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On The Exact Distribution Of Linear Combinations Of Order Statistics

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ON THE EXACT DISTRIBUTION OF LINEAR COMBINATIONS OF ORDER STATISTICS

by

Ernest Roy Mead
Department of Mathematics

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Faculty of Graduate Studies
The University of Western Ontario
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ABSTRACT

In the distribution theory of linear combinations of order statistics, the case for which the order statistics are from the uniform population should be very fundamental because of the property of the probability integral transform which takes the order statistics from any continuous distribution into the order statistics of the uniform distribution on the interval \((0, 1)\). The exact distribution of any one linear combination of \(n\) uniform order statistics with real coefficients is found and the limiting distribution is shown to be normal under certain conditions on the coefficients. Also the exact joint distribution of \(k(\leq n)\) linear combinations of \(n\) uniform order statistics with real coefficients is found and the limiting distribution is shown to be multivariate normal under certain conditions on the coefficients.

It is also illustrated how it is possible to obtain the exact distribution of linear combinations of exponential
order statistics from the distribution obtained for the uniform case.

The general results obtained for the uniform distribution are applied to find the exact distribution for some particular linear combinations.
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CHAPTER 1

INTRODUCTION

1.1 General Introduction and Statement of the Problem

There has been considerable interest shown in the recent literature with regard to distributions of linear combinations of functions of order statistics. Several investigations of the limiting distributions of linear functions of the order statistics have been made in which the aim was to find conditions on the functions and the underlying distribution such that the limiting distribution would be normal. For example, there have been papers by Govindarajulu (1966, 1968), Chernoff, Gastwirth, and Johns (1967), Moore (1968), and Stigler (1968). Also, the paper by Pyke (1965) contained an extensive survey on the closely related theory of the distributions of functions of spacings.

There have also been investigations into the exact distributions of linear combinations of order statistics. If the functions of the order statistics all reduce to the identity function, the linear combinations of the order
statistics can be considered equivalently in terms of linear combinations of the corresponding spacings. In either case, most of the investigations have been concerned with very particular linear combinations and often for certain sample sizes. For example, Mackay and Pearson (1933) considered the range of the normal distribution for samples of size three. Rider (1959) found the distribution of the quasi-ranges for a sample from the exponential population. The most extensive research has been for the case in which the order statistics or the corresponding spacings (or coverages) came from the uniform distribution on the interval \( (0,1) \). This is partially on account of the use of the probability integral transform which transforms the order statistics of any continuous distribution to those of the uniform distribution. Darling (1953) found the characteristic function corresponding to a linear combination of quite general functions of uniform spacings in terms of a contour integral, although he used the result mainly to examine the limiting distribution of certain proposed test statistics. Barton and David (1955) found the distribution of the sum of any \( k \leq n \) consecutive uniform order statistics from a sample of size \( n \). For the case of linear combinations of the uniform order statistics, or the corresponding coverages, with general real coefficients, several approaches
have been presented recently. Some of these can be found in Ali (1968) and in Dempster and Kleye (1968). The latter consider the problem from a geometric point of view with certain restrictions on the coefficients.

For the case in which more than one linear combination is found from the order statistics, there seems to have been very little attention given in the literature. Niven (1963) considered the joint distribution of the sample mean and sample range for the uniform distribution and the exponential distribution for samples of size three and four and the normal distribution for samples of size three. This thesis is a direct extension of the paper by Ali (1968) and considers the exact joint distribution of k linear combinations with real coefficients when the order statistics come from the uniform distribution on the interval (0,1).

The Statement of the Problem

Consider the n order statistics \( u_1, u_2, \ldots, u_n \) (0 < \( u_1 < u_2 < \ldots < u_n < 1 \)) corresponding to a random sample of size \( n \) from the uniform distribution on the interval (0,1). With these n order statistics, form \( k (1 \leq k \leq n) \) linear combinations

\[
\begin{align*}
y_1 &= c_{11} u_1 + c_{12} u_2 + \ldots + c_{1n} u_n \\
y_2 &= c_{21} u_1 + c_{22} u_2 + \ldots + c_{2n} u_n \\
&\vdots \\
y_k &= c_{k1} u_1 + c_{k2} u_2 + \ldots + c_{kn} u_n
\end{align*}
\]
where the coefficients \( c_{ij} \) \( (i = 1, 2, \ldots, k; j = 1, 2, \ldots, n) \)
are any real numbers. The problem is then to find the exact joint distribution of \( y_1, y_2, \ldots, y_k \).

In Chapter 2, a method of obtaining the exact distribution of any one linear combination of \( n \) uniform order statistics by inversion of the characteristic function is presented (cf. Ali (1968)). This method is generalized in Chapter 3 in order to find the joint distribution of any two linear combinations of uniform order statistics. Then in Chapter 4, the joint distribution for the general number \( k(\leq n) \) of linear combinations of uniform order statistics is found. The limiting distributions for the linear combinations of Chapters 2, 3 and 4 are examined in Chapter 5 and are found to be normal under certain conditions on the coefficients.

In Chapter 6 it is illustrated that it is possible to obtain the distributions of linear combinations of exponential order statistics using the distributions obtained for the uniform population. It is also pointed out that sometimes the distribution of the circular serial correlation coefficients can be found using the uniform results. Finally, in Chapter 7 several examples are given which illustrate how the results of Chapter 3 can be used to find the joint distribution of any two particular linear combinations of uniform order statistics.
1.2 Some Useful Notation and Previous Results

Throughout the following chapters the uniform distribution that is constantly referred to is defined on the unit interval \((0,1)\) so that its distribution is represented by the probability density function

\[
f(x) = \begin{cases} 
1 & 0 < x < 1 \\
0 & \text{elsewhere.}
\end{cases}
\]

The order statistics corresponding to a random sample of size \(n\) from this distribution will always be denoted by

\[0 < u_1 < u_2 < \ldots < u_n < 1.
\]

The following notation will be helpful and will be used throughout unless specifically noted otherwise:

\[
\sum \text{ represents } \sum_{j=0}^{n};
\]

\[\Pi \text{ represents } \Pi_{j=0}^{n};\]

\[
\sum' \text{ represents } \sum_{i_s=0}^{n}, i_s \neq i_j, j=1,2,\ldots,s-1;\]

\[
\sum' \text{ represents } \sum_{i_1}^{i_s} \sum_{i_2}^{i_s} \ldots \sum_{i_s}^{i_s};\]
\[ \sum_{p_0, p_1, \ldots, p_n} b_0 b_1 \ldots b_n \] represents the summation over all non-negative integral values of \( p_0, p_1, \ldots, p_n \) such that \( p_0 + p_1 + \ldots + p_n = p \), where \( b_0, b_1, \ldots, b_n \) are fixed.

Also, for real \( x \), the symbols \( \lambda(x) \) and \( |x| \) are defined to be

\[
\lambda(x) = \begin{cases} 
1 & x > 0 \\
0 & x < 0 
\end{cases}
\]

and

\[
|x| = \begin{cases} 
-x & x < 0 \\
x & x > 0 
\end{cases}
\]

respectively.

Partial fraction expansions will be used several times and the formula used will basically be

\[
\frac{1}{n} \prod_{i=0}^{n} (x - b_i) = \frac{1}{n} \prod_{j=0}^{n} \frac{1}{b_i - b_j} \cdot \frac{1}{x - b_i} \quad (1.2.1)
\]

where \( b_0, b_1, \ldots, b_n \) are distinct.
A considerable amount of divided difference theory will be used in this thesis. The symbol $D(g(x); b_0, b_1, \ldots, b_n)$ will be used to represent the $n$th divided difference of the function $g(x)$ with respect to the real arguments $b_0, b_1, \ldots, b_n$. Then the following facts will be useful (cf. Steffensen (1927)):

1.2.1 $D(g(x); b_0, b_1, \ldots, b_n)$ is symmetrical in all the $n+1$ arguments $b_0, b_1, \ldots, b_n$.

1.2.2 $D(g(x); b_0, b_1, \ldots, b_n)$

$$= \int_0^1 \int_0^{t_1} \ldots \int_0^{t_{n-2}} \int_0^{t_{n-1}} g^{(n)}(n) \left[ b_0 + \sum_{i=1}^{n} (b_i - b_{i-1}) t_i \right] dt_n \ldots dt_1$$

$$= \int_0^1 \int_0^{t_1} \ldots \int_0^{t_3} \int_0^{t_2} g^{(n)}(n) \left[ b_0 + \sum_{i=1}^{n} (b_i - b_{i-1}) t_{n+1} \right] dt_1 \ldots dt_n$$

where $g^{(n)}$ represents the $n$th derivative of $g$ with respect to the whole argument.

1.2.3 If none of the arguments $b_0, b_1, \ldots, b_n$ coincide,

$$D(g(x); b_0, b_1, \ldots, b_n) = \sum_{\substack{n \\ \Pi \ (b_j - b_s) \ \text{for} \ s=0, \ s \neq j}} g(b_j) \ .$$
1.2.4 \[ D(x^{n+p}; b_0, b_1, \ldots, b_n) = \begin{cases} 
0 & \text{if } -n \leq p < 0 \\
1 & \text{if } p = 0 \\
\left( \sum_{p=0}^{p} b_0 b_1 \ldots b_n \right) & \text{if } p > 0.
\end{cases} \]

1.2.5 If \( P(x) \) is a polynomial of degree less than \( n \),

\[ D(P(x); b_0, b_1, \ldots, b_n) = 0. \]

1.2.6 If \( P(x) \) is a polynomial of degree \( n \) and if the coefficient of \( x^n \) is 1, then

\[ D(P(x); b_0, b_1, \ldots, b_n) = 1. \]
CHAPTER 2

THE DISTRIBUTION OF ANY ONE LINEAR COMBINATION OF
ORDER STATISTICS FROM THE UNIFORM DISTRIBUTION

2.1 Introduction

In this chapter, the exact distribution of a linear combination of order statistics from the uniform distribution on the interval (0,1) is discussed. Some examples are also given for illustration.

2.2 The Distribution of a Linear Combination of Uniform Order Statistics

THEOREM 2.1 Let \( u_1 < u_2 < \ldots < u_n \) be the order statistics corresponding to a random sample of size \( n \) from the uniform distribution on the interval (0,1). Then for any set of real coefficients \( c_1, c_2, \ldots, c_n \), the distribution function \( F_n(z) \) of the linear combination \( Z_n = \sum_{i=1}^{n} c_i u_i \) is

\[
F_n(z) = D(g(x); z-a_0, z-a_1, \ldots, z-a_n) \tag{2.2.1}
\]
where \( g(u) = \left[ \frac{1}{2}(u + |u|) \right]^n \), \( a_j = \sum_{i=0}^{n} c_i \), \( (j = 1, \ldots, n) \)
and \( a_0 = 0 \).

This result was proved by Ali (1968) in two ways. The one method was by induction and the other by inversion of the characteristic function. It is the latter method that is generalized to the case of several linear combinations and will be presented in Section 2.3 for the sake of completeness. The distribution function presented in the above theorem can be written explicitly. If none of the arguments \( a_j (j = 0, 1, \ldots, n) \) coincide, it follows from 1.2.3 that

\[
F_n(z) = \sum_{s=0}^{n} g(z-a_j) / \prod_{s=0, s \neq j} ((z-a_j) - (z-a_s))
\]

\[
= \sum_{s=0}^{n} \left\{ \left( z-a_j \right) + \left| z-a_j \right| \right\} / \prod_{s=0, s \neq j} (a_s-a_j).
\] (2.2.2)

On the other hand, if \( m \) of the arguments coincide, say

\[ a_{i_1} = a_{i_2} = \ldots = a_{i_m} \]

where \( i_1, i_2, \ldots, i_m \) are distinct integers such that \( 0 \leq i_j \leq n \) \( (j = 1, 2, \ldots, m) \), then the divided difference
can be evaluated by letting (ε > 0)

\[ a_{i_2} = a_{i_1} + \varepsilon \]
\[ a_{i_3} = a_{i_1} + 2\varepsilon \]
\[ \vdots \]
\[ a_{i_m} = a_{i_1} + (m-1)\varepsilon \]

and considering the limit of (2.2.2) as ε → 0 just as in Steffensen (1927). Ali (1968) has given a convenient form of notation for this case. For convenience, it will in general be assumed that \( a_0, a_1, \ldots, a_n \) are distinct unless specified otherwise.

**COROLLARY 2.1** The density function \( f_n(z) \) corresponding to the distribution function (2.2.2) is

\[ f_n(z) = \frac{1}{n} \left[ \frac{1}{2} \left( (z - a_j) + |z - a_j| \right) \right]^{n-1} \prod_{s=0}^{n} (a_s - a_j). \]

2.3 **Inversion of the Characteristic Function**

As pointed out in the previous section, the Inversion Formula for characteristic functions will be used to derive the distribution of the linear combination of uniform order statistics \( Z_n = \sum_{i=1}^{n} c_i u_i \). First of all it is convenient to
redefine the problem in terms of the coverages of the uniform distribution. Let

\[ v_0 = 1 - u_n \]
\[ v_1 = u_1 \]
\[ v_2 = u_2 - u_1 \]
\[ \ldots \]
\[ v_n = u_n - u_{n-1} \]

Then \( v_0, v_1, \ldots, v_n \) are the coverages corresponding to the uniform order statistics \( u_1 < u_2 < \ldots < u_n \) and \( v_1, v_2, \ldots, v_n \) have a probability density function equal to \( \frac{1}{n} \) over the simplex defined by \( \sum_{i=1}^{n} v_i \leq 1, v_i > 0 \) \( (i = 1, 2, \ldots, n) \). The problem of finding the distribution of \( Z_n = \sum_{i=1}^{n} c_i u_i \) can now be considered to be the problem of finding the distribution of \( Z_n = \sum_{i=1}^{n} a_i v_i = \sum_{i=0}^{n} a_i v_i \) where \( a_0 = 0 \) and \( a_1, a_2, \ldots, a_n \) are defined in terms of \( c_1, c_2, \ldots, c_n \) as they were previously. The moments and characteristic function of \( Z_n \) are now found independently of any preceding derivations.

**Lemma 2.1** The \( p^{th} \) moment of the linear combination \( Z_n \) is

\[ E(Z_n^p) = \frac{\binom{n}{p} \sum a^p \cdot 1 \ldots a^n}{\binom{n+p}{p}} \]
PROOF Using the expression \( Z_n = \sum_{i=1}^{n} a_i v_i \),

\[
E(Z_n^p) = E[(a_1 v_1 + \ldots + a_n v_n)^p]
\]

\[
= E[\sum_p \left( \begin{array}{c} p \\ p \end{array} \right) \left( a_1 \ldots a_n v_1 \ldots v_n \right)^p]
\]

\[
= \sum_p \left( \begin{array}{c} p \\ p \end{array} \right) a_1 \ldots a_n \mathbb{E}(v_1 \ldots v_n)
\]

\[
= \sum_p \left( \begin{array}{c} p \\ p \end{array} \right) a_1 \ldots a_n \left( \begin{array}{c} p_1 \ldots p_n \\ p_1 \ldots p_n \end{array} \right) \left( a_1 \ldots a_n \right) \frac{p_n |n|}{|n+p|}
\]

by using known results for the Dirichlet distribution (cf. Wilks (1962), p. 177) with parameters all equal to one. Thus

\[
E(Z_n^p) = \frac{|n|}{|n+p|} \sum_p \left( \begin{array}{c} p \\ p \end{array} \right) a_1 \ldots a_n
\]

\[
= \frac{|n|}{|n+p|} \sum_p \left( \begin{array}{c} p_0 p_1 \ldots p_n \\ p_0 a_1 \ldots a_n \end{array} \right)
\]

where \((0)^0\) is considered to be 1 and \(a_0 = 0\). The lemma is proved.
COROLLARY 2.1.2 The mean of the linear combination $Z_n$ is

$$\bar{a}_n = \frac{\sum a_j}{(n+1)}.$$

COROLLARY 2.3.2 The variance of the linear combination $Z_n$ is

$$\sigma_n^2 = \frac{\sum (a_j - \bar{a}_n)^2}{(n+1)(n+2)}.$$

PROOF

From Lemma 2.1.2.1

$$E(Z_n^2) = \frac{\sum_{i=0}^{n-1} \sum_{j=0}^{n-1} a_i a_j}{(n+2)}$$

$$= 2\left(\sum a_j^2 + \sum_{i<j} a_i a_j\right)/(n+1)(n+2)$$

$$= \left\{\sum a_j^2 + (\sum a_j^2 + 2 \sum_{i<j} a_i a_j)\right\}/(n+1)(n+2)$$

$$= (\sum a_j^2 + (\sum a_j^2)^2)/(n+1)(n+2)$$

$$= (\sum a_j^2 + (n+1)\sum a_j^2)/(n+1)(n+2).$$

Thus

$$\sigma_n^2 = E(Z_n^2) - \bar{a}_n^2.$$
\[
\frac{1}{(n+1)(n+2)} \left( \sum_{j} a_j^2 + (n+1)^2 a_n^2 \right) - a_n^2
\]

\[
= \left( \sum_{j} a_j^2 - (n+1)a_n^2 \right) / (n+1)(n+2)
\]

\[
= \sum (a_j - \bar{a})^2 / (n+1)(n+2)
\]

and the corollary is proved.

**Lemma 2.2** The characteristic function of the linear combination \( Z_n \) can be represented by

\[
\phi(t) = \frac{1}{n} (it)^{-n} \sum_{j} e^{iat} \sum_{s=0}^{n} \frac{e^{i \left( a_j - a_s \right)}}{\prod_{s \neq j} (a_j - a_s)}
\]

if \( a_i \neq a_j \) (\( i \neq j \)). The other cases follow from

the continuity theorem for characteristic functions.
PROOF Since \( Z_n \) has a finite range distribution, the moments determine the characteristic function and distribution exactly. Hence, it follows from Lemma 2.1 that

\[
\phi(t) = \sum_{p=0}^{\infty} \frac{\mathbb{E}(Z_n^p)(it)^p}{p!}
\]

\[= \sum_{p=0}^{\infty} \left\{ \frac{\binom{n}{p}}{\binom{n+p}{p}} \prod_{i=1}^{p} a_i^{p_i} \right\} (it)^p
\]

\[= \sum_{p=0}^{\infty} \frac{\binom{n}{n+p}}{\binom{n+p}{p}} D(x^{n+p}; a_0, a_1, \ldots, a_n) (it)^p
\]

using 1.2.4,

\[= \sum_{k=n}^{\infty} \binom{n}{k} D(x^k; a_0, a_1, \ldots, a_n) (it)^{k-n}
\]

\[= \sum_{k=0}^{\infty} \binom{n}{k} D(x^k; a_0, a_1, \ldots, a_n) (it)^{k-n}
\]

because from 1.2.4, \( D(x^k; a_0, a_1, \ldots, a_n) = 0 \) for \( k < n \).

Then if none of the \( a_i \) (\( i = 0, 1, \ldots, n \)) are coincident,
\[ \phi(t) = \sum_{k=0}^{\infty} \binom{n}{k} (it)^{k-n} \cdot \sum_{\substack{a_j^i \in \mathbb{N} \setminus \{a_j\} \setminus \{a_s\} \setminus \{a_j - a_s\} \setminus \{a_j - a_s\} \setminus \{a_j - a_s\} \setminus \{a_j - a_s\} \setminus \{a_j - a_s\}}} \]

\[ = \binom{n}{(it)}^{-n} \sum_{\substack{a_j^i \in \mathbb{N} \setminus \{a_j\} \setminus \{a_s\} \setminus \{a_j - a_s\} \setminus \{a_j - a_s\} \setminus \{a_j - a_s\} \setminus \{a_j - a_s\} \setminus \{a_j - a_s\} \setminus \{a_j - a_s\}}} \frac{1}{a_j - a_s} \cdot \sum_{k=0}^{\infty} (ia_j t)^k / k^{k-n} \]

\[ = \binom{n}{(it)}^{-n} \sum_{\substack{a_j^i \in \mathbb{N} \setminus \{a_j\} \setminus \{a_s\} \setminus \{a_j - a_s\} \setminus \{a_j - a_s\} \setminus \{a_j - a_s\} \setminus \{a_j - a_s\} \setminus \{a_j - a_s\} \setminus \{a_j - a_s\} \setminus \{a_j - a_s\} \setminus \{a_j - a_s\} \setminus \{a_j - a_s\} \setminus \{a_j - a_s\} \setminus \{a_j - a_s\} \setminus \{a_j - a_s\} \setminus \{a_j - a_s\} \setminus \{a_j - a_s\} \setminus \{a_j - a_s\}}} e^{ia_j t} / (a_j - a_s) \cdot (2.3.1) \]

Hence the lemma is proved.

Notice that the assumption \( a_o = 0 \) is not very restrictive and can be dropped. Suppose that for \( a_o' \neq 0 \),

\[ Z_n = a_o' v_o + a_1' v_1 + \ldots + a_n' v_n \]

Then, using the definition of the coverages

\[ Z_n = a_o' (1 - v_1 - \ldots - v_n) + a_1' v_1 + \ldots + a_n' v_n \]

\[ = a_o' + (a_1' - a_o') v_1 + \ldots + (a_n' - a_o') v_n \]
\[ E(e^{it\zeta}) = e^{ia_0't} \cdot \frac{\ln (it)^n}{(it)^n} \sum_{s=0}^{n} \frac{e^{n}}{(a_j - a_o') (a_s - a_o')} \]

Thus the characteristic function has the same form whether \( a_o = 0 \) or not, and the moments and distribution will also hold for general \( a_o \).

The following lemma will be required in the proof of the theorem to follow.

**Lemma 2.3** If \( \Gamma \) is a contour consisting of the real axis from \( -\infty \) to \( -c (c > 0) \), the semicircle (above the real axis) of radius \( c \) and centre at the origin, and the real axis from \( c \) to \( \infty \), then for a real 'a' and integral \( n > 0 \),

\[
\int_{\Gamma} \frac{e^{iaz}}{z^n} dz = \begin{cases} 
-2\pi i^n a^{n-1}/|n-1| & \text{if } a < 0 \\
0 & \text{if } a > 0 
\end{cases}
\]
PROOF  Assume $a > 0$. Let $\Gamma_R$ consist of the portion of
$\Gamma$ between $-R$ and $R$ ($R$ large and $R > 0$) and let $\Omega_R$ consist of
the semicircle from $R$ to $-R$ above the real axis.

Since $e^{iaz}/z^n$ is analytic everywhere except at the point
$z = 0$ which lies outside $\Gamma_R \cup \Omega_R$, by Cauchy's Integral Theorem

$$\int_{\Gamma_R \cup \Omega_R} z^{-n} e^{iaz} \, dz = 0$$

and

$$\int_{\Gamma_R} z^{-n} e^{iaz} \, dz = - \int_{\Omega_R} z^{-n} e^{iaz} \, dz.$$

Let $z = Re^{i\theta}$, $0 \leq \theta \leq \pi$. Then

$$\left| \int_{\Omega_R} z^{-n} e^{iaz} \, dz \right| = \left| \int_{0}^{\pi} \frac{e^{iaRe^{i\theta}}}{(Re^{i\theta})^n} Rei\theta \, d\theta \right|$$

$$\leq \int_{0}^{\pi} \frac{|e^{iaR(cos\theta + i sin\theta)}|}{(Re^{i\theta})^n} Rd\theta$$

$$= \int_{0}^{\pi} \frac{e^{-aR sin\theta}}{R^{n-1}} \, d\theta.$$

But $a > 0$, $R > 0$ and $sin \theta > 0$ for $0 \leq \theta \leq \pi$. Hence, if $n > 1$
\[
\lim_{R \to \infty} \frac{1}{R^n} \int_0^\pi e^{-aR \sin \theta} \, d\theta = 0.
\]

If \( n = 1 \), note that Jordan's Inequality (cf. Copson (1935), p. 136) says that for \( 0 \leq \theta \leq \frac{\pi}{2} \),

\[
\frac{2\theta}{\pi} \leq \sin \theta \leq \theta.
\]

Thus

\[
\int_0^\pi e^{-aR \sin \theta} \, d\theta = 2 \int_0^{\pi/2} e^{-aR \sin \theta} \, d\theta
\]

\[
\leq 2 \int_0^{\pi/2} e^{-aR 2\theta/\pi} \, d\theta
\]

\[
= \left. \frac{-\pi}{aR} e^{-aR 2\theta/\pi} \right|_0^{\pi/2}
\]

\[
= \frac{\pi}{aR} (1 - e^{-aR})
\]

and

\[
\lim_{R \to \infty} \int_0^\pi e^{-aR \sin \theta} \, d\theta = 0.
\]
Hence
\[ \int_{\gamma_R} z^{-n} e^{iaz} \, dz = \lim_{R \to \infty} \int_{\gamma_R} z^{-n} e^{iaz} \, dz = 0. \]

Now assume that \( a < 0 \). Let \( \Omega'_R \) consist of the semicircle from \(-R\) to \( R \) below the real axis. Then by a well-known theorem, in conjunction with the Residue Theorem,

\[ \int_{-\gamma_R} \int_{\gamma_R} z^{-n} e^{iaz} \, dz = 2\pi i \cdot \frac{d^{n-1}}{dz^{n-1}} \left( \frac{e^{iaz}}{|n-1|} \right) \bigg|_{z=0} \]
\[ = 2\pi i \cdot \frac{n-1}{n} a^{n-1} e^{iaz} \bigg/ |n-1| \bigg|_{z=0} \]
\[ = 2\pi i a^{n-1} / |n-1| \]

and
\[ \int_{\gamma_R} z^{-n} e^{iaz} \, dz = \int_{\Omega'_R} z^{-n} e^{iaz} \, dz - 2\pi i a^{n-1} / |n-1| \]

Let \( z = Re^{i\theta}, \pi \leq \theta \leq 2\pi \). Then
\[ \left| \int_{\Omega'_R} z^{-n} e^{iaz} \, dz \right| = \left| \int_{\pi}^{2\pi} \frac{e^{iaRe^{i\theta}}}{(Re^{i\theta})^n} iRe^{i\theta} \, d\theta \right| \]
\[
\begin{align*}
\left. \int_{\pi}^{2\pi} \frac{e^{iaR(\cos \theta + i \sin \theta)}}{(Re^{i\theta})^n} \, R d\theta \right. \\
= \int_{\pi}^{2\pi} \frac{e^{-aR \sin \theta}}{R^{n-1}} \, d\theta
\end{align*}
\]

But \( a < 0, \, R > 0, \) and \( \sin \theta \leq 0 \) for \( \pi \leq \theta \leq 2\pi. \) So if \( n > 1, \)

\[
\lim_{R \to \infty} \frac{1}{R^{n-1}} \int_{\pi}^{2\pi} e^{-aR \sin \theta} \, d\theta = 0.
\]

If \( n = 1, \) consider that \( \sin \theta \leq -\frac{2\theta}{\pi} + 2 \) for \( \pi \leq \theta \leq \frac{3\pi}{2}. \) Thus

\[
\int_{\pi}^{2\pi} e^{-aR \sin \theta} \, d\theta \\
= 2 \int_{\pi}^{\frac{3\pi}{2}} e^{-aR \sin \theta} \, d\theta \\
\leq 2 \int_{\pi}^{\frac{3\pi}{2}} e^{-aR \left( -\frac{2\theta}{\pi} + 2 \right)} \, d\theta \\
= 2 e^{-2aR} \frac{2}{2aR} e^{2aR\theta/\pi} \left. \frac{3\pi}{2} \right|_{\pi} \\
= \frac{\pi}{aR} (e^{aR-1})
\]

and

\[
\lim_{R \to \infty} \int_{\pi}^{2\pi} e^{-aR \sin \theta} \, d\theta = 0.
\]

Hence

\[
\left. \int_{\Gamma_{R}} z^{-n} e^{iaz} \, dz \right| \Gamma_{R} = \lim_{R \to \infty} \int_{\Gamma_{R}} z^{-n} e^{iaz} \, dz \\
= -2\pi i n^{-1}/|n-1|.
\]

Thus the lemma is proved.
In the following theorem the variable \( t \) will be real or complex according to its use. Also remember that in the expression used for \( \phi(t) \), the coefficients of all the terms for which \( t \) appears in the denominator when the exponential factor is expanded were artificially introduced and are zero; that is, the apparent singularity at \( t = 0 \) is not really a singularity of the characteristic function.

