{d1}$, a dual-price strategy is always sub-optimal unless $P_2\overline{F_2} = P_1\overline{F_1}$.

Proof. The proof of Lemma 6.2.9 requires a comparison of the expected profit functions of the cases shown in Table 6.2.4. These expected profit functions, derived using the method outlined in Section 6.2.4, are as follows:

$$E[\Pi_{Case2}] = P_1\overline{F_1}\left(3 - 2Prob_{d1} - (1 - Prob_{d1})\left(3\overline{F_1} - \overline{F_1}^2\right)\right) - c. \quad (6.2.98)$$

$$E[\Pi_{Case3}] = P_1\overline{F_1}\left(3 - 2Prob_{d1} - \overline{F_1}^2(1 - Prob_{d1})\right) - 2c. \quad (6.2.99)$$

$$E[\Pi_{Case4}] = P_1\overline{F_1}(3 - 2Prob_{d1}) - 3c. \quad (6.2.100)$$

$$E[\Pi_{Case5}] = P_2\overline{F_2}\left(3 - 2Prob_{d1} - (1 - Prob_{d1})\left(3\overline{F_2} - \overline{F_2}^2\right)\right) - c. \quad (6.2.101)$$

$$E[\Pi_{Case6}] = \left(P_2\overline{F_2} - P_1\overline{F_1}\right)(1 - Prob_{d1})\left(3\overline{F_1} - \overline{F_1}^2\right) + P_1\overline{F_1}(3 - 2Prob_{d1}) - P_2\overline{F_1}\overline{F_2}^2(1 - Prob_{d1}) - 2c. \quad (6.2.102)$$

$$E[\Pi_{Case7}] = \left(P_2\overline{F_2} - P_1\overline{F_1}\right)\left(\overline{F_1}^2\right)(1 - Prob_{d1}) + P_1\overline{F_1}(3 - 2Prob_{d1}) - 3c. \quad (6.2.103)$$

$$E[\Pi_{Case8}] = P_2\overline{F_2}\left(3 - 2Prob_{d1} - \overline{F_2}^2(1 - Prob_{d1})\right) - 2c. \quad (6.2.104)$$

$$E[\Pi_{Case9}] = \left(P_2\overline{F_2} - P_1\overline{F_1}\right)(1 - Prob_{d1})\left(3\overline{F_1} - \overline{F_1}^2\right) + P_1\overline{F_1}(3 - 2Prob_{d1}) - 3c. \quad (6.2.105)$$

$$E[\Pi_{Case10}] = P_2\overline{F_2}(3 - 2Prob_{d1}) - 3c. \quad (6.2.106)$$

For a dual-price strategy (Case 6, 7 or 9) to be optimal, it must simultaneously have a higher expected profit than all of the available single-inventory and price strategies

(Cases 2, 3, 4, 5, 8 and 10). Shown below are comparisons of the expected profit functions of the dual-price strategies with the expected profit functions of selected single-price strategies.

The first comparisons are between the dual-price strategy of Case 6 and the single-price strategies of Cases 3 and 8.

For Case 6 ($Q_1 = 1, Q_2 = 1$) to be superior to Case 3 ($Q_1 = 2, Q_2 = 0$),

$$E[\Pi_{Case6}] \geq E[\Pi_{Case3}]. \tag{6.2.107}$$

Substitution of (6.2.100) and (6.2.102) into (6.2.107) gives:

$$\left(P_2\overline{F_2} - P_1\overline{F_1}\right)\left(1 - Prob_{d1}\right)\left(3\overline{F_1} - \overline{F_1}^2\right)$$

$$+ P_1\overline{F_1}\left(3 - 2Prob_{d1}\right) - P_2\overline{F_1}\,\overline{F_2}^2\left(1 - Prob_{d1}\right) - 2c \tag{6.2.108}$$

$$\geq P_1\overline{F_1}\left(3 - 2Prob_{d1} - \overline{F_1}^2\left(1 - Prob_{d1}\right)\right) - 2c\,.$$

Rearranging (6.2.108) gives:

$$\left(P_2\overline{F_2} - P_1\overline{F_1}\right)\left(3\overline{F_1}\left(1 - Prob_{d1}\right)\right) + P_2\overline{F_1}\,\overline{F_2}\left(\overline{F_2} - \overline{F_1}\right)\left(1 - Prob_{d1}\right)$$

$$+ P_1\overline{F_1}^2\left(2\overline{F_1} - 3\right) \geq 0\,. \tag{6.2.109}$$

In (6.2.109), $\overline{F_2} - \overline{F_1}$ is negative, making the second term on the left hand side non-positive. Also, $2\overline{F_1} - 3$ is negative, making the third term on the left hand side non-positive. Since these two terms are non-positive, for (6.2.109) to be true, the first term on the left hand side must be non-negative. This requires that $P_2\overline{F_2} - P_1\overline{F_1}$ be non-negative, or

$$P_2\overline{F_2} \geq P_1\overline{F_1}\,. \tag{6.2.110}$$

For Case 6 ($Q_1 = 1, Q_2 = 1$) to be superior to Case 8 ($Q_1 = 0, Q_2 = 2$),

$$E[\Pi_{Case6}] \geq E[\Pi_{Case8}]\,. \tag{6.2.111}$$

Substitution of (6.2.102) and (6.2.104) into (6.2.111) and rearranging gives:

$$\left(P_2\overline{F_2} - P_1\overline{F_1}\right)\left(\left(1 - Prob_{d1}\right)\left(3\overline{F_1} - \overline{F_1}^2\right) - \left(3 - 2Prob_{d1}\right)\right) \geq P_2\overline{F_2}^2\left(\overline{F_1} - \overline{F_2}\right). \tag{6.2.112}$$

Note that the right-hand side of (6.2.112) is non-negative. To begin the process of determining if the same (or opposite) can be said for the left-hand side, denote the second bracketed term on the left-hand side of (6.2.112) as

$$y(\overline{F_1} \mid Prob_{d1}) = \left(1 - Prob_{d1}\right)\left(3\overline{F_1} - \overline{F_1}^2\right) - \left(3 - 2Prob_{d1}\right). \tag{6.2.113}$$

Taking the first order conditions of (6.2.113) gives:

$$\frac{\partial}{\partial \overline{F_1}} y(\overline{F_1} \mid Prob_{d1}) = \left(1 - Prob_{d1}\right)\left(3 - 2\overline{F_1}\right) = 0\,. \tag{6.2.114}$$

Solving (6.2.114) gives a stationary point at $\overline{F_1}^* = 3/2$. To determine if this is a maximum or a minimum, consider the second partial derivative of (6.2.113):

$$\frac{\partial^2}{\partial \overline{F_1}^2} y(\overline{F_1} \mid Prob_{d1}) = -2\left(1 - Prob_{d1}\right) \tag{6.2.115}$$

which is negative for $0 \leq Prob_{d1} < 1$. Therefore, $y(\overline{F_1} \mid Prob_{d1})$ is a maximum at $\overline{F_1}^* = 3/2$. Substitution into $y(\overline{F_1} \mid Prob_{d1})$ gives:

$$y(\overline{F_1}^* \mid Prob_{d1}) = -\frac{\left(3 + Prob_{d1}\right)}{4}\,, \tag{6.2.116}$$

which is negative for $0 \leq Prob_{d1} \leq 1$.

Since $y(\overline{F_1} \mid Prob_{d1})$ is strictly negative and the right hand side of (6.2.112) is non-negative, this requires that

$$P_2\overline{F_2} \le P_1\overline{F_1}\,. \tag{6.2.117}$$

The non-trivial condition that satisfies both (6.2.110) and (6.2.117) is

$$P_2\overline{F_2} = P_1\overline{F_1}\,. \tag{6.2.118}$$

Because (6.2.118) must be true for the expected profit from the dual-price model (Case 6, where $Q_1 = 1, Q_2 = 1$) to be at least as high as the expected profits from two of the single-price models (Case 3, where $Q_1 = 2, Q_2 = 0$, and Case 8, where $Q_1 = 0, Q_2 = 2$), this dual-price strategy is always sub-optimal unless $P_2\overline{F_2} = P_1\overline{F_1}$.

The next comparisons are between the dual-price strategy of Case 7 and the single-price strategies of Cases 4 and 10.

For Case 7 ($Q_1 = 2, Q_2 = 1$) to be superior to Case 4 ($Q_1 = 3, Q_2 = 0$),

$$E[\Pi_{Case7}] \ge E[\Pi_{Case4}]\,. \tag{6.2.119}$$

Substitution of (6.2.100) and (6.2.103) into (6.2.119) gives:

$$\left(P_2\overline{F_2} - P_1\overline{F_1}\right)\left(\overline{F_1}^{\,2}\right)\left(1 - Prob_{d1}\right) + P_1\overline{F_1}\left(3 - 2Prob_{d1}\right) - 3c \ge P_1\overline{F_1}\left(3 - 2Prob_{d1}\right) - 3c\,. \tag{6.2.120}$$

Expanding and rearranging (6.2.120) gives:

$$\overline{F_1}^{\,2}\left(P_2\overline{F_2} - P_1\overline{F_1}\right)\left(1 - Prob_{d1}\right) \ge 0\,. \tag{6.2.121}$$

For (6.2.121) to be true, $P_2\overline{F_2} - P_1\overline{F_1}$ must be non-negative, or

$$P_2\overline{F_2} \geq P_1\overline{F_1}\ . \tag{6.2.122}$$

For Case 7 ($Q_1 = 2, Q_2 = 1$) to be superior to Case 10 ($Q_1 = 0, Q_2 = 3$),

$$E[\Pi_{Case7}] \geq E[\Pi_{Case10}]. \tag{6.2.123}$$

Substitution of (6.2.103) and (6.2.106) into (6.2.123) gives:

$$\left(P_2\overline{F_2} - P_1\overline{F_1}\right)\left(\overline{F_1}^{\,2}\right)\left(1 - Prob_{d1}\right) + P_1\overline{F_1}\left(3 - 2Prob_{d1}\right) - 3c \geq P_2\overline{F_2}\left(3 - 2Prob_{d1}\right) - 3c\ . \tag{6.2.124}$$

Expanding and rearranging (6.2.124) gives:

$$\left(P_2\overline{F_2} - P_1\overline{F_1}\right)\left(\overline{F_1}^{\,2}\left(1 - Prob_{d1}\right) - 2Prob_{d1} - 3\right) \geq 0\,. \tag{6.2.125}$$

Since the second term in brackets in (6.2.125) is non-positive, this requires that $P_2\overline{F_2} - P_1\overline{F_1}$ be non-positive, or

$$P_2\overline{F_2} \leq P_1\overline{F_1}\ . \tag{6.2.126}$$

The non-trivial condition that satisfies both (6.2.122) and (6.2.126) is

$$P_2\overline{F_2} = P_1\overline{F_1}\ . \tag{6.2.127}$$

Because (6.2.127) must be true for the expected profit from the dual-price model (Case 7, where $Q_1 = 2, Q_2 = 1$) to be at least as high as the expected profits from two of the single-price models (Case 4, where $Q_1 = 3, Q_2 = 0$, and Case 10, where $Q_1 = 0, Q_2 = 3$), this dual-price strategy is always sub-optimal unless $P_2\overline{F_2} = P_1\overline{F_1}$.

The final comparisons are between the dual-price strategy of Case 9 and the single-price strategies of Cases 4 and 10.

For Case 9 ($Q_1 = 1, Q_2 = 2$) to be superior to Case 4 ($Q_1 = 3, Q_2 = 0$),

$$E[\Pi_{Case9}] \geq E[\Pi_{Case4}]. \tag{6.2.128}$$

Substitution of (6.2.100) and (6.2.105) into (6.2.128) gives:

$$\left(P_2\overline{F_2} - P_1\overline{F_1}\right)\left(1 - Prob_{d1}\right)\left(3\overline{F_1} - \overline{F_1}^2\right) + P_1\overline{F_1}\left(3 - 2Prob_{d1}\right) - 3c \geq P_1\overline{F_1}\left(3 - 2Prob_{d1}\right) - 3c\,. \tag{6.2.129}$$

Expanding and rearranging (6.2.129) gives:

$$\left(P_2\overline{F_2} - P_1\overline{F_1}\right)\left(1 - Prob_{d1}\right)\left(3\overline{F_1} - \overline{F_1}^2\right) \geq 0\,. \tag{6.2.130}$$

Since $3\overline{F_1} - \overline{F_1}^2$ is non-negative, for (6.2.130) to be true, $P_2\overline{F_2} - P_1\overline{F_1}$ must also be non-negative, or

$$P_2\overline{F_2} \geq P_1\overline{F_1}\,. \tag{6.2.131}$$

For Case 9 ($Q_1 = 1, Q_2 = 2$) to be superior to Case 10 ($Q_1 = 0, Q_2 = 3$),

$$E[\Pi_{Case9}] \geq E[\Pi_{Case10}]. \tag{6.2.132}$$

Substitution of (6.2.105) and (6.2.106) into (6.2.132) gives:

$$\left(P_2\overline{F_2} - P_1\overline{F_1}\right)\left(1 - Prob_{d1}\right)\left(3\overline{F_1} - \overline{F_1}^2\right) + P_1\overline{F_1}\left(3 - 2Prob_{d1}\right) - 3c \geq P_2\overline{F_2}\left(3 - 2Prob_{d1}\right) - 3c\,. \tag{6.2.133}$$

Rearranging (6.2.133) gives:

$$\left(P_2\overline{F_2} - P_1\overline{F_1}\right)\left(\left(1 - Prob_{d1}\right)\left(3\overline{F_1} - \overline{F_1}^2\right) - \left(3 - 2Prob_{d1}\right)\right) \geq 0. \tag{6.2.134}$$

Note that $\left(1 - Prob_{d1}\right)\left(3\overline{F_1} - \overline{F_1}^2\right) - \left(3 - 2Prob_{d1}\right)$ is strictly negative as proven above, therefore, $P_2\overline{F_2} - P_1\overline{F_1}$ must be non-positive, or

$$P_2\overline{F_2} \leq P_1\overline{F_1}. \tag{6.2.135}$$

The non-trivial condition that satisfies both (6.2.131) and (6.2.135) is

$$P_2\overline{F_2} = P_1\overline{F_1}. \tag{6.2.136}$$

Because (6.2.136) must be true for the expected profit from the dual-price model (Case 9, where $Q_1 = 1, Q_2 = 2$) to be at least as high as the expected profits from two of the single-price models (Case 4, where $Q_1 = 3, Q_2 = 0$, and Case 10, where $Q_1 = 0, Q_2 = 3$), this dual-price strategy is always sub-optimal unless $P_2\overline{F_2} = P_1\overline{F_1}$.

For each of the three dual-price models (Cases 6, 7 and 9) to have an expected profit at least as high as the expected profit from two of the selected single-price models (Cases 3, 4, 8 and 10), the necessary condition is $P_2\overline{F_2} = P_1\overline{F_1}$, otherwise the dual-price models are sub-optimal. *Q.E.D.*

The second step in the proof of Theorem 6.2.6 is to show that under the optimality condition found in Lemma 6.2.3, at least one, single-price strategy has the same expected profit as each dual-price strategy. This is shown in the proof of Lemma 6.2.10.

Lemma 6.2.10. If there are two possible sizes of customer base, $d_1 = 1$ *with probability* $Prob_{d1}$*, and* $d_2 = 3$ *with probability* $1 - Prob_{d1}$*, and if* $P_2\overline{F_2} = P_1\overline{F_1}$*, then there exist single-price strategies with expected profits at least as high as the expected profit of each dual-price strategy.*

Proof. The proof of Lemma 6.2.10 begins with the substitution of $P_2\overline{F_2} = P_1\overline{F_1}$ into the expected profit functions for Cases 2 – 10 giving:

$$E[\Pi_{Case2}] = P_1\overline{F_1}\left(3 - 2Prob_{d1} - 3\overline{F_1}(1 - Prob_{d1}) + \overline{F_1}^2(1 - Prob_{d1})\right) - c, \tag{6.2.137}$$

$$E[\Pi_{Case3}] = P_1\overline{F_1}\left(3 - 2Prob_{d1} - \overline{F_1}^2(1 - Prob_{d1})\right) - 2c, \tag{6.2.138}$$

$$E[\Pi_{Case4}] = P_1\overline{F_1}(3 - 2Prob_{d1}) - 3c, \tag{6.2.139}$$

$$E[\Pi_{Case5}] = P_1\overline{F_1}\left(3 - 2Prob_{d1} - 3\overline{F_2}(1 - Prob_{d1}) + \overline{F_2}^2(1 - Prob_{d1})\right) - c, \tag{6.2.140}$$

$$E[\Pi_{Case6}] = P_1\overline{F_1}\left(3 - 2Prob_{d1} - \overline{F_1}\,\overline{F_2}(1 - Prob_{d1})\right) - 2c, \tag{6.2.141}$$

$$E[\Pi_{Case7}] = P_1\overline{F_1}(3 - 2Prob_{d1}) - 3c, \tag{6.2.142}$$

$$E[\Pi_{Case8}] = P_1\overline{F_1}\left(3 - 2Prob_{d1} - \overline{F_2}^2(1 - Prob_{d1})\right) - 2c, \tag{6.2.143}$$

$$E[\Pi_{Case9}] = P_1\overline{F_1}(3 - 2Prob_{d1}) - 3c, \text{ and} \tag{6.2.144}$$

$$E[\Pi_{Case10}] = P_1\overline{F_1}(3 - 2Prob_{d1}) - 3c. \tag{6.2.145}$$

Consider the expected profit functions for the dual-price strategy of Case 6 and the single-price strategy of Case 8 (compare Equations 52 and 54). Since $\overline{F_2} \le \overline{F_1}$, the expected profit from this single-price strategy will be at least as high as the expected profit of this dual-price strategy.

Now, note that the expected profit functions for Cases 7 and 9 (dual-price strategies) are identical to the expected profit functions for Cases 4 and 10 (single-price strategies).

Therefore, under the condition that $P_2\overline{F_2} = P_1\overline{F_1}$, there exist single-price strategies that give the expected profits at least as high as each of the dual-price strategies. *Q.E.D.*

There exists a necessary condition, $P_2\overline{F_2} = P_1\overline{F_1}$, under which each dual-price strategy simultaneously gives at least the same expected profit as all of the single-price strategies. Under this condition, there exist single-price strategies with expected profits at least as high as the expected profits of each of the dual-price strategies. On the assumption that simplicity is preferred, *ceteris paribus*, a single-price strategy is optimal. *Q.E.D.*

In this chapter, four sets of expected profit functions were derived to determine if a dual-price strategy is of more benefit to a seller than single-price strategies for small populations. In all four cases (two involving deterministic customer bases and two involving random customer bases), the optimal strategy is to use only a single price and corresponding inventory level to maximize expected profit. Future work is to include determining if the results of this chapter hold for larger population sizes, and if so, how can this be reconciled with the results of the previous chapter that indicate multiple-pricing strategies are sometimes optimal.

CHAPTER 7

THE NEWSVENDOR PROBLEM WITH PRICING AND SECONDARY REVENUES

7.1 INTRODUCTION

In the newsvendor problem with pricing models presented Chapters 5 and 6, revenues were only realized by the sale of one kind of item. In this chapter, we consider the case where the seller of an inventory of homogeneous items (primary items) has the opportunity to earn secondary revenues from the customers, provided they have already purchased a primary item. Here, the seller decides on a single selling price P and corresponding inventory level Q for primary items, before knowing exact demand (denoted $X(P)$ as in previous chapters), which is determined by size of customer base d, and random reservation price, RP. The work in this chapter builds upon the demand model of Chapter 3. If a customer has a reservation price at least as high as P and a primary item is available, the customer will pay P for a primary item, and provide an additional amount of profit on secondary items, S. It is assumed that S is a random variable, with a distribution known to the seller, and is treated not as a number of units sold, but as a dollar amount of profit from a single customer. The seller incurs a cost of c for each primary item made available for sale, regardless of whether the item is ultimately sold or not. As practical examples, consider the sale of warrantees on new electronics, or luggage handling fees on airline tickets. For other examples on the NPP and secondary revenues, see Fort (2004), Marburger (1997) and Rosen and Rosenfield (1997).

In this chapter, the profit function for a seller described above is derived. Then, the corresponding expected profit and variance functions are derived, and used with three examples to illustrate the sensitivity of total profit to changes in different variables.

7.2 PROFIT FUNCTION – RANDOM SECONDARY REVENUE

First we derive an expression for the seller's total profit when secondary profits can be realized after a customer has purchased a primary item.

