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Scales Of Logarithmic Summability

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SCALES OF LOGARITHMIC SUMMABILITY

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ABSTRACT

In this thesis scales of methods of summability L_p and ℓ_p are defined and investigated. The former scale starts with a method L_o (we say $s_n \to s(L_o)$ if

$$\lim_{x\to 1-} \frac{1}{\log \frac{1}{1-x}} \sum_{n=1}^{\infty} \pi_o(n) s_n x^n = s, \text{ where } \pi_o(n) = \frac{1}{n}\Big)$$ which

is equivalent to the standard logarithmic method of summability L, which itself includes the well known Abel method in the sense that any series summable by the Abel method is also summable L. The ℓ_p methods are $\overline{N}$ (or weighted means) methods of summability. A tauberian theorem, which is a generalisation of a known result for the L method, is proved for both scales of methods. Next, inclusion theorems for both scales, and between the two scales of methods, are established. Finally, a condition is obtained which ensures that the Fourier series of a function is L_p summable.

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INTRODUCTORY NOTE

This thesis is divided into 4 chapters which are subdivided into sections. The sections are numbered consecutively throughout the thesis - thus the first section of Chapter 3 is section 7 (written §7). The numbering of the equations, theorems, lemmas, etc. is consecutive within a section. References within the thesis are given by section number and number of the item - thus theorem (6.1) is the first theorem in section 6. However, if the item appears in the section in which it is referred to the section number is omitted - thus equation 3 refers to the third equation in the section under consideration.

The symbol Ξ is used to denote the end of a proof.

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CHAPTER 1

In §1 the concept of summability is briefly discussed and various known methods of summability are given. The methods of summability which are the main concern of the thesis are introduced in §2. In §3 the basic properties of the methods given in §2 are discussed. The final section of the chapter contains proofs of some lemmas which are used subsequently.

§1. DEFINITIONS.

Capital letters such as H, K, ..., are used to denote positive numbers independent of the variables under consideration, but are not necessarily the same at each occurrence. We adopt the following familiar conventions: [see for example 6, p.2]

$f = O(\phi)$ means $|f| < H\phi$

$f = O_L(\phi)$ {or $O_R(\phi)$} means $f > -H\phi$ {or $< H\phi$}

$f = o(\phi)$ means $f/\phi \to 0$

$f \sim \phi$ means $f/\phi \to 1$

A method of summability P is a method of assigning a "sum" say s to a series $\sum_{k=0}^{\infty} a_k$. We say $\sum_{k=0}^{\infty} a_k$ is *summable* P to s and write

$$\sum_{k=0}^{\infty} a_k = s \ (P) \quad .$$

We shall also say s is the P-*limit* of s_n, where $s_n = \sum_{k=0}^{n} a_k$, and write

$$s_n \to s \ (P) \quad .$$

A method of summability is said to be *regular*, if it sums every convergent series to its ordinary sum.

Given two summability methods P, Q we write $P \supseteq Q$ and say P *includes* Q if any series summable Q is summable P to the same sum, if in addition there is a series summable P but not summable Q we say P *strictly includes* Q and write $P \supset Q$. If $P \supseteq Q$ and $Q \supseteq P$ the two methods are said to be *equivalent* and we write $P \simeq Q$.

For reference purposes we give definitions of some well known summability methods which we will need to refer to later.

(1) *The method* ℓ.

If $t_o = s_o$, $t_1 = s_1$ and $t_n = \frac{1}{\log n} \sum_{k=o}^{n} \frac{s_k}{1+k} \to s$,

then $s_n \to s$ (ℓ) [see 10, and 6, p.59, p.87].

(2) *The method* A *(Abel)*.

If $\sum_{n=o}^{\infty} a_n x^n$ is convergent for $|x| < 1$, $f(x)$ is its sum and $\lim_{x\to 1-} f(x) = s$, or if equivalently

$\lim_{x\to 1-} (1-x) \sum_{n=o}^{\infty} s_n x^n = s$ then $s_n \to s$ (A) or $\sum_{n=o}^{\infty} a_n = s$ (A)

[see 6, p.7, p.81].

(3) *The method* L .

If $\lim_{x\to 1-} \frac{-1}{\log(1-x)} \sum_{n=o}^{\infty} \frac{s_n}{n+1} x^{n+1} = s$ then $s_n \to s$ (L)

[see 4, and 6, p.81].

(4) *The power series method.*

Given a power series $c(x) = \sum_{n=o}^{\infty} c_n x^n$ with real non-negative coefficients having a radius of convergence $r_c > o$, we say $s_n \to s$ (C) if $\lim_{x\to r_c-} \frac{1}{c(x)} \sum_{n=o}^{\infty} c_n s_n x^n = s$.

This method is regular if either $r_c = \infty$ or $\sum_{n=0}^{\infty} c_n(r_c)^n = \infty$ [see 3, Theorem 1].

(5) *The method* $(\overline{N},q_n)$.

If $q_n \geq 0$, $q_0 > 0$, $\sum_{n=0}^{\infty} q_n = \infty$ and

$$t_n = \frac{\sum_{k=0}^{n} q_k s_k}{\sum_{k=0}^{n} q_k} \to s ,$$

then $s_n \to s$ $(\overline{N},q_n)$ [see 6, p.57].

We also, for convenience, define a method $(\overline{N}_*,q_n)$.

(6) *The method* $(\overline{N}_*,q_n)$.

If $q_n \geq 0$, $\sum_{n=0}^{\infty} q_n = \infty$ and

$$t_n = \frac{\sum_{k=0}^{n} q_k s_k}{\sum_{k=0}^{n} q_k} \to s ,$$

we will say $s_n \to s$ $(\overline{N}_*,q_n)$.

This differs from the usual $\bar{N}$ definition {see (5)} only in that we do not insist that t_n be defined for all n but for all sufficiently large n. We show later {Theorem (3.2)} that this is equivalent to an $\bar{N}$ method.

§2. NEW SUMMABILITY METHODS.

Throughout the thesis we will suppose that:

p and n denote non-negative integers;

$$\pi_p(x) = \begin{cases} \dfrac{1}{\log_o x \log_1 x \ldots \log_p x} & \text{for } x \geq e_p, \\ o & \text{otherwise,} \end{cases}$$

where $\log_o x = x$ for $x \geq e_o = 1$,

and $\log_{n+1} x \doteq \log(\log_n x)$

for $x \geq e_{n+1} = \exp e_n$;

$$\sigma_p(x) = \sum_{n=o}^{\infty} \pi_p(n)\, x^n \qquad (-1 < x < 1);$$

$$t_p(n) = \frac{1}{\log_{p+1} n} \sum_{k=o}^{n} \pi_p(k) s_k \quad (n \geq e_{p+1});$$

$$\sigma_p(x;s) = \frac{1}{\sigma_p(x)} \sum_{n=0}^{\infty} \pi_p(n) s_n x^n \quad (-1 < x < 1).$$

We now introduce the following summability methods.

(1) *The method* L_p

We shall say the series $\sum_{k=0}^{\infty} a_k$ is summable L_p to s,

and write $\sum_{k=0}^{\infty} a_k = s\ (L_p)$ or $s_n \to s\ (L_p)$, if

$$\lim_{x\to 1-} \sigma_p(x;s) = s.$$

(2) *The method* ℓ_p.

If $t_p(n) \to s$ as $n \to \infty$ we shall say the series $\sum_{k=0}^{\infty} a_k$

is summable ℓ_p to s, and write $\sum_{k=0}^{\infty} a_k = s\ (\ell_p)$ or

$s_n \to s\ (\ell_p)$.

§3. BASIC RESULTS.

The first theorem is well known but its proof has been included for the sake of completeness [see for example 3].

THEOREM 1. *The* L_p *method is regular.*

PROOF. Suppose $s_n \to s$; then for m a positive integer, we have

$$\limsup_{x \to 1-} |\sigma_p(x;s)-s| \le \limsup_{x \to 1-} \frac{1}{\sigma_p(x)} \sum_{n=o}^{\infty} \pi_p(n) |s_n - s| x^n$$

$$= \limsup_{x \to 1-} \frac{1}{\sigma_p(x)} \sum_{n=m}^{\infty} \pi_p(n) |s_n - s| x^n$$

$$\le \sup_{n \ge m} |s_n - s|$$

which tends to zero as $m \to \infty$. Hence $\sigma_p(x;s) \to s$ as $x \to 1-$ and L_p is regular∎

In order to establish a similar result for the ℓ_p methods it is only necessary to show that these methods are equivalent to $\overline{N}$ methods which Hardy has shown are regular [see 6, Theorem 12].

It is evident that

(1) $$\ell_p \simeq (\overline{N}_*, \pi_p(n)) \quad .$$

We now relate the $\overline{N}_*$ methods to regular $\overline{N}$ methods.

THEOREM 2. *If* $r_o > o$, $r_n \geq o$, $\sum_{n=o}^{\infty} q_n = \infty$, $q_n \geq o$ *and* $r_n = q_n$ *for* $n \geq N_o$, *then* $(\overline{N}_*, q_n) \simeq (\overline{N}, r_n)$.

