8-20-2024 3:30 PM

Osculating Curves

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A thesis submitted in partial fulfillment of the requirements for the Master of Science degree in Applied Mathematics

Abstract

Consider a complex analytic curve X in $\mathbb{C}^2$, along with a specific point $p \in X$. The primary concern arises in approximating geometrically the curve X precisely at the point p. Analogously, in introductory calculus, students learn to compute the tangent line to the graph of a function $y = f(x)$ at a given point $p = (x_*, f(x_*))$ by utilizing the derivative of f at x_*.

For analytical convenience, we assume a local representation of the curve X using a power series expansion. This representation centers the point p at the origin $(0,0) \in \mathbb{C}^2$. Thus, our mathematical input becomes a Taylor series:

$$f(x) = \sum_{i=1}^{\infty} c_i x^i \ .$$

Alternatively, one may conceptualize our input as a "germ" or a "jet" of sufficiently high order. This assumption of a local representation is grounded in the application of the implicit function theorem. The theorem, coupled with the specific nature of the questions posed, allows us to work with a localized description of X that relies on only finitely many coefficients $\{c_i\}_{i=1}^{\infty}$.

An alternative perspective on defining a plane curve involves considering it as the set of points where an implicit polynomial function $F(x,y) \in \mathbb{C}[x,y]$ equals zero. This set of points is termed a "variety," denoted as $V(F(x,y)) = \{(x,y) \in \mathbb{C}^2 \mid F(x,y) = 0\}$.

Given a degree d and a sequence $\mathbf{c} = (c_1, c_2, \ldots)$ defining the power series expansion, our objective is to ascertain the "best approximation" of the curve Γ through a curve expressed as $V(F(x,y))$, where $F(x,y)$ has a degree no greater than d. This specific curve is referred to as the "degree d osculating curve" of the Taylor series, or equivalently, the "degree d osculating curve" of X at the point p.

Keywords: Approximation , Osculation

Summary for Lay Audience

A curve is said to osculate a second curve if the two touch only at a point. A straight line tangent to a curve is a familiar example. The straight line osculates the curve at the point where it touches the curve. A second example will be familiar to some calculus students: an osculating circle. In this case, the circle not only touches a given curve, but also matches the curvature of the curve at the point of touching.

Osculating curves approximate the curve they are touching in the neighbourhood of the contact point. For this reason, they are used a lot in Computer Aided Design (CAD) to speed up calculations and to ensure that curves and surfaces remain smooth at places where they join.

The thesis develops a new way of calculating osculating curves, without being restricted to straight lines or circles. This allows formulae of greater generality than before to be computed.

Co-Authorship Statement

In Chapter 4 of this thesis, the content is derived from a collaborative research paper prepared for presentation at the *Computer Algebra in Scientific Computing (CASC) 2024* conference. The paper was co-authored by Sepideh Bahrami, Taylor Brysiewicz, David J. Jeffrey, and Marc Moreno Maza. The research and findings presented in this chapter are the result of joint efforts and contributions from all listed authors.

Dedication

I dedicate this thesis to my parents, Elham and Mohammad, whose constant support and love have been my greatest strength; to my sister, Reyhaneh, whose encouragement has always lifted my spirits; and to my friends, whose support and belief in me have made this journey special. I also want to thank a dear friend whose influence has deeply changed my life. This work is a reflection of the support and inspiration from all of you.

Acknowledgement

I want to thank my supervisors, Marc Moreno Maza and David Jeffrey, for their invaluable guidance and support throughout my research. Their expertise and encouragement were essential to the completion of this thesis. I am deeply grateful for their help and advice.

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Chapter 1

Introduction

The local analysis of curves and surfaces is a fundamental aspect of complex geometry and has broad applications in various scientific and engineering disciplines. In this area, commonly used techniques include power series expansions, the computation of tangent spaces, and intersection multiplicity numbers.

In the study of curves and surfaces, the idea of curvature is an important part of understanding how curves (resp. surfaces) deviate from being a straight line (resp. plane). This idea extends to osculation, which is interesting both for mathematical reasons and for its practical applications. An osculating curve is a local approximation to another curve at a specific point. This means it closely follows the curvature and direction of the original curve at that point. Understanding osculating curves helps in approximating complex curves by simpler ones, which is crucial in applications such as computer graphics and numerical methods.

The osculating curve provides information about the curvature of the original curve at a point. Curvature measures how quickly a curve changes direction as you move along it. This information is essential in many practical applications, including designing roads, analyzing shapes in engineering, and understanding wave patterns.

Computer-aided design (CAD) refers to software that allows engineers to design things, for example, the body of a car. Osculating curves are important when designing shapes as they help with the visualization and analysis of shapes. For instance, the body of a car bends smoothly around the outside of the car. Osculating curves help designers ensure smooth joins, meaning the curvature changes smoothly between different sections of a surface. In CAD, surfaces are often defined by curves, and making sure that these curves blend smoothly into each other without sudden changes in curvature is important for creating designs that are aesthetically pleasing and easy to manufacture. Surfaces can be really complicated, and osculating curves can be used to approximate these complicated surfaces by simpler ones that approximately match the local curvature and direction of the surface being designed. This is very useful in simplifying and optimizing designs without losing important geometric details.

Modern factories use machines that are computer-controlled. Computer Numerically Controlled (CNC) machines and 3D printing use osculating curves. Tool paths need to follow the shapes of the surfaces they are making accurately. Osculating curves help in generating tool paths that ensure close cutting or printing at high speeds, and they also help prevent errors caused by sudden changes in surface curvature.

Many CAD systems use parametric modeling and mathematical parameters. Osculating curves provide a mathematical framework for defining and manipulating curves and surfaces parametrically. This allows engineers to change designs by adjusting parameters that control the curvature and shape of curves. We see that osculating curves have many applications.

Many studies have been conducted in this field. Philip Franklin's foundational work on Osculating Curves and Surfaces [8] explains the key ideas behind these mathematical concepts and shows how they can be used in many fields. Franklin starts by defining osculating curves as the best possible approximations of other curves at specific points, using higher-order derivatives to capture their shape precisely. He then expands these ideas to include osculating planes, circles, and higher-dimensional surfaces like spheres and hypersurfaces.

Before Franklin, Arthur Cayley used a method involving higher-order contact between curves to study osculating circles and conics [4]. Specifically, he explored the concept of contact points where a curve and a conic section (like an ellipse or hyperbola) share higher-order derivatives at a specific point. This method involves using power series expansions and Taylor series to approximate the curve locally and determine the best-fitting conic section that has multiple points of contact (up to five-point contact) with the original curve.

By considering higher-order derivatives and the associated power series, Cayley was able to define osculating conics that provide a very accurate approximation of the curve at the given point. This approach allows for precise modeling and analysis of the curve's behavior in a small neighborhood around the point of contact.

Balay-Wilson and Brysiewicz's paper on "Points of Ninth Order on Cubic Curves" extends these ideas by providing a necessary and sufficient condition for points on a cubic to be associated with an infinite family of other cubics with nine-point contact at that point. They parameterize this family of cubics using the osculating quadratic, offering new insights into higher-order contact on cubic curves. Their method involves geometric analysis to establish the conditions under which a cubic curve will have points of ninth-order contact, leading to a parameterization of the family of such curves. This work significantly advances the understanding of high-order contacts and their applications in algebraic geometry [3].

This thesis focuses on the geometric approximation of complex analytic curves in $\mathbb{C}^2$. We assume that such a curve is given to us locally at a point p by the expansion of a power series f. Our goal is to find the best way to approximate the plane curve X using another curve of a specific form, $\mathcal{V}(F(x,y))$, where $F(x,y)$ is a polynomial with a degree of at most d, where d is a prescribed positive integer. This kind of curve is called a "degree d osculating curve" of X at the point p. Therefore, our project takes a different approach from previous works.

The target polynomial $F(x,y)$ is obtained by solving a linear system given by a matrix whose coefficients are polynomials in the coefficients of the (truncated) power series f. A first contribution of this thesis is a study of the combinatorial properties of this matrix, see Chapter 3.

One non-obvious question in our approach is to prove that this matrix is generically non-singular. This is done in Chapter 4 with the introduction and study of what we call *osculating spaces*. This is a second contribution of this thesis.

Using our method, we can identify the osculating curves at smooth points on a curve with the highest contact order. However, there are instances where osculating curves of degree d exhibit a higher contact order than expected. The points where this higher contact occurs are known as d-extactic points.

Cayley also explained the concept of sextactic points of a plane curve in one of his papers [5]. These are points where the contact order is higher than what we expect.

Additionally, the paper "A Uniform Approach for the Fast Computation of Matrix-Type Padé Approximates" [6] presents an innovative method for efficiently computing Matrix-Type Padé Approximates (MTPAs). These approximates are crucial in numerical analysis and scientific computing. By developing a unified computational framework, the authors enhance the speed and accuracy of generating MTPAs, addressing challenges similar to those faced in modeling and approximating complex geometrical structures. This connection shows how advanced mathematical methods are useful in both geometric modeling and matrix approximation. They both aim to offer efficient and precise solutions for various scientific and engineering problems.

Cayley wrote two important papers on osculating curves [4], [5]. He was a famous English mathematician who wrote over 900 papers on mathematics, including matrix theory, algebra, and geometry. He is famous for the Cayley-Hamilton theorem, which states that every square matrix satisfies its own characteristic equation. Cayley learned of the work of George Salmon when the two sat next to each other in a lecture by Hamilton on quaternions [12]. Cayley worked as a lawyer for 15 years and only worked on mathematics at night. He later became a professor at Cambridge and worked full-time on mathematics. He died and was buried in Cambridge, but his gravestone was lost in the 1980s [11].

Modern mathematicians who have worked on problems of curves include Vladimir Arnold, a Ukrainian-born mathematician who won a Wolf Prize for his work on dynamical systems, differential equations, and singularity theory. He wrote a paper on extactic points (the title of the paper misspells extactic as extatic) [2].