**THEOREM 2.2** The density function of the linear combination \( Z_n \) is

\[
f_n(z) = n! \left[ \frac{1}{2} (z - a_j) + |z - a_j| \right]^{n-1} \frac{1}{\Pi_{s=0}^{n} (a_s - a_j)}.
\]

**PROOF** First of all, note that since \( Z_n \) has a finite range distribution, its characteristic function \( \phi(t) \) is analytic everywhere (cf. Lukacs (1960), p. 139).

From Lemma 2.2 and the Inversion Formula for characteristic functions,

\[
f_n(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itz} \ln(it)^{-n} \frac{e^{ia_j t}}{\Pi_{s=0}^{n} (a_j - a_s)} dt
\]
\[
\frac{1}{2\pi i} \int_{\Gamma} e^{i\tau z} |n(\tau)|^{-n} \sum_{s=0}^{n} \frac{e^{i\alpha_j \tau}}{n \Pi (a_j - a_s)} \, \tau
\]

using the fact that the integrand is analytic everywhere in the complex plane in order that the contour of integration could be changed to \( \Gamma \) without altering the value of the integral. \( \Gamma \) is the same contour of integration as that defined in Lemma 2.3. Thus

\[
f_n(z) = \frac{|n|}{2\pi i^n} \sum_{s=0}^{n} \frac{1}{n \Pi (a_j - a_s)} \int_{\Gamma} \frac{e^{i\tau (a_j - z)}}{\tau^n} \, \tau
\]

because with the new contour the integral exists for each term of the summation. So using Lemma 2.3 and defining the density arbitrarily over the set of measure zero for which \( a_j - z = 0 \) (\( j = 0, 1, \ldots, n \)),

\[
f_n(z) = -n \sum_{(a_j - z) \leq 0} \frac{(a_j - z)^{n-1}}{n \Pi (a_j - a_s)}
\]

\[
= n \left[ \frac{1}{2} (z - a_j) + |z - a_j| \right]^{n-1} / \left\{ \Pi (a_s - a_j) \right\}_{s \neq j}
\]

which is the required result.
2.4 Some Examples

The distribution of the sum of the $n$ order statistics, the distribution of the first order statistic, and the distribution of the $n^{th}$ order statistic will be found using the results of this chapter. These examples are just a few of many which are convenient because the results have been known for a long time and provide a good comparison.

First of all, consider the sum of the $n$ uniform order statistics $u_1 < u_2 < \ldots < u_n$. Thus

$$Z_n = \sum_{i=1}^{n} u_i$$

$$= \sum_{i=1}^{n} (n-i+1) v_i,$$

so $c_1 = c_2 = \ldots = c_n = 1$ and $a_o = 0$, $a_1 = n$, $a_2 = n-1$, $\ldots$, $a_n = 1$. Since for this case none of $a_o$, $a_1$, $\ldots$, $a_n$ coincide, this is an example in which formula (2.2.2) can be applied directly to obtain

$$F_n(z) = \sum_{j=1}^{n+1} \frac{1}{n+1} \left[ \frac{1}{2} \left( z - (n-j+1) \right) + \left| z - (n-j+1) \right| \right]^n$$

$$= \sum_{s=1}^{n} \left[ (n-s+1) - (n-j+1) \right]_{s \neq j}$$
where \( a_0 \) has been relabelled as \( a_{n+1} \) for convenience of notation. This expression can be rewritten as follows:

\[
\begin{align*}
\frac{f_n(z)}{n+1} &= \sum_{j=1}^{n+1} \left[ \frac{1}{2} \left( -(n-j+1) + |z-(n-j+1)| \right) \right]^n \prod_{s=1 \atop s \neq j}^{n+1} (j-s) \\
&= \frac{z^n + (z-1)^n + (z-2)^2 + \cdots + (z-j)^n}{n \prod_{n-l}^{n-2} (-1)^{n-l}(l-1)^j} \\
&= \frac{1}{n} \left( z^n - \binom{n}{1}(z-1)^n + \binom{n}{2}(z-2)^n - \binom{n}{3}(z-3)^n + \cdots \right)
\end{align*}
\]

where the summation will continue as long as \( z, z-1, z-2, \ldots \) are positive. This result can be compared with the one in Cramer (1946) p. 245 because \( Z_n = \sum_{i=1}^{n} u_i \) is also just the sum of \( n \) independent uniform random variables.

Now consider the distribution of the first order statistic so that

\[
Z_n = u_1 = v_1
\]

In this case, \( a_0 = 0, a_1 = 1, a_2 = a_3 = \ldots = a_n = 0 \) and the coefficients are all coincident except for \( a_1 = 1 \). So as suggested in Section 2.2, consider \( (\varepsilon > 0) \)
\[
a_0' = a_0 \\
a_1' = a_1 \\
a_2' = a_0 + \varepsilon \\
a_3' = a_0 + 2\varepsilon \\
\ldots \ldots \ldots \\
a_n' = a_0 + (n-1)\varepsilon.
\]

Then the distribution function of \( \sum a_j v_j \) is

\[
F_{n,\varepsilon}(z) = \frac{1}{2} \left( \left| z - a_j' \right| + \left| z - a_j' \right| \right) / \prod_{s=0, s \neq j}^{n} (a_s' - a_j').
\]

So

\[
F_{n,\varepsilon}(z) = 0 \quad \text{if } z \leq 0;
\]

\[
F_{n,\varepsilon}(z) = \frac{1}{2} \left( z - a_j' \right)^n / \prod_{j=0, j \neq 1}^{n} (a_s' - a_j')
\]

\[
= 1 - (z - a_j')^n / \prod_{s=0, s \neq 1}^{n} (a_s' - a_1') \quad \text{from 1.2.3 and 1.2.6},
\]

\[
= 1 - (1 - z)^n / (1 - \varepsilon)(1 - 2\varepsilon) \ldots (1 - (n-1)\varepsilon) \quad \text{if } 0 < z < 1
\]

where \( \varepsilon \) has been chosen such that \((n-1)\varepsilon < z\) for the particular value of \( z \); also, using 1.2.6.
\[
F_{n,\epsilon}(z) = \sum_{j=0}^{n} (z-a_j)^n/ \prod_{s=0, s\neq j}^{n} (a_s' - a_j')
= 1 \quad \text{for } z \geq 1.
\]

Then considering \(\lim_{\epsilon \to 0} F_{n,\epsilon}(z)\), it follows that

\[
F_n(z) = \begin{cases} 
0, & z \leq 0 \\
1-(1-z)^n, & 0 < z < 1 \\
1, & z \geq 1
\end{cases}
\]

which is just the result expected for the distribution of \(v_1\) (cf. Pyke (1965)).

Finally, assume that \(c_0 = c_1 = \ldots = c_{n-1} = 0\) and \(c_n = 1\).

So \(a_0 = 0, a_1 = a_2 = \ldots = a_n = 1\) and the distribution of the \(n\)th order statistic:

\[
Z_n = u_n = \sum_{j=1}^{n} v_j
\]

is being considered. Since \(a_1, \ldots, a_n\) are coincident, consider \((\epsilon > 0)\)
\[ a'_0 = a_0 \]
\[ a'_1 = a_1 \]
\[ a'_2 = a_1 + \epsilon \]
\[ a'_3 = a_1 + 2\epsilon \]
\[ \ldots \]
\[ a'_n = a_1 + (n-1)\epsilon. \]

Then the distribution of \( \sum_{j} a'_j v_j \) is

\[ F_{n,\epsilon}(z) = \left[ \frac{1}{2}(z-a'_j + |z-a'_j|) \right]^{n} / \prod_{s=0, s\neq j}^{n} (a'_s - a'_j), \]

and if \( z \leq 0 \)

\[ F_{n,\epsilon}(z) = 0; \]

if \( 0 < z < 1 \)

\[ F_{n,\epsilon}(z) = z^n / \prod_{s=1}^{n} (a'_s + (s-1)\epsilon) \]

\[ = z^n / (1+\epsilon)(1+2\epsilon)\ldots(1+(n-1)\epsilon); \]

if \( z \geq 1 \)

\[ F_{n,\epsilon}(z) = \sum (z-a'_j)^n / \prod_{s=0, s\neq j}^{n} (a'_s - a'_j) \]

\[ = 1 \]

follows from 1.2.6 where care has been taken in selecting \( \epsilon \) such that \( 1 + (n-1)\epsilon < z \) for the particular value of \( z \).
Then by considering the limit of $F_{n,\varepsilon}(z)$ as $\varepsilon \to 0$, it follows that

$$F_n(z) = \begin{cases} 0 & , \quad z \leq 0 \\ z^n & , \quad 0 < z < 1 \\ 1 & , \quad z \geq 1 \end{cases}$$

This is, of course, a well-known result.

In a similar manner the results of this chapter can be used to find the exact distribution of any linear combination of the uniform order statistics.
CHAPTER 3

THE JOINT DISTRIBUTION

OF MORE THAN ONE LINEAR COMBINATION

OF ORDER STATISTICS FROM THE UNIFORM DISTRIBUTION

3.1 Introduction

In this chapter and the one to follow, the joint distribution of more than one linear combination of order statistics from the uniform distribution will be considered.

If \( u_1 < u_2 < \ldots < u_n \) represent the order statistics corresponding to a random sample of size \( n \) from the uniform distribution on the interval \((0, 1)\), then \( y_1, y_2, \ldots, y_k \) will represent \( k \) linear combinations defined by \( (k \leq n) \)

\[
y_1 = c_{11} u_1 + c_{12} u_2 + \ldots + c_{1n} u_n \\
y_2 = c_{21} u_1 + c_{22} u_2 + \ldots + c_{2n} u_n \\
\hspace{2cm} \ldots \ldots \ldots \ldots \ldots \\
y_k = c_{k1} u_1 + c_{k2} u_2 + \ldots + c_{kn} u_n
\]
where the coefficients $c_{ij}$ $(i = 1, 2, \ldots, k; j = 1, 2, \ldots, n)$ are real numbers. For the purposes of these chapters it is convenient to represent $y_1, y_2, \ldots, y_k$ in terms of the coverages $v_0, v_1, \ldots, v_n$ as defined in Chapter 2, so that

$$
y_1 = a_{10}v_0 + a_{11}v_1 + \ldots + a_{1n}v_n
$$

$$
y_2 = a_{20}v_0 + a_{21}v_1 + \ldots + a_{2n}v_n
$$

$$
\vdots
$$

$$
y_k = a_{k0}v_0 + a_{k1}v_1 + \ldots + a_{kn}v_n
$$

where $a_{ij} = \sum_{s=j}^{n} c_{is}$, $a_{io} = 0$ $(i = 1, 2, \ldots, k; j = 1, 2, \ldots, n)$. As was pointed out previously, the fact that the coefficients of $v_0$ are zero does not really create a restriction on the problem and they could equally as well be considered as nonzero.

Although there have been a number of attempts at finding the distribution of a single linear combination of order statistics as pointed out in Chapter 1, there seem to have been very few attempts to find the joint distribution of more than one linear combination. Niven (1963) considered a few examples including the exact joint distribution of the sample mean and range for the uniform case, but only for sample sizes $n = 3$ and $n = 4$. In this
chapter the exact joint distribution of any two suitable combinations for any sample size is found. The method used will be an extension of the one appearing in Section 2.3. First of all, the characteristic function for the more general case of \( k \) linear combinations will be derived. It will also be of use in Chapter 4 where the distribution of the general case for \( k \) linear combinations is considered.

3.2 The Joint Characteristic Function

**Lemma 3.1** The joint characteristic function of \( y_1, y_2, \ldots, y_k \) can be represented by

\[
\phi(t_1, \ldots, t_k) = \frac{1}{n} \sum_{i_1}^{n} \frac{e^{i(a_{i1}t_1 + \ldots + a_{ki1}t_k)}}{\prod_{i_2}^{n} \left[ (a_{i1} - a_{i2})t_1 + \ldots + (a_{ki1} - a_{ki2})t_k \right]}.
\]

(3.2.1)

if

\[
\begin{vmatrix}
1 & a_{i1} & \ldots & a_{(k-1)i1} & a_{ki1} \\
1 & a_{i2} & \ldots & a_{(k-1)i2} & a_{ki2} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & a_{ik} & \ldots & a_{(k-1)i_k} & a_{ki_k} \\
1 & a_{i(k+1)} & \ldots & a_{(k-1)i(k+1)} & a_{ki(k+1)}
\end{vmatrix} \neq 0
\]

for all distinct \( i_1, i_2, \ldots, i_{k+1} \). The other cases follow from the continuity theorem.
PROOF  It follows from the definition of characteristic function that

$$\phi(t_1, \ldots, t_k) = \mathbb{E}\left\{ e^{i(t_1v_1 + t_2v_2 + \ldots + t_kv_k)} \right\} = \mathbb{E}\left\{ e^{i[t_1(a_{11}v_1 + a_{12}v_2 + \ldots + a_{1n}v_n) + + t_2(a_{21}v_1 + a_{22}v_2 + \ldots + a_{2n}v_n) + \ldots + t_k(a_{kl}v_1 + a_{k2}v_2 + \ldots + a_{kn}v_n)]} \right\} = \mathbb{E}\left\{ e^{i[v_1(a_{11}t_1 + a_{21}t_2 + \ldots + a_{kl}t_k) + + v_2(a_{12}t_1 + a_{22}t_2 + \ldots + a_{k2}t_k) + \ldots + v_n(a_{1n}t_1 + a_{2n}t_2 + \ldots + a_{kn}t_k)]} \right\} = \prod_{i=1}^{n} \frac{e^{i(a_{il}t_1 + a_{2i}t_2 + \ldots + a_{ki}t_k)}}{\Pi'[i(a_{il}t_1 + \ldots + a_{ki}t_k) - i(a_{i2}t_1 + \ldots + a_{ki}t_k)]}$$

from (2.3.1),

$$= \prod_{i=1}^{n} \frac{e^{i(a_{il}t_1 + a_{2i}t_2 + \ldots + a_{ki}t_k)}}{\Pi'[(a_{il} - a_{i2})t_1 + (a_{2i} - a_{i2})t_2 + \ldots + (a_{ki} - a_{ki})t_k]}$$

provided

$$a_{il}t_1 + a_{2i}t_2 + \ldots + a_{ki}t_k \neq a_{i2}t_1 + a_{2i}t_2 + \ldots + a_{ki}t_k$$

for $i_1, i_2 = 0, 1, \ldots, n$ ($i_1 \neq i_2$). Thus the lemma is proved.
In this chapter, the variables $t_1, t_2, \ldots, t_k$ will be used to represent real or complex variables depending on the context.

**Lemma 3.2** The characteristic function of $y_1, y_2, \ldots, y_k$ is analytic everywhere in the complex $k$-dimensional space.

**Proof** The characteristic function represented by the expression

$$
\phi(t) = \frac{1}{n} \sum_{i=1}^{n} \frac{e^{ia_i t}}{\prod_{l=1}^{i_2} (a_{i_l}^t - a_{i_2}^t) } , \quad a_{i_1} \neq a_{i_2} \text{ when } i_1 \neq i_2 ,
$$

where $a_0, a_1, \ldots, a_n$ are real constants, is analytic everywhere in the complex plane, because it is the characteristic function of a finite range distribution (cf. Lukacs (1960), p. 139). Let $c_2, c_3, \ldots, c_k$ represent $t_2, \ldots, t_k$ fixed at arbitrary values. Then

$$
\phi(t_1, c_2, \ldots, c_k) = \frac{1}{n} \sum_{i=1}^{n} \frac{e^{i(a_{i_1} t_1^* + a_{i_2} c_2^* + \ldots + a_{i_k} c_k^*)}}{\prod_{l=1}^{i_2} [(a_{i_1} - a_{i_2}) t_1 + (a_{i_2} - a_{i_1}) c_2 + \ldots + (a_{i_k} - a_{i_2}) c_k]}
$$
\[ \prod_{i=1}^{n} \frac{e^{i(a_{1i}t_1+a_{2i}c_2^*+\ldots+a_{ki}c_k^*)}}{\Pi_i^e [(a_{1i}t_1+a_{2i}c_2^*+\ldots+a_{ki}c_k^*)-(a_{1i}t_1+a_{2i}c_2^*+\ldots+a_{ki}c_k^*)]} \]

Thus the characteristic function with \( t_2, \ldots, t_k \) fixed and represented by \( \phi(t_1, c_2, \ldots, c_k) \) is analytic everywhere in the complex plane. Similarly \( \phi(c_1, t_2, c_3, \ldots, c_k), \ldots, \phi(c_1, \ldots, c_{k-1}, t_k) \) are each analytic everywhere in the complex plane. Thus \( \phi(t_1, \ldots, t_k) \) is analytic everywhere in the complex \( k \)-dimensional space (cf. Bochner and Martin (1948), p. 140) and the lemma is proved.

3.3 The Joint Density of Two Linear Combinations

The following integral will be required in this section.

**Lemma 3.3** If \( \Gamma \) is a contour consisting of the real axis from \(-\infty \) to \(-c (c > 0)\), the semicircle (above the real axis and the point \( b \)) of radius \( c \) and centre at the origin, and the real axis from \( c \) to \( \infty \), then for real 'a'

\[ \int_{\Gamma} \frac{e^{iaz}}{z+b} \, dz \begin{cases} = 0 & \text{if } a > 0 \\ = -2\pi ie^{-iab} & \text{if } a < 0 \end{cases} \]
PROOF Assume \( a > 0 \). Let \( \Gamma_R \) consist of the portion of \( \Gamma \) between \(-R\) and \( R \) (\( R \) large and \( R > 0 \)) and let \( \Omega_R \) be the semicircle from \( R \) to \(-R\) above the real axis. Since \( e^{iaz}/(z+b) \) is analytic within and on the contour \( \Gamma_R \cup \Omega_R \), by Cauchy's Integral Theorem

\[
\int_{\Gamma_R \cup \Omega_R} \frac{e^{iaz}}{z+b} \, dz = 0
\]

and then

\[
\int_{\Gamma_R} \frac{e^{iaz}}{z+b} \, dz = -\int_{\Omega_R} \frac{e^{iaz}}{z+b} \, dz.
\]

Let \( z = Re^{i\theta}, \ 0 \leq \theta \leq \pi \). Then

\[
\left| \int_{\Omega_R} \frac{e^{iaz}}{z+b} \, dz \right| = \left| \int_{0}^{\pi} \frac{e^{iaRe^{i\theta}}}{Re^{i\theta}+b} \cdot Re^{i\theta} \, d\theta \right|
\]

\[
\leq \int_{0}^{\pi} \left| \frac{e^{iaR(\cos \theta+i \sin \theta)}}{Re^{i\theta}+b} \right| \cdot Rd\theta
\]

\[
= \int_{0}^{\pi} \frac{e^{-aR\sin \theta}}{\left| Re^{i\theta}+b \right|} \cdot Rd\theta
\]

\[
\leq \frac{R}{|R-|b||} \int_{0}^{\pi} e^{-aR\sin \theta} \, d\theta
\]

\[
= \frac{2R}{|R-|b||} \int_{0}^{\pi} \frac{1}{2} e^{-aR\sin \theta} \, d\theta
\]
But using Jordan's Inequality, just as in Lemma 2.3,

\[
\lim_{R \to \infty} \int_0^\pi e^{-aR\sin \theta} d\theta = 0.
\]

Therefore

\[
\lim_{R \to \infty} \int_{-R}^R \frac{e^{iaz}}{z+b} \, dz = \int_{-R}^R \frac{e^{iaz}}{z+b} \, dz
\]

\[
= 0 \text{ for } a > 0.
\]

Now assume that \( a < 0 \). Let \( \Omega_R^1 \) consist of the semi-circle from \(-R\) to \( R \) below the real axis. Then the Residue Theorem gives

\[
\int_{\Gamma_R \cup \Omega_R^1} \frac{e^{iaz}}{z+b} \, dz = 2\pi i \left\{ \lim_{z \to b} (z+b) \frac{e^{iaz}}{z+b} \right\}
\]

\[
= 2\pi i e^{-iab}
\]

So

\[
\int_{\Gamma_R} \frac{e^{iaz}}{z+b} \, dz = \int_{\Omega_R^1} \frac{e^{iaz}}{z+b} \, dz - 2\pi i e^{-iab}.
\]

Let \( z = Re^{i\theta}, \pi < \theta < 2\pi \). Then
\[ \left| \int_{\Omega_R} \frac{e^{iaz}}{z+b} \, dz \right| = \left| \int_{\pi}^{2\pi} \frac{e^{iaRe^{i\theta}}}{Re^{i\theta} + b} Rie^{i\theta} \, d\theta \right| \]

\[ \leq \left| \int_{\pi}^{2\pi} \frac{e^{iaR(\cos\theta+i\sin\theta)}}{Re^{i\theta} + b} \, Rd\theta \right| \]

\[ \leq \frac{R}{|R-|b|} \left| \int_{\pi}^{2\pi} e^{-aR\sin\theta} \, d\theta \right| \]

\[ = \frac{2R}{|R-|b|} \left| \int_{\pi}^{3\pi} e^{-aR\sin\theta} \, d\theta \right| . \]

However, as in Lemma 2.3, for \( a < 0 \),

\[ \lim_{R \to \infty} \int_{\pi}^{3\pi} e^{-aR\sin\theta} \, d\theta = 0 . \]

Thus

\[ \lim_{R \to \infty} \int_{\Omega_R} \frac{e^{iaz}}{z+b} \, dz = \int_{\Gamma} \frac{e^{iaz}}{z+b} \, dz \]

\[ = -2\pi ie^{-iab} \quad \text{for} \quad a < 0, \]

and the lemma is proved.

For the following theorem it is necessary to keep in mind that the characteristic function of the two linear combinations \( y_1 \) and \( y_2 \) is analytic everywhere in the complex
two-dimensional space and thus the integrand of the
Inversion Formula is analytic everywhere in the complex
two-dimensional space. So, in considering the double
integral of the Inversion Formula, the contour of integration
for $t_1$ when $t_2$ is fixed can be distorted, if necessary, to
a shape similar to the $\Gamma$ defined in Lemma 2.3 or Lemma 3.3
without changing the value of the integral. Then after
performing the integration with respect to $t_1$, the
integrand is analytic everywhere in the complex plane of
$t_2$ (cf. Kaplan (1966), p. 171). Thus, if necessary, it
is now possible to distort the contour of integration for
$t_2$, as was done for $t_1$, again without changing the value
of the integral. In the theorem, the integrand is usually
written as a sum of terms which are not analytic every-
where, but which when added together and simplified form
the analytic integrand which is referred to constantly.
The symbol $\lambda(x)$ is zero or one as defined in Chapter 1.

**THEOREM 3.1** The joint density function of $y_1$ and $y_2$ can be
represented by
\[
\begin{align*}
&f(y_1, y_2) = \frac{n}{n-2} \sum' \lambda(y_1 - a_{i_1}) \lambda(y_2 - a_{i_2}) \\
&\quad \cdot \frac{\begin{vmatrix} 1 & a_{i_1} & a_{2i_1} \\ 1 & a_{i_2} & a_{2i_2} \\ 1 & y_1 & y_2 \end{vmatrix}}{\begin{vmatrix} 1 & a_{i_1} \\ 1 & a_{i_2} \end{vmatrix}} \\
&= \begin{vmatrix} 1 & a_{i_1} & a_{2i_1} \\ 1 & a_{i_2} & a_{2i_2} \\ 1 & a_{i_3} & a_{2i_3} \end{vmatrix}^{n-2} \\
&\quad \cdot \begin{vmatrix} 1 & a_{i_1} & a_{2i_1} \\ 1 & a_{i_2} & a_{2i_2} \\ 1 & a_{i_3} & a_{2i_3} \end{vmatrix}
\end{align*}
\]

(3.3.1)

if

\[
\begin{vmatrix} 1 & a_{i_1} & a_{2i_1} \\ 1 & a_{i_2} & a_{2i_2} \\ 1 & a_{i_3} & a_{2i_3} \end{vmatrix} \neq 0
\]

and

\[
\begin{vmatrix} 1 & a_{i_1} \\ 1 & a_{i_2} \end{vmatrix} \neq 0
\]

for all distinct \(i_1, i_2, i_3\).

**Proof** From Lemma 3.1 for \(k = 2\) and the Inversion Formula for characteristic functions, if
\[
\begin{vmatrix}
1 & a_{11} & a_{21} \\
1 & a_{12} & a_{22} \\
1 & a_{13} & a_{23}
\end{vmatrix} \neq 0
\]

for \( i_1, i_2, i_3 = 0, 1, \ldots, n \) (\( i_1, i_2, i_3 \) distinct),

\[
f(y_1, y_2) = \frac{n}{(2\pi)^{2n}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-it_1(y_1-a_{11})-it_2(y_2-a_{21})} \prod_{i_2} \left[ \frac{1}{(a_{11} - a_{12})t_1 + (a_{21} - a_{22})t_2} \right] \, dt_1 \, dt_2. \quad (3.3.2)
\]

Now assume that

\[
\begin{vmatrix}
1 & a_{11} \\
1 & a_{12}
\end{vmatrix} \neq 0
\]

for all distinct \( i_1, i_2 \), so that the partial fraction expansion given in (1.2.1) can be applied considering \( t_2 \) fixed and \( t_1 \) as the variable. Thus

\[
\left\{ \prod_{i_2} \left[ \frac{1}{(a_{11} - a_{12})t_1 + (a_{21} - a_{22})t_2} \right] \right\}^{-1}
\]
\[
\begin{align*}
&= \{ \prod_{i_2} (a_{i_1} - a_{i_2}) \left[ t_1 + \left( \frac{a_{2i_1} - a_{2i_2}}{a_{i_1} - a_{i_2}} \right) t_2 \right] \}^{-1} \\
&= \frac{1}{n} \prod_{i_4 = \frac{0}{1}} (a_{i_1} - a_{i_4}) \cdot \sum_{i_2} \prod_{i_3} \left[ \frac{a_{2i_1} - a_{2i_3}}{a_{i_1} - a_{i_3}} - \frac{a_{2i_1} - a_{2i_2}}{a_{i_1} - a_{i_2}} \right] t_2 t_1 + \left( \frac{a_{2i_1} - a_{2i_2}}{a_{i_1} - a_{i_2}} \right) t_2 \\
&= \frac{1}{n} \prod_{i_4 = \frac{0}{1}} (a_{i_1} - a_{i_4}) \cdot \sum_{i_2} \prod_{i_3} \left[ \begin{array}{ccc}
1 & a_{i_1} & a_{2i_1} \\
1 & a_{i_2} & a_{2i_2} \\
1 & a_{i_3} & a_{2i_3}
\end{array} \right] t_2 t_1 + \left( \frac{a_{2i_1} - a_{2i_2}}{a_{i_1} - a_{i_2}} \right) t_2 \\
&= \frac{(-1)^{n-2}}{n} \prod_{i_2} \left| \begin{array}{cc}
1 & a_{i_1} \\
1 & a_{i_2}
\end{array} \right| ^{n-2} \cdot \prod_{i_3} \left[ \begin{array}{ccc}
1 & a_{i_1} & a_{2i_1} \\
1 & a_{i_2} & a_{2i_2} \\
1 & a_{i_3} & a_{2i_3}
\end{array} \right] t_2 t_1 + \left( \frac{a_{2i_1} - a_{2i_2}}{a_{i_1} - a_{i_2}} \right) t_2
\end{align*}
\]
From Lemma 3.2, the integrand of (3.3.2) as a whole is analytic everywhere. Thus the contour of integration of \( t_1 \) for \( t_2 \) fixed can be distorted to a contour \( \Gamma \) similar to the one in Lemma 3.3, where all the singularities of the individual terms of (3.3.3) lie below the contour. But now the integral with respect to \( t_1 \) exists for each term and it follows from (3.3.2) and (3.3.3) that

\[
f(y_1, y_2) = \frac{|n|}{(2\pi)^2 i^n} \int_{-\infty}^{\infty} \left( \sum_{i_1 i_2} \left( -1 \right)^{n-2} \frac{\left| \begin{array}{cc} 1 & a_{i_1 i_1} \\ 1 & a_{i_1 i_2} \end{array} \right|}{\left| \begin{array}{cc} 1 & a_{i_1 i_1} \\ 1 & a_{i_1 i_2} \\ 1 & a_{i_1 i_3} \end{array} \right|} t_2 \right) e^{\frac{-i t_1 (y_1 - a_{i_1 1}) - i t_2 (y_2 - a_{i_2 1})}{\left( \frac{a_{i_1 i_1} - a_{i_2 i_1}}{a_{i_1 i_1} - a_{i_2 i_2}} \right) i_2}} dt_1 dt_2.
\]

\[
= \frac{|n|}{(2\pi)^2 i^n} \int_{-\infty}^{\infty} \left( \sum_{i_1 i_2} \left( -1 \right)^{n-2} \frac{\left| \begin{array}{cc} 1 & a_{i_1 i_1} \\ 1 & a_{i_1 i_2} \end{array} \right|}{\left| \begin{array}{cc} 1 & a_{i_1 i_1} \\ 1 & a_{i_1 i_2} \\ 1 & a_{i_1 i_3} \end{array} \right|} e^{\frac{-i t_2 (y_2 - a_{i_2 1})}{\left( \frac{a_{i_1 i_1} - a_{i_2 i_1}}{a_{i_1 i_1} - a_{i_2 i_2}} \right) i_2}} dt_1 dt_2.
\]
\[
\left. \int \left[ \frac{e^{-it_1(y_1-a_{li_1})}}{t_1+\frac{a_{2i_1}}{a_{li_1}a_{li_2}}} \right] \frac{dt_1}{dt_2} \right|^{n-2}_{1} \lambda(y_1-a_{li_1}) \left( -2\pi i \right) \epsilon \left( y_1-a_{li_1} \right) \left( -2\pi i \right) \epsilon \left( y_1-a_{li_1} \right)
\]

by using Lemma 3.3 and arbitrarily defining the integral over the set of measure zero for which \( y_1-a_{li_1} = 0 \) (\( i_1 = 0, 1, \ldots, n \)),

\[
= \frac{n}{(2\pi)^{2n}} \int_{-\infty}^{\infty} \left[ \sum_{i_1,i_2} \frac{(-1)^{n-1} \lambda(y_1-a_{li_1})}{|1+a_{li_1}a_{li_2}+\ldots+a_{li_1}a_{li_2}a_{li_3}|} \right] \frac{dt_1}{dt_2} \right|^{n-2}_{1} \lambda(y_1-a_{li_1}) \left( -2\pi i \right) \epsilon \left( y_1-a_{li_1} \right) \left( -2\pi i \right) \epsilon \left( y_1-a_{li_1} \right)
\]
\[
\begin{vmatrix}
1 & a_{11} & a_{21} \\
1 & a_{12} & a_{22} \\
1 & y_1 & y_2 \\
\end{vmatrix} dt_2
\]

because
\[
-\text{i}t_2 \left[ (y_2 - a_{21}) - (y_1 - a_{11}) \frac{a_{21} - a_{22}}{a_{11} - a_{12}} \right]
\]

\[
= -\text{i}t_2 \left[ \frac{(a_{11} - a_{12})(y_2 - a_{21}) - (y_1 - a_{11})(a_{21} - a_{22})}{(a_{11} - a_{12})} \right]
\]

\[
= -\text{i}t_2 \left[ \begin{vmatrix}
1 & a_{11} & a_{21} \\
1 & a_{12} & a_{22} \\
1 & y_1 & y_2 \\
\end{vmatrix} \right] \frac{1}{1} \frac{1}{a_{11}} 
\]

