GKM theory of rationally smooth group embeddings

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Abstract

This thesis is concerned with the study of rationally smooth standard group embeddings. We prove that the equivariant cohomology of any of these compactifications can be described, via GKM-theory, as certain ring of piecewise polynomial functions. Moreover, building on previous work of Renner ([R3]), we show that the embeddings under consideration come equipped with both a canonical decomposition into rational cells and a filtration by equivariantly formal closed subvarieties.

The techniques developed in this monograph supply a method for constructing free module generators on the equivariant cohomology of $\mathbb{Q}$-filtrable GKM-varieties. Our findings extend the earlier work of Arabia ([Ar]) and Guillemin-Kogan ([GK]) on equivariant characteristic classes.

In the last two chapters of this work, inspired by the papers of Brion ([Br4]) and Renner ([R7]), we compute explicitly the GKM characters associated to any standard group embedding. Our major result describes the equivariant cohomology of rationally smooth standard group embeddings in terms of roots, idempotents, and underlying monoid data.

Keywords: Equivariant cohomology, GKM theory, rationally smooth, algebraic monoids, group embeddings, filtrable spaces, equivariant Euler classes, $\mathcal{J}$-irreducible monoids, toric varieties.

To my parents

Bertha and Faustino

and my sisters

Lourdes and Susan.

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Richard Paul Gonzales Vilcarromero.

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Introduction

It has been proved that a smooth projective variety, upon which an algebraic torus acts with finitely many fixed points, can be decomposed into invariant affine cells [BB1]. This method for breaking down a space into pieces, also known as **BB-theory**, allows us to compute important topological invariants, especially **Betti numbers**. On the other hand, Borel has developed an algebraic method, **equivariant cohomology**, to study spaces equipped with group actions. Borel's method has dramatically deepened our understanding of how topology interacts with group theory. The interplay between these two methods is of fundamental importance for the theory of group embeddings.

A **group embedding** X is a compactification of an algebraic group G endowed with a $G \times G$-action that extends the natural two-sided action of G on itself. It is worth emphasizing that this is a generalization of the notion of **toric varieties**, objects that have been studied extensively in algebraic geometry for nearly forty years ([D, F, DP, BDP, Cox]). One can obtain substantial information about the topology of a group embedding by restricting one's attention to the induced action of a maximal torus T of G. Renner has recently developed a large part of the theory of **rationally smooth** standard group embeddings ([R3, R4, R5, R6]). These objects are characterized by the fact that they satisfy **local Poincaré duality** (Definition 2.1.1). Furthermore, one can find a canonical cellular decomposition (like the cells we obtain from BB-theory) for such spaces. Indeed, it turns out that they can be decomposed into **rational cells** (Definition 2.1.8). This is quite relevant since it allows us to compute topological invariants (e.g. Betti numbers) for standard group embeddings (Corollary 2.3.3). On the other hand, **GKM theory** makes it possible to describe the cohomology of group embeddings in terms of T-fixed points and weighted T-invariant curves. In fact, it is an ideal method for studying group embeddings. For a comprehensive overview of why this should be the case, see [Br2, CS, EG1, GKM, GZ, U, VV].

The main purpose of GKM theory is to identify the image of the functorial map

$$i^* : H_T^*(X) \to H_T^*(X^T),$$

assuming certain technical conditions are met. These conditions can be verified explicitly for a large, interesting and growing class of group embeddings. In particular, using the theory of reductive monoids, we can identify explicitly and combinatorially the salient GKM data (T-fixed points and weighted T-curves) that are needed to quantify the sought-after image of i^* (Theorem 4.3.4).

It was shown by Renner in [R5] that there is a useful combinatorial characterization of rationally smooth embeddings. These objects constitute a much larger class of embeddings than the smooth ones. In fact, most of the techniques used in the study of smooth varieties have a natural extension to the rationally smooth case, e.g. BB-decomposition, GKM theory, etc.

This monograph has three main objectives. The primary objective is to verify that GKM-theory is directly applicable in the study of rationally smooth standard group embeddings. Previously, this has been carried out only in the case of smooth embeddings. The second goal is to describe the GKM-graph of a rationally smooth standard group embedding and use it to calculate its equivariant cohomology. The final aim is to generalize the aforementioned techniques to the study of more general spaces, in particular to spaces which admit a decomposition into rationally smooth cells ($\mathbb{Q}$-filtrable varieties, Definition 2.3.4). We develop the necessary topological framework to undertake these tasks. Furthermore, we provide a complete description of the equivariant cohomology of any rationally smooth standard group embedding; thus increasing the effectiveness of GKM theory as a tool in embedding theory.

Let $K_T^*(X)$ be the equivariant K-theory of X, that is, the Grothendieck group of isomorphism classes of T-equivariant (algebraic) vector bundles over X.

The following theorem was inspired on the work of Atiyah ([At3]), Hsiang ([Hs]) and Chang-Skjelbred ([CS]). Its cohomological version is one of the fundamental results in GKM-theory.

GKM Theorem ([Br2],[VV], [U]). *Let X be a smooth complex projective variety with a torus action containing only a finite number of fixed points and T-invariant curves. Then $K_T^*(X;\mathbb{C})$ is a free $K_T^*(pt;\mathbb{C})$-module of rank $|X^T|$. Moreover,*

$$K_T(X;\mathbb{C}) \simeq \{(f_1,\dots,f_n) \in \bigoplus_{x\in X^T} R_x \mid f_i \cong f_j \operatorname{mod}(1-e^{-\chi_{i,j}})\},$$

where R_x is a copy of the representation ring of the torus $R[T]$, and x_i, x_j are the two fixed points in the closure of the one dimensional orbit $C_{i,j}$ and $\chi_{i,j}$ is the character associated to $C_{i,j}$.

GKM theory for equivariant Chow rings was implemented by Brion ([Br2]), building on previous work of Edidin and Graham ([EG1]). Later on, Vistoli and Vezzosi ([VV]) proved an analogue of GKM theory for the equivariant algebraic K-theory of smooth projective varieties. Brion ([Br4]) had also described the required GKM data for a large class of smooth group compactifications, namely, regular embeddings ([BDP]). Uma ([U]) finally showed that the equivariant K-theory ring of a regular embedding can be understood as a generalized Stanley-Reisner ring.

On the other extreme of the spectrum, Rosu and Knutson ([RK]), using a sheaf-theoretical approach, sucessfully applied GKM-theory to the study of smooth manifolds and topological equivariant K-theory.

The approach taken in this monograph differs from the ones in the literature at two major points. First, it is more elementary. We work mostly with rational singular cohomology, avoiding the use of sofisticated sheaf-theoretical devices whenever possible. Secondly, we use a different cellular decomposition. Our major technical tool here is the notion of rational cell (Definition 2.1.8). The advantage of this concept relies on the fact that it allows for an equal treatment of singular and smooth varieties.

To summarize, in this monograph we develop the appropriate setting in which a cohomological version of the GKM Theorem holds for standard group embeddings, spaces that are, for the most part, singular. Most importantly, we identify explicitly the salient GKM-data (i.e. fixed points and invariant curves), and use it to provide a complete description of the equivariant cohomology ring of any rationally smooth standard group embedding (Theorem 4.3.4). Our methods also yield a recipe for finding a suitable set of module generators in terms of equivariant Euler classes (Theorem 2.6.9).

Thesis Organization

Chapter 1: This chapter is basically a survey of the well-established concepts and definitions that are relevant to this monograph. The chapter starts with a quick overview of Equivariant Cohomology, using as a guide the classical references of Borel ([Bo1]) and Quillen ([Q]). Next, the most important Localization Theorems in topological transformation groups are stated ([Hs]). The core of this chapter is dedicated to GKM-theory and the notions of T-skeletal actions and GKM-varieties ([GKM]). Finally, the equivariant cohomology of flag varieties and simplicial toric varieties is studied.

Chapter 2: Here we devote ourselves to the study of rational cells, our basic building blocks. After describing their most remarkable topological properties, we define $\mathbb{Q}$-filtrable varieties, spaces that come equipped with a paving by rational cells. Sections 2.1, 2.3 and 2.6 contain new developments. This chapter concludes by supplying a method for building canonical free module generators on the equivariant cohomology of any $\mathbb{Q}$-filtrable GKM variety (Theorem 2.6.9). Our findings extend the earlier works of Arabia([Ar]), Brion ([Br5]), and Guillemin-Kogan ([GK]).

Chapter 3: This chapter begins the study of Standard Group Embeddings (Definition 3.2.1). We show that they are $T \times T$-skeletal varieties. Even more so, we describe the fixed points and invariant curves in terms of the Renner monoid and certain roots. Notably, our computations do not depend on any special property of the reductive monoid in consideration. We conclude this Chapter by showing that rationally smooth standard group embeddings have also a canonical $\mathbb{Q}$-filtration (Theorem 3.2.13). That is to say, they are GKM-varieties as well. The explicit calculation of the $T \times T$-characters is done in the next chapter. Most results here are new, notably, Theorem 3.2.3, Theorem 3.2.7, Theorem 3.2.8 and Theorem 3.2.13.

Chapter 4: The most important chapter of this thesis. In the first two sections, we compute, in very explicit terms, all the GKM-characters associated to the $T \times T$-invariant curves of a standard group embedding. Once again, these calculations turn out to be independent of any particular property of the underlying reductive monoid. Moreover, we classify these curves and characters in terms of combinatorial monoid data. In the second half of this chapter, we specify our findings to the case of rationally smooth standard embeddings. Our main theorem, Theorem 4.3.4, gives the ultimate description of the equivariant cohomology of rationally smooth standard embeddings in terms of roots, idempotents, and the Renner monoid. All the results in this final chapter, with very few exceptions, are new. The most remarkable results are Theorem 4.1.1, Theorem 4.3.4, Corollary 4.3.5 and Theorem 4.3.6. As a closing remark, we illustrate the theory thus developed with some particular examples in Section 4.4.

Chapter 1

Equivariant Cohomology

This chapter is essentially a recollection of the well-established concepts and definitions that are relevant to this monograph. The classical references are [Bo1], [Q], [CS], [Hs], [GKM], [AP] and [Br3].

1.1 The Borel construction

Let G be a compact Lie group and let X be a G-space, that is, a topological space endowed with a continuous action of G. For the purposes of this section, all spaces are assumed to be Hausdorff and paracompact.

Let $G \hookrightarrow EG \to BG$ be a universal bundle for G. Consider the diagonal action of G on $EG \times X$ and form the associated fiber space $X_G := (EG \times X)/G$ over BG with typical fiber X. It is crucial to notice that although G may not act freely on X, it acts freely on $EG \times X$, for it does so on EG. Hence, in the following diagram

$$X \hookrightarrow X_G \xrightarrow{p_X} BG \,,$$

the map p_X, induced by the canonical projection $EG \times X \to EG$, is a fibration. It is usual to denote X_G as $EG \times_G X$ too, so we will use both notations alike.

The **equivariant cohomology** of the G-space X is defined by

$$H^*_G(X;\Lambda) := H^*(X_G;\Lambda)$$

where by $H^*(-;\Lambda)$ we mean singular cohomology with coefficients in the commutative ring Λ. This construction was introduced by Borel in [Bo1]. Notice that $H^*_G(X;\Lambda)$ is, via p^*_X, an algebra over $H^*_G(pt;\Lambda)$.

Throughout this monograph cohomology is considered with rational coefficients. So, for simplicity, $H^*_G(X;\mathbb{Q})$ will be written as $H^*_G(X)$. When $X = pt$, it is usual to write H^*_G instead of $H^*_G(pt)$.

It can be shown that $H^*_G(X)$ is independent of the choice of universal bundle $EG \to BG$, so that equivariant cohomology becomes a contravariant functor from the category of pairs (G, X) to the category of graded anti-commutative Λ-algebras. See [Bo1] and [Q] for details.

Example 1.1.1. Let $T = (S^1)^m$ be a compact torus. Then $BT = (\mathbb{C}\mathrm{P}^\infty)^m$, and consequently $H^*_T(pt) = H^*(BT) = \mathbb{Q}[x_1, \ldots, x_m]$, where $deg(x_i) = 2$. A more intrinsic description of $H^*_T(pt)$ is as follows. Denote by $\Xi(T)$ the character group of T consisting of all continuous group homomorphisms $T \to S^1$. Any $\chi \in \Xi(T)$ defines a one-dimensional complex representation of T with space $\mathbb{C}_\chi$. Here T acts on $\mathbb{C}_\chi$ via $t \cdot z := \chi(t)z$. Consider the associated complex line bundle

$$L(\chi) := (E_T \times_T \mathbb{C}_\chi \to BT)$$

and its first Chern class $c(\chi) \in H^2(BT)$. Let S be the symmetric algebra over $\mathbb{Q}$ of the group $\Xi(T)$. Then S is a polynomial ring on m generators of degree 1, and the map $\chi \to c(\chi)$ extends to a ring isomorphism

$$c : S \to H^*_T(pt)$$

which doubles degrees. The map c is refered in the literature as the *characteristic homomorphism*.

From the copious list of properties of equivariant cohomology, we just mention briefly a few of them here. The reader is urged to consult [Bo1] or [Q] for a complete treatment of equivariant cohomology.

One salient property of equivariant cohomology is the **induction formula**. Let K be a closed subgroup of G and let X be a K-space. Consider the natural action of K on $G \times X$, and form the quotient space $G \times_K X := (G \times X)/K$. Define a G action on $G \times_K X$ by putting $g[g', x] = [gg', x]$. Then

$$H^*_G(G \times_K X) \simeq H^*_K(X).$$

The induction formula is also valid for locally compact Lie groups.

Remark 1.1.2. Let K be a closed subgroup of G, and let Y be a G-space. There is a homeomorphism between the G-spaces $G \times_K Y$ and $(G/K) \times Y$ given by $(g, x) \mapsto (g, g^{-1}x)$. Taking such homeomorphism into account, consider the case when K is a maximal compact torus, say $(S^1)^n$, of an algebraic torus $G = (\mathbb{C}^*)^n$. Because $(\mathbb{C}^*)^n/(S^1)^n \simeq (\mathbb{R}^+)^n$ is contractible, the induction formula then yields

$$H^*_K(-) \simeq H^*_G(-).$$

This equivalence of functors is relevant for our purposes. It states that equivariant cohomology makes no distinction between actions of compact tori and algebraic tori. For a concrete application of this observation, see Theorem 1.4.7.

Let H be a closed subgroup of G. Then

$$(G/H)_G = EG \times_G (G/H) = (EG \times_G G)/H = (EG)/H = BH;$$

in other words,

$$H^*_G(G/H) = H^*(BH),$$

for each closed subgroup $H \subset G$.

Equivariant maps between homogeneous G-spaces are given by $G/H \to G/K$, for pairs of subgroups $H \subset K$. Thus we have equivariant morphisms

$$H_G^*(G/K) = H^*(BK) \longrightarrow H^*(BH) = H_G^*(G/H)$$

for each pair $H \subset K$.

Remark 1.1.3. Let G be a compact connected Lie group. Let T be a maximal compact torus of G. Under these assumptions, G/T is connected and admits a Bruhat decomposition. In fact, G/T is homeomorphic to the flag variety of the complexification of G. To see this, let $G^{\mathbb{C}}$ be the complexification of G; then $G^{\mathbb{C}}$ is a connected reductive group. Let B be a Borel subgroup of $G^{\mathbb{C}}$ containing the compact torus T. Then, by the Iwasawa decomposition, we have $G^{\mathbb{C}} = GB$ and $G \cap B = T$. Consequently, the map $G/T \to G^{\mathbb{C}}/B$ is a homeomorphism. By the Bruhat decomposition, the flag variety $G^{\mathbb{C}}/B$ has a paving by $|W|$ cells, each of them being isomorphic to a complex affine space. Therefore, $H^*(G/T)$ vanishes in odd degrees, and the topological Euler characteristic $\chi(G/T)$ is equal to $|W|$.

Remark 1.1.4. It follows from the long exact sequence of homotopy groups associated to the fibration

$$T \hookrightarrow ET \longrightarrow BT$$

that BT is simply connected. Likewise, replacing T by G in the fibration above renders BG as simply connected.

1.2 Spectral sequences

Let G be a compact Lie group and let X be a G-space.

1.2.1 Leray-Serre spectral sequences

These are the spectral sequences associated to the diagram

$$BG \xleftarrow[p_X]{} EG \times_G X \xrightarrow{f_X} X/G\ ,$$

where p_X and f_X are the maps induced by the projections of $EG \times X$ onto its factors.

(1) The map p_X gives rise to the Serre spectral sequence

$$E_2^{s,t} = H^s(BG; H^t(X)) \Longrightarrow H_G^{s+t}(X).$$

(2) In turn, the map f_X produces the Leray-Serre spectral sequence

$$E_2^{s,t} = H^s(X/G; \mathcal{H}_G^t) \Longrightarrow H_G^{s+t}(X).$$

The sheaf $\mathcal{H}_G^t$ is the sheaf on X/G associated to the presheaf $V \mapsto H_G^t(f_X^{-1}V)$. One checks that the stalk of $\mathcal{H}_G^t$ at $[y] \in X/G$ is $H_G^t(f_X^{-1}y)$. See [Q] for the details.

Remark 1.2.1. Certainly the map p_X above is a fibration. On the other hand, the same cannot be postulated about f_X. Indeed, for any $[x] \in X/G$, the fibre $f_X^{-1}([x])$ equals EG/G_x, the classifying space of G_x. So there is no canonical fibre, as the fibres depend on the particular choice of point $[x]$ in X/G. However, some global properties of f_X can still be deduced from this. For instance, if the stabilizer G_x is finite for any $x \in X$, then f_X would be a map with $\mathbb{Q}$-acyclic fibres.

Lemma 1.2.2. *Let G be a compact Lie group and X be a G-space. Suppose that G acts on X with finite isotropy groups. Then,*

$$H_G^*(X) \simeq H^*(X/G).$$

Proof. As it was discussed on Remark 1.2.1, the fibres of map $f_X : X_G \to X/G$ are the various BG_x, for $[x] \in X/G$. Since the isotropy groups G_x are all finite, then BG_x is $\mathbb{Q}$-acyclic. The result now follows from the Leray-Serre spectral sequence (2) above. □

Lemma 1.2.3. *If G acts trivially on X, then*

$$H_G^* \otimes_{\mathbb{Q}} H^*(X) \xrightarrow{\simeq} H_G^*(X).$$

Proof. Since $EG \times_G X = BG \times X$, this follows from the Künneth formula. □

Lemma 1.2.4. *Let G be a compact connected Lie group, T be a maximal torus, N be the normalizer of T in G, and $W = N/T$ be the Weyl group of G. Then*

$$H^*(G/N) \simeq H^*(G/T)^W \simeq H^*(pt);$$

that is, G/N is $\mathbb{Q}$-acyclic. In symbols, $G/N \sim_{\mathbb{Q}} pt$.

Proof. Since $W \hookrightarrow G/T \twoheadrightarrow G/N$ is a finite covering, it follows that

$$H^*(G/N) \simeq H^*(G/T)^W$$

and, by counting cells, $\chi(G/T) = |W| \cdot \chi(G/N)$. Moreover, Remark 1.1.3 asserts that $H^{odd}(G/T) = 0$ and $\dim_{\mathbb{Q}} H^*(G/T) = \chi(G/T) = |W|$. Consequently,

$$H^{odd}(G/N) \simeq H^{odd}(G/T)^W = 0$$

together with

$$\dim_{\mathbb{Q}} H^*(G/N) = \chi(G/N) = \frac{1}{|W|} \cdot \chi(G/T) = 1.$$

In short, $G/N \sim_{\mathbb{Q}} pt$. □

Lemma 1.2.5. *Let G be a compact connected Lie group, T be a maximal torus, N the normalizer of T and W the Weyl group acting as an automorphism group of T. Then,*

$$H^*(BG) \simeq H^*(BN) \simeq H^*(BT)^W.$$

Moreover, BG has vanishing odd cohomology.

Proof. Since the fiber bundle $G/N \hookrightarrow BN \xrightarrow{\pi} BG$ has $\mathbb{Q}$-acyclic fibres and $\pi_1(BG, *) = 0$, it follows easily from the Serre spectral sequence that the map $\pi^* : H^*(BG) \to H^*(BN)$ is an isomorphism. Hence,

$$H^*(BG) \simeq H^*(BN) \simeq H^*(BT)^W,$$

where the second isomorphism comes from the fact that $BT \to BN$ is a covering map, with W acting as deck transformations.

Finally, the explicit description of $H^*(BT)$ (Example 1.1.1) implies that BT has no odd cohomology. Given that $H^*(BG) = H^*(BT)^W$, then BG has no odd cohomology either. □

Example 1.2.6. Let $G = U(n)$ be the compact subgroup of $GL(n, \mathbb{C})$ consisting of unitary matrices. Then $T^n = \{\mathrm{diag}(e^{2\pi i\theta_1}, \dots, e^{2\pi i\theta_n})\}$ is a maximal torus and $W = S_n$ acts on T^n by permuting the θ_j's. Recall that $H^*(BT^n) \simeq \mathbb{Q}[x_1, \dots, x_n]$ and W acts $H^*(BT^n)$ by permutations of the x_j's. Hence

$$H^*(BG) \simeq \mathbb{Q}[x_1, \dots, x_n]^W \simeq \mathbb{Q}[c_1, \dots, c_n]$$

is exactly the ring of symmetric polynomials, and the universal Chern classes $c_1, \dots, c_n$ are respectively the elementary symmetric polynomials.

1.2.2 Eilenberg-Moore spectral sequence

Let X be a given G-space and K be a closed subgroup of G. Then the restriction of the G-action to K makes X into a K-space. What is the relationship between

$H^*_G(X)$ and $H^*_K(X)$?

The following is a commutative diagram of fibrations:

$$\begin{tikzcd}
 & X \arrow[r, equal] \arrow[d] & X \arrow[d] \\
G/K \arrow[r] \arrow[d, equal] & X_K \arrow[r] \arrow[d] & X_G \arrow[d] \\
G/K \arrow[r] & BK \arrow[r] & BG.
\end{tikzcd}$$

Recall that we may assume EK and EG to be the same space.

For any pullback of a fibration, Eilenberg-Moore constructed a spectral sequence $\{E_n, d_n\}$ such that

$$E_n \Longrightarrow H^*(X_K) = H^*_K(X),$$

$$E_2^{p,q} = Tor^{p,q}_{H^*(BG)}(H^*(BK), H^*(X_G))$$

Example 1.2.7. If $K = \{id\}$, then the above spectral sequence reduces to

$$E_2^{p,q} = Tor^{p,q}_{H^*(BG)}(H^*(pt), H^*(X_G))\,, \qquad E_n \Rightarrow H^*(X).$$

Moreover, if $H^*_G(X)$ is a free H^*_G-module, then

$$\mathbb{Q} \otimes_{H^*_G} H^*_G(X) \simeq H^*(X).$$

Example 1.2.8. Let X, Y be two G-spaces. Then $X \times Y$ is a $(G \times G)$-space and its restriction to the diagonal subgroup $\Delta : G \to G \times G$ makes $X \times Y$ into a G-space. Hence, the spectral sequence gives

$$E_2^{p,q} = Tor^{p,q}_{H^*(BG \times BG)}(H^*(BG), H^*(X_G \times Y_G)),$$

along with

$$E_n \Rightarrow H^*_G(X \times Y).$$

This is the Künneth spectral sequence of equivariant cohomology.

Proposition 1.2.9. *Let G be a compact connected Lie group and let $T \subset G$ be a maximal torus with normalizer N and with Weyl group $W = N/T$; let X be a G-space. When working with rational coefficients, the following hold:*

(i) The group W acts on $H_T^*(X)$ and we have an isomorphism

$$H_G^*(X) \simeq H_T^*(X)^W.$$

In particular, $H_G^*(pt)$ is isomorphic to S^W where S denotes the symmetric algebra of the character group $\Xi(T)$ (ocurring in degree 2), and S^W the ring of W-invariants in S.

(ii) The map

$$S \simeq H_G^*(G/T) \longrightarrow H^*(G/T)$$

is surjective and induces an isomorphism $S/(S_+^W) \to H^*(G/T)$ where (S_+^W) denotes the ideal of S generated by all homogeneous W-invariants of positive degree.

(iii) We have an isomorphism

$$S \otimes_{S^W} H_G^*(X) \simeq H_T^*(X).$$

In particular, $H_T^*(G/T)$ is isomorphic to $S \otimes_{S^W} S$.

Proof. The proof is obtained by putting together all the data obtained from our previous results. First, consider the fibre bundle $G/T \hookrightarrow BT \to BG$. Recall that both G/T and BG have vanishing odd cohomology, as it can be seen from Remark 1.1.3 and Lemma 1.2.5. Thus, the Serre spectral sequence associated to the given fibration degenerates and yields

$$H^*(BT) \simeq H^*(BG) \otimes H^*(G/T).$$

In other words, $H^*(BT)$ is a free module over $H^*(BG)$. This implies (ii).

On the other hand, there is a pullback diagram

$$\begin{tikzcd} G/T \arrow[r] & X_T \arrow[r] \arrow[d] & X_G \arrow[d] \\ G/T \arrow[r] & BT \arrow[r] & BG \end{tikzcd}$$

from which it follows at once (due to the Eilenberg-Moore spectral sequence) that

$$H^*(X_T) \simeq H^*(BT) \otimes_{H^*(BG)} H^*(X_G).$$

So (iii) holds.

Now remember that $G/N \sim_{\mathbb{Q}} pt$. Hence the fibration diagram

$$G/N \hookrightarrow X_N \to X_G$$

yields $H^*_G(X) \simeq H^*_N(X)$. Finally, the covering

$$W \to X_T \to X_N$$

gives

$$H^*_N(X) \simeq H^*_T(X)^W,$$

and (i) is obtained. □

Corollary 1.2.10. *There is a graded W-submodule R of H^*_T, isomorphic to the regular representation of W, such that*

$$H^*_T \simeq R \otimes (H^*_T)^W$$

as graded $(H^*_T)^W$-modules.

Proof. Proposition 1.2.9 (ii) asserts that $S = H^*_T$ is a free S^W-module. Moreover, it provides the factorization $S = S^W \otimes H^*(G/T)$. That is, $H^*(G/T) = S/(S^W_+)$. A well-known result of Leray ([Bo3], Proposition 20.2) now implies that the representation of W in $H^*(G/T)$ is isomorphic to the regular representation. Setting $R = H^*(G/T)$ concludes the proof. □

1.3 Localization theorems for torus actions

Given a compact torus $K = (S^1)^n$, denote by H_K^* the ring $H_K^*(pt)$. Cohomology is always considered with rational coefficients.

Proposition 1.3.1 (Borel, [Hs]). *Let K be the circle group, X be a finite dimensional K-space, X^K be the fixed point set. Then*

(i) $H_K^*(X, X^K) \simeq H^*((X - X^K)/K)$ is a torsion H_K^*-module.

(ii) the kernel and cokernel of $H_K^*(X) \to H_K^*(X^K) = H_K^* \otimes_{\mathbb{Q}} H^*(X^K)$ are both torsion H_K^*-modules. □

Let $S \subset H_K^*$ be the multiplicative system $H_K^* \setminus \{0\}$. For a given K-space X, denote by X^K the fixed point set. The following is a classical theorem due to Borel ([Bo1]).

Theorem 1.3.2. *Let K be a compact torus and X be a paracompact K-space. Suppose $H_K^*(X)$ is a finite H_K^*-module. Then the localized restriction homomorphism*

$$S^{-1}H_K^*(X) \longrightarrow S^{-1}H_K^*(X^K) = H^*(X^K) \otimes_{\mathbb{Q}} (S^{-1}H_K^*)$$

is an isomorphism. □

Localization is explored systematically by Segal ([Seg2]) and by Atiyah and Segal ([ASe1]) in the context of fixed point theorems for equivariant K-theory.

We now focus our attention to the case of a compact torus K acting on a topological space X satisfying the hypothesis of Theorem 1.3.2.

