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# Monotone Functions On General Measure Spaces

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# Abstract

Given a measure space and a totally ordered collection of measurable sets, called an ordered core, the notion of a core decreasing function is introduced and used to generalize monotone functions to general measure spaces. The least core decreasing majorant construction, the level function construction, and the greatest core decreasing minorant, already known for functions on the real line, are extended to this general setting. A functional description of these constructions is provided and is shown to be closely related to the pre-order relation of functions induced by integrals over the ordered core.

For an ordered core, the down space construction of a Banach function space is defined as a variant of the Köthe dual restricted to core decreasing functions. Concrete descriptions of the duals of the down spaces are provided. The down spaces of $L^1$ and $L^\infty$ are shown to form an exact Calderón couple with divisibility constant 1; a complete description of the exact interpolation spaces for the couple is given in terms of level functions; and the down spaces of universally rearrangement invariant spaces (u.r.i.) are shown to be precisely those interpolation spaces that have the Fatou property. The dual couple is also an exact Calderón couple with divisibility constant 1; a complete description of the exact interpolation spaces for the couple is given in terms of least core decreasing majorants; and the duals of down spaces of u.r.i. spaces are shown to be precisely those interpolation spaces that have the Fatou property.

Integral operators whose kernel operators satisfy a monotonicity condition on their level sets are shown to induce an ordered core. Certain weighted norm inequalities are shown to remain valid when the weights are replaced with core decreasing functions. Boundedness of an abstract formulation of Hardy operators between Lebesgue spaces over general measure spaces is studied and, when the domain is $L^1$, shown to be equivalent to the existence of a Hardy inequality on the half line with general Borel measures.

**Keywords:** Ordered core, core decreasing function, Calderón couple, interpolation space, level function, down space, Hardy inequalities.

# Summary for Lay Audience

A fundamental feature of real numbers is that they form a total order, for any pair of distinct real numbers, one is bigger than the other. Consequentially, it is natural to define monotone functions as an assignation of numbers that preserve (or reverse) this order. In this thesis, we extend monotone functions to collections of elements that admit a notion of volume but do not have a predetermined order among the elements. Instead, we rely on a collection of subsets that take the role of the intervals $\{[0, x]\}_{x>0}$ in the real line, called an *ordered core*. We use ordered cores to define monotone functions in this more abstract setting and extend some tools related to decreasing functions, previously only available on the real line, to this more abstract setting.

We work with *function spaces*. For a fixed function, assign it a size by measuring its interaction with all the decreasing functions. The space we produce through this process is the *down space*. We describe them completely and study some of their properties. We focus on *duality* and *interpolation*.

The dual space is a collection of functions over our original space that satisfy certain properties. In the case of finite dimensional spaces (collections of column vectors of $n$-entries), we may identify the dual space with the collection of row vectors of $n$-entries. For our Down spaces, we also give a concrete description of their duals.

For interpolation, we consider a concrete pair of function spaces corresponding to the down spaces of $L^1$ ($L^1\downarrow$) and $L^\infty$ ($L^\infty\downarrow$). We consider intermediate collections of functions that can be written as the sum of a function that is not 'too wide' and a function that is not 'too tall'. If we know the behavior of an operation on $L^1$ and $L^\infty$, we also understand the operation on any intermediate collection. In this thesis, we give a complete characterization of the function spaces that are intermediate between $L^1\downarrow$ and $L^\infty\downarrow$.

We finish with an application of our theory of monotone functions in the study of Hardy inequalities.

# Co-Authorship Statement

Chapter 1 is an introductory chapter detailing the background required for the remaining chapters, it provides background references containing most of the proofs. I am the sole author of this chapter.

Chapters 2, 3, and 4 are mostly based on published work [24]. The research and writing was done in close collaboration between me and Dr. Gord Sinnamon.

Chapter 5 is based on my submitted work in [23].

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# Contents

# List of Appendices

# Glossary of Notation

Numbers that follow each entry are the page numbers on which the notation first occurs.

| | | |
|---|---|---|
| $(U, \Sigma, \mu)$ | A general measure space with $\sigma$-algebra $\Sigma$ and a $\sigma$-finite measure $\mu$ | 2 |
| $\leq_{\mathcal{A}}$ | The preorder relation induced by $\mathcal{A}$ on the elements of $U$ | 20 |
| $(\leftarrow, u]_{\mathcal{A}}$ | The collection $x \in U$ such that $x \leq_{\mathcal{A}} u$ | 20 |
| $(\leftarrow, u)_{\mathcal{A}}$ | The collection $x \in U$ such that $x \leq_{\mathcal{A}} u$ holds and $u \leq_{\mathcal{A}} x$ fails | 20 |
| $[u]_{\mathcal{A}}$ | The collection $x \in U$ such that $x \leq_{\mathcal{A}} u$ holds and $u \leq_{\mathcal{A}} x$ holds | 20 |
| $\lvert x \rvert_a$ | Distance of $x$ from $a$ in a metric space | 22 |
| $\sigma(C)$ | The smallest $\sigma$-ring containing the family of sets $C$ | 1 |
| $\chi_A$ | The characteristic function of the set $A$ | 4 |
| $\mu_f$ | The distribution function of $f$ with respect to the measure $\mu$ | 6 |
| $\mu \perp \tau$ | The measures $\mu$ and $\tau$ are mutually singular | 7 |
| $A_n \uparrow A$ | The sequence of sets $\{A_n\}$ increases to $A$ | 7 |
| $A_n \downarrow A$ | The sequence of sets $\{A_n\}$ decreases to $A$ | 71 |
| $\mathcal{A}$ | An ordered core over the space $(U, \Sigma, \mu)$ | 20 |
| $B(a; r)$ | Open metric ball centered at $a$ with radius $r$ | 22 |
| $B[a; r]$ | Closed metric ball centered at $a$ with radius $r$ | 22 |
| $\mathcal{B}$ | The ordered core $\{\emptyset\} \cup \{[0, x] : x > 0\}$ over the space $([0, \infty), \sigma(\mathcal{B}), \lambda)$ | 41 |
| $C \approx D$ | There exist strictly positive constants $a$ and $b$ such that $aC \leq D \leq bC$. In context, $C$ and $D$ depend on certain parameters, and $a, b$ may be chosen independent of these parameters. | 17 |
| $f \preccurlyeq_{\mathcal{A}} g$ | $\int_A f \, d\mu \leq \int_A g \, d\mu$ for each $A \in \mathcal{A}$ | 51 |
| $f \leq_{\downarrow} g$ | $\int_{[0,x]} f \, d\lambda \leq \int_{[0,x]} g \, d\lambda$ for each $x \geq 0$ | 12 |
| $f_n \uparrow f$ | The sequence of functions $\{f_n\}$ increases to $f$ pointwise almost everywhere | 4 |
| $f^*$ | The nonincreasing rearrangement of $f$ | 6 |
| $f^o$ | The level function of $f$ | 13 |
| $\widetilde{f}$ | The least core decreasing majorant of $f$ | 13 |
| $\underline{f}$ | The greatest core decreasing minorant of $f$ | 60 |
| $\mathcal{M}$ | The enriched ordered core for $\mathcal{A}$ | 25 |
| $L(\Sigma)$ | The collection of $\Sigma$-measurable functions $f : U \to [-\infty, \infty]$ | 2 |
| $L^+(\Sigma)$ | The collection of $\Sigma$-measurable functions $f : U \to [0, \infty]$ | 5 |
| $L^+(\mathcal{A})$ | The collection of $\sigma(\mathcal{A})$-measurable functions $f : U \to [0, \infty]$ | 20 |
| $L^{\uparrow}(\mathcal{A})$ | The collection of core increasing functions $f : U \to [0, \infty]$ | 27 |
| $L^{\downarrow}(\mathcal{A})$ | The collection of core decreasing functions $f : U \to [0, \infty]$ | 24 |
| $L^p_\mu$ | The Lebesgue space with respect to a measure $\mu$ | 2 |
| $L^0_\mu$ | The collection of $\mu$-measurable functions | 2 |
| $L^p$ | The Lebesgue space with respect to the Lebesgue measure | 16 |
| $L_\rho$ | The function space with respect to the function seminorm $\rho$ | 5 |
| $L_{\rho'}$ | The associate space of $L_\rho$ | 5 |
| $L^1_{\mathrm{Loc},\mathcal{A}}$ | The collection of locally integrable functions with respect to $\mathcal{A}$ | 30 |

| | | |
|---|---|---|
| $Q$ | Transition map from $([0,\infty),\sigma(\mathcal{B}),\lambda)$ to $(U,\Sigma,\mu)$ with core $\mathcal{A}$ | 45 |
| $R$ | Transition map from $(U,\Sigma,\mu)$ with core $\mathcal{A}$ to $([0,\infty),\sigma(\mathcal{B}),\lambda)$ | 44 |
| $V^+$ | The collection of positive elements of a vector lattice $V$ | 3 |
| $X\downarrow$ | The down space of $X$ | 15 |
| $X^o$ | The function space induced by the norm $\|f\|_{X^o} = \|f^o\|_X$ | 73 |
| $\widetilde{X}$ | The function space induced by the norm $\|f\|_{\widetilde{X}} = \|\widetilde{f}\|_X$ | 73 |

# Introduction

Monotone functions on $\mathbb{R}$ are very well-understood objects compared to general measurable functions. A systematic study of these functions has shed light on many problems and, throughout the years, a wide variety of techniques and applications have been developed to work with monotone functions. For instance, the level function construction that first appeared in the works of Lorentz [14] and Halperin [8] has the following properties:

Let $1 \leq p \leq \infty$ and $1/p + 1/p' = 1$. If $f$ is a Lebesgue measurable function on $[0, \infty)$, there exists a nonnegative, nonincreasing function $f^o$, called the *level function* of $f$, such that

$$\int_0^\infty |f|g \leq \int_0^\infty f^o g$$

holds for all nonnegative, nonincreasing $g$, and

$$\|f^o\|_{L^p} = \sup\left\{ \int_0^\infty |f|g : \|g\|_{L^{p'}} \leq 1, g \geq 0, g \text{ nonincreasing}\right\}.$$

Here $\|\cdot\|_{L^p}$ denotes the usual $L^p$ norm.

This improves Hölder's inequality (to a so-called D-type Hölder inequality) in the presence of monotonicity since we have $\int_0^\infty |f|g \leq \|f^o\|_{L^p}\|g\|_{L^{p'}}$ whenever $g$ is nonnegative and nonincreasing. This construction has been used to give formulas for the dual spaces of abstract Lorentz spaces [12], to prove weighted Hardy inequalities [26, 32], to characterize boundedness of the Fourier transform in Lorentz spaces [18, 28], to transfer monotonicity (from kernel to weight) in weighted norm inequalities for general positive integral operators [29] and to provide equivalent norms for traditional and abstract Cesáro spaces that facilitate interpolation of these spaces and of their duals [13].

The down spaces, a variant of the Köthe dual when restricted to the set of decreasing functions, for the $L^p$ spaces, are given by the norm

$$\|f\|_{L^p\downarrow} = \sup\left\{\int_0^\infty |f|\, g : \|g\|_{L^{p'}} \leq 1 \text{ and } g \text{ is nonincreasing}\right\}.$$

A careful study of the down spaces is provided in [26, 16, 17], including a complete characterization of their interpolation spaces and their relationship with u.r.i. Banach function spaces.

These powerful tools are currently available only for functions defined on $(\mathbb{R}, \lambda)$, where the natural order on $\mathbb{R}$ determines the collection of nonnegative, nonincreasing functions. The object of our research is to make these tools available for functions on general measure spaces in which a highly customizable notion of order is used to determine monotonicity. This is done by a careful examination of the concept of a measure space with an *ordered core*, which was introduced in [31] to study abstract Hardy operators. We use this ordered core to define *core decreasing functions*, which will take the role of monotone functions in our general measure spaces. We use these tools to extend level functions, D-type Hölder inequalities, down spaces with their interpolation theory, and the transferring monotonicity technique to our more general setting.

This thesis is based on published work in [24] and submitted work in [23].

### Organization of the thesis

Chapter 1 contains no original results, instead, we review most of the mathematical prerequisites needed for the later chapters. We begin with a brief review of results in measure theory and vector lattices not usually covered in a first graduate course. We discuss the definition of Banach function spaces with special care about the nonincreasing rearrangement and universally rearrangement invariant (u.r.i.) spaces. We give a brief summary of the real method of interpolation as well as a fundamental theorem of Calderón. The rest of the chapter is devoted to giving more detail into the theory of monotone functions over the real line.

Our research begins in Chapter 2, where we provide the technical tools required for the later results. We study the properties of ordered cores, define and study core decreasing functions, and finish with a careful analysis of mappings between measure spaces with ordered cores. We reserve a more categorical point of view of these mappings for the appendix.

In Chapter 3 we study a preorder relation on nonnegative measurable functions induced by the ordered core. We extend the level function construction, the least core decreasing majorant and the greatest core decreasing minorant to the setting of ordered cores. We provide a pointwise and functional description of these constructions and relate them with the preorder induced by the core.

Chapter 4 contains the theory of down spaces over a measure space with an ordered core. We describe their associate spaces and give a full description of all their interpolation spaces using the $K$-method of interpolation. We show that the fundamental compatible couple of down spaces corresponding to $L^1$ and $L^\infty$ form a Calderón couple and relate their interpolation spaces with down spaces for u.r.i. spaces. We also describe all interpolation spaces for the dual couple.

In Chapter 5 we exhibit an application of the high customization provided by the theory of monotonicity in general measure spaces developed before. We study integral kernel operators satisfying a monotonicity condition to induce an ordered core and extend the transferring monotonicity technique to these integral operators. As an application, we provide a characterization of the boundedness of the abstract Hardy operator from $L^1_\mu \to L^q_\tau$. We exhibit new proofs and extensions of weight characterizations of Hardy inequalities in metric measure spaces.

# Chapter 1

# Preliminaries

## 1.1 Basic results in measure theory

We consider $X$ to be any set and record some basic constructions. The first is a minimal collection of subsets of $U$ for which a premeasure can be extended to a measure.

**Definition 1.1.1** *A non-empty collection $S$ of subsets of $X$ is called a semiring if it is closed under finite intersections and if for all $A, B \in S$ there exists a finite disjoint collection $\{C_j\}_{j=1}^k$ of sets in $S$ such that*

$$B \setminus A = \bigcup_{j=1}^{k} C_j.$$

*A premeasure is a map $\rho : S \to [0, \infty)$ satisfying:*

- *If $\emptyset \in S$, then $\rho(\emptyset) = 0$.*
- *(Finitely additive) If $\{C_j\}_{j=1}^k$ is a disjoint collection of sets in $S$ and $\bigcup_{j=1}^k C_j \in S$ then $\rho(S) = \sum_{j=1}^k \rho(C_j)$.*
- *(Countably monotone) If $E \in S$ and $\{C_j\}_{j\geq 1}$ is a sequence in $S$ such that $E \subseteq \bigcup_j C_j$, then $\rho(E) \leq \sum_{j=1}^{\infty} \rho(C_j)$.*

*A premeasure $\rho$ is said to be $\sigma$-finite if $\cup S = \bigcup_{j=1}^{\infty} C_j$ for some collection $\{C_j\}_{j\geq 1}$ such that $\rho(C_j) < \infty$.*

We are interested in the extension of premeasures to measures, we consider their natural domain.

**Definition 1.1.2** *A non-empty collection $R$ of subsets of $X$ is a $\sigma$-ring if it is closed under set differences and countable unions. For any non-empty collection $S$ of subsets of $X$, we denote by $\sigma(S)$ the smallest $\sigma$-ring containing $S$. If $X$ belongs to a $\sigma$-ring $R$, we call $R$ a $\sigma$-algebra.*

The following result establishes a property of the generated $\sigma$-rings.

**Theorem 1.1.3** *If $S$ is any non-empty collection of subsets of $X$ and $E \in \sigma(S)$, then there exists a countable subcollection $S_0$ such that $E \in \sigma(S_0)$.*

**Proof:** See [7][Page 24].

∎

**Theorem 1.1.4** *(Caratheodory-Hahn) Let $\rho : S \to [0,\infty)$ be a $\sigma$-finite premeasure defined on a semiring $S$. Denote by $\rho^*$ the induced outer measure and by $\overline{\rho}$ the measure defined on $\rho^*$-measurable sets. Then, $\overline{\rho}$ is the unique measure extending $\rho$ to $\sigma(S)$.*

**Proof:** See [19, p.356].

∎

A consequence of the previous theorem is that to prove the equality of $\sigma$-finite measures, it is enough to prove that they coincide on a generating semiring.

We will make use of the following version of the change of variables formula.

**Theorem 1.1.5** *Given a $\sigma$-finite measure space $(Y, \mathcal{T}, \tau)$, a measurable space $(U, \Sigma)$ and a measurable function $\varphi : (Y, \mathcal{T}, \tau) \to (U, \Sigma)$, the set function $\varphi_*(\tau) : \Sigma \to [0,\infty)$ defined by*

$$\varphi_*(\tau)(E) = \tau(\varphi^{-1}(E)),$$

*defines a measure. A $\Sigma$- measurable function $g : U \to [-\infty,\infty]$ is integrable with respect to $\varphi_*(\tau)$ precisely when $g \circ \varphi$ is integrable with respect to $\tau$. In addition, the formula*

$$\int_{\varphi(Y)} g \, d(\varphi_*(\tau)) = \int_Y g \circ \varphi \, d\tau \tag{1.1}$$

*holds. We call the measure $\varphi_*(\tau)$ the pushforward measure.*

**Proof:** See [4, Theorem 3.6.1].

∎

From now on, we suppose that all the measure spaces involved are $\sigma$-finite. Given a measure space $(U, \Sigma, \mu)$, the collection of equivalence classes of $\Sigma$-measurable functions equal up to a set of $\mu$-measure zero will be denoted $L(\Sigma)$ or $L^0_\mu$.

For a real number $p \in (0,\infty]$ and a measurable real valued function $f$ we denote

$$\|f\|_{L^p_\mu} = \left(\int_U |f|^p \, d\lambda\right)^{\frac{1}{p}} \text{ if } p < \infty, \quad \text{and} \quad \|f\|_{L^\infty_\mu} = \text{ess sup}\{|f(s)| : s \in U\}.$$

The collections of functions for which these quantities are finite are called the Lebesgue spaces and are denoted $L^p_\mu$, it is well known that these are Banach spaces for $p \in [1,\infty]$ and quasi-Banach spaces for $p \in (0,1)$.

We will make use of Hölder's inequality and Minkowski's integral inequality.

**Proposition 1.1.6** *Let $p \in (1,\infty)$, $f, g$ measurable functions and $\frac{1}{p} + \frac{1}{p'} = 1$. Then*

$$\int_U |fg| \, d\lambda \leq \left(\int_U |f|^p \, d\lambda\right)^{\frac{1}{p}} \left(\int_U |g|^{p'} \, d\lambda\right)^{\frac{1}{p'}} \tag{1.2}$$

*and*

$$\left(\int_U \left(\int_Y |f| \, d\tau\right)^p d\lambda\right)^{1/p} \leq \int_Y \left(\int_U |f|^p \, d\lambda\right)^{1/p} d\tau \tag{1.3}$$

**Proof:** See [10, Theorems 188 and 202].

∎

## 1.2 Positive operators

The following results from the theory of vector lattices will be needed, we follow the exposition from [1].

**Definition 1.2.1** *A $\mathbb{R}$-vector space $V$ is a vector lattice if it is equipped with a partial order relation denoted '$\leq$' that is compatible with the algebraic operations. That is*

$$\text{If } x, y \in V \text{ and } x \leq y \text{ then } x + z \leq y + z, \quad \forall z \in V.$$

$$\text{If } x, y \in V \text{ and } x \leq y \text{ then } \alpha x \leq \alpha y, \quad \forall \alpha \in \mathbb{R}^+.$$

*And if for any $x, y \in V$, there exists a supremum denoted $x \vee y \in V$.*

*The collection of vectors $x \in V$ such that $x \geq 0$ is denoted by $V^+$.*

*Any vector $x \in V$ can be uniquely decomposed in its positive and negative parts. That is,*

$$x = x^+ - x^-,$$

*where $x^+ = x \vee 0$ and $x^- = (-x) \vee 0$.*

For our purposes, a fundamental example of vector lattice is given by $V = L^0_\mu$ for a $\sigma$-finite measure space $(U, \Sigma, \mu)$. The order relation is up to inequality $\mu$-almost everywhere. The supremum of two functions is given by $f \vee g = \max\{f, g\}$.

We consider the class of linear maps that preserve the order relations.

**Definition 1.2.2** *Given two vector lattices $V, W$. A linear map $T : V \to W$ is positive if $T(x) \in W^+$ for every $x \in V^+$.*

We will focus on vector lattices with some extra properties.

**Definition 1.2.3** *A vector lattice $V$ is archimedean if for any $x \in V^+$ the sequence $\left\{\frac{1}{n}x\right\}_{n>0}$ satisfies the following:*

$$\text{If } z \in V : z \leq \frac{1}{n}x, \quad \forall n, \quad \text{then } z \leq 0.$$

Examples of archimedean vector lattices include the $L^p$ spaces for $p \in [1, \infty]$. The following theorem will be useful in extending linear operators defined on positive elements.

**Theorem 1.2.4** *(Kantorovich) Let $V, W$ be vector lattices, with $W$ being archimedean. Suppose that $T : V^+ \to W^+$ is an additive mapping, that is, $T(x + y) = T(x) + T(y)$ holds for all $x, y \in V^+$. Then $T$ has a unique extension to a positive operator from $V$ to $W$. Moreover, the unique extension is given by*

$$T(x) = T(x^+) - T(x^-).$$

**Proof:** See [1, Theorem 1.10].

$\blacksquare$

The definition of supremum need not be constrained to a finite collection of vectors.

**Definition 1.2.5** *Given a set $A \subseteq V$, we say that $A$ is bounded above if there exists $z \in V$ such that $y \leq z$ for all $y \in A$. We say that $x = \sup A$ if $y \leq x$ for all $y \in V$ and for any $z \in V$ if $y \leq z$ for all $y \in A$, then $x \leq z$. A vector lattice $V$ is Dedekind complete if every set $A$ bounded above in $V$ has a supremum in $V$.*

Examples of Dedekind complete vector lattices include the $L^p$ spaces for $p \in [1, \infty)$. It also follows from the definition, that any Dedekind complete vector lattice is also archimedean.

We are interested in the following operator version of the Hahn-Banach theorem.

**Theorem 1.2.6** *(Hahn-Banach-Kantorovich) Let $V$ be a $\mathbb{R}$-vector space, $W$ be a Dedekind complete vector lattice and $\rho : V \to W$ satisfying*

$$\rho(x + y) \leq \rho(x) + \rho(y), \quad \textit{and} \quad \rho(\alpha x) = \alpha\rho(x),$$

*for all $x, y \in V$ and $\alpha \in V$. If $H$ is a linear subspace of $V$ and $f : H \to W$ is a linear map satisfying $f(x) \leq \rho(x)$ for all $x \in H$, then there exists an operator $h : V \to W$ such that*

- *$h(x) = f(x)$ for all $x \in H$.*
- *$h(x) \leq \rho(x)$ for all $x \in V$.*

**Proof:** See [1, Theorem 1.25].

∎

## 1.3 Banach function spaces

We focus on a particular instance of vector lattices called Banach function spaces, we follow the exposition from [33].

Over a $\sigma$-finite measure space $(U, \Sigma, \mu)$, we denote by $L^+(\Sigma)$ the set of all nonnegative measurable functions on $U$. For a sequence $\{f_n\} \in L^+(\Sigma)$ we write $f_n \uparrow f$ whenever the sequence $\{f_n\}$ is increasing and converges to $f$ $\mu$-almost everywhere.

**Definition 1.3.1** *A mapping $\rho : L^+(\Sigma) \to [0, \infty]$ is a function seminorm if for any $f, g, \{f_n\}_{n\in\mathbb{N}} \in L^+(\Sigma)$, $c \in (0, \infty)$ and any measurable $E \subseteq R$, the following holds:*

- *$\rho(f) = 0$ if $f = 0$, $\rho(cf) = c\rho(f)$, $\rho(f + g) \leq \rho(f) + \rho(g)$.*
- *$0 \leq g \leq f$ implies $\rho(g) \leq \rho(f)$.*

*If in addition $\rho(f) = 0$ implies $f = 0$ $\mu$-almost everywhere, then $\rho$ is called a function norm. A function seminorm has the Riesz-Fischer property if*

$$\sum_{n=1}^{\infty} \rho(f_n) < \infty, \quad \textit{implies} \quad \rho\Big(\sum_{n=1}^{\infty} f_n\Big) \leq \sum_{n=1}^{\infty} \rho(f_n).$$

*A function seminorm has the Fatou property if*

$$f_n \uparrow f, \quad \textit{implies} \quad \rho(f_n) \uparrow \rho(f).$$

*A function seminorm is saturated if for any set $E$ of positive measure there exists a subset $F \subseteq E$ such that $\mu(F) > 0$ and $\rho(\chi_F) < \infty$.*

For a given function norm $\rho$ we define the function space $L_\rho$ as the collection of all measurable functions $f$ such that $\rho(|f|)$ is finite. The following result relates the completeness of $L_\rho$ with the Riesz-Fischer and Fatou properties.

