Solving Partial Differential Equations Using the Finite Difference Method and the Fourier Spectral Method

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1 Introduction

This paper examines the finite difference methods and spectral methods for solving partial differential equations (PDE). The finite difference method for solving partial differential equations involves approximating derivatives as the difference between values of functions f_i at grid points x_i. This method is used to obtain a numerical solution to an equation. The Fourier spectral method involves writing the solution of the PDE as a Fourier series sum of sinusoidal functions[4].

2 The Finite Difference Method

The finite difference method is used to obtain a numerical solution to a partial differential equation in a bounded domain. The solution to the PDE is replaced with an approximation using a finite number of points in the domain. Increasing the number of points generally increases the accuracy of the numerical solution[1]. The finite difference method involves three methods of approximation, forward difference, backward difference, and central difference.
Forward difference is an explicit method for solving PDEs. It is used to approximate f_i using the current point x_i and the next grid point.

$$f_i = f(x_{i+1}) - f(x_i)$$

Backward difference uses the current grid point and the previous.

$$f_i = f(x_{i-1}) - f(x_i)$$

Central difference uses the grid points on either side of the point x_i.

$$f_i = f(x_{i+1}) - f(x_{i-1})$$

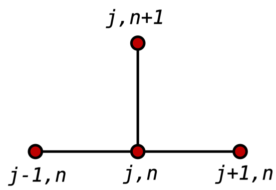

Figure 1: The finite difference method for the heat equation [2]

3 Finite Difference Application with the Heat Equation

Consider the heat equation, and assume a uniform grid

$$\frac{\delta f}{\delta t} = \kappa \frac{\delta^2 f}{\delta^2 x}$$

We have that

$$f_i(t) = f(x_i, t)$$

where f_i is a function of t, and f is a function of (x_i, t).

Taking the time derivative results in

$$\frac{df_i}{dt} = \frac{\delta f(x_i,t)}{\delta t} = \kappa \frac{\delta^2 f(x_i,t)}{\delta x^2}$$

To approximate a second order differential equation using the finite difference method, take the average slope of approximations.

$$\frac{\delta^2 f}{\delta x^2} = \frac{\delta}{\delta x}\frac{\delta f}{\delta x} = \frac{\frac{f_{i+1}-f_i}{\Delta x} - \frac{f_i - f_{i-1}}{\Delta x}}{\Delta x}$$

Which simplifies to

$$\frac{f_{i+1}+f_{i-1}-2f_i}{\Delta x^2}$$

Thus

$$\frac{df_i}{dt} = \kappa \frac{f_{i+1}+f_{i-1}-2f_i}{\Delta x^2}$$

For $i = 1, 2, 3, \ldots$.

4 Fourier Spectral Method

Spectral methods may be applied when the problem involves periodic boundary conditions. This paper focuses on the Fourier spectral method. In this method, the Fourier series can be used as a basis function set[5]. The Fourier spectral method is a technique for solving boundary value problems[3]. It involves applications of Fourier transforms and differentiation of the Fourier series expansion which is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx).$$

5 Solving the Heat Equation With Periodic Boundary Conditions Using Separation of Variables Example

Given

$$4U_t = U_{xx}$$

With initial value

$$u(x,0) = 2\sin(\frac{\pi x}{2}) \text{ - } \sin(\pi x) + 4\sin(2\pi x)$$

and boundary values

$$u(0,t) = 0, u(2,t) = 0.$$

Let

$$u(x,t) = X(x)T(t).$$

We have

$$4X(x)T'(t) = X''(x)T(t)$$

Diving by $X(x)T(t)$ gives

$$\frac{4T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = \lambda.$$

Where λ is a constant. First consider

$$\frac{4T'(t)}{T(t)} = \lambda.$$

Isolating for $T'(t)$ gives

$$T'(t) = \frac{\lambda}{4}T(t)$$
$$\frac{dT}{dt} = \frac{\lambda}{4}T$$
$$\frac{dT}{T} = \frac{\lambda}{4}dt$$

Integrating both sides gives

$$\int \frac{dT}{T} = \int \frac{\lambda}{4}dt$$
$$ln(T) = \frac{\lambda}{4}t + C$$

Solving for $T(t)$ gives

$$T(t) = Ce^{\frac{\lambda t}{4}}$$

Additionally, we have

$$X'' = \lambda x, \text{ where } x(0) = 0, \text{ and } x(2) = 0.$$

Next we solve for eigenvalues starting with $\lambda = 0$. When $\lambda = 0$, we have that $X'' = 0$. Thus $X(x) = AX + B$. Evaluating with initial conditions gives

$$0 = X(0) = B, \text{ so } B = 0$$
$$0 = X(2) = 2A, \text{ so } A = 0.$$

Thus, the solution is trivial for $\lambda = 0$.
Suppose $\lambda > 0$. Let $\lambda = u^2 > 0$. We have

$$X''(x) - u^2 X(x) = 0, \text{ where}$$
$$X(x) = e^{rx}$$
$$X'(x) = re^{rx}$$
$$\text{X''(x)} = \text{r}^2 e^{rx}$$