In summary, the study of osculating curves is rich with historical context and modern applications. From the foundational work of Cayley and Franklin to the cutting-edge research in computational methods, understanding osculating curves provides essential insights into the geometry of curves and surfaces, with significant implications for both theoretical mathematics and practical engineering.

Chapter 2

Background

Given a complex analytic curve $X \subset \mathbb{C}^2$ and a point $p \in X$, a natural problem is to geometrically approximate X at p. Assuming that p is the origin, the curve X can be described by a Taylor series

$$f(x) = \sum_{i=1}^{\infty} c_i x^i. \tag{2.1}$$

An alternative way of defining a plane curve is as the vanishing locus of an implicit polynomial function $F(x,y) \in \mathbb{C}[x,y]$.

$$\mathbf{V}(F(x,y)) = \{(x,y) \in \mathbb{C}^2 \mid F(x,y) = 0\}.$$

Given $d \in \mathbb{N}$ and a sequence $\mathbf{c} = (c_1, c_2, \ldots)$ defining (2.1), our goal is to determine the *best approximation* of X by a curve of the form $\mathbf{V}(F(x,y))$ where $F(x,y)$ has degree at most d. [1] Such a curve is called a *degree d osculating curve of X at p*. For example, the degree 1 osculating curve of X at p is simply the tangent line of X at p. Given, a formula for this curve is

$$F(x,y) = c_1 x - y.$$

In this section, we review different notions related to our goal. In Section 2.1, we start with the well-known subject of osculating circles, taught in undergraduate calculus courses. In Section 2.2, we recall basic concepts and results about algebraic curves. Section 2.3 is dedicated to the notions of a *contact order* and *osculating curve* and in section 2.4 we see examples of computing calculating circles in Maple.

2.1 Osculating circles

Multivariable calculus explores into the study of functions of several variables and their geometric interpretation. Central to this discipline is the exploration of space curves, which represent paths or trajectories in three-dimensional space. These curves find extensive applications across various fields, including physics, engineering, and computer graphics, where they serve as fundamental tools for describing the motion of objects and analyzing dynamic systems. One critical aspect of studying space curves is understanding their curvature at specific points, a concept elucidated by the osculating circle. The osculating circle provides a geometric approximation of the curve's behaviour at a given point, offering valuable insights into its local curvature. Mastery of space curves and the computation of their osculating circles are integral to advancing mathematical analyses and problem-solving methodologies in multivariable calculus. [14]

In the following, we'll solve an example that will familiarize us more with the concept of the osculating circle.

Example 1 *How can we find the osculating circle of the parabola $y = x^2$ at the origin?*

Let $\gamma(t)$ be a regular parametric plane curve, where t is the parameter. This determines the unit tangent vector $\mathbf{T}(t)$, the unit normal vector $\mathbf{N}(t)$, the signed curvature $k(t)$, and the radius of curvature $R(t)$ at each point where t is defined:

$$\mathbf{T}(t) = \gamma'(t), \quad \mathbf{T}'(t) = k(t)\mathbf{N}(t), \quad R(t) = \frac{1}{|k(t)|}.$$

Suppose that P is a point on γ where $k(t) \neq 0$. The corresponding center of curvature is the point Q at distance R along $\mathbf{N}$, in the same direction if k is positive and in the opposite direction if k is negative. The circle with center at Q and radius R is called the osculating circle to the curve γ at the point P.

The curvature κ of a curve is a measure of how sharply that curve bends at a given point. For a smooth curve defined by a vector-valued function $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$, the curvature at a point is given by:

$$\kappa = \frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|^3}$$

where $\mathbf{v}(t) = \frac{d\mathbf{r}}{dt}$ is the velocity vector, $\mathbf{a}(t) = \frac{d\mathbf{v}}{dt}$ is the acceleration vector, $\|\cdot\|$ denotes the magnitude of a vector, and $\times$ represents the cross product.

In two dimensions, for a curve given by $y = f(x)$, the curvature can also be expressed as:

$$\kappa = \frac{|f''(x)|}{(1 + (f'(x))^2)^{3/2}}$$

The curvature of the parabola at the origin is $\kappa = \frac{1}{y} = \frac{1}{1} = 1$. Hence, the radius of the osculating circle at the origin is $\rho = \frac{1}{\kappa} = 1$, and its center is $(0,1)$. Therefore, the equation of the osculating circle in terms of its center $(0,1)$ and radius 1 is given by:

$$(x-0)^2 + (y-1)^2 = 1$$

which simplifies to:

$$x^2 + (y-1)^2 = 1$$

2.2 Basic facts about algebraic curves

In the study of algebraic curves in the projective plane $\mathbb{P}^2$ over the complex numbers $\mathbb{C}$, fundamental concepts such as singular and smooth points, tangent lines, and intersection multiplicities play crucial roles [10, 9]. An algebraic curve C in $\mathbb{P}^2$ is defined by a homogeneous polynomial $F(x,y)$ in $\mathbb{C}[x,y]$. Singular points on C are characterized by simultaneous vanishing of its partial derivatives F_x and F_y, while smooth points have well-defined tangent lines. The intersection multiplicity of two curves C and D at a point p captures the algebraic complexity of their intersection locally [10]. Bézout's Theorem provides a key result: it states that the total number of intersection points, accounting for multiplicities, between two curves C and C' equals the product of their degrees [10, 9]. This theorem underscores the foundational link between algebraic geometry and the algebra of polynomials, elucidating the geometric properties of curves in $\mathbb{P}^2$.

Let $\mathbb{P}^2$ denote the projective plane over $\mathbb{C}$, where a point $p \in \mathbb{P}^2$ is represented by homogeneous coordinates $(x:y)$. If $\mathbb{C}[x,y]$ is the polynomial ring in two variables x and y and $F(x,y) \in \mathbb{C}[x,y]$ is a homogeneous polynomial, we define the algebraic curve, or simply the curve C, as

$$V(F) = \{(x,y) \in \mathbb{P}^2 \mid F(x,y) = 0\}.$$

For $F \in \mathbb{C}[x,y]$, we denote its partial derivative with respect to x and y by F_x and F_y. A point p on a curve $C = V(F)$ is called singular if

$$F_x(p) = F_y(p) = 0.$$

While a non-singular point is called smooth. Given a curve $C = V(F)$ and a point $p \in C$, we denote the tangent to C at p by T_p. For a smooth point p, there is a unique tangent at p given as

$$T_p = V(xF_x(p) + yF_y(p)).$$

Definition 1 *Let $C,D \in \mathbb{C}[x,y]$ and p be a maximal ideal. We define the intersection multiplicity of C and D at p to be*

$$I_p(C,D) = dim \frac{\mathbb{C}[x,y]_p}{\langle C,D \rangle}.$$

where the dimension is taken to be the dimension as a $\mathbb{C}$ vector space.

Theorem 1 *(Bézout's Theorem) Let C and C' be two curves in $\mathbb{P}^2$ without common components of degrees d and d' respectively. Let $C \cap C' = \{p_1, \dots, p_n\}$. Then*

$$\sum_{i=1}^{n} (C \cdot C')_{p_i} = d \cdot d'.$$

In other words, two curves intersect in precisely the product of their degrees number of points, counting multiplicity.

2.3 Contact order and osculating curve

Throughout this manuscript, we assume that X has been given locally as the graph of the sum of a power series (2.1). Fix a positive integer d. The space of degree d algebraic curves in $\mathbb{C}^2$ is a projective space of (projective) dimension

$$N_d = \binom{d+2}{2} - 1 \tag{2.2}$$

since a generic degree d polynomial in two variables

$$F(x,y) = \sum_{i+j\leq d} a_{i,j}x^i y^j \tag{2.3}$$

has (N_d+1)-many monomials, and scaling $F(x,y)$ does not change where it vanishes.

With Definition 2, we adapt the well-known notion of contact order (see the landmark textbook [7] by Richard Courant) to our context.

Definition 2 *Let again $F \in \mathbb{C}[x,y]$ be a polynomial and $f \in \mathbb{C}[[x]]$ be a power series. Assume that both the constant terms of F and f are null. Let k be a positive integer. We say that the curve $\mathbf{V}(F(x,y))$ has* contact order *k with the graph Γ of $x \mapsto f(x)$ at the origin if*

$$\begin{aligned} &F(x,f(x)) \equiv 0 \ mod\ x^j \qquad for\ j=1,\ldots,k \\ &F(x,f(x)) \not\equiv 0 \ mod\ x^{k+1} \end{aligned}$$

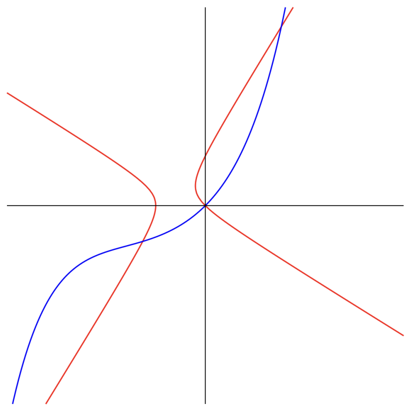
If the curve $\mathbf{V}(F(x,y))$ does not have contact order k with the graph Γ of $x \mapsto f(x)$ at the origin for any positive k, then we say that $\mathbf{V}(F(x,y))$ has contact order 0 with the graph Γ of $x \mapsto f(x)$ at the origin. In any case, we say that $F(x,y)$ approximates *$f(x)$ to order k at the origin, and we write*

$$k = C(\mathbf{V}(F),X) = C(F,f),$$

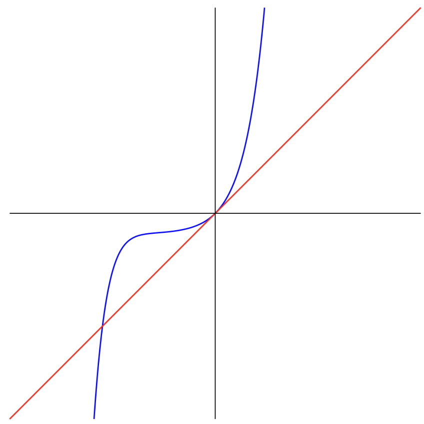
whenever the curve $\mathbf{V}(F(x,y))$ has contact order k with the graph Γ of $x \mapsto f(x)$ at the origin.