But the integrand as a whole still represents an analytic function with regard to \(t_2\) (cf. Kaplan (1966), p. 171). Thus the contour of integration can be distorted to the contour \(\Gamma\) as in Lemma 2.3 which avoids the point \(t_2 = 0\). Then the integral of each term in the summation exists and it follows that

\[
f(y_1, y_2) = \frac{|n|}{2\pi i^{n-1}} \sum' \frac{(-1)^{n-1}}{i_1 i_2} \begin{vmatrix}
1 & a_{11} & a_{21} \\
1 & a_{12} & a_{22} \\
1 & y_1 & y_2 \\
\end{vmatrix} \lambda(y_1 - a_{11})
\]
\[
\begin{align*}
\int_{t_2}^{t_{n-1}} e^{-it_2} \left| \begin{array}{cc}
1 & a_{i1} \\
n-1 & a_{i2} \\
1 & y_1 \\
n & y_2 \\
\end{array} \right| dt_2 \\
= (-1)^{n-1} \sum'_{i_1i_2} \left| \begin{array}{cc}
1 & a_{i1} \\
n-1 & a_{i2} \\
1 & y_1 \\
n & y_2 \\
\end{array} \right| \lambda(y_1-a_{i1}) \\
= \frac{n}{2\pi i n-1} \left| \begin{array}{cc}
1 & a_{i1} \\
n-1 & a_{i2} \\
1 & y_1 \\
n & y_2 \\
\end{array} \right| \left| \begin{array}{cc}
1 & a_{i1} \\
n-1 & a_{i2} \\
1 & y_1 \\
n & y_2 \\
\end{array} \right|^{-1} \\
= \frac{\lambda}{(-2\pi i)^{n-1}} \left| \begin{array}{cc}
1 & a_{i1} \\
n-1 & a_{i2} \\
1 & y_1 \\
n & y_2 \\
\end{array} \right| \left| \begin{array}{cc}
1 & a_{i1} \\
n-1 & a_{i2} \\
1 & y_1 \\
n & y_2 \\
\end{array} \right|^{-1} \\
= \frac{n}{n-2} \sum'_{i_1i_2} \lambda(y_1-a_{i1}) \left| \begin{array}{cc}
1 & a_{i1} \\
n-1 & a_{i2} \\
1 & y_1 \\
n & y_2 \\
\end{array} \right| \left| \begin{array}{cc}
1 & a_{i1} \\
n-1 & a_{i2} \\
1 & y_1 \\
n & y_2 \\
\end{array} \right|^{-1} \\
\end{align*}
\]

by using Lemma 2.3,
which is the result required by the theorem.

Note that the conditions of the preceding theorem are not necessary in order to use the method used in the theorem. They were only needed in order to write explicitly the density for a general class of combinations. For example, the assumption

\[
\begin{vmatrix}
1 & a_{11}\\
1 & a_{12}
\end{vmatrix} \neq 0
\]

for all distinct \(i_1, i_2\) was required in order that the partial fraction decomposition could be applied in the general manner in which it was. However, if one of these determinants had been zero and if it were known which one, the method could have been applied in exactly the same way. The only difference between the final answer and (3.3.1) would be that the terms preceded by

\[
\lambda \left[ \begin{vmatrix}
1 & a_{11} & a_{21} \\
1 & a_{12} & a_{22} \\
1 & y_1 & y_2
\end{vmatrix} \right]
\]

where

\[
\begin{vmatrix}
1 & a_{11} \\
1 & a_{12}
\end{vmatrix} = 0
\]
would not appear. Thus if $\lambda(\omega)$ is considered to be zero, the formula (3.3.1) can be applied in this case as well. Of course, if more than one of the determinants

$$\begin{vmatrix} 1 & a_{11} \\ 1 & a_{12} \end{vmatrix}$$

is zero, then at least one of the determinants

$$\begin{vmatrix} 1 & a_{11} & a_{21} \\ 1 & a_{12} & a_{22} \\ 1 & a_{13} & a_{23} \end{vmatrix}$$

is zero and the other assumption fails. However even in this case the formula (3.3.1) can be used to evaluate the distribution. Just as in the univariate case of one linear combination in Chapter 2 in which the result could be obtained for coincident arguments by using $\varepsilon(>0)$ appropriately and considering the limit as $\varepsilon \to 0$, so here can the coefficients be adjusted by $\varepsilon$'s in such a way that the determinants are not zero, and then the limit taken in order to arrive at the distribution. Illustrations for these cases along with several facts that simplify the calculations when confronted with actual combinations will be given in Chapter 7.
3.4 The Marginal Distributions

If the result obtained in Theorem 3.1 is the joint density function for the linear combinations \( y_1 \) and \( y_2 \), then the marginal densities of \( y_1 \) and \( y_2 \) should just be the densities for \( y_1 \) and \( y_2 \) obtained in Chapter 2. In this section it is shown that this is so. Without loss in generality, it is possible to consider just the marginal density of \( y_1 \). The following lemmas will be useful.

**Lemma 3.4** For a fixed value of \( i_1 \),

\[
\begin{vmatrix}
1 & a_{1i_1} & a_{2i_1} \\
1 & a_{1i_2} & a_{2i_2} \\
1 & y_1 & y_2 \\
\end{vmatrix}
\]

\[
\sum_{i_2} \begin{vmatrix}
1 & a_{1i_1} & a_{2i_1} \\
1 & a_{1i_2} & a_{2i_2} \\
1 & y_1 & y_2 \\
\end{vmatrix}
\]

\[
\begin{vmatrix}
1 & a_{1i_1} \\
1 & y_1 \\
\end{vmatrix}
\]

\[
\begin{vmatrix}
1 & a_{1i_1} & a_{2i_1} \\
1 & a_{1i_2} & a_{2i_2} \\
1 & a_{1i_3} & a_{2i_3} \\
\end{vmatrix}
\]

\[
\begin{vmatrix}
1 & a_{1i_1} \\
1 & y_1 \\
\end{vmatrix}
\]

\[
\begin{vmatrix}
1 & a_{1i_1} & a_{2i_1} \\
1 & a_{1i_2} & a_{2i_2} \\
1 & a_{1i_3} & a_{2i_3} \\
\end{vmatrix}
\]

where

\[
\begin{vmatrix}
1 & a_{1i_1} & a_{2i_1} \\
1 & a_{1i_2} & a_{2i_2} \\
1 & a_{1i_3} & a_{2i_3} \\
\end{vmatrix} \neq 0
\]

and

\[
\begin{vmatrix}
1 & a_{1i_1} \\
1 & a_{1i_2} \\
\end{vmatrix} \neq 0
\]

for \( i_1, i_2, i_3 \) distinct.
PROOF  

Using the relation

\[
\begin{vmatrix}
1 & a_{11} & a_{21} \\
1 & a_{12} & a_{22} \\
1 & a_{13} & a_{23}
\end{vmatrix}
= \begin{vmatrix}
1 & a_{11} \\
1 & a_{12} \\
1 & a_{13}
\end{vmatrix}
\begin{vmatrix}
a_{21} \\
a_{22} \\
a_{23}
\end{vmatrix}
- \begin{vmatrix}
1 & a_{11} \\
1 & a_{12} \\
1 & a_{13}
\end{vmatrix}
\begin{vmatrix}
a_{21} \\
a_{22} \\
a_{23}
\end{vmatrix}
\]

it follows that

\[
\sum_{i_2}^{n'} \begin{vmatrix}
1 & a_{11} \\
1 & a_{12} \\
1 & a_{13}
\end{vmatrix}
\begin{vmatrix}
a_{21} \\
a_{22} \\
a_{23}
\end{vmatrix}
\]

\[
= \sum_{i_2}^{n'} \begin{vmatrix}
1 & a_{11} \\
1 & a_{12} \\
1 & a_{13}
\end{vmatrix}
\begin{vmatrix}
a_{21} \\
a_{22} \\
a_{23}
\end{vmatrix}
- \begin{vmatrix}
1 & a_{11} \\
1 & a_{12} \\
1 & a_{13}
\end{vmatrix}
\begin{vmatrix}
a_{21} \\
a_{22} \\
a_{23}
\end{vmatrix}
\]

\[
= \frac{1}{\prod_{i_4=1}^{n} a_{1i_4}} \begin{vmatrix}
1 & a_{11} \\
1 & a_{12} \\
1 & a_{13}
\end{vmatrix}
\begin{vmatrix}
a_{21} \\
a_{22} \\
a_{23}
\end{vmatrix}
\]

\[
= \frac{1}{\prod_{i_4=1}^{n} a_{1i_4}} \sum_{i_2}^{n'} \begin{vmatrix}
1 & a_{21} \\
1 & a_{22} \\
1 & a_{23}
\end{vmatrix}
\begin{vmatrix}
a_{11} \\
a_{12} \\
a_{13}
\end{vmatrix}
\]

\[
= \frac{1}{\prod_{i_4=1}^{n} a_{1i_4}} \begin{vmatrix}
1 & a_{21} \\
1 & a_{22} \\
1 & a_{23}
\end{vmatrix}
\begin{vmatrix}
a_{11} \\
a_{12} \\
a_{13}
\end{vmatrix}
\]

\[
= 0, \quad \text{if } \begin{vmatrix} a_{11} & a_{21} \end{vmatrix} \neq 0,
\]

\[
\begin{vmatrix} a_{11} & a_{21} \end{vmatrix} \neq 0.
\]
\[
\begin{bmatrix}
1 & a_{i1} \\
1 & y_1
\end{bmatrix}^{n-1} = D((x-z)^{n-1}; b_0, b_1, \ldots, b_{i1-1}, b_{i1}, b_{i1+1}, \ldots, b_n)
\]

where
\[
\begin{bmatrix}
1 & a_{2i1} \\
1 & y_2
\end{bmatrix} ; \quad b_j = \begin{bmatrix}
1 & a_{2i1} \\
1 & b_j
\end{bmatrix}, j = 0, 1, \ldots, i_1-1, i_1, i_1+1, \ldots, n.
\]

However, from 1.2.6,

\[
D((x-z)^{n-1}; b_0, b_1, \ldots, b_{i1-1}, b_{i1}, b_{i1+1}, \ldots, b_n) = 1
\]

and relation (3.4.1) follows immediately.

**Lemma 3.5** For a fixed value of \(i_1\),

\[
\sum_{i_2}^{\Pi} \begin{bmatrix}
1 & a_{i1} & a_{i2} \\
1 & y_1 & y_2
\end{bmatrix}^{n-2} = 0 \quad (3.4.2)
\]

where
\[
\begin{bmatrix}
1 & a_{i1} & a_{i2} \\
1 & b_{i1} & b_{i2}
\end{bmatrix} \neq 0
\]
and

\[
\begin{vmatrix}
  1 & a_{li_1} \\
  1 & a_{li_2}
\end{vmatrix} \neq 0
\]

for \( i_1, i_2, i_3 \) distinct.

**Proof.** Notice the similarity of this result to the divided difference relation for distinct \( a_0, a_1, \ldots, a_n \) where from 1.2.3 and 1.2.5

\[
D(x^{n-1}; a_0, a_1, \ldots, a_n) = \sum_{s \in \mathbb{N}_n \setminus \{a_j - a_s\}_{s \neq i}}^{a_{n-1}} \frac{a_j}{n} (a_j - a_s)
\]

\[= 0.\]

Consider that

\[
\begin{vmatrix}
  1 & a_{li_1} & a_{2i_1} \\
  1 & a_{li_2} & a_{2i_2} \\
  1 & y_1 & y_2
\end{vmatrix}^{n-2}
\]

\[
\begin{vmatrix}
  1 & a_{li_1} & a_{2i_1} \\
  1 & a_{li_2} & a_{2i_2} \\
  1 & a_{li_3} & a_{2i_3}
\end{vmatrix}
\]

\[
= \sum_{i' \subset \{2, \ldots, n\} \setminus \{i_2, i_3\}} \begin{vmatrix}
  1 & a_{li_1} & a_{2i_1} \\
  1 & a_{li_2} & a_{2i_2} \\
  1 & y_1 & y_2
\end{vmatrix}^{n-2}
\]

\[
\begin{vmatrix}
  1 & a_{li_1} & a_{2i_1} \\
  1 & a_{li_2} & a_{2i_2} \\
  1 & a_{li_3} & a_{2i_3}
\end{vmatrix}
\]

\[
= \sum_{i' \subset \{2, \ldots, n\} \setminus \{i_2, i_3\}} \begin{vmatrix}
  1 & a_{li_1} & a_{2i_1} \\
  1 & a_{li_2} & a_{2i_2} \\
  1 & y_1 & y_2
\end{vmatrix}^{n-2}
\]

\[
\begin{vmatrix}
  1 & a_{li_1} & a_{2i_1} \\
  1 & a_{li_2} & a_{2i_2} \\
  1 & a_{li_3} & a_{2i_3}
\end{vmatrix}
\]

\[
= \sum_{i' \subset \{2, \ldots, n\} \setminus \{i_2, i_3\}} \begin{vmatrix}
  1 & a_{li_1} & a_{2i_1} \\
  1 & a_{li_2} & a_{2i_2} \\
  1 & y_1 & y_2
\end{vmatrix}^{n-2}
\]

\[
\begin{vmatrix}
  1 & a_{li_1} & a_{2i_1} \\
  1 & a_{li_2} & a_{2i_2} \\
  1 & a_{li_3} & a_{2i_3}
\end{vmatrix}
\]

\[
= \sum_{i' \subset \{2, \ldots, n\} \setminus \{i_2, i_3\}} \begin{vmatrix}
  1 & a_{li_1} & a_{2i_1} \\
  1 & a_{li_2} & a_{2i_2} \\
  1 & a_{li_3} & a_{2i_3}
\end{vmatrix}
\]
\[
\begin{align*}
&= \left| \begin{array}{c}
1 \ a_{i1} \\
1 \ y_1
\end{array} \right|_{1}^{n-2} \\
&= \left| \begin{array}{c}
1 \ a_{i1} \\
1 \ y_1
\end{array} \right| - \left| \begin{array}{c}
1 \ a_{i1} \\
1 \ y_1
\end{array} \right|
\end{align*}
\]

where \[ \left| \begin{array}{c}
1 \ a_{i1} \\
1 \ y_1
\end{array} \right| \neq 0 \]
where

\[
\begin{vmatrix}
1 & a_{2i_1} \\
1 & y_2 \\
1 & y_1 \\
\end{vmatrix}
\quad \text{and} \quad
\begin{vmatrix}
1 & a_{2i_j} \\
1 & a_{2i_{i_1}} \\
1 & a_{i_j} \\
\end{vmatrix}, j = 0, 1, \ldots, i_1 - 1, i_1 + 1, \ldots, n.
\]

Hence from 1.2.5 it follows that (3.4.2) holds because

\[
D((z-x)^{n-2}; b_0, b_1, \ldots, b_{i_1 - 1}, b_{i_1 + 1}, \ldots, b_n) = 0.
\]

**THEOREM 3.2**  The marginal density of \( y_1 \) is given by

\[
f_1(y_1) = n \sum_{i_1} \lambda(y_1 - a_{i_1})(y_1 - a_{i_1})^{n-1}/(a_{i_2} - a_{i_1})^{n-1} \quad (3.4.3)
\]

for \( a_{i_1} \neq a_{i_2} (i_1 \neq i_2) \).

**PROOF**  From Theorem 3.1 it follows that

\[
f_1(y_1) = \int_{-\infty}^{\infty} f(y_1, y_2) dy_2
\]
\[ \frac{n}{n-2} \sum_{i_1}^{\infty} \lambda(y_1-a_{i1}) \int_{-\infty}^{\gamma_2} \left( \begin{array}{c|c|c} 1 & a_{i1} & a_{2i1} \\ \hline 1 & a_{i1} & a_{2i1} \\ y_1 & y_1 & y_2 \\ \hline 1 & a_{i1} & a_{2i2} \\ \hline y_1 & y_1 & y_2 \\ \hline 1 & a_{i1} & a_{2i2} \\ \hline 1 & a_{i1} & a_{2i3} \\ \hline 1 & a_{i1} & a_{2i3} \\ \hline y_2 & y_2 & y_2 \\ \hline 1 & a_{i1} & a_{2i3} \\ \hline \end{array} \right)^{n-2} \right) dy_2. \]

But for each value of \( y_1 \) it is possible to find a value of \( y_2 \), say \( y_2^* \), such that for \( y_2 \geq y_2^* \)

\[ \lambda \begin{array}{c|c|c} 1 & a_{i1} & a_{2i1} \\ \hline 1 & a_{i1} & a_{2i1} \\ y_1 & y_1 & y_2 \\ \hline 1 & a_{i1} & a_{2i2} \\ \hline \end{array} = 1 \]

for all \( i_1, i_2 \) \( (i_1 \neq i_2) \). Thus, with the help of Lemma 3.5,

\[ f_1(y_1) \]

\[ \frac{n}{n-2} \sum_{i_1}^{\infty} \lambda(y_1-a_{i1}) \int_{-\infty}^{\gamma_2} \left( \begin{array}{c|c|c} 1 & a_{i1} & a_{2i1} \\ \hline 1 & a_{i1} & a_{2i1} \\ y_1 & y_1 & y_2 \\ \hline 1 & a_{i1} & a_{2i2} \\ \hline y_1 & y_1 & y_2 \\ \hline 1 & a_{i1} & a_{2i2} \\ \hline 1 & a_{i1} & a_{2i3} \\ \hline 1 & a_{i1} & a_{2i3} \\ \hline y_2 & y_2 & y_2 \\ \hline 1 & a_{i1} & a_{2i3} \\ \hline \end{array} \right)^{n-2} \right) dy_2. \]
\[
\frac{n}{n-2} \sum_{i_1}^{n} \lambda(y_{1_{i_1}} - a_{1_{i_1}}) \sum_{i_2}^{n} \begin{bmatrix} y_2^* \\ y_2^{**} \end{bmatrix} \begin{bmatrix} 1 & a_{1_{i_1}} & a_{2_{i_1}} \\ 1 & a_{1_{i_2}} & a_{2_{i_2}} \\ 1 & y_{1_{i_1}} & y_{2_{i_1}} \end{bmatrix} \begin{bmatrix} 1 \\ \Pi' \end{bmatrix} \begin{bmatrix} 1 \\ a_{1_{i_2}} & a_{2_{i_2}} \\ i_3 & a_{1_{i_3}} & a_{2_{i_3}} \end{bmatrix} \begin{bmatrix} 1 \\ a_{1_{i_2}} & a_{2_{i_2}} \end{bmatrix} \\
\begin{bmatrix} 1 & a_{2_{i_1}} \\ 1 & a_{2_{i_2}} \end{bmatrix} y_1 - \begin{bmatrix} a_{1_{i_2}} & a_{2_{i_2}} \\ a_{1_{i_2}} & a_{2_{i_2}} \end{bmatrix} \begin{bmatrix} 1 & a_{1_{i_1}} \\ 1 & a_{1_{i_2}} \end{bmatrix} \\
\frac{1}{n-2} \sum_{i_1}^{n-1} \lambda(y_{1_{i_1}} - a_{1_{i_1}}) \begin{bmatrix} n-2 \\ y_2^* \\ y_2^{**} \end{bmatrix} \begin{bmatrix} 1 & a_{1_{i_1}} & a_{2_{i_1}} \\ 1 & a_{1_{i_2}} & a_{2_{i_2}} \\ 1 & y_{1_{i_1}} & y_{2_{i_1}} \end{bmatrix} \begin{bmatrix} 1 & a_{1_{i_1}} & a_{2_{i_1}} \\ \Pi' & a_{1_{i_2}} & a_{2_{i_2}} \\ i_3 & a_{1_{i_3}} & a_{2_{i_3}} \end{bmatrix} \begin{bmatrix} 1 & a_{1_{i_2}} & a_{2_{i_2}} \\ 1 & a_{1_{i_2}} & a_{2_{i_2}} \\ 1 & y_{1_{i_1}} & y_{2_{i_1}} \end{bmatrix} \\
\begin{bmatrix} 1 & a_{1_{i_1}} & a_{2_{i_1}} \\ 1 & a_{1_{i_2}} & a_{2_{i_2}} \end{bmatrix} y_1 - \begin{bmatrix} a_{1_{i_2}} & a_{2_{i_2}} \\ a_{1_{i_2}} & a_{2_{i_2}} \end{bmatrix} \begin{bmatrix} 1 & a_{1_{i_1}} \\ 1 & a_{1_{i_2}} \end{bmatrix}
\]
Hence, it follows from Lemma 3.4 that

\[ f_1(y_1) = n \sum_{i_1} \lambda (y_1-a_{1i_1})(y_1-a_{1i_1})^{n-1}/\prod_{i_2} (a_{1i_2}-a_{1i_1}) \]

and the theorem is proved.

The result just obtained obviously agrees with the result obtained in Theorem 2.2. That is, if the only linear combination being considered is

\[ y_1 = a_{10}v_0 + a_{11}v_1 + \ldots + a_{1n}v_n, \]

then

\[ f_n(y_1) = n \prod_{s=0, s\neq j}^{n} \left[ \frac{1}{2} \left( (y_1-a_{1j}) + |y_1-a_{1j}| \right) \right]^{n-1} / \prod_{s=0}^{n} (a_{1s}-a_{1j}) \]

is the same as (3.4.3) since

\[ \lambda (y_1-a_{1j})(y_1-a_{1j})^{n-1} \]

is equivalent to the expression

\[ \left[ \frac{1}{2} \left( (y_1-a_{1j}) + |y_1-a_{1j}| \right) \right]^{n-1} \]

3.5 An Alternate Summation Rule

The summation rule appearing in Theorem 3.1 is just the one which happened to arise due to the method of proof used. However other summation rules are possible and the following
one will be useful later when considering linear combinations of exponential variables:

\[ f(y_1, y_2) = \frac{1}{n-2} \sum_{i_1, i_2} \text{sgn} \begin{vmatrix} a_{111} & a_{2i1} \\ a_{1i2} & a_{2i2} \end{vmatrix} \begin{vmatrix} l_{111} & a_{2i1} \\ l_{1i2} & a_{2i2} \end{vmatrix} \begin{vmatrix} y_1 & y_2 \\ \Pi'_{i_3} & a_{2i3} \end{vmatrix} \begin{vmatrix} 1 & a_{111} & a_{2i1} \\ 1 & a_{1i2} & a_{2i2} \\ 1 & a_{1i3} & a_{2i3} \end{vmatrix} \begin{vmatrix} n-2 \\ 1 \\ y_1 \\ y_2 \\ \Pi'_{i_3} & a_{2i3} \end{vmatrix} \] (3.5.1)

where the summation is over all the distinct pairs \((i_1, i_2), (i_1, i_2 = 1, \ldots, n; i_1 \neq i_2)\) for which \((y_1, y_2)\) falls in the triangle with vertices \((0, 0), (a_{111}, a_{2i1}), (a_{1i2}, a_{2i2})\).