Lemma 7.2.1. The total profit for a seller who receives profit from the sale of secondary items in addition to the revenues from the sale of primary items is:

$$\Pi_{Total}(P,Q)=\begin{cases} Px+\sum_{i=1}^{x}s_i-cQ, & x\le Q \\ PQ+\sum_{i=1}^{Q}s_i-cQ, & x>Q \end{cases}. \tag{7.2.1}$$

where x *denotes the realized value of* $X(P)$*, and* s_i *is the realized value of* S *for customer i, who has already purchased a primary item.*

Proof. The total profit for the seller is made up of profit from the sale of primary items and profit from the sale of secondary items. Consider first the profit obtained from the sale of primary items. Denoting realized demand as x (i.e.; the realized value of $X(P)$), if x is less than the number of items made available for sale, the revenue is Px. If x is greater than the number of items made available for sale, the revenue is PQ. Note that in both scenarios, the incurred cost is cQ, as the seller pays for each item made available for sale, regardless of whether or not it is sold. From these two possible revenue scenarios, the profit from the sale of primary items is:

$$\Pi_{Pri}(P,Q)=\begin{cases} Px-cQ, & x\le Q \\ PQ-cQ, & x>Q \end{cases}. \tag{7.2.2}$$

The profit from the sale of secondary items is the total of all of these profits earned from individual customers who have already purchased a primary item ($\min((Q, X(P)))$. The profit earned from each individual customer is random, and the total profit from secondary revenues is:

$$\Pi_{Sec}=\begin{cases} \sum_{i=1}^{x} s_i, & x\le Q \\ \sum_{i=1}^{Q} s_i, & x>Q \end{cases}. \tag{7.2.3}$$

As total profit is made up of the revenue from the sale of primary items (less inventory costs) in addition to the profit from the sale of secondary items, combining (7.2.2) and (7.2.3) gives (7.2.1). *Q.E.D.*

7.3 EXPECTED PROFIT FUNCTION – RANDOM SECONDARY REVENUE

Next, we derive and expression for expected profit for the seller described in the introduction.

Lemma 7.3.1. The expected total profit for a seller who receives profit from the sale of secondary items in addition to the revenues from the sale of primary items is:

$$E[\Pi_{Total}(P,Q)]=E[Z_1]\left(P+E[S]\right)-cQ\,, \tag{7.3.1}$$

where Z_1 denotes the number of primary items that are sold.

Proof. Begin with the total profit function given in (7.2.1):

$$\Pi_{Total}(P,Q) = \begin{cases} Px + \sum_{i=1}^{x} s_i - cQ, & x \le Q \\ PQ + \sum_{i=1}^{Q} s_i - cQ, & x > Q \end{cases}. \tag{7.2.1}$$

and rewrite as

$$\Pi_{Total}(P,Q) = P\min\big(Q, X(P)\big) + \sum_{i=1}^{\min((Q, X(P))} s_i - cQ. \tag{7.3.2}$$

Suppress the notation for ease of reading, and denote $\min\big(Q, X\big)$ as random variable Z_1, and $\sum_{i=1}^{\min(Q, X)} s_i$ as random variable Z_2. Rewriting (7.3.2) gives:

$$\Pi_{Total} = PZ_1 + Z_2 - cQ. \tag{7.3.3}$$

Take the expectation of both sides:

$$E[\Pi_{Total}] = E[PZ_1 + Z_2 - cQ]. \tag{7.3.4}$$

Since the expectation of a sum is the sum of expectations,

$$E[\Pi_{Total}] = E[PZ_1] + E[Z_2] - E[cQ]. \tag{7.3.5}$$

Taking the constant P out of the first term gives:

$$E[\Pi_{Total}] = PE[Z_1] + E[Z_2] - E[cQ]. \tag{7.3.6}$$

As Z_2 is the sum of realized values of a random variable S, the total number of which is also random, use Wald's Equation to express the second term on the right-hand side as:

$$E[Z_2] = E[Z_1]E[S]. \tag{7.3.7}$$

Substitution of (7.3.7) into (7.3.6) gives:

$$E[\Pi_{Total}] = PE[Z_1] + E[Z_1]E[S] - E[cQ]. \tag{7.3.8}$$

The last term on the right-hand side is the expectation of a constant which is just the constant itself, therefore:

$$E[\Pi_{Total}] = PE[Z_1] + E[Z_1]E[S] - cQ\,. \tag{7.3.9}$$

Rearranging gives (7.3.1). *Q.E.D.*

7.4 VARIANCE – RANDOM SECONDARY REVENUE

Next, we derive an expression for the variance in total profit for the seller described in the introduction.

Lemma 7.4.1. The variance in total profit for a seller who receives profit from the sale of secondary items in addition to the revenues from the sale of primary items is:

$$Var\left(\Pi_{Total}(P, Q)\right) = Var\left(S\right)E[Z_1] + \left(E[S + P]\right)^2 Var\left(Z_1\right), \tag{7.4.1}$$

where Z_1 denotes the number of primary items that are sold.

Proof. Begin with the total profit function given in (7.2.1):

$$\Pi_{Total}(P, Q) = \begin{cases} Px + \sum_{i=1}^{x} s_i - cQ, & x \le Q \\ PQ + \sum_{i=1}^{Q} s_i - cQ, & x > Q \end{cases}. \tag{7.2.1}$$

Rewrite (7.2.1) as

$$\Pi_{Total}(P, Q) = P\min\left(Q,\ X(P)\right) + \sum_{i=1}^{\min\left(Q,\ X(P)\right)} s_i - cQ\,. \tag{7.4.2}$$

The notation is suppressed for ease of reading. Since secondary profits are realized every time a primary item is purchased (and revenue of P is realized), (7.4.2) can be written as:

$$\Pi_{Total}(P, Q) = \sum_{i=1}^{\min(Q, X)} (P + s_i) - cQ\,. \tag{7.4.3}$$

Denote $\min(Q, X)$ as Z_1. Taking the variance of both sides of (7.4.3) gives:

$$Var(\Pi_{Total}) = Var\left(\sum_{i=1}^{Z_1} (P + s_i) - cQ\right). \tag{7.4.4}$$

Note that the total cost, cQ, is a constant. Therefore,

$$Var\left(\sum_{i=1}^{Z_1} (P + s_i) - cQ\right) = Var\left(\sum_{i=1}^{Z_1} (P + s_i)\right). \tag{7.4.5}$$

From the Law of Total Variance,

$$Var\left(\sum_{i=1}^{Z_1} (P + s_i)\right) = Var(S + P)E[Z_1] + (E[S + P])^2 Var(Z_1). \tag{7.4.6}$$

Since P is a constant, (7.4.6) becomes

$$Var(\Pi_{Total}) = Var(S)E[Z_1] + (E[S + P])^2 Var(Z_1), \tag{7.4.7}$$

as in (7.4.1). *Q.E.D.*

7.5 NUMERICAL EXAMPLES – RANDOM SECONDARY REVENUE

To demonstrate the sensitivity of profit on certain variables in the model, four numerical examples are provided. In each example, the values of the variables needed to calculate profit (as in Equation 7.2.1) are held constant, with the exception of the values of one of the variables. For a given set of values, demand (the number of customers willing to pay price P for a primary item) as well as total profit from secondary items are randomly generated and used to calculate profit. The process is repeated 100 000 times for each set of values, and the results are plotted as a probability function (p.f.) to show the relative frequency of different total profits being observed. Each figure includes at least four probability functions for comparison, with each probability function corresponding to a different value of a given variable.

The general set of conditions, which applies to each example unless otherwise stated, is as follows. The size of the customer base is deterministic, $d = 50$. The cost of making each primary item available for sale is deterministic, $c = 10$. The maximum amount that a randomly-selected customer is willing to pay for an item is random and follows a normal distribution as $RP \sim N(\mu_{RP} = 50, \sigma_{RP} = 10)$. The profit received from a customer who has already purchased a primary item is random and follows a normal distribution as $S \sim N(\mu_S = 50, \sigma_S = 10)$. The number of primary items made available for sale by the seller is $Q = 40$. The selling price of a primary item is $P = 50$.

7.5.1 SENSITIVITY TO CHANGES IN EXPECTED RESERVATION PRICE

The first example demonstrates the sensitivity of total profit to changes in expected reservation price. The results of the simulation are plotted in Figure 7.5.1. Here, four probability functions demonstrate the effect of different expected reservation prices on total profit. The probability functions are generated using various expected values of reservation price (30, 40, 50 and 60). Two characteristics of the functions are worth noting: relative location and shape.

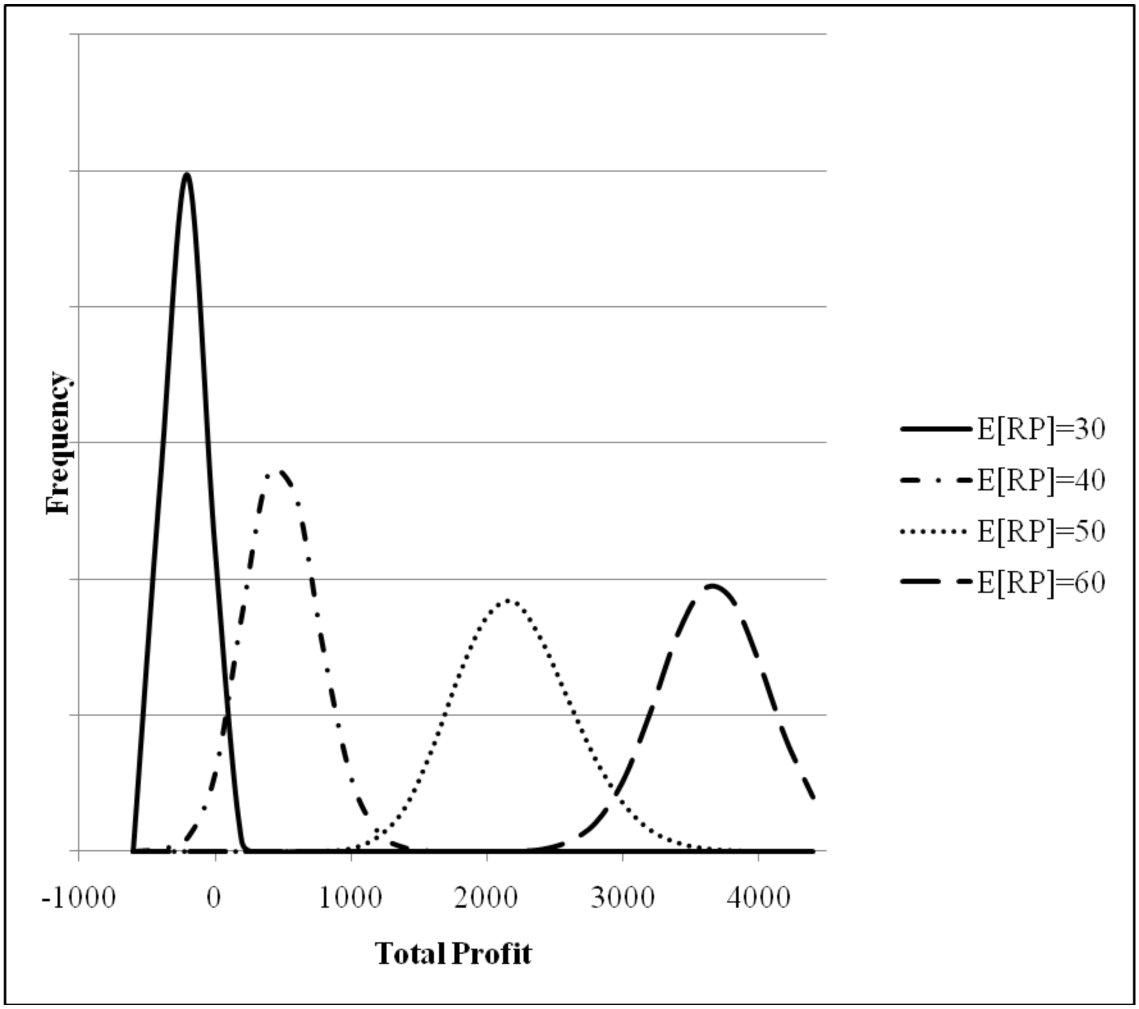

Figure 7.5.1. Sensitivity of Total Profit to Changes in Expected Reservation Price.

To understand Figure 7.5.1, consider first how changing expected reservation price might affect total profit, as given by (7.2.1):

$$\Pi_{Total}(P,Q)=\begin{cases} Px+\sum_{i=1}^{x} s_i - cQ, & x \le Q \\ PQ+\sum_{i=1}^{Q} s_i - cQ, & x > Q \end{cases}. \qquad (7.2.1)$$

As expected reservation price increases, so does the likelihood of selling more primary items (and gaining secondary profits as well). Referring to (7.2.1), this means that realized demand for items increases, and either more of the available items are sold (x increases up to, and including Q), or they are all sold with unmet demand ($x > Q$). Therefore, it is not surprising to see the probability functions shift to the right as expected reservation price increases.

By inspection, as expected reservation price increases, so too does expected profit. This can also be explained analytically, using the expression for expected profit:

$$E[\Pi_{Total}(P, Q)] = E[Z_1](P + E[S]) - cQ \, , \qquad (7.3.1)$$

Here, Z_1 denotes the number of items sold, which increases as demand increases (up to the point of no remaining inventory).

Finally, consider how variance in total profit changes as reservation price increases. Refer to the expression for variance from Section 7.4:

$$Var(\Pi_{Total}(P, Q)) = Var(S)E[Z_1] + (E[S + P])^2 Var(Z_1), \qquad (7.4.1)$$

Although it appears from (7.4.1) that variance should increase continually as more customers are willing to buy an item, if expected reservation price increases too much, demand will exceed supply too often, and stock-outs will occur often, actually decreasing variance in the number of items sold. This phenomenon is seen in the p.f. for an expected reservation price of 60.

7.5.2 SENSITIVITY TO CHANGES IN EXPECTED SECONDARY PROFIT

The second example demonstrates the sensitivity of total profit to changes in expected secondary profits. The results of the simulation are plotted in Figure 7.5.2.

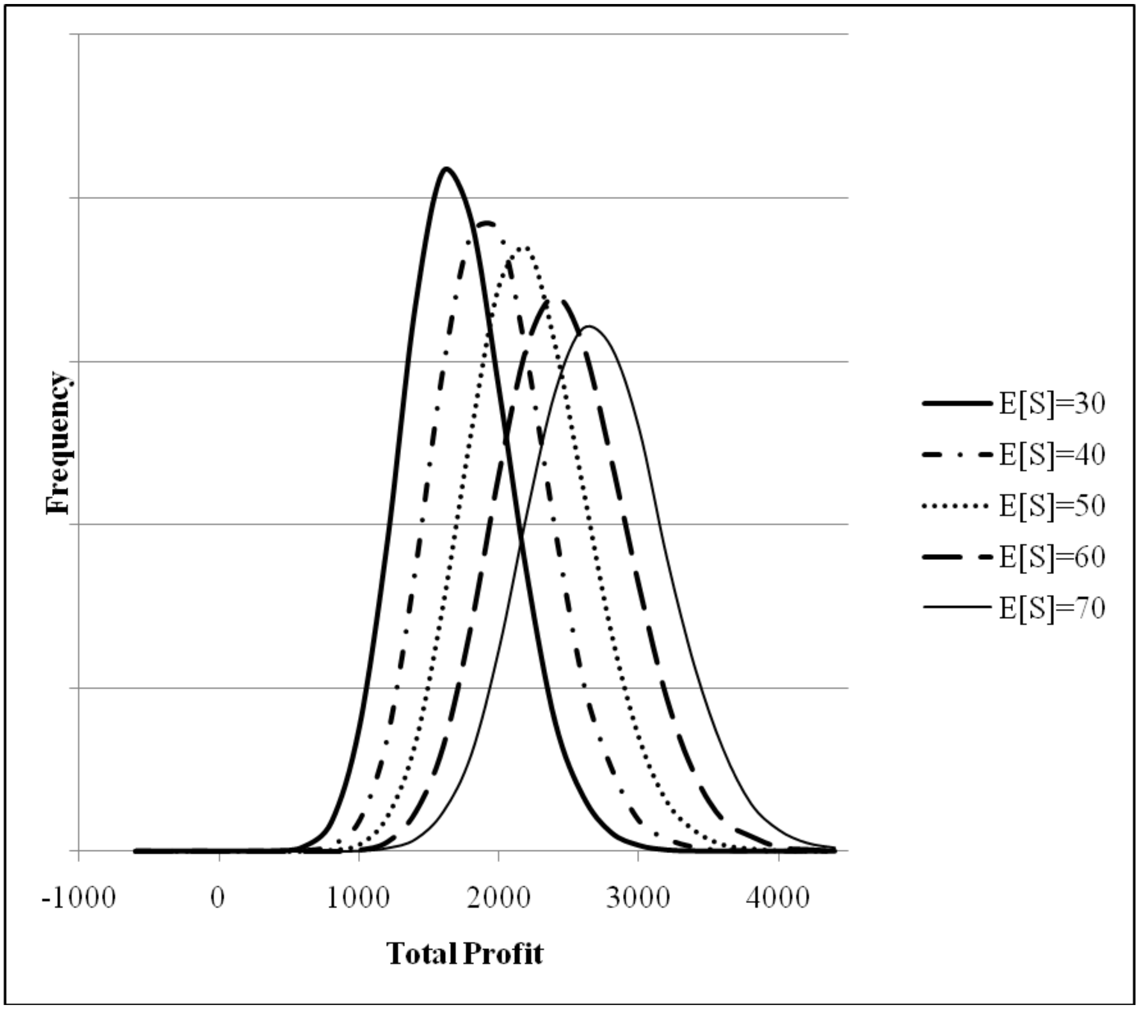

Figure 7.5.2. Sensitivity of Total Profit to Changes in Expected Secondary Profit.

Here, the parameters are those listed as the general conditions, with expected secondary profit taking on values of 30, 40, 50, 60 and 70 per customer. The effect of different secondary profits is not nearly as large as those seen in Example 1. While increasing the expected secondary profit does shift the profit function in the positive direction, the overall effect of increased expected profit is small, with an increased variance. Again, these phenomena can be explained using the expressions derived earlier.

First, consider first how changing expected secondary profit might affect total profit, as given by (7.2.1):

$$\Pi_{Total}(P, Q) = \begin{cases} Px + \sum_{i=1}^{x} s_i - cQ, & x \le Q \\ PQ + \sum_{i=1}^{Q} s_i - cQ, & x > Q \end{cases}. \tag{7.2.1}$$

Regardless of how many primary items are sold, if the realized values of S increase ($s_1, s_2, \ldots,$) so too will total profit, *ceteris paribus*.

By inspection of Figure 7.5.2, as expected secondary profit increases, so too does expected total profit. This can also be explained analytically, using the expression for expected profit:

$$E[\Pi_{Total}(P, Q)] = E[Z_1](P + E[S]) - cQ\,, \tag{7.3.1}$$

Here, expected total profit increases linearly with expected secondary profit, up to the point where no more customers can purchase a primary item (and therefore contribute secondary profit to the seller).

Finally, consider how variance in total profit changes as expected secondary profit increases. Refer to the expression for variance from Section 7.4:

$$Var(\Pi_{Total}(P, Q)) = Var(S)E[Z_1] + (E[S + P])^2 Var(Z_1), \tag{7.4.1}$$

The increased variance in total profit seen in Figure 7.5.2 is explained by the $E[S + P]$ term on the right-hand side of (7.4.1). As expected secondary profit increases, so too do both sides of (7.4.1).

7.5.3 SENSITIVITY TO CHANGES IN SELLING PRICE

The third example demonstrates the sensitivity of total profit to changes in selling prices. The results of the simulation are plotted in Figure 7.5.3.