Denote by X_1 the set

$$X_1 = \{x \in X | \operatorname{codim}(K_x) \leq 1\},$$

that is, X_1 is the set of points consisting of 0 and 1 dimensional orbits of K. Let δ be the connecting homomorphism in the long exact sequence for the equivariant cohomology of the pair (X_1, X^K). The following is a topological version of the localization theorem. It was first proved by Chang and Skjelbred ([CS]). Another proof can be found in [GKM], Theorem 6.3.

Theorem 1.3.3. *Suppose $H^*_K(X)$ is a free module over H^*_K. Then the sequence*

$$0 \longrightarrow H^*_K(X) \xrightarrow{\gamma} H^*_K(X^K) \xrightarrow{\delta} H^*_K(X_1, X^K)$$

is exact, and in particular the equivariant cohomology of X may be identified as the submodule of the equivariant cohomology of the fixed point set which is given by ker(δ). Additionally, δ is compatible with the cup product and so the sequence above determines the ring structure of $H^*_K(X)$. □

1.4 GKM theory

GKM theory is a relatively recent tool that owes its name to the work of Goresky, Kottwitz and MacPherson [GKM]. This theory encompasses techniques that date back to the early works of Atiyah ([At3], [ASe1]), Segal ([Seg1]), Borel ([Bo1]) and Chang-Skjelbred ([CS]).

1.4.1 Equivariant formality

Definition 1.4.1. Suppose a compact torus $K = (S^1)^r$ acts on a (possibly singular) space X. Let $p_X : X_K \longrightarrow BK$ be the fibration associated to the Borel construction. We say that X is **equivariantly formal** if the spectral sequence

$$E_2^{p,q} = H^p(BK; H^q(X)) \Longrightarrow H_K^{p+q}(X)$$

for this fibration degenerates at E_2.

Lemma 1.4.2. *Let X be a K-space whose ordinary rational cohomology vanishes in odd degrees. Then X is equivariantly formal.*

Proof. Recall from Example 1.1.1 that the classifying space of K, namely $(\mathbb{C}\mathrm{P}^\infty)^r$, has cohomology only in even degrees. After placing this information in the E_2-term of the Serre spectral sequence, one notices that all the differentials are zero. Hence the spectral sequence degenerates. □

Lemma 1.4.3. *Let X be a K-space. Then X is equivariantly formal if and only if its K-equivariant cohomology is a free module over H_K^*. More precisely, X is equivariantly formal if and only if*

$$H_K^*(X) \simeq H^*(X) \otimes_{\mathbb{Q}} H_K^*$$

as H_K^*-modules.

Proof. If X is equivariantly formal, the result follows immediately from the degeneration of the Leray spectral sequence at its second term. The other direction follows from the Eilenberg-Moore spectral sequence and was shown in Example 1.2.7. □

Example 1.4.4. Let M be a symplectic manifold with a Hamiltonian K-action. The results of Kirwan ([K]) yield $H_K(M) \simeq H^*(M) \otimes_{\mathbb{Q}} H_K^*$. So by Lemma 1.4.3, M is equivariantly formal. Likewise, any space with a K-invariant CW-decomposition into *even* cells is equivariantly formal (Lemma 1.4.2).

We will show that the class of equivariantly formal spaces also includes rationally smooth standard group embeddings (Theorem 3.2.13).

The following result can be found in [GKM], Theorem 1.6.2.

Proposition 1.4.5. *Let X be a K-space. Then, X is equivariantly formal if and only if the edge homomorphism*

$$H_K^*(X) \longrightarrow H^*(X)$$

is surjective. In this case, the ordinary rational cohomology is given by extension of scalars,

$$H^*(X) \simeq H_K^*(X) \otimes_{H_K^*} \mathbb{Q}.$$

$\square$

Corollary 1.4.6. *Let X be a K-space. If X is equivariantly formal, then X is equivariantly formal with respect to any subtorus K' of K.*

Proof. Since the map $H_K^*(X) \longrightarrow H^*(X)$, induced by restriction to the fibre, is surjective (Proposition 1.4.5) and factors through $H_{K'}^*(X)$, the result follows from applying Proposition 1.4.5 to the map $H_{K'}^*(X) \longrightarrow H^*(X)$. $\square$

The theorem below characterizes equivariant formality when the fixed point set is finite.

Theorem 1.4.7. *Denote by T a compact torus or an algebraic complex torus. Let X be a compact T-space with a finite number of fixed points. Then, the following are equivalent:*

a) X is equivariantly formal.

b) $H_T^*(X, \mathbb{Q})$ is a free $H_T^*(pt)$-module of rank $|X^T|$, the number of fixed points.

c) The singular rational cohomology of X vanishes in odd degrees.

Proof. Due to our earlier Remark 1.1.2, equivariant cohomology makes no distinction between actions of compact tori and algebraic tori. Bearing this in mind, one simply notices that the equivalence between statements (a) and (b) has already been established in Lemma 1.4.3. As for the claim about the rank, it is enough to use Theorem 1.3.2.

For the direction $b) \Rightarrow c)$ we proceed in two steps. First, since $H_T^*(X)$ is a free $H_T^*(pt)$-module, the Eilenberg-Moore spectral sequence implies that

$$H^*(X) \simeq H^*(pt) \otimes_{H_T^*(pt)} H_T^*(X),$$

or, in other words, that we have the identification of rings

$$H^*(X, \mathbb{Q}) \simeq H_T^*(X, \mathbb{Q})/(H_T^+(pt, \mathbb{Q})),$$

where $(H_T^+(pt, \mathbb{Q}))$ denotes the ideal of $H_T^*(X, \mathbb{Q})$ generated by the images of homogeneous elements of $H_T^*(pt, \mathbb{Q})$ of positive degree. Second, the freeness of $H_T^*(X)$, together with the Localization Theorem (Theorem 1.3.3), imply that $H_T^*(X)$ *injects* into $H_T^*(X^T) = \bigoplus_{|X^T|} H_T^*(pt)$. Given that $H_T^*(pt) = H^*(BT) = \mathbb{Q}[x_1, \ldots, x_{rank(T)}]$, where each x_i is a cohomology class in degree 2, it follows that $H_T^*(X)$ is zero in odd degrees. This observation, together with the first part, leads to $H^{odd}(X) = 0$.

Finally, (a) follows readily from (c), as shown in Lemma 1.4.2. $\square$

Example 1.4.8 (Non-equivariantly formal space)**.** The circle $K = S^1$ acts on $\mathbb{C}\mathrm{P}^1$, the Riemann sphere, by rotation with fixed points at the North and South poles. Let X be three copies of $\mathbb{C}\mathrm{P}^1$ joined at these fixed points so as to form a "ring". Figure 1.1 depicts the situation.

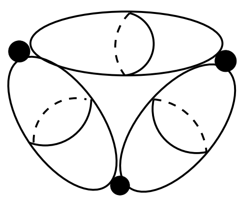

Figure 1.1: A projective variety which is not equivariantly formal [T].

The space X is a projective variety. To see this, consider $\mathbb{C}\mathrm{P}^2$ with the $\mathbb{C}^*$-action given by $t[x_0 : x_1 : x_2] := [x_0 : tx_1 : t^2x_2]$. Then X is isomorphic to the union of the canonical lines $x_0 = 0$, $x_1 = 0$, and $x_2 = 0$, with the induced $\mathbb{C}^*$-action. Notice that X has only three fixed points. Moreover, $H^1(X) = \mathbb{Q}$. Indeed, by excision, $H^1(X, X^T) = \mathbb{Q}^3$. Whence the long exact sequence of the pair (X, X^T) yields $H^1(X) = \mathbb{Q}$. Theorem 1.4.7 now assures that X is not equivariantly formal.

1.4.2 T-Skeletal Actions

Suppose X is a (possibly singular) complex projective algebraic variety with an algebraic action of a complex torus $T = (\mathbb{C}^*)^n$. Let $K = (S^1)^n \subset T$ denote the compact subtorus. We use complex coefficients throughout this subsection.

The equivariant cohomology $H^*_K(X;\mathbb{C})$ is an algebra: it is a ring under the cup product and it is a module over the symmetric algebra $\mathbf{S} = H^*(BK;\mathbb{C}) \simeq \mathbb{C}[\mathfrak{t}^*]$ of polynomial functions on the Lie algebra $\mathfrak{t}$ of K. Furthermore, Remark 1.1.2 allows to identify the functors $H^*_T(-)$ and $H^*_K(-)$.

Definition 1.4.9. Let X be a projective algebraic T-variety. Let $\mu : T \times X \to X$ be the action map. We say that μ is a ***T*-skeletal action** if

1. X^T is finite, and

2. The number of one-dimensional orbits of T on X is finite.

In this context, X is called a ***T*-skeletal variety**.

Let X be a normal projective T-skeletal variety. Then X has an equivariant embedding into a projective space with a linear action of T ([Su], Theorem 1). Denote by $x_1, \ldots, x_r$ the fixed points of X and by $E_1, E_2, \ldots, E_\ell$ the one-dimensional T-orbits. If X is equivariantly formal, there is an explicit formula for its equivariant cohomology algebra: Each 1-dimensional T-orbit E_j is a copy of $\mathbb{C}^*$ with two fixed points (called them x_{j_0} and x_{j_∞}) in its closure. So $\overline{E_j} = E_j \cup \{x_{j_0}\} \cup \{x_{j_\infty}\}$ is an embedded Riemann sphere which may be singular at the fixed points. The K-action rotates this sphere according to some character $\chi_j : K \to \mathbb{C}^*$. This character is uniquely determined up to sign (permuting the two fixed points changes χ_j to its opposite). The kernel of χ_j may be identified with the Lie algebra of the stabilizer of any point $e \in E_j$. In symbols,

$$\mathfrak{t}_j = \ker \chi_j = \mathrm{Lie}(\mathrm{Stab}_K(e)) \subset \mathfrak{t}.$$

Remark 1.4.10. Since $K = (S^1)^n$ is a dense subset (in the Zariski topology) of $T = (\mathbb{C}^*)^n$, it follows that $X^K = X^T$.

Let us denote by $K_j \subset K$ the stabilizer of any point in the orbit E_j, for $1 \leq j \leq \ell$. As we have seen, $\mathfrak{t}_j = Lie(K_j)$. For each j define

$$\beta_j : \bigoplus_{i=1}^{r} \mathbb{C}[\mathfrak{t}^*] \longrightarrow \mathbb{C}[\mathfrak{t}_j^*]$$

to be the map given by

$$\beta_j(f_1, \ldots, f_r) = f_{j0}|_{\mathfrak{t}_j} - f_{j\infty}|_{\mathfrak{t}_j}$$

where $\partial E_j = \{x_{j_0}, x_{j_\infty}\}$. Reversing the labels will change β_j by a sign but the kernel will be preserved.

Theorem 1.4.11 ([CS], [GKM])**.** *Let X be a normal projective T-skeletal variety. Suppose that X is equivariantly formal. Then the restriction mapping*

$$H_T^*(X) \longrightarrow H_T^*(X^T) \simeq \bigoplus_{x_i \in X^T} \mathbb{C}[\mathfrak{t}^*]$$

is injective, and its image is the subalgebra

$$H = \left\{ (f_1, f_2, \ldots, f_r) \in \bigoplus_{i=1}^{r} \mathbb{C}(\mathfrak{t}^*) \;\middle|\; f_{j0}|\mathfrak{t}_j = f_{j\infty}|\mathfrak{t}_j \quad \text{for } 1 \leq j \leq \ell \right\}$$

consisting of polynomial functions $(f_1, f_2, \ldots, f_r)$ such that for each 1-dimensional orbit E_j, the functions f_{j_0} and f_{j_∞} agree on the subalgebra $\mathfrak{t}_j$. In short,

$$H_K^*(X) \simeq \bigcap_{j=1}^{\ell} \operatorname{Ker}(\beta_j).$$

Proof. From the Localization Theorem (Theorem 1.3.3) it follows that $H_K^*(X) = \operatorname{Ker}(\delta)$, where $\delta : H_K^*(X^T) \to H_K^*(X_1, X^T)$. Here X_1 denotes the closure of the union of the 1-dimensional T-orbits. Let E_j be one of such orbits with closure $\overline{E}_j$ containing the fixed points $\partial E_j = \{x, y\}$. Let $T_j = Stab_T(z)$, where $z \in E_j$. Since T is abelian, T_j does not depend on the choice of point z. From the fibration

$$EK/K_j \hookrightarrow (E_j \times EK)/K \longrightarrow E_j/K \simeq *$$

it follows that

$$H_K^*(E_j) \simeq H^*(BK_j) \simeq H^*(BT_j) \simeq \mathbb{C}[\mathfrak{t}_j^*];$$

that is, $H_K^*(E_j)$ is zero in odd degrees. We can cover $\overline{E}_j$ by two equivariant open subsets, namely $U_1 = \overline{E}_j - \{x\}$ and $U_2 = \overline{E}_j - \{y\}$. Notice that $U_1 \cap U_2 = \mathbb{C}^*$. The Mayer-Vietoris exact sequence associated to this covering agrees with the long exact sequence of the pair $(\overline{E}_j, \partial E_j)$. Since both $H_K^i(\overline{E}_j)$ and $H_K^i(E_j)$ are zero for odd i, the long exact sequences split into short exact sequences,

$$\begin{tikzcd}
0 \arrow[r] & H_K^i(\overline{E}_j) \arrow[r] & H_K^i(x) \oplus H_K^i(y) \arrow[r, "\delta"] & H_K^{i+1}(\overline{E}_j, x \cup y) \arrow[r] & 0 \\
0 \arrow[r] & H_K^i(\overline{E}_j) \arrow[r, "\alpha"] \arrow[u, "\simeq"] & H_K^i(U_2) \oplus H_K^i(U_1) \arrow[r, "\beta"] \arrow[u] & H_K^i(E_j) \arrow[r] \arrow[u, "\simeq"] & 0
\end{tikzcd}$$

where $\beta : \mathbb{C}[\mathfrak{t}^*] \oplus \mathbb{C}[\mathfrak{t}^*] \to \mathbb{C}[\mathfrak{t}_j^*]$ is given by

$$\beta(f, g) = f|_{\mathfrak{t}_j} - g|_{\mathfrak{t}_j}.$$

Applying this computation to each one-dimensional orbit provides the final result. $\square$

Remark 1.4.12. Let K be a maximal torus of a compact connected Lie group G. Suppose that X is a G space. Then, by Proposition 1.2.9 (i), the G-equivariant cohomology of X is given by the invariants under the Weyl group, namely,

$$H_G^*(X) \simeq (H_K^*(X))^W.$$

The formula of Theorem 1.4.11 is compatible with the action of W given that W permutes the fixed points $x_1, \ldots, x_r$ and the one-dimensional orbits $E_1, \ldots, E_\ell$. So Theorem 1.4.11 can be used to calculate the G-equivariant cohomology of X as well.

If X is a normal projective T-skeletal variety, then it is possible to define a ring $PP_T^*(X)$ of **piecewise polynomial functions**. Indeed, let $R = \bigoplus_{x \in X^T} R_x$, where R_x is a copy of the polynomial algebra H_T^*. We then define $PP_T^*(X)$ as the subalgebra of R defined by

$$PP_T^*(X) = \{(f_1, ..., f_n) \in \bigoplus_{x \in X^T} R_x \mid f_i \equiv f_j \; mod(\chi_{i,j})\}$$

where x_i and x_j are the two fixed points in the closure of the one-dimensional T-orbit $\mathcal{C}_{i,j}$, and $\chi_{i,j}$ is the character of T associated with $\mathcal{C}_{i,j}$.

Theorem 1.4.11 suggests the next definition.

Definition 1.4.13. Let X be a complex algebraic variety equipped with a torus action $\mu : T \times X \to X$. We say that μ is a ***GKM*-action** if it is T-skeletal and X is equivariantly formal. In this situation, we call the pair (X, μ) a ***GKM*-variety**. When the reference to μ is clear from the context, we simply say that X is a GKM-variety.

In this new terminology, Theorem 1.4.11 reads

Theorem 1.4.14. *Let (X, μ) be a normal projective GKM-variety. Then the equivariant cohomology of X is isomorphic to the ring of piecewise polynomial functions $PP_T^*(X)$.* □

Theorem 1.4.15. *Let X be a normal projective variety with a T-skeletal action*

$$\mu : T \times X \to X.$$

Then (X, μ) is a GKM-variety if and only if the singular rational cohomology of X vanishes in odd degrees.

Proof. This is a partial translation of Theorem 1.4.7 into our new terminology. □

Remark 1.4.16. Smooth projective varieties with T-skeletal actions are GKM-varieties. See Lemma 2.3.6.

Building on previous work of Edidin and Graham ([EG1]), Brion established GKM theory for equivariant Chow rings ([Br2]). Later on, Vistoli and Vezzosi ([VV]) proved an analogue of GKM theory for the equivariant algebraic K-theory of smooth projective varieties. Brion ([Br4]) had also described the required GKM data for a large class of smooth group compactifications, namely, regular embeddings ([BDP]). Uma ([U]) finally showed that the equivariant K-theory ring of a regular embedding can be understood as a generalized Stanley-Reisner ring. We aim at a generalization of these results to the case of rationally smooth standard group embeddings. Besides showing that rationally smooth standard embeddings are GKM-varieties, we also provide a very explicit description of their equivariant cohomology (Chapters 3 and 4).

1.5 Examples

1.5.1 Equivariant cohomology of flag varieties

Let G be a connected semisimple algebraic group over $\mathbb{C}$. Let B denote a maximal connected solvable subgroup of G, i.e. a Borel subgroup. Let $T \subset B$ denote a maximal torus in G. It is well known ([Bo2]) that B can be written as $B = TU$, where U is the unipotent radical of B. Let $\Xi(T)$ be the character group of T.

Recall that T acts on U by inner automorphisms, $u \mapsto tut^{-1}$. This action induces an action of T on the tangent space $\mathfrak{u}$ of U. Consequently, $\mathfrak{u}$ decomposes into weight spaces indexed by certain characters $\Phi^+ \subset \Xi(T)$, known as (positive) **roots**:

$$\mathfrak{u} = \oplus_{\alpha\in\Phi^+}\mathfrak{u}_\alpha,$$

where the $\mathfrak{u}_\alpha$'s are one-dimensional invariant subspaces. We let $\Phi = \Phi^+ \cup -\Phi^+$ The next result appears in [Bo2] and [Hu].

Theorem 1.5.1. *a) $dim(\mathfrak{u}_\alpha) = 1$, for each $\alpha \in \Phi^+$.*

b) There is a unique, closed T-stable subgroup U_α of U whose tangent space at the identity of U is $\mathfrak{u}_\alpha$.

c) There is a unique Borel subgroup B^-, called the Borel subgroup opposite to B (relative to T), such that $T \subset B^-$ and $B \cap B^- = T$.

d) If U^- is the unipotent radical of B^-, the set of weights of T on $\mathfrak{u}^-$ is $-\Phi^+$.

e) The unipotent radical U of B is isomorphic, as an algebraic variety, to $\prod_{\alpha>0} U_\alpha$, where the product may be taken in any order. Analogously, $U^- \simeq \prod_{\alpha<0} U_\alpha$.

f) G is generated as a group by the groups U_α, $\alpha \in \Phi$, and T.

g) Φ generates a subgroup of finite index in $\Xi(T)$. □

Example 1.5.2. Let $G = SL(n, \mathbb{C})$. Then B equals the set of upper-triangular matrices with determinant one. The group T consists of diagonal matrices with determinant one and U is the group of unipotent upper triangular matrices. In this setting, the opposite Borel subgroup B^- is equal to the set of lower-triangular matrices with determinant one. One checks that $\Phi^+ = \{\alpha_{i,j} \,|\, i > j\}$ and $\Phi^- = \{\alpha_{i,j} \,|\, i < j\}$. Here, $\alpha_{i,j}(t_1, \dots, t_n) = t_i t_j^{-1}$ and $U_{i,j} = \{I_n + aE_{i,j} \,|\, a \in \mathbb{C}\}$, where $E_{i,j}$ is the elementary matrix with one non-zero entry in the (i, j)-position.

The homogeneous space G/B is called the **flag variety** of G. It is a projective variety ([Bo2]). Notice that T acts on G/B with a finite number of fixed points, namely $(G/B)^T \simeq W$. It follows from the Bruhat decomposition, $G = \sqcup_{w\in W} BwB$, that the flag variety G/B admits a paving by affine cells of the form $B[w] = BwB/B$, indexed over $w \in W$. Each one of these cells is isomorphic to an affine space $\mathbb{C}^{\ell(w)}$, where $\ell(w)$ is the length of w. Since these cells are even dimensional, then G/B has trivial cohomology in odd degrees (Lemma 1.2.4). Thus, the hypothesis of Theorem 1.4.7 hold, and we conclude that G/B is equivariantly formal. We will see below that G/B is actually a GKM-variety (Definition 1.4.13), so to describe its cohomology, it suffices to collect the necessary GKM-data.

T-invariant curves and the Bruhat graph. The Weyl group is generated by reflections $\{s_\alpha\}_{\alpha\in\Phi}$, where s_α corresponds to reflection with respect to the hyperplane defined by α. Let $\mathcal{G}_{s_\alpha}$ denote the copy of $SL(2,\mathbb{C})$ in G generated by U_α and $U_{-\alpha}$. The following is a result of Carrell ([C]).

Proposition 1.5.3. *The flag variety G/B is a GKM-variety. In fact, every closed T-invariant curve in G/B has the form $\mathcal{G}_{s_\alpha}w$, for some w in W and reflection s_α. Consequently, every T-invariant curve is non-singular. Moreover, $(\mathcal{G}_{s_\alpha}w)^T = \{w, s_\alpha w\}$, so $\mathcal{G}_{s_\alpha}x \subset X(w)$ if and only if $x, s_\alpha x \leq w$, where $X(w) = \overline{BwB/B}$ is a Schubert variety in G/B.* □

Let $i : (G/B)^T \to G/B$ be the inclusion of the fixed point set, and identify $(G/B)^T$ with W. Let $S = H_T^*(pt)$. Then, $H_T^*((G/B)^T)$ identifies with the ring $S[W]$ as an S-algebra with compatible action of W.

Theorem 1.5.4 ([C], [Br2])**.** *The image of*

$$i^* : H_T^*(G/B) \to S[W]$$

consists of all $\sum_{w\in W} f_w w$ such that $f_w \cong f_{s_\alpha w} (\mathrm{mod}\,\alpha)$ whenever $w \in W$ and $\alpha \in \Phi^+$.

Proof. After taking into account the GKM-data collected in Proposition 1.5.3, the result follows immediately from Theorem 1.4.11. $\square$

The previous results have analogues for arbitrary algebraic homogeneous spaces G/P, where G is a connected reductive group and P a parabolic subgroup. In particular, one can describe the T-curves in G/P in terms of the reflections $s_\alpha \in \Phi$. For a proof of the next Lemma, see [C] or Lemma 2.2 of [CK].

Lemma 1.5.5. *Let x be a T-fixed point of G/P. Then every closed irreducible T-stable curve C passing through x has the form $C = \overline{U_\alpha x}$ for some $\alpha \in \Phi$. Moreover, $C^T = \{x, s_\alpha x\}$, and each such C is smooth.* $\square$

The smoothness follows from the fact that C admits a transitive action of the subgroup of G generated by U_α and $U_{-\alpha}$.

Lemma 1.5.5 will be of relevance to the discussion in Chapters 3 and 4.

1.5.2 Equivariant cohomology of simplicial toric varieties

We begin with some notation and results concerning toric varieties. More details can be found in [D] and [F].

Denote by T a d-dimensional torus, by $M = \mathrm{Hom}(T, \mathbb{C}^*)$ its character group and by $N = Hom(\mathbb{C}^*, T)$ the group of one parameter subgroups of T. There is a natural pairing $M \times N \to \mathbb{Z}$: $(m, n) \mapsto \langle m, n\rangle$, where $\langle m, n\rangle$ is the integer such that $m(n(t)) = t^{\langle m,n\rangle}$ for all $t \in \mathbb{C}^*$.

Let X be a toric variety; that is, X is a normal T-variety with a dense orbit isomorphic to T. Recall that X is determined by its fan Σ in $N \otimes \mathbb{R}$. The cones of Σ parametrize the orbits in X; we denote by $\sigma \to \Omega_\sigma$ this parametrization, and by $V(\sigma)$ the closure of Ω_σ in X. Then $\Omega_\sigma = T/T_\sigma$ where T_σ is the subtorus of T with character lattice $M/M \cap \sigma^\perp$ and with lattice of one-parameter subgroups N_σ (the subgroup of N generated by $N \cap \sigma$). In consequence, the dimension of Ω_σ is the codimension of σ. It follows that the T-action on X is T-skeletal.

For each Ω_σ there is a unique T-stable open affine subset X_σ of X which contains Ω_σ as a closed subset. In fact, there is a T-equivariant retraction $r_\sigma : X_\sigma \to \Omega_\sigma$ which renders X_σ as T-equivariantly isomorphic to $T \times_{T_\sigma} S_\sigma$, where S_σ is an affine, T_σ-toric variety with a fixed point.

A toric variety X is called **simplicial** if each cone of its fan is generated by linearly independent vectors; equivalently, X has quotient singularities by finite groups (see [D]). In this case, we will describe the equivariant cohomology ring $H^*_T(X) \otimes \mathbb{Q}$ in terms of piecewise polynomial functions.

Remark 1.5.6. It is a well-known result of Danilov ([D]) that any complete simplicial toric variety has zero cohomology in odd degrees. In other words, any complete simplicial toric variety is a GKM-variety.

Proposition 1.5.7 ([BV])**.** *Notation being as above, the map $r^*_\sigma : H^*_T(X_\sigma) \to H^*_T(\Omega_\sigma) \simeq S^*(M_\mathbb{Q}/\sigma^\perp)$ is an isomorphism of graded algebras over $S^*(M_\mathbb{Q})$, the symmetric algebra on the character ring of T. In addition, for any face τ of σ, the diagram*

$$\begin{tikzcd} H^*_T(X_\sigma) \arrow[r] \arrow[d] & S^*(M_\mathbb{Q}/\sigma^\perp) \arrow[d] \\ H^*_T(X_\tau) \arrow[r] & S^*(M_\mathbb{Q}/\tau^\perp) \end{tikzcd}$$

commutes, where the left (resp. right) vertical arrow is defined by inclusion of X_τ in X_σ (resp. by the map $M_\mathbb{Q}/\sigma^\perp \to M_\mathbb{Q}/\tau^\perp$). $\square$

Piecewise polynomial functions. Denote by R_Σ the set of all families $(f_\sigma)_{\sigma\in\Sigma}$ such that $f_\sigma \in S^*(M_\mathbb{Q}/\sigma^\perp)$ and that, for all $\tau \in \sigma$, the image of f_σ in $S^*(M_\mathbb{Q}/\tau^\perp)$ is equal to f_τ. Then R_Σ is an algebra over $S^*(M_\mathbb{Q})$: the algebra of continuous, piecewise polynomial functions on Σ.

For $f \in R_\Sigma$, decompose f_σ into the sum of its homogeneous components $f_{\sigma,n}$. Then for fixed n, the family $(f_{\sigma,n})_{\sigma\in\Sigma}$ is in R_Σ. This defines a grading $R_\Sigma = \oplus_{n=0}^{\infty} R_{\Sigma,n}$ of the algebra R_Σ.

Assume that the fan Σ is simplicial. For $\sigma \in \Sigma$, consider the restriction map $H_T^*(X) \to H_T^*(X_\sigma)$, $u \mapsto u_\sigma$. By Proposition 1.5.7, we can identify u_σ with an element of $S^*(M_{\mathbb{Q}}/\sigma^\perp)$, so that the family $(u_\sigma)_{\sigma\in\Sigma}$ is in R_Σ. The following result is due to Brion and Vergne ([BV], [Br2]).