Definition 3 *We call an* osculating curve of degree d to $f(x)$ *any algebraic curve of degree d with maximal contact order with $f(x)$ at the origin.*

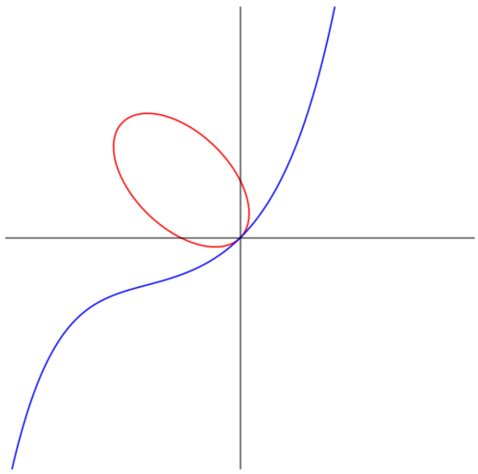
Example 2 *Below, we illustrate conics which osculate the graph of $f(x) = x+x^2+x^3+x^4+x^5$ at the origin to orders $0,1,\ldots,5$.*



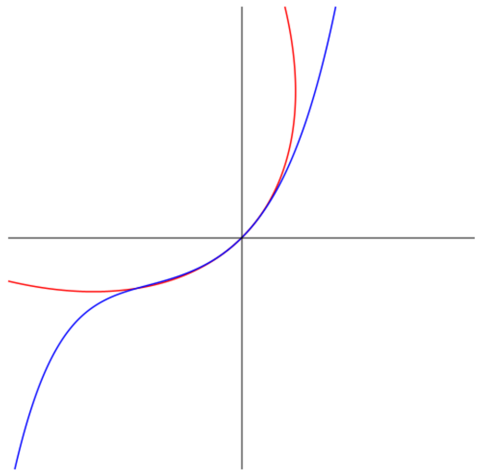
(a) Contact Order 0
$F(x,y) := x^2 + xy + y^2 + x + y$



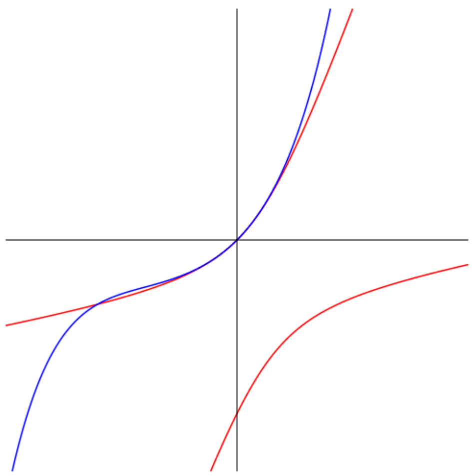
(b) Contact Order 1
$F(x,y) := x + y + yx + x^2 + y^2$



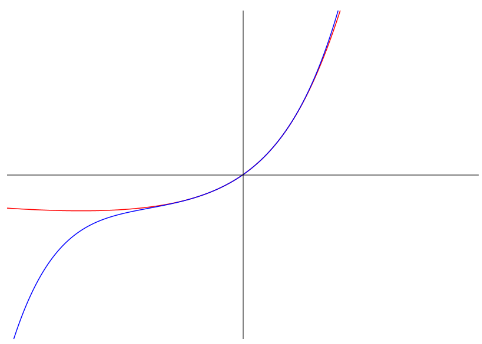
(c) Contact Order 2
$F(x,y) = x - y + yx + x^2 + y^2$



(d) Contact Order 3
$F(x,y) := 6x - 6y + yx + x^2 + y^2$



(e) Contact Order 4
$F(x,y) := 6y + 2y^2 - 6yx - 6x + x^2$



(f) Contact Order 5
$F(x,y) := -18x + 18y - 8yx - 2x^2 + y^2$

Figure 2.1: Osculating curves with contact orders from 0 to 5 for $y = x + x^2 + x^3 + x^4 + x^5$

2.4 Osculating circles and evolutes in Maple

The evolute of a curve is the locus of the centres of curvature of the curve. Using Maple allows these calculations to be done easily. We begin by defining an ellipse and calculating the tangent and normal vectors at a point, chosen to give an interesting plot. Some of the output is stopped to save space. Notation follows the textbooks.

```
> r:=<2*cos(t),sin(t)>:
> dr:=diff(r,t):
> T:=simplify(dr/Norm(dr,2));
```

$$T := \begin{bmatrix} \frac{-2\sin(t)}{\sqrt{-3\cos(t)^2+4}} \\ \frac{\cos(t)}{\sqrt{-3\cos(t)^2+4}} \end{bmatrix}$$

```
> dT:=simplify( diff(T,t) ):
> N:=simplify( dT/Norm(dT,2) ):
> kappa:=simplify( Norm(dT,2)/Norm(dr,2) );
```

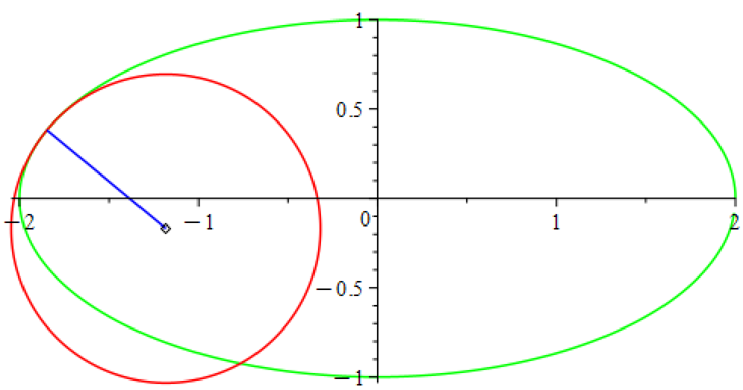
$$\kappa := \frac{4\sqrt{2}}{(5-3\cos(2t))^{3/2}}$$

The variable `RofC` is the radius of curvature, and `CofC` is the centre of curvature.

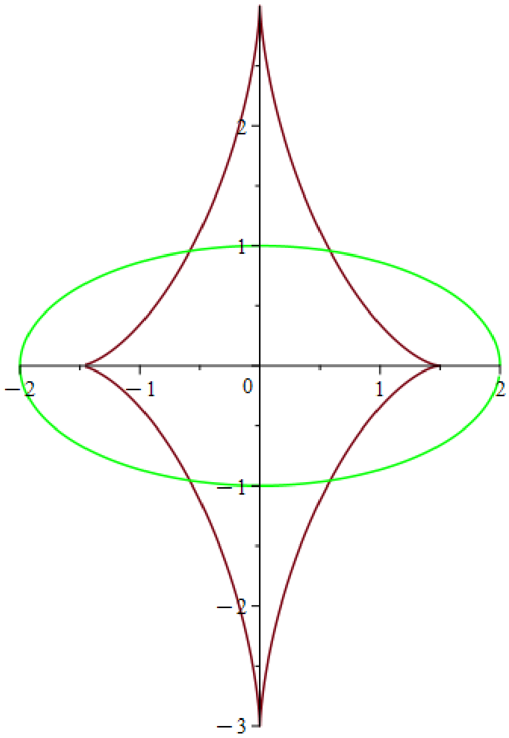
```
RofC:=1/kappa:
CofCE:=simplify(rE+RofC*NE);
```

$$CofC := \begin{bmatrix} 3\cos(t)^3/2 \\ -3\sin(t)^3 \end{bmatrix}$$

Evaluating all quantities at $t = 7\pi/8$ gives the plot



Since the centre of curvature was computed for general t, the evolute comes "for free" by plotting the variable `CofC` as a function of t.



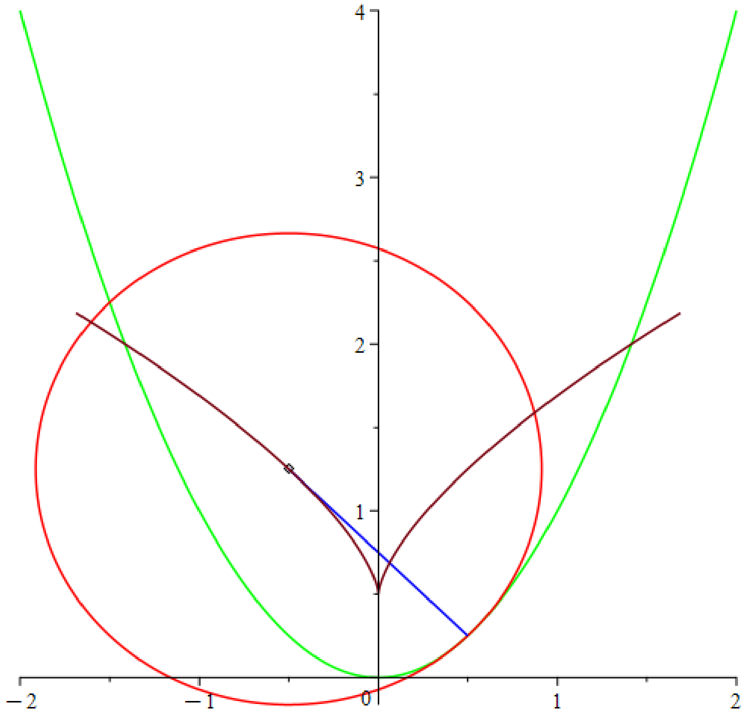
The commands to calculate osculating circles and evolutes can be collected into Maple procedures.

```
OscCircle:=proc(r,t,p,s,x,y)
  local dr,T,dT,N,kappa,RofC,CofC,Oc,Ocp,Ocimp;
  description "Given a curve r(t)=<x(t),y(t)>, as a Maple Vector,
   and a point t=p,",
  "return an equation for the osculating circle at a=x(p),b=y(p)",
  "The point t=p can be symbolic, giving the general case",
  "Both a parametric form and an implicit form are returned";
  uses LinearAlgebra;
  #
  # Input Args
  # r : Vector expression in parameter (t assumed in proc)
  # t : Parameter used to describe r(t)
  # p : Value of t giving a point on curve r(t)
  # s : Parameter used to describe osculating circle on output
  # x,y : unassigned variables used to return implicit equation
             for osculating circle
  #
  dr:=diff(r,t);
  # unit tangent vector
  T:=simplify(dr/Norm(dr,2));
  # and now the principal unit normal
  dT:=simplify(diff(T,t));
```

```
  N:= simplify(dT/Norm(dT,2));
  # From this we can find the curvature
  kappa:= simplify(Norm(dT,2)/Norm(dr,2));
  # Radius of Curvature or Radius of Circle
  RofC:=1/kappa;
  CofC:= simplify(r + RofC*N);
  #
  Oc:=CofC + RofC*<sin(s),cos(s)>;
  Ocp:= simplify(eval(Oc, t=p));
  Ocimp:= simplify(eval((x-CofC[1])^2 + (y-CofC[2])^2 - RofC^2, t=p));
  return([Ocp,Ocimp]);
end proc:
```

```
Evolute:=proc(r,t) local dr,T,dT,N,kappa,RofC,CofC,Oc,Ocp,Ocimp;
description "Given a curve r(t)=<x(t),y(t)>, as a Maple Vector,",
"return an equation for the evolute";
uses LinearAlgebra;
#
# Input Args
# r : Vector expression in parameter (t assumed in proc)
# t : Parameter used to describe r(t)
#
 dr:= diff(r,t);
# unit tangent vector
 T:= simplify(dr/Norm(dr,2));
# and now the principal unit normal
 dT:= simplify( diff(T,t) );
 N:= simplify( dT/Norm(dT,2) );
# From this we can find the curvature
 kappa:= simplify( Norm(dT,2)/Norm(dr,2) );
# Radius of Curvature or Radius of Circle
 RofC:=1/kappa;
 CofC:= simplify(r+RofC*N);
#
end proc:
```