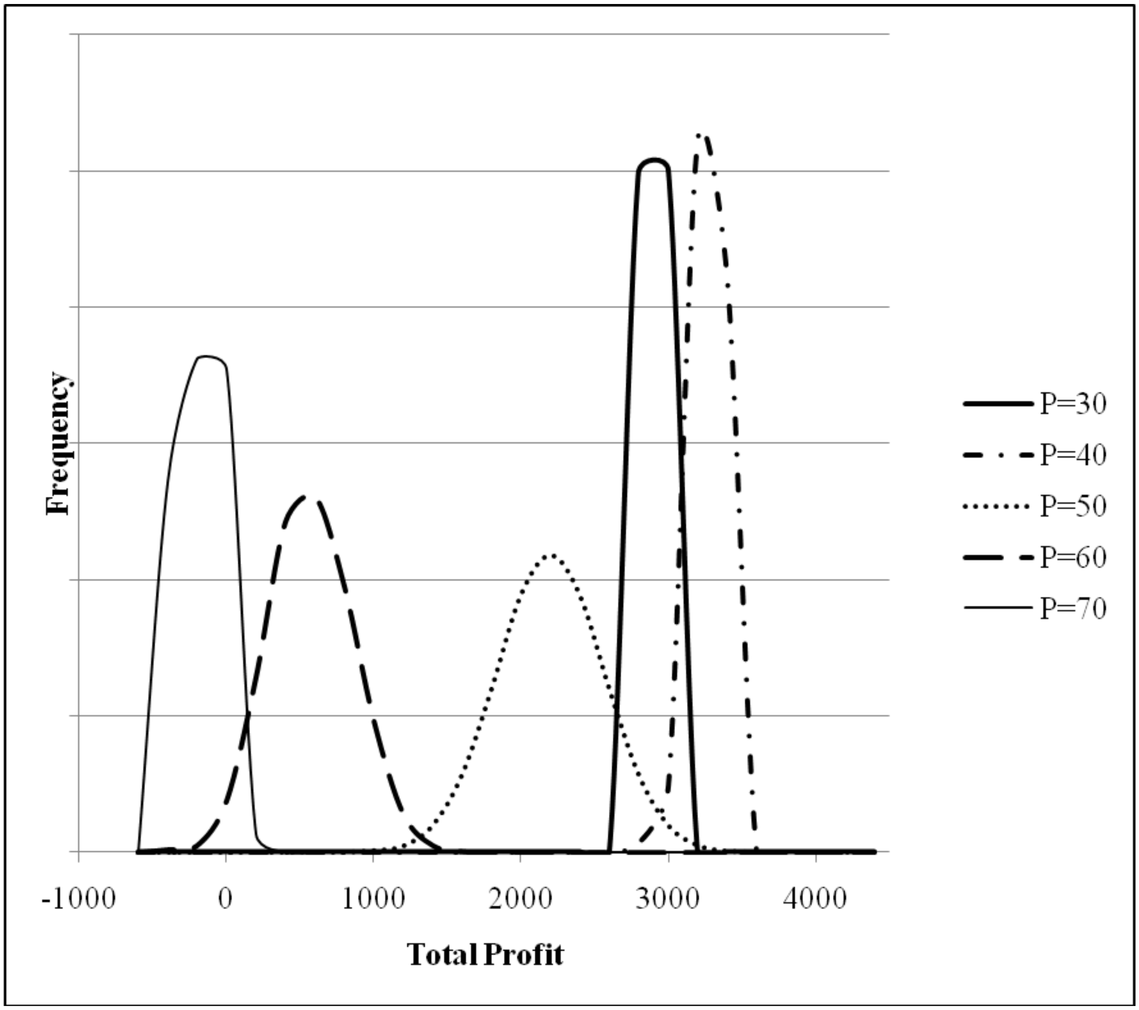

Figure 7.5.3. Sensitivity of Total Profit to Changes in Selling Price.

In Figure 7.5.3, five functions demonstrate the effect of different selling prices on profit. Here, the parameters are those listed as the general conditions, with selling price P taking on values of 30, 40, 50, 60 and 70. In this specific example, at a low selling price (P =

30), profits are high, as many customers are willing to pay P = 40 for a ticket. The result is not only high revenues from ticket sales, but as a result more customers available to purchase secondary items. At a high selling price (P = 60), profits are low, as fewer customers are willing to pay P = 60 for a ticket. The result is not only low revenues from ticket sales, but as a result fewer customers available to purchase secondary items.

While an increase in price leads appears to lead to an obvious increase in total profit,

$$\Pi_{Total}(P, Q) = \begin{cases} Px + \sum_{i=1}^{x} s_i - cQ, & x \le Q \\ PQ + \sum_{i=1}^{Q} s_i - cQ, & x > Q \end{cases}. \quad (7.2.1)$$

at some point demand will suffer and an increase in P will not be enough to offset the corresponding decrease in $X(P)$. Similar reasoning applies to the differences in expected profit, as is evident by inspection of Figure (7.5.3).

Consider finally, the differences in variance in expected profit, as in

$$Var(\Pi_{Total}(P, Q)) = Var(S)E[Z_1] + (E[S+P])^2 Var(Z_1), \quad (7.4.1)$$

With increases in selling price, the $E[S+P]$ term on the right-hand side of (7.4.1) will increase variance in total profit. However, if primary items are priced such that either no one is willing to buy one, or conversely, everyone is, there is no variance in sales ($Var(Z_1)$ approaches zero), and the variance in total profit is due to variance in secondary profits.

7.6 THE NEWSVENDOR PROBLEM WITH PRICING AND SECONDARY REVENUES – DETERMINISTIC SECONDARY REVENUE

We now consider the case of a seller with an inventory of homogeneous, primary items, who also makes available optional secondary items to those who have already purchased a primary item. As an example, consider an electronics retailer who offers warrantees at an additional price. The seller sets the selling price of the primary item and inventory level, P_1 and Q, respectively. In addition, the seller sets the selling price of a secondary item, P_2, before realizing exact demand. No inventory decision or constraint on secondary items is required. The population of customers can be broken into two groups as follows. A certain fraction of the customers are only interested in purchasing a primary item ("Group 1") and have a reservation price distribution of $F_{RP1}(\cdot)$. Denote the fraction of customers in the population who want only a primary item as Φ. The remainder of the customers are interested in purchasing the primary item only if they can purchase a secondary item as well ("Group 2"), and have a reservation price distribution of $F_{RP2}(\cdot)$. If a customer in Group 1 has a reservation price at least high as P_1, he will want to purchase an item. If a customer in Group 2 has a reservation price at least as high as $P_1 + P_2$, he will want to purchase both a primary and secondary item.

7.6.1 EXPECTED PROFIT

Here we derive an expected profit function for the seller described above, and conclude with an example. Consider first, expected revenue when there is no unmet demand.

Lemma 7.6.1. If $x \le Q$ customers have reservation prices at least as high as P_1, and y customers are in Group 1, the expected revenue from Group 1 is yP_1 and the expected revenue from Group 2 is $(P_1 + P_2)(x - y)\left(\frac{1 - F_{RP2}(P_1 + P_2)}{1 - F_{RP2}(P_1)}\right)$.

Proof. Consider first, the expected revenue from Group 1. Since there is no unmet demand, everyone in this group (*y*) buys a primary item, giving a revenue of yP_1.

If *y* customers are in Group 1, that leaves a total of $x - y$ customers in Group 2. Of these, the number of customers from Group 2 who are able to afford both a primary and secondary item is random, between 0 and $x - y$, and follows a binomial distribution with parameters $x - y$ and $\frac{1 - F_{RP2}(P_1 + P_2)}{1 - F_{RP2}(P_1)}$. Given this, the expected number of customers in who will buy both a primary and secondary item is $(x - y)\left(\frac{1 - F_{RP2}(P_1 + P_2)}{1 - F_{RP2}(P_1)}\right)$. Multiplying by the corresponding revenue from each customer, $P_1 + P_2$, completes the proof. *Q.E.D.*

Lemma 7.6.2. If $x \le Q$ customers have reservation prices at least as high as P_1, the expected revenue from Group 1 is $x\Phi P_1$ and the expected revenue from Group 2 is $x(P_1 + P_2)(1 - \Phi)\left(\frac{1 - F_{RP2}(P_1 + P_2)}{1 - F_{RP2}(P_1)}\right)$.

Proof. From Lemma 7.6.1, the revenue from *y* customers in Group 1 is yP_1. For a population of size *x*, the number of customers who will buy a primary item takes on values between zero and *x*, and follows a binomial distribution with parameters *x* and Φ. Therefore, the expected revenue from Group 1, $E[Rev_{G1}(P_1 \mid x)]$, is given by:

$$E[Rev_{G1}(P_1 \mid x)] = \sum_{y=0}^{x} yP_1 \binom{x}{y} \Phi^y (1-\Phi)^{x-y} \tag{7.6.1}$$

Factoring P_1 leaves the expected value of a random variable that follows a binomial distribution with parameters *x* and Φ. Therefore, the expected revenue from Group 1 is $x\Phi P_1$.

From Lemma 7.6.1, the expected revenue from *y* customers in Group 2 is $(P_1 + P_2)(x-y)\left(\frac{1-F_{RP2}(P_1+P_2)}{1-F_{RP2}(P_1)}\right)$. For a population of size *x*, the number of customers who will buy both a primary item takes on values between zero and *x*, and follows a binomial distribution with parameters *x* and Φ. Therefore, the expected revenue from Group 2, $E[Rev_{G2}(P_1, P_2 \mid x)]$, is given by:

$$E[Rev_{G2}(P_1, P_2 \mid x)] = \sum_{y=0}^{x} (P_1 + P_2)(x-y)\left(\frac{1-F_{RP2}(P_1+P_2)}{1-F_{RP2}(P_1)}\right)\binom{x}{y} \Phi^y (1-\Phi)^{x-y}, \tag{7.6.2}$$

Expanding gives:

$$\begin{aligned} E[Rev_{G2}(P_1, P_2 \mid x)] &= x(P_1 + P_2)\left(\frac{1-F_{RP2}(P_1+P_2)}{1-F_{RP2}(P_1)}\right)\sum_{y=0}^{x} \binom{x}{y} \Phi^y (1-\Phi)^{x-y} \\ &\quad -(P_1 + P_2)\left(\frac{1-F_{RP2}(P_1+P_2)}{1-F_{RP2}(P_1)}\right)\sum_{y=0}^{x} y\binom{x}{y} \Phi^y (1-\Phi)^{x-y}. \end{aligned} \tag{7.6.3}$$

The first summation is the sum of all probabilities from a random variable that follows a binomial distribution, therefore (7.6.2) becomes

$$E[Rev_{G2}(P_1, P_2 \mid x)] = x(P_1 + P_2)\left(\frac{1-F_{RP2}(P_1+P_2)}{1-F_{RP2}(P_1)}\right) - (P_1 + P_2)\left(\frac{1-F_{RP2}(P_1+P_2)}{1-F_{RP2}(P_1)}\right)\sum_{y=0}^{x} y\binom{x}{y}\Phi^{y}(1-\Phi)^{x-y}. \tag{7.6.4}$$

The second term is the expected value of a random variable that follows a binomial distribution with parameters *x* and Φ. Therefore (7.6.3) becomes:

$$E[Rev_{G2}(P_1, P_2 \mid x)] = x(P_1 + P_2)\left(\frac{1-F_{RP2}(P_1+P_2)}{1-F_{RP2}(P_1)}\right) - x(P_1 + P_2)\Phi\left(\frac{1-F_{RP2}(P_1+P_2)}{1-F_{RP2}(P_1)}\right). \tag{7.6.5}$$

Collecting like terms and rearranging (7.6.5) gives:

$$E[Rev_{G2}(P_1, P_2 \mid x)] = x(P_1 + P_2)\left(\frac{1-F_{RP2}(P_1+P_2)}{1-F_{RP2}(P_1)}\right)(1-\Phi) \tag{7.6.6}$$

which completes the proof. *Q.E.D.*

Lemma 7.6.3. If $x > Q$ customers have reservation prices at least as high as P_1, there are y customers in Group 1, there are z customers in Group 2 who have reservation prices at least as high as $P_1 + P_2$, and there is no unmet demand ($y + z \le Q$), the expected revenue is given by $yP_1 + z(P_1 + P_2)$.

Proof. Since there is no unmet demand, every customer in Group 1 (y) will spend P_1 each, and every customer in Group 2 with a reservation price at least as high as $P_1 + P_2$ (z) will spend $P_1 + P_2$. *Q.E.D.*

Lemma 7.6.4. If $x > Q$ customers have reservation prices at least as high as P_1, there are y customers in Group 1, there are z customers in Group 2 who have reservation prices at least as high as $P_1 + P_2$, and there is unmet demand ($y + z > Q$), expected revenue is given by $QP_1 + QP_2\left(\frac{z}{y+z}\right)$.

Proof. Since there will be unmet demand, the total number of customers who make purchases will be Q, out of a total of $y + z$. Denote the number of customers from Group 2 who make a purchase as *j*, giving a revenue of $j(P_1 + P_2)$. This leaves $Q - j$ customers from Group 1 who make a purchase, giving a revenue of $P_1(Q - j)$. The expected revenue is given by:

$$\sum_{j=0}^{\min(z,Q)} \left(P_1(Q-j) + j(P_1+P_2)\right) \frac{\binom{z}{j}\binom{y}{Q-j}}{\binom{y+z}{Q}}. \tag{7.6.7}$$

Expansion of (7.6.7) gives:

$$QP_1 \sum_{j=0}^{\min(z,Q)} \frac{\binom{z}{j}\binom{y}{Q-j}}{\binom{y+z}{Q}} + P_2 \sum_{j=0}^{\min(z,Q)} j \frac{\binom{z}{j}\binom{y}{Q-j}}{\binom{y+z}{Q}}. \tag{7.6.8}$$

Since the first summation in (7.6.8) is a sum of probabilities over all possible values of *j*, (7.6.8) can be written as:

$$QP_1 + P_2 \sum_{j=0}^{\min(z,Q)} j \frac{\binom{z}{j}\binom{y}{Q-j}}{\binom{y+z}{Q}}. \tag{7.6.9}$$

The remaining summation gives the expected value of a random variable that follows a hypergeometric distribution with parameters z, y and Q. Therefore, (7.6.9) can be written as

$$QP_1 + QP_2 \frac{z}{y+z} \tag{7.6.10}$$

which completes the proof. *Q.E.D.*

Denote $Bin(x, d, Prob)$ as a probability taken from a binomial distribution calculated as:

$$\binom{d}{x}(Prob)^x(1-Prob)^{d-x}.$$

Lemma 7.6.5. If $x > Q$ *customers have reservation prices at least as high as* P_1*, and there are y customers in Group 1, then the expected revenue is given by:*

$$\sum_{z=0}^{Q-y} \left(yP_1 + z(P_1+P_2)\right)Bin\left(z, x-y, \frac{1-F_{RP2}(P_1+P_2)}{1-F_{RP2}(P_1)}\right)$$

$$+ \sum_{z=Q-y+1}^{x-y}\left(QP_1 + QP_2\left(\frac{z}{y+z}\right)\right)Bin\left(z, x-y, \frac{1-F_{RP2}(P_1+P_2)}{1-F_{RP2}(P_1)}\right)$$

when $y \le Q$ and

$$+\sum_{z=0}^{x-y}\left(QP_1+QP_2\left(\frac{z}{y+z}\right)\right)Bin\left(z,\,x-y,\,\frac{1-F_{RP2}(P_1+P_2)}{1-F_{RP2}(P_1)}\right)$$

when $y > Q$.

Proof. The result of Lemma 7.6.5 is obtained using the definition of an expected value. When $y \le Q$, and *z* can take on values between zero and $Q-y$, then there is no possibility of unmet demand in Group 2. The possible revenues are from Lemma 7.6.3 $(yP_1 + z(P_1+P_2))$. The size of the population from which *z* customers can make purchases is the size of Group 2 ($x-y$), and the probability that someone from this group is willing to pay P_1+P_2 is given by the conditional probability $\frac{1-F_{RP2}(P_1+P_2)}{1-F_{RP2}(P_1)}$. Using these parameters gives the first term. When $y \le Q$, and there is possibility of unmet demand in Group 2, *z* can take on values between $Q-y+1$ and $x-y$. The possible revenues are from Lemma 7.6.4 ($QP_1+QP_2\frac{z}{y+z}$). Using these parameters gives the second term. Finally, when $y > Q$, and there may or may not be unmet demand in Group 2, *z* can take on values between zero and $x-y$. The possible revenues are from Lemma 7.6.4 ($QP_1+QP_2\frac{z}{y+z}$). Using these parameters gives the final term. *Q.E.D.*

Now consider the revenue when *x* customers have reservation prices at least as high as P_1.

Lemma 7.6.6. If $x > Q$ customers have reservation prices at least as high as P_1, the expected revenue is given by:

$$\sum_{y=0}^{Q}\sum_{z=0}^{Q-y}\left(yP_1 + z(P_1+P_2)\right)Bin\left(z,\, x-y,\, \frac{1-F_{RP2}(P_1+P_2)}{1-F_{RP2}(P_1)}\right)Bin(y,\, x,\, \Phi)$$

$$+\,Q\sum_{y=0}^{Q}\sum_{z=Q-y+1}^{x-y}\left(P_1 + P_2\left(\frac{z}{y+z}\right)\right)Bin\left(z,\, x-y,\, \frac{1-F_{RP2}(P_1+P_2)}{1-F_{RP2}(P_1)}\right)Bin(y,\, x,\, \Phi)$$

$$+\,Q\sum_{y=Q+1}^{x}\sum_{z=0}^{x-y}\left(P_1 + P_2\left(\frac{z}{y+z}\right)\right)Bin\left(z,\, x-y,\, \frac{1-F_{RP2}(P_1+P_2)}{1-F_{R2}(P_1)}\right)Bin(y,\, x,\, \Phi).$$

Proof. From Lemma 7.6.5, the expected revenue when $y \le Q$ is used in the expected revenue calculation above, with *y* taking on values between zero and Q, The size of the population from which *y* customers can make purchases is *x*, and the probability that someone from this population is interested in purchasing only a primary item is Φ. Using these parameters gives the first two terms. Also from Lemma 7.6.5, the expected revenue when $y > Q$ is used in the expected revenue calculation above, with *y* taking on values between zero and $Q+1$ and *x*. Using these parameters gives the final term. *Q.E.D.*

Finally, consider the expected profit for the seller.

Theorem 7.6.1. The expected profit for the seller when Q items are made available for sale and the prices for primary and secondary items are P_1 and P_2 respectively, is given by:

$$\sum_{x=0}^{Q} x\left[P_1\Phi + (P_1+P_2)(1-\Phi)\left(\frac{1-F_{RP}(P_1+P_2)}{1-F_{RP}(P_1)}\right)\right]$$

$$*Bin(y,x,\Phi)*Bin(x,d,\Phi(1-F_{RP1}(P_1))+(1-\Phi)(1-F_{RP2}(P_1)))$$

$$+\sum_{x=Q+1}^{d}\sum_{y=0}^{Q}\sum_{z=0}^{Q-y}(yP_1+z(P_1+P_2))Bin\left(z,x-y,\frac{1-F_{RP2}(P_1+P_2)}{1-F_{RP2}(P_1)}\right)$$

$$*Bin(y,x,\Phi)*Bin(x,d,\Phi(1-F_{RP1}(P_1))+(1-\Phi)(1-F_{RP2}(P_1)))$$

$$+Q\sum_{x=Q+1}^{d}\sum_{y=0}^{Q}\sum_{z=Q-y+1}^{x-y}\left(P_1+P_2\left(\frac{z}{y+z}\right)\right)Bin\left(z,x-y,\frac{1-F_{RP2}(P_1+P_2)}{1-F_{RP2}(P_1)}\right) \tag{7.6.11}$$

$$*Bin(y,x,\Phi)*Bin(x,d,\Phi(1-F_{RP1}(P_1))+(1-\Phi)(1-F_{RP2}(P_1)))$$

$$+Q\sum_{x=Q+1}^{d}\sum_{y=Q+1}^{x}\sum_{z=0}^{x-y}\left(P_1+P_2\left(\frac{z}{y+z}\right)\right)Bin\left(z,x-y,\frac{1-F_{RP2}(P_1+P_2)}{1-F_{RP2}(P_1)}\right)$$

$$*Bin(y,x,\Phi)*Bin(x,d,\Phi(1-F_{RP1}(P_1))+(1-\Phi)(1-F_{RP2}(P_1)))$$

$$-cQ.$$

Proof. Lemma 7.6.2 provides the expected revenue when $x \le Q$. Here, *x* can take on values between zero and Q. The size of the population from which *x* customers can make purchases is *d*. The probability that someone from Group 1 will have a reservation price at least as high as P_1 is $1-F_{RP1}(P_1)$. The probability that someone from Group 2 will have a reservation price at least as high as P_1 is $1-F_{RP2}(P_1)$. Therefore, the expected probability that a randomly-selected customer will have a reservation price at least as high as P_1 is $\Phi(1-F_{RP1}(P_1))+(1-\Phi)(1-F_{RP2}(P_1))$. Using the definition of expected value and these parameters gives the first term. Lemma 7.6.6 provides the expected revenue when $x > Q$. Here, *x* can take on values between $Q+1$ and d. Using the definition of expected value and these parameters gives the next three expected revenue terms. Subtracting the total cost to the seller, cQ, completes the proof. *Q.E.D.*

7.6.2 NUMERICAL EXAMPLE

Here we provide an example to demonstrate the problem. Consider a customer base of size $d = 10$, where customers either have a reservation price distribution of $RP1 \sim U[0, 7]$, with probability $\Phi = 0.5$, or a reservation price distribution of $RP2 \sim U[0, 10]$, with probability $1 - \Phi = 0.5$. The seller incurs a unit cost of $c = 1$ for every primary item made available for sale. By searching over a grid of allowable values, the maximum expected profit for various Totals (calculated as $P_1 + P_2$) are determined using (7.6.11) and shown in Figure 7.6.1.