Theorem 1.5.8. *Let $X = X_\Sigma$ be a simplicial toric variety. Then*
(i) the map

$$\begin{aligned} H_T^*(X) &\longrightarrow R_\Sigma \\ u &\longmapsto (u_\sigma)_{\sigma\in\Sigma} \end{aligned}$$

is an isomorphism of graded algebras over $S^*(M_{\mathbb{Q}})$.
(ii) If, besides, X is complete, then the map

$$H_T^*(X)/M_{\mathbb{Q}} H_T^*(X) \to H^*(X)$$

is an isomorphism. □

Alternatively, the equivariant cohomology of simplicial toric varieties can be described as a Stanley-Reisner ring ([BDP]). (See also [U] for a K-theory analogue of this result.) In Chapter 4 we provide yet another description using descent systems.

Chapter 2

Rationally smooth

In this chapter we define our most important topological tool: rational cells. After describing some of their remarkable features, we define $\mathbb{Q}$-filtrable varieties, spaces that come equipped with a paving by rational cells. We conclude this chapter supplying a method for building canonical free module generators on the equivariant cohomology of any $\mathbb{Q}$-filtrable GKM variety.

Sections 2.2, 2.4 and 2.5 contain, predominantly, known results. In contrast, Sections 2.1, 2.3 and 2.6 contain new developments. Salient new results are Lemma 2.3.1, Theorem 2.3.5 and Theorem 2.6.9.

2.1 Rational cells

Definition 2.1.1. Let X be a complex algebraic variety of dimension n. We say that X is **rationally smooth at** x, if there exists a neighborhood U of x (in the complex topology) such that for all $y \in U$ we have

$$H^m(X, X - \{y\}) = (0) \quad \text{if} \quad m \neq 2n, \text{ and}$$

$$H^{2n}(X, X - \{y\}) = \mathbb{Q}.$$

We say that X is **rationally smooth** if X is rationally smooth at every $x \in X$.

The set of rationally smooth points is open for the complex topology and contains all smooth points. Quotients of smooth varieties by finite groups are rationally smooth (Proposition 2.1.4 (iii)). Other examples of rationally smooth varieties are unibranched curves.

A complex algebraic variety is rationally smooth if and only if it is a rational cohomology manifold. Rationally smooth projective varieties satisfy the Poincaré duality theorem with rational coefficients. The interested reader should consult [M], where McCrory gives a characterization of rational cohomology manifolds.

Example 2.1.2. The singular variety obtained from identifying the points 0 and ∞ in $\mathbb{CP}^1$ (that is, the "pinched torus" or projective nodal curve $y^2z = x^2(x+z)$ in $\mathbb{C}\mathrm{P}^2$) is not rationally smooth. In effect, the cohomology of the pair $(X, X - \{0\})$ coincides with the cohomology of the pair $(U, U - \{0\})$, where U is the affine variety $xy = 0$. For such a pair, it is easily seen that

$$H^k(U, U - \{0\}) = \begin{cases} 0 & \text{if} \quad k \neq 1, 2 \\ \mathbb{Q} & \text{if} \quad k = 1 \\ \mathbb{Q}^2 & \text{if} \quad k = 2. \end{cases}$$

Example 2.1.3. By Proposition 2.1.4 (iii) below, any V-manifold or *orbifold* is rationally smooth. Examples of this kind are provided by the so called **weighted projective spaces** $\mathbb{P}(q_0, \ldots, q_n)$, where the q_j are non-negative integers, the weights. Basically, $\mathbb{P}(q_0, \ldots, q_n)$ is defined as the quotient of $\mathbb{P}^n$ by the coordinate-wise action of the product $\mu_{q_0} \times \ldots \times \mu_{q_n}$ of the q_j-th roots of unity μ_j, $j = 0, \ldots, n$. It can also be described as the quotient of $\mathbb{C}^{n+1} - \{0\}$ by the action of $\mathbb{C}^*$ given by

$$t \cdot (z_0, \ldots, z_n) = (t^{q_0} z_0, \ldots, t^{q_n} z_n).$$

The natural quotient map is denoted

$$p : \mathbb{C}^{n+1} - \{0\} \longrightarrow \mathbb{P}(q_0, \ldots, q_n).$$

Let U_j be the set of all points in $\mathbb{C}^{n+1}$ subject to the condition $z_j = 1$. It is easy to see that U_j is isomorphic to $\mathbb{C}^n$. Further, the subgroup $\mu(q_j) \subset \mathbb{C}^*$ leaves U_j invariant. Consequently, $p(U_j)$ can be identified with the quotient space $V_j = U_j/\mu(q_j)$. These V_j's form the standard open affine covering of $\mathbb{P}(q_0, \ldots, q_n)$ as a V-manifold.

Let X be an algebraic variety of dimension n and let x be a point of X. We say that X is **irreducible at** x if there is only one irreducible component of X containing x. The following is a result of Brion [Br5].

Proposition 2.1.4. *Let X be an algebraic variety of dimension n and let $x \in X$.*

(i) The dimension of the vector space $H^{2n}(X, X - \{x\})$ is the number of n-dimensional irreducible components of X through x.

(ii) If X is rationally smooth at x, then it is irreducible at x.

(iii) Let $\pi : X \to Y$ be the quotient by the action of a finite group G. If X is rationally smooth at x, then Y is rationally smooth at $\pi(x)$.

(iv) Let $\pi : X \to Y$ be a smooth morphism. Then X is rationally smooth at x if and only if Y is rationally smooth at $\pi(x)$. □

Let T be a complex algebraic torus.

Definition 2.1.5. Let X be an algebraic variety with a T-action and a fixed point x. We say that x is an **attractive fixed point** if there exists a one-parameter subgroup $\lambda : \mathbb{C}^* \to T$ and a neighborhood U of x, such that $\lim\limits_{t\to 0} \lambda(t) \cdot y = x$ for all points y in U.

There is an important characterization of attractive fixed points. A proof of the following result can be found in [Br5], Proposition A2.

Proposition 2.1.6. *For a torus T acting on a variety X with a fixed point x, the following conditions are equivalent:*

(i) The weights of T in the tangent space $T_x(X)$ are contained in an open half space.

(ii) There exists a one-parameter subgroup $\lambda : \mathbb{C}^* \to T$ such that, for all y in a neighborhood of x, we have $\lim\limits_{t\to 0} \lambda(t)\, y = x$.

If (ii) holds, then the set

$$X_x := \{y \in X \mid \lim_{t\to 0} \lambda(t)\, y = x\}$$

is the unique affine T-invariant open neighborhood of x in X. Moreover, X_x admits a closed T-equivariant embedding into T_xX. $\square$

Lemma 2.1.7. *Let X be an irreducible affine variety with a T-action and an attractive fixed point $x_0 \in X$. Then X is rationally smooth at x_0 if and only if X is rationally smooth everywhere.*

Proof. If X is rationally smooth everywhere, then it is rationally smooth at x_0. For the converse, we use Proposition 2.1.6 (ii) and the affineness of X to guarantee the existence of a one-parameter subgroup $\lambda : \mathbb{C}^* \to T$ such that

$$X = \{y \in X \mid \lim_{t\to 0} \lambda(t)\, y = x_0\}.$$

In symbols, $x_0 \in \overline{\mathbb{C}^* \cdot y}$, for any $y \in X$. Now consider the complex topology on X. We claim that any non-empty open T-stable subset of X containing x_0 is all of

X. In effect, let U be a T-stable neighborhood of x_0. Then, for any $y \in X$, there exists $s_y \in \mathbb{C}^*$, such that $s_y \cdot y \in U$. Indeed, because x_0 is attractive, one can find a sequence $\{t_n\} \subset \mathbb{C}^*$ such that $t_n \cdot y$ converges to x_0. That is, there exists N with the property that $t_N \cdot y$ belongs to U. Setting $s_y = t_N$ yields $s_y \cdot y \in U$. However, U is T-stable, and therefore it contains the entire orbit $\mathbb{C}^* \cdot y$. In short, $y \in U$ or, equivalently, $U = X$.

Hence, the non-empty open T-stable subset of rationally smooth points of X is, *a fortiori*, equal to X. □

Definition 2.1.8. Let X be an irreducible affine variety with a T-action and an attractive fixed point $x_0 \in X$. If X is rationally smooth at x_0 (and thus everywhere), we refer to (X, x_0) as a **rational cell**.

It follows from Definition 2.1.8 and Proposition 2.1.6 that if (X, x_0) is a rational cell, then

$$X = \{y \in X \mid \lim_{t \to 0} \lambda(t)\, y = x_0\},$$

for a suitable one-parameter subgroup λ. Notably, $\{x_0\}$ is the unique closed T-orbit in X.

Example 2.1.9. Certainly $\mathbb{C}^n$ is a rational cell with the usual $\mathbb{C}^*$-action by scalar multiplication. Here the origin is the unique attractive fixed point.

Example 2.1.10. Let $V = \{xy = z^2\} \subset \mathbb{C}^3$. The standard $\mathbb{C}^*$-action by scalar multiplication makes V a rational cell with $(0,0,0)$ as its attractive fixed point. This is clear once we observe that V is the quotient of $\mathbb{C}^2$ by the finite group with two elements, where the non-trivial element acts on $(s,t) \in \mathbb{C}^2$ via $(s,t) \mapsto (-s,-t)$. So Proposition 2.1.4 (iii) implies that V is rationally smooth. Compare Example 2.1.17.

Example 2.1.11. A normal variety is not necessarilly rationally smooth. For instance, consider the hypersurface $H \subset \mathbb{C}^4$ defined by $\{xy = uv\}$. Because the singular locus of H, namely $\{(0,0,0,0)\}$, has codimension three, it follows that H is normal ([Sha], p. 128, comments after Theorem II.5.1.3). Nevertheless, H is not rationally smooth at the origin. To see this, let $T = (\mathbb{C}^*)^2$ act on H via $(t,s) \cdot (x,y,u,v) = (tx, ts^2y, su, st^2v)$. Then H has the origin as its unique attractive fixed point. Moreover, H contains four T-invariant curves (the four coordinate axes) passing through $(0,0,0,0)$. If H were rationally smooth at the origin, then, by a result of Brion (Corollary 2.4.6), the dimension of H would equal the number of its T-invariant curves. This is a contradiction, since H is only three dimensional.

Definition 2.1.12. Let Z be a rationally smooth complex projective variety. Let n be the (complex) dimension of Z. We say that Z is a **rational cohomology complex projective space** if there is a ring isomorphism

$$H^*(Z) \simeq \mathbb{Q}[t]/(t^{n+1}),$$

where $deg(t) = 2$.

The following can be found in [BD], Theorem 1.

Lemma 2.1.13. *Let Z be a complex projective algebraic variety of dimension n. Then $H^*(Z)$ contains a subring isomorphic to $H^*(\mathbb{C}\mathrm{P}^n)$.*

Proof. Let $j : Z \hookrightarrow \mathbb{C}\mathrm{P}^M$ be the inclusion mapping and consider $\omega \in H^2(\mathbb{C}\mathrm{P}^M)$ the canonical generator. Take $\alpha = j^*(\omega) \in H^2(Z)$. Then $\alpha^{n+1} = 0$ and $\alpha^k \neq 0$ for all $k \leq n$. To see this, remember that $j_*([Z]) \in H^*_{2n}(\mathbb{C}\mathrm{P}^M)$ is the fundamental class of Z in $\mathbb{C}\mathrm{P}^M$ and thus *non-zero*. By Poincaré duality, the Kronecker pairing implies

$$\langle \omega^n, j_*[Z] \rangle = \langle j^*\omega^n, [Z] \rangle \neq 0,$$

or, said another way, $j^*\omega^n$ cannot be zero either.

In other words,

$$\mathbb{Q}[\alpha]/(\alpha^{n+1})$$

is a subring of $H^*(Z)$. $\square$

Corollary 2.1.14. *Let Z be an n-dimensional rationally smooth projective variety. If Z has the same rational homology groups of $\mathbb{C}\mathrm{P}^n$, then there is a ring isomorphism*

$$H^*(Z) \simeq \mathbb{Q}[\alpha]/(\alpha^{n+1}).$$

In other words, Z is a rational cohomology complex projective space if and only if Z has the same Betti numbers of $\mathbb{C}\mathrm{P}^n$. $\square$

Let (X, x) be a rational cell. Then, by Proposition 2.1.6, X admits a closed T-equivariant embedding into T_xX. Set $\dot{X}$ to be $X - \{x\}$. Choose an injective one-parameter subgroup $\lambda : \mathbb{C}^* \to T$ as in Definition 2.1.8. Then all weights of the $\mathbb{C}^*$-action on T_xX via λ are positive. Thus, the quotient

$$\mathbb{P}(X) := \dot{X}/\mathbb{C}^*$$

exists and is a projective variety. Indeed, it is a closed subvariety of $\mathbb{P}(T_xX)$, a weighted projective space. We can view $\mathbb{P}(X)$ as an algebraic version of the link of X at x.

The following result, except for parts (b) and (c), is due to Brion ([Br5]). The idea of the proof of part (b) is due to Renner.

Theorem 2.1.15. *Let (X, x_0) be a rational cell of dimension n. Then,*

a) X is contractible.

b) $X - \{x_0\}$ is homeomorphic to $\mathbb{S}(X) \times \mathbb{R}^+$, where $\mathbb{S}(X) := X - \{x_0\}/\mathbb{R}^+$ is a compact topological space.

c) $X - \{x_0\}$ deformation retracts to $\mathbb{S}(X)$. In addition, X is rationally smooth at x_0 if and only if $X - \{x_0\}$, and thus $\mathbb{S}(X)$, is a rational cohomology sphere $\mathbb{S}^{2n-1}$.

d) The space $\mathbb{P}(X) = X - \{x_0\}/\mathbb{C}^*$ is a rationally smooth complex projective variety of dimension $n-1$. Furthermore, X is rationally smooth if and only if $\mathbb{P}(X)$ is a rational cohomology complex projective space $\mathbb{CP}^{n-1}$.

Proof. a) The action of $\mathbb{C}^*$ on X extends to a map $\mathbb{C} \times X \to X$ sending $0 \times X$ to x_0 and restricting to the identity $1 \times X \to X$.

b) From Proposition 2.1.6, we know that X admits a closed T-equivariant embedding into $T_{x_0}X \simeq \mathbb{C}^d$, which identifies x_0 with 0. Choosing a one-parameter subgroup $\lambda : \mathbb{C}^* \to T$ as in Definition 2.1.8 yields a $\mathbb{C}^*$-action on $\mathbb{C}^d$ with only positive weights $m_1, \ldots, m_d$. Specifically, $\lambda \in \mathbb{C}^*$ acts on $\mathbb{C}^d$ via

$$\lambda \cdot (z_1, \ldots, z_d) = (\lambda^{m_1} z_1, \ldots, \lambda^{m_d} z_d).$$

Next, define an $\mathbb{R}^+$-equivariant map $N : \mathbb{C}^d \to \mathbb{R}$ by

$$N(z_1, \ldots, z_d) = \sqrt{\sum_{i=1}^{d} (z_i \overline{z_i})^{1/m_i}}.$$

Clearly, for $\lambda \in \mathbb{C}$ and $z \in \mathbb{C}^d$, the definition favors $N(\lambda \cdot z) = |\lambda| N(z)$ (here $\lambda \cdot z$ means $(\lambda^{m_1} z_1, \ldots, \lambda^{m_d} z_d)$).

Since $\mathbb{R}^+$ acts freely on $X - \{0\} \subseteq \mathbb{C}^d - \{0\}$, the quotient map

$$X - \{0\} \to \mathbb{S}(X)$$

is a principal $\mathbb{R}^+$-fibration. Note that $\mathbb{R}^+$ acts transitively on each fibre. We claim that this fibration is trivial, i.e.

$$X - \{0\} \simeq \mathbb{S}(X) \times \mathbb{R}^+.$$

To prove the claim, we just need to provide a global section s. In fact, we can do so canonically. Let $s : \mathbb{S}(X) \to X - \{0\}$ be the map defined by

$$s([x]) = \frac{1}{N(x)} \cdot x.$$

This map is well defined (given that we are using the $\mathbb{C}^*$-action mentioned above) and not only defines a global section, but also a homeomorphism between $\mathbb{S}(X)$ and $X \cap N^{-1}(1)$, where $N^{-1}(1)$ is the "unit" sphere. Thus, $\mathbb{S}(X)$ is compact.

c) The first claim follows immediately from part b). As for the second assertion, remember that X is contractible. Thus, the following long exact sequence

$$\ldots \longrightarrow H^*(X, X-\{x_0\}) \longrightarrow H^*(X) \longrightarrow H^*(X-\{x_0\}) \longrightarrow H^{*+1}(X, X-\{x_0\}) \longrightarrow \ldots$$

splits into short exact sequences

$$0 \longrightarrow H^*(X - \{x_0\}) \longrightarrow H^{*+1}(X, X - \{x_0\}) \longrightarrow 0.$$

Therefore X is rationally smooth if and only if $X - \{x_0\}$ is a rational homology sphere of dimension $2n - 1$.

d) $\mathbb{C}^*$ acts on $X - \{x_0\}$ with finite stabilizers (since x_0 is the unique fixed point). It follows from Proposition A5 of [Br5] that $X - \{x_0\}$ is covered by $\mathbb{C}^*$-stable open subsets U admitting an equivariant morphism $p : U \to \mathbb{C}^*/\Gamma$, where $\Gamma \subset \mathbb{C}^*$ is a finite subgroup (depending on U). Let Y be the fibre of p at the base point of $\mathbb{C}^*/\Gamma$. Then, $Y \subset X$ is a locally closed Γ-stable subvariety, and U is equivariantly isomorphic to the quotient

$$(\mathbb{C}^* \times Y)/\Gamma$$

where Γ acts diagonally on $\mathbb{C}^* \times Y$. This a version of the slice theorem.

Thus, $\mathbb{P}(X)$ is covered by the quotients Y/Γ. Noticeably, $\mathbb{C}^* \times Y$ is rationally smooth, because $X - \{x_0\}$ is rationally smooth and the map $\mathbb{C}^* \times Y \to X$ sending

(t, y) to ty is étale (Proposition 2.1.4, (iv)). Thus, Y is rationally smooth (by the Kunneth formula) and so is the quotient Y/Γ (by Proposition 2.1.4, (iii)). Therefore, $\mathbb{P}(X)$ is rationally smooth.

Finally, since $X - \{x_0\}/S^1 \to X - \{x_0\}/\mathbb{C}^*$ induces an isomorphism in rational cohomology, it is enough to work with $\tilde{P} = X - \{x_0\}/S^1$. Observe that S^1 acts on $X - \{x_0\}$ with finite isotropy groups. So the map $\pi : X - \{x_0\} \to \tilde{P}$ is a proper map with fibres isomorphic to S^1. More precisely, each fibre $\pi^{-1}([x])$ is of the form S^1/Γ_x, where Γ_x is a finite subgroup of S^1. Next, the Gysin sequence associated to π looks as follows

$$\ldots \longrightarrow H^m(X - \{x_0\}) \longrightarrow H^{m-1}(\tilde{P}) \longrightarrow H^{m+1}(\tilde{P}) \longrightarrow H^{m+1}(X - \{x_0\}) \longrightarrow \ldots$$

Therefore, $X - \{x_0\}$ is a rational homology sphere of dimension $2n - 1$ if and only if $\tilde{P}$ (and so $\mathbb{P}(X)$) is a rational cohomology complex projective space of (complex) dimension $n - 1$. □

Corollary 2.1.16. *Keeping the same notation as in Theorem 2.1.15, the rational cell X is homeomorphic to the open cone over $\mathbb{S}(X)$. Moreover, $\mathbb{P}(X)$ is equivariantly formal.*

Proof. The first assertion follows at once from Theorem 2.1.15, part (c), and uniform convergence. As for the second, it is enough to remember that, by Theorem 2.1.15 again, $\mathbb{P}(X)$ is a rational complex projective space and thus has no cohomology in odd degrees. Lemma 1.4.2 concludes the proof. □

Example 2.1.17. Let W be the affine variety $\{(x, y, z) \in \mathbb{C}^3 \mid z^2 = 2xy\}$. All points in W can be described by the following parametric equations: $x = s^2$, $y = t^2$ and $z = \sqrt{2}st$, where $t, s \in \mathbb{C}$. This representation, however, is not unique. In fact, (s, t) and (s', t') give the same point if and only if $(s, t) = (s', t')$ or $(s, t) = (-s', -t')$. In other words, $W \simeq \mathbb{C}^2/\{\pm 1\}$. Hence, W is isomorphic to the variety V of Example 2.1.10. On the other hand, note that

$$|x|^2 + |y|^2 + |z|^2 = (|s^2| + |t^2|)^2.$$

That is, the intersection of W with the unit sphere in $\mathbb{C}^3$ is homeomorphic to $S^3/_{(s,t)\sim(-s,-t)} = \mathbb{R}P^3$. Equivalently, $W \setminus \{(0,0,0)\}/\mathbb{R}^+$ is homeomorphic to $\mathbb{R}P^3$, a rational 3-sphere. Next, consider the usual $\mathbb{C}^*$-action on $\mathbb{C}^3$ given by scalar multiplication. Because W is an invariant subvariety, we conclude, with the aid of Theorem 2.1.15, that W is a rational cell. Alternatively, this shows that the variety V of Example 2.1.10 is a rational cell. Neither W nor V are topological manifolds, for they are cones over $\mathbb{R}P^3$.

Proposition 2.1.18. *Let (X, x_0) be a rational cell of dimension n. Denote by X^+ its one point compactification. Then X^+ is simply connected and has the rational homotopy type of $\mathbb{S}^{2n}$, the Euclidean $2n$-sphere.*

Proof. First, observe that X^+ is path-connected. As a consequence of Theorem 2.1.15, we can write X^+ as a union of two open cones D_0 and D_∞; namely, $D_0 = S \times [0,1)/S \times \{0\}$ and $D_\infty = S \times (\epsilon, \infty]/S \times \{\infty\}$, where S stands for $\mathbb{S}(X) = (X \setminus \{x_0\})/\mathbb{R}_+$, and ϵ is a positive number less than 1. Given that $X - \{x_0\}$ is path-connected, the intersection $D_0 \cap D_\infty = S \times (\epsilon, 1)$ is path-connected as well. In summary, X^+ can be written as the union of two contractible open subsets with path-connected intersection. Thus, by van Kampen's theorem, X^+ itself is simply connected. To finish the proof, we need to show that X^+ is a rational cohomoloy $2n$-sphere. This is a simple exercise, using the Mayer-Vietoris sequence of the the cover $\{D_0, D_\infty\}$. $\square$

Example 2.1.19. Rationally smooth torus embeddings ([D]). These are exactly the simplicial toric varieties (see Section 1.5.2). In fact, rationally smooth torus embeddings admit a decomposition into rational cells (see Chapter 3).

Example 2.1.20 (Schubert varieties)**.** Let G be a semisimple group and let G/B be its flag variety. We know, from Section 1.5.1, that G/B admits a T-invariant decomposition into affine cells; namely

$$G/B = \bigsqcup_{w \in W} C_w,$$

where $C_w = B[w] = BwB/B$ is isomorphic to $\mathbb{C}^{\ell(w)}$. Here $\ell(w)$ is the length of w. Let X_w be the Zariski closure of C_w in G/B. In this context, C_w is called a **Schubert cell** and X_w is the corresponding **Schubert variety**. In general, Schubert varieties are far from being smooth or even rationally smooth. However, there is a fundamental result (see [Hu]) which says that

$$X_w = \bigsqcup_{v \in W, v \leq w} C_v,$$

where $v \leq w$ in the Bruhat order of W. Based on this result, and Corollary 2.3.3, one concludes that Schubert varieties have trivial cohomology in odd degrees. They also contain a finite number of T-invariant curves and fixed points (see [C]), so Schubert varieties are GKM-varieties (Definition 1.4.13).

Lemma 2.1.21 (One-dimensional rational cells)**.** *Let (X, x) be a rational cell of dimension one. Then*

1. *X is a cone over a topological circle.*

2. *X is homeomorphic to $\mathbb{C}$.*

3. *If, additionally, X is normal, then X is isomorphic to $\mathbb{C}$ as an algebraic variety.*

Proof. Without loss of generality, we can assume that T acts faithfully on X. Thus, T is isomorphic to $\mathbb{C}^*$. Now assertions (1) and (2) can be proved as follows. Since X is one-dimensional, then the singular locus is an invariant discrete set. Nonetheless, x_0 is the unique attractive fixed point, and $\mathbb{C}^*$ is *connected*, so the singular locus is either empty or consists of only one point, namely, x_0. As a result, $X \setminus \{x_0\}$ is smooth. Next notice that X has two $\mathbb{C}^*$-orbits: the attractive fixed point x_0, and a dense open orbit of the form $\mathbb{C}^*/\Gamma$, where Γ is a finite group. Hence, X is homeomorphic to $\mathbb{C}$ and it is a cone over the circle S^1/Γ.

Finally, if we also assume that X is normal and one-dimensional, then *a fortiori* X is smooth ([Har]). This proves (3). □

Lemma 2.1.22. *Let (X, x) be a rational cell. Suppose x is a smooth point. Then X is isomorphic to its tangent space at x.*

Proof. By Proposition 2.1.6, we know that X admits an equivariant closed embedding into T_xX. If x is a smooth point, then both X and T_xX have the same dimension. For affine varieties this can only happen if $X = T_xX$. □

2.2 Filtrations of topological spaces

2.2.1 Algebraic torus actions

Let X be a projective algebraic variety with a $\mathbb{C}^*$-action. Let $X^T = \bigcup_{i=1}^r X_i$ be the decomposition of the fixed point set into irreducible components. Define, for $i = 1, \ldots, r$, the set

$$W_i^s = \bigcup_{a \in X_i} W^s(a),$$

where $W^s(a) = \{x \in X \,|\, \lim\limits_{t \to 0} t \cdot x = a\}$. Analogously, define

$$W_i^u = \bigcup_{a \in X_i} W^u(a),$$

for $i = 1, \ldots, r$, where, this time, $W^u(a)$ denotes the set $\{x \in X | \lim_{t\to\infty} t \cdot x = a\}$. Then W_i^s and W_i^u will be called the stable and unstable subvarieties of X corresponding to X_i, respectively. It follows from [BB1] that $\{W_i^s\}$, $\{W_i^u\}$ are decompositions of X into locally closed subvarieties. These decompositions will be called stable and unstable, respectively. Following the terminology of [BB2], the subvarieties W_i^s and W_i^u will be called cells of the decompositions.

Remark 2.2.1. Assume that X is irreducible. Because the stable and unstable decompositions are locally closed, it follows that there is exactly one i (resp. j) such that W_i^s (resp. W_j^u) is open in X.

Example 2.2.2. In general the BB-decomposition of a projective variety is not a stratification; that is, it may happen that the closure of a BB-cell is not the union of cells, even if we assume our T-variety X to be smooth, as the following example of Bialynicki-Birula ([BB2]) shows. Let $\mathbb{C}^*$ act on $\mathbb{C}\mathrm{P}^2$ via

$$t \cdot [x_0, x_1, x_2] = [x_0, tx_1, t^2x_2].$$

The induced $\mathbb{C}^*$-action on the tangent space $T_{e_1}\mathbb{C}\mathrm{P}^2$ at $e_1 = [0, 1, 0]$ is of the form $t \cdot [y_1, y_2] = [t^{-1}y_1, ty_2]$. Let $\phi : X \to \mathbb{C}\mathrm{P}^2$ be the blowing up of e_1. Since e_1 is fixed under the action, we have an induced action of $\mathbb{C}^*$ on X. There are exactly two fixed points of the action contained in $\phi^{-1}(e_1) \simeq \mathbb{C}\mathrm{P}^1$, they correspond to two invariant one-dimensional subspaces of $T_{e_1}\mathbb{C}\mathrm{P}^2$. Let p_1 be the point representing the subspace spanned by $[1, 0]$ and p_2 the one corresponding to the subspace spanned by $[0, 1]$. Then, for the $\mathbb{C}^*$-action on X we have:

$$W^u(p_2) = \{[[y_1, y_2]] \in \phi^{-1}(e_1) \mid y_2 \neq 0\},$$

$$W^s(p_1) = \{[[y_1, y_2]] \in \phi^{-1}(e_1) \mid y_1 \neq 0\}.$$

Clearly,

$$\overline{W^u(p_2)} = \overline{W^s(p_1)} = \phi^{-1}(e_1)$$

together with

$$W^u(p_1) = \hat{y_1},$$

where $\hat{y_1}$ is the lifting of the y_1-axis of $T_{e_1}\mathbb{C}\mathrm{P}^2$ to X. Needless to say, $W^u(p_1) \neq \{p_1\}$. Hence, $\overline{W^u(p_2)} = W^u(p_2) \cup \{p_1\}$ and $\overline{W^u(p_2)} \cap W^u(p_1) \neq \emptyset$. However, $\overline{W^u(p_2)}$ does not contain $W^u(p_1)$. Thus, the unstable decomposition of X is not a stratification.