It is now shown that (3.3.1) and (3.5.1) are equivalent. From (3.3.1)

\[ f(y_1, y_2) = \frac{n}{n-2} \sum_{i_1=0}^{n} \sum_{i_2=0}^{n} \lambda(y_1-a_{1i1}) \lambda \begin{vmatrix} 1 & a_{1i1} & a_{2i1} \\ 1 & a_{1i2} & a_{2i2} \\ 1 & y_1 & y_2 \end{vmatrix} \begin{vmatrix} 1 & a_{111} & a_{2i1} \\ 1 & a_{1i2} & a_{2i2} \\ 1 & y_1 & y_2 \end{vmatrix} \begin{vmatrix} \Pi'_{i_3} & a_{2i3} \\ 1 & a_{1i3} & a_{2i3} \end{vmatrix} \begin{vmatrix} n-2 \\ 1 \\ y_1 \\ y_2 \\ \Pi'_{i_3} & a_{2i3} \end{vmatrix} \begin{vmatrix} 1 & a_{111} & a_{2i1} \\ 1 & a_{1i2} & a_{2i2} \\ 1 & a_{1i3} & a_{2i3} \end{vmatrix} \]
\[
\frac{n}{n-2}
\]

\[
\sum_{i_1 \leq 0} \sum_{i_2 \geq 0} [\lambda(y_1 - a_{1l_1}) - \lambda(y_1 - a_{1l_2})] \lambda \left[ \begin{array}{c}
1 & a_{1l_1} & a_{2l_1} \\
1 & a_{1l_2} & a_{2l_2} \\
y_1 & y_2
\end{array} \right] \\
\left[ \begin{array}{c}
1 & a_{1l_1} & a_{2l_1} \\
1 & a_{1l_2} & a_{2l_2} \\
y_1 & y_2
\end{array} \right]^{n-2} \\
\left[ \begin{array}{c}
1 & a_{1l_1} & a_{2l_1} \\
1 & a_{1l_2} & a_{2l_2} \\
i_3 & a_{1l_1} & a_{2l_1}
\end{array} \right]
\]

\[
+ \frac{n}{n-2}
\]

\[
\sum_{i_1 = 1} \sum_{i_2 = 1} [\lambda(y_1 - a_{1l_1}) - \lambda(y_1 - a_{1l_2})] \lambda \left[ \begin{array}{c}
1 & a_{1l_1} & a_{2l_1} \\
1 & a_{1l_2} & a_{2l_2} \\
y_1 & y_2
\end{array} \right] \\
\left[ \begin{array}{c}
1 & a_{1l_1} & a_{2l_1} \\
1 & a_{1l_2} & a_{2l_2} \\
y_1 & y_2
\end{array} \right]^{n-2} \\
\left[ \begin{array}{c}
1 & a_{1l_1} & a_{2l_1} \\
1 & a_{1l_2} & a_{2l_2} \\
i_3 & a_{1l_1} & a_{2l_1}
\end{array} \right]
\]

\[
\sum_{i_1 = 1} \lambda(y_1 - a_{1l_1}) - \lambda(y_1 - a_{1l_0}) \lambda \left[ \begin{array}{c}
1 & a_{1l_1} & a_{2l_1} \\
1 & a_{1l_0} & a_{2l_0} \\
y_1 & y_2
\end{array} \right] \\
\left[ \begin{array}{c}
1 & a_{1l_1} & a_{2l_1} \\
1 & a_{1l_0} & a_{2l_0} \\
y_1 & y_2
\end{array} \right]^{n-2} \\
\left[ \begin{array}{c}
1 & a_{1l_1} & a_{2l_1} \\
i_3 & a_{1l_1} & a_{2l_1}
\end{array} \right]
\]
But it follows from Lemma 3.5 that for \( i_1 \neq 0 \)

\[
\begin{vmatrix}
1 & a_{i1} & a_{2i1} \\
1 & a_{i0} & a_{2i0} \\
1 & y_1 & y_2
\end{vmatrix}
= - \sum_{i_2=1}^{n} \left[ \begin{vmatrix}
1 & a_{i1} & a_{2i1} \\
1 & a_{i0} & a_{2i0} \\
1 & a_{i3} & a_{2i3}
\end{vmatrix}
\right]_{i_2 \neq i_1}
\]

and thus

\[
f(y_1, y_2) = \frac{n}{n-2}
\]
\[
= \frac{n}{n-2} \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \begin{bmatrix}
\lambda(y_1-a_{i1}) - \lambda(y_1-a_{i2})
\end{bmatrix}
\begin{bmatrix}
1 & a_{i1} & a_{2i1} \\
1 & a_{i2} & a_{2i2} \\
y_1 & y_1 & y_2
\end{bmatrix}
\begin{bmatrix}
1 & a_{i1} \\
1 & a_{i2}
\end{bmatrix}
\]

- \[
\sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \begin{bmatrix}
\lambda(y_1-a_{i1}) - \lambda(y_1-a_{i2})
\end{bmatrix}
\begin{bmatrix}
1 & a_{i1} & a_{2i1} \\
1 & a_{i2} & a_{2i2} \\
y_1 & y_1 & y_2
\end{bmatrix}
\begin{bmatrix}
1 & a_{i1} \\
1 & a_{i2}
\end{bmatrix}
\]

+ \[
\sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \begin{bmatrix}
\lambda(y_1-a_{i1}) - \lambda(y_1-a_{i2})
\end{bmatrix}
\begin{bmatrix}
1 & a_{i1} & a_{2i1} \\
1 & a_{i2} & a_{2i2} \\
y_1 & y_1 & y_2
\end{bmatrix}
\begin{bmatrix}
1 & a_{i1} \\
1 & a_{i2}
\end{bmatrix}
\]

= \frac{n}{n-2} \sum_{(i_1,i_2) \neq (i_1,i_2)} \text{sgn} \begin{vmatrix}
\begin{bmatrix}
a_{i1} & a_{2i1} \\
a_{i2} & a_{2i2}
\end{bmatrix}
\begin{bmatrix}
1 & a_{i1} & a_{2i1} \\
1 & a_{i2} & a_{2i2} \\
y_1 & y_1 & y_2
\end{bmatrix}
\begin{bmatrix}
1 & a_{i1} \\
1 & a_{i2}
\end{bmatrix}
\end{vmatrix}
\]
where it is easy to check that the last step holds by simply considering the possible values that the bracketed expressions can assume. These values happen to be \( \pm 1 \) (according as

\[
\begin{vmatrix}
  a_{11} & a_{21} \\
  a_{12} & a_{22}
\end{vmatrix}
\]

is \( \pm 1 \) within the triangle with vertices \((0,0), (a_{11},a_{21}), (a_{12},a_{22})\), and zero elsewhere. Thus (3.5.1) has been obtained from (3.3.1).
CHAPTER 4

THE DISTRIBUTION OF k LINEAR COMBINATIONS OF UNIFORM ORDER STATISTICS

4.1 Introduction

The result of the last chapter for the case of two linear combinations is generalized to the case of k linear combinations. That is, the joint density function for \( y_1, y_2, \ldots, y_k \) is found, where in terms of the coverages for the uniform distribution

\[
\begin{align*}
y_1 &= a_{10} v_0 + a_{11} v_1 + \cdots + a_{1n} v_n \\
y_2 &= a_{20} v_0 + a_{21} v_1 + \cdots + a_{2n} v_n \\
&\vdots \\
y_k &= a_{k0} v_0 + a_{k1} v_1 + \cdots + a_{kn} v_n
\end{align*}
\]

The derivation of this general density function requires the use of several large determinants involving the coefficients of the coverages as displayed above. Many
of these determinants are of similar form and can be represented by the last two elements on their main diagonal. Thus define the following symbols:

\[
\Delta(a(j-1)i_j, a_{st}) = \begin{vmatrix}
1 & a_{i1} & a_{2i1} & \cdots & a_{(j-1)i1} & a_{si1} \\
1 & a_{i2} & a_{2i2} & \cdots & a_{(j-1)i2} & a_{si2} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & a_{ij} & a_{2ij} & \cdots & a_{(j-1)ij} & a_{sij} \\
1 & a_{it} & a_{2it} & \cdots & a_{(j-1)it} & a_{sit}
\end{vmatrix}
\]

and

\[
\Delta(a(j-1)i_j, y_s) = \begin{vmatrix}
1 & a_{i1} & a_{2i1} & \cdots & a_{(j-1)i1} & a_{si1} \\
1 & a_{i2} & a_{2i2} & \cdots & a_{(j-1)i2} & a_{si2} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & a_{ij} & a_{2ij} & \cdots & a_{(j-1)ij} & a_{sij} \\
y_1 & y_2 & \cdots & y_{(j-1)} & y_s
\end{vmatrix}
\]

where \(j, s, \text{and} t\) are nonnegative integers such that \(s > j - 1\) and \(t > j\). If \(J = 0\) or if \(J = 1\), then \(\Delta(a(j-1)i_j, a_{si})\) represents 1 and \(\Delta(1, a_{si})\) respectively.

Also, an extension of the relation
\[
\begin{vmatrix}
1 & a_{11} & a_{12} & a_{13} \\
1 & a_{21} & a_{22} & a_{23} \\
1 & a_{31} & a_{32} & a_{33}
\end{vmatrix}
= \begin{vmatrix}
1 & a_{11} & a_{21} \\
1 & a_{12} & a_{22} \\
1 & a_{13} & a_{23}
\end{vmatrix}
- \begin{vmatrix}
1 & a_{11} & 0 \\
1 & a_{12} & 0 \\
1 & a_{13} & 0
\end{vmatrix}
\] (4.1.1)

is required. For the case of the \( j \)-dimensional determinant \( \Delta(a_{j-2}i_{(j-1)}, a_{j-1}i_j) \) it is

\[
\Delta(a_{j-2}i_{(j-1)}, a_{j-1}i_j) \cdot \Delta(a_{j-4}i_{(j-3)}, a_{j-3}i_{(j-2)})
\]

\[
= \Delta(a_{j-3}i_{(j-2)}, a_{j-2}i_{(j-1)}) \cdot \Delta(a_{j-3}i_{(j-2)}, a_{j-1}i_j)
\]

\[
- \Delta(a_{j-3}i_{(j-2)}, a_{j-2}i_{(j-1)}) \cdot \Delta(a_{j-3}i_{(j-2)}, a_{j-1}i_{(j-1)})
\] (4.1.2)

(cf. Hanus (1903), p.60). In addition, the symbol \( \lambda_m (m = 1, 2, \ldots, k) \) will be defined to be 1 or 0 according as the expression

\[
\begin{vmatrix}
1 & a_{11} & \cdots & a_{m1} \\
\vdots & \ddots & \cdots & \vdots \\
1 & a_{1m} & \cdots & a_{mm} \\
1 & y_1 & \cdots & y_m
\end{vmatrix}
\]

is \( \geq 0 \) or \( < 0 \) for \( m = 1, 2, \ldots, k \).
4.2 The Joint Density \( f(y_1, \ldots, y_K) \)

As in Chapter 3 for the case of two linear combinations, it is again necessary to note that the characteristic function of the \( K \) linear combinations in the general case is analytic everywhere and thus the integrand of the Inversion Formula is analytic everywhere in the complex \( k \)-dimensional space. Thus it is possible, if convenient, to distort the contour of integration for \( t_1 \) to a contour \( \Gamma \) similar to those defined in Lemmas 2.3 and 3.3 and the value of the integral will not be affected. Then since the integrand is still analytic everywhere in the complex \((k-1)\)-dimensional space of \( t_2, t_3, \ldots, t_k \) (cf. Kaplan (1966), p. 171) after integration with respect to \( t_1 \), the same type of contour distortion can be applied to the contour for \( t_2 \), and then for \( t_3 \) and so on until the \( K \)-fold integral has been evaluated. Again as in Chapter 3, the expressions appearing in the integrands will be sums of nonanalytic functions, which upon expansion and simplification would reveal the actual analytic expression that the integrand is. It is this analytic integrand that is being referred to in the proofs where the actual expression which appears has singularities in the individual terms. It is these singularities that are being avoided by distorting the contours of integration, in order that the integration can be performed on each term.
**Lemma 4.1** For a given $n$ and fixed $t_{k+1}$,

\[
\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \sum_{l_1}^{l} \sum_{l_2}^{l_2'} \prod_{j=1}^{k} (y_j - a_{ji_1}) t_j - i(y_{k+1} - a_{(k+1)}i_1) t_{k+1} \cdot e^{-i \sum_{j=1}^{k} (a_{ji_1} - a_{ji_2}) t_j + (a_{(k+1)}i_1 - a_{(k+1)}i_2) t_{k+1}} \cdot dt_1 \cdots dt_k
\]

\[
= (-1)^{n+k-2} (2\pi i)^k.
\]

\[
\sum_{i_1 \cdots i_{k+1}}^{\prime} \prod_{m=1}^{k} \lambda_m \begin{bmatrix}
\alpha_{i_1 i_1} & \cdots & \alpha_{i_1 i_k} & | & n-k-1 \\
\alpha_{i_1 (k+1)} & \cdots & \alpha_{i_1 (k+1)} & | & \alpha_{i_1 (k+1)} \\
\alpha_{i_2 (k+2)} & \cdots & \alpha_{i_2 (k+2)} & | & \alpha_{i_2 (k+2)} \\
\alpha_{i_3 (k+2)} & \cdots & \alpha_{i_3 (k+2)} & | & \alpha_{i_3 (k+2)} \\
\vdots & \cdots & \vdots & \ddots & \vdots \\
\alpha_{i_{k+1} (k+2)} & \cdots & \alpha_{i_{k+1} (k+2)} & | & \alpha_{i_{k+1} (k+2)} \\
\alpha_{i_{k+1} (k+2)} & \cdots & \alpha_{i_{k+1} (k+2)} & | & \alpha_{i_{k+1} (k+2)} \\
\alpha_{i_1 (k+1)} & \cdots & \alpha_{i_1 (k+1)} & | & \alpha_{i_1 (k+1)} \\
\alpha_{i_2 (k+1)} & \cdots & \alpha_{i_2 (k+1)} & | & \alpha_{i_2 (k+1)} \\
\alpha_{i_3 (k+1)} & \cdots & \alpha_{i_3 (k+1)} & | & \alpha_{i_3 (k+1)} \\
\vdots & \cdots & \vdots & \ddots & \vdots \\
\alpha_{i_{k+1} (k+1)} & \cdots & \alpha_{i_{k+1} (k+1)} & | & \alpha_{i_{k+1} (k+1)} \\
\end{bmatrix}
\]

\[
\cdot \begin{bmatrix}
1 & \alpha_{i_1 i_1} & \cdots & a_{k+1} & 1-n-k-1 \\
1 & \alpha_{i_1 (k+1)} & \cdots & a_{k+1} & 1-n-k-1 \\
1 & \alpha_{i_1 (k+2)} & \cdots & a_{k+1} & 1-n-k-1 \\
\vdots & \cdots & \vdots & \ddots & \vdots \\
1 & \alpha_{i_1 (k+2)} & \cdots & a_{k+1} & 1-n-k-1 \\
1 & \alpha_{i_2 i_j} & \cdots & a_{j+1} & 1-n-k-1 \\
1 & \alpha_{i_2 (k+1)} & \cdots & a_{j+1} & 1-n-k-1 \\
\vdots & \cdots & \vdots & \ddots & \vdots \\
1 & \alpha_{i_2 (k+2)} & \cdots & a_{j+1} & 1-n-k-1 \\
1 & \alpha_{i_3 i_j} & \cdots & a_{j+1} & 1-n-k-1 \\
1 & \alpha_{i_3 (k+1)} & \cdots & a_{j+1} & 1-n-k-1 \\
\vdots & \cdots & \vdots & \ddots & \vdots \\
1 & \alpha_{i_3 (k+2)} & \cdots & a_{j+1} & 1-n-k-1 \\
\vdots & \cdots & \vdots & \ddots & \vdots \\
1 & \alpha_{i_i i_j} & \cdots & a_{j+1} & 1-n-k-1 \\
1 & \alpha_{i_i (j+1)} & \cdots & a_{j+1} & 1-n-k-1 \\
\end{bmatrix}
\]

for all positive integers $k(<n)$, provided

\[
\begin{bmatrix}
1 & \alpha_{i_1 i_1} & \cdots & a_{j+j} & 1-n-k-1 \\
\vdots & \cdots & \cdots & \cdots & \cdots \\
1 & \alpha_{i_i i_1} & \cdots & a_{j+j} & 1-n-k-1 \\
\end{bmatrix} \neq 0 \quad \text{and} \quad \begin{bmatrix}
1 & \alpha_{i_1 i_1} & \cdots & a_{j+j} & 1-n-k-1 \\
\vdots & \cdots & \cdots & \cdots & \cdots \\
1 & \alpha_{i_i i_1} & \cdots & a_{j+j} & 1-n-k-1 \\
\end{bmatrix} \neq 0
\]

for $j = 1, 2, \ldots, k$ and for all distinct $i, i_2, \ldots, i_{j+1}$.
PROOF This lemma will be proved using a type of induction.

Assume that \( k = 1 \) and that \( t_2 \neq 0 \) is fixed. Then, since

\[
\begin{vmatrix}
1 & a_{i1} \\
1 & a_{i2}
\end{vmatrix} \neq 0
\]

and

\[
\begin{vmatrix}
1 & a_{i1} & a_{i2} \\
1 & a_{i2} & a_{i3} \\
1 & a_{i3} & a_{i2}
\end{vmatrix} \neq 0
\]

for all distinct \( i_1, i_2, i_3 \),

\[
\int_{-\infty}^{\infty} \sum_{i_1 i_2} \frac{e^{-it_1(y_1-a_{i1})-it_2(y_2-a_{i2})}}{\Pi'[(a_{i1} - a_{i2})t_1 + (a_{i2} - a_{i3})t_2]} \, dt_1
\]

\[
= \int_{-\infty}^{\infty} \sum_{i_1 i_2} \left( \prod_{i_3} \begin{vmatrix}
1 & a_{i1} & a_{i2} \\
1 & a_{i2} & a_{i3} \\
1 & a_{i3} & a_{i2}
\end{vmatrix} \right)^{-n-2} \prod_{i_1} \frac{e^{-it_1(y_1-a_{i1})-it_2(y_2-a_{i2})}}{t_1 + \left( \frac{a_{i2} - a_{i3}}{a_{i1} - a_{i2}} \right) t_2} \, dt_1
\]

using the partial fraction expansion (1.2.1) similar to its use in Theorem 3.1. However, the integrand is essentially the integrand for the Inversion Formula for
two linear combinations. Hence, using Lemma 3.2, it is seen that the integrand that is being represented by the above integrand is analytic everywhere in the complex plane of $t_1$. Thus the contour of integration can be changed to the contour $\Gamma$ described in Lemma 3.3, where $\Gamma$ lies above all the singularities associated with $t_1$ when the terms of the summation are considered individually. Then the integral of each term in the sum can be evaluated. And so, with the help of Lemma 3.3

$$
\sum_{i_1i_2}^\prime \frac{(a_{i_1} - a_{i_2})^{n-2}e^{-it_2(y_2-a_{i_1})}}{\prod_{i_3}^{\prime} \begin{vmatrix} 1 & a_{i_1} & a_{i_2} \\ a_{i_1} & 1 & a_{i_2} \\ a_{i_2} & a_{i_3} & 1 \end{vmatrix} t_2} \int_\Gamma \frac{-it_1(y_1-a_{i_1})}{t_1 + \frac{a_{i_2} - a_{i_3}}{a_{i_1} - a_{i_2}} t_2} dt_1
$$

$$
= \sum_{i_1i_2}^\prime \frac{(a_{i_1} - a_{i_2})^{n-2}e^{-it_2(y_2-a_{i_1})}}{\prod_{i_3}^{\prime} \begin{vmatrix} 1 & a_{i_1} & a_{i_2} \\ a_{i_1} & 1 & a_{i_2} \\ a_{i_2} & a_{i_3} & 1 \end{vmatrix} i(y_1-a_{i_1}) \frac{a_{i_2} - a_{i_3}}{a_{i_1} - a_{i_2}} t_2} \cdot \lambda(y_1-a_{i_1})(-2\pi i)e^{\lambda(y_1-a_{i_1})(-2\pi i)}
$$
using (4.1.1). If \( t_2 \) had been fixed at zero, the integration would have produced the same result as the above expression when simplified and evaluated at zero. Thus (4.2.1) holds for \( k = 1 \).

Assume that (4.2.1) is valid for \( k<n \) as it appears in the theorem and consider the case for \( k+1 \). That is, for \( t_{k+2} \) fixed

\[
\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left[ \prod_{i=1}^{k+1} \left( \sum_{j=1}^{k+1} (a_{ji_1} - a_{ji_2}) t_j + (a_{(k+2)i_1} - a_{(k+2)i_2}) t_{k+2} \right) \right] \, dt_1 \cdots dt_{k+1}
\]
\[
\int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \sum_{i_1}^{K} \exp \left[ -i \sum_{j=1}^{K} (y_j - a_{ji})^* t_j - i \left( y_{k+1}^* t_{k+1} + y_{k+2}^* t_{k+2} \right) \right] \right. \\
\left. (a(k+1)^* i_{k+1}^* + a(k+2)^* i_{k+2}^*) \right] \prod_{l=1}^{K} \left\{ \sum_{j=1}^{K} (a_{jl} - a_{jl}^*) t_j + \right. \\
\left. \{ (a(k+1)^* i_{k+1}^* + a(k+2)^* i_{k+2}^*) - (a(k+1)^* i_{k+1}^* + a(k+2)^* i_{k+2}^*) \} \right\} ]^{-1} \\
\cdot dt_1 \ldots dt_K \\
= \int_{\Gamma} \left[ (-1)^{n+K-2}(2\pi i)^K \right. \\
\cdot \sum_{i_1 \ldots i_{k+1}} \sum_{m=1}^{K} \left( \prod_{\lambda=1}^{m} i_{\lambda} \right) \frac{\Delta^{n-K-1}(a(k-1)i_{k}^*, a_{k(i(k+1))})}{\Delta \left( a_{k(i(k+1)), a(k+1)i_{(k+2)}^* t_{k+1} + a(k+2)i_{(k+2)}^* t_{k+2} } \right)} \\
\left. \cdot \frac{\Delta (a_{k(i(k+1))} y_{k+1}^* t_{k+1} + y_{k+2}^* t_{k+2})}{\Delta (a(k-1)i_{k}^* a_{k(i(k+1))})} \right] dt_{k+1}
\]

by using the assumption that (4.2.1) holds and letting

\( y_{k+1}^* t_{k+1} + y_{k+2}^* t_{k+2} \) replace \( y_{k+1}^* t_{k+1} \) and \( a(k+1)^* t_{k+1} + a(k+2)^* t_{k+2} \) replace \( a(k+1)^* t_{k+1} \) in (4.2.1). Also, as pointed out previously (cf. Kaplan (1966), p. 171), the integrand
considered as a whole is an analytic function everywhere in the complex plane of $t_{k+1}$, so the contour of integration with respect to $t_{k+1}$ could be changed to $\Gamma$ as defined in Lemma 3.3 with all the singularities associated with $t_{k+1}$ ($t_{k+2}$ being a fixed value) lying below $\Gamma$. Now it is possible to evaluate the integral for each term in the preceding expression. Thus, the expression can be written

$$(-1)^{n+k-2}(2\pi i)^k \sum_{i_1 \ldots i_{k+1}} \left[ \prod_{m=1}^{k+1} \frac{\Delta(a_{ki}(k+1), y_{k+2}, t_{k+2})}{\Delta(a_{ki}(k+1), a_{ki}(k+1))} \right] \cdot e^{i \frac{\Delta(a_{ki}(k+1), y_{k+1})}{\Delta(a_{ki}(k+1), a_{ki}(k+1))} t_{k+1}} \prod_{i_{k+2}} \left[ \Delta(a_{ki}(k+1), y_{k+1}) t_{k+1} + \Delta(a_{ki}(k+1), a_{ki}(k+2), a_{ki}(k+2)) t_{k+2} \right] \, dt_{k+1}$$

But

$$\prod_{i_{k+2}} \left[ \Delta(a_{ki}(k+1), a_{ki}(k+1)) t_{k+1} + \Delta(a_{ki}(k+1), a_{ki}(k+2)) t_{k+2} \right]$$
\[
\frac{1}{\Delta(a_{ki(k+1)}, a(k+1)i(k+4))} = \sum_{i_{k+4} = 0}^{n} \frac{\Delta(a_{ki(k+1)}, a(k+1)i(k+4))}{t_{k+1} + \frac{\Delta(a_{ki(k+1)}, a(k+1)i(k+4))}{\Delta(a_{ki(k+1), a(k+1)i(k+2))}} t_{k+2} \]
\]

\[
\frac{1}{\Delta(a_{ki(k+1)}, a(k+1)i(k+4))} = \sum_{i_{k+4} = 0}^{n} \frac{\Delta(a_{ki(k+1)}, a(k+1)i(k+4))}{t_{k+1} + \frac{\Delta(a_{ki(k+1), a(k+1)i(k+2))}}{\Delta(a_{ki(k+1), a(k+1)i(k+2))}} t_{k+2} \]
\]

by using (1.2.1),
\[ \sum'_{i_{k+2}} \frac{1}{\prod_{i_{k+3}} \left[ \frac{\Delta(a_{k+1}^i, a_{k+2}^i, a_{k+3}^i) \cdot \Delta(a_{k-1}^{i_k}, a_{k+1}^{i_k})}{\Delta(a_{k+1}^i, a_{k+1}^{i_{k+1}}(k+2)) \cdot \Delta(a_{k+1}^{i_k}, a_{k+1}^{i_{k+1}}(k+3))} t_{k+2} \right]} \]

\[ t_{k+1} + \frac{\Delta(a_{k+1}^{i_{k+1}} a_{k+2}^{i_{k+2}})}{\Delta(a_{k+1}^{i_{k+1}} a_{k+1}^{i_{k+2}})} t_{k+2} \]

by using (4.1.2),

\[ \sum' \frac{\Delta^{n-k-2}(a_{k+1}^{i_{k+1}} a_{k+1}^{i_{k+2}})}{\prod_{i_{k+3}} \left[ \frac{\Delta(a_{k+1}^{i_{k+2}}, a_{k+2}^{i_{k+2}}, a_{k+3}^{i_{k+3}}) \Delta(a_{k-1}^{i_k}, a_{k+1}^{i_k}(k+1)) t_{k+2} \right]} \]

\[ t_{k+1} + \frac{\Delta(a_{k+1}^{i_{k+1}} a_{k+2}^{i_{k+2}})}{\Delta(a_{k+1}^{i_{k+1}} a_{k+1}^{i_{k+2}})} t_{k+2} \]
and
\[
\begin{align*}
&-i \frac{\Delta(a_{k+1}, y_{k+1})}{\Delta(a_{k-1}i_k, a_{k+1})} t_{k+1} \\
&\cdot e^{\frac{\Delta(a_{k+1}, a_{k+2})}{\Delta(a_{k-1}i_k, a_{k+1})} t_{k+2}} \\
&\cdot \Delta(n_{k-1}(a_{k+1}i_k, a_{k+1})) e^{-2\pi i \lambda_{k+1} e}
\end{align*}
\]

from Lemma 3.3. Hence (4.2.2) becomes

\[
(-1)^{n+k-2}(2\pi i)^k \sum'_{i_1 \cdot \cdot \cdot i_{k+1}} m=1^k \lambda_m \\
\cdot \frac{\Delta(a_{k+1}, y_{k+2})}{\Delta(a_{k-1}i_k, a_{k+1})} t_{k+2} \\
\cdot \Delta(n_{k-1}(a_{k-1}i_k, a_{k+1})) e
\]

\[
\cdot \sum'_{i_{k+2}} \prod_{i_{k+3}} \left[ \Delta(a_{k+1}i_{k+2}, a_{k+2}i_{k+3}) \Delta(a_{k+1}i_k, a_{k+1}) t_{k+2} \right] \\
\cdot \Delta(n_{k-2}(a_{k+1}, a_{k+1}i_{k+2}) e^{-2\pi i \lambda_{k+1} e}
\]

\[
= (-1)^{n+k-1} (2\pi i)^{k+1} \cdot \\
\sum_{i_1 \ldots i_{k+2}} \left( \prod_{m=1}^{k+1} \lambda \right)^{n-k-2} (a_{ki(k+1)}^*, a(k+1)i(k+2)) \\
\prod_{i_{k+3}} \left[ \Delta(a(k+1)i(k+2), a(k+2)i(k+3)) \right] t_{k+2} \\
- i \frac{\Delta(a(k+1)i(k+2), Y_{k+2})}{\Delta(a_{ki(k+1)}^*, a(k+1)i(k+2))} t_{k+2}
\]

because

\[
\frac{\Delta(a_{ki(k+1)}^*, Y_{k+2})}{\Delta(a(k-1)i_k^*, a_{ki(k+1)})} \\
- \frac{\Delta(a_{ki(k+1)}^*, Y_{k+1})\Delta(a_{ki(k+1)}^*, a(k+2)i(k+2))}{\Delta(a(k-1)i_k^*, a_{ki(k+1)})\Delta(a_{ki(k+1)}^*, a(k+1)i(k+2))} \\
= \frac{\Delta(a(k-1)i_k^*, a_{ki(k+1)})\Delta(a(k+1)i(k+2), Y_{k+2})}{\Delta(a(k-1)i_k^*, a_{ki(k+1)})\Delta(a_{ki(k+1)}^*, a(k+1)i(k+2))}
\]

using (4.1.2),

\[
= \frac{\Delta(a(k+1)i(k+2), Y_{k+2})}{\Delta(a_{ki(k+1)}^*, a(k+1)i(k+2))}.
\]
But (4.2.3) is just (4.2.1) with \( k \) replaced by \( k+1(<n) \), so the induction type of step is complete and the lemma is proved.

**THEOREM 4.1** The joint density function of \( y_1, y_2, \ldots, y_k \) can be represented by

\[
\frac{n}{n-k} \sum' \left( \frac{\prod_{m=1}^{k} \lambda_m}{\lambda_{l_1} \cdots \lambda_{l_k}} \right)
\]

\[
\begin{vmatrix}
1 & a_{l_1} & \cdots & a_{k_l_1} & \cdots & a_{k_l_k} & \cdots & a_{k_l_k} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
1 & a_{l_1} & \cdots & a_{k_l_1} & \cdots & a_{k_l_k} & \cdots & a_{k_l_k} \\
\end{vmatrix}
\]

\[
(4.2.4)
\]

if

\[
\begin{vmatrix}
1 & a_{l_1} & \cdots & a_{j_l_1} \\
\cdots & \cdots & \cdots & \cdots \\
1 & a_{l_1(j+1)} & \cdots & a_{j_l(j+1)}
\end{vmatrix} \neq 0
\]

for all distinct \( i_1, \ldots, i_{(j+1)}, j = 1, 2, \ldots, k \).