**Theorem 1.3.2** *The normed linear space $L_\rho$ is complete (i.e., $L_\rho$ is a Banach space) if and only if $\rho$ has the Riesz-Fischer property. If $\rho$ has the Fatou property, then it has the Riesz-Fischer property, and so, $L_\rho$ is complete.*

**Proof:** See [33, Theorem 1, page 444] and [33, Theorem 2, page 445].
∎

We now consider our natural notion of dual space, it will be given by another seminorm.

**Definition 1.3.3** *Given a function seminorm $\rho$, the associate seminorm is defined by*

$$\rho'(f) = \sup\left\{ \int_U fg\, d\mu : g \in L^+(\Sigma) \text{ and } \rho(g) \leq 1 \right\}$$

*The associate space (or Köthe dual) is the function space $L_{\rho'}$.*

**Example 1.3.4** *Let $p \in [1, \infty]$ and $\rho(f) = \|f\|_{L^p_\mu}$. Then $L_\rho = L^p_\mu$ and $\rho'(f) = \|f\|_{L^{p'}_\mu}$. Notice that this holds even for the case $p = \infty$, where it does not hold for the topological dual space.*

Some results regarding these associate spaces are recorded next.

**Proposition 1.3.5** *Given a function seminorm $\rho$ and $f, g \in L^+(\Sigma)$, the following hold:*

1. *The function seminorm $\rho'$ has the Fatou property.*
2. *(Hölder's inequality) We have*
$$\int_U fg\, d\mu \leq \rho(f)\rho'(g). \tag{1.4}$$
3. *We have $\rho''(f) \leq \rho(f)$ and $\rho'''(f) = \rho'(f)$.*
4. *We have the embedding $L_\rho \subseteq L_{\rho''}$ and the equality $L_{\rho'} = L_{\rho'''}$ with identical norms.*
5. *The function seminorm $\rho$ is saturated if and only if $\rho'$ is a norm.*
6. *A function seminorm has the Fatou property if and only if $\rho'' = \rho$.*

**Proof:** Item (1) is [33, Theorem 1, page 457]. Items (2) and (3) are proved in [33, Theorem 2, page 457]. Item (4) is an immediate consequence of (3). Item (5) is [33, Theorem 4, page 458].

Regarding item (6), if $\rho'' = \rho$ then $\rho$ has the Fatou property as a consequence of item (1). The converse is proven in [3, Theorem 2.7, page 10], notice that in [3], the Fatou property is part of the definition of a Banach function space.
∎

### 1.3.1 Rearrangement invariant spaces

We follow the notation in the previous section.

**Definition 1.3.6** *Let $f$ be a measurable function. The distribution function $\mu_f : [0,\infty) \to [0,\infty]$ is given by*

$$\mu_f(\lambda) = \mu(\{x \in U : |f(x)| > \lambda\}).$$

We can use the distribution functions to make comparisons between functions defined on different measure spaces.

**Definition 1.3.7** *Let $(U, \Sigma, \mu)$ and $(Y, \mathcal{T}, \tau)$ be $\sigma$-finite measure spaces. We say that $f \in L(\Sigma)$ and $g \in L(\mathcal{T})$ are equimeasurable if they have the same distribution function, that is, if $\mu_f(\lambda) = \tau_g(\lambda)$ for all $\lambda \geq 0$.*

The generalized inverse of the previous function is given next.

**Definition 1.3.8** *The nonincreasing rearrangement of $f$ is the function $f^* : [0,\infty) \to [0,\infty]$ defined by*

$$f^*(t) = \inf\{\lambda : \mu_f(\lambda) \leq t\}.$$

Some properties of the rearrangement are recorded in the following proposition.

**Proposition 1.3.9** *Let $f, g, \{f_n\}_{n\in\mathbb{N}} \in L(\Sigma)$, then:*

1. *The function $f^*$ is nonincreasing and right continuous.*
2. *If $|f| \leq |g|$ $\mu$-almost everywhere, then $f^* \leq g^*$.*
3. *If $f_n \uparrow f$ $\lambda$-a.e, then $f_n^* \uparrow f^*$.*
4. *If $t_1, t_2 > 0$, then*
$$(f+g)^*(t_1+t_2) \leq f^*(t_1) + g^*(t_2).$$
5. *If $\mu(\lambda)$ and $f^*(t)$ are finite, then*
$$f^*(\mu_f(\lambda)) \leq \lambda \quad \text{and} \quad \mu_f(f^*(t)) \leq t.$$
6. *For $p \in (0,\infty)$ we have*
$$\int_U |f|^p \, d\mu = \int_0^\infty (f^*)^p(t)\, dt.$$
*And $\|f\|_{L^\infty_\mu} = f^*(0)$.*

**Proof:** Items (1) to (5) are proved in [3, Proposition 1.7, page 41]. Item (6) is proved in [3, Proposition 1.8, page 43].

∎

The following are function spaces that are well behaved with respect to the nonincreasing rearrangement.

**Definition 1.3.10** *A universally rearrangement invariant (u.r.i.) space over $U$ is a Banach function space over $U$ such that for $f, g \in L(\Sigma)$ and $\rho(g) < \infty$:*

$$\text{If} \int_0^t f^* \leq \int_0^t g^*, \text{ for all } t > 0, \quad \text{then} \quad \rho(f) \leq \rho(g).$$

It follows directly from the definition that if $f$ and $g$ are equimeasurable, then $\rho(f) = \rho(g)$, whenever $\rho$ is a u.r.i. space. More properties are recorded below.

**Proposition 1.3.11** *Let $\rho$ be a u.r.i. function norm, then:*

- *If there exists a function $f$ not $\mu$-a.e. zero such that $\rho(f) < \infty$, then $\rho$ is saturated.*
- *The associate norm $\rho'$ is u.r.i.*

**Proof:** To prove the first statement, suppose that $E$ is a set of positive finite measure. We will show that $\rho(\chi_E) < \infty$, which shows that $\rho$ is saturated. Since $U$ is $\sigma$-finite, there exists a sequence of sets $\{U_n\}$ of finite measure, such that $U_n \uparrow U$. Since $|f| > 0$ on a set of positive measure, there exists some $n, \epsilon$ such that the set

$$F = \{u \in U_n : |f(u)| > \epsilon\}$$

has positive finite measure. If $\mu(E) \leq \mu(F)$, then

$$\int_0^t (\chi_E)^* = \min(t, \mu(E)) \leq \min(t, \mu(F)) = \int_0^t (\chi_F)^* \leq \int_0^t \frac{1}{\epsilon}(\epsilon\chi_F)^* \leq \int_0^t \frac{1}{\epsilon}(f)^* = \int_0^t (f/\epsilon)^*,$$

above we used the fact that $\epsilon\chi_F \leq |f|$. Since $\rho(f/\epsilon) < \infty$ and $\rho$ is u.r.i., it follows that $\rho(\chi_E) < \infty$.

In the case that $\mu(F) < \mu(E)$, we have

$$\int_0^t (\chi_E)^* = \begin{cases} t, & t \leq \mu(F) \\ t, & t \in (\mu(F), \mu(E)] \\ \mu(E), & t > \mu(E) \end{cases} \leq \begin{cases} \frac{\mu(E)}{\mu(F)} t, & t \leq \mu(F) \\ \mu(E), & t \in (\mu(F), \mu(E)] \\ \mu(E), & t > \mu(E) \end{cases} = \frac{\mu(E)}{\mu(F)} \int_0^t (\chi_F)^*$$
$$\leq \frac{\mu(E)}{\mu(F)} \int_0^t (f/\epsilon)^* = \int_0^t \left(\frac{\mu(E)}{\epsilon\mu(F)} f\right)^*.$$

Once more, we get $\rho(\chi_E) \leq \rho(f\mu(E)/(\epsilon\mu(F))) < \infty$. This shows that $\rho$ is saturated.

The statement that $\rho'$ is u.r.i. is [15, Theorem 11.11(i)].

∎

**Example 1.3.12** *Item (6) in Proposition 1.3.9, shows that the $L^p_\mu$ spaces are u.r.i. for all $p \in [1, \infty]$.*

## 1.4 Interpolation spaces and the K-functional

The objects to be studied are couples of Banach spaces, roughly speaking, a couple is compatible when there is a way to compare elements that could belong to different spaces.

**Definition 1.4.1** *A pair of Banach spaces $(X_0, X_1)$ is a **compatible couple** if the two spaces are continuously embedded in a Hausdorff topological vector space.*

Any subspace $Y$ of a Banach space $X$ induces a compatible couple by making $X_0 = Y$, $X_1 = X$, and the inclusion maps as the continuous embeddings.

A very important example is given by couples of $L^p_\mu$ spaces over a sigma-finite measure space $(U, \Sigma, \mu)$. They are all embedded in a space of measurable functions equipped with convergence in measure.

For any compatible couple $(X_0, X_1)$ their intersection and sum (defined in the bigger space $X$) are Banach spaces.

**Definition 1.4.2** *The spaces $X_0 + X_1$ and $X_0 \cap X_1$ have the norms*

$$||x||_{X_0+X_1} = \inf\{||x_0||_{X_0} + ||x_1||_{X_1} : x = x_0 + x_1\};$$

$$||x||_{X_0\cap X_1} = \max\{||x||_{X_0}, ||x||_{X_1}\}.$$

Clearly, we have the inclusions $X_0 \cap X_1 \hookrightarrow X_i \hookrightarrow X_0 + X_1$ for $i = 1, 2$, however, there can be more spaces satisfying these conditions.

**Definition 1.4.3** *An intermediate space is a space continuously embedded between $X_0 \cap X_1$ and $X_0 + X_1$.*

### 1.4.1 Interpolation pairs

Here we study operators between compatible couples

**Definition 1.4.4** *Given two compatible couples $(X_0, X_1)$ and $(Y_0, Y_1)$ an admissible operator is a linear map $T : X_0 + X_1 \to Y_0 + Y_1$ such that $T|_{X_i} \in B(X_i, Y_i)$ for $i = 0, 1$. The norm of the admissible map $||T||$ is the maximum of the operator norms of the restricted maps $T|_{X_i}$. If $||T|| \leq 1$ we say that $T$ is a contraction.*

Given a compatible couple, we distinguish the intermediate spaces that remain stable with respect to admissible operators.

**Definition 1.4.5** *Given two compatible couples $(X_0, X_1)$ and $(Y_0, Y_1)$, an interpolation pair is a pair of intermediate spaces $(X, Y)$ such that $T(X) \subseteq Y$ for every admissible operator $T$.*

*If $X_0 = Y_0, X_1 = Y_1$ and $X = Y$ then $X$ is an interpolation space. It is an exact interpolation space if $||T||_{X\to X} \leq ||T||$, for every admissible map.*

The next result begins to show the relationship between u.r.i. spaces and interpolation spaces for the couple $(L^1_\mu, L^\infty_\mu)$.

**Theorem 1.4.6** *If $\rho$ is a u.r.i. function norm, then $L_\rho$ is an exact interpolation space between $L^1_\mu$ and $L^\infty_\mu$.*

**Proof:** See [6, Theorem 3].

∎

As a consequence of the theorem above, we note that the $L^p_\mu$ spaces are interpolation spaces for the couple $(L^1_\mu, L^\infty_\mu)$, more generally any u.r.i. space is an exact interpolation space.

A main goal in Interpolation of Operators: Given a pair of compatible Banach spaces, describe all possible interpolation spaces.

## 1.5 The K-method of interpolation

We study a method of generating interpolation spaces for a given compatible couple. For our purposes, we assume that the elements of the Banach spaces are functions defined on a $\sigma$-finite measure space $(U, \Sigma, \mu)$.

**Definition 1.5.1** *For a compatible couple $(X_0, X_1)$ the K-functional is defined by the formula*

$$K(f, t, X_0, X_1) = \inf \{\|f_0\|_{X_0} + t\|f_1\|_{X_1} : f = f_0 + f_1\}.$$

**Remark** Notice that $K(f, 1, X_0, X_1) = \|f\|_{X_0+X_1}$.

We will need some basic properties of the K-functional.

**Proposition 1.5.2** *For any compatible couple $(X_0, X_1)$ and $f \in X_0 + X_1$. The K-functional is a nonnegative concave function.*

*If $W : (X_0, X_1) \to (Y_0, Y_1)$ is an admissible contraction, then*

$$K(Wf, t, Y_0, Y_1) \leq K(f, t, X_0, X_1). \tag{1.5}$$

*If $(X_0, X_1)$ is a compatible couple of Banach function spaces over the same measure space, then*

$$K(f, t, X_0, X_1) = \inf \{\|f_0\|_{X_0} + t\|f_1\|_{X_1} : |f| = f_0 + f_1, 0 \leq f_0, 0 \leq f_1\}. \tag{1.6}$$

**Proof:** The concavity of the $K$-functional is proved in [3, Proposition 1.2, page 294]. The inequality (1.5) follows from [3, Theorem 1.1, page 301].

To prove formula (1.6), suppose that $f = f_0 + f_1$ and $|f| = g_0 + g_1$ with $f_0, g_0 \in X_0$ and $f_1, g_1 \in X_1$. Define $u_i = \mathrm{sgn}(f) f_i$ and $w_i = \mathrm{sgn}(f) g_i$ for $i \in \{0, 1\}$. Observe that $|f| = u_0 + u_1$, $f = w_0 + w_1$, $|u_i| \leq |f_i|$ and $|w_i| \leq |g_i|$, therefore

$$K(|f|, t; X_0, X_1) \leq \|u_0\|_{X_0} + t\|u_1\|_{X_1} \leq \|f_0\|_{X_0} + t\|f_1\|_{X_1},$$

and

$$K(f, t; X_0, X_1) \leq \|w_0\|_{X_0} + t\|w_1\|_{X_1} \leq \|g_0\|_{X_0} + t\|g_1\|_{X_1}.$$

Taking infimum over all decompositions $f = f_0 + f_1$ and $|f| = g_0 + g_1$ shows the equality $K(|f|, t; X_0, X_1) = K(f, t; X_0, X_1)$.

To complete the proof, for a decomposition $|f| = g_0 + g_1$, define the functions

$$h_0(x) = \begin{cases} g_0(t), & \text{if } g_0(t) \geq 0 \text{ and } g_1(t) \geq 0, \\ g_0(t) + g_1(t), & \text{if } g_0(t) \geq 0 \text{ and } g_1(t) < 0, \\ 0, & \text{if } g_0(t) < 0. \end{cases}$$

and

$$h_1(x) = \begin{cases} g_1(t), & \text{if } g_0(t) \geq 0 \text{ and } g_1(t) \geq 0, \\ g_0(t) + g_1(t), & \text{if } g_0(t) < 0 \text{ and } g_1(t) \geq 0, \\ 0, & \text{if } g_1(t) < 0. \end{cases}$$

Notice that $g_0(t) < 0$ and $g_1(t) < 0$ do not occur at the same time, since $g_0(t) + g_1(t) \geq 0$. We have $h_0(t) + h_1(t) = g_0(t) + g_1(t) = |f|$ with $|h_0| \leq |g_0|$, $|h_1| \leq |g_1|$, $0 \leq h_0$, and $0 \leq h_1$. Thus,

$$\inf\{\|f_0\|_{X_0} + t\|f_1\|_{X_1} : |f| = f_0 + f_1, 0 \leq f_0, 0 \leq f_1\} \leq \|h_0\|_{X_0} + t\|h_1\|_{X_0} \leq \|g_0\|_{X_0} + t\|g_1\|_{X_0},$$

taking infimum over all decompositions $|f| = g_0 + g_1$ yields the inequality

$$\inf\{\|f_0\|_{X_0} + t\|f_1\|_{X_1} : |f| = f_0 + f_1, 0 \leq f_0, 0 \leq f_1\} \leq K(|f|, t; X_0, X_1).$$

The reverse inequality follows immediately from the definition of the $K$-functional, therefore

$$\inf\{\|f_0\|_{X_0} + t\|f_1\|_{X_1} : |f| = f_0 + f_1, 0 \leq f_0, 0 \leq f_1\} = K(|f|, t; X_0, X_1) = K(f, t; X_0, X_1),$$

completing the proof.

∎

In practice, it will be useful to have a description of the $K$-functional that does not involve all possible decompositions of the form $x = x_0 + x_1$. Finding this representation is in general a very difficult problem, but it is possible to compute in some cases.

**Theorem 1.5.3** *Let $X_0 = L^1_\mu$ and $X_1 = L^\infty_\mu$, then*

$$K(f, t, X_0, X_1) = \int_0^t f^*.$$

**Proof:** See [5], page 338 and 341.

∎

A Banach function space $\Phi$ of functions over $(0, \infty)$ with the measure $dt/t$ is a *parameter of the K-method* if it contains the function $t \mapsto \min(1, t)$. With a parameter of the K-method we can generate interpolation spaces in the following way.

**Proposition 1.5.4** *For a compatible couple $(X_0, X_1)$ and a parameter of the K-method $\Phi$, define a map from $X_0 + X_1 \to [0, \infty]$ by*

$$\|f\|_{(X_0,X_1)_\Phi} = \|K(f, \cdot, X_0, X_1)\|_\Phi.$$

*The space of all $f \in X_0 + X_1$ for which $\|f\|_{(X_0,X_1)_\Phi}$ is finite, is an exact interpolation space of $(X_0, X_1)$.*

**Proof:** See [Proposition 3.3.1][5].

∎

As a consequence of Theorem 1.5.3, we can rewrite Definition 1.3.10 in terms of the $K$-functional for $(L^1_\mu, L^\infty_\mu)$. We can say that a function norm $\rho$ is u.r.i. if the condition

$$K(f,t,L^1_\mu,L^\infty_\mu) \leq K(f,t,L^1_\mu,L^\infty_\mu)$$

implies $\rho(f) \leq \rho(g)$. The monotonicity condition for a Banach function space (and thus for a parameter of the K-method $\Phi$) shows that $(L^1_\mu, L^\infty_\mu)_\Phi$ is a u.r.i. space. It is natural to ask if the converse is true, i.e. if a u.r.i. space is an exact interpolation space, or if an exact interpolation space is a u.r.i. space. The following theorem answers this question.

**Theorem 1.5.5** ***(Calderón)*** *The following are equivalent:*

- *$Z$ is a u.r.i. space over $(U, \Sigma, \mu)$.*
- *$Z = (L^1_\mu, L^\infty_\mu)_\Phi$ with identical norms.*
- *$Z$ is an exact interpolation space of $(L^1_\mu, L^\infty_\mu)$.*

**Proof:** See [3, Theorem 2.1, page 116] and [6].

It is not true in general, that every exact interpolation space is generated by the K-method. However, for some compatible couples like $(L^1_\mu, L^\infty_\mu)$, it will be the case that every exact interpolation space is generated by the K-method. To state this result we need two definitions, in the first one we distinguish the compatible couples for which we can decompose a function in terms of their K-functional.

**Definition 1.5.6** *A compatible couple $(X_0, X_1)$ is said to be divisible if there exists a constant $\gamma \in (0,\infty)$ such that for any sequence of nonnegative, concave functions $\{\omega_j\}_{j\in\mathbb{N}^+}$ such that $\sum_{j=1}^\infty \omega(1) < \infty$ and*

$$K(f,t,X_0,X_1) \leq \sum_{j=1}^{\infty} \omega_j(t), \quad \text{for all } t > 0,$$

*there exists a sequence $\{f_j\}_{j\in\mathbb{N}^+}$ such that*

$$K(f_j,t,X_0,X_1)(t) \leq \gamma\omega_j(t), \quad \text{for all } t \text{ and } j, \tag{1.7}$$

*and the series $\sum_{j=1}^\infty f_j$ converges to $f$ in $X_0 + X_1$.*

*The smallest constant $\gamma$ satisfying (1.7) is called the K-divisibility constant of $(X_0, X_1)$. Note that the divisibility constant is always at least* 1.

The second key definition is the following.

**Definition 1.5.7** *A couple compatible couple $(X_0, X_1)$ is called an exact Calderón couple if given the condition $K(f,t,X_0,X_1) \leq K(g,t,X_0,X_1)$, then there exists admissible operator $T$ such that $Tg = f$.*

We see that the couple $(L^1_\mu, L^\infty_\mu)$ is an exact Calderón couple and some properties of the operator $T$ from the definition above.

**Theorem 1.5.8** *Let $f, g \in L^1_\mu + L^\infty_\mu$ such that*

$$\int_0^t f^* \leq \int_0^t g^*, \quad \textit{for all } t > 0.$$

*Then, there exists an admissible contraction $T$ from $L^1_\mu + L^\infty_\mu$ to itself, such that $Tg = f$. Moreover, if $h$ is a nonnegative nonincreasing function, then $Th$ is nonnegative and nonincreasing.*

**Proof:** See [2, Theorem 5].
∎

The following result relates the Calderón property with $K$-divisibility.

**Theorem 1.5.9** *Let $(X_0, X_1)$ be an exact Calderón couple with divisibility constant 1, then if $Z$ is an exact interpolation space for $(X_0, X_1)$, there exists a parameter of the $K$-method $\Phi$ such that $(X_0, X_1)_\Phi = Z$ with identical norms.*

**Proof:** See [Propositions 3.3.1 and 4.4.5][5].
∎

One of the main results of this dissertation is Theorem 4.2.9, which exhibits a compatible couple of Banach spaces that is a Calderón couple with divisibility constant 1.

## 1.6 Monotonicity on the half line

We consider the notion of nondecreasing functions almost everywhere (See [26, Definition 2.1 and Theorem 2.4]).

**Definition 1.6.1** *Given a Borel measure $\lambda$ on $[0, \infty)$ we say that a function $f$ is nonincreasing almost everywhere if there exists a nonincreasing function $g$ such that $f = g$ up to a set of $\lambda$-measure is zero.*

The usual order relation for measurable functions is pointwise, that is, $f \leq g$ if the set $\{x \in [0, \infty) : g(x) > f(x)\}$ has zero $\lambda$-measure. Over the set of locally integrable nonnegative functions, we can define a second partial order, we denote $f \leq_\downarrow g$ if

$$\int_{[0,x]} f \, d\lambda \leq \int_{[0,x]} g \, d\lambda, \text{ for all } x > 0.$$

Notice that requiring local integrability ensures that the quantities involved are finite. Transitivity and reflexivity of $\leq_\downarrow$ is immediate. Antisymmetry follows from an application of Theorem 1.1.4, showing that if $f \leq_\downarrow g$ and $g \leq_\downarrow f$, then the measures

$$E \mapsto \int_E f d\lambda, \quad \text{and} \quad E \mapsto \int_E g d\lambda,$$

coincide. We will explore this type of order in more detail in Chapter 3.

The set of nonnegative locally integrable functions has two important constructions in the partial orders '$\le_\downarrow$' and '$\le$'.

The least decreasing majorant of $f$ is defined by

$$\widetilde{f}(x) = \text{ess sup}\{f(t) : t \ge x\}.$$

This function is a nonincreasing majorant of $f$ (see [26, Lemma 2.3]). It is optimal for the order '$\le$' in the sense that if $g$ is another nonincreasing majorant of $f$, then $\widetilde{f}(x) \le g(x)$. We can also describe this least decreasing majorant as a linear functional in the following way

$$\int_{[0,\infty)} \widetilde{f} g \, d\lambda = \sup \left\{ \int_{[0,\infty)} f h \, d\lambda : h \in L^+_\lambda \text{ and } h \le_\downarrow g \right\}.$$

Notice that the second partial order '$\le_\downarrow$' appears in this functional description. A proof of the above functional description is provided in [29, Theorem 2.1], however, we will give an alternative proof in a more general setting in Theorem 3.2.5.

There is another construction for a minimal decreasing majorant of $f$ in the partial order '$\le_\downarrow$', which needs the following definition.

**Definition 1.6.2** *(Level function) For any Borel measurable function $f$, we say that a nonincreasing function $f^o$ is a level function of $f$ if for any nonincreasing function $g$ we have*

$$\int_{[0,\infty)} f^o g \, d\lambda = \sup \left\{ \int_{[0,\infty)} |f| h \, d\lambda : h \in L^\downarrow_\lambda \text{ and } h \le_\downarrow g \right\}.$$

The next result shows the existence of $f^o$ and it being a least decreasing majorant with respect to the partial order '$\le_\downarrow$'.

**Proposition 1.6.3** *Let $\lambda$ be a $\sigma$-finite Borel measure over $[0, \infty)$ and $f$ be a nonnegative measurable function. Then there exists a nonincreasing function $f^o$ satisfying:*

1. *If $0 \le f \le g$, then $f^o \le g^o$.*
2. *If $0 \le f_n \uparrow f$, then $f_n^o \uparrow f^o$.*
3. *If $g$ is nonincreasing, then*

$$\int_{[0,\infty)} f^o g \, d\lambda = \sup \left\{ \int_{[0,\infty)} f h \, d\lambda : h \in L^\downarrow_\lambda \text{ and } h \le_\downarrow g \right\}.$$

4. *If $f$ is locally integrable, then $|f| \le_\downarrow f^o$ and if $h$ is a nonincreasing locally integrable function satisfying $|f| \le_\downarrow h$, then $f^o \le_\downarrow h$. That is, $f^o$ is the least decreasing majorant (for '$\le_\downarrow$') of $|f|$.*

**Proof:** For bounded functions, the internal structure of $f^o$ is shown in [26, Theorem 4.4]. Item (1) is proved in [26, Theorem 4.4]. Item (2) is proved in [26, Lemma 5.3]. The construction is extended by monotonicity to unbounded functions (see [27, Definition 2.3]) and items (1) and (2) are still valid.