2.2.2 Filtrable spaces

Definition 2.2.3. Let X be a complex algebraic variety endowed with a $\mathbb{C}^*$-action. A **BB-decomposition** $\{W_i^s\}$ (resp. $\{W_i^u\}$) is said to be **filtrable** if there exists a finite decreasing sequence $X_0 \supset X_1 \supset \ldots \supset X_m$ of closed subvarieties of X such that:

a) $X_0 = X$, $X_m = \emptyset$,

b) For each $j = 0, \ldots, m-1$, the "stratum" $X_j - X_{j+1}$ is a cell of the decomposition $\{W_i^s\}$ (resp. $\{W_i^u\}$).

Remark 2.2.4. If the BB-decomposition is a stratification, then it is filtrable.

The following result is due to Bialynicki-Birula ([BB2]). We include the proof here for the reader's convenience.

Theorem 2.2.5. *Let X be a normal projective algebraic variety with a torus action. Then the stable and unstable decompositions are filtrable.*

Proof. Since X is normal and projective, Sumihiro's results ([Su]) imply that there exists an equivariant embedding of X into $\mathbb{C}\mathrm{P}^s$ with a linear action of $\mathbb{C}^*$. The decompositions of $\mathbb{C}\mathrm{P}^s$ determined by the action are filtrable. This can be shown as follows. Without loss of generality, we can assume that the $\mathbb{C}^*$-action on $\mathbb{C}\mathrm{P}^s$ is diagonal and

$$t \cdot [x_0, \ldots, x_s] = [t^{n_0}x_0, \ldots, t^{n_s}x_s],$$

where $n_0, \ldots, n_s$ are integers and $n_j \leq n_{j+1}$, for $j = 0, \ldots, s-1$. Let

$$n_0 = \ldots = n_{j_1-1} < n_{j_1} = \ldots = n_{j_2-1} < n_{j_2} = \ldots < n_{j_q} = \ldots = n_s,$$

and let H_i be the projective subspace of $\mathbb{C}\mathrm{P}^s$ defined by equations $x_0 = \ldots = x_{j_1-1} = 0$. Moreover, let P_i be the projective subspace of $\mathbb{C}\mathrm{P}^s$ defined by equations $x_0 = \ldots = x_{j_i-1} = x_{j_{(i+1)}} = \ldots = x_s = 0$, for $i = 0, \ldots, q$. Then,

$$\bigcup P_i = (\mathbb{C}\mathrm{P}^s)^{\mathbb{C}^*}, \qquad H_i \supset H_{i+1}, \qquad H_0 = \mathbb{C}\mathrm{P}^s, \qquad H_{q+1} = \emptyset,$$

and the difference $H_i - H_{i+1}$ is the cell of the stable decomposition of $\mathbb{C}\mathrm{P}^s$ composed of those points x such that $\lim\limits_{t \to 0} tx \in P_i$.

In order to show that the stable decomposition $\{W_i^s\}$ of X is also filtrable notice first that

$$X^{\mathbb{C}^*} = (\mathbb{C}\mathrm{P}^s)^{\mathbb{C}^*} \cap X = \bigcup P_i \cap X$$

and

$$P_i \cap P_{i'} = \emptyset,$$

for $i \neq i'$. Hence, irreducible components of $P_i \cap X$, for $i = 1, \ldots, q$, coincide with irreducible components of $X^{\mathbb{C}^*}$. Moreover, the intersection $(H_i - H_{i+1}) \cap X$ is composed of all such points $x \in X$ that satisfy the condition $\lim\limits_{t \to 0} tx \in P_i \cap X$. In other words, $(H_i \cap X) - (H_{i+1} \cap X)$ is a union of some cells of the stable decomposition, say

$$(H_i \cap X) - (H_{i+1} \cap X) = W_{i_1}^s \cup \ldots \cup W_{i_l}^s.$$

Since, for $j \neq k$, we have

$$(W_{i_j}^s \cap P_i) \cap (W_{i_k}^s \cap P_i) = \emptyset$$

and $W_{i_j}^s \cap P_i$ is closed (as an irreducible component of $X^{\mathbb{C}^*}$), then the intersection

$$\overline{W_{i_j}^s} \cap \overline{W_{i_k}^s},$$

for $j \neq k$, is contained in $H_{i+1} \cap X$. Therefore, the union

$$H_{i+1} \cap X \cup W^s_{i_1} \cup \ldots \cup W^s_{i_r}$$

is closed, for $i = 1, \ldots, l$.

Suppose that we have already defined a sequence $X_0 \supset \ldots \supset X_p$ of closed subschemes of X such that $X_0 = X$, $X_p = H_i \cap X$ and $X_j - X_{j+1}$ is a cell of the stable decomposition, for $j = 0, \ldots, p-1$. Then we put

$$X_{p+j} = (H_{i+1} \cap X) \cup W^s_{i_1} \cup \ldots \cup W^s_{i_{l-j}},$$

for $j = 1, \ldots, l$. This proves that the stable decomposition $\{W^s_i\}$ of X is filtrable. The same result also holds for the unstable decomposition. □

Remark 2.2.6. Jurkiewicz ([J]) gives an example of a $\mathbb{C}^*$-action on a complete nonsingular toric variety X for which the stable decomposition is not filtrable. Hence, Theorem 2.2.5 is not applicable to *non-projective* complete varieties.

2.3 Homology and Betti numbers of $\mathbb{Q}$-filtrable spaces

Lemma 2.3.1. *Let X be an n-dimensional complex projective algebraic variety with a $\mathbb{C}^*$-action. Suppose X can be decomposed as the disjoint union*

$$X = Y \sqcup C,$$

where Y is a closed stable subvariety and C is an open rational cell containing a fixed point of X, say c_0, as its unique attractive fixed point. Then,

$$H^k(X, Y) = \begin{cases} 0 & \text{if} \quad k \neq 2n \\ \mathbb{Q} & \text{if} \quad k = 2n. \end{cases}$$

Furthermore, if Y has vanishing odd cohomology, then

$$H^k(X,\mathbb{Q}) = \begin{cases} H^k(Y,\mathbb{Q}) & \text{if} \quad k \neq 2n \\ H^{2n}(Y,\mathbb{Q}) \oplus \mathbb{Q} & \text{if} \quad k = 2n. \end{cases}$$

Proof. Let $H^*_c(-)$ denote cohomology with compact supports. It is well-known that $H^*(X) = H^*_c(X)$ and $H^*(Y) = H^*_c(Y)$, because X and Y are complex projective varieties. Moreover, by Corollary B.14 of [PS], one has

$$H^*(X,Y) \simeq H^*_c(X-Y) = H^*_c(C).$$

Given that C is a rational cell, and a cone over a rational cohomology sphere of dimension $2n-1$ (Corollary 2.1.16), it follows easily that

$$H^*_c(C) = H^*(C, C - \{c_0\}) = \begin{cases} 0 & \text{if} \quad k \neq 2n \\ \mathbb{Q} & \text{if} \quad k = 2n. \end{cases}$$

So the first claim is proved.

As for the second assertion, consider the long exact sequence of the pair (X,Y), namely,

$$\ldots \longrightarrow H^{*-1}(Y) \longrightarrow H^*(X,Y) \longrightarrow H^*(X) \longrightarrow H^*(Y) \longrightarrow H^{*+1}(X,Y) \longrightarrow \ldots.$$

By our previous remarks, this long exact sequence can be rewritten as

$$\ldots \longrightarrow H^{*-1}(Y) \longrightarrow H^*_c(C) \longrightarrow H^*(X) \longrightarrow H^*(Y) \longrightarrow H^{*+1}_c(C) \longrightarrow \ldots.$$

If Y has no cohomology in odd degrees, then the long exact sequence splits, yielding the identifications $H^i(X) = H^i(Y)$, when $i \neq 2n$, and

$$H^{2n}(X) = H^{2n}(Y) \oplus H^{2n}_c(C) = H^{2n}(Y) \oplus \mathbb{Q}.$$

The proof is now complete. □

Corollary 2.3.2. *Keeping the notation of Lemma 2.3.1, attaching a $2n$-dimensional rational cell produces no changes in cohomology up to degree $2n-2$. Furthermore, if Y has no cohomology in odd degrees, then X has no odd cohomology either, and there is a short exact sequence of the form*

$$0 \longrightarrow H_c^{2n}(C) \longrightarrow H^{2n}(X) \longrightarrow H^{2n}(Y) \to 0.$$

Proof. We simply observe that the long exact sequence of the pair (X, Y) gives

$$H^k(X) \simeq H^k(Y)$$

for $k \leq 2n-2$. Besides, we also obtain the exact sequence

$$0 \longrightarrow H^{2n-1}(X) \longrightarrow H^{2n-1}(Y) \longrightarrow H_c^{2n}(C) = \mathbb{Q} \longrightarrow H^{2n}(X) \longrightarrow H^{2n}(Y) \longrightarrow 0.$$

So in general $H^{2n-1}(X)$ injects into $H^{2n-1}(Y)$. In case we assume Y to have vanishing odd cohomology, we obtain X with vanishing odd cohomology as well, and a "lifting of generators" sequence:

$$0 \longrightarrow H_c^{2n}(C) \longrightarrow H^{2n}(X) \longrightarrow H^{2n}(Y) \to 0.$$

□

Corollary 2.3.3. *Let X be a normal complex projective variety endowed with a $\mathbb{C}^*$-action and a finite number of fixed points. Suppose that X can be written as a disjoint union of rational cells, each one containing a fixed point of X as its unique attractive fixed point. Then X has vanishing odd cohomology over the rationals, and the dimension of its cohomology group in degree $2k$ equals the number of rational cells of complex dimension k. Furthermore, X is equivariantly formal and $\chi(X) = |X^T|$.*

Proof. Since the BB-decomposition on X is filtrable, the result follows from the previous lemma as we move up in the filtration by attaching one rational cell at the time. This process is systematic and preserves cohomology in lower degrees at each step. □

Let T be an algebraic torus acting on a variety X. A one-parameter subgroup $\lambda : \mathbb{C}^* \to T$ is called *generic* if $X^{\mathbb{C}^*} = X^T$, where $\mathbb{C}^*$ acts on X via λ. Generic one-parameter subgroups always exist. Note that the BB-cells of X, obtained using λ, are T-invariant.

Our results in this section suggest the following definition.

Definition 2.3.4. Let X be a projective variety equipped with a T-action. We say that X is **$\mathbb{Q}$-filtrable** if

1. X is normal,
2. the fixed point set X^T is finite, and
3. there exists a generic one-parameter subgroup $\lambda : \mathbb{C}^* \to T$ for which the associated BB-decomposition of X consists of T-invariant *rational cells.*

Theorem 2.3.5. *Let X be a normal projective T-variety. Suppose that X is $\mathbb{Q}$-filtrable. Then*

(a) X admits a filtration into closed subvarieties X_i, $i = 0, \ldots, m$, such that

$$\emptyset = X_0 \subset X_1 \subset \ldots \subset X_{m-1} \subset X_m = X.$$

(b) each cell $C_i = X_i \setminus X_{i-1}$ is a rational cell, for $i = 1, \ldots, m$.

(c) For each $i = 1, \ldots, m$, the singular rational cohomology of X_i vanishes in odd degrees. In other words, each X_i is equivariantly formal.

(d) If, in addition, the T-action on X is T-skeletal, then each X_i is a GKM-variety.

Proof. Assertions (a) and (b) are a direct consequence of Definition 2.3.4 and Proposition 2.2.5. Applying Corollary 2.3.3 and Theorem 1.4.7 at each step of the filtration yields claim (c). For statement (d), we argue as follows. Notice that all the X_i's have vanishing odd cohomology, as it is guaranteed by (c). Moreover, since the X_i's are T-invariant and the T-action on X is T-skeletal, then each X_i contains only a finite number of fixed points and T-invariant curves. In consequence, Theorem 1.4.15 applied to each X_i gives (d). $\square$

Next, we state a result of Bialynicki-Birula ([BB1]).

Lemma 2.3.6. *Let X be a smooth projective variety on which a torus acts with a finite number of fixed points. Then X is filtrable and its integral cohomology is zero in odd degrees. In particular, smooth projective varieties with a T-skeletal torus action are GKM-varieties.*

Proof. It follows from the results of [BB1] that X can be decomposed into cells W_i isomorphic to affine spaces $\mathbb{C}^{n_i}$. Clearly X is normal, and so it is filtrable by Theorem 2.2.5. Finally, using Corollary 2.3.3 and Theorem 2.3.5, we verify the claims. $\square$

2.4 Equivariant Normalization Lemma

Let us start with a few technical propositions. For a proof, the reader is invited to consult [Br5], Propositions A3 and A4.

Proposition 2.4.1. *Let X be an affine variety with a $\mathbb{C}^*$-action and an attractive fixed point x. Then there exists a $\mathbb{C}^*$-module V and a finite equivariant surjective morphism $\pi : X \to V$ such that $\pi^{-1}(0) = \{x\}$.* $\square$

Proposition 2.4.2. *Let X be a connected variety with a nontrivial action of a torus T and a fixed point x. Then there exists a closed irreducible T-stable curve $C \subset X$ which contains x as an isolated fixed point.* $\square$

The following is a result of Brion ([Br5]) on rational smoothness and torus actions.

Theorem 2.4.3. *Let X be an irreducible affine T-variety with an attractive fixed point x. Then X is rationally smooth at x if and only if the following conditions hold:*

(i) A punctured neighborhood of x in X is rationally smooth.

(ii) $X^{T'}$ is rationally smooth at x for each subtorus $T' \subset T$ of codimension one.

(iii) $\dim(X) = \sum_{T'} \dim(X^{T'})$, where the (finite) sum runs over all codimension-one subtori for which $X^{T'} \neq X^T$. □

Let X be an affine T-variety with an attractive fixed point x. Then, by Proposition 2.1.6, X admits a closed equivariant embedding into its tangent space T_xX. Notice that there are only a finite number of codimension-one subtori $T_1, \dots, T_m$ of T for which $X^{T_j} \neq X^T$. Certainly, each one of them is contained in the kernel of a weight of T in T_xX. On the other hand, T acts on each X^{T_i} through its quotient $T/T_i \simeq \mathbb{C}^*$. Because x is an attractive fixed point of X, we can assume, without loss of generality, that x is an attractive fixed point of each X^{T_i}, for the induced action of $\mathbb{C}^* \simeq T/T_i$.

We are now ready to state what we call the Equivariant Normalization Theorem for rational cells. It is due to Brion ([Br3]) and Arabia ([Ar]).

Theorem 2.4.4. *Let (X, x) be a rational cell. Then there exists a T-module V and an equivariant finite surjective map $\pi : X \to V$ such that $\pi(x) = 0$ and $V^T = \{0\}$.*

It is worth pointing out that some of the arguments to appear next are well-known constructions in algebraic geometry.

Proof of Theorem 2.4.4. We follow closely Brion's construction ([Br3], Theorem 18). Since x is an attractive fixed point, there exists an equivariant embedding ι of X into T_xX, its tangent space at x. In other words, all the weights of T in T_xX lie in an open half space of $\mathfrak{t}^*$. As it was emphasized before, there is only a finite collection of codimension-one subtori, say $T_1, \ldots, T_m$, for which $X^{T_j} \neq X^T$. Let T_i be one of them. Under the present circumstances, given that x is attractive, we can also assume that x is an attractive fixed point of X^{T_i}, for the induced action of $\mathbb{C}^* \simeq T/T_i$. Hence, by Proposition 2.4.1, there exists a T-equivariant finite surjective map $\pi_i : X^{T_i} \to V_i$, where V_i is some T-module with a trivial action of T_i. Notice that T acts on both X^{T_i} and V_i through the same character.

By construction X^{T_i} is T-stable and closed in X, so we can extend π_i to an equivariant morphism

$$\pi_i : X \to V_i.$$

Synchronizing efforts via the product map, we obtain an equivariant morphism

$$\pi : X \to V,$$

where V is the direct sum of the V_i, sum taken over all the T_i's above. Notice that x, being an attractive fixed point, lies in the closure all the T-orbits in X. In particular, x is contained in all the irreducible components of $\pi^{-1}(0)$ (i.e. $\pi^{-1}(0)$ is connected).

We now claim that the morphism π is finite. Indeed, $\{x\} = \pi^{-1}(0)$. For otherwise, $\pi^{-1}(0)$ would contain a T-stable curve upon which T acts through a non-trivial character (Proposition 2.4.2). Certainly this is impossible, because π restricts to a finite morphism on each X^{T_i}.

To conclude the proof, recall that, by definition, V satisfies

$$\dim(V) = \sum_{T_i} \dim(X^{T_i}).$$

Since X is rationally smooth at x, Theorem 2.4.3 (iii) dictates that X and V must have the same dimension. In conclusion, π is both dominant and surjective. $\square$

Remark 2.4.5. It is clear from the proof of Theorem 2.4.4 that if X is smooth, then the map $\pi : X \to V$ can be chosen to be an isomorphism.

We now specialize a result of Brion ([Br5]) to rational cells.

Corollary 2.4.6. *Let (X, x) be a rational cell. Suppose that the number of closed irreducible T-stable curves on X is finite. Let $n(X, x)$ be this number. Then*

$$n(X, x) = dim(X).$$

Proof. Each closed irreducible T-stable curve C_i is the fixed point set of a unique codimension-one torus, say T_i. Since there are only a finite number of codimension-one tori, say $T_1, \ldots, T_m$, for which $X^{T_i} \neq X^T$, then it follows from the proof of Theorem 2.4.4 that the equality below holds:

$$dim(X) = \sum_{j=1}^{m} dim(X^{T_j}) = \sum_{j=1}^{m} dim(C_j) = n(X, x).$$

We are done. $\square$

2.5 Equivariant Euler classes

Denote by T an algebraic torus.

Let (Y, y_0) be a rational cell of dimension n. Recall that $\mathbb{S}(Y) = [Y - \{y_0\}]/\mathbb{R}^+$ is a rational cohomology sphere S^{2n-1} and that Y is homeomorphic to the (open) cone over $\mathbb{S}(Y)$ (Theorem 2.1.15 and Corollary 2.1.16).

The Borel construction (Section 1.1) yields the fibration

$$\mathbb{S}(Y) \hookrightarrow \mathbb{S}(Y)_T \longrightarrow BT.$$

Observe that the E_2-term of the corresponding Serre spectral sequence consists of only two lines, namely,

$$E_2^{p,q} = H^p(BT) \otimes H^q(\mathbb{S}(Y)) \neq 0 \text{ only when } q = 0 \text{ and } q = 2n-1.$$

Let $\mathrm{Eu}_T(y_0, Y) \in H^{2n}(BT)$ be the transgression of the generator $\lambda_Y \in H^{2n-1}(\mathbb{S}(Y))$. We call $\mathrm{Eu}_T(y_0, Y)$ the **equivariant Euler class of Y at y_0**.

It follows from [Hs], Theorem IV.6, that $\mathrm{Eu}_T(y_0, Y)$ splits into the product of linear polynomials, namely

$$\mathrm{Eu}_T(y_0, Y) = \omega_1^{k_1} \cdots \omega_s^{k_s},$$

where $w_i \in H^2(BT) \simeq \Xi(T) \otimes \mathbb{Q}$. Here $\Xi(T)$ stands for the character group of T, and the isomorphism is given by assigning to each character χ the first Chern class of the line bundle $ET \times_T \mathbb{C} \to BT$, where T acts on $\mathbb{C}$ by $t \cdot z = \chi(t)z$. Likewise, the results of Hsiang ([Hs], Chapter V.1) yield the following identification

$$H_T^*(\mathbb{S}(Y)) \simeq H_T^*(pt)/\langle \mathrm{Eu}_T(y_0, Y) \rangle,$$

where $\langle \mathrm{Eu}_T(y_0, Y) \rangle$ denotes the principal ideal of $H_T^*(pt)$ generated by $\mathrm{Eu}_T(y_0, Y)$.

Since Y is a cone over $\mathbb{S}(Y)$, then $H_c^*(Y) \simeq H^*(Y, Y - \{y_0\}) \simeq \mathbb{Q}$, where $H_c^*(-)$ denotes cohomology with compact supports. Using the Serre spectral sequence, one notices that these isomorphisms are also valid in equivariant cohomology:

$$H_{T,c}^*(Y) \simeq H_T^*(Y, Y - \{y_0\}) \simeq H_T^*.$$

Let $\mathcal{T}_Y$ be canonical generator of $H_T^*(Y, Y - \{y_0\})$. This generator can be described by the commutative diagram

$$\begin{tikzcd}
H_T^*(Y, Y - \{y_0\}) \arrow[rr, "i^*"] \arrow[d, shift left, "\int_{[Y]}"] & & H_T^*(Y) \arrow[d, "res"] \\
H_T^*(y_0) \arrow[u, shift left, "\Phi_Y^*"] \arrow[rr, dashed, "\times(\mathrm{Eu}_T(y_0,Y))"] & & H_T^*(y_0),
\end{tikzcd}$$

where Φ_Y^* is multiplication by $\mathcal{T}_Y$. In other words, $\mathcal{T}_Y$ is the unique class in $H_T^*(Y, Y - \{y_0\})$ whose restriction to $H_T^*(pt)$ coincides with $\mathrm{Eu}_T(y_0, Y)$. It is customary in the literature to call $\mathcal{T}_Y$ the **Thom class of** Y. Let us bear in mind that the map Φ_Y^* raises degree by $2n$. Clearly, $H_T^*(Y, Y - \{y_0\}) \simeq H_c^*(Y) \otimes H_T^*(pt)$ and so, $H_{T,c}^j(Y) = 0$ for $j < 2n$. As for the integral appearing here, it is, by definition, the inverse of Φ_Y^*.

Let $\mathcal{Q}_T$ be the quotient field of H_T^*. If $\mu \in H_{T,c}^*(Y)$, then

$$\mathrm{Eu}_T(y_0, Y) \wedge \int_{[Y]} \mu = \mu_{y_0},$$

where μ_{y_0} denotes restriction of the class μ to y_0. Hence, the identity

$$\frac{1}{\mathrm{Eu}_T(y_0, Y)} = \frac{1}{\mu_{y_0}} \int_{[Y]} \mu,$$

holds in $\mathcal{Q}_T$, for every non-zero μ in $H_T^*(Y, Y - \{y_0\})$.

More generally, let X be a complex algebraic variety with a T-action and an isolated fixed point x. Suppose that X is rationally smooth at x and that x is attractive. By Proposition 2.1.6, there exists an open affine neighborhood X_x of x such that X_x is a rational cell. Thus one defines

$$\mathrm{Eu}_T(x, X) := \mathrm{Eu}(x, X_x).$$

In fact, if we only assume that x is a rationally smooth point of X, the previous definition still makes sense, since we can choose X_x to be a conical neighborhood of x. When working with complex algebraic varieties, such neighborhoods always exist ([Ar]).

From these remarks, it follows that if x is a rationally smooth point of X, then $\mathrm{Eu}_T(x, X)$ is a polynomial, and splits into a product of linear factors.

In case the isolated fixed point $x \in X$ is not necessarily rationally smooth, Arabia ([Ar]) has shown that we can still define an Euler class $\mathrm{Eu}_T(x, X)$. The key ingredient here is that, by the localization theorem, the map

$$i^* : H_T^*(X, X - \{x\}) \to H_T^*(x)$$

is an isomorphism modulo H_T^*-torsion. Therefore, the function that assigns to a torsion-free element $\mu \in H_T^*(X, X - \{x\})$ the fraction $\frac{1}{\mu_x} \int_X \mu \in \mathcal{Q}_T$ is *constant.*

Definition 2.5.1. Let X be a T-variety. Suppose that $x \in X^T$ is an isolated fixed point. The fraction

$$\frac{1}{\mathrm{Eu}_T(x, X)} := \frac{1}{\mu_x} \int_X \mu \in \mathcal{Q}_T,$$

where μ is any torsion-free element of $H_T^*(X, X - \{x\})$, is called the *inverse of the equivariant Euler class of X at x.* When this fraction is non-zero, we denote its inverse by $\mathrm{Eu}_T(x, X)$ and call it the *Equivariant Euler class of X at x.*

Example 2.5.2. When $X = \mathbb{C}^n$, $x = 0$, and the algebraic torus T acts linearly on $\mathbb{C}^n$, one proves

$$\mathrm{Eu}_{\mathbf{T}}(0, \mathbb{C}^n) = (-1)^n \prod_{\alpha \in \mathcal{A}} \alpha,$$

where $\mathcal{A}$ is the collection of weights. Furthermore, if the weights in $\mathcal{A}$ are pairwise linearly independent, then the associated complex projective space $\mathbb{P}(\mathbb{C}^n_{\mathcal{A}})$ has exactly n T-fixed points: the lines $\mathbb{C}_{\alpha_i}$. One also verifies that

$$\mathrm{Eu}_{\mathbf{T}}([\mathbb{C}_{\alpha_i}], \mathbb{P}(\mathbb{C}^n_{\mathcal{A}})) = \prod_{j \neq i} (\alpha_i - \alpha_j).$$

See [Ar], Remark 2.4.1-1.

Proposition 2.5.3 (Localization formula, [Ar])**.** *Let X be a complex projective variety. Suppose that a torus T acts on X with only a finite number of fixed points. Then*

$$\int_X \mu = \sum_{x \in X^T} \frac{\mu|_x}{\mathrm{Eu}_T(x, X)},$$

for any $\mu \in H_T^*(X)$. *Furthermore, taking* $\mu = 1$ *yields*

$$\sum_{x \in X^T} \frac{1}{\mathrm{Eu}_T(x, X)} = 0.$$

□

Theorem 2.5.4 ([Ar], [Br3]). *Let* (X, x) *be a rational cell of dimension* d. *Let* $\pi : X \to \mathbb{C}^n$ *be the equivariant normalization map from Theorem 2.4.4. Then*

(a) The induced morphism in cohomology

$$\pi^* : H_c^{2d}(\mathbb{C}^n) \longrightarrow H_c^{2d}(X)$$

is an isomorphism and satisfies $\int_Y \pi^*(\mu) = deg(\pi) \int_{\mathbb{C}^n} \mu$, *where* $deg(\pi)$ *is the cardinality of a generic fibre of* π. *This formula also holds in equivariant cohomology, in particular*

$$\mathrm{Eu}_T(0, \mathbb{C}^n) = deg(\pi) \cdot \mathrm{Eu}_T(x_0, X).$$

(b) $\mathrm{Eu}_T(X, x) = c \prod_{T_i} \mathrm{Eu}_T(X^{T_i}, x)$, *where* c *is a positive rational number, and the product runs over the finite number of codimension-one subtori* T_i *of* T *for which* $X^{T_i} \neq X^T$.

