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The Fundamental Groupoid in Discrete Homotopy Theory

Udit Ajit Mavinkurve, *Western University*

Supervisor: Kapulkin, Krzysztof R., *The University of Western Ontario*
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Abstract

Discrete homotopy theory is a homotopy theory designed for studying graphs and for detecting combinatorial, rather than topological, "holes." Central to this theory are the discrete homotopy groups, defined using maps out of grids of suitable dimensions. Of these, the discrete fundamental group in particular has found applications in various areas of mathematics, including matroid theory, subspace arrangements, and topological data analysis.

In this thesis, we introduce the discrete fundamental groupoid, a multi-object generalization of the discrete fundamental group, and use it as a starting point to develop some robust computational techniques. A new notion of covering graphs allows us to extend the existing theory of universal covers to all graphs, and to prove a classification theorem for coverings. We also prove a discrete version of the Seifert–van Kampen theorem, generalizing a previous result of H. Barcelo et al. We then use it to solve the realization problem for the discrete fundamental group through a purely combinatorial construction.

One of the biggest open problems in the subject currently is determining whether the cubical nerve functor provides an equivalence between the discrete homotopy theory of graphs and the classical homotopy theory of spaces. We propose a new line of attack towards this open problem, by breaking it into more tractable problems comparing the homotopy theories of the respective n-types, for each integer $n \geq 0$. We also solve this problem for the first nontrivial case, $n = 1$.

Keywords: discrete homotopy theory, graphs, discrete fundamental groupoid, covering graphs, discrete Seifert–van Kampen theorem, homotopy n-types

Summary for lay audience

An important way to study the functioning of a large-scale network — for instance, with a view towards fixing glitches in its behaviour — is to study its "shape". A specific example of such a large-scale network might be the human connectome, a network where the nodes are individual neurons in the human brain and the edges are synapses between these neurons.

Topology is the area of mathematics concerned with the study of global properties that are invariant under continuous deformations. However, networks are not continuous objects; they are discrete. Thus, it becomes important to adapt techniques from topology and make them suitable for studying discrete objects such as networks.

There are many ways to do this. In this thesis, we follow the approach known as *discrete homotopy theory*, initiated by Barcelo, Laubenbacher, and collaborators in [KL98, BBdLL06]. In this approach, associated to each network, there is an algebraic object called the *discrete fundamental group* that essentially counts the number of functional "holes" in the network. We study a generalized version of this object, called the *discrete fundamental groupoid*. Working in this generalized setting allows us to develop several robust techniques to calculate these algebraic objects.

For instance, if we know the number of holes in two halves of a network (e.g. in the left and the right hemispheres of the human brain) as well as the number of holes that are common to both halves (e.g. in the corpus callosum), we should ideally be able to count the total number of holes in the network by adding up the numbers for the two halves and then subtracting the number common to both halves, since we counted these holes twice. For this counting principle to work, we might want to impose the condition that the entirety of

any hole in the network lies neatly in one or both of the two halves, as opposed to a situation where a portion of a hole lies in one half and the rest in the other half. This is the content of the discrete Seifert–van Kampen theorem proved in [BKLW01]. However, this condition is found to be too restrictive and is not met in several examples we care about. We prove a refinement of the discrete Seifert–van Kampen theorem where we relax the imposed condition and require instead that every hole in the network can be subdivided into several smaller holes such that each of these smaller holes lies neatly in one or both of the two halves. This relaxed condition is found to be met in a larger class of examples.

We develop several such results that aid in computing these network invariants.

Co-Authorship statement

Chapters 2 and 3 comprise material from the paper [KM23] coauthored with my supervisor, Krzysztof Kapulkin. Chapters 4 and 5 comprise material from a forthcoming paper, also coauthored with Krzysztof Kapulkin. In these joint papers, it can be assumed that both authors contributed equally to the development of ideas, although as the junior author, I did most of the writing.

Sections 2.1–2.5, 4.1–4.2, and 5.1 are primarily concerned with exposition of background material, as opposed to original work.

In memory of Mohabat Tarkeshian.

Acknowledgements

This thesis was produced on the traditional lands of the Anishinaabek, Haudenosaunee, Lūnaapéewak, and Chonnonton Nations.

I would like to thank my advisor, Chris Kapulkin, for his guidance throughout my time as a PhD student. It would be impossible to recount all the things he has taught me, within and outside of mathematics. Thank you, Chris, for always accommodating my neuroses and for ensuring that I met important deadlines. The last five years have been anything but dull.

I would like to thank Dan Christensen and Graham Denham for their continued mentorship, feedback, and goodwill. In particular, I would like to thank Dan for suggesting Lemma 3.2.4 and for pointing out an error in an early version of Theorem 3.3.5. I would also like to thank the anonymous referee of [KM23] for a careful reading and several valuable comments.

On a personal level, I am sure I have relied on the kindness of too many people to name. Let me take this opportunity to thank my closest friends, Raman Sarabha, Prakash Singh, and Mohabat Tarkeshian, for supporting me through some very bleak times. I have also been helped in various ways, big and small, by Pranav Chakravarthy, Jeremy Gamble, Chris Hellmann, James Leslie, Apurva Nakade, Joanne Quigley, Alex Rolle, Luis Scoccola, Dinesh Valluri, and Marios Velivasakis.

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Contents

Chapter 1

Introduction

Discrete homotopy theory is an emerging branch of mathematics that extends concepts and techniques from the classical homotopy theory of topological spaces to the study of discrete structures such as simplicial complexes and graphs. If we were to naively treat these discrete structures as topological spaces and allow continuous deformations, we would end up losing valuable combinatorial information. Thus, it becomes important to develop a new notion of homotopy that respects the discrete nature of these structures.

One such discrete notion of homotopy was developed by Barcelo, Laubenbacher, and collaborators in [KL98, BBdLL06]. Their theory built on, and made rigorous, several ideas implicit in the work of Ronald Atkin [Atk74, Atk76] and so, they named it A-homotopy theory in Atkin's honour. Whereas Atkin was motivated by the analysis of social and technological networks, the same notion of homotopy also comes up in very different contexts, e.g., in the work of Maurer [Mau73] on basis graphs of matroids, and the work of Lovász [Lov77] on spanning trees of graphs. Thus, discrete homotopy theory was known to have a wide variety of applications, both within and outside of mathematics from the very beginning. We refer the reader to [BL05] for a survey of applications, including further connections to subspace arrangements and time series analysis.

More recently, discrete homotopy theory has seen significant development, partly driven by new applications in topological data analysis [BCW14, MZ19] and partly by the introduction of new techniques from abstract homotopy theory [CK22].

The work in this thesis represents the natural next step towards building a robust foundation of computational tools within discrete homotopy theory and developing the theory necessary to understand to what extent discrete homotopy theory coincides with the classical homotopy theory of spaces.

Before we review the literature on discrete homotopy theory and discuss our original contributions to this field, it would be helpful to be more precise about what we mean by discrete homotopy theory.

1.1 What is discrete homotopy theory?

The objects of interest are *graphs*. For us, a *graph* X is a set X_V equipped with a reflexive and symmetric relation $X_E \subseteq X_V \times X_V$. The elements of X_V are called *vertices* of the graph, and two vertices x and x' in X are connected by an *edge* if the ordered pair (x, x') is in X_E, in which case we write $x \sim x'$. In our figures, we will suppress the unique loop present at each vertex, and represent each pair of oppositely directed edges by a single undirected edge.

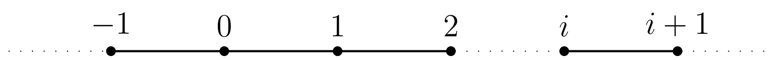
There is an abundance of all sorts of graphs, but for now, we will note the following examples:

1. the *finite interval* graph I_n of length n which has, as vertices, the integers

$0, 1, \ldots, n$, and an edge $i \sim j$ whenever $|i - j| \leq 1$;

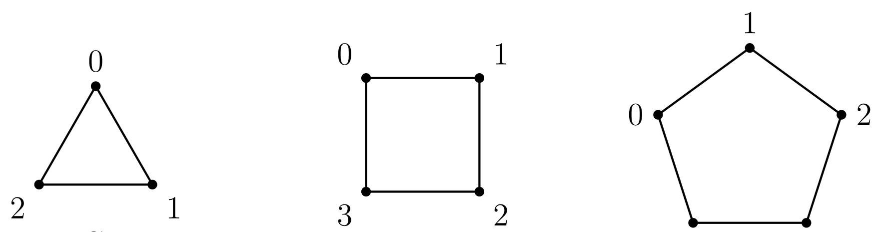
I_0 I_1 I_2 I_3

2. the *infinite interval* graph I_∞ which has, as vertices, all integers $i \in \mathbb{Z}$, and an edge $i \sim j$ whenever $|i - j| \leq 1$;



I_∞

3. the *n-cycle* graph C_n, which is obtained from I_n by identifying the vertex 0 with the vertex n.



C_3 C_4 C_5

A *graph map* $f\colon X \to Y$ is a function $f\colon X_V \to Y_V$ on the underlying set of vertices that preserves the relation. In particular, since our graphs are reflexive, this allows edges in the domain to be collapsed to a single vertex in the codomain.

For example, we have for each $n \geq 0$, the injective graph maps $l, r\colon I_n \to I_{n+1}$ given by $l(i) = i$ and $r(i) = i + 1$ for $0 \leq i \leq n$. We also have the surjective graph maps $s\colon C_{n+1} \to C_n$ given by

$$s(i) = \begin{cases} i & \text{if } 0 \leq i \leq n-1 \\ 0 & \text{if } i = n. \end{cases}$$

We write Graph for the category of graphs and graph maps. It is closely related to the category of metric spaces and 1-Lipschitz functions, which features heavily in topological data analysis [BCW14, MZ19].

Given two graphs X and Y, we can create a new graph $X \,\square\, Y$, called the *box product* of X and Y, as follows:

$$\begin{aligned} (X \,\square\, Y)_V &= X_V \times Y_V \\ (X \,\square\, Y)_E &= \left\{ (x,y) \sim (x',y') \;\middle|\; \begin{array}{r} \text{either } x \sim x' \text{ and } y = y' \\ \text{or } x = x' \text{ and } y \sim y' \end{array} \right\} \end{aligned}$$

The box product is one of many possible ways to define product graphs. We refer the reader to [IK00] for a survey of other ways of defining product graphs.

With the finite interval graphs and the box product operation in hand, we can define homotopies between graph maps. Given two graphs maps $f, g\colon X \to Y$, an *A-homotopy* $H\colon f \Rightarrow g$ of length $n \in \mathbb{N}$ is a graph map of the form $H\colon X \,\square\, I_n \to Y$ such that $H(-,0) = f$ and $H(-,n) = g$. Note that the finite interval graphs I_n are playing the role that the unit interval $[0,1]$ plays in the classical homotopy theory of spaces, and similarly the box product is playing the role that the product topology plays.

This leads to a notion of *A-homotopy equivalences* in the usual way. A graph map $f\colon X \to Y$ is an *A-homotopy equivalence* if there exists a graph map $g\colon Y \to X$ along with A-homotopies $g \circ f \Rightarrow \mathrm{id}_X$ and $f \circ g \Rightarrow \mathrm{id}_Y$.

One can show that the unique map $I_n \to I_0$ is an A-homotopy equivalence for any $n \in \mathbb{N}$, whereas the unique map $I_\infty \to I_0$ is not. Perhaps more surprisingly, the unique map $C_n \to I_0$ is an A-homotopy equivalence for $n = 3$ and $n = 4$. Note that if we were treating graphs as topological spaces, then the infinite interval graph I_∞ would be homeomorphic to the real line $\mathbb{R}$, which is contractible, and all the cycle graphs C_n would be homeomorphic to the unit circle S^1, which is not. This shows us that discrete homotopy theory diverges from the usual theory of treating graphs as 1-dimensional CW complexes.

Given a graph X along with a distinguished vertex x_0, we can define its *nth discrete homotopy group*, denoted $A_n(X, x_0)$, as follows:

$$A_n(X, x_0) = \left\{ f \colon I_\infty^{\square\, n} \to X \;\;|\;\; f(\mathbf{i}) = x_0 \text{ for all but finitely many } \mathbf{i} \in I_\infty^{\square\, n} \right\} / \sim_*$$

where we are quotienting by A-homotopies that also take the value x_0 at all but finitely many vertices in $I_\infty^{\square\, n+1}$. The group operation is given by concatenating the finite non-constant regions. We note that this definition is entirely analogous to the definition of homotopy groups of pointed spaces, which are defined using continuous maps out of the unit n-cube $[0, 1]^n$ that map the boundary of the n-cube to the basepoint.

We say that a graph map $f \colon X \to Y$ is a *weak A-homotopy equivalence* if the induced map $f_* \colon A_n(X, x) \to A_n(Y, fx)$ is an isomorphism for all $n \geq 0$ and all vertices $x \in X$. It can be shown that the graph maps $s \colon C_{n+1} \to C_n$ defined earlier are weak A-homotopy equivalences for all $n \geq 5$. Note that they are not A-homotopy equivalences, since any graph map of the form $C_n \to C_{n+1}$ fails to be surjective and is A-homotopic to a constant map. The unique map $I_\infty \to I_0$ is another example of a weak A-homotopy equivalence that is not an A-homotopy equivalence. This suggests that the class of weak A-homotopy equivalences, rather than the class of A-homotopy equivalences, is the correct notion for weak equivalences in Graph.

Discrete homotopy theory, then, is the study of the category Graph, equipped with the class of weak A-homotopy equivalences.

1.2 Literature review

The systematic study of a discrete homotopy theory for simple, undirected graphs can be traced back to the work of Malle [Mal83], where we can find definitions of what are now popularly known as A-homotopies, A-homotopy equivalences, and the discrete fundamental group of a graph. Malle proves that if a graph contains no 3- or 4-cycles, then its discrete fundamental group is isomorphic to the classical fundamental group of the graph treated as a 1-dimensional CW complex. He also characterizes connected graphs with trivial discrete fundamental group, using the notion of pseudoplanar nets. In fact, our definition of a *net* (see Definition 3.3.1) is inspired from Malle's pseudoplanar nets.

In [BKLW01], Barcelo et al. improve on Malle's result by proving that the discrete fundamental group of a graph is isomorphic to the quotient of the classical fundamental group of the graph (treated as a 1-dimensional CW complex) by the normal subgroup generated by the 3- and 4-cycles. They also define the higher discrete homotopy groups, and prove a discrete version of the Seifert–van Kampen theorem. We note that although most of this paper deals with simplicial complexes and their A-groups, all the constructions factor through the category of graphs.

Graphs take center stage in [BBdLL06], where the authors introduce cubical methods to study discrete homotopy theory. They associate to each graph

Γ, a CW complex X_Γ, constructed from Γ (treated as a 1-dimensional CW complex) by attaching an n-cell for every graph map of the form $I_1^{\square\, n} \to X$. They emphasize that this CW complex should be understood as the geometric realization of a certain cubical set associated to the graph, and conjecture that, for each integer $n \geq 0$, the nth discrete homotopy group of Γ is isomorphic to the nth classical homotopy group of X_Γ. Note that this further generalizes the earlier results comparing the discrete fundamental group of a graph with the classical fundamental group of the graph treated as a 1-dimensional CW complex. This conjecture is later settled by Carranza and Kapulkin in [CK22].

In the meantime, a discrete homology theory for graphs is developed in [BCW14]. Although most of this paper deals with metric spaces, all the constructions factor through the category of graphs. The authors formulate a discrete anlalogue of the Eilenberg–Steendrod axioms and show that the discrete homology theory satisfies these axioms. They also prove a discrete analogue of the Hurewicz theorem in dimension 1. The higher-dimensional version of this discrete Hurewicz theorem is proven in [CK22].

A different, but related, homology theory for graphs is developed in [GLMY14]. In [BGJW19], the authors offer a comparison of these two homology theories. In particular, they construct a graph for which these two homology theories do not coincide.

A theory of covering graphs is partially developed in [Har19] as a technique to compute the discrete fundamental group of the n-cycle graphs for $n \geq 5$ without relying on comparisons with the classical homotopy theory of spaces. The central notion in this treatment is that of a *local isomorphism*. The theory of local isomorphisms as covering graphs is well-behaved only when the base graph does not contain any 3- or 4-cycles. For instance, in [BGJW21], the authors give a construction for a simply connected cover over any base graph. But this construction has the desired universal property only if the base graph does not contain any 3- or 4-cycles. In the same paper, the authors also prove that graphs without 3- or 4-cycles have trivial discrete homology groups in degrees ≥ 2. An analogue of this result for discrete homotopy groups is proven by Lutz in [Lut21], i.e. graphs without 3- or 4-cycles have trivial discrete homotopy groups in degrees ≥ 2. Thus, graphs without any 3- or 4-cycles behave like 1-dimensional CW complexes and are not particularly interesting.

Another important contribution of [BGJW21] is the construction of graphs with nontrivial discrete homology groups in any given non-negative degree. In [Lut21], Lutz provides a similar construction of graphs with nontrivial discrete homotopy groups in any given non-negative degree.

1.3 Summary of results

1.3.1 A Seifert–van Kampen theorem for graphs

The discrete homotopy groups $A_n(X, x_0)$ of a pointed graph (X, x_0) are easy to define, but generally hard to compute. A discrete analogue of the Seifert–van Kampen theorem is thus a crucial tool to have in our computational toolbox. Recall that the classical Seifert–van Kampen theorem gives a sufficient condition for when the fundamental group of the union of two spaces can be computed in terms of their individual fundamental groups. Using the language of pushouts, it can be stated as follows:

Theorem (Seifert–van Kampen)**.** *Consider a pushout square*

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & X_1 \arrow[d] \\ X_2 \arrow[r] & X \arrow[ul, phantom, very near start, "\ulcorner"] \end{tikzcd}$$

of pointed, connected spaces. If the maps $X_0 \to X_1$ and $X_0 \to X_2$ are both open embeddings, then the above pushout square is preserved by the fundamental group functor $\pi_1 \colon \mathsf{Top}_* \to \mathsf{Grp}$.

The requirement that the maps $X_0 \to X_1$ and $X_0 \to X_2$ must both be open embeddings may appear straightforward, but is hard to translate to the discrete setting of graphs. Here is the analogous statement in discrete homotopy theory:

Theorem (Theorem 3.3.5)**.** *Consider a pushout square*

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & X_1 \arrow[d] \\ X_2 \arrow[r] & X \arrow[ul, phantom, very near start, "\ulcorner"] \end{tikzcd}$$

of pointed, connected graphs. If every map $h \colon I_1 \square I_1 \to X$ satisfies the following net resolution condition:

$$h \text{ admits a net } (H, s) \text{ such that each cell of } H \text{ factors through } X_1 \text{ or } X_2$$

then the above pushout square is preserved by the discrete fundamental group functor $A_1 \colon \mathsf{Graph}_* \to \mathsf{Grp}$.

Here, a net of h is a pair (H, s) consisting of a map $H \colon I_m \square I_n \to X$ for some $m, n \in \mathbb{N}$ together with an order-preserving, surjective map $s \colon I_{2m+2n} \to$

I_4 such that $H \circ \partial_{m,n} = h \circ \partial_{1,1} \circ s$, where $\partial_{m,n} \colon I_{2m+2n} \to I_m \,\square\, I_n$ is the map that wraps around the interval of length $2m + 2n$ along the boundary of the $m \times n$ grid graph $I_m \,\square\, I_n$. A cell of H is any map $I_1 \,\square\, I_1 \to I_m \,\square\, I_n$ composed with H.

Theorem 3.3.5 improves upon a similar result of [BKLW01] which requires that every map $h \colon I_1 \,\square\, I_1 \to X$ itself factors through X_1 or X_2. However, this stricter condition is not met in several examples of interest (see e.g. Fig. 1.1), where we still expect the pushout square to be preserved by the functor $A_1 \colon \mathsf{Graph}_* \to \mathsf{Grp}$. These examples are found to satisfy our net resolution condition, which is a weaker requirement.

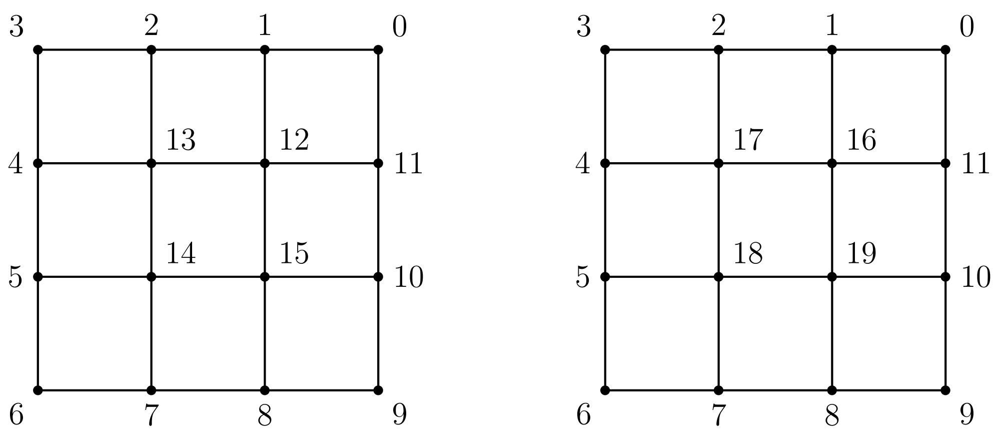


Figure 1.1: An example of a graph whose discrete fundamental group can be computed using Theorem 3.3.5, but not using [BKLW01, Thm. 2.8]. Vertices with the same label are identified. For a more detailed discussion, see Example 3.3.7.

1.3.2 Realization problem for the discrete fundamental group

One might ask whether every group can be realized as the discrete fundamental group of some graph. We answer this question in the affirmative by providing, for any group G, together with a choice of presentation $G = \langle S \mid R \rangle$, an explicit construction of a graph $X_{S,R}$, whose discrete fundamental group we then compute using Theorem 3.3.5.

Theorem (Theorem 3.4.5)**.** *Given any group G with presentation $G = \langle S \mid R \rangle$, we have:*

$$A_1(X_{S,R}, x_0) \cong G, \qquad \textit{for any } x_0 \in X_{S,R}.$$

This construction is a key step towards constructing Eilenberg–MacLane graphs, i.e. graphs that have exactly one non-trivial discrete homotopy group. Note that, except for a few special cases, constructing Eilenberg–MacLane graphs remains an open problem.

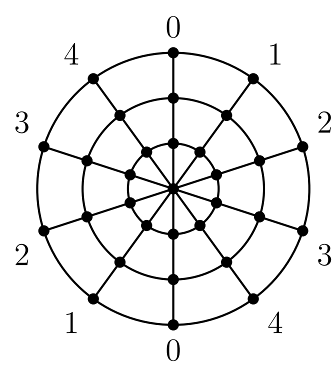


Figure 1.2: The graph $X_{\{a\},\{a^2\}}$ with discrete fundamental group $\mathbb{Z}/2\mathbb{Z}$. Vertices with the same label are identified.

1.3.3 Theory of covering graphs

The theory of covering graphs was partially developed in [BGJW21, Har19] as a technique to compute the discrete fundamental group of the n-cycle graphs for $n \geq 5$. The central notion in these treatments was that of a *local isomorphism.*

Given a graph X and a vertex x, the *(star) neighbourhood* N_x of x in X is defined as follows:

$$
\begin{aligned}
(N_x)_V &= \{x' \in X_V \mid x' \sim x \text{ in } X\} \\
(N_x)_E &= \{x' \sim x'' \text{ in } X \mid x' = x \text{ or } x'' = x\}\,.
\end{aligned}
$$

A graph map $f\colon X \to Y$ is a *local isomorphism* if, for every vertex $x \in X$, the restriction $f|_{N_x}\colon N_x \to N_{f(x)}$ is an isomorphism.

The main drawback of working with local isomorphisms is that they don't enjoy the path-homotopy lifting property unless the base graph is known to contain no 3- or 4-cycles. This is a serious drawback since such graphs have no nontrivial higher homotopy groups [Lut21]. Most graphs that we might want to work with, such as the graphs $X_{S,R}$ constructed above, have many 3- and 4-cycles. To rectify this issue, we introduce a new notion of coverings.

Definition (Definition 3.2.9)**.** *A map $f\colon X \to Y$ is a* covering *if, in addition to being a local isomorphism, it has the right lifting property against the open box inclusion $\sqsubset\colon I_3 \to I_1 \,\square\, I_1$. That is, given any commutative square as follows:*

$$
\begin{tikzcd}
I_3 \arrow[r] \arrow[d, "\sqsubset"'] & X \arrow[d, "f"] \\
I_1 \,\square\, I_1 \arrow[r] \arrow[ur, dotted] & Y
\end{tikzcd}
$$

there exists a map $I_1 \,\square\, I_1 \to Y$ that makes both triangles commute.

Unlike local isomorphisms, coverings enjoy the path-homotopy lifting property without any restriction on the base graph. The new definition also allows us to give a more complete development of the theory of covering graphs. In particular, we are able to prove the following classification result for covering graphs:

Theorem (Corollary 3.2.23)**.** *For any pointed, connected graph (X, x_0), there is a Galois correspondence between connected coverings over (X, x_0) and the lattice of subgroups of $A_1(X, x_0)$ ordered by reverse inclusion.*

Finally, defining a *universal cover* of a pointed graph (Y, y_0) to be any pointed covering $p\colon (X, x_0) \to (Y, y_0)$ where the covering graph X is simply connected, we are able to prove the following:

Theorem (Theorem 3.2.29)**.** *Every pointed graph (X, x_0) admits a universal cover $(\widetilde{X}_{x_0}, [c_{x_0}])$. Furthermore, the universal cover is unique up to a unique isomorphism.*

Note that these statements have verbatim analogues for topological spaces.

1.3.4 Comparing homotopy theories

One might ask whether, or to what extent, the discrete homotopy theory of graphs (i.e. the study of the category Graph of graphs equipped with the class of weak A-homotopy equivalences) coincides with the classical homotopy theory of spaces (i.e. the study of the category Top of topological spaces equipped with the class of weak homotopy equivalences). Let us make this question more precise, using the language of *fibration categories*.

A *fibration category* is a category equipped with a class of weak equivalences and a class of fibrations, subject to certain axioms (see Definition 5.1.1). The extra structure provided by the fibrations is useful for many constructions in homotopy theory, such as for computing homotopy limits. It also gives a convenient description of the homotopy category, i.e. the category obtained by formally inverting all the weak equivalences. We compare fibration categories by means of exact functors (see Definition 5.1.9). An exact functor between two fibration categories is a *weak equivalence* if the induced functor on homotopy categories is an equivalence of categories.

In [CK22], the authors describe a fibration category structure on Graph, where the weak equivalences are the weak A-homotopy equivalences. They also show that the *cubical nerve functor* $N \colon \mathsf{Graph} \to \mathsf{Kan}$ is an exact functor of fibration categories, where Kan denotes the fibration category of cubical Kan complexes and serves as an alternative model for the classical homotopy theory of spaces. What is not known is whether the cubical nerve functor $N \colon \mathsf{Graph} \to \mathsf{Kan}$ is a weak equivalence of fibration categories. This is currently one of the biggest open problems in the field. Indeed, if true, it will allow us to import powerful results and techniques from classical homotopy theory, such as the Blakers-Massey theorem, over to the discrete setting. In this thesis, we provide a new line of attack towards this problem. The following theorem provides the context required to describe our proposed strategy.

Theorem (Theorems 4.3.10 and 5.3.2)**.**

1. *For each integer $n \geq 0$, we have a model category structure*[1] *on the category* cSet *of cubical sets, where the weak equivalences are cubical maps $f \colon X \to Y$ that induce isomorphisms $f_* \colon \pi_k(X, x) \to \pi_k(Y, fx)$*

[1] A model category is similar to a fibration category, but more convenient to work with since we have both fibrations and cofibrations. Given any model category $\mathcal{M}$, the full subcategory $\mathcal{M}^{\mathrm{fib}}$ of fibrant objects forms a fibration category with weak equivalences and fibrations inherited from $\mathcal{M}$.

for every $0 \leq k \leq n$ *and every* $x \in X$*, and such that the cubical nerve of every graph is a fibrant object. We denote this model category by* $\mathsf{cSet}_{n\text{-types}}$*.*

2. *For each integer* $n \geq 0$*, we have a fibration category structure on the category* Graph*, where the weak equivalences are graph maps* $f\colon X \to Y$ *that induce isomorphisms* $f_*\colon A_k(X, x) \to A_k(Y, fx)$ *for every* $0 \leq k \leq n$ *and every* $x \in X$*. We denote this fibration category by* $\mathsf{Graph}_{n\text{-types}}$*.*

3. *For each integer* $n \geq 0$*, the cubical nerve functor* $N\colon \mathsf{Graph}_{n\text{-types}} \to \mathsf{cSet}^{\mathrm{fib}}_{n\text{-types}}$ *is an exact functor of fibration categories.*

We may now ask whether the cubical nerve functor $N\colon \mathsf{Graph}_{n\text{-types}} \to \mathsf{cSet}^{\mathrm{fib}}_{n\text{-types}}$ is a weak equivalence of fibration categories for any given integer $n \geq 0$. We expect this problem to more tractable, especially since there are algebraic models for homotopy n-types of spaces.

For instance, for $n = 1$, the fibration category $\mathsf{cSet}^{\mathrm{fib}}_{1\text{-types}}$ is known to be weakly equivalent to the classical fibration category structure on the category Gpd of groupoids (see, e.g., [Hat02, Exercise 4.3.4]). Using the computational techniques developed for the discrete fundamental groupoid of a graph, we prove the following.

Theorem (Theorem 5.3.11)**.** *The fundamental groupoid functor* $\Pi_1\colon \mathsf{Graph}_{1\text{-types}} \to \mathsf{Gpd}$ *is a weak equivalence of fibration categories.*

Chapter 2

Discrete Homotopy Theory

In this chapter, we review the background material on discrete homotopy theory that will be used throughout the thesis, and also prove some preliminary (but original) results comparing different formulations of the A-homotopy groups. In Section 2.1, we recall the definition of a graph and a graph map, list some important examples, and prove that the category of graphs is (co)complete.

In Section 2.2, we recall the definition of the box product of graphs (often also called the Cartesian product), which is distinct from the categorical product (often also called the strong product). Like the categorical product, the box product also defines a closed symmetric monoidal structure on the category of graphs, but the box product generally has fewer edges than the categorical product. The box product forms the basis of the branch of discrete homotopy theory often called A-theory, which is the focus of this thesis.

In Section 2.3, we recall the definitions of combinatorial paths in a graph, defined either as graph maps out of the finite interval graphs, or as graph maps out of infinite interval graphs that are constant almost everywhere. Both definitions give rise to the same notion of path-connectedness, which interacts well with the box product; see Proposition 2.3.10.

In Section 2.4, we recall the definitions of A-homotopies between graph maps, which naturally leads to the notion of A-homotopy equivalences between graphs.

In Section 2.5, we recall the definitions of path-homotopies between paths in a graph, and of the A-homotopy groups of a graph, whose underlying sets are given by the set of path-components of iterated loop graphs. This leads us the notion of weak A-homotopy equivalences, i.e. graph maps inducing isomorphisms on all A-homotopy groups, as well as the weaker notion of n-equivalences, i.e. graph maps inducing isomorphisms on the first n A-homotopy groups. The material in Sections 2.1, 2.2, 2.3, 2.4 and 2.5 is adapted from the papers [BKLW01] and [BBdLL06] and we do not make any claims of originality.

Finally, in Section 2.6, we compare different formulations of the A-homotopy groups. Historically, e.g., in [BBdLL06, BL05], the A-homotopy groups of a graph have been defined using maps out of an infinite grid graph and requiring that these maps are constant outside of a fixed finite region. Alternatively, one could define the A-homotopy groups using maps out of finite grids, without stipulating any additional conditions. Each of these approaches has its own set of advantages and disadvantages. What is not immediately clear in this second approach is what identifications need to be made when defining maps on grids of different sizes. To describe them, we adapt the notion of reparametrization by a shrinking map, originally used in [GLMY14], to the setting of undirected graphs; see Definition 2.6.1 below. The equivalence of the two approaches is then established in Theorem 2.6.14.

2.1 The category of graphs

We begin by recalling the definition of a graph and a graph map.

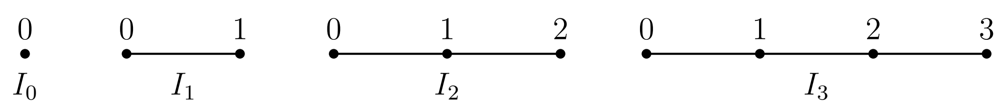
Definition 2.1.1.

1. A *graph* $X = (X_V, X_E)$ is a set X_V equipped with a reflexive and symmetric relation $X_E \subseteq X_V \times X_V$. Elements of X_V are called *vertices* of the graph, and two vertices x and x' in X are connected by an *edge* if $(x, x') \in X_E$, in which case we write $x \sim x'$ and say x and x' are adjacent. In our figures, we will suppress the unique loop present at each vertex, and represent each pair of oppositely directed edges by a single undirected edge.

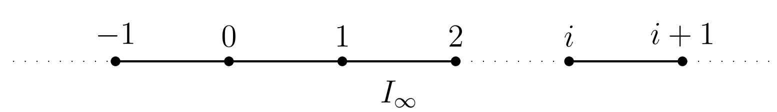
2. A *graph map* $f\colon X \to Y$ is a set-function $f\colon X_V \to Y_V$ that preserves the relation. We write Graph for the the category of graphs and graph maps.

Notation 2.1.2.

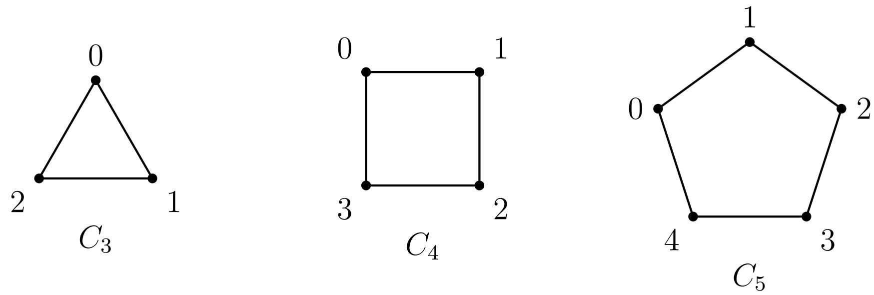
1. The *finite interval* graph I_n of length $n \in \mathbb{N}$ has, as vertices, the integers $0, 1, \ldots, n$, and an edge $i \sim j$ whenever $|i - j| \leq 1$.



2. The *infinite interval* graph I_∞ has, as vertices, all integers $i \in \mathbb{Z}$, and an edge $i \sim j$ whenever $|i - j| \leq 1$.



3. The *n-cycle* graph C_n has, as vertices, the integers mod n, and an edge $i \sim j$ whenever $|i - j| \leq 1$.



Proposition 2.1.3. *The forgetful functor $(-)_V \colon \mathsf{Graph} \to \mathsf{Set}$ that maps a graph to its underlying vertex set admits both adjoints.*

$$\begin{tikzcd} \mathsf{Graph} \arrow[r, "(-)_V"] & \mathsf{Set} \arrow[l, bend right, "\mathsf{discrete}"', "\perp"] \arrow[l, bend left, "\mathsf{complete}", "\perp"'] \end{tikzcd}$$

The left adjoint to $(-)_V$ maps a set X to the discrete graph on X, given by the pair $(X, \{(x, x) \mid x \in X\})$, while the right adjoint to $(-)_V$ maps a set X to the complete graph on X, given by the pair $(X, X \times X)$. In particular, $(-)_V$ preserves limits and colimits. $\square$

Proposition 2.1.4. *The category Graph is (co)complete.*

Proof. Let J be any small category and let $F \colon \mathsf{J} \to \mathsf{Graph}$ be any functor. The graph $\lim F$ is constructed as follows: the underlying vertex set $(\lim F)_V$ is the limit in Set of the composite functor $(-)_V \circ F \colon \mathsf{J} \to \mathsf{Set}$. This set comes equipped with set-functions $\lambda_j \colon (\lim F)_V \to (Fj)_V$. The relation $(\lim F)_E$ is given by:

$$(\lim F)_E = \{(x, x') \mid x, x' \in (\lim F)_V,\ (\lambda_j(x), \lambda_j(x')) \in (Fj)_E\ \forall\ j \in \mathsf{J}\}\,.$$

The graph $\operatorname{colim} F$ is constructed as follows: the underlying vertex set $(\operatorname{colim} F)_V$ is the colimit in Set of the composite functor $(-)_V \circ F \colon \mathsf{J} \to \mathsf{Set}$. This set comes equipped with set-functions $\lambda_j \colon (Fj)_V \to (\operatorname{colim} F)_V$. The relation $(\operatorname{colim} F)_E$ is given by:

$$(\operatorname{colim} F)_E = \{(\lambda_j(x), \lambda_j(x')) \mid (x, x') \in (Fj)_E,\ j \in \mathsf{J}\}.$$

It is straightforward to verify that both these graphs satisfy their respective universal properties. □

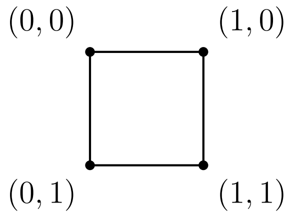
2.2 Monoidal structure on the category of graphs

To define the notion of homotopy, we need to review the definition of the product of graphs. Although Graph has the categorical product, in the context of discrete homotopy theory, we work with a different monoidal product, known as the box product.

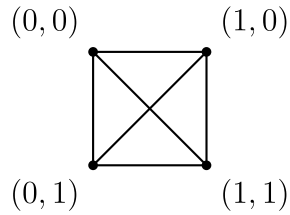
Definition 2.2.1. The *box product* $X \,\square\, Y$ of two graphs X and Y is defined as follows:

$$(X \,\square\, Y)_V = X_V \times Y_V$$
$$(X \,\square\, Y)_E = \left\{ (x,y) \sim (x',y') \;\middle|\; \begin{array}{r} \text{either } x \sim x' \text{ and } y = y' \\ \text{or } x = x' \text{ and } y \sim y' \end{array} \right\}$$

The box product differs from the categorical product in Graph. For instance, the box product $I_1 \,\square\, I_1$ is depiected below on the left, whereas the categorical product $I_1 \times I_1$ is depicted below on the right.