PROOF. Choose $n \geq N_o$ and let

$$t_n = \frac{\sum_{k=o}^{n} q_k s_k}{\sum_{k=o}^{n} q_k} ,$$

$$u_n = \frac{\sum_{k=o}^{n} r_k s_k}{\sum_{k=o}^{n} r_k} ,$$

then

$$u_n = \frac{\sum_{k=o}^{N_o-1} r_k s_k - \sum_{k=o}^{N_o-1} q_k s_k}{\sum_{k=o}^{N_o-1} r_k + \sum_{k=N_o}^{n} q_k} + \frac{\sum_{k=o}^{n} q_k s_k}{\sum_{k=o}^{n} q_k} \cdot \frac{\sum_{k=o}^{n} q_k}{\sum_{k=o}^{N_o-1} r_k + \sum_{k=N_o}^{n} q_k} ,$$

i.e. $u_n = E_n + t_n F_n$.

Since $E_n \to o$ and $F_n \to 1$ as $n \to \infty$ the theorem follows Ξ

Now (1) and Theorem 2 give us:

THEOREM 3. *The* ℓ_p *method is regular.*

In connection with $\overline{N}$ methods Hardy has proved [6, Theorem 14],

THEOREM 4. (Hardy) *If* $d_n > o$, $q_n > o$, $\sum_{n=o}^{\infty} d_n = \infty$, $\sum_{n=o}^{\infty} q_n = \infty$, $D_n = \sum_{k=o}^{n} d_k$, $Q_n = \sum_{k=o}^{n} q_k$, *and either*

$$(2) \qquad q_{n+1}/q_n \leq d_{n+1}/d_n \quad ,$$

or

$$d_{n+1}/d_n \leq q_{n+1}/q_n$$

(3) *and*

$$D_n/d_n \leq H\, Q_n/q_n \quad ,$$

then $(\overline{N},q_n) \supseteq (\overline{N},d_n)$.

The next two theorems relate the methods L_o and ℓ_o to the known summability methods L and ℓ respectively.

THEOREM 5. $L_o \simeq L$.

PROOF. Borwein has shown that

(4) $s_n \to s$ (L) if and only if $s_{n+1} \to s$ (L)

[4, Theorem 1].

Suppose $s_n \to s$ (L), then by (4)

$$\lim_{x\to 1-} \frac{-1}{\log(1-x)} \sum_{n=0}^{\infty} \frac{s_{n+1}}{n+1} x^{n+1} = s,$$

and since

$$\sigma_0(x) = \sum_{n=1}^{\infty} \frac{x^n}{n} = \log \frac{1}{1-x}$$

we have

$$\lim_{x\to 1-} \frac{-1}{\log(1-x)} \left(\log \frac{1}{1-x}\right) \frac{1}{\sigma_0(x)} \sum_{n=1}^{\infty} \frac{s_n}{n} x^n = s ,$$

and so $s_n \to s$ (L_0) .

Now if $s_n \to s$ (L_0) we obtain, by a reversal of the above argument, $s_{n+1} \to s$ (L) and by (4) $s_n \to s$ (L) Ξ

LEMMA 1. $s_n \to s$ (ℓ_p) *if and only if* $s_{n+1} \to s$ (ℓ_p).

PROOF. In view of equation (1) it suffices to show $s_n \to s$ $(\overline{N}_*, \pi_p(n))$ if and only if $s_{n+1} \to s$ $(\overline{N}_*, \pi_p(n))$.

For sufficiently large n, let

$$u_n = \frac{\sum_{k=o}^{n} \pi_p(k) s_{k+1}}{\sum_{k=o}^{n} \pi_p(k)} .$$

Hence $s_{n+1} \to s\ (\overline{N}_*, \pi_p(n))$ is equivalent to $s_n \to s\ (\overline{N}_*, \pi_p(n-1))$.

We define $(\overline{N}, q_n)$ by

$$(5) \qquad q_n = \begin{cases} \pi_p(n) & n \geq k-1 , \\ \pi_p(k) & n < k-1 , \end{cases}$$

and $(\overline{N}, r_n)$ by

$$(6) \qquad r_n = \begin{cases} \pi_p(n-1) & n \geq k , \\ \dfrac{\pi_p(k)\pi_p(k)}{\pi_p(k+1)} & n < k , \end{cases}$$

where $k = [e_p] + 1$.

By Theorem 2 we have

$$(\overline{N}, q_n) \simeq (\overline{N}_*, \pi_p(n))$$

and

$$(\overline{N}, r_n) \simeq (\overline{N}_*, \pi_p(n-1)).$$

From (5) and (6) it can be easily shown that

(7) $$r_{n+1}/r_n \leq q_{n+1}/q_n \quad ,$$

and

(8) $$R_n/r_n \leq H\ Q_n/q_n \quad .$$

In view of Theorem 4, (7) implies $(\overline{N},r_n) \supseteq (\overline{N},q_n)$, and (7) and (8) together imply $(\overline{N},q_n) \supseteq (\overline{N},r_n)$. The theorem follows.

LEMMA 2 (a). (Ishiguro) *If* $s_n \to s$ (ℓ) *then* $s_n = o(n.\log n)$ *and* $a_n = o(n.\log n)$.

2 (b). *If* $s_n \to s$ (ℓ_p) *then* $s_n = o\left(\frac{1}{\pi_{p+1}(n)}\right)$ *and* $a_n = o\left(\frac{1}{\pi_{p+1}(n)}\right)$.

PROOF (a). This is due to Ishiguro [10, Theorem 4].

(b). For $n-1 \geq e_{p+1}$ we have ,

$$s_n = \frac{1}{\pi_p(n)}\,[t_p(n)\log_{p+1}n - t_p(n-1)\log_{p+1}(n-1)] \;;$$

hence

$$\pi_{p+1}(n)s_n = t_p(n) - t_p(n-1)\,\frac{\log_{p+1}(n-1)}{\log_{p+1}n} \to o \,,$$

and so,

$$\pi_{p+1}(n)a_n = \pi_{p+1}(n)s_n - \frac{\pi_{p+1}(n)}{\pi_{p+1}(n-1)}\cdot\pi_{p+1}(n-1)s_{n-1} \to o \quad \Xi$$

THEOREM 6. $\ell_o \simeq \ell$.

PROOF. Suppose $s_n \to s$ (ℓ_o) , then, by Lemma 1, $s_{n-1} \to s$ (ℓ_o), i.e.

$$(9) \qquad \frac{1}{\log n}\sum_{k=1}^{n} \frac{s_{k-1}}{k} \to s \ .$$

Hence

$$\frac{1}{\log n}\sum_{k=o}^{n}\frac{s_k}{1+k} = \frac{1}{\log n}\sum_{k=1}^{n}\frac{s_{k-1}}{k} + \frac{s_n}{(1+n)\log n} \to s$$

by (9) and Lemma 2 (b).

Thus $\ell \supseteq \ell_o$, and the equivalence follows by using Lemma 2 (a) and reversing the above argument$_{\Xi}$

§4. SOME LEMMAS.

The first lemma is easy but basic.

LEMMA 1. *If* $\lim_{x\to a} \frac{f(x)}{g(x)} = c > o$ *and* $\lim_{x\to a} g(x) = \infty$ *where* $-\infty \leq a \leq \infty$, *then* $\lim_{x\to a} \frac{\log_{p+1} f(x)}{\log_{p+1} g(x)} = 1.$

PROOF. Since

$$\lim_{x\to a} \frac{\log f(x)}{\log g(x)} = \lim_{x\to a} \frac{\log \frac{f(x)}{g(x)}}{\log g(x)} + 1 = 1,$$

the result holds for the case p = o, and the general case can be established by induction$_{\Xi}$

The next lemma is an application of the following theorem which is due to Agnew [1, Theorem 1.1].

THEOREM 1. (Agnew) *Let* a > o. *Then the relation*

$$(1)\ \limsup_{x\to 1-} \left| \sum_{k=1}^{\infty} x^k s_k - \sum_{k=1}^{[a/\log x^{-1}]} s_k \right| \leq A(a) \limsup_{n\to\infty} |ns_n|$$

holds for each series $\sum_{n=1}^{\infty} s_n$ *of real or complex terms for which* $\limsup |ns_n| < \infty$ *where* $A(a)$ *is the constant defined by*

$$A(a) = \gamma + \log a + 2\int_a^{\infty} x^{-1}e^{-x}dx,$$ *where* γ *is Euler's constant.*

Moreover $A(a)$ *is the best constant in the following sense. There is a real series* $\sum_{n=1}^{\infty} s_n$ *such that* $o < \limsup|ns_n| < \infty$ *and the members of* (1) *are equal.*

LEMMA 2. $\sigma_p(x) \sim \log_{p+1} \frac{1}{1-x}$ *as* $x \to 1-$.

PROOF. For $p = o$ the result is obvious. For $p > o$, since $\limsup_{n\to\infty} n\pi_p(n) = o$, we obtain from Theorem 1 that

$$\limsup_{x \to 1-} \left| \sum_{k=o}^{\infty} x^k \pi_p(k) - \sum_{k=o}^{[1/\log x^{-1}]} \pi_p(k) \right| \le o.$$

Also it is familiar that $\sum_{k=o}^{n} \pi_p(k) \sim \log_{p+1} n$ as $n \to \infty$.