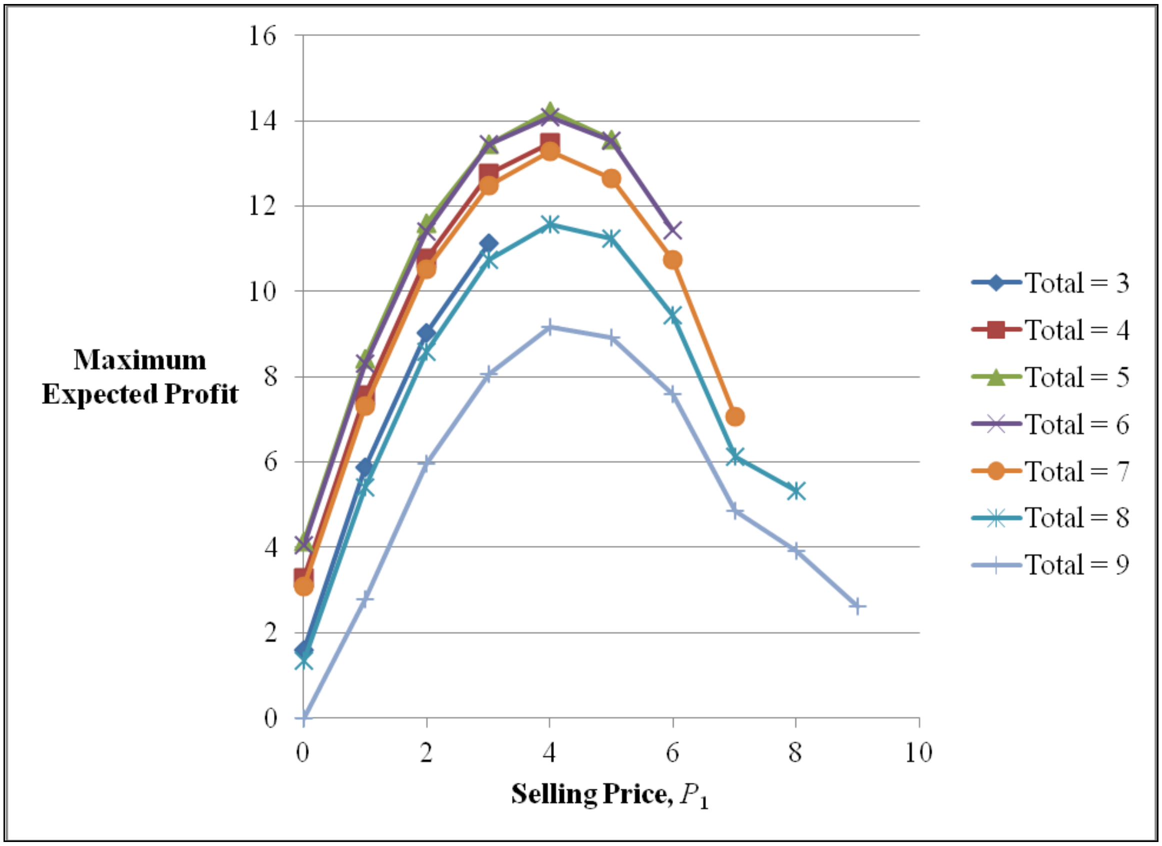

Figure 7.6.1. Maximum Expected Profit as a Function of Selling Price, P_1.

By inspection, the optimal Maximum Expected Profit of 14.2 is at Total=5, where $P_1 = 4$ and $P_2 = 1$.

CHAPTER 8

CONCLUSIONS AND FUTURE RESEARCH

8.1 CONCLUSIONS AND FUTURE RESEARCH

In Chapter 3, we have considered a realistic model of consumer buying behavior for homogeneous items. Our assumptions are that demand can be modeled using size of customer base and distribution of reservation prices of the customers (the most a customer is willing to pay for a single item). Based on this, we have shown that an additive model with price-dependent uncertainty is appropriate, and the linear model is appropriate only under very restrictive conditions. We provided forms of the elasticity in the general model of demand, and showed that maximum variance in the general model of demand occurs when the selling price is the median of the reservation prices. The expression for elasticity provided in this chapter would be useful for the business practitioner who prices his product according to price elasticity of demand (often targeted at unity).

In Chapter 4, we used the general model of demand from Chapter 3 to derive expected profit functions when the seller uses a single-price strategy, for both deterministic and random sizes of customer bases. Numerical examples were provided.

In Chapter 5, we used the general model of demand from Chapter 3 to derive expected profit functions when the sellers uses a multiple-price strategy, with a laddered structure of prices. We provided models for deterministic and random sizes of customer bases, for both discrete and continuous cases. From the dual-price model, we derived an expression for the optimal inventory level of items to be made available for sale at the higher price. We also expanded the dual-price model to include the Bandwagon Effect, and provided a numerical example. We show how optimal solutions can be found analytically in cases where the seller makes relatively few price (and corresponding inventory level) decisions, but as the number of decisions increases, the difficulty in finding optimal solutions can become prohibitive. To address this, we provide a dynamic programming model that can be used to find optimal solutions. Even in these larger cases, restricting the number of possibilities for allowable inventory and prices makes finding optimal solutions easier. Future work related to this chapter includes expanding on the work done on the Bandwagon Effect, to see if analytical results similar to those found using the dual-price model can be obtained. The results of this chapter, especially those on the topic of the Bandwagon Effect, would be useful for a seller of tickets to performance events (e.g.; sports or theatre), where "demand creates demand."

In Chapter 6, we consider the case of a seller facing small sizes of customer base (both deterministic and random). Expected profit functions are derived to determine under which circumstances, if any, we can show that a multiple-price strategy is superior to a single-price strategy. A first attempt at finding analytical optimal results demonstrated that regardless of whether or not the size of the customer base is known exactly, the seller is best off setting a single selling price and corresponding inventory level. Future work in this area includes determining whether or not the conclusions from this chapter are applicable for larger population sizes, and if not, why not.

In Chapter 7, we provided a model for a seller who receives secondary profits from customers who have already purchased a primary item. We derived expected profit and variance expressions and used numerical examples to show how they could be used to provide insight into the random total profit that the seller could expect. Numerical examples were used to demonstrate the sensitivity of expected profit to changes in expected reservation price, expected secondary profit and selling price. Here, we assumed that the secondary revenues from the customers was simply a random quantity with a known distribution. To expand the problem, we also considered the case where a fraction of the customers are interested in only a primary product (with a known reservation price distribution) and the remaining customers are interested in a purchasing a primary item only if they can afford a secondary item as well (each with different reservation price distributions). We derive an expected profit function for this case, and provide a numerical example. Future work in this area includes deriving analytical results for both forms of the newsvendor problem with pricing and secondary revenues. Again, a seller of tickets to a performance event would find the results of this chapter useful for maximizing expected total profit.

APPENDIX A

SOLUTION TO EXAMPLE FROM DANA (1999)

This appendix provides the method used in Dana (1999) to solve the example discussed in Section 4.3.

The example is as follows. A seller of homogeneous items decides on selling prices and corresponding inventory levels before knowing the exact size of the customer base, and how much each customer is willing to pay for an item. There are two possible sizes of customer base, $d_1 = 100$ and $d_2 = 400$, which are equally likely. Denote the probability of the high-demand state being realized as $Prob_{d2}$, which is 50%. The maximum amount each customer is willing to pay for an item (reservation price) is random, and follows a uniform distribution as $RP \sim U[0, P_{max} = 100]$. In addition, the seller incurs a unit cost of $c = 20$ for each item made available for sale, regardless of whether or not the item ultimately sold. In demonstrating the solution by Dana (p. 639), subscript “1” (“2”) indicates the low (high) price and corresponding inventory level.

The first step is to determine P_1^* and Q_1^*. This is done by setting marginal revenue equal to marginal cost for the low-demand state to find Q_1^*, and then using that inventory level in the low-demand state demand curve to find P_1^*.

Begin by deriving the demand equation for the low-demand state (D = d_1 = 100). For reservation prices that follow a uniform distribution, the demand equation is downward sloping and linear in P_1, as given by:

$$Q_1(P_1) = d_1 - \frac{d_1}{P_{max}} P_1. \tag{A1}$$

Rewriting (A1) as the inverse demand function gives:

$$P_1(Q_1) = P_{max} - \frac{P_{max}}{d_1} Q_1. \tag{A2}$$

Multiplying (A2) by Q_1 gives the revenue curve for the low-demand state:

$$R_1 = Q_1 P_1(Q_1) = Q_1\left(P_{max} - \frac{P_{max}}{d_1} Q_1 \right) \tag{A3}$$

Taking the first derivative of (A3) with respect to Q_1 gives:

$$\frac{\partial}{\partial Q_1} R_1 = \frac{\partial}{\partial Q_1}\left(Q_1\left(P_{max} - \frac{P_{max}}{d_1} Q_1 \right)\right) \tag{A4}$$

which is the marginal revenue equation for the low-demand state, MR_1:

$$MR_1 = P_{max} - \frac{2P_{max}}{d_1} Q_1. \tag{A5}$$

Setting the right hand side of (A5) equal to the marginal cost of making an item available for sale, c, provides an equation for finding the optimal number of items to be made available at the low price, Q_1^*:

$$P_{max} - \frac{2P_{max}}{d_1} Q_1^* = c. \tag{A6}$$

Solving (A6) for Q_1^* gives the optimal quantity of items to be made available for sale at price P_1:

$$Q_1^* = \frac{d_1}{2P_{max}}\left(P_{max} - c\right) \tag{A7}$$

Using the information given in the example in (A7) gives:

$$Q_1^* = \frac{100}{2(100)}\left(100 - 20\right) = 40\,. \tag{A8}$$

Optimal price P_1^* is found by substituting the right hand side of (A8) into the original demand equation (A2):

$$P_1^* = P_{max} - \frac{P_{max}}{d_1}\left(\frac{d_1}{2P_{max}}\left(P_{max} - c\right)\right) \tag{A9}$$

Simplifying and rearranging (A9) gives:

$$P_1^* = \frac{P_{max} + c}{2} \tag{A10}$$

Using (A10) and the values of the parameters given in the example gives:

$$P_1^* = \frac{100 + 20}{2} = 60\,. \tag{A11}$$

Note that for the solution $P_1^* = 60$ and $Q_1^* = 40$, there are no "leftover" customers if the demand state is $d_1 = 100$. That is, if the selling price is set at 60 and there are 100 customers in the population, the expected demand is 40 since $d_1\left(1 - F_{RP}(P_1^*)\right) = 100\left(1 - \frac{60 - 0}{100 - 0}\right) = 40$, which is the solution for Q_1^*. Therefore, in the low-demand state, everyone who is expected to be willing to pay for an item is able to purchase one.

However, in the high-demand state ($d_2 = 400$), the expected number of customers willing to pay 60 is $d_2\left(1 - F_{RP}(P_1^*)\right) = 400\left(1 - \frac{60-0}{100-0}\right) = 160$, which means there will be $160 - 40 = 120$ "leftover" customers remaining who were willing to pay at least 60. To provide items for this residual population, an additional number of items are made available for sale at another price. The optimal number of items, Q_2^*, and the price at which to sell them, P_2^*, are found using a procedure similar to that which was used to find P_1^* and Q_1^*. The significant difference here, is that a residual demand equation must be used, as some of the original $d_2 = 400$ customers have already purchased an item at price P_1^*, leaving behind a fraction of customers who were willing to pay P_1^* for an item, but could not buy one as they had all been sold. Denote that fraction of "leftover" customers as *a*, which in the example is calculated as $120/160 = 0.75$.

Begin with the demand equation for the high-demand state ($D = d_2 = 400$), assuming that the entire customer base is still available to purchase an item. For reservation prices that follow a uniform distribution, the demand equation, $Q_{2,\,Full}(P_2)$, is downward sloping and linear in P_2, as given by:

$$Q_{2,\,Full}(P_2) = d_2 - \frac{d_2}{P_{max}} P_2 . \qquad \text{(A12)}$$

The residual demand equation, $Q_2(P_2)$, is found by multiplying the high-demand state demand equation by the fraction of "leftover" customers:

$$Q_2(P_2) = aQ_{2,\,Full}(P_2) . \qquad \text{(A13)}$$

Substituting (A12) into (A13) gives the explicit form of the residual demand equation:

$$Q_2(P_2) = a\left(d_2 - \frac{d_2}{P_{max}} P_2\right). \tag{A14}$$

Rewriting (A14) as the inverse residual demand equation, $P_2(Q_2)$, gives:

$$P_2(Q_2) = P_{max} - \frac{P_{max}}{ad_2} Q_2. \tag{A15}$$

To find the residual revenue equation for the high-demand state, R_2, multiply (A15) by Q_2 to give:

$$R_2 = Q_2 P_2(Q_2) = Q_2\left(P_{max} - \frac{P_{max}}{ad_2} Q_2\right) \tag{A16}$$

Taking the first derivative of (A16) with respect to Q_2 gives:

$$\frac{\partial}{\partial Q_2} R_2 = \frac{\partial}{\partial Q_2}\left(Q_2\left(P_{max} - \frac{P_{max}}{ad_2} Q_2\right)\right) \tag{A17}$$

which is the residual marginal revenue equation for the high-demand state, MR_2:

$$MR_2 = P_{max} - \frac{2P_{max}}{ad_2} Q_2. \tag{A18}$$

However, recall that the probability of realizing the high-demand state is $Prob_{d2}$, which means that the expected residual marginal revenue is only (A18) multiplied by $Prob_{d2}$. Therefore, the expression used to find Q_2^* is:

$$Prob_{D_2}\left(P_{max} - \frac{2P_{max}}{ad_2} Q_2^*\right) = c \tag{A19}$$

Solving (A19) for Q_2^* gives the equation to calculate the optimal number of items to be made available for sale at second price:

$$Q_2^* = \frac{ad_2}{2P_{max}}\left(P_{max} - \frac{c}{Prob_{d2}}\right). \tag{A20}$$

Using the values given in the example in (A20) gives:

$$Q_2^* = \frac{0.75(400)}{2(100)}\left(100 - \frac{20}{0.5}\right) = 90. \tag{A21}$$

The optimal second price to charge, P_2^*, is found by substituting the right hand side of (A20) into the residual demand equation (A15) to give:

$$P_2^* = P_{max} - \frac{P_{max}}{ad_2}\left(\frac{ad_2}{2P_{max}}\left(P_{max} - \frac{c}{Prob_{d2}}\right)\right) \tag{A22}$$

Simplifying (A22) gives:

$$P_2^* = \frac{P_{max} + \dfrac{c}{Prob_{d2}}}{2} \tag{A23}$$

Substituting the values given in the example into (A23) gives the optimal second selling price as:

$$P_2^* = \frac{100 + \dfrac{20}{0.5}}{2} = 70. \tag{A24}$$

APPENDIX B

DERIVATIONS OF EXPECTED PROFIT FUNCTIONS

B1 DETERMINISTIC CUSTOMER BASE WITH $d = 2$

This section contains the derivations of the expected profit functions for a seller facing a deterministic customer base of d = 2.

In this example, the seller chooses from a set of inventory and corresponding price combinations where $Q_{Total} \leq 2$. These inventory combinations are shown in Table B1.1. Note that Q_i is the number of units of inventory to be made available for sale at price P_i, i = 1, 2.

Table B1.1. Inventory Combinations for Example 1

		Q_1		
		0	1	2
Q_2	0	Case 1	Case 2	Case 3
	1	Case 4	Case 5	
	2	Case 6		

While Case 1 is included in Table B1.1 for completeness, the decision to make no items available for sale results in a trivial expected profit of zero, and is not considered for the remainder of the example.

As there are two customers in the population, the probabilities to be used in (6.2.3) are joint probabilities, calculated for various combinations of reservation prices relative to the selling prices. The joint probabilities are given in Table B1.2. Recall that $\overline{F_i}$ denotes

$1 - F_{RP}(P_i)$, the probability that a randomly-chosen customer has a reservation price at least as high as price P_i.

Table B1.2. Joint Probabilities for Example 1

		Customer 1		
		$RP_1 < P_1$	$P_1 \le RP_1 < P_2$	$P_2 \le RP_1$
Customer 2	$RP_2 < P_1$	$\left(1-\overline{F_1}\right)^2$	$\left(1-\overline{F_1}\right)\left(\overline{F_1}-\overline{F_2}\right)$	$\left(1-\overline{F_1}\right)\overline{F_2}$
	$P_1 \le RP_2 < P_2$	$\left(1-\overline{F_1}\right)\left(\overline{F_1}-\overline{F_2}\right)$	$\left(\overline{F_1}-\overline{F_2}\right)^2$	$\overline{F_2}\left(\overline{F_1}-\overline{F_2}\right)$
	$P_2 \le RP_2$	$\left(1-\overline{F_1}\right)\overline{F_2}$	$\overline{F_2}\left(\overline{F_1}-\overline{F_2}\right)$	$\overline{F_2}^{\,2}$

For ease of reference, the joint probabilities in Table B1.2 are denoted as indicated in Table B1.3.

Table B1.3. Notation for Joint Probabilities for $d = 2$.

		Customer 1		
		$RP_1 < P_1$	$P_1 \le RP_1 < P_2$	$P_2 \le RP_1$
Customer 2	$RP_2 < P_1$	r_1	r_2	r_3
	$P_1 \le RP_2 < P_2$	r_4	r_5	r_6
	$P_2 \le RP_2$	r_7	r_8	r_9

Shown below are the expected profit calculations for the various combinations of inventory decisions shown in Table B1.1.

For Case 2, $Q_1 = 1, Q_2 = 0$. For this inventory decision, the possible revenues that can be realized are shown in Table B1.4.

Table B1.4. Possible Revenues for Example 1, Case 2.

		Customer 1		
		$RP_1 < P_1$	$P_1 \le RP_1 < P_2$	$P_2 \le RP_1$
Customer 2	$RP_2 < P_1$	0	P_1	P_1
	$P_1 \le RP_2 < P_2$	P_1	P_1	P_1
	$P_2 \le RP_2$	P_1	P_1	P_1

Multiplying the joint probabilities from Table B1.3 by their respective revenues in Table B1.4, taking the summation of the products and subtracting the total cost gives the expected profit for Case 2:

$$E[\Pi_{Case2}] = 0r_1 + P_1(r_2 + r_3 + r_4 + r_5 + r_6 + r_7 + r_8 + r_9) - 1c\,. \qquad \text{(B1.1)}$$

Substitution of the joint probabilities from Table B1.2 into (B1.1) and simplifying gives:

$$E[\Pi_{Case2}] = P_1\overline{F_1}\left(2 - \overline{F_1}\right) - c\,. \qquad \text{(B1.2)}$$

For Case 3, $Q_1 = 2, Q_2 = 0$. For this inventory decision, the possible revenues that can be realized are shown in Table B1.5.

Table B1.5. Possible Revenues for Example 1, Case 3.

		Customer 1		
		$RP_1 < P_1$	$P_1 \le RP_1 < P_2$	$P_2 \le RP_1$
Customer 2	$RP_2 < P_1$	0	P_1	P_1
	$P_1 \le RP_2 < P_2$	P_1	$2P_1$	$2P_1$
	$P_2 \le RP_2$	P_1	$2P_1$	$2P_1$

Multiplying the joint probabilities from Table B1.3 by their respective revenues in Table B1.5, taking the summation of the products and subtracting the total cost gives the expected profit for Case 3:

$$E[\Pi_{Case3}] = 0r_1 + P_1(r_2 + r_3 + r_4 + r_7) + 2P_1(r_5 + r_6 + r_8 + r_9) - 2c. \tag{B1.3}$$

Substitution of the joint probabilities from Table B1.2 into (B1.3) and simplifying gives:

$$E[\Pi_{Case3}] = 2P_1\overline{F_1} - 2c. \tag{B1.4}$$

For Case 4, $Q_1 = 0, Q_2 = 1$. For this inventory decision, the possible revenues that can be realized are shown in Table B1.6.

Table B1.6. Possible Revenues for Example 1, Case 4.

		Customer 1		
		$RP_1 < P_1$	$P_1 \le RP_1 < P_2$	$P_2 \le RP_1$
Customer 2	$RP_2 < P_1$	0	0	P_2
	$P_1 \le RP_2 < P_2$	0	0	P_2
	$P_2 \le RP_2$	P_2	P_2	P_2

Multiplying the joint probabilities from Table B1.3 by their respective revenues in Table B1.6, taking the summation of the products and subtracting the total cost gives the expected profit for Case 4:

$$E[\Pi_{Case4}] = 0(r_1 + r_2 + r_4 + r_5) + P_2(r_3 + r_6 + r_7 + r_8 + r_9) - 1c. \tag{B1.5}$$

Substitution of the joint probabilities from Table B1.2 into (B1.5) and simplifying gives:

$$E[\Pi_{Case4}] = P_2\overline{F_2}(2 - \overline{F_2}) - c. \tag{B1.6}$$

For Case 5, $Q_1 = 1, Q_2 = 1$. For this inventory decision, the possible revenues that can be realized are shown in Table B1.7.