$I_1 \square I_1$



$I_1 \times I_1$

Given two graphs X and Y, we have the maps $\pi_X \colon X \square Y \to X$ and $\pi_Y \colon X \square Y \to Y$ given by the following composites:

$$\pi_X = \left(X \square Y \xrightarrow{\mathrm{id}_X \, \square \, !} X \square I_0 \xrightarrow{\cong} X \right)$$

$$\pi_Y = \left(X \square Y \xrightarrow{! \, \square \, \mathrm{id}_Y} I_0 \square Y \xrightarrow{\cong} Y \right)$$

Thus, we have a map from the box product to the categorical product:

$$(\pi_X, \pi_Y) : X \square Y \to X \times Y$$

We can check that this map is always bijective on vertices and injective on edges.

Definition 2.2.2. The *internal hom* $\hom^{\square}(X, Y)$ of two graphs X and Y is defined as follows:

$$\begin{aligned} \hom^{\square}(X,Y)_V &= \mathsf{Graph}\,(X,Y) \\ \hom^{\square}(X,Y)_E &= \left\{ f \sim g \;\middle|\; \begin{array}{l} f, g \in \mathsf{Graph}\,(X,Y) \text{ and} \\ \forall\, x \in X, f\,(x) \sim g\,(x) \end{array} \right\} \end{aligned}$$

One can then easily verify the following:

Proposition 2.2.3. *The category* $\left(\mathsf{Graph}, \square, I_0, \hom^{\square}(-,-)\right)$ *is a closed symmetric monoidal category. In other words, given graphs* X, Y, Z*, we have a bijection*

$$\mathsf{Graph}\,(X \square Y, Z) \cong \mathsf{Graph}\left(X, \hom^{\square}(Y, Z)\right)$$

that is natural in X*,* Y*, and* Z*.* □

In [KK23], the authors prove that, apart from the categorical product, the box product is the only closed symmetric monoidal structure on Graph.

2.3 Paths in a graph

We now define combinatorial paths and recall some of their basic properties.

Definition 2.3.1. Let X be a graph and $x, x' \in X_V$ be vertices.

1. A *path* $\gamma\colon x \rightsquigarrow x'$ in X of *length* $n \in \mathbb{N}$ is a map $\gamma\colon I_n \to X$ such that $\gamma(0) = x$ and $\gamma(n) = x'$.

2. A *path* $\gamma\colon x \rightsquigarrow x'$ in X of *length* ∞ is a map $\gamma\colon I_\infty \to X$ for which there exist integers $N_-, N_+ \in \mathbb{Z}$ such that $\gamma(i) = x$ for all $i \leq N_-$ and $\gamma(i) = x'$ for all $i \geq N_+$.

Definition 2.3.2. Two vertices x and x' in a graph X are *path-connected* if there exists a path from x to x'. A graph X is *path-connected* if every pair of vertices in X is path-connected.

Definition 2.3.3. Given any vertex x in a graph X, the *constant path* $\mathrm{c}_x\colon x \rightsquigarrow x$ of length $n \in \mathbb{N} \cup \{\infty\}$ is given by the composite

$$\mathrm{c}_x = \left(I_n \xrightarrow{!} I_0 \xrightarrow{x} X \right)$$

Definition 2.3.4. Let X be a graph and $x, x' \in X_V$ be vertices.

1. Given a path $\gamma\colon x \rightsquigarrow x'$ in X of finite length n, its *inverse path* $\overline{\gamma}\colon x' \rightsquigarrow x$, defined by a map $\overline{\gamma}\colon I_n \to X$, is given by $\overline{\gamma}(i) = \gamma(n-i)$.

2. Given a path $\gamma\colon x \rightsquigarrow x'$ in X of length ∞, its *inverse path* $\overline{\gamma}\colon x' \rightsquigarrow x$, defined by a map $\overline{\gamma}\colon I_\infty \to X$, is given by $\overline{\gamma}(i) = \gamma(-i)$.

Definition 2.3.5. Let X be a graph and $x, x', x'' \in X_V$ be vertices.

1. Given two paths $\gamma\colon x \rightsquigarrow x'$ and $\sigma\colon x' \rightsquigarrow x''$ in X of finite lengths m and n respectively, their *concatenation* $\gamma * \sigma\colon x \rightsquigarrow x''$, defined by a map $\gamma * \sigma\colon I_{m+n} \to X$, is given by

$$(\gamma * \sigma)(i) = \begin{cases} \gamma(i) & \text{if } i \leq m \\ \sigma(i-m) & \text{if } i \geq m \end{cases}$$

2. Given two paths $\gamma\colon x \rightsquigarrow x'$ and $\sigma\colon x' \rightsquigarrow x''$ in X, both of length ∞, their *concatenation* $\gamma * \sigma\colon x \rightsquigarrow x''$, defined by a map $\gamma * \sigma\colon I_\infty \to X$, is given by

$$(\gamma * \sigma)(i) = \begin{cases} \gamma(i) & \text{if } i \leq M_+ \\ \sigma(i - M_+ + N_-) & \text{if } i \geq M_+ \end{cases}$$

   where M_+ is the smallest integer such that $\gamma(i) = x'$ for all $i \geq M_+$ and N_- is the largest integer such that $\sigma(i) = x'$ for all $i \leq N_-$.

Definition 2.3.6. Let $n \in \mathbb{N}$ and for $i = 1, \ldots, n$, let $\mathrm{e}_i\colon I_1 \to I_n$ be the graph map given by

$$\mathrm{e}_i(j) = \begin{cases} i-1 & \text{if } j = 0 \\ i & \text{if } j = 1 \end{cases}$$

Given a path $\gamma\colon x \rightsquigarrow x'$ in X of finite length n, its *i-th edge* is given by the path $\mathrm{e}_i^*(\gamma) = \gamma \circ \mathrm{e}_i \colon \gamma(i-1) \rightsquigarrow \gamma(i)$ of length 1, where $i = 1, \ldots, n$.

Proposition 2.3.7. *Path-connectedness of vertices in a graph X is the equivalence relation on X_V generated by adjacency of vertices.* □

Given a graph map $f\colon X \to Y$ and a path $\gamma\colon x \rightsquigarrow x'$ in X, the composite $f \circ \gamma$ is a path in Y from $f(x)$ to $f(x')$. Thus, if two vertices in X are path-connected, then their respective images under f are path-connected in Y.

Definition 2.3.8. The *set of path-components* $\pi_0 X$ of a graph X is the set of equivalence classes of vertices of X under path-connectedness. A graph map $f\colon X \to Y$ induces a set-function $f_*\colon \pi_0 X \to \pi_0 Y$. The assignment $f \mapsto f_*$ is clearly functorial, and we have a well-defined functor $\pi_0\colon \mathsf{Graph} \to \mathsf{Set}$.

Proposition 2.3.9. *The functor $\pi_0\colon \mathsf{Graph} \to \mathsf{Set}$ is left adjoint to the functor $\mathsf{Set} \to \mathsf{Graph}$ that maps a set X to the discrete graph on X.*

$$\begin{tikzcd} \mathsf{Set} \arrow[r, "\mathsf{discrete}" description] & \mathsf{Graph} \arrow[l, bend right=40, "\pi_0"'] \arrow[l, bend left=40, "(-)_V"] \end{tikzcd}$$

□

Given two graphs X and Y, we can apply π_0 to the maps $\pi_X\colon X \mathbin{\square} Y \to X$ and $\pi_Y\colon X \mathbin{\square} Y \to Y$ and then take their product in Set to obtain the following set-function:

$$\pi_0(X \mathbin{\square} Y) \to \pi_0 X \times \pi_0 Y$$

Proposition 2.3.10. *The functor $\pi_0\colon \mathsf{Graph} \to \mathsf{Set}$ is strong monoidal. In particular, the set-function $\pi_0(X \mathbin{\square} Y) \to \pi_0 X \times \pi_0 Y$ is a bijection that is natural in the graphs X and Y.*

Proof. Clearly the functor $\pi_0 \colon \mathsf{Graph} \to \mathsf{Set}$ preserves units, since $\pi_0 I_0 \cong \{*\}$. In order to check that the set-function $\pi_0 (X \mathbin{\square} Y) \to \pi_0 X \times \pi_0 Y$ is a bijection, we need to check that given two vertices x, x' that are path-connected in X and two vertices y, y' that are path-connected in Y, the pairs (x, y) and (x', y') are path-connected in the box-product $X \mathbin{\square} Y$. Let $\gamma \colon x \rightsquigarrow x'$ be a path in X and let $\sigma \colon y \rightsquigarrow y'$ be a path in Y. Then, the concatenation $(\gamma \mathbin{\square} \mathrm{c}_y) * (\mathrm{c}_{x'} \mathbin{\square} \sigma)$ is a path from (x, y) to (x', y') in $X \mathbin{\square} Y$. $\square$

2.4 Homotopy theory of graphs

We are now ready to define homotopies and homotopy equivalences. Before doing so, we make a small observation that helps motivate these notions.

Observe that two maps $f, g \colon X \to Y$ are adjacent in the graph $\hom^{\square}(X, Y)$ if and only if there exists a graph map $H \colon I_1 \to \hom^{\square}(X, Y)$ such that $H(0) = f$ and $H(1) = g$, or equivalently, a graph map $H \colon X \mathbin{\square} I_1 \to Y$ such that $H(-, 0) = f$ and $H(-, 1) = g$.

Definition 2.4.1. Let X and Y be two graphs and $f, g \colon X \to Y$ be two graph maps.

1. An *A-homotopy* $H \colon f \Rightarrow g$ of length $n \in \mathbb{N}$ is a map $H \colon X \mathbin{\square} I_n \to Y$ such that $H(-, 0) = f$ and $H(-, n) = g$.

2. An *A-homotopy* $H \colon f \Rightarrow g$ of length ∞ is a map $H \colon X \mathbin{\square} I_\infty \to Y$ for which there exist integers $N_-, N_+ \in \mathbb{Z}$ such that $H(-, i) = f$ for all $i \leq N_-$ and $H(-, i) = g$ for all $i \geq N_+$.

When such an A-homotopy exists, we say that f and g are A-homotopic and write $f \sim_A g$. Equivalently, an A-homotopy $H \colon f \Rightarrow g$ between two graph maps $f, g\colon X \to Y$ is a path $H\colon f \rightsquigarrow g$ in the graph $\hom^{\square}(X, Y)$. It follows that $\sim_A$ is an equivalence relation on the set $\mathsf{Graph}\,(X, Y)$.

Definition 2.4.2. A graph map $f\colon X \to Y$ is an *A-homotopy equivalence* if there exists some graph map $g\colon X \to Y$ such that $g \circ f \sim_A \mathrm{id}_X$ and $f \circ g \sim_A \mathrm{id}_Y$.

Example 2.4.3. The unique map $I_n \to I_0$ is an A-homotopy equivalence for every $n \in \mathbb{N}$. In contrast, the unique map $I_\infty \to I_0$ is *not* an A-homotopy equivalence.

Example 2.4.4. The unique map $C_n \to I_0$ is an A-homotopy equivalence for $n = 3, 4$.

2.5 Homotopy groups of graphs

Definition 2.5.1. Let X be a graph and $x_0, x_1 \in X_V$ be two vertices. For $m \in \mathbb{N} \cup \{\infty\}$, the m-path graph $P_m X\,(x_0, x_1)$ is the full subgraph of $\hom^{\square}(I_m, X)$ with vertices given by paths of length m in X from x_0 to x_1.

Observe that two paths $\gamma, \sigma\colon x_0 \rightsquigarrow x_1$ are adjacent in the graph $P_m X\,(x_0, x_1)$ if and only if there exists a graph map $H\colon I_1 \to P_m X\,(x_0, x_1)$ such that $H\,(0) = \gamma$ and $H\,(1) = \sigma$, or equivalently, if and only if there exists a graph map $H\colon I_m \,\square\, I_1 \to X$ such that $H\,(-, 0) = \gamma$ and $H\,(-, 1) = \sigma$.

Definition 2.5.2. Let X be a graph, $x_0, x_1 \in X$ be two vertices, and $\gamma, \sigma\colon x_0 \rightsquigarrow x_1$ be two paths in X of length m for some $m \in \mathbb{N} \cup \{\infty\}$.

1. A *path-homotopy* $H\colon \gamma \Rightarrow \sigma$ of length $n \in \mathbb{N}$ is a map $H\colon I_m \,\square\, I_n \to X$ such that $H\,(-, 0) = \gamma$ and $H\,(-, n) = \sigma$, and such that each $H\,(-, i)$ is a path from x_0 to x_1 in X of length m.

2. A *path-homotopy* $H\colon \gamma \Rightarrow \sigma$ of length ∞ is a map $H\colon I_m \square I_\infty \to X$ for which there exist integers $N_-, N_+ \in \mathbb{Z}$ such that $H(-,i) = \gamma$ for all $i \leq N_-$ and $H(-,i) = \sigma$ for all $i \geq N_+$, and such that each $H(-,i)$ is a path from x_0 to x_1 in X of length m.

When such a path-homotopy exists, we say that γ and σ are path-homotopic and write $\gamma \sim_m \sigma$.

Equivalently, a path-homotopy $H\colon \gamma \Rightarrow \sigma$ between two paths $\gamma, \sigma\colon x_0 \rightsquigarrow x_1$ in X, both of length m, is a path $H\colon \gamma \rightsquigarrow \sigma$ in the graph $P_m X(x_0, x_1)$. It follows that $\sim_m$ is an equivalence relation on the set $P_m X(x_0, x_1)_V$.

Definition 2.5.3. For a pointed graph (X, x) and $m \in \mathbb{N} \cup \{\infty\}$, the m-loop graph $\Omega_m(X, x)$ is given by $P_m X(x, x)$. It has a distinguished vertex given by the constant path at x of length m. This gives an endofunctor $\Omega_m\colon \mathsf{Graph}_* \to \mathsf{Graph}_*$.

Iterating the endofunctor $\Omega_m\colon \mathsf{Graph}_* \to \mathsf{Graph}_*$, we get:

$$\Omega_m^d\colon \mathsf{Graph}_* \to \mathsf{Graph}_*, \qquad \text{for } d \in \mathbb{N}$$

Proposition 2.5.4. *For any pointed graph (X, x), $m \in \mathbb{N} \cup \{\infty\}$ and $d \in \mathbb{N}$, the graph $\Omega_m^d(X, x)$ is isomorphic to the full subgraph of $\hom^\square(I_m^{\square d}, X)$ with vertices given by the maps $\varphi\colon I_m^{\square d} \to X$ for which each component*

$$\varphi_i\colon I_m \to \hom^\square(I_m^{\square d-1}, X), \qquad \textit{for } i = 1, \ldots, d$$

given by

$$t \mapsto \left(\left(t_1, \ldots, \widehat{t_i}, \ldots, t_d\right) \mapsto \varphi(t_1, \ldots, t, \ldots, t_d)\right)$$

is a path of length m from c_x to c_x, where c_x is the constant map at x. $\square$

Definition 2.5.5. For $d \geq 1$, and for each $i = 1, \ldots, d$, we define a binary operation $\cdot_i$ on the vertex set of the graph $\Omega^d_\infty(X,x)$ as follows: given $\varphi, \psi \in \Omega^d_\infty(X,x)$, let $\varphi \cdot_i \psi \in \Omega^d_\infty(X,x)$ be the element of $\hom^\square(I_\infty^{\square d}, X)$ corresponding to the concatenation $\varphi_i * \psi_i$, where $\varphi_i, \psi_i \colon I_\infty \to \hom^\square(I_\infty^{\square d-1}, X)$ are the paths $\mathrm{c}_x \rightsquigarrow \mathrm{c}_x$ of length ∞ given by

$$t \mapsto \Big(\big(t_1, \ldots, \widehat{t_i}, \ldots, t_d\big) \mapsto \varphi\,(t_1, \ldots, t, \ldots, t_d)\Big)$$

and

$$t \mapsto \Big(\big(t_1, \ldots, \widehat{t_i}, \ldots, t_d\big) \mapsto \psi\,(t_1, \ldots, t, \ldots, t_d)\Big)$$

respectively.

Warning 2.5.6. The binary operation $\cdot_i$ does *not* define a graph map of the form

$$\Omega^d_\infty(X,x) \;\square\; \Omega^d_\infty(X,x) \to \Omega^d_\infty(X,x)\,.$$

Definition 2.5.7. For $d \geq 1$, we define the *d-th A-homotopy group* $A_d(X,x)$ of a pointed graph (X,x) to be the set $\pi_0\Omega^d_\infty(X,x)$ of path-components of the graph $\Omega^d_\infty(X,x)$. For $d \geq 1$, and for each $i = 1, \ldots, d$, the binary operation $\cdot_i$ induces a group operation on homotopy groups $\cdot_i \colon A_d(X,x) \times A_d(X,x) \to A_d(X,x)$. A pointed graph map $f \colon (X,x) \to (Y,fx)$ induces a group homomorphism $f_* \colon A_d(X,x) \to A_d(Y,fx)$. The assignment $f \mapsto f_*$ is functorial, and we have a well-defined functor $A_d \colon \mathsf{Graph}_* \to \mathsf{Grp}$, where Graph_* denotes the category of pointed graphs and pointed graph maps. For $d \geq 2$ and $1 \leq i < j \leq d$, the group operations $\cdot_i$ and $\cdot_j$ satisfy the interchange law: given any $a, b, c, d \in \Omega^d_\infty(X,x)$, we have a path in $\Omega^d_\infty(X,x)$ from $(a \cdot_i b) \cdot_j (c \cdot_i d)$ to $(a \cdot_j c) \cdot_i (b \cdot_j d)$. It follows by the Eckmann-Hilton argument that these group operations coincide and are abelian.

The first A-homotopy group $A_1(X, x)$ of a pointed graph (X, x) is also known as its *discrete fundamental group*, although we will drop the adjective "discrete" henceforth and simply call it the *fundamental group*.

Definition 2.5.8. A graph map $f\colon X \to Y$ is a *weak A-homotopy equivalence* if it induces a bijection $f_*\colon \pi_0 X \to \pi_0 Y$ and a group isomorphism $f_*\colon A_d(X, x) \to A_d(Y, fx)$ for every $d > 0$ and every vertex $x \in X$. Let $\mathcal{W}$ denote the class of weak A-homotopy equivalences in Graph.

Definition 2.5.9. Given any integer $n \geq 0$, a graph map $f\colon X \to Y$ is an *n-equivalence* if it induces a bijection $f_*\colon \pi_0 X \to \pi_0 Y$ and a group isomorphism $f_*\colon A_d(X, x) \to A_d(Y, fx)$ for every $0 < d \leq n$ and every vertex $x \in X$. Let $\mathcal{W}_n$ denote the class of n-equivalences in Graph.

2.6 Finite vs. infinite paths

In the preceding section, we defined homotopy groups using infinite paths (Definition 2.5.7), but we could have used paths of finite length instead. While this statement is clear on the intuitive level, its proof is quite technical. In this section, we develop the necessary tools and prove that the two definitions (via 'finite' vs 'infinite' grids) of homotopy groups agree.

Note that the length of a path in a graph is a property of the map, and not its image. To emphasize this point, we introduce the notion of reparametrization of a path.

Definition 2.6.1. Let $m, n \in \mathbb{N} \cup \{\infty\}$. A *shrinking* map is a surjective and order-preserving graph map $s\colon I_m \to I_n$. Given a path $\gamma\colon x_0 \rightsquigarrow x_1$ of length n in a graph X and a shrinking map $s\colon I_m \to I_n$, the composite $\gamma \circ s$ is a path in X of length m from x_0 to x_1, traversing the same sequence of edges as γ. We say that the paths $\gamma \circ s$ and γ are *reparametrizations* of each other.

Lemma 2.6.2.

1. *Every path of finite length admits a reparametrization of length ∞.*

2. *Every path of length ∞ admits a reparametrization of finite length.*

Proof. Given a path $\gamma\colon x_0 \rightsquigarrow x_1$ in X of finite length n, we can choose the shrinking map $s\colon I_\infty \to I_n$ given by

$$s\,(i) = \begin{cases} 0 & \text{if } i \leq 0 \\ i & \text{if } 0 \leq i \leq n \\ n & \text{if } n \leq i \end{cases}$$

Then, the path $\gamma \circ s$ is a reparametrization of γ having length ∞.

Given a path $\gamma\colon x_0 \rightsquigarrow x_1$ in X of length ∞, there exist integers $N_-, N_+ \in \mathbb{Z}$ such that $\gamma\,(i) = x_0$ for all $i \leq N_-$ and $\gamma\,(i) = x_1$ for all $i \geq N_+$. Then, letting $n = N_+ - N_-$, we can choose a shrinking map $s\colon I_\infty \to I_n$ and a map $\widetilde{\gamma}\colon I_n \to X$ given by:

$$s\,(i) = \begin{cases} 0 & \text{if } i \leq N_- \\ i - N_- & \text{if } N_- \leq i \leq N_+ \\ n & \text{if } N_+ \leq i \end{cases}$$

and

$$\widetilde{\gamma}\,(i) = \gamma\,(i + N_-)\,.$$

Note that $\gamma = \widetilde{\gamma} \circ s$. Hence, $\widetilde{\gamma}$ is a reparametrization of γ having finite length

n. $\square$

Given any subset $\mathcal{I} \subseteq \mathbb{N} \cup \{\infty\}$, we have an equivalence relation $\sim_{\mathcal{I}}$ on the disjoint union $\coprod_{n\in\mathcal{I}} P_n X\,(x_0, x_1)$ that is generated by reparametrizations. That is, given a path $\gamma\colon x_0 \rightsquigarrow x_1$ of length n and a shrinking map $s\colon I_m \to I_n$, for $m, n \in \mathcal{I}$, we identify $\gamma \sim_{\mathcal{I}} \gamma \circ s$ and take the smallest equivalence relation containing these pairs.

Definition 2.6.3. For a graph X and two vertices x_0 and x_1, the *$\mathcal{I}$-indexed path graph* $P_{\mathcal{I}} X\,(x_0, x_1)$ is the quotient of $\coprod_{n\in\mathcal{I}} P_n X\,(x_0, x_1)$ under the equivalence relation defined above.

$$P_{\mathcal{I}} X\,(x_0, x_1) = \left(\coprod_{n\in\mathcal{I}} P_n X\,(x_0, x_1)\right) / \sim_{\mathcal{I}}$$

Suppose we have an inclusion $\mathcal{I} \subseteq \mathcal{J}$, where $\mathcal{I}, \mathcal{J}$ are both subsets of $\mathbb{N} \cup \{\infty\}$. Then, given any two paths $\gamma, \sigma\colon x_0 \rightsquigarrow x_1$ that are identified in $P_{\mathcal{I}} X\,(x_0, x_1)$, they are also identified in $P_{\mathcal{J}} X\,(x_0, x_1)$. Thus, the inclusion $\coprod_{n\in\mathcal{I}} P_n X\,(x_0, x_1) \hookrightarrow \coprod_{n\in\mathcal{J}} P_n X\,(x_0, x_1)$ induces a map

$$P_{\mathcal{I}} X\,(x_0, x_1) \longrightarrow P_{\mathcal{J}} X\,(x_0, x_1)$$

Proposition 2.6.4. *The following maps are isomorphisms:*

$$P_{\{\infty\}} X\,(x_0, x_1) \longrightarrow P_{\mathbb{N}\cup\{\infty\}} X\,(x_0, x_1)$$

and

$$P_{\mathbb{N}} X\,(x_0, x_1) \longrightarrow P_{\mathbb{N}\cup\{\infty\}} X\,(x_0, x_1)$$

Proof. We first prove surjectivity and then injectivity of both maps. We have the following commutative squares:

$$\begin{tikzcd}
P_\infty X(x_0,x_1) \arrow[r, hook] \arrow[d] \arrow[dr, dotted] & \coprod_{n\in\mathbb{N}\cup\{\infty\}} P_n X(x_0,x_1) \arrow[d] \\
P_{\{\infty\}} X(x_0,x_1) \arrow[r] & P_{\mathbb{N}\cup\{\infty\}} X(x_0,x_1)
\end{tikzcd}$$

and

$$\begin{tikzcd}
\coprod_{n\in\mathbb{N}} P_n X(x_0,x_1) \arrow[r, hook] \arrow[d] \arrow[dr, dotted] & \coprod_{n\in\mathbb{N}\cup\{\infty\}} P_n X(x_0,x_1) \arrow[d] \\
P_{\mathbb{N}} X(x_0,x_1) \arrow[r] & P_{\mathbb{N}\cup\{\infty\}} X(x_0,x_1)
\end{tikzcd}$$

By Lemma 2.6.2, the maps represented by the dotted diagonal arrows in both commutative squares are surjective. It follows that the maps $P_{\{\infty\}} X(x_0,x_1) \to P_{\mathbb{N}\cup\{\infty\}} X(x_0,x_1)$ and $P_{\mathbb{N}} X(x_0,x_1) \to P_{\mathbb{N}\cup\{\infty\}} X(x_0,x_1)$ are both surjective.

Injectivity of $P_{\{\infty\}} X(x_0,x_1) \to P_{\mathbb{N}\cup\{\infty\}} X(x_0,x_1)$, follows from the fact that for any shrinking map $s\colon I_m \to I_\infty$ where $m \in \mathbb{N}\cup\{\infty\}$, we must have $m = \infty$.

Given any shrinking map $s\colon I_\infty \to I_n$ where $n \in \mathbb{N}$, we have a corresponding shrinking map $\widetilde{s}\colon I_M \to I_n$ where $M \in \mathbb{N}$ defined as follows. Let $M_- \in \mathbb{Z}$ be such that $s(M_-)\,0$ and $M_+ \in \mathbb{Z}$ be such that $s(M_+) = n$. Let $M = M_+ - M_-$ and let $\widetilde{s}\colon I_M \to I_n$ be given by $\widetilde{s}(i) = s(i + M_-)$. Then, for any path $\gamma\colon x_0 \rightsquigarrow x_1$ of length n, the paths $\gamma \circ s$ and $\gamma \circ \widetilde{s}$ are identified in $P_{\mathbb{N}\cup\{\infty\}} X(x_0,x_1)$. That is, we can replace any shrinking map of the form $s\colon I_\infty \to I_n$ by a shrinking map of the form $\widetilde{s}\colon I_M \to I_n$ where $M \in \mathbb{N}$. Hence, the map $P_{\mathbb{N}} X(x_0,x_1) \to P_{\mathbb{N}\cup\{\infty\}} X(x_0,x_1)$ is injective. □

Lemma 2.6.5. *Given a path $\gamma\colon x_0 \rightsquigarrow x_1$ in X of length ∞ and a shrinking map $s\colon I_\infty \to I_\infty$, there exists a path-homotopy $\gamma \Rightarrow \gamma \circ s$, or equivalently a path $\gamma \rightsquigarrow \gamma \circ s$ in $P_\infty X(x_0,x_1)$.*

Proof. We first consider two special cases before moving on to the general case.

First, suppose the shrinking map $s\colon I_\infty \to I_\infty$ is of the form

$$s(i) = \begin{cases} i & \text{if } i \leq k \\ i-1 & \text{if } i \geq k+1 \end{cases}$$

for some $k \in \mathbb{Z}$ and let $H\colon I_\infty \otimes I_1 \to I_\infty$ be given by $H(i,0) = i$ and $H(i,1) = s(i)$. Then, $\gamma \circ H$ is a path-homotopy from γ to $\gamma \circ s$.

Next, suppose the shrinking map $s\colon I_\infty \to I_\infty$ is of the form $s(i) = i + k$ for some $k \in \mathbb{Z}$. Let $H\colon I_\infty \otimes I_k \to I_\infty$ be given by $H(i,j) = i + j$, so that $H(i,0) = i$ and $H(i,k) = s(i)$. Then, $\gamma \circ H$ is a path-homotopy from γ to $\gamma \circ s$.

We can now consider the general case where the shrinking map $s\colon I_\infty \to I_\infty$ is arbitrary. Since γ is a path of length ∞, there exist $N_-, N_+ \in \mathbb{Z}$ such that $\gamma(i) = x_0$ for all $i \leq N_-$ and $\gamma(i) = x_1$ for all $i \geq N_+$. Let $M_-, M_+ \in \mathbb{Z}$ be such that $s(M_-) = N_-$ and $s(M_+) = N_+$.

Define $s'\colon I_\infty \to I_\infty$ as follows:

$$s'(i) = \begin{cases} i - M_- + N_- & \text{if } i \leq M_- \\ s(i) & \text{if } M_- \leq i \leq M_+ \\ i - M_+ + N_+ & \text{if } M_+ \leq i \end{cases}$$

Then, s' is also a shrinking map. Furthermore, s' can be obtained as the composite of a finite sequence of shrinking maps of the forms considered in the previous two special cases. Thus, we have a path-homotopy $\gamma \Rightarrow \gamma \circ s'$. Finally, we observe that $\gamma \circ s' = \gamma \circ s$. $\square$

Proposition 2.6.6. *The set-function*

$$\pi_0 P_\infty X\,(x_0, x_1) \longrightarrow \pi_0 P_{\{\infty\}} X\,(x_0, x_1)$$

induced by the quotient map $P_\infty X\,(x_0, x_1) \to P_{\{\infty\}} X\,(x_0, x_1)$ *is a bijection.*

Proof. We need to check that whenever two vertices in $P_{\{\infty\}} X\,(x_0, x_1)$ are path-connected, their preimages in $P_\infty X\,(x_0, x_1)$ are also path-connected. It suffices to consider vertices connected by an edge in $P_{\{\infty\}} X\,(x_0, x_1)$.

Let $[\gamma] \sim [\sigma]$ be an edge in $P_{\{\infty\}} X\,(x_0, x_1)$. Then, there exists an edge $\widetilde{\gamma} \sim \widetilde{\sigma}$ in $P_\infty X\,(x_0, x_1)$ such that $[\gamma] = [\widetilde{\gamma}]$ and $[\sigma] = [\widetilde{\sigma}]$. By Lemma 2.6.5, we have paths $\gamma \rightsquigarrow \widetilde{\gamma}$ and $\sigma \rightsquigarrow \widetilde{\sigma}$ in $P_\infty X\,(x_0, x_1)$. Thus, there is a path $\gamma \rightsquigarrow \sigma$ in $P_\infty X\,(x_0, x_1)$. $\square$

This allows us to use the path graph $P_{\{\infty\}} X\,(x_0, x_1)$, or indeed the path graph $P_{\mathbb{N}} X\,(x_0, x_1)$ which is isomorphic to it, in place of $P_\infty X\,(x_0, x_1)$, as long as we are only concerned with the set of path-connected components of these graphs. We expect that each of these models will be useful in different contexts. For instance, the path graph $P_{\mathbb{N}} X\,(x_0, x_1)$ will come in handy when we need to use induction arguments on path-lengths.

Definition 2.6.7. For a pointed graph (X, x) and a non-empty subset $\mathcal{I} \subseteq \mathbb{N} \cup \{\infty\}$, the $\mathcal{I}$-indexed loop graph $\Omega_{\mathcal{I}}\,(X, x)$ is given by $P_{\mathcal{I}} X\,(x, x)$. It has a distinguished vertex given by the equivalence class of the constant path at x. This gives an endofunctor $\Omega_{\mathcal{I}} \colon \mathsf{Graph}_* \to \mathsf{Graph}_*$.

Iterating the endofunctor $\Omega_{\mathcal{I}} \colon \mathsf{Graph}_* \to \mathsf{Graph}_*$, we get:

$$\Omega_{\mathcal{I}}^d \colon \mathsf{Graph}_* \to \mathsf{Graph}_*, \qquad \text{for } d \in \mathbb{N}.$$

Suppose we have an inclusion $\mathcal{I} \subseteq \mathcal{J}$, where $\mathcal{I}$, $\mathcal{J}$ are both non-empty subsets of $\mathbb{N} \cup \{\infty\}$. Then, we have natural transformations

$$\Omega^d_{\mathcal{I}} \Rightarrow \Omega^d_{\mathcal{J}}, \qquad \text{for } d \in \mathbb{N}.$$

Proposition 2.6.8. *For every pointed graph (X, x), the following maps are isomorphisms:*

$$\Omega^d_{\mathbb{N}}(X, x) \to \Omega^d_{\mathbb{N}\cup\{\infty\}}(X, x) \qquad \text{and} \qquad \Omega^d_{\{\infty\}}(X, x) \to \Omega^d_{\mathbb{N}\cup\{\infty\}}(X, x)\,. \quad \square$$

Notation 2.6.9. Given a vector $\mathbf{n} = (n_1, \ldots, n_d) \in (\mathbb{N} \cup \{\infty\})^d$, let $I_{\mathbf{n}}$ denote the graph $I_{n_1} \square \cdots \square I_{n_d}$, and $\mathbf{n} \setminus n_i$ denote the vector $(n_1, \ldots, \widehat{n_i}, \ldots, n_d) \in (\mathbb{N} \cup \{\infty\})^{d-1}$. Let $\Omega_{\mathbf{n}} \colon \mathsf{Graph}_* \to \mathsf{Graph}_*$ denote the endofunctor obtained as the composite $\Omega_{n_d} \circ \cdots \circ \Omega_{n_1}$.

Proposition 2.6.10. *For any pointed graph (X, x) and vector $\mathbf{n} = (n_1, \ldots, n_d) \in (\mathbb{N} \cup \{\infty\})^d$, the graph $\Omega_{\mathbf{n}}(X, x)$ is isomorphic to the full subgraph of $\hom^{\square}(I_{\mathbf{n}}, X)$ with vertices given by maps $\varphi \colon I_{\mathbf{n}} \to X$ for which each component*

$$\varphi_i \colon I_{n_i} \to \hom^{\square}(I_{\mathbf{n}\setminus n_i}, X), \qquad \text{for } i = 1, \ldots, d$$

given by

$$t \mapsto \Big(\big(t_1, \ldots, \widehat{t_i}, \ldots, t_d\big) \mapsto \varphi\,(t_1, \ldots, t, \ldots, t_d)\Big)$$

is a path of length n_i from c_x to c_x, where c_x is the constant map at x. $\square$

Given any subset $\mathcal{I} \subseteq \mathbb{N} \cup \{\infty\}$ and $d \in \mathbb{N}$, we have an equivalence relation $\sim_{\mathcal{I}^d}$ on the disjoint union $\coprod_{\mathbf{n} \in \mathcal{I}^d} \Omega_{\mathbf{n}}(X, x)$ that is generated by (higher-order) reparametrizations. That is, given an element $\varphi \in \Omega_{\mathbf{n}}(X, x)$ and shrinking maps $s_i \colon I_{m_i} \to I_{n_i}$, $i = 1, \ldots, d$, for $\mathbf{m}, \mathbf{n} \in \mathcal{I}^d$, we identify $\varphi \sim_{\mathcal{I}^d} \varphi \circ (s_1 \mathbin{\square} \cdots \mathbin{\square} s_d)$ and take the smallest equivalence relation containing these pairs.

Proposition 2.6.11. *For any pointed graph (X, x), non-empty subset $\mathcal{I} \subseteq \mathbb{N} \cup \{\infty\}$ and $d \in \mathbb{N}$, we have:*

$$\Omega^d_{\mathcal{I}}(X, x) \cong \left(\coprod_{\mathbf{n} \in \mathcal{I}^d} \Omega_{\mathbf{n}}(X, x) \right) / \sim_{\mathcal{I}^d} . \qquad \square$$

Lemma 2.6.12. *Given $\varphi \in \Omega^d_\infty(X, x)$ and shrinking maps $s_1, \ldots, s_d \colon I_\infty \to I_\infty$, there exists a path $\varphi \rightsquigarrow \varphi \circ (s_1 \mathbin{\square} \cdots \mathbin{\square} s_d)$ in $\Omega^d_\infty(X, x)$.*

Proof. Note that $(s_1 \mathbin{\square} \cdots \mathbin{\square} s_d)$ equals the following composite:

$$(s_1 \mathbin{\square} \mathrm{id} \mathbin{\square} \cdots \mathbin{\square} \mathrm{id}) \circ (\mathrm{id} \mathbin{\square} s_2 \mathbin{\square} \mathrm{id} \mathbin{\square} \cdots \mathbin{\square} \mathrm{id}) \circ \cdots \circ (\mathrm{id} \mathbin{\square} \cdots \mathbin{\square} \mathrm{id} \mathbin{\square} s_d)$$

Thus, it suffices to prove that there exists a path $\varphi \rightsquigarrow \varphi \circ (\mathrm{id} \mathbin{\square} \cdots \mathbin{\square} s_i \mathbin{\square} \cdots \mathbin{\square} \mathrm{id})$ for any $\varphi \in \Omega^d_\infty(X, x)$ and any $i = 1, \ldots, d$.

Let $\varphi_i \colon I_\infty \to \hom^{\square}(I_\infty^{\square d-1}, X)$ be given by

$$t \mapsto \left(\left(t_1, \ldots, \widehat{t_i}, \ldots, t_d \right) \mapsto \varphi(t_1, \ldots, t, \ldots, t_d) \right).$$

By Lemma 2.6.5, there is a path-homotopy $\varphi_i \Rightarrow \varphi_i \circ s_i$, or equivalently a path $\varphi \rightsquigarrow \varphi \circ (\mathrm{id} \mathbin{\square} \cdots \mathbin{\square} s_i \mathbin{\square} \cdots \mathbin{\square} \mathrm{id})$ in $\Omega^d_\infty(X, x_0)$. $\square$

Proposition 2.6.13. *The set-function*

$$\pi_0 \Omega^d_\infty(X, x) \longrightarrow \pi_0 \Omega^d_{\{\infty\}}(X, x)$$

induced by the quotient map $\Omega^d_\infty(X, x) \to \Omega^d_{\{\infty\}}(X, x)$ is a bijection.

Proof. We need to check that whenever two vertices in $\Omega^d_{\{\infty\}}(X,x)$ are path-connected, their preimages in $\Omega^d_\infty(X,x)$ are also path-connected. It suffices to consider vertices connected by an edge in $\Omega^d_{\{\infty\}}(X,x)$.