Item (3) is [29, Theorem 2.3]. Item (4) is proved in [29, Lemma 2.2].

∎

There is also another description of the level function, to state it we consider a class of averaging operators, for a countable collection $\mathcal{I}$ of disjoint intervals of positive measure, defined by

$$Jf(x) = \begin{cases} \frac{1}{\lambda(I)} \int_I f\,d\lambda, & \text{if } x \in I \in \mathcal{I} \\ f(x), & \text{otherwise.} \end{cases}$$

**Proposition 1.6.4** *If $f$ is bounded and vanishes outside of an interval $[0, M]$ for some $M > 0$, then there exists an averaging operator $J_f$ such that $f^o = (J_f)f$ $\lambda$-a.e. The operator $J_f$ is a contraction on any u.r.i. space.*

**Proof:** The equality $J_f f = f^o$ follows from [29, Proposition 1.5]. To prove the second statement. The estimates

$$\left|\frac{1}{\lambda(I)}\int_I f\,d\lambda\right| \leq \frac{1}{\lambda(I)}\int_I |f|\,d\lambda \leq \frac{1}{\lambda(I)}\int_I \|f\|_{L^\infty_\lambda}\,d\lambda = \|f\|_{L^\infty_\lambda},$$

$$\begin{aligned} \int\limits_{[0,\infty)} |Jf|\,d\lambda &= \int_{\mathcal{I}} |Jf|\,d\lambda + \int_{x\notin\mathcal{I}} |f|\,d\lambda \leq \sum_{I\in\mathcal{I}} \int_I \left(\frac{1}{\lambda(I)}\int_I |f|\,d\lambda\right) d\lambda + \int_{x\notin\mathcal{I}} |f|\,d\lambda \\ &= \sum_{I\in\mathcal{I}} \int_I |f|\,d\lambda + \int_{x\notin\mathcal{I}} |f|\,d\lambda = \int\limits_{[0,\infty)} |f|\,d\lambda = \|f\|_{L^1_\lambda}. \end{aligned}$$

Show that $J_f$ is an admissible map with norm 1. Theorem 1.5.5 proves that $J_f$ is a contraction on any u.r.i. space.

∎

### 1.6.1 Down spaces

The inequality (1.4) cannot be improved without restricting the function $g$ in the sense that for a fixed $f$ there is a function $g$ such that the ratio of the two sides is as close to 1 as desired. However, if we know that $g$ is nonincreasing, then some improvement of the inequality can be expected. Consider the following norm

**Definition 1.6.5** ***(Down norm)*** *For a $\lambda$-measurable function $f$, let*

$$\rho_\downarrow(f) = \sup\left\{ \int\limits_{[0,\infty)} |f|\,g\,d\lambda : \rho'(g) \leq 1 \textit{ and } g \textit{ is nonincreasing} \right\}. \tag{1.8}$$

Had we not restricted $g$ to be nonincreasing, this is just the function norm $\rho''$, which coincides with $\rho$ in the presence of the Fatou property. In general, we have the inequality

$$\rho_{\downarrow}(f) \leq \rho''(f) \leq \rho(f).$$

We get the improvement of the inequality (1.4):

$$\int_{[0,\infty)} |f|\, g\, d\lambda \leq \rho_{\downarrow}(f)\rho'(g)$$

for all nonincreasing nonnegative functions $g$.

**Definition 1.6.6** ***(Down space)*** *Given a function norm $\rho$, the down space $L_\rho\!\downarrow$ is the collection of functions for which $\rho_{\downarrow}(f)$ is finite.*

Since $\rho_{\downarrow}(f) \leq \rho''(f)$, it follows that $L_\rho \subseteq L_{\rho''} \subseteq L_\rho\!\downarrow$. Some examples of down spaces are given next.

**Proposition 1.6.7** *We have the equality $L^1_\lambda = L^1_\lambda\!\downarrow$ with identical norms. The space $L^\infty_\lambda\!\downarrow$ has the norm*

$$\|f\|_{L^\infty_\lambda\downarrow} = \sup_{x\geq 0} \frac{1}{\lambda([0,x])} \int_{[0,x]} |f|\; d\lambda.$$

**Proof:** We will provide a proof of a generalized case in Theorem 4.1.3.

∎

If $\rho$ is a u.r.i. function norm we have the following relationship of the norm and the level function.

**Theorem 1.6.8** *Let $\rho$ be a u.r.i. function norm and $f \in L^+(\Sigma)$. Then,*

$$\rho_{\downarrow}(f) = \rho(f^o).$$

**Proof:** See [27, Theorem 2.2].

∎

We will consider another space, using the least decreasing minorant.

**Definition 1.6.9** *Given a function norm $\rho$ the space $\widetilde{L_\rho}$ is the collection of functions for which the norm*

$$\|f\|_{\widetilde{L_\rho}} = \|\widetilde{f}\|_{L_\rho},$$

*is finite.*

Since $f \leq \widetilde{f}$, it follows that $\widetilde{L_\rho} \subseteq L_\rho$. The relationship between the spaces $\widetilde{L_\rho}$ and $L_\rho\!\downarrow$ is given in the next result.

**Theorem 1.6.10** *Let $\rho$ be a u.r.i. function norm, then $(L_\rho\!\downarrow)' = \widetilde{L_{\rho'}}$ with identical norms.*

**Proof:** See [30].

∎

It follows from the last theorem that $\widetilde{L^\infty_\lambda} = (L^1_\lambda\downarrow)' = (L^1_\lambda)' = L^\infty_\lambda$. The interpolation properties of the compatible couples $(L^1_\lambda, L^\infty_\lambda\downarrow)$ and $(\widetilde{L^1_\lambda}, L^\infty_\lambda)$ are summarized in the following result.

**Theorem 1.6.11** *Let $\lambda$ be a Borel measure on $[0,\infty)$ then the following results hold:*

1. *For any $f \in L^1_\lambda + L^\infty_\lambda\downarrow$,*

$$K(f, t, L^1_\lambda, L^\infty_\mu\downarrow) = \int_0^t (f^o)^* = K(f^o, t, L^1_\lambda, L^\infty_\mu), \quad \textit{for all } t > 0. \tag{1.9}$$

2. *For any $g \in \widetilde{L^1_\lambda} + L^\infty_\lambda$,*

$$K(g, t, \widetilde{L^1_\lambda}, L^\infty_\mu) = \int_0^t (\widetilde{g})^* = K(\widetilde{g}, t, L^1_\lambda, L^\infty_\mu), \quad \textit{for all } t > 0. \tag{1.10}$$

3. *The couple $(L^1_\lambda, L^\infty_\lambda\downarrow)$ is an exact Calderón couple with divisibility constant* 1.

4. *The couple $(\widetilde{L^1_\lambda}, L^\infty_\lambda)$ is an exact Calderón couple.*

**Proof:** Item (1) is proved in [16, Theorem 5.4]. Item (2) follows from [25, Theorem 1]. For item (3), the Calderón property follows from [16, Theorem 4.6] and the divisibility constant is proved in [16, Corollaries 3.9 and 4.7]. Item (4) follows follows from [17, Theorem 4.3].

∎

## 1.7 Hardy inequalities

An important inequality due to Hardy ([9]) is

$$\left(\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)\,dt\right)^p dx\right)^{1/p} \le \frac{p}{p-1}\left(\int_0^\infty f^p\,dx\right)^{1/p}, \quad \text{for all } f \in L^+.$$

This inequality shows the boundedness of the Hardy operator $f \mapsto \frac{1}{x}\int_0^x f$ between $L^p$ to itself.

We will concern ourselves with different generalizations of this inequality. The first result gives necessary and sufficient conditions for the boundedness of the Hardy operator with two general measures.

**Theorem 1.7.1** *Let $\lambda, \nu$ be $\sigma$-finite Borel measures on $[0,\infty)$. Then, the best constant $C$ in the inequality*

$$\left(\int_{[0,\infty)} \left(\int_{[0,x]} f\,d\lambda\right)^q d\nu\right)^{1/q} \le C\left(\int_{[0,\infty)} f^p\,d\lambda\right)^{1/p}, \quad \textit{for all } f \in L^+, \tag{1.11}$$

*satisfies:*

1. *If* $1 < p \leq q < \infty$, *then*

$$C \approx \sup_{x \geq 0} \Big( \int_{[x,\infty)} d\nu \Big)^{1/q} \Big( \int_{[0,x]} d\lambda \Big)^{1/p'}.$$

2. *If* $1 < q < p < \infty$, *then*

$$C \approx \Big( \int_{[0,\infty)} \Big( \int_{[x,\infty)} d\nu \Big)^{r/q} \Big( \int_{[0,x]} d\lambda \Big)^{r/q'} d\lambda(x) \Big)^{1/r}.$$

3. *If* $0 < q < 1 < p < \infty$, *then*

$$C \approx \Big( \int_{[0,\infty)} \Big( \int_{[0,x]} d\lambda \Big)^{r/p'} \Big( \int_{[x,\infty)} d\nu \Big)^{r/p} d\nu(x) \Big)^{1/r}.$$

*Here* $1/p + 1/p' = 1$, $1/q + 1/q' = 1$ *and* $1/r + 1/p = 1/q$.

**Proof:** See [26, Theorem 7.1] with all the measures vanishing on $(-\infty, 0)$. Be mindful of a typo in the exponents for part (3).

$\blacksquare$

In the theorem above, the domain is the space $L^p_\lambda$ for $p > 1$. The case $p = 1$ must be treated separately. The next result explores this case.

**Theorem 1.7.2** *Let* $\mu, \nu, \eta$ *be Borel measures on* $[0, \infty)$ *and* $0 < q < 1 = p$, *then the best constant for which the inequality*

$$\Big( \int_{[0,\infty)} \Big( \int_{[0,x]} f \, d\lambda \Big)^q d\nu(x) \Big)^{1/q} \leq C \int_{[0,\infty)} f \, d\eta, \tag{1.12}$$

*holds for all nonnegative measurable functions satisfies*

$$C \approx \Big( \int_{[0,\infty)} \Big( \int_{[0,x]} \frac{1}{\underline{w}} d\nu \Big)^{\frac{q}{1-q}} d\nu(x) \Big)^{1/q}. \tag{1.13}$$

*Where* $\underline{w}(x) = \operatorname{ess\,inf}_\lambda \{w(t) : t \in [0, x]\}$, $d\eta = d\lambda^\perp + w d\lambda$ *and* $\lambda^\perp \perp \lambda$.

**Proof:** See [11, Theorem 3.1].

$\blacksquare$

We consider a very general class of Hardy operators introduced in [31], generalizing the domain from the half line to general measure spaces.

**Definition 1.7.3** *Let* $(U, \Sigma, \mu)$ *and* $(Y, \mathcal{T}, \tau)$ *be* $\sigma$*-finite measure spaces. We call* $B : Y \to \Sigma$ *a core map if:*

1. *The range of* $B$ *is a totally ordered subset of* $\Sigma$.

2. *For all $E \in \Sigma$, the map $y \to \mu(E \cap B(y))$ is $\mathcal{T}$-measurable and takes finite values.*

3. *There exists a countable set $Y_0 \subseteq Y$ such that $\cup_{y\in Y} B(y) = \cup_{y\in Y_0} B(y)$.*

*Given a core map, we say that the operator*

$$Tf(y) = \int_{B(y)} f\, d\mu,$$

*is an abstract Hardy operator.*

Given a core map, an inequality of the form

$$\Big(\int_Y \Big(\int_{B(y)} f\, d\mu\Big)^q d\tau(y)\Big)^{1/q} \le C\Big(\int_U f^p\, d\mu\Big)^{1/p},$$

for all positive measurable functions $f$ is called an *Abstract Hardy inequality*. Notice that setting $Y = U = [0,\infty)$ and $B(y) = [0, y]$ recovers the half line case. Abstract Hardy inequalities are not the most general Hardy inequalities we will consider. The next result shows that, provided $p > 1$, it is possible to reduce a three-measure Hardy inequality to an abstract Hardy inequality.

**Theorem 1.7.4** *Let $p \in (1,\infty)$, $q \in (0,\infty)$, $(U,\Sigma,\mu)$, $(Y,\mathcal{T},\tau)$ be $\sigma$-finite measure spaces, $B : Y \to \Sigma$ a core map. Decompose $\mu$ in the form $d\mu = d(\eta^\perp) + u\, d\eta$ with respect to $\eta$ such that $\eta^\perp \perp \eta$ and $u \in L^+(\Sigma)$. Define the measure $\nu$ by $d\nu = u^{p'} d\eta$. Suppose $\nu(B(y)) < \infty$ and $\eta^\perp(B(y)) = 0$ $\tau$-almost everywhere on $Y$. Then, $B$ is also a core map with respect to the measure $\nu$ and the best constant in the inequality*

$$\Big(\int_Y \Big(\int_{B(y)} f\, d\mu\Big)^q d\tau(y)\Big)^{1/q} \le C\Big(\int_U f^p\, d\eta\Big)^{1/p}, \tag{1.14}$$

*is also the best constant in the abstract Hardy inequality*

$$\Big(\int_Y \Big(\int_{B(y)} f d\nu\Big)^q d\tau(y)\Big)^{1/q} \le C\Big(\int_U f^p\, d\nu\Big)^{1/p}, \quad \text{for all } f \in L^+(\Sigma).$$

**Proof:** See [31, Theorem 5.1].

∎

Given an abstract Hardy inequality, the next theorem shows an equivalence with an inequality over the half line.

**Theorem 1.7.5** *Let $p \in [1,\infty]$, $q \in (0,\infty]$, $(U,\Sigma,\mu)$, $(Y,\mathcal{T},\tau)$ be $\sigma$-finite measure spaces, $B : Y \to \Sigma$ a core map. The best constant in the inequality*

$$\Big(\int_Y \Big(\int_{B(y)} f d\mu\Big)^q d\tau(y)\Big)^{1/q} \le C\Big(\int_U f^p\, d\mu\Big)^{1/p}, \quad \text{for all } f \in L^+(\Sigma)$$

*with appropriate adjustments to the formulas for the norms when $p = \infty$ or $q = \infty$, is the same as the best constant in the inequality*

$$\left( \int_0^\infty \left( \int_0^{b(x)} f(t)\,dt \right)^q dx \right)^{1/q} \leq C \Big( \int_U f(x)\,dx \Big)^{1/p}, \quad \textit{for all } f \in L^+.$$

*Where $b = (\mu \circ B)^*$ and the rearrangement is taken with respect to $\tau$. The function $b$ is called the normal form parameter.*

**Proof:** See [31, Theorem 2.4].

$\blacksquare$

Notice that the reduction from three measures to two measures done in Theorem 1.7.4 does not apply to the case $p = 1$. We will reserve the study for that case in Section 5.2.

# Chapter 2

# Ordered Cores

## 2.1 Basic properties

Let $(U, \Sigma, \mu)$ be a $\sigma$-finite measure space. We will not assume an order relation among the elements of the space. Instead, we will rely on a distinguished collection of measurable sets to establish the monotonicity properties.

**Definition 2.1.1** *Let $(U, \Sigma, \mu)$ be a $\sigma$-finite measure space. We say that $\mathcal{A} \subseteq \Sigma$ is an ordered core provided*

1. *The collection $\mathcal{A}$ contains the empty set.*
2. *(Total order) For any $E_1, E_2 \in \mathcal{A}$ either $E_1 \subseteq E_2$ or $E_2 \subseteq E_1$.*
3. *(Finite measure) $\mu(E) < \infty$ for any $E \in \mathcal{A}$.*

*If, in addition, $\cup\mathcal{A} = \cup A_0$ for some countable sub-collection $A_0 \subseteq \mathcal{A}$ we say that $\mathcal{A}$ is $\sigma$-bounded. If $\cup\mathcal{A} = U$ we say that $\mathcal{A}$ is full.*

*Any ordered core induces a relation $\leq_{\mathcal{A}}$ defined by $u \leq_{\mathcal{A}} v$ if for all $E \in \mathcal{A}$, $v \in E$ implies $u \in E$. The collection of non-negative measurable functions over $\sigma(\mathcal{A})$ will be denoted by $L^+(\mathcal{A})$.*

Observe that the order relation is transitive and reflexive, but in general, it need not be anti-symmetric. And it is total in the sense that for any pair $u, v \in U$ we have $u \leq_{\mathcal{A}} v$ or $v \leq_{\mathcal{A}} u$.

Based on this order we establish the following definitions.

**Definition 2.1.2** *Let $(U, \Sigma, \mu)$ be a $\sigma$-finite measure space and $\mathcal{A}$ be an ordered core.*

1. *The expression $u <_{\mathcal{A}} v$ means that $u \leq_{\mathcal{A}} v$ holds but $v \leq_{\mathcal{A}} u$ fails.*
2. *For any $u \in U$ the symbol $(\leftarrow, u]_{\mathcal{A}}$ denotes the set $\{s \in U : s \leq_{\mathcal{A}} u\}$ and the symbol $(\leftarrow, u)_{\mathcal{A}}$ denotes the set $\{s \in U : s <_{\mathcal{A}} u\}$.*
3. *For any $u \in U$ the symbol $[u]_{\mathcal{A}}$ denotes the set $\{s \in U : u \leq_{\mathcal{A}} s$ and $s \leq_{\mathcal{A}} u\}$.*

We establish some basic properties of the relation.

**Proposition 2.1.3** *Let $(U, \Sigma, \mu)$ be a $\sigma$-finite measure space, $\mathcal{A}$ be an ordered core and $u, v \in U$. The following statements are equivalent.*

1. $v \in (\leftarrow, u)_{\mathcal{A}}$.
2. *There exists $E \in \mathcal{A}$ such that $v \in E$ and $u \notin E$.*

**Proof:** To show item (1) → item (2): Let $v \in (\leftarrow, u)_{\mathcal{A}}$. By definition $u \leq_{\mathcal{A}} v$ fails and item (2) is the negation of $u \leq_{\mathcal{A}} v$. Conversely, if item (2) holds, then $u \leq_{\mathcal{A}} v$ fails. Since the relation $\leq_{\mathcal{A}}$ is total, $v \leq_{\mathcal{A}} u$ holds. Thus (1) holds.

∎

As a consequence, we have a description of the sets $(\leftarrow, u)_{\mathcal{A}}$ and $(\leftarrow, u]_{\mathcal{A}}$ in terms of unions and intersections.

**Corollary 2.1.4** *Let $(U, \Sigma, \mu)$ be a $\sigma$-finite measure space, $\mathcal{A}$ an ordered core and $u \in U$. Then,*

$$(\leftarrow, u)_{\mathcal{A}} = \bigcup_{E \in \mathcal{A}, u \notin E} E, \quad (\leftarrow, u]_{\mathcal{A}} = \bigcap_{E \in \mathcal{A}, u \in E} E.$$

**Proof:** The implication (1) → (2) in Proposition 2.1.3 shows $(\leftarrow, u)_{\mathcal{A}} \subseteq \bigcup_{E \in \mathcal{A}, u \notin E} E$. The statement $\bigcup_{E \in \mathcal{A}, u \notin E} E \subseteq (\leftarrow, u)_{\mathcal{A}}$ follows from the implication (2) → (1) of Proposition 2.1.3. This completes the proof of the first statement.

For the second statement, let $v \in (\leftarrow, u]_{\mathcal{A}}$, then by definition of the order relation $\leq_{\mathcal{A}}$, if $u \in E \in \mathcal{A}$, it follows that $v \in E$. This shows that $(\leftarrow, u]_{\mathcal{A}} \subseteq \bigcap_{E \in \mathcal{A}, u \in E} E$. To prove equality, let $v \notin (\leftarrow, u]_{\mathcal{A}}$, then $v \nleq_A u$, hence $u \in (\leftarrow, v)_{\mathcal{A}}$. By Proposition 2.1.3 there exists $E \in \mathcal{A}$ such that $u \in E$ and $v \notin E$. Therefore $v \notin \bigcap_{E \in \mathcal{A}, u \in E} E$, which completes the proof.

∎

It is worth noting that the sets introduced in the previous definition need not be measurable. An example of this phenomenon is exhibited in Section 2.1.1. We can force these sets to be measurable by considering a subclass of ordered cores. Some of the constructions developed in later sections take simpler forms when restricted to this subclass.

**Definition 2.1.5** *Let $(U, \Sigma, \mu)$ be a $\sigma$-finite measure space, $\mathcal{A}$ is called a separable ordered core if for any $C \subseteq \mathcal{A}$ the sets $\cup C$ and $\cap C$ are $\sigma(\mathcal{A})$-measurable.*

### 2.1.1 Examples

In this subsection, we establish some examples of ordered cores and their induced order relations. We begin with the simplest example:

**Example 2.1.6** *Let $U = [0, \infty)$, $\Sigma$ the Borel $\sigma$-algebra, $\mu$ any Borel measure finite on compact sets and $\mathcal{A} = \{\emptyset\} \cup \{[0, x] : x > 0\}$. If $x > 0$ the sets introduced in Definition 2.1.2 are*

$$(\leftarrow, x]_{\mathcal{A}} = [0, x], \ (\leftarrow, x)_{\mathcal{A}} = [0, x) \text{ and } [x]_{\mathcal{A}} = \{x\}.$$

The next example exhibits the effect of changing the ordered core.

**Example 2.1.7** *Let* $U = [0, \infty)$*,* $\Sigma$ *the Borel* $\sigma$*-algebra,* $\mu$ *any Borel measure finite on compact sets and* $\mathcal{A} = \{\emptyset\} \cup \{[0, n] : n \in \mathbb{N}\}$*. If* $n$ *is a positive integer and* $n - 1 < x \leq n$*, then the sets introduced in Definition 2.1.2 are*

$$(\leftarrow, x]_{\mathcal{A}} = [0, n], \ (\leftarrow, x)_{\mathcal{A}} = [0, n-1] \text{ and } [x]_{\mathcal{A}} = (n-1, n].$$

*Also* $[0] = \{0\}$.

Any set can induce an ordered core on the half-line, as exhibited in the next example, this generalizes the previous two examples.

**Example 2.1.8** *Let* $U \subseteq [0, \infty)$*,* $\Sigma$ *the Borel* $\sigma$*-algebra,* $\mu$ *any Borel measure finite on compact sets,* $S \subseteq U$*, Borel measurable and* $\mathcal{A} = \{\emptyset\} \cup \{[0, x] \cap U : x \in S\}$*.*

Perhaps the simplest example outside of the half-line comes from the Euclidean spaces.

**Example 2.1.9** *Let* $U = \mathbb{R}^d$*,* $\Sigma$ *the Borel* $\sigma$*-algebra,* $\mu$ *any Borel measure finite on compact sets and* $\mathcal{A} = \{\emptyset\} \cup \{B[0; r] : r > 0\}$*, where* $B[0; r] = \left\{s \in \mathbb{R}^d : |s| \leq r\right\}$*. If* $x \in \mathbb{R}^d$*, the sets introduced in Definition 2.1.2 are*

$$(\leftarrow, x]_{\mathcal{A}} = \left\{s \in \mathbb{R}^d : |s| \leq |x|\right\}, \ (\leftarrow, x)_{\mathcal{A}} = \left\{s \in \mathbb{R}^d : |s| < |x|\right\} \text{ and } [x]_{\mathcal{A}} = \left\{s \in \mathbb{R}^d : |s| = |x|\right\}.$$

**Example 2.1.10** *Consider a metric measure space, that is a set* $\mathbb{X}$ *together with a metric* $d$ *and a measure* $\mu$ *defined on the Borel sets induced by the metric such that* $\mu(B[a; r]) < \infty$ *for every* $r > 0$ *and* $a \in \mathbb{X}$*. The collection* $\mathcal{A} = \{B[a; r]\}_{r>0} \cup \{\emptyset\}$ *is an ordered core. If* $x \in \mathbb{X}$*, the sets introduced in Definition 2.1.2 are*

$$(\leftarrow, x]_{\mathcal{A}} = B[a; |x|_a], \ (\leftarrow, x)_{\mathcal{A}} = B(a; |x|_a) \text{ and } [x]_{\mathcal{A}} = B[a; |x|_a] \setminus B(a; |x|_a),$$

*where* $|x|_a$ *denotes the distance form* $a$ *to* $x$*.*

In the next examples, an ordered core is induced by a measurable function

**Example 2.1.11** *Let* $(U, \Sigma, \mu)$ *be a* $\sigma$*-finite measure space,* $\varphi : U \to \mathbb{C}$ *be a measurable function with distribution function taking finite values, then the set*

$$\mathcal{A} = \{\emptyset\} \cup \{\{s \in U : |\varphi(s)| > r\} : r > 0\}$$

*is an ordered core, and for any* $u \in U$*, the sets introduced in Definition 2.1.2 are*

$$(\leftarrow, u]_{\mathcal{A}} = \{s \in U : |\varphi(u)| \leq |\varphi(s)|\}, \ (\leftarrow, u)_{\mathcal{A}} = \{s \in U : |\varphi(u)| < |\varphi(s)|\}$$

*and*

$$[u]_{\mathcal{A}} = \{s \in U : |\varphi(s)| = |\varphi(u)|\}.$$

*This construction recovers the metric measure space example, by considering* $\varphi_a(s) = \frac{1}{dist(a,s)}$ *and* $\varphi_a(a) = 0$*.*

The ordered cores in the previous examples are separable. The next example shows that this needs not be the case.