Table B1.7. Possible Revenues for Example 1, Case 5.

		Customer 1		
		$RP_1 < P_1$	$P_1 \le RP_1 < P_2$	$P_2 \le RP_1$
Customer 2	$RP_2 < P_1$	0	P_1	P_1
	$P_1 \le RP_2 < P_2$	P_1	P_1	P_1
	$P_2 \le RP_2$	P_1	$P_1 + P_2$	$P_1 + P_2$

Multiplying the joint probabilities from Table B1.3 by their respective revenues in Table B1.7, taking the summation of the products and subtracting the total cost gives the expected profit for Case 5:

$$E[\Pi_{Case5}] = 0r_1 + P_1(r_2 + r_3 + r_4 + r_5 + r_6 + r_7) + (P_1 + P_2)(r_8 + r_9) - 2c\,. \tag{B1.7}$$

Substitution of the joint probabilities from Table B1.2 into (B1.7) and simplifying gives:

$$E[\Pi_{Case5}] = P_1\overline{F_1}(2 - \overline{F_1}) + P_2\overline{F_1}\overline{F_2} - 2c\,. \tag{B1.8}$$

For Case 6, $Q_1 = 0, Q_2 = 2$. For this inventory decision, the possible revenues that can be realized are shown in Table B1.8.

Table B1.8. Possible Revenues for Example 1, Case 6.

		Customer 1		
		$RP_1 < P_1$	$P_1 \le RP_1 < P_2$	$P_2 \le RP_1$
Customer 2	$RP_2 < P_1$	0	0	P_2
	$P_1 \le RP_2 < P_2$	0	0	P_2
	$P_2 \le RP_2$	P_2	P_2	$2P_2$

Multiplying the joint probabilities from Table B1.3 by their respective revenues in Table B1.8, taking the summation of the products and subtracting the total cost gives the expected profit for Case 6:

$$E[\Pi_{Case6}] = 0(r_1 + r_2 + r_4 + r_5) + P_2(r_3 + r_6 + r_7 + r_8) + 2P_2 r_9 - 2c\,. \tag{B1.9}$$

Substitution of the joint probabilities from Table B1.2 into (B1.9) and simplifying gives:

$$E[\Pi_{Case6}] = 2P_2\overline{F_2} - 2c\,. \tag{B1.10}$$

B2 DETERMINISTIC CUSTOMER BASE WITH $d = 3$

This section contains the derivations of the expected profit functions for a seller facing a deterministic customer base of d = 3.

In this example, the seller chooses from a set of inventory and corresponding price combinations where $Q_{Total} \leq 3$. These inventory combinations are shown in Table B2.1. Note that Q_i is the number of units of inventory to be made available for sale at price P_i, i = 1, 2.

Table B2.1. Inventory Combinations for Example 2

		Q_1			
		0	1	2	3
Q_2	0	Case 1	Case 2	Case 3	Case 4
	1	Case 5	Case 6	Case 7	
	2	Case 8	Case 9		
	3	Case 10			

While Case 1 is included in Table B2.1 for completeness, the decision to make no items available for sale results in a trivial expected profit of zero, and is not considered for the remainder of the example.

As there are two customers in the population, the probabilities to be used in (6.2.3) are joint probabilities, calculated for various combinations of reservation prices relative to the selling prices. The joint probabilities are given in Tables B2.2a – B2.2c. Recall that $\overline{F_i}$ denotes $1-F_{RP}(P_i)$, the probability that a randomly-chosen customer has a reservation price at least as high as price P_i.

Table B2.2a. Joint Probabilities for Example 2, $RP_3 < P_1$.

		Customer 1		
		$RP_1 < P_1$	$P_1 \le RP_1 < P_2$	$P_2 \le RP_1$
Customer 2	$RP_2 < P_1$	$\left(1-\overline{F_1}\right)^3$	$\left(1-\overline{F_1}\right)^2\left(\overline{F_1}-\overline{F_2}\right)$	$\left(1-\overline{F_1}\right)^2\overline{F_2}$
	$P_1 \le RP_2 < P_2$	$\left(1-\overline{F_1}\right)^2\left(\overline{F_1}-\overline{F_2}\right)$	$\left(\overline{F_1}-\overline{F_2}\right)^2\left(1-\overline{F_1}\right)$	$\overline{F_2}\left(\overline{F_1}-\overline{F_2}\right)\left(1-\overline{F_1}\right)$
	$P_2 \le RP_2$	$\left(1-\overline{F_1}\right)^2\overline{F_2}$	$\overline{F_2}\left(\overline{F_1}-\overline{F_2}\right)\left(1-\overline{F_1}\right)$	$\overline{F_2}^2\left(1-\overline{F_1}\right)$

Table B2.2b. Joint Probabilities for Example 2, $P_1 < RP_3 < P_2$.

		Customer 1		
		$RP_1 < P_1$	$P_1 \le RP_1 < P_2$	$P_2 \le RP_1$
Customer 2	$RP_2 < P_1$	$\left(1-\overline{F_1}\right)^2\left(\overline{F_1}-\overline{F_2}\right)$	$\left(1-\overline{F_1}\right)\left(\overline{F_1}-\overline{F_2}\right)$	$\left(1-\overline{F_1}\right)\left(\overline{F_1}-\overline{F_2}\right)\overline{F_2}$
	$P_1 \le RP_2 < P_2$	$\left(1-\overline{F_1}\right)\left(\overline{F_1}-\overline{F_2}\right)^2$	$\left(\overline{F_1}-\overline{F_2}\right)^3$	$\overline{F_2}\left(\overline{F_1}-\overline{F_2}\right)^2$
	$P_2 \le RP_2$	$\left(1-\overline{F_1}\right)\left(\overline{F_1}-\overline{F_2}\right)\overline{F_2}$	$\overline{F_2}\left(\overline{F_1}-\overline{F_2}\right)^2$	$\overline{F_2}^2\left(\overline{F_1}-\overline{F_2}\right)$

Table B2.2c. Joint Probabilities for Example 2, $P_2 < RP_3$.

		Customer 1		
		$RP_1 < P_1$	$P_1 \le RP_1 < P_2$	$P_2 \le RP_1$
Customer 2	$RP_2 < P_1$	$\left(1-\overline{F_1}\right)^2\overline{F_2}$	$\left(1-\overline{F_1}\right)\left(\overline{F_1}-\overline{F_2}\right)\overline{F_2}$	$\left(1-\overline{F_1}\right)\overline{F_2}^2$
	$P_1 \le RP_2 < P_2$	$\left(1-\overline{F_1}\right)\left(\overline{F_1}-\overline{F_2}\right)\overline{F_2}$	$\left(\overline{F_1}-\overline{F_2}\right)^2\overline{F_2}$	$\overline{F_2}^2\left(\overline{F_1}-\overline{F_2}\right)$
	$P_2 \le RP_2$	$\left(1-\overline{F_1}\right)\overline{F_2}^2$	$\overline{F_2}^2\left(\overline{F_1}-\overline{F_2}\right)$	$\overline{F_2}^3$

For ease of reference, the joint probabilities in Table B2.2 are denoted as indicated in Tables B2.3a – B2.3c.

Table B2.3a. Notation for Joint Probabilities for Example 2, $RP_3 < P_1$.

		Customer 1		
		$RP_1 < P_1$	$P_1 \le RP_1 < P_2$	$P_2 \le RP_1$
Customer 2	$RP_2 < P_1$	r_1	r_2	r_3
	$P_1 \le RP_2 < P_2$	r_4	r_5	r_6
	$P_2 \le RP_2$	r_7	r_8	r_9

Table B2.3b. Notation for Joint Probabilities for Example 2, $P_1 < RP_3 < P_2$.

		Customer 1		
		$RP_1 < P_1$	$P_1 \le RP_1 < P_2$	$P_2 \le RP_1$
Customer 2	$RP_2 < P_1$	r_{10}	r_{11}	r_{12}
	$P_1 \le RP_2 < P_2$	r_{13}	r_{14}	r_{15}
	$P_2 \le RP_2$	r_{16}	r_{17}	r_{18}

Table B2.3c. Notation for Joint Probabilities for Example 2, $P_2 < RP_3$.

		Customer 1		
		$RP_1 < P_1$	$P_1 \le RP_1 < P_2$	$P_2 \le RP_1$
Customer 2	$RP_2 < P_1$	r_{19}	r_{20}	r_{21}
	$P_1 \le RP_2 < P_2$	r_{22}	r_{23}	r_{24}
	$P_2 \le RP_2$	r_{25}	r_{26}	r_{27}

Shown below are the expected profit calculations for the various combinations of inventory decisions shown in Table B2.1.

For Case 2, $Q_1 = 1, Q_2 = 0$. For this inventory decision, the possible revenues that can be realized are shown in Tables B2.4a – B2.4c.

Table B2.4a. Possible Revenues for Example 2, Case 2, $RP_3 < P_1$.

		Customer 1		
		$RP_1 < P_1$	$P_1 \le RP_1 < P_2$	$P_2 \le RP_1$
Customer 2	$RP_2 < P_1$	0	P_1	P_1
	$P_1 \le RP_2 < P_2$	P_1	P_1	P_1
	$P_2 \le RP_2$	P_1	P_1	P_1

Table B2.4b. Possible Revenues for Example 2, Case 2, $P_1 < RP_3 < P_2$.

		Customer 1		
		$RP_1 < P_1$	$P_1 \le RP_1 < P_2$	$P_2 \le RP_1$
Customer 2	$RP_2 < P_1$	P_1	P_1	P_1
	$P_1 \le RP_2 < P_2$	P_1	P_1	P_1
	$P_2 \le RP_2$	P_1	P_1	P_1

Table B2.4c. Possible Revenues for Example 2, Case 2, $P_2 < RP_3$.

		Customer 1		
		$RP_1 < P_1$	$P_1 \le RP_1 < P_2$	$P_2 \le RP_1$
Customer 2	$RP_2 < P_1$	P_1	P_1	P_1
	$P_1 \le RP_2 < P_2$	P_1	P_1	P_1
	$P_2 \le RP_2$	P_1	P_1	P_1

Multiplying the joint probabilities from Tables B2.3a – B2.3c by their respective revenues in Tables B2.4a – B2.4c, taking the summation of the products and subtracting the total cost gives the expected profit for Case 2:

$$E[\Pi_{Case2}] = 0r_1 + P_1(1 - r_1) - 1c\,. \qquad \text{(B2.1)}$$

Substitution of the joint probabilities from Tables B2.2a – B2.2c into (B2.1) and simplifying gives:

$$E[\Pi_{Case2}] = P_1\overline{F_1}\left(3 - 3\overline{F_1} + \overline{F_1}^{\,2}\right) - c\,. \qquad \text{(B2.2)}$$

For Case 3, $Q_1 = 2, Q_2 = 0$. For this inventory decision, the possible revenues that can be realized are shown in Tables B2.5a – B2.5c.

Table B2.5a. Possible Revenues for Example 2, Case 3, $RP_3 < P_1$.

		Customer 1		
		$RP_1 < P_1$	$P_1 \le RP_1 < P_2$	$P_2 \le RP_1$
Customer 2	$RP_2 < P_1$	0	P_1	P_1
	$P_1 \le RP_2 < P_2$	P_1	$2P_1$	$2P_1$
	$P_2 \le RP_2$	P_1	$2P_1$	$2P_1$

Table B2.5b. Possible Revenues for Example 2, Case 3, $P_1 < RP_3 < P_2$.

		Customer 1		
		$RP_1 < P_1$	$P_1 \le RP_1 < P_2$	$P_2 \le RP_1$
Customer 2	$RP_2 < P_1$	P_1	$2P_1$	$2P_1$
	$P_1 \le RP_2 < P_2$	$2P_1$	$2P_1$	$2P_1$
	$P_2 \le RP_2$	$2P_1$	$2P_1$	$2P_1$

Table B2.5c. Possible Revenues for Example 2, Case 3, $P_2 < RP_3$.

		Customer 1		
		$RP_1 < P_1$	$P_1 \le RP_1 < P_2$	$P_2 \le RP_1$
Customer 2	$RP_2 < P_1$	P_1	$2P_1$	$2P_1$
	$P_1 \le RP_2 < P_2$	$2P_1$	$2P_1$	$2P_1$
	$P_2 \le RP_2$	$2P_1$	$2P_1$	$2P_1$

Multiplying the joint probabilities from Tables B2.3a – B2.3c by their respective revenues in Tables B2.5a – B2.5c, taking the summation of the products and subtracting the total cost gives the expected profit for Case 3:

$$E[\Pi_{Case3}] = 0r_1 + P_1(r_2 + r_3 + r_4 + r_7 + r_{10} + r_{19}) + 2P_1(1 - (r_1 + r_2 + r_3 + r_4 + r_7 + r_{10} + r_{19})) - 2c\,. \tag{B2.3}$$

Substitution of the joint probabilities from Tables B2.2a – B2.2c into (B2.3) and simplifying gives:

$$E[\Pi_{Case3}] = P_1\overline{F}_1\left(3 - \overline{F}_1^{\,2}\right) - 2c\,. \tag{B2.4}$$

For Case 4, $Q_1 = 3, Q_2 = 0$. For this inventory decision, the possible revenues that can be realized are shown in Tables B2.6a – B2.6c.

Table B2.6a. Possible Revenues for Example 2, Case 4, $RP_3 < P_1$.

		Customer 1		
		$RP_1 < P_1$	$P_1 \le RP_1 < P_2$	$P_2 \le RP_1$
Customer 2	$RP_2 < P_1$	0	P_1	P_1
	$P_1 \le RP_2 < P_2$	P_1	$2P_1$	$2P_1$
	$P_2 \le RP_2$	P_1	$2P_1$	$2P_1$

Table B2.6b. Possible Revenues for Example 2, Case 4, $P_1 < RP_3 < P_2$.

		Customer 1		
		$RP_1 < P_1$	$P_1 \le RP_1 < P_2$	$P_2 \le RP_1$
Customer 2	$RP_2 < P_1$	P_1	$2P_1$	$2P_1$
	$P_1 \le RP_2 < P_2$	$2P_1$	$3P_1$	$3P_1$
	$P_2 \le RP_2$	$2P_1$	$3P_1$	$3P_1$

Table B2.6c. Possible Revenues for Example 2, Case 4, $P_2 < RP_3$.

		Customer 1		
		$RP_1 < P_1$	$P_1 \le RP_1 < P_2$	$P_2 \le RP_1$
Customer 2	$RP_2 < P_1$	P_1	$2P_1$	$2P_1$
	$P_1 \le RP_2 < P_2$	$2P_1$	$3P_1$	$3P_1$
	$P_2 \le RP_2$	$2P_1$	$3P_1$	$3P_1$

Multiplying the joint probabilities from Tables B2.3a – B2.3c by their respective revenues in Tables B2.6a – B2.6c, taking the summation of the products and subtracting the total cost gives the expected profit for Case 4:

$$E[\Pi_{Case4}] = 0r_1 + P_1(r_2 + r_3 + r_4 + r_7 + r_{10} + r_{19}) \\ + 2P_1(r_5 + r_6 + r_8 + r_9 + r_{11} + r_{12} + r_{13} + r_{16} + r_{20} + r_{21} + r_{22} + r_{25}) \\ + 3P_1(r_{14} + r_{15} + r_{17} + r_{18} + r_{23} + r_{24} + r_{26} + r_{27}) - 3c. \quad \text{(B2.5)}$$

Substitution of the joint probabilities from Tables B2.2a – B2.2c into (B2.5) and simplifying gives:

$$E[\Pi_{Case4}] = 3P_1\overline{F_1} - 3c. \quad \text{(B2.6)}$$

For Case 5, $Q_1 = 0, Q_2 = 1$. For this inventory decision, the possible revenues that can be realized are shown in Tables B2.7a – B2.7c.

Table B2.7a. Possible Revenues for Example 2, Case 5, $RP_3 < P_1$.

<table>
<tr><td colspan="2" rowspan="2"></td><td colspan="3">Customer 1</td></tr>
<tr><td>$RP_1 < P_1$</td><td>$P_1 \le RP_1 < P_2$</td><td>$P_2 \le RP_1$</td></tr>
<tr><td rowspan="3">Customer 2</td><td>$RP_2 < P_1$</td><td>0</td><td>0</td><td>P_2</td></tr>
<tr><td>$P_1 \le RP_2 < P_2$</td><td>0</td><td>0</td><td>P_2</td></tr>
<tr><td>$P_2 \le RP_2$</td><td>P_2</td><td>P_2</td><td>P_2</td></tr>
</table>

Table B2.7b. Possible Revenues for Example 2, Case 5, $P_1 < RP_3 < P_2$.

<table>
<tr><td colspan="2" rowspan="2"></td><td colspan="3">Customer 1</td></tr>
<tr><td>$RP_1 < P_1$</td><td>$P_1 \le RP_1 < P_2$</td><td>$P_2 \le RP_1$</td></tr>
<tr><td rowspan="3">Customer 2</td><td>$RP_2 < P_1$</td><td>0</td><td>0</td><td>P_2</td></tr>
<tr><td>$P_1 \le RP_2 < P_2$</td><td>0</td><td>0</td><td>P_2</td></tr>
<tr><td>$P_2 \le RP_2$</td><td>P_2</td><td>P_2</td><td>P_2</td></tr>
</table>

Table B2.7c. Possible Revenues for Example 2, Case 5, $P_2 < RP_3$.

		Customer 1		
		$RP_1 < P_1$	$P_1 \le RP_1 < P_2$	$P_2 \le RP_1$
Customer 2	$RP_2 < P_1$	P_2	P_2	P_2
	$P_1 \le RP_2 < P_2$	P_2	P_2	P_2
	$P_2 \le RP_2$	P_2	P_2	P_2

Multiplying the joint probabilities from Tables B2.3a – B2.3c by their respective revenues in Tables B2.7a – B2.7c, taking the summation of the products and subtracting the total cost gives the expected profit for Case 5:

$$E[\Pi_{Case5}] = 0(r_1 + r_2 + r_4 + r_5 + r_{10} + r_{11} + r_{13} + r_{14}) \\ P_2(1 - (r_1 + r_2 + r_4 + r_5 + r_{10} + r_{11} + r_{13} + r_{14})) - 1c\,. \tag{B2.7}$$

Substitution of the joint probabilities from Tables B2.2a – B2.2c into (B2.7) and simplifying gives:

$$E[\Pi_{Case5}] = P_2 \overline{F}_2 \left(3 - 3\overline{F}_2 + \overline{F}_2^{\,2}\right) - c\,. \tag{B2.8}$$

For Case 6, $Q_1 = 1, Q_2 = 1$. For this inventory decision, the possible revenues that can be realized are shown in Tables B2.8a – B2.8c.

Table B2.8a. Possible Revenues for Example 2, Case 6, $RP_3 < P_1$.

		Customer 1		
		$RP_1 < P_1$	$P_1 \le RP_1 < P_2$	$P_2 \le RP_1$
Customer 2	$RP_2 < P_1$	0	P_1	P_1
	$P_1 \le RP_2 < P_2$	P_1	P_1	P_1
	$P_2 \le RP_2$	P_1	$P_1 + P_2$	$P_1 + P_2$

Table B2.8b. Possible Revenues for Example 2, Case 6, $P_1 < RP_3 < P_2$.

		Customer 1		
		$RP_1 < P_1$	$P_1 \le RP_1 < P_2$	$P_2 \le RP_1$
Customer 2	$RP_2 < P_1$	P_1	P_1	P_1
	$P_1 \le RP_2 < P_2$	P_1	P_1	P_1
	$P_2 \le RP_2$	P_1	$P_1 + P_2$	$P_1 + P_2$

Table B2.8c. Possible Revenues for Example 2, Case 6, $P_2 < RP_3$.