Let $[\varphi] \sim [\psi]$ be an edge in $\Omega^d_{\{\infty\}}(X,x)$. Then, there exists an edge $\widetilde{\varphi} \sim \widetilde{\psi}$ in $\Omega^d_\infty(X,x)$ with $[\widetilde{\varphi}] = [\varphi]$ and $\left[\widetilde{\psi}\right] = [\psi]$. By Lemma 2.6.12, we have paths $\varphi \rightsquigarrow \widetilde{\varphi}$ and $\psi \rightsquigarrow \widetilde{\psi}$ in $\Omega^d_\infty(X,x)$. Thus, there is a path $\varphi \rightsquigarrow \psi$ in $\Omega^d_\infty(X,x)$. $\square$

As a corollary, we obtain the main theorem of this section:

Theorem 2.6.14. *The d-th A-homotopy group $A_d(X,x)$ of a pointed graph (X,x) can be equivalently defined as the set of path-components of any of the following graphs:*

- $\Omega^d_\infty(X,x)$ *of graph maps from the infinite grid to (X,x) stabilizing in all directions;*
- $\Omega^d_{\{\infty\}}(X,x)$ *which is the quotient of the graph $\Omega^d_\infty(X,x)$ above by the equivalence relation generated by reparametrization; or*
- $\Omega^d_{\mathbb{N}}(X,x)$ *of graph maps from finite grids to (X,x) quotiented by the equivalence relation generated by reparametrization.*

Note that, for $d \geq 2$, the group operation on $A_d(X,x)$ must necessarily descend from the binary operations on the vertex set of $\Omega^d_\infty(X,x)$, since none of the other models admit the correct binary operation. The only exception to this is the case $d = 1$, where $\Omega_{\mathbb{N}}(X,x)$ admits a product structure in Graph. In particular, Proposition 2.6.13 cannot be strengthened to a statement about isomorphic groups for $d \geq 2$.

Chapter 3

The Fundamental Groupoid

In this chapter, we introduce the discrete fundamental groupoid of a graph, a multi-object generalization of the discrete fundamental group defined in Section 2.5, and use it as a starting point to develop robust computational techniques, such as the theory of covering graphs and a discrete version of the Seifert–van Kampen theorem. In Section 3.1, we define the discrete fundamental groupoid $\Pi_1 X$ of a graph X and verify that Π_1 is a well-defined functor from the category of graphs to the category of groupoids. We then show that it is monoidal with respect to the box product and deduce from this that the discrete fundamental groupoid is homotopy invariant.

In Section 3.2, we develop the theory of covering graphs, which mirrors the familiar theory of covering spaces from algebraic topology. We introduce a new notion of coverings, distinct from the notion of local isomorphisms previously studied in [Har19, BGJW21]. This new definition allows us to overcome the restriction to base graphs not containing 3- or 4-cycles which hampers the applicability of previous iterations of covering graph theories. In particular, we prove that every graph admits a universal cover (Theorem 3.2.29).

We also prove a classification theorem for covering graphs (Theorem 3.2.21), from which we can deduce the familiar Galois correspondence between pointed connected coverings and subgroups of the discrete fundamental group (Corollary 3.2.23).

In Section 3.3, we tackle the question: when does the discrete fundamental group preserve pushouts? This is the question that would be answered by any candidate for a Seifert–van Kampen theorem in discrete homotopy theory. The statement found in literature ([BKLW01, Thm. 2.8]) requires that every 3- or 4-cycle in the pushout graph must be contained entirely in one of the two graphs that we are taking a pushout of. However, this condition is not met in several examples of interest, where we still expect the pushout to be preserved. We prove that it is sufficient to require instead that every 3- or 4-cycle in the pushout graph admits a "subdivision" into smaller 3- or 4-cycles, each of which is itself contained in one of the two graphs that we are taking a pushout of (see Theorem 3.3.5). In fact, we deduce such a statement from a stronger version for the fundamental groupoid (Theorem 3.3.4).

Then, in Section 3.4, we use the discrete Seifert–van Kampen theorem to solve the realization problem for the discrete fundamental group, which asks: can every group be realized as the discrete fundamental group of some graph? We answer this in the affirmative by providing, for any group G with a choice of presentation $\langle S \mid R \rangle$, a completely combinatorial construction of a graph $X_{S,R}$ that has G as its discrete fundamental group. This construction is a key step towards constructing the Eilenberg–MacLane graphs $K(G,1)$, i.e. graphs whose discrete fundamental group is G and whose higher discrete homotopy groups are all trivial.

3.1 Definition and basic properties

In this section, we define the fundamental groupoid $\Pi_1 X$ of a graph X (see Definition 3.1.4). Its objects are the vertices of X, and its hom-sets are given by the set of path-components of the path-graphs constructed in Definition 2.6.3. Taking advantage of the 'finite' grid approach of Section 2.6, we define the operations making $\Pi_1 X$ into a groupoid using maps between the path-graphs that encode concatentation and inversion of paths, and verify that it gives a well-defined functor from the category of graphs to the category of groupoids. We then show that the fundamental groupoid functor is monoidal with respect to the box product and hence, homotopy invariant.

Recall the definition of concatenation of paths having finite length. For $m, n \in \mathbb{N}$ and vertices $x, x', x'' \in X$, we have a well-defined graph map

$$*\colon P_m X\left(x, x'\right) \times P_n X\left(x', x''\right) \to P_{m+n} X\left(x, x''\right)$$

given by

$$(\gamma, \sigma) \mapsto \gamma * \sigma$$

Given two shrinking maps $s\colon I_{m'} \to I_m$ and $t\colon I_{n'} \to I_n$, where $m, m', n, n' \in \mathbb{N}$, their *concatenation* is the shrinking map $s * t\colon I_{m'+n'} \to I_{m+n}$ given by

$$(s * t)(i) = \begin{cases} s(i) & \text{if } 0 \leq i \leq m' \\ t(i - m') + m & \text{if } m' \leq i \leq m' + n' \end{cases}$$

Given paths $\gamma\colon x \rightsquigarrow x'$ and $\sigma\colon x' \rightsquigarrow x''$ in X of lengths m, n respectively, and shrinking maps $s\colon I_{m'} \to I_m$ and $t\colon I_{n'} \to I_n$, we have

$$(\gamma \circ s) * (\sigma \circ t) = (\gamma * \sigma) \circ (s * t)\,.$$

Thus, we obtain an induced graph map

$$*\colon P_{\mathbb{N}}X(x,x') \times P_{\mathbb{N}}X(x',x'') \to P_{\mathbb{N}}X(x,x'')$$

given by

$$([\gamma],[\sigma]) \mapsto [\gamma * \sigma]$$

Note that concatenation of paths of length ∞ does not define a graph map.

Lemma 3.1.1. *Let X be any graph. Then, the following diagrams of graph maps commute:*

$$\begin{tikzcd} P_{\mathbb{N}}X(x,x') \,\square\, P_{\mathbb{N}}X(x',x'') \,\square\, P_{\mathbb{N}}X(x'',x''') \arrow[r, "* \,\square\, \mathrm{id}"] \arrow[d, "\mathrm{id} \,\square\, *"'] & P_{\mathbb{N}}X(x,x'') \,\square\, P_{\mathbb{N}}X(x'',x''') \arrow[d, "*"] \\ P_{\mathbb{N}}X(x,x') \,\square\, P_{\mathbb{N}}X(x',x''') \arrow[r, "*"] & P_{\mathbb{N}}X(x,x''') \end{tikzcd}$$

$$\begin{tikzcd} P_{\mathbb{N}}X(x,x') \times I_0 \arrow[r, "\mathrm{id}\times \mathrm{c}_{x'}"] \arrow[dr, "\cong"'] & P_{\mathbb{N}}X(x,x') \times P_{\mathbb{N}}X(x',x') \arrow[d, "*"] \\ & P_{\mathbb{N}}X(x,x') \end{tikzcd}$$

$$\begin{tikzcd} P_{\mathbb{N}}X(x,x) \times P_{\mathbb{N}}X(x,x') \arrow[d, "*"] & I_0 \times P_{\mathbb{N}}X(x,x') \arrow[l, "\mathrm{c}_x \times \mathrm{id}"'] \arrow[dl, "\cong"] \\ P_{\mathbb{N}}X(x,x') & \end{tikzcd}$$

for all $x, x', x'', x''' \in X$. $\square$

For $n \in \mathbb{N}$ and vertices $x, x' \in X$, we have a well-defined graph map

$$\overline{(\ )}\colon P_nX(x,x') \to P_nX(x',x)$$

given by taking the inverse path $\gamma \mapsto \overline{\gamma}$ in the sense of Definition 2.3.4.

Given a shrinking map $s\colon I_{n'} \to I_n$, where $n, n' \in \mathbb{N}$, its *reverse* is the shrinking map $\overline{s}\colon I_{n'} \to I_n$ given by $\overline{s}(i) = n - s(n' - i)$. Putting it together, given any path $\gamma\colon x \rightsquigarrow x'$ in X of length n and a shrinking map $s\colon I_{n'} \to I_n$, we have

$$\overline{\gamma \circ s} = \overline{\gamma} \circ \overline{s}.$$

which gives an induced graph map

$$\overline{(\ )}\colon P_{\mathbb{N}}X(x, x') \to P_{\mathbb{N}}X(x', x)$$

given by $[\gamma] \mapsto [\overline{\gamma}]$.

Lemma 3.1.2. *Let X be any graph. Then, the following diagram of set-functions commutes:*

$$\begin{tikzcd}
\pi_0 P_{\mathbb{N}}X(x', x) \times \pi_0 P_{\mathbb{N}}X(x, x') \arrow[d, "*"'] & \pi_0 P_{\mathbb{N}}X(x, x') \arrow[l, "(\overline{(\ )}, \mathrm{id})"'] \arrow[r, "(\mathrm{id}, \overline{(\ )})"] \arrow[d] & \pi_0 P_{\mathbb{N}}X(x, x') \times \pi_0 P_{\mathbb{N}}X(x', x) \arrow[d, "*"] \\
\pi_0 P_{\mathbb{N}}X(x', x') & \{*\} \arrow[l, "{[\mathrm{c}_{x'}]}"'] \arrow[r, "{[\mathrm{c}_{x}]}"] & \pi_0 P_{\mathbb{N}}X(x, x)
\end{tikzcd}$$

for all $x, x' \in X$. □

Definition 3.1.3. Given any path $\gamma\colon x \rightsquigarrow x'$ in a graph X having finite length, let its *path-homotopy class* be its equivalence class in $\pi_0 P_{\mathbb{N}}X(x, x')$.

Definition 3.1.4. The *fundamental groupoid* $\Pi_1 X$ of a graph X is defined as follows:

$$\Pi_1 X = \begin{cases} \text{objects:} & \text{vertices } x, x', \dots \text{ of } X \\ \text{morphisms:} & \text{path-homotopy classes } [\gamma] : x \to x' \text{ of} \\ & \text{paths } \gamma\colon x \rightsquigarrow x' \text{ in } X \text{ of finite length} \end{cases}$$

In other words, we have:

$$\operatorname{ob}\Pi_1 X = X_V$$

and for every $x, x' \in \operatorname{ob}\Pi_1 X$, we have:

$$\Pi_1 X\,(x, x') = \pi_0 P_{\mathbb{N}} X\,(x, x')\,.$$

Composition is given by $[\sigma] \circ [\gamma] = [\gamma * \sigma]$, which is well-defined since it is obtained by applying π_0 to the map

$$*\colon P_{\mathbb{N}} X\,(x, x') \times P_{\mathbb{N}} X\,(x', x'') \to P_{\mathbb{N}} X\,(x, x'')$$

$$([\gamma]\,, [\sigma]) \mapsto [\gamma * \sigma]\,.$$

Inverses are given by $[\gamma]^{-1} = [\overline{\gamma}]$, which is well-defined since it is obtained by applying π_0 to the map

$$\overline{(\;)}\colon P_{\mathbb{N}} X\,(x, x') \to P_{\mathbb{N}} X\,(x', x)$$

$$[\gamma] \mapsto [\overline{\gamma}]\,.$$

Then, $\Pi_1 G$ is a well-defined groupoid.

Given any graph map $f\colon X \to Y$, we have a well-defined functor $\Pi_1 f\colon \Pi_1 X \to \Pi_1 Y$ that maps a morphism $[\gamma] : x \to x'$ in $\Pi_1 X$ to the morphism $[f \circ \gamma] : f(x) \to f(x')$ in $\Pi_1 Y$. Furthermore, the assignment $f \mapsto \Pi_1 f$ is itself functorial, and we obtain a well-defined functor

$$\Pi_1\colon \mathsf{Graph} \to \mathsf{Gpd}.$$

Proposition 3.1.5. *For any pointed graph (X, x), we have a group isomor-*

phism

$$A_1(X,x) \cong \Pi_1 X(x,x)$$

If X is path-connected, we have an equivalence of categories $A_1(X,x) \simeq \Pi_1 X$.

□

3.1.1 Monoidality

Given two graphs X and Y, we have the following groupoids:

$$\Pi_1(X \,\square\, Y) \qquad \text{and} \qquad \Pi_1 X \times \Pi_1 Y$$

They have identical objects: pairs (x,y) where x is a vertex in X and y is a vertex in Y. However, they might a priori differ in their morphisms. Our next goal is to show that these two groupoids are in fact isomorphic.

We begin by observing that a morphism $(x,y) \to (x',y')$ in $\Pi_1(X \,\square\, Y)$ is given by the path-homotopy class $[\gamma]$ of a path $\gamma\colon (x,y) \rightsquigarrow (x',y')$ in the box product $X \,\square\, Y$. On the other hand, a morphism $(x,y) \to (x',y')$ in $\Pi_1 X \times \Pi_1 Y$ is given by a pair $([\sigma],[\tau])$ where $[\sigma]$ is the path-homotopy class of a path $\sigma\colon x \rightsquigarrow x'$ in X and $[\tau]$ is the path-homotopy class of a path $\tau\colon y \rightsquigarrow y'$ in Y.

Definition 3.1.6. Given graphs X and Y, let

$$\Psi_{X,Y}\colon \Pi_1(X \,\square\, Y) \to \Pi_1 X \times \Pi_1 Y$$

be the functor whose components are given by Π_1 applied to $\pi_X\colon X \,\square\, Y \to X$ and $\pi_Y\colon X \,\square\, Y \to Y$, respectively. It maps a morphism $[\gamma]: (x,y) \to (x',y')$ to the morphism $([\pi_X\gamma],[\pi_Y\gamma]): (x,y) \to (x',y')$.

These functors, as X and Y vary, satisfy certain coherence conditions and equip $\Pi_1 \colon \mathsf{Graph} \to \mathsf{Gpd}$ with the structure of a colax monoidal functor. We want to show that the $\Psi_{X,Y}$ are isomorphisms, which would mean that $\Pi_1 \colon \mathsf{Graph} \to \mathsf{Gpd}$ is, in fact, a strong monoidal functor.

Definition 3.1.7. Given any two graphs X and Y, let

$$\Phi_{X,Y} \colon \Pi_1 X \times \Pi_1 Y \to \Pi_1 (X \mathbin{\square} Y)$$

be the functor that maps a morphism $([\sigma], [\tau]) \colon (x, y) \to (x', y')$ in $\Pi_1 X \times \Pi_1 Y$ to the morphism $[(\sigma \mathbin{\square} y) * (x' \mathbin{\square} \tau)] \colon (x, y) \to (x', y')$ in $\Pi_1 (X \mathbin{\square} Y)$.

It is straightforward to verify that $\Psi_{X,Y} \circ \Phi_{X,Y}$ equals the identity functor on the groupoid $\Pi_1 X \times \Pi_1 Y$. It only remains to show that $\Phi_{X,Y} \circ \Psi_{X,Y}$ equals the identity functor on the groupoid $\Pi_1 (X \mathbin{\square} Y)$. In particular, we need to show that given any path $\gamma \colon (x, y) \rightsquigarrow (x', y')$ in the box product $X \mathbin{\square} Y$, we have:

$$[(\pi_X \gamma \mathbin{\square} y) * (x' \mathbin{\square} \pi_Y \gamma)] = [\gamma] \, .$$

Lemma 3.1.8. *Let $m, n \in \mathbb{N}$. Given any two paths $p, q \colon (0, 0) \rightsquigarrow (m, n)$ of length $m + n$ in the graph $I_m \mathbin{\square} I_n$, we have $[p] = [q]$ in $\pi_0 P_{\mathbb{N}} (I_m \mathbin{\square} I_n) ((0, 0), (m, n))$.*

Proof. Recall that we have a deformation retract of $I_m \mathbin{\square} I_n$ onto the vertex $(0, 0)$ that can be obtained by first contracting along the second variable and then contracting along the first variable. Explicitly, we have a graph map $H \colon I_m \mathbin{\square} I_n \mathbin{\square} I_{m+n} \to I_m \mathbin{\square} I_n$ given by

$$H (i, j, k) = \begin{cases} (i, \min (j, n - k)) & \text{if } 0 \leq k \leq n \\ (\min (i, m + n - k), 0) & \text{if } n \leq k \leq m + n \end{cases}$$

that satisfies:

$$H(-,-,0) = \mathrm{id}_{I_m \,\square\, I_n}, \qquad H(-,-,m+n) = \mathrm{c}_{(0,0)},$$

and

$$H(0,0,k) = (0,0), \qquad \text{for all } k = 1, \ldots, m+n.$$

Then, the composite

$$I_{2m+2n} \,\square\, I_{m+n} \xrightarrow{(p*\overline{q}) \,\square\, \mathrm{id}_{I_{m+n}}} I_m \,\square\, I_n \,\square\, I_{m+n} \xrightarrow{H} I_m \,\square\, I_n$$

defines a path-homotopy $p * \overline{q} \Rightarrow \mathrm{c}_{(0,0)}$, or equivalently a path $p * \overline{q} \rightsquigarrow \mathrm{c}_{(0,0)}$ in $\Omega_{2m+2n}(I_m \,\square\, I_n, (0,0))$. Hence, we have:

$$\begin{aligned} [p] &= \left[p * \mathrm{c}_{(k,l)}\right] \\ &= [p * \overline{q} * q] \\ &= \left[\mathrm{c}_{(0,0)} * q\right] \\ &= [q] \end{aligned}$$

in $\pi_0 P_{\mathbb{N}}(I_m \,\square\, I_n)((0,0),(m,n))$. $\square$

Lemma 3.1.9. *Given any path $\gamma\colon (x,y) \rightsquigarrow (x',y')$ of finite length n in $X \,\square\, Y$, we have:*

$$[(\pi_X \gamma \,\square\, y) * (x' \,\square\, \pi_Y \gamma)] = [\gamma]$$

in $\pi_0 P_{\mathbb{N}}(X \,\square\, Y)((x,y),(x',y'))$.

Proof. Replacing the path γ with a reparametrization of γ if necessary, we can suppose, without loss of generality, that $\gamma(i) \neq \gamma(i-1)$ for any $i = 1, \ldots, n$.

Let $i_1, \ldots, i_k \in \{1, \ldots, n\}$ be all the indices for which the maps

$\mathrm{e}_{i_1}^*(\gamma), \ldots, \mathrm{e}_{i_k}^*(\gamma) : I_1 \to X \,\Box\, Y$ are constant in the second variable (see Definition 2.3.6 for the notation e_i^*). Let $j_1, \ldots, j_l \in \{1, \ldots, n\}$ be all the indices for which the maps $\mathrm{e}_{j_1}^*(\gamma), \ldots, \mathrm{e}_{j_l}^*(\gamma) : I_1 \to X \,\Box\, Y$ are constant in the first variable.

We define a path $\sigma \colon x \rightsquigarrow x'$ in X of length k as follows:

$$\sigma = \left(\pi_X \circ \mathrm{e}_{i_1}^*(\gamma)\right) * \cdots * \left(\pi_X \circ \mathrm{e}_{i_k}^*(\gamma)\right)$$

and a path $\tau \colon y \rightsquigarrow y'$ in Y of length l as follows:

$$\tau = \left(\pi_Y \circ \mathrm{e}_{j_1}^*(\gamma)\right) * \cdots * \left(\pi_Y \circ \mathrm{e}_{j_l}^*(\gamma)\right)$$

We define a shrinking map $s \colon I_n \to I_k$ as follows:

$$s(i) = \begin{cases} 0 & \text{if } i = 0 \\ s(i-1) & \text{if } \mathrm{e}_i^*(\gamma) \text{ is constant in the first variable} \\ s(i-1)+1 & \text{if } \mathrm{e}_i^*(\gamma) \text{ is not constant in the first variable} \end{cases}$$

and a shrinking map $t \colon I_n \to I_l$ as follows:

$$t(i) = \begin{cases} 0 & \text{if } i = 0 \\ t(i-1) & \text{if } \mathrm{e}_i^*(\gamma) \text{ is constant in the second variable} \\ t(i-1)+1 & \text{if } \mathrm{e}_i^*(\gamma) \text{ is not constant in the second variable} \end{cases}$$

Finally, we define a path $p\colon (0,0) \rightsquigarrow (k,l)$ of length $k+l=n$ in $I_k \,\Box\, I_l$ as follows:

$$p(i) = (s(i), t(i))$$

By construction, we have the following:

$$\pi_X \circ \gamma = \sigma \circ s, \qquad \pi_Y \circ \gamma = \tau \circ t, \qquad \text{and} \qquad \gamma = (\sigma \,\square\, \tau) \circ p.$$

Then, we have:

$$\begin{aligned} ((\pi_X \circ \gamma) \,\square\, y) * (x' \,\square\, (\pi_Y \circ \gamma)) &= ((\pi_X \circ \gamma) \,\square\, (\pi_Y \circ \gamma)) \circ p_{\perp,n,n} \\ &= (\sigma \circ s) \,\square\, (\tau \circ t) \circ p_{\perp,n,n} \\ &= (\sigma \,\square\, \tau) \circ p_{\perp,k,l} \circ (s * t) \end{aligned}$$

where $p_{\perp,n,n}$ represents the path $(\mathrm{id}_{I_n} \,\square\, 0) * (n \,\square\, \mathrm{id}_{I_n})$ in $I_n \,\square\, I_n$ and $p_{\perp,k,l}$ represents the path $(\mathrm{id}_{I_k} \,\square\, 0) * (k \,\square\, \mathrm{id}_{I_l})$ in $I_k \,\square\, I_l$.

By Lemma 3.1.8, we have $[p_{\perp,k,l}] = [p]$ in $\pi_0 P_{\mathbb{N}} (I_k \,\square\, I_l) ((0,0), (k,l))$.

Consequently, we have:

$$\begin{aligned} [((\pi_X \circ \gamma) \,\square\, y) * (x' \,\square\, (\pi_Y \circ \gamma))] &= [(\sigma \,\square\, \tau) \circ p_{\perp,k,l} \circ (s * t)] \\ &= [(\sigma \,\square\, \tau) \circ p \circ (s * t)] \\ &= [\gamma \circ (s * t)] \\ &= [\gamma]. \end{aligned}$$

□

Theorem 3.1.10. *The functor* $\Pi_1 \colon \mathsf{Graph} \to \mathsf{Gpd}$ *is strong monoidal. In particular, given any two graphs* X *and* Y*, the functors*

$$\Psi_{X,Y} \colon \Pi_1 (X \,\square\, Y) \to \Pi_1 X \times \Pi_1 Y$$

and

$$\Phi_{X,Y} \colon \Pi_1 X \times \Pi_1 Y \to \Pi_1 (X \,\square\, Y)$$

are inverse to each other. □

We can now deduce the enrichment of the category of graphs over the category of groupoids.

Theorem 3.1.11. *The category* Graph *admits an enrichment over* Gpd *with hom-groupoids given by* $\Pi_1\left(\hom^{\square}(X, Y)\right)$*. Given this enrichment, the fundamental groupoid functor* $\Pi_1 \colon \mathsf{Graph} \to \mathsf{Gpd}$ *is a* Gpd*-enriched functor.*

Proof. This follows from a straightforward application of the change of base lemma for enriched categories [Rie14, Lem. 3.4.3] to the Graph, which is enriched over itself by Proposition 2.2.3 as a closed monoidal category, and to the functor $\Pi_1 \colon \mathsf{Graph} \to \mathsf{Gpd}$, which is strong monoidal by Theorem 3.1.10. □

A category enriched over Gpd is precisely a $(2, 1)$-category, with 2-cells given by morphisms in the hom-groupoids. In the case of Graph, where the hom-groupoids are given by $\Pi_1\left(\hom^{\square}(X, Y)\right)$, the 2-cells are given by path-homotopy classes of paths in the graph $\hom^{\square}(X, Y)$ or, equivalently, path-homotopy classes of homotopies between graph maps.

Corollary 3.1.12. Graph *admits the structure of a* $(2, 1)$*-category, with* 0*-cells given by graphs,* 1*-cells given by graph maps, and* 2*-cells given by path-homotopy classes of homotopies between graph maps. Furthermore,* $\Pi_1 \colon \mathsf{Graph} \to \mathsf{Gpd}$ *is a* $(2, 1)$*-functor and hence, maps homotopies between graph maps to natural isomorphisms between the induced functors.* □

3.2 Covering graphs

In this section, we develop the theory of covering graphs, a discrete analogue of the theory of covering spaces from algebraic topology.

We begin by introducing local isomorphisms and covering maps, giving several equivalent definitions of each of these notions (Lemmas 3.2.5 and 3.2.8), and studying their basic properties, e.g., the path and homotopy lifting properties (Lemmas 3.2.7 and 3.2.13, respectively). Note that the notion that we refer to as the 'local isomorphism' below was previously studied under the name 'covering map' in [BGJW21, Har19], the name which we reserve for a stronger notion.

We then turn our attention to the category of coverings of a fixed graph and show that it is equivalent to the category of Set-valued functors on the fundamental groupoid of the graph (Theorem 3.2.21). From that, we deduce the familiar Galois correspondence between connected coverings of a pointed connected graph and the poset of subgroups of its fundamental group (Corollary 3.2.23).

We conclude this section with the construction of the universal cover of a pointed graph (Theorem 3.2.29), which we take to be the initial object in the category of pointed coverings.

We note here that the key results of this section, including Theorems 3.2.21 and 3.2.29 and Corollary 3.2.23, require the full strength of our definition of a covering graph. When working with local isomorphisms instead, one needs to impose an additional assumption that the base graph has no 3- and 4-cycles, which by the result of Lutz [Lut21, Thm. 1.1] limits the theory to graphs with no non-trivial homotopy groups above dimension 1.

3.2.1 Local isomorphisms and covering maps

To define local isomorphisms and covering graphs, we need some preliminary definitions. The first of these is a well-known categorical notion of lifting properties.

Definition 3.2.1. Let $\mathcal{C}$ be any category. A morphism $p\colon C \to D$ in $\mathcal{C}$ has the *right lifting property (RLP)* against a morphism $i\colon A \to B$ in $\mathcal{C}$ if for every commutative square of the form

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "i"'] & C \arrow[d, "p"] \\ B \arrow[r, "v"] & D \end{tikzcd}$$

there exists a morphism $h\colon B \to C$, called a *lift*, such that the following diagram commutes:

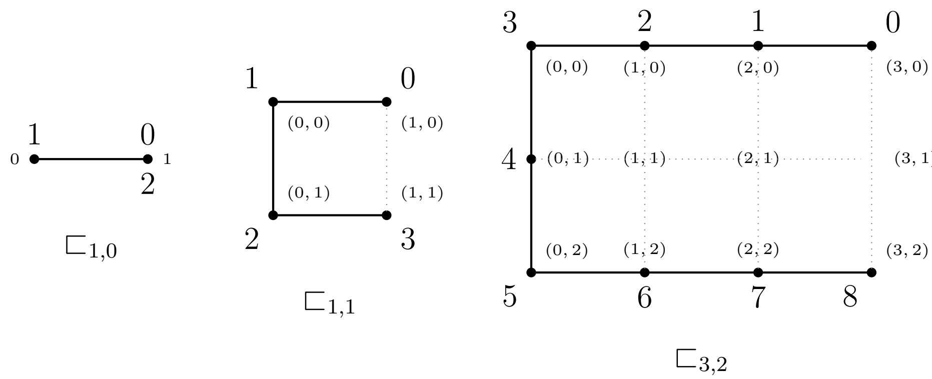
$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "i"'] & C \arrow[d, "p"] \\ B \arrow[r, "v"] \arrow[ur, dotted, "h"] & D \end{tikzcd}$$

Furthermore, if the lift $h\colon B \to C$ is unique, then we say that p has the *unique RLP* against i.

Next, we define certain graph maps that we will be interested in taking lifts against.

Definition 3.2.2. For any $m, n \in \mathbb{N}$, let $\sqsubset_{m,n}\colon I_{2m+n} \to I_m \,\square\, I_n$ be the graph map given by:

$$\sqsubset_{m,n}(i) = \begin{cases} (m-i, 0) & \text{if } 0 \leq i \leq m \\ (0, i-m) & \text{if } m \leq i \leq m+n \\ (i-m-n, n) & \text{if } m+n \leq i \leq 2m+n \end{cases}$$



Definition 3.2.3. The *(star) neighbourhood* N_x of a vertex x in a graph X is a subgraph of X given as follows:

$$(N_x)_V = \{x' \in X_V \mid x' \sim x \text{ in } X\}$$
$$(N_x)_E = \{x' \sim x'' \mid x' = x \text{ or } x'' = x\}$$

Lemma 3.2.4. *Let $p\colon Y \to X$ be a graph map.*

1. *The map p has the RLP against $\sqsubset_{1,0}\colon I_1 \to I_2$ if and only if the restriction $p|_{N_y}\colon N_y \to N_{p(y)}$ is injective for all $y \in Y$.*

2. *The map p has the RLP against $0\colon I_0 \to I_1$ if and only if the restriction $p|_{N_y}\colon N_y \to N_{p(y)}$ is surjective for all $y \in Y$.*

Proof. 1. ($\Rightarrow$) Let $y \in Y$ and $x = p(y) \in X$. Suppose there exist $y', y'' \in N_y$ such that $p(y') = p(y'') = x' \in N_x$. We can define a graph map $\lambda\colon I_2 \to Y$ where

$$\lambda(0) = y', \qquad \lambda(1) = y, \qquad \lambda(2) = y''$$

giving us the following lifting problem:

$$\begin{tikzcd} I_2 \arrow[r, "\lambda"] \arrow[d, "\sqsubset_{1,0}"'] & Y \arrow[d, "p"] \\ I_1 \arrow[r, "e"] & X \end{tikzcd}$$

where $e\colon I_1 \to X$ is given by $e(0) = x$ and $e(1) = x'$. By the hypothesis, the above lifting problem has a solution, which implies that

$$y' = \lambda(0) = \lambda(2) = y''$$

Thus, $p|_{N_y}$ is injective.

($\Leftarrow$) Consider a lifting problem between $\sqsubset_{1,0}\colon I_2 \to I_1$ and $p\colon Y \to X$ as follows:

$$\begin{tikzcd} I_2 \arrow[r, "u"] \arrow[d, "\sqsubset_{1,0}"'] & Y \arrow[d, "p"] \\ I_1 \arrow[r, "v"] & X \end{tikzcd}$$

Then, we have two vertices $u(0), u(2) \in N_{u(1)}$ such that $p \circ u(0) = p \circ u(2) = v(1) \in N_{p \circ u(1)} = N_{v(0)}$. By the hypothesis, $p|_{N_{u(1)}}$ is injective, and hence, we must have $u(0) = u(2)$. Thus, the above lifting problem has a solution.

2. ($\Rightarrow$) Let $y \in Y$ and $x = p(y) \in X$. Given any $x' \in N_x$, let $e\colon I_1 \to X$ be the graph map given by $e(0) = x$ and $e(1) = x'$, and consider the following lifting problem:

$$\begin{tikzcd} I_0 \arrow[r, "y"] \arrow[d, "0"'] & Y \arrow[d, "p"] \\ I_1 \arrow[r, "e"] & X \end{tikzcd}$$

By hypothesis, there exists a solution to the lifting problem, say $\widetilde{e}\colon I_1 \to Y$. Then, letting $y' = \widetilde{e}\,(1)$ we have $y' \in N_y$ and also

$$p\,(y') = (p \circ \widetilde{e})\,(1) = e\,(1) = x'$$

Thus, $p|_{N_y}$ is surjective.

($\Leftarrow$) Consider a lifting problem between $0\colon I_0 \to I_1$ and $p\colon Y \to X$ as follows:

$$\begin{array}{ccc} I_0 & \xrightarrow{u} & Y \\ {\scriptstyle 0}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ I_1 & \xrightarrow{v} & X \end{array}$$

Then, we have a vertex $v\,(1) \in N_{v(0)} = N_{p\circ u(0)}$. By the hypothesis, $p|_{N_{u(0)}}$ is surjective. Hence, there exists some vertex $y' \in N_{u(0)}$ such that $p\,(y') = v\,(1)$. Thus, the above lifting problem has a solution $\widetilde{v}\colon I_1 \to Y$ given by $\widetilde{v}\,(0) = u\,(0)$ and $\widetilde{v}\,(1) = y'$. $\square$

Lemma 3.2.5. *Let $p\colon Y \to X$ be a graph map. The following are equivalent:*

1. *p has the unique RLP against $0\colon I_0 \to I_n$ for all $n \in \mathbb{N}$.*

2. *p has the unique RLP against $0\colon I_0 \to I_1$.*

3. *p has the RLP against $0\colon I_0 \to I_1$ and $\sqsubset_{1,0}\colon I_2 \to I_1$.*

4. *For every vertex $y \in Y$, the restriction $p|_{N_y}\colon N_y \to N_{p(y)}$ is an isomorphism.*

Proof.

(1) $\Rightarrow$ (2) is immediate.

(2) $\Rightarrow$ (3) p has the RLP against $0\colon I_0 \to I_1$ by assumption. So, it suffices to show that it has the RLP against $\sqsubset_{1,0}\colon I_2 \to I_1$. To this end, consider a lifting problem as follows:

$$\begin{tikzcd} I_2 \arrow[r, "u"] \arrow[d, "\sqsubset_{1,0}"'] & Y \arrow[d, "p"] \\ I_1 \arrow[r, "v"] & X \end{tikzcd}$$

Note that $\sqsubset_{1,0}(1) = 0$ and $\sqsubset_{1,0}(0) = 1 = \sqsubset_{1,0}(2)$. Thus, the above lifting problem admits a solution $h\colon I_1 \to Y$ if and only if we have $u(0) = u(2)$.

To see that $u(0) = u(2)$, we consider the following diagram:

$$\begin{tikzcd} I_0 \arrow[r, "1"] \arrow[d, "0"'] & I_2 \arrow[r, "u"] \arrow[d, dashed, "\sqsubset_{1,0}"] & Y \arrow[d, "p"] \\ I_1 \arrow[r, equal] & I_1 \arrow[r, "v"] & X \end{tikzcd}$$

This defines a lifting problem between $0\colon I_0 \to I_1$ and $p\colon Y \to X$. There exist two distinct solutions to the lifting problem given by the left part of the diagram, namely $\mathrm{e}_2\colon I_1 \to I_2$ and $\overline{\mathrm{e}_1}\colon I_1 \to I_2$. It follows that $u \circ \mathrm{e}_2$ and $u \circ \overline{\mathrm{e}_1}$ are both solutions to the lifting problem between $0\colon I_0 \to I_1$ and $p\colon Y \to X$. But, since p was assumed to have the unique RLP against $0\colon I_0 \to I_1$, they must be equal. So, we have:

$$u(0) = (u \circ \overline{\mathrm{e}_1})(1) = (u \circ \mathrm{e}_2)(1) = u(2)$$

as required.

(3) $\Rightarrow$ (4) This follows from Lemma 3.2.4.

(4) ⇒ (1) Consider the following lifting problem:

$$\begin{tikzcd} I_0 \arrow[r, "y"] \arrow[d, "0"'] & Y \arrow[d, "p"] \\ I_n \arrow[r, "\gamma"] & X \end{tikzcd}$$

We want to construct the unique lift $\widetilde{\gamma}\colon I_n \to Y$. We construct it inductively. Note that we must have $\widetilde{\gamma}(0) = y$, giving us the base case. Suppose we have constructed $\widetilde{\gamma}(i-1) \in Y$ for some $i \in \{1, \ldots, n\}$. Let $y_{i-1} = \widetilde{\gamma}(i-1)$ and $x_{i-1} = \gamma(i-1)$. Then, since $p|_{N_{y_{i-1}}}\colon N_{y_{i-1}} \to N_{x_{i-1}}$ is an isomorphism, there is a unique vertex $y_i \in N_{y_{i-1}}$ such that $p(y_i) = \gamma(i) \in N_{x_{i-1}}$. Then, we must define $\widetilde{\gamma}(i) = y_i$. Thus, the unique lift $\widetilde{\gamma}\colon I_n \to Y$ is given by the following recursive formula:

$$\widetilde{\gamma}(i) = \begin{cases} y & \text{if } i = 0 \\ \left(p|_{N_{\widetilde{\gamma}(i-1)}}\right)^{-1}(\gamma(i)) & \text{if } i = 1, \ldots, n \end{cases} \qquad \square$$

Definition 3.2.6. A map $p\colon Y \to X$ is a *local isomorphism* if it satisfies any of the equivalent conditions in Lemma 3.2.5.

We emphasize, again, that our definition of a local isomorphism captures what other authors, e.g., [BGJW21, Har19] refer to as a covering graph. For us, a covering graph is a stronger notion, presented in Definition 3.2.9.

Lemma 3.2.7 (Path lifting property, cf. [Har19, Thm. 3.0.9])**.** *Let $p\colon Y \to X$ be a local isomorphism. Given any path $\gamma\colon x \rightsquigarrow x'$ in X and any vertex $y \in p^{-1}(x)$, there exists a unique path $\widetilde{\gamma}$ in Y starting at y such that $p \circ \widetilde{\gamma} = \gamma$.*

Proof. This follows from the fact that local isomorphisms have the unique RLP against $0\colon I_0 \to I_n$ for all $n \in \mathbb{N}$. $\square$

Lemma 3.2.8. *Let $p\colon Y \to X$ be a graph map. The following are equivalent:*

(1) The induced map $p_*\colon \hom^{\square}(I_n, Y) \to \hom^{\square}(I_n, X)$ is a local isomorphism for all $n \in \mathbb{N}$.