Proof. By construction, $\pi : X \to \mathbb{C}^n$ is an equivariant finite surjective map of affine varieties. Therefore, it is a covering map outside of a closed subvariety $Z \subset \mathbb{C}^n$. Let $U = \mathbb{C}^n \setminus Z$. Then $\pi : \pi^{-1}(U) \to U$ is a covering map. Notice that the dimension of Z is strictly less than the dimension of $\mathbb{C}^n$, so the long exact sequence of the pair $(X, \pi^{-1}(Z))$ yields $H_c^{2n}(\pi^{-1}(U)) \simeq H_c^{2n}(X)$. Now statement (a) follows from the corresponding statement about the covering map $\pi : \pi^{-1}(U) \to U$.

In order to prove assertion (b), let us keep in mind that the equality

$$deg(\pi)\mathrm{Eu}_T(x_0, X) = \mathrm{Eu}_T(0, \mathbb{C}^n)$$

has been granted by part (a). Also, from the proof of Theorem 2.4.4, we know that there is a finite surjective map $\pi_i : X^{T_i} \to \mathbb{C}^{k_i}$ associated to each codimension-one subtorus T_i for which $X^{T_i} \neq X^T$. What is more, every X^{T_i} is rationally smooth at x. So applying part (a) on the various X^{T_i} yields

$$\mathrm{Eu}_T(0, \mathbb{C}^{k_i}) = deg(\pi_i)\mathrm{Eu}_T(x_0, X^{T_i}).$$

Denote by d the degree of π and by d_i the degree of π_i. Because Euler classes are multiplicative ([Ar]), it follows that

$$\mathrm{Eu}_T(0, \mathbb{C}^n) = \prod_i \mathrm{Eu}_T(0, \mathbb{C}^{k_i}).$$

But the latter term equals $\prod_i d_i \mathrm{Eu}_T(x_0, X^{T_i})$. Matching the expressions above finally concedes

$$\mathrm{Eu}_T(x_0, X) = \left(\frac{\prod_i d_i}{d}\right) \cdot \prod_i \mathrm{Eu}_T(x_0, X^{T_i}) = c \cdot \prod_i \mathrm{Eu}_T(x_0, X^{T_i}).$$

□

Corollary 2.5.5. *Let (X, x) be a rational cell of dimension n. Suppose that X contains only a finite number of closed irreducible T-curves $\mathcal{C}_i$, $i = 1, \ldots, n$. Let χ_i be the character associated with the action of T on $\mathcal{C}_i$. Then*

$$\mathrm{Eu}(x_0, X) = c \cdot \chi_1 \cdots \chi_n,$$

where c is a positive rational number.

Proof. In this case, $X^{T_i} = \mathcal{C}_i$. The result can now be deduced from Theorem 2.5.4 (b) and Example 2.5.2. □

2.6 Module generators for $H_T^*(X)$

Let X be a $\mathbb{Q}$-filtrable GKM-variety. In other words, X is a normal projective T-variety with only a finite number of fixed points and T-invariant curves. Moreover, there exists a BB-decomposition of X as a disjoint union of rational cells, say $(C_1, x_1), \ldots, (C_m, x_m)$, each one containing $x_i \in X^T$ as its unique attractive fixed point. This decomposition induces a filtration of X

$$\emptyset = X_0 \subset X_1 \subset X_2 \ldots \subset X_m = X$$

by closed invariant subvarieties X_i, so that each difference $X_i \setminus X_{i-1}$ equals C_i, for $i = 1, \ldots, m$. The key observation here is provided by Theorem 2.3.5. It states that every X_i is equivariantly formal and is made up of rational cells. In consequence, GKM-theory can be applied to each X_i. We will refer to X_i as the *i-th filtered piece of X*, and m will be called the *length of the filtration*.

Denote by $x_1, \ldots, x_m$ the fixed points of X. The filtration induces a total ordering of the fixed points, namely,

$$x_1 < x_2 < \ldots < x_m.$$

Let (C_i, x_i) be a rational cell of X. From the previous section, we know that

$$H^*_{T,c}(C_i) \simeq H^*_T(C_i, C_i - \{x_i\}) \simeq H^*_T(x_i),$$

where the second isomorphism is provided by the Thom class $\mathcal{T}_i$, a well-known element of $H^*_T(C_i, C_i - \{x_i\})$. When restricted to $H^*_T(x_i)$, the Thom class $\mathcal{T}_i$ becomes a product of linear polynomials: the Euler class $\mathrm{Eu}(c_i, C_i)$.

In section 2.3 we built non-equivariant short exact sequences of the form

$$0 \longrightarrow H^{2k}_c(C_i) \longrightarrow H^{2k}(X_i) \longrightarrow H^{2k}(X_{i-1}) \longrightarrow 0 \,,$$

for every i. Since the spaces involved have zero cohomology in odd degrees, then these short exact sequences naturally generalize to the equivariant case, so we also have equivariant short exact sequences

$$0 \longrightarrow H^{2k}_{T,c}(C_i) \longrightarrow H^{2k}_T(X_i) \longrightarrow H^{2k}_T(X_{i-1}) \longrightarrow 0\ ,$$

for each i. On the other hand, by equivariant formality, the singular equivariant cohomology of each X_i injects into $H^*_T(X_i^T) = \oplus_{j\leq i} H^*_T(x_j)$.

In summary, for each i, we have the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^*_{T,c}(C_{i+1}) & \longrightarrow & H^*_T(X_{i+1}) & \longrightarrow & H^*_T(X_i) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H^*_T(x_{i+1}) & \longrightarrow & \oplus_{j\leq i+1} H^*_T(x_j) & \longrightarrow & \oplus_{j\leq i} H^*_T(x_j) & \longrightarrow & 0 \end{array}$$

where the vertical maps are all injective. Indeed, such maps correspond to the various restrictions to fixed point sets. We will use this diagram to build cohomology generators. The next two lemmas are inspired in Theorem 2.3 and Proposition 4.1 of [HHH], where Kac-Moody flag varieties are studied.

Lemma 2.6.1. *Let X be a $\mathbb{Q}$-filtrable variety. Then there exists a non-canonical isomorphism of H^*_T-modules*

$$H^*_T(X) \simeq \bigoplus_{x_i\in X^T} \mathrm{Eu}_T(C_i, x_i) H^*_T(pt),$$

which is compatible with restriction to the various i-th filtered pieces $X_i \subset X$.

Proof. We argue by induction on the length of the filtration. The case $m = 1$ is simple, because it corresponds to $X = \{x_1\}$, a singleton. Assuming that we have proved the assertion for m, let us prove the case $m + 1$. Substitute $i = m$ in the commutative diagram above. Then

$$H^*_T(X_{m+1}) = H^*_T(X) \simeq H^*_{T,c}(C_{m+1}) \oplus H^*_T(X_m).$$

By induction, $H_T^*(X_m) \simeq \prod_{i\leq m} \mathrm{Eu}_T(C_i, x_i)H_T^*(pt)$. So the claim for $m+1$ follows directly from the equivalence between $H_{T,c}^*(C_{m+1})$ and $\mathrm{Eu}_T(C_{m+1}, x_{m+1})H_T^*(pt)$. $\square$

The isomorphism of the previous Lemma is not canonical because the cellular decomposition of X depends on a particular choice of generic one-parameter subgroup.

Given a class $\mu \in H_T^*(X)$, denote by $\mu(x_i)$ its restriction to the fixed point x_i.

Lemma 2.6.2. *Let X be a projective T-variety. Assume that X is $\mathbb{Q}$-filtrable and let $x_1 < x_2 < \ldots < x_m$ be the order relation on X^T compatible with the filtration of X. For each i, let $\varphi_i \in H_T^*(X)$ be a class such that*

$$\varphi_i(x_j) = 0 \textit{ for } j < i,$$

and

$$\varphi_i(x_i) \textit{ is a generator of the ideal } \mathrm{Eu}_T(i, C_i)H_T^*.$$

Then the classes $\{\varphi_i\}$ generate $H_T^*(X)$ freely as a module over $H_T^*(pt)$.

Proof. Since X is equivariantly formal, we know that $H_T^*(X)$ injects into $H_T^*(X^T)$ and is a free H_T^*-module of rank $m = |X^T|$. First, we show that the φ_i's are linearly independent. Arguing by contradiction, suppose there is a non-trivial linear combination such that

$$\sum_{i=0}^{m} f_i\varphi_i = 0,$$

with $f_i \in H_T^*$. Let k be the minimum of the set $\{i \,|\, f_i \neq 0\}$. Then we have

$$f_k\varphi_k + f_{k+1}\varphi_{k+1} + \ldots f_m\varphi_m = 0$$

where $f_k \neq 0$. Let us restrict this linear combination to x_k. Then

$$f_k\varphi_k(x_k) + f_{k+1}\varphi_{k+1}(x_k) + \ldots f_m\varphi_m(x_k) = 0.$$

But $\varphi_\ell(x_k) = 0$ for all $\ell > k$. Thus we obtain

$$f_k \varphi(x_k) = 0.$$

However, $\varphi(x_k)$ is a non-zero multiple of the Euler class $\mathrm{Eu}(x_k, C_k)$ and, as such, it is non-zero. We conclude that f_k must be zero. This is a contradiction.

To conclude the proof, we need to show that the φ_i's generate $H_T^*(X)$ as a module. But this is a routine exercise, using induction on the length of the filtration of X (the base case being trivial). The commutative diagram of page 62 then disposes of the inductive step. $\square$

As for the existence of classes satisfying Lemma 2.6.2, we will show that they can always be constructed on GKM-varieties. First, we need two technical lemmas.

Lemma 2.6.3. *Let X be a normal projective T-variety with finitely many fixed points. Choose a generic one-parameter subgroup and write X as $X = C \sqcup Y$, where*

$$C = \{z \in X \mid \lim_{t \to 0} tz = x\}$$

is the stable cell of $x \in X^T$, and Y is closed and T-stable. Then any closed irreducible T-stable curve that passes through x is contained in the Zariski closure of C.

Proof. Let ℓ be a closed irreducible T-stable curve passing through x. Recall that ℓ is the closure of a one-dimensional orbit Tz. Moreover, $\ell = \overline{Tz}$ has two fixed points, namely, x and a fixed point $y_{i(\ell)}$ contained *necessarily* in Y. We claim that $z \in C$. For otherwise, $\lim_{t \to 0} tz = y_{i(\ell)}$, which implies that z belongs to the stable subvariety of $y_{i(\ell)}$. Since Y is T-invariant and closed, then $\ell = \overline{Tz} \subset Y$. That is, $x \in \partial\ell$ would belong to Y, which is absurd. Thus $z \in C$.

The fact that C is also T-stable gives the inclusion $Tz \subset C$. We conclude that $\ell = \overline{Tz} \subset \overline{C}$. $\square$

Lemma 2.6.4. *Let X be a normal projective variety on which a torus acts with a finite number of fixed points and one-dimensional orbits. Suppose X is equivariantly formal and there is a generic one-parameter subgroup such that X can be written as a disjoint union $X = C \sqcup Y$, where*

$$C = \{z \in X \mid \lim_{t \to 0} tz = x\}$$

is a rational cell with unique attractive fixed point $x \in X^T$, and Y is closed and T-stable. Then the cohomology class $\tau \in \oplus_{w \in X^T} H^*_T(w)$, defined by

$$\tau(x) = \mathrm{Eu}(x, C) \text{ and } \tau(y) = 0 \text{ for all } y \in Y^T,$$

belongs to the image of $H^*_T(X)$ in $H^*_T(X^T)$.

Proof. The hypotheses imply that X is a GKM-variety. As a result, the equivariant cohomology of X can be described by the GKM-relations of Theorem 1.4.11. So, to prove the lemma, it is enough to verify that τ satisfies such relations.

Because τ restricts to zero at every fixed point except x, we need only show that

$$\tau(x) = \tau(x) - \tau(y) = \mathrm{Eu}_T(x, C)$$

is divisible by χ_i whenever the fixed points $x \in C$ and $y_i \in Y^T$ are joined by a T-curve ℓ_i in X, and T acts on ℓ_i through χ_i. Let p be the total number of ℓ_i's.

By Lemma 2.6.3, the curve ℓ_i is contained in the Zariski closure $\overline{C}$ of C. In fact, $\ell_i \setminus \{x, y_i\} \subset C$. Also, it follows from Corollary 2.4.6 that $p = dim(C)$. Thus, using Corollary 2.5.5, we conclude that $\mathrm{Eu}_T(x, C)$ is a non-zero multiple of the χ_i's. In short, τ belongs to $H^*_T(X)$. □

It is noticeable that, in the previous lemmas, no assumption on the irreducibility of X has been made. Surely we allow for some flexibility in this matter, since the various filtered pieces X_i of a $\mathbb{Q}$-filtrable space X need not be irreducible.

Theorem 2.6.5. *Let X be a $\mathbb{Q}$-filtrable GKM-variety. Then cohomology generators $\{\varphi_i\}$ of $H_T^*(X)$ with the properties described in Lemma 2.6.2 exist.*

Proof. We proceed by induction on m, the length of the filtration of X. If $m = 1$, then $X = \{x_1\}$ and the statement is clear, since we can just choose $\varphi_1 = 1$. Assuming we have proved the statement for varieties with a filtration of length m, let us prove the case when the length is $m + 1$. First, notice that $X_{m+1} = X$ and, by the inductive hypothesis, there are classes $\varphi_1, \ldots, \varphi_m \in H_T^*(X_m)$ which satisfy the desired properties in $H_T^*(X_m)$. Using the commutative diagram of page 62, we can lift them to classes $\tilde{\varphi_1}, \ldots, \tilde{\varphi_m}$ which still satisfy the required conditions, though this time they lie in $H_T^*(X_{m+1}) = H_T^*(X)$. In consequence, we just need to construct a class $\varphi_{m+1} \in H_T^*(X)$ with the sought-after qualities. So set $\varphi_{m+1}(x_{m+1}) = \mathrm{Eu}(x_{m+1}, C_{m+1})$ and $\varphi_{m+1}(x_j) = 0$ for all $j \leq m$. By Lemma 2.6.4, this class surely belongs to $H_T^*(X)$. Thus the result also holds for varieties with a filtration of length $m + 1$. This proves the inductive step and concludes the argument. $\square$

Definition 2.6.6. Let X be a $\mathbb{Q}$-filtrable T-variety. Fix an ordering of the fixed points, say $x_1 < x_2 < \ldots < x_m$. Given $\mu \in H_T^*(X)$, we define its **local index at** x_i, denoted $I_i(\mu)$, by the following formula:

$$I_i(\mu) = \int_{X_i} p_i^*(\mu),$$

where $p_i : X_i \to X$ denotes the inclusion of the i-th filtered piece into X. It follows from the definition that assigning local indices yields an H_T^*-linear morphism

$$I_i : H_T^*(X) \to H_T^*(pt).$$

Using the localization formula (Proposition 2.5.3), one can easily prove the following

Lemma 2.6.7. *The local index of μ at x_i satisfies*

$$I_i(\mu) = \sum_{j \leq i} \frac{\mu(x_j)}{\mathrm{Eu}(x_j, X_i)},$$

where $\mu(x_j)$ denotes the restriction of μ to x_j. □

Corollary 2.6.8. *Let $x_i \in X^T$, be a fixed point. Suppose that $\mu \in H_T^*(X)$ is a cohomology class that satisfies $\mu(x_j) = 0$ for all $j < i$. Then*

$$\mu(x_i) = I_i(\mu)\mathrm{Eu}(x_i, X_i).$$

□

Our most important result in this Section is the following generalization of the work of Guillemin and Kogan ([GK]) to $\mathbb{Q}$-filtrable GKM-varieties.

Theorem 2.6.9. *Let X be a $\mathbb{Q}$-filtrable GKM-variety. Let $x_1 < x_2 < \ldots < x_m$ be the order relation on X^T compatible with the filtration of X. Then there exists a unique class $\theta_i \in H_T^*(X)$ with the following properties:*

(i) $I_i(\theta_i) = 1$,

(ii) $I_j(\theta_i) = 0$ *for all* $j \neq i$,

(iii) the restriction of θ_i to $x_j \in X^T$ is zero for all $j < i$, and

(iv) $\theta_i(x_i) = \mathrm{Eu}_T(i, C_i)$.

Moreover, the θ_i's generate $H_T^*(X)$ freely as a module over $H_T^*(pt)$.

Proof. By Theorem 2.6.5, choose a set of free generators $\{\varphi_i\}$ which satisfy the properties described in Lemma 2.6.2, together with the additional condition $\varphi_i(x_i) = \mathrm{Eu}(i, C_i)$.

Given i, notice that $I_j(\varphi_i) = 0$, for all $j < i$, and $I_i(\varphi_i) = 1$. We will show that we can modify these φ_i's accordingly to obtain the generators θ_i. In fact, given $i \in \{1, \dots, m\}$, the only obstruction to setting $\theta_i = \varphi_i$ is that $I_j(\varphi_i)$ can be non-zero for some $j > i$.

Let $i \in \{1, \dots, m\}$. If $I_j(\varphi_i) = 0$ for all $j > i$, then let $\theta_i = \varphi_i$. Otherwise, proceed as follows. Let k_0 be the minimum of all $k > i$ such that $I_k(\varphi_i) \neq 0$. Define $\Psi_i = \varphi_i - I_{k_0}(\varphi_i)\varphi_{k_0}$. Let us compute the local indices of Ψ_i. Clearly, if $j < i$, we have $I_j(\Psi_i) = 0$. Also, if $j = i$, then $I_i(\Psi_i) = 1$. It is worth noticing that Ψ_i restricts to 0 at each x_j with $j < i$. Now if j satisfies $i < j \leq k_0$, then $I_j(\Psi_i) = 0$. So, arguing by induction, we can provide a class $\widetilde{\Psi_i}$ such that $I_j(\widetilde{\Psi_i}) = 0$ for all $j \neq i$, and $I_i(\widetilde{\Psi_i}) = 1$. Thus, set $\theta_i = \widetilde{\Psi_i}$. Working on each i at a time, we conclude that there exist classes θ_i satisfying conditions (i)-(iv) of the Theorem.

Let us now prove uniqueness. Suppose there are classes $\{\theta_i\}$ and $\{\theta_i'\}$ satisfying all the properties of the theorem. Fix i and let $\tau = \theta_i - \theta_i'$. It is clear that τ is an element of $H_T^*(X)$ whose local index $I_j(\tau)$ is zero for all j. Suppose that τ is not zero. Then, since $H_T^*(X)$ injects into $H_T^*(X^T)$, there should be a k such that $\tau(x_k) \neq 0$. Take the minimum of all k's for which $\tau(x_k) \neq 0$. Denote this minimum by s. Then, by Corollary 2.6.8, one would have $\tau(x_s) = I_s(\tau)\mathrm{Eu}(x_s, X_s) = 0$. This is absurd. Therefore $\tau = 0$. Since i can be chosen arbitrarily, we conclude that $\theta_i = \theta_i'$ for all i.

Finally, notice that properties (iii) and (iv) together with Lemma 2.6.2 imply that the θ_i's freely generate $H_T^*(X)$. We are done. $\square$

Chapter 3

Standard Group Embeddings

In this chapter we start our study of rationally smooth standard group embeddings. We show that they are in fact GKM-varieties with a canonical $\mathbb{Q}$-filtration (Theorem 3.2.13). Therefore, all the machinery developed previously can be put into effect to attain a concrete description of their equivariant cohomology. Our results, in this and the subsequent chapter, increase the applicability of GKM theory in the study of group embeddings.

Notable new results are Theorem 3.2.3, Theorem 3.2.7, Theorem 3.2.8 and Theorem 3.2.13.

3.1 Preliminaries

In what follows, all algebraic varieties and groups are considered over the base field $\mathbb{C}$ of complex numbers. Let G be a connected reductive group.

Definition 3.1.1. Let X be an algebraic variety. We say that X is an **embedding of** G if

1. X is a $G \times G$-variety.

2. There is a point $x \in X$ such that $\mathcal{O}_x$, the $G \times G-$orbit of x, is open and dense in X and $\mathcal{O}_x \simeq (G \times G)/\Delta G$; in other words, the two sided action of G on itself, $((a, b), g) \mapsto agb^{-1}$, extends to X.

Let X_1 and X_2 be two embeddings of G. A morphism between them is defined to be a morphism of $G \times G$-varieties $\phi : X_1 \to X_2$ with the property that the diagram

$$\begin{tikzcd} G \arrow[r, hook] \arrow[d, "id"] & X_1 \arrow[d, "\phi"] \\ G \arrow[r, hook] & X_2 \end{tikzcd}$$

commutes.

A morphism between two G-embeddings, if it exists, is unique. We can give a structure of partially ordered set to the collection of embeddings of a group G by setting $X_1 \geq X_2$ if a morphism $\phi : X_1 \to X_2$ exists.

Because of [GKM], it is possible to calculate the equivariant cohomology of many topological spaces using a combinatorial/numerical apparatus known as GKM **data**. This amounts to identifying certain fixed points, curves and characters and then defining the associated ring $PP_T^*(X)$ of *piecewise polynomial functions* (Theorem 1.4.14). It is useful to determine conditions under which there is a canonical isomorphism

$$H_T^*(X; \mathbb{Q}) \cong PP_T(X). \qquad (*)$$

This is certainly the case if X is a smooth, projective variety with a T-skeletal action (Lemma 2.3.6). But there are other conditions that guarantee an isomorphism as in (*) above, for example, when X is a GKM-variety (Theorem 1.4.14) or a $\mathbb{Q}$-filtrable, T-skeletal variety (Theorem 2.3.5).

In the case of group embeddings, it is possible to determine $PP_{T\times T}^*(X)$ in terms of combinatorial data obtained directly from the underlying two-sided action

$$G \times G \times X \to X.$$

We will see in Section 3.2 that in many cases this embedding X can be obtained from a reductive monoid M as $X = \mathbb{P}_\epsilon(M) := [M \setminus \{0\}]/\mathbb{C}^*$, where ϵ is a central, attractive, 1-parameter subgroup of the unit group of M. The purpose of this chapter is to write out the GKM data of $X = \mathbb{P}_\epsilon(M)$ (i.e. fixed points and invariant curves) in terms of M (Section 3.2.1).

3.1.1 Algebraic Monoids

Our main reference here is [R8].

Definition 3.1.2. A linear **algebraic monoid** M is an affine, algebraic variety together with an associative morphism $\mu : M \times M \to M$ and an identity element $1 \in M$ for μ. An affine algebraic monoid M is called **reductive** if it is irreducible, normal, and its unit group is a reductive algebraic group. A reductive monoid is called **semisimple** if it has a zero element, and its unit group has a one-dimensional center.

Throughout this monograph, all algebraic monoids are assumed to be irreducible and linear.

Let M be an algebraic monoid. Denote by G its unit group and by T a maximal torus of G.

An algebraic monoid M comes equipped with a natural $G \times G$-action given by $(g, h) \cdot a = gah^{-1}$. Let $\mathcal{U}(M)$ be the set of orbits $\mathcal{O} = GaG$ which contain an idempotent. The set of idempotents in M is typically denoted by $E(M)$.

Definition 3.1.3. Let M be an algebraic monoid. We say that M is **regular** if $M = GE(M)$.

The next three results can be found in [R8].

Theorem 3.1.4. *Let M be an algebraic monoid with zero. Then the following conditions are equivalent:*

1. *M is regular,*
2. *$\mathcal{U}(M)$ is* the set *of $G \times G$-orbits in M.* □

Theorem 3.1.5. *Let M be an algebraic monoid with zero. Then, M is reductive if and only if M is regular.* □

Theorem 3.1.6. *Let M be a reductive monoid with zero. Let G be its group of units. Then the set of $G \times G$-orbits is finite, and every $G \times G$-orbit contains an idempotent.* □

From now on, we concentrate on reductive monoids.

Let M be a reductive monoid with 0. The results of Putcha ([Pu]) and Renner ([R8]) provide a characterization of the Zariski closure of T in M, namely,

$$\overline{T} = C_M(T) = \{x \in M \mid xt = tx, \ \forall t \in T\}.$$

Notice that $\overline{T}$ is a reductive monoid. Furthermore, $\overline{T}$ is an affine toric variety.

The set of $G \times G$-orbits, $\mathcal{U}(M)$, is often called the set of $\mathcal{J}$**-classes**. In fact, $\mathcal{U}(M)$ is a finite poset:

$$MaM \leq MbM \Leftrightarrow GaG \subset \overline{GbG}.$$

One defines a partial order on $E(\overline{T})$, the set of idempotents of $\overline{T}$, by declaring $f \leq e$ if and only if $ef = f = fe$.

In this context, there are two important results of Putcha ([Pu]) and Renner ([R8]) that we state here.

Theorem 3.1.7. *Any idempotent of M is conjugate to one in $\overline{T}$, that is,*

$$E(M) = \bigcup_{g \in G} gE(\overline{T})g^{-1}.$$

Additionally, if $e, f \in E(\overline{T})$ are conjugate under G, then they are also conjugate under W. □

Theorem 3.1.8. *Let M be a reductive monoid with zero. Suppose e and f are idempotents of M. Then $GeG = GfG$ if and only if e and f are conjugate under G.* □

All the structures just described are strongly intertwined, as the following theorem shows.

Theorem 3.1.9. *Let M be a reductive monoid. Then, there are bijections*

$$\mathcal{U}(M) \longleftrightarrow E(M)/G \longleftrightarrow E(\overline{T})/W$$

given by

$$GeG \longleftrightarrow \{geg^{-1} \mid g \in G\} \longleftrightarrow \{wew^{-1} \mid w \in W\}$$

for $e \in E(\overline{T})$, where $E(M)/G$ denotes the set of G-conjugacy classes in $E(M)$ and $E(\overline{T})/W$ denotes the set of W-conjugacy classes in $E(\overline{T})$.

Proof. It follows from Theorems 3.1.6 and 3.1.7 that any $G \times G$-orbit can be written as GeG, for some idempotent $e \in E(\overline{T})$. Now the map on the left is both well-defined and bijective in virtue of Theorems 3.1.7 and 3.1.8. Finally, the map on the right is a well-defined bijection due to Theorem 3.1.7. □

Fix a Borel subgroup B of G. Define Λ, the **cross section lattice** of M relative to T and B, by the following formula

$$\Lambda := \{e \in E(\overline{T}) \mid Be = eBe\}.$$

It turns out that Λ can be identified with the set of $G\times G$-orbits in M. Therefore,

$$M = \bigsqcup_{e\in\Lambda} GeG$$

and WeW has a unique minimal element: there exists a unique $\nu \in WeW$ for which $B\nu = \nu B$.

On the other hand, because of Theorem 3.1.10, we can also identify Λ with the set of $W-$orbits in $E(\overline{T}) = \{e \in T \,|\, e^2 = e\}$.

Let $R = \overline{N_G(T)} \subset M$. Then, for all $x \in R$, one has $xT = Tx$ and $x = wt$, where $w \in N_G(T)$ and $t \in \overline{T}$. Concisely, $R = \{x \in M \,|\, Tx = xT\}$.

The **Renner monoid**, $\mathcal{R}$, is defined to be $\mathcal{R} := R/T$. It is a finite regular monoid. More concretely, any $x \in \mathcal{R}$ can be written as $x = fu$, where $f \in E(\overline{T})$ and $u \in W$. Besides,

$$\mathcal{R} = \bigsqcup_{e\in\Lambda} WeW,$$

where Λ is the cross-section lattice. See [R8] for the details.

We should also emphasize that the Renner monoid $\mathcal{R}$ corresponds to the set of $B \times B$-orbits in M. In fact, there is an analogue of the Bruhat decomposition for reductive monoids:

$$M = \bigsqcup_{r\in\mathcal{R}} BrB.$$

Denote by $\mathcal{R}_k$ the set of elements of rank k in $\mathcal{R}$, that is,

$$\mathcal{R}_k = \{x \in \mathcal{R} \,|\, \dim Tx = k\,\}.$$

Analogously, one defines $\Lambda_k \subset \Lambda$ and $E_k \subset E(\overline{T})$.