**PROOF** From Lemma 3.1 and the Inversion Formula for characteristic functions.
\[ f(y_1, \ldots, y_k) = \frac{n}{(2\pi)^{\frac{k}{2}}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{i_1}^{k} \left[ \frac{\sum_{j=1}^{k} (y_j - a_{j1}) t_j}{\sum_{j=1}^{k} (a_{ji1} - a_{ji2}) t_j + (a_{k1} - a_{k2}) t_k} \right] dt_1 \cdots dt_k \]

\[ = \frac{n}{(2\pi)^{\frac{k}{2}}} \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \right. \]

\[ \left. \left[ \sum_{i_1}^{k-1} (y_j - a_{j1}) t_k - (y_k - a_{k1}) t_k \right] \prod_{i_2}^{k-1} \left[ \sum_{j=1}^{k} (a_{ji1} - a_{ji2}) t_j + (a_{k1} - a_{k2}) t_k \right] \right] dt_k \cdots dt_{k+1} dt_k \]

\[ = \frac{n}{(2\pi)^{\frac{k}{2}}} \cdot \int_{-\infty}^{\infty} \left[ (-1)^{n+k-3} (2\pi i)^{\frac{k-3}{2}} \sum_{i_1}^{k-2} \left[ \prod_{m=1}^{\frac{k-2}{2}} \lambda_m \right]^{\frac{n-k}{2}} \frac{(a_{k-2} - a_{k-1})^{i_k}}{\prod_{i_{k+1}}^{k} \Delta(a_{k-1} i_k, a_{k1} i_k)} \right. \]

\[ \times \frac{\Delta(a_{k-1} i_k, y_k)}{-i \Delta(a_{k-2} i_{k-1}, a_{k1} i_k)} t_k \]

\[ \times \frac{e^{-i \Delta(a_{k-2} i_{k-1}, a_{k1} i_k)} t_k}{t_k^{n-k+1}} \left. \int_{-\infty}^{\infty} \right] dt_k \]
where the last line is obtained using Lemma 4.1. But the expression represented by the integrand is analytic everywhere in the complex plane of $t_k$. Thus the contour of integration can be changed to the contour $\Gamma$ as described in Lemma 2.3 and the integration can then be performed term by term. So

$$f(y_1, \ldots, y_k) = \frac{|n(-1)^{n+k-3}}{2\pi i^{n-k+1}} \sum_{i_1 \ldots i_k} \left\{ \prod_{m=1}^{k-1} \frac{\lambda_m}{\pi} \right\} \frac{\Delta^{n-k}\left(a_{k-2}i_{k-1}, a_{k-1}i_k\right)}{\Delta(a_{k-1}i_k, a_{k}i_{k+1})} \cdot$$

$$\cdot \frac{\Delta(a_{k-1}i_k, y_k)}{-i \Delta(a_{k-2}i_{k-1}, a_{k-1}i_k)} \cdot \frac{t_k^{n-k+1}}{t_k^n} \cdot dt_k$$

$$= \frac{|n(-1)^{n+k-3}}{2\pi i^{n-k+1}} \sum_{i_1 \ldots i_k} \left\{ \prod_{m=1}^{k-1} \frac{\lambda_m}{\pi} \right\} \frac{\Delta^{n-k}(a_{k-2}i_{k-1}, a_{k-1}i_k)}{\Delta(a_{k-1}i_k, a_{k}i_{k+1})} \cdot$$

$$\cdot \frac{(-2\pi)i^{n-k+1} \lambda_k}{|n-k|} \cdot \left[ \frac{\Delta(a_{k-1}i_k, y_k)}{\Delta(a_{k-2}i_{k-1}, a_{k-1}i_k)} \right]^{n-k}$$

using Lemma 2.3, and arbitrarily defining the integral over the set of measure zero for which $\Delta(a_{k-1}i_k, y_k) = 0$, ...
\begin{equation}
\frac{n}{n-k} \prod_{i=1}^{n-k} \frac{1}{\lambda_m} \sum'_{i_k} \left( \prod_{m=1}^{k} \lambda_m \right) \frac{\Delta^{n-k}(a_{(k-1)i_k}, y_k)}{\prod'_{i_k} \Delta(a_{(k-1)i_k,i_k}^+a_{ki}(k+1))} \right)
\end{equation}

which is the density (4.2.4) required for the theorem.

Remarks similar to those made in Section 3.3 about the use of Theorem 3.1 when the conditions are not satisfied can be made about the use of Theorem 4.1 when all the conditions do not hold.

4.3 The Marginal Distributions

In this section, general results corresponding to those of Lemmas 3.4 and 3.5 from Section 3.4 are stated. With these lemmas it is possible to show that the marginal distribution of any subset of the linear combinations \( y_1, y_2, \ldots, y_k \) corresponds to the ordinary joint distribution of these linear combinations, just as it obviously must.

**Lemma 4.2** For fixed values of \( i_1, \ldots, i_{k-1} \),

\begin{equation}
\sum'_{i_k} \frac{\Delta^{n-k}(a_{(k-1)i_k}, y_k)}{\prod'_{i_k} \Delta(a_{(k-1)i_k,i_k}^+a_{ki}(k+1))} = 0 \quad (4.3.1)
\end{equation}

where
\[ \Delta(a_{(j-1)i}, a_{j(i+1)}) \neq 0 \]

for all distinct \( i_1, \ldots, i_{j+1}; j = k-2, k-1, k. \)

**PROOF** The proof is similar to the proof of Lemma 3.5, except that the general relation (4.1.2) must be used instead of (4.1.1).

**Lemma 4.3** For fixed values of \( i_1, \ldots, i_{k-1}, \)

\[
\sum_{i_k} \frac{\Delta^{n-k-1}(a_{(k-1)i_k}, y_k)}{\Delta(a_{(k-2)i(k-1)}, a_{(k-1)i_k}) \prod_{i_{k+1}} \Delta(a_{(k-1)i_k}, a_{ki(k+1)})}
\]

\[
= \frac{\Delta^{n-k+1}(a_{(k-2)i(k-1)}, y_{k-1})}{\prod_{i_k} \Delta(a_{(k-2)i(k-1)}, a_{(k-1)i_k})}
\]

(4.3.2)

where

\[ \Delta(a_{(j-1)i}, a_{j(i+1)}) \neq 0 \]

for all distinct \( i_1, \ldots, i_{j+1}; j = k-2, k-1, k. \)

**PROOF** The proof is similar to the proof of Lemma 3.4.
THEOREM 4.2  
The marginal density of $y_1, \ldots, y_{k-s+1}$ is given by

$$f_{12\ldots k-s+1}(y_1, \ldots, y_{k-s+1}) = f(y_1, \ldots, y_{k-s+1})$$

for $s = 2, \ldots, k$, where $f(y_1, \ldots, y_{k-s+1})$ is of the form (4.2.4).

PROOF  
In finding the marginal density by integrating out the variables $y_{k-s+2}, \ldots, y_k$ in the joint density $f(y_1, \ldots, y_k)$, the same argument as the one used in Theorem 3.2 can be used. The theorem follows upon performance of the integration. Note that there is no loss in generality in assuming the marginal distribution of the first $k-s+1$ variables is required. If they had not been the first $k-s+1$ variables, they could have been relabelled in such a way that they would be.
CHAPTER 5

THE LIMITING DISTRIBUTION OF LINEAR COMBINATIONS OF UNIFORM ORDER STATISTICS

5.1 Introduction

In this chapter, the linear combinations of the uniform order statistics will continue to be considered in the equivalent form of linear combinations of the corresponding coverages. Using this form, the linear combinations from Chapters 2, 3 and 4 are shown to have a limiting normal distribution under certain conditions on the coefficients.

The following notation will be helpful for this chapter:

\[ \sum_{\alpha \cdots \alpha}^{\Sigma} \]

\[ l \quad u \]
indicates summation over all possible sets \( \{\alpha_1, \ldots, \alpha_u\} \) of \( u \) distinct non-negative integers \( \leq n \);

\[
\sum d(\alpha_1 \ldots \alpha_u)
\]

is similar to the preceding expression except that if the terms are not completely symmetric the summation also includes all permutations of the sets that produce new distinct terms;

\[
\sum d \left( \begin{array}{c}
\alpha_1 \\
\alpha_1
\end{array} \right)
\]

represents the summation taken over the same values of \( \alpha_1, \ldots, \alpha_u \) as in the summation

\[
\sum d(\alpha_1 \ldots \alpha_u) \alpha_1 \ldots \alpha_u
\]

As usual, \( f(n) = O(g(n)) \) means that

\[
\lim_{n \to \infty} \left| \frac{f(n)}{g(n)} \right| \leq K
\]
for some constant $K > 0$; $f(n) = o(g(n))$ means that

$$\lim_{n \to \infty} f(n)/g(n) = 0;$$

$[x]$ represents the greatest integer $\leq x$.

The following material from the paper by Wald and Wolfowitz (1944) will be used in the proofs:

1) CONDITION $W$: Let $H_n = (h_1, \ldots, h_n)$ $(n = 1, 2, \ldots)$ be sequences of real numbers and let

$$\mu_r(H_n) = \frac{1}{n} \sum_{i=1}^{n} (h_i - \sum_{j=1}^{n} h_j)^r$$

for all integral values of $r$. The sequences $H_n(n = 1, 2, \ldots)$ are said to satisfy condition $W$ if, for all integral $r > 2$,

$$\frac{\mu_r(H_n)}{(\mu_2(H_n))^{\frac{r}{2}}} = o(1).$$

2) If in addition to obeying the condition $W$,

$$\mu_1(H_n) = o \text{ and } \mu_2(H_n) = 1,$$

then

$$\sum_{\alpha_1 \ldots \alpha_{k+r}} h_{\alpha_1} \ldots h_{\alpha_k}^{i_1} h_{\alpha_{k+1}}^{i_2} \ldots h_{\alpha_{k+r}}^{i_r} = o \left( n^{[\frac{k}{2}] + r} \right)$$

(5.1.1)

where $k, r, i_1, \ldots, i_r (i_j > 1, j = 1, 2, \ldots, r)$ are
fixed values independent of \( n \) and the summation is over all possible sets of \( u \) distinct positive integers \( \leq n \).

3) Also

\[
\sum_{a_1, \ldots, a_u} h_{a_1}^2 \ldots h_{a_u}^2 = n^u + o(n^u) . \tag{5.1.2}
\]

all different

5.2 The Limiting Distribution of One Linear Combination

From Corollaries 2.1.2 and 2.2.2 of Chapter 2, it is known that for the linear combination

\[
Z_n = a_o v_o + a_1 v_1 + \ldots + a_n v_n ,
\]

the mean and variance are

\[
\bar{a}_n = \{a_j/(n+1)
\]

and

\[
v(Z_n) = \sigma_n^2 = \{ (a_j - \bar{a}_n)^2 / (n+1)(n+2) \}.
\]

**Theorem 6.1** If \( a_o, a_1, \ldots, a_n \) satisfy condition \( W \) and if \( n \) is large enough that \( \sigma_n^2 \neq 0 \), then
\[ \lim_{n \to \infty} \Pr \left( \frac{Z_n - E(Z_n)}{(V(Z_n))^{\frac{1}{2}}} \leq z \right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{t^2}{2}} \, dt. \]

**Proof**

Consider the random variable defined by

\[ Z'_n = \frac{Z_n - E(Z_n)}{((n+2)V(Z_n))^{\frac{1}{2}}} \]

The characteristic function of this random variable is

\[ E(e^{itZ'_n}) = E \left( e^{it(Z_n-\bar{a}_n)/(n+2)^{\frac{1}{2}}\sigma_n} \right) \]

\[ = e^{-i\bar{a}_n t/(n+2)^{\frac{1}{2}}\sigma_n} \cdot \phi(t/(n+2)^{\frac{1}{2}}\sigma_n) \]

(\text{where } \phi(t) \text{ is the characteristic function of Lemma 2.2})

\[ = e^{-i\bar{a}_n t/(n+2)^{\frac{1}{2}}\sigma_n} \cdot \ln \left( \sum_{\substack{s=0 \atop s \neq j}}^{n} \frac{e^{i\left(\frac{1}{\sigma_n}((a_j-\bar{a}_n)/(n+2)^{\frac{1}{2}}\sigma_n)\right)t}}{\prod_{\substack{s=0 \atop s \neq j}}^{n} (a_j - a_s)} \right) \]

\[ = \frac{\ln}{(it)^n} \sum_{\substack{s=0 \atop s \neq j}}^{n} e^{i\left((a_j-a_n)/(n+2)^{\frac{1}{2}}\sigma_n\right)t} \frac{1}{\prod_{\substack{s=0 \atop s \neq j}}^{n} \left(\frac{1}{(a_j - a_s)/(n+2)^{\frac{1}{2}}\sigma_n} - \frac{1}{(a_s-\bar{a}_n)/(n+2)^{\frac{1}{2}}\sigma_n}\right)} \]

\[ = \frac{\ln}{(it)^n} \sum_{\substack{s=0 \atop s \neq j}}^{n} e^{ib_j t} \]

\[ \prod_{\substack{s=0 \atop s \neq j}}^{n} (b_j - b_s) \]
by letting \( b_j = (a_j - \bar{a}_n)/(n+2)^2a_n \), \( j = 0, 1, \ldots, n \). Note that \( \sum b_j = 0 \) and \( \sum b_j^2 = n+1 \) and that since the sequence \( a_0, a_1, \ldots, a_n \) satisfies the condition \( W \), so does the sequence \( b_0, b_1, \ldots, b_n \).

From Corollaries 2.1.1, 2.1.2, and Lemma 2.1

\[
E(Z'_n) = \frac{n}{n+p} \left( b_0^p b_1^p \cdots b_n^p \right) \quad (5.2.1)
\]

\[
E(Z'_n) = 0
\]

\[
V(Z'_n) = \sum b_j^2/(n+1)(n+2) = 1/(n+2).
\]

But (5.2.1) can be written in the form

\[
E(Z'_n) = \frac{n}{n+p} \sum' \left[ \sum (a_{i_1} \cdots a_{k+r}) b_{i_1} \cdots b_{i_r} \right] \quad (5.2.4)
\]

where \( \sum' \) indicates that the summation is taken over all \( r, k, i_1, \ldots, i_r \) such that \( k + i_1 + \ldots + i_r = p \) and \( i_j > 1 \) (\( j = 1, \ldots, r \)). Let \( N = n+1 \). Then it follows from (5.1.1) that

\[
E(Z'_n) = \frac{n}{n+p} \cdot O \left[ \max \left( \left[ \frac{k}{2} \right] + r \right) \right] N
\]

because \( \sum' \) is a finite summation and the inner summation is no more than a finite number times the summation in (5.1.1).
Also, it is easy to see that the maximum value of $\lfloor \frac{k}{2} \rfloor + r$ is $\lfloor \frac{p}{2} \rfloor$. Thus
\[
E(Z_n^p) = \ln \frac{ln(p)}{n+p} \cdot O\left( N^{\frac{p}{2}} \right).
\]

Now assume that $p$ is odd. Then
\[
E(Z_n^p)/(V(Z_n))^\frac{p}{2} = \ln \frac{ln(p)}{n+p} \cdot O\left( N^{\frac{p}{2}}/(n+2) \right)^\frac{-p}{2}
\]
\[
= O\left( N^{\frac{p}{2}}/(n+o(N))^\frac{1}{2} \right)^\frac{-p}{2}
\]
\[
= o\left( N^{-\frac{p}{2}} \right)
\]
\[
= o(1).
\]

So
\[
E\left\{ \left( \frac{Z_n - E(Z_n)}{V(Z_n)} \right)^p \right\} = o(1) \quad \text{if } p \text{ is odd.}
\]

Assume that $p$ is even and equal to $2s$. When $p$ is even not only is the maximum value of $\lfloor \frac{k}{2} \rfloor + r$ equal to $\lfloor \frac{p}{2} \rfloor$, but it is attained only if $k$ is even and $i_1 = i_2 = \ldots = i_r = 2$.

Thus the expression (5.2.4) can be written as
\[ E(Z^n) \]
\[
\frac{n^p}{n+p} \left\{ \sum_{j=0}^{s} \left( d(\alpha_1 \ldots \alpha_{2j+r}) \beta_1 \ldots \beta_2a_{2j}a_{2j+1} \ldots b_{2j+r}^{2j} \right) + O(N^{s-1}) \right\}
\]

where \( j + r = s \) and \( O(N^{s-1}) \) allows for all the terms for which the maximum value of \( \lfloor \frac{k}{2} \rfloor + r \) is not attained. Note that for \( j = s \) there are no squared factors, but \( 2s \) linear factors. Then put \( r \) in terms of \( j \) and \( s \) so that

\[ E(Z^n) \]
\[
\frac{n^p}{n+p} \sum_{j=0}^{s} \left( d(\alpha_1 \ldots \alpha_{2j+s}) \beta_1 \ldots \beta_2a_{2j}a_{2j+1} \ldots b_{2j+s}^{2j} + O(N^{s-1}) \right)
\]
\[
= \frac{n^p}{n+p} \sum_{j=0}^{s} \left( d(\alpha_1 \ldots \alpha_{s+j}) \beta_1 \ldots \beta_2a_{s-j}a_{s-j+1} \ldots b_{s+j} + O(N^{s-1}) \right)
\]
\[
= \frac{n^p}{n+p} \sum_{j=0}^{s} B(s-j, 2j) + O(N^{s-1}) \tag{5.2.5}
\]

where

\[ B(s-j, 2j) = \sum_{\alpha_1 \ldots \alpha_{s-j}} \beta_1 \ldots \beta_2a_{s-j}a_{s-j+1} \ldots b_{s+j} \]

for \( j = 0, 1, \ldots, s \), the \( s-j \) indicating the number of squared factors and \( 2j \) indicating the number of linear factors. It would now be advantageous to express
B(s-j, 2j) for j = 1, 2, ..., s in terms of B(s, o) because B(s, o) consists entirely of squared factors as in (5.1.2).

But for any j ≠ 0,

\[
\sum d(a_1 \ldots a_{j+s}) \frac{a_j}{a_{s-j}} a_{s-j+1} \ldots a_{j+s} =
\]

\[
\frac{\binom{N-s+j}{2j}}{\binom{N-s+j}{2j-1}(N-j-s+1)} \cdot
\]

\[
\sum d(b_1^2 \ldots b_{s-j}^2 a_{s-j+1} \ldots a_{j+s-1}^2) a_1 a_{s-j} a_{s-j+1} \ldots a_{j+s-1}
\]

\[
\cdot (\sum b_i - b_1 - \ldots - b_{j+s-1})
\]

where the summation on the right-hand side obviously contains all the terms that appear in the left-hand side, but the former also contains a certain number of permutations and thus requires the combinatorial type coefficient. However \( \sum b_i = 0 \) and

\[
\frac{\binom{N-s+j}{2j}}{\binom{N-s+j}{2j-1}(N-j-s+1)} = \frac{1}{2j}.
\]

Thus (j ≠ 0)
\[ B(s-j, 2j) = -\frac{1}{2j} \cdot \]

\[ d(b_1^2 \ldots b_j^2 b_{s-j}^2 \ldots b_{s-j+1}^2) a_1 a_{s-j} a_{s-j+1} \ldots a_{j+s-1} \]

\[ \cdot (b_1 + \ldots + b_{j+s-1}) \]

\[ = -\frac{1}{2j} d(b_1^2 \ldots b_j^2 \sum_{s-j} b_{s-j+1}^2 \ldots b_{s-j+1}^2) a_1 a_{s-j} a_{s-j+1} \ldots a_{j+s-1} \]

\[ \cdot (b_{s-j+1} + \ldots + b_{j+s-1}) + O(N^{s-1}) \]

where \( O(N^{s-1}) \) replaces all the terms that appear with cubed factors,

\[ = -\frac{1}{2j} \left( \frac{\sum_{s-j+1}^N (2j-1)}{\sum_{s-j+1}^N (2j-2) N-j-s+1} \right) \]

\[ d(a_1 \ldots a_{j+s-1}) a_1 a_{s-j+1} a_{s-j+2} \ldots a_{j+s-1} + O(N^{s-1}) \]
where again the combinatorial coefficient was needed because of the change in the rule of summation,

\[ = - \frac{\frac{s-j+1}{2j}}{2j} \cdot B(s-j+1, 2j-2) + O(N^{s-1}) \]

Thus it is easy to see that the number of linear factors can be reduced two at a time until all the factors are squares. That is,

\[ B(s-j, 2j) = \left\{ -\frac{\frac{s-j+1}{2j}}{2j} \right\} \cdot B(s-j+1, 2j-2) + O(N^{s-1}) \]

\[ = \left\{ -\frac{\frac{s-j+1}{2j}}{2j} \right\} \cdot \left\{ -\frac{\frac{s-j+2}{2j-2}}{2j} \right\} \cdot B(s-j+2, 2j-4) + O(N^{s-1}) \]

\[ = \left\{ -\frac{\frac{s-j+1}{2j}}{2j} \right\} \cdot \left\{ -\frac{\frac{s-j+2}{2j-2}}{2j} \right\} \cdot \ldots \cdot \left\{ -\frac{s-1}{4} \right\} \cdot \left\{ -\frac{s}{2} \right\} \cdot B(s, 0) + O(N^{s-1}) \]

\[ = \left\{ -\frac{1}{2} \right\} ^J \frac{\frac{s}{s-j}}{s-j} \cdot B(s, 0) + O(N^{s-1}) \]

\[ = \left\{ -\frac{1}{2} \right\} ^J \frac{s}{s-j} \cdot B(s, 0) + O(N^{s-1}). \] (5.2.6)

Therefore, from (5.2.5) and (5.2.6) it follows that

\[ E(Z_n^p) = \frac{n^p}{n+p} \sum_{j=0}^{s} \left\{ (-\frac{1}{2})^J \cdot B(s, 0) + O(N^{s-1}) \right\} + O(N^{-s-1}) \]

\[ = \frac{n^p}{n+p} \left( 1-\frac{1}{2} \right)^s \cdot B(s, 0) + O(N^{-s-1}) \]
\[
\begin{align*}
\frac{|n|p}{|n+p|} & \left(\frac{1}{2}\right)^s \left( d(a_1 \ldots a_s) b_{\alpha_1}^2 \ldots b_{\alpha_s}^2 \right) + o(N^{-s-1}) \\
= \frac{|p|}{2^s |s|} & \left( \sum^\| \right. \frac{b_{\alpha_1}^2 \ldots b_{\alpha_s}^2}{a_1 \ldots a_s} \left. \right) \frac{|n|}{|n+p|} + o(N^{-s-1}) \\
& \text{all different} \\
= \frac{|p|}{2^s |s|} & (N^s + o(N^s))(N^{-2s} + o(N^{-2s})) + o(N^{-s}) \\
\end{align*}
\]

by using (5.1.2),

\[
\frac{|2s|}{2^s |s|} \cdot N^{-s} + o(N^{-s}) .
\]

So

\[
E(\frac{Z_n^p}{(V(Z_n))^2})^2 = \frac{(\frac{|2s|}{2^s |s|})N^{-s} + o(N^{-s})}{(N^{-1} + o(N^{-1}))^s} = \frac{|2s|}{2^s |s|} + o(1) .
\]

That is,

\[
E_\cdot \left\{ \left( \frac{Z_n - E(Z_n)}{(V(Z_n))^2} \right)^p \right\} = \frac{|2s|}{2^s |s|} + o(1)
\]

if \( p \) is even and equal to \( 2s \).
It has been shown that the moments of the distribution of \( \frac{1}{n} (Z_n - E(Z_n)) / (V(Z_n))^{1/2} \) converge to the moments of the standard normal distribution if the coefficients satisfy condition \( W \). These moments determine the normal distribution uniquely (cf. Kendall & Stuart (1952), p. 115) and thus the limiting distribution of \( \frac{1}{n} (Z_n - E(Z_n)) / (V(Z_n))^{1/2} \) is normal with mean zero and variance one and the theorem is proved.

5.3 The Multivariate Limiting Distribution

In this section, the limiting distribution corresponding to the exact distribution of Chapter 4 is found. Here, the linear combinations \( y_1, y_2, \ldots, y_k \) will be denoted by \( y_{1n}, y_{2n}, \ldots, y_{kn} \) where

\[
\begin{aligned}
y_{1n} &= a_{10}v_0 + a_{11}v_1 + \ldots + a_{1n}v_n \\
y_{2n} &= a_{20}v_0 + a_{21}v_1 + \ldots + a_{2n}v_n \\
&\vdots \\
y_{kn} &= a_{k0}v_0 + a_{k1}v_1 + \ldots + a_{kn}v_n
\end{aligned}
\]

As in the previous section it is not necessary that \( a_{10}, \ldots, a_{k0} \) are zero, nor is it necessary that the coefficients do not change with varying \( n \). That is, the
coefficients $a_{11}, \ldots, a_{ik}$ could be denoted by $a_{1i}, \ldots, a_{i2}$ and the same results follow if the latter set of coefficients satisfies the same conditions which were required for the former. Also in this section, the mean $E(y_{in})$, $(i = 1, 2, \ldots, k)$ will be represented by $\bar{a}_{in}$.

**Lemma 5.1** Consider the variables

$$y_{sn} = \sum a_{sj}v_j \quad \text{and} \quad y_{tn} = \sum a_{tj}v_j$$

for $s, t = 1, 2, \ldots, k (s \neq t)$. Then the correlation between $y_{sn}$ and $y_{tn}$ is given by

$$\rho_{stn} = \frac{\left(\sum (a_{sj} - \bar{a}_{sn})(a_{tj} - \bar{a}_{tn}) \right)}{\left(\sum (a_{sj} - \bar{a}_{sn})^2 \right)^{1/2}} \left(\sum (a_{tj} - \bar{a}_{tn})^2 \right)^{1/2}$$

*(5.3.1)*

**Proof**

$$\rho_{stn} = \frac{E(y_{sn} - E(y_{sn}))(y_{tn} - E(y_{tn}))}{\left(\text{Var}(y_{sn}) \cdot \text{Var}(y_{tn})\right)^{1/2}}$$

*(5.3.2)*

But

$$E(y_{sn} - E(y_{sn}))(y_{tn} - E(y_{tn}))$$

$$= E(y_{sn}y_{tn}) - E(y_{sn})E(y_{tn})$$
\[ E \left\{ \left( \sum_{s,j} a_s a_j v_j \right) \left( \sum_{t,j} a_t a_j v_j \right) \right\} - \bar{a}_{sn} \bar{a}_{tn} \]

\[ = E \left( \sum_{s,j} a_s a_t v_j \sum_{i \neq j} a_s a_t v_i \right) - \bar{a}_{sn} \bar{a}_{tn} \]

\[ = \frac{2}{(n+1)(n+2)} \left( \sum_{s,j} a_s a_t v_j + \frac{1}{(n+1)(n+2)} \sum_{i \neq j} a_s a_t v_i \right) - \bar{a}_{sn} \bar{a}_{tn} \]

by considering the Dirichlet distribution as in Lemma 2.1,

\[ = \frac{1}{(n+1)(n+2)} \left\{ \sum_{s,j} a_s a_t v_j + (\sum_{s,j} a_s a_t v_j) \right\} - \bar{a}_{sn} \bar{a}_{tn} \]

\[ = \frac{1}{(n+1)(n+2)} \left\{ \sum_{s,j} a_s a_t v_j + \bar{a}_{sn} \bar{a}_{tn} \left( (n+1)^2 - (n+1)(n+2) \right) \right\} \]

\[ = \frac{1}{(n+1)(n+2)} \left\{ \sum_{s,j} a_s a_t v_j - (n+1) \bar{a}_{sn} \bar{a}_{tn} \right\} \]

\[ = \frac{1}{(n+1)(n+2)} \sum (a_s - \bar{a}_{sn}) (a_t - \bar{a}_{tn}) \]

Thus (5.3.1) follows immediately from (5.3.2) and
Corollary 2.1.2.