(1') p has the unique RLP against $0 \square \mathrm{id}\colon I_0 \square I_n \to I_1 \square I_n$ for all $n \in \mathbb{N}$.

(2) The induced map $p_*\colon \hom^{\square}(I_n, Y) \to \hom^{\square}(I_n, X)$ is a local isomorphism for $n = 0, 1$.

(2') p has the unique RLP against $0 \square \mathrm{id}\colon I_0 \square I_n \to I_1 \square I_n$ for $n = 0, 1$.

(3) p has the unique RLP against $0\colon I_0 \to I_1$ and $\sqsubset_{1,1}\colon I_3 \to I_1 \square I_1$.

(4) p has the RLP against $0\colon I_0 \to I_1$, $\sqsubset_{1,0}\colon I_2 \to I_1$, and $\sqsubset_{1,1}\colon I_3 \to I_1 \square I_1$.

Proof. By Proposition 2.2.3, we have a bijective correspondence between the following lifting problems:

$$\begin{tikzcd} I_0 \square I_n \arrow[r] \arrow[d, "0 \square \mathrm{id}"'] & Y \arrow[d, "p"] & \leftrightsquigarrow & I_0 \arrow[r] \arrow[d, "0"'] & \hom^{\square}(I_n, Y) \arrow[d, "p_*"] \\ I_1 \square I_n \arrow[r] & X & & I_1 \arrow[r] & \hom^{\square}(I_n, X) \end{tikzcd}$$

Furthermore, either lifting problem admits a (unique) solution if and only if its counterpart admits a (unique) solution. Thus, we have the equivalences (1) $\Leftrightarrow$ (1′) and (2) $\Leftrightarrow$ (2′).

(1′) $\Rightarrow$ (2′) is immediate.

(2′) $\Rightarrow$ (3) Note that for $n = 0$, the map $0 \square \mathrm{id}\colon I_0 \square I_n \to I_1 \square I_n$ equals the map $0\colon I_0 \to I_1$. So, it suffices to show that p has the unique RLP against

$\sqsubset_{1,1}\colon I_3 \to I_1 \square I_1$. To this end, consider a lifting problem as follows:

$$\begin{tikzcd}
I_3 \arrow[r, "u"] \arrow[d, "\sqsubset_{1,1}"'] & Y \arrow[d, "p"] \\
I_1 \square I_1 \arrow[r, "v"] & X
\end{tikzcd}$$

Note that this lifting problem has a unique solution if and only if we have $u(0) \sim u(3)$ in Y.

Consider the following diagram:

$$\begin{tikzcd}
I_0 \square I_1 \cong I_1 \arrow[r, "\mathrm{e}_2"] \arrow[d, "0 \square \mathrm{id}"'] & I_3 \arrow[r, "u"] \arrow[d, dashed, "\sqsubset_{1,1}"] & Y \arrow[d, "p"] \\
I_1 \square I_1 \arrow[r, equal] & I_1 \square I_1 \arrow[r, "v"] & X
\end{tikzcd}$$

This defines a lifting problem between $0 \square \mathrm{id}\colon I_0 \square I_1 \to I_1 \square I_1$ and p. By the assumption, this lifting problem has a unique solution. So, there exists a map $h\colon I_1 \square I_1 \to Y$ that makes the above diagram commute. Thus, we have $h(1,1) \sim h(1,0)$ in Y. We will now show that $h(1,0) = u(0)$ and that $h(1,1) = u(3)$.

We know that $h(0,0) = u(1)$ and that $p \circ h = v$. Thus, both $u \circ \overline{\mathrm{e}_1}\colon I_1 \to Y$ and $h \circ (\mathrm{id} \square 0) : I_1 \square I_0 \cong I_1 \to Y$ are solutions to the following lifting problem:

$$\begin{tikzcd}
I_0 \arrow[r, "u(1)"] \arrow[d, "0"'] & Y \arrow[d, "p"] \\
I_1 \arrow[r, "v \circ (\mathrm{id} \square 0)"] & X
\end{tikzcd}$$

Since p has the unique RLP against $0\colon I_0 \to I_1$ by the assumption, the two solutions must coincide and in particular, we have:

$$h\,(1,0) = (h \circ (\mathrm{id} \,\square\, 0))\,(1) = (u \circ \overline{\mathrm{e}_1})\,(1) = u\,(0)$$

A similar argument shows that $h\,(1,1) = u\,(3)$.

$(3) \Rightarrow (1')$ Consider the following lifting problem:

$$\begin{tikzcd} I_0 \,\square\, I_n \arrow[r, "u"] \arrow[d, "0 \,\square\, \mathrm{id}"'] & Y \arrow[d, "p"] \\ I_1 \,\square\, I_n \arrow[r, "v"] & X \end{tikzcd}$$

We want to construct a lift $h \colon I_1 \,\square\, I_n \to Y$ and show that it is unique.

Note that any solution $h \colon I_1 \,\square\, I_n \to Y$ must necessarily satisfy the condition $h\,(0,i) = u\,(0,i)$ for $i = 0, \ldots, n$.

For each $i = 0, \ldots, n$, consider the following diagram:

$$\begin{tikzcd} I_0 \,\square\, I_0 \arrow[r, "\mathrm{id} \,\square\, i"] \arrow[d, "0 \,\square\, \mathrm{id}"'] & I_0 \,\square\, I_n \arrow[r, "u"] \arrow[d, dashed, "0 \,\square\, \mathrm{id}"] & Y \arrow[d, "p"] \\ I_1 \,\square\, I_0 \arrow[r, "\mathrm{id} \,\square\, i"] & I_1 \,\square\, I_n \arrow[r, "v"] & X \end{tikzcd}$$

Since p has the unique RLP against $0 \,\square\, \mathrm{id} \colon I_0 \,\square\, I_0 \to I_1 \,\square\, I_0$ by hypothesis, there exists a unique lift $h_i \colon I_1 \cong I_1 \,\square\, I_0 \to Y$.

Note that if a solution $h \colon I_1 \,\square\, I_n \to Y$ to the original lifting problem exists, then the composite $h \circ (\mathrm{id} \,\square\, i)$ is also a solution to the second lifting problem. By uniqueness of solutions, we must have $h \circ (\mathrm{id} \,\square\, i) = h_i$. In other words, we must necessarily define $h\,(1,i) = h_i\,(1)$ for $i = 0, \ldots, n$.

It only remains to verify that for each $i = 1, \ldots, n$, we have $h\,(1, i-1) \sim$

$h(1, i)$ in Y.

To this end, let $\widetilde{h}_i \colon I_3 \to Y$ be the map given by the concatenation $\widetilde{h}_i = \overline{h_{i-1}} * u_i * h_i$, where $u_i \colon I_1 \to Y$ is the composite $u_i = u \circ (\mathrm{id} \mathbin{\square} \mathrm{e}_i) : I_1 \cong I_0 \mathbin{\square} I_1 \to Y$, for $i = 1, \ldots, n$. Observe that:

$$\widetilde{h}_i(0) = h_{i-1}(1) = h(i-1, 1), \qquad \widetilde{h}_i(3) = h_i(1) = h(i, 1)$$

We can consider the following lifting problem for $i = 1, \ldots, n$:

$$\begin{tikzcd} I_3 \arrow[r, "\widetilde{h}_i"] \arrow[d, "\sqsubset_{1,1}"'] & Y \arrow[d, "p"] \\ I_1 \mathbin{\square} I_1 \arrow[r, "v \circ (\mathrm{id} \,\square\, \mathrm{e}_i)"] & X \end{tikzcd}$$

By hypothesis, p has the unique RLP against $\sqsubset_{1,1} \colon I_3 \to I_1 \mathbin{\square} I_1$, and hence we have $\widetilde{h}_i(0) \sim \widetilde{h}_i(3)$ in Y, as required.

(3) $\Leftrightarrow$ (4) This follows from Lemma 3.2.5 and the observation that any solution to a lifting problem against $\sqsubset_{1,1} \colon I_3 \to I_1 \mathbin{\square} I_1$ is necessarily unique. $\square$

Definition 3.2.9. A map $p \colon Y \to X$ is a *covering map* if it satisfies any of the equivalent conditions in Lemma 3.2.8. In such a case, the graph X is called the *base graph* and the graph Y is called the *covering graph*.

Remark 3.2.10. Every covering map is a local isomorphism. A local isomorphism is a covering map if and only if it also has the RLP against $\sqsubset_{1,1} \colon I_3 \to I_1 \mathbin{\square} I_1$. In particular, any local isomorphism $p \colon Y \to X$ where the base graph X contains no 3-cycles or 4-cycles is a covering map.

Example 3.2.11. The graph map $p \colon I_\infty \to C_n;\ i \mapsto i \pmod{n}$ is a local isomorphism for all $n \geq 3$, but a covering map only for $n \geq 5$. In particular, this shows how our definition of a covering graph differs from the one given in [BKLW01, Har19].

Example 3.2.12. The graph maps $C_{3n} \to C_3;\ i \mapsto i \pmod 3$ and $C_{4n} \to C_4;\ i \mapsto i \pmod 4$ are both local isomorphisms for every $n \geq 1$, but covering maps only for $n = 1$.

We now prove the homotopy lifting property of covering graphs. This property is known to fail for local isomorphisms if the base graph contains a 3- or 4-cycle. In fact, our definition of a covering graph was greatly influenced by closely examining Hardeman's proof of the analogous property for local isomorphisms under the additional assumption that the base graph does not contain any 3- or 4-cycles [Har19, Thm. 3.0.10].

Lemma 3.2.13 (Homotopy lifting property)**.** *Let $p\colon Y \to X$ be a covering map. Given any two paths $\gamma, \sigma\colon x \rightsquigarrow x'$ in X, a vertex $y \in p^{-1}(x)$, and two paths $\widetilde{\gamma}, \widetilde{\sigma}$ in Y starting at y such that $p \circ \widetilde{\gamma} = \gamma$ and $p \circ \widetilde{\sigma} = \sigma$, if we have $[\gamma] = [\sigma]$ in $\pi_0 P_{\mathbb{N}} X(x, x')$, then the paths $\widetilde{\gamma}$ and $\widetilde{\sigma}$ have the same endpoint $y' \in p^{-1}(x')$ and we have $[\widetilde{\gamma}] = [\widetilde{\sigma}]$ in $\pi_0 P_{\mathbb{N}} Y(y, y')$.*

Proof. Since we have $[\gamma] = [\sigma]$ in $\pi_0 P_{\mathbb{N}} X(x, x')$, there must exist a path-homotopy $H\colon I_m \square I_n \to X$ such that

$$H(-,0) = \gamma \circ s, \qquad H(-,n) = \sigma \circ t$$

and

$$H(0,-) = \mathrm{c}_x, \qquad H(m,-) = \mathrm{c}_{x'}$$

where s and t are some shrinking maps.

Consider the following lifting problem:

$$\begin{tikzcd} I_0 \,\square\, I_n \arrow[r, "c_y"] \arrow[d, "0 \,\square\, \mathrm{id}"'] & Y \arrow[d, "p"] \\ I_m \,\square\, I_n \arrow[r, "H"] & X \end{tikzcd}$$

Since $p\colon Y \to X$ is a covering map, the induced map $p_*\colon \hom^{\square}(I_n, Y) \to \hom^{\square}(I_n, X)$ is a local isomorphism for all $n \in \mathbb{N}$. Hence, the induced map $p_*\colon \hom^{\square}(I_n, Y) \to \hom^{\square}(I_n, X)$ has the unique RLP against $0\colon I_0 \to I_m$ for all $m, n \in \mathbb{N}$. Equivalently, $p\colon Y \to X$ has the unique RLP against $0 \,\square\, \mathrm{id}\colon I_0 \,\square\, I_n \to I_m \,\square\, I_n$ for all $m, n \in \mathbb{N}$. Thus, the lifting problem described earlier has a unique solution. That is, we have a unique map $\widetilde{H}\colon I_m \,\square\, I_n \to Y$ such that:

$$p \circ \widetilde{H} = H, \text{ and } \qquad \widetilde{H}(0, -) = c_y.$$

Also note that we have:

$$\widetilde{H}(-, 0) = \widetilde{\gamma} \circ s, \qquad \widetilde{H}(-, n) = \widetilde{\sigma} \circ t.$$

Observe that the path $\widetilde{H}(m, -)\colon (\widetilde{\gamma} \circ s)(n) \rightsquigarrow (\widetilde{\sigma} \circ t)(n)$ satisfies $p \circ \widetilde{H}(m, -) = c_{x'}$. Thus, it must also be a constant path at some $y' \in p^{-1}(x')$. It follows that both $\widetilde{\gamma}$ and $\widetilde{\sigma}$ must have their endpoints at the same vertex $y' \in p^{-1}(x')$ and that the map $\widetilde{H}$ defines a path-homotopy $\widetilde{\gamma} \circ s \Rightarrow \widetilde{\sigma} \circ t$. $\square$

3.2.2 The category of coverings over a fixed base graph

Our next goal is to define and study properties of the category of coverings of a fixed graph.

Definition 3.2.14. Let $p\colon Y \to X$ and $p'\colon Y' \to X$ be two covering maps over the same base X. A *morphism* $f\colon p \to p'$ *of coverings over* X is a graph map $f\colon Y \to Y'$ such that the following diagram commutes:

$$\begin{tikzcd} Y \arrow[rr, "f"] \arrow[dr, "p"'] & & Y' \arrow[dl, "p'"] \\ & X & \end{tikzcd}$$

For any graph X, coverings over X and morphisms between them form a category, denoted $\mathsf{Cov}\,(X)$. It is the full subcategory of the slice $\mathsf{Graph}\downarrow X$ on covering maps.

Given a graph map $g\colon X' \to X$, we have a functor $g^*\colon \mathsf{Cov}\,(X) \to \mathsf{Cov}\,(X')$. It maps a covering $p\colon Y \to X$ over X to the pullback $g^*p\colon g^*Y \to X'$, which is a covering over X'. It is straightforward to verify that g^* is a functor. Furthermore, the assignment $(g\colon X' \to X) \mapsto (g^*\colon \mathsf{Cov}\,(X) \to \mathsf{Cov}\,(X'))$ is also functorial, and we have a well-defined functor

$$\mathsf{Cov}\colon \mathsf{Graph}^{\mathrm{op}} \to \mathsf{Cat}$$

The next lemma is a well-known categorical fact about cancellation of morphisms with certain lifting property, which we chose to include here for completeness.

Lemma 3.2.15. *Let $\mathcal{C}$ be any category. Consider the following commutative triangle in $\mathcal{C}$:*

$$\begin{tikzcd} X \arrow[rr, "q"] \arrow[dr, "p"'] & & Y \arrow[dl, "r"] \\ & Z & \end{tikzcd}$$

If p and r have the unique RLP against some fixed morphism $i\colon A \to B$, then so does q.

Proof. Consider a lifting problem between i and q as follows:

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "i"'] & X \arrow[d, "q"] \arrow[dr, "p"] & \\ B \arrow[r, "v"] & Y \arrow[r, "r"] & Z \end{tikzcd}$$

Since p has the unique RLP against i, there exists a unique morphism $h\colon B \to X$ such that $h \circ i = u$ and $p \circ h = r \circ v$.

Observe that both $q \circ h$ and v are solutions to the following lifting problem between i and r:

$$\begin{tikzcd} A \arrow[r, "q \circ u"] \arrow[d, "i"'] & Y \arrow[d, "r"] \\ B \arrow[r, "r \circ v"] & Z \end{tikzcd}$$

But since r has the unique RLP against i, we must have $q \circ h = v$.

Since we have both $h \circ i = u$ and $q \circ h = v$, it follows that $h\colon B \to X$ is a solution to the original lifting problem between i and q. Furthermore, it is unique since any solution to the lifting problem between i and q is also a solution to the corresponding lifting problem between i and p, which admits a unique solution. $\square$

Recalling that covering maps are defined to be those maps that have the unique RLP against $0\colon I_0 \to I_1$ and $\sqsubset_{1,1}\colon I_3 \to I_1 \square I_1$, we have the following result:

Proposition 3.2.16. *Let $f : p \to p'$ be a morphism in $\mathsf{Cov}\,(X)$ between two coverings $p\colon Y \to X$ and $p' : Y' \to X$. Then the underlying graph map $f : Y \to Y'$ is itself a covering map.* $\square$

Given a covering graph over X, we will construct a Set-valued functor on the fundamental groupoid $\Pi_1 X$. The next set of definitions elaborate on this construction.

Definition 3.2.17.

1. Given a local isomorphism $p\colon Y \to X$ and a path $\gamma\colon x \rightsquigarrow x'$ in X, the *unwinding of* γ is a set-function

$$\mathsf{unw}_{\gamma}\colon p^{-1}(x) \to p^{-1}(x')$$

given by

$$y \mapsto \text{ the end-point of the unique lift } \widetilde{\gamma} \text{ of } \gamma \text{ starting at } y$$

where $\widetilde{\gamma}$ is the unique path in Y that starts at y and satisfies $p \circ \widetilde{\gamma} = \gamma$ (cf. Lemma 3.2.7).

If p is a covering map, by Lemma 3.2.13, the function $\mathsf{unw}_{\gamma}\colon p^{-1}(x) \to p^{-1}(x')$ only depends on the path-homotopy class of γ and so, we may denote it by $\mathsf{unw}_{[\gamma]}\colon p^{-1}(x) \to p^{-1}(x')$.

2. Given any covering map $p\colon Y \to X$, let

$$\mathsf{Fib}_X(p) : \Pi_1 X \to \mathsf{Set}$$

be the functor that maps a vertex $x \in X$ to the fiber $p^{-1}(x)$ over x, and a morphism $[\gamma] : x \to x'$ in X to the set-function $\mathsf{unw}_{[\gamma]}\colon p^{-1}(x) \to p^{-1}(x')$.

3. Given any graph X, let

$$\mathsf{Fib}_X\colon \mathsf{Cov}(X) \to \mathsf{Set}^{\Pi_1 X}$$

be the functor that maps an object $p \in \mathsf{Cov}(X)$ to the functor

$\mathsf{Fib}_X(p) : \Pi_1 X \to \mathsf{Set}$, and a morphism $f : p \to p'$ in $\mathsf{Cov}(X)$ to the induced natural transformation $\mathsf{Fib}_X(f) : \mathsf{Fib}_X(p) \Rightarrow \mathsf{Fib}_X(p')$ whose components are given by the restrictions $f|_{p^{-1}(x)} : p^{-1}(x) \to (p')^{-1}(x)$ for $x \in X$.

4. Given any morphism $g : X' \to X$ in Graph, the following naturality square commutes up to a natural isomorphism of functors:

$$\begin{tikzcd}
\mathsf{Cov}(X) \arrow[r, "\mathsf{Fib}_X"] \arrow[d, "g^*"'] & \mathsf{Set}^{\Pi_1 X} \arrow[d, "(-)\circ g_*"] \\
\mathsf{Cov}(X') \arrow[r, "\mathsf{Fib}_{X'}"] & \mathsf{Set}^{\Pi_1 X'}
\end{tikzcd}$$

Let $\mathsf{Fib} : \mathsf{Cov} \Rightarrow \mathsf{Set}^{\Pi_1(-)}$ be the pseudo-natural transformation whose components are given by $\mathsf{Fib}_X : \mathsf{Cov}(X) \to \mathsf{Set}^{\Pi_1 X}$.

In the reverse direction, given a Set-valued functor on the fundamental groupoid $\Pi_1 X$ of a graph X, we can construct a covering graph over X.

Definition 3.2.18. Given any graph X, the *total graph* $\mathsf{Tot}_X F$ of a functor $F : \Pi_1 X \to \mathsf{Set}$ is defined as follows:

$$(\mathsf{Tot}_X F)_V = \coprod_{x \in X} Fx$$

$$(\mathsf{Tot}_X F)_E = \left\{ y \sim y' \;\middle|\; \begin{array}{l} x \sim x' \text{ in X, where } y \in Fx \text{ and } y' \in Fx', \text{ and} \\ (F[e])(y) = y', \text{ where } e : I_1 \to X \text{ is given by} \\ e(0) = x \text{ and } e(1) = x' \end{array} \right\}$$

It comes equipped with a graph map $p : \mathsf{Tot}_X F \to X$ that maps $y \in \mathsf{Tot}_X F$ to $x \in X$ if $y \in Fx$.

Proposition 3.2.19. *Given any graph X and functor $F\colon \Pi_1 X \to \mathsf{Set}$, the graph map $p\colon \mathsf{Tot}_X F \to X$ is a covering map.*

Proof. We need to verify that $p\colon \mathsf{Tot}_X F \to X$ has the unique RLP against $0\colon I_0 \to I_1$ and against $\sqsubset_{1,1}\colon I_3 \to I_1 \square I_1$.

We first consider the following lifting problem against $0\colon I_0 \to I_1$:

$$\begin{tikzcd} I_0 \arrow[r, "y"] \arrow[d, "0"'] & \mathsf{Tot}_X F \arrow[d, "p"] \\ I_1 \arrow[r, "e"] & X \end{tikzcd}$$

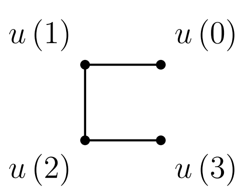
where $e(0) = p(y) = x$ and $e(1) = x'$ for some $x, x' \in X$ such that $x \sim x'$ in X.

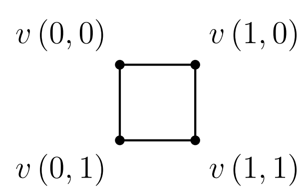
Note that any solution $\widetilde{e}\colon I_1 \to Y$ to the above lifting problem must satisfy $\widetilde{e}(0) = y$ and $\widetilde{e}(1) = F([e])(y)$. Since these two conditions define a well-defined graph map, we have a unique solution to the above lifting problem.

Next, we consider the lifting problem:

$$\begin{tikzcd} I_3 \arrow[r, "u"] \arrow[d, "\sqsubset_{1,1}"'] & \mathsf{Tot}_X F \arrow[d, "p"] \\ I_1 \square I_1 \arrow[r, "v"] & X \end{tikzcd}$$

We can visualize the maps $u\colon I_3 \to \mathsf{Tot}_X F$ and $v\colon I_1 \square I_1 \to X$ as follows:





This lifting problem admits a unique solution if and only if we have $u(0) \sim u(3)$ in $\mathsf{Tot}_X F$.

Note that we have $v(1,0) \sim v(1,1)$. Thus, it suffices to show that $F([e])(u(0)) = u(3)$ where $e\colon I_1 \to X$ is given by $e(0) = v(1,0)$ and $e(1) = v(1,1)$.

Consider the path $\gamma\colon I_3 \to X$ given by the composite

$$\gamma = \left(I_3 \xrightarrow{\sqsubset_{1,1}} I_1 \square I_1 \xrightarrow{v} X \right)$$

Note that $[\gamma] = [e]$.

Thus, we have:

$$\begin{aligned} F[e](u(0)) &= (F[\gamma])(u(0)) \\ &= F[\mathrm{e}_3^*(\gamma)] \circ F[\mathrm{e}_2^*(\gamma)] \circ F[\mathrm{e}_1^*(\gamma)](u(0)) \\ &= F[\mathrm{e}_3^*(\gamma)] \circ F[\mathrm{e}_2^*(\gamma)](u(1)) \\ &= F[\mathrm{e}_3^*(\gamma)](u(2)) \\ &= u(3) \end{aligned}$$

□

Definition 3.2.20.

1. Given any graph X, let

$$\mathsf{Tot}_X\colon \mathsf{Set}^{\Pi_1 X} \to \mathsf{Cov}(X)$$

be the functor that maps a functor $F\colon \Pi_1 X \to \mathsf{Set}$ to the covering map $\mathsf{Tot}_X F \to X$, and a natural transformation $\alpha\colon F \Rightarrow G$ between two functors $F, G\colon \Pi_1 X \to \mathsf{Set}$ to the induced morphism $\mathsf{Tot}_X(\alpha) : \mathsf{Tot}_X(F) \to \mathsf{Tot}_X(G)$ of coverings over X given by the disjoint union $\coprod_{x\in X} \alpha_x$.

2. Given any morphism $g\colon X' \to X$ in Graph, the following naturality

square commutes up to a natural isomorphism:

$$\begin{tikzcd}
\mathsf{Set}^{\Pi_1 X} \arrow[r, "\mathsf{Tot}_X"] \arrow[d, "(-)\circ g_*"'] & \mathsf{Cov}\,(X) \arrow[d, "g^*"] \\
\mathsf{Set}^{\Pi_1 X'} \arrow[r, "\mathsf{Tot}_{X'}"] & \mathsf{Cov}\,(X')
\end{tikzcd}$$

Let $\mathsf{Tot}\colon \mathsf{Set}^{\Pi_1(-)} \Rightarrow \mathsf{Cov}$ be the pseudo-natural transformation whose components are given by $\mathsf{Tot}_X\colon \mathsf{Set}^{\Pi_1 X} \to \mathsf{Cov}\,(X)$.

Theorem 3.2.21. *For each graph X, the functors*

$$\mathsf{Fib}_X\colon \mathsf{Cov}\,(X) \to \mathsf{Set}^{\Pi_1 X}$$

and

$$\mathsf{Tot}_X\colon \mathsf{Set}^{\Pi_1 X} \to \mathsf{Cov}\,(X)$$

define an equivalence of categories.

Proof. Straightforward after unfolding the definitions. □

If the graph X is path-connected, we may replace the functor category $\mathsf{Set}^{\Pi_1 X}$ by a smaller model $\mathsf{Set}^{A_1(X,x)}$, i.e. the category of $A_1\,(X,x)$-sets, for any vertex $x \in X$, by Proposition 3.1.5.

Corollary 3.2.22. *For a pointed connected graph (X,x), we have an equivalence of categories*

$$\mathsf{Cov}(X) \simeq \mathsf{Set}^{A_1(X,x)}$$

given by postcomposing the functors Fib_X and Tot_X with the equivalence of Proposition 3.1.5. □

This allows us to recover the usual Galois correspondence between subgroups of the fundamental group of a pointed connected space and the connected coverings thereof.

Corollary 3.2.23. *For a pointed connected graph* (X, x)*, the functors* Fib_X *and* Tot_X *restrict to give an equivalence of categories between the opposite of the poset of subgroups of* $A_1(X, x)$ *and the category of connected covering graphs of* (X, x)*, i.e.,*

$$\left\{ \begin{array}{c} \text{subgroups of } A_1(X,x) \\ \text{ordered by } \subseteq \end{array} \right\}^{\mathrm{op}} \simeq \left\{ \begin{array}{c} \text{connected coverings} \\ \text{of } (X,x) \end{array} \right\}.$$

A connected cover $p\colon (Y, y_0) \to (X, x_0)$ *corresponds to the subgroup* $p_*(A_1(Y, y_0)) \leq A_1(X, x_0)$*.*

Proof. By restricting the equivalence of Corollary 3.2.22 to connected objects in both categories (i.e., objects X such that $\mathrm{Hom}(X, -)$ preserves coproducts), we obtain an equivalence between the categories of transitive $A_1(X, x)$-sets and of connected covering graphs of (X, x). The former is however equivalent to that of quotients of $A_1(X, x)$ by its subgroups, and hence equivalent to the opposite of the poset of subgroups of $A_1(X, x)$. □

3.2.3 Universal covers

We next turn our attention to the construction of the universal cover of a pointed graph (X, x_0), which is the initial object in the category of pointed covering graphs of (X, x_0). To define it, we need to upgrade the definition of coverings and their maps (Definitions 3.2.9 and 3.2.14) to the pointed setting.

Definition 3.2.24.

1. Let (X, x_0) be a pointed graph. A pointed graph map $p\colon (Y, y_0) \to (X, x_0)$ where the underlying map $p\colon Y \to X$ is a covering is called a *pointed covering.*

2. Given two pointed coverings $p\colon (Y, y_0) \to (X, x_0)$ and $p'\colon (Y', y_0') \to (X, x_0)$ over the same pointed base (X, x_0), a *morphism* $f\colon p \to p'$ *of pointed coverings over* (X, x_0) is a pointed map $f\colon (Y, y_0) \to (Y', y_0')$ such that the underlying map $f\colon Y \to Y'$ is a morphism of coverings over X.

3. For any pointed graph (X, x_0), pointed coverings over X and morphisms between them form a category, denoted $\mathsf{Cov}\,(X, x_0)$. It is the full subcategory of $\mathsf{Graph}_* \downarrow (X, x_0)$ on pointed covering maps.

4. An initial object in the category $\mathsf{Cov}\,(X, x_0)$ is called a *universal cover.*

Recall that a graph X is *simply connected* if it is path-connected and if the fundamental group $A_1\,(X, x_0)$ is trivial for any vertex $x_0 \in X$.

Proposition 3.2.25. *Let* $p\colon (Y, y_0) \to (X, x_0)$ *be a pointed covering. If* Y *is simply connected,* p *is a universal cover.*

Proof. Let $p'\colon (Y', y_0') \to (X, x_0)$ be some pointed covering. We want to show that there exists a unique morphism of pointed coverings $f\colon p \to p'$ over (X, x_0).

$$\begin{tikzcd} Y \arrow[rr, dotted, "f"] \arrow[dr, "p"'] & & Y' \arrow[dl, "p'"] \\ & X & \end{tikzcd}$$

Let $y_1 \in Y$ be any vertex. Since Y is path-connected, there exists a path $\widetilde{\gamma}_1\colon y_0 \rightsquigarrow y_1$. Let $x_1 = p\,(y_1)$ and $\gamma_1 = p \circ \widetilde{\gamma}_1$. Then, γ_1 is a path $x_0 \rightsquigarrow x_1$ in X.

Since p' is a covering, there exists a unique path $\widetilde{\gamma}'_1$ in Y' that starts at y' and satisfies the condition $p' \circ \widetilde{\gamma}'_1 = \gamma_1$. Let y'_1 be the end-point of the path $\widetilde{\gamma}'_1$.

Note that $y'_1 \in Y'$ only depends on $y_1 \in Y$ and the path-homotopy class of the path $\widetilde{\gamma}_1$ in Y. However, since Y is simply-connected, $y'_1 \in Y'$ only depends on $y_1 \in Y$ and we have a well-defined mapping $f\colon Y \to Y'$ given by $y_1 \mapsto y'_1$.

We need to check that f is a graph map. Suppose we have $y_1 \sim y_2$ in Y. Let $\widetilde{e}\colon I_1 \to Y$ be given by $\widetilde{e}\,(0) = y_1$ and $\widetilde{e}\,(1) = y_2$. Then, given any path $\widetilde{\gamma}_1\colon y_0 \rightsquigarrow y_1$, we have a path $\widetilde{\gamma}_2 = \widetilde{\gamma}_1 * \widetilde{e}\colon y_0 \rightsquigarrow y_2$. Let $x_1 = p\,(y_1)$, $x_2 = p\,(y_2)$, $\gamma_1 = p \circ \widetilde{\gamma}_1$, $\gamma_2 = p \circ \widetilde{\gamma}_2$, and $e = p \circ \widetilde{e}$. Then, $\gamma_2 = \gamma_1 * e$ is a path $x_0 \rightsquigarrow x_2$ in X. By construction, $f\,(y_1)$ is the end-point of the unique path $\widetilde{\gamma}'_1$ in Y' that starts at y'_0 and satisfies the condition $p' \circ \widetilde{\gamma}'_1 = \gamma_1$, and $f\,(y_2)$ is the end-point of the unique path $\widetilde{\gamma}'_2$ in Y' that starts at y'_0 and satisfies the condition $p' \circ \widetilde{\gamma}'_2 = \gamma_2$. Let $\widetilde{e}'$ be the unique path in Y' that starts at y'_1 and satisfies the condition $p' \circ \widetilde{e}' = e$. Then, by uniqueness of lifts, we must have $\widetilde{\gamma}'_2 = \widetilde{\gamma}'_1 * \widetilde{e}'$. Since $\widetilde{e}'$ is a path $f\,(y_1) \rightsquigarrow f\,(y_2)$ of length 1, we have $f\,(y_1) \sim f\,(y_2)$ in Y'.

Finally, we note that if we had any morphism $g\colon p \to p'$ over (X, x_0), then for any $y_1 \in Y$, the vertex $g\,(y_1)$ is the end-point of the path $g \circ \widetilde{\gamma}_1$, where $\widetilde{\gamma}_1$ is any path $y_0 \rightsquigarrow y_1$. Since $p' \circ g = p$, the path $g \circ \widetilde{\gamma}_1$ is the unique path in Y' that starts at y'_0 and satisfies $p' \circ g \circ \widetilde{\gamma}_1 = p \circ \widetilde{\gamma}_1$. Thus, we must have $g = f$, thereby proving the uniqueness of f. $\square$

From the Galois correspondence of Corollary 3.2.22, we can deduce the existence of a universal cover over any pointed graph (X, x_0). We now proceed to give an explicit description of this universal cover.

Definition 3.2.26. Given any pointed graph (X, x_0), we construct the graph

$\widetilde{X}_{x_0}$ as follows:

$$\left(\widetilde{X}_{x_0}\right)_V = \{\text{path-homotopy class } [\gamma] \mid \gamma \text{ is a path in } X \text{ starting at } x_0\}$$

$$\left(\widetilde{X}_{x_0}\right)_E = \left\{ [\gamma] \sim [\gamma * e] \;\middle|\; \begin{array}{l} \gamma \text{ is a path in } X \text{ starting at } x_0\text{, and} \\ e \text{ is a path of length 1 in } X, \\ \text{starting at the end-point of } \gamma \end{array} \right\}$$

It has a distinguished vertex given by the path-homotopy class of the constant path at x_0. It also comes equipped with a pointed graph map $p\colon \left(\widetilde{X}_{x_0}, [\mathrm{c}_{x_0}]\right) \to (X, x_0)$ that maps $[\gamma] \in \widetilde{X}_{x_0}$ to the end-point of γ in X.

Remark 3.2.27. Observe that the fiber of $p\colon \widetilde{X}_{x_0} \to X$ over the base point $x_0 \in X$ is precisely the fundamental group $A_1(X, x_0)$.

Proposition 3.2.28. *Given any pointed graph (X, x_0), the map $p\colon \left(\widetilde{X}_{x_0}, [\mathrm{c}_{x_0}]\right) \to (X, x_0)$ is a universal cover.*

Proof. We will first verify that $p\colon \widetilde{X}_{x_0} \to X$ is a covering map and then check that $\widetilde{X}_{x_0}$ is simply connected.

Consider the following lifting problem:

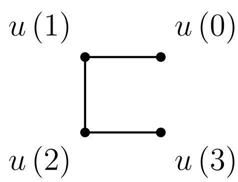
$$\begin{tikzcd} I_0 \arrow[r, "{[\gamma]}"] \arrow[d, "0"'] & \widetilde{X}_{x_0} \arrow[d, "p"] \\ I_1 \arrow[r, "e"] & X \end{tikzcd}$$

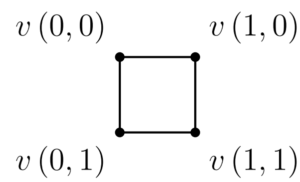
where $e\colon I_1 \to X$ is a path of length 1 in X starting at the endpoint of γ. This lifting problem admits a unique solution $\widetilde{e}\colon I_1 \to \widetilde{X}_{x_0}$, given by $\widetilde{e}(0) = [\gamma]$ and $\widetilde{e}(1) = [\gamma * e]$. Hence, p has the unique RLP against $0\colon I_0 \to I_1$.

Next, consider the following lifting problem:

$$\begin{tikzcd} I_3 \arrow[r, "u"] \arrow[d, "\sqsubset_{1,1}"'] & \widetilde{X}_{x_0} \arrow[d, "p"] \\ I_1 \square I_1 \arrow[r, "v"] & X \end{tikzcd}$$

We can visualize the maps $u \colon I_3 \to \widetilde{X}_{x_0}$ and $v \colon I_1 \square I_1 \to X$ as follows:





This lifting problem admits a unique solution if and only if we have $u\,(0) \sim u\,(3)$ in $\widetilde{X}_{x_0}$.

Suppose $u\,(0) = [\gamma]$ for some path $\gamma \colon x_0 \rightsquigarrow v\,(1,0)$ in X. Also consider the path $\sigma \colon v\,(1,0) \rightsquigarrow v\,(1,1)$ in X given by the composite

$$\sigma = \left(I_3 \xrightarrow{\sqsubset_{1,1}} I_1 \square I_1 \xrightarrow{v} X \right)$$

Then, $u\,(3) = [\gamma * \sigma]$.

Let $e \colon v\,(1,0) \rightsquigarrow v\,(1,1)$ be the path of length 1 in X given by

$$e = \left(I_1 \cong I_0 \square I_1 \xrightarrow{1 \,\square\, \mathrm{id}} I_1 \square I_1 \xrightarrow{v} X \right)$$

Then, $u\,(3) = [\gamma * \sigma] = [\gamma * e]$. Thus, we have $u\,(0) \sim u\,(3)$ in $\widetilde{X}_{x_0}$. Hence, p has the unique RLP against $\sqsubset_{1,1} \colon I_3 \to I_1 \square I_1$. Thus, p is a covering map.

We will now show that $\widetilde{X}_{x_0}$ is path-connected. Given any vertex $[\gamma] \in \widetilde{X}_{x_0}$, where $\gamma\colon I_n \to X$ is a path in X that starts at x_0 ,let $\gamma_t\colon I_n \to X$ be given by

$$\gamma_t(i) = \begin{cases} \gamma(i) & \text{if } i \le t \\ \gamma(t) & \text{if } i \ge t \end{cases}$$

Then, the map $\widetilde{\gamma}\colon I_n \to \widetilde{X}_{x_0}$ given by $\widetilde{\gamma}(i) = [\gamma_t]$ defines a path $[\mathrm{c}_{x_0}] \rightsquigarrow [\gamma]$ in $\widetilde{X}_{x_0}$. Thus, the graph $\widetilde{X}_{x_0}$ is path-connected.