These programs can be applied to a parabola. An osculating circle at $t = 1/2$ is plotted, together with its centre and a radius, and the evolute is also plotted. Notice that the evolute passes through the centre of curvature as expected. The circle is off-centre by construction, but the parabola and evolute are symmetric about the y-axis.


4
3
2
1
0
−2
−1
1
2

Chapter 3

Characterization and properties of osculating curves

Equipped with the definition of contact order (Definition 2), the condition that the degree d polynomial

$$F(x,y) = \sum_{1 i+j \leq d} a_{i,j} x^i y^j,$$

with $a_{0,0} = 0$, approximates the power series

$$f(x) = \sum_{i=1}^{\infty} c_i x^i,$$

to order N_d, with $N_d = \binom{d+2}{2} - 1$, is defined by a system of *polynomial equations* in the N_d unknown variables $\mathbf{a} = \{a_{i,j}\}_{1 \leq i+j \leq d}$ and the parameters $\mathbf{c} = \{c_1, \ldots, c_{N_d - 1}\}$. In this system, each variable $a_{i,j}$ appears to degree 1. Moreover, the condition that, for $1 \leq j \leq N_d$.

$$F(x, f(x)) \equiv 0 \bmod x^j \tag{3.1}$$

holds, involves only the first $j-1$ terms $\{c_1, \ldots, c_{j-1}\}$. Note that, for $j = 1$, the above condition is trivially true.

Notation 1 *We denote by $h_j(\boldsymbol{a}, \boldsymbol{c}) \in \mathbb{C}[\boldsymbol{c}][\boldsymbol{a}]$ the polynomial equation given by the vanishing of $[x^j]F(x, f(x))$, that is, the coefficient of x^j in the composition $F(x, f(x))$.*

Therefore, the coefficients of F are the solutions in $\mathbb{P}_{\mathbf{a}}^{N_d - 1}$ to the homogeneous polynomial system H_c with $N_d - 1$ (non-trivally true) equations in N_d variables, where

$$H_c = \{h_k(\mathbf{a}, \mathbf{c})\}_{k=1}^{N_d - 1} \subset \mathbb{Q}[\mathbf{c}][\mathbf{a}]. \tag{3.2}$$

In Section 3.1, we exhibit properties for the polynomials $h_k(\mathbf{a}, \mathbf{c})$. Then, in Section 3.2, we study the structure of the matrix encoding the linear system defining the unknown variables $a_{i,j}$.

3.1 Combinatorial properties of the polynomials $h_k(\mathbf{a},\mathbf{c})$

We start this section with an example.

Example 3 *In this example, we want to see the general form of the matrix for finding the osculating conic for a given power series. In this context, we have:*

$$F(x,y) = a_{00} + a_{10}x + a_{20}x^2 + a_{01}y + a_{11}xy + a_{02}y^2, \text{ and } H_2 = \{[x^k]F(x, f(x))\}_{k=0}^4$$

$$\begin{aligned} F(x, f(x)) = a_{00} &+ a_{10}x + a_{01}(c_1x + c_2x^2 + c_3x^3 + \dots) \\ &+ a_{20}x^2 + a_{11}x(c_1x + c_2x^2 + c_3x^3 + \dots) \\ &+ a_{02}(c_1x + c_2x^2 + c_3x^3 + \dots)^2 \end{aligned}$$

$$\begin{aligned} [x^0]F(x, f(x)) &= a_{00} \\ [x^1]F(x, f(x)) &= a_{10} + a_{01}c_1 \\ [x^2]F(x, f(x)) &= a_{01}c_2 + a_{20} + a_{11}c_1 + a_{02}c_1^2 \\ [x^3]F(x, f(x)) &= a_{01}c_3 + a_{11}c_2 + 2a_{02}c_1c_2 \\ [x^4]F(x, f(x)) &= a_{02}c_2^2 + 2a_{02}c_1c_3 + a_{11}c_3 + a_{01}c_4 \end{aligned}$$

Now we construct the matrix of coefficients to solve a linear equation system for finding them.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & c_1 & 0 & 0 \\ 0 & 0 & 1 & c_2 & c_1 & c_1^2 \\ 0 & 0 & 0 & c_3 & c_2 & 2c_1c_2 \\ 0 & 0 & 0 & c_4 & c_3 & 2c_1c_3 + c_2^2 \end{bmatrix} \begin{bmatrix} a_{00} \\ a_{10} \\ a_{20} \\ a_{01} \\ a_{11} \\ a_{02} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

After constructing the matrix, we solve the system using Cramer's rule. The solutions, as function of c_1, c_2, c_3, and c_4, are given by:

$$\begin{aligned} a_{0,0} &= 0 \\ a_{1,0} &= \frac{c_1c_2^3}{c_1^2c_2c_4 - c_1^2c_3^2 + c_1c_2^3 - c_1c_2^2c_3 + c_2^4 - 2c_1c_2c_4 + 2c_1c_3^2 - c_2^3 + c_2^2c_3 + c_2c_4 - c_3^2} \\ a_{0,1} &= \frac{-c_2^3}{c_1^2c_2c_4 - c_1^2c_3^2 + c_1c_2^3 - c_1c_2^2c_3 + c_2^4 - 2c_1c_2c_4 + 2c_1c_3^2 - c_2^3 + c_2^2c_3 + c_2c_4 - c_3^2} \\ a_{1,1} &= \frac{-2c_1c_2c_4 + 2c_1c_3^2 + c_2^2c_3}{c_1^2c_2c_4 - c_1^2c_3^2 + c_1c_2^3 - c_1c_2^2c_3 + c_2^4 - 2c_1c_2c_4 + 2c_1c_3^2 - c_2^3 + c_2^2c_3 + c_2c_4 - c_3^2} \\ a_{2,0} &= \frac{c_1^2c_2c_4 - c_1^2c_3^2 - c_1c_2^2c_3 + c_2^4}{c_1^2c_2c_4 - c_1^2c_3^2 + c_1c_2^3 - c_1c_2^2c_3 + c_2^4 - 2c_1c_2c_4 + 2c_1c_3^2 - c_2^3 + c_2^2c_3 + c_2c_4 - c_3^2} \\ a_{0,2} &= \frac{c_2c_4 - c_3^2}{c_1^2c_2c_4 - c_1^2c_3^2 + c_1c_2^3 - c_1c_2^2c_3 + c_2^4 - 2c_1c_2c_4 + 2c_1c_3^2 - c_2^3 + c_2^2c_3 + c_2c_4 - c_3^2} \end{aligned}$$

In Chapter 4, we shall prove that there exists a polynomial F of degree d satisfying the equations given by Relation (3.1) thus, which has contact order at least N_d with $f(x)$. Moreover, we shall see that

(1) the osculating curve of degree d to $f(x)$ is unique, and

(2) for almost all values of $\mathbf{c}$, the osculating curve of degree d to $f(x)$ has contact order N_d with $f(x)$.

Proving these results will require the study of particular algebraic varieties called that we call *osculating spaces*.

In the present chapter we will focus on combinatorial aspects of the formula for the osculating curve of a generic power series by a degree d polynomial. More precisely, we analyze the polynomials $h_k(\mathbf{a},\mathbf{c})$ appearing as coefficients of x in the composition $F(x,f(x))$. As seen in Example 3 the structure of these polynomials is special in several ways. First, each polynomial is bi-homogeneous in the $\mathbf{a}$ and $\mathbf{c}$ variables in that each monomial has the same degree in the $\mathbf{a}$ and $\mathbf{c}$ variables separately. The degree in the $\mathbf{a}$ variables is always one. Moreover, these polynomials are all *quasihomogeneous* in that each monomial involved has the same *quasidegree*. The quasi-degree of a monomial $m = a_{ij}c_1^{\gamma_1}c_2^{\gamma_2}\cdots c_{N_d-1}^{\gamma_{N_d-1}}$ is

$$\mathrm{qdeg}(m) = 1\cdot\gamma_1 + 2\cdot\gamma_2 + \cdots + (N_d-1)\cdot\gamma_{N_d-1}.$$

To precisely describe these patterns, we introduce the following definitions and then state a fundamental technical theorem.