**Example 2.1.12** *Let $V$ and $W$ be disjoint copies of $\omega_1$, the first uncountable ordinal ordered by inclusion. Their smallest elements are denoted $0_V$ and $0_W$ respectively. Let $U = V\dot\cup W$ be their disjoint union and express any subset as $E\dot\cup F$, where $E \subseteq U$ and $F \subseteq W$. Let*

$$\Sigma_0 = \sigma_{cc}(U) \quad \text{and} \quad \Sigma_1 = \{E\dot\cup F : E \in \sigma_{cc}(V), F \in \sigma_{cc}(W)\}.$$

*Here $\sigma_{cc}(S)$ denotes the $\sigma$-algebra generated by the countable subsets of $S$.*

*Define the measure $\mu_1$ on $\Sigma_1$ by*

$$\mu_1(E\dot\cup F) = \delta_{0_V}(E) + \delta_{0_W}(F) + \begin{cases} 0, & E, F \text{ countable}; \\ 1, & V \setminus E, F \text{ countable}; \\ 2, & E, W \setminus F \text{ countable}; \\ 3, & V \setminus E, W \setminus F \text{ countable}; \end{cases}$$

*and let $\mu_0$ be the restriction of $\mu_1$ to $\Sigma_0$. Note that $\mu_0$ and $\mu_1$ are finite, complete measures. We introduce the core*

$$\mathcal{A} = \{\alpha\dot\cup\emptyset : \alpha \text{ is a countable ordinal}\} \cup \{V\dot\cup(W \setminus \alpha) : \alpha \text{ is a countable ordinal}\}.$$

*This collection is an ordered core on the measure spaces $(U, \Sigma_0, \mu_0)$ and $(U, \Sigma_1, \mu_1)$. Notice that $\Sigma_0 = \sigma(\mathcal{A})$. The induced order is $\alpha \leq_{\mathcal{A}} \beta$ if $\alpha \in V$ and $\beta \in W$ or*

$$\alpha \leq_{\mathcal{A}} \beta \quad \text{if} \quad \begin{cases} \alpha \subseteq \beta, & \alpha \in V \text{ and } \beta \in V; \\ \beta \subseteq \alpha, & \alpha \in W \text{ and } \beta \in W. \end{cases}$$

*For $\alpha \in V$ we have*

$$(\leftarrow, \alpha]_{\mathcal{A}} = \alpha + 1, \quad \text{and} \quad (\leftarrow, \alpha)_{\mathcal{A}} = \alpha,$$

*where $\alpha + 1$ denotes the successor ordinal. For $\beta \in W$ we have*

$$(\leftarrow,\beta]_{\mathcal{A}} = V\dot\cup(W \setminus \beta), \quad \text{and} \quad (\leftarrow,\beta)_{\mathcal{A}} = V\dot\cup(W \setminus (\beta + 1)).$$

*Notice that the core is not separable, with respect to $\Sigma_0$, since the set*

$$\bigcup_{\alpha\in V} \alpha\dot\cup\emptyset = V\dot\cup\emptyset$$

*is not $\Sigma_0$-measurable, which is exactly the same as not being $\sigma(\mathcal{A})$-measurable.*

## 2.2 Core decreasing functions

We define a collection of functions that behaves similarly to the cone of decreasing functions on the half-line. The main definition is the following.

**Definition 2.2.1** *Let $(U, \Sigma, \mu)$ be a $\sigma$-finite measure space and $\mathcal{A}$ be an ordered core. A function $f : U \to [-\infty, \infty]$ is called decreasing (relative to $\mathcal{A}$) if for all $u, v \in U$, $u \leq_{\mathcal{A}} v$ implies $f(v) \leq f(u)$. A nonnegative, decreasing, $\sigma(\mathcal{A})$-measurable function is called core decreasing. A function $f$ is core decreasing $\mu$-almost everywhere if there exists a core decreasing function $g$ such that $f = g$ $\mu$-almost everywhere. The collection of equivalence classes of core decreasing functions is denoted $L^{\downarrow}(\mathcal{A})$.*

The collection of nonnegative constant functions is always contained in $L^{\downarrow}(\mathcal{A})$, therefore there always exist core decreasing functions.

It is immediate from the definition that the collection $\{\chi_E : E \in \mathcal{A}\}$ belongs to $L^{\downarrow}(\mathcal{A})$. In fact, the characteristic function of any $\sigma(\mathcal{A})$-measurable set $M$ is core decreasing whenever $M$ satisfies the condition: for all $u, v \in U$, if $v \in M$ and $u \leq_{\mathcal{A}} v$, then $u \in M$.

A fundamental collection of sets with the property above is characterized in the following lemma.

**Lemma 2.2.2** *Suppose $M \in \sigma(\mathcal{A})$ and $\mu(M) < \infty$. The following are equivalent*

*(a) There is a countable nonempty subset $C$ of $\mathcal{A}$ such that $M = \cup C$ or $M = \cap C$.*

*(b) There is a nonempty subset $C$ of $\mathcal{A}$ such that $M = \cup C$ or $M = \cap C$.*

*(c) For all $u, v \in U$, if $v \in M$ and $u \leq_{\mathcal{A}} v$, then $u \in M$.*

**Proof:** It is clear that (a) implies (b). We now show that (b) implies (c). Let $v \in \cup C$ for some $C$ and $u \leq_{\mathcal{A}} v$, then there exists $A \in C$ such that $v \in A$. It follows from the relation '$\leq_{\mathcal{A}}$' that $u \in A$ as well, thus $u \in \cup C$. For the remaining case, suppose that $v \in \cap C$ for some $C$ and $u \leq_{\mathcal{A}} v$, then for all $E \in C$ we have $v \in E$ so $u \in V$. This shows that $u \in \cap C$ and proves the implication.

It remains to show that (c) implies (a). Suppose that $M \in \sigma(\mathcal{A})$ and satisfies (c), by Theorem 1.1.3 there exists a countable subset $A_0 \subseteq \mathcal{A}$ such that $M \in \sigma(A_0)$. Define the collections

$$\mathcal{L} = \{E \in A_0 : E \subseteq M\}, \quad \mathcal{N} = \{E \in A_0 : M \subseteq E\}.$$

Clearly $\mathcal{L}$ and $\mathcal{N}$ are countable subsets of $\mathcal{A}$, thus the sets $L_0 = \cup\mathcal{L}$ and $N_0 = \cap\mathcal{N}$ belong to $\sigma(A_0)$.

We show that $\mathcal{L} \cup \mathcal{N} = A_0$. Suppose that there exists $E \in A_0$ such that both $E \subseteq M$ and $M \subseteq E$ both fail. Choose $u \in M \setminus E$ and $v \in E \setminus M$. Since $v \in E$ and $u \notin E$, then by Proposition 2.1.3 we have $v \in (\leftarrow, u)_{\mathcal{A}}$, hence $v \leq_{\mathcal{A}} u$. Since $u \in M$, then the statement $v \notin M$ contradicts the fact that $M$ satisfies (c).

Let $U_0 = \cup A_0$, $C = N_0 \setminus L_0$ and define the collection

$$\mathcal{K} = \{E \in \sigma(A_0) : C \subseteq U_0 \setminus E \text{ or } C \subseteq E\}.$$

If $E \in \mathcal{L}$, then $C \subseteq U_0 \setminus E$. If $E \in \mathcal{N}$, then $C \subseteq E$. Therefore $A_0 \subseteq \mathcal{K}$. We now show that $\mathcal{K}$ is a $\sigma$-algebra over $U_0$.

It is clear that $\emptyset \in \mathcal{K}$ and by construction $\mathcal{K}$ is closed under complements. Suppose $\{E_n\}$ is a sequence in $\mathcal{K}$. If $C \subseteq E_n$ for some $n \in \mathbb{N}$, then $C \subseteq \cup_n E_n$. If $C \subseteq E_n$ fails for every $n \in \mathbb{N}$, then by construction, we have that $C \subseteq U_0 \setminus E_n$ for all $n \in \mathbb{N}$. It follows that

$$C \subseteq \cap_n (U_0 \setminus E_n) = U_0 \setminus (\cup_n E_n).$$

It follows that $M \in \sigma(A_0) \subseteq \mathcal{K}$. But $L_0 \subseteq M \subseteq N_0$, so if $C \subseteq M$, then $N_0 \subseteq L_0 \cup C \subseteq M \subseteq N_0$, and if $C \subseteq U_0 \setminus M$, then $L_0 \subseteq M \subseteq N_0 \setminus C = L_0$. So (a) holds with either $C = \mathcal{L}$ or $C = \mathcal{N}$. This completes the proof.

∎

With the previous lemma, we can enrich the core $\mathcal{A}$, by adding all the $\sigma(\mathcal{A})$-measurable sets with finite $\mu$-measure that do not change the relation $\leq_{\mathcal{A}}$.

**Theorem 2.2.3** *Let $(U, \Sigma, \mu)$ be a $\sigma$-finite measure space, $\mathcal{A}$ be an ordered core and $\mathcal{M}$ be the collection of all $M \in \sigma(\mathcal{A})$ of finite measure satisfying the conditions in Lemma 2.2.2. Then $\mathcal{M}$ is an ordered core of $U$, $\mathcal{A} \subseteq \mathcal{M}$ and the relations $\leq_{\mathcal{A}}$ and $\leq_{\mathcal{M}}$ coincide. If $\mathcal{A}$ is $\sigma$-bounded, so is $\mathcal{M}$. If $\mathcal{A}$ is full, so is $\mathcal{M}$. In addition, $\mathcal{M}$ is closed under countable intersections and under countable unions provided the result has finite measure.*

**Proof:** It is clear from the construction of $\mathcal{M}$ that $\mathcal{A} \subseteq \mathcal{M}$ and each set has finite measure. Also, $\emptyset \in \mathcal{M}$ is clear. To show that $\mathcal{M}$ is totally ordered by inclusion: let $N, M \in \mathcal{M}$ with $N \nsubseteq M$. Let $x \in N \setminus M$. If $y \in M$ then if $x \leq_{\mathcal{A}} y$ we would have $x \in M$ by item (c) in Lemma 2.2.2. Therefore $y \leq_{\mathcal{A}} x$ and by item (c) in Lemma 2.2.2 we get that $y \in N$. Therefore $M \subseteq N$ and shows total order, thus $\mathcal{M}$ is an ordered core.

Since $\mathcal{A} \subseteq \mathcal{M}$, then $u \leq_{\mathcal{M}} v$ implies $u \leq_{\mathcal{A}} v$. Conversely, if $u \leq_{\mathcal{A}} v$ and $v \in M$, then by item (c) in Lemma 2.2.2 we get that $u \in V$, that is $u \leq_{\mathcal{M}} v$. Therefore the relations $\leq_{\mathcal{M}}$ and $\leq_{\mathcal{A}}$ coincide.

Since $\mathcal{A} \subseteq \mathcal{M}$ then it is clear that $\mathcal{A}$ being full implies $\mathcal{M}$ is full. By construction each $M \in \mathcal{M}$ is contained in $\cup\mathcal{A}$, it follows that $\cup\mathcal{M} \subseteq \cup\mathcal{A}$. Therefore, if $\mathcal{A}$ is $\sigma$-bounded, there exists $A_n \in \mathcal{A}$ such that $\cup A_n = \mathcal{M}$. It follows that $\mathcal{M}$ is also $\sigma$-bounded.

Let $\{M_n\}$ be a sequence in $\mathcal{M}$. If $M = \cap_n M_n = \emptyset$ then there is nothing to prove. Since the sequence is countable, then $M \in \sigma(\mathcal{A})$ and it has finite measure since $\mu(M) \leq \mu(M_1) < \infty$. If $u \leq_{\mathcal{A}} v$ and $v \in M$ then $v \in M_n$ for each $n \in \mathbb{N}$. By item (c) in Lemma 2.2.2 we conclude that $u \in M_n$ so $u \in M$ and we conclude that $M \in \mathcal{M}$. This shows that $\mathcal{M}$ is closed under countable intersections.

Similarly, if $M = \cup_{M_n}$ and $\mu(M) < \infty$, then if $u \leq_{\mathcal{A}} v$ and $v \in M$, then $v \in M_n$ for some $n \in \mathcal{M}$. Again, it follows that $u \in M_n$ so $u \in M$, which completes the proof.

∎

We define the enriched core as follows.

**Definition 2.2.4** *Let $(U, \Sigma, \mu)$ be a $\sigma$-finite measure space, $\mathcal{A}$ be an ordered core. The collection of sets $\mathcal{M} \subseteq \sigma(\mathcal{A})$ described in Lemma 2.2.2 is called the enriched core of $\mathcal{A}$.*

We can characterize the set of core decreasing functions as increasing limits of simple functions for sets in the enriched core $\mathcal{M}$ constructed above.

**Lemma 2.2.5** *Suppose $\mathcal{A}$ is $\sigma$-bounded and $f : U \to [0, \infty)$. Then $f \in L^{\downarrow}(\mathcal{A})$ if and only if it is the pointwise limit of an increasing sequence of simple functions of the form*

$$\sum_{k=1}^{n} \alpha_k \chi_{M_k},$$

*for $\alpha_k > 0$ and $M_k \in \mathcal{M}$ for each $k$.*

**Proof:** Suppose that $f$ is core decreasing. Since $\mathcal{A}$ is $\sigma$-bounded, there exists a sequence $\{A_n\}_{n\in\mathbb{N}} \in \mathcal{A}$ such that $A_n \uparrow \cup\mathcal{A}$.

For each $n, k \in \mathbb{N}^+$, the set

$$M_{n,k} = \{u \in A_n : f(u) \geq k2^{-n}\},$$

is $\sigma(\mathcal{A})$-measurable. Let $u \in M_{n,k}$ and $v \leq_{\mathcal{A}} u$. Since $f$ is core decreasing, we get $k2^{-n} \leq f(u) \leq f(v)$, thus $v \in M_{n,k}$. We also have $\mu(M_{n,k}) \leq \mu(A_n) < \infty$. It follows that $M_{n,k} \in \mathcal{M}$ by Lemma 2.2.2.

Define the sequence of functions $f_n$ by

$$f_n(u) = \sum_{k=1}^{n2^n} 2^{-n}\chi_{M_{n,k}}(u) = 2^{-n}\lfloor 2^n \min(f(u), n)\rfloor \chi_{A_n}(u),$$

where $\lfloor s\rfloor$ denotes the greatest integer less than or equal to $s$. To see that both representations of $f_n$ are equivalent, first notice that both expressions vanish if $u \notin A_n$. If $u \in M_{n,k}$ for all $k \in \{1, \ldots, n2^n\}$, then both expressions evaluate to $n$. If $u \notin M_{n,k}$ for all $k \in \{1, \ldots, n2^n\}$, then $f(u) < 2^{-n}$, so both expressions evaluate to zero. For the final case, let

$$k_0 = \max\{k \in \{1, \ldots, (n-1)2^n\} : u \in M_{n,k}\},$$

then $k_0 2^{-n} \leq f(u) < (k_0 + 1)2^{-n}$, then the right-hand side evaluates to $k_0 2^{-n}$. Based on the observation that $u \in M_{n,k}$ for all $k = 1, \ldots, k_0$ we get that the left-hand side also evaluates to $k_0 2^{-n}$.

The first representation shows that $f_n$ is a linear combination of the desired form. The second one shows that $f_n \uparrow f$. To see this, we show that $\{f_n\}$ is an increasing sequence:

$$f_{n+1}(u) = 2^{-(n+1)}\lfloor 2^{n+1} \min(f(u), n+1)\rfloor \chi A_{n+1}(u) \geq 2^{-n}\lfloor 2^n \min(f(u), n+1)\rfloor \chi A_{n+1}(u),$$

using the inequality $\lfloor 2s\rfloor \geq 2\lfloor s\rfloor$. It follows that $f_{n+1}(u) \geq f_n(u)$. Using the inequality $\lfloor s\rfloor \leq s \leq \lfloor s\rfloor + 1$ we get

$$f_n(u) \leq \min(f(u), n) \leq f_n + \frac{1}{2^n},$$

letting $n \to \infty$ yields $f_n \uparrow f$ and completes the proof of the first implication.

Conversely, it is clear from 2.2.2 that $\chi_{M_k}$ is core decreasing for any $M_k \in \mathcal{M}$. Positive scaling and addition of core decreasing functions is core decreasing. Taking increasing limits of core decreasing functions keeps the measurability requirement and respects inequalities, therefore it remains core decreasing and completes the proof.

∎

**Example 2.2.6** *Consider the ordered core over* $[0, \infty)$ *defined in Example 2.1.6. The core decreasing functions are nonnegative decreasing (in the usual sense) functions. Considering the ordered core from Example* 2.1.7 *changes the set of core decreasing functions, to decreasing functions that are constant on every set* $(n-1, n]$ *for each* $n \in \mathbb{N}^+$.

*More generally, consider the core from Example 2.1.8 whenever* $S$ *is closed. In this case, the core decreasing functions are nonnegative decreasing functions that are constant on each connected component of* $U \setminus S$. *Notice, that since* $S$ *is closed, there are countably many such connected components.*

**Example 2.2.7** *Consider the ordered core over $\mathbb{R}^d$ defined in Example 2.1.9. The core decreasing functions are nonnegative radially decreasing functions.*

*Similarly, over a metric measure space with the ordered core defined in 2.1.10, the core decreasing functions are radially decreasing functions with respect to the fixed point $a \in \mathbb{X}$. In other words, functions satisfying $f(u) \leq f(v)$ whenever $dist(a, v) \leq dist(a, u)$.*

**Example 2.2.8** *Consider the ordered core defined in Example 2.1.11. The core decreasing functions are the nonnegative functions $f$ that are similarly ordered to $|\varphi|$; that is,*

$$|\varphi(u)| \leq |\varphi(v)| \Longleftrightarrow f(u) \leq f(v), \quad \forall u, v \in U.$$

We finish the examples by noting that a function being decreasing with respect to the ordered core is not the same as the function being core decreasing.

**Example 2.2.9** *Consider the measure space and ordered core from Example 2.1.12. Let $f = \chi_{V \dot{\cup} \emptyset}$. Then, $f(\alpha) = 1$ if $\alpha \in V$ and $f(\alpha) = 0$ if $\alpha \in W$, therefore $f$ is decreasing with respect to the core $\mathcal{A}$, but it is not $\Sigma_0$-measurable.*

*Therefore, over $(U, \Sigma_0, \mu)$, $f$ is a decreasing function with respect to $\mathcal{A}$ that is not measurable. Over the measure space $(U, \Sigma_1, \mu)$, the function $f$ is measurable and decreasing with respect to $\mathcal{A}$, but not core decreasing, since it is not $\sigma(\mathcal{A})$-measurable.*

We can define core increasing functions similarly.

**Definition 2.2.10** *Let Let $(U, \Sigma, \mu)$ be a $\sigma$-finite measure space, $\mathcal{A}$ be an ordered core. A function $f : U \to [-\infty, \infty]$ is called increasing (relative to $\mathcal{A}$) if for all $u, v \in U$, $u \leq_{\mathcal{A}} v$ implies $f(u) \leq f(v)$. A nonnegative increasing $\sigma(\mathcal{A})$-measurable function is called core increasing. The collection of core increasing functions is denoted $L^{\uparrow}(\mathcal{A})$.*

Suppose that $\mathcal{A}$ is an ordered core over a finite measure space $(U, \Sigma, \mu)$, then the collection of sets

$$\widehat{\mathcal{A}} = \{\emptyset\} \cup \{U \setminus A : A \in \mathcal{A}\},$$

is also an ordered core. Notice that $\sigma(\mathcal{A}) = \sigma(\widehat{\mathcal{A}})$, so it has the same measurable functions and we get that the preorder gets reversed, that is,

$$u \leq_{\mathcal{A}} v \Longleftrightarrow v \leq_{\widehat{\mathcal{A}}} u, \ \forall u, v \in U.$$

Thus, we get the correspondence

$$f \text{ is core increasing with respect to } \mathcal{A} \Longleftrightarrow f \text{ is core decreasing with respect to } \widehat{\mathcal{A}},$$

for any $\sigma(\mathcal{A})$-measurable function $f$.

We need to do an approximation argument to get the analogous result to Lemma 2.2.5 for core increasing functions.

**Lemma 2.2.11** *Suppose $\mathcal{A}$ is a $\sigma$-bounded full ordered core and $f : U \to [0, \infty)$. Then $f \in L^{\uparrow}(\mathcal{A})$ if and only if it is the pointwise limit of an increasing sequence of simple functions of the form*

$$\sum_{k=1}^{n} \alpha_k \chi_{(U \setminus M_k)},$$

*for $\alpha_k > 0$ and $M_k \in \mathcal{M}$ for each $k$.*

**Proof:** Let $\{A_n\} \in \mathcal{A}$ be a sequence satisfying $A_n \uparrow U$. Let $f \in L^{\uparrow}(\mathcal{A})$ and set $f_n = f\chi_{A_n}$. The function $f_n$ is core increasing with respect to the core $\{A_n \cap A : A \in \mathcal{A}\}$, since $\mu(A_n) < \infty$, the collection $\{A_n \setminus (A_n \cap A) : A \in \mathcal{A}\}$ is an ordered core and $f_n$ is core increasing with respect to it. Therefore, Lemma 2.2.5 yields a sequence $g_{n,m} \uparrow_m f_n$ where

$$g_{n,m} = \sum_{k=1}^{r_m} \alpha_{n,k} \chi_{(A_n \setminus M_{m,k})},$$

for some $\alpha_{n,k} > 0$, $M_{m,k} \in \mathcal{M}$ and $r_m \in \mathbb{N}$.

Define the function $h_{n,m} = \sum_{k=1}^{r_m} \alpha_{n,k} \chi_{(U \setminus M_{m,k})}$ and note that $h_{n,m}$ and $g_{n,m}$ coincide on $A_n$. Define

$$w_n = \max\{h_{j,j} : j \leq n\},$$

then it follows that $w_n \uparrow f$.

∎

## 2.3 Morphisms of ordered cores

In this section we study maps between ordered cores called *core morphisms*, the main result of this section is that any core morphism induces a linear map between a large vector space of functions, that is well-behaved when restricted to the cone of nonnegative measurable functions.

**Definition 2.3.1** *Let $(U, \Sigma, \mu), (T, \mathcal{T}, \tau)$ be $\sigma$-finite measure spaces, the collection $\mathcal{A} \subseteq \mathcal{T}$ be an ordered core. Then a map $r : \mathcal{A} \to \mathcal{T}$ is called a core morphism provided:*

1. *There exists a constant $c > 0$ such that $\tau(r(B) \setminus r(A)) \leq c\mu(B \setminus A)$ for every $A, B \in \mathcal{A}$.*

2. *The map $r$ is order-preserving with respect to inclusion.*

3. *The set $r(\emptyset)$ satisfies $\tau(r(\emptyset)) = 0$.*

Notice that $\{r(A) \setminus r(\emptyset) : A \in \mathcal{A}\} \cup \{\emptyset\}$ is an ordered core.

We will need the following measure-theoretic results about the $\sigma$-ring generated by the core.

**Lemma 2.3.2** *Let $(U, \Sigma, \mu)$ be a $\sigma$-finite measure space, $\mathcal{A}$ an ordered core. Then the collection*

$$\mathcal{A}^+ = \{B \setminus A : B, A \in \mathcal{A}\}$$

*is a semiring.*

*Moreover, for any $A, B, \{B_k\}, \{A_k\} \in \mathcal{A}$*

$$(B \setminus A) \setminus \bigcup_{k=1}^{n} (B_k \setminus A_k) = \bigcup_{k=1}^{m} (B'_k \setminus A'_k), \tag{2.1}$$

*for some disjoint collection $\{(B'_k \setminus A'_k)\}_{k=1}^m$ in $\mathcal{A}^+$.*

**Proof:** Let $A_1, A_2, B_1, B_2 \in \mathcal{A}$. To avoid that $B_1 \setminus A_1$ and $B_2 \setminus A_2$ are empty, we may assume that $A_1 \subseteq B_1$, $A_2 \subseteq B_2$. After a relabelling, we may assume that $B_1 \subseteq B_2$. If $B_1 \subseteq A_2$ or $B_1 = A_1$, then

$$(B_2 \setminus A_2) \cap (B_1 \setminus A_1) = \emptyset, \quad \text{and} \quad (B_2 \setminus A_2) \setminus (B_1 \setminus A_1) = B_2 \setminus A_2.$$

In other cases, since $B_1 \subseteq A_2$ fails, by the total order of $\mathcal{A}$, we may assume that $A_2 \subseteq B_1$. Then

$$(B_2 \setminus A_2) \cap (B_1 \setminus A_1) = B_1 \setminus (A_2 \cup A_1), \quad \text{and} \quad (B_2 \setminus A_2) \setminus (B_1 \setminus A_1) = (B_2 \setminus B_1) \cup (A_j \setminus A_k),$$

where $A_j = A_1 \cup A_2$ and $A_k = A_1 \cap A_2$. Notice that in every case $(B_2 \setminus A_2) \cap (B_1 \setminus A_1) \in \mathcal{A}$ and $(B_2 \setminus A_2) \setminus (B_1 \setminus A_1)$ is a disjoint union of finite elements in $\mathcal{A}^+$. Therefore $\mathcal{A}^+$ is a semiring.