In order to prove that the fundamental group $A_1\left(\widetilde{X}_{x_0}, [\mathrm{c}_{x_0}]\right)$ is trivial, we need to make an observation about the above construction that we will use later. Given any path $\gamma\colon x_0 \rightsquigarrow x_1$ in X, observe that the path $\widetilde{\gamma}$ that we constructed above is the unique path in $\widetilde{X}_{x_0}$ that starts at the base vertex $[\mathrm{c}_{x_0}]$ and also satisfies $p \circ \widetilde{\gamma} = \gamma$. In particular, the unwinding of γ

$$\mathsf{unw}_\gamma\colon p^{-1}(x_0) \to p^{-1}(x_1)$$

maps the base vertex $[\mathrm{c}_{x_0}]$ to the vertex $[\gamma] \in p^{-1}(x_1)$. That is,

$$\mathsf{unw}_\gamma([\mathrm{c}_{x_0}]) = [\gamma], \qquad \text{for any path } \gamma\colon x_0 \rightsquigarrow x_1 \text{ in } X$$

We now proceed to prove that $A_1\left(\widetilde{X}_{x_0}, [\mathrm{c}_{x_0}]\right)$ is trivial. Observe that since $p\colon \widetilde{X}_{x_0} \to X$ is a covering map, the induced group homomorphism

$$A_1(p) : A_1\left(\widetilde{X}_{x_0}, [\mathrm{c}_{x_0}]\right) \to A_1(X, x_0)$$

is injective. Thus, it suffices to prove that the image of the group homomorphism $A_1(p)$ is trivial. That is, given any loop $\widetilde{\omega}\colon [\mathrm{c}_{x_0}] \rightsquigarrow [\mathrm{c}_{x_0}]$ in $\widetilde{X}_{x_0}$, we want to show that the loop $\omega = p \circ \widetilde{\omega}\colon x_0 \rightsquigarrow x_0$ is path-homotopic to the constant path c_{x_0} in X.

Since p is a covering map, $\widetilde{\omega}$ is the unique path in $\widetilde{X}_{x_0}$ that starts at the base vertex $[\mathrm{c}_{x_0}]$ and satisfies the condition $p \circ \widetilde{\omega} = \omega$. Hence, the unwinding of ω

$$\mathsf{unw}_\omega \colon p^{-1}(x_0) \to p^{-1}(x_0)$$

maps $[\mathrm{c}_{x_0}]$ to the end-point of $\widetilde{\omega}$, viz. $[\mathrm{c}_{x_0}]$. But we have already argued that $\mathsf{unw}_\omega([\mathrm{c}_{x_0}]) = [\omega]$. Thus, we must have $[\omega] = [\mathrm{c}_{x_0}]$ in $A_1(X, x_0)$ as required. $\square$

We are now ready to state (and prove) the main theorem of this section.

Theorem 3.2.29.

1. *Every pointed graph (X, x_0) admits a universal cover.*

2. *A pointed covering $p\colon (Y, y_0) \to (X, x_0)$ is universal if and only if Y is simply connected.*

Proof. In the previous proposition, we proved that every pointed graph (X, x_0) admits the universal cover $p\colon \left(\widetilde{X}_{x_0}, [\mathrm{c}_{x_0}]\right) \to (X, x_0)$. (This also follows from the Galois correspondence of Corollary 3.2.22.) Furthermore, we proved that the graph $\widetilde{X}_{x_0}$ is simply connected. Since initial objects are unique up to a unique isomorphism, it follows that given any universal cover $p\colon (Y, y_0) \to (X, x_0)$, the graph Y must be simply connected. $\square$

Definition 3.2.30. Given a pointed graph (X, x_0), let $\mathsf{Fib}_{x_0}\colon \mathsf{Cov}(X) \to \mathsf{Set}$ denote the functor that maps a covering map $p\colon Y \to X$ to the fiber $p^{-1}(x_0)$ of p over x_0 and a morphism of coverings $f\colon p \to p'$ over X to the restriction $f|_{p^{-1}(x_0)}\colon p^{-1}(x_0) \to (p')^{-1}(x_0)$.

Proposition 3.2.31. *Let (X, x_0) be a pointed graph.*

1. *The functor* Fib_{x_0} *is representable, and it is represented by the universal cover* $\widetilde{X}_{x_0} \to X$*. In particular, given any cover* $Y \to X$ *over* X*, the hom-set* $\mathsf{Cov}(X)\left(\widetilde{X}_{x_0}, Y\right)$ *is in bijection with the fiber* $\mathsf{Fib}_{x_0}(Y)$ *of* Y *over the base-point* x_0*.*

2. *The fundamental group* $A_1(X, x_0)$ *is isomorphic to the automorphism group* $\mathsf{Aut}_{\mathsf{Set}^{\mathsf{Cov}(X)}}(\mathsf{Fib}_{x_0})$ *of the functor* $\mathsf{Fib}_{x_0} \colon \mathsf{Cov}(X) \to \mathsf{Set}$ *in the functor category* $\mathsf{Set}^{\mathsf{Cov}(X)}$*.*

Proof. We prove each part in turn.

1. Given any $y_0 \in \mathsf{Fib}_x(Y)$, we have a unique morphism of pointed coverings $\left(\widetilde{X}_{x_0}, [\mathrm{c}_{x_0}]\right) \to (Y, y_0)$. On the other hand, given any morphism of coverings $f \colon \widetilde{X}_{x_0} \to Y$, we have a unique element $y_0 = f([\mathrm{c}_{x_0}])$ in $\mathsf{Fib}_{x_0}(Y)$.

2. By the Yoneda lemma, we have the following bijection:

$$\mathsf{Set}^{\mathsf{Cov}(X)}(\mathsf{Fib}_{x_0}, \mathsf{Fib}_{x_0}) \cong \mathsf{Cov}(X)\left(\widetilde{X}_{x_0}, \widetilde{X}_{x_0}\right)$$

   Moreover, $\mathsf{Cov}(X)\left(\widetilde{X}_{x_0}, \widetilde{X}_{x_0}\right)$ is in bijection with $\mathsf{Fib}_{x_0}\left(\widetilde{X}_{x_0}\right)$, which is precisely the fundamental group $A_1(X, x_0)$.

   Since every morphism of coverings $\widetilde{X}_{x_0} \to \widetilde{X}_{x_0}$ must be invertible, we have:

$$\mathsf{Aut}_{\mathsf{Cov}(X)}\left(\widetilde{X}_{x_0}\right) = \mathsf{Cov}(X)\left(\widetilde{X}_{x_0}, \widetilde{X}_{x_0}\right)$$

   and similarly, we have:

$$\mathsf{Aut}_{\mathsf{Set}^{\mathsf{Cov}(X)}}(\mathsf{Fib}_{x_0}) = \mathsf{Set}^{\mathsf{Cov}(X)}(\mathsf{Fib}_{x_0}, \mathsf{Fib}_{x_0})$$

Furthermore, we can check that all the bijections preserve the group structure and are, in fact, group isomorphisms. □

Example 3.2.32. Recall that the map $p\colon I_\infty \to C_n;\ i \mapsto i \pmod n$ is a covering map for $n \geq 5$. Since I_∞ is simply connected, this is a universal cover. The automorphism group of $p\colon I_\infty \to C_n$ is $\mathbb{Z}$, with an automorphism $i \mapsto i + nk$ for each $k \in \mathbb{Z}$. Thus, we have:

$$A_1(C_n, *) \cong \begin{cases} 0 & \text{for } n = 3, 4 \\ \mathbb{Z} & \text{for } n \geq 5 \end{cases}$$

3.3 Seifert–Van Kampen theorem for graphs

This section establishes an analogue of the familiar Seifert–van Kampen theorem from algebraic topology, a version of which was previously proven in discrete homotopy theory in [BKLW01]. Our statement is a strengthening of the result found therein, obtained by refining the condition on the pushout square to be preserved.

Our preliminary statement of the Seifert–van Kampen theorem (Theorem 3.3.2) for the fundamental groupoid contains the technical heart of the proof. We subsequently give two more statements: Theorem 3.3.4, which is a refined version more convenient to use in practice, and Theorem 3.3.5, which specializes the theorem to the fundamental group. We conclude this section with an example application of our refined Seifert–van Kampen Theorem 3.3.5 to which the statement of [BKLW01] would not have applied (Example 3.3.7).

We begin with a preliminary definition required to establish our technical assumption for the statement of the Seifert–van Kampen theorem.

Definition 3.3.1.

1. For any $m, n \in \mathbb{N}$, let the *boundary* $\partial_{m,n} \colon I_{2m+2n} \to I_m \,\square\, I_n$ be the graph map given by:

$$\partial_{m,n}(i) = \begin{cases} (m-i, 0) & \text{if } 0 \leq i \leq m \\ (0, i-m) & \text{if } m \leq i \leq m+n \\ (i-m-n, n) & \text{if } m+n \leq i \leq 2m+n \\ (m, 2n+2n-i) & \text{if } 2m+n \leq i \leq 2m+2n \end{cases}$$

The following figure provides visual depictions of two such boundary maps.

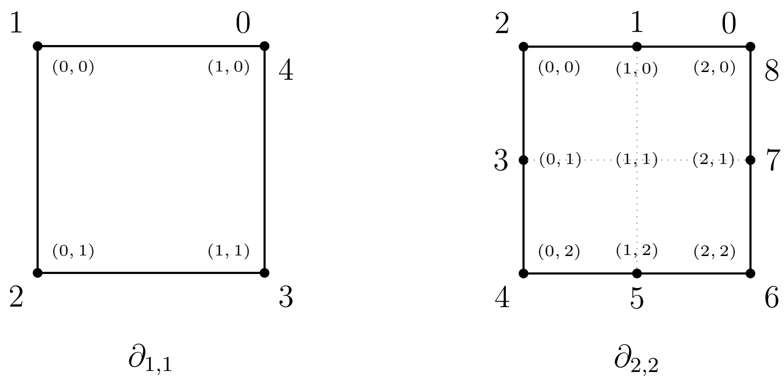


Figure 3.1: Two examples of the the boundary map $\partial_{m,n} \colon I_{2m+2n} \to I_m \,\square\, I_n$, depicted by overlaying the interval graph I_{2m+2n} on top of the grid graph $I_m \,\square\, I_n$.

2. Let X be a graph and $h \colon I_1 \,\square\, I_1 \to X$ be any map. A *net of* h is a pair (H, s) consisting of a map $H \colon I_m \,\square\, I_n \to X$ for some $m, n \in \mathbb{N}$, together with a shrinking map $s \colon I_{2m+2n} \to I_4$, such that the following diagram

commutes:

$$\begin{array}{ccccc} I_{2m+2n} & \xrightarrow{\partial_{m,n}} & I_m \,\square\, I_n & \xrightarrow{H} & X \\ {\scriptstyle s}\downarrow & & & & \| \\ I_4 & \xrightarrow{\partial_{1,1}} & I_1 \,\square\, I_1 & \xrightarrow{h} & X \end{array}$$

3. Let X be a graph and $H\colon I_m \,\square\, I_n \to X$ be any map. For any $i = 1, \ldots, m$ and $j = 1, \ldots, n$, the (i,j)-*cell of* H, denoted by $H_{i,j}$, is given by the following composite:

$$H_{i,j} = \left(I_1 \,\square\, I_1 \xrightarrow{\mathrm{e}_i \,\square\, \mathrm{e}_j} I_m \,\square\, I_n \xrightarrow{H} X \right)$$

Theorem 3.3.2. *Consider a pushout square in* Graph *as follows:*

$$\begin{array}{ccc} X_0 & \longrightarrow & X_1 \\ \downarrow & \ulcorner & \downarrow \\ X_2 & \longrightarrow & X \end{array}$$

If every map $h\colon I_1 \,\square\, I_1 \to X$ *satisfies the following net resolution condition:*

$$h \textit{ admits a net } (H,s) \textit{ such that each cell } H_{i,j} \textit{ of } H \textit{ factors through } X_1 \textit{ or } X_2 \tag{N}$$

then the pushout square is preserved by the functor $\Pi_1\colon \mathsf{Graph} \to \mathsf{Gpd}$. *That is, we have the following pushout square in* Gpd*:*

$$\begin{array}{ccc} \Pi_1 X_0 & \longrightarrow & \Pi_1 X_1 \\ \downarrow & \ulcorner & \downarrow \\ \Pi_1 X_2 & \longrightarrow & \Pi_1 X. \end{array}$$

In order to prove this theorem, we will need the following lemma characterizing functors out of the fundamental groupoid of a graph:

Lemma 3.3.3. *Let X be a graph and $\mathcal{G}$ be a groupoid. Further suppose we are given the following data:*

- *a function $F\colon X_V \to \operatorname{ob}\mathcal{G}$*
- *graph maps $F^{(1)}_{x,x'}\colon P_1X\left(x,x'\right) \to \mathcal{G}\left(Fx,Fx'\right)_{\mathsf{discrete}}$, one for each pair of vertices $x,x' \in X$*

subject to the following conditions:

- *$F^{(1)}_{x,x}\left(\mathrm{c}_x\right) = \mathrm{id}_{Fx}$ for each $x \in X$*
- *$F^{(1)}_{x',x}\left(\overline{e}\right) = F^{(1)}_{x,x'}\left(e\right)^{-1}$ for each path $e\colon x \rightsquigarrow x'$ of length 1*

Given any path $\gamma\colon x \rightsquigarrow x'$ of length $n \in \mathbb{N}$, let $F^{(n)}_{x,x'}\left(\gamma\right) \in \mathcal{G}\left(Fx,Fx'\right)$ be given by:

$$F^{(n)}_{x,x'}\left(\gamma\right) = F^{(1)}_{x_{n-1},x_i}\left(\mathrm{e}^*_n\left(\gamma\right)\right) \circ \cdots \circ F^{(1)}_{x_0,x_1}\left(\mathrm{e}^*_1\left(\gamma\right)\right)$$

where $x_i = \gamma\left(i\right)$ for $i = 0,\ldots,n$. Then, we have the following:

1. *Given any path $\gamma\colon x \rightsquigarrow x'$ of length m and any path $\sigma\colon x' \rightsquigarrow x''$ of length n, we have:*
$$F^{(m+n)}_{x,x''}\left(\gamma * \sigma\right) = F^{(n)}_{x',x''}\left(\sigma\right) \circ F^{(m)}_{x,x'}\left(\gamma\right)$$
2. *Given any path $\gamma\colon x \rightsquigarrow x'$ of length n, we have:*
$$F^{(n)}_{x',x}\left(\overline{\gamma}\right) = F^{(n)}_{x,x'}\left(\gamma\right)^{-1}$$
3. *Given any path $\gamma\colon x \rightsquigarrow x'$ of length n and any shrinking map $s\colon I_m \to I_n$, we have:*
$$F^{(m)}_{x,x'}\left(\gamma \circ s\right) = F^{(n)}_{x,x'}\left(\gamma\right)$$

4. *Given any map* $H\colon I_m \,\square\, I_n \to X$ *where each cell* $H_{i,j}$ *satisfies the condition:*

$$F^{(4)}_{x_{i,j},x_{i,j}}\left(H_{i,j}\circ\partial_{1,1}\right)=\mathrm{id}_{Fx_{i,j}}$$

where $x_{i,j}=\left(H_{i,j}\circ\partial_{1,1}\right)(0)$*, we have:*

$$F^{(2m+2n)}_{x,x}\left(H\circ\partial_{m,n}\right)=\mathrm{id}_{Fx},$$

where $x=\left(H\circ\partial_{m,n}\right)(0)$*.*

5. *If, for each map* $h\colon I_1 \,\square\, I_1 \to X$*, we have*

$$F^{(4)}_{x,x}\left(h\circ\partial_{1,1}\right)=\mathrm{id}_{Fx}$$

where $x=\left(h\circ\partial_{1,1}\right)(0)$*, then*

$$F^{(n)}_{x,x'}\colon P_nX\left(x,x'\right)\to\mathcal{G}\left(Fx,Fx'\right)_{\mathsf{discrete}}$$

is a well-defined graph map for all $n\in\mathbb{N}$ *and all* $x,x'\in X$*.*

6. *If, for each map* $h\colon I_1 \,\square\, I_1 \to X$*, we have*

$$F^{(4)}_{x,x}\left(h\circ\partial_{1,1}\right)=\mathrm{id}_{Fx}$$

where $x=\left(h\circ\partial_{1,1}\right)(0)$*, then we have a well-defined functor* $F\colon\Pi_1X\to\mathcal{G}$ *that maps* $x\in X$ *to* $Fx\in\operatorname{ob}\mathcal{G}$ *and a morphism* $[\gamma]:x\to x'$*, where* $\gamma\colon x\rightsquigarrow x'$ *is a path in* X *of length* n *to* $F^{(n)}_{x,x'}(\gamma)$*.*

Proof. 1. This follows from the observation that:

$$\mathrm{e}_i^*(\gamma * \sigma) = \begin{cases} \mathrm{e}_i^*(\gamma) & \text{for } i = 1, \ldots, m \\ \mathrm{e}_{i-m}^*(\sigma) & \text{for } i = m+1, \ldots, m+n \end{cases}$$

2. This follows from the observation that:

$$\mathrm{e}_i^*(\overline{\gamma}) = \overline{\mathrm{e}_{n+1-i}^*(\gamma)}$$

3. This follows from the observation that:

$$\mathrm{e}_i^*(\gamma \circ s) = \begin{cases} \mathrm{c}_{(\gamma \circ s)(i)} & \text{if } s(i) = s(i-1) \\ \mathrm{e}_{s(i)}^*(\gamma) & \text{if } s(i) = s(i-1) + 1 \end{cases}$$

4. Given any map $H\colon I_m \,\square\, I_n \to X$, we have a diagram in $\mathcal{G}$ as follows:

$$\begin{tikzcd}
FH(0,0) \arrow[r] \arrow[d] & FH(1,0) \arrow[r] \arrow[d] & \cdots \arrow[r] & FH(m-1,0) \arrow[r] \arrow[d] & FH(m,0) \arrow[d] \\
FH(0,1) \arrow[r] \arrow[d] & FH(1,1) \arrow[r] \arrow[d] & \cdots \arrow[r] & FH(m-1,1) \arrow[r] \arrow[d] & FH(m,1) \arrow[d] \\
\vdots \arrow[d] & \vdots \arrow[d] & \ddots & \vdots \arrow[d] & \vdots \arrow[d] \\
FH(0,n-1) \arrow[r] \arrow[d] & FH(1,n-1) \arrow[r] \arrow[d] & \cdots \arrow[r] & FH(m-1,n-1) \arrow[r] \arrow[d] & FH(m,n-1) \arrow[d] \\
FH(0,n) \arrow[r] & FH(1,n) \arrow[r] & \cdots \arrow[r] & FH(m-1,n) \arrow[r] & FH(m,n)
\end{tikzcd}$$

where every arrow is invertible. If any particular cell $H_{i,j}\colon I_1 \,\square\, I_1 \to X$

satisfies the condition

$$F^{(4)}_{x_{i,j},x_{i,j}}\left(H_{i,j}\circ\partial_{1,1}\right)=\mathrm{id}_{Fx_{i,j}}$$

where $x_{i,j}=\left(H_{i,j}\circ\partial_{1,1}\right)(0)$, then it means that the corresponding square in the above diagram is commutative. But if every square in the above diagram commutes, then the outer boundary of the diagram also commutes. Hence, the condition

$$F^{(2m+2n)}_{x,x}\left(H\circ\partial_{m,n}\right)=\mathrm{id}_{Fx},$$

where $x=\left(H\circ\partial_{m,n}\right)(0)$, is satisfied.

5. In order to check that

$$F^{(n)}_{x,x'}\colon P_nX\left(x,x'\right)\to\mathcal{G}\left(Fx,Fx'\right)_{\mathsf{discrete}}$$

is a well-defined graph map, we need to verify that

$$F^{(n)}_{x,x'}\left(\gamma\right)=F^{(n)}_{x,x'}\left(\sigma\right)$$

for any two paths $\gamma,\sigma\colon x\rightsquigarrow x'$ of length n that are adjacent in the graph $P_nX\left(x,x'\right)$.

Let $H\colon I_n\,\square\,I_1\to X$ be a map given by $H\left(-,0\right)=\gamma$ and $H\left(-,1\right)=\sigma$. Since every cell $H_{i,j}\colon I_1\,\square\,I_1\to X$ satisfies the condition

$$F^{(4)}_{x_{i,j},x_{i,j}}\left(H_{i,j}\circ\partial_{1,1}\right)=\mathrm{id}_{Fx_{i,j}}$$

where $x_{i,j} = (H_{i,j} \circ \partial_{1,1})(0)$, we have:

$$F^{(2n+2)}_{x',x'}(H \circ \partial_{n,1}) = \mathrm{id}_{Fx'}$$

Noting that

$$H \circ \partial_{n,1} = \overline{\gamma} * \mathrm{c}_x * \sigma * \mathrm{c}_{x'}$$

we are done.

6. We have well-defined graph maps

$$F^{(n)}_{x,x'} \colon P_n X(x,x') \to \mathcal{G}(Fx, Fx')_{\mathsf{discrete}}$$

that satisfy

$$F^{(m)}_{x,x'}(\gamma \circ s) = F^{(n)}_{x,x'}(\gamma)$$

for every path $\gamma \colon x \rightsquigarrow x'$ of length n and every shrinking map $s \colon I_m \to I_n$.

Thus, we have well-defined graph maps

$$F_{x,x'} \colon P_{\mathbb{N}} X(x,x') \to \mathcal{G}(Fx, Fx')_{\mathsf{discrete}}$$

for each $x, x' \in X$. Equivalently, we have well-defined set-functions:

$$F_{x,x'} \colon \pi_0 P_{\mathbb{N}} X(x,x') \to \mathcal{G}(Fx, Fx')$$

Furthermore, we have $F_{x,x}[\mathrm{c}_x] = \mathrm{id}_{Fx}$ for each $x \in X$ and $F_{x,x''}[\gamma * \sigma] = F_{x',x''}[\sigma] \circ F_{x,x'}[\gamma]$ for each pair of paths $\gamma \colon x \rightsquigarrow x'$ and $\sigma \colon x' \rightsquigarrow x''$.

Thus, we have a well-defined functor $F \colon \Pi_1 X \to \mathcal{G}$.

□

We can now prove Theorem 3.3.2.

Proof of Theorem 3.3.2. Let $\mathcal{G}$ be any groupoid, and let $F_1 \colon \Pi_1 X_1 \to \mathcal{G}$ and $F_2 \colon \Pi_1 X_2 \to \mathcal{G}$ be functors such that the outer square in the following diagram commutes:

$$\begin{tikzcd}
\Pi_1 X_0 \arrow[r] \arrow[d] & \Pi_1 X_1 \arrow[d] \arrow[ddr, bend left, "F_1"] & \\
\Pi_1 X_2 \arrow[r] \arrow[drr, bend right, "F_2"'] & \Pi_1 X \arrow[dr, dotted, "F"] & \\
& & \mathcal{G}
\end{tikzcd}$$

We want to show that there exists a unique functor $F \colon \Pi_1 X \to \mathcal{G}$ such that the above diagram commutes.

By Lemma 3.3.3, it suffices to provide the following data:

- a function $F \colon X_V \to \operatorname{ob} \mathcal{G}$
- graph maps $F^{(1)}_{x,x'} \colon P_1 X\,(x, x') \to \mathcal{G}\,(Fx, Fx')_{\mathsf{discrete}}$, one for each pair of vertices $x, x' \in X$

subject to the following conditions:

- $F^{(1)}_{x,x}\,(\mathrm{c}_x) = \mathrm{id}_{Fx}$ for each $x \in X$
- $F^{(1)}_{x',x}\,(\overline{e}) = F^{(1)}_{x,x'}\,(e)^{-1}$ for each path $e \colon x \rightsquigarrow x'$ of length 1
- for each map $h \colon I_1 \square I_1 \to X$, we have

  $$F^{(4)}_{x,x}\,(h \circ \partial_{1,1}) = \mathrm{id}_{Fx}$$

  where $x = (h \circ \partial_{1,1})\,(0)$

The function $F\colon X_V \to \operatorname{ob}\mathcal{G}$ is uniquely determined by F_1 and F_2 since we have the following pushout square in Set:

$$\begin{tikzcd} (X_0)_V \arrow[r] \arrow[d] & (X_1)_V \arrow[d] \\ (X_2)_V \arrow[r] \arrow[ur, phantom, "\ulcorner"] & X_V \end{tikzcd}$$

Similarly, since every path of length 1 in X factors through either X_1 or X_2, the functions $F^{(1)}_{x,x'}\colon P_1X\,(x,x') \to \mathcal{G}\,(Fx, Fx')$ are also uniquely determined by F_1 and F_2.

The first two conditions can also be easily verified. It only remains to verify the third condition.

By the hypothesis, every map $h\colon I_1 \,\square\, I_1 \to X$ admits a net (H, s) such that each cell $H_{i,j}$ of H factors through either X_1 or X_2. It follows that each cell $H_{i,j}$ satisfies the condition

$$F^{(4)}_{x_{i,j},x_{i,j}}\,(H_{i,j} \circ \partial_{1,1}) = \mathrm{id}_{Fx_{i,j}}$$

where $x_{i,j} = (H_{i,j} \circ \partial_{1,1})\,(0)$. By part (4) of Lemma 3.3.3, it follows that we have:

$$F^{(2m+2n)}_{x,x}\,(H \circ \partial_{m,n}) = \mathrm{id}_{Fx},$$

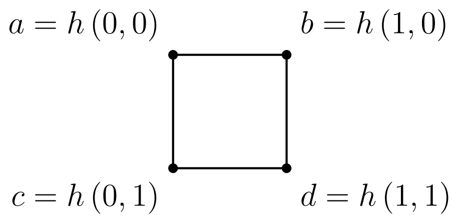
where $x = (H \circ \partial_{m,n})\,(0)$. But since (H, s) is a net of h, it follows that:

$$H \circ \partial_{m,n} = h \circ \partial_{1,1} \circ s$$

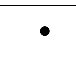
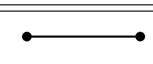
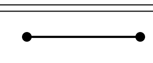
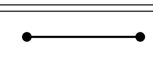
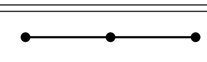
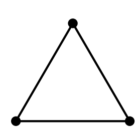
Thus, we have $F^{(2m+2n)}_{x,x}\,(h \circ \partial_{1,1} \circ s) = \mathrm{id}_{Fx}$. By part (3) of Lemma 3.3.3, it follows that $F^{(4)}_{x,x}\,(h \circ \partial_{1,1}) = \mathrm{id}_{Fx}$, as required. $\square$

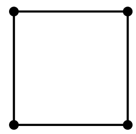
In practice, one does not need to check the net resolution condition (N) of Theorem 3.3.2 for all maps $I_1 \square I_1 \to X$, since for many of them it is trivially satisfied. In Theorem 3.3.4, we give a refined version of Theorem 3.3.2 that only requires checking the possibly problematic situations.

The table below gives all possible shapes that the image of a map $h\colon I_1 \square I_1 \to X$ can take. For convenience, we will name the vertices as follows:



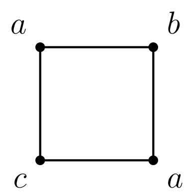
Then, up to symmetry of $I_1 \square I_1$, a map $h\colon I_1 \square I_1 \to X$ can take the following distinct shapes:

Image	Diagram	Conditions on a, b, c, d
single vertex		$a = b = c = d$
single edge		$a = c$ and $b = d$
single edge		$a = d$ and $b = c$
single edge		$a = b = c$
pair of edges		$a = d$
a 3-cycle		$a = b$

a 4-cycle		$\varnothing$

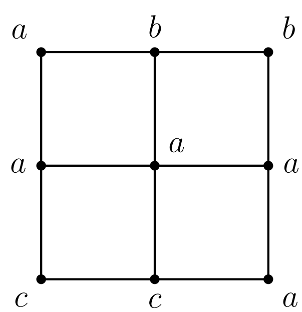
Note that whenever the map $h\colon I_1 \,\square\, I_1 \to X$ itself factors through one of X_1 or X_2, it trivially satisfies the net resolution condition (N). This applies to all maps $h\colon I_1 \,\square\, I_1 \to X$ whose image in X is either a single vertex or a single edge.

Let us also consider the case when the image of a map $h\colon I_1 \,\square\, I_1 \to X$ is a pair of edges. If h does not itself factor through either X_1 or X_2, then we must have that one of the edges factors through X_1 while the other factors through X_2. We can visualize such a map as follows:



a pair of edges in X, with $a \sim b$ in X_1 and $a \sim c$ in X_2

Even so, h satisfies the condition (N), with a net $H\colon I_2 \,\square\, I_2 \to X$ that can be visualized as follows:



a net for h

Thus, it suffices to check the net resolution condition (N) for those maps $h\colon I_1 \square I_1 \to X$ whose image is a 3- or 4-cycle in X.

Theorem 3.3.4 (Seifert–van Kampen Theorem for Fundamental Groupoid)**.** *Consider a pushout square in* Graph *as follows:*

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & X_1 \arrow[d] \\ X_2 \arrow[r] & X \arrow[ul, phantom, very near start, "\ulcorner"] \end{tikzcd}$$

If every map $h\colon I_1 \square I_1 \to X$ whose image in X is a 3- or 4-cycle satisfies the following net resolution condition:

$$h \text{ admits a net } (H, s) \text{ such that each cell } H_{i,j} \text{ of } H \text{ factors through } X_1 \text{ or } X_2 \tag{N}$$

then the pushout square is preserved by the functor $\Pi_1\colon \mathsf{Graph} \to \mathsf{Gpd}$. □

Theorem 3.3.5 (Seifert–van Kampen Theorem for Fundamental Group)**.** *Consider a pushout square of pointed connected graphs as follows:*

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & X_1 \arrow[d] \\ X_2 \arrow[r] & X \arrow[ul, phantom, very near start, "\ulcorner"] \end{tikzcd}$$

If every map $h\colon I_1 \square I_1 \to X$ whose image in X is a 3- or 4-cycle satisfies the following net resolution condition:

$$h \text{ admits a net } (H, s) \text{ such that each cell } H_{i,j} \text{ of } H \text{ factors through } X_1 \text{ or } X_2 \tag{N}$$

then we have the following pushout square in Grp*:*

$$\begin{tikzcd} A_1(X_0, x_0) \arrow[r] \arrow[d] & A_1(X_1, x_1) \arrow[d] \\ A_1(X_2, x_2) \arrow[r] & A_1(X, x) \arrow[ul, phantom, very near start, "\ulcorner"] \end{tikzcd}$$

□

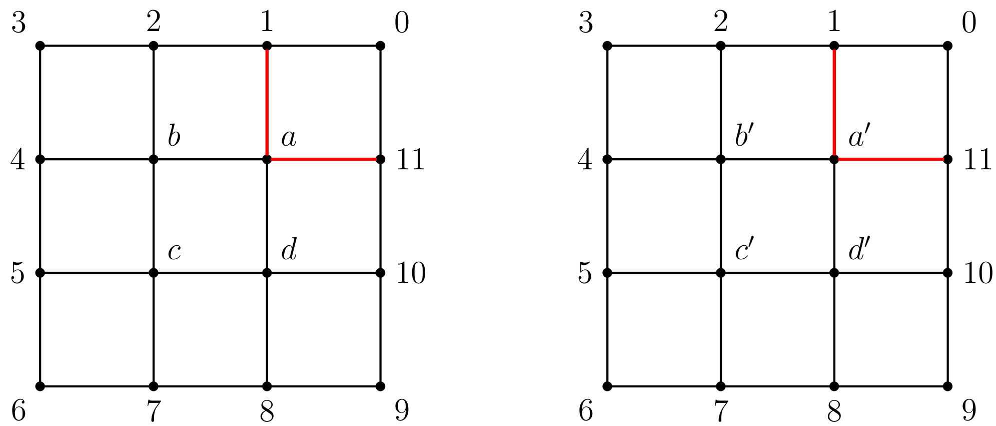
Example 3.3.6. For any $m \in \mathbb{N}$ and $n \geq 5$, the fundamental group of the graph $\bigvee_{i=1}^{m} C_n$ is isomorphic to the free group of rank m. We can prove this by induction on m and by applying Theorem 3.3.5 to the following pushout diagram, since $\bigvee_{i=1}^{m} C_n$ has no 3- or 4-cycles:

$$\begin{tikzcd} I_0 \arrow[r, "0"] \arrow[d, "0"'] & \bigvee_{i=1}^{m-1} C_n \arrow[d] \\ C_n \arrow[r] & \bigvee_{i=1}^{m} C_n \arrow[ul, phantom, very near start, "\ulcorner"] \end{tikzcd}$$

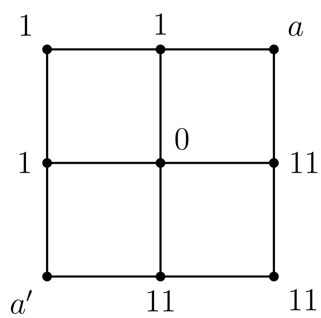
Example 3.3.7. Note that the boundary map $\partial_{m,n} \colon I_{2m+2n} \to I_m \,\square\, I_n$ satisfies $\partial_{m,n}(0) = \partial_{m,n}(2m+2n)$ and hence descends to a *boundary inclusion* $\partial_{m,n} \colon C_{2m+2n} \hookrightarrow I_m \,\square\, I_n$. Let $Y_{m,n}$ denote the graph given by the following pushout:

$$\begin{tikzcd} C_{2m+2n} \arrow[r, hook, "\partial_{m,n}"] \arrow[d, hook, "\partial_{m,n}"'] & I_m \,\square\, I_n \arrow[d] \\ I_m \,\square\, I_n \arrow[r] & Y_{m,n} \arrow[ul, phantom, very near start, "\ulcorner"] \end{tikzcd}$$

For $m, n \geq 3$, the graph $Y_{m,n}$ has precisely four 4-cycles that are not contained entirely in either copy of $I_m \,\square\, I_n$. However, each of these 4-cycles satisfies the net resolution condition (N) via a nontrivial net. The figure below showcases a nontrivial net for one of these 4-cycles in $Y_{3,3}$.



The graph $Y_{3,3}$ - each pair of identically labelled vertices is identified.



a net for the highlighted 4-cycle in $Y_{3,3}$

Thus, we can apply Theorem 3.3.5 to conclude that $Y_{m,n}$ is simply connected for $m, n \geq 3$.

3.4 Application: graphs with prescribed fundamental group

In this section, we present a construction that applies the Seifert–van Kampen Theorem 3.3.5 to produce an example of a graph with a prescribed fundamental group. After preliminary definitions of cones and disks, we give the construction in Construction 3.4.4 and prove its correctness in Theorem 3.4.5.

Definition 3.4.1. The *cone* $\mathsf{Cone}_n(X)$ of height $n \in \mathbb{N}$ on a graph X is given by the following pushout:

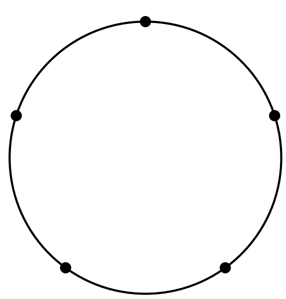
$$\begin{tikzcd} X \square I_0 \arrow[r, "\mathrm{id} \square 0"] \arrow[d, "!"'] & X \square I_n \arrow[d] \\ I_0 \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & \mathsf{Cone}_n(X) \end{tikzcd}$$

Let $*$ denote the identification class of $(x, 0)$ in $\mathsf{Cone}_n(X)$ for all $x \in X$.

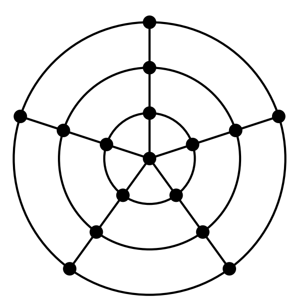
Proposition 3.4.2. *Given any graph X and $n \in \mathbb{N}$, the cone $\mathsf{Cone}_n(X)$ is contractible.*

Proof. Consider the graph map $\widetilde{H} \colon X \square I_n \square I_n \to \mathsf{Cone}_n(X)$ given by the formula $\widetilde{H}(x, i, j) = [x, \min(i, n-j)]$. Since $\widetilde{H}(x, 0, j) = *$ for all $x \in X$ and $j \in I_n$, it descends to a map $H \colon \mathsf{Cone}_n(X) \square I_n \to \mathsf{Cone}_n(X)$. Note that $H(-,-,0) = \mathrm{id}_{\mathsf{Cone}_n(X)}$ and $H(-,-,n) = c_*$. Thus, the graph $\mathsf{Cone}_n(X)$ is contractible. $\square$

Definition 3.4.3. For $m, n \in \mathbb{N}$, the (m,n)*−disk* $D_{m,n}$, with m spokes and of radius n, is given by $\mathsf{Cone}_n(C_m)$. The *boundary of the* (m,n)*-disk* is the map $\partial_{m,n} \colon C_m \to D_{m,n}$ given by $\partial_{m,n}(i) = (i, n)$.



C_5



$D_{5,3}$

Construction 3.4.4. Let F_S be the free group generated by a set S. Given any word $r = s_1^{d_1} \cdots s_k^{d_k} \in F_S$, with $s_i \neq s_{i+1}$ for $1 \leq i \leq k-1$, we can define its *degree* as follows:

$$\mathsf{deg}\,(r) = |d_1| + \cdots + |d_k|\,.$$

We can then define a map

$$\omega_r = C_{5\cdot\mathsf{deg}(r)} \longrightarrow \bigvee_{s\in S} C_5$$

corresponding the word r as follows: first, wrap d_1 times around the C_5 corresponding to $s_1 \in S$ (clockwise if $d_1 > 0$ and counter-clockwise if $d_1 < 0$), then d_2 times around the C_5 corresponding to $s_2 \in S$, and so on.

Given any group G, with a presentation

$$G = \langle S \mid R \rangle$$

let the graph $X_{S,R}$ be given by following pushout:

$$\begin{tikzcd}
\coprod_{r\in R} C_{5\cdot\mathsf{deg}(r)} \arrow[r, "(\omega_r)_{r\in R}"] \arrow[d, "\coprod_{r\in R}\partial_{5\cdot\mathsf{deg}(r),3}"'] & \bigvee_{s\in S} C_5 \arrow[d] \\
\coprod_{r\in R} D_{5\cdot\mathsf{deg}(r),3} \arrow[r] & X_{S,R} \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

Theorem 3.4.5. *Given any group G, with a presentation*

$$G = \langle S \mid R \rangle = \langle s_1, \ldots, s_m \mid r_1, \ldots, r_n \rangle\,,$$

we have:

$$A_1\,(X_{S,R}, x) \cong G$$

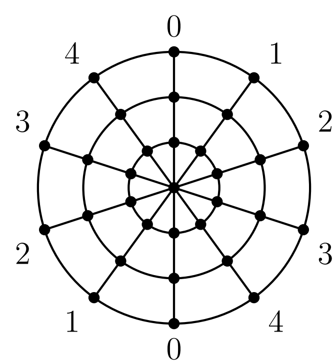
for any $x \in X_{S,R}$.