		Customer 1		
		$RP_1 < P_1$	$P_1 \le RP_1 < P_2$	$P_2 \le RP_1$
Customer 2	$RP_2 < P_1$	P_1	$P_1 + P_2$	$P_1 + P_2$
	$P_1 \le RP_2 < P_2$	$P_1 + P_2$	$P_1 + P_2$	$P_1 + P_2$
	$P_2 \le RP_2$	$P_1 + P_2$	$P_1 + P_2$	$P_1 + P_2$

Multiplying the joint probabilities from Tables B2.3a – B2.3c by their respective revenues in Tables B2.8a – B2.8c, taking the summation of the products and subtracting the total cost gives the expected profit for Case 6:

$$\begin{aligned} E[\Pi_{Case6}] &= 0r_1 + P_1(r_2 + r_3 + r_4 + r_5 + r_6 + r_7 + r_{10} + r_{11} + r_{12} + r_{13} + r_{14} + r_{15} + r_{16} + r_{19}) \\ &\quad + (P_1 + P_2)(r_8 + r_9 + r_{17} + r_{18} + r_{20} + r_{21} + r_{22} + r_{23} + r_{24} + r_{25} + r_{26} + r_{27}) - 2c. \end{aligned} \tag{B2.9}$$

Substitution of the joint probabilities from Tables B2.2a – B2.2c into (B2.9) and simplifying gives:

$$E[\Pi_{Case6}] = P_1\overline{F_1}\left(3 - 3\overline{F_1} + \overline{F_1}^2\right) + P_2\overline{F_2}\left(3\overline{F_1} - \overline{F_1}^2 - \overline{F_1}\,\overline{F_2}\right) - 2c. \tag{B2.10}$$

For Case 7, $Q_1 = 2, Q_2 = 1$. For this inventory decision, the possible revenues that can be realized are shown in Tables B2.9a – B2.9c.

Table B2.9a. Possible Revenues for Example 2, Case 7, $RP_3 < P_1$.

		Customer 1		
		$RP_1 < P_1$	$P_1 \le RP_1 < P_2$	$P_2 \le RP_1$
Customer 2	$RP_2 < P_1$	0	P_1	P_1
	$P_1 \le RP_2 < P_2$	P_1	$2P_1$	$2P_1$
	$P_2 \le RP_2$	P_1	$2P_1$	$2P_1$

Table B2.9b. Possible Revenues for Example 2, Case 7, $P_1 < RP_3 < P_2$.

		Customer 1		
		$RP_1 < P_1$	$P_1 \le RP_1 < P_2$	$P_2 \le RP_1$
Customer 2	$RP_2 < P_1$	P_1	$2P_1$	$2P_1$
	$P_1 \le RP_2 < P_2$	$2P_1$	$2P_1$	$2P_1$
	$P_2 \le RP_2$	$2P_1$	$2P_1$	$2P_1$

Table B2.9c. Possible Revenues for Example 2, Case 7, $P_2 < RP_3$.

		Customer 1		
		$RP_1 < P_1$	$P_1 \le RP_1 < P_2$	$P_2 \le RP_1$
Customer 2	$RP_2 < P_1$	$2P_1$	$2P_1$	$2P_1$
	$P_1 \le RP_2 < P_2$	$2P_1$	$2P_1 + P_2$	$2P_1 + P_2$
	$P_2 \le RP_2$	$2P_1$	$2P_1 + P_2$	$2P_1 + P_2$

Multiplying the joint probabilities from Tables B2.3a – B2.3c by their respective revenues in Tables B2.9a – B2.9c, taking the summation of the products and subtracting the total cost gives the expected profit for Case 7:

$$E[\Pi_{Case7}] = 0r_1 + P_1(r_2 + r_3 + r_4 + r_7 + r_{10} + r_{19})$$
$$+ 2P_1(r_5 + r_6 + r_8 + r_9 + r_{11} + r_{12} + r_{13} + r_{14} + r_{15} + r_{16} + r_{17} + r_{18} + r_{20} + r_{21} + r_{22} + r_{25}) \qquad \text{(B2.11)}$$
$$+ (2P_1 + P_2)(r_{23} + r_{24} + r_{26} + r_{27}) - 3c\,.$$

Substitution of the joint probabilities from Tables B2.2a – B2.2c into (B2.11) and simplifying gives:

$$E[\Pi_{Case7}] = P_1\overline{F_1}\left(3-\overline{F_1}^2\right)+P_2\overline{F_1}^2\overline{F_2}-3c\,. \tag{B2.12}$$

For Case 8, $Q_1 = 0, Q_2 = 2$. For this inventory decision, the possible revenues that can be realized are shown in Tables B2.10a – B2.10c.

Table B2.10a. Possible Revenues for Example 2, Case 8, $RP_3 < P_1$.

		Customer 1		
		$RP_1 < P_1$	$P_1 \le RP_1 < P_2$	$P_2 \le RP_1$
Customer 2	$RP_2 < P_1$	0	0	P_2
	$P_1 \le RP_2 < P_2$	0	0	P_2
	$P_2 \le RP_2$	P_2	P_2	$2P_2$

Table B2.10b. Possible Revenues for Example 2, Case 8, $P_1 < RP_3 < P_2$.

		Customer 1		
		$RP_1 < P_1$	$P_1 \le RP_1 < P_2$	$P_2 \le RP_1$
Customer 2	$RP_2 < P_1$	0	0	P_2
	$P_1 \le RP_2 < P_2$	0	0	P_2
	$P_2 \le RP_2$	P_2	P_2	$2P_2$

Table B2.10c. Possible Revenues for Example 2, Case 8, $P_2 < RP_3$.

		Customer 1		
		$RP_1 < P_1$	$P_1 \le RP_1 < P_2$	$P_2 \le RP_1$
Customer 2	$RP_2 < P_1$	P_2	P_2	$2P_2$
	$P_1 \le RP_2 < P_2$	P_2	P_2	$2P_2$
	$P_2 \le RP_2$	$2P_2$	$2P_2$	$2P_2$

Multiplying the joint probabilities from Tables B2.3a – B2.3c by their respective revenues in Tables B2.10a – B2.10c, taking the summation of the products and subtracting the total cost gives the expected profit for Case 8:

$$E[\Pi_{Case8}] = 0(r_1 + r_2 + r_4 + r_5 + r_{10} + r_{11} + r_{13} + r_{14})$$
$$+ P_2(r_3 + r_6 + r_7 + r_8 + r_{12} + r_{15} + r_{16} + r_{17} + r_{19} + r_{20} + r_{22} + r_{23}) \quad \text{(B2.13)}$$
$$+ 2P_2(r_9 + r_{18} + r_{21} + r_{24} + r_{25} + r_{26} + r_{27}) - 2c\,.$$

Substitution of the joint probabilities from Tables B2.2a – B2.2c into (B2.13) and simplifying gives:

$$E[\Pi_{Case8}] = P_2\overline{F}_2\left(3 - \overline{F}_2^{\,2}\right) - 2c\,. \quad \text{(B2.14)}$$

For Case 9, $Q_1 = 1, Q_2 = 2$. For this inventory decision, the possible revenues that can be realized are shown in Tables B2.11a – B2.11c.

Table B2.11a. Possible Revenues for Example 2, Case 9, $RP_3 < P_1$.

		Customer 1		
		$RP_1 < P_1$	$P_1 \le RP_1 < P_2$	$P_2 \le RP_1$
Customer 2	$RP_2 < P_1$	0	P_1	P_1
	$P_1 \le RP_2 < P_2$	P_1	P_1	$P_1 + P_2$
	$P_2 \le RP_2$	P_1	P_1	$P_1 + P_2$

Table B2.11b. Possible Revenues for Example 2, Case 9, $P_1 < RP_3 < P_2$.

		Customer 1		
		$RP_1 < P_1$	$P_1 \le RP_1 < P_2$	$P_2 \le RP_1$
Customer 2	$RP_2 < P_1$	P_1	P_1	P_1
	$P_1 \le RP_2 < P_2$	P_1	P_1	$P_1 + P_2$
	$P_2 \le RP_2$	P_1	P_1	$P_1 + P_2$

Table B2.11c. Possible Revenues for Example 2, Case 9, $P_2 < RP_3$.

		Customer 1		
		$RP_1 < P_1$	$P_1 \le RP_1 < P_2$	$P_2 \le RP_1$
Customer 2	$RP_2 < P_1$	P_1	$P_1 + P_2$	$P_1 + P_2$
	$P_1 \le RP_2 < P_2$	$P_1 + P_2$	$P_1 + P_2$	$P_1 + 2P_2$
	$P_2 \le RP_2$	$P_1 + P_2$	$P_1 + P_2$	$P_1 + 2P_2$

Multiplying the joint probabilities from Tables B2.3a – B2.3c by their respective revenues in Tables B2.11a – B2.11c, taking the summation of the products and subtracting the total cost gives the expected profit for Case 9:

$$
\begin{aligned}
E[\Pi_{Case9}] = 0r_1 \\
&+ P_1(r_2 + r_3 + r_4 + r_5 + r_7 + r_8 + r_{10} + r_{11} + r_{12} + r_{13} + r_{14} + r_{15} + r_{16} + r_{17} + r_{19}) \\
&+ (P_1 + P_2)(r_6 + r_9 + r_{15} + r_{18} + r_{20} + r_{21} + r_{22} + r_{23} + r_{25} + r_{26}) \\
&+ (P_1 + 2P_2)(r_{24} + r_{27}) - 3c\ . \qquad \text{(B2.15)}
\end{aligned}
$$

Substitution of the joint probabilities from Tables B2.2a – B2.2c into (B2.15) and simplifying gives:

$$E[\Pi_{Case9}] = P_1\overline{F_1}\left(3 - 3\overline{F_1} + \overline{F_1}^{\,2}\right) + P_2\overline{F_2}\left(3\overline{F_1} - \overline{F_1}^{\,2}\right) - 3c\ . \qquad \text{(B2.16)}$$

For Case 10, $Q_1 = 0, Q_2 = 3$. For this inventory decision, the possible revenues that can be realized are shown in Tables B2.12a – B2.12c.

Table B2.12a. Possible Revenues for Example 2, Case 10, $RP_3 < P_1$.

		Customer 1		
		$RP_1 < P_1$	$P_1 \le RP_1 < P_2$	$P_2 \le RP_1$
Customer 2	$RP_2 < P_1$	0	0	P_2
	$P_1 \le RP_2 < P_2$	0	0	P_2
	$P_2 \le RP_2$	P_2	P_2	$2P_2$

Table B2.12b. Possible Revenues for Example 2, Case 10, $P_1 < RP_3 < P_2$.

		Customer 1		
		$RP_1 < P_1$	$P_1 \le RP_1 < P_2$	$P_2 \le RP_1$
Customer 2	$RP_2 < P_1$	0	0	P_2
	$P_1 \le RP_2 < P_2$	0	0	P_2
	$P_2 \le RP_2$	P_2	P_2	$2P_2$

Table B2.12c. Possible Revenues for Example 2, Case 10, $P_2 < RP_3$.

		Customer 1		
		$RP_1 < P_1$	$P_1 \le RP_1 < P_2$	$P_2 \le RP_1$
Customer 2	$RP_2 < P_1$	P_2	P_2	$2P_2$
	$P_1 \le RP_2 < P_2$	P_2	P_2	$2P_2$
	$P_2 \le RP_2$	$2P_2$	$2P_2$	$3P_2$

Multiplying the joint probabilities from Tables B2.3a – B2.3c by their respective revenues in Tables B2.12a – B2.12c, taking the summation of the products and subtracting the total cost gives the expected profit for Case 10:

$$\begin{aligned} E[\Pi_{Case10}] = 0&(r_1 + r_2 + r_4 + r_5 + r_{10} + r_{11} + r_{13} + r_{14}) \\ &+ P_2(r_3 + r_6 + r_7 + r_8 + r_{12} + r_{15} + r_{16} + r_{17} + r_{19} + r_{20} + r_{22} + r_{23}) \\ &+ 2P_2(r_9 + r_{18} + r_{21} + r_{24} + r_{25} + r_{26}) + 3P_2 r_{27} - 3c. \end{aligned} \quad \text{(B2.17)}$$

Substitution of the joint probabilities from Tables B2.2a – B2.2c into (B2.17) and simplifying gives:

$$E[\Pi_{Case10}] = 3P_2\overline{F}_2 - 3c. \quad \text{(B2.18)}$$

B3 STOCHASTIC CUSTOMER BASE WITH $d_1 = 1$ AND $d_2 = 2$

This section contains the derivations of the expected profit functions for a seller facing a stochastic customer base of $d_1 = 1$ with probability $Prob_{d1}$ and $d_2 = 2$ with probability $1 - Prob_{d1}$.

In this example, the seller chooses from a set of inventory and corresponding price combinations where $Q_{Total} \leq 2$. These inventory combinations are shown in Table B3.1. Note that Q_i is the number of units of inventory to be made available for sale at price P_i, i = 1, 2.

Table B3.1. Inventory Combinations for Example 3

		Q_1		
		0	1	2
Q_2	0	Case 1	Case 2	Case 3
	1	Case 4	Case 5	
	2	Case 6		

While Case 1 is included in Table B3.1 for completeness, the decision to make no items available for sale results in a trivial expected profit of zero, and is not considered for the remainder of the example.

Since there can be either one or two customers in the population, two sets of probabilities are needed for the expected profit calculations given in Chapter 7. These (joint) probabilities are calculated for various combinations of reservation prices relative to the selling prices. The probabilities for the case of a single customer are presented in Table

B3.2a, with corresponding notation presented in Table B3.2b. Recall that $\overline{F_i}$ denotes $1-F_{RP}(P_i)$, the probability that a randomly-chosen customer has a reservation price at least as high as price P_i.

Table B3.2a. Probabilities for Example 3 for $d_1 = 1$.

Range	$RP_1 < P_1$	$P_1 \le RP_1 < P_2$	$P_2 \le RP_1$
Probability	$1-\overline{F_1}$	$\overline{F_1}-\overline{F_2}$	$\overline{F_2}$

Table B3.2b. Notation for Probabilities for Example 3 for $d_1 = 1$.

Range	$RP_1 < P_1$	$P_1 \le RP_2 < P_2$	$P_2 \le RP_2$
Probability	q_1	q_2	q_3

The joint probabilities for the case of a two customers are presented in Table B3.3a, with corresponding notation presented in Table B3.3b.

Table B3.3a. Joint Probabilities for Example 3 for $d_2 = 2$.

		Customer 1		
		$RP_1 < P_1$	$P_1 \le RP_1 < P_2$	$P_2 \le RP_1$
Customer 2	$RP_2 < P_1$	$\left(1-\overline{F_1}\right)^2$	$\left(1-\overline{F_1}\right)\left(\overline{F_1}-\overline{F_2}\right)$	$\left(1-\overline{F_1}\right)\overline{F_2}$
	$P_1 \le RP_2 < P_2$	$\left(1-\overline{F_1}\right)\left(\overline{F_1}-\overline{F_2}\right)$	$\left(\overline{F_1}-\overline{F_2}\right)^2$	$\overline{F_2}\left(\overline{F_1}-\overline{F_2}\right)$
	$P_2 \le RP_2$	$\left(1-\overline{F_1}\right)\overline{F_2}$	$\overline{F_2}\left(\overline{F_1}-\overline{F_2}\right)$	$\overline{F_2}^2$

Table B3.3b. Notation for Joint Probabilities for Example 3 for $d_2 = 2$.

		Customer 1		
		$RP_1 < P_1$	$P_1 \le RP_1 < P_2$	$P_2 \le RP_1$
Customer 2	$RP_2 < P_1$	r_1	r_2	r_3
	$P_1 \le RP_2 < P_2$	r_4	r_5	r_6
	$P_2 \le RP_2$	r_7	r_8	r_9

Shown below are the expected profit calculations for the various combinations of inventory decisions (Cases) shown in Table B3.1.

For Case 2, $Q_1 = 1, Q_2 = 0$. For this inventory decision, the possible revenues that can be realized are shown in Tables B3.4a and B3.4b.

Table B3.4a. Possible Revenues for Example 3, Case 2, $d_1 = 1$

Range	$RP_1 < P_1$	$P_1 \le RP_2 < P_2$	$P_2 \le RP_2$
Revenue	0	P_1	P_1

Table B3.4b. Possible Revenues for Example 3, Case 2, $d_2 = 2$

		Customer 1		
		$RP_1 < P_1$	$P_1 \le RP_1 < P_2$	$P_2 \le RP_1$
Customer 2	$RP_2 < P_1$	0	P_1	P_1
	$P_1 \le RP_2 < P_2$	P_1	P_1	P_1
	$P_2 \le RP_2$	P_1	P_1	P_1

Multiplying the (joint) probabilities from Tables B3.2b and B3.3b by their respective revenues in Tables B3.4a and B3.4b, taking the summation of the products and subtracting the total cost gives the expected profit for Case 2:

$$E[\Pi_{Case2}] = Prob_{d1}\left(0q_1 + P_1\left(q_2 + q_3\right)\right) \\ + \left(1 - Prob_{d1}\right)\left(0r_1 + P_1\left(r_2 + r_3 + r_4 + r_5 + r_6 + r_7 + r_8 + r_9\right)\right) - 1c\,. \tag{B3.1}$$

Substitution of the (joint) probabilities from Tables B3.2a and B3.3a into (B3.1) and simplifying gives:

$$E[\Pi_{Case2}] = P_1\overline{F_1}\left(2 - Prob_{d1}\right) - P_1\overline{F_1}^2\left(1 - Prob_{d1}\right) - c\,. \tag{B3.2}$$

For Case 3, $Q_1 = 2, Q_2 = 0$. For this inventory decision, the possible revenues that can be realized are shown in Tables B3.5a and B3.5b.

Table B3.5a. Possible Revenues for Example 3, Case 3, $d_1 = 1$

Range	$RP_1 < P_1$	$P_1 \le RP_2 < P_2$	$P_2 \le RP_2$
Revenue	0	P_1	P_1

Table B3.5b. Possible Revenues for Example 3, Case 3, $d_2 = 2$

		Customer 1		
		$RP_1 < P_1$	$P_1 \le RP_1 < P_2$	$P_2 \le RP_1$
Customer 2	$RP_2 < P_1$	0	P_1	P_1
	$P_1 \le RP_2 < P_2$	P_1	$2P_1$	$2P_1$
	$P_2 \le RP_2$	P_1	$2P_1$	$2P_1$

Multiplying the (joint) probabilities from Tables B3.2b and B3.3b by their respective revenues in Tables B3.5a and B3.5b, taking the summation of the products and subtracting the total cost gives the expected profit for Case 3:

$$E[\Pi_{Case3}] = Prob_{d1}\left(0q_1 + P_1\left(q_2 + q_3\right)\right) + \left(1 - Prob_{d1}\right)\left(0r_1 + P_1\left(r_2 + r_3 + r_4 + r_7\right) + 2P_1\left(r_5 + r_6 + r_8 + r_9\right)\right) - 2c\,. \qquad \text{(B3.3)}$$

Substitution of the (joint) probabilities from Tables B3.2a and B3.3a into (B3.3) and simplifying gives:

$$E[\Pi_{Case3}] = P_1\overline{F_1}\left(2 - Prob_{d1}\right) - 2c\,. \qquad \text{(B3.4)}$$

For Case 4, $Q_1 = 0, Q_2 = 1$. For this inventory decision, the possible revenues that can be realized are shown in Tables B3.6a and B3.6b.

Table B3.6a. Possible Revenues for Example 3 for Case 4, $d_1 = 1$

Range	$RP_1 < P_1$	$P_1 \le RP_2 < P_2$	$P_2 \le RP_2$
Revenue	0	0	P_2

Table B3.6b. Possible Revenues for Example 3 for Case 4, $d_2 = 2$

		Customer 1		
		$RP_1 < P_1$	$P_1 \le RP_1 < P_2$	$P_2 \le RP_1$
Customer 2	$RP_2 < P_1$	0	0	P_2
	$P_1 \le RP_2 < P_2$	0	0	P_2
	$P_2 \le RP_2$	P_2	P_2	P_2

Multiplying the (joint) probabilities from Tables B3.2b and B3.3b by their respective revenues in Tables B3.6a and B3.6b, taking the summation of the products and subtracting the total cost gives the expected profit for Case 4:

$$E[\Pi_{Case4}] = Prob_{d1}\left(0(q_1 + q_2) + P_1 q_3\right) + \left(1 - Prob_{d1}\right)\left(0(r_1 + r_2 + r_4 + r_5) + P_2(r_3 + r_6 + r_7 + r_8 + r_9)\right) - 2c. \tag{B3.5}$$

Substitution of the (joint) probabilities from Tables B3.2a and B3.3a into (B3.5) and simplifying gives:

$$E[\Pi_{Case4}] = P_2 \overline{F_2}\left(2 - Prob_{d1}\right) - P_2 \overline{F_2}^{\,2}\left(1 - Prob_{d1}\right) - c. \tag{B3.6}$$

For Case 5, $Q_1 = 1, Q_2 = 1$. For this inventory decision, the possible revenues that can be realized are shown in Tables B3.7a and B3.7b.