Definition 4 *For $j,\delta,n\in\mathbb{N}$ we let $R^n_{j,\delta}$ be all (n)-tuples of natural numbers $\boldsymbol{r}=(r_1,\ldots,r_n)\in\mathbb{N}^n$ such that $\sum_{t=1}^n r_t = j$ and $\sum_{t=1}^n t\cdot r_t = \delta$. Given a collection of variables $\mathbf{c}=(c_1,\ldots,c_n)$ we write*

$$q_{j,\delta}(c_1,\ldots,c_n) = \sum_{\boldsymbol{r}=(r_1,\ldots,r_n)\in R^n_{j,\delta}} \boldsymbol{c}^{\boldsymbol{r}}$$

where $\boldsymbol{c}^{\boldsymbol{r}} = c_1^{r_1}\cdot c_2^{r_2}\cdots c_n^{r_n}$. .

The polynomial $q_{j,\delta}(c_1,\ldots,c_n)$ is the partial Bell polynomial *$B_{j,\delta}(c_1,\ldots,c_{j-\delta+1})$ evaluated at $c_i=0$ for $i>n$.*

What follows is essentially a rederivation of the beautiful *Faá di Bruno's formula* on derivatives of a composition of differentiable functions, or equivalently, the coefficients of a composition of power series as in [13].

Theorem 2 *Fix $d\in\mathbb{N}$ and set*

$$F(x,y) = \sum_{i+j\le d} a_{ij}x^iy^j, \qquad f(x) = \sum_{i=1}^{\infty} c_ix^i.$$

Then for $F(x,f(x)) = \sum_{k\ge1} h_k(\boldsymbol{a},\boldsymbol{c})x^k$ we have

$$\frac{\partial h_k}{\partial a_{ij}} = [a_{ij}]h_k(\boldsymbol{a},\boldsymbol{c}) = [a_{ij}][x^k]F(x,f(x)) = \sum_{\boldsymbol{r}\in R^{N_d-1}_{j,k-i}} \boldsymbol{c}^{\boldsymbol{r}} = q_{j,k-i}(c_1,\ldots,c_{N_d-1})$$

Proof This is a direct analysis of the composition $F(x, f(x))$. Recall that

$$F(x,y) = \sum_{i+j\leq d} a_{ij}x^i y^j, \qquad f(x) = \sum_{i=1}^{N_d-1} c_i x^i$$

and so the coefficient $[x^k]F(x, f(x))$ of x^k in $F(x, f(x))$ is clearly linear in the $\mathbf{a}$ variables. Consequently, we analyze $[a_{ij}]h_k(\mathbf{a},\mathbf{c}) = [a_{ij}][x^k]F(x, f(x))$.

We have:

$$\begin{aligned}
[a_{ij}]h_k(\mathbf{a},\mathbf{c}) &= [a_{ij}][x^k]F(x, f(x)) \\
&= [a_{ij}][x^k] \sum_{i+j\leq d} a_{ij}x^i (f(x))^j \\
&= [a_{ij}][x^k] \sum_{i+j\leq d} a_{ij}x^i \left(\sum_{n=1}^{\infty} c_i x^i\right)^j \\
&= [a_{ij}][x^k] \sum_{i+j\leq d} a_{ij}x^i \left(\sum_{\delta\geq 1} x^\delta \sum_{\mathbf{r}=(r_1,\ldots,r_{N_d-1})\in R^{N_d-1}_{j,\delta}} c_1^{r_1}\cdots c_{N_d-1}^{r_{N_d-1}}\right) \\
&= [a_{ij}][x^k] \sum_{i+j\leq d} a_{ij}x^i \left(\sum_{\delta\geq 1} x^\delta \sum_{\mathbf{r}\in R^{N_d-1}_{j,\delta}} \mathbf{c}^{\mathbf{r}}\right) \\
&= [a_{ij}][x^k] \sum_{i+j\leq d} a_{ij} \sum_{\delta\geq 1} x^{\delta+i} \sum_{\mathbf{r}\in R^{N_d-1}_{j,\delta}} \mathbf{c}^{\mathbf{r}} \\
&= [x^k] \sum_{\delta\geq 1} x^{\delta+i} \sum_{\mathbf{r}\in R^{N_d-1}_{j,\delta}} \mathbf{c}^{\mathbf{r}}
\end{aligned}$$

The coefficient of x^k in this final equation clearly occurs when $\delta = k - i$. Thus, we have

$$[a_{ij}]h_k(\mathbf{a},\mathbf{c}) = [a_{ij}][x^k]F(x, f(x)) = \sum_{\mathbf{r}\in R^{N_d-1}_{j,k-i}} \mathbf{c}^{\mathbf{r}} = q_{j,k-i}(c_1,\ldots,c_{N_d-1})$$

as desired.

The formula given in Theorem 2 allows us to write down, in terms of the $\mathbf{c}$ parameters, a *linear* system of equations in the $\mathbf{a}$ variables. We write M_d for this $(N_d-1)\times N_d$ matrix whose kernel is the vector space spanned by the d-th osculating curve of $f(x)$. We remark that the columns of M_d are indexed by the pairs (i, j) such that $i + j \leq d$ and the rows of M_d are indexed by $1,2,\ldots,N_d-1$. Then, Theorem 2 gives a formula for the $(k,(i, j))$-th entry of M_d and the following corollary is obtained as a result of applying Cramer's rule to the matrix equation

$$\overline{M}_d \cdot \mathbf{a} = [1;0;\cdots 0]^t$$

where $\overline{M}_d$ the $N_d \times N_d$ matrix obtained by appending a row of 1's above $\overline{M}$, labeled as the 0-th row.

Example 4 *The matrix $\overline{M_3}$ is the 9×9 matrix*

	(1,0)	(2,0)	(3,0)	(0,1)	(1,1)	(2,1)	(0,2)	(1,2)	(0,3)
0	1	1	1	1	1	1	1	1	1
1	$q_{0,0}$	$q_{-1,0}$	$q_{-2,0}$	$q_{1,1}$	$q_{0,1}$	$q_{-1,1}$	$q_{1,2}$	$q_{0,2}$	$q_{1,3}$
2	$q_{1,0}$	$q_{0,0}$	$q_{-1,0}$	$q_{2,1}$	$q_{1,1}$	$q_{0,1}$	$q_{2,2}$	$q_{1,2}$	$q_{2,3}$
3	$q_{2,0}$	$q_{1,0}$	$q_{0,0}$	$q_{3,1}$	$q_{2,1}$	$q_{1,1}$	$q_{3,2}$	$q_{2,2}$	$q_{3,3}$
4	$q_{3,0}$	$q_{2,0}$	$q_{1,0}$	$q_{4,1}$	$q_{3,1}$	$q_{2,1}$	$q_{4,2}$	$q_{3,2}$	$q_{4,3}$
5	$q_{4,0}$	$q_{3,0}$	$q_{2,0}$	$q_{5,1}$	$q_{4,1}$	$q_{3,1}$	$q_{5,2}$	$q_{4,2}$	$q_{5,3}$
6	$q_{5,0}$	$q_{4,0}$	$q_{3,0}$	$q_{6,1}$	$q_{5,1}$	$q_{4,1}$	$q_{6,2}$	$q_{5,2}$	$q_{6,3}$
7	$q_{6,0}$	$q_{5,0}$	$q_{4,0}$	$q_{7,1}$	$q_{6,1}$	$q_{5,1}$	$q_{7,2}$	$q_{6,2}$	$q_{7,3}$
8	$q_{7,0}$	$q_{6,0}$	$q_{5,0}$	$q_{8,1}$	$q_{7,1}$	$q_{6,1}$	$q_{8,2}$	$q_{7,2}$	$q_{8,3}$

For $i < 0$ we have that $q_{i,j} = 0$ and for $i > 0$ we have $q_{i,0} = 0$. For $j = 0$ the only nonzero $q_{i,j}$ is when $i = 0$ in which case $q_{0,0} = 1$. We further remark that $q_{i,1} = c_i$. Hence, this matrix simplifies to

	(1,0)	(2,0)	(3,0)	(0,1)	(1,1)	(2,1)	(0,2)	(1,2)	(0,3)
0	1	1	1	1	1	1	1	1	1
1	1			c_1			$q_{1,2}$		$q_{1,3}$
2		1		c_2	c_1		$q_{2,2}$	$q_{1,2}$	$q_{2,3}$
3			1	c_3	c_2	c_1	$q_{3,2}$	$q_{2,2}$	$q_{3,3}$
4				c_4	c_3	c_2	$q_{4,2}$	$q_{3,2}$	$q_{4,3}$
5				c_5	c_4	c_3	$q_{5,2}$	$q_{4,2}$	$q_{5,3}$
6				c_6	c_5	c_4	$q_{6,2}$	$q_{5,2}$	$q_{6,3}$
7				c_7	c_6	c_5	$q_{7,2}$	$q_{6,2}$	$q_{7,3}$
8				c_8	c_7	c_6	$q_{8,2}$	$q_{7,2}$	$q_{8,3}$

The following example illustrates the pattern discussed, where each row of the matrix corresponds to coefficients c_i and powers of c_1 up to c_8. We observe how the parameters and their powers propagate across subsequent rows and columns, demonstrating a structured pattern based on the arrangement of these elements.