Proof. The key observation is that every 3-cycle or 4-cycle in $X_{S,R}$ necessarily factors through one of the disks $D_{5\cdot\mathsf{deg}(r_j),3}$. Thus, we can apply Theorem 3.3.5 to get the required result. $\square$

Example 3.4.6. Taking $G = \mathbb{Z}/2\mathbb{Z}$, with the presentation $\langle a \mid a^2 \rangle$, the graph $X_{\{a\},\{a^2\}}$ is given by the following pushout:

$$\begin{tikzcd} C_{10} \arrow[r, "\omega_{a^2}"] \arrow[d, "\partial_{10,3}"'] & C_5 \arrow[d] \\ D_{10,3} \arrow[r] & X_{\{a\},\{a^2\}} \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

We can visualize this graph as follows, with each pair of identically labelled vertices identified:



The graph $X_{\{a\},\{a^2\}}$ with fundamental group $\mathbb{Z}/2\mathbb{Z}$

Chapter 4

Model Categories of Cubical Sets

The use of cubical methods in discrete homotopy theory goes back to the paper [BBdLL06] where the authors first defined what is now known as the cubical nerve of a graph (see [CK22]). In this chapter, we take a step back from discrete homotopy theory to construct model category structures for homotopy n-types of cubical sets, i.e. model categories where the weak equivalences are cubical maps inducing isomorphisms on the first n cubical homotopy groups. While we expect these results to be of independent interest, they will be useful to us when we return to discrete homotopy theory in Chapter 5.

In Section 4.1, we review some general theory of model categories. We recall the definition of a model category originally due to Quillen [Qui67], as well as some fundamental aspects of model category theory, such as the construction of the homotopy category and comparing model categories via Quillen adjunctions. Next, we review techniques for constructing model categories. These techniques include the small object argument, which gives an explicit description of functorial factorizations, Cisinski theory, which allows for the easy construction of model structures on the category of presheaves over an EZ-Reedy category, and finally, a technique for transferring a model structure along an adjunction. The material in this section is adapted from various sources, e.g. [Hov99, Rie14, Cis19, HKRS17].

In Section 4.2, we review the basic theory of cubical sets – structures similar to simplicial sets, the difference being that these are built up of cubes rather than simplices. We describe the Grothendieck model structure on the category of cubical sets, due to Cisinski, which models the theory of ∞-groupoids, and is the cubical analogue of the Quillen model structure on the category of simplicial sets. Indeed, there is a Quillen equivalence between these two model categories, given by the triangulation adjunction $T : \mathsf{cSet} \rightleftarrows \mathsf{sSet} : U$. We then review the definition of cubical homotopy groups and prove that the weak equivalences in the Grothendieck model structure are cubical maps inducing isomorphisms on all cubical homotopy groups. The material in this section is adapted from various sources, e.g. [Cis06, CK23].

Equipped with this background, in Section 4.3, we construct two distinct, but Quillen equivalent, model structures on the category of cubical sets where the weak equivalences are cubical maps inducing isomorphisms on the first n cubical homotopy groups. The first of these is the more obvious model – it is the left Bousfield localization of the Grothendieck model structure of Section 4.2, i.e. the model structure obtained by keeping the same class of cofibrations and enlarging the class of weak equivalences. This model can also be constructed using the Cisinski theory of Section 4.1, and we opt for this presentation instead. The second model is obtained by transferring the first one along the $(n+1)$-coskeleton functor. Ultimately, this is the model that will be used in Chapter 5. Finally, we observe that both of these models have simplicial analogues that have been studied previously. We prove that all four models are Quillen equivalent via a zig-zag of Quillen equivalences.

4.1 Background on model categories

4.1.1 Definitions and basic results

We begin by reviewing some preliminary definitions which are fundamental to the theory of model categories.

Definition 4.1.1. Given two morphisms $i\colon A \to B$ and $f\colon X \to Y$ in a category $\mathcal{C}$, a *lifting problem* between i and f is a commutative square as follows:

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "i"'] & X \arrow[d, "f"] \\ B \arrow[r, "v"] \arrow[ur, dotted, "h"] & Y \end{tikzcd}$$

A *lift* or a *solution* $h\colon B \to X$ to this lifting problem is a morphism $h\colon B \to X$ that makes both triangles commute. If every lifting problem between i and f admits a solution, we say that i has the *left lifting property (LLP)* with respect to f, or equivalently, that f has the *right lifting property (RLP)* with respect to i, and we write $i \boxslash f$.

Definition 4.1.2. Given a class S of morphisms in a category $\mathcal{C}$, a morphism i in $\mathcal{C}$ has the *LLP with respect to S* if it has the LLP with respect to every morphism in S. The class of all morphisms with the LLP with respect to S is denoted ${}^{\boxslash}S$. Similarly, a morphism f in $\mathcal{C}$ has the *RLP with respect to S* if it has the RLP with respect to every morphism in S. The class of all morphisms with the RLP with respect to S is denoted $S^{\boxslash}$.

Definition 4.1.3. A class S of morphisms in a category $\mathcal{C}$ is *saturated* if it is closed under retracts, closed under pushouts, and closed under transfinite compositions. Given a set I of morphisms in $\mathcal{C}$, the *saturation* of I is the smallest saturated class of morphisms in $\mathcal{C}$ containing I.

Lemma 4.1.4 ([Rie14, Lem. 11.1.4])**.** *Given any class S of morphisms in a category $\mathcal{C}$, the class ${}^{\boxslash}S$ is saturated.* $\square$

Definition 4.1.5. A *weak factorization system* on a category $\mathcal{C}$ is a pair $(\mathcal{L}, \mathcal{R})$ of classes of morphisms in $\mathcal{C}$ subject to the following axioms:

1. Both $\mathcal{L}$ and $\mathcal{R}$ are closed under retracts;
2. The class $\mathcal{L}$ is contained in ${}^{\boxslash}\mathcal{R}$; and
3. Every morphism in $\mathcal{C}$ can be factored as a morphism in $\mathcal{L}$ followed by a morphism in $\mathcal{R}$.

We are now ready to define model categories.

Definition 4.1.6. A *model category* is a locally small category $\mathcal{M}$ equipped with three wide subcategories: a subcategory $\mathcal{W}$ of *weak equivalences* (denoted by $\xrightarrow{\sim}$), a subcategory $\mathcal{C}$ of *cofibrations* (denoted by $\rightarrowtail$), and a subcategory $\mathcal{F}$ of *fibrations* (denoted by $\twoheadrightarrow$), subject to the following axioms.

1. The category $\mathcal{M}$ has finite limits and finite colimits;
2. Weak equivalences satisfy the 2-out-of-3 property. That is, given two composable morphisms in $\mathcal{M}$ as follows:

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

if any two among the morphisms f, g and gf are weak equivalences, then so is the third; and

3. The pairs $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ are both weak factorization systems on $\mathcal{M}$.

An *acyclic cofibration* is a morphism that is both a weak equivalence and a cofibration, and an *acyclic fibration* is a morphism that is both a weak equivalence and a fibration. An object $X \in \mathcal{M}$ is *cofibrant* if the unique map $\varnothing \to X$ is a cofibration, and *fibrant* if the unique map $X \to 1$ is a fibration. (Here, $\varnothing$ and 1 denote the initial and the terminal objects in $\mathcal{M}$ respectively.)

The model categories that will be of interest to us will often come equipped with the following additional structure.

Definition 4.1.7. A *functorial fibrant replacement* in a model category $\mathcal{M}$ is an endofunctor $\widehat{(-)} \colon \mathcal{M} \to \mathcal{M}$ together with a natural transformation $\mathrm{id}_{\mathcal{M}} \to \widehat{(-)}$ such that for every object $X \in \mathcal{M}$, the object $\widehat{X}$ is fibrant and the morphism $X \to \widehat{X}$ is an acyclic cofibration.

In a sense, the essential part of a model structure is the class of weak equivalences. Given any category $\mathcal{C}$ along with a wide subcategory W of weak equivalences, it is possible to construct a category $\mathcal{C}[W^{-1}]$ by freely inverting the weak equivalences. However, the category $\mathcal{C}[W^{-1}]$ thus constructed may not be locally small, even if $\mathcal{C}$ is. A model category structure has, in addition to weak equivalences, a class of fibrations and a class of cofibrations, and this provides a way out of the above problem, by allowing us to study the homotopy theory of objects that are both fibrant and cofibrant.

Definition 4.1.8. Let $\mathcal{M}$ be a model category.

1. A *cylinder object* of an object $X \in \mathcal{M}$ is a factorization $X \sqcup X \rightarrowtail CX \xrightarrow{\sim} X$ of the codiagonal map $\nabla\colon X \sqcup X \to X$ as a cofibration followed by a weak equivalence.

2. A *path object* of an object $Y \in \mathcal{M}$ is a factorization $Y \xrightarrow{\sim} PY \twoheadrightarrow Y \times Y$ of the diagonal map $\Delta\colon Y \to Y \times Y$ as a weak equivalence followed by a fibration.

3. A left homotopy between two morphisms $f, g\colon X \to Y$ in $\mathcal{M}$ via a cylinder object $X \sqcup X \rightarrowtail CX \xrightarrow{\sim} X$ is a commutative triangle of the form:

$$\begin{tikzcd} & X \sqcup X \arrow[dl, tail] \arrow[dr, "{[f,g]}"] & \\ CX \arrow[rr, "H"] & & Y \end{tikzcd}$$

   If such a left homotopy exists, we say that f and g are *left homotopic* and we write $f \sim_l g$.

4. A *right homotopy* between two morphisms $f, g\colon X \to Y$ in $\mathcal{M}$ via a path object $Y \xrightarrow{\sim} PY \twoheadrightarrow Y \times Y$ is a commutative triangle of the form

$$\begin{tikzcd} X \arrow[rr, "H"] \arrow[dr, "{[f,g]}"'] & & PY \arrow[dl, two heads] \\ & Y \times Y & \end{tikzcd}$$

   If such a right homotopy exists, we say that f and g are *right homotopic* and we write $f \sim_r g$.

The axioms of a model category ensure that every object admits a cylinder object and a path object.

Proposition 4.1.9 ([Hov99, Cor. 1.2.6])**.** *Given a model category* $\mathcal{M}$*, and objects* $X, Y \in \mathcal{M}$*, where* X *is cofibrant and* Y *is fibrant, the relations* $\sim_l$ *and* $\sim_r$ *coincide and define an equivalence relation on the set* $\mathcal{M}(X, Y)$ *that is compatible with composition. Furthermore, they are independent of the choice of cylinder or path objects.* □

In the situation of the above proposition, we drop the subscripts and denote the relation by $\sim$.

Definition 4.1.10. The *homotopy category* $\operatorname{Ho}\mathcal{M}$ of a model category $\mathcal{M}$ is defined as follows:

- the objects of $\operatorname{Ho}\mathcal{M}$ are the objects of $\mathcal{M}$ that are both fibrant and cofibrant
- $\operatorname{Ho}\mathcal{M}(X, Y) := \mathcal{M}(X, Y)/\sim$ for every $X, Y \in \operatorname{Ho}\mathcal{M}$.

By the previous proposition, this is a well-defined category.

Theorem 4.1.11 (Quillen, see [Hov99, Thm. 1.2.10])**.** *Given a model category* $\mathcal{M}$*, there exists an equivalence of categories* $\operatorname{Ho}\mathcal{M} \simeq \mathcal{M}[\mathcal{W}^{-1}]$*, where* $\mathcal{M}[\mathcal{W}^{-1}]$*.* □

The appropriate notion of a morphism between two model categories is a pair of adjoint functors that is compatible with the defining weak factorization systems.

Definition 4.1.12.

1. An adjunction $L : \mathcal{M} \rightleftarrows \mathcal{N} : R$ between two model categories is a *Quillen adjunction* if any of the following equivalent conditions hold:

   (a) L preserves cofibrations and acyclic cofibrations;

(b) R preserves fibrations and acyclic fibrations;

(c) L preserves cofibrations and R preserves fibrations;

(d) L preserves acyclic cofibrations and R preserves acyclic fibrations.

When this happens, we say L is a *left Quillen functor* and R is a *right Quillen functor*.

2. A Quillen adjunction $L : \mathcal{M} \rightleftarrows \mathcal{N} : R$ between two model categories is a *Quillen equivalence* if for every cofibrant object $X \in \mathcal{M}$ and every fibrant object $Y \in \mathcal{N}$, a morphism $X \to RY$ is a weak equivalence in $\mathcal{M}$ if and only if its adjunct $LX \to Y$ is a weak equivalence in $\mathcal{N}$.

3. Two model categories are *Quillen equivalent* if there is a zig-zag of Quillen equivalences between them.

We will need the following result on recognizing Quillen equivalences.

Proposition 4.1.13 ([DKLS20, Cor. 1.14])**.** *Let $L : \mathcal{M} \rightleftarrows \mathcal{N} : R$ be a Quillen adjunction between two model categories. If R preserves and reflects weak equivalences, then the adjunction is a Quillen equivalence if and only if, for every cofibrant object $X \in \mathcal{M}$, the unit $X \to RLX$ is a weak equivalence.* $\square$

4.1.2 Constructing model categories

In this section, we review some standard techniques for constructing model categories, which will be used throughout this chapter. We begin with the small object argument, which provides a technique to construct weak factorization systems.

Definition 4.1.14. Let I be a set of morphisms in a locally small, cocomplete category $\mathcal{C}$. A *relative I-cell complex* is a transfinite composition of pushouts of coproducts of elements of I. We write $\mathrm{cell}(I)$ for the class of all relative I-cell complexes. We write $\mathrm{cof}(I)$ for the class of retracts of elements of $\mathrm{cell}(I)$.

Definition 4.1.15. A set of morphisms I in a locally small, cocomplete category $\mathcal{C}$ *permits the small object argument* if there exists a cardinal κ such that for any element $i\colon A \to B$ of I, the functor $\mathcal{C}(A, -)\colon \mathcal{C} \to \mathsf{Set}$ preserves κ-filtered colimits.

Theorem 4.1.16 (Small object argument, [Hov99, Thm. 2.1.14])**.** *Let $\mathcal{C}$ be a locally small, cocomplete category, and let I be a set of morphisms in $\mathcal{C}$ that permits the small object argument. Then, every morphism in $\mathcal{C}$ can be factored functorially as an element of* $\mathrm{cell}(I)$ *followed by an element of $I^{\boxslash}$.* $\square$

Corollary 4.1.17 ([Hov99, Cor. 2.1.15])**.** *Let $\mathcal{C}$ be a locally small, cocomplete category, and let I be a set of morphisms in $\mathcal{C}$ that permits the small object argument. Then, we have ${}^{\boxslash}(I^{\boxslash}) = \mathrm{cof}(I)$. Furthermore, ${}^{\boxslash}(I^{\boxslash})$ is the saturation of the set I.* $\square$

We will be interested in model categories where both the defining weak factorization systems are constructed via the small object argument.

Definition 4.1.18. A model category $\mathcal{M}$ is *cofibrantly generated* if there exist sets I and J of morphisms in $\mathcal{M}$ such that:

- both I and J permit the small object argument;
- the cofibrations of $\mathcal{M}$ are the elements of $\mathrm{cof}(I)$, or equivalently, the acyclic fibrations of $\mathcal{M}$ are the elements of $I^{\boxslash}$;
- the acyclic cofibrations of $\mathcal{M}$ are the elements of $\mathrm{cof}(J)$, or equivalently, the fibrations of $\mathcal{M}$ are the elements of $J^{\boxslash}$.

The elements of I are called the *generating cofibrations* and the elements of J are called the *generating acyclic cofibrations*.

An advantage of working with cofibrantly generated model categories is that they admit functorial fibrant replacement.

Remark 4.1.19. Let $\mathcal{M}$ be a cofibrantly generated model category and J be a set of generating acyclic cofibrations. Given any object $X \in \mathcal{M}$, the unique morphism from X to the terminal object of $\mathcal{M}$ can be factored functorially as an element of $\operatorname{cell}(J)$ followed by an element of $J^{\boxslash}$.

$$\begin{tikzcd} X \arrow[rr, "!"] \arrow[dr, tail, "\sim", "\operatorname{cell}(J) \ni"'] & & 1 \\ & \widehat{X} \arrow[ur, "\in J^{\boxslash}"'] & \end{tikzcd}$$

The mapping $X \mapsto \widehat{X}$ is a functorial fibrant replacement in $\mathcal{M}$.

Next, we review the machinery of Cisinski theory, which allows for the easy construction of model structures on presheaf categories. We will be interested in working with presheaves over particularly well-behaved categories, called EZ-Reedy categories.

Definition 4.1.20. A *Reedy category* is a small category $\mathcal{A}$ equipped with a degree function $\deg\colon \operatorname{ob}\mathcal{A} \to \mathbb{N}$ on objects and two wide subcategories $\mathcal{A}_-$ and $\mathcal{A}_+$ such that the following conditions are satisfied:

1. for every non-identity morphism $f\colon a \to b$ in $\mathcal{A}_-$, we have $\deg a > \deg b$;

2. for every non-identity morphism $f\colon a \to b$ in $\mathcal{A}_+$, we have $\deg a < \deg b$; and

3. every morphism in $\mathcal{A}$ can be factored uniquely as a morphism in $\mathcal{A}_-$ followed by a morphism in $\mathcal{A}_+$.

Definition 4.1.21. A Reedy category $\mathcal{A}$ is *Eilenberg-Zilber* (or *EZ-Reedy* for short) if every morphism $f\colon a \to b$ in $\mathcal{A}_-$ admits a section (that is, a map $s\colon b \to a$ such that $fs = \mathrm{id}_b$) and if given any two morphisms $f, f'\colon a \to b$ having the same set of sections, we have $f = f'$.

We will now review some basic properties of EZ-Reedy categories. Let us fix an EZ-Reedy category $\mathcal{A}$.

Notation 4.1.22. Given any integer $n \geq -1$, let $\mathcal{A}_{\leq n}$ denote the full subcategory of $\mathcal{A}$ on the objects with degree $\leq n$. The inclusion $i_n\colon \mathcal{A}_{\leq n} \to \mathcal{A}$ induces a forgetful functor

$$i_n^*\colon \mathsf{Set}^{\mathcal{A}^{\mathrm{op}}} \to \mathsf{Set}^{\mathcal{A}_{\leq n}^{\mathrm{op}}}.$$

This functor admits both adjoints:

$$\begin{tikzcd}
\mathsf{Set}^{\mathcal{A}^{\mathrm{op}}} \arrow[r, "i_n^*" description] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] & \mathsf{Set}^{\mathcal{A}_{\leq n}^{\mathrm{op}}} \arrow[l, bend right=40, "i_{n*}"'] \arrow[l, bend left=40, "i_{n!}"]
\end{tikzcd}$$

We write sk_n and cosk_n for the composite functors

$$\mathrm{sk}_n = i_{n*} \circ i_n^*\colon \mathsf{Set}^{\mathcal{A}^{\mathrm{op}}} \to \mathsf{Set}^{\mathcal{A}^{\mathrm{op}}} \qquad \text{and} \qquad \mathrm{cosk}_n = i_{n!} \circ i_n^*\colon \mathsf{Set}^{\mathcal{A}^{\mathrm{op}}} \to \mathsf{Set}^{\mathcal{A}^{\mathrm{op}}}$$

respectively. These are called the *n-skeleton* and the *n-coskeleton* functors respectively. These in turn form an adjoint pair:

$$\begin{tikzcd}
\mathsf{Set}^{\mathcal{A}^{\mathrm{op}}} \arrow[r, shift left=2, "\mathrm{sk}_n"] \arrow[r, phantom, "\perp"] & \mathsf{Set}^{\mathcal{A}^{\mathrm{op}}} \arrow[l, shift left=2, "\mathrm{cosk}_n"]
\end{tikzcd}$$

Given any object $a \in \mathcal{A}$, we write よ_a for the representable presheaf $\mathcal{A}(-, a) \colon \mathcal{A}^{\mathrm{op}} \to \mathsf{Set}$ and we write $\partial\text{よ}_a$ for the *boundary* of the representable presheaf よ_a, given by

$$\partial\text{よ}_a := \mathrm{sk}_{\deg a - 1}\text{よ}_a$$

Definition 4.1.23. Let $X \colon \mathcal{A}^{\mathrm{op}} \to \mathsf{Set}$ be a presheaf over $\mathcal{A}$. For any $a \in \mathcal{A}$, an element $x \in X(a)$ is *degenerate* if there exists a non-identity map $f \colon a \to b$ in $\mathcal{A}_-$ and an element $y \in X(b)$ such that $yf = x$. The element x is *non-degenerate* if it is not degenerate.

The following result tells us that any arbitrary presheaf over an EZ-category is built up inductively, by attaching representable presheaves along their boundary, similar to how CW complexes are built up inductively, by attaching n-disks along their boundary $(n-1)$-spheres, for all $n \in \mathbb{N}$.

Theorem 4.1.24 (Skeletal induction, [Cis19, Thm. 1.3.8])**.** *Let $X \hookrightarrow Y$ be a monomorphism of presheaves over $\mathcal{A}$. For each $n \geq 0$, we have a pushout square as follows:*

$$\begin{tikzcd}
\coprod\limits_{y\in\Sigma} \partial\text{よ}_a \arrow[r] \arrow[d, hook] & X \cup \mathrm{sk}_{n-1}Y \arrow[d] \\
\coprod\limits_{y\in\Sigma} \text{よ}_a \arrow[r] \arrow[ur, phantom, "\ulcorner" very near end] & X \cup \mathrm{sk}_{n}Y
\end{tikzcd}$$

where Σ denotes the set of all non-degenerate elements $y \in Y(a)$, which do not belong to X, and such that $\deg a = n$. □

Corollary 4.1.25. *The class of monomorphisms in the presheaf category* $\mathsf{Set}^{\mathcal{A}^{\mathrm{op}}}$ *is the saturation of the set*

$$\{\partial\text{よ}_a \hookrightarrow \text{よ}_a \;\mid\; a \in \mathcal{A}\}$$

of all boundary inclusions. □

There are two key ingredients that need to be fed into Cisinski's machinery for constructing a model structure on the category $\mathsf{Set}^{\mathcal{A}^{\mathrm{op}}}$: a functorial cylinder, and a set of generating anodyne maps.

Definition 4.1.26. A *functorial cylinder* on $\mathcal{A}$ consists of an endofunctor $(-)\otimes I$ on the presheaf category $\mathsf{Set}^{\mathcal{A}^{\mathrm{op}}}$, together with natural transformations $\partial^0, \partial^1 \colon \mathrm{id} \to (-) \otimes I$ and $\sigma \colon (-) \otimes I \to \mathrm{id}$ such that:

- ∂^0 and ∂^1 are sections of σ;
- for every presheaf $X \colon \mathcal{A}^{\mathrm{op}} \to \mathsf{Set}$, the map $(\partial^0_X, \partial^1_X) \colon X \sqcup X \to X \otimes I$ is a monomorphism;
- $(-) \otimes I$ preserves small colimits and monomorphisms;
- for every monomorphism $X \hookrightarrow Y$ in $\mathsf{Set}^{\mathcal{A}^{\mathrm{op}}}$ and for $\varepsilon \in \{0,1\}$, the following square is a pullback:

$$\begin{tikzcd} X \arrow[r, "\partial^\varepsilon_X"] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & X \otimes I \arrow[d, hook] \\ Y \arrow[r, "\partial^\varepsilon_Y"] & Y \otimes I \end{tikzcd}$$

Let us fix a functorial cylinder $(-) \otimes I$ on $\mathcal{A}$.

Definition 4.1.27. Given two maps $f, g \colon X \to Y$ in $\mathsf{Set}^{\mathcal{A}^{\mathrm{op}}}$, an *elementary homotopy* from f to g is a map $H \colon X \otimes I \to Y$ such that the following diagram commutes:

$$\begin{tikzcd} & X \sqcup X \arrow[dl, hook', "(\partial^0_X, \partial^1_X)"'] \arrow[dr, "{[f,g]}"] & \\ X \otimes I \arrow[rr, "H"] & & Y \end{tikzcd}$$

Given two presheaves $X, Y \in \mathsf{Set}^{\mathcal{A}^{\mathrm{op}}}$, let $[X, Y]$ denote the quotient of the hom-set $\mathsf{Set}^{\mathcal{A}^{\mathrm{op}}}(X, Y)$ by the equivalence relation generated by elementary homotopies. Two maps $f, g \colon X \to Y$ in $\mathsf{Set}^{\mathcal{A}^{\mathrm{op}}}$ are *homotopic* if they have the same equivalence class in $[X, Y]$. A map $f \colon X \to Y$ in $\mathsf{Set}^{\mathcal{A}^{\mathrm{op}}}$ is a *homotopy equivalence* if there exists a map $g \colon Y \to X$ such that gf is homotopic to id_X and fg is homotopic to id_Y.

Notation 4.1.28. Given a monomorphism $X \hookrightarrow Y$ in $\mathsf{Set}^{\mathcal{A}^{\mathrm{op}}}$ and $\varepsilon \in \{0, 1\}$, let $X \otimes I \cup Y \otimes \{\varepsilon\}$ and $X \otimes I \cup Y \otimes \partial I$ be defined the following pushout squares:

$$
\begin{tikzcd}
X \arrow[r, "\partial^{\varepsilon}_X"] \arrow[d, hook] & X \otimes I \arrow[d] \\
Y \arrow[r] & X \otimes I \cup Y \otimes \{\varepsilon\} \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}
\qquad
\begin{tikzcd}
X \sqcup X \arrow[r, "(\partial^0_X, \partial^1_X)"] \arrow[d, hook] & X \otimes I \arrow[d] \\
Y \sqcup Y \arrow[r] & X \otimes I \cup Y \otimes \partial I \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}
$$

We now turn our attention to the second ingredient: a set of generating anodyne maps.

Construction 4.1.29. Let S be a set of monomorphisms in $\mathsf{Set}^{\mathcal{A}^{\mathrm{op}}}$. We will construct a set $\Lambda_I(S)$ of monomorphisms in $\mathsf{Set}^{\mathcal{A}^{\mathrm{op}}}$ by the following inductive process. We begin by defining:

$$\Lambda^0_I(S) := S \cup \{\partial \text{よ}_a \otimes I \cup \text{よ}_a \otimes \{\varepsilon\} \hookrightarrow \text{よ}_a \times I \mid a \in \mathcal{A},\ \varepsilon = 0, 1\}.$$

Given a set $\Lambda^n_I(S)$ for any $n \in \mathbb{N}$, we define

$$\Lambda^{n+1}_I(S) := \{X \otimes I \cup Y \otimes \partial I \hookrightarrow Y \otimes I \mid X \hookrightarrow Y \in \Lambda^n_I(S)\}.$$

Finally, we define

$$\Lambda_I(S) := \bigcup_{n\geq 0} \Lambda_I^n(S).$$

Definition 4.1.30. Let S be a set of monomorphisms in $\mathsf{Set}^{\mathcal{A}^{\mathrm{op}}}$.

1. A map $f\colon X \to Y$ in $\mathsf{Set}^{\mathcal{A}^{\mathrm{op}}}$ is (S,I)-*anodyne* if it belongs to the saturation of the set $\Lambda_I(S)$. We write $\mathrm{An}_I(S)$ for the class of all (S,I)-anodyne maps

2. A map $f\colon X \to Y$ in $\mathsf{Set}^{\mathcal{A}^{\mathrm{op}}}$ is a *naive fibration* if it has the RLP with respect to every (S,I)-anodyne map. A presheaf $X \in \mathsf{Set}^{\mathcal{A}^{\mathrm{op}}}$ is *fibrant* if the unique map from X to the terminal presheaf is a naive fibration.

3. A map $f\colon X \to Y$ in $\mathsf{Set}^{\mathcal{A}^{\mathrm{op}}}$ is a *weak equivalence* if, for every fibrant preshead Z, the induced map

$$f^*\colon [Y,Z] \to [X,Z]$$

is a bijection.

4. A map $f\colon X \to Y$ in $\mathsf{Set}^{\mathcal{A}^{\mathrm{op}}}$ is a *fibration* if it has the RLP with respect to every map that is both a monomorphism and a weak equivalence.

We are now ready to state the main theorem of Cisinski theory.

Theorem 4.1.31 ([Cis19, Thm. 2.4.19])**.** *The category of* $\mathsf{Set}^{\mathcal{A}^{\mathrm{op}}}$ *admits a cofibrantly generated model structure where the cofibrations are the monomorphisms, and the fibrations and the weak equivalences are defined as above. Moreover, a map* $f\colon X \to Y$ *in* $\mathsf{Set}^{\mathcal{A}^{\mathrm{op}}}$ *between fibrant presheaves is a fibration if and only if it is a naive fibration.* □

We will also need to use the following preliminary results that go into proving the above theorem.

Proposition 4.1.32 ([Cis19, Prop. 2.4.25]). *Every (S, I)-anodyne map is a weak equivalence.* □

Proposition 4.1.33 ([Cis19, Prop. 2.4.26]). *A map $f\colon X \to Y$ in $\mathsf{Set}^{A^{\mathrm{op}}}$ between two fibrant presheaves X and Y is a weak equivalence if and only if it is a homotopy equivalence.* □

Next, we review a theorem which allows us to induce one model structure from another using an adjunction between their respective categories.

Definition 4.1.34. Let $(\mathcal{M}, \mathcal{W}, \mathcal{C}, \mathcal{F})$ be a model category, let $\mathcal{N}$ be a complete and cocomplete category, and let $L : \mathcal{M} \rightleftarrows \mathcal{N} : R$ be an adjunction. A morphism $f\colon X \to Y$ in $\mathcal{N}$ is a *fibration* (resp. a *weak equivalence* if its image under R in $\mathcal{M}$ is a fibration (resp. a weak equivalence). A morphism $f\colon X \to Y$ in $\mathcal{N}$ is a *cofibration* if it has the LLP with respect to every morphism in $\mathcal{N}$ that is both a fibration and a weak equivalence. If these classes of maps define a model structure on $\mathcal{N}$, then it is called the *right-transferred model structure* from $\mathcal{M}$ along R.

Theorem 4.1.35 (Transfer theorem, [HKRS17, Cor. 3.3.4]). *Let* $(\mathcal{M}, \mathcal{W}, \mathcal{C}, \mathcal{F})$ *be a cofibrantly generated model category, let $\mathcal{N}$ be a locally presentable category, and let $L : \mathcal{M} \rightleftarrows \mathcal{N} : R$ be an adjunction. The right-transferred model structure on $\mathcal{N}$ exists if and only if the following acyclicity condition holds:*

$$^{\boxslash}R^{-1}\mathcal{F} \subseteq R^{-1}\mathcal{W}. \tag{A}$$

□

4.2 Background on cubical sets

4.2.1 The box category and cubical sets

We begin by defining the box category $\Box$ and reviewing some of its basic properties. For instance, $\Box$ is an EZ-Reedy category. The presheaves over $\Box$ are called cubical sets.

Notation 4.2.1. Let $[1]$ denote the poset $\{0 \leq 1\}$. For each $n \geq 0$, we have a poset $[1]^n$, whose elements are binary strings of length n, and where the partial order is given component-wise. Let $\Box$ be the subcategory of Cat defined as follows. The set of objects is given by:

$$\mathrm{ob}(\Box) := \{[1]^n = \{0 \leq 1\}^n \text{ for all } n \geq 0\}.$$

The maps in $\Box$ are generated under composition by the following three kinds of maps:

- faces $\partial^n_{i,\varepsilon} \colon [1]^{n-1} \to [1]^n$ for $i = 1, \ldots, n$ and $\varepsilon = 0, 1$, given by

$$(x_1, \ldots, x_{n-1}) \mapsto (x_1, \ldots, x_{i-1}, \varepsilon, x_i, \ldots, x_{n-1});$$

- degeneracies $\sigma^n_i \colon [1]^{n+1} \to [1]^n$ for $i = 1, \ldots, n+1$, given by

$$(x_1, \ldots, x_{n+1}) \mapsto (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1}); \text{ and}$$

- connections $\gamma^n_{i,\varepsilon} \colon [1]^{n+1} \to [1]^n$ for $i = 1, \dots, n$ and $\varepsilon = 0, 1$, given by

$$(x_1, \dots, x_{n+1}) \mapsto \begin{cases} (x_1, \dots, x_{i-1}, \max(x_i, x_{i+1}), x_{i+2}, \dots, x_{n+1}) & \text{if } \varepsilon = 0, \\ (x_1, \dots, x_{i-1}, \min(x_i, x_{i+1}), x_{i+2}, \dots, x_{n+1}) & \text{if } \varepsilon = 1. \end{cases}$$

These maps satisfy the following cubical identities:

$$\partial_{j,\eta}\partial_{i,\varepsilon} = \partial_{i+1,\varepsilon}\partial_{j,\eta} \text{ for } j \leq i \qquad \gamma_{j,\eta}\gamma_{i,\varepsilon} = \begin{cases} \gamma_{i,\varepsilon}\gamma_{j+1,\eta} & \text{for } j > i \\ \gamma_{i,\varepsilon}\gamma_{i+1,\varepsilon} & \text{for } j = i, \eta = \varepsilon \end{cases}$$

$$\sigma_i\sigma_j = \sigma_j\sigma_{i+1} \quad \text{for } j \leq i \qquad \gamma_{j,\eta}\partial_{i,\varepsilon} = \begin{cases} \partial_{i-1,\varepsilon}\gamma_{j,\eta} & \text{for } j < i-1 \\ \mathrm{id} & \text{for } j = i-1,\ i,\ \varepsilon = \eta \\ \partial_{i,\varepsilon}\sigma_i & \text{for } j = i-1,\ i,\ \varepsilon = 1-\eta \\ \partial_{i,\varepsilon}\gamma_{j-1,\eta} & \text{for } j > i \end{cases}$$

$$\sigma_j\partial_{i,\varepsilon} = \begin{cases} \partial_{i-1,\varepsilon}\sigma_j & \text{for } j < i \\ \mathrm{id} & \text{for } j = i \\ \partial_{i,\varepsilon}\sigma_{j-1} & \text{for } j > i \end{cases} \qquad \sigma_j\gamma_{i,\varepsilon} = \begin{cases} \gamma_{i-1,\varepsilon}\sigma_j & \text{for } j < i \\ \sigma_i\sigma_i & \text{for } j = i \\ \gamma_{i,\varepsilon}\sigma_{j+1} & \text{for } j > i. \end{cases}$$

Theorem 4.2.2 ([GM03, Thm. 5.1])**.** *Every morphism in $\Box$ can be expressed uniquely as a composite of the form*

$$(\partial_{c_1,\eta_1} \cdots \partial_{c_r,\eta_r})(\gamma_{b_1,\varepsilon_1} \cdots \gamma_{b_q,\varepsilon_q})(\sigma_{a_1} \cdots \sigma_{a_p})$$

where $1 \leq a_1 < \cdots < a_p$, $1 \leq b_1 \leq \cdots \leq b_q$, with $b_i < b_{i+1}$ if $\varepsilon_i = \varepsilon_{i+1}$, and $c_1 > \cdots > c_r \geq 1$. $\Box$

Proposition 4.2.3 ([CKM20, Prop. 1.8])**.** *The category $\Box$ admits the structure of an EZ-Reedy category (see Definition 4.1.21), where:*

- $\deg[1]^n = n$,
- $\square_+$ *is generated under composition by the face maps, and*
- $\square_-$ *is generated under composition by the degeneracy and the connection maps.* $\square$

Definition 4.2.4. A *cubical set* is a presheaf $X \colon \square^{\mathrm{op}} \to \mathsf{Set}$. We write $X_n := X([1]^n)$, and we write cubical operators on the right — for instance, given an n-cube $x \in X_n$ of X, we write $x\partial_{1,0}$ for the $(1,0)$-face of x. We write cSet for the category of cubical sets and cubical maps.

Notation 4.2.5.

- For $n \geq 0$, the *combinatorial n-cube*, denoted $\square^n$, is the representable functor $\square(-,[1]^n) \colon \square^{\mathrm{op}} \to \mathsf{Set}$.
- For $n \geq 0$, the *boundary of the n-cube*, denoted $\partial\square^n$, is the subobject of $\square^n$ defined by:
$$\partial\square^n := \mathrm{sk}_{n-1}\square^n = \bigcup_{\substack{i=1,\ldots,n \\ \varepsilon=0,1}} \mathrm{im}\,\partial_{i,\varepsilon}.$$
The canonical inclusion $\partial\square^n \hookrightarrow \square^n$ is called the *boundary inclusion of the n-cube.*
- For $n > 0$, $i = 1, \ldots, n$, and $\varepsilon = 0, 1$, the (i,ε)*-open box*, denoted $\sqcap^n_{i,\varepsilon}$, is the suboject of $\square^n$ defined by:
$$\sqcap^n_{i,\varepsilon} := \bigcup_{(j,\eta)\neq(i,\varepsilon)} \mathrm{im}\,\partial_{j,\eta}.$$
The canonical inclusion $\sqcap^n_{i,\varepsilon} \hookrightarrow \square^n$ is called the (i,ε)*-open box inclusion of the n-cube.*

4.2.2 Homotopy theory of cubical sets

We will now describe the Grothendieck model structure on the category of cubical sets, due to Cisinski, which models the theory of ∞-groupoids, and is the cubical analogue of the Quillen model structure on the category of simplicial sets. We also describe the triangulation adjunction $T : \mathsf{cSet} \rightleftarrows \mathsf{sSet} : U$, which gives a Quillen equivalence between these two model categories.

We begin by reviewing the *geometric product of cubical sets*.