The lemma now follows by Lemma 1 Ξ

The next lemma is an improvement on a lemma due to Ishiguro [11, Lemma 1].

LEMMA 3. *If* $g(x)$ *is bounded in* $[o, 1]$ *and continuous on the left at* $x = 1$ *and* ε *is any positive number, then there exist two polynomials,* $R(x)$ *and* $T(x)$ *such that,*

$$(2) \qquad R(x) \le g(x) \le T(x) \quad \text{for} \quad o \le x \le 1$$

and

$$(3) \qquad o \le g(x) - R(x) < \varepsilon, \quad o \le T(x) - g(x) < \varepsilon,$$

for $o < \tau \le x \le 1$ *where* τ *is a constant,* $o < \tau < 1$.

PROOF. Let ε be an arbitrary positive number. Then there is a δ such that

$$|g(x) - g(1)| < \frac{\varepsilon}{4} \quad \text{for } 1 - \delta < x \le 1.$$

Let $\tau = 1 - \delta/2$, H be a constant such that $g(x) \le H$, and let

$$f(x) = \begin{cases} H + \frac{\varepsilon}{4} , & o \leq x < 1 - \delta , \\ \frac{2g(1) - 2H + \frac{\varepsilon}{2}}{\delta}(x - 1 + \delta) + H + \frac{\varepsilon}{4} , & 1-\delta \leq x \leq \tau , \\ g(1) + \frac{\varepsilon}{2} , & \tau < x \leq 1 . \end{cases}$$

Then f(x) is a continuous function such that

$$f(x) \geq g(x) + \frac{\varepsilon}{4} \qquad \text{for} \quad o \leq x \leq 1 ,$$

and

$$o \leq f(x) - g(x) < 3\,\frac{\varepsilon}{4} \quad \text{for} \quad \tau \leq x \leq 1 .$$

By Weierstrass' Approximation Theorem [see 2, p.529] there exists a polynomial T(x) such that

$$|f(x) - T(x)| < \frac{\varepsilon}{4} \quad \text{for} \quad o \leq x \leq 1.$$

Similarly we may construct the polynomial $R(x)_{\Xi}$

CHAPTER 2.

This chapter contains basic tauberian theorems for ℓ_p and L_p summability.

§5. TAUBERIAN THEOREM FOR ℓ_p SUMMABILITY.

The first theorem is the basic result for ℓ_p summability. Though it can be derived from a result due to K. Chandrasekharan and S. Minakshisundaram [5, Theorem 1.88], a simpler proof, modelled on Kwee's proof of the case p = o [13, Lemma 3], is given.

THEOREM 1. *If* $\sum_{n=o}^{\infty} a_n = s \ (\ell_p)$ *and if the following Tauberian condition holds:*

(1) $\liminf (s_n - s_m) \geq o$ *when* $n > m \to \infty$

and $\log_{p+2} n - \log_{p+2} m \to o$,

then $\sum_{n=o}^{\infty} a_n$ *converges.*

PROOF. Assume, without loss of generality, that $s = o$, and let $N = [e_{p+2}] + 1$. Then for $n > m \geq N$

$$t_p(n)\log_{p+1}n - t_p(m)\log_{p+1}m = s_{m+1}\pi_p(m+1) + \dots + s_n\pi_p(n).$$

Let ε be an arbitrary positive number. By (1) there are numbers $M = M(\varepsilon) \geq N$ and $\delta = \delta(\varepsilon) > o$ such that:

if $n > m \geq M$ and $\log_{p+2}n - \log_{p+2}m \leq \delta$,

then $s_n - s_m \geq -\varepsilon$, and hence

$$(s_m-\varepsilon)\sum_{k=m+1}^{n}\pi_p(k) \leq t_p(n)\log_{p+1}n-t_p(m)\log_{p+1}m \leq (s_n+\varepsilon)\sum_{k=m+1}^{n}\pi_p(k);$$

$$\text{i.e.}\quad s_m - \varepsilon \leq \left(t_p(n)\frac{\log_{p+1}n}{\log_{p+1}m} - t_p(m)\right)\frac{\log_{p+1}m}{\sum_{k=m+1}^{n}\pi_p(k)},$$

$$\text{and}\quad s_n + \varepsilon \geq \left(t_p(n)\frac{\log_{p+1}n}{\log_{p+1}m} - t_p(m)\right)\frac{\log_{p+1}m}{\sum_{k=m+1}^{n}\pi_p(k)}.$$

Keeping ε fixed and letting $n > m \to \infty$ subject to

$$\frac{1}{2}\delta \leq \log_{p+2} n - \log_{p+2} m \leq \delta ,$$

we get
$$\limsup_{m \to \infty} s_m \leq \varepsilon$$

and
$$\liminf_{n \to \infty} s_n \geq -\varepsilon ,$$

since
$$t_n \to o ,$$

$$\varepsilon^{\delta} \geq \frac{\log_{p+1} n}{\log_{p+1} m} \geq e^{\delta/2} > 1 + \frac{1}{2}\delta ,$$

and

$$\frac{\log_{p+1} m}{\sum_{k=m+1}^{n} \pi_p(k)} \sim \frac{\log_{p+1} m}{\log_{p+1} n - \log_{p+1} m} = \frac{1}{\dfrac{\log_{p+1} n}{\log_{p+1} m} - 1}$$

$$= O(1) .$$

It follows that $\lim_{n\to\infty} s_n = o$ Ξ

LEMMA 1. *If* $a_n = O_L(\pi_{p+1}(n))$ *for* $n \geq e_{p+2}$, *then* (1) *is satisfied.*

PROOF. There is a positive number H such that for $k \geq e_{p+2}$

$$a_k \geq -H\pi_{p+1}(k)$$

so that for $n > m \geq e_{p+2}$,

$$s_n - s_m = \sum_{k=m+1}^{n} a_k \geq -H \cdot \sum_{k=m+1}^{n} \pi_{p+1}(k)$$

$$\sim -H(\log_{p+2}n - \log_{p+2}m)$$

Hence $\liminf (s_n - s_m) \geq o$ when $n > m \to \infty$ and $\log_{p+2}n - \log_{p+2}m \to o$; and condition (1) is satisfied.

§6. TAUBERIAN THEOREM FOR L_p SUMMABILITY.

The first lemma is similar to a result proved by Ishiguro [11, Lemma 2].

LEMMA 1. *If* g(x) *is bounded in* [o, 1] *and continuous on the left at* x = 1 *and if* $s_n \geq o$ *and* $s_n \to s$ (L_p),

$$\text{(1)} \quad \textit{then} \lim_{x\to 1-} \frac{1}{\sigma_p(x)} \sum_{n=o}^{\infty} \pi_p(n) s_n x^n g(x^n) = s.g(1) .$$

PROOF. By Lemmas (4.1) and (4.2), we have, for $c \geq o$,

$$\lim_{x\to 1-} \frac{1}{\sigma_p(x)} \sum_{n=o}^{\infty} \pi_p(n) s_n x^n \,.\, x^{cn}$$

$$= \lim_{x\to 1-} \frac{\sigma_p(x^{c+1})}{\sigma_p(x)} \cdot \frac{1}{\sigma_p(x^{c+1})} \sum_{n=o}^{\infty} \pi_p(n) s_n x^{(c+1)n}$$

$$= s \,.\, \lim_{x\to 1-} \frac{\sigma_p(x^{c+1})}{\sigma_p(x)}$$

$$= s. \lim_{x\to 1-} \frac{\log_{p+1} \frac{1}{1-x^{c+1}}}{\log_{p+1} \frac{1}{1-x}} = s .$$

Thus (1) is true whenever g(x) is a polynomial.

By Lemma (4.3) we have,

$$\frac{1}{\sigma_p(x)} \sum_{n=o}^{\infty} \pi_p(n) s_n x^n g(x^n) \le \frac{1}{\sigma_p(x)} \sum_{n=o}^{\infty} \pi_p(n) s_n x^n \; T(x^n)$$

and so

$$\limsup_{x \to 1-} \frac{1}{\sigma_p(x)} \sum_{n=o}^{\infty} \pi_p(n) s_n x^n g(x^n) \le \lim_{x \to 1-} \frac{1}{\sigma_p(x)} \sum_{n=o}^{\infty} \pi_p(n) s_n x^n T(x^n)$$

$$= s.T(1)$$

$$\le s(g(1) + \varepsilon) .$$

Hence

$$\limsup_{x \to 1-} \frac{1}{\sigma_p(x)} \sum_{n=o}^{\infty} \pi_p(n) s_n x^n g(x^n) \le s.g(1) \quad ,$$

and similarly

$$\liminf_{x \to 1-} \frac{1}{\sigma_p(x)} \sum_{n=o}^{\infty} \pi_p(n) s_n x^n g(x^n) \ge s.g(1) \Xi$$

The next lemma is a partial converse to Theorem (7.1).