We prove the formula (2.1) by induction on $n$. The base case $n = 1$ was already proved, so we focus on the inductive hypothesis. If there exists some $k \in \{1, \ldots, n\}$ such that $A_k = B_k$, then after a relabelling we may assume $A_n = B_n$ and thus

$$(B \setminus A) \setminus \bigcup_{k=1}^{n} (B_k \setminus A_k) = (B \setminus A) \setminus \bigcup_{k=1}^{n-1} (B_k \setminus A_k) = \bigcup_{k=1}^{m} (B'_k \setminus A'_k),$$

by the induction hypothesis, so we may assume that $A_k \subset B_k$ for all $k \in \{1, \ldots, n\}$.

If the collection $\{(B_k \setminus A_k)\}_{k=1}^n$ is not disjoint, then after a relabelling we may assume that the sequence $\{B_k\}$ is increasing by inclusion. If $(B_n \setminus A_n)$ and $(B_{n-1} \setminus A_{n-1})$ are disjoint, then if $B \subseteq B_{n-1}$, we get that $(B \setminus A) \cap (B_n \setminus A_n) = \emptyset$, thus

$$(B \setminus A) \setminus \bigcup_{k=1}^{n} (B_k \setminus A_k) = (B \setminus A) \setminus \bigcup_{k=1}^{n-1} (B_k \setminus A_k) = \bigcup_{k=1}^{m} (B'_k \setminus A'_k),$$

by the induction hypothesis. Therefore we may assume that $B_{n-1} \subset B$.

If $(B_n \setminus A_n)$ and $(B_{n-1} \setminus A_{n-1})$ are disjoint, we use the induction hypothesis twice to get

$$\begin{aligned}(B \setminus A) \setminus \bigcup_{k=1}^{n} (B_k \setminus A_k) &= \left( (B_{n-1} \setminus A) \setminus \bigcup_{k=1}^{n-1} (B_k \setminus A_k) \right) \bigcup \left( (B \setminus B_{n-1}) \setminus (B_n \setminus A_n) \right) \\ &= \left( \bigcup_{k=1}^{m_1} (B'_k \setminus A'_k) \right) \bigcup \left( \bigcup_{k=1}^{m_2} (B''_k \setminus A''_k) \right).\end{aligned}$$

The resulting union is disjoint, completing the proof in this case.

The remaining case is when $(B_n \setminus A_n)$ and $(B_{n-1} \setminus A_{n-1})$ are not disjoint. Define the sequences

$$C_k = A_k, \quad D_k = B_k, \quad \text{if} \quad k \in \{1, \dots, n-2\}, \quad \text{and} \quad C_{n-1} = A_{n-1} \cap A_n,\ D_{n-1} = B_n.$$

Then

$$(B \setminus A) \setminus \bigcup_{k=1}^{n} (B_k \setminus A_k) = (B \setminus A) \setminus \bigcup_{k=1}^{n-1} (D_k \setminus C_k) = \bigcup_{k=1}^{m} (B'_k \setminus A'_k),$$

by the induction hypothesis. Therefore, we may assume that the collection $\{(B_k \setminus A_k)\}_{k=1}^{n}$ is disjoint.

If $(B_k \setminus A_k) \cap (B \setminus A)$ is empty for some $k \in \{1, \dots, n\}$ then after relabelling we may assume that $(B_n \setminus A_n) \cap (B \setminus A) = \emptyset$ and we get

$$(B \setminus A) \setminus \bigcup_{k=1}^{n} (B_k \setminus A_k) = (B \setminus A) \setminus \bigcup_{k=1}^{n-1} (B_k \setminus A_k) = \bigcup_{k=1}^{m} (B'_k \setminus A'_k),$$

by the induction hypothesis. So we may assume that $(B_k \setminus A_k) \cap (B \setminus A) \neq \emptyset$ for all $k \in \{1, \dots, n\}$.

If $B_n \subseteq B$ we have

$$(B \setminus A) \setminus \bigcup_{k=1}^{n} (B_k \setminus A_k) = (B \setminus B_n) \cup \left( (A_n \setminus A) \setminus \bigcup_{k=1}^{n-1} (B_k \setminus A_k) \right) = (B \setminus B_n) \cup \bigcup_{k=1}^{m} (B'_k \setminus A'_k),$$

by the induction hypothesis.

Similarly if $B \subseteq B_n$ we have

$$(B \setminus A) \setminus \bigcup_{k=1}^{n} (B_k \setminus A_k) = (A_n \setminus A) \setminus \bigcup_{k=1}^{n-1} (B_k \setminus A_k) = \bigcup_{k=1}^{m} (B'_k \setminus A'_k),$$

by the induction hypothesis. This completes the proof of equation (2.1).

∎

We now introduce a large vector space of functions which will serve as the natural domain for our induced linear maps.

**Definition 2.3.3** *Let $(U, \Sigma, \mu)$ be a $\sigma$-finite measure space, $\mathcal{A}$ an ordered core we define*

$$L^1_{Loc,\mathcal{A}} = \left\{ f \in L^0_\mu : \int_E |f| \, d\mu < \infty, \text{ for all } E \in \mathcal{A} \right\}.$$

We show some vector lattice properties of this vector space.

**Proposition 2.3.4** *Let $(U, \Sigma, \mu)$ be a $\sigma$-finite measure space and $\mathcal{A}$ be a $\sigma$-bounded full ordered core. Then the space $L^1_{Loc,\mathcal{A}}$ is a Dedekind complete vector lattice with respect to the order relation of pointwise inequality $\mu$-almost everywhere.*

**Proof:** If $U \in \mathcal{A}$, then $L^1_{\text{Loc},\mathcal{A}} = L^1_\mu$ which is a Dedekind complete vector lattice. So, we may assume that $U \notin \mathcal{A}$.

It is clear that $L^1_{\text{Loc},\mathcal{A}}$ is a vector lattice. To show that it is Dedekind complete, let $\{f_\alpha\}_{\alpha\in I}$ a set in $L^1_{\text{Loc},\mathcal{A}}$ bounded above, that is there exists $h \in L^1_{\text{Loc},\mathcal{A}}$ such that $f_\alpha \le h$ for all $\alpha \in I$.

Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{A}$ such that $A_n \uparrow U$. Define the sequence of sets $\{X_n\}$ by

$$X_1 = A_1 \quad \text{and} \quad X_n = A_n \setminus A_{n-1}.$$

Denote by $f_{n,\alpha}$ the restriction of $f_\alpha$ to $X_n$. Since the collection $\{f_{n,\alpha}\}_{\alpha\in I}$ belongs to $L^1_\mu(X_n)$ and is bounded above by the restriction of $h$ to $X_n$, then there exists a function $g_n \in L^1_\mu(X_n)$ such that $g_n$ is the supremum (in vector lattice sense) of $\{f_{n,\alpha}\}_{\alpha\in I}$.

Define the function

$$g = \sum_{n=1}^{\infty} g_n \chi_{X_n}.$$

To see that this function belongs to $L^1_{\text{Loc},\mathcal{A}}$, let $E \in \mathcal{A}$ and let $N \in \mathbb{N}$ satisfy $E \subseteq A_N$. The set $A_N$ exists, otherwise $U = \cup_n A_n \subseteq E$, which we assume is not possible. Then,

$$\int_E |g|\, d\mu \le \int_{A_N} |g|\, d\mu = \sum_{n=1}^{N} \int_{A_n} |g_n|\, d\mu = \sum_{n=1}^{N} \|g_n\|_{L^1_\mu(X_n)} < \infty.$$

Thus $g \in L^1_{\text{Loc},\mathcal{A}}$. To see that $g$ is an upper bound of $\{f_\alpha\}_{\alpha\in I}$, suppose there exists some $f_\alpha$ such that $f_\alpha \le g$ fails. Then, there exists some $j \in \mathbb{N}$ and $\epsilon > 0$ such that the set

$$\left\{s \in X_j : f_\alpha(s) - g(s) \ge \epsilon\right\},$$

has positive measure. It follows that $f_{j,\alpha} \le g_j$ fails, which is a contradiction. This shows that $g$ is an upper bound of $\{f_\alpha\}_{\alpha\in I}$.

To show that $g$ is the supremum of $\{f_\alpha\}_{\alpha\in I}$. Let $h \in L^1_{\text{Loc},\mathcal{A}}$ be an upper bound of $\{f_\alpha\}_{\alpha\in I}$ and suppose that $g \le h$ fails, then there exists some $j \in \mathbb{N}$ and $\epsilon > 0$ such that the set

$$\left\{s \in X_j : g_\alpha(s) - h(s) \ge \epsilon\right\},$$

has positive measure. Let $h_j$ be the restriction of $h$ to $X_j$. It follows that $g_j \le h_j$ fails, contradicting the fact that $g_j$ is the supremum of $\{f_{j,\alpha}\}_{\alpha\in}$ in $L^1_\mu(X_j)$. This shows that $g$ is the supremum of $\{f_\alpha\}_{\alpha\in I}$ and completes the proof.

∎

The next lemma will be useful to show equality on functions in this vector space, whenever the ordered core is full and $\sigma$-bounded.

**Lemma 2.3.5** *Let $(U, \Sigma, \mu)$ be a $\sigma$-finite measure space and $\mathcal{A}$ a $\sigma$-bounded ordered core. Then:*

1. *If $\eta, \nu$ are two measures over $(\cup\mathcal{A}, \sigma(\mathcal{A}))$ and $\eta(A) = \nu(A) < \infty$ for all $A \in \mathcal{A}$, then $\nu = \mu$.*

2. *If $\mathcal{A}$ is full, $f, g \in L^+(\mathcal{A}) \cap L^1_{Loc,\mathcal{A}}$ and $\int_E f\, d\mu = \int_E g\, d\mu$ for all $E \in \mathcal{A}$, then $f = g$ $\mu$-almost everywhere.*

3. *If $\mathcal{A}$ is full; $f, g \in L^1_{Loc,\mathcal{A}}$; $f \le g$ almost everywhere and $\int_E f\, d\mu = \int_E g\, d\mu$ for all $E \in \mathcal{A}$, then $f = g$ $\mu$-almost everywhere.*

**Proof:**

1. By Lemma 1.1.1 the set $\mathcal{A}^+$ is a semiring. Since $\eta(A) = \nu(A) < \infty$ for any $A \in \mathcal{A}$ and $\mathcal{A}$ is $\sigma$-bounded, it follows that $\eta(B \setminus A) = \nu(B \setminus A)$ for all $(B \setminus A) \in \mathcal{A}^+$ and $\nu$ and $\mu$ are $\sigma$-finite. Therefore the restriction to $\mathcal{A}^+$ is a $\sigma$-finite premeasure, it follows from Theorem 1.1.4 that $\eta = \nu$.

2. For each $E \in \sigma(\mathcal{A})$, set $\eta(E) = \int_E f\, d\mu$ and $\nu(E) = \int_E g\, d\mu$. Since these two measures take the same finite value on each $E \in \mathcal{A}$, then by item (1), they are the same measure over $\sigma(\mathcal{A})$.

   Let $\{A_m\} \in \mathcal{A}$ such that $A_m \uparrow U$ and for each $n, m \in \mathbb{N}^+$ set
   $$E_{n,m} = \left\{ u \in A_m : f(u) - g(u) \ge \frac{1}{n} \right\}.$$
   Since $f, g$ are $\sigma(\mathcal{A})$-measurable, then $E_{n,m} \in \sigma(\mathcal{A})$, thus
   $$\int\limits_{E_{n,m}} g\, d\mu = \int\limits_{E_{n,m}} f\, d\mu \le \int\limits_{E_{n,m}} \left( g(u) + \frac{1}{n} \right) d\mu(u) = \int\limits_{E_{n,m}} g\, d\mu + \frac{1}{n}\mu(E_{n,m}).$$
   Since $\int\limits_{E_{n,m}} g\, d\mu$ is finite, it follows that $\mu(E_{n,m}) = 0$. Let $n \to \infty$ and then $m \to \infty$ to get that $g \ge f$ $\mu$-a.e. A symmetric argument shows that $f \ge g$ $\mu$-a.e. and completes the proof of item (2).

3. Let $h = g - f$. By assumption $h \ge 0$ and $\int_E h\, d\mu = \int_E g\, d\mu - \int_E f\, d\mu = 0$. Let $\{A_n\} \in \mathcal{A}$ satisfy $A_n \uparrow U$, then by the monotone convergence theorem
   $$\int_U |h|\, d\mu = \sup_n \int_{A_n} h\, d\mu = 0,$$
   thus $h = 0$ $\mu$-a.e. so $f = g$ almost everywhere.

∎

The main result is that any core morphism induces a linear map that maintains integrals over core sets while maintaining a pointwise bound.

**Theorem 2.3.6** *Let $(U, \Sigma, \mu), (T, \mathcal{T}, \tau)$ be $\sigma$-finite measure spaces, the collection $\mathcal{A} \subseteq \Sigma$ be an ordered core and $r : \mathcal{A} \to \mathcal{T}$ be a core morphism with constant $c$, then there exists a map $R : L^1_{Loc,r(\mathcal{A})} \cup L^+_\tau \to L^1_{Loc,\mathcal{A}} \cup L^+_\mu$ such that for any $f \in L^1_{Loc,r(\mathcal{A})} \cup L^+_\tau$:*

1. *The restricted map $R : L^\infty_\tau \to L^\infty_\mu$, is linear and satisfies $\|Rf\|_{L^\infty_\mu} \le c\|f\|_{L^\infty_\tau}$.*

2. *If $f \ge 0$ $\tau$-almost everywhere , then $Rf \ge 0$ $\mu$-almost everywhere.*

3. *For any sequence* $\{f_n\} \in L^+(\mathcal{T})$ *such that* $f_n \uparrow f$ $\tau$*-almost everywhere, then* $Rf_n \uparrow Rf$ $\mu$*-almost everywhere.*

4. $R$ *is additive and positive homogeneous on* $L^+(\mathcal{T})$*.*

5. $R$ *is linear on* $L^1_{Loc,r(\mathcal{A})}$*, mapping to* $L^1_{Loc,\mathcal{A}}$*.*

6. $Rf \in L(\mathcal{A})$ *and* $Rf = 0$ *outside* $\cup\mathcal{A}$*.*

7. $|Rf| \leq R|f|$ $\mu$*-almost everywhere.*

8. *For all* $A, B \in \mathcal{A}$*,*

$$\int_{B\setminus A} Rf\, d\mu = \int_{r(B)\setminus r(A)} f\, d\tau.$$

9. *The restricted linear map* $R : L^1_\tau \to L^1_\mu$ *satisfies* $\|Rf\|_{L^1_\mu} \leq \|f\|_{L^1_\tau}$*, with equality if* $f \geq 0$ $\tau$*-a.e.*

10. *If* $c = 1$*,* $A \in \mathcal{A}$ *and* $\mu(A) = \tau(r(A))$*, then* $R(\chi_{r(A)}) = \chi_A$ $\mu$*-almost everywhere.*

11. *If* $c = 1$*,* $f \in L^+(\tau)$ *and* $g \in L^+(r(\mathcal{A}))$*, then* $R(fg) = RfRg$ $\mu$*-almost everywhere.*

To prove this result we will make use of the following lemma, the proof is adapted from [31, Lemmas 4.3 and 4.4].

**Lemma 2.3.7** *Let* $(U, \Sigma, \mu), (T, \mathcal{T}, \tau)$ *be* $\sigma$*-finite measure spaces, the collection* $\mathcal{A} \subseteq \Sigma$ *be an ordered core and a core morphism* $r : \mathcal{A} \to \mathcal{T}$*. Let* $\mathcal{A}^+ = \{B \setminus A : A, B \in \mathcal{A}\}$ *be the semiring of sets generated by* $\mathcal{A}$*.*

*For each* $f \in L^+_\tau \cap L^\infty_\tau$ *set*

$$\rho_f(B \setminus A) = \int_{r(B)\setminus r(A)} f\, d\tau. \tag{2.2}$$

*Then,* $\rho_f$ *is a premeasure on the semi-ring* $\mathcal{A}^+$*.*

**Proof:** First notice that if $B = A$ then $r(B) = r(A)$, hence $\rho_f(B \setminus A) = 0$.

We now show that $\rho_f$ is a well-defined function. For that, suppose that $A_1, B_1, A_2, B_2 \in \mathcal{A}$ satisfy $B_1 \setminus A_1 = B_2 \setminus A_2$. If either $B_1 = A_1$ or $B_2 = A_2$ then $\rho_f(B_1 \setminus A_1) = 0 = \rho_f(B_2 \setminus A_2)$ by the previous argument. Without loss of generality, by the total order of $\mathcal{A}$, we may assume that $B_1 \subseteq B_2$. Let $x \in B_2 \setminus A_2$, then $x \in B_1 \setminus A_1$, from the total order it follows that $A_2 \subset B_1$. This shows that $B_2 \subseteq B_1$ and yields equality. To show that $A_1 = A_2$, let $x \in A_1 \setminus A_2$, then $x \in (B_2 \setminus A_2) \setminus (B_1 \setminus A_1)$ and yields a contradiction. Therefore $A_1 \subseteq A_2$ and a symmetric argument show equality. Thus, if $A_1 \subset B_1$ and $A_2 \subset B_2$, then $B_1 \setminus A_1 = B_2 \setminus A_2$ implies $B_1 = B_2$ and $A_1 = A_2$, so the function $\rho_f$ is well defined.

We now show that $\rho_f$ is a premeasure. It was already shown that $\rho_f(\emptyset) = 0$. To show that $\rho_f$ is finitely additive. Let $B, A, \{B_k\}, \{C_k\} \in \mathcal{A}$ satisfy

$$B \setminus A = \bigcup_{k=1}^{n} (B_k \setminus A_k), \tag{2.3}$$

where the sets $\{(B_k \setminus A_k)\}_{k=1}^n$ are disjoint.

We show that $\rho_f(B \setminus A) = \sum_{k=1}^n \rho_f(B_k \setminus A_k)$ by induction on $n$. If $n = 1$, then $\rho_f(B \setminus A) = \rho_f(B_1 \setminus A_1)$, since $\rho_f$ is well defined.

To prove the inductive step: If there exist some $k_0 \in \{1, \dots, n\}$ such that $B_{k_0} = A_{k_0}$, then we can reorder the sequence such that $B_n = A_n$ to get

$$\rho_f(B \setminus A) = \bigcup_{k=1}^{n-1} \rho_f(B_k \setminus A_k) = \bigcup_{k=1}^{n-1} \rho_f(B_k \setminus A_k) + 0 = \bigcup_{k=1}^{n} \rho_f(B_k \setminus A_k).$$

Therefore, we may assume that $B_k \neq A_k$ for all $k \in \{1, \dots, n\}$. After a relabelling, we may also assume that the sequence $\{B_k\}$ is increasing by inclusion. If $B \subset B_n$, then there exists some $x \in B_n \setminus (A_n \cup B)$ and equation (2.3) fails. If $B_n \subset B$, then there exists $x \in B \setminus B_k$ for all $k \in \{1, \dots, n\}$ and equation (2.3) fails again. This shows that $B_n = B$. Also, if $A_k \subset A$ for some $k \in \{1, \dots, n\}$ then there exists some $x \in B_k \setminus A_k$ and $x \notin B \setminus A$, arriving at the same contradiction.

If $B_n = B_{n-1}$ then the sets $B_n \setminus A_n$ and $B_{n-1} \setminus A_{n-1}$ are not disjoint, therefore we may assume that $B_{n-1} \subset B_n$. If $A_n \subset B_{n-1}$, then we contradict the fact that $B_n \setminus A_n$ and $B_{n-1} \setminus A_{n-1}$ are disjoint, therefore we may assume that $B_{n-1} \subseteq A_n$. To show that we have equality, let $x \in A_n$, if $x \in A$ then $x \in A_{n-1} \subset B_{n-1}$. If $x \notin A$ then $x \in B \setminus A$, by equation (2.3), there exists some $k \in \{1, \dots, n-1\}$ such that $x \in (B_k \setminus A_k)$, thus $x \in B_{n-1}$. So we get the equality $B_{n-1} = A_n$ and we can define a new sequence

$$C_k = B_k \setminus A_k, \quad \text{if} \quad k \in \{1, \dots, n-2\}, \quad \text{and} \quad C_{n-1} = B_n \setminus A_{n-1}.$$

The induction hypothesis yields

$$\begin{aligned}
\rho_f(B \setminus A) &= \sum_{k=1}^{n-1} \rho_f(C_k) = \sum_{k=1}^{n-2} \rho_f(B_k \setminus A_k) + \rho_f(B_n \setminus A_{n-1}) \\
&= \sum_{k=1}^{n-2} \rho_f(B_k \setminus A_k) + \int\limits_{r(B_n)\setminus r(A_{n-1})} f\,d\tau = \sum_{k=1}^{n-2} \rho_f(B_k \setminus A_k) + \int\limits_{r(B_{n-1})\setminus r(A_{n-1})} f\,d\tau + \int\limits_{r(B_n)\setminus r(A_n)} f\,d\tau \\
&= \sum_{k=1}^{n} \rho_f(B_k \setminus A_k).
\end{aligned}$$

To show that $\rho_f$ is countably monotone, we prove that it is monotone for finite sequences. Suppose that $B \setminus A \subseteq \bigcup_{k=1}^n (B_k \setminus A_k)$, we show by induction on $n$ that

$$\rho_f(B \setminus A) \leq \sum_{k=1}^{n} \rho_f(B_k \setminus A_k). \tag{2.4}$$

If $n = 1$, then $B \subseteq B_1$ and $A_1 \subseteq A$, hence $r(B) \subseteq r(B_1)$ and $r(A_1) \subseteq r(A)$, thus

$$\rho_f(B \setminus A) = \int\limits_{r(B)\setminus r(A)} f\,d\tau \leq \int\limits_{r(B_1)\setminus r(A_1)} f\,d\tau = \rho_f(B_1 \setminus A_1).$$

To prove the induction step: If there exists some $k \in \{1, \ldots, n\}$ such that $(B \setminus A) \cap (B_k \setminus A_k) = \emptyset$, then we may reorder the sequence to get $(B \setminus A) \cap (B_n \setminus A_n) = \emptyset$. The inductive hypothesis yields

$$\rho_f(B \setminus A) \leq \sum_{k=1}^{n-1} \rho_f(B_k \setminus A_k) \leq \sum_{k=1}^{n} \rho_f(B_k \setminus A_k).$$

Hence, we may assume that $(B \setminus A) \cap (B_k \setminus A_k) \neq \emptyset$, for all $k \in \{1, \ldots, n\}$.

If the sequence $\{(B_k \setminus A_k)\}_{k=1}^n$ is not disjoint, we may reorder the sequence to get $(B_n \setminus A_n) \cap (B_{n-1} \setminus A_{n-1}) \neq \emptyset$. Define the sequences

$$C_k = A_k, \quad D_k = B_k, \quad \text{if} \quad k \in \{1, \ldots, n-2\}, \qquad \text{and} \qquad C_{n-1} = A_{n-1} \cap A_n,\ D_{n-1} = B_n.$$

By the inductive hypothesis

$$\begin{aligned}
\rho_f(B \setminus A) &\leq \sum_{k=1}^{n-1} \rho_f(D_k \setminus C_k) = \sum_{k=1}^{n-2} \rho_f(B_k \setminus A_k) + \rho_f\big(B_n \setminus (A_{n-1} \cap A_n)\big) \\
&= \sum_{k=1}^{n-2} \rho_f(B_k \setminus A_k) + \int\limits_{r(B_n) \setminus r(A_{n-1} \cap A_n)} f \, d\tau \\
&\leq \sum_{k=1}^{n-2} \rho_f(B_k \setminus A_k) + \int\limits_{r(B_{n-1}) \setminus r(A_{n-1})} f \, d\tau + \int\limits_{r(B_n) \setminus r(A_n)} f \, d\tau \\
&= \sum_{k=1}^{n} \rho_f(B_k \setminus A_k).
\end{aligned}$$

Thus we may assume assume that the sequence $\{(B_k \setminus A_k)\}_{k=1}^n$ is disjoint and $(B \setminus A) \cap (B_k \setminus A_k) \neq \emptyset$. Then we may reorder the sequence $\{B_k\}$ increasingly. Then, we must have that $A_n \subseteq B \subseteq B_n$ and $A_1 \subseteq A \subseteq B_1$, otherwise either $(B \setminus A) \subseteq \bigcup_{k=1}^n (B_k \setminus A_k)$ or there exists $k \in \{1, \ldots, n\}$ for which $(B \setminus A) \cap (B_k \setminus A_k) = \emptyset$.

Therefore

$$B \setminus A = (B_1 \setminus A) \cup (B_n \setminus B) \cup \bigcup_{k=2}^{n-1} (B_k \setminus A_k).$$

Finite additivity yields

$$\rho_f(B \setminus A) = \rho_f(B_1 \setminus A) + \rho_f(B_n \setminus B) + \sum_{k=2}^{n-1} \rho_f(B_k \setminus A_k) \leq \sum_{k=1}^{n} \rho_f(B_k \setminus A_k).$$

This proves that $\rho_f$ is finitely monotone.