Example 5

$$\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & c_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & c_2 & c_1 & 0 & c_1^2 & 0 & 0 \\
0 & 0 & 0 & 1 & c_3 & c_2 & c_1 & 2c_1c_2 & c_1^2 & c_1^3 \\
0 & 0 & 0 & 0 & c_4 & c_3 & c_2 & 2c_1c_3+c_2^2 & 2c_1c_2 & 3c_1^2c_2 \\
0 & 0 & 0 & 0 & c_5 & c_4 & c_3 & 2c_1c_4+2c_2c_3 & 2c_1c_3+c_2^2 & 3c_1^2c_3+3c_1c_2^2 \\
0 & 0 & 0 & 0 & c_6 & c_5 & c_4 & 2c_1c_5+2c_2c_4+c_3^2 & 2c_1c_4+2c_2c_3 & 3c_1^2c_4+6c_1c_2c_3+c_2^3 \\
0 & 0 & 0 & 0 & c_7 & c_6 & c_5 & 2c_1c_6+2c_2c5+2c_3c_4 & 2c_1c_5+2c_2c_4+c_3^2 & 3c_1^2c_5+6c_1c_2c_4+3c_1c_3^2+3c_2^2c_3 \\
0 & 0 & 0 & 0 & c_8 & c_7 & c_6 & 2c_1c_7+2c_2c_6+2c_3c_5+c_4^2 & 2c_1c_6+2c_2c_5+2c_3c_4 & 3c_1^2c_6+6c_1c_2c_5+6c_1c_3c_4+3c_2^2c_4+3c_2c_3^2
\end{pmatrix}$$

3.2 Structural patterns in the matrix $\overline{M}_d$

As we can see in Example 3, the terms of every numerator can be found in the denominator. Recall that the columns of the matrix $\overline{M}_d$ are indexed by the $a_{i,j}$'s. We observe that this matrix has a specific pattern based on the $a_{i,j}$'s. More precisely, we have the following.

Proposition 1 *The following properties hold:*

(1) in the column indexed by $a_{i,j}$, all the elements from row 1 excluded to row $i+j+1$ included are zero.

(2) in the column indexed by $a_{i,j}$, all non-zero entries below row $i+j+1$ are polynomials homogeneous of degree j. Also, each entry in each column must have a specific property based on partitions for j to $r+1$ with multi-degrees of polynomials of each array.

The proof has been mentioned in previous sections (see Theorem 2).
Furthermore, regarding the solutions, a discernible pattern emerges. By applying Cramer's rule, each solution is represented as a fraction. Here, the denominator corresponds to the determinant of the matrix, while each numerator takes the form of a homogeneous polynomial.

- of degree $e_{i,j} = \binom{d+2}{2} - j - 2$, and quasi-degree $\binom{d+2}{2} + i$
- whose terms are all terms of the denominator.

Recall that the matrix of the coefficients is non-singular. Recall that $a_{i,j}(c)$ represents the coefficient at position (i,j) in the solution vector, dependent on the parameter c.

We denote by $\phi(i,j)$, the pair $(n_{i,j}, q_{i,j})$ where $n_{i,j}$ is the degree of the numerator of $a_{i,j}(c)$ and $q_{i,j}$ is the quasi-degree of the numerator of $a_{i,j}(c)$.

Proposition 2 *The pairs $\phi(i,j)$ are pairwise different.*

Proof To prove that ϕ is injective, we need to show that for any distinct pairs (i_1, j_1) and (i_2, j_2) in $\mathbb{Z}^2$, $\phi(i_1, j_1) \neq \phi(i_2, j_2)$.

Consider two distinct pairs (i_1, j_1) and (i_2, j_2) in $\mathbb{Z}^2$. Without loss of generality, assume $i_1 \neq i_2$.

We have:

$$\phi(i_1, j_1) = \left(\binom{d+2}{2} + i_1, \binom{d+2}{2} - j_1 - 2\right)$$
$$\phi(i_2, j_2) = \left(\binom{d+2}{2} + i_2, \binom{d+2}{2} - j_2 - 2\right)$$

Since $i_1 \neq i_2$, the first coordinates of $\phi(i_1, j_1)$ and $\phi(i_2, j_2)$ are different. Hence, $\phi(i_1, j_1) \neq \phi(i_2, j_2)$.

Similarly, if $j_1 \neq j_2$, the second coordinates of $\phi(i_1, j_1)$ and $\phi(i_2, j_2)$ are different. Hence, $\phi(i_1, j_1) \neq \phi(i_2, j_2)$.

Thus, for any distinct pairs (i_1, j_1) and (i_2, j_2), we have $\phi(i_1, j_1) \neq \phi(i_2, j_2)$, proving that ϕ is injective. □

Chapter 4

Osculating spaces of plane curves

4.1 Osculating spaces

As in Section 2.3 and thus as in Definition 2, we denote by X an analytic curnve given locally at $p = (0,0)$ as the graph of the sum of a power series:

$$f(x) = \sum_{i=1}^{\infty} c_i x^i. \tag{4.1}$$

For $d \in \mathbb{N}$, the generic polynomial of degree d in $\mathbb{C}[x,y]$ has the form

$$F(x,y) = \sum_{0 \le i+j \le d} a_{i,j} x^i y^j. \tag{4.2}$$

Recall that we denote by $V(F(x,y))$ its vanishing locus and that we assume that $p \in V(F(x,y))$ holds, which implies $a_{0,0} = 0$.

Definition 5 *The* osculating space *of X of order k and degree d is the $\mathbb{C}$-vector space*

$$V_{d,k} = \{F \in \mathbb{C}[x,y] \mid deg(F) \le d,\ C(F,f) \ge k\}.$$

Writing $F(x,y)$ as in (4.2) and $f(x)$ as in (4.1), the composition $F(x, f(x))$ notably has coefficients which are polynomials in the indeterminates $\mathbf{a}$ and $\mathbf{c}$:

$$F(x, f(x)) = \sum_{j=0}^{\infty} h_j(\mathbf{a},\mathbf{c}) x^j.$$

In particular, the only c_i involved in $h_j(\mathbf{a},\mathbf{c})$ are $\{c_1, c_2, \ldots, c_j\}$ and each $h_j(\mathbf{a},\mathbf{c})$ is linear in the variables $\mathbf{a}$. These coefficients $h_j(\mathbf{a},\mathbf{c})$ represent the conditions in Definition 2 and so, given $\mathbf{c}$, we obtain an explicit representation of $V_{d,k}$ as the solution set to a linear system of equations in the variables $\mathbf{a}$.

Lemma 1 *Given a fixed $f(x)$ which represents X locally near $p = (0,0)$, the osculating space $V_{d,k}$ is the solution set to the homogeneous linear system of k equations $h_0(\mathbf{a},\mathbf{c}) = h_1(\mathbf{a},\mathbf{c}) = h_2(\mathbf{a},\mathbf{c}) = \cdots h_{k-1}(\mathbf{a},\mathbf{c}) = 0$ in $(N_d + 1)$-many variables $\mathbf{a}$.*

Proof That these equations cut out the osculating space $V_{d,k}$ follows immediately from Definitions 2 and 5. Linearity in $\mathbf{a}$ is easily observed since the polynomials $h_j(\mathbf{a},\mathbf{c})$ come from the composition

$$F(x,f(x)) = \sum_{0\le i+j\le d} a_{ij}x^i(f(x))^j = \sum_{j=0}^{\infty} h_j(\mathbf{a},\mathbf{c})x^j.$$

Osculating spaces respect the inclusion $V_{d,k+1} \subseteq V_{d,k}$ and the corresponding difference in dimensions can be at most one: $V_{d,k+1}$ is the intersection of the vector space $V_{d,k}$ with the hyperplane $\mathbf{V}(h_k)$ through the origin. We remark that

- The inclusion $V_{d,k+1} \subseteq V_{d,k}$ need not be strict (Example 6).
- An osculating space may contain no irreducible curves (Example 7).
- The chain $\mathbb{C}_{\mathbf{a}}^{N_d+1} = V_{d,0} \supseteq V_{d,1} \supseteq \cdots$ may not terminate (Example 8).

If the chain does terminate at some $V_{d,m}$, then $V_{d,m}$ must have dimension one by the above remarks. Equivalently, $V_{d,m}$ represents a unique curve $\mathscr{V}(F(x,y))$.

Definition 6 *If $V_{d,m} = span(F(x,y))$ for some $m \in \mathbb{N}$ and irreducible polynomial $F(x,y) \in \mathbb{C}[x,y]$, we call $F(x,y)$ the osculating curve of X at p.*

Remark 1 *Directly from its definition, there are two ways for an osculating curve of degree d to not exist. The first is that $V_{d,k}$ has dimension larger than 1 for all k. Since $V_{d,0}$ is finite-dimensional, this means that the vector spaces $V_{d,k}$ eventually stabilize at some sufficiently large k. Equivalently, the statement "any polynomial of degree at most d meeting X to order at least k at p meets X to infinite order at p". Equivalently, $y = f(x)$ describes, locally, a curve $X = \mathscr{V}(F)$ of degree $d' \le d$. If $d = d'$ then $V_{d,d^2+1} = span(F)$ has dimension 1 by Bézout's theorem. Hence, it must be that $d' < d$ and the vector spaces V_{d',d'^2+1} consist of reducible curves having X as a component. The second way is that when $dim(V_{d,k}) = 1$, we have that $V_{d,k}$ consists of only reducible curves. This is the case for Example 7, for instance.*

Theorem 3 *The vector space $V_{d,k}$ is non-trivial for $k \le N_d$. Generically, $dim(V_{d,k}) = N_d + 1 - k$. For generic $\mathbf{c}$ and any d, an osculating curve of degree d exists.*

Proof The first part is the rank-nullity theorem: a homogeneous system of k equations in $N_d + 1$ variables always has a nontrivial solution when $k \le N_d$.