For any given idempotent $e \in E(M)$, one can define the following opposite parabolic subgroups of G:

$$P_e = C^r_G(e) = \{g \in G \,|\, ge = ege\},$$

and

$$P_e^- = C_G^\ell(e) = \{g \in G \mid eg = ege\},$$

they are called right and left centralizer of e, respectively. Their intersection,

$$C_G(e) = \{g \in G \mid ge = eg\},$$

is called the centralizer of e in G. It can be shown ([Pu]) that $C_G(e)$ is a common Levi factor of P_e and P_e^-; so $C_G(e)$ is a connected reductive subgroup of G.

Theorem 3.1.10 ([R8])**.** *Let M be a reductive monoid with unit group G and cross section lattice Λ. Let $e \in \Lambda$.*

1. *Define $eMe = \{x \in M \mid x = exe\}$. Then eMe is a reductive algebraic monoid with unit group $H_e := e \cdot C_G(e)$ and unit element e. A cross section lattice of eMe is*
$$e\Lambda = \{f \in \Lambda \mid ef = f\}.$$
2. *Define $M_e = \overline{\{x \in G \mid ex = xe = e\}^\circ}$. Then M_e is a reductive algebraic monoid with zero $e \in M$ and unit group $G_e = \{x \in G \mid ex = xe = e\}^\circ$. A cross section lattice for M_e is*
$$\Lambda_e = \{f \in \Lambda \mid fe = e\}.$$

□

The following is a result of Rittatore ([Ri]).

Theorem 3.1.11. *Reductive monoids are exactly the affine embeddings of reductive groups. The commutative reductive monoids are exactly the affine embeddings of tori.* □

3.2 Monoids and Standard Group Embeddings

Definition 3.2.1. Let M be a reductive monoid with unit group G and zero element $0 \in M$. There exists a central one-parameter subgroup $\epsilon : \mathbb{C}^* \to G$ with image Z, that converges to 0 ([Br7], Lemma 1.1.1). Then $\mathbb{C}^*$ acts attractively on M via ϵ, and hence the quotient

$$\mathbb{P}_\epsilon(M) = [M \setminus \{0\}]/\mathbb{C}^*$$

is a normal projective variety. See Section 1.3 of [Br5]. Notice also that $G \times G$ acts on $\mathbb{P}_\epsilon(M)$ via

$$G \times G \times \mathbb{P}_\epsilon(M) \to \mathbb{P}_\epsilon(M),\ (g, h, [x]) \mapsto [gxh^{-1}].$$

Furthermore, $\mathbb{P}_\epsilon(M)$ is a group embedding of the reductive group G/Z. In the sequel, $X = \mathbb{P}_\epsilon(M)$ will be called a **Standard Group Embedding**.

Let B be a Borel subgroup of G. Recall that M contains a finite number of $G \times G$-orbits and $B \times B$-orbits, indexed by Λ and $\mathcal{R}$, respectively. It is clear that $X = \mathbb{P}_\epsilon(M)$ inherits such property as well. Indeed, the set of $G \times G$-orbits of X is indexed by $\Lambda \setminus \{0\}$. Similarly, the $B \times B$-orbits of X are indexed by $\mathcal{R} \setminus \{0\}$.

When M is semisimple (in which case ϵ is essentially unique), we write $\mathbb{P}(M)$ for $\mathbb{P}_\epsilon(M)$. Indeed, for such a monoid, $Z \simeq \mathbb{C}^*$ is the connected center of the unit group G of M. Thus, a semisimple monoid with unit group G can be thought of as an affine cone over some projective embedding $\mathbb{P}(M)$ of the semisimple group $G_0 = G/Z$.

For an up-to-date description of these and other embeddings, see [AB].

Example 3.2.2. Let G_0 be a semisimple algebraic group over the complex numbers and let $\rho : G_0 \to \mathrm{End}(V)$ be a representation of G_0. Define Y_ρ to be the Zariski closure of $G = [\rho(G_0)]$ in $\mathbb{P}(\mathrm{End}(V))$, the projective space associated with $\mathrm{End}(V)$.

Finally, let X_ρ be the normalization of Y_ρ. By definition, X_ρ is an standard group embedding of G. Notice that M_ρ, the Zariski closure of $\mathbb{C}^*\rho(G_0)$ in $\mathrm{End}(V)$, is a *semisimple monoid* whose group of units is $\mathbb{C}^*\rho(G_0)$. Embeddings of this type will be studied in more detail in Section 4.4.

The purpose of this section is to write out the GKM data of $X = \mathbb{P}_\epsilon(M)$ (i.e. the $T\times T$-fixed points and $T\times T$-invariant curves) in terms of the standard combinatorial invariants of M. In fact, we will show that any standard group embedding

$$X = \mathbb{P}_\epsilon(M)$$

contains only a *finite* number of $T\times T$-fixed points and $T\times T$-invariant curves. This calculation does not depend on any special property of M. Thus there is no harm in such a calculation even though it does not always yield a recipe for $H^*_T(\mathbb{P}_\epsilon(M))$. Later on, we specialize it to the case of a rationally smooth embedding.

3.2.1 GKM Data of a Standard Group Embedding

Let M be a reductive monoid with unit group G and zero element $0 \in M$. Let $\epsilon : \mathbb{C}^* \twoheadrightarrow Z$ be an attractive one-parameter subgroup in the center of G. Consider the standard group embedding $X = \mathbb{P}_\epsilon(M)$. Our first step is to identify the following two sets.

1. $\{x \in M \mid dim TxT = 1\}$.
2. $\{x \in M \mid dim TxT = 2\}$.

The first class will determine the set $X^{T\times T}$ of $T \times T$-fixed points and the second one will determine the set $\mathcal{C}(X, T \times T)$ of $T \times T$-fixed curves.

Fixed Points

As always, let $\mathcal{R} = \{x \in M \mid Tx = xT\}/T = \overline{N_G(T)}/T$ be the Renner monoid and let $\mathcal{R}_1 = \{x \in \mathcal{R} \mid dim(Tx) = 1\}$ be the set of rank-one elements of $\mathcal{R}$. We will identify $\mathcal{R}_1$ with its image in $\mathbb{P}_\epsilon(M)$ and simply write $\mathcal{R}_1 \subseteq \mathbb{P}_\epsilon(M)$.

Theorem 3.2.3. *$\mathcal{R}_1 \subseteq \mathbb{P}_\epsilon(M)$ is the set of fixed points of $T \times T$ acting on $\mathbb{P}_\epsilon(M)$. Hence, there is only a finite number of $T \times T$-fixed points in $X = \mathbb{P}_\epsilon(M)$.*

Proof. The set of fixed points of $T \times T$ on $\mathbb{P}_\epsilon(M)$ corresponds to

$$\{x \in M \mid dim(TxT) = 1\}.$$

Notice that if $dim(Tx) = 1$ then $Tx = Zx$. Similarly, if $dim(xT) = 1$ then $xT = Zx$. These remarks, together with the fact that $Tx \cup xT \subseteq TxT$, yield the equality

$$\{x \in M \mid dim(TxT) = 1\} = \{x \in M \mid Tx = xT \text{ and } dim(Tx) = 1\},$$

where the latter set is precisely $\mathcal{R}_1$. □

Fixed Curves

Proposition 3.2.4. *Let $x \in M$ and assume that $x \neq 0$. Then the following are equivalent.*

1. *$dimTxT = 2$.*
2. *Either $dim(xT) = 2$ and $Tx \subseteq xT$, $xT = TxT$; or $dim(Tx) = 2$ and $xT \subseteq Tx$, $Tx = TxT$; or $dim(TxT) = 2$ and $Tx = xT = TxT$.*

Proof. It is simple to check that *2.* implies *1.* For the converse, assume that *1.* holds. Now $Tx \cup xT \subseteq TxT$. If $dim(Tx) = dim(xT) = 1$ then $Tx = Zx = xT$, where $Z \subseteq T$ is the given attractive one-parameter subgroup of the center of G.

But then $dim(TxT) = 1$, a contradiction. Hence at least one of Tx or xT is two-dimensional. If $dim(Tx) = 2$ then $Tx \subseteq TxT$ yet they have the same dimension. Thus $Tx = TxT$. If $dim(xT) = 2$ then we end up with $xT = TxT$. □

Corollary 3.2.5. *Exactly one of the following assertions is true for $x \in M$ such that $dim(TxT) = 2$.*

1. $xT \subset Tx = TxT$ *and* $dim(xT) = 1$.

2. $Tx \subset xT = TxT$ *and* $dim(Tx) = 1$.

3. $xT = Tx = TxT$. □

The following is a result of Renner ([R3]). We include a proof for the convenience of the reader.

Lemma 3.2.6. *Let M be a reductive monoid with zero and unit group G. Let $T \subseteq G$ be a maximal torus. Choose a central one-parameter subgroup $\epsilon : \mathbb{C}^* \to G$, with image Z, that converges to 0. Then*

$$\{x \in M\backslash\{0\} \mid Zx = Tx\} = \bigsqcup_{e \in E_1(\overline{T})} eG.$$

Consequently, if $X = \mathbb{P}_\epsilon(M) = (M\backslash\{0\})/\mathbb{C}^*$ and $eX = (eM\backslash\{0\})/\mathbb{C}^* \simeq eG/Z$ then

$$X^T = \bigsqcup_{e \in E_1(\overline{T})} eX$$

for the action $T \times X \to X$ given by $(t, [x]) \rightsquigarrow [tx]$. Similar results hold for the right action $([x], t) \rightsquigarrow [xt]$ of T on X.

Proof. We reproduce Renner's argument ([R3]). Let $x \in M \setminus \{0\}$ be such that $Zx = Tx$. Since $x \neq 0$ by Theorem 3.4 of [R3] there is an $e \in E_1$ such that $ex \neq 0$ (that M is semisimple is not needed here). By the monoid Bruhat decomposition

[R1] we can write $x = brb'$ where $b, b' \in B$ and $r \in \mathcal{R}$. Then we let $y = xb'^{-1} = br$. Write $r = fw$ where $f \in E(\overline{T})$ and $w \in W$. Then $fy = fbr = fbfr = fcr = fcw$ for some $c \in C_B(f)$. In particular $fy \in fG$. Thus, by Proposition 3.22 of [R8], if $f \notin E_1$ then $dim(Tfy) > 1$. Thus $Zfy \subsetneq Tfy$. Thus $Zy \subsetneq Ty$ since $dim(Ty) \geq dim(Tfy)$. This is impossible. We conclude that $f = e \in E_1$. Thus, if $t \in T$ and $tbe = be$, then $tebe = etbe = ebe$. In particular $te = e$. But $dim\{t \in T \mid tbe = be\} = dim\{t \in T \mid te = e\} = dimT - 1$. In particular $T_e \subseteq \{t \in T \mid tbe = be\}$, and consequently $e \in \{t \in \overline{T} \mid tbe = be\}$. Thus $ebe = be$. Therefore $y \in eM$, and finally $x = yb' \in eM$. $\square$

Theorem 3.2.7. *Notation being as above, there are three types of closed irreducible $T \times T$-curves in $X = \mathbb{P}_\epsilon(M)$.*

1. *$\overline{U_\alpha ew}$, $s_\alpha \notin C_W(e)$ and $w \in W$ (fixed pointwise by T on the right).*

2. *$\overline{weU_\alpha}$, $s_\alpha \notin C_W(e)$ and $w \in W$ (fixed pointwise by T on the left).*

3. *$\overline{Tx} = \overline{xT}$ where $x \in \mathcal{R}_2 = \{x \in \mathcal{R} \mid dim(Tx) = 2\}$.*

Thus, there is only a finite number of $T \times T$-invariant curves in $X = \mathbb{P}_\epsilon(M)$.

Proof. Keeping the numeration of Corollary 3.2.5, we know that the $T \times T-$curves of $X = \mathbb{P}_\epsilon(M)$ fall into three classes. The first two types correspond, as Lemma 3.2.6 dictates, to curves that are fixed pointwise by T on either the left or the right. The former collection lies on $X^T = \bigsqcup_{e \in E_1(\overline{T})} eG/Z$. Moreover, due to the Bruhat decomposition, for each $e \in E_1(\overline{T})$ the following identity holds

$$eG/Z = G/P_e = \bigsqcup_{r \in eW} [r]B_u,$$

where B_u is the unipotent radical of B.

Our task is to find all the T-curves of eG/Z, where e varies over all the rank-one idempotents of $\overline{T}$. So fix an idempotent $e \in E_1(\overline{T})$. It follows from the results of Carrell (Lemma 1.5.5), that the T-curves of eG/Z are of the form $[r]U_\alpha$, for some root α such that $s_\alpha \notin C_W(f)$ and $f = w^{-1}ew$. Indeed, since f is a rank-one idempotent, then $s_\alpha \in C_W(f)$ if and only if $U_\alpha f = fU_\alpha = \{f\}$ ([R1], Lemma 5.1). Because there is no essential difference between e and f, we conclude that a $T \times T$-curve, TxT, is fixed pointwise on the left by T if and only if $TxT = wfU_\alpha$, where $\alpha \notin C_W(f)$, $f \in E_1(\overline{T})$, and $w \in W$. A similar argument disposes of the case when a $T \times T$-curve is fixed pointwise by T on the right.

Finally, if $Tx = xT = TxT$ and $\dim(Tx) = 2$, then $x \in \mathcal{R}_2$. Identifying $x \in \mathcal{R}_2$ with its image $[x]$ in $X = \mathbb{P}_\epsilon(M)$, it is clear that $T[x]T$ is a $T \times T$-curve in X. □

Let us state Theorem 3.2.3 and Theorem 3.2.7 in a more compact form.

Theorem 3.2.8. *Let $X = \mathbb{P}_\epsilon(M)$ be a standard group embedding. Then its natural $T \times T$-action*

$$\mu : T \times T \times \mathbb{P}_\epsilon(M) \to \mathbb{P}_\epsilon(M), \quad (s,t,[x]) \mapsto [sxt^{-1}]$$

is $T \times T$-skeletal. □

So it is quite relevant to ask whether μ is a GKM-action. We will show that this is in fact the case for *rationally smooth standard group embeddings*, the theme of our next section.

3.2.2 GKM Theory of Standard Group Embeddings

Let M be a reductive monoid with zero element $0 \in M$ and unit group $G \subseteq M$. Let $Z \subseteq G$ be a central one-parameter-subgroup with $0 \in \overline{Z}$. As before, define

$$\mathbb{P}_\epsilon(M) = [M \setminus \{0\}]/Z.$$

The next result was first proved in [R5].

Theorem 3.2.9. *The following are equivalent.*

1. $X = \mathbb{P}_\epsilon(M)$ *is rationally smooth.*

2. $M \setminus \{0\}$ *is rationally smooth.*

3. *For any minimal, nonzero, idempotent* e *of* M, M_e *is rationally smooth.*

4. *For any maximal torus* T *of* G, $\overline{T} \setminus \{0\}$ *is rationally smooth.* □

Notice, in particular, that the condition does not depend on Z. Theorem 3.2.8 provides a combinatorial/numerical description of rationally smooth embeddings. See [R5].

Let us recapitulate. We know, from the previous section, that $X = \mathbb{P}_\epsilon(M)$ admits a $T \times T$-skeletal action. Our goal is to determine when this action is also a GKM-action. Since X contains only a finite number of fixed points, Theorem 1.4.7 asserts that our task consists on finding a subclass of group embeddings with vanishing odd cohomology.

In Chapter 2, we worked with an important class of spaces with no odd cohomology: $\mathbb{Q}$-filtrable spaces. We will show in this section that if $\mathbb{P}_\epsilon(M)$ is rationally smooth, then it is $\mathbb{Q}$-filtrable. Put simply, rationally smooth standard group embeddings admit BB-decompositions into rational cells.

Let $X = \mathbb{P}_\epsilon(M)$ be a standard group embedding. Renner has shown that X comes equipped with the following "cell" decomposition:

$$X = \bigsqcup_{r \in \mathcal{R}_1} C_r,$$

where $\mathcal{R}_1 = X^{T \times T}$. Even more is true, as the next theorem asserts.

Theorem 3.2.10. *The decomposition*

$$\mathbb{P}_\epsilon(M) = \bigsqcup_{r\in\mathcal{R}_1} C_r$$

is the BB-decomposition associated to a generic one-parameter subgroup. Moreover, if $\mathbb{P}_\epsilon(M)$ is rationally smooth, then the C_r's are rational cells.

Proof. We need only verify the second assertion, because the first one has been established in [R3] (Theorem 3.4) and [R7] (Theorem 4.3). With this purpose in mind, we call the reader's attention to the fact that, in the terminology of [R3], M is quasismooth (Definition 2.2 of [R3]) if and only if $M \setminus \{0\}$ is rationally smooth. The equivalence between these two notions follows from Theorem 2.1 of [R3] and Theorems 2.1, 2.3, 2.4 and 2.5 of [R5].

Next, by Lemma 4.6 and Theorem 4.7 of [R3], each C_r equals

$$U_1 \times C_r^* \times U_2,$$

where the U_i's are affine spaces. Moreover, if we write $r \in \mathcal{R}_1$ as $r = ew$, with $e \in E_1(\overline{T})$ and $w \in W$, then $C_r = C_e^* w$. So it is enough to show that C_e^* is rationally smooth, for $e \in E_1(\overline{T})$.

By Theorem 5.1 of [R3], it follows that, if $X = \mathbb{P}_\epsilon(M)$ is rationally smooth, then

$$C_e^* = [f_e M(e)]/\mathbb{C}^*,$$

for some unique $f_e \in E(\overline{T})$, where $M(e) = M_e\mathbb{C}^*$ and M_e is rationally smooth (Theorem 2.5 of [R5]). Furthermore, the proof of Theorem 5.1 of [R3] also implies that $[e]$ is the zero element of the rationally smooth, reductive, affine monoid $M(e)/\mathbb{C}^*$. Additionally,

$$C_e^* = \{x \in M(e)/\mathbb{C}^* \mid \lim_{s\to 0} sx = [e]\},$$

for some generic one-parameter subgroup. Using Lemma 3.2.11 below, one concludes that C_e^* is rationally smooth. $\square$

Lemma 3.2.11. *Let M be a reductive monoid with zero. Suppose that zero 0 is a rationally smooth point of M. Let $f \in E(M)$, be an idempotent of M. Then $0 \in fM$ is a rationally smooth point of the closed subvariety fM.*

Proof. By Lemma 1.1.1 of [Br7], one can find a one-parameter subgroup $\lambda : \mathbb{C}^* \to T$, with image S, such that $\lambda(0) = f$. Notice that

$$fM = \{x \in M \mid \lambda(t)x = x\,, \forall\, t \in \mathbb{C}^*\}.$$

That is, fM is the fixed point set of the subtorus S of T. Thus, by Theorem 1.1 of [Br5], one concludes that 0 is also a rationally smooth point of fM. □

Corollary 3.2.12. *Let $X = \mathbb{P}_\epsilon(M)$ be a standard group embedding. If X is rationally smooth, then X is $\mathbb{Q}$-filtrable and so it has no cohomology in odd degrees.*

Proof. Under the given assumptions, X is projective, normal, and admits a BB-decomposition into rational cells (Theorem 3.2.9). We have compiled all the necessary data to appeal to Corollary 2.3.3 and conclude that X is $\mathbb{Q}$-filtrable. □

In consequence, GKM-theory works for rationally smooth standard group embeddings. Furthermore, since rationally smooth standard embeddings are $\mathbb{Q}$-filtrable, i.e. they can be filtered by closed subvarieties

$$\emptyset = X_0 \subset X_1 \subset \ldots \subset X_m = X,$$

where X_i is obtained from X_{i-1} by attaching a rational cell, one obtains the applicability of GKM-theory at each step of the filtration; even though the various X_i's are not necessarily rationally smooth. This approach is more flexible than the general approach (by comparing singular cohomology with intersection cohomology), used, for instance, in the proof of Theorem 3.3.3. Such flexibility should have become apparent from our study of these filtrations in Section 2.6.

To conclude this section, let us summarize our results.

Theorem 3.2.13. *Let M be a reductive monoid with zero. Let $\epsilon : \mathbb{C}^* \to Z$ be an attractive one-parameter subgroup in the center of G. Suppose that the standard group embedding $X = \mathbb{P}_\epsilon(M)$ is rationally smooth. Then the action μ of $T \times T$ on X, given by*

$$\mu : T \times T \times X \to X, \quad (s, t, [x]) \mapsto [sxt^{-1}],$$

is a GKM-action. Furthermore, X admits a filtration by closed invariant subvarieties

$$\emptyset = X_0 \subset X_1 \subset \ldots \subset X_m = X,$$

where each X_i is a GKM-variety, and each difference $X_i \setminus X_{i-1}$ is a rational cell.

Proof. By Corollary 3.2.12 and Theorem 3.2.8, X is a $\mathbb{Q}$-filtrable, GKM-variety. An straightforward application of Theorem 2.3.5 gives the rest. □

3.3 Vanishing of odd cohomology. The H-polynomial approach

The following is a collection of results due to Renner. See [R3] and [R5] for details. We include them here for the sake of completeness. Basically, Theorem 3.3.3 gives an alternative proof of the fact that any rationally smooth standard embedding $\mathbb{P}(M)$, where M is semisimple, has zero cohomology in odd degrees.

Definition 3.3.1. Let M be a semisimple monoid with monoid $\mathcal{R}$ of $B \times B$-orbits. Define $H(\mathcal{R})$, the *H-polynomial of* $\mathcal{R}$, as follows.

$$H(\mathcal{R}) = \sum_{x \in \mathcal{R}} (t-1)^{r(x)} t^{l(x)-r(x)}$$

where $r(x) = dim(Tx)$ is the *rank* of x and $l(x) = dim(BxB)$ is its *length*. We then let

$$H(M) = (t-1)^{-1}(H(\mathcal{R}) - 1).$$

$H(M)$ is called the *H-polynomial of M*. If $M = M_\rho = \overline{K^*\rho(G)}$, for some irreducible representation of G, we sometimes write H_ρ for $H(M_\rho)$.

Remark 3.3.2. This is indeed a polynomial since, for any $x \in \mathcal{R} \setminus \{0\}$, $r(x) > 0$. The other thing to notice here is that $H(M)$ depends only on the projective variety $\mathbb{P}(M) = [M \setminus \{0\}]/K^*$. So if $\mathbb{P}(M) \cong \mathbb{P}(N)$ then $H(M) = H(N)$. Furthermore, if there is morphism $M_1 \to M_2$ which is finite and dominant, then $H(M_1) = H(M_2)$.

Theorem 3.3.3. *Let M be a semisimple algebraic monoid such that $M \setminus \{0\}$ is rationally smooth. Then*

$$H(M)(t^2) = P_X(t)$$

where $X = [M \setminus \{0\}]/K^*$.

Proof. By our assumptions on M, X is rationally smooth. Hence by the results of McCrory in [M], $H^*(X) \cong IH^*(X)$. Thus $IP_X(t) = P_X(t)$. So it suffices to show that $H(M)(t^2) = IP_X(t)$.

Let $x \in X$. Then, without loss of generality, $x = [e]$, where $e \in M \setminus \{0\}$ is an idempotent. Then from Theorem 1.1 of [BJ]

$$IP_{X,x}(t) = \tau_{\leq d_x - 1}((1-t^2)IP_{\mathbb{P}(S_x)}(t))$$

where S_x is the appropriate slice and $d_x = dim(S_x)$. One checks, using the local structure of reductive monoids [Br7], that $S_x = M_e$. By the results of [R5], $M_e \sim_0 \Pi M_{n_i}(K)$, which is a rational cell. Hence, by Lemma 1.3 of [Br5], $\mathbb{P}(S_x)$ is a rational homology projective space of dimension $d_x - 1$. Thus $IP_{X,x}(t) = \tau_{\leq d_x - 1}((1-t^2)IP_{\mathbb{P}(S_x)}(t)) = 1$. Consequently, the formula in Theorem 1.1 of [BJ] simplifies to a summation with summands of the form $P_{(G\times G)x}(t)$, as in (5.1.5) of [BJ]. Thus

$$IP_X(t) = \sum_x P_{(G\times G)x}(t),$$

where the sum is taken over a set of representatives of the $G \times G$-orbits of X. But this is the same formula that one obtains by combining the $B \times B$-orbits into one summand for each $G \times G$-orbit, in the formula for $H(M)(t^2)$. $\square$

Chapter 4

GKM data of a Rationally Smooth Standard Group Embedding

It has been shown in Theorem 3.2.13 that the equivariant cohomology of a rationally smooth standard group embedding can be described in terms of GKM-theory. In this chapter, for each $T \times T$-invariant curve, we obtain the associated GKM-character explicitly. Theorem 4.3.4 gives the ultimate description of $H^*_{T\times T}(\mathbb{P}_\epsilon(M))$ in terms of certain characters and the Renner monoid, a finite combinatorial invariant associated to the monoid M.

We also describe the relation between $H^*_{T\times T}(\mathbb{P}_\epsilon(M))$ and $H^*_T(\mathbb{P}_\epsilon(\overline{T}))$, the associated torus embedding. Finally, we provide a few concrete examples.

The most remarkable new results in this Chapter are Theorem 4.1.1, Theorem 4.3.4, Corollary 4.3.5 and Theorem 4.3.6.

4.1 Classification of GKM-curves

Let M be a reductive monoid with zero and unit group G. Let T be a maximal torus and $\epsilon : \mathbb{C}^* \to Z$ be an attractive one-parameter subgroup in the center of G. Consider the standard group embedding $X = \mathbb{P}_\epsilon(M)$. Most of the calculations here do not depend on whether $\mathbb{P}_\epsilon(M)$ is rationally smooth.

Recall that the set of $T \times T$-fixed points in X corresponds to

$$\mathcal{R}_1 = \{x \in \mathcal{R} \mid dim(Tx) = dim(xT) = 1\}.$$

From Theorem 3.2.7, we also know that there are three types of $T \times T$-curves in X:

1. Curves that are fixed pointwise by T on the right: $\overline{U_\alpha ew}$, $e \in E_1(\overline{T})$, $s_\alpha \notin C_W(e)$, and $w \in W$.

2. Curves that are fixed pointwise by T on the left: $\overline{weU_\alpha}$, $e \in E_1(\overline{T})$, $s_\alpha \notin C_W(e)$, and $w \in W$.

3. $\overline{Tx} = \overline{xT} = \overline{TxT}$ where $x \in \mathcal{R}_2 = \{x \in \mathcal{R} \mid dim(Tx) = 2\}$.

But which pair of fixed points, i.e. elements of $\mathcal{R}_1$, is joined by each of these curves? Preserving the given order, we obtain

1. ew and $s_\alpha ew$

2. we and wes_α

3. The two elements $r, s \in \mathcal{R}_1$ such that $r, s \in \overline{TxT}$.

Theorem 4.1.1. *The set of $T \times T$ - curves in $X = \mathbb{P}_\epsilon(M)$ is identified as follows, by pairs of $T \times T$-fixed points. Here $Ref(W)$ refers to the set of reflections of W and we assume there is an ambient Borel subgroup (to get the ordering on $\mathcal{R}$).*

1. $\{(x, sx) \mid x \in \mathcal{R}_1, s \in Ref(W) \text{ and } x < sx\}$.

2. $\{(x, xs) \mid x \in \mathcal{R}_1, s \in Ref(W) \text{ and } x < xs\}$.

3. $\mathcal{R}_2 \cong \{A \subseteq \mathcal{R}_1 \mid |A| = 2 \text{ and } A = \{ex, fx\} \text{ for some } e, f \in E_1(\overline{T}) \text{ and some } x \in \mathcal{R}_2\}$.