Let $\otimes \colon \square \times \square \to \square$ denote the functor given by the assignment $([1]^m, [1]^n) \mapsto [1]^{m+n}$. Post-composing it with the Yoneda embedding $\square \hookrightarrow \mathsf{cSet}$ and taking its left Kan extension along $\square \times \square \hookrightarrow \mathsf{cSet} \times \mathsf{cSet}$, we obtain the *geometric product* of cubical sets.

$$\begin{tikzcd}
\square \times \square \arrow[r, "\otimes"] \arrow[d, hook] & \square \arrow[r, hook] & \mathsf{cSet} \\
\mathsf{cSet} \times \mathsf{cSet} \arrow[urr, "\otimes"'] & &
\end{tikzcd}$$

This defines a monoidal structure on cSet. For any fixed cubical set X, the functor $X \otimes (-) \colon \mathsf{cSet} \to \mathsf{cSet}$ admits a right adjoint which we denote by $\hom_R(X, -)$.

Remark 4.2.6. The functor $(-) \otimes \square^1 \colon \mathsf{cSet} \to \mathsf{cSet}$ defines a functorial cylinder on $\square$ (see Definition 4.1.26).

This naturally leads to the following definitions of *elementary homotopies* and *homotopy equivalences*.

Definition 4.2.7. Given two cubical maps $f, g \colon X \to Y$, an *elementary homotopy* from f to g is a map $H \colon X \otimes \square^1 \to Y$ such that the following diagram commutes:

$$\begin{tikzcd} X \otimes \square^0 \arrow[d, "\partial_{1,0}"'] \arrow[dr, "f"] & \\ X \otimes \square^1 \arrow[r, "H"] & Y \\ X \otimes \square^0 \arrow[u, "\partial_{1,1}"] \arrow[ur, "g"'] & \end{tikzcd}$$

Given two cubical sets X and Y, let $[X, Y]$ denote the quotient of the hom-set $\mathsf{cSet}(X, Y)$ by the equivalence relation generated by elementary homotopies. Two cubical maps $f, g \colon X \to Y$ are *homotopic* if they have the same equivalence class in $[X, Y]$. A cubical map $f \colon X \to Y$ is a *homotopy equivalence* if there exists a cubical map $g \colon Y \to X$ such that gf is homotopic to id_X and fg is homotopic to id_Y.

The following result is a consequence of the fact that $\square$ is an EZ-Reedy category.

Lemma 4.2.8. *The class of monomorphisms in* cSet *is the saturation of the set*

$$I = \{\partial\square^n \hookrightarrow \square^n \ \mid \ n \geq 0\}$$

of all boundary inclusions.

Proof. This follows from Corollary 4.1.25 and Proposition 4.2.3. □

We are now ready to describe the various classes of maps in the Grothendieck model structure on cSet.

Definition 4.2.9.

1. A cubical map $f \colon X \to Y$ is *anodyne* if it belongs to the saturation of the set

$$J = \left\{\sqcap^n_{i,\varepsilon} \hookrightarrow \square^n \ \mid \ n > 0,\ i = 1, \ldots, n,\ \varepsilon = 0, 1\right\}$$

of all open box inclusions. Note that the class of all anodyne maps is precisely $\mathrm{An}_{\square^1}(\varnothing)$ (see Definition 4.1.30).

2. A cubical map $f\colon X \to Y$ is a *Kan fibration* if it has the right lifting property with respect to every anodyne map. A cubical set X is a *Kan complex* if the unique map $X \to \square^0$ is a Kan fibration. We write Kan for the full subcategory of cSet on Kan complexes.

3. A cubical map $f\colon X \to Y$ is a weak homotopy equivalence if, for every Kan complex Z, the induced map

$$f^*\colon [Y, Z] \to [X, Z]$$

is a bijection.

We have the following preliminary result:

Proposition 4.2.10. *A cubical map $f\colon X \to Y$ between two Kan complexes X and Y is a weak homotopy equivalence if and only if it is a homotopy equivalence.* $\square$

Proof. This follows from Proposition 4.1.33. $\square$

And finally, the main theorem of this section:

Theorem 4.2.11 (Cisinski, see [DKLS20, Thm. 1.34])**.** *The category* cSet *admits a cofibrantly generated model structure where*

- *the cofibrations are the monomorphisms,*
- *the fibrations are the Kan fibrations, and*
- *the weak equivalences are the weak homotopy equivalences.*

We refer to this model structure as the Grothendieck model structure. $\square$

Let $T \colon \square \to \mathsf{sSet}$ denote the functor given by the assignment $[1]^n \mapsto (\Delta^1)^n$. Taking its left Kan extension along the Yoneda embedding $\square \hookrightarrow \mathsf{cSet}$, we obtain the *triangulation functor* $T \colon \mathsf{cSet} \to \mathsf{sSet}$ and its right adjoint $U \colon \mathsf{sSet} \to \mathsf{cSet}$ given by $(UX)_n = \mathsf{sSet}((\Delta^1)^n, X)$.

$$\begin{tikzcd} \square \arrow[r, "{[1]^n \mapsto (\Delta^1)^n}"] \arrow[d, hook] & \mathsf{sSet} \arrow[dl, shift left, "U"] \\ \mathsf{cSet} \arrow[ur, shift left, "T"] & \end{tikzcd}$$

Theorem 4.2.12 ([DKLS20, Thm. 6.26])**.** *The adjunction $T \dashv U$ is a Quillen equivalence between the Grothendieck model structure on* cSet *and the Quillen model structure on* sSet*.* $\square$

4.2.3 Homotopy groups of cubical sets

We now review the definitions of homotopy groups of cubical sets, following [CK23].

Definition 4.2.13. Given a Kan complex X, its *set of path components*, denoted $\pi_0 X$, is the set $[\square^0, X]$.

Definition 4.2.14. Given a Kan complex X and a 0-cube $x \in X_0$, the *loop space* $\Omega(X, x)$ is defined by the following pullback:

$$\begin{tikzcd} \Omega(X,x) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \hom_R(\square^1, X) \arrow[d, "{(\partial^*_{1,0}, \partial^*_{1,1})}"] \\ \square^0 \arrow[r, "{(x,x)}"] & X \times X \end{tikzcd}$$

Note that $\Omega(X,x)$ is also a Kan complex, and it has a distinguished 0-cube given by $x\sigma_1 \in \Omega(X,x)_0$. For $n \geq 0$, let $\Omega^n(X,x)$ be defined as follows:

$$\Omega^n(X,x) := \begin{cases} (X,x) & \text{if } n = 0 \\ \Omega(\Omega^{n-1}(X,x)) & \text{if } n > 0. \end{cases}$$

The *nth homotopy group* $\pi_n(X,x)$ of a pointed Kan complex (X,x) is given by:

$$\pi_n(X,x) := \pi_0 \Omega^n(X,x).$$

Definition 4.2.15. Let $n \geq 0$. Given a cubical set X and a 0-cube $x \in X_0$, its *nth homotopy group* $\pi_n(X,x)$ is the nth homotopy group of its fibrant replacement in the Grothendieck model structure on cSet.

Our next goal is to show that the homotopy groups of a cubical set X agree with the simplicial homotopy groups of its triangulation TX.

Let $|{-}|_\square \colon \square \to \mathsf{Top}$ denote the functor given by the assignment $[1]^n \mapsto [0,1]^n$. Taking its left Kan extension along the Yoneda embedding $\square \hookrightarrow \mathsf{cSet}$, we obtain the *geometric realization functor* $|{-}|_\square \colon \mathsf{cSet} \to \mathsf{Top}$ and its right adjoint $\mathrm{Sing}_\square \colon \mathsf{Top} \to \mathsf{cSet}$ given by $\mathrm{Sing}_\square(X)_n = \mathsf{Top}([0,1]^n, X)$.

$$\begin{tikzcd} \square \arrow[r, "{[1]^n \mapsto [0,1]^n}"] \arrow[d, hook] & \mathsf{Top} \arrow[dl, shift left, "\mathrm{Sing}_\square"] \\ \mathsf{cSet} \arrow[ur, shift left, "|{-}|_\square"] & \end{tikzcd}$$

Proposition 4.2.16. *Given any cubical set X, we have a bijection $\pi_0 X \cong \pi_0 TX$ and a group isomorphism $\pi_n(X,x) \cong \pi_n(Tx,x)$ for every $n > 0$ and every 0-cube $x \in X_0$.*

Proof. Let $\widetilde{X}$ be a fibrant replacement of X in the Grothendieck model structure on cSet, and let $\widetilde{T\widetilde{X}}$ be a fibrant replacement of $T\widetilde{X}$ in the Quillen model structure on sSet. Then, we have:

$$\begin{aligned}
\pi_n(X,x) &= \pi_n(\widetilde{X},x) \\
&\cong \pi_n(|\widetilde{X}|_\square,x) \quad \text{by [CK23, Thm. 3.25]} \\
&\cong \pi_n(|T\widetilde{X}|_\Delta,x) \quad \text{by [CK23, Lem. 2.23]} \\
&\cong \pi_n(|\widetilde{T\widetilde{X}}|_\Delta,x) \quad \text{by [Qui67, Prop. 2.3.5]} \\
&\cong \pi_n(\widetilde{T\widetilde{X}},x) \quad \text{by [Hov99, Prop. 3.6.3]} \\
&= \pi_n(T\widetilde{X},x) \\
&\cong \pi_n(TX,x) \quad \text{by [DKLS20, Thm. 6.26].}
\end{aligned}$$

where $|-|_\Delta \colon \mathsf{sSet} \to \mathsf{Top}$ denotes the geometric realization of simplicial sets. $\square$

Finally, we state Whitehead's theorem for cubical sets, which gives a characterization of the weak equivalences in the Grothendieck model structure on cSet in terms of homotopy groups.

Theorem 4.2.17 ([CK23, Thm. 4.7])**.** *A cubical map $f \colon X \to Y$ between two Kan complexes is a homotopy equivalence if and only if it induces a bijection $f_* \colon \pi_0 X \to \pi_0 Y$ and an isomorphism $f_* \colon \pi_n(X,x) \to \pi_n(Y,fx)$ for every $n > 0$ and every 0-cube $x \in X_0$.* $\square$

Corollary 4.2.18. *A cubical map $f \colon X \to Y$ is a weak homotopy equivalence if and only if it induces a bijection $f_* \colon \pi_0 X \to \pi_0 Y$ and an isomorphism $f_* \colon \pi_n(X,x) \to \pi_n(Y,fx)$ for every $n > 0$ and every 0-cube $x \in X_0$.*

Proof. Let $\widetilde{f} \colon \widetilde{X} \to \widetilde{Y}$ be a functorial fibrant replacement of $f \colon X \to Y$ in the

Grothendieck model structure on cSet. Then, f is a weak homotopy equivalence if and only if $\widetilde{f}$ is. The required result then follows from Proposition 4.2.10 and Theorem 4.2.17. $\square$

4.3 Homotopy n-types of cubical sets

We are finally ready to construct model structures on the category cSet where the weak equivalences are cubical maps that induce isomorphisms on the first n homotopy groups, for some integer $n \geq 0$.

Let us fix an integer $n \geq 0$.

4.3.1 The Cisinski model structure

First, we construct a model structure on cSet using Cisinski theory. We begin by defining the various classes of maps involved.

Definition 4.3.1.

1. A cubical map $f \colon X \to Y$ is *n-anodyne* if it belongs to the saturation of the set

$$J_n = \left\{ \sqcap^k_{i,\varepsilon} \hookrightarrow \square^k \;\middle|\; k > 0,\ i = 1, \ldots, k,\ \varepsilon = 0, 1 \right\} \cup \left\{ \partial\square^k \hookrightarrow \square^k \;\middle|\; k \geq n+2 \right\}.$$

   Note that the class of all n-anodyne maps is precisely $\mathrm{An}_{\square^1}(\{\partial\square^{n+2} \hookrightarrow \square^{n+2}\})$ (see Definition 4.1.30).

2. A cubical map $f \colon X \to Y$ is a *naive n-fibration* if it has the right lifting property with respect to every n-anodyne map. A cubical set X is *n-fibrant* if the unique map $X \to \square^0$ is a naive n-fibration.

3. A cubical map $f\colon X \to Y$ is an *n-equivalence* if, for every n-fibrant cubical set Z, the induced map

$$f^*\colon [Y, Z] \to [X, Z]$$

is a bijection.

4. A cubical map $f\colon X \to Y$ is an *n-fibration* if it has the right lifting property with respect to every map that is both a monomorphism and an n-equivalence.

Note that every naive n-fibration is, in particular, a Kan fibration. Thus, every n-fibrant cubical set is a Kan complex. It follows that every weak homotopy equivalence is, in particular, an n-equivalence.

Lemma 4.3.2. *Every n-anodyne map is both a monomorphism and an n-equivalence. Equivalently, every n-fibration is a naive n-fibration.*

Proof. Since the class of monomorphisms is saturated and since every map in J_n is a monomorphism, it follows that every n-anodyne map is a monomorphism. The fact that every n-anodyne map is an n-equivalence follows from Proposition 4.1.32. $\square$

The following result gives an alternative characterization of n-fibrant cubical sets.

Proposition 4.3.3. *A cubical set X is n-fibrant if and only if it is a Kan complex with $\pi_k(X, x) \cong 0$ for all $k \geq n+1$ and all 0-cubes $x \in X_0$.*

Proof. Given a Kan complex X, [CK23, Thm. 3.22] implies that the map $X \to \square^0$ has the right lifting property with respect to the boundary inclusion $\partial\square^{k+1} \hookrightarrow \square^{k+1}$ if and only if $\pi_k(X, x) \cong 0$ for every 0-cube $x \in X$. $\square$

Proposition 4.3.4. *A cubical map $f\colon X \to Y$ between two n-fibrant objects X and Y is an n-equivalence if and only if it is a homotopy equivalence.*

Proof. This follows from Proposition 4.1.33. □

Theorem 4.3.5. *The category cSet admits a cofibrantly generated model structure where:*

- *the cofibrations are the monomorphisms,*
- *the fibrant objects are the n-fibrant cubical sets, and*
- *the weak equivalences are the n-equivalences.*

Moreover, a cubical map $f\colon X \to Y$ between n-fibrant cubical sets X and Y is an n-fibration if and only if it is a naive n-fibration. We refer to this model structure as the Cisinski model structure for n-types of cubical sets, and denote it by $\mathrm{L}_n\mathsf{cSet}$. It is the left Bousfield localization of the Grothendieck model structure on cSet at the class of n-equivalences.

Proof. This follows from Theorem 4.1.31. □

Next, we want to give a characterization of n-equivalences in terms of homotopy groups, analogous to Corollary 4.2.18. We begin with a technical lemma.

Lemma 4.3.6. *Let $f\colon X \to Y$ be a cubical map that is a bijection on the k-cubes for each $0 \le k \le n+1$. Then, the induced map $f_*\colon \pi_0 X \to \pi_0 Y$ is a bijection, and the induced group homomorphism $f_*\colon \pi_k(X, x) \to \pi_k(Y, fx)$ is an isomorphism for each $0 < k \le n$ and each 0-cube $x \in X_0$.*

Proof. We begin by fixing some notation. Consider the following commuting diagram:

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d, "f_0"'] & X_1 \arrow[r] \arrow[d, "f_1"'] & X_2 \arrow[r] \arrow[d, "f_2"'] & \cdots \\ Y_0 \arrow[r] & Y_1 \arrow[r] & Y_2 \arrow[r] & \cdots \end{tikzcd}$$

where $X_0 = X$, $Y_0 = Y$, $f_0 = f$, and the maps $X_j \to X_{j+1}$ and $Y_j \to Y_{j+1}$ are defined by the following pushouts:

$$\begin{tikzcd} \coprod\limits_{s \in S_j} \sqcap^{k_s}_{i_s, \varepsilon_s} \arrow[r] \arrow[d, hook] & X_j \arrow[d] \\ \coprod\limits_{s \in S_j} \square^{k_s} \arrow[r] & X_{j+1} \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \coprod\limits_{t \in T_j} \sqcap^{k_t}_{i_t, \varepsilon_t} \arrow[r] \arrow[d, hook] & Y_j \arrow[d] \\ \coprod\limits_{t \in T_j} \square^{k_t} \arrow[r] & Y_{j+1} \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

where

$$S_j = \{\sqcap^k_{i,\varepsilon} \to X_j \ \mid \ k > 0,\ i = 1, \ldots, k,\ \varepsilon = 0, 1\}$$

and

$$T_j = \{\sqcap^k_{i,\varepsilon} \to Y_j \ \mid \ k > 0,\ i = 1, \ldots, k,\ \varepsilon = 0, 1\}.$$

Thus, $\widetilde{X} := \operatorname{colim} X_j$ and $\widetilde{Y} := \operatorname{colim} Y_j$ are the functorial fibrant replacements of X and Y respectively, in the Grothendieck model structure on cSet.

We will show that each of the maps $f_j \colon X_j \to Y_j$ is a bijection on cubes of dimensions $\leq n+1$. From this, it will follow that the induced map $\widetilde{f} \colon \widetilde{X} \to \widetilde{Y}$ is also a bijection on the cubes of dimensions $\leq n+1$. Since $\widetilde{X}$ and $\widetilde{Y}$ are Kan complexes, it follows that $\widetilde{f}$ induces an isomorphism on the homotopy groups in degrees $\leq n$ (see [CK23, Cor. 3.16]).

We proceed by induction on j. The base case ($j = 0$) is true by assumption. For the induction step, we suppose that $f_j \colon X_j \to Y_j$ is a bijection on cubes of dimensions $\leq n+1$.

Consider the sets

$$S_{j,\leq n+1} = \{\sqcap^k_{i,\varepsilon} \to X_j \ \mid \ 0 < k \leq n+1,\ i = 1, \ldots, k,\ \varepsilon = 0, 1\},$$

$$S_{j,\geq n+2} = \{\sqcap^k_{i,\varepsilon} \to Y_j \ \mid \ k \geq n+2,\ i = 1, \ldots, k,\ \varepsilon = 0, 1\},$$

$$T_{j,\leq n+1} = \{\sqcap^k_{i,\varepsilon} \to X_j \ \mid \ 0 < k \leq n+1,\ i = 1, \ldots, k,\ \varepsilon = 0, 1\}, \text{ and}$$

$$T_{j,\geq n+2} = \{\sqcap^k_{i,\varepsilon} \to Y_j \ \mid \ k \geq n+2,\ i = 1, \ldots, k,\ \varepsilon = 0, 1\}.$$

Then, we have

$$S_j = S_{j,\leq n+1} \sqcup S_{j,\geq n+2} \qquad \text{and} \qquad T_j = T_{j,\leq n+1} \sqcup T_{j,\geq n+2}.$$

Furthermore, since $f_j \colon X_j \to Y_j$ is a bijection on cubes of dimensions $\leq n+1$, we also have a bijection

$$S_{j,\leq n+1} \cong T_{j,\leq n+1}.$$

Thus, any k-cubes that are glued to X_j to obtain X_{j+1} are also glued to Y_j to obtain Y_{j+1} (and vice-versa), for all $k \leq n+1$. Thus, the induced map $f_{j+1} \colon X_{j+1} \to Y_{j+1}$ is also a bijection on cubes of dimensions $\leq n+1$, completing the induction step. □

Theorem 4.3.7. *A cubical map $f \colon X \to Y$ is an n-equivalence if and only if it induces a bijection $f_* \colon \pi_0 X \to \pi_0 Y$ and an isomorphism $f_* \colon \pi_k(X, x) \to \pi_k(Y, fx)$ for every $0 < k \leq n$ and every 0-cube $x \in X_0$.*

Proof. We begin by fixing some notation. Consider the following sequence:

$$X_0 \xrightarrow{g_0} X_1 \xrightarrow{g_1} X_2 \xrightarrow{g_2} \cdots$$

where $X_0 = X$ and the maps $g_j \colon X_j \to X_{j+1}$ are defined by the following pushouts:

$$\begin{tikzcd}
\coprod\limits_{s \in S_j} \sqcap^{k_s}_{i_s,\varepsilon_s} \arrow[r] \arrow[d, hook] & X_j \arrow[d, "g_j"] \\
\coprod\limits_{s \in S_j} \square^{k_s} \arrow[r] & X_{j+1} \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

where

$$S_j = \{\sqcap^k_{i,\varepsilon} \to X_j \ \mid \ k > 0,\ i = 1, \ldots, k,\ \varepsilon = 0, 1\}$$

if j is even, and

$$\begin{tikzcd}
\coprod\limits_{t \in T_j} \partial\square^{k_t} \arrow[r] \arrow[d, hook] & X_j \arrow[d, "g_j"] \\
\coprod\limits_{t \in T_j} \square^{k_t} \arrow[r] & X_{j+1} \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

where

$$T_j = \{\partial\square^k \to X_j \ \mid \ k \geq n+2\}$$

if j is odd.

Let $\widetilde{X} := \operatorname{colim} X_j$ and let $g \colon X \to \widetilde{X}$ be the induced map. Then, $\widetilde{X}$ is n-fibrant and the map $g \colon X \to \widetilde{X}$ is n-anodyne. Furthermore, each $g_j \colon X_j \to X_{j+1}$ induces an isomorphism on the homotopy groups in degrees $\leq n$. Indeed, if j is even, then g_j is anodyne and in particular, a weak homotopy equivalence. Thus, in this case, it induces an isomorphism on all homotopy groups by Corollary 4.2.18. If j is odd, then g_j is a bijection on cubes of dimensions $\leq n+1$. Thus, in this case, it induces an isomorphisms on the homotopy groups in degrees $\leq n$ by Lemma 4.3.6. It follows that $g \colon X \to \widetilde{X}$ also induces isomorphisms on the homotopy groups in degrees $\leq n$.

Applying the same construction to Y instead of X, we obtain an n-fibrant cubical set $\widetilde{Y}$ along with a map $h\colon Y \to \widetilde{Y}$ that is n-anodyne and that induces isomorphisms on the homotopy groups in degrees $\leq n$. We also obtain an induced map $\widetilde{f}\colon \widetilde{X} \to \widetilde{Y}$ that makes the following square commute:

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "f"'] & \widetilde{X} \arrow[d, "\widetilde{f}"] \\ Y \arrow[r, "h"] & \widetilde{Y} \end{tikzcd}$$

Since g and h are both n-anodyne maps and, in particular, n-equivalences (see Lemma 4.3.2), and since the class of n-equivalences satisfies the 2-out-of-3 property, f is an n-equivalence if and only if $\widetilde{f}$ is an n-equivalence. Since $\widetilde{X}$ and $\widetilde{Y}$ are both n-fibrant, by Proposition 4.3.4, $\widetilde{f}$ is an n-equivalence if and only if it is a homotopy equivalence. By Theorem 4.2.17, $\widetilde{f}$ is a homotopy equivalence if and only if it induces isomorphisms on all homotopy groups. Since $g\colon X \to \widetilde{X}$ and $h\colon Y \to \widetilde{Y}$ both induce isomorphisms on all homotopy groups in degrees $\leq n$, and since $\widetilde{X}$ and $\widetilde{Y}$ have trivial homotopy groups in degrees above n (see Proposition 4.3.3), $\widetilde{f}$ induces isomorphisms on all homotopy groups if and only if f induces isomorphisms on all homotopy groups in degrees $\leq n$. $\square$

4.3.2 The transferred model structure

We now construct a second model structure on cSet where the weak equivalences are n-equivalences, using the transfer theorem. Let us begin with some identities regarding the $(n+1)$-skeleton functor $\mathrm{sk}_{n+1}\colon \mathsf{cSet} \to \mathsf{cSet}$ applied to boundary and open box inclusions.

Lemma 4.3.8. *We have the following identities:*

$$\mathrm{sk}_{n+1}(\partial\Box^k \hookrightarrow \Box^k) = \begin{cases} \partial\Box^k \hookrightarrow \Box^k & \text{if } k \leq n+1 \\ \mathrm{id}_{\mathrm{sk}_{n+1}\Box^k} & \text{if } k \geq n+2 \end{cases}$$

and

$$\mathrm{sk}_{n+1}(\sqcap^k_{i,\varepsilon} \hookrightarrow \Box^k) = \begin{cases} \sqcap^k_{i,\varepsilon} \hookrightarrow \Box^k & \text{if } k \leq n+1 \\ \sqcap^{n+1}_{i,\varepsilon} \hookrightarrow \partial\Box^{n+1} & \text{if } k = n+2 \\ \mathrm{id}_{\mathrm{sk}_{n+1}\Box^k} & \text{if } k \geq n+3. \end{cases}$$

Proof. This follows from Theorem 4.1.24 and Proposition 4.2.3. □

We want to transfer the model structure constructed in Theorem 4.3.5 via the $(n+1)$-coskeleton functor $\mathrm{cosk}_{n+1} \colon \mathsf{cSet} \to \mathsf{cSet}$. Thus, we will be dealing with the following classes of maps:

Definition 4.3.9 (cf. Definition 4.1.34)**.**

1. A cubical map $f \colon X \to Y$ is a *transferred naive n-fibration* if $\mathrm{cosk}_{n+1} f \colon \mathrm{cosk}_{n+1} X \to \mathrm{cosk}_{n+1} Y$ is a naive n-fibration. A cubical set X is *transferred n-fibrant* if the unique map $X \to \Box^0$ is a transferred naive n-fibration, or equivalently if $\mathrm{cosk}_{n+1} X$ is n-fibrant.

2. A cubical map $f \colon X \to Y$ is a *transferred n-fibration* if $\mathrm{cosk}_{n+1} f \colon \mathrm{cosk}_{n+1} X \to \mathrm{cosk}_{n+1} Y$ is an n-fibration.

3. A cubical map $f \colon X \to Y$ is a *transferred n-cofibration* if it has the left lifting property with respect to every map that is both a transferred n-fibration and an n-equivalence.

Our next goal is to prove the following theorem:

Theorem 4.3.10. *The category* cSet *admits a cofibrantly generated model structure where:*

- *the cofibrations are the transferred* n*-cofibrations,*
- *the fibrations are the transferred* n*-fibrations, and*
- *the weak equivalences are the* n*-equivalences.*

We refer to this model structure as the transferred model structure for n*-types of cubical sets, and denote it by* $\mathsf{cSet}_{n\text{-types}}$*. It is right-transferred from the Cisinski model structure for* n*-types of cubical sets along the* $(n+1)$*-coskeleton functor* $\operatorname{cosk}_{n+1} \colon \mathsf{cSet} \to \mathsf{cSet}$*.*

At first glance, it seems as though we did not transfer the class of n-equivalences. The following couple of results tell us that the class of n-equivalences remains unchanged upon transferring along the $(n+1)$-coskeleton functor.

Lemma 4.3.11. *For every cubical set* X*, the canonical map* $\eta_X \colon X \to \operatorname{cosk}_{n+1}X$ *is an* n*-equivalence.*

Proof. The map $\eta_X \colon X \to \operatorname{cosk}_{n+1}X$ is a bijection on the k-cubes for each $0 \leq k \leq n+1$. The required result then follows from Lemma 4.3.6 and Theorem 4.3.7. □

Proposition 4.3.12. *A cubical map* $f \colon X \to Y$ *is an* n*-equivalence if and only if* $\operatorname{cosk}_{n+1}f \colon \operatorname{cosk}_{n+1}X \to \operatorname{cosk}_{n+1}Y$ *is an* n*-equivalence.*

Proof. Consider the naturality square:

$$\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[d, "f"'] & \operatorname{cosk}_{n+1}X \arrow[d, "\operatorname{cosk}_{n+1}f"] \\
Y \arrow[r, "\eta_Y"] & \operatorname{cosk}_{n+1}Y
\end{tikzcd}$$

By Lemma 4.3.11, both the horizontal maps in the above diagram are n-equivalences. Since the class of n-equivalences satisfies the 2-out-of-3 property, we have the required result. $\square$

Next, we characterize the transferred *naive* n-fibrations in terms of their lifting properties.

Proposition 4.3.13. *A cubical map $f\colon X \to Y$ is a transferred naive n-fibration if and only if it has the right lifting property with respect to the set*

$$\begin{aligned} J'_n = &\left\{\sqcap^k_{i,\varepsilon} \hookrightarrow \square^k \;\mid\; 0 < k \le n+1,\ i = 1, \dots, k,\ \varepsilon = 0, 1\right\} \\ &\cup \left\{\sqcap^{n+2}_{i,\varepsilon} \hookrightarrow \partial\square^{n+2} \;\mid\; i = 1, \dots, k,\ \varepsilon = 0, 1\right\}. \end{aligned}$$

Proof. Observe that given two cubical maps $i\colon A \to B$ and $f\colon X \to Y$, f has the right lifting property with respect to $\mathrm{sk}_{n+1} i$ if and only if $\mathrm{cosk}_{n+1} f$ has the right lifting property with respect to i. The required result then follows from Lemma 4.3.8. $\square$

Corollary 4.3.14. *Every Kan fibration is, in particular, a transferred naive n-fibration. Every Kan complex is a transferred n-fibrant cubical set.* $\square$

In general, we do not have a characterization of transferred n-fibrations in terms of their lifting properties. The exception is when the domain and codomain are transferred n-fibrant cubical sets.

Proposition 4.3.15. *A cubical map $f\colon X \to Y$ between two transferred n-fibrant cubical sets X and Y is a transferred n-fibration if and only if it is a transferred naive n-fibration.*

Proof. Since $\mathrm{cosk}_{n+1} X$ and $\mathrm{cosk}_{n+1} Y$ are n-fibrant, the map $\mathrm{cosk}_{n+1} f\colon \mathrm{cosk}_{n+1} X \to$

$\mathrm{cosk}_{n+1}Y$ is an n-fibration if and only if it is a naive n-fibration (see Theorem 4.3.5). $\square$

We also have the following characterization of maps that are both a transferred n-fibration and an n-equivalence.

Proposition 4.3.16. *A cubical map $f\colon X \to Y$ is both a transferred n-fibration and an n-equivalence if and only if it has right lifting property with respect to the set*

$$I'_n = \left\{ \partial\square^k \hookrightarrow \square^k \;\mid\; 0 \leq k \leq n+1 \right\}.$$

Equivalently, a cubical map $i\colon A \to B$ is a transferred n-cofibration if and only if it belongs to the saturation of I'_n.

Proof. By Proposition 4.3.12, a map $f\colon X \to Y$ is both a transferred n-fibration and an n-equivalence if and only if $\mathrm{cosk}_{n+1}f\colon \mathrm{cosk}_{n+1}X \to \mathrm{cosk}_{n+1}Y$ is both an n-fibration and an n-equivalence. By Theorem 4.3.5 and Lemma 4.2.8, we know that $\mathrm{cosk}_{n+1}f\colon \mathrm{cosk}_{n+1}X \to \mathrm{cosk}_{n+1}Y$ is both an n-fibration and an n-equivalence if and only if it has right lifting property with respect to the set

$$I = \{\partial\square^n \hookrightarrow \square^n \;\mid\; n \geq 0\}$$

of all boundary inclusions. Since $\mathrm{cosk}_{n+1}f$ has the right lifting property with respect to a cubical map $i\colon A \to B$ if and only if f has the right lifting property with respect to $\mathrm{sk}_{n+1}i$, the required result then follows from Lemma 4.3.8. $\square$

We will need the following lemma in order to verify the acyclicity condition 4.1.35(A).

Lemma 4.3.17. *For every cubical set X, the canonical map $\varepsilon_X \colon \mathrm{sk}_{n+1}X \to X$ is n-anodyne.*

Proof. The map $\varepsilon_X \colon \mathrm{sk}_{n+1}X \to X$ is a transfinite composite of the following sequence:

$$\mathrm{sk}_{n+1}X \longrightarrow \mathrm{sk}_{n+2}X \longrightarrow \mathrm{sk}_{n+3}X \longrightarrow \cdots$$

and each map in this sequence is n-anodyne (see Theorem 4.1.24). □

Proposition 4.3.18. *Let $i \colon A \to B$ be a cubical map with left lifting property with respect to all transferred n-fibrations. Then, i is an n-equivalence.*

Proof. From the definitions, it follows that a cubical map $f \colon X \to Y$ is a transferred n-fibration if and only it has right lifting property against $\mathrm{sk}_{n+1}j$ for every map $j \colon C \to D$ that is both a monomorphism and an n-equivalence. Consider the naturality square:

$$\begin{tikzcd}
\mathrm{sk}_{n+1}C \arrow[r, "\varepsilon_C"] \arrow[d, "\mathrm{sk}_{n+1}j"'] & C \arrow[d, "j"] \\
\mathrm{sk}_{n+1}D \arrow[r, "\varepsilon_D"] & D
\end{tikzcd}$$

By Lemma 4.3.17, both the horizontal maps in the above diagram are n-anodyne, and hence, both monomorphisms and n-equivalences. Thus, if j is a monomorphism and an n-equivalence, so is $\mathrm{sk}_{n+1}j$. Since the class of maps that are both monomorphisms and n-equivalences is saturated (see Theorem 4.3.5), it follows that any map $i \colon A \to B$ that has the left lifting property with respect to all transferred n-fibrations is both a monomorphism and an n-equivalence. □

Proof of Theorem 4.3.10. Proposition 4.3.18 allows us to apply Theorem 4.1.35 to right-transfer the Cisinski model structure for n-types of cubical sets (see Theorem 4.3.5) along the adjunction $\mathrm{sk}_{n+1} \dashv \mathrm{cosk}_{n+1}$. □

Finally, we prove that this model structure is Quillen equivalent to the model structure constructed in Theorem 4.3.5 via the skeleton-coskeleton adjunction.

Theorem 4.3.19. *The adjunction*

$$\begin{tikzcd} \mathrm{L}_n\mathsf{cSet} \arrow[r, shift left=2, "\mathrm{sk}_{n+1}"] \arrow[r, phantom, "\perp"] & \mathsf{cSet}_{n\text{-types}} \arrow[l, shift left=2, "\mathrm{cosk}_{n+1}"] \end{tikzcd}$$

is a Quillen equivalence between the Cisinski and the transferred model structures for n-types of cubical sets.

Proof. From Theorem 4.3.10, we know that the adjunction is a Quillen adjunction between these model structures. Furthermore, by Proposition 4.3.12, we also know that cosk_{n+1} preserves and reflects n-equivalences. Thus, it suffices to prove that for every cubical set X, the unit $X \to \mathrm{cosk}_{n+1}\mathrm{sk}_{n+1}X$ of the adjunction $\mathrm{sk}_{n+1} \dashv \mathrm{cosk}_{n+1}$ is an n-equivalence. Observe that the functor $i^*_{n+1} \circ i_{n+1*} \colon \mathsf{cSet}_{\leq n+1} \to \mathsf{cSet}_{\leq n+1}$ equals the identity functor on $\mathsf{cSet}_{\leq n+1}$. It follows that

$$\mathrm{cosk}_{n+1} \circ \mathrm{sk}_{n+1} = i_{n+1!} \circ i^*_{n+1} \circ i_{n+1*} \circ i^*_{n+1} = i_{n+1!} \circ i^*_{n+1} = \mathrm{cosk}_{n+1}.$$

The required result then follows from Lemma 4.3.11. □

4.3.3 Equivalence with simplicial models

In the previous two sections, we constructed two distinct model structures on the category of cubical sets where the weak equivalences were n-equivalences. Both of these models have simplicial analogues that have been studied previously. We begin by describing the simplicial analogue to the model structure of Theorem 4.3.5.

Definition 4.3.20.

1. A simplicial map $f\colon X \to Y$ is a *naive n-fibration* if it has the right lifting property with respect to the set
$$\{\Lambda^n_i \hookrightarrow \Delta^n \mid n > 0,\ 0 \leq i \leq n\} \cup \left\{\partial\Delta^k \hookrightarrow \Delta^k \mid k \geq n+2\right\}.$$
A simplicial set X is *n-fibrant* if the unique map $X \to \Delta^0$ is a naive n-fibration.

2. A simplicial map $f\colon X \to Y$ is an *n-equivalence* if it induces a bijection $f_*\colon \pi_0 X \to \pi_0 Y$ and an isomorphism $f_*\colon \pi_k(X, x) \to \pi_k(Y, fx)$ for every $0 < k \leq n$ and every 0-simplex $x \in X_0$.

3. A simplicial map $f\colon X \to Y$ is an *n-fibration* if it has the right lifting property with respect to every map that is both a monomorphism and an n-equivalence.

Theorem 4.3.21 ([Cis06, § 9.2])**.** *The category* sSet *admits a cofibrantly generated model structure where:*

- *the cofibrations are the monomorphisms,*

- *the fibrant objects are the n-fibrant simplicial sets, and*

- *the weak equivalences are the n-equivalences.*

Moreover a simplicial map $f\colon X \to Y$ between n-fibrant simplicial sets X and Y is an n-fibration if and only if it is a naive n-fibration. We refer to this model structure as the Cisinski model structure for n-types of simplicial sets, and denote it by $\mathrm{L}_n\mathsf{sSet}$. It is the left Bousfield localization of the Quillen model structure on sSet at the class of n-equivalences. □

Next, we describe the simplicial analogue to the model structure of Theorem 4.3.10.

Theorem 4.3.22 ([EDHP95, Thm. 2.3])**.** *The category sSet admits a cofibrantly generated model structure where:*

- *the cofibrations are the maps belonging to the saturation of the set*

$$\left\{\partial\Delta^k \hookrightarrow \Delta^k \;\mid\; 0 \leq k \leq n+1\right\},$$

- *the fibrations are the maps with right lifting property with respect to the set*

$$\left\{\Lambda^k_i \hookrightarrow \Delta^k \;\mid\; 0 < k \leq n+1,\ 0 \leq i \leq k\right\}$$
$$\cup \left\{\Lambda^{n+2}_i \hookrightarrow \partial\Delta^{n+2} \;\mid\; 0 \leq i \leq n+2\right\},$$

  and

- *the weak equivalences are n-equivalences.*

We refer to this model structure as the Elvira-Hernandez model structure for n-types of simplicial sets and denote it by $\mathsf{sSet}_{n\text{-types}}$. It is right-transferred from *the Cisinski model structure for n-types of simplicial sets along the $(n+1)$-coskeleton functor* $\operatorname{cosk}_{n+1} \colon \mathsf{sSet} \to \mathsf{sSet}$.