For the second part, we assume $\mathbf{c}$ is generic and proceed by induction on k: our induction hypothesis is $\dim(V_{d,k}) = \dim(V_{d,k-1}) + 1$. The base case $k = 1$ is seen to be true since $\mathbb{C}_{\mathbf{a}}^{N_d+1} = V_{d,0}$ is the affine cone over $\mathbb{P}_{\mathbf{a}}^{N_d}$. A polynomial vanishes $p = (0,0)$ if and only if the constant term is zero, so $\dim(V_{d,0}) = N_d + 1 = \dim(V_{d,1}) + 1$. We proceed by induction. Note that $\dim(V_{d,k}) - \dim(V_{d,k+1})$ is either zero or one since we are including only one additional homogeneous equation: $h_k(\mathbf{a},\mathbf{c})$. Observe that c_k does not appear in $h_j(\mathbf{a},\mathbf{c})$ for any $j < k$, but does appear in $h_k(\mathbf{a},\mathbf{c})$ in the unique term $a_{0,1}c_k$. Suppose towards a contradiction that the equality $\dim(V_{d,k}) = \dim(V_{d,k+1})$ holds, so there exists a dependency

$$h_k(\mathbf{a},\mathbf{c}) = \lambda_0 h_0(\mathbf{a},\mathbf{c}) + \cdots + \lambda_{k-1}h_{k-1}(\mathbf{a},\mathbf{c}).$$

Consider $\mathbf{c} \mapsto \mathbf{c} + \alpha e_k$ for generic $\alpha \in \mathbb{C}$. Applying this transformation, the right hand side of the above dependency is unchanged, while the left hand side differs by $a_{0,1}\alpha c_k$, implying that $a_{0,1} \in \text{span}(h_0, \ldots, h_{k-1})$. However, by assumption, $a_{0,1} \neq 0$ since the graph of $y = f(x)$ does not have a vertical tangent at $x = 0$. We conclude by induction that $\dim(V_{d,k}) = N_d + 1 - k$ and so $\dim(V_{d,N_d}) = 1$.

Finally, we show that V_{d,N_d} is spanned by an *irreducible* polynomial for generic $\mathbf{c}$. Suppose towards contradiction that $V_{d,N_d} = \text{span}(F_1(x,y) \cdot F_2(x,y))$ where $\deg(F_1) = d_1 < d$ and $\deg(F_2) = d - d_1$. Since contact order is additive under unions, $C(F_1F_2, f) = C(F_1, f) + C(F_2, f)$. By the proofs given above the contact order with generic X obtainable by a polynomial of some degree is bounded: $C(F_1, f) \leq N_{d_1}$ and $C(F_2, f) \leq N_{d-d_1}$. Some algebra reveals that

$$C(F_1, F_2, f) = C(F_1, f) + C(F_2, f) \leq N_{d_1} + N_{d-d_1} = N_d + d_1(d_1 - d) < N_d,$$

where the last strict inequality follows from $d_1 - d < 0$.

4.2 Algorithms

We now describe algorithms for computing the osculating spaces and curves of any degrees and contact orders associated to an analytic curve X at a point p. As before, our algorithms given below all assume that X is represented as the graph of a power series $f(x)$ near $x = 0$ and that $p = (0,0)$.

Algorithm 1 Osculating space

Input:
- Natural numbers k and d
- The Taylor polynomial $f_{k-1}(x)$ of $f(x)$ of degree $k-1$

Output: A representation of the osculating space $V_{d,k}$ of $y = f(x)$ at $p = (0,0)$

1. Construct a generic degree d polynomial $F_d(x,y) = \sum_{0 \leq i+j \leq d} a_{ij}x^iy^j$ with the a_{ij} left as indeterminants
2. Obtain the first k coefficients $h_j(\mathbf{a}, \mathbf{c})$ of $F_d(x, f_{k-1}(x)) = \sum_{j=0}^{\infty} h_j(\mathbf{a}, \mathbf{c})x^j$.
3. Write the matrix M of the system $h_0(\mathbf{a}, \mathbf{c}) = \cdots = h_{k-1}(\mathbf{a}, \mathbf{c}) = 0$.
4. Compute any representation V for the nullspace of M
5. **return** V

Regarding correctness, the steps of Algorithm 1 directly follow from the definitions given in the above section. The only exception is that we compute $F_d(x, f_{k-1}(x))$ instead of $F_d(x, f(x))$. We remark that this does not introduce any errors since the polnomials $h_0, \ldots, h_{k-1}$ only involve $c_1, \ldots, c_{k-1}$.

Algorithm 2 computes the osculating curve of X at p of some degree d, provided that it exists. It relies on Algorithm 1. Algorithm 2 terminates by the assumption that an osculating curve exists. The only other requirement of Algorithm 2 is that the representation of the osculating spaces returned by the sub-procedure Algorithm 1 allows for the computation of their dimensions (used in step 2) and a basis (used in step 5).

Algorithm 2 Osculating curve

Input: • A natural number d
• An oracle for the Taylor polynomials $f_k(x)$ of $f(x)$
Assume: The osculating curve of degree d of X at p exists
Output: A polynomial $F(x,y)$ defining the osculating curve of degree d of X at p

```
Set k = N_d and compute V = V_{d,k}
while dim(V) > 1 do
    k = k + 1
    V = V_{d,k}
return A basis F(x,y) for V
```

4.3 A specific example and a general example

The steps taken by the Maple implementation are shown in a detailed example. To compute the osculating conic of the graph of $f(x) = e^x - 1$ at $p = (0,0)$:

(1) Expand $e^x - 1$ as a Taylor series.

$$f(x) = x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \tfrac{1}{24}x^4 + \tfrac{1}{120}x^5 + \cdots . \qquad (4.3)$$

(2) Truncate to a Taylor polynomial of the required degree.

$$f_4(x) = x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \tfrac{1}{24}x^4. \qquad (4.4)$$

(3) Form a generic polynomial of degree $d = 2$.

$$F(x,y) = a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{10}x + a_{01}y\ , \qquad (4.5)$$

where we have ensured that $a_{00} = 0$.

(4) Compute $N_d = 5$. which is obtained from (2.2).

(5) Substitute $y = f_4(x)$ into (4.5) and collect powers of x.

$$F(x,f_4(x)) = (a_{01}+a_{10})x + \left(\tfrac{1}{2}a_{01}+a_{02}+a_{11}+a_{20}\right)x^2 + \left(\tfrac{1}{6}a_{01}+a_{02}+\tfrac{1}{2}a_{11}\right)x^3 + \left(\tfrac{1}{24}a_{01}+\tfrac{7}{12}a_{02}+\tfrac{1}{6}a_{11}\right)x^4 + \left(\tfrac{1}{120}a_{01}+\tfrac{1}{4}a_{02}+\tfrac{1}{24}a_{11}\right)x^5 + O(x^6)$$

Note that the expression for $F(x,f_4(x))$ derived by Maple continues up to terms in x^{12}, and the expression here is truncated to save space. Packages like [15] are particularly helpful as they avoid computing irrelevant coefficients.

(6) Assign elements in the matrix M. Row i contains coefficients of x^i. Column j contains coefficients of $a_{01}, a_{02}, a_{10}, a_{11}, a_{2,0}$.

$$M = [1\ 0\ 1\ 0\ 0; \tfrac{1}{2}\ 1\ 0\ 1\ 1; \tfrac{1}{6}\ 1\ 0\ \tfrac{1}{2}\ 0; \tfrac{1}{24}\ \tfrac{7}{12}\ 0\ \tfrac{1}{6}\ 0]\ , \qquad (4.6)$$

where the coefficients of x^5 and higher are not used. Notice that

$$F(x,f_4(x)) = \begin{bmatrix} x & x^2 & x^3 & x^4 \end{bmatrix} M \left[a_{01}, a_{02}, a_{10}, a_{11}, a_{20}\right]^T + O(x^5)\ .$$

(7) We want the coefficients of $\begin{bmatrix} x & x^2 & x^3 & x^4 \end{bmatrix}$ to equal zero. Thus we want

$$M\left[a_{01}, a_{02}, a_{10}, a_{11}, a_{20}\right]^T = 0 \, ,$$

which defines the null space of the matrix M.

(8) Calculate the null space of M using Maple: $\begin{bmatrix} 9 & -\frac{1}{2} & 9 & 4 & 1 \end{bmatrix}^t$.

(9) Since this is one dimensional and the corresponding conic is irreducible, we return the osculating conic.

Note that all of these steps can be performed while leaving the coefficients of $f_4(x)$ as indeterminants: $f_4(x) = c_1x + c_2x^2 + c_3x^3 + c_4x^4$. The coefficients obtained from the composition $F(x, f_4(x))$ are

$$h_0 = a_{00},\ h_1 = a_{10} + a_{01}c_1, h_2 = a_{01}c_2 + a_{20} + a_{11}c_1 + a_{02}c_1^2$$

$$h_3 = a_{01}c_3 + a_{11}c_2 + 2a_{02}c_1c_2, h_4 = a_{02}c_2^2 + 2a_{02}c_1c_3 + a_{11}c_3 + a_{01}c_4$$

The matrix equation we obtain in our Maple implementation is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & c_1 & 0 & 0 & 0 \\ 0 & 0 & c_2 & 1 & c_1 & c_1^2 \\ 0 & 0 & c_3 & 0 & c_2 & 2c_1c_2 \\ 0 & 0 & c_4 & 0 & c_3 & c_2^2 + 2c_1c_3 \end{bmatrix} \begin{bmatrix} a_{00} \\ a_{10} \\ a_{20} \\ a_{01} \\ a_{11} \\ a_{02} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We remark that each entry is both (1) homogeneous in the **c**'s and (2) quasi-homogeneous in the **c**'s if one assigns the weight i to the variable c_i. This homogeneity is a consequence of the beautiful Faá di Bruno's formula on derivatives of a composition of differentiable functions, or equivalently, the coefficients of a composition of power series. The nullspace of the matrix above is spanned by

$$(\underbrace{0}_{a_{0,0}}, \underbrace{c_1c_2^3}_{a_{1,0}}, \underbrace{-c_2^3}_{a_{0,1}}, \underbrace{c_2^4 - c_1c_2^2c_3 - c_1^2c_3^2 + c_1^2c_2c_4}_{a_{2,0}}, \underbrace{c_2^2c_3 + 2c_1c_3^2 - 2c_1c_2c_4}_{a_{1,1}}, \underbrace{-c_3^2 + c_2c_4}_{a_{0,2}})$$

This proves the formula for the osculating conic of a generic analytic curve.