Proof. First we recall that the Renner monoid $\mathcal{R}$ is partially ordered by the relation $x \leq y$ if $BxB \subseteq \overline{ByB}$. This is a generalization of the Bruhat-Chevalley order from group theory to the case of reductive monoids. See [R8], Definition 8.32. Bearing this in mind, Assertions *1.* and *2.* follow from the fact that if $x \neq sx$ and $s \in Ref(W)$, then either $x < sx$ or else $sx < x$ ([R8], Section 8.6). For *3.* we proceed as follows. Recall that any $x \in \mathcal{R}_2$ can be written as $x = fu$, where $f \in E_2(\overline{T})$ is a rank-two idempotent, and $u \in W$. Since u is invertible, it is enough to prove the statement for $x = f$. Now notice that $(f\overline{T} \setminus \{0\})/\mathbb{C}^*$ is isomorphic to $\mathbb{CP}^1$ ([Br5], Corollary 1.4.1). Thus there are exactly two fixed points, they correspond to the unique rank-one idempotents $e, e' \in E_1(\overline{T})$ such that $ef \neq 0$ and $e'f \neq 0$. □

Any $T \times T$-fixed point is contained in a closed $G \times G$-orbit. The curves identified in *1.* and *2.* of Theorem 4.1.1 are the ones that are contained in closed $G \times G$-orbits. The curves identified in *3.* of Theorem 4.1.1 are those that are not contained in any closed $G \times G$-orbit. In [Br4] these curves are further separated into whether or not the corresponding fixed points are in the same closed $G \times G$-orbit. This distinction will become relevant in the next section when we identify the character associated with each $T \times T$-curve of type *3.*

Notice that the description in *3.* above is just a convenient, indirect way of identifying the elements of $\mathcal{R}_2$ as pairs of $T \times T$ - fixed points. Notice also that, for each $x \in \mathcal{R}_2$, there are exactly two elements $e, f \in E(\mathcal{R}_1)$ such that $ex \neq 0$ and $fx \neq 0$.

Example 4.1.2. We illustrate Theorem 4.1.1 with the example $M = M_n(K)$. Let $E_{i,j}$ denote an elementary matrix. We then obtain (with the ordering as in Theorem 4.1.1)

1. $\{(E_{i,j}, E_{i,k}) \mid j \neq k\}$.

2. $\{(E_{i,j}, E_{k,j}) \mid i \neq k\}$.

3. $\{(E_{i,j}, E_{k,l}) \mid i \neq k \text{ and } j \neq l\}$.

In each case the associated curve is the $T \times T$-orbit of the sum of the given pair of elementary matrices. In case *1.* the two elementary matrices are in the same row. In case *2.* the two elementary matrices are in the same column. Case *3.* determines the remaining cases.

4.2 The Associated Characters

We now identify the character $\theta_x = (\lambda_x, \rho_x)$ of $T \times T$ associated with the $T \times T$-curve $c = [TxT] \in \mathcal{C}(X, T)$. Recall that this character, unique up to sign, has been described in Definition 1.4.9.

As discussed previously (Theorems 3.2.7 and 4.1.1), there are three different types of $T \times T$-curves. In this section we focus mainly on the third type, that is, when $c = [TxT]$ and $x \in \mathcal{R}_2$. The other $T \times T$-curves (where either $Tx = TxT$ or $xT = TxT$) will also be discussed, but recall that these are essentially T-curves on the complete homogeneous space G/P_e, with $e \in E_1$ (Lemma 1.5.5).

So let $x \in \mathcal{R}_2$. Since we are working on the monoid level, the initial step in our discussion is to calculate the map

$$m_x : T \times T \to TxT, \ \ (s, t) \rightsquigarrow sxt.$$

We then compose m_x with the canonical map $\pi_x : TxT \to TxT/Z \cong \mathbb{C}^*$ to obtain

$$\theta_x = \pi_x \circ m_x$$

where $Z \subseteq G$ is the given central, attractive, 1-parameter subgroup of the unit group G of M. Notice that θ_x depends on the choice of isomorphism $TxT/Z \cong \mathbb{C}^*$. The other isomorphism $TxT/Z \cong \mathbb{C}^*$ yields θ_x^{-1}. In the calculation of θ_x it is important to keep track of this ambiguity. It is also useful to consider the map

$$t_x : T \to Tx, t \rightsquigarrow tx$$

and the character $\lambda_x = \pi_x \circ t_x$. Notice that $TxT = Tx$, so we wish to express $\theta_x : T \times T \to \mathbb{C}^*$ as a composition

$$T \times T \to T \times T \to T \to Tx \to \mathbb{C}^*$$

involving the multiplication $T \times T \to T$, the action of W on T, and these other quantities: t_x, π_x, λ_x.

Also we assess the effect of the $W \times W$-action

$$W \times W \times \mathcal{C}(X, T \times T) \to \mathcal{C}(X, T \times T) \quad , (v, w, c) \rightsquigarrow vcw^{-1}$$

on the associated characters. This will effectively reduce the calculation of θ_x, with $x \in \mathcal{R}_2$, to calculating θ_x for a set of representatives of the $W \times W$-orbits of $\mathcal{R}_2$.

Explicit computations

Denote by $\Xi(T)$ the character group of T.

Let $x \in \mathcal{R}_2$. Then we can write $x = fu = ug$, where $u \in W$ and $f, g \in E_2(\overline{T})$. An elementary calculation yields that

$$m_x : T \times T \to TxT = xT, \ \ (s, t) \rightsquigarrow sxt$$

is given by $m_x(s,t) = st^u x$ where, by definition, $t^u = utu^{-1}$. Recall that $\lambda_x = \pi_x \circ t_x$, where $t_x : T \to Tx, t \rightsquigarrow tx$, and $\pi_x : TxT \to TxT/Z \cong K^*$.

Lemma 4.2.1. *Write $\theta_x = (\lambda_x, \rho_x) \in \Xi(T \times T) = \Xi(T) \oplus \Xi(T)$. Then*

1. $\lambda_x = \lambda_f$.

2. $\rho_x = \lambda_g = \lambda_f \circ int(u)$, *where* $int(u)(t) = utu^{-1}$.

Proof. Consider $m : T \times T \to Tf,\ \ (s,t) \rightsquigarrow st^u f$. Then $m(s,t) \in Zf$ if and only if $m_x(s,t) \in Zx$. Thus $ker(\pi_f \circ m) = ker(\pi_x \circ m_x)$. So $\lambda_x = \lambda_f$ and $\rho_x = \lambda_f \circ int(u)$. But m is also the product of $(s,1) \rightsquigarrow sf$ and $(1,t) \rightsquigarrow t^u f$. The first of these is λ_f and the second of these is $\lambda_f \circ int(u)$. But $t^u f \in Zf$ if and only if $tg \in Zg$ since $ugu^{-1} = f$. Thus $ker(\lambda_x \circ int(u)) = ker(\lambda_g)$. We conclude that $\theta_x = (\lambda_x, \rho_x) = (\lambda_f, \lambda_g) = (\lambda_f, \lambda_f \circ int(u))$.

Notice that we can also write it as $m_x : T \times T \to TxT = xT,\ \ m_x(s,t) = sxt = xs^{u^{-1}}$. The resulting calculation then yields $\theta_x = (\lambda_x, \rho_x) = (\lambda_f, \lambda_g) = (\lambda_g \circ int(u^{-1}), \lambda_g)$. □

Notice that either $\theta_x = (\lambda_x, \lambda_x \circ int(u))$ or $\theta_x = (\lambda_x^{-1}, \lambda_x^{-1} \circ int(u))$ depending on the orientation.

Lemma 4.2.2. *Let $x \in \mathcal{R}_2$, so that $x = fu = ug$ where $u \in W$ and $f, g \in E_2(\overline{T})$, and write $\theta_x = (\lambda_f, \lambda_g)$ with $\lambda_g = \lambda_f \circ int(u)$ (as in Lemma 4.2.1).*

1. *Let $y = xw$, where $w \in W$. Then $\theta_y = (\lambda_f, \lambda_g \circ int(w)) = (\lambda_x, \rho_x \circ int(w))$.*

2. *Let $y = wx$, where $w \in W$. Then $\theta_y = (\lambda_f \circ int(w^{-1}), \lambda_g) = (\lambda_x \circ int(w^{-1}), \rho_x)$.*

Proof. Assume that $y = xw$, and let $h = (uw)^{-1} fuw$. Then $\theta_y = (\lambda_f, \lambda_h)$ where $\lambda_h = \lambda_f \circ int(uw) = \lambda_f \circ int(u) \circ int(w) = \lambda_g \circ int(w)$.

Assume that $y = wx$, and let $h = wfw^{-1}$. Then $\theta_y = (\lambda_h, \lambda_g)$ where $\lambda_h = \lambda_f \circ int(w^{-1})$ (since $h = wfw^{-1}$). □

Let $x \in \mathcal{R}_2$, and write $x = fu$, where $f \in E(\overline{T})$ and $u \in W$. The H-class of x, denoted by H_x, is defined to be $H_x := \{sx \mid s \in C_W(f)\}$. See [R8].

Lemma 4.2.3. *The following are equivalent for $x \in \mathcal{R}_2$.*

1. *The H-class of x contains two elements.*

2. *The two $T \times T$-fixed points in $X = \mathbb{P}_\epsilon(M)$, in the closure of TxT, are in the same $W \times W$-orbit.*

Proof. Let $x \in \mathcal{R}_2$ and let $a, b \in \overline{TxT}$ be the two $T \times T$-fixed "points" in $\overline{TxT}$. Assume that $H_x = \{x, y\}$. Then there exist $s, u \in W$ and $f, g \in E_2(\overline{T})$ such that $x = fu = ug$ and $y = fsu = sug$. In particular, $sf = fs \neq f$, and $s^2 = 1$ (for otherwise, $fs^2u = s^2ug$ would be another element in the H-class of x). Notice also that $y = fut = utg$ where $t = u^{-1}su$. In any case, the two fixed points $a, b \in \overline{TxT}$ are $a = f_1x = f_1u$ and $b = f_2x = f_2u$ where f_1, f_2 are the two rank-one idempotents below f. One checks that $b = sat$ and $a = sbt$. Indeed, $sat = sf_1ut = sf_1uu^{-1}su = sf_1su = f_2u = b$. Notice that $sf_1s = f_2$ since $sf = fs \neq f$.

Now let $x = fu \in \mathcal{R}_2$ and assume that $f_1x = f_1u$ and $f_2x = f_2u$ are in the same $W \times W$-orbit. Then f_1 and f_2 are in the same $W \times W$-orbit. That is, f_1 and f_2 are conjugate (Theorem 3.1.8). Furthermore, Corollary 8.9 and Proposition 10.9 of [Pu] assert that f_1 and f_2 are conjugate by an element $s \in C_W(f) = \{v \in W \mid vf = fv\}$. One then checks that $y = sx$ is the other element in the H-class of x. $\square$

Remark 4.2.4. In the proof of the Lemma above we mentioned that $s^2 = 1$. In fact, in this situation we can claim more: s is a reflection. For that let's look at the induced action of $int(s)$ on $f\overline{T} - \{0\}/Z \simeq \mathbb{C}\mathrm{P}^1$. Since $int(s)$ is an automorphism, the induced map is either $z \mapsto z$ or $z \mapsto z^{-1}$. The former is impossible because, as we saw above, $sf = fs \neq f$ and $sf_1s = f_2$, that is, $int(s)$ permutes the points

$0 = f_1$ and $\infty = f_2$ of $\mathbb{C}\mathrm{P}^1$. Therefore, by looking at the commutative diagram

$$\begin{tikzcd} Tf \arrow[r, "int(s)"] \arrow[d, "\pi"] & Tf \arrow[d, "\pi"'] \\ Tf/Z \arrow[r, "z\mapsto z^{-1}"] & Tf \end{tikzcd}$$

we conclude that s, when restricted to Tf, is a reflection. Finally, given that the natural map $T \to Tf$ is s-equivariant, it follows that s itself is a reflection in W. So $s = s_{\alpha_f}$, for some root α_f in $\Phi \subseteq \Xi(T)$.

Lemma 4.2.5. *Let $x, y \in \mathcal{R}_2$ be distinct and assume that $H_x = \{x, y\}$. Write $x = fu$ and $y = fs_{\alpha_f}u$, as in the proof of Lemma 4.2.3 and Remark 4.2.4. Then $\lambda_f \circ int(s_{\alpha_f}) = \lambda_f^{-1}$. Consequently,*

$$\theta_x = (\lambda_x, \rho_x) \implies \theta_y = (\lambda_x, \rho_x^{-1}).$$

Furthermore, $\lambda_x = \alpha_f$ and $\rho_x = \alpha_f \circ int(u)$ are roots of G with respect to T.

Proof. From Lemma 4.2.1 we obtain $\lambda_g = \lambda_f \circ int(u)$, as well as $\lambda_g = \lambda_f \circ int(s_{\alpha_f} u)$. But $int(s_{\alpha_f} u) = int(s_{\alpha_f}) \circ int(u)$. Thus, either $\lambda_f = \lambda_f \circ int(s_{\alpha_f})$ or else $\lambda_f^{-1} = \lambda_f \circ int(s_{\alpha_f})$ since these characters are unoriented. We must rule out the former case. This amounts to looking at the map induced on fT/Z from the restriction $int(s_{\alpha_f}) : fT \to fT$. By the remark above, $int(s_{\alpha_f})[ft] = [ft^{-1}]$, for all $t \in T$. Thus, $\lambda_f^{-1} = \lambda_f \circ int(s_{\alpha_f})$. Finally, by Remark 4.2.4 again, it follows that $\lambda_x = \lambda_f = \alpha_f$ and $\rho_x = \alpha_f \circ int(u)$ are roots. $\square$

Example 4.2.6. Let $M = M_n(K)$ and let T be the set of invertible, diagonal matrices. One checks that

$$\mathcal{R}_2 = \{E_{i,j} + E_{k,l} \mid i \neq k \text{ and } j \neq l\}.$$

where $E_{i,j}$ denotes the elementary matrix with a one in the (i, j)-position and and zeros elsewhere. Let $\underline{s} = (s_1, ..., s_n) \in T$ denote the obvious diagonal matrix. A simple calculation yields that, for $\underline{s}, \underline{t} \in T$ and $x = E_{i,j} + E_{k,l}$,

$$\theta_x(\underline{s}, \underline{t}) = s_i s_k^{-1} t_j t_l^{-1}.$$

The other element $y \in \mathcal{R}_2$, in the H-class of $x = E_{i,j} + E_{k,l}$, is $y = E_{k,j} + E_{i,l}$. Thus,

$$\theta_y(\underline{s}, \underline{t}) = s_i s_k^{-1} t_l t_j^{-1}.$$

In the terminology of Lemma 4.2.1, $\theta_x = (\lambda_x, \rho_x)$ where $\lambda_x = \alpha_{i,k}$ and $\rho_x = \alpha_{j,l}$. Similarly, $\lambda_y = \alpha_{i,k}$ and $\rho_x = \alpha_{l,j}$.

We now discuss the remaining cases (where either $Tx = TxT$ or $xT = TxT$). Again our treatment is somewhat terse because the whole issue reduces to the well-documented situation discussed in [C].

Lemma 4.2.7. *Let $x = ew \in \mathcal{R}_1$ and let $\alpha \in \Phi$ be such that $U_\alpha x \neq \{x\}$. Then, for $s, t \in T$ and $u \in U_\alpha$,*

$$suxt^{-1} = sus^{-1} z_x(s,t) x$$

where $z_x : T \times T \to Z$. Thus, the character of the action of $T \times T$ on

$$C(x, \alpha) = \overline{U_\alpha x} \subseteq \mathbb{P}_\epsilon(M)$$

is the root $(\alpha, 1)$.

Proof. Starting from $suxt^{-1}$, one obtains $suxt^{-1} = sus^{-1}sewt^{-1}w^{-1}w$. Since the quantities $(t^{-1})^w := wt^{-1}w^{-1}$ and e commute, then the term on the right hand side of the identity above becomes $sus^{-1}(s(t^{-1})^w)ew$. This latter expression is, quite simply, equal to $sus^{-1}s(t^{-1})^w ex$. On the other hand, observe that $Te = Ze$, because e is a rank-one idempotent of $\overline{T}$. In other words, $s(t^{-1})^w e = z_x(s,t)e$ where $z_x(s,t) \in Z$. From this, it follows that

$$suxt^{-1} = sus^{-1} z_x(s,t) x = sus^{-1} x z_x(s,t).$$

Hence,

$$s(uxZ)t^{-1} = sus^{-1}xZ,$$

and the result follows. $\square$

4.3 GKM-graph

Let Λ be the cross section lattice of M. Recall that Λ corresponds to the partially ordered set of $G \times G$-orbits in M. Under this identification, closed $G \times G$-orbits in $\mathbb{P}_\epsilon(M)$ correspond to idempotents $e \in \Lambda_1$. See the comments after Definition 3.2.1.

Proposition 4.3.1. *Let M be a reductive monoid with zero and G be its unit group. Let $e \neq 0$ be an idempotent of $E(T)$. Consider $\mathbb{P}_\epsilon(M)$ as above. Then the $G \times G$ orbit of $[e]$ in X fits into the fibration sequence*

$$H_e/\mathbb{C}^* \hookrightarrow G[e]G \xrightarrow{\pi} G/P_e \times G/P_e^- \ .$$

Here $H_e := e \cdot C_G(e)$. In particular, if e has rank one, then

$$G[e]G \simeq G/P_e \times G/P_e^-,$$

for, in this case, $eMe \simeq \mathbb{C}$, $H_e \simeq e \times \mathbb{C}^*$ and $P_e \cdot e = \mathbb{C}^* \cdot e$.

Proof. Notice that $Stab_{G\times G}(e)$, the $G \times G$-stabilizer of $e \in M$, is contained in the subgroup $P_e \times P_e^-$. To see this, let $(g, h) \in Stab_{G\times G}(e)$. Then $geh^{-1} = e$, that is $egeh^{-1} = e^2$, but e is an idempotent, so $egeh^{-1} = e$. The latter yields $ege = eh$, and the term on the right hand side equals ge, by assumption. We conclude that $ege = ge$. Analogously, $eh = ehe$.

Since $Stab_{G\times G}(e) \subset P_e \times P_e^-$, the map π is the natural map of homogeneous spaces, and therefore it is a fibration with fibre $(P_e \times P_e^-)/Stab_{G\times G}(e)$. But the fibre it is easily seen to be isomorphic to $e \cdot C_G(e)$, where

$$C_G(e) = \{g \in G \,|\, ge = eg\}.$$

After taking the quotient by the $\mathbb{C}^*$-action, we obtain the result. □

Proposition 4.3.2. *Let $G[e]G$ be a closed $G\times G$ orbit in X (in other words, $e \in \Lambda_1$). Then $H^*_{T\times T}(G[e]G)$ consists of all maps $\varphi : WeW \to H^*_T \otimes H^*_T$ such that*

i) $\varphi(ew) \cong \varphi(s_\alpha ew) \mod (\alpha, 1)$ *for* $s_\alpha \notin C_W(e)$.

ii) $\varphi(we) \cong \varphi(wes_\alpha) \mod (1, \alpha)$ *for* $s_\alpha \notin C_W(e)$.

Proof. It follows from Proposition 4.3.1 that $G[e]G$ is isomorphic to the complete homogeneous space $G/P_e \times G/P_e^-$ with vanishing odd cohomology. The $T\times T$-fixed points of $G[e]G$ are then given by WeW. By Lemma 1.5.5, the $T \times T$-curves of $G[e]G$ are given by $U_\alpha ew$, with $s_\alpha \notin C_W(e)$ and weU_α, with $s_\alpha \notin C_W(e)$. Curves of the former type join the fixed points ew and $s_\alpha ew$. As for the latter type, they join we to wes_α. Theorem 1.4.11 now yields the result. □

Recall the notation of Lemma 4.2.5.

Lemma 4.3.3 ([R8])**.** *Let $x = fu$ be an element of $\mathcal{R}$, the Renner monoid of M. Denote by H_x its H-class. If $x \in \mathcal{R}_2$, then either H_x has two elements or $H_x = \{x\}$. In the former case, $H_x = \{x, y\}$, where $y = s_{\alpha_f} x$ and $s_{\alpha_f} \in C_W(f)$ is the reflection for which $s_{\alpha_f} f = f s_{\alpha_f} \neq f$. In the latter case, any element $s \in C_W(f)$ satisfies $sf = fs = f$.* □

We now state the major result of this monograph. For the analogous result in the case of (smooth) regular compactifications, see Theorem 3.1.1 of [Br4].

Theorem 4.3.4. *Let $X = \mathbb{P}_\epsilon(M)$ be a rationally smooth standard group embedding. Then the following hold:*

1. *X is equivariantly formal for singular cohomology. Indeed, X has no odd cohomology over $\mathbb{Q}$ and the map induced by restriction to the fixed point set,*

$$H^*_{T\times T}(X) \longrightarrow H^*_{T\times T}(X^{T\times T}),$$

   is injective.

2. *The natural map $H^*_{T\times T}(X) \longrightarrow H^*_{T\times T}(\bigsqcup_{e\in\Lambda_1} G[e]G) = \bigoplus_{e\in\Lambda_1} H^*_{T\times T}(G[e]G)$ is injective. In fact, its image consists of all tuples $(\varphi_e)_{e\in\Lambda_1}$, indexed over Λ_1 and with $\varphi_e \in H^*_{T\times T}(G[e]G)$, subject to the additional conditions:*

   (a) *If $f \in E_2(\overline{T})$ and $H_f = \{f, s_{\alpha_f} f\}$, with $s_{\alpha_f} f = f s_{\alpha_f} \neq f$, then*

$$\varphi_{e_f}(f_1 u) \equiv \varphi_{e_f}(f_2 u) \bmod (\alpha_f, \alpha_f \circ \mathrm{int}(u)),$$

   for all $u \in W$. Here, f_1 and $f_2 = s_{\alpha_f} \cdot f_1 \cdot s_{\alpha_f}$ are the two idempotents in $E_1(\overline{T})$ below f, the root α_f corresponds to the reflection s_{α_f}, and $e_f \in \Lambda_1$ is the unique element of Λ_1 which is conjugate to f_1.

   (b) *If $f \in E_2(\overline{T})$ and $H_f = \{f\}$, then*

$$\varphi_{e_1}(f_1 u) \equiv \varphi_{e_2}(f_2 u) \bmod (\lambda_f, \lambda_f \circ int(u)),$$

   for all $u \in W$. Here, λ_f is the character of T defined in Lemma 4.2.1, the idempotents f_1, f_2 are the unique idempotents below f, and $e_i \in \Lambda_1$ is conjugate to f_i, for $i = 1, 2$.

Proof. Claim 1. is simply a restament of Theorem 3.2.13. So we now focus on Assertion 2.

First, notice that all the $T \times T$-fixed points of X are contained in the (disjoint) union of the closed orbits. So we have a commutative triangle

$$\begin{tikzcd} H^*_{T\times T}(X) \arrow[rr, "i^*"] \arrow[dr, "j^*"'] & & H^*_{T\times T}(X^{T\times T}) \\ & \bigoplus_{e\in\Lambda_1} H^*_{T\times T}(GeG) \arrow[ur, "k^*"'] & \end{tikzcd}$$

where all maps are induced by inclusions. The injectivity of i^* yields at once the injectivity of j^*.

We can say even more. Since $GeG \simeq G/P_e \times G/P_e^-$ (Proposition 4.3.1), we conclude that each closed orbit is equivariantly formal. What is more, $X^{T\times T} = \mathcal{R}_1$ is also the fixed point set of $L = \bigsqcup_{e\in\Lambda_1} GeG$. Thus, k^* is injective. Now notice that L contains all the curves of type 1 and 2 in X. These curves, in addition, describe the equivariant cohomology of L (Proposition 4.3.2).

To conclude the proof, we just need to show that the curves of type 3 give assertions 2(a) and 2(b). So let $x = fu \in \mathcal{R}_2$ be one of these curves. By Lemma 4.3.3, the H-class H_x of x contains either one or two elements.

If $H_x = \{x, s_{\alpha_f}x\}$, then Lemma 4.2.3 implies that the two fixed points of $[TxT]$, namely f_1x and f_2x, lie in the same closed $G\times G$-orbit. Here recall that f_1, f_2 are the two idempotents below f. Moreover, f_2 is conjugate to f_1 via s_{α_f}, namely, $f_2 = s_{\alpha_f}\cdot f_1\cdot s_{\alpha_f}$. We now use Lemma 4.2.5 to write the associated character θ_x as

$$\theta_x = (\alpha_f, \alpha_f \circ int(u)),$$

where α_f is the root associated to the reflection s_{α_f}. Since Λ_1 indexes all closed $G\times G$-orbits in X, there exists a unique $e_x \in \Lambda_1$ such that f_1 and e_x are conjugate. Assertion 2 (a) is now proved.

Finally, if $H_x = \{x\}$, then f_1 and f_2 are not conjugate (Lemma 4.2.3). That is, f_1x and f_2x lie in different closed $G\times G$-orbits. Since $x = fu$, Lemma 4.2.1 finishes the proof. $\square$

Together with Proposition 2.6.9, the result above provides a complete combinatorial description of the equivariant cohomology of any rationally smooth standard embedding.

As it was pointed out before, Brion ([Br4], Theorem 3.1.1) has obtained a result analogous to Theorem 4.3.4 for regular compactifications of G. These compactifications are characterized, among other properties, by the fact that they are smooth varieties and possess a finite number of closed $G \times G$-orbits, all of them isomorphic to $G/B \times G/B$. There are three main differences between the embeddings studied by Brion in [Br4] and our standard group embeddings. First, standard group embeddings are, in general, singular. Second, the closed $G \times G$-orbits of a standard group embedding are usually of the form $G/P_e \times G/P_e^-$, where P_e and P_e^- are opposite parabolic subgroups (Proposition 4.3.1). Such homogeneous spaces are not necessarily isomorphic to $G/B \times G/B$. Finally, the results of Renner ([R2], Corollary 3.4) assert that any normal projective group embedding of a semisimple group G is standard. That is, standard group embeddings form a very natural class from the viewpoint of embedding theory. This class is larger than the class of regular compactifications. In particular, our Theorem 4.3.4 implies Theorem 3.1.1 of [Br4] for the case of projective regular embeddings.

These observations should help the reader to not only understand the importance and scope of our main Theorem 4.3.4, but also put our results in perspective.

It follows from Proposition 1.2.9 (i) that the $G \times G$-equivariant cohomology of X is obtained by means of the following formula

$$H^*_{G\times G}(X) \simeq (H^*_{T\times T}(X))^{W\times W}.$$

For the case in hand, we can be more precise, as the following result shows.

Corollary 4.3.5. *Let $X = \mathbb{P}_\epsilon(M)$ be a rationally smooth standard group embedding. Then the ring $H^*_{G\times G}(X)$ consists of all tuples $(\Psi_e)_{e\in\Lambda_1}$, where*

$$\Psi_e : WeW \to (H^*_T \otimes H^*_T)^{C_W(e)\times C_W(e)},$$

such that

(a) If $f \in E_2(\overline{T})$ and $H_f = \{f, s_{\alpha_f} f\}$, then

$$\Psi_e(f_1) \equiv \Psi_e(f_2) \bmod (\alpha_f, \alpha_f),$$

where $e \in \Lambda_1$ is conjugate to f_1, $f_2 = s_{\alpha_f} \cdot f_1 \cdot s_{\alpha_f}$, the reflection $s_{\alpha_f} \in C_W(f)$ is associated with the root α_f, and $f_i \leq f$.

(b) If $f \in E_2$ and $H_f = \{f\}$, then

$$\Psi_e(f_1) \equiv \Psi_{e'}(f_2) \bmod (\lambda_f, \lambda_f),$$

where $\lambda_f \in \Xi(T)$, and $f_1, f_2 \leq f$ are conjugate to e and e', respectively.