LEMMA 2. *If* $s_n \to s \; (L_p)$ *and* $s_n \ge - H$ *then* $s_n \to s \; (\ell_p)$.

PROOF. The proof is similar to Ishiguro's proof of the case p = o [11, Theorem 2].

$$\text{Let } g(x) = \begin{cases} o & \text{for} \quad o \le x < \frac{1}{e}, \\ \frac{1}{x} & \text{for} \quad \frac{1}{e} \le x \le 1, \end{cases}$$

so that $g(1) = 1$ and $g(x^n) = o$ if $n > \dfrac{1}{\log \frac{1}{x}} = A(x)$.

Hence, by Lemma 1,

$$\lim_{x\to 1-} \frac{1}{\sigma_p(x)} \sum_{n \le A(x)} \pi_p(n)(s_n + M) = s + M .$$

Putting $x = e^{-1/n}$, we get

$$\lim_{n\to\infty} \frac{1}{\sigma_p(e^{-1/n})} \sum_{k=o}^{n} \pi_p(k)(s_k + M)$$

$$= \lim_{n\to\infty} \frac{\log_{p+1} n}{\sigma_p(e^{-1/n})} \cdot \frac{1}{\log_{p+1} n} \sum_{k=o}^{n} \pi_p(k)(s_k + M)$$

$$= \lim_{n\to\infty} \frac{1}{\log_{p+1} n} \sum_{k=o}^{n} \pi_p(k)(s_k + M) = s + M ,$$

since, by Lemmas (4.1) and (4.2),

$$\lim_{n\to\infty} \frac{\log_{p+1} n}{\sigma_p(e^{-1/n})} = \lim_{n\to\infty} \frac{\log_{p+1} n}{\log_{p+1} 1/(1-e^{-1/n})} = 1 \quad \Xi$$

The next two lemmas are due essentially to Vijayaraghavan [see 6, Theorem 238 and 239 and also 14, Chapter II, Theorem 9].

LEMMA 3. *If*

(2) $\liminf (s(t) - s(u)) > -\infty$ *when* $t > u \to \infty$ *and* $\Phi(t) - \Phi(u) \to o$, *where* $s(t) = s_n$ *for* $n \leq t < n + 1$, *and where*

(3) Φ *is an increasing continuous non-negative function in* $[o, \infty]$ *such that* $\Phi(u) \to \infty$ *as* $u \to \infty$, *then there are positive numbers* H_1 *and* H_2 *such that*

$$s(a) - s(b) > - H_1(\Phi(a) - \Phi(b)) - H_2$$

whenever $a \geq b \geq 1$.

PROOF. For some positive constant K_1 there are positive numbers U and δ such that

$$s(t) - s(u) > - K_1$$

when $t > u \geq U$, and $\Phi(t) - \Phi(u) \leq \delta$.

Now if $u < U$, and $\Phi(t) - \Phi(u) \leq \delta$

then either $t < U$

or $\Phi(t) - \Phi(U) \leq \delta$

and since $\Phi(t)$ increases to ∞, it follows that t is bounded above by a number which depends only on U and δ.

Hence there is a positive constant H_2 such that

$$s(t) - s(u) > - H_2$$

for $t > u \geq o$, and $\Phi(t) - \Phi(u) \leq \delta$.

Now let $\{b_i\}$ be a sequence of positive numbers defined as follows:

$$b_o = b$$

$$\Phi(b_i) = \Phi(b_{i-1}) + \delta \qquad (i \geq 1).$$

Let $b_r \leq a \leq b_{r+1}$, then

$$s(a) - s(b) = \sum_{i=o}^{r-1} \{s(b_{i+1}) - s(b_i)\} + s(a) - s(b_r)$$

$$> - (r+1) H_2 .$$

Now $r\delta = \Phi(b_r) - \Phi(b_o) \leq \Phi(a) - \Phi(b)$,

and we have

$$s(a) - s(b) > - H_1 (\Phi(a) - \Phi(b)) - H_2$$

where $H_1 = \frac{H_2}{\delta}$ Ξ

LEMMA 4. *Let* $\tau(x) = \sum_{n=o}^{\infty} c_n(x)s_n$ *for* $x > o$. *Suppose conditions (2) and (3) of the previous lemma are satisfied. In addition suppose:*

(4) $\Phi(u) - \Phi(u-1) \to o$ *as* $u \to \infty$,

(5) $c_n(x) \geq o$ $(x > o)$,

(6) $c_n(x) \to o$ *as* $x \to \infty$,

(7) $\sum_{n=o}^{\infty} c_n(x) = 1 \qquad (x > o)$,

(8) $\sum_{n=o}^{M} c_n(x) \to o$ *when* $x > M \to \infty$ *and* $\phi(x) - \phi(M) \to \infty$,

(9) $\sum_{n=M}^{\infty} c_n(x) \, (\phi(n) - \phi(M)) \to o$ *when* $M > x \to \infty$

and $\phi(M) - \phi(x) \to \infty$,

(10) $\tau(x)$ *is bounded as* $x \to \infty$.

Then s_n *is bounded.*

[Note: This omits a condition specified in Hardy's Theorem 239]

PROOF. Conditions (5), (6) and (7) are sufficient to ensure that $\tau(x)$ is a totally regular transformation i.e. is regular and if $s_n \to \pm \infty$ then $\tau(x) \to \pm \infty$ [see H.1, Theorems 5 and 10]. Hence by (10) s_n does not tend to $+ \infty$ or $- \infty$ as $n \to \infty$.

We assume the theorem is false i.e. $\limsup_{n \to \infty} |s_n| = \infty$ and let

(11) $f_1(t) = \max_{n \leq t} s_n \quad , \quad f_2(t) = \max_{n \leq t} (-s_n)$.

It is clear that $f_1(t)$ and $f_2(t)$ increase with t and at least one tends to ∞ . We consider two possibilities:

(i) $f_1(n) \geq f_2(n)$ for infinitely many n,

or

(ii) $f_1(n) < f_2(n)$ for all n sufficiently large.

Case (i).

Clearly $f_1(t) \to \infty$, and for any positive R there are integers M such that

(12) $s_M = f_1(M) > 2R, \quad f_1(M) \geq f_2(M).$

Let M_1 be the least such M. Choose K_1 to be the least integer K $> M_1$ such that

(13) $$s_K \leq \frac{1}{2} s_{M_1} \quad ,$$

the existence of such K's for sufficiently large R are ensured by the fact that s_n does not tend to ∞ . By Lemma 3

$$s_{K_1} - s_{M_1} > - H_1(\Phi(K_1) - \Phi(M_1)) - H_2 \quad ,$$

i.e. $$H_1(\Phi(K_1) - \Phi(M_1)) > s_{M_1} - s_{K_1} - H_2 \geq R - H_2 \ ,$$

and so $\Phi(K_1) - \Phi(M_1) \to \infty$ as $R \to \infty$.

Define x to be any number between M_1 and K_1 such that

(14) $\Phi(x) = \frac{1}{2}\{\Phi(K_1) + \Phi(M_1)\}$;

then

(15) $\Phi(x) - \Phi(M_1) \to \infty$, $\Phi(K_1) - \Phi(x) \to \infty$, as $R \to \infty$.

Let

$$\tau_1(x) = \sum_{n=o}^{M_1-1} c_n(x)s_n \quad ,$$

$$\tau_2(x) = \sum_{n=M_1}^{K_1-1} c_n(x)s_n \quad ,$$

$$\tau_3(x) = \sum_{n=K_1}^{\infty} c_n(x)s_n$$

then $\tau(x) = \tau_1(x) + \tau_2(x) + \tau_3(x)$.

We estimate $\tau_1(x)$, $\tau_2(x)$ and $\tau_3(x)$ in turn, letting $\delta(R)$ signify any function of R which tends to o as $R \to \infty$.

Now

$$\tau_1(x) \geq - f_2(M_1) \sum_{n=o}^{M_1-1} c_n(x) \geq - f_1(M_1) \sum_{n=o}^{M_1} c_n(x)$$

$$\geq - f_1(M_1)\delta(R)$$

by (11), (12) and (8) .

Since K_1 is the first $K > M_1$ which satisfies (13) ,

$$\tau_2(x) > \tfrac{1}{2}f_1(M_1) \sum_{n=M_1+1}^{K_1-1} c_n(x) = \tfrac{1}{2}f_1(M_1) \left(1 - \sum_{n=o}^{M_1} c_n(x) - \sum_{n=K_1}^{\infty} c_n(x)\right) .$$

If $M > x \to \infty$, $\Phi(M) - \Phi(x) \to \infty$, choose integral M' so that $M > M' > x$,

$$\Phi(M) - \Phi(M') > 1$$

$$\text{and} \quad \Phi(M') - \Phi(x) \to \infty .$$

This is always possible for sufficiently large x in view of (3) and (4).

Hence, by (9), we get

$$(16) \quad \sum_{n=M}^{\infty} c_n(x) < \sum_{n=M}^{\infty} c_n(x)(\Phi(n) - \Phi(M'))$$

$$< \sum_{n=M'}^{\infty} c_n(x)(\Phi(n) - \Phi(M')) \to o,$$

when $M > x \to \infty$ and $\Phi(M) - \Phi(x) \to \infty$.