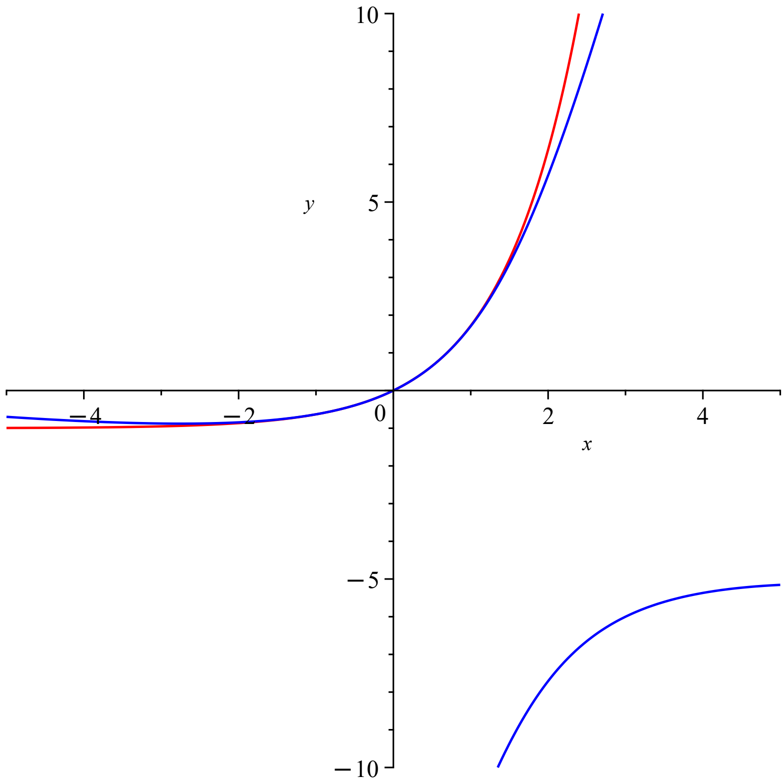


Figure 4.1: Osculating conic of $f_2 = e^x - 1 - \frac{x^5}{270}$

We just computed the osculating conic for the function $f = e^x - 1$ at the origin in degree $d = 5$. A plot of the function and the osculating conic is shown in Figure 4.1. The conic is a hyperbola, and its two branches are shown in the figure. This raises the question of when an osculating conic is hyperbolic and when it is ellipsoidal. To answer this, the function $f = e^x - 1$ can be modified to increase its curvature. The osculating conic for $f_1 = e^x - 1 + x^2/2$ at the origin is $T_1 = 61x^2 + 10xy + y^2 + 72x - 72y$, an ellipse. The transition from hyperbola to ellipse takes place at $f_c = e^x - 1 + x^2/6$, when the conic becomes $T_c = 5x^2 + 3xy + 12x - 12y$.

Another interesting modification is $f_2 = e^x - 1 - \frac{x^5}{270}$. Since the conic is computed using only terms up to $O(x^4)$, this function has the same osculating conic as f. However, if the Taylor series in x around the origin are computed, we get:

$$f = x + \frac{2}{3}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6 + O(x^7) \tag{4.7}$$

$$T = x + \frac{2}{3}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{216}x^5 - \frac{1}{432}x^6 + O(x^7) \tag{4.8}$$

$$f_2 = x + \frac{2}{3}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{216}x^5 + \frac{1}{720}x^6 + O(x^7) \tag{4.9}$$

Thus, f_2 and T match up to 5 terms. This situation is sometimes termed “extactic”, meaning that the overlap between the curves is higher than necessary.

4.4 Additional examples

We conclude our computational journey with several illustrative examples.

Example 6 *Consider the problem of computing the osculating conic associated to the Taylor series* $y = f(x) = x + x^2 + 2x^3 + 3x^4 + 2x^5$. *Running Algorithm 2 returns, after two iterations of the loop, that the osculating conic is the hyperbola defined by* $F(x,y) = -x + y + 2x^2 - 4xy + y^2$. *This polynomial spans both* $V_{2,5}$ *and* $V_{2,6}$, *implying that the conic has a sextactic contact with the graph of* $f(x)$. *This power series and the osculating conic appear in Figure 4.2.*

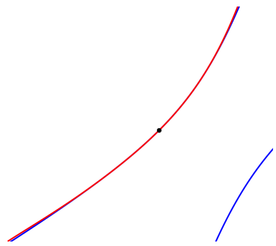


Figure 4.2: The graph of a power series (red) and its osculating conic (blue). The osculating conic is a hyperbola (note both branches) and meets the graph of the power series to contact order six rather than the expected order of five at the origin.

Example 7 *Consider the graph of* $f(x) = \sin(x)$, *with Taylor series* $y = f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$. *The point* $p = (0,0)$ *on this graph is an inflection point. Equivalently, the tangent line defined by* $F(x,y) = x - y$ *meets the graph to order* 3. *The square of the tangent line* $G(x,y) = (x-y)^2$ *thus meets to order* 6. *One can check that we have* $V_{2,5} = V_{2,6} = span(G(x,y))$. *Consequently, the graph of* $\sin(x)$ *admits* no *osculating conic at the origin.*

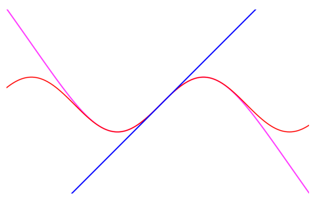


Figure 4.3: The graph of the sine function (red) and its osculating (tangent) line at $p = (0,0)$ (blue). Since p is an inflection point, the tangent line meets to order three and its square to order six. The pink curve is the osculating cubic which meets to order ten, rather than the expected order of nine.

Beyond the non-existence of an osculating conic to the graph of $y = \sin(x)$ *at the origin, we compute* $V_{3,N_3} = V_{3,9}$ *to be dimension two instead of the expected dimension of one. We have*

$$V_{3,9} = span((x-y)^3, -42000x + 42000y + 4437x^3 + 3159x^2y - 729xy^2 + 133y^3)$$

Since this osculating space has dimension two, $V_{3,10}$ *has dimension at least one and so there exists a cubic curve which approximates* $y = \sin(x)$ *geometrically near the origin better than one would expect. Indeed, one may compute that* $V_{3,10} = span(-42000x + 42000y + 4437x^3 + 3159x^2y - 729xy^2 + 133y^3)$. *See Figure 4.3 for depictions of these curves.*

Example 8 *Consider the curve defined by* $y(1-x)=1$*, or equivalently* $1-y+xy=0$*. Through the origin, it is represented by the Taylor series for* $f(x)=\frac{x}{1-x}$*:*

$$y = f(x) = x + x^2 + x^3 + x^4 + x^5 + \cdots$$

Running Algorithm 2 on $d=2$ *and* $f_4(x) = x + \cdots + x^4$ *produces the basis*

$$F(x,y) = x - y + xy$$

for $V_{2,5}$*. Indeed, this implies that* $F(x,y)$ *is the osculating conic of the curve by Definition 6. This is an example where* $V_{2,k} = V_{2,5}$ *for all* $k \geq 5$*.*

Chapter 5

Conclusion

This research focused on improving the way we find osculating curves, which touch and closely follow another curve at a point. By using a new method involving power series and matrices, we achieved more accurate results beyond just straight lines and circles.

Osculating curves are crucial because they provide detailed information about the shape of a curve at a specific point. Our method has practical applications in various fields, such as designing cars, airplanes, and other shapes in Computer Aided Design (CAD). Accurate osculating curves ensure smooth and precise designs, leading to better products and more efficient manufacturing processes.

For example, in automotive design, using accurate osculating curves can improve aerodynamics, resulting in more fuel-efficient vehicles. In the aerospace industry, they can help create more efficient and safe aircraft designs. By making it easier to find these curves, we enhance our ability to understand and manipulate shapes.

This work builds on earlier research and improves how we use mathematics to study curves and surfaces. It shows how advanced math techniques can solve real-world problems in engineering and design. Overall, this thesis helps advance tools and methods for analyzing and working with complex curves, opening up possibilities for future innovations.

Bibliography

[1] Lars Ahlfors. *Complex Analysis: An Introduction to the Theory of Analytic Functions of One Complex Variable*. McGraw-Hill Education, 1979.

[2] Vladimir I. Arnold. The geometry of spherical curves and the algebra of extactic points. *Russian Mathematical Surveys*, 51(6):1019–1083, 1996.

[3] Leah Balay-Wilson and Taylor Brysiewicz. Points of ninth order on cubic curves. *Rose-Hulman Undergraduate Mathematics Journal*, 15(1):1, 2014.

[4] Arthur Cayley. On the conic of five-pointic contact at any point of a plane curve. *Philosophical Transactions of the Royal Society of London*, 149:371–400, 1859.

[5] Arthur Cayley. On the sextactic points of a plane curve. *Philosophical Transactions of the Royal Society of London*, 155:545–578, 1865.

[6] Aldo Conca, Simone Naldi, Giorgio Ottaviani, and Bernd Sturmfels. Taylor polynomials of rational functions. *arXiv:2304.00712*, 2023.

[7] Richard Courant. *Differential and Integral Calculus, Volume 1*. John Wiley & Sons, 1941.

[8] Philip Franklin. Osculating curves and surfaces. *Transactions of the American Mathematical Society*, 28:400–416, 1926.

[9] William Fulton. *Algebraic Curves*. Springer, New York, NY, 2008.

[10] Robin Hartshorne. *Algebraic Geometry*. Springer, New York, NY, 1977.

[11] D. J. J. O'Connor. The lost gravestone of arthur cayley, 2024. Accessed July 2024.

[12] John J. O'Connor and Edmund F. Robertson. Arthur cayley, 2024. Accessed July 2024.

[13] Erik Postma and Marc Moreno Maza. Substituting units into multivariate power series. *Maple Transactions*, 2(1):1–15, 2022.

[14] James Stewart. *Calculus*. Brooks Cole, 2015.

[15] Juan Pablo González Trochez, Marc Moreno Maza, Matt Calder, and Erik Postma. Algorithms for multivariate laurent series. *ACM Communications in Computer Algebra*, 56(2):64–67, 2022.

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