Table B3.7a. Possible Revenues for Example 3 for Case 5, $d_1 = 1$

Range	$RP_1 < P_1$	$P_1 \le RP_2 < P_2$	$P_2 \le RP_2$
Revenue	0	P_1	P_1

Table B3.7b. Possible Revenues for Example 3 for Case 5, $d_2 = 2$

		Customer 1		
		$RP_1 < P_1$	$P_1 \le RP_1 < P_2$	$P_2 \le RP_1$
Customer 2	$RP_2 < P_1$	0	P_1	P_1
	$P_1 \le RP_2 < P_2$	P_1	P_1	P_1
	$P_2 \le RP_2$	P_1	$P_1 + P_2$	$P_1 + P_2$

Multiplying the (joint) probabilities from Tables B3.2b and B3.3b by their respective revenues in Tables B3.7a and B3.7b, taking the summation of the products and subtracting the total cost gives the expected profit for Case 5:

$$E[\Pi_{Case5}] = Prob_{d1}\left(0q_1 + P_1\left(q_2 + q_3\right)\right)$$
$$+\left(1 - Prob_{d1}\right)\left(0r_1 + P_1\left(r_2 + r_3 + r_4 + r_5 + r_6 + r_7\right) + \left(P_1 + P_2\right)\left(r_8 + r_9\right)\right) - 2c. \quad \text{(B3.7)}$$

Substitution of the (joint) probabilities from Tables B3.2a and B3.3a into (B3.7) and simplifying gives:

$$E[\Pi_{Case5}] = P_1\overline{F_1}\left(2 - Prob_{d1}\right) - P_1\overline{F_1}^2\left(1 - Prob_{d1}\right) + P_2\overline{F_1}\,\overline{F_2}\left(1 - Prob_{d1}\right) - 2c. \quad \text{(B3.8)}$$

For Case 6, $Q_1 = 0, Q_2 = 2$. For this inventory decision, the possible revenues that can be realized are shown in Tables B3.8a and B3.8b.

Table B3.8a. Possible Revenues for Example 3 for Case 6, $d_1 = 1$

Range	$RP_1 < P_1$	$P_1 \le RP_2 < P_2$	$P_2 \le RP_2$
Revenue	0	0	P_2

Table B3.8b. Possible Revenues for Example 3 for Case 6, $d_2 = 2$

		Customer 1		
		$RP_1 < P_1$	$P_1 \le RP_1 < P_2$	$P_2 \le RP_1$
Customer 2	$RP_2 < P_1$	0	0	P_2
	$P_1 \le RP_2 < P_2$	0	0	P_2
	$P_2 \le RP_2$	P_2	P_2	$2P_2$

Multiplying the (joint) probabilities from Tables B3.2b and B3.3b by their respective revenues in Tables B3.8a and B3.8b, taking the summation of the products and subtracting the total cost gives the expected profit for Case 6:

$$E[\Pi_{Case6}] = Prob_{d1}\left(0(q_1+q_2)+P_1q_3\right) \\ +\left(1-Prob_{d1}\right)\left(0(r_1+r_2+r_4+r_5)+P_2(r_3+r_6+r_7+r_8)+2P_2r_9\right)-2c\,. \quad \text{(B3.9)}$$

Substitution of the (joint) probabilities from Tables B3.2a and B3.3a into (B3.9) and simplifying gives:

$$E[\Pi_{Case6}] = P_2\overline{F_2}\left(2-Prob_{d1}\right)-2c\,. \quad \text{(B3.10)}$$

B4 STOCHASTIC CUSTOMER BASE WITH $d_1 = 1$ AND $d_2 = 3$

This section contains the derivations of the expected profit functions for a seller facing a stochastic customer base of $d_1 = 1$ with probability $Prob_{d1}$ and $d_2 = 3$ with probability $1 - Prob_{d1}$.

In this example, the seller chooses from a set of inventory and corresponding price combinations where $Q_{Total} \leq 3$. These inventory combinations are shown in Table B4.1. Note that Q_i is the number of units of inventory to be made available for sale at price P_i, i = 1, 2.

Table B4.1. Inventory Combinations for Example 4.

		Q_1			
		0	1	2	3
Q_2	0	Case 1	Case 2	Case 3	Case 4
	1	Case 5	Case 6	Case 7	
	2	Case 8	Case 9		
	3	Case 10			

While Case 1 is included in Table B4.1 for completeness, the decision to make no items available for sale results in a trivial expected profit of zero, and is not considered for the remainder of the example.

Since there can be either one or three customers in the population, two sets of probabilities are needed for the expected profit calculations given in Chapter 7. These (joint) probabilities are calculated for various combinations of reservation prices relative to the selling prices. The probabilities for the case of a single customer are presented in Table B4.2a, with corresponding notation presented in Table B4.2b. Recall that $\overline{F_i}$ denotes $1 - F_{RP}(P_i)$, the probability that a randomly-chosen customer has a reservation price at least as high as price P_i.

Table B4.2a. Probabilities for Example 4 for Example 4 $d_1 = 1$.

Range	$RP_1 < P_1$	$P_1 \le RP_1 < P_2$	$P_2 \le RP_1$
Probability	$1 - \overline{F_1}$	$\overline{F_1} - \overline{F_2}$	$\overline{F_2}$

Table B4.2b. Notation for Probabilities for Example 4 for $d_1 = 1$.

Range	$RP_1 < P_1$	$P_1 \le RP_2 < P_2$	$P_2 \le RP_2$
Probability	q_1	q_2	q_3

The (joint) probabilities for the case of a three customers are presented in Tables B4.3a – B4.3c, with corresponding notation presented in Tables B4.4a – B4.4c.

Table B4.3a. Joint Probabilities for Example 4 for $RP_3 < P_1$.

		Customer 1		
		$RP_1 < P_1$	$P_1 \le RP_1 < P_2$	$P_2 \le RP_1$
Customer 2	$RP_2 < P_1$	$(1-\overline{F}_1)^3$	$(1-\overline{F}_1)^2(\overline{F}_1-\overline{F}_2)$	$(1-\overline{F}_1)^2\overline{F}_2$
	$P_1 \le RP_2 < P_2$	$(1-\overline{F}_1)^2(\overline{F}_1-\overline{F}_2)$	$(\overline{F}_1-\overline{F}_2)^2(1-\overline{F}_1)$	$\overline{F}_2(\overline{F}_1-\overline{F}_2)(1-\overline{F}_1)$
	$P_2 \le RP_2$	$(1-\overline{F}_1)^2\overline{F}_2$	$\overline{F}_2(\overline{F}_1-\overline{F}_2)(1-\overline{F}_1)$	$\overline{F}_2^{\,2}(1-\overline{F}_1)$

Table B4.3b. Joint Probabilities for Example 4 for $P_1 < RP_3 < P_2$.

		Customer 1		
		$RP_1 < P_1$	$P_1 \le RP_1 < P_2$	$P_2 \le RP_1$
Customer 2	$RP_2 < P_1$	$(1-\overline{F}_1)^2(\overline{F}_1-\overline{F}_2)$	$(1-\overline{F}_1)(\overline{F}_1-\overline{F}_2)$	$(1-\overline{F}_1)(\overline{F}_1-\overline{F}_2)\overline{F}_2$
	$P_1 \le RP_2 < P_2$	$(1-\overline{F}_1)(\overline{F}_1-\overline{F}_2)^2$	$(\overline{F}_1-\overline{F}_2)^3$	$\overline{F}_2(\overline{F}_1-\overline{F}_2)^2$
	$P_2 \le RP_2$	$(1-\overline{F}_1)(\overline{F}_1-\overline{F}_2)\overline{F}_2$	$\overline{F}_2(\overline{F}_1-\overline{F}_2)^2$	$\overline{F}_2^{\,2}(\overline{F}_1-\overline{F}_2)$

Table B4.3c. Joint Probabilities for Example 4 for $P_2 < RP_3$.

		Customer 1		
		$RP_1 < P_1$	$P_1 \le RP_1 < P_2$	$P_2 \le RP_1$
Customer 2	$RP_2 < P_1$	$(1-\overline{F}_1)^2\overline{F}_2$	$(1-\overline{F}_1)(\overline{F}_1-\overline{F}_2)\overline{F}_2$	$(1-\overline{F}_1)\overline{F}_2^{\,2}$
	$P_1 \le RP_2 < P_2$	$(1-\overline{F}_1)(\overline{F}_1-\overline{F}_2)\overline{F}_2$	$(\overline{F}_1-\overline{F}_2)^2\overline{F}_2$	$\overline{F}_2^{\,2}(\overline{F}_1-\overline{F}_2)$
	$P_2 \le RP_2$	$(1-\overline{F}_1)\overline{F}_2^{\,2}$	$\overline{F}_2^{\,2}(\overline{F}_1-\overline{F}_2)$	$\overline{F}_2^{\,3}$

Table B4.4a. Notation for Joint Probabilities for Example 4 for $RP_3 < P_1$.

		Customer 1		
		$RP_1 < P_1$	$P_1 \le RP_1 < P_2$	$P_2 \le RP_1$
Customer 2	$RP_2 < P_1$	r_1	r_2	r_3
	$P_1 \le RP_2 < P_2$	r_4	r_5	r_6
	$P_2 \le RP_2$	r_7	r_8	r_9

Table B4.4b. Notation for Joint Probabilities for Example 4 for $P_1 < RP_3 < P_2$.

		Customer 1		
		$RP_1 < P_1$	$P_1 \le RP_1 < P_2$	$P_2 \le RP_1$
Customer 2	$RP_2 < P_1$	r_{10}	r_{11}	r_{12}
	$P_1 \le RP_2 < P_2$	r_{13}	r_{14}	r_{15}
	$P_2 \le RP_2$	r_{16}	r_{17}	r_{18}

Table B4.4c. Notation for Joint Probabilities for Example 4 for $P_2 < RP_3$.

		Customer 1		
		$RP_1 < P_1$	$P_1 \le RP_1 < P_2$	$P_2 \le RP_1$
Customer 2	$RP_2 < P_1$	r_{19}	r_{20}	r_{21}
	$P_1 \le RP_2 < P_2$	r_{22}	r_{23}	r_{24}
	$P_2 \le RP_2$	r_{25}	r_{26}	r_{27}

Shown below are the expected profit calculations for the various combinations of inventory decisions shown in Table B4.1.

For Case 2, $Q_1 = 1, Q_2 = 0$. For this inventory decision, the possible revenues that can be realized are shown in Tables B4.4a – B4.4d.

Table B4.5a. Possible Revenues for Example 4, Case 2, $d_1 = 1$

Range	$RP_1 < P_1$	$P_1 \le RP_2 < P_2$	$P_2 \le RP_2$
Revenue	0	P_1	P_1

Table B4.5b. Possible Revenues for Example 4, Case 2, $d_2 = 3$, $RP_3 < P_1$.

		Customer 1		
		$RP_1 < P_1$	$P_1 \le RP_1 < P_2$	$P_2 \le RP_1$
Customer 2	$RP_2 < P_1$	0	P_1	P_1
	$P_1 \le RP_2 < P_2$	P_1	P_1	P_1
	$P_2 \le RP_2$	P_1	P_1	P_1

Table B4.5c. Possible Revenues for Example 4, Case 2, $d_2 = 3$, $P_1 < RP_3 < P_2$.

		Customer 1		
		$RP_1 < P_1$	$P_1 \le RP_1 < P_2$	$P_2 \le RP_1$
Customer 2	$RP_2 < P_1$	P_1	P_1	P_1
	$P_1 \le RP_2 < P_2$	P_1	P_1	P_1
	$P_2 \le RP_2$	P_1	P_1	P_1

Table B4.5d. Possible Revenues for Example 4, Case 2, $d_2 = 3$, $P_2 < RP_3$.

		Customer 1		
		$RP_1 < P_1$	$P_1 \le RP_1 < P_2$	$P_2 \le RP_1$
Customer 2	$RP_2 < P_1$	P_1	P_1	P_1
	$P_1 \le RP_2 < P_2$	P_1	P_1	P_1
	$P_2 \le RP_2$	P_1	P_1	P_1

Multiplying the (joint) probabilities from Tables B4.2b and B4.4a - B4.4c by their respective revenues in Tables B4.5a – B4.5d, taking the summation of the products and subtracting the total cost gives the expected profit for Case 2:

$$E[\Pi_{Case2}] = Prob_{d1}\left(0q_1 + P_1\left(q_2 + q_3\right)\right) + \left(1 - Prob_{d1}\right)\left(0r_1 + P_1\left(1 - r_1\right)\right) - 1c. \qquad \text{(B4.1)}$$

Substitution of the (joint) probabilities from Tables B4.2a and B4.3a - B4.3c into (B4.1) and simplifying gives:

$$E[\Pi_{Case2}] = P_1\overline{F_1}\left(3 - 2Prob_{d1} - \left(1 - Prob_{d1}\right)\left(3\overline{F_1} - \overline{F_1}^2\right)\right) - c. \qquad \text{(B4.2)}$$

For Case 3, $Q_1 = 2, Q_2 = 0$. For this inventory decision, the possible revenues that can be realized are shown in Tables B4.6a – B4.6d.

Table B4.6a. Possible Revenues for Example 4, $d_1 = 1$

Range	$RP_1 < P_1$	$P_1 \le RP_2 < P_2$	$P_2 \le RP_2$
Revenue	0	P_1	P_1

Table B4.6b. Possible Revenues for Example 4, $d_2 = 3$, $RP_3 < P_1$.

		Customer 1		
		$RP_1 < P_1$	$P_1 \le RP_1 < P_2$	$P_2 \le RP_1$
Customer 2	$RP_2 < P_1$	0	P_1	P_1
	$P_1 \le RP_2 < P_2$	P_1	$2P_1$	$2P_1$
	$P_2 \le RP_2$	P_1	$2P_1$	$2P_1$

Table B4.6c. Possible Revenues for Example 4, $d_2 = 3$, $P_1 < RP_3 < P_2$.

		Customer 1		
		$RP_1 < P_1$	$P_1 \le RP_1 < P_2$	$P_2 \le RP_1$
Customer 2	$RP_2 < P_1$	P_1	$2P_1$	$2P_1$
	$P_1 \le RP_2 < P_2$	$2P_1$	$2P_1$	$2P_1$
	$P_2 \le RP_2$	$2P_1$	$2P_1$	$2P_1$

Table B4.6d. Possible Revenues for Example 4, $d_2 = 3$, $P_2 < RP_3$.

		Customer 1		
		$RP_1 < P_1$	$P_1 \le RP_1 < P_2$	$P_2 \le RP_1$
Customer 2	$RP_2 < P_1$	P_1	$2P_1$	$2P_1$
	$P_1 \le RP_2 < P_2$	$2P_1$	$2P_1$	$2P_1$
	$P_2 \le RP_2$	$2P_1$	$2P_1$	$2P_1$

Multiplying the (joint) probabilities from Tables B4.2b and B4.4a - B4.4c by their respective revenues in Tables B4.6a – B4.6c, taking the summation of the products and subtracting the total cost gives the expected profit for Case 3:

$$E[\Pi_{Case3}] = Prob_{d1}\left(0q_1 + P_1\left(q_2 + q_3\right)\right) \\ + \left(1 - Prob_{d1}\right)\left(0r_1 + P_1\left(r_2 + r_3 + r_4 + r_7 + r_{10} + r_{19}\right)\right) \tag{B4.3}$$

$$+\left(1-Prob_{d1}\right)\left(2P_1\left(1-\left(r_1+r_2+r_3+r_4+r_7+r_{10}+r_{19}\right)\right)\right)-2c\,.$$

Substitution of the (joint) probabilities from Tables B4.2a and B4.3a - B4.3c into (B4.3) and simplifying gives:

$$E[\Pi_{Case3}] = P_1\overline{F}_1\left(3-2Prob_{d1}-\overline{F}_1^{\,2}\left(1-Prob_{d1}\right)\right)-2c\,. \qquad \text{(B4.4)}$$

For Case 4, $Q_1 = 3, Q_2 = 0$. For this inventory decision, the possible revenues that can be realized are shown in Tables B4.7a – B4.7d.

Table B4.7a. Possible Revenues for Example 4 for Case 4, $d_1 = 1$

Range	$RP_1 < P_1$	$P_1 \le RP_2 < P_2$	$P_2 \le RP_2$
Revenue	0	P_1	P_1

Table B4.7b. Possible Revenues for Example 4 for Case 4, $d_2 = 3$, $RP_3 < P_1$.

		Customer 1		
		$RP_1 < P_1$	$P_1 \le RP_1 < P_2$	$P_2 \le RP_1$
Customer 2	$RP_2 < P_1$	0	P_1	P_1
	$P_1 \le RP_2 < P_2$	P_1	$2P_1$	$2P_1$
	$P_2 \le RP_2$	P_1	$2P_1$	$2P_1$

Table B4.7c. Possible Revenues for Example 4 for Case 4, $d_2 = 3$, $P_1 < RP_3 < P_2$.

		Customer 1		
		$RP_1 < P_1$	$P_1 \le RP_1 < P_2$	$P_2 \le RP_1$
Customer 2	$RP_2 < P_1$	P_1	$2P_1$	$2P_1$
	$P_1 \le RP_2 < P_2$	$2P_1$	$3P_1$	$3P_1$
	$P_2 \le RP_2$	$2P_1$	$3P_1$	$3P_1$

Table B4.7d. Possible Revenues for Example 4 for Case 4, $d_2 = 3$, $P_2 < RP_3$.

		Customer 1		
		$RP_1 < P_1$	$P_1 \le RP_1 < P_2$	$P_2 \le RP_1$
Customer 2	$RP_2 < P_1$	P_1	$2P_1$	$2P_1$
	$P_1 \le RP_2 < P_2$	$2P_1$	$3P_1$	$3P_1$
	$P_2 \le RP_2$	$2P_1$	$3P_1$	$3P_1$

Multiplying the (joint) probabilities from Tables B4.2b and B4.4a - B4.4c by their respective revenues in Tables B4.7a – B4.7d, taking the summation of the products and subtracting the total cost gives the expected profit for Case 4:

$$\begin{aligned} E[\Pi_{Case4}] &= Prob_{d1}\left(0q_1 + P_1(q_2 + q_3)\right) \\ &+ (1 - Prob_{d1})\left(0r_1 + P_1(r_2 + r_3 + r_4 + r_7 + r_{10} + r_{19})\right) \\ &+ (1 - Prob_{d1})\left(2P_1(r_5 + r_6 + r_8 + r_9 + r_{11} + r_{12} + r_{13} + r_{16} + r_{20} + r_{21} + r_{22} + r_{25})\right) \\ &+ (1 - Prob_{d1})\left(3P_1(r_{14} + r_{15} + r_{17} + r_{18} + r_{23} + r_{24} + r_{26} + r_{27})\right) - 3c. \end{aligned} \tag{B4.5}$$

Substitution of the (joint) probabilities from Tables B4.2a and B4.3a - B4.3c into (B4.5) and simplifying gives:

$$E[\Pi_{Case4}] = P_1\overline{F_1}(3 - 2Prob_{d1}) - 3c. \tag{B4.6}$$

For Case 5, $Q_1 = 0, Q_2 = 1$. For this inventory decision, the possible revenues that can be realized are shown in Tables B4.8a – B4.8d.

Table B4.8a. Possible Revenues for Example 4 for Case 5, $d_1 = 1$

Range	$RP_1 < P_1$	$P_1 \le RP_2 < P_2$	$P_2 \le RP_2$
Revenue	0	0	P_2

Table B4.8b. Possible Revenues for Example 4 for Case 5, $d_2 = 3$, $RP_3 < P_1$.

		Customer 1		
		$RP_1 < P_1$	$P_1 \le RP_1 < P_2$	$P_2 \le RP_1$
Customer 2	$RP_2 < P_1$	0	0	P_2
	$P_1 \le RP_2 < P_2$	0	0	P_2
	$P_2 \le RP_2$	P_2	P_2	P_2

Table B4.8c. Possible Revenues for Example 4 for Case 5, $d_2 = 3$, $P_1 < RP_3 < P_2$.

		Customer 1		
		$RP_1 < P_1$	$P_1 \le RP_1 < P_2$	$P_2 \le RP_1$
Customer 2	$RP_2 < P_1$	0	0	P_2
	$P_1 \le RP_2 < P_2$	0	0	P_2
	$P_2 \le RP_2$	P_2	P_2	P_2

Table B4.8d. Possible Revenues for Example 4 for Case 5, $d_2 = 3$, $P_2 < RP_3$.