Proof. This model is constructed directly in [EDHP95]. We observe that the cofibrations and the weak equivalences in their model structure coincide with those of the model structure obtained by right-transferring $\mathrm{L}_n\mathsf{sSet}$ along $\operatorname{cosk}_{n+1} \colon \mathsf{sSet} \to \mathsf{sSet}$ (cf. Proposition 4.3.12 and Proposition 4.3.16). □

We also have a simplicial analogue of Theorem 4.3.19.

Theorem 4.3.23. *The adjunction*

$$\begin{tikzcd} \mathrm{L}_n\mathsf{sSet} \arrow[r, shift left=2, "\mathrm{sk}_{n+1}"] \arrow[r, phantom, "\perp"] & \mathsf{sSet}_{n\text{-types}} \arrow[l, shift left=2, "\operatorname{cosk}_{n+1}"] \end{tikzcd}$$

is a Quillen equivalence between the Cisinski and the Elvira-Hernandez model structures for n-types of simplicial sets.

Proof. Analogous to the proof of Theorem 4.3.19. □

Finally, we compare the cubical models with their simplicial analogues.

Proposition 4.3.24. *A cubical map $f \colon X \to Y$ is an n-equivalence if and only if the simplicial map $Tf \colon TX \to TY$ is an n-equivalence.*

Proof. This follows from Proposition 4.2.16 and Theorem 4.3.7. □

Theorem 4.3.25. *The adjunction*

$$\begin{tikzcd} \mathrm{L}_n\mathsf{cSet} \arrow[r, shift left=2, "T"] \arrow[r, phantom, "\perp"] & \mathrm{L}_n\mathsf{sSet} \arrow[l, shift left=2, "U"] \end{tikzcd}$$

is a Quillen equivalence between the Cisinski model structures for n-types of cubical sets and of simplicial sets.

Proof. We know that the adjunction $T \dashv U$ is a Quillen equivalence between the Grothendieck model structure on cSet and the Quillen model structure on sSet (Theorem 4.2.12). The Cisinski model structure for n-types of cubical sets is obtained by taking the left Bousfield localization of the Grothendieck model structure on cSet at the class of n-equivalences, whereas the Cisinski model structure for n-types of simplicial sets is obtained by taking the left Bousfield localization of the Quillen model structure on sSet. The required result then follows from Proposition 4.3.24 and [Hir03, Thm. 3.3.20]. $\square$

Corollary 4.3.26. *We have the following zigzag of Quillen equivalences:*

$$\begin{tikzcd} \mathsf{cSet}_{n\text{-types}} \arrow[r, shift right, "\mathrm{cosk}_{n+1}"'] \arrow[r, phantom, "\perp"] & \mathrm{L}_n\mathsf{cSet} \arrow[l, shift right, "\mathrm{sk}_{n+1}"'] \arrow[r, shift left, "T"] \arrow[r, phantom, "\perp"] & \mathrm{L}_n\mathsf{sSet} \arrow[l, shift left, "U"] \arrow[r, shift left, "\mathrm{sk}_{n+1}"] \arrow[r, phantom, "\perp"] & \mathsf{sSet}_{n\text{-types}}. \arrow[l, shift left, "\mathrm{cosk}_{n+1}"] \end{tikzcd}$$

Proof. This follows from collecting the Quillen equivalences of Theorems 4.3.19, 4.3.25 and 4.3.23. $\square$

Chapter 5

Fibration Categories of Graphs

In this chapter, we return to discrete homotopy theory and lay the foundation for the study of graphs up to n-equivalences (see Definition 2.5.9). Specifically, we construct fibration category structures on Graph where the weak equivalences are the n-equivalences. Our approach follows that of [CK22], where the authors construct a fibration category structure on Graph where the weak equivalences are the weak A-homotopy equivalences (see Definition 2.5.8) and the fibrations are graph maps that get sent to Kan fibrations in cSet via the cubical nerve functor.

First, in Section 5.1, we review some general theory of fibration categories. We recall the definition of a fibration category, originally due to Brown [Bro73], which generalizes the notion of a model category. The standard example of a fibration category is given by the full subcategory of fibrant objects in a model category, and we list some examples arising from the model structures on cSet constructed in Chapter 4. We then review the construction of the homotopy category and the notion of a morphism between fibration categories. The exposition in this section closely follows that of Szumiło [Szu16, Szu17].

In Section 5.2, we review the definition of the cubical nerve of a graph as well as some important results from [CK22], including the fibration category structure on Graph where the weak equivalences are the weak A-homotopy equivalences.

Finally, in Section 5.3, we construct the fibration category of Graph where the weak equivalences are the n-equivalences. We conclude the chapter by proving that, for $n = 1$, this fibration category of graphs is weakly equivalent to the classical fibration category of groupoids via the fundamental groupoid functor.

5.1 Background on fibration categories

In a model category, the class of cofibrations is completely determined by the fibrations and the weak equivalences. This should lead us to think that it is possible to replicate much of model category theory with just "half a model structure". This idea is formalized in the notion of a fibration category.

Definition 5.1.1 (Brown, [Szu16, Def. 1.1])**.** A *fibration category* is a category $\mathcal{C}$ equipped with two subcategories: a subcategory of *weak equivalences* (denoted by $\xrightarrow{\sim}$) and a subcategory of *fibrations* (denoted by $\twoheadrightarrow$), subject to the following axioms. (Here, an *acyclic fibration* is a morphism that is both a weak equivalence and a fibration.)

(F1) Weak equivalences satisfy the 2-out-of-6 property. That is, given three composable morphisms in $\mathcal{C}$ as follows:

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$$

if the composites gf and hg are weak equivalences, then so are f, g, and

h (and hence, also hgf);

(F2) Every isomorphism in $\mathcal{C}$ is an acyclic fibration;

(F3) The category $\mathcal{C}$ has a terminal object 1, and every object $X \in \mathcal{C}$ is fibrant. That is, the unique morphism $X \to 1$ is a fibration;

(F4) Fibrations and acyclic fibrations are stable under pullbacks along arbitrary morphisms in $\mathcal{C}$ (in particular, these pullbacks exist in $\mathcal{C}$);

(F5) Every morphism in $\mathcal{C}$ can be factored as a weak equivalence followed by a fibration.

Example 5.1.2. Given any model category $\mathcal{M}$, the full subcategory $\mathcal{M}^{\mathrm{fib}}$ of fibrant objects in $\mathcal{M}$ forms a fibration category with weak equivalences and fibrations inherited from $\mathcal{M}$.

Example 5.1.3 (cf. Theorem 4.2.11)**.** The category Kan of Kan complexes admits a fibration category structure where the fibrations are the Kan fibrations and the weak equivalences are the weak homotopy equivalences.

Example 5.1.4 (cf. Theorem 4.3.5)**.** The category $(\mathrm{L}_n\mathsf{cSet}_{(\infty,0)})^{\mathrm{fib}}$ of n-fibrant cubical sets admits a fibration category structure where the fibrations are the (naive) n-fibrations and the weak equivalences are the n-equivalences.

Example 5.1.5 (cf. Theorem 4.3.10)**.** The category $\mathsf{cSet}^{\mathrm{fib}}_{n\text{-types}}$ of transferred n-fibrant cubical sets admits a fibration category structure where the fibrations are the transferred (naive) n-fibrations and the weak equivalences are the n-equivalences.

Remark 5.1.6. By Proposition 4.3.3 and Corollary 4.3.14, we have the following chain of inclusions of categories:

$$(\mathrm{L}_n \mathsf{cSet}_{(\infty,0)})^{\mathrm{fib}} \subseteq \mathsf{Kan} \subseteq \mathsf{cSet}^{\mathrm{fib}}_{n\text{-types}} \subseteq \mathsf{cSet}.$$

As with model categories, we would like to study the homotopy theory of objects that are well-behaved. In a model category, the well-behaved objects are those that are both fibrant and cofibrant. In a fibration category, every object is fibrant. But since we do not have access to cofibrations, we do not have cofibrant objects. We remedy this by considering all possible "replacements" of the object that we would have liked to be cofibrant. For instance, in the following definition, a *right homotopy* is not given by a map $X \to PY$, but by a zig-zag $X \overset{\sim}{\twoheadleftarrow} W \to PY$.

Definition 5.1.7 ([Szu17, Def. 1.2]). Let $\mathcal{C}$ be a fibration category.

1. A *path object* of an object $Y \in \mathcal{C}$ is a factorization $Y \xrightarrow{\sim} PY \twoheadrightarrow Y \times Y$ of the diagonal map $\Delta \colon Y \to Y \times Y$ as a weak equivalence followed by a fibration.

2. A *right homotopy* between two morphisms $f, g \colon X \to Y$ in $\mathcal{C}$ via a path object $Y \xrightarrow{\sim} PY \twoheadrightarrow Y \times Y$ is a commutative square of the form

$$\begin{tikzcd} W \arrow[r, "H"] \arrow[d, two heads, "\sim"'] & PY \arrow[d, two heads] \\ X \arrow[r, "{[f,g]}"] & Y \times Y \end{tikzcd}$$

   If such a right homotopy exists, we say that f and g are *right homotopic* and we write $f \sim_r g$.

The following result gives a convenient description of the category $\mathrm{Ho}\,\mathcal{C}$

obtained by formally inverting the weak equivalences in a fibration category $\mathcal{C}$.

Theorem 5.1.8 ([Szu17, Thm. 1.4])**.** *Let $\mathcal{C}$ be a fibration category.*

1. *Every morphism $\varphi \in \operatorname{Ho}\mathcal{C}(X, Y)$ can be expressed as a right fraction $fs^{-1} = \left(X \xleftarrow{\sim} \widetilde{X} \to Y\right)$, where $s\colon \widetilde{X} \xrightarrow{\sim} X$ is a weak equivalence in $\mathcal{C}$ and $f\colon \widetilde{X} \to Y$ is a morphism in $\mathcal{C}$.*

2. *Two right fractions fs^{-1} and gt^{-1} are equal in $\operatorname{Ho}\mathcal{C}(X, Y)$ if and only if there exist weak equivalences u and v such that $su \sim_r tv$ and $fu \sim_r gv$.*

3. *If $\varphi \in \operatorname{Ho}\mathcal{C}(X, Y)$ and $\psi \in \operatorname{Ho}\mathcal{C}(Y, Z)$ can be expressed as $fs^{-1} = \left(X \xleftarrow{\sim} \widetilde{X} \to Y\right)$ and $gt^{-1} = \left(Y \xleftarrow{\sim} \tilde{Y} \to Z\right)$ respectively, and if there exists a right fraction $hu^{-1} = \left(\widetilde{X} \xleftarrow{\sim} W \to \tilde{Y}\right)$ such that $fu \sim_r th$, then the composite $\psi\varphi \in \operatorname{Ho}\mathcal{C}(X, Z)$ can be expressed as $(gh)(su)^{-1}$.* $\square$

The appropriate notion of a morphism between two fibration categories is that of an exact functor.

Definition 5.1.9 ([Szu16, Def. 1.2, Def. 2.1])**.**

1. A functor $F\colon \mathcal{C} \to \mathcal{D}$ between fibration categories is *exact* if it preserves fibrations, acyclic fibrations, pullbacks along fibrations, and the terminal object.

2. An exact functor $F\colon \mathcal{C} \to \mathcal{D}$ between fibration categories is a *weak equivalence* if it induces an equivalence of categories $\operatorname{Ho}\mathcal{C} \to \operatorname{Ho}\mathcal{D}$.

Example 5.1.10. Given a Quillen adjunction $L : \mathcal{C} \rightleftarrows \mathcal{D} : R$ between two model categories, the right Quillen functor $R\colon \mathcal{D} \to \mathcal{C}$ restricts to an exact functor $R\colon \mathcal{D}^{\text{fib}} \to \mathcal{C}^{\text{fib}}$ of fibration categories. Moreover, if $L \dashv R$ is a Quillen equivalence, then $R\colon \mathcal{D}^{\text{fib}} \to \mathcal{C}^{\text{fib}}$ is a weak equivalence of fibration categories.

Example 5.1.11 (cf. Corollary 4.3.26)**.** We have the following zig-zag of weak equivalences of fibration categories:

$$\mathsf{cSet}^{\mathrm{fib}}_{n\text{-types}} \xrightarrow{\mathrm{cosk}_{n+1}} (\mathrm{L}_n\mathsf{cSet}_{(\infty,0)})^{\mathrm{fib}} \xleftarrow{U} (\mathrm{L}_n\mathsf{sSet}_{(\infty,0)})^{\mathrm{fib}} \xleftarrow{\mathrm{cosk}_{n+1}} \mathsf{sSet}^{\mathrm{fib}}_{n\text{-types}}.$$

We will need the following result on recognizing weak equivalences of fibration categories.

Proposition 5.1.12 ([Cis10, Thm. 3.19])**.** *An exact functor $F \colon \mathcal{C} \to \mathcal{D}$ between fibration categories is a weak equivalence if and only if it satisfies the following approximation properties:*

(A1) *F reflects weak equivalences. That is, a morphism $f \colon X \to Y$ in $\mathcal{C}$ is a weak equivalence in $\mathcal{C}$ if $Ff \colon FX \to FY$ is a weak equivalence in $\mathcal{D}$;*

(A2) *Given any morphism $f \colon B \to FY$ in $\mathcal{D}$, there exists a morphism $\widehat{f} \colon X \to Y$ in $\mathcal{C}$ such that there is a commutative square of the form:*

$$\begin{tikzcd} A \arrow[r, "\sim"] \arrow[d, "\sim"'] & FX \arrow[d, "F\widehat{f}"] \\ B \arrow[r, "f"] & FY \end{tikzcd}$$

where $A \xrightarrow{\sim} B$ and $A \xrightarrow{\sim} FX$ are weak equivalences in $\mathcal{D}$. □

5.2 The cubical nerve of a graph

In this section, we review the definition of the cubical nerve of a graph, as well as some of its important properties.

Definition 5.2.1.

1. Given a graph X and an integer $n \geq 0$, a graph map $f\colon I_\infty^{\square n} \to X$ is *stable in all directions* if there exists an integer $M \geq 0$ such that for each $i = 1, \ldots, n$ and $\varepsilon = 0, 1$ we have:

$$f(t_1, \ldots, t_{i-1}, (2\varepsilon{-}1)t_i, t_{i+1}, \ldots, t_n) = f(t_1, \ldots, t_{i-1}, (2\varepsilon{-}1)M, t_{i+1}, \ldots, t_n)$$

whenever $t_i \geq M$.

2. The *nerve* $\mathrm{N}X$ of a graph X is a cubical set whose n-cubes are given by:

$$(\mathrm{N}X)_n = \Big\{ f\colon I_\infty^{\square n} \to X \;\mid\; f \text{ is stable in all directions} \Big\}$$

and whose cubical operators are given as follows:

- the map $\partial_{i,\varepsilon}\colon (\mathrm{N}X)_n \to (\mathrm{N}X)_{n-1}$ for $i = 1, \ldots, n$ and $\varepsilon = 0, 1$ is given by

$$f\partial_{i,\varepsilon}(t_1, \ldots, t_{n-1}) = f(t_1, \ldots, t_{i-1}, (2\varepsilon - 1)M, t_i, \ldots, t_{n-1});$$

where $M \geq 0$ is some integer such that for each $i = 1, \ldots, n$ and $\varepsilon = 0, 1$ we have:

$$f(t_1, \ldots, t_{i-1}, (2\varepsilon{-}1)t_i, t_{i+1}, \ldots, t_n) = f(t_1, \ldots, t_{i-1}, (2\varepsilon{-}1)M, t_{i+1}, \ldots, t_n)$$

whenever $t_i \geq M$.

- the map $\sigma_i\colon (\mathrm{N}X)_n \to (\mathrm{N}X)_{n+1}$ for $i = 1, \ldots, n+1$ is given by

$$f\sigma_i(t_1, \ldots, t_{n+1}) = f(t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_{n+1});$$

- the map $\gamma_{i,\varepsilon} \colon (\mathrm{N}X)_{n+1} \to (\mathrm{N}X)_n$ for $i = 1, \dots, n$ and $\varepsilon = 0, 1$ is given by

$$f\gamma_{i,\varepsilon}(t_1, \dots, t_{n+1}) = \begin{cases} f(t_1, \dots, t_{i-1}, \max(t_i, t_{i+1}), t_{i+2}, \dots, t_{n+1}) & \text{if } \varepsilon = 0, \\ f(t_1, \dots, t_{i-1}, \min(t_i, t_{i+1}), t_{i+2}, \dots, t_{n+1}) & \text{if } \varepsilon = 1. \end{cases}$$

This defines the *nerve functor* $\mathrm{N} \colon \mathsf{Graph} \to \mathsf{cSet}$.

Theorem 5.2.2.

1. *The nerve functor* $\mathrm{N} \colon \mathsf{Graph} \to \mathsf{cSet}$ *preserves all finite limits.*
2. *The nerve* $\mathrm{N}X$ *of any graph* X *is a Kan complex.*
3. *We have a bijection* $\pi_0 X \cong \pi_0 \mathrm{N}X$ *for every* $X \in \mathsf{Graph}$ *and an isomorphism* $A_n(X, x) \cong \pi_n(\mathrm{N}X, x)$ *for every* $(X, x) \in \mathsf{Graph}_*$ *and every* $n > 0$.

Proof. (1) is [CK22, Prop. 3.8]. (2) is [CK22, Thm. 4.5]. (3) is [CK22, Thm. 4.6] □

The nerve functor allows us to induce a fibration category structure on Graph from the Grothendieck model structure on cSet.

Theorem 5.2.3 ([CK22, Thm. 5.9, Prop. 5.12])**.** *The category* Graph *admits a fibration category structure where*

- *the fibrations are maps which are sent to Kan fibrations under the nerve functor* $\mathrm{N} \colon \mathsf{Graph} \to \mathsf{Kan}$*, and*
- *the weak equivalences are the weak A-homotopy equivalences.*

Given this structure on Graph*, the nerve functor* $\mathrm{N} \colon \mathsf{Graph} \to \mathsf{Kan}$ *is an exact functor of fibration categories.* □

5.3 Homotopy n-types of graphs

Let us fix an integer $n \geq 0$. Recall that $\mathcal{W}_n$ denotes the class of n-equivalences in Graph. We would like to construct a fibration category structure on Graph where the class of weak equivalences is precisely $\mathcal{W}_n$.

Note that the nerve functor $\mathrm{N}\colon \mathsf{Graph} \to \mathsf{cSet}$ does not take values in the category $(\mathrm{L}_n\mathsf{cSet}_{(\infty,0)})^{\mathrm{fib}}$ of n-fibrant cubical sets. One possible way of dealing with this could be to restrict the domain to the full subcategory of Graph on graphs X whose nerve $\mathrm{N}X$ is an n-fibrant cubical set. In other words, we could consider the full subcategory of Graph on graphs X for which we have $A_k(X, x) \cong 0$ for every $k \geq n+1$ and every vertex $x \in X$. Unfortunately, this category is currently not well understood, and even the existence of such graphs is not clear except in some very special cases.

On the other hand, the nerve of every graph is a transferred n-fibrant cubical set, and we may view the nerve functor as taking values in the catgeory $\mathsf{cSet}^{\mathrm{fib}}_{n\text{-types}}$ transferred n-fibrant cubical sets.

Definition 5.3.1. A graph map $f\colon X \to Y$ is an *n-fibration* if its image $\mathrm{N}f\colon \mathrm{N}X \to \mathrm{N}Y$ under the nerve functor $\mathrm{N}\colon \mathsf{Graph} \to \mathsf{cSet}^{\mathrm{fib}}_{n\text{-types}}$ is a transferred (naive) n-fibration. Or equivalently, if $\mathrm{N}f$ has the right lifting property with respect to the set

$$\begin{aligned} J'_n = &\left\{ \sqcap^k_{i,\varepsilon} \hookrightarrow \square^k \;\middle|\; 0 < k \leq n+1,\ i = 1, \ldots, k,\ \varepsilon = 0, 1 \right\} \\ &\cup \left\{ \sqcap^{n+2}_{i,\varepsilon} \hookrightarrow \partial\square^{n+2} \;\middle|\; i = 1, \ldots, k,\ \varepsilon = 0, 1 \right\}. \end{aligned}$$

Theorem 5.3.2. *The category* Graph *admits a fibration category structure where*

- *the fibrations are the n-fibrations, and*

- *the weak equivalences are the n-equivalences.*

We denote this fibration category structure by $\mathsf{Graph}_{n\text{-types}}$. *Given this structure on* Graph*, the nerve functor* $\mathrm{N}\colon \mathsf{Graph}_{n\text{-types}} \to \mathsf{cSet}^{\mathrm{fib}}_{n\text{-types}}$ *is an exact functor of fibration categories.*

Proof. (F1) and (F2) follow from definitions of n-equivalences and n-fibrations. (F3) follows from Theorem 5.2.2 and Corollary 4.3.14. Since Graph is complete (Proposition 2.1.4), all pullbacks exist, and are preserved by the nerve functor (Theorem 5.2.2). Thus, (F4) follows from the fact that the fibrations and the acyclic fibrations in $\mathsf{cSet}^{\mathrm{fib}}_{n\text{-types}}$ are stable under pullbacks. Finally, since every graph map can be factored as a weak A-homotopy equivalence (which is, in particular, an n-equivalence) followed by a map that is sent to a Kan fibration under the nerve functor (which is, in particular, an n-fibration), we have (F5). The nerve functor $\mathrm{N}\colon \mathsf{Graph}_{n\text{-types}} \to \mathsf{cSet}^{\mathrm{fib}}_{n\text{-types}}$ clearly preserves fibrations and acyclic fibrations. By Theorem 5.2.2, it also preserves all pullbacks, as well as the terminal object. □

5.3.1 The case $n = 1$.

We now turn our attention to the case $n = 1$. What can we say about the fibration category $\mathsf{Graph}_{1\text{-types}}$? We begin by reviewing the classical fibration category structure on the category Gpd of groupoids.

Definition 5.3.3. A functor $F\colon \mathcal{C} \to \mathcal{D}$ of groupoids is an *isofibration* if for any object $c \in \mathcal{C}$ and any isomorphism $\psi\colon Fc \to d$ in $\mathcal{D}$, there exists an isomorphism $\varphi\colon c \to c'$ in $\mathcal{C}$ such that $F\varphi = \psi$.

Theorem 5.3.4 ([JT91, Thm. 2])**.** *The category* Gpd *of groupoids admits a fibration category structure where:*

- *the fibrations are the isofibrations, and*
- *the weak equivalences are the equivalences of categories.* □

Theorem 5.3.5. *The fundamental groupoid functor* $\Pi_1 \colon \mathsf{Graph}_{1\text{-types}} \to \mathsf{Gpd}$ *is an exact functor of fibration categories.*

We will prove this theorem in stages. First, we prove that the fundamental groupoid functor $\Pi_1 \colon \mathsf{Graph} \to \mathsf{Gpd}$ preserves fibrations.

Proposition 5.3.6. *If a graph map* $f \colon X \to Y$ *is a 1-fibration, then the functor* $\Pi_1 f \colon \Pi_1 X \to \Pi_1 Y$ *is an isofibration.*

Proof. Consider an object $x \in \Pi_1 X$ and an isomorphism $[\gamma] \colon f(x) \to y$ in $\Pi_1 Y$. We can pick a representative path $\gamma \colon f(x) \rightsquigarrow y$ in Y and consider the following lifting problem:

$$\begin{tikzcd} \sqcap^1_{i,0} \arrow[r, "x"] \arrow[d, hook] & \mathrm{N}X \arrow[d, "\mathrm{N}f"] \\ \square^1 \arrow[r, "\gamma"] & \mathrm{N}Y \end{tikzcd}$$

Since f is a 1-fibration of graphs, this lifting problem has a solution, say $\widetilde{\gamma} \colon \square^1 \to \mathrm{N}X$. That is, there exists a path $\widetilde{\gamma} \colon x \rightsquigarrow x'$ in X such that $f \circ \widetilde{\gamma} = \gamma$. Thus, there exists an isomorphism $[\widetilde{\gamma}] \colon x \to x'$ in $\Pi_1 X$ such that $\Pi_1 f[\widetilde{\gamma}] = [\gamma]$. □

Next, we want to prove that the fundamental groupoid functor $\Pi_1 \colon \mathsf{Graph} \to \mathsf{Gpd}$ preserves pullbacks along 1-fibrations.

Consider an arbitrary pullback square in Graph

$$\begin{tikzcd} X \times_Z Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

We have the following two groupoids:

$$\Pi_1 (X \times_Z Y) \qquad \text{and} \qquad \Pi_1 X \times_{\Pi_1 Z} \Pi_1 Y$$

They have identical objects: pairs (x, y) where x is a vertex in X and y is a vertex in Y subject to the condition $f(x) = g(y)$. However, they might a priori differ in their morphisms.

A morphism $(x, y) \to (x', y')$ in $\Pi_1 (X \times_Z Y)$ is given by the path-homotopy class $[\gamma]$ of a path $\gamma \colon (x, y) \rightsquigarrow (x', y')$ in the pullback graph $X \times_Z Y$, which in turn is determined by a pair (η, τ) where $\eta \colon x \rightsquigarrow x'$ is a path in X and $\eta \colon y \rightsquigarrow y'$ is a path in Y subject to the condition $f \circ \eta = g \circ \tau$ as paths in Z.

On the other hand, a morphism $(x, y) \to (x', y')$ in $\Pi_1 X \times_{\Pi_1 Z} \Pi_1 Y$ is given by a pair $([\eta], [\tau])$ where $[\eta]$ is the path-homotopy class of a path $\eta \colon x \rightsquigarrow x'$ in X and $[\tau]$ is the path-homtopy class of a path $\tau \colon y \rightsquigarrow y'$ in Y, subject to the condition $[f \circ \eta] = [g \circ \tau]$ as morphisms in $\Pi_1 Z$.

By the universal property of pullbacks, we have a canonical functor

$$\Psi \colon \Pi_1 (X \times_Z Y) \to \Pi_1 X \times_{\Pi_1 Z} \Pi_1 Y$$

that maps a morphism $[(\eta, \tau)]$ in $\Pi_1 (X \times_Z Y)$ to a morphism $([\eta], [\tau])$ in $\Pi_1 X \times_{\Pi_1 Z} \Pi_1 Y$. We want to show that this functor is an isomorphism of groupoids under the additional hypothesis that $f \colon X \to Z$ is a 1-fibration of graphs.

We will need the following couple of lemmas.

Lemma 5.3.7. *Consider a pullback square in* Graph

$$\begin{tikzcd} X \times_Z Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

where $f\colon X \to Z$ *is a 1-fibration of graphs. Given any morphism* $([\eta], [\tau])\colon (x, y) \to (x', y')$ *in* $\Pi_1 X \times_{\Pi_1 Z} \Pi_1 Y$*, it is always possible to choose representative paths* η' *and* τ' *from the path-homotopy classes* $[\eta]$ *and* $[\tau]$ *respectively, such that* $f \circ \eta' = g \circ \tau'$ *as paths in* Z*.*

Proof. We start by choosing arbitrary representative paths η and τ from the path-homotopy classes $[\eta]$ and $[\tau]$ respectively. Since the pair $([\eta], [\tau])$ is a morphism in $\Pi_1 X \times_{\Pi_1 Z} \Pi_1 Y$, we have $[f \circ \eta] = [g \circ \tau]$ as morphisms in $\Pi_1 Z$. Thus, we can also choose a path-homotopy $H\colon f \circ \eta \Rightarrow g \circ \tau$ in Z.

Consider the following lifting problem:

$$\begin{tikzcd} \sqcap^1_{1,0} \arrow[r, "x"] \arrow[d, hook] & NX \arrow[d, "Nf"] \\ \square^1 \arrow[r, "g \circ \tau"] & NZ \end{tikzcd}$$

Since $f\colon X \to Z$ is a 1-fibration of graphs, this lifting problem admits a solution $\widetilde{\tau}\colon \square^1 \to NX$. That is, we have a path $\widetilde{\tau}$ in X that starts at x and satisfies $f \circ \widetilde{\tau} = g \circ \tau$. Let $x'' \in X$ be the end-point of $\widetilde{\tau}$.

Next, we consider the following lifting problem:

$$\begin{tikzcd} \sqcap^2_{1,1} \arrow[r, "{[\eta, x, \widetilde{\tau}]}"] \arrow[d, hook] & NX \arrow[d, "Nf"] \\ \square^2 \arrow[r, "H"] & NZ \end{tikzcd}$$

Once again, since $f\colon X \to Z$ is a 1-fibration of graphs, this lifting problem admits a solution $\widetilde{H}\colon \square^2 \to NX$. That is, we have a map $\widetilde{H}\colon I_\infty \,\square\, I_\infty \to X$ that stabilizes in all directions, and satisfies the following conditions:

$$\widetilde{H}\partial_{2,0} = \eta, \quad \widetilde{H}\partial_{1,0} = x\sigma_1^1, \qquad \widetilde{H}\partial_{2,1} = \widetilde{\tau}, \qquad f \circ \widetilde{H} = H$$

Let $\alpha = \widetilde{H}\partial_{1,1}$. Then, α is a path from x' to x'' in X, satisfying $f \circ \alpha = c_{f(x')} = c_{g(y')}$.

Letting $\eta' = \widetilde{\tau} * \overline{\alpha}$ and $\tau' = \tau$, we observe that we have a path-homotopy $\eta \Rightarrow \eta'$ in X, a path-homotopy $\tau \Rightarrow \tau'$ in Y and furthermore, we have:

$$f \circ \eta' = f \circ (\widetilde{\tau} * \overline{\alpha}) = f \circ \widetilde{\tau} * f \circ \overline{\alpha} = g \circ \tau * c_{g(y')} = g \circ \tau = g \circ \tau'. \qquad \square$$

Lemma 5.3.8. *Consider a pullback square in* Graph

$$\begin{tikzcd} X \times_Z Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

where $f\colon X \to Z$ is a 1-fibration of graphs. Suppose we have two paths $\eta, \eta'\colon x \rightsquigarrow x'$ in X and two paths $\tau, \tau'\colon y \rightsquigarrow y'$ in Y, subject to the following conditions: $f \circ \eta = g \circ \tau$ as paths in Z, $f \circ \eta' = g \circ \tau'$ as paths in Z, $[\eta] = [\eta']$ as morphisms in $\Pi_1 X$ and $[\tau] = [\tau']$ as morphisms in $\Pi_1 Y$. Then, we have $[(\eta, \tau)] = [(\eta', \tau')]$ as morphisms in $\Pi_1 (X \times_Z Y)$.

Proof. We start by choosing arbitrary path-homotopies $H\colon \eta \Rightarrow \eta'$ in X and $G\colon \tau \Rightarrow \tau'$ in Y. Let $z = f(x) = g(y)$ and $z' = f(x') = g(y')$. Let $\alpha = f \circ \eta = g \circ \tau$ and $\alpha' = f \circ \eta' = g \circ \tau'$. Then, we can define a map $\Theta\colon \partial\square^3 \to NZ$ as follows:

$$\Theta\mid_{\operatorname{im}\partial^3_{1,0}} = z\sigma^1_1\sigma^2_2 \qquad \Theta\mid_{\operatorname{im}\partial^3_{1,1}} = z'\sigma^1_1\sigma^2_2$$
$$\Theta\mid_{\operatorname{im}\partial^3_{2,0}} = \alpha\sigma^2_2 \qquad \Theta\mid_{\operatorname{im}\partial^3_{2,1}} = \alpha'\sigma^2_2$$
$$\Theta\mid_{\operatorname{im}\partial^3_{3,0}} = f \circ H \qquad \Theta\mid_{\operatorname{im}\partial^3_{3,1}} = g \circ G$$

We can also define a map $\theta \colon \sqcap^3_{3,1} \to NX$ as follows:

$$\theta\mid_{\operatorname{im}\partial^3_{1,0}} = x\sigma^1_1\sigma^2_2 \qquad \theta\mid_{\operatorname{im}\partial^3_{1,1}} = x'\sigma^1_1\sigma^2_2$$
$$\theta\mid_{\operatorname{im}\partial^3_{2,0}} = \eta\sigma^2_2 \qquad \theta\mid_{\operatorname{im}\partial^3_{2,1}} = \eta'\sigma^2_2$$
$$\theta\mid_{\operatorname{im}\partial^3_{3,0}} = H$$

Consider the following lifting problem:

$$\begin{tikzcd} \sqcap^3_{3,1} \arrow[r, "\theta"] \arrow[d, hook] & NX \arrow[d, "Nf"] \\ \partial\square^3 \arrow[r, "\Theta"] & NZ \end{tikzcd}$$

Since $f \colon X \to Z$ is a 1-fibration of graphs, this lifting problem admits a solution $\widetilde{\Theta} \colon \partial\square^3 \to NX$. Let $H' = \widetilde{\Theta}\mid_{\operatorname{im}\partial^3_{3,1}}$. Then, H' is a path-homotopy $\eta \Rightarrow \eta'$ that satisfies $f \circ H' = g \circ G$. Thus, the pair $(H', G) : I_\infty \square I_\infty \to X \times_Z Y$ defines a path-homotopy $(\eta, \tau) \Rightarrow (\eta', \tau')$ in $X \times_Z Y$. It follows that $[(\eta, \tau)] = [(\eta', \tau')]$ as morphisms in $\Pi_1 (X \times_Z Y)$. $\square$

We are now ready to prove that the fundamental groupoid functor $\Pi_1 \colon \mathsf{Graph} \to \mathsf{Gpd}$ preserves pullbacks along fibrations.

Proposition 5.3.9. *Consider a pullback square in* Graph

$$\begin{tikzcd} X \times_Z Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

where $f\colon X \to Z$ is a 1-fibration of graphs. Then, the canonical functor

$$\Psi\colon \Pi_1(X \times_Z Y) \to \Pi_1 X \times_{\Pi_1 Z} \Pi_1 Y$$

is an isomorphism of groupoids.

Proof. We will construct a functor

$$\Phi\colon \Pi_1 X \times_{\Pi_1 Z} \Pi_1 Y \to \Pi_1(X \times_Z Y)$$

that is inverse to Ψ. Given a morphism $([\eta],[\tau]):\ (x,y) \to (x',y')$ in $\Pi_1 X \times_{\Pi_1 Z} \Pi_1 Y$, by Lemma 5.3.7, we can choose representative paths η' and τ' from the path-homotopy classes $[\eta]$ and $[\tau]$ respectively, such that $f \circ \eta' = g \circ \tau'$. The pair (η',τ') then defines a path in the pullback graph $X \times_Z Y$. Let Φ map the morphism $([\eta],[\tau]):\ (x,y) \to (x',y')$ in $\Pi_1 X \times_{\Pi_1 Z} \Pi_1 Y$ to the morphism $[(\eta',\tau')]:\ (x,y) \to (x',y')$ in $\Pi_1(X \times_Z Y)$.

By Lemma 5.3.8, this assignment is well-defined in the sense that it is independent of the specific choice of the representative paths η' and τ'. Furthermore, this assignment is functorial.

Since $[\eta] = [\eta']$ and $[\tau] = [\tau']$, we have $\Psi \circ \Phi = \mathrm{id}_{\Pi_1 X \times_{\Pi_1 Z} \Pi_1 Y}$.

On the other hand, given a morphism $[(\eta,\tau)]:\ (x,y) \to (x',y')$ in $\Pi_1(X \times_Z Y)$, any choice of representative path (η,τ) in $X \times_Z Y$ gives us a choice of representative paths η in X and τ in Y that satisfy the condition $f \circ \eta = g \circ \tau$. Thus, we have $\Phi \circ \Psi = \mathrm{id}_{\Pi_1 X \times_{\Pi_1 Z} \Pi_1 Y}$. $\square$

Proof of Theorem 5.3.5. The functor $\Pi_1\colon \mathsf{Graph}_{\text{1-types}} \to \mathsf{Gpd}$ clearly preserves weak equivalences and the terminal object. By Proposition 5.3.6, it preserves fibrations. Thus, it also preserves acyclic fibrations. By Proposition 5.3.9, it preserves pullbacks along fibrations. $\square$

Finally, we will prove that the fundamental groupoid functor $\Pi_1 \colon \mathsf{Graph} \to \mathsf{Gpd}$ is a weak equivalence of fibration categories by verifying the approximation properties of Proposition 5.1.12.

Proposition 5.3.10. *Given any functor of groupoids $F \colon \mathcal{G} \to \Pi_1 Y$, there exists a graph map $f \colon X \to Y$ such that there is a commutative square of the form:*

$$\begin{tikzcd} \mathcal{G}' \arrow[r, "\sim"] \arrow[d, "\sim"'] & \Pi_1 X \arrow[d, "\Pi_1 f"] \\ \mathcal{G} \arrow[r, "F"] & \Pi_1 Y \end{tikzcd}$$

where $\mathcal{G}' \xrightarrow{\sim} \mathcal{G}$ and $\mathcal{G}' \xrightarrow{\sim} \Pi_1 X$ are equivalences of categories.

Proof. Suppose $\mathcal{G} = \coprod_{i \in \mathcal{I}} \mathcal{G}_i$ where each $\mathcal{G}_i$ is a connected component of $\mathcal{G}$. For each $i \in \mathcal{I}$, let G_i be the automorphism group of some object $g_i \in \mathcal{G}_i$. By Theorem 3.4.5, there exists a connected graph X_i such that $A_1(X_i, x_i) \cong G_i$ for any vertex $x_i \in X_i$. Let $X = \coprod_{i \in \mathcal{I}} X_i$, and let $\mathcal{G}' = \coprod_{i \in \mathcal{I}} G_i$. Then, we have an inclusion $\mathcal{G}' \hookrightarrow \mathcal{G}$ and a functor $\mathcal{G}' \to \Pi_1 X$ that maps the object g_i to some $x_i \in X_i$. Both these functors are equivalences of categories. For each $i \in \mathcal{I}$, we define a graph map $f_i \colon X_i \to Y$ to be the constant map at the vertex $F(g_i) \in Y$. And finally, we define $f \colon X \to Y$ to be f_i on each connected component X_i. $\square$

Theorem 5.3.11. *The fundamental groupoid functor $\Pi_1 \colon \mathsf{Graph}_{\text{1-types}} \to \mathsf{Gpd}$ is a weak equivalence of fibration categories.*

Proof. Observe that $\Pi_1 \colon \mathsf{Graph} \to \mathsf{Gpd}$ reflects weak equivalences. The required result then follows from Proposition 5.3.10 and Proposition 5.1.12. $\square$

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