**THEOREM 5.2** Consider the linear combinations

\[ y_{in} = \sum_{i,j} a_{ij} v_j \quad (i = 1, 2, \ldots, k) \]
If for every $i = 1, 2, \ldots, k$, the sequence $a_{i0}, a_{i1}, \ldots, a_{in}$ of real numbers satisfies the condition $W$, then

$$\begin{bmatrix}
\frac{y_{1n} - E(y_{1n})}{\sqrt{\text{Var}(y_{1n})}}, & \ldots, & \frac{y_{kn} - E(y_{kn})}{\sqrt{\text{Var}(y_{kn})}}
\end{bmatrix}$$

have a limiting $k$-variate normal distribution if the correlations $\rho_{stn}$ ($s, t = 1, \ldots, k; s \neq t$) approach limits as $n \to \infty$. The distribution has means zero, variances one, and correlations $\rho_{st}$ where

$$\rho_{st} = \lim_{n \to \infty} \rho_{stn} .$$

**PROOF** It follows from Corollary 2.1.1, Corollary 2.1.2, and Lemma 5.1 that

$$E(y_{in}) = \frac{\sum a_{ij}}{N} \quad (i = 1, \ldots, k)$$

$$\text{Var}(y_{in}) = \frac{\sum (a_{ij} - \bar{a}_{in})^2}{N(N+1)} \quad (i = 1, \ldots, k)$$

$$\rho_{stn} = \frac{\sum (a_{sj} - \bar{a}_{sn})(a_{tj} - \bar{a}_{tn})}{\left\{ \sum (a_{sj} - \bar{a}_{sn})^2 \sum (a_{tj} - \bar{a}_{tn})^2 \right\}^{1/2}} \quad (s, t = 1, \ldots, k; s \neq t).$$

Without loss in generality assume that
\[ \sum a_{ij} = 0 \quad (i = 1, \ldots, k) \quad (5.3.3) \]

and

\[ \sum a_{ij}^2 = (n+1)^2 \quad (i = 1, \ldots, k). \quad (5.3.4) \]

Then

\[ \rho_{stn} = \frac{\sum a_{sj} a_{tj}}{(n+1)^2}, \]

the variances are asymptotically one, and \( y_{in} \) is asymptotically equivalent to \( \frac{y_{in} - E(y_{in})}{\sqrt{\text{Var}(y_{in})}} \), \( (i = 1, \ldots, k) \).

Now consider the linear combination

\[ y'_n = \delta_1 y_{1n} + \delta_2 y_{2n} + \ldots + \delta_k y_{kn} \]

where \( \delta_1, \delta_2, \ldots, \delta_k \) are real numbers. Then

\[ \text{Var}(y'_n) = \sum_{j=1}^{k} \delta_j^2 \text{Var}(y_{jn}) + 2 \sum_{i<j} \delta_i \delta_j \rho_{ijn}, \]

so \( \lim \text{Var}(y'_n) \) exists if \( \lim \rho_{stn} \) exists for \( s, t = 1, \ldots, k; \) \( s \neq t. \)

If \( \lim \text{Var}(y'_n) = 0 \), then \( y'_n \) converges in probability \( n \to \infty \)

to zero and \( y'_n \) has a degenerate normal distribution.
If \( \lim_{n \to \infty} \text{Var}(y_n') = c \), for some constant \( c > 0 \), consider the expression

\[
\frac{1}{N^2 + 1} \left\{ \frac{1}{N^2} \sum (\delta_1 a_{1j} + \delta_2 a_{2j} + \ldots + \delta_k a_{kj})^r \right\} ^{\frac{1}{r}}.
\]

(5.3.5)

The numerator is

\[
\frac{1}{N^2 + 1} \sum (\delta_1 a_{1j} + \delta_2 a_{2j} + \ldots + \delta_k a_{kj})^r
\]

\[
= \frac{1}{N^2 + 1} \sum \left\{ \sum^* \left\{ \binom{r}{r_1 r_2 \ldots r_k} \delta_1^{r_1} \ldots \delta_k^{r_k} \frac{1}{N^2 + 1} \sum_{l=1}^k a_{lj} \right\} \right\}
\]

\[
= \sum^* \left\{ \binom{r}{r_1 r_2 \ldots r_k} \delta_1^{r_1} \ldots \delta_k^{r_k} \frac{1}{N^2 + 1} \sum_{l=1}^k a_{lj} \right\}
\]

\[
\leq \sum^* \left\{ \binom{r}{r_1 r_2 \ldots r_k} \delta_1^{r_1} \ldots \delta_k^{r_k} \frac{1}{N^2 + 1} \prod_{s=1}^{k-1} \left\{ \sum_{l=1}^{2^s r_s} \frac{1}{2^s} \right\} \right\}
\]

\[
\cdot \left\{ \sum_{a_{kj}}^{2^{(k-1)r_k}} \frac{1}{2^{k-1}} \right\}
\]

by repeated application of the Schwarz Inequality,
= \sum_{r}^{\ast} \left[ \frac{r}{r_{1} \ldots r_{k}} \right]^{r_{1} \ldots r_{k}} \delta_{r_{1} \ldots r_{k}}

= \prod_{s=1}^{k-1} \left\{ \frac{2^{s} r_{s}}{2^{(s-1)r_{s}} + 1} \right\}^{\frac{1}{2^{s}} \sum_{s}^{N} a_{s j}} \cdot \left\{ \frac{(2^{(k-1)r_{k}})}{2^{(k-1)r_{k}} + 1} \right\}^{\frac{1}{2^{k-1}} \sum_{k}^{N} a_{k j}}

= \sum_{r}^{\ast} \left[ \frac{r}{r_{1} \ldots r_{k}} \right]^{r_{1} \ldots r_{k}} \delta_{r_{1} \ldots r_{k}} \cdot \mathcal{O}(1)

= \mathcal{O}(1)

by noting that under the assumptions (5.3.3) and (5.3.4),
the condition \( W \) reduces to

\[
\frac{\sum_{\mu} a_{\mu j}}{r_{N}^{2} + 1} = \mathcal{O}(1)
\]

for all \( r = 3, 4, \ldots \).

Now consider the denominator of (5.3.5). Since
\( y'_{n} \) can be written as

\[
y'_{n} = b_{0} v_{0} + b_{1} v_{1} + \ldots + b_{n} v_{n}
\]
where \( b_j = \sum_{i=1}^{k} \delta_i a_{ij} \), \((J = 0, 1, \ldots, n)\) and \( \sum b_j = 0 \), then

\[
\text{Var}(y_n') = \frac{\sum b_j^2}{N(N+1)}.
\]

Therefore the denominator of (5.3.5) is just

\[
\left( \frac{N+1}{N} \text{Var}(y_n') \right)^{\frac{r}{2}}
\]

which is asymptotically equal to \( c^{\frac{r}{2}} \).

Thus (5.3.5) becomes

\[
\frac{1}{\frac{r}{N^2} + 1} \frac{\sum b_j^2}{\left( \frac{\sum b_j^2}{N^2} \right)^{\frac{r}{2}}} = 0(1) \text{ for } r = 3, 4, \ldots,
\]

that is, \( \delta_1 a_{1j} + \ldots + \delta_k a_{kj} \) satisfies condition \( W \) and from the result found in Theorem 5.1, \( y_n' \) has a limiting normal distribution with mean zero and variance

\[
\sum_{j=1}^{k} \delta_j^2 + 2 \sum_{i<j} \delta_i \delta_j \rho_{ij}.
\]

The theorem now follows with an argument due to Fraser (1957), p. 242. Let \( Y_1, \ldots, Y_k \) denote \( k \) random
variables having a $k$-variate normal distribution with means zero, variance one, and correlation matrix $((\rho_{st}))$, $\rho_{ss} = 1$. Then the distribution of

$$\delta_1 Y_1 + \ldots + \delta_k Y_k$$

is the same as the limiting distribution of

$$\delta_1 y_{1n} + \ldots + \delta_k y_{kn}$$

for all real $\delta_1, \ldots, \delta_k$. If it were known, first of all, that $y_{1n}, \ldots, y_{kn}$ had a limiting distribution, say that of $Z_1, \ldots, Z_k$, then

$$\delta_1 Z_1 + \ldots + \delta_k Z_k$$

would have the same distribution as the limiting distribution of

$$\delta_1 y_{1n} + \ldots + \delta_k y_{kn}$$

But then

$$\delta_1 Y_1 + \ldots + \delta_k Y_k$$
and

\[ \delta_{1}Z_{1} + \ldots + \delta_{k}Z_{k} \]

have identical distributions for all \( \delta_{1}, \ldots, \delta_{k} \) and so \( Y_{1}, \ldots, Y_{k} \) and \( Z_{1}, \ldots, Z_{k} \) have identical distributions, i.e. \( Y_{1}, \ldots, Y_{k} \) and the limiting distribution of \( Y_{1n}, \ldots, Y_{kn} \) are identical. Secondly, if it were known that \( Y_{1n}, \ldots, Y_{kn} \) did not have a limiting distribution, two subsequences on \( n \) could be found which would have different limiting distributions. But this leads to a contradiction of the fact that

\[ \delta_{1}Y_{1n} + \ldots + \delta_{k}Y_{kn} \]

has the same limiting distribution as the distribution of

\[ \delta_{1}Y_{1} + \ldots + \delta_{k}Y_{k} \]

Thus the theorem is proved.
CHAPTER 6

LINEAR COMBINATIONS OF EXPONENTIAL VARIABLES

6.1 Introduction

Let \( w_1 < w_2 < \ldots < w_n \) denote the \( n \) order statistics corresponding to a random sample of size \( n \) from the exponential population with distribution function \( 1 - e^{-x} \), \( 0 < x < \infty \). Define

\[
\begin{align*}
x_1 &= nw_1 \\
x_2 &= (n-1)[w_2 - w_1] \\
x_3 &= (n-2)[w_3 - w_2] \\
& \vdots \\
x_n &= [w_n - w_{n-1}].
\end{align*}
\]

Then it is well-known (cf. Pyke (1965), p. 400) that \( x_1, x_2, \ldots, x_n \) are independent and identically distributed with exponential distribution function \( 1 - e^{-x} \), \( 0 < x < \infty \).

Thus to find the distribution of a linear combination of exponential order statistics,
\[ X_n = \sum_{i=1}^{n} c_i w_i \]

where \( c_1, c_2, \ldots, c_n \) are real coefficients, it is sufficient to consider a linear combination of independent exponential variates because

\[ \sum_{i=1}^{n} c_i w_i = \sum_{i=1}^{n} \left( \frac{a_i}{n - i + 1} \right) x_i \]

where \( a_i = \sum_{j=i}^{n} c_j \), \( i = 1, 2, \ldots, n \). With this relationship in mind, there will be no loss in generality in this chapter if only linear combinations of independent exponential variables are considered. Also note that if \( x_0 \) is another exponential random variable distributed identical to but independent of \( x_1, \ldots, x_n \), then the random variables defined by

\[ v_1 = \frac{x_1}{S}, \quad v_2 = \frac{x_2}{S}, \quad \ldots, \quad v_n = \frac{x_n}{S} \]

where \( S = \sum_{i=0}^{n} x_i \), have the same distribution as the coverages \( v_1, \ldots, v_n \) of the uniform distribution as defined previously and \( S \) is distributed independently of
\( v_1, \ldots, v_n \) (cf. Karlin (1966), p. 242). Thus

\[
Z_n = \sum_{i=1}^{n} b_i v_i = \sum_{i=1}^{n} b_i \frac{x_i}{S}
\]

for real constants \( b_1, \ldots, b_n \) where \( S \) and \( Z_n \) are independent. In this chapter, it is illustrated how this relationship can be used to obtain the distribution of \( \sum_{i=1}^{n} b_i x_i \) or the distribution of two such combinations from the distributions obtained in Chapters 2 and 3. A connection with serial correlation is also illustrated.

6.2 The Distribution of a Single Combination of Exponential Variables.

Again, for convenience, the coefficients \( b_0, b_1, \ldots, b_n \) \((b_0=0)\) will be assumed to be noncoincident. Otherwise they could be considered in a manner similar to that mentioned in Chapter 2.

**Theorem 6.1** The density function of the linear combination

\[
\sum_{i=1}^{n} b_i x_i
\]

is

\[
g(x) = \sum_{\{j: x/b_j > 0\}} \frac{|b_j|^{-1} b_j n^{-1} e^{-x/b_j}}{\prod_{r=1, r \neq j}^{n} (b_j - b_r)}
\]
PROOF

Since \( S = \sum_{i=1}^{n} x_i \) where \( x_0, x_1, \ldots, x_n \) are independent exponential with identical characteristic functions \((1-i\lambda)^{-1}\), the characteristic function of \( S \) is \((1-i\lambda)^{-n}\). But this is just the characteristic function corresponding to the probability density function

\[
h(s) = \frac{s^{n-1}e^{-s}}{|n|}, \quad 0 < s < \infty \quad (6.2.1)
\]

Thus using the fact that

\[
\sum_{i=1}^{n} b_i x_i = S \cdot Z_n
\]

where \( S \) and \( Z_n \) are independent

\[
g(x) = \int_{0}^{\infty} h(s) \cdot f(\frac{x}{s}) \cdot |J| \cdot ds
\]

\[
= \int_{0}^{\infty} \frac{s^{n-1}e^{-s}}{|n|} \left\{ \sum_{r=0}^{n} \left( \frac{1}{2} (|\frac{x}{s} - b_j| + |\frac{x}{s} - b_j|) \right)^{n-1} \right\} \frac{1}{s} ds
\]

from (6.2.1) and Corollary 2.1 with \( b_0 = 0 \),

\[
= \frac{1}{|n-1|} \sum \frac{1}{|n| (b_r - b_j)} \int_{0}^{\infty} s^{n-1}e^{-s} \left\{ \frac{1}{2} (|\frac{x}{s} - b_j| + |\frac{x}{s} - b_j|) \right\}^{n-1} ds.
\]
Now assume \( x < 0 \). Then obviously \( \frac{x}{s} - b_j < 0 \) if \( b_j > 0 \).

Thus

\[
g(x) = \frac{1}{|n-1|} \sum_{\{j: b_j < 0\}} \prod_{\substack{r=0 \atop r \neq j}}^{n} (b_r - b_j) \int_{\frac{x}{b_j}}^{\infty} e^{-s(s-b_j)^{n-1}ds}
\]

\[
= \frac{1}{|n-1|} \sum_{\{j: b_j < 0\}} \prod_{\substack{r=0 \atop r \neq j}}^{n} (b_r - b_j) \int_{\frac{x}{b_j}}^{\infty} e^{-s+\frac{x}{b_j}} (s-x/b_j)^{n-1}ds
\]

\[
= -\frac{1}{|n-1|} \sum_{\{j: b_j < 0\}} \prod_{\substack{r=0 \atop r \neq j}}^{n} (b_r - b_j) \frac{b_j - x/b_j}{b_j} \cdot |n-1|
\]

\[
= \sum_{\{j: b_j < 0\}} \left| b_j \right|^{n-1} b_j^{n-1} e^{-x/b_j}
\]

because \( b_j < 0 \) implies that \( -b_j = |b_j| \).

Now assume \( x > 0 \) and note that using 1.2.5 of Chapter 1, the density \( f(x/s) \) could equally as well
have been written as

\[ f(\frac{x}{s}) = (-1)^n \sum \frac{1}{\prod_{r=0}^{n} (b_r - b_j)_{r \neq j}} \left[ \frac{1}{2} \left( b_j - \frac{x}{s} \right) + b_j - \frac{x}{s} \right]^{n-1} \]

Then obviously \((b_j - \frac{x}{s}) < 0\) if \(b_j < 0\), so

\[ g(x) = \frac{(-1)^n}{(n-1)!} \sum_{\{j : b_j > 0\}} \frac{1}{\prod_{r=0}^{n} (b_r - b_j)_{r \neq j}} \int_{\frac{x}{b_j}}^{\infty} e^{-s(sb_j - x)^{n-1}} ds \]

\[ = \frac{(-1)^n}{(n-1)!} \sum_{\{j : b_j > 0\}} \frac{b_j e^{-x/b_j}}{\prod_{r=0}^{n} (b_r - b_j)_{r \neq j}} \int_{\frac{x}{b_j}}^{\infty} e^{-s+x/b_j (s-x/b_j)^{n-1}} ds \]

\[ = \sum_{\{j : b_j > 0\}} \frac{b_j^{n-1} e^{-x/b_j}}{\prod_{r=0}^{n} (b_r - b_j)_{r \neq j}} \]

\[ = \sum_{\{j : b_j > 0\}} \frac{|b_j|^{-1} b_j^{n-1} e^{-x/b_j}}{\prod_{r=1}^{n} (b_j - b_r)_{r \neq j}} \]
because the $b_j$'s being considered are positive. Combining the two cases for $x > 0$ and $x < 0$, it follows that

$$g(x) = \sum_{\{j : x/b_j > 0\}} \frac{|b_j|^{-1} b_j^{-n-1} e^{-x/b_j}}{n \prod_{r=1}^{n} (b_j - b_r)}$$

and the theorem is proved.

This theorem could have been proved using the Inversion Formula for characteristic functions. An integral similar to the one evaluated in the following lemma would have been required.

**LEMMA 6.1** For real $x$, $b$, $c$ and $a$ ($b \neq 0$, $a \neq 0$)

$$\int_{-\infty}^{\infty} \frac{e^{-ixt}}{b+ic+iat} \, dt = \begin{cases} 2\pi \frac{b}{|b| |a|} e^{x(b+ic)/a} & \text{if } \frac{bx}{a} < 0 \\ 0 & \text{if } \frac{bx}{a} > 0 \end{cases}$$

**PROOF** The proof of this lemma is similar to that of Lemma 2.3 and Lemma 3.3. First of all, note that
\[
\lim_{t \to -\infty} (t + \frac{b+ic}{ia}) \cdot e^{-ixt} \frac{1}{b+ic+iat} = -\frac{i}{a} e^{x(b+ic)} \\
(6.2.2)
\]

and also that

\[-\frac{b+ic}{ia} = -\frac{c}{a} + \frac{b}{a}i\]

so the singularity is in the upper half plane or lower half plane according as \(b/a > 0\) or \(b/a < 0\). Let \(\Omega_R\) be the large semicircle above the real axis from \(R(R > 0)\) to \(-R\), centre at the origin. Similarly let \(\Omega_R^\prime\) be the semicircle below the real axis from \(-R\) to \(R\). Denote the real axis from \(-R\) to \(R\) by \(\Gamma_R\).

Assume \(x > 0\) and \(b/a > 0\). Then \(-(c/a)+(b/a)i\) falls above the real axis and

\[
\int_{\Gamma_R \cup \Omega_R^\prime} e^{-ixz} \frac{1}{b+ic+iaz} \, dz = 0.
\]

So

\[
\int_{\Gamma} \frac{e^{-ixt}}{b+ic+iat} \, dt = \int_{\Omega_R^\prime} e^{-ixz} \frac{1}{b+ic+iaz} \, dz.
\]

Let \(z = Re^{i\theta}, \pi < \theta < 2\pi\). Then
\[
\left| \int_{\Omega_R} \frac{e^{-ixz}}{b+ic+iaz} \, dz \right| = \left| \int_{\pi}^{2\pi} \frac{e^{-ixRe^{i\theta}}}{b+ic+iaRe^{i\theta}} R e^{i\theta} \, d\theta \right|
\]
\[
\leq R \int_{\pi}^{2\pi} \frac{e^{xR\sin \theta}}{|b+ic+iaRe^{i\theta}|} \, d\theta
\]
\[
\leq \frac{R}{|b+ic|-|a|R} \int_{\pi}^{2\pi} e^{xR\sin \theta} \, d\theta
\]

But as shown in Lemma 2.3,
\[
\lim_{R \to \infty} \int_{\pi}^{2\pi} e^{xR\sin \theta} \, d\theta = 0.
\]

Thus
\[
\int_{-\infty}^{\infty} \frac{e^{-ixt}}{b+ic+iat} \, dt = 0 \quad \text{if } x > 0 \text{ and } b/a > 0.
\]

Assume \( x > 0 \) and \( b/a < 0 \) and consider the same closed contour as above. Then
\[
\int_{-\Gamma_R} \frac{e^{-ixz}}{b+ic+iaz} \, dz = \frac{2\pi}{a} e^{\frac{b+ic}{a}}
\]

by using the Residue Theorem and (6.2.2). But just as above, the integral over \( \Omega_R \) approaches zero as \( R \to \infty \). Thus
\[
\int_{-\infty}^{\infty} \frac{e^{-ixt}}{b+ic+iat} \, dt = -\frac{2\pi}{a} e^{x \frac{b+ic}{a}} \quad \text{if } x > 0 \text{ and } b/a < 0.
\]

In an entirely similar manner, by first assuming \( x < 0 \) and \( b/a > 0 \) and then assuming \( x < 0 \) and \( b/a < 0 \), it is possible to show, by considering the contour \( \Gamma \cup \Omega \), that

\[
\int_{-\infty}^{\infty} \frac{e^{-ixt}}{b+ic+iat} \, dt = \begin{cases} 
\frac{2\pi}{a} e^{x \frac{b+ic}{a}} & \text{if } x < 0 \text{ and } b/a > 0, \\
0 & \text{if } x < 0 \text{ and } b/a < 0.
\end{cases}
\]

Combining these results with the previous two it follows that

\[
\int_{-\infty}^{\infty} \frac{e^{-ixt}}{b+ic+iat} \, dt = \begin{cases} 
2\pi \frac{b}{|b|a} e^{x \frac{b+ic}{a}} & \text{if } \frac{bx}{a} < 0 \\
0 & \text{if } \frac{bx}{a} > 0.
\end{cases}
\]

**COROLLARY 6.1** For real \( x \) and \( a \),

\[
\int_{-\infty}^{\infty} \frac{e^{-ixt}}{|a|} \, dt = \begin{cases} 
\frac{2\pi}{|a|} e^{-x/a} & \text{if } \frac{x}{a} > 0 \\
0 & \text{if } \frac{x}{a} < 0.
\end{cases}
\]
This corollary can be used in the inversion of the characteristic function of the combination

\[
\frac{1}{\prod_{i=1}^{n} b_i x_i}.
\]

Just note that, because the \( x_1, \ldots, x_n \) are independent exponential,

\[
\phi(t) = \prod_{j=1}^{n} E(e^{itb_jx_j}) = \prod_{j=1}^{n} (1-ib_jt)^{-1} = \sum_{j=1}^{n} \prod_{r=1}^{n} \frac{b_j^{n-1}}{(b_j-b_r)} \cdot \frac{1}{1-ib_jt}
\]

using (1.2.1) where \( b_0, b_1, \ldots, b_n \) are distinct. Then the Inversion Formula for characteristic functions gives

\[
g(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \left[ \sum_{j=1}^{n} \frac{b_j^{n-1}}{\prod_{r=1}^{n} (b_j-b_r)} \cdot \frac{1}{1-ib_jt} \right] dt
\]

\[
= \frac{1}{2\pi} \sum_{j=1}^{n} \frac{b_j^{n-1}}{\prod_{r=1}^{n} (b_j-b_r)} \int_{-\infty}^{\infty} \frac{e^{-itx}}{1-ib_jt} dt
\]
\[ \sum_{\{j : x/b_j > 0\}} \frac{|b_j|^{-1} b_j^{n-1} e^{-x/b_j}}{\Pi_{r=1}^{n} (b_j - b_r)_{j \neq r}} \]

from Corollary 6.1.

6.3 The Joint Distribution of Two Linear Combinations of Exponential Variables

Define

\[ z_1 = b_{11} x_1 + b_{12} x_2 + \ldots + b_{1n} x_n \]
\[ z_2 = b_{21} x_1 + b_{22} x_2 + \ldots + b_{2n} x_n \]

where \( x_1, x_2, \ldots, x_n \) are the independent exponential random variables as defined before and where \( b_{i1}, b_{i2}, \ldots, b_{in} \) \( (i = 1, 2) \) are real constants. In this section it will be illustrated that the joint density \( g(z_1, z_2) \) of \( z_1 \) and \( z_2 \) can be found using the density \( f(y_1, y_2) \) with \( y_1 \) and \( y_2 \) defined to be

\[ y_1 = b_{11} v_1 + b_{12} v_2 + \ldots + b_{1n} v_n \]
\[ y_2 = b_{21} v_1 + b_{22} v_2 + \ldots + b_{2n} v_n \]

where \( v_1, v_2, \ldots, v_n \) are the coverages of the uniform distribution. Let \( \text{sgn} \ x \) be defined so that \( x \ \text{sgn} \ x = |x| \) and let \( b_{10} = 0, i = 1, 2. \)
THEOREM 6.2  The joint probability density function of $z_1$ and $z_2$ can be represented by

$$g(z_1, z_2) = \sum_{(i_1, i_2)} \text{sgn} \begin{vmatrix} b_{1i_1} & b_{2i_1} \\ b_{1i_2} & b_{2i_2} \end{vmatrix} \left| \begin{vmatrix} b_{1i_1} & b_{2i_1} \\ b_{1i_2} & b_{2i_2} \end{vmatrix} \right|^{n-2} \begin{vmatrix} b_{1i_1} & b_{2i_1} \\ b_{1i_2} & b_{2i_2} \end{vmatrix}^{-1} \begin{vmatrix} b_{1i_1} & b_{2i_1} \\ b_{1i_2} & b_{2i_2} \end{vmatrix}^{1} \begin{vmatrix} z_1 \\ z_2 \end{vmatrix}^{1}$$

(6.3.1)

provided

$$\begin{vmatrix} 1 & b_{1i_1} & b_{2i_1} \\ 1 & b_{1i_2} & b_{2i_2} \\ 1 & b_{1i_3} & b_{2i_3} \end{vmatrix} \neq 0$$

for all distinct $i_1, i_2, i_3$ ($i_1, i_2, i_3 = 0, 1, \ldots, n$), where the summation is over all the distinct pairs $(i_1, i_2)$, $(i_1, i_2 = 1, \ldots, n; i_1 \neq i_2)$ for which $(z_1, z_2)$ lies in the region formed between the lines with endpoints at the origin and passing through the points $(a_{1i_1}, a_{2i_1})$ and $(a_{1i_2}, a_{2i_2})$.

PROOF  As pointed out in the introduction,

$$y_1 = \sum_{i=1}^{n} b_{1i} x_i = \sum_{i=1}^{n} b_{1i} x_i / S = z_1 / S$$
\[ y_2 = \sum_{i=1}^{n} b_{2i} v_i = \sum_{i=1}^{n} b_{2i} x_i / S = z_2 / S \]

where \( S \) has the density

\[ h(s) = \frac{s^n e^{-s}}{|n|} \]

and \( y_1 \) and \( y_2 \) are independent of \( S \). Thus

\[ g(z_1, z_2) = \int_{0}^{\infty} h(s) f(z_1/s, z_2/s) |J| ds \]

\[ = \int_{0}^{\infty} s^n e^{-s} \left[ \frac{|n|}{|n-2|} \frac{1}{\prod_{i_1}^{n-2} sgn(b_{li_1} b_{2i_1})} \frac{1}{\prod_{i_2}^{n-2} sgn(b_{li_2} b_{2i_2})} \right] ds \]

where the summation is over all the distinct pairs \((i_1, i_2), (i_1, i_2 = 1, \ldots, n; i_1 \neq i_2)\) for which \((z_1/s, z_2/s)\) lies in the triangle with vertices \((0,0), (b_{li_1} b_{2i_1}), (b_{li_2} b_{2i_2})\). Here the density \( f(y_1, y_2) \) has been written in the form illustrated in Section 3.5. It is easy to see that \((z_1/s, z_2/s)\) belongs to this triangle if and only if \((z_1, z_2)\) falls between the lines as in the statement of the theorem and at the same time
\[ s > S^* = \frac{\begin{vmatrix} 1 & b_{21} \\ z_1 & b_{21} \\ l & b_{21} \\ \end{vmatrix} - \begin{vmatrix} 1 & b_{11} \\ z_2 & b_{11} \\ l & b_{11} \\ \end{vmatrix}}{\begin{vmatrix} b_{11} & b_{21} \\ z_1 & b_{21} \\ l & b_{21} \\ \end{vmatrix}} \]

Thus

\[ g(z_1, z_2) = \frac{1}{n-2} \int_0^\infty s^{n-2} e^{-s} \left[ \sum_{i_1, i_2} \left( \frac{b_{1i_1} b_{2i_1}}{b_{1i_2} b_{2i_2}} \right) \right] ds \]

\[ = \frac{1}{n-2} \sum_{i_1, i_2} \text{sgn} \begin{vmatrix} b_{1i_1} b_{2i_1} \\ b_{1i_2} b_{2i_2} \\ \end{vmatrix} \begin{vmatrix} b_{1i_1} b_{2i_1} \\ b_{1i_2} b_{2i_2} \\ 1 & b_{1i_1} b_{2i_1} \\ \end{vmatrix}^{n-2} \]

\[ \begin{vmatrix} b_{1i_1} b_{2i_1} \\ b_{1i_2} b_{2i_2} \\ 1 & b_{1i_1} b_{2i_1} \\ \end{vmatrix}^{n-2} \]

\[ \begin{vmatrix} b_{1i_1} b_{2i_1} \\ b_{1i_2} b_{2i_2} \\ 1 & b_{1i_1} b_{2i_1} \\ \end{vmatrix}^{n-2} \]

\[ \begin{vmatrix} b_{1i_1} b_{2i_1} \\ b_{1i_2} b_{2i_2} \\ 1 & b_{1i_1} b_{2i_1} \\ \end{vmatrix}^{n-2} \]
\[ \begin{align*}
\sum_{(i_1', i_2')} \text{sgn} & \begin{vmatrix}
\begin{array}{cc}
  b_{11} & b_{21} \\
  b_{12} & b_{22}
\end{array}
\end{vmatrix} \\
\begin{vmatrix}
\begin{array}{cc}
  b_{11} & b_{21} \\
  b_{12} & b_{22}
\end{array}
\end{vmatrix} \\
\begin{vmatrix}
\begin{array}{cc}
  b_{11} & b_{21} \\
  b_{12} & b_{22}
\end{array}
\end{vmatrix}
\end{align*} \]

which agrees with (6.3.1) as required by the theorem.