		Customer 1		
		$RP_1 < P_1$	$P_1 \le RP_1 < P_2$	$P_2 \le RP_1$
Customer 2	$RP_2 < P_1$	P_2	P_2	P_2
	$P_1 \le RP_2 < P_2$	P_2	P_2	P_2
	$P_2 \le RP_2$	P_2	P_2	P_2

Multiplying the (joint) probabilities from Tables B4.2b and B4.4a - B4.4c by their respective revenues in Tables B4.8a – B4.8d, taking the summation of the products and subtracting the total cost gives the expected profit for Case 5:

$$E[\Pi_{Case5}] = Prob_{d1}\left(0(q_1+q_2)+P_1q_3\right)$$

$$+\left(1-Prob_{d1}\right)\left(0(r_1+r_2+r_4+r_5+r_{10}+r_{11}+r_{13}+r_{14})\right) \qquad \text{(B4.7)}$$

$$+\left(1-Prob_{d1}\right)\left(P_2\left(1-(r_1+r_2+r_4+r_5+r_{10}+r_{11}+r_{13}+r_{14})\right)\right)-c\,.$$

Substitution of the (joint) probabilities from Tables B4.2a and B4.3a - B4.3c into (B4.7) and simplifying gives:

$$E[\Pi_{Case5}] = P_2\overline{F_2}\left(3 - 2Prob_{d1} - (1 - Prob_{d1})\left(3\overline{F_2} - \overline{F_2}^{\,2}\right)\right) - c. \quad \text{(B4.8)}$$

For Case 6, $Q_1 = 1, Q_2 = 1$. For this inventory decision, the possible revenues that can be realized are shown in Tables B4.9a – B4.9d.

Table B4.9a. Possible Revenues for Example 4 for Case 6, $d_1 = 1$

Range	$RP_1 < P_1$	$P_1 \le RP_2 < P_2$	$P_2 \le RP_2$
Revenue	0	P_1	P_1

Table B4.9b. Possible Revenues for Example 4 for Case 6, $d_2 = 3$, $RP_3 < P_1$.

		Customer 1		
		$RP_1 < P_1$	$P_1 \le RP_1 < P_2$	$P_2 \le RP_1$
Customer 2	$RP_2 < P_1$	0	P_1	P_1
	$P_1 \le RP_2 < P_2$	P_1	P_1	P_1
	$P_2 \le RP_2$	P_1	$P_1 + P_2$	$P_1 + P_2$

Table B4.9c. Possible Revenues for Example 4 for Case 6, $d_2 = 3$, $P_1 < RP_3 < P_2$.

		Customer 1		
		$RP_1 < P_1$	$P_1 \le RP_1 < P_2$	$P_2 \le RP_1$
Customer 2	$RP_2 < P_1$	P_1	P_1	P_1
	$P_1 \le RP_2 < P_2$	P_1	P_1	P_1
	$P_2 \le RP_2$	P_1	$P_1 + P_2$	$P_1 + P_2$

Table B4.9d. Possible Revenues for Example 4 for Case 6, $d_2 = 3$, $P_2 < RP_3$.

		Customer 1		
		$RP_1 < P_1$	$P_1 \le RP_1 < P_2$	$P_2 \le RP_1$
Customer 2	$RP_2 < P_1$	P_1	$P_1 + P_2$	$P_1 + P_2$
	$P_1 \le RP_2 < P_2$	$P_1 + P_2$	$P_1 + P_2$	$P_1 + P_2$
	$P_2 \le RP_2$	$P_1 + P_2$	$P_1 + P_2$	$P_1 + P_2$

Multiplying the (joint) probabilities from Tables B4.2b and B4.4a - B4.4c by their respective revenues in Tables B4.9a – B4.9d, taking the summation of the products and subtracting the total cost gives the expected profit for Case 6:

$$\begin{aligned} E[\Pi_{Case6}] &= Prob_{d1}\left(0q_1 + P_1\left(q_2 + q_3\right)\right) \\ &+ \left(1 - Prob_{d1}\right)\left(P_1\left(r_2 + r_3 + r_4 + r_5 + r_6 + r_7 + r_{10} + r_{11} + r_{12} + r_{13} + r_{14} + r_{15} + r_{16} + r_{19}\right)\right) \\ &+ \left(1 - Prob_{d1}\right)\left(P_1 + P_2\right)\left(r_8 + r_9 + r_{17} + r_{18} + r_{20} + r_{21} + r_{22} + r_{23} + r_{24} + r_{25} + r_{26} + r_{27}\right) \\ &- 2c\,. \end{aligned} \tag{B4.9}$$

Substitution of the (joint) probabilities from Tables B4.2a and B4.3a - B4.3c into (B4.9) and simplifying gives:

$$\begin{aligned} E[\Pi_{Case6}] &= \left(P_2\overline{F_2} - P_1\overline{F_1}\right)\left(1 - Prob_{d1}\right)\left(3\overline{F_1} - \overline{F_1}^{\,2}\right) \\ &+ P_1\overline{F_1}\left(3 - 2Prob_{d1}\right) - P_2\overline{F_1}\,\overline{F_2}^{\,2}\left(1 - Prob_{d1}\right) - 2c\,. \end{aligned} \tag{B4.10}$$

For Case 7, $Q_1 = 2, Q_2 = 1$. For this inventory decision, the possible revenues that can be realized are shown in Tables B4.10a – B4.10d.

Table B4.10a. Possible Revenues for Example 4 for Case 7, $d_1 = 1$

Range	$RP_1 < P_1$	$P_1 \le RP_2 < P_2$	$P_2 \le RP_2$
Revenue	0	P_1	P_1

Table B4.10b. Possible Revenues for Example 4 for Case 7, $d_2 = 3$, $RP_3 < P_1$.

		Customer 1		
		$RP_1 < P_1$	$P_1 \le RP_1 < P_2$	$P_2 \le RP_1$
Customer 2	$RP_2 < P_1$	0	P_1	P_1
	$P_1 \le RP_2 < P_2$	P_1	$2P_1$	$2P_1$
	$P_2 \le RP_2$	P_1	$2P_1$	$2P_1$

Table B4.10c. Possible Revenues for Example 4 for Case 7, $d_2 = 3$, $P_1 < RP_3 < P_2$.

		Customer 1		
		$RP_1 < P_1$	$P_1 \le RP_1 < P_2$	$P_2 \le RP_1$
Customer 2	$RP_2 < P_1$	P_1	$2P_1$	$2P_1$
	$P_1 \le RP_2 < P_2$	$2P_1$	$2P_1$	$2P_1$
	$P_2 \le RP_2$	$2P_1$	$2P_1$	$2P_1$

Table B4.10d. Possible Revenues for Example 4 for Case 7, $d_2 = 3$, $P_2 < RP_3$.

		Customer 1		
		$RP_1 < P_1$	$P_1 \le RP_1 < P_2$	$P_2 \le RP_1$
Customer 2	$RP_2 < P_1$	$2P_1$	$2P_1$	$2P_1$
	$P_1 \le RP_2 < P_2$	$2P_1$	$2P_1 + P_2$	$2P_1 + P_2$
	$P_2 \le RP_2$	$2P_1$	$2P_1 + P_2$	$2P_1 + P_2$

Multiplying the (joint) probabilities from Tables B4.2b and B4.4a - B4.4c by their respective revenues in Tables B4.10a – B4.10d, taking the summation of the products and subtracting the total cost gives the expected profit for Case 7:

$$\begin{aligned} E[\Pi_{Case7}] = {} & Prob_{d1}\left(0q_1 + P_1\left(q_2 + q_3\right)\right) \\ & + \left(1 - Prob_{d1}\right)\left(0r_1 + P_1\left(r_2 + r_3 + r_4 + r_7 + r_{10} + r_{19}\right)\right) \\ & + \left(1 - Prob_{d1}\right)\left(2P_1\right)\left(1 - \left(r_2 + r_3 + r_4 + r_7 + r_{10} + r_{19} + r_{23} + r_{24} + r_{26} + r_{27}\right)\right) \\ & + \left(1 - Prob_{d1}\right)\left(2P_1 + P_2\right)\left(r_{23} + r_{24} + r_{26} + r_{27}\right) - 3c. \end{aligned} \tag{B4.11}$$

Substitution of the (joint) probabilities from Tables B4.2a and B4.3a - B4.3c into (B4.11) and simplifying gives:

$$E[\Pi_{Case7}] = \left(P_2\overline{F_2} - P_1\overline{F_1}\right)\left(\overline{F_1}^{\,2}\right)\left(1 - Prob_{d1}\right) + P_1\overline{F_1}\left(3 - 2Prob_{d1}\right) - 3c. \tag{B4.12}$$

For Case 8, $Q_1 = 0, Q_2 = 2$. For this inventory decision, the possible revenues that can be realized are shown in Tables B4.11a – B4.11d.

Table B4.11a. Possible Revenues for Example 4 for Case 8, $d_1 = 1$

Range	$RP_1 < P_1$	$P_1 \le RP_2 < P_2$	$P_2 \le RP_2$
Revenue	0	0	P_2

Table B4.11b. Possible Revenues for Example 4 for Case 8, $d_2 = 3$, $RP_3 < P_1$.

		Customer 1		
		$RP_1 < P_1$	$P_1 \le RP_1 < P_2$	$P_2 \le RP_1$
Customer 2	$RP_2 < P_1$	0	0	P_2
	$P_1 \le RP_2 < P_2$	0	0	P_2
	$P_2 \le RP_2$	P_2	P_2	$2P_2$

Table B4.11c. Possible Revenues for Example 4 for Case 8, $d_2 = 3$, $P_1 < RP_3 < P_2$.

		Customer 1		
		$RP_1 < P_1$	$P_1 \le RP_1 < P_2$	$P_2 \le RP_1$
Customer 2	$RP_2 < P_1$	0	0	P_2
	$P_1 \le RP_2 < P_2$	0	0	P_2
	$P_2 \le RP_2$	P_2	P_2	$2P_2$

Table B4.11d. Possible Revenues for Example 4 for Case 8, $d_2 = 3$, $P_2 < RP_3$.

		Customer 1		
		$RP_1 < P_1$	$P_1 \le RP_1 < P_2$	$P_2 \le RP_1$
Customer 2	$RP_2 < P_1$	P_2	P_2	$2P_2$
	$P_1 \le RP_2 < P_2$	P_2	P_2	$2P_2$
	$P_2 \le RP_2$	$2P_2$	$2P_2$	$2P_2$

Multiplying the (joint) probabilities from Tables B4.2b and B4.4a - B4.4c by their respective revenues in Tables B4.11a – B4.11d, taking the summation of the products and subtracting the total cost gives the expected profit for Case 8:

$$E[\Pi_{Case8}] = Prob_{d1}\left(0(q_1+q_2)+P_1q_3\right)$$
$$+\left(1-Prob_{d1}\right)\left(0\left(r_1+r_2+r_4+r_5+r_{10}+r_{11}+r_{13}+r_{14}+r_{15}\right)\right)$$
$$+\left(1-Prob_{d1}\right)\left(P_2\left(1-\left(r_3+r_6+r_7+r_8+r_{12}+r_{15}+r_{16}+r_{17}+r_{19}+r_{20}+r_{22}+r_{23}\right)\right)\right)$$
$$+\left(1-Prob_{d1}\right)\left(2P_2\left(r_9+r_{18}+r_{21}+r_{24}+r_{25}+r_{26}+r_{27}\right)\right)-2c. \tag{B4.13}$$

Substitution of the (joint) probabilities from Tables B4.2a and B4.3a - B4.3c into (B4.13) and simplifying gives:

$$E[\Pi_{Case8}] = P_2\overline{F_2}\left(3-2Prob_{d1}-\overline{F_2}^{\,2}\left(1-Prob_{d1}\right)\right)-2c. \tag{B4.14}$$

For Case 9, $Q_1=1, Q_2=2$. For this inventory decision, the possible revenues that can be realized are shown in Tables B4.12a – B4.12d.

Table B4.12a. Possible Revenues for Example 4 for Case 9, $d_1=1$

Range	$RP_1 < P_1$	$P_1 \le RP_2 < P_2$	$P_2 \le RP_2$
Revenue	0	P_1	P_1

Table B4.12b. Possible Revenues for Example 4 for Case 9, $d_2=3$, $RP_3 < P_1$.

		Customer 1		
		$RP_1 < P_1$	$P_1 \le RP_1 < P_2$	$P_2 \le RP_1$
Customer 2	$RP_2 < P_1$	0	P_1	P_1
	$P_1 \le RP_2 < P_2$	P_1	P_1	P_1+P_2
	$P_2 \le RP_2$	P_1	P_1	P_1+P_2

Table B4.12c. Possible Revenues for Example 4 for Case 9, $d_2=3$, $P_1 < RP_3 < P_2$.

		Customer 1		
		$RP_1 < P_1$	$P_1 \le RP_1 < P_2$	$P_2 \le RP_1$
Customer 2	$RP_2 < P_1$	P_1	P_1	P_1
	$P_1 \le RP_2 < P_2$	P_1	P_1	P_1+P_2
	$P_2 \le RP_2$	P_1	P_1	P_1+P_2

Table B4.12d. Possible Revenues for Example 4 for Case 9, $d_2 = 3$, $P_2 < RP_3$.

		Customer 1		
		$RP_1 < P_1$	$P_1 \le RP_1 < P_2$	$P_2 \le RP_1$
Customer 2	$RP_2 < P_1$	P_1	$P_1 + P_2$	$P_1 + P_2$
	$P_1 \le RP_2 < P_2$	$P_1 + P_2$	$P_1 + P_2$	$P_1 + 2P_2$
	$P_2 \le RP_2$	$P_1 + P_2$	$P_1 + P_2$	$P_1 + 2P_2$

Multiplying the (joint) probabilities from Tables B4.2b and B4.4a - B4.4c by their respective revenues in Tables B4.12a – B4.12d, taking the summation of the products and subtracting the total cost gives the expected profit for Case 9:

$$E[\Pi_{Case9}] = Prob_{d1}\left(0q_1 + P_1\left(q_2 + q_3\right)\right)$$
$$+\left(1 - Prob_{d1}\right)\left(P_1\left(r_2 + r_3 + r_4 + r_5 + r_7 + r_{10} + r_{11} + r_{12} + r_{13} + r_{14} + r_{16} + r_{17} + r_{19}\right)\right)$$
$$+\left(1 - Prob_{d1}\right)\left(P_1 + P_2\right)\left(r_6 + r_8 + r_9 + r_{15} + r_{18} + r_{20} + r_{21} + r_{22} + r_{23} + r_{25} + r_{26}\right) \quad \text{(B4.15)}$$
$$+\left(1 - Prob_{d1}\right)\left(P_1 + 2P_2\right)\left(r_{26} + r_{27}\right) - 3c.$$

Substitution of the (joint) probabilities from Tables B4.2a and B4.3a - B4.3c into (B4.15) and simplifying gives:

$$E[\Pi_{Case9}] = \left(P_2\overline{F}_2 - P_1\overline{F}_1\right)\left(1 - Prob_{d1}\right)\left(3\overline{F}_1 - \overline{F}_1^{\,2}\right) + P_1\overline{F}_1\left(3 - 2Prob_{d1}\right) - 3c. \quad \text{(B4.16)}$$

For Case 10, $Q_1 = 0, Q_2 = 3$. For this inventory decision, the possible revenues that can be realized are shown in Tables B4.13a – B4.13d.

Table B4.13a. Possible Revenues for Example 4 for Case 10, $d_1 = 1$

Range	$RP_1 < P_1$	$P_1 \le RP_2 < P_2$	$P_2 \le RP_2$
Revenue	0	0	P_2

Table B4.13b. Possible Revenues for Example 4 for Case 10, $d_2 = 3$, $RP_3 < P_1$.

		Customer 1		
		$RP_1 < P_1$	$P_1 \le RP_1 < P_2$	$P_2 \le RP_1$
Customer 2	$RP_2 < P_1$	0	0	P_2
	$P_1 \le RP_2 < P_2$	0	0	P_2
	$P_2 \le RP_2$	P_2	P_2	$2P_2$

Table B4.13c. Possible Revenues for Example 4 for Case 10, $d_2 = 3$, $P_1 < RP_3 < P_2$.

		Customer 1		
		$RP_1 < P_1$	$P_1 \le RP_1 < P_2$	$P_2 \le RP_1$
Customer 2	$RP_2 < P_1$	0	0	P_2
	$P_1 \le RP_2 < P_2$	0	0	P_2
	$P_2 \le RP_2$	P_2	P_2	$2P_2$

Table B4.13d. Possible Revenues for Example 4 for Case 10, $d_2 = 3$, $P_2 < RP_3$.

		Customer 1		
		$RP_1 < P_1$	$P_1 \le RP_1 < P_2$	$P_2 \le RP_1$
Customer 2	$RP_2 < P_1$	P_2	P_2	$2P_2$
	$P_1 \le RP_2 < P_2$	P_2	P_2	$2P_2$
	$P_2 \le RP_2$	$2P_2$	$2P_2$	$3P_2$

Multiplying the (joint) probabilities from Tables B4.2b and B4.4a - B4.4c by their respective revenues in Tables B4.13a – B4.13d, taking the summation of the products and subtracting the total cost gives the expected profit for Case 10:

$$\begin{aligned} E[\Pi_{Case10}] = {} & Prob_{d1}\left(0(q_1+q_2)+P_1q_3\right) \\ & +\left(1-Prob_{d1}\right)\left(0\left(r_1+r_2+r_4+r_5+r_{10}+r_{11}+r_{13}+r_{14}\right)\right) \\ & +\left(1-Prob_{d1}\right)\left(P_2\left(1-\left(r_3+r_6+r_7+r_8+r_{12}+r_{15}+r_{16}+r_{17}+r_{19}+r_{20}+r_{22}+r_{23}\right)\right)\right) \\ & +\left(1-Prob_{d1}\right)\left(2P_2\left(r_9+r_{18}+r_{21}+r_{24}+r_{25}+r_{26}\right)+3P_2r_{27}\right)-3c\,. \end{aligned} \quad \text{(B4.17)}$$

Substitution of the (joint) probabilities from Tables B4.2a and B4.3a - B4.3c into (B4.17) and simplifying gives:

$$E[\Pi_{Case10}] = P_2 \overline{F_2}\left(3 - 2Prob_{d1}\right) - 3c\ . \tag{B4.18}$$

REFERENCES

Becker, G. (1991) "A Note on Restaurant Pricing and Other Examples of Social Influences on Price." *The Journal of Political Economy* 99 (5), pp. 1109 – 1116.

Dana, J. (1999) "Equilibrium Price Dispersion under Demand Uncertainty: the Roles of Costly Capacity and Market Structure." *Rand Journal of Economics* 30 (4), pp. 632 – 660.

Dana, J. and Petruzzi, N. (2001) "Note: The Newsvendor Model with Endogenous Demand." *Management Science* 47 (11), pp. 1488 – 1497.

DeSerpa, A. And Faith, R. (1996) "'Bru-u-u-uce:' The Simple Economics of Mob Goods." *Public Choice* 89, pp. 77 – 91.

Fort, R. (2004) "Inelastic Sports Pricing." *Managerial and Decision Economics* 25, pp. 87 – 94.

Karlin, S. and Carr, C. R. (1962) "Prices and Optimal Inventory Policy." Ch. 10 in *Studies in Applied Probability and Management Science,* K. Arrow, S. Karlin and H. Scarf (eds.), Stanford University Press, Stanford, CA, pp. 159 – 172.

Kunreuther, H. and Richard, J. F. (1971) "Optimal Pricing and Inventory Decisions for Non-Seasonal Items." *Econometrica* 39 (1), pp. 173 – 175.

Marburger, D. (1997) "Optimal Ticket Pricing for Performance Goods." *Managerial and Decision Economics* 18, pp. 375 – 381.

Mills, E. S. (1959) "Uncertainty and Price Theory." *Quarterly Journal of Economics* 73, pp. 117 – 130.

Nevins, A. J. (1966) "Some Effects of Uncertainty: Simulation of a Model of Price." *Quarterly Journal of Economics* 80, pp. 73 – 87.

Petruzzi, N. and Dada, M. (1999) "Pricing and the Newsvendor Problem: A Review with Extensions." *Operations Research* 47 (2), pp. 183 – 194.

Rosen S. and Rosenfield, A. M. (1997) "Ticket Pricing." *Journal of Law and Economics* 40 (2), pp. 351 – 376.

Samuelson, W. F. and Marks, S. G. (2010) "Managerial Economics, 6th Ed." John Wiley and Sons, Hoboken, NJ, p. 76.

Thomas, L. J. (1974) "Price and Production Decisions with Random Demand." *Operations Research* 22 (3), pp. 513 – 518.

Whitin, T. M. (1955) "Inventory Control and Price Theory." *Management Science* 2, pp. 61 – 68.

Young, L. (1978) "Price, Inventory and the Structure of Uncertain Demand." *New Zealand Operations Research* 6 (2), pp. 157 – 177.

Zhan, R. L. and Shen, Z. M. (2005) "Newsvendor Problem with Pricing: Properties, Algorithms, and Simulation." In *Proceedings of the 2005 Winter Simulation Conference*, M. E. Kuhl, N. M. Steiger, F. B. Armstrong and J. A. Joines (eds.), pp. 1743 – 1748.

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REFEREED JOURNAL PUBLICATIONS

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