It follows that $\tau_2(x) > (\frac{1}{2} - \delta(R))f_1(M_1)$.

If $n \geq K_1$, by Lemma 3,

$$s_n - s_{K_1-1} > - H_1(\Phi(n) - \Phi(K_1-1)) - H_2 \ ;$$

also

$$s_{K_1-1} > \frac{1}{2} s_{M_1} > R > H_2+1 \quad \text{for large } R.$$

So, for large R,

$$s_n > - H_1 . \Phi(n) + H_1 . \Phi(K_1-1) + 1$$

and by (3) and (4),

$$s_n > - H_1(\Phi(n) - \Phi(K_1)) \ ;$$

consequently

$$\tau_3(x) > - H_1 \sum_{n=K_1}^{\infty} c_n(x)(\Phi(n)-\Phi(K_1)) > - \delta(R),$$

by (9).

We now obtain

$$\tau(x) > -f_1(M_1)\delta(R) + (\tfrac{1}{2} - \delta(R))f_1(M_1) - \delta(R)$$

which tends to infinity with R in contradiction to (10).

Thus case (i) leads to a contradiction.

Case (ii) .

In this case we have $f_1(n) < f_2(n)$ for all n sufficiently large, so that $f_2(n) \to \infty$. Choose the least integer K_1 such that

$$(17)\quad f_2(n) > f_1(n) \quad (n \geq K_1), \quad s_{K_1} = -f_2(K_1) < -2R,$$

and then the greatest integer $M_1 < K_1$ for which

$$s_{M_1} \geq \tfrac{1}{2} s_{K_1} = -\tfrac{1}{2} f_2(K_1) ;$$

the existence of such M_1's are ensured by the fact that s_n does not tend to $-\infty$.

Thus

$$(18)\quad s_{M_1} \geq -\tfrac{1}{2}f_2(K_1), \quad s_n < -\tfrac{1}{2}f_2(K_1) \quad (M_1 < n \leq K_1).$$

Then

$$s_{K_1} - s_{M_1} > -H_1(\Phi(K_1) - \Phi(M_1)) - H_2 \quad \text{by Lemma 3,}$$

and

$$s_{K_1} - s_{M_1} \leq \frac{1}{2} s_{K_1} < -R \;;$$

so that

$$H_1 \; (\Phi(K_1) - \Phi(M_1)) > R - H_2 \quad ,$$

and $\Phi(K_1) - \Phi(M_1) \to \infty$ when $R \to \infty$. Hence (15) is still true if x is defined as in (14).

Let

$$\tau_1(x) = \sum_{n=o}^{M_1} c_n(x)s_n \quad ,$$

$$\tau_2(x) = \sum_{n=M_1+1}^{K_1} c_n(x)s_n \quad ,$$

$$\tau_3(x) = \sum_{n=K_1+1}^{\infty} c_n(x)s_n \quad ;$$

then $\qquad \tau(x) = \tau_1(x) + \tau_2(x) + \tau_3(x)$.

Now

$$\tau_1(x) \leq f_1(M_1)\sum_{n=o}^{M_1} c_n(x) \leq f_1(K_1)\sum_{n=o}^{M_1} c_n(x)$$

$$\leq f_2(K_1)\sum_{n=o}^{M_1} c_n(x) \leq f_2(K_1)\delta(R) ,$$

by (11), (17), and (8) ;

$$\tau_2(x) < -\tfrac{1}{2}f_2(K_1)\sum_{n=M_1+1}^{K_1} c_n(x)$$

$$= -\tfrac{1}{2}f_2(K_1)\left(1-\sum_{n=o}^{M_1} c_n(x)-\sum_{n=K_1+1}^{\infty} c_n(x)\right)$$

$$\leq -\left(\tfrac{1}{2} - \delta(R)\right)f_2(K_1) ,$$

by (18), (8), and (16);

$$\tau_3(x) \leq \sum_{n=K_1+1}^{\infty} c_n(x)f_1(n) \leq \sum_{n=K_1+1}^{\infty} c_n(x)f_2(n)$$

$$= \sum_{n=K_1+1}^{\infty} c_n(x)(f_2(K_1) + f_2(n) - f_2(K_1)),$$

but $- s_n - f_2(K_1) = - s_n + s_{K_1} < H_1(\Phi(n) - \Phi(K_1)) + H_2$

for $n > K_1$, and therefore

$$f_2(n) - f_2(K_1) < H_1(\phi(n) - \phi(K_1)) + H_2 \quad .$$

Hence

$$\tau_3(x) \leq f_2(K_1) \sum_{n=K_1+1}^{\infty} c_n(x) + H_1 \sum_{n=K_1+1}^{\infty} c_n(x)(\phi(n) - \phi(K_1))$$

$$+ H_2 \sum_{n=N_1+1}^{\infty} c_n(x) \leq f_2(K_1)\delta(R)$$

by (9) and (16).

We now obtain

$$\tau(x) \leq -(\tfrac{1}{2} - \delta(R))f_2(K_1)$$

and so $\tau(x) \to -\infty$ when $R \to \infty$. This is again a contradiction, and the theorem is proved$_{\Xi}$

We are now in a position to prove the central theorem of the thesis. The proof is based on Kwee's proof [13, Theorem A], of the case p = o.

THEOREM 1. *If* $\sum_{n=o}^{\infty} a_n = s(L_p)$ *and if condition* (5.1) *holds then* $\sum_{n=o}^{\infty} a_n$ *converges.*

PROOF. Let $\Phi(u) = \begin{cases} \log_{p+2} u & \text{for } u \geq e_{p+2}, \\ \dfrac{u}{e_{p+2}} & \text{for } o \leq u < e_{p+2}, \end{cases}$

and for $x > o$ let

$$\tau(x) = \frac{1}{\sigma_p(e^{-1/x})} \sum_{n=o}^{\infty} \pi_p(n) s_n e^{-n/x} = \sum_{n=o}^{\infty} c_n(x) s_n ,$$

where

$$c_n(x) = \frac{\pi_p(n) e^{-n/x}}{\sigma_p(e^{-1/x})} .$$

We now show that the conditions of Lemma 4 are satisfied. Condition (3) is evidently fulfilled and in view of Lemma (4.1) it is clear that condition (4) is satisfied.

Condition (2) is satisfied because of condition (5.1).

Condition (5) is obvious.

Condition (6): $\lim_{x\to\infty} c_n(x) = \lim_{x\to\infty} \pi_p(n) \dfrac{e^{-n/x}}{\sigma_p(e^{-1/x})}$

$$= \lim_{x\to\infty} \frac{\pi_p(n) e^{-n/x}}{\log_{p+1} 1/(1-e^{-1/x})} \text{ (by Lemma (4.2))}$$

$$= o .$$

Condition (7): $\sum_{n=o}^{\infty} c_n(x) = \dfrac{1}{\sigma_p(e^{-1/x})} \sum_{n=o}^{\infty} \pi_p(n)(e^{-1/x}) = 1.$

Condition (8): Letting $x > M \to \infty$ subject to

$$\log_{p+2} x - \log_{p+2} M \to \infty \quad ,$$

or equivalently, $$\frac{\log_{p+1} M}{\log_{p+1} x} \to o \; ,$$

we have,

$$o \leq \sum_{n=o}^{M} c_n(x) = \frac{\sum_{n=o}^{M} \pi_p(n)e^{-n/x}}{\sum_{n=o}^{M} \pi_p(n)} \cdot \frac{\sum_{n=o}^{M} \pi_p(n)}{\sigma_p(e^{-1/x})}$$

$$\leq \frac{\sum_{n=o}^{M} \pi_p(n)}{\sigma_p(e^{-1/x})}$$

$$\sim \frac{\log_{p+1} M}{\log_{p+1} x} \cdot \frac{\log_{p+1} x}{\log_{p+1}(1/1-e^{-1/x})}$$

$$\to o \quad ,$$

by Lemmas (4.1) and (4.2) .

Condition (9): We will show $\sum_{n=M}^{\infty} c_n(x)\Phi(n) \to o$ when $M > x \to \infty$ which is stronger than (9). Since $\pi_p(t)e^{-t/x} \log_{p+2} t$ is a decreasing function of t for $t \geq e_{p+2}$, we have,

for $M - 1 \geq e_{p+2}$,

$$o \leq \sum_{n=M}^{\infty} c_n(x)\phi(n) = \frac{1}{\sigma_p(e^{-1/x})} \sum_{n=M}^{\infty} \pi_p(n)e^{-n/x} \log_{p+2} n$$

$$\leq \frac{1}{\sigma_p(e^{-1/x})} \int_{M-1}^{\infty} \pi_p(t)e^{-t/x} \log_{p+2} t \, dt$$

$$\leq \frac{x\pi_p(M-1) \log_{p+2}(M-1)}{\sigma_p(e^{-1/x})} = \lambda_p \text{, say .}$$

In the case $p > o$ it is evident that $\lambda_p \to o$ when $M > x \to \infty$ and in the remaining case we have, for $M > x \to \infty$,

$$\lambda_o = \frac{x}{M-1} \frac{\log_2(M-1)}{\sigma_o(e^{-1/x})}$$

$$\sim \frac{x}{M-1} \frac{\log_2(M-1)}{\log 1/(1-e^{-1/x})} \qquad \text{by Lemma (4.2)}$$

$$\sim \frac{x}{\log x} \cdot \frac{\log_2(M-1)}{M-1} \qquad \text{by Lemma (4.1)}$$

$$\leq \frac{M}{\log M} \frac{\log_2(M-1)}{M-1} \to o \text{ .}$$

Condition (10): $\lim_{x\to\infty} \tau(x) = \lim_{t\to 1-} \frac{1}{\sigma_p(t)} \sum_{n=o}^{\infty} \pi_p(n) s_n t^n = s$

since $s_n \to s(L_p)$, and so condition (10) holds.