Just as in the case of one linear combination, (6.3.1) can be checked by deriving it using the Inversion Formula for characteristic functions. First of all note that the characteristic function of \( z_1 \) and \( z_2 \) is

\[ \phi(t_1, t_2) = E(e^{it_1 z_1 + it_2 z_2}) \]

\[ = E(e^{it_1(b_{11}x_1 + \ldots + b_{1n}x_n) + it_2(b_{21}x_1 + \ldots + b_{2n}x_n)}) \]

\[ = E(e^{i(b_{11}t_1 + b_{21}t_2)x_1 + \ldots + i(b_{1n}t_1 + b_{2n}t_2)x_n}) \]

\[ = \prod_{j=1}^{n} E(e^{i(b_{1j}t_1 + b_{2j}t_2)x_j}) \]

because \( x_1, \ldots, x_n \) are independent,

\[ = \prod_{j=1}^{n} \frac{1}{(1 - ib_{1j}t_1 - ib_{2j}t_2)} \]
provided

\[
\begin{vmatrix}
 b_{1i_1} & b_{2i_1} \\
 b_{1i_2} & b_{2i_2}
\end{vmatrix} \neq 0,
\]

for \(i_1, i_2 = 1, \ldots, n \) (\(i_1 \neq i_2\)). Then using the Inversion Formula for characteristic functions

\[
g(z_1, z_2) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{-iz_1 t_1 - iz_2 t_2}}{(1 - ib_{1i_1} t_1 - ib_{2i_2} t_2)} \ dt_1 dt_2
\]

\[
= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-iz_1 t_1 - iz_2 t_2}
\]

\[
\prod_{i_1=1}^{n} \prod_{i_2=1}^{n} \begin{bmatrix}
 b_{1i_1} & b_{2i_1} \\
 b_{1i_2} & b_{2i_2}
\end{bmatrix}^{-1} \cdot \frac{1}{1 - ib_{2i_1} t_2 - ib_{li_1} t_1} dt_1 dt_2
\]

using (1.2.1),

\[
= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \sum_{i_1=1}^{n} \frac{b_{li_1} e^{-iz_2 t_2}}{b_{li_1}^{-1}} \prod_{i_2=1}^{n} \begin{bmatrix}
 b_{1i_1} & b_{2i_1} \\
 b_{1i_2} & b_{2i_2}
\end{bmatrix}^{-1} \cdot \frac{1}{1 - ib_{2i_1} t_2 - ib_{li_1} t_1} dt_1 dt_2
\]

\[
\cdot \frac{-z}{b_{li_1}} e^{2\pi \lambda_{li_1} t_2} dt_2
\]
using Lemma 6.1 and letting

\[ \lambda_{i_1} = \begin{cases} 
1 & \text{if } z_1/b_{li_1} \geq 0 \\
0 & \text{if } z_1/b_{li_1} < 0
\end{cases} \]

\[ = \frac{1}{2\pi} \sum_{i_1=1}^{n} \sum_{i_2=1}^{\mathbb{N}} \lambda_{i_1} \text{sgn}(b_{li_1}) \begin{vmatrix}
\begin{array}{cc}
\text{b}_{li_1} & \text{b}_{2i_1} \\
\text{b}_{li_2} & \text{b}_{2i_2}
\end{array}
\end{vmatrix}
\begin{vmatrix}
\begin{array}{c}
\text{n-2} - \frac{z_1}{b_{li_1}} \\
\text{e}
\end{array}
\end{vmatrix}
\begin{vmatrix}
\begin{array}{c}
\text{b}_{li_1} \\
\text{b}_{li_2}
\end{array}
\end{vmatrix}
\begin{vmatrix}
\begin{array}{c}
\text{b}_{2i_1} \\
\text{b}_{2i_2}
\end{array}
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\text{b}_{2i_1} \\
\text{b}_{2i_2}
\end{array}
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\begin{vmatrix}
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\text{b}_{li_2}
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\text{b}_{2i_2}
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\text{b}_{li_2}
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\begin{vmatrix}
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\begin{vmatrix}
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\text{b}_{li_1} \\
\text{b}_{li_2}
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\begin{vmatrix}
\begin{array}{c}
\text{b}_{2i_1} \\
\text{b}_{2i_2}
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\text{b}_{li_1} \\
\text{b}_{li_2}
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\text{b}_{2i_2}
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\begin{vmatrix}
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\text{b}_{li_1} \\
\text{b}_{li_2}
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\begin{vmatrix}
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\text{b}_{2i_1} \\
\text{b}_{2i_2}
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\begin{vmatrix}
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\text{b}_{li_2}
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\text{b}_{2i_1} \\
\text{b}_{2i_2}
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\begin{vmatrix}
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\text{b}_{li_1} \\
\text{b}_{li_2}
\end{array}
\end{vmatrix}
\begin{vmatrix}
\begin{array}{c}
\text{b}_{2i_1} \\
\text{b}_{2i_2}
\end{array}
\end{vmatrix}
\begin{vmatrix}
\begin{array}{c}
\text{b}_{li_1} \\
\text{b}_{li_2}
\end{array}
\end{vmatrix}
\begin{vmatrix}
\begin{array}{c}
\text{b}_{2i_1} \\
\text{b}_{2i_2}
\end{array}
\end{vmatrix}
\end{vmatrix}
\begin{vmatrix}
\begin{array}{c}
\text{t}_{2}
\end{array}
\end{vmatrix}
\right)
\]
\[
\begin{bmatrix}
2\pi \text{sgn} & b_{1i_1} & l \\
& b_{1i_2} & l \\
& b_{1i_1} & b_{2i_1} \\
& b_{1i_2} & b_{2i_2} \\
\end{bmatrix} \cdot \lambda_{1i_1} \lambda_{1i_2} \cdot \begin{bmatrix}
\frac{b_{2i_1}}{b_{1i_1}} \\
\frac{b_{2i_1}}{b_{1i_2}} \\
\frac{b_{1i_1}}{b_{1i_1}} \\
\frac{b_{1i_2}}{b_{2i_2}} \\
\end{bmatrix} + \begin{bmatrix}
\frac{b_{1i_1}}{b_{1i_1}} \\
\frac{b_{1i_2}}{b_{1i_2}} \\
\frac{b_{1i_1}}{b_{2i_1}} \\
\frac{b_{1i_2}}{b_{2i_2}} \\
\end{bmatrix}
\]

from Lemma 6.1, where \( \lambda_{1i_1} \lambda_{1i_2} \) is one or zero according as

\[
\begin{vmatrix}
\frac{b_{2i_1}}{b_{1i_1}} \\
\frac{b_{2i_1}}{b_{1i_2}} \\
\frac{b_{1i_1}}{b_{1i_1}} \\
\frac{b_{1i_2}}{b_{2i_2}} \\
\end{vmatrix} = \begin{vmatrix}
\frac{b_{2i_1}}{b_{1i_1}} \\
\frac{b_{2i_1}}{b_{1i_2}} \\
\frac{b_{1i_1}}{b_{2i_1}} \\
\frac{b_{1i_2}}{b_{2i_2}} \\
\end{vmatrix}
\]

is \( > 0 \) or \( < 0 \).

\[
= \sum_{i_1=1}^{n} \sum_{\substack{i_2=1 \atop i_2 \neq i_1}}^{n} \lambda_{1i_1} \lambda_{1i_2} \text{sgn} \begin{vmatrix}
b_{1i_1} & b_{2i_1} \\
b_{1i_2} & b_{2i_1} \\
b_{1i_1} & b_{2i_1} \\
b_{1i_2} & b_{2i_2} \\
\end{vmatrix} \cdot \begin{vmatrix}
b_{1i_1} & b_{2i_1} & l \\
b_{1i_2} & b_{2i_1} & 1 \\
b_{1i_1} & b_{2i_1} & l \\
b_{1i_2} & b_{2i_2} & l \\
\end{vmatrix}
\]

\[
\begin{vmatrix}
\frac{b_{2i_1}}{b_{1i_1}} \\
\frac{b_{2i_1}}{b_{1i_2}} \\
\frac{b_{1i_1}}{b_{2i_1}} \\
\frac{b_{1i_2}}{b_{2i_2}} \\
\end{vmatrix}
\]

is \( > 0 \) or \( < 0 \).

\[
= \sum_{i_3=1}^{n} \sum_{\substack{i_1=1 \atop i_3 \neq i_1, i_2}}^{n} \lambda_{1i_3} \lambda_{1i_2} \text{sgn} \begin{vmatrix}
b_{1i_1} & b_{2i_1} \\
b_{1i_2} & b_{2i_1} \\
b_{1i_1} & b_{2i_1} \\
b_{1i_2} & b_{2i_3} \\
\end{vmatrix} \cdot \begin{vmatrix}
b_{1i_1} & b_{2i_1} & l \\
b_{1i_2} & b_{2i_1} & l \\
b_{1i_1} & b_{2i_1} & l \\
b_{1i_2} & b_{2i_2} & l \\
\end{vmatrix}
\]

\[
\begin{vmatrix}
\frac{b_{2i_1}}{b_{1i_1}} \\
\frac{b_{2i_1}}{b_{1i_2}} \\
\frac{b_{1i_1}}{b_{2i_1}} \\
\frac{b_{1i_2}}{b_{2i_2}} \\
\end{vmatrix}
\]

is \( > 0 \) or \( < 0 \).

\[
\begin{vmatrix}
\frac{b_{2i_1}}{b_{1i_1}} \\
\frac{b_{2i_1}}{b_{1i_2}} \\
\frac{b_{1i_1}}{b_{2i_1}} \\
\frac{b_{1i_2}}{b_{2i_2}} \\
\end{vmatrix}
\]

is \( > 0 \) or \( < 0 \).
\[
\begin{align*}
\frac{\sum_{i=1}^n \sum_{i_2=1}^{i_1} (\lambda_{i_1} \lambda_{i_1 i_2} \text{sgn } b_{1i_1} - \lambda_{i_2} \lambda_{i_2 i_1} \text{sgn } b_{1i_2}) \cdot \text{sgn} \begin{pmatrix}
\begin{array}{c|c}
| b_{1i_1} & 1 \\
\hline
b_{1i_2} & 1 \\
\end{array} \\
\hline
\begin{array}{c|c}
| b_{2i_1} & 1 \\
\hline
b_{2i_2} & 1 \\
\end{array}
\end{pmatrix}}{\begin{pmatrix}
| b_{1i_1} & b_{2i_1} & 1 \\
\hline
| b_{1i_2} & b_{2i_2} & 1 \\
\hline
z_1 & z_2 & 1
\end{pmatrix}}
\end{align*}
\]
\[ q(z_1, z_2) = \lambda \left( \frac{z_1}{b_{li_1}} \right) \lambda \left( \frac{b_{li_1} b_{2i_1}}{z_1 b_{li_2}} \right) \frac{b_{li_1}}{b_{li_2}} \operatorname{sgn} \left( \frac{b_{li_1}}{b_{li_2}} \right) \]

\[ -\lambda \left( \frac{z_1}{b_{li_2}} \right) \lambda \left( \frac{b_{li_2} b_{2i_2}}{z_1 b_{li_2}} \right) \frac{b_{li_1}}{b_{li_2}} \operatorname{sgn} \left( \frac{b_{li_1}}{b_{li_2}} \right) . \]

Assume \((b_{li_1}, b_{2i_1})\) and \((b_{li_2}, b_{2i_2})\) are in the same half-plane formed by the \(z_2\)-axis. Then \(b_{li_1} b_{li_2} > 0\),

\[ \lambda(z_1/b_{li_1}) = \lambda(z_1/b_{li_2}) , \]

\[ \operatorname{sgn} \left( \frac{b_{li_1}}{b_{li_2}} \right) = \operatorname{sgn} \left( \frac{b_{li_2}}{b_{li_2}} \right) , \]

and

\[ q(z_1, z_2) = \lambda \left( \frac{z_1}{b_{li_1}} \right) \operatorname{sgn} \left( \frac{b_{li_1}}{b_{li_2}} \right) . \]
\[
\lambda \left( \begin{bmatrix}
\frac{b_{11}}{z_1} & \frac{b_{21}}{z_2} \\
\frac{b_{11}}{b_{21}} & \frac{b_{11}}{b_{21}} \\
\frac{b_{12}}{b_{22}} & \frac{b_{12}}{b_{22}} \\
\end{bmatrix}
- \lambda \left( \begin{bmatrix}
\frac{b_{12}}{z_1} & \frac{b_{22}}{z_2} \\
\frac{b_{12}}{b_{22}} & \frac{b_{12}}{b_{22}} \\
\frac{b_{12}}{b_{22}} & \frac{b_{12}}{b_{22}} \\
\end{bmatrix}
\right) \right)
\]

Thus \( q(z_1, z_2) = 0 \) unless \((z_1, z_2)\) is in the same half-plane (formed by the \(z_2\)-axis) as \((b_{11}, b_{21})\) and \((b_{12}, b_{22})\), and

\[
\begin{vmatrix}
\frac{b_{11}}{z_1} & \frac{b_{21}}{z_2} \\
\frac{b_{12}}{z_1} & \frac{b_{22}}{z_2} \\
\end{vmatrix}
\text{ and }
\begin{vmatrix}
\frac{b_{12}}{z_1} & \frac{b_{22}}{z_2} \\
\frac{b_{12}}{z_1} & \frac{b_{22}}{z_2} \\
\end{vmatrix}
\]

have opposite signs. Then, noting that

\[
\begin{vmatrix}
\frac{b_{11}}{z_1} & \frac{b_{21}}{z_2} \\
\frac{b_{11}}{z_1} & \frac{b_{21}}{z_2} \\
\end{vmatrix}
> 0
\]

if and only if \((z_1, z_2)\) lies above the line through \((0, 0)\) and \((b_{11}, b_{21})\), it is possible to see that \((z_1, z_2)\) must lie between the lines through \((0, 0), (b_{11}, b_{21})\) and \((0, 0), (b_{12}, b_{22})\) as described in Theorem 6.2 in order that \( q(z_1, z_2) = 1 \).
Now assume \((b_{li1}, b_{2i1})\) and \((b_{li2}, b_{2i2})\) are in opposite half-planes as formed by the \(z_2\)-axis and assume without loss in generality that \((b_{li1}, b_{2i1})\) is in the right-hand half-plane. Then
\[
\begin{vmatrix}
b_{li1} & 1 \\
b_{li2} & 1 \\
1 & b_{li1}
\end{vmatrix} > 0 \quad \text{and} \quad \begin{vmatrix}b_{li1} & 1 \\
b_{li2} & 1 \\
1 & b_{li2}\end{vmatrix} < 0
\]
and obviously only one of the terms of \(q(z_1, z_2)\) can be nonzero at a time. Assume \(z_1 > 0\). Then
\[
q(z_1, z_2) = \lambda \left( \frac{z_1}{b_{li1}} \right) \lambda \begin{vmatrix}b_{li1} & b_{2i1} \\z_1 & z_2 \\b_{li1} & b_{2i1} \end{vmatrix} \begin{vmatrix}b_{li2} & b_{2i2} \\z_1 & z_2 \\b_{li2} & b_{2i2} \end{vmatrix}
\]
and is one only if \((z_1, z_2)\) lies in the right-hand half-plane and on the same side of the line through \((0,0), (b_{li1}, b_{2i1})\) as \((b_{li2}, b_{2i2})\). It is zero otherwise. Assume \(z_1 < 0\). Then
\[
q(z_1, z_2) = -\lambda \left( \frac{z_1}{b_{li2}} \right) \lambda \begin{vmatrix}b_{li2} & b_{2i2} \\z_1 & z_2 \\b_{li1} & b_{2i1} \end{vmatrix} \begin{vmatrix}b_{li2} & b_{2i2} \\z_1 & z_2 \\b_{li2} & b_{2i2} \end{vmatrix} (-1)
\]
and is one only if \((z_1, z_2)\) lies in the left-hand half-plane and on the same side of the line through \((0,0)\), \((b_{11}, b_{21})\) as \((b_{11}, b_{21})\). It is zero otherwise.

Combining these results, it follows that if \((z_1, z_2)\) falls between the lines through \((0,0)\), \((b_{11}, b_{21})\) and \((0,0), (b_{12}, b_{22})\) as described in the theorem, then \(q(z_1, z_2) = 1\). Otherwise \(q(z_1, z_2) = 0\). Thus (6.3.1) and (6.3.2) are equivalent.

6.4 A Connection with Serial Correlation

Consider the circular serial correlation coefficients defined by

\[
r_j = \frac{\sum_{i=1}^{N} (X_i - \overline{X})(X_{i+j} - \overline{X})}{\sum_{i=1}^{N} (X_i - \overline{X})^2}
\]

where \(X_1, X_2, \ldots, X_N\) are independent standard normal variables with \(X_{N+i} = X_i\). If \(N\) is odd, these circular serial correlation coefficients can be expressed (cf. Watson (1956)) as

\[
r_j = \frac{b_{j0}x_0 + b_{j1}x_1 + \ldots + b_{jn}x_n}{x_0 + x_1 + \ldots + x_n}
\]

where \(x_0, x_1, \ldots, x_n\) are the independent exponential variables defined before and where \((n+1) = (N-1)/2\). The coefficients \(b_{j0}, b_{j1}, \ldots, b_{jn}\) are simply the distinct latent roots.
arising from the matrix of the quadratic form of the numerator of the $r_j$. Then, because of the property considered in the introduction of this chapter, the $r_j$ have the same joint distribution as the corresponding linear combinations of the uniform spacings $v_0, v_1, \ldots, v_n$. Thus when $N$ is odd the circular serial correlation coefficients can be written as

\[ r_1 = b_{10}v_0 + b_{11}v_1 + \ldots + b_{1n}v_n \]
\[ r_2 = b_{20}v_0 + b_{21}v_1 + \ldots + b_{2n}v_n \]
\[ \ldots \]
\[ r_k = b_{k0}v_0 + b_{k1}v_1 + \ldots + b_{kn}v_n \]

and their joint distribution can be found using Theorem 4.1. By considering the alternate summation rule in Section 3.5 it is easy to see that the result obtained is the same as that given by Watson (1956).
CHAPTER 7

APPLICATION OF THE FORMULA FOR $f(y_1, y_2)$

7.1 Introduction

Several examples are given in this chapter, for which formula (3.3.1) is used to find the joint density of two linear combinations of uniform order statistics. If the conditions of Theorem 3.1 are satisfied, the application of the formula is straightforward. However, there are some properties of the formula which when noted and made use of can speed the calculation of the density for the specific example being considered. This is on account of the fact that the formula (3.3.1) was written in a form that made it more suitable for general mathematical manipulation and was not written as the shortest summation possible. For example, in the form given in Theorem 3.1, several terms contribute to the density in regions for which the density is actually zero,
however they cancel and there is no point in calculating these terms when finding the joint distribution of a particular set of linear combinations. Also, many cases exist for which the conditions of the Theorem 3.1 are not satisfied. However, as mentioned before, the distribution can be found in these cases by adjusting the coefficients using a small $\varepsilon > 0$ in such a way that the conditions of Theorem 3.1 are satisfied. Then, just as in Chapter 2 for the case of a single linear combination, the distribution can be found by considering the limit as $\varepsilon \to 0$. Here, depending on the reason for the determinants being zero, it is sometimes possible to get the distribution mechanically without actually using the $\varepsilon$ (or the associated derivatives) development.

7.2 Examples

First of all, a case in which all the conditions of Theorem 3.1 are satisfied is examined. Consider a random sample of size two from the uniform distribution and let

\[ y_1 = u_1 + u_2 \]
\[ y_2 = u_2 - u_1 \]
or in terms of the coverages

\[
y_1 = 2v_1 + v_2
\]
\[
y_2 = v_2
\]

Then it follows from (3.3.1) that

\[
f(y_1, y_2) = \frac{|2|}{0} \chi(y_1 - a_{10}) \lambda \left( \begin{bmatrix}
1 & a_{10} & a_{20} \\
1 & a_{11} & a_{21} \\
1 & y_1 & y_2
\end{bmatrix} \right) \cdot \frac{1}{\left| \begin{bmatrix}
1 & a_{10} & a_{20} \\
1 & a_{11} & a_{21} \\
1 & a_{12} & a_{22}
\end{bmatrix} \right|} \]

\[
+ \lambda(y_1 - a_{10}) \lambda \left( \begin{bmatrix}
1 & a_{10} & a_{20} \\
1 & a_{12} & a_{22} \\
1 & y_1 & y_2
\end{bmatrix} \right) \cdot \frac{1}{\left| \begin{bmatrix}
1 & a_{10} & a_{20} \\
1 & a_{12} & a_{22} \\
1 & a_{11} & a_{21}
\end{bmatrix} \right|} \]

\[
+ \lambda(y_1 - a_{11}) \lambda \left( \begin{bmatrix}
1 & a_{11} & a_{21} \\
1 & a_{10} & a_{20} \\
1 & y_1 & y_2
\end{bmatrix} \right) \cdot \frac{1}{\left| \begin{bmatrix}
1 & a_{11} & a_{21} \\
1 & a_{10} & a_{20} \\
1 & a_{12} & a_{22}
\end{bmatrix} \right|} \]
\[
+ \lambda(y_1-a_{11}) \lambda \left( \begin{array}{ccc} 1 & a_{11} & a_{21} \\ 1 & a_{12} & a_{22} \\ 1 & y_1 & y_2 \end{array} \right) \cdot \begin{array}{c} 1 \\ 1 & a_{11} \\ 1 & a_{12} \end{array}
\]

\[
+ \lambda(y_1-a_{12}) \lambda \left( \begin{array}{ccc} 1 & a_{12} & a_{22} \\ 1 & a_{10} & a_{20} \\ 1 & y_1 & y_2 \end{array} \right) \cdot \begin{array}{c} 1 \\ 1 & a_{12} \\ 1 & a_{10} \end{array}
\]

\[
+ \lambda(y_1-a_{12}) \lambda \left( \begin{array}{ccc} 1 & a_{12} & a_{22} \\ 1 & a_{11} & a_{21} \\ 1 & y_1 & y_2 \end{array} \right) \cdot \begin{array}{c} 1 \\ 1 & a_{12} \\ 1 & a_{11} \end{array}
\]

\[
= \frac{|2|}{2} \left[ \lambda(y_1)\lambda(y_2)-\lambda(y_1)\lambda(y_2-y_1)-\lambda(y_1-2)\lambda(y_2)+\lambda(y_1-2)\lambda(y_2+y_1-2) \right.

+ \lambda(y_1-1)\lambda(y_2-y_1)-\lambda(y_1-1)\lambda(y_2+y_1-2) \right]
by evaluating the determinants,

$$=[\lambda(y_1) - \lambda(y_1 - 1)][\lambda(y_2) - \lambda(y_2 - y_1)] + [\lambda(y_1 - 1) - \lambda(y_1 - 2)]$$

$$\cdot [\lambda(y_2) - \lambda(y_2 + y_1 - 2)]$$

$$= \begin{cases} 
1 & 0 < y_2 < y_1 < 1 \\
1 & 0 < y_2 < 2 - y_1 < 1 \\
0 & \text{elsewhere.} 
\end{cases} \quad (7.2.1)$$

That is, the density is just 1 in the triangle with vertices (0,0), (1,1) and (2,0) illustrated in Diagram 1, and zero elsewhere.

However this result could have been obtained with less calculation by noting that (3.3.1) is zero outside the
triangle with vertices (0,0), (1,1) and (2,0).
This is easily seen either by using Lemma 3.5 or by direct
consideration of the definition of \( y_1 \) and \( y_2 \). Then noting
that only the term with \( i_1 = 0, i_2 = 1 \) contributes to the
value of the density within the triangle, it follows that the
density is

\[
\begin{bmatrix}
1 & 2 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
a_{10} & a_{20} \\
a_{11} & a_{21} \\
a_{12} & a_{22}
\end{bmatrix}
\]

inside the triangle

\[
f(y_1,y_2) =
\begin{cases}
1 & \text{inside the triangle} \\
0 & \text{elsewhere}
\end{cases}
\]

In the general problem, if a polygon is formed by
joining all the points \((a_{1i_1}, a_{2i_1}), (i_1 = 0, 1, \ldots, n)\), in
pairs, it is possible to show with Lemma 3.5, that the
density outside the polygon is zero and that the density
at any point \((y_1, y_2)\) inside the polygon includes all the
terms (i.e. distinct sets of values for \(i_1, i_2\)) for which

\[
a_{11} \leq y_1 \leq a_{12}
\]
and

\[ y_2 > \left( \frac{a_{2i_2} - a_{2i_1}}{a_{1i_2} - a_{1i_1}} \right) y_1 - \left( \frac{a_{1i_1} a_{2i_2} - a_{1i_2} a_{2i_1}}{a_{1i_2} - a_{1i_1}} \right). \]

That is, given \((y_1, y_2)\), the density \(f(y_1, y_2)\) can be found by just calculating the terms \(i_1, i_2\) such that the line between \((a_{1i_1}, a_{2i_1})\) and \((a_{1i_2}, a_{2i_2})\) falls below the point \((y_1, y_2)\), where the point which lies to the left (i.e. \((a_{1i_1}, a_{2i_1})\)) of the other is always considered first (i.e. the line between \((a_{1i_2}, a_{2i_2})\) and \((a_{1i_1}, a_{2i_1})\) is not considered as a separate line). Note that this simplifies the use of formula (3.3.1) considerably. For example, in the next case considered, the formula (3.3.1) contains twelve terms, but only three of these need to be calculated.