Substituting the equations above into the original gives

$$r^2 e^{rx} - u^2 e^{rx} = 0.$$

Dividing by e^x gives

$$r^2 - u^2 = 0$$
$$r^2 = u^2$$
$$r = \pm u$$

So

$$X(x) = C_1 e^{ux} + C_2 e^{-ux}$$

Evaluating with initial condition $X(0) = 0$ gives

$$C_1 + C_2 = 0$$
$$C_1 = -C_2$$
$$,$$

and for $X(2) = 0$

$$C_1 e^{2u} + C_2 e^{-2u} = 0$$
$$C_1 e^{2u} - C_1 e^{-2u} = 0$$
$$C_1(e^{2u} - e^{-2u}) = 0$$
$$C_1 = -C_2 = 0$$
$$.$$

Thus, the solution is trivial for $\lambda > 0$.
Finally, suppose $\lambda < 0/$. Let $\lambda = -u^2 < 0$. We have

$$X''(x) + u^2 X(x) = 0.$$

Let $X(x) = e^{rx}$. Then we have $r^2 + u^2 = 0$. Thus, $r = \pm iu$. We have

$$X(x) = C_1 e^{iux} + C_2 e^{-iux}$$
$$X(x) = C_1 sin(ux) + C_2 cos(ux)$$

Applying the boundary condition $X(0) = 0$ gives

$$C_1 sin(0) + C_2 cos(0) = 0$$
$$.$$

So $C_2 = 0$. Next, $X(2) = 0$.

$$X(2) = C_1 sin(2u) = 0$$

It must be the case that

$$sin(2u) = 0.$$

So

$$2u = n\pi \text{ for } n = 1, 2, 3, \ldots.$$

Then, $u = \frac{n\pi}{2}$. So $\lambda_n = -(\frac{n\pi}{2})^2$, and $X_n(x) = sin(\frac{n\pi x}{2})$. We have that

$$U_n(x,t) = X_n(x)T_n(t) = sin(\frac{n\pi x}{2})exp(\frac{-n^2\pi^2}{16}t).$$

So

$$u(x,t) = \sum_{n=1}^{\infty} b_n sin(\frac{n\pi x}{2})exp(\frac{-n^2 pi^2}{16}t).$$

$$\int_0^L sin(\frac{n\pi x}{L})sin(\frac{m\pi x}{L})dx = \begin{cases} 0 & \text{if } m \neq n \\ \frac{L}{2} & \text{if } m = n \end{cases}$$

Solve for b_n.

$$b_n = \frac{2}{L}\int_0^L f(x)sin(\frac{n\pi x}{L})dx = \int_0^2 f(x)sin(\frac{n\pi x}{2})dx.$$

We have that

$$f(x) = 2sin(\frac{\pi x}{2}) - sin(\pi x) + 4sin(2\pi x)$$

and

$$b_n = 2\int_0^2 sin(\frac{\pi x}{2})sin(\frac{n\pi x}{2})dx - \int_0^2 sin(\pi x)sin(\frac{n\pi x}{2})dx + 4\int_0^2 sin(2)sin(\frac{n\pi x}{2})dx$$

Separating by cases gives

$$\int_0^2 sin(\frac{\pi x}{2})sin(\frac{n\pi x}{2})dx = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{else} \end{cases}$$

$$\int_0^2 sin(\pi x)sin(\frac{n\pi x}{2})dx == \begin{cases} 1 & \text{if } n = 2 \\ 0 & \text{else} \end{cases}$$

$$\int_0^2 sin(\pi x)sin(\frac{n\pi x}{2})dx == \begin{cases} 1 & \text{if } n = 4 \\ 0 & \text{else} \end{cases}$$

So

$$b_1 = 2\int_0^2 sin(\frac{\pi x}{2})sin(\frac{n\pi x}{2})dx = 2$$

$$b_2 = 0 - \int_0^2 sin(\pi x)sin(\frac{n\pi x}{2})dx + 0 = -1$$

$$b_4 = 0 - 0 + 4\int_0^2 sin(2\pi x)sin(\frac{n\pi x}{2})dx = 4$$

$$b_n = 0, \text{ otherwise}$$

Thus, we have

$$u(x,t) = 2sin(\frac{\pi x}{2})exp(\frac{\pi^2}{16}t) - sin(\pi x)exp(\frac{-\pi^2}{4}t) - sin(2\pi x)exp(-\pi^2 t).$$

6 Conclusion

The methods discussed in this paper are powerful methods for obtaining a numerical solution to partial differential equations. The finite difference method involves approximating the differential operator by a system of differential quotients. The Fourier spectral method involves the approximation of a solution as an expansion in terms of a spectral function.

7 Bibliography

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