Thus, by Lemma 4, s_n is bounded and, by Lemma 2, $s_n \to s(\ell_p)$ and so, by Theorem (5.1), $\sum_{n=o}^{\infty} a_n$ converges Ξ

In view of Theorem 1, Theorem (5.1) and Lemma (5.1) we now have the following useful corollary.

COROLLARY 1. *If a series* $\sum_{n=o}^{\infty} a_n$ *is* L_p *or* ℓ_p *summable and if* $a_n = O_L(\pi_{p+1}(n))$ *for* $n \geq e_{p+2}$, *then the series converges.*

The p=o case of the above includes "o" tauberian results established by Ishiguro [12].

CHAPTER 3

This chapter contains various inclusion theorems between the L_p and ℓ_p summability methods.

§7. INCLUSION RESULTS

The first theorem may be deduced from a general theorem [see 8, p.181] but a simpler proof is given for completeness.

THEOREM 1. $L_p \supseteq \ell_p$.

PROOF. Suppose $s_n \to s \; (\ell_p)$,

then

$$t_p(n)\log_{p+1}n - t_p(n-1)\log_{p+1}(n-1) = \pi_p(n)s_n \quad \text{for } n \geq e_{p+1} ,$$

and

$$\lim_{x\to 1-} \sigma_p(x;s) = \lim_{x\to 1-} \frac{(1-x)\sum_{n=e_{p+1}}^{\infty} x^n t_p(n)\log_{p+1}n}{\sigma_p(x)}$$

$$= \lim_{x \to 1-} \frac{\sum_{n=e_{p+1}}^{\infty} x^n t_p(n) \log_{p+1} n}{\sum_{n=o}^{\infty} \left[\sum_{m=o}^{n} \pi_p(m)\right] x^m}$$

$$\doteq \lim_{x \to 1-} \frac{\sum_{n=o}^{\infty} \gamma_n c_n x^n}{\sum_{n=o}^{\infty} c_n x^n} \quad ,$$

where

$$c_n = \sum_{m=o}^{n} \pi_p(m) \quad ,$$

$$\gamma_n = \begin{cases} o & n < e_{p+1} \ , \\ \dfrac{t_p(n) \log_{p+1} n}{c_n} & n \geq e_{p+1} \ ; \end{cases}$$

and since $\gamma_n \to s$, we have $\lim_{x \to 1-} \sigma_p(x;s) = s$ [see definition 1.4] Ξ

The next lemma is essentially due to Borwein.

LEMMA 1. *If* $x \geq e_p$, $y > o$, *then*

$$(\log_p x)^{-y} = \int_o^{\infty} e^{-xt} \lambda_{p,y}(t) dt \ ,$$

where $\lambda_{p,y}(t)$ *is defined by the recursive formulae:*

$$\lambda_{o,y}(t) = \frac{t^{y-1}}{\Gamma(y)} ,$$

$$\lambda_{r+1,y}(t) = \frac{1}{\Gamma(y)} \int_o^\infty u^{y-1}\lambda_{r,u}(t)du \quad (r=o,1,2,\ldots).$$

PROOF. The lemma is true for p = o, since, when $x \geq e_o = 1$,

$$(\log_o x)^{-y} = x^{-y} = \frac{1}{\Gamma(y)} \int_o^\infty e^{-xt}t^{y-1}dt = \int_o^\infty e^{-xt}\lambda_{o,y}(t)dt.$$

Assume the lemma is true for p = r. Then, for $x \geq e_{r+1}$ we have,

$$(\log_{r+1}x)^{-y} = \frac{1}{\Gamma(y)} \int_o^\infty e^{-u\log_{r+1}x}\, u^{y-1}du$$

$$= \frac{1}{\Gamma(y)} \int_o^\infty (\log_r x)^{-u}\, u^{y-1}du$$

$$= \frac{1}{\Gamma(y)} \int_o^\infty u^{y-1}du \int_o^\infty e^{-xt}\lambda_{r,u}(t)dt$$

$$= \int_o^\infty e^{-xt}dt\, \frac{1}{\Gamma(y)} \int_o^\infty u^{y-1}\lambda_{r,u}(t)du$$

$$= \int_o^\infty e^{-xt}\lambda_{r+1,y}(t)dt,$$

the inversion in the order of integration being justified by Fubini's theorem since all the functions concerned are non-negative and Lebesgue measurable. The lemma is thus established by induction Ξ

The case p=1 of the next lemma is due to Hardy [6, p.268].

LEMMA 2. *If* $n \geq e_p$, $y > o$, *then*

$$(\log_p n)^{-y} = \int_o^1 t^n \emptyset(t)dt,$$

where the function $\emptyset$ *is non-negative and independent of* n.

PROOF. By Lemma 1,

$$(\log_p n)^{-y} = \int_o^\infty e^{-nx}\lambda_{p,y}(x)dx = \int_o^1 t^n \emptyset(t)dt,$$

where $\emptyset(t) = t^{-1}\lambda_{p,y}(\log \frac{1}{t})$ Ξ

THEOREM 2. *There is a series summable* ℓ_{p+1} *but not summable* L_p *i.e.* $L_p \not\supseteq \ell_{p+1}$.

PROOF. Let N be the integer such that $N - 1 < e_{p+1} \leq N$, and let

$$a_n = \begin{cases} \pi_p(n)(\log_{p+1} n)^{-1-i} & \text{for } n \geq e_{p+1}, \\ 0 & \text{for } n < e_{p+1}, \end{cases}$$

where $i = \sqrt{-1}$.

Then

$$s_{n-1} - i[(\log_{p+1} n)^{-i} - (\log_{p+1} N)^{-i}]$$

$$= \sum_{k=N}^{n-1} \pi_p(k)(\log_{p+1} k)^{-1-i} - \int_N^n (\log_{p+1} t)^{-1-i} \pi_p(t)\,dt$$

$$= \sum_{k=N}^{n-1} \phi_k ,$$

where

$$\phi_k = \int_k^{k+1} \left[\int_k^t \left(- \frac{d}{dx} \pi_p(x)(\log_{p+1} x)^{-1-i} \right) dx \right] dt$$

$$= \int_k^{k+1} \left[\int_k^t (\pi_p(x))^2 (\log_{p+1} x)^{-1-i} \left[\sum_{r=0}^{p} \frac{\pi_r(x)}{\pi_p(x)} + (1+i)(\log_{p+1} x)^{-1} \right] dx \right] dt$$

$$= \int_k^{k+1} \left[\int_k^t O\left(\frac{1}{x^2} \right) dx \right] dt$$

$$= O\left(\frac{1}{k^2} \right) .$$

Hence $\sum_{k=N}^{\infty} \phi_k$ converges, and so

$$s_{n-1} - i(\log_{p+1} n)^{-i} \text{ tends to a finite limit as } n \to \infty .$$

Since

$$s_n = s_{n-1} + \pi_p(n)\,(\log_{p+1} n)^{-1-i} ,$$

we have $s_n = i(\log_{p+1} n)^{-i} + k_n$ where k_n tends to a finite limit as $n \to \infty$.

Consequently $\{s_n\}$ is bounded but does not converge, and as $a_n = O(\pi_{p+1}(n))$, it follows from a Corollary (6.1) that $\sum_{n=o}^{\infty} a_n$ is not L_p summable.

We now show $\sum_{n=o}^{\infty} a_n$ is ℓ_{p+1} summable. For $m \geq N$, we have,

$$t_{p+1}(m) = \frac{1}{\log_{p+2} m} \sum_{n=N}^{m} \pi_{p+1}(n) \left(\frac{(\log_{p+1} n)^{-i}}{i} + k_n \right) .$$

$$= \frac{1}{i \log_{p+2} m} \sum_{n=N}^{m} \pi_p(n)(\log_{p+1} n)^{-1-i}$$

$$+ \frac{1}{\log_{p+2} m} \sum_{n=o}^{m} \pi_{p+1}(n)\, k_n$$

$$= \frac{1}{i \log_{p+2} m} \cdot s_m + \frac{1}{\log_{p+2} m} \sum_{n=o}^{m} \pi_{p+1}(n)\, k_n \ ,$$

and hence $t_{p+1}(m)$ tends to a finite limit as $m \to \infty$ Ξ

In connection with the methods defined in (1.4) Borwein has proved the following theorem [3, Theorem A].