Finally, to prove that $\rho_f$ is countably monotone, fix $\epsilon > 0$ and suppose that $B \setminus A \subseteq \bigcup_{k=1}^\infty (B_k \setminus A_k)$. Notice that for each $n \in \mathbb{N}$:

$$B \setminus A \subseteq \left( \bigcup_{k=1}^{n} (B_k \setminus A_k) \right) \cup \left( (B \setminus A) \setminus \left( \bigcup_{k=1}^{n} (B_k \setminus A_k) \right) \right).$$

Since $(B\setminus A)\setminus\Big(\bigcup_{k=1}^{n}(B_k\setminus A_k)\Big)$ is a decreasing sequence of sets with finite $\mu$-measure and empty intersection, we may pick $n_0$ such that

$$\mu\Bigg((B\setminus A)\setminus\Bigg(\bigcup_{k=1}^{n_0}(B_k\setminus A_k)\Bigg)\Bigg)<\frac{\epsilon}{c\|f\|_{L_\tau^\infty}}.$$

By Lemma 2.3.2, there exists a disjoint collection $\{(B'_k\setminus A'_k)\}_{k=1}^{m}\in\mathcal{A}^+$ such that

$$\Bigg((B\setminus A)\setminus\Bigg(\bigcup_{k=1}^{n_0}(B_k\setminus A_k)\Bigg)\Bigg)=\bigcup_{k=1}^{m}(B'_k\setminus A'_k).$$

Notice that since the sequence is disjoint, then $\sum_{k=1}^{m}\mu(B'_k\setminus A'_k)\leq\frac{\epsilon}{c\|f\|_{L_\tau^\infty}}$. Since

$$B\setminus A\subseteq\Bigg(\bigcup_{k=1}^{n_0}(B_k\setminus A_k)\Bigg)\cup\bigcup_{k=1}^{m}(B'_k\setminus A'_k),$$

and the right-hand side is a finite union in $\mathcal{A}^+$, finite monotonicity shows that

$$\begin{aligned}\rho_f(B\setminus A)&\leq\sum_{k=1}^{n_0}\rho_f(B_k\setminus A_k)+\sum_{k=1}^{m}\rho_f(B'_k\setminus A'_k)=\sum_{k=1}^{n_0}\rho_f(B_k\setminus A_k)+\sum_{k=1}^{m}\int\limits_{r(B'_k)\setminus r(A'_k)}f\,d\tau\\&\leq\sum_{k=1}^{n_0}\rho_f(B_k\setminus A_k)+\|f\|_{L_\tau^\infty}\sum_{k=1}^{m}\tau\big(r(B'_k)\setminus r(A'_k)\big)\\&\leq\sum_{k=1}^{n_0}\rho_f(B_k\setminus A_k)+\|f\|_{L_\tau^\infty}c\sum_{k=1}^{m}\mu(B'_k\setminus A'_k)\\&\leq\sum_{k=1}^{n_0}\rho_f(B_k\setminus A_k)+\epsilon.\end{aligned}$$

We have used the fact that $r$ is a core morphism in the second to last line. Therefore

$$\rho_f(B\setminus A)\leq\sum_{k=1}^{\infty}\rho_f(B_k\setminus A_k)+\epsilon.$$

Let $\epsilon\to 0$ to complete the proof.

∎

**Proof of Theorem 2.3.6:** We define the operator $R$ on positive and bounded functions first.

If $f=0$ set $Rf=0$, otherwise fix $f\in L_\tau^\infty\cap L_\tau^+$. Lemma 2.2 defines a premeasure $\rho_f$ over the semi-ring $A^+$. By the Caratheodory-Hahn Theorem, there is a measure $\overline{\rho_f}$ on $(\cup\mathcal{A},\sigma(\mathcal{A}))$ that extends $\rho_f$ and coincides with the outer measure induced by $\rho_f$ on $\sigma(\mathcal{A})$-measurable sets. For any $A,B\in\mathcal{A}$ we have

$$\rho_f(B\setminus A)=\int\limits_{r(B)\setminus r(A)}f\,d\tau\leq\|f\|_{L_\tau^\infty}\tau\big(r(B)\setminus r(A)\big)\leq c\|f\|_{L_\tau^\infty}\mu(B\setminus A).\tag{2.5}$$

Letting $A = \emptyset$ shows that $\rho_f(B) < \infty$. Since $\mathcal{A}$ is $\sigma$-bounded, then $\rho_f$ is a $\sigma$-finite pre-measure, thus the extension $\overline{\rho_f}$ is unique and $\sigma$-finite. We proceed to show that it is absolutely continuous with respect to $\mu$.

If $E \in \sigma(\mathcal{A})$ and $\mu(E) = 0$, then for any $\epsilon > 0$, there exists a set $G = \cup_{j\in\mathbb{N}}(B_j \setminus A_j)$ with $B_j, A_j \in \mathcal{A}$ for all $j \in \mathbb{N}$, such that $E \subseteq G$ and $\mu(G) < \frac{\epsilon}{c\|f\|_{L^\infty_\tau}}$. By equation (2.5) we get

$$\overline{\rho_f}(E) \leq \sum_{j\in\mathbb{N}} \rho_f(B_j \setminus A_j) \leq \sum_{j\in\mathbb{N}} c\|f\|_{L^\infty_\tau}\mu(B_j \setminus A_j) = c\|f\|_{L^\infty_\tau}\mu(G) < \epsilon.$$

Letting $\epsilon \to 0$ shows that $\overline{\rho_f}(E) = 0$ and proves that $\overline{\rho_f}$ is a $\sigma$-finite measure, absolutely continuous with respect to $\mu$. Therefore, the Radon-Nikodym Theorem provides a unique ($\mu$-a.e) nonnegative $\sigma(\mathcal{A})$-measurable function $h$ such that

$$\overline{\rho_f}(E) = \int_E h\, d\mu, \quad \forall E \in \sigma(\mathcal{A}).$$

Define $Rf = h\chi_{(\cup\mathcal{A})}$. We have the operator $R$ defined on $L^+_\tau \cap L^\infty_\tau$ taking values on the real vector space $V = \{g \in L^1_{\text{Loc},\mathcal{A}} : g = g\chi_{(\cup\mathcal{A})}\}$ and satisfying $Rf \geq 0$.

To show that $R$ is positive homogeneous, let $\alpha, \beta \geq 0$ and $f_1, f_2 \in L^+_\tau \cap L^\infty_\tau$, notice that for each $A \in \mathcal{A}$:

$$\begin{aligned}\int_A R(\alpha f_1 + \beta f_2)\, d\mu = \overline{\rho_{\alpha f_1+\beta f_2}}(A \setminus \emptyset) &= \int_{r(A)\setminus r(\emptyset)} (\alpha f_1 + \beta f_2)\, d\tau = \alpha \int_{r(A)\setminus r(\emptyset)} f_1\, d\tau + \beta \int_{r(A)\setminus r(\emptyset)} f_2\, d\tau \\ &= \alpha\overline{\rho_{f_1}}(A \setminus \emptyset) + \beta\overline{\rho_{f_2}}(A \setminus \emptyset) = \alpha \int_A Rf_1\, d\mu + \beta \int_A Rf_2\, d\mu \\ &= \int_A (\alpha Rf_1 + \beta Rf_2)\, d\mu.\end{aligned}$$

An application of Lemma 2.3.5 shows that $R(\alpha f_1 + \beta f_2)$ and $\alpha R(f_1) + \beta R(f_2)$ coincide on $\cup\mathcal{A}$ $\mu$-a.e. It follows that $R(\alpha f_1 + \alpha f_2) = \alpha R(f_1) + \beta R(f_2)$ over $U$ up to a set of zero $\mu$-measure.

An application of Theorem 1.2.4, with $V = W = L^\infty_\tau$ provides an $\mathbb{R}$-linear extension of $R$ to $L^\infty_\tau$.

Notice that, in this case

$$|Rf| = \left|R(f^+) - R(f^-)\right| \leq \left|R(f^+)\right| + \left|R(f^-)\right| = R(f^+) + R(f^-) = R\,|f|\,.$$

Monotonicity of the norm $L^\infty_\tau$ implies that to prove (1) we need to consider nonnegative functions only. Suppose that $f \geq 0$, let $\{A_m\}_{m\in\mathbb{N}} \in \mathcal{A}$ satisfy $\cup_m A_m = \cup\mathcal{A}$ and for each $n, m \in \mathbb{N}$ define

$$E_{n,m} = \left\{x \in A_m : Rf(x) > c\|f\|_{L^\infty_\tau} + \frac{1}{n}\right\}.$$

Since $Rf$ is $\sigma(\mathcal{A})$-measurable, for every $\delta > 0$, there exists a pairwise disjoint sequence $\{B_j \setminus A_j\}$

such that $E_{n,m} \subseteq G_\delta = \cup(B_j \setminus A_j)$ and $\mu(G) < \delta + \mu(E_{n,m})$. Integration yields

$$\mu(E_{n,m})\left(c\|f\|_{L^\infty_\tau} + \frac{1}{n}\right) < \int_{E_n} Rf\,d\mu \le \int_{G_\delta} Rf\,d\mu = \sum_j \int_{B_j\setminus A_j} Rf\,d\mu = \sum_j \int_{r(B_j)\setminus r(A_j)} f\,d\tau$$
$$\le \|f\|_{L^\infty_\tau} \sum_j \tau(r(B_j)\setminus r(A_j)) \le c\|f\|_{L^\infty_\tau} \sum_j \mu(B_j \setminus A_j) = c\|f\|_{L^\infty_\tau}\mu(G_\delta)$$
$$\le c\|f\|_{L^\infty_\tau}(\delta + \mu(E_{n,m})).$$

Let $\delta \to 0$ to get $\mu(E_{n,m})\left(c\|f\|_{L^\infty_\tau} + \frac{1}{n}\right) \le \mu(E_{n,m})\left(c\|f\|_{L^\infty_\tau}\right)$. This is impossible unless $\mu(E_{n,m}) = 0$.

Let $m \to \infty$ and then $n \to \infty$ to get $\|Rf\|_{L^\infty_\mu} \le c\|f\|_{L^\infty_\tau}$ and complete the proof of (1).

We proceed to extend the operator $R$ to $L^+_\tau$.

We will need the following observation: If $\{f_n\}, f \in L^\infty_\tau$ and $f_n \uparrow f$, then $Rf_n \uparrow Rf$. Since $R$ is linear and positive on $L^\infty_\tau$, then $Rf_n$ is an increasing sequence in $L^+_\mu$ and the limit function is $\sigma(\mathcal{A})$-measurable and nonnegative.

Let $A \in \mathcal{A}$, then the Monotone Convergence Theorem yields,

$$\int_A Rf\,d\mu = \int_{r(A)\setminus r(\emptyset)} f\,d\tau = \lim_{n\to\infty} \int_{r(A)\setminus r(\emptyset)} f_n\,d\tau = \lim_{n\to\infty} \int_{r(A)\setminus r(\emptyset)} Rf_n\,d\tau = \int_{r(A)\setminus r(\emptyset)} \lim_{n\to\infty} Rf_n\,d\tau.$$

The equality above and Lemma 2.3.5 shows that $\lim_{n\to\infty} Rf_n = Rf$.

We now extend $R$ to $L^+_\tau$. Let $f_n \uparrow f$ with $f_n \in L^\infty_\tau \cap L^+_\tau$. By the same argument as before $\lim_{n\to\infty} Rf_n$ is $\sigma(\mathcal{A})$-measurable and vanishes outside $\cup\mathcal{A}$. Define $Rf = \sup_n Rf_n$. By the previous observation, this definition of $Rf$ coincides with the previous one whenever $f \in L^\infty_\tau$.

It remains to show that $Rf$ is independent of the choice of sequence. Let $\{f_n\}, \{g_n\} \in L^\infty_\tau \cap L^+_\tau$ satisfy $f_n \uparrow f$ and $g_n \uparrow f$. For each fixed $m$, $h_n = \min(f_n, g_m)$ defines an increasing sequence in $L^\infty_\tau \cap L^+_\tau$ that satisfies $h_n \uparrow g_m$. It follows, by the previous observation, that $Rh_n \uparrow_n Rg_m$. Since $h_n \le f_n$, then,

$$Rg_m = \lim_{n\to\infty} Rh_n \le \lim_{n\to\infty} Rf_n.$$

Letting $m \to \infty$ yields $\lim_{n\to\infty} Rg_n \le \lim_{n\to\infty} Rf_n$. Reversing the roles of $f_n$ and $g_n$ yields the opposite inequality and shows that $Rf$ is well defined. This completes the proof of item (2).

To prove item (3), let $\{f_n\} \in L^+(\mathcal{T})$ satisfy $f_n \uparrow f$. Then the sequence $g_n = \min(n, f_n)$ also satisfies $g_n \uparrow f$ and $g_n \in L^\infty_\tau$. By the previous observation $Rg_n \uparrow Rf$. However, $Rg_n \le Rf_n$, since $g_n \le f_n$. Therefore $Rf_n \uparrow Rf$ and proves (3).

To prove item (4), let $\alpha, \beta \ge 0$ and $f, g \in L^+(\mathcal{T})$ and sequences $\{f_n\}, \{g_n\} \in L^\infty_\tau \cap L^+(\mathcal{T})$ increasing to $f, g$. Then, using the linearity of $R$ in $L^\infty_\tau$ and the independence of approximating sequence:

$$R(\alpha f + \beta g) = \lim_{n\to\infty} R(\alpha f_n + \beta g_n) = \lim_{n\to\infty} \alpha R(f_n) + \beta R(g_n) = \alpha R(f) + \beta R(g).$$

The above proves item (4).

Let $A, B \in \mathcal{A}$ and $f \in L^+(\tau)$, then

$$\int_{B\setminus A} Rf\,d\mu = \sup_n \int_{B\setminus A} Rf_n\,d\mu = \sup_n \int_{r(B)\setminus r(A)} f_n\,d\tau = \int_{r(B)\setminus r(A)} f\,d\tau.$$

Letting $A = \emptyset$ above, shows that if $f \in L^1_{\mathrm{Loc},r(\mathcal{A})}$, then $Rf \in L^+(\tau) \cap L^1_{\mathrm{Loc},\mathcal{A}}$. An application of Theorem 1.2.4 with $W = L^1_{\mathrm{Loc},\mathcal{A}}$ and $V = L^1_{\mathrm{Loc},r(\mathcal{A})}$ extends $R$ to a positive linear operator on $L^1_{\mathrm{Loc},r(\mathcal{A})}$ and proves item (5).

Item (6) follows from the construction of $R$ based on increasing sequences and linear combinations of functions vanishing outside $\cup\mathcal{A}$. Item (7) follows from the same observation done for the case in $L^\infty_\tau$. Item (8) has already been established for positive functions, thus it follows from the linearity of the integral. Letting $A_n \uparrow \cup\mathcal{A}$ and item (8) show that $\|Rf\|_{L^1_\mu} = \|f\|_{L^1_\tau}$ for nonnegative functions $f$, thus item (7) and monotonicity of the norm in $L^1_\mu$ prove (9).

To prove item (10). Notice that

$$\int_A R(\chi_{r(A)})\, d\mu = \int_{r(A)} \chi_{r(A)}\, d\tau = \tau(r(A)) = \mu(A).$$

above, we have used the fact that $r(\emptyset)$ must be a null set. Since $\|R(\chi_{r(A)})\|_{L^1_\mu} = \|\chi_{r(A)}\|_{L^1_\tau} = \tau(r(A)) = \mu(A)$, it follows that $R(\chi_{r(A)})$ is zero $\mu$-a.e. outside $A$. Since $\|R(\chi_{r(A)})\|_{L^\infty_\mu} \leq \|\chi_{r(A)}\|_{L^\infty_\tau} = 1$, then $R(\chi_{r(A)}) \leq \chi_A$. Therefore,

$$\|\chi_A - R(\chi_{r(A)})\|_{L^1_\mu} = \int_A (\chi_A - R(\chi_{r(A)}))\, d\mu = 0.$$

This shows that $\chi_A = R(\chi_{r(A)})$ $\mu$-a.e and completes the proof of item (10).

Finally, to prove item (11), fix $f \in L^\infty_\tau \cap L^+(\mathcal{T})$, $B \in \mathcal{A}$ and define the maps $\nu, \eta : \sigma(r(A)) \to [0, \infty)$ as follows

$$\nu(E) = \int_B Rf R\chi_E\, d\mu, \quad \eta(E) = \int_B R(f\chi_E)\, d\mu, \qquad \text{for each } E \in \sigma(r(A)).$$

Clearly $\nu(\emptyset) = 0 = \eta(\emptyset)$ and if $\cup_k E_k$ is a disjoint union in $\sigma(r(\mathcal{A}))$, then

$$\nu(E) = \int_B Rf R\left(\sup_n \sum_{k=1}^n \chi_{Ek}\right) d\mu = \sup_n \sum_{k=1}^n \int_B Rf R\left(\chi_{Ek}\right)\, d\mu = \sum_{k=1}^\infty \nu(E_k).$$

$$\eta(E) = \int_B R\left(\sup_n \sum_{k=1}^n f\chi_{Ek}\right) d\mu = \sup_n \sum_{k=1}^n \int_B R\left(f\chi_{Ek}\right)\, d\mu = \sum_{k=1}^\infty \eta(E_k).$$

Therefore $\nu, \eta$ are measures on $\sigma(r(\mathcal{A}))$, since $Rf \in L^\infty_\mu \cap L^+(\mathcal{T})$ and $\mu(B) < \infty$, these are finite measures. For each $A \in \mathcal{A}$, since $\mathcal{A}$ is totally ordered and $r$ is order preserving, we see that $r(A) \cap r(B) = r(A \cap B)$. Therefore,

$$\nu(r(A)) = \int_B Rf R(\chi_{r(A)})\, d\mu = \int_B \chi_A Rf\, d\mu = \int_{B\cap A} Rf\, d\mu = \begin{cases} \int_{r(A)} f\, d\tau, & \text{if } A \subseteq B. \\ \int_{r(B)} f\, d\tau, & \text{otherwise.} \end{cases}$$

and

$$\eta(r(A)) = \int_B R(f\chi_{r(A)})\, d\mu = \int_{r(B)} f\chi_{r(A)}\, d\tau = \int_{B\cap A} Rf\, d\mu = \begin{cases} \int_{r(A)} f\, d\tau, & \text{if } A \subseteq B. \\ \int_{r(B)} f\, d\tau, & \text{otherwise.} \end{cases}$$

By Lemma 2.3.5 $\nu, \eta$ are the same measure, therefore for any $E \in \sigma(r(\mathcal{A}))$ the equation

$$\int_B RfR(\chi_E)\, d\mu = \int_B R(f\chi_E)\, d\mu$$

holds. Since $B \in \mathcal{A}$ was arbitrary, another application of Lemma 2.3.5 with the functions $RfR(\chi_E)$ and $R(f\chi_E)$ shows that they coincide $\mu$-almost everywhere.

Any nonnegative measurable function in $\sigma(r(\mathcal{A}))$ is the increasing limit of a sequence of linear combinations of functions $\chi_E$ with $E \in \sigma(r(\mathcal{A}))$. Thus, by items (3) and (4), we get that $R(fg) = R(f)R(g)$ for any $g \in L^+(\sigma(r(\mathcal{A})))$ and $f \in L^\infty_\tau$. If $f$ is not bounded above, approximate it by an increasing sequence $\{f_n\} \in L^\infty_\tau$, and another application of item (3) finishes the proof of item (11).

∎

The following example shows that the hypothesis on items (2,10,11) is necessary

**Example 2.3.8** *Consider the core $\mathcal{A}$ defined in Example 2.1.9 over $\mathbb{R}^2$ and $(T, \mathcal{T}, \tau)$ be the half line with the Lebesgue measure. Define the core morphism $r$ by*

$$r([0,x]) = B\left(0, \sqrt{\frac{x}{\pi}}\right), \quad \textit{for all } x > 0.$$

*Consider the sets*

$$H = \{(\rho\cos(\theta), \rho\sin(\theta)) : 0 < \rho < \theta,\ \theta \in [0,\pi]\}, K = \{(\rho\cos(\theta), \rho\sin(\theta)) : 0 < \rho < \pi, \theta \in [0,\pi]\},$$

*$L = \{(x, -y) : (x, y \in K)\}$ and the functions $f = \chi_H, g = \chi_K$, and $h = \chi_L$. Note that all of the functions belong to $L^1_{Loc,\mathcal{A}}$, $f = fg$ and the core morphism preserves measure.*

*Since*

$$\int_{B(0,\rho)} f = m(H \cap B(0,\rho)) = \min\left(\frac{\pi^3}{3}, \frac{\rho^3}{6} + \frac{\rho^2(\pi-\rho)}{2}\right) = \int_0^{\pi\rho^2} \left(\frac{1}{2} - \frac{\sqrt{t}}{2\pi\sqrt{\pi}}\right)\chi_{(0,\pi^3)}(t)\, dt,$$

*therefore, by Lemma 2.3.5, $Rf(t) = \left(\frac{1}{2} - \frac{\sqrt{t}}{2\pi\sqrt{\pi}}\right)\chi_{(0,\pi^3)}(t)$. Similarly $Rg(t) = \frac{1}{2}\chi_{(0,\pi^3)}(t) = Rh(t)$. The function $f$ shows that the image of a characteristic function need not be a simple function. Also, $R(fg) = Rf \neq (Rf)(Rg)$ shows that $R$ need not be multiplicative. And since $R(g-h) = 0$, it shows that for functions taking both positive and negative values, the map $R$ can map non-zero functions to zero.*

## 2.4 An induced measure on the half-line

In Example 2.3.8 we had a core morphism between a measure space with an ordered core and the half line, mapping each core set to the measure of each set. We can generalize this core-morphism to any $\sigma$-bounded ordered core, we will be mapping to the half line with an induced Borel measure by the ordered core.

For this purpose, we will employ the enriched core described in Lemma 2.2.2.

We begin by exploring one of the properties that hold for enriched cores.

**Theorem 2.4.1** *Let $(U, \Sigma, \mu)$ be a $\sigma$-finite measure space, $\mathcal{A}$ be an ordered core and $\mathcal{M}$ be its enriched core from Lemma 2.2.2. Then, the set*

$$\{\mu(M) : M \in \mathcal{M}\}$$

*is the closure of the set*

$$\{\mu(A) : A \in \mathcal{A}\}.$$

*in* $[0, \infty)$.

**Proof:**

First we show that the closure of $\{\mu(A) : A \in \mathcal{A}\}$ is contained in $\{\mu(M) : A \in \mathcal{M}\}$.

Let $x$ be in the closure of $\{\mu(A) : A \in \mathcal{A}\}$. If $x \in \{\mu(A) : A \in \mathcal{A}\}$, then there is nothing to show, since $\mathcal{A} \subseteq \mathcal{M}$.

If $x \notin \{\mu(A) : A \in \mathcal{A}\}$, then there exists a sequence $\{A_n\} \in \mathcal{A}$ such that $\mu(A_n) \uparrow x$ or $\mu(A_n) \downarrow x$ and the sequence is strictly increasing or decreasing by inclusion.

If $\mu(A_n) \uparrow x$, set $E = \cup_n A_n$. Then $E \in \mathcal{M}$ by Lemma 2.2.2 and by the monotone convergence theorem $\mu(E) = x$.

If $\mu(A_n) \downarrow x$, set $E = \cap_n A_n$. Then $E \in \mathcal{M}$ by Lemma 2.2.2 and by the dominated convergence theorem $\mu(E) = x$. This shows that the closure of $\{\mu(A) : A \in \mathcal{A}\}$ is contained in $\{\mu(M) : M \in \mathcal{M}\}$.

Conversely, suppose that $x = \mu(M)$ for some $M \in \mathcal{M}$. Then, by Lemma 2.2.2 we may choose a sequence $\{A_n\} \in \mathcal{A}$ such that $M = \cap_n A_n$ or $M = \cup_n A_n$. In the first case, the monotone convergence theorem shows that $\mu(A_n) \uparrow \mu(M)$ and in the second case, the dominated convergence theorem shows that $\mu(A_n) \downarrow \mu(M)$ and completes the proof.

∎

The following example shows that, when a core is not enriched, the set $\{\mu(A) : A \in \mathcal{A}\}$ need not be a closed subset of $[0, \infty)$.

**Example 2.4.2** *Let $U = [0, 3]$ and $\mu$ be the Lebesgue measure. Consider the ordered core $\mathcal{A} = \{\emptyset, U\} \cup \{[0, x] : x \in (1, 2)\}$. Then*

$$\{\mu(A) : A \in \mathcal{A}\} = \{0, 3\} \cup (1, 2).$$

*Here the enriched core is $\mathcal{M} = \{\emptyset, U\} \cup \{[0, x] : x \in [1, 2]\} \cup \{[0, x) : x \in (1, 2]\}$ and*

$$\{\mu(M) : M \in \mathcal{M}\} = \{0, 3\} \cup [1, 2].$$

We now construct a Borel measure on $[0, \infty)$, that encodes the monotonicity properties of the enriched ordered core.

Let $\mathcal{B} = \{\emptyset\} \cup \{[0, x] : x \geq 0\}$. This collection generates the Borel $\sigma$-algebra. We will construct a measure $\lambda$ on $[0, \infty)$ such that $\mathcal{B}$ is a full, $\sigma$-bounded, ordered core on $([0, \infty), \sigma(\mathcal{B}), \lambda)$.

**Theorem 2.4.3** *Let $\Gamma = \mu(\mathcal{M})$, and the functions*

$$a(x) = \sup([0, x] \cap \Gamma), \quad b(x) = \inf([x, \infty] \cap \Gamma),$$

*where* $\inf \emptyset = \infty$.