Let \(y_1\) and \(y_2\) be the sample mean and sample range respectively for a sample of size three from the uniform distribution. That is,

\[ y_1 = \frac{(u_1 + u_2 + u_3)}{3} \]
\[ y_2 = u_3 - u_1 \]

or

\[ y_1 = v_1 + \frac{2}{3} v_2 + \frac{1}{3} v_3 \]
\[ y_2 = v_2 + v_3. \]
Noting that all the conditions of Theorem 3.1 are satisfied,

\[ f(y_1, y_2) \]

\[
= 6 \sum_{i_1 = o}^{3} \sum_{i_2 = o}^{3} \lambda (y_1 - a_{i1}) \lambda \left( \begin{array}{cc}
1 & a_{i1} & a_{2i1} \\
 a_{i1} & a_{2i1} \\
y_1 & y_2
\end{array} \right) \\
\lambda \left( \begin{array}{cc}
1 & a_{i1} \\
 a_{i1} \\
y_1 & y_2
\end{array} \right)
\]

\[
= 6 \left\{ \lambda (y_1) [\lambda (y_2) y_2 - \lambda (y_2 - \frac{3}{2} y_1) (2y_2 - 3y_1) + \lambda (y_2 - 3y_1) (y_2 - 3y_1)] \\
- \lambda (y_1 - 1) [\lambda (y_2) y_2 + \lambda (y_2 + 3y_1 - 3) (y_2 + 3y_1 - 3) + \lambda (y_2 + \frac{3}{2} y_1 - \frac{3}{2}) (3 - 3y_1 - 2y_2)] \\
+ \lambda (y_1 - \frac{2}{3}) [\lambda (y_2 - \frac{3}{2} y_1) (2y_2 - 3y_1) + \lambda (y_2 + 3y_1 - 3) (y_2 + y_1 - 1) + \lambda (y_2 - 1) (3 - 3y_2)] \\
+ \lambda (y_1 - \frac{1}{3}) [\lambda (y_2 - 3y_1) (3y_1 - y_2) - \lambda (y_2 + \frac{3}{2} y_1 - \frac{3}{2}) (2y_2 + 3y_1 - 3) + \lambda (y_2 - 1) (3y_2 - 3)] \right\}
\]

by simply evaluating all the determinants. Then with Diagram 2 in mind, it is easy to check that

\[
f(y_1, y_2) = \begin{cases}
6y_2 & \text{in (1)} \\
6(3y_1 - y_2) & \text{in (2)} \\
18(1 - y_2) & \text{in (3)} \\
6(3 - y_2 - 3y_1) & \text{in (4)} \\
0 & \text{elsewhere.}
\end{cases}
\]
However, utilizing the previous discussion, this density could have been obtained more quickly. Just consider any point \((y_1, y_2)\) in the region (1) and note that \(y_1\) lies between 0 and 1 and above the line between \((0,0)\) and \((1,0)\) only. Thus the density in region (1) is

\[
6 \begin{bmatrix}
1 & 0 & 0 \\
1 & 1 & 0 \\
1 & y_1 & y_2 \\
1 & 0 & 0 \\
1 & 1 & 0 \\
1 & 2/3 & 1
\end{bmatrix} = 6y_2.
\]

Then consider any point \((y_1, y_2)\) in region (2). \(y_1\) lies between 0 and 1 and above the line between \((0,0)\) and \((1,0)\), but also between 0 and \(2/3\) and above the line between \((0,0)\)
and \((\frac{2}{3}, 1)\). Thus the density in region (2) is

\[
\begin{align*}
6 \begin{vmatrix}
1 & 0 & 0 \\
1 & 1 & 0 \\
1 & \frac{2}{3} & 1
\end{vmatrix} & + 6 \begin{vmatrix}
1 & 0 & 0 \\
1 & \frac{2}{3} & 1 \\
1 & \frac{1}{3} & 1
\end{vmatrix} = 6(3y_1-y_2).
\end{align*}
\]

Similarly, the density in region (3) receives a contribution only from the terms for which \(i_1 = 0\), \(i_2 = 1\) and \(i_1 = 0\), \(i_2 = 2\) and \(i_1 = 3\), \(i_2 = 1\). Upon evaluation, these three terms are

\[
6y_2 + (-12y_2 + 18y_1) + (-12y_2 - 18y_1 + 18) = 18(1-y_2).
\]

For region (4), the only terms necessary are those for which \(i_1 = 0\), \(i_2 = 1\) and \(i_1 = 3\), \(i_2 = 1\). These have already been used in the preceding calculations and thus the density in region (4) is

\[
6y_2 + (-12y_2 - 18y_1 + 18) = 18(3-y_2 - 3y_1).
\]

Note that in all these calculations with pairs of points, the point on the left was always considered first i.e. as \((a_{1i_1}, a_{2i_1})\). Also note that only three terms \((i_1 = 0, i_2 = 1;\)
$i_1=0, i_2=2; i_1=3, i_2=1$ had to be calculated in order to obtain the complete density (7.2.2). The result obtained compares with the one obtained by Niven (1963).

Next consider the joint distribution of the two order statistics corresponding to a random sample of size two from the uniform distribution; that is,

$$y_1 = u_1$$
$$y_2 = u_2$$

or

$$y_1 = v_1$$
$$y_2 = v_1 + v_2.$$

Note that

$$\begin{vmatrix}
  1 & a_{10} \\
  1 & a_{12}
\end{vmatrix} = 0,$$

but as pointed out in Section 3.3, this creates no problem provided one considers $\lambda(\infty) = 0$ in the formula (3.3.1).

This can be easily seen for this example by noting that the characteristic function of $y_1$ and $y_2$ is

$$\phi(t_1, t_2) = E(e^{it_1y_1 + it_2y_2})$$
$$= E(e^{iv_1(t_1 + t_2) + iv_2t_2})$$
$$= \frac{1}{t_2} \left[ \frac{1}{t_1(t_1+t_2)} + \frac{e^{it_1+t_2}}{t_1(t_1+t_2)} - \frac{e^{it_2}}{t_2t_1} \right].$$
Then when the partial fraction expansion is applied to the integrand of the Inversion Formula, the first and last terms do not require the expansion and thus the terms in formula (3.3.1) with coefficients

\[
\lambda \left( \begin{array}{c}
1 \\
1 \\
l \\
1 \\
l \\
1 \\
\end{array} \right) \quad \text{and} \quad \lambda \left( \begin{array}{c}
1 \\
1 \\
\end{array} \right)
\]

do not appear. Applying formula (3.3.1) and evaluating the determinants,

\[
f(y_1, y_2)
\]

\[
= 2[\lambda(y_1)\lambda(y_2-y_1)-\lambda(y_1)y_1(\lambda(\infty)-\lambda(y_1-1)\lambda(y_2-y_1))+\lambda(y_1-1)\lambda(y_2-1)
\]

\[
+ \lambda(y_1)\lambda(\infty) - \lambda(y_1)\lambda(y_2-1)]
\]

\[
= 2[\lambda(y_1)-\lambda(y_1-1)][\lambda(y_2-y_1)-\lambda(y_2-1)]
\]

\[
= \begin{cases} 
2 & 0 < y_1 < y_2 < 1 \\
0 & \text{elsewhere.}
\end{cases}
\]

But this is just the well-known result that was expected, where the region of positive density is the triangular region illustrated in Diagram 3.
Just as in the previous examples the density could have been found quickly by simply noting that only the line between \((0,0)\) and \((1,1)\) lies below the points in the triangle and thus only the term for \(i_1 = 0, i_2 = 1\) needs to be calculated. That is,

\[
f(y_1, y_2) = \begin{cases} 
  \frac{2}{a_{10} a_{20}} & \text{in the triangle} \\
  \frac{1}{a_{11} a_{21}} & \\
  \frac{1}{a_{12} a_{22}} & \\
  0 & \text{elsewhere} \\
\end{cases}
\]
As remarked in the introduction and in Section 3.3, formula (3.3.1) is still useful if some of the three-dimensional determinants are zero. Consider the joint distribution of the sample median and sample range for a sample of size five from the uniform distribution. Then

\[ y_1 = u_3 \]
\[ y_2 = u_5 - u_1 \]

or

\[ y_1 = v_1 + v_2 + v_3 \]
\[ y_2 = v_2 + v_3 + v_4 + v_5. \]

Obviously, the region of positive density is the square illustrated in Diagram 4.

For convenience, in this example represent the determinants
\[
\begin{vmatrix}
1 & a_{111} \\
1 & a_{112}
\end{vmatrix}
\]

and

\[
\begin{vmatrix}
1 & a_{111} & a_{211} \\
1 & a_{112} & a_{212} \\
1 & a_{113} & a_{213}
\end{vmatrix}
\]

by \((i_1 \ i_2)\) and \((i_1 \ i_2 \ i_3)\) respectively. Then note that

\[(0 \ 4) = (0 \ 5) = (1 \ 2) = (1 \ 3) = (2 \ 3) = (4 \ 5) = 0\]

and

\[(0 \ 2 \ 3) = (0 \ 4 \ 5) = (1 \ 2 \ 3) = (1 \ 4 \ 5) = (2 \ 3 \ 4) = (2 \ 3 \ 5) = (2 \ 4 \ 3) = (3 \ 4 \ 5) = 0.\]

As in the previous example, any terms in formula (3.3.1) with \(i_1\) and \(i_2\) such that \((i_1 \ i_2) = 0\) could be omitted. The problem in this example though, is that in addition some of the determinants \((i_1 \ i_2 \ i_3)\) are zero and appear in the denominator of certain terms of formula (3.3.1). This difficulty arises because of coincident points. The points \((0, 1)\) and \((1, 1)\) each appear twice for various values of \(i_1\) and \(i_2\). However, if these points \((a_{13}, a_{23})\) and \((a_{15}, a_{25})\) are adjusted such that the new points \((a'_{13}, a'_{23})\) and \((a'_{15}, a'_{25})\) are \((a_{13} + 3e, a_{23} + 4e)\) and \((a_{15} + e, a_{25} + 2e)\) respectively (for small \(\delta > 0\)), then
(0 1 2) = (0 1 4) = (0 2 4) = (4 1 2) = 1
(0 1 3') = 1 + 4ε
(0 1 5') = 1 + 2ε
(0 2 3') = ε
(0 2 5') = 1 + ε
(0 3' 4) = 1 + 3ε
(0 3' 5') = 1 + 4ε + 3ε^2 = (1 + 3ε)(1 + ε)

and so on. Thus none of (i_1, i_2, i_3) are zero, and formula (3.3.1) can be applied to this modified problem. The modified region of positive density will now appear as in Diagram 5.

Now assume that the density is required for an arbitrary value of y_1 and an arbitrary value of y_2 such that (y_1, y_2)
falls in region \((2)\) of the original diagram. Then choose the \(\varepsilon > 0\) small enough, such that the point \((y_1, y_2)\) falls in the region illustrated in Diagram 5. Thus the terms appearing in the density evaluated at this point will only be those associated with the lines between \((0,0)\) and \((1,0)\); \((0,0)\) and \((1,1)\); \((0,0)\) and \((1+3\varepsilon, 1+4\varepsilon)\). That is

\[
f_\varepsilon(y_1, y_2) = \frac{5}{13} \left[ \begin{array}{c} 1 \\ a_{10} \\ a_{20} \\ 3 \\ 1 \\ a_{11} \\ a_{21} \\ 1 \\ y_1 \\ y_2 \end{array} \right] \frac{1}{(0 1 2)(0 1 3')(0 1 4)(0 1 5')} \]

\[
+ \frac{1}{(0 2 1)(0 2 3')(0 2 4)(0 2 5')} \left[ \begin{array}{c} 1 \\ a_{10} \\ a_{20} \\ 3 \\ 1 \\ a_{12} \\ a_{22} \\ 1 \\ y_1 \\ y_2 \end{array} \right] \frac{1}{(0 3' 1)(0 3' 2)(0 3' 4)(0 3' 5')} \]

\[
= 20 \left[ \frac{y_2^3}{(1+2\varepsilon)(1+4\varepsilon)} - \frac{(y_2 - y_1)^3}{\varepsilon(1+\varepsilon)} + \frac{((1+3\varepsilon)y_2-(1+4\varepsilon)y_1)^3}{\varepsilon(1+4\varepsilon)(1+3\varepsilon)^2(1+\varepsilon)} \right] \]

\[
= 20 \left[ \frac{y_2^3}{(1+2\varepsilon)(1+4\varepsilon)} - \frac{1}{\varepsilon(1+\varepsilon)} \right] \cdot \left\{ \frac{((1+4\varepsilon)(1+3\varepsilon)^2(y_2 - y_1)^3 + ((1+3\varepsilon)y_2 - (1+4\varepsilon)y_1)^3}{(1+4\varepsilon)(1+3\varepsilon)^2} \right\} \]
\[
20 \left[ \frac{y_2^3}{(1+2\varepsilon)(1+4\varepsilon)} - \frac{\varepsilon}{\varepsilon(1+\varepsilon)} \right] \\
\cdot \left\{ \frac{y_2^3(1+3\varepsilon)^2 + y_1^3(1+4\varepsilon)(2+7\varepsilon) - 3y_2y_1^2(1+4\varepsilon)(1+3\varepsilon)}{(1+4\varepsilon)(1+3\varepsilon)^2} \right\}
\]

Then

\[
\lim_{\varepsilon \to 0} f_\varepsilon(y_1, y_2) = 20(y_2^3 - y_2^3 - 2y_1^3 + 3y_2y_1^2)
\]

\[
= 60y_2y_1^2 - 40y_1^3.
\]

Now assume that the density is required for an arbitrary point \((y_1, y_2)\) in the region (3) of the original diagram. Then choose \(\varepsilon > 0\) small enough that this point falls in the region as illustrated. There are five lines lying below this region and thus it is seen that

\[
f_\varepsilon(y_1, y_2) = \frac{5}{3} \left[ \frac{1}{(0 \ 1 \ 2)(0 \ 1 \ 3')(0 \ 1 \ 4)(0 \ 1 \ 5')} \right]
\]

\[
\cdot \left[ \frac{u}{(0 \ 2 \ 1)(0 \ 2 \ 3')(0 \ 2 \ 4)(0 \ 2 \ 5')} + (0 \ 3' \ 1)(0 \ 3' \ 2)(0 \ 3' \ 4)(0 \ 3' \ 5') \right]
\]
\[
\begin{vmatrix}
1 & a_{14} & a_{24} \\
1 & a_{11} & a_{21} \\
1 & y_1 & y_2 \\
\end{vmatrix}^3
\]
\[
(4 \ 1 \ 0)(4 \ 1 \ 2)(4 \ 1 \ 3')(4 \ 1 \ 5')
\]

\[
\begin{vmatrix}
1 & a_{15} & a_{25} \\
1 & a_{11} & a_{21} \\
1 & y_1 & y_2 \\
\end{vmatrix}^3
\]
\[
(5' \ 1 \ 0)(5' \ 1 \ 2)(5' \ 1 \ 3')(5' \ 1 \ 4)
\]

The first three terms are the same as those that were considered for region (2), so just consider the other two terms. Thus

\[
\begin{vmatrix}
1 & a_{14} & a_{24} \\
1 & a_{11} & a_{21} \\
1 & y_1 & y_2 \\
\end{vmatrix}^3
\]
\[
(4 \ 1 \ 0)(4 \ 1 \ 2)(4 \ 1 \ 3')(4 \ 1 \ 5')
\]

\[
\begin{vmatrix}
1 & a_{15} & a_{25} \\
1 & a_{11} & a_{21} \\
1 & y_1 & y_2 \\
\end{vmatrix}^3
\]
\[
(5' \ 1 \ 0)(5' \ 1 \ 2)(5' \ 1 \ 3')(5' \ 1 \ 4)
\]

\[
= 20 \left[ -\frac{(y_2 + y_1 - 1)^3}{(1 + 7\epsilon)(3\epsilon)} + \frac{[y_2(1-\epsilon) + (y_1-1)(1+2\epsilon)]^3}{(1+2\epsilon)(1-\epsilon)(1+6\epsilon+2\epsilon^2)(3\epsilon)} \right]
\]
$$\lim \epsilon (y_1, y_2) = (60y_2y_1^2 - 40y_1^3) + 20(y_2 + y_1 - 1)^2(2y_1 - y_2 - 2)$$

in region (3).

For an arbitrary point \((y_1, y_2)\) in region (4), it is easy to see that
\[
f_\epsilon(y_1, y_2) = \frac{1}{13} \left[ \begin{array}{c} 1 \ a_{10} \ a_{20} \\ 1 \ a_{11} \ a_{21} \\ 1 \ y_1 \ y_2 \end{array} \right] ^3 (0 \ 1 \ 2)(0 \ 1 \ 3')(0 \ 1 \ 4)(0 \ 1 \ 5') \\
+ \frac{\left[ \begin{array}{c} 1 \ a_{14} \ a_{24} \\ 1 \ a_{11} \ a_{21} \\ 1 \ y_1 \ y_2 \end{array} \right] ^3 (4 \ 1 \ 0)(4 \ 1 \ 2)(4 \ 1 \ 3')(4 \ 1 \ 5')}{(5' \ 1 \ 0)(5' \ 1 \ 2)(5' \ 1 \ 3')(5' \ 1 \ 4)} \\
+ \frac{\left[ \begin{array}{c} 1 \ a_{15} \ a_{25} \\ 1 \ a_{11} \ a_{21} \\ 1 \ y_1 \ y_2 \end{array} \right] ^3}{(1+2\epsilon)(1+4\epsilon)} + \frac{(y_2+y_1-1)^2(2y_1-y_2-2)+o(1)}{1+14\epsilon+o(\epsilon)} \\
= 20 \left[ \frac{y_2^3}{(1+2\epsilon)(1+4\epsilon)} + \frac{(y_2+y_1-1)^2(2y_1-y_2-2)+o(1)}{1+14\epsilon+o(\epsilon)} \right]
\]

by making use of the calculations for the other regions.

Thus

\[
\lim_{\epsilon \to 0} f_\epsilon(y_1, y_2) = 20[y_2^3 + (y_2+y_1-1)^2(2y_1-y_2-2)]
\]

and it has been illustrated that the joint density of the sample median and sample range for a sample of size five from the uniform distribution is,
\[ f(y_1, y_2) = \begin{cases} 
20y_2^3 & \text{in (1)} \\
20y_1^2(3y_2-2y_1) & \text{in (2)} \\
20[y_1^2(3y_2-2y_1)+(y_2+y_1-1)^2(2y_1-y_2-2)] & \text{in (3)} \\
20[y_2^3+(y_2+y_1-1)^2(2y_1-y_2-2)] & \text{in (4)} \\
0 & \text{elsewhere}.
\]

It is not always necessary to go through the procedure with \( e \) that was used in the preceding example. Consider the joint distribution of the first two order statistics corresponding to a sample of size 3 from the uniform distribution; i.e.

\[ y_1 = u_1 \]
\[ y_2 = u_2 \]

and in terms of coverages

\[ y_1 = v_1 + 0v_3 \]
\[ y_2 = v_1 + v_2 + 0v_3. \]

For these combinations

\[
\begin{vmatrix}
1 & a_{10} \\
1 & a_{12}
\end{vmatrix} = 0,
\begin{vmatrix}
1 & a_{10} \\
1 & a_{13}
\end{vmatrix} = 0,
\begin{vmatrix}
1 & a_{12} \\
1 & a_{13}
\end{vmatrix} = 0
\]
and

\[
\begin{vmatrix}
1 & a_{10} & a_{20} \\
1 & a_{11} & a_{21} \\
1 & a_{13} & a_{23}
\end{vmatrix} = 0, \quad \begin{vmatrix}
1 & a_{10} & a_{20} \\
1 & a_{12} & a_{22} \\
1 & a_{13} & a_{23}
\end{vmatrix} = 0,
\]

the latter occurring on account of the fact that the points \((a_{10}, a_{20})\) and \((a_{13}, a_{23})\) are coincident. Diagram 6 illustrates the region of positive density for \(y_1\) and \(y_2\).

**DIAGRAM 6**

\[
\begin{array}{c}
\text{(0,1)} \\
\hline
\text{y}_2 \\
\hline
\text{(0,0)} \\
\hline
\text{y}_1
\end{array}
\]

The line between \((a_{10}, a_{20})\) and \((a_{11}, a_{21})\) and the line between \((a_{13}, a_{23})\) and \((a_{11}, a_{21})\) (actually the same line) are the only two lines lying below the triangular region. So if it were not for the fact that the terms for \(i_1 = 0\), \(i_2 = 1\) and \(i_1 = 3\), \(i_2 = 1\) have zero determinants in their denominators, the density could be found immediately.
However this time, instead of actually going through the procedure of adjusting the coefficients using $\varepsilon$ and considering limits, note that from Lemma 3.5, once the coefficients have been adjusted,

\[
\begin{vmatrix}
1 & a_{11} & a_{21} \\
1 & a_{10} & a_{20} \\
1 & y_1 & y_2
\end{vmatrix}
\begin{vmatrix}
1 & a_{11} & a_{21} \\
1 & a_{13} & a_{23} \\
1 & y_1 & y_2
\end{vmatrix}
+ \begin{vmatrix}
1 & a_{11} & a_{21} \\
1 & a_{10} & a_{20} \\
1 & a_{12} & a_{22}
\end{vmatrix}
\begin{vmatrix}
1 & a_{11} & a_{21} \\
1 & a_{13} & a_{23} \\
1 & a_{10} & a_{20}
\end{vmatrix}
\begin{vmatrix}
1 & a_{12} & a_{22} \\
1 & a_{12} & a_{22} \\
1 & a_{13} & a_{23}
\end{vmatrix}

= - \begin{vmatrix}
1 & a_{11} & a_{21} \\
1 & a_{12} & a_{22} \\
1 & y_1 & y_2
\end{vmatrix}
\begin{vmatrix}
1 & a_{11} & a_{21} \\
1 & a_{12} & a_{22} \\
1 & a_{10} & a_{20}
\end{vmatrix}
\begin{vmatrix}
1 & a_{13} & a_{23} \\
1 & a_{13} & a_{23} \\
1 & a_{13} & a_{23}
\end{vmatrix}

Then when the limit of the right hand side is taken as $\varepsilon \to 0$, no zero determinants will appear. Thus
\[ f(y_1, y_2) = \begin{cases} 
\begin{vmatrix} 
1 & a_{11} & a_{21} \\
1 & a_{12} & a_{22} \\
1 & y_1 & y_2 
\end{vmatrix} \\
-6 \\
\begin{vmatrix} 
1 & a_{11} & a_{21} \\
1 & a_{12} & a_{22} \\
1 & a_{10} & a_{20} 
\end{vmatrix} \\
\begin{vmatrix} 
1 & a_{11} & a_{21} \\
1 & a_{12} & a_{22} \\
1 & a_{13} & a_{23} 
\end{vmatrix} 
\end{cases} 
\]

\[ 0 < y_1 < y_2 < 1 \]

\[ f(y_1, y_2) = \begin{cases} 
6(1-y_2) & 0 < y_1 < y_2 < 1 \\
0 & \text{elsewhere} 
\end{cases} 
\]

which of course is a well-known result. So in this example, Lemma 3.5 could be used to avoid the mechanics of the approach although this is really what is being done. This trick will not work for the example which preceded this one. It will work for the following one.

Consider the linear combinations which represent the second and third order statistics for a random sample of size 3 from the uniform distribution; i.e.:

\[ y_1 = u_2 \]
\[ y_2 = u_3, \]

or in terms of coverages

\[ y_1 = v_1 + v_2 \]
\[ y_2 = v_1 + v_2 + v_3. \]
For this case

\[
\begin{vmatrix}
1 & a_{10} & 0 \\
1 & a_{11} & 0 \\
1 & a_{12} & 0 \\
\end{vmatrix} = 0, \\
\begin{vmatrix}
1 & a_{11} & a_{20} \\
1 & a_{21} & a_{20} \\
1 & a_{12} & a_{22} \\
\end{vmatrix} = 0.
\]

and

Also by considering the region of positive density as illustrated in Diagram 7

\[
\begin{vmatrix}
1 & a_{10} & a_{20} \\
1 & a_{11} & a_{21} \\
1 & a_{12} & a_{22} \\
\end{vmatrix} = 0, \\
\begin{vmatrix}
1 & a_{11} & a_{21} \\
1 & a_{12} & a_{22} \\
1 & a_{13} & a_{23} \\
\end{vmatrix} = 0.
\]

it is seen that the terms which contribute should be those for which \( i_1 = 0, i_2 = 1 \) and \( i_1 = 0, i_2 = 2 \) (the points \((a_{11}, a_{21})\) and \((a_{12}, a_{22})\) being coincident), that is
However a zero determinant value is entering in the denominators. But just as in the preceding example, if the determinant had not been zero, Lemma 3.5 indicates that the above two terms could be replaced by

\[
\begin{vmatrix}
1 & a_{10} & a_{20} \\
1 & a_{11} & a_{21} \\
1 & a_{12} & a_{22}
\end{vmatrix}
\begin{vmatrix}
y_1 \\
y_2
\end{vmatrix}
= 6y_1.
\]

Thus

\[
f(y_1, y_2) = \begin{cases} 
3y_1 & 0 < y_1 < y_2 < 1 \\
0 & \text{elsewhere.}
\end{cases}
\]

This is another well-known result.
Now consider the two linear combinations which represent the sample mean and sample range of a sample of size 4 from the uniform distribution,

\[ y_1 = (u_1 + u_2 + u_3 + u_4) / 4 \]

\[ y_2 = u_4 - u_1 \]

which in terms of coverages can be written

\[ y_1 = v_1 + \frac{3}{4} v_2 + \frac{1}{2} v_3 + \frac{1}{4} v_4 \]

\[ y_2 = v_2 + v_3 + v_4 . \]

For these linear combinations,

\[
\begin{vmatrix}
1 & a_{12} & a_{22} \\
1 & a_{13} & a_{23} \\
1 & a_{14} & a_{24}
\end{vmatrix}
= 0
\]

and thus all the conditions of Theorem 3.1 are not satisfied. However as before the density can still be found using (3.3.1). The difficulty can be seen in considering the region of positive density for \( y_1 \) and \( y_2 \) illustrated in Diagram 8. The zero determinant arises because the three points \( (\frac{1}{4}, 1) \), \( (\frac{1}{2}, 1) \) and \( (\frac{3}{4}, 1) \) lie in a straight line. However the terms associated with the lines through these points do not contribute in the region of positive density for \( y_1 \) and \( y_2 \) and thus
can be ignored. This could be shown by considering one of the points to be changed slightly using $\varepsilon$, such that they no longer lie in a straight line and then proceeding with limiting values. However, this procedure would not affect the final evaluation of the density for the other regions, because taking limits will just return all the expressions involved in these regions to their original form. Thus the term associated with the line through the two points $(0,0)$ and $(1,0)$ is $i_1 = 0, i_2 = 1$ which has the value $12y_2^2$. Similarly the line through $(0,0)$ and $(\frac{3}{4}, 1)$ leads to $i_1 = 0, i_2 = 2$ and the value $-96(\frac{3}{4}y_2 - y_1)^2$. Then $(0,0)$ and $(\frac{1}{2}, 1)$ give $i_1 = 0, i_2 = 3$ and $192(\frac{1}{2}y_2 - y_1)^2$; $(\frac{1}{4}, 1)$ and $(1, 0)$ give $i_1 = 4, i_2 = 1$ and $-96(\frac{3}{4}y_2 + y_1 - 1)^2$; $(\frac{1}{2}, 1)$ and $(1, 0)$ give $i_1 = 3, i_2 = 1$ and $192(\frac{1}{2}y_2 + y_1 - 1)^2$. Thus
\[ f(y_1, y_2) = \begin{cases} 
12y_2^2 & \text{in (1)} \\
12y_2^2 - 96(\frac{3}{4}y_2-y_1)^2 & \text{in (2)} \\
12y_2^2 - 96(\frac{3}{4}y_2-y_1)^2 + 192(\frac{1}{2}y_2-y_1)^2 & \text{in (3)} \\
12y_2^2 - 96(\frac{3}{4}y_2-y_1)^2 + 192(\frac{1}{2}y_2-y_1)^2 - 96(\frac{3}{4}y_2+y_1-1)^2 & \text{in (4)} \\
12y_2^2 - 96(\frac{3}{4}y_2-y_1)^2 - 96(\frac{3}{4}y_2+y_1-1)^2 & \text{in (5)} \\
12y_2^2 - 96(\frac{3}{4}y_2-y_1)^2 - 96(\frac{3}{4}y_2+y_1-1)^2 + 192(\frac{1}{2}y_2+y_1-1)^2 & \text{in (6)} \\
12y_2^2 - 96(\frac{3}{4}y_2+y_1-1)^2 + 192(\frac{1}{2}y_2+y_1-1)^2 & \text{in (7)} \\
12y_2^2 - 96(\frac{3}{4}y_2+y_1-1)^2 & \text{in (8)} \\
0 & \text{elsewhere.} 
\end{cases} \]

That is

\[ f(y_1, y_2) = \begin{cases} 
12y_2^2 & \text{in (1)} \\
6(24y_1y_2-16y_1^2-7y_2^2) & \text{in (2)} \\
6(4y_1-y_2)^2 & \text{in (3)} \\
48(1-y_2)(4y_1+y_2-2) & \text{in (4)} \\
48(-2y_2^2-4y_1^2+3y_2^2+4y_1-2) & \text{in (5)} \\
48(-y_2^2+4y_1y_2-y_2-4y_1+2) & \text{in (6)} \\
6(4y_1+y_2-4)^2 & \text{in (7)} \\
6(-7y_2^2-16y_1^2-16-24y_1y_2+24y_2^2+32y_1 & \text{in (8)} \\
0 & \text{elsewhere} 
\end{cases} \]
This result also checks with that obtained by Niven (1963). This example was just another illustrating how Lemma 3.5 could be used to help find the distribution when the conditions of Theorem 3.1 were not satisfied, this time on account of three points lying in a straight line. It just happened that in this example the terms that were affected contributed only to the region in which the density was zero.
REFERENCES