THEOREM 3. *If* χ *is a real function of bounded variation in* [o, 1] *and if*

(1) $c_n = d_n \int_o^1 t^n d\chi(t) \geq \delta d_n \int_o^1 t^n |d\chi(t)|$ $(1 \geq \delta > o, n=N, N+1, \ldots)$

(2) $r_c = r_d > o$ *and* C *is regular,*

then $C \supseteq D$.

Using this result we can prove the following lemma.

LEMMA 3. $L_{p+1} \supseteq L_p$.

PROOF. By Lemma 2, for $n \geq e_{p+1}$,

$$\frac{\pi_{p+1}(n)}{\pi_p(n)} = (\log_{p+1} n)^{-1} = \int_0^1 t_n \, \emptyset(t)dt ,$$

where $\emptyset(t)$ is non-negative and independent of n, and hence, by Theorem 3, $L_{p+1} \supseteq L_p$ ∎

THEOREM 4. $L_p \supset \ell_p$.

PROOF. We consider a series used to show the existence of a series summable by the Abel method A, but not summable by any Cesaro method [6, Theorem 56].

Let

$$e^{1/(1+x)} = \sum_{n=0}^{\infty} a_n x^n .$$

It is known that a_n is not $O(n^r)$ for any r, and hence, by Lemma (3.2b), $\sum_{n=0}^{\infty} a_n$ is not summable ℓ_p. Since the series is summable A, and [see 6, p.81] $A \subset L \simeq L_0 \subseteq L_p$, the theorem can now be deduced from Theorem 1 ∎

In view of Theorems 2 and 4 we are able to strengthen Lemma 3 to:

THEOREM 5. $L_{p+1} \supset L_p$.

The final theorem in this chapter is:

THEOREM 6. $\ell_{p+1} \supset \ell_p$.

PROOF. The inclusion $\ell_{p+1} \supseteq \ell_p$ follows immediately from equation (3.1) and Theorems (3.2) and (3.4). The stronger inclusion may be deduced from Theorem 2, however, a direct proof is easy.

Consider $$s_n = (-1)^n \frac{1}{\pi_{p+1}(n)} .$$

Then $s_n \to o\ (\ell_{p+1})$ i.e. $\sum_{n=o}^{\infty} a_n$ is summable ℓ_{p+1}, but

$s_n \neq o\left(\frac{1}{\pi_{p+1}(n)}\right)$; hence, by Lemma (3.2b) , $\sum_{n=o}^{\infty} a_n$ is not ℓ_p summable Ξ

CHAPTER 4

In this chapter a criterion is obtained for the Fourier series of a given function f(x) to be L_p summable.

8. PRELIMINARY RESULTS.

The first lemma is elementary.

LEMMA 1. *If* $2 \sup_{n\geq o} \left|\sum_{k=o}^{n} a_k\right| = H < \infty$ *and if* b_n *is monotonic decreasing to o, then* $\sum_{n=o}^{\infty} a_n b_n$ *is convergent and* $\left|\sum_{n=N}^{\infty} a_n b_n\right| \leq Hb_N$.

PROOF. $\left|\sum_{n=N}^{M} a_n b_n\right| = \left|\sum_{n=N}^{M} (s_n - s_{n-1}) b_n\right| = \left|\sum_{n=N}^{M} s_n b_n - \sum_{n=N}^{M} s_{n-1} b_n\right|$

$$= \left|\sum_{n=N}^{M} s_n (b_n - b_{n+1}) - s_{N-1} b_N + s_M b_{M+1}\right|$$

$$\leq \left|\sum_{n=N}^{M} s_n (b_n - b_{n+1})\right| + \left|s_{N-1} b_N\right| + \left|s_M b_{M+1}\right|$$

$$\leq \sup_{N\leq k\leq M} |s_k| \sum_{n=N}^{M} (b_n - b_{n+1}) + |s_{N-1}b_N| + |s_M b_{M+1}|$$

$$\leq \frac{H}{2}(b_N - b_{M+1}) + \frac{H}{2} b_N + \frac{H}{2} b_{M+1}$$

$$= Hb_N \quad \Xi$$

LEMMA 2. *If* $k > o$ *then* $\pi_p(x^k) = O\left(\frac{\pi_p(x)}{x^{k-1}}\right)$ *as* $x \to \infty$.

PROOF. If $p = o$ the result is trivial.

Since $\frac{\log_p x}{\log_p x^k} = \frac{\log_{p-1}(\log x)}{\log_{p-1}(k \log x)}$ is equal to

$\frac{1}{k}$ if $p = 1$, and $\to 1$ as $x \to \infty$ for $p \geq 2$ (by Lemma (4.1)),

we have $\frac{x \log x \ldots \log_p x}{x^k \log x^k \ldots \log_p x^k} \cdot x^{k-1} = O(1)$ as $x \to \infty$ Ξ

LEMMA 3. *Uniformly for* $t > o$,

$$\frac{1}{t} \sum_{n=o}^{\infty} \pi_p(n) \sin nt\, x^n = O\left(\frac{\pi_p(\frac{1}{1-x})}{(1-x)^2}\right) \text{ as } x \to 1- .$$

PROOF. For x close enough to 1 to ensure that

$\pi_p\left(\left(\frac{1}{1-x}\right)^{1/2}\right) > o$, we have

$$\left|\frac{1}{t}\sum_{n=o}^{\infty} \pi_p(n)\sin nt\, x^n\right| \leq \sum_{n=o}^{\infty} n\pi_p(n)x^n$$

$$= \sum_{n\leq(\frac{1}{1-x})^{1/2}} n\pi_p(n)x^n + \sum_{n>(\frac{1}{1-x})^{1/2}} n\pi_p(n)x^n$$

$$\leq \left(\frac{1}{1-x}\right)^{1/2} + \left(\frac{1}{1-x}\right)^{1/2} \cdot \pi_p\left(\left(\frac{1}{1-x}\right)^{1/2}\right) \cdot \sum_{n>(\frac{1}{1-x})^{1/2}} x^n$$

$$\leq \left(\frac{1}{1-x}\right)^{1/2} + \left(\frac{1}{1-x}\right)^{3/2} \pi_p\left(\left(\frac{1}{1-x}\right)^{1/2}\right) .$$

Since $\left(\frac{1}{1-x}\right)^{1/2} \frac{(1-x)^2}{\pi_p(\frac{1}{1-x})} < \frac{(1-x)^{3/2}}{1-x}\left(\log \frac{1}{1-x}\right)^p \to o$ as $x \to 1-$,

we have

$$\left(\frac{1}{1-x}\right)^{1/2} + \left(\frac{1}{1-x}\right)^{3/2} \pi_p\left(\left(\frac{1}{1-x}\right)^{1/2}\right)$$

$$= o\left[\frac{\pi_p(\frac{1}{1-x})}{(1-x)^2}\right] + \left(\frac{1}{1-x}\right)^{3/2} \pi_p\left(\left(\frac{1}{1-x}\right)^{1/2}\right)$$

$$= o\left(\frac{\pi_p(\frac{1}{1-x})}{(1-x)^2}\right) + (\frac{1}{1-x})^{3/2}\ O\left(\frac{\pi_p(\frac{1}{1-x})}{(\frac{1}{1-x})^{-1/2}}\right) \quad \text{(by Lemma 2)}$$

$$= O\left(\frac{\pi_p(\frac{1}{1-x})}{(1-x)^2}\right) \quad \text{as } x \to 1- \quad \Xi$$

LEMMA 4. *If* δ *is an arbitrary number less than* $\frac{1}{e_p^{\,2}}$ *, then, uniformly for* $o \leq x < 1$, $o < t < \delta$,

$$\frac{t}{\pi_p(\frac{1}{t})} \sum_{n=o}^{\infty} \pi_p(n) \sin nt\ x^n = O(1) .$$

PROOF. $$\left|\frac{t}{\pi_p(\frac{1}{t})} \sum_{n=o}^{\infty} \pi_p(n) \sin nt\ x^n\right|$$

$$\leq \left|\frac{t}{\pi_p(\frac{1}{t})} \sum_{n\leq t^{-1/2}} \pi_p(n) \sin nt\ x^n\right| + \left|\frac{t}{\pi_p(\frac{1}{t})} \sum_{n>t^{-1/2}} n\pi_p(n)\frac{\sin nt}{n}\ x^n\right|$$

$$\leq \frac{t^2}{\pi_p(\frac{1}{t})} \cdot t^{-1/2} + \frac{t^{1/2}\pi_p(t^{-1/2})}{\pi_p(\frac{1}{t})} \cdot H$$

by Lemma 1 and the fact that $\sum_{k=1}^{n} \frac{\sin kt}{k}$ is uniformly bounded in $[o,\delta]$, since $o < \delta < \pi$ [see 7, p.29] .