*Then, there exists a $\sigma$-finite Borel measure $\lambda$, supported on $\Gamma$, such that*

$$\lambda([0,x]) = a(x), \quad x > 0.$$

*Moreover, $\lambda([0,x]) = x$ if and only if $x \in \Gamma$.*

*Finally, for every nonnegative Borel measurable function $\varphi$,*

$$\int_{[0,\infty)} \varphi\, d\lambda = \int_0^{\sup\Gamma} \varphi \circ b(x)\, dx = \int_\Gamma \varphi(x)\, dx + \sum (d-c)\varphi(d), \tag{2.6}$$

*where the sum is taken over the connected components $(c,d)$ of the complement of $\Gamma$.*

**Proof:** It is clear from the definition that for each $x \geq 0$ we have $a(x) \leq x \leq b(x)$ and $a(x) = x = b(x)$ whenever $x \in \Gamma$. Conversely, by Theorem 2.4.1, $\Gamma$ is closed, so $a(x) \in \Gamma$ and $b(x) \in \Gamma \cup \{\infty\}$. Therefore $a(x) = x$ implies $x \in \Gamma$ and the same statement holds for $b(x)$. Therefore $a(x) = x = b(x)$ when $x \in \Gamma$, $a(x) < b(x)$ when $x \notin \Gamma$ and by construction $(a(x), b(x))$ is a connected component of the complement of $\Gamma$.

Since $b$ is non-decreasing, it is Borel measurable. Let $\lambda$ be the push-forward measure induced by $b$. Thus

$$\lambda(E) = m\left(b^{-1}(E)\right), \quad \forall E \in \sigma(\mathcal{B}),$$

where $m$ is the Lebesgue measure.

Therefore, for all $x \geq 0$, $b^{-1}([0,x])$ is Borel measurable. For all $x \geq 0$, $a(x) \in \Gamma$ so $b(a(x)) = a(x)$. If $b(t) < \infty$, then $b(t) \in \Gamma$, so $a(b(t)) = b(t)$. We claim that for all $x \geq 0$, $b^{-1}([0,x]) = [0, a(x)]$. To see this, if $x \leq a(x)$, then $b(t) \leq b(a(x)) = a(x) \leq x$. Conversely, if $b(t) \leq x$, then $t \leq b(t) = a(b(t)) \leq a(x)$.

Hence

$$a(x) = m\left(b^{-1}([0,x])\right) = \lambda([0,x]).$$

Since $a(x) = x$ if and only if $x \in \Gamma$, then we have $x = \lambda([0,x])$ if and only if $x \in \Gamma$. This also shows that $\lambda$ is $\sigma$-finite.

We now show that $\lambda$ is supported on $\Gamma$. Since $\Gamma$ is closed then its complement has countably many connected components. If $\sup\Gamma < \infty$ then for the unbounded component we have

$$\lambda(\sup\Gamma, \sup\Gamma + n] = \lambda[0, \sup\Gamma + n] - \lambda[0, \sup\Gamma] = a(\sup\Gamma + n) - a(\sup\Gamma) = \sup\Gamma - \sup\Gamma = 0.$$

It follows that $\lambda(\sup\Gamma, \infty] = 0$.

If $(c,d)$ is a bounded connected component of the complement of $\Gamma$, then

$$\lambda((c,d)) \leq \lambda([0,d]) - \lambda([0,c]) = a(d) - a(c) = c - c = 0.$$

It follows that $\lambda([0,\infty) \setminus \Gamma) = 0$.

Observe that if $(c,d)$ is a connected component of the complement of $\Gamma$, then $b(x) = d$ for all $x \in (c,d)$. It was already shown that $b(x) = x$ when $x \in \Gamma$.

Therefore, for all $\varphi \in L^+(\mathcal{B})$, an application of Theorem 1.1 yields

$$\begin{aligned}\int_{[0,\infty)} \varphi \, d\lambda &= \int_{\Gamma} \varphi \, d\lambda = \int_{b^{-1}(\Gamma)} \varphi \circ b(x) \, dx = \int_0^{\sup \Gamma} \varphi \circ b(x) \, dx \\ &= \int_{\Gamma} \varphi \circ b(x) \, dx + \sum \int_c^d \varphi \circ b(x) \, dx = \int_{\Gamma} \varphi(x) \, dx + \sum \int_c^d \varphi(d) \, dx \\ &= \int_{\Gamma} \varphi(x) \, dx + \sum (d-c)\varphi(d).\end{aligned}$$

This completes the proof.

∎

**Definition 2.4.4** *Given a $\sigma$-finite measure space $(U, \Sigma, \mu)$ with an ordered core $\mathcal{A}$, then the measure $\lambda$ given by Theorem 2.4.3 is the measure induced by $\mathcal{A}$.*

For future reference, we record a rearrangement formula for the collection $L^{\downarrow}(\mathcal{B})$ of core decreasing functions.

**Lemma 2.4.5** *Let $\lambda$ be the induced measure by $\mathcal{A}$, the function b be the one defined on Theorem 2.4.3 and $\varphi \in L^{\downarrow}(\mathcal{B})$. Then,*

$$\varphi \circ b = \varphi^*.$$

*Where $^*$ denotes the nonincreasing rearrangement with respect to the measure $\lambda$.*

**Proof:** Let the symbol $^\#$ denote the nonincreasing rearrangement with respect to the Lebesgue measure and suppose that

$$\varphi = \sum_{k=1}^{K} \alpha_k \chi_{[0, x_K]},$$

for some $x_1 \geq \cdots \geq x_K \geq x_{K+1} > 0$ and $\alpha_1, \ldots, \alpha_K$ some real numbers. For each $j \in \{1, \ldots, K\}$ the function $\varphi$ takes the value $\sum_{k=1}^{j} \alpha_k$ on the set $(x_{j+1}, x_j]$. By Theorem 2.4.3 we have that

$$\lambda((x_{j+1}, x_j]) = \lambda([0, x_j]) - \lambda([0, x_{j+1}]) = a(x_j) - a(x_{j+1}),$$

where the function $a$ is the one defined on Theorem 2.4.3. In that Theorem we find that $b(z) \leq x$ if and only if $z \leq a(x)$. Therefore, $\varphi \circ b$ takes that same value on the set $(a(x_{j+1}), a(x_j)]$, which has Lebesgue measure $a(x_j) - a(x_{j+1})$. These are the only values that the functions $\varphi$ and $\varphi \circ b$ take, so $(\varphi \circ b)^\# = \varphi^*$.

For a general $\varphi \in L^{\downarrow}(\mathcal{B})$, by Lemma 2.2.5, $\varphi$ is an increasing limit of simple functions $\varphi_n$ of the previous form. The result follows from (3) in Proposition 1.3.9.

∎

### 2.4.1 Transition maps

We explore the close relation between measurable functions on $(U, \Sigma, \mu)$ and the induced measure space $([0, \infty), \lambda)$. This is done by applying Theorem 2.3.6, the maps produced take the role of transition maps between the original space and the induced measure space. In this section, we explore the properties of the transition maps and study their behavior on monotone functions.

**Proposition 2.4.6** *Let $(U, \Sigma, \mu)$, be a $\sigma$-finite measure space with ordered core $\mathcal{A}$, enriched core $\mathcal{M}$, and induced measure $\lambda$. Then, there exists a map $R : L^1_{Loc,\mathcal{A}} \cup L^+_\mu \to L^1_{Loc,\lambda} \cup L^+_\lambda$ such that for any $f \in L^1_{Loc,\mathcal{A}} \cup L^+_\mu$:*

1. *The restricted linear map $R : L^\infty_\mu \to L^\infty_\lambda$ satisfies $\|Rf\|_{L^\infty_\lambda} \leq \|f\|_{L^\infty_\mu}$.*
2. *If $f \geq 0$ $\mu$-almost everywhere, then $Rf \geq 0$ $\lambda$-almost everywhere.*
3. *For any sequence $\{f_n\} \in L^+(\mathcal{A})$ such that $f_n \uparrow f$ $\mu$-almost everywhere, $Rf_n \uparrow Rf$ $\lambda$-almost everywhere.*
4. *$R$ is additive and positive homogeneous on $L^+(\mathcal{A})$.*
5. *$R$ is linear on $L^1_{Loc,\mathcal{A}}$, mapping to $L^1_{Loc,\mathcal{B}}$.*
6. *$Rf \in L(\mathcal{B})$ and $Rf = 0$ outside $\Gamma$.*
7. *$|Rf| \leq R|f|$ $\lambda$-almost everywhere.*
8. *For all $M \in \mathcal{M}$,*
$$\int_M f \, d\mu = \int_{[0,\mu(M)]} Rf \, d\lambda.$$
9. *The restricted linear map $R : L^1_\mu \to L^1_\lambda$ satisfies $\|Rf\|_{L^1_\lambda} \leq \|f\|_{L^1_\mu}$, with equality if $f \geq 0$ $\mu$-a.e.*
10. *If $M \in \mathcal{M}$, then $R\chi_M = \chi_{[0,\mu(M)]}$ $\lambda$-a.e.*
11. *If $f \in L^+(\Sigma)$ and $g \in L^+(\mathcal{A})$, then $R(fg) = RfRg$ $\mu$-almost everywhere.*

**Proof:** For each $x \in \Gamma$, choose $M_x \in \mathcal{M}$ such that $\mu(M_x) = x$. Consider the map $r : \mathcal{B} \to \mathcal{M}$ defined by
$$r([0, x]) = M_{a(x)}, \quad \forall x \geq 0.$$
and $r(\emptyset) = \emptyset$. Here $a(x)$ is the function defined in Theorem 2.4.3. If $x > y$ then $a(x) \geq a(y)$. If $a(x) > a(y)$ then $M_y \subset M_x$ by the total ordering of $\mathcal{M}$ and the fact that $\mu(M_y) < \mu(M_x)$. Therefore $y < x$ implies $r(y) \subseteq r(x)$ and shows that $r$ is order-preserving.

Let $x, y \geq 0$ and suppose that $y \leq x$. Then,
$$\begin{aligned}\mu\Big(r([0, y]) \setminus r([0, x])\Big) &= \mu\Big(M_y \setminus M_x\Big) = \mu(M_y) - \mu(M_x) = a(y) - a(x) = \lambda([0, y]) - \lambda([0, x]) \\ &= \lambda([0, y] \setminus [0, x]).\end{aligned}$$

If $y < x$, the equation holds trivially.

Therefore the map $r$ is a core morphism with constant $c = 1$. Theorem 2.3.6 provides the existence of the map $R$. Also notice that $L^1_{\text{Loc},\mathcal{A}} = L^1_{\text{Loc},\mathcal{M}}$.

To prove (8), let $M \in \mathcal{M}$. Let $M_x \in \mathcal{M}$ be the set $r([0, \mu(M)])$. Then $M_x = M$ up to a set of $\mu$-measure zero. Hence

$$\int_M f\,d\mu = \int_{M_x \setminus \emptyset} f\,d\mu = \int_{r\left([0,\mu(M)]\right)\setminus r(\emptyset)} f\,d\mu = \int_{[0,\mu(M)]\setminus\emptyset} Rf\,d\lambda = \int_{[0,\mu(M)]} Rf\,d\lambda.$$

Similarly, to prove item (10), note that $\chi_M = \chi_{M_x}$ $\mu$-a.e. Thus Theorem 2.3.6 shows that $R\chi_M = R\chi_{M_x} = \chi_{[0,\mu(M)]}$.

The remaining items (1) to (11) follow directly from the corresponding items in Theorem 2.3.6, the definition of $r$, and the observation that $c = 1$.

∎

**Proposition 2.4.7** *Let $(U, \Sigma, \mu)$, be a $\sigma$-finite measure space with ordered core $\mathcal{A}$, enriched core $\mathcal{M}$, and the induced measure $\lambda$. Then, there exists a map $Q : L^1_{Loc,\lambda} \cup L^+_\lambda \to L^1_{Loc,\mathcal{A}} \cup L^+_\mu$ such that for any $\varphi \in L^1_{Loc,\lambda} \cup L^+_\lambda$:*

1. *The restricted linear map $Q : L^\infty_\lambda \to L^\infty_\mu$ satisfies $\|Q\varphi\|_{L^\infty_\mu} \le \|\varphi\|_{L^\infty_\lambda}$.*
2. *If $\varphi \ge 0$ $\lambda$-almost everywhere, then $Q\varphi \ge 0$ $\mu$-almost everywhere.*
3. *For any sequence $\{\varphi_n\} \in L^+(\mathcal{B})$ such that $\varphi_n \uparrow \varphi$ $\lambda$-almost everywhere, $Q\varphi_n \uparrow Q\varphi$ $\mu$-almost everywhere.*
4. *$Q$ is additive and positive homogeneous on $L^+(\mathcal{B})$.*
5. *$Q$ is linear on $L^1_{Loc,\lambda}$, mapping to $L^1_{Loc,\mathcal{A}}$.*
6. *$Q\varphi \in L(\mathcal{A})$ and $Q\varphi$ is zero outside $\cup\mathcal{A}$.*
7. *$|Q\varphi| \le Q\,|\varphi|$ $\mu$-almost everywhere.*
8. *For all $M \in \mathcal{M}$,*
$$\int_{[0,\mu(M)]} \varphi\,d\lambda = \int_M Q\varphi\,d\mu.$$
9. *The restricted linear map $Q : L^1_\lambda \to L^1_\mu$ satisfies $\|Q\varphi\|_{L^1_\mu} = \|\varphi\|_{L^1_\lambda}$, with equality if $\varphi \ge 0$ $\lambda$-almost everywhere.*
10. *If $M \in \mathcal{M}$, then $Q\chi_{[0,\mu(M)]} = \chi_M$ $\mu$-a.e.*
11. *If $\varphi, \psi \in L^+(\mathcal{B})$, then $Q(\varphi\psi) = Q\varphi Q\psi$ $\mu$-almost everywhere.*

**Proof:** Define the set function $r : \mathcal{M} \to \mathcal{B}$ by $r(\emptyset) = \emptyset$ and

$$r(M) = [0, \mu(M)], \quad \forall M \in \mathcal{M} \setminus \{\emptyset\}.$$

Since $\mathcal{M}$ is totally ordered, the monotonicity of $\mu$ makes the map $r$ order preserving. For $M, N \in \mathcal{M}$ such that $N \subseteq M$ we have

$$\begin{aligned}\lambda(r(M) \setminus r(N)) &= \lambda([0, \mu(M)] \setminus [0, \mu(N)]) = \lambda([0, \mu(M)]) - \lambda([0, \mu(N)]) = \mu(M) - \mu(N) \\ &= \mu(M \setminus N).\end{aligned}$$

Therefore, $r$ is a core morphism with constant $c = 1$. Theorem 2.3.6 provides the existence of the map $Q$. Also notice that $L^1_{\mathrm{Loc},\mathcal{A}} = L^1_{\mathrm{Loc},\mathcal{M}}$.

Items (1) to (11) follow directly from the corresponding items in Theorem 2.3.6, the definition of $r$, and the observation that $c = 1$. Only (11) requires comment. The corresponding statement in Theorem 2.3.6 requires that $\psi$ is measurable in the $\sigma$-algebra generated by $\{[0, x] : x \in \Gamma\}$. This forces $\psi$ to be constant on every connected component of the complement of $\Gamma$. But $\lambda$ is supported on $\Gamma$, so we may assume that $\psi$ is constant $\lambda$-a.e. on every connected component of the complement of $\Gamma$.

∎

The next theorem explores the close relations between the transition maps $R$ and $Q$.

**Theorem 2.4.8** *Let $R, Q$ be as in Propositions 2.4.6 and 2.4.7. Then:*

1. *If $\varphi \in L^+(\mathcal{B}) \cup L^1_{Loc,\lambda}$, then $RQ\varphi = \varphi$ $\lambda$-a.e;*
2. *If $f \in L^+(\mathcal{A}) \cup L^1_{Loc,\mathcal{A}}$, then $QRf = f$ $\mu$-a.e;*
3. *If $f \in L^+(\mathcal{A})$, $\varphi \in L^+(\mathcal{B})$ and $M \in \mathcal{M}$ then*

$$\int_M f(Q\varphi)\,d\mu = \int_{[0,\mu(M)]} (Rf)\varphi\,d\lambda \quad \text{and} \quad \int_U f(Q\varphi)\,d\mu = \int_{[0,\infty)} (Rf)\varphi\,d\lambda;$$

4. *If $\varphi \in L^{\downarrow}(\mathcal{B})$, $Q\varphi \in L^{\downarrow}(\mathcal{A})$ and $\varphi^* = (Q\varphi)^*$;*
5. *If $f \in L^{\downarrow}(\mathcal{A})$, $Rf \in L^{\downarrow}(\mathcal{B})$ and $f^* = (Rf)^*$;*
6. *If $g \in L^{\downarrow}(\mathcal{A})$, then*

$$\begin{aligned}&\left\{h \in L^{\downarrow}(\mathcal{A}) : \int_M h\,d\mu \le \int_M g\,d\mu,\ \forall M \in \mathcal{M}\right\} \\ &\qquad = \left\{Q\psi : \psi \in L^{\downarrow}(\mathcal{B}),\ \int_{[0,x]} \psi\,d\lambda \le \int_{[0,x]} Rg\,d\lambda,\ \forall x \ge 0\right\}.\end{aligned}$$

*The symbol $^*$ denotes the nonincreasing rearrangement from Section 1.3.1.*

**Proof:** To prove (1): Let $\varphi \in L^+(\mathcal{B}) \cup L^1_{\mathrm{Loc},\lambda}$. Let $x \in [0,\infty)$ and consider the map $a(x)$ from Theorem 2.4.3 and $M_{a(x)} \in \mathcal{M}$ satisfying $\mu(M_{a(x)}) = a(x)$. Then using item (3) in Propositions 2.4.6 and 2.4.7 and the fact that $[0,x] = [0,a(x)]$ $\lambda$-a.e we get

$$\int_{[0,x]} RQ\varphi\, d\lambda = \int_{[0,a(x)]} RQ\varphi\, d\lambda = \int_{M_{a(x)}} Q\varphi\, d\mu = \int_{[0,a(x)]} \varphi\, d\lambda = \int_{[0,x]} \varphi\, d\lambda.$$

Therefore Lemma 2.3.5 yields equality $\lambda$-a.e and proves item (1).

To prove (2): Let $f \in L^+(\mathcal{A}) \cup L^1_{\mathrm{Loc},\mu}$. Let $A \in \mathcal{A}$. Then using item (3) in Propositions 2.4.6 and 2.4.7 we get

$$\int_A QRf\, d\mu = \int_{[0,\mu(A)]} Rf\, d\lambda = \int_A f\, d\mu.$$

Therefore Lemma 2.3.5 yields equality $\mu$-a.e and proves item (2).

To prove (3): Let $f \in L^+(\mathcal{A})$, $\varphi \in L^+(\mathcal{B})$ and $M \in \mathcal{M}$. Then $Q\varphi \in \sigma(\mathcal{A})$. Therefore, by (11) in Proposition 2.4.6, we have $R(fQ\varphi) = (Rf)(RQ\varphi)$ and by item (1) we get $R(fQ\varphi) = (Rf)\varphi$, thus

$$\int_M f(Q\varphi)\, d\mu = \int_{[0,\mu(M)]} R(f(Q\varphi))\, d\lambda = \int_{[0,\mu(M)]} (Rf)\varphi\, d\lambda.$$

Since $Q\varphi$ is zero outside $\cup\mathcal{A}$, we can reduce the integration to this set. Similarly, we can reduce the integrals involving $Rf$ to $\Gamma$. Since $\mathcal{A}$ is $\sigma$-bounded, take an increasing sequence $\{A_n\}_{n\in\mathbb{N}} \in \mathcal{A}$ such that $A_n \uparrow \cup\mathcal{A}$. The monotone convergence theorem yields

$$\int_U f(Q\varphi)\, d\mu = \int_{\cup\mathcal{A}} f(Q\varphi)\, d\mu = \sup_n \int_{A_n} f(Q\varphi)\, d\mu = \sup_n \int_{[0,\mu(A_n)]} (Rf)\varphi\, d\lambda = \int_\Gamma (Rf)\varphi\, d\lambda$$
$$= \int_{[0,\infty)} (Rf)\varphi\, d\lambda.$$

This completes the proof of item (3).

To prove (4): Let $\varphi \in L^{\downarrow}(\mathcal{B})$, then it is an increasing limit of functions of the form $\sum_{k=1}^n \alpha_k \chi_{[0,x_k]}$. Without loss of generality, we may assume that $\{x_k\}$ is decreasing. By the construction of $\lambda$, we may consider functions of the form $\sum_{k=1}^n \alpha_k \chi_{[0,a(x_k)]}$. By item (3) in Proposition 2.4.7 we have that $Q\varphi$ is the increasing limit of $\sum_{k=1}^n \alpha_k Q\chi_{[0,a(x_k)]} = \sum_{k=1}^n \alpha_k \chi_{M_{a(x_k)}}$. Thus, $Q\varphi \in L^{\downarrow}(\mathcal{A})$ by Lemma 2.2.5.

To finish proving item (4), note that by item (3) in Proposition 1.3.9, it suffices to show that $\varphi^* = (Q\varphi)^*$ holds for functions of the form $\varphi = \sum_{k=1}^n \alpha_k \chi_{[0,x_i}$ with $\{x_i\} \in \Gamma$ and decreasing. The only non-zero values they take are $\sum_{k=1}^j \alpha_k$, for $j \in \{1,\dots,n\}$, with $\varphi$ taking that value on the set $(x_{i+1}, x_i]$ and $Q\varphi$ on the set $M_{x_i} \setminus M_{x_{i+1}}$. Therefore,

$$\lambda((x_{i+1}, x_i]) = x_i - x_{i+1} = \mu(M_{x_i} \setminus M_x).$$

This shows that the distribution functions of $\varphi$ and $Q\varphi$ coincide, therefore $\varphi^* = (Q\varphi)^*$ and completes the proof of item (4).

To prove item (5): Let $f \in L^{\downarrow}(\mathcal{A})$. By Lemma 2.2.5, the function $f$ is an increasing sequence of functions of the form

$$\sum_{k=1}^{n} \alpha_k \chi_{M_k} = \sum_{k=1}^{n} \alpha_k Q(\chi_{[0,\mu(M_k)]}) = Q\left(\sum_{k=1}^{n} \alpha_k \chi_{[0,\mu(M_k)]}\right).$$

for some $\{\alpha_k\} \in (0,\infty)$, $M_k \in \mathcal{M}$. An application of $R$ yields

$$R\left(\sum_{k=1}^{n} \alpha_k \chi_{M_k}\right) = RQ\left(\sum_{k=1}^{n} \alpha_k \chi_{[0,\mu(M_k)]}\right) = \sum_{k=1}^{n} \alpha_k \chi_{[0,\mu(M_k)]},$$

by item (1). By item (3) in Proposition 2.4.6 and Lemma 2.2.5, we get $Rf \in L^{\downarrow}(\mathcal{B})$. Moreover, by item (3) we have

$$\left(\sum_{k=1}^{n} \alpha_k \chi_{M_k}\right)^* = \left(Q\left(\sum_{k=1}^{n} \alpha_k \chi_{[0,\mu(M_k)]}\right)\right)^* = \left(\sum_{k=1}^{n} \alpha_k \chi_{[0,\mu(M_k)]}\right)^* = \left(R\left(\sum_{k=1}^{n} \alpha_k \chi_{M_K}\right)\right)^*.$$

Then item (3) in Proposition 1.3.9 yields $(Rf)^* = f^*$ and proves item (5).

Fix $g \in L^{\downarrow}(\mathcal{A})$. To show item (6), we show each set contains the other. Let $h \in L^{\downarrow}(\mathcal{A})$, then item (2) shows that $h = QRh$. Let $\psi = Rh$ and $x \geq 0$, then

$$\int_{[0,x]} \psi\, d\lambda = \int_{[0,x]} Rh\, d\lambda = \int_{M_{a(x)}} h\, d\mu \leq \int_{M_{a(x)}} g\, d\mu = \int_{[0,a(x)]} Rg\, d\lambda = \int_{[0,x]} Rg\, d\lambda.$$

This proves "$\subseteq$". Conversely, let $\psi \in L^{\downarrow}(\mathcal{B})$ and set $h = Q\psi$, then for any $M \in \mathcal{M}$:

$$\int_M h\, d\mu = \int_{[0,\mu(M)]} Rh\, d\lambda = \int_{[0,\mu(M)]} RQ\psi\, d\lambda = \int_{[0,\mu(M)]} \psi\, d\lambda \leq \int_{[0,\mu(M)]} Rg\, d\lambda = \int_M g\, d\mu.$$

This proves equality of the sets and completes the proof.

∎

We finish this chapter by describing a decomposition of the measure space $(\cup_{\mathcal{A}}, \sigma(\mathcal{A}), \mu)$ into two sets which correspond, via the map $R$, to the atomic and non-atomic parts of the measure $\lambda$.