Proof. Let $e \in \Lambda_1$. The closed orbit $G[e]G$ is isomorphic to $G/P_e \times G/P_e^-$. Since $P_e = C_G(e) \rtimes U_e$, where $C_G(e)$ is the centralizer of e in G, and $U(e)$ is the unipotent part of P_e. Moreover, $U(e) = \mathcal{R}_u(P(e))$ and $C_G(e)$ is a closed connected reductive subgroup, called the Levi subgroup of $P(e)$. It follows, by the results of Brion ([Br3]) that

$$H^*(BP_e) \simeq H^*(BC_G(e)) \simeq H^*(BT)^{C_W(e)}.$$

Consequently,

$$\begin{aligned} H^*_{G\times G}(G[e]G) &\simeq H^*(BP_e) \otimes H^*(BP_e) \\ &\simeq H^*(BC_G(e)) \otimes H^*(BC_G(e)) \\ &\simeq H^*(BT)^{C_W(e)} \otimes H^*(BT)^{C_W(e)} \\ &= (H^*_T \otimes H^*_T)^{C_W(e)\times C_W(e)}. \end{aligned}$$

Notice that $(u,v) \in W \times W$ acts on a tuple (f_r) in $H^*_{T\times T}(\mathcal{R}_1) = \oplus_{r\in\mathcal{R}_1} H^*_{T\times T}$ via

$$(u,v)\cdot(f_r) := ((u,v)\cdot f_{u\,r\,v^{-1}}).$$

Since restriction of Ψ_e to $(u,v)\cdot e = uev^{-1}$ is equal to $(u,v)\cdot\Psi_e(e)$, for all $(u,v) \in W \times W$, then relations 2(a) and 2(b) from Theorem 4.3.4 reduce to the proposed descriptions (a) and (b). $\square$

Associated to $X = \mathbb{P}_\epsilon(M)$, there is a standard torus embedding $\mathcal{Y}$ of T/Z, namely,

$$\mathcal{Y} = \mathbb{P}_\epsilon(\overline{T}) = [\overline{T} \setminus \{0\}]/\mathbb{C}^*.$$

By construction, $\mathcal{Y}$ is a normal projective torus embedding and $\mathcal{Y} \subseteq X$.

Our next theorem allows to compare the equivariant cohomologies of $X = \mathbb{P}_\epsilon(M)$ and the associated torus embedding $\mathcal{Y} \subseteq X$. The situation here contrasts deeply with the corresponding one for regular embeddings ([Br4], Corollary 3.1.2; [U], Corollary 2.2.3). It is worth emphasizing that the idea of comparing the embeddings $\mathcal{Y}$ and X goes back to [LP].

Theorem 4.3.6. *The inclusion of the associated torus embedding $\iota : \mathcal{Y} \hookrightarrow X$ induces an injection:*

$$\iota^* : H^*_{G\times G}(X) \hookrightarrow H^*_{T\times T}(\mathcal{Y})^W \simeq (H^*_T(\mathcal{Y}) \otimes H^*_T)^W,$$

where the W-action on $H^*_{T\times T}(\mathcal{Y})$ is induced from the action of $\mathrm{diag}(W)$ *on $\mathcal{Y}$. Furthermore, ι^* is an isomorphism if and only if $C_W(e) = \{1\}$ for every $e \in \Lambda_1$.*

Proof. Since X is rationally smooth, then $\mathcal{Y}$ is rationally smooth as well (Theorem 3.2.9). Therefore, we have the following commutative diagram

$$\begin{tikzcd} H^*_{T\times T}(X) \arrow[r, hook] \arrow[d, "\iota^*"] & H^*_{T\times T}(X^{T\times T}) \arrow[d, "\iota^*"] \\ H^*_{T\times T}(\mathcal{Y}) \arrow[r, hook] & H^*_{T\times T}(\mathcal{Y}^{T\times T}), \end{tikzcd}$$

where the horizontal maps are injective, because both standard group embeddings are equivariantly formal.

On the other hand, recall that Λ_1 provides a set of representatives of both the $W \times W$-orbits in $X^{T\times T} = \mathcal{R}_1$ and the W-orbits in $\mathcal{Y}^{T\times T} = E_1(T)$. Thus, after taking invariants, we obtain an injection

$$H^*_{T\times T}(\mathcal{R}_1)^{W\times W} = \bigoplus_{e\in\Lambda_1} (H^*_{T\times T})^{C_W(e)\times C_W(e)} \hookrightarrow H^*_{T\times T}(E_1(T))^W = \bigoplus_{e\in\Lambda_1} (H^*_{T\times T})^{C_W(e)}.$$

Placing this information into the commutative diagram above shows that the restriction map

$$\iota^* : (H^*_{T\times T}(X))^{W\times W} \longrightarrow H^*_{T\times T}(\mathcal{Y})^W$$

is injective.

Observe that $H^*_{T\times T}(\mathcal{Y})^W \simeq (H^*_T(\mathcal{Y}) \otimes H^*_T)^W$. Truly, we have a split exact sequence

$$\begin{tikzcd} 1 \arrow[r] & diag(T) \arrow[r] & T \times T \arrow[r, "(t_1,t_2)\mapsto t_1t_2^{-1}"] & T \arrow[l, dashed, bend left] \arrow[r] & 1, \end{tikzcd}$$

where the splitting is given by $t \mapsto (t,1)$. It follows that $T \times T$ is canonically isomorphic to $diag(T) \times (T \times 1)$. Furthermore, by definition, $diag(T)$ acts trivially on $\mathcal{Y}$. As a consequence, we have a ring isomorphism $H^*_{T\times T}(\mathcal{Y}) \simeq H^*_{diag(T)} \otimes H^*_T(\mathcal{Y})$. This isomorphism is further W-invariant since the W-action on the cohomology rings is induced from the action of $diag(W)$ on $\mathcal{Y}$.

To prove the second part of the Theorem, we adapt to our situation an argument of Littelmann and Procesi ([LP], Theorem 2.3).

Firstly, assuming that i^* is also surjective, we need to show that $C_W(e) = \{1\}$ for all $e \in \Lambda_1$. Since X is equivariantly formal, then $H^*_{G\times G}(X)$ is a free $(H^*_{T\times T})^{W\times W}$-module. And $H^*_{T\times T}(\mathcal{Y})$ is a free $H^*_{T\times T}$-module, for the same reason. By Corollary 1.2.10 one can choose a graded $W \times W$-submodule R of $H^*_{T\times T}$, isomorphic to the regular representation of $W \times W$, such that

$$H^*_{T\times T} \simeq R \otimes (H^*_{T\times T})^{W\times W}$$

as graded $(H^*_{T\times T})^{W\times W}$-module. Accordingly, $H^*_{T\times T}(\mathcal{Y})^{W\times W}$ is in a natural way a free $(H^*_{T\times T})^{W\times W}$-module.

Notice that the rank of $H^*_{G\times G}(X)$, as a $H^*_{G\times G}$-module, equals $|\mathcal{R}_1|$, the number of $T \times T$-fixed points. This is just a consequence of the fact that X has no odd cohomology (Proposition 1.4.5). Since, by assumption, ι^* is a graded isomorphism of free $(H^*_{T\times T})^{W\times W}$-modules, we conclude that the ranks of $H^*_{G\times G}(X)$ and $H^*_{T\times T}(\mathcal{Y})^W$ must be the same. The next step consists in finding out a more intrisic formula for the rank of the latter module, so as to compare it with $|\mathcal{R}_1|$.

Let $\mathcal{I}$ denote the ideal in $(H^*_{T\times T})^{W\times W}$ of elements of strictly positive degree. Recall that we can find a graded W-stable submodule U of $H^*_{T\times T}(\mathcal{Y})$ such that the morphism

$$U \otimes H^*_{T\times T} \longrightarrow H^*_{T\times T}(\mathcal{Y})$$

is a W-equivariant isomorphism of graded $H^*_{T\times T}$-modules. Because $\mathcal{Y}$ is equivariantly formal, we can actually set U to be $H^*(\mathcal{Y})$ (Lemma 1.4.3). The dimension of U is the Euler characteristic of $\mathcal{Y}$, and hence equal to $|E_1|$, the number of $T \times T$-fixed points in $\mathcal{Y}$. So

$$H^*_{T\times T}(\mathcal{Y})^W / \mathcal{I} H^*_{T\times T}(\mathcal{Y})^W$$

is isomorphic to $(U \otimes R)^W$ as W-representation. Since R decomposes into the direct sum of $|W|$-copies of the regular representation of W, then Lemma 4.3.7 below shows that $\dim (U \otimes R)^W = |E_1||W|$. Consequently,

$$\dim H^*_{T\times T}(\mathcal{Y})^W / \mathcal{I} H^*_{T\times T}(\mathcal{Y})^W = |E_1||W|,$$

which, by the graded Nakayama Lemma, also coincides with the rank of $H^*_{T\times T}(\mathcal{Y})^W$ as a free $(H^*_{T\times T})^{W\times W}$-module.

In summary, the surjectivity of ι^* implies that $|\mathcal{R}_1| = |E_1||W|$. Now Lemma 4.3.8 below finally yields $C_W(e) = \{1\}$ for all $e \in \Lambda_1$.

For the converse, suppose that $C_W(e) = \{1\}$ for all $e \in \Lambda_1$. We need to show that i^* is surjective. To achieve our goal, we modify slightly an argument of [LP], Section 4.1, and Brion [Br4], Corollary 3.1.2. Define the variety

$$\mathcal{N} = \bigcup_{w \in W} w\mathcal{Y}.$$

We claim that this union is, in fact, a disjoint union. Indeed, observe that $\mathcal{N}$ contains all the $T \times T$-fixed points of X. That is, $\mathcal{N}$ has $|\mathcal{R}_1|$ fixed points. On the other hand, each $w\mathcal{Y}$ has $|E_1|$ fixed points (for its corresponding T-action). Now, if it were the case that there is a pair of distinct subvarieties $w\mathcal{Y}$ and $w'\mathcal{Y}$ with non-empty intersection, then this intersection should also contain $T \times T$-fixed points. But then a simple counting argument would yield $|\mathcal{R}_1| < |E_1||W|$. This is impossible, by our assumptions and Lemma 4.3.8. Hence,

$$\mathcal{N} = \bigsqcup_{w \in W} w\mathcal{Y}.$$

Clearly, $\mathcal{N}$ is rationally smooth and equivariantly formal (because each $w\mathcal{Y}$ is so, for $w \in W$). Moreover, since $\mathcal{N}$ contains all the $T \times T$-fixed points of X, then the restriction map

$$H^*_{T\times T}(X) \to H^*_{T\times T}(\mathcal{N})$$

is injective.

It follows from Theorem 4.1.1 that all the $T \times T$-curves of X are contained either in closed $G \times G$-orbits (curves of type 1. and 2.) or in $\mathcal{N}$ (curves of type 3.).

As a consequence, Theorem 1.4.11 can also be applied to $\mathcal{N}$. After taking $W \times W$-invariants (compare Corollary 4.3.5), we see that the restriction to $\mathcal{N}$ induces an isomorphism

$$H^*_{T\times T}(X)^{W\times W} \simeq H^*_{T\times T}(\mathcal{N})^{W\times W} \simeq \left(\bigoplus_{w\in W} H^*_{T\times T}(\mathcal{Y})\right)^{W\times W} \simeq H^*_{T\times T}(\mathcal{Y})^W.$$

The proof is now complete. □

Lemma 4.3.7 ([LP])**.** *If N is a finite group, and U and V are two finite dimensional representations of N such that V is the sum of copies of the regular representation of N, then*

$$\dim (V \otimes U)^N = \frac{\dim V \cdot \dim U}{|N|}.$$

□

Lemma 4.3.8. *Let $\mathcal{R}_1$ be the set of rank one elements of the Renner monoid $\mathcal{R}$. Then $|\mathcal{R}_1| = |E_1| \cdot |W|$ if and only if $C_W(e) = 1$ for every $e \in \Lambda_1$.*

Proof. We know, by Theorem 3.1.10, that Λ_1 can be identified with a set of representatives of the $W \times W$-orbits in $\mathcal{R}_1$. Likewise, Λ_1 also corresponds to a set of representatives of the W-orbits in E_1. Let k be the cardinality of Λ_1 and let $e_1, \ldots, e_k$ be a complete list of the elements of Λ_1. Since we are dealing with elements of rank one, it is easy to see that $We_iW \simeq (W/C_W(e_i)) \times (W/C_W(e_i))$, for all $i = 1, \ldots, k$. Thus

$$|\mathcal{R}_1| = \sum_i |We_iW| = \sum_i |W/C_W(e_i)|^2.$$

On the other hand, the orbit $We_i \subset E_1$ satisfies $We_i \simeq W/C_W(e_i)$. This implies the following formula

$$|E_1| = \sum_i |We_i| = \sum_i |W/C_W(e_i)|.$$

Now recall that $\mathcal{R}_1 = E_1W = WE_1$. In other words, $|\mathcal{R}_1| \leq |E_1||W|$ and so

$$\sum_i |W/C_W(e_i)|^2 \leq \sum_i |W/C_W(e_i)||W|.$$

Therefore, $|\mathcal{R}_1| = |E_1||W|$ if and only if

$$\sum_i \left(|W/C_W(e_i)||W| - |W/C_W(e_i)|^2\right) = 0.$$

Notice that the latter condition is equivalent to having $|W/C_W(e_i)| = |W|$ for every i, because $|W| - |W/C_W(e_i)| \geq 0$. It is now clear that $|\mathcal{R}_1| = |E_1||W|$ if and only if $|C_W(e_i)| = 1$ for all $i = 1, \ldots, k$. $\square$

4.4 Examples

Recall that if (W, S) is a Weyl group and $J \subset S$, then W^J is the set of minimal length representatives for the cosets of W_J in W, where W_J is the subgroup of W generated by J. In particular, the canonical composition

$$W^J \to W \to W/W^J$$

is bijective.

4.4.1 $\mathcal{J}$-irreducible Monoids

A reductive monoid M with $0 \in M$ is called **$\mathcal{J}$-irreducible** if $M \backslash \{0\}$ has exactly one minimal $G \times G$-orbit. Any $\mathcal{J}$-irreducible monoid is also semisimple. See [PR], or Section 7.3 of [R8] for a systematic discussion of this important class of reductive monoids, and for a proof of the following Theorem.

Theorem 4.4.1. *Let M be a reductive monoid. The following are equivalent.*

1. *M is $\mathcal{J}$-irreducible.*

2. *There is an irreducible rational representation $\rho : M \to End(V)$ which is finite as a morphism of algebraic varieties.*

3. *If $\overline{T} \subseteq M$ is the Zariski closure in M of a maximal torus $T \subseteq G$ then the Weyl group W of T acts transitively on the set of minimal nonzero idempotents of $\overline{T}$.*

By the results of Section 4 of [PR], if M is $\mathcal{J}$-irreducible, there is a unique, minimal, nonzero idempotent $e_1 \in E(\overline{T})$ such that $e_1 B = e_1 B e_1$, where B is the given Borel subgroup containing T. That is, $\Lambda_1 = \{e_1\}$. If M is $\mathcal{J}$-irreducible we say that M is **$\mathcal{J}$-irreducible of type** J if, for this idempotent e_1,

$$J = \{s \in S \mid se_1 = e_1 s\},$$

where S is the set of simple involutions relative to T and B. The set J can be determined in terms of any irreducible representation satisfying condition *2* of Theorem 4.4.1. See [PR] for the details.

As above we let $S \subseteq W$ be the set of simple involutions of W relative to T and B. We can regard S as the set of vertices of a graph with edges $\{(s,t) \mid st \neq ts\}$. Thus we may speak of the connected components of any subset of S.

The following result was first recorded in [PR]. It describes the $G \times G$-orbit structure of a $\mathcal{J}$-irreducible monoid of type $J \subseteq S$.

Theorem 4.4.2. *Let M be a $\mathcal{J}$-irreducible monoid of type $J \subseteq S$.*

1. *There is a canonical one-to-one order-preserving correspondence between the set of $G \times G$-orbits acting on M and the set of W-orbits acting on the set of idempotents of $\overline{T}$. This set is canonically identified with $\Lambda = \{e \in E(\overline{T}) \mid eB = eBe\}$.*

2. *$\Lambda \setminus \{0\} \cong \{I \subseteq S \mid$ no connected component of I is contained entirely in $J\}$ in such a way that e corresponds to $I \subseteq S$ if $I = \{s \in S \mid se = es \neq e\}$. If we let $\Lambda_2 = \{e \in \Lambda \mid dim(Te) = 2\}$ then this bijection identifies Λ_2 with $S \setminus J$.*

3. *If $e \in \Lambda \backslash \{0\}$ corresponds to I, as in 2 above, then $C_W(e) = W_K$ where $K = I \cup \{s \in J \mid st = ts \text{ for all } t \in I\}$.*

In fact, Λ is completely determined by J. See [R8] for a systematic discussion of $\mathcal{J}$-irreducible monoids, in particular Lemma 7.8 of [R8]. Notice also that part *1* of Theorem 4.4.2 is true for any reductive monoid (compare Theorem 3.1.10 and the remarks following it).

Let M be a $\mathcal{J}$-irreducible monoid of type $J \subseteq S$ and let $\overline{T}$ be the closure in M of a maximal torus T of G. By part *b)* of Theorem 5.4 of [R8], $\overline{T}$ is a normal variety. Define

$$X(J) = [\overline{T} \backslash \{0\}]/\mathbb{C}^*.$$

The terminology is justified since $X(J)$ depends only on J and not on M or λ ([PR]).

Rationally smooth embeddings obtained from $\mathcal{J}$-irreducible monoids have been classified by Renner in [R5]. The reader will find there a detailed list of all the subsets J for which $X(J)$ is rationally smooth.

Definition 4.4.3. Let (W, S) be a Weyl group and let $J \subseteq S$ be a proper subset. Define

$$S^J = (W_J(S \setminus J)W_J) \cap W^J.$$

We refer to (W^J, S^J) as the **descent system** associated with $J \subseteq S$.

Proposition 4.4.4. *There is a canonical identification*

$$S^J \cong \{g \in E_2 \mid ge_1 = e_1\}.$$

For a proof, see [R4].

The following table, first recorded in [R4], provides the reader with a summary-translation between the monoid jargon and the Bruhat poset jargon.

Reductive Monoid Jargon	**Bruhat Order Jargon**
$e_1 \in \Lambda_1 = \{e_1\}$	$1 \in W^J$
$e = e_v \in E_1$	The $v \in W^J$ with $e = ve_1v^{-1}$
$e_v \leq e_w$ in E_1, i.e. $e_vBe_w \neq 0$	$w \leq v$ in W^J
$E_2 = \{g \in E \mid dim(gT) = 2\}$	$(u, v) \in W^J \times W^J$ such that $u < v$ and $u^{-1}v \in S^JW_J$
$\{g \in E_2 \mid gB = gBg\}$	$S \setminus J$
$\{g \in E_2 \mid ge_1 = e_1 \}$	$S^J = (W_J(S \setminus J)W_J) \cap W^J$
$\{g \in E_2 \mid ge_1 = e_1, g \sim g_s\}$	$S^J_s = (W_JsW_J) \cap W^J$
$E_2(e_w) = \{g \in E_2 \mid ge_w = e_w\}$	$\{v \in W^J \mid w^{-1}v \in S^JW_J\}$
$\Gamma(e_w) = \{g \in E_2(e_w) \mid ge' = e'$ for some $e' < e_w\}$	$A^J(w) = \{r \in S^J \mid w < wr\}$
$\Gamma_s(e_w) = \Gamma(e_w) \cap \{g \in E_2 \mid g \sim g_s \}$	$A^J_s(w) = \{r \in S^J_s \mid w < wr\}$

For $X = \mathbb{P}(M)$, where M is a $\mathfrak{J}$-irreducible monoid, there are no GKM-curves satisfying the properties of Theorem 4.3.4 (2b), since curves of that type join necessarily fixed points in different closed $G \times G$-orbits. We can make our Theorem 4.3.4 more precise in this context.

Theorem 4.4.5. *Let $X = \mathbb{P}(M)$ be a $\mathfrak{J}$-irreducible rationally smooth standard group embedding of type J. Let e_1 be the unique rank-one idempotent for which $\Lambda_1 = \{e_1\}$. Then the natural morphism $H^*_{T\times T}(X) \to H^*_{T\times T}(G[e_1]G)$ is injective. Furthermore, the image consists of all maps $\varphi \in H^*_{T\times T}(G[e_1]G)$, subject to the condition that, for every $g \in S^J = \{g \in E_2(\overline{T}) \mid ge_1 = e_1g\}$, and $(u, v) \in W \times W$, the following holds:*

$$\varphi(u\, e_1\, u^{-1}\, v) \equiv \varphi(u\, \alpha_g\, e_1\, \alpha_g\, u^{-1}\, v) \bmod (\alpha_g \circ int(u^{-1}), \alpha_g \circ int(u^{-1}) \circ int(v)),$$

where α_g is the root associated to the reflection s_α for which $s_\alpha g = gs_\alpha \neq g$.

Proof. Since there is only one closed $G \times G$-orbit, namely $G[e_1]G$, then the first assertion is a direct consequence of Theorem 4.3.4 (2). Also, recall that there are no curves of type 3, so we just need to focus on translating Theorem 4.3.4, (2a), into our situation. Let $f \in E_2(\overline{T})$. Then there are exactly two rank-one idempotents f_1, f_2, such that $f_1 f = f_1$, $f_2 f = f_2$ and $f_2 = s_\alpha f_1 s_\alpha$, where $s_\alpha f = s_\alpha f \neq f$. On the other hand, because $\Lambda_1 = \{e_1\}$, then $f_1 = u e_j u^{-1}$, for some $u \in W$. The latter implies that $g = u^{-1} f u$ is an idempotent of $\overline{T}$ such that $g e_1 = e_1$. Using the Bruhat-Monoid jargon chart, one easily concludes that $g \in S^J$. In short, any $f \in E_2(\overline{T})$ such that $fe = e$ for some $e \in W^J \simeq E_1(\overline{T})$ is conjugate to an element of S^J. This observation and Theorem 4.3.4, (2a), yield the result. $\square$

Corollary 4.4.6. *Let $X = \mathbb{P}(M)$ be a $\mathfrak{J}$-irreducible rationally smooth standard group embedding of type J. Let e_1 be the unique rank-one idempotent for which $\Lambda_1 = \{e_1\}$. Then the ring $H^*_{G\times G}(X)$ consists of all tuples Ψ, where*

$$\Psi : We_1W \simeq W^J \times W^J \longrightarrow (H^*_{T\times T})^{W_J \times W_J},$$

such that

$$\varphi(e_1) \equiv \varphi(\alpha_g\, e_1\, \alpha_g) \bmod (\alpha_g, \alpha_g),$$

for every $g \in S^J$.

Proof. Simply translate Corollary 4.3.4 into this situation, making use of Theorem 4.4.5. $\square$

The wonderful compactification

The wonderful compactification ([DP]) corresponds to taking $J = \emptyset$. Let $\Lambda_1 = \{e\}$. In this case, our Theorem 4.4.5 yields a different proof of the results of [Br4] and [U].

Theorem 4.4.7. *Let $X = \mathbb{P}(M)$ be the wonderful compactification of a semisimple group G. Then $H^*_{T\times T}(X)$ consists of all maps $\varphi \in H^*_{T\times T}(G/B \times G/B)$ such that*

$$\varphi(u\, e\, u^{-1}\, v) \equiv \varphi(u\, \alpha\, e\, \alpha\, u^{-1}\, v) \bmod (\alpha \circ int(u^{-1}), \alpha \circ int(u^{-1}) \circ int(v)),$$

for every root $\alpha \in S$ and $(u, v) \in W \times W$.

Proof. For the wonderful compactification, we have $GeG \simeq G/B \times G/B$. In addition, since $J = \emptyset$, then $\Lambda_2 = S$ and $S^J = S$. These observations and Theorem 4.4.5 finally imply the result. $\square$

A familiar object: $\mathbb{P}^{(n+1)^2-1}(\mathbb{C})$

This corresponds to the case when (W, S) is of type A_n. In fact, for this case, one has $M = M_{n+1}$, $G = GL_{n+1}$, $G/\mathbb{C}^* = SL_{n+1}$, $W \simeq S_{n+1}$ and $J = \{s_2, \ldots, s_n\}$. Thus, $X = \mathbb{P}^{(n+1)^2-1}$ and so X is rationally smooth.

In this case, $e_1 = (a_{ij})$, with $a_{11} = 1$ and $a_{ij} = 0$ for any $(i, j) \neq (1, 1)$.

Let

$$W =< s_1, ...s_n >$$

be the Weyl group of type A_n (so that $W \cong S_{n+1}$), and let

$$J = \{s_2, ..., s_n\} \subseteq S = \{s_1, ..., s_n\}.$$

Then $J \subseteq S$ is combinatorially smooth. One checks that

$$W^J = \{1, s_1, s_2s_1, s_3s_2s_1, ..., s_ns_{n-1} \cdots s_2s_1\}.$$

Notice that

$$1 < s_1 < s_2s_1 < ... < s_ns_{n-1} \cdots s_1.$$

In this very special example we obtain that $S^J = W^J \backslash \{1\}$. Besides, $G[e]G = \mathbb{P}^n \times \mathbb{P}^n$. Considering the previous remarks, Theorem 4.4.5 reads as follows:

Theorem 4.4.8. *$H^*_{T\times T}(\mathbb{P}^{(n+1)^2-1})$ injects into $H^*_{T\times T}(\mathbb{P}^n \times \mathbb{P}^n)$ and it consists of all maps $\varphi \in H^*_{T\times T}(\mathbb{P}^n \times \mathbb{P}^n)$ subject to the condition that, for every $g \in S^J$ and $(u, v) \in S_n \times S_n$, the following holds:*

$$\varphi(ue_1u^{-1}v) \equiv \varphi(u\alpha_g e_1 \alpha_g u^{-1}v) \bmod (\alpha_g \circ int(u^{-1}), \alpha_g \circ int(u^{-1}) \circ int(v)).$$

Here $\alpha_g = t_1 \cdot t_{j+1}^{-1}$ is the root $\alpha_1 \circ int(s_2) \circ \ldots \circ int(s_j)$, for each $g = s_j \cdots s_1$ with $j \geq 1$, $g \neq 1$, and $\alpha_1 = t_1 t_2^{-1}$. □

4.4.2 Rationally smooth torus embeddings $X(J)$

Let M be a $\mathfrak{J}$-irreducible monoid of type J. We denote by $X(J)$ the associated projective torus embedding, that is,

$$X(J) = (\overline{T} - \{0\})/Z.$$

Since $X(J)$ is a torus embedding, all closed $T\times T$-orbits are isomorphic to points. In fact, $T[e]T \simeq [e]$ for every $e \in E_1(\overline{T})$.

The $T \times T$-fixed points in $X(J)$ correspond to $W^J \simeq E_1(\overline{T})$.

The collection of $T \times T$-curves of $X(J)$, say $C(X(J), T \times T)$, corresponds to the set of rank-two idempotents $E_2(\overline{T})$. Furthermore, $C(X(J), T \times T)$ can be identified with the set

$$\{(u, v) \in W^J \times W^J \mid u < v \text{ and } u^{-1}v \in S^J W_J\}.$$

In this case, there are no $T\times T$-curves joining fixed points in the same closed $T\times T$-orbit.

This information together with Theorem 4.3.4 yield the following result.

Theorem 4.4.9. *Let $X(J)$ be the projective torus embedding associated to a rationally smooth standard group embedding $\mathbb{P}(M)$, where M is a $\mathfrak{J}$-irreducible monoid* *of type J. Then $H^*_{T\times T}(X(J)) \simeq H^*_T \otimes H^*_T(X(J))$. Moreover, $H^*_T(X(J))$ consists of all maps*

$$\varphi : W^J \to H^*_T$$

such that $\varphi(u) \equiv \varphi(v) \bmod (\chi_{u,v})$, whenever $u < v$ and $u^{-1}v \in S^J W_J$. Here $\chi_{u,v}$ equals $\lambda_{f_{u,v}}$, where $f_{u,v}$ is the unique idempotent in $E_2(\overline{T})$ such that both $u \cdot f_{u,v} \neq 0$ and $v \cdot f_{u,v} \neq 0$. $\square$

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