The required result follows by Lemma 2 Ξ

LEMMA 5. *If, for any arbitrary positive* δ , $\emptyset(u)$ *is Lebesgue integrable in* (o,δ) *and*

$$\int_t^\delta \frac{|\emptyset(u)| \ \pi_p(\frac{1}{u})du}{u^2} = o(\log_{p+1} \frac{1}{t}) \text{ as } t \to o+ ,$$

then

$$\int_o^t |\emptyset(u)|du = o\left(\frac{t^2}{\pi_{p+1}(\frac{1}{t})}\right) \quad as \quad t \to o+ .$$

PROOF. For t sufficiently small,

$$\int_o^t |\emptyset(u)|du = \lim_{h\to o+}\int_h^t \frac{|\emptyset(u)|\pi_p(\frac{1}{u})u^2}{u^2\pi_p(\frac{1}{u})} du$$

$$= \lim_{h\to o+}\left[- \frac{u^2}{\pi_p(\frac{1}{u})} \int_u^\delta \frac{|\emptyset(x)|}{x^2} \pi_p(\frac{1}{x}) \ dx\Big|_h^t + \int_h^t\left(\int_u^\delta \frac{|\emptyset(x)\pi_p(\frac{1}{x})|}{x^2}\right) d\left(\frac{u^2}{\pi_p(\frac{1}{u})}\right)\right] ,$$

$$= o\left(\frac{t^2}{\pi_{p+1}(\frac{1}{t})}\right) + \lim_{h\to o+} \int_h^t \varepsilon(u)\log_{p+1}(\tfrac{1}{u})d\left(\frac{u^2}{\pi_p(\frac{1}{u})}\right) ,$$

where $\varepsilon(u) \to o$ as $t \to o+$.

Now for sufficiently small t,

$$\lim_{h\to o+}\left|\int_h^t |\varepsilon(u)|\log_{p+1}(\tfrac{1}{u})d\left(\frac{u^2}{\pi_p(\frac{1}{u})}\right)\right|$$

$$\leq \sup_{o\leq u\leq t}|\varepsilon(u)| \lim_{h\to o+}\left|\int_h^t \log_{p+1}(\tfrac{1}{u})d\left(\frac{u^2}{\pi_p(\frac{1}{u})}\right)\right|$$

$$= \sup_{o\leq u\leq t}|\varepsilon(u)| \lim_{h\to o+}\left|\left[\frac{\log_{p+1}(\frac{1}{u}).u^2}{\pi_p(\frac{1}{u})}\right|_h^t + \int_h^t \frac{u^2}{\pi_p(\frac{1}{u})}\cdot\frac{\pi_p(\frac{1}{u})}{u^2}\,du\right]\right| ,$$

$$= o\left(\frac{t^2}{\pi_{p+1}(\frac{1}{t})}\right) \quad \text{as } t \to o+ _\Xi$$

§9. APPLICATION TO FOURIER SERIES.

Throughout this section we suppose that the function f(x) is periodic with period 2π and is Lebesgue integrable over the interval $(o, 2\pi)$.

Let its Fourier series be

$$\frac{1}{2} a_o + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) .$$

Fixing x_o, we write

$$\emptyset(t) = \emptyset_{x_o}(t) = \{f(x_o + t) + f(x_o - t) - 2s\}$$

and

$$S_n(x_o) = \frac{1}{2} a_o + \sum_{k=1}^{n} (a_k \cos kx_o + b_k \sin kx_o) .$$

THEOREM 1. *If*

$$\int_t^\delta \frac{|\emptyset(u)|\pi_p(\frac{1}{u})\, du}{u^2} = o(\log_{p+1} \frac{1}{t}) \text{ as } t \to o+$$

for any arbitrary positive $\delta < \frac{1}{e_p^2}$ *, then the Fourier series of* $f(x)$ *is summable* L_p *to* s *at* x_o .

PROOF. The case $p = o$ of this theorem, with an additional condition, has been proved in a slightly different manner by Hsiang [9] .

We have

$$S_n(x_o) - s = \frac{1}{\pi} \int_o^\pi \emptyset(t) \frac{\sin nt}{t} dt + \varepsilon_n, \text{ [see 15, p. 55]}$$

where $\varepsilon_n \to o$.

Hence, for $o < x < 1$,

$$\frac{1}{\sigma_p(x)} \sum_{n=o}^{\infty} (S_n(x_o) - s)\, \pi_p(n)\, x^n$$

$$= \frac{1}{\pi\sigma_p(x)} \int_o^{\pi} \frac{\phi(t)}{t} \left(\sum_{n=o}^{\infty} \pi_p(n) \sin nt\, x^n \right) dt$$

$$+ \frac{1}{\sigma_p(x)} \sum_{n=o}^{\infty} \pi_p(n) \varepsilon_n x^n ,$$

the inversion being justified, since

$$\int_o^{\pi} \frac{|\phi(t)|}{t} \left(\sum_{n=o}^{\infty} |\pi_p(n) \sin nt\, x^n| \right) dt \leq \int_o^{\pi} |\phi(t)| \sum_{n=o}^{\infty} n\pi_p(n) x^n dt < \infty .$$

In view of the regularity of the L_p method, it follows that

$$\frac{1}{\sigma_p(x)} \sum_{n=o}^{\infty} (S_n(x_o)-s)\pi_p(n)x^n = \frac{1}{\pi\sigma_p(x)} \{J_1(x)+J_2(x)+J_3(x)\} + o(1) ,$$

where

$$J_1(x) = \int_o^{1-x} \frac{\phi(t)}{t} \left[\sum_{n=o}^{\infty} \pi_p(n) \sin nt\, x^n \right] dt,$$

$$J_2(x) = \int_{1-x}^{\delta} \frac{\phi(t)\pi_p(\frac{1}{t})}{t^2} \cdot \frac{t}{\pi_p(\frac{1}{t})} \left[\sum_{n=o}^{\infty} \pi_p(n) \sin nt\, x^n \right] dt ,$$

$$J_3(x) = \int_{\delta}^{\pi} \frac{\phi(t)}{t} \left[\sum_{n=o}^{\infty} \pi_p(n) \sin nt\, x^n \right] dt .$$

First by Lemmas (8.3) and (8.5),

$$J_1(x) = o\left(\frac{(1-x)^2}{\pi_{p+1}(\frac{1}{1-x})} \right) O\left(\frac{\pi_p(\frac{1}{1-x})}{(1-x)^2} \right)$$

$$= o(\log_{p+1} \frac{1}{1-x}) \quad \text{as } x \to 1- \quad .$$

Next

$$J_2(x) = o(\log_{p+1} \frac{1}{1-x}) \quad \text{as } x \to 1- \quad ,$$

by Lemma (8.4).

Finally, by Lemma (8.1) and the uniform boundedness of $\sum_{n=1}^{m} \sin nt$ in (δ, π) [see 7, p.28],

$$|J_3(x)| \leq \frac{1}{\delta} \int_\delta^\pi |\phi(t)| \left| \sum_{n=o}^{\infty} \pi_p(n) \sin nt \, x^n \right| dt$$

$$< \frac{H}{\delta} \int_o^\pi |\phi(t)| dt$$

$$= o(\log_{p+1} \frac{1}{1-x}) \quad \text{as } x \to 1- \quad \Xi$$

In connection with the preceding theorem we make the following

REMARK. *If*

$$\int_t^\delta \frac{|\phi(u)|\pi_p(\frac{1}{u})du}{u^2} = o(\log_{p+1}\frac{1}{t}) \qquad as \qquad t \to o+$$

then

$$\int_t^\delta \frac{|\phi(u)|\pi_{p+1}(\frac{1}{u})du}{u^2} = o(\log_{p+2}\frac{1}{t}) \qquad as \qquad t \to o+ .$$

PROOF. $$\int_t^\delta \frac{|\phi(u)|\pi_{p+1}(\frac{1}{u})du}{u^2} = \int_t^\delta \frac{|\phi(u)|\pi_p(\frac{1}{u})du}{u^2\log_{p+1}\frac{1}{u}}$$

$$= -\frac{1}{\log_{p+1}\frac{1}{u}} \int_u^\delta \frac{|\phi(v)|\pi_p(\frac{1}{v})dv}{v^2}\Bigg|_t^\delta$$

$$- \int_t^\delta \left(\int_u^\delta \frac{|\phi(v)|\pi_p(\frac{1}{v})dv}{v^2} \right) \frac{\pi_p(\frac{1}{u})du}{u^2(\log_{p+1}\frac{1}{u})^2}$$

$$= o(1) - \int_t^\delta o\left(\frac{\pi_{p+1}(\frac{1}{u})}{u^2}\right) du$$

$$= o(\log_{p+2} \frac{1}{t}) \quad .$$

Since for t and t' sufficiently close to o, we have

$$\int_t^\delta \varepsilon(u) \frac{\pi_p(\frac{1}{u})}{u^2(\log_{p+1} \frac{1}{u})} du, \qquad \text{where} \quad \varepsilon(u) \to o \quad \text{as} \quad u \to o$$

$$= \int_t^{t'} \varepsilon(u) \frac{\pi_{p+1}(\frac{1}{u})}{u^2} du + \int_{t'}^\delta \varepsilon(u) \frac{\pi_{p+1}(\frac{1}{u})}{u^2} du$$

$$= o(\log_{p+2} \frac{1}{t}) + O(1) \quad \Xi$$

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