**Lemma 2.4.9** *Let $(U,\Sigma,\mu), \mathcal{A}, \mathcal{B}, \lambda, R, Q$ as in the previous theorem. Then, there exist sets denoted $U_C, U_D \in \sigma(\mathcal{A})$ such that*

$$\cup\mathcal{A} = U_C \cup U_D \quad \text{and} \quad U_C \cap U_D = \emptyset,$$

*such that for any $f \in L^+(\Sigma) \cup L^1_{Loc,\mathcal{A}}$:*

$$\int_{U_C} f\, d\mu = \int_{\Gamma} Rf(x)\, dx, \quad \text{and} \quad \int_{U_D} f\, d\mu = \sum Rf(d)(d-c),$$

*where the sum is taken over the connected components* $(c,d)$ *of the complement of* $\Gamma$. *If, in addition,* $f \in L^+(\mathcal{A}) \cup L^1_{Loc,\mathcal{A}}$, *then*

$$f\chi_{\cup A} = f\chi_{U_C} + \sum_{j\in J} \alpha_j \chi_{(M_j\setminus N_j)}, \quad \mu\text{-a.e.} \tag{2.7}$$

*for some countable disjoint collection* $\{(M_j \setminus N_j)\}_{j\in J}$ *with* $M_j, N_j \in \mathcal{M}$ *and some* $\alpha_j > 0$.

*If* $V_C, V_D \in \sigma(\mathcal{A})$ *satisfies the properties above, then* $U_C = V_C$ *and* $U_D = V_D$ *up to a set of* $\mu$*-measure zero.*

**Proof:** Since $\lambda$ is a $\sigma$-finite measure, the collection $I = \{x \in \Gamma : \lambda(\{x\}) > 0\}$ is countable. Also note that this set is the same as $\{b(x) > 0 : a(x) < b(x)\}$ by Theorem 2.4.3. Let

$$G = \bigcup \{(a(x), b(x)] : x \in I\} = \bigcup \{[0, b(x)] \setminus [0, a(x)] : x \in I\}.$$

The union is countable and disjoint, therefore

$$Q\chi_G = \sum_{x\in I} Q\Big(\chi_{[0,b(x)]} - \chi_{[0,a(x)]}\Big) = \sum_{x\in I}\Big(\chi_{M_{b(x)}} - \chi_{M_{a(x)}}\Big) = \chi_{U_D},$$

where $U_D = \bigcup_{x\in I} (M_{b(x)} \setminus M_{a(x)}) \in \sigma(\mathcal{A})$. Define the set $U_C = \cup\mathcal{A} \setminus U_D$.

Notice that $\chi_{U_D}$ is $\sigma(\mathcal{A})$-measurable, therefore item (11) in Lemma 2.4.6 yields

$$\int_{U_D} f\,d\mu = \int_U f\chi_{U_D}\,d\mu = \int_\Gamma RfR(\chi_{U_D})\,d\lambda = \int_G Rf\,d\lambda = \sum_{x\in I} \int_{(a(x),b(x)]} Rf\,d\lambda$$
$$= \sum_{x\in I} Rf(b(x))(b(x) - a(x)) = \sum Rf(d)(d-c).$$

Since $\int_U f\,d\mu = \int_{U_C} f\,d\mu + \int_{U_D} f\,d\mu$, Theorem 2.4.3 shows that

$$\int_{U_C} f\,d\mu = \int_\Gamma Rf(x)\,dx.$$

Notice that $Q1 = \chi_{\cup\mathcal{A}}$, by linearity of $Q$ we have that $Q\chi_{\Gamma\setminus G} = \chi_{\cup\mathcal{A}} - \chi_{U_D} = \chi_{U_C}$. If $f$ is $\sigma(\mathcal{A})$-measurable, then $f\chi_{\cup A} = QRf$ and using Theorem 2.4.8:

$$\begin{aligned} f\chi_{\cup\mathcal{A}} &= Q(Rf\chi_G + Rf\chi_{\Gamma\setminus G}) = Q(Rf\chi_G) + Q(Rf)Q(\chi_{\Gamma\setminus G}) = Q(Rf\chi_G) + f\chi_{U_C} \\ &= Q\Big(\sum Rf\chi_{(c,d]}\Big) + f\chi_{U_C} = Q\Big(\sum Rf(d)\chi_{(c,d]}\Big) + f\chi_{U_C} = \sum Rf(d)Q(\chi_{(c,d]}) + f\chi_{U_C} \\ &= f\chi_{U_C} + \sum Rf(d)\chi_{(M_d\setminus M_c)}. \end{aligned}$$

Here, we used the fact that for a connected component $(c,d)$ in the complement of $\Gamma$ we have that $\chi_{(c,d]} = \chi_{\{d\}}$ $\lambda$-a.e. Then we can make $\alpha_j = Rf(d)$, $M_j = M_d$ and $N_j = M_c$ to prove equation (2.7).

To show uniqueness of the decomposition, note that $\int_{U_C} f\,d\mu = \int_{V_C} f\,d\mu$ for any $f \in L^1_{\text{Loc},\mathcal{A}}$. Making $f = \chi_A$ for $A \in \mathcal{A}$ yields

$$\int_A \chi_{U_C}\,d\mu = \int_{U_C} f\,d\mu = \int_{V_C} f\,d\mu = \int_A \chi_{V_C}\,d\mu.$$

Therefore Lemma 2.3.5 yields $\chi_{U_C} = \chi_{V_C}$ $\mu$-a.e. Hence $U_C = V_C$ up to a set of $\mu$-measure zero, since the decomposition of $U$ was disjoint, then $U_D$ and $V_D$ also differ by a set of $\mu$-measure zero. This completes the proof.

∎

In the case that $\mathcal{A}$ is full and separable, the decomposition takes a more explicit form.

**Corollary 2.4.10** *Let $(U, \Sigma, \mu), \mathcal{A}, \mathcal{B}, \lambda, R, Q$ as in the previous theorem with $\mathcal{A}$ full and separable. Then there exists a countable collection of distinct equivalence classes $\{[u_j]_{\mathcal{A}}\}$ such that $\mu([u_j]_{\mathcal{A}}) > 0$ and a set $U_C \in \sigma(\mathcal{A})$ such that $U = U_C \cup \bigcup_j [u_j]_{\mathcal{A}}$, a disjoint union. Also for each $f \in L^+(\Sigma) \cup L^1_{Loc,\mathcal{A}}$:*

$$\int_{U_C} f \, d\mu = \int_{\Gamma} Rf(x) \, dx$$

*If in addition $f \in L^+(\sigma(\mathcal{A})) \cup L^1_{Loc,\mathcal{A}}$:*

$$f = f\chi_{U_C} + \sum_j \alpha_j \chi_{[u_j]_{\mathcal{A}}},$$

*where $\alpha_j = f(u_j)$ and does not depend on the choice of representative.*

**Proof:** For each $f \in L^+(\mathcal{A}) \cup L^1_{\mathrm{Loc},\mathcal{A}}$, Lemma 2.4.9 provides a decomposition of the form

$$f = f\chi_{U_C} + \sum_{j \in J} \alpha_j \chi_{(M_j \setminus N_j)}.$$

For each $j$ define the sets

$$L_j = \bigcup \{E \in \mathcal{M} : \mu(E) < \mu(M_j)\}, \quad S_j = \bigcap \{E \in \mathcal{M} : \mu(E) > \mu(N_j)\}.$$

Since $\mathcal{A}$ is separable, then $L_j, S_j \in \mathcal{M}$, also $N_j \subseteq L_j$ and $S_j \subseteq M_j$. Since $(\mu(N_j), \mu(M_j))$ is a connected component of the complement of $\Gamma$, then $\mu(E) < \mu(M_j)$ implies that $\mu(E) \leq \mu(N_j)$, thus $\mu(L_j) = \mu(N_j)$. Similarly $\mu(S_j) = \mu(M_j)$.

It follows that $\chi_{(M_j \setminus N_j)} = \chi_{(S_j \setminus L_j)}$ up to a set of $\mu$-measure zero. Moreover, by construction, if $u_j \in S_j \setminus L_j$, then $[u_j]_{\mathcal{A}} = (S_j \setminus L_j)$.

Therefore

$$f = f\chi_{U_C} + \sum_{j \in J} \alpha_j \chi_{[u_j]_{\mathcal{A}}}.$$

Evaluation on each $u_j$ yields $f(u_j) = \alpha_j$ and completes the proof.

∎

# Chapter 3

# Monotone envelopes

In this chapter, we introduce two very important constructions that generalize the least-decreasing majorant and the level function to the space $(U, \Sigma, \mu)$ with an ordered core $\mathcal{A}$. We will study a new preorder relation on locally integrable $\sigma(\mathcal{A})$-measurable functions, we use this preorder to give a functional characterization of the least-decreasing majorant, greatest decreasing minorant, and level function.

## 3.1 Two different preorder relations

We introduce two different preorder relations on a very large collection of nonnegative functions

**Definition 3.1.1** *Let $f, g \in L^1_{Loc,\mathcal{A}} \cap L^+_\mu$. We write $f \preccurlyeq_{\mathcal{A}} g$ whenever*

$$\int_E f\, d\mu \le \int_E g\, d\mu,\ \forall E \in \mathcal{A}.$$

First, we note that we can use the core $\mathcal{A}$ or its enriched core $\mathcal{M}$ in the definition without changing the preorder. To see this, suppose that $f \preccurlyeq_{\mathcal{A}} g$ and let $M \in \mathcal{M}$ then $\mu(M) \in \Gamma$ and there exists a sequence $x_n \to \mu(M)$ such that $x_n = \mu(E_n)$ for some $E_n \in \mathcal{A}$. If we can extract an increasing subsequence, the monotone convergence theorem shows that

$$\int_M f\, d\mu = \int_{[0,\mu(M)]} Rf\, d\lambda = \sup_n \int_{[0,\mu(E_n)]} Rf\, d\lambda = \sup_n \int_{E_n} f\, d\mu \le \sup_n \int_{E_n} g\, d\mu$$
$$= \sup_n \int_{[0,\mu(E_n)]} Rg\, d\lambda = \int_{[0,\mu(M)]} Rg\, d\lambda = \int_M g\, d\mu.$$

In the case that a decreasing subsequence of $x_n$ can be extracted, the dominated convergence theorem applies and a similar argument shows that $\int_M f\, d\mu \le \int_M g\, d\mu$.

It is clear that $f \le g$ implies $f \preccurlyeq_{\mathcal{A}} g$, however, the converse may fail, as it is shown in the following example.

**Example 3.1.2** *Let $(U, \Sigma, \mu)$ be the ordered core given in Example 2.1.9 with $d = 2$. Consider $f = \frac{1}{2}$ and $g = \chi_{H^+}$ where $H^+ = \{(x, y) : y \geq 0\}$. Clearly $Rf = Rg$ so $f \preccurlyeq_{\mathcal{A}} g$ and $g \preccurlyeq_{\mathcal{A}} f$ but $f \leq g$ and $g \leq f$ both fail.*

The previous example also shows that $f \preccurlyeq_{\mathcal{A}} g$ and $g \preccurlyeq_{\mathcal{A}} f$ need not imply $f = g$. However, we do have a partial order relation when restricted to a large subset $\sigma(\mathcal{A})$-measurable functions.

**Proposition 3.1.3** *Let $f, g \in L^+(\mathcal{A}) \cap L^1_{Loc,\mathcal{A}}$ such that $f \preccurlyeq_{\mathcal{A}} g$ and $g \preccurlyeq_{\mathcal{A}} f$, then $f = g$ $\mu$-almost everywhere.*

**Proof:** If both $f \preccurlyeq_{\mathcal{A}} g$ and $g \preccurlyeq_{\mathcal{A}} f$, then $\int_A f\,d\mu = \int_A g\,d\mu$ for all $A \in \mathcal{A}$. The result follows directly from Lemma 2.3.5.

∎

The following proposition explores the preorder '$\preccurlyeq_{\mathcal{A}}$' and multiplication by core decreasing functions.

**Proposition 3.1.4** *Let $f, g \in L^+(\Sigma)$, then $f \preccurlyeq_{\mathcal{A}} g$ if and only if $\int_U fh\,d\mu \leq \int_U gh\,d\mu$ for all $h \in L^{\downarrow}(\mathcal{A})$.*

**Proof:** Suppose $f \preccurlyeq_{\mathcal{A}} g$ and let $h \in L^{\downarrow}(\mathcal{A})$. Lemma 2.2.5 shows that $h$ is the increasing limit of functions of the form $\sum_{k=1}^{n} \alpha_k \chi_{M_k}$ for some $\alpha_k > 0$ and $M_k \in \mathcal{M}$. Then,

$$\int_U f\left(\sum_{k=1}^{n} \alpha_k \chi_{M_k}\right) d\mu = \sum_{k=1}^{n} \alpha_k \int_{M_k} f\,d\mu \leq \sum_{k=1}^{n} \alpha_k \int_{M_k} g\,d\mu = \int_U g\left(\sum_{k=1}^{n} \alpha_k \chi_{M_k}\right).$$

The monotone convergence theorem proves $\int_U fh\,d\mu \leq \int_U gh\,d\mu$.

Conversely, let $M \in \mathcal{M}$ and $h = \chi_M$ then

$$\int_M f\,d\mu = \int_U fh\,d\mu \leq \int_U gh\,d\mu = \int_M g\,d\mu.$$

∎

## 3.2 The least core decreasing majorant

We define a projection operator from $L(\Sigma)$ onto the set of core decreasing functions. This projection will be optimal in the sense of the partial order '$\leq$' $\mu$-almost everywhere.

**Definition 3.2.1** *Let $f \in L(\Sigma)$, we say that $\widetilde{f}$ is a least core decreasing majorant of $f$ if $|f| \leq \widetilde{f} \in L^{\downarrow}(\mathcal{A})$, and for every $g \in L^{\downarrow}(\mathcal{A})$ satisfying $|f| \leq g$ $\mu$-a.e, we have $\widetilde{f} \leq g$ $\mu$-a.e.*

Notice that, by Proposition 3.1.3, if a least core decreasing majorant exists, then it must be unique $\mu$-almost everywhere. The next lemma shows that a least core decreasing majorant always exists.

**Lemma 3.2.2** *Every $\Sigma$-measurable function $g$ has a least core decreasing majorant, denoted $\widetilde{g}$, which is unique up to $\mu$ almost everywhere equality. If $g_n \in L^+(\Sigma)$ and $g_n \uparrow g$ $\mu$-a.e, then $\widetilde{g_n} \uparrow \widetilde{g}$ $\mu$-a.e.*

**Proof:** Suppose that $|g| \leq C < \infty$ and let $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathcal{A}$ such that $A_n \uparrow U$. Set

$$\alpha_n = \inf\left\{\int_{A_n} h\, d\mu : h \in L^{\downarrow}(\mathcal{A}) \text{ and } h \geq |g|\right\}.$$

Note that the constant function $C$ belongs to $L^{\downarrow}(\mathcal{A})$, is an upper bound for $|g|$ and $\int_{A_n} C\, d\mu = c\mu(A_n) < \infty$. Hence the numbers $\alpha_n$ are finite.

For each $n \in \mathbb{N}^+$, there exists $h_n \in L^{\downarrow}(\mathcal{A})$ such that $h_n \geq |g|$ and $\alpha_n + 1/n > \int_{A_n} h_n\, d\mu$. Since the pointwise minimum of finitely many core decreasing functions is core decreasing, we may assume that $\{h_n\}$ is a decreasing sequence. Let $h = \inf_n h_n$, which is clearly a core decreasing majorant of $g$.

To show that $h$ is the least core decreasing majorant of $g$, let $w$ be another core decreasing majorant, then so is $\min\{h, w\}$, thus

$$\int_{A_n} h\, d\mu \leq \int_{A_n} h_n\, d\mu < \alpha_n + 1/n \leq \int_{A_n} \min\{w, h\}\, d\mu + 1/n.$$

Then $1/n \geq \int_{A_n} (h - \min\{w, h\})\, d\mu \geq 0$. Let $n \to \infty$ to get $\min\{w, h\} = h$ almost everywhere. This completes the proof in the case that $g$ is bounded.

For the unbounded case, define $g_m = \min\{m, |g|\}$ and let $\widetilde{g_m}$ be its least core decreasing majorant which exists since $g_m$ is bounded. Since $\widetilde{g_m} \geq \min\{m, |g|\} \geq \min\{m-1, |g|\} = g_{m-1}$, we have $\widetilde{g_m} \geq \widetilde{g_{m-1}}$. Therefore, $\{\widetilde{g_m}\}_{m\in\mathbb{N}}$ is an increasing sequence.

Let $h = \sup_{m\in\mathbb{N}} \widetilde{g_m}$. Since each $\widetilde{g_m}$ is bounded below by $\min|g|, m$, $h \geq |g|$, thus $h$ is a core decreasing majorant of $|g|$. If $w$ is another core decreasing majorant of $|g|$, then $\min(m, w)$ is a core decreasing majorant of $|g_m|$, thus $\min(m, w) \geq \widetilde{g_m}$. Let $m \to \infty$ to get that $w \geq h$ and prove that $h$ is the least core decreasing majorant of $g$.

For the final property, let $f_n \uparrow g$ with $f_n \in L^+(\Sigma)$. Then, for each $n \geq 1$, $f_n \leq g \leq \widetilde{g}$. Since $\widetilde{g}$ is a core decreasing majorant of $f_n$, we get $\widetilde{f_n} \leq \widetilde{g}$. By the same argument, we get that $\{\widetilde{f_n}\}$ is an increasing sequence. Denote $w = \sup_n \widetilde{f_n}$. It is core decreasing and satisfies $w \leq \widetilde{g}$. However, since $f_n \leq \widetilde{f_n}$, letting $n \to \infty$ yields $g \leq w$. Thus, $w$ is a core decreasing majorant of $g$ and we get $\widetilde{g} \leq w$, completing the proof.

∎

The proof of Lemma 3.2.2 does not describe the least core decreasing majorant. We address this in the remainder of this chapter. First, we give a pointwise description for the case that the core is separable. Then we provide a functional description, which is valid for every ordered core.

**Theorem 3.2.3** *Let $(U, \Sigma, \mu)$ be a $\sigma$-finite measure space with a separable full ordered core $\mathcal{A}$ and let $f \in L(\Sigma)$. Then*

$$\widetilde{f}(u) = \operatorname{ess\,sup}_{t \geq_{\mathcal{A}} u} |f(t)| = \operatorname{ess\,sup}_{t \notin (\leftarrow, u)_{\mathcal{A}}} |f(t)|.$$

**Proof:** Define $g(u) = \text{ess sup}_{t \notin (\leftarrow,u)_{\mathcal{A}}} |f(t)|$. Then g is nonnegative and if $t \leq_{\mathcal{A}} x$ then $g(x) \leq g(t)$. We show first that $g \in L^{+}(\mathcal{A})$.

Fix $\alpha > 0$ and define the set $O = g^{-1}(\alpha, \infty)$. Since $x \in (\leftarrow, x]_{\mathcal{A}}$ for all $x \geq 0$, it follows that $O \subseteq \bigcup_{x \in O} (\leftarrow, x]_{\mathcal{A}}$. We wish to show the converse.

Let $x \in O$ and $t \leq_{\mathcal{A}} x$, then $(\leftarrow, t)_{\mathcal{A}} \subseteq (\leftarrow, x)_{\mathcal{A}}$. Thus, $g(t) \geq g(x) > \alpha$ and it shows that $t \in O$. It follows that $O = \bigcup_{x \in O} (\leftarrow, x]_{\mathcal{A}}$.

Since the core is separable and the sets $(\leftarrow, x]_{\mathcal{A}} \in \mathcal{M}$, then $O \in \mathcal{M}$. This shows that $g$ is $\sigma(\mathcal{A})$-measurable and we see that $g$ is core decreasing. It remains to show that $|f| \leq g$ and $g \leq \widetilde{f}$ both hold $\mu$-a.e.

By virtue of Corollary 2.4.10, it suffices to show that $|f| \leq g \leq \widetilde{f}$ $\mu$-almost everywhere on $U_C$ and on each of the countably many equivalence classes $\{[u_j]_{\mathcal{A}}\}$ where $\mu([u_j]_{\mathcal{A}}) > 0$.

Fix one of those equivalence classes $[u_j]_{\mathcal{A}}$ and let $z \in [u_j]_{\mathcal{A}}$. Since $g$ and $\widetilde{f}$ belong to $L^{+}(\mathcal{A})$, they are constant on $[u_j]_{\mathcal{A}}$ by Corollary 2.4.10. By definition of essential supremum, we have

$$\begin{aligned} 0 &= \mu\Big( \{t \notin (\leftarrow, z)_{\mathcal{A}} : |f(t)| > g(z)\} \Big) \geq \mu\Big( \Big\{ t \in [u_j]_{\mathcal{A}} : |f(t)| > g(z) \Big\} \Big) \\ &= \mu\Big( \Big\{ t \in [u_j]_{\mathcal{A}} : |f(t)| > g(t) \Big\} \Big). \end{aligned}$$

It follows that $|f| \leq g$ $\mu$-a.e. on $[u_j]_{\mathcal{A}}$. To prove this on $U_C$: Fix $\epsilon > 0$, $n, m \in \mathbb{N}^{+}$, choose $\{A_m\} \in \mathcal{A}$ satisfying $A_m \uparrow U$ and set

$$S_{m,n} = \{z \in U_C \cap A_m : |f(z)| - g(z) > \epsilon \text{ and } n\epsilon \leq |f(z)| < (n+1)\epsilon\}.$$

Define the continuous function $\phi : [0, \infty) \to [0, \infty)$ by

$$\phi(t) = \int_t^{\infty} R(\chi_{S_{m,n}})(x) \chi_{\Gamma}(x)\, dx.$$

Since $S_{m,n} \subseteq A_m$, if $t > \mu(A_m)$, then we have $\phi(t) = 0$. Since $S_{m,n} \subseteq U_C$, an application of Corollary 2.4.10 yields

$$\mu(S_{m,n}) = \int_{U_C} \chi_{S_{m,n}}\, d\mu = \int_{\Gamma} R(\chi_{S_{m,n}})(x)\, dx = \phi(0).$$

Suppose that $\mu(S_{m,n}) > 0$. By continuity of $\phi$, there exist $t_1, t_2 \in (0, \mu(A_m))$, such that $\phi(t_1) = \frac{\mu(S_{m,n})}{2}$ and $\phi(t_2) = \frac{\mu(S_{m,n})}{3}$. Thus, there exists $E \in \mathcal{M}$ such that $t_1 \leq \mu(E) \leq t_2$ and it follows that

$$\begin{aligned} \mu(S_{m,n} \cap E) &= \int_{U_C} \chi_{S_{m,n}} \chi_E\, d\mu = \int_{\Gamma \cap [0, \mu(E)]} R(\chi_{S_{m,n}})(x)\, dx \geq \int_{\Gamma \cap [0, t_1]} R(\chi_{S_{m,n}})(x)\, dx \\ &= \int_0^{\infty} R(\chi_{S_{m,n}})(x) \chi_{\Gamma}(x)\, dx - \int_{t_1}^{\infty} R(\chi_{S_{m,n}})(x) \chi_{\Gamma}(x)\, dx = \phi(0) - \phi(t_1) \\ &= \frac{\mu(S_{m,n})}{2}. \end{aligned}$$

Also,

$$\begin{aligned} \mu(S_{m,n} \setminus E) &= \int_{U_C} \chi_{S_{m,n}} (1 - \chi_E)\, d\mu = \int_{\Gamma \cap [\mu(E), \infty)} R(\chi_{S_{m,n}})(x)\, dx \geq \int_{\Gamma \cap [t_2, \infty)} R(\chi_{S_{m,n}})(x)\, dx \\ &= \phi(t_2) = \frac{\mu(S_{m,n})}{3}. \end{aligned}$$

Let $z \in S_{m,n} \cap E$. If $t \in S_{m,n} \setminus E$ we have $z \leq_{\mathcal{A}} t$, and it follows that

$$g(z) = \text{ess sup}_{z \leq_{\mathcal{A}} t} |f(t)| \geq \text{ess sup}_{S_{m,n} \setminus E} |f(t)| \geq n\epsilon.$$

But $z \in S_{m,n} \cap E$ implies

$$g(z) < |f(z)| - \epsilon \leq (n+1)\epsilon - \epsilon = n\epsilon.$$

Thus $n\epsilon < g(z) \leq n\epsilon$, this is a contradiction. Therefore $\mu(S_{m,n}) = 0$ for each $m, n$. Let $m \to \infty$ to get that

$$\{z \in U_C : |f(z)| - g(z) > \epsilon\}$$

is a null set. Let $\epsilon \to 0$ to get $|f| \leq g$ up to a set of $\mu$-measure zero.

Since $g$ is a core decreasing majorant of $|f|$, then $\widetilde{f} \leq g$ almost everywhere.

Since $\widetilde{f}$ is core decreasing, then if $z \leq_{\mathcal{A}} t$, $|f|(t) \leq \widetilde{f}(t) \leq \widetilde{f}(z)$. Hence

$$\widetilde{f}(z) \geq \text{ess sup}_{z \leq_{\mathcal{A}} t} |f(t)| = g(z).$$

Thus $\widetilde{f} = g$ and the proof is complete.

$\blacksquare$

The next example shows that the formula provided in Theorem 3.2.3 need not hold for non-separable ordered cores.

**Example 3.2.4** *Consider the ordered core introduced in Example 2.1.12 and let* $f = 1 - \chi_{0_W}$. *Notice that* $f = \chi_E$ *for the core set* $E = V \dot{\cup} (W \setminus 0)$. *Therefore it is a core decreasing function, hence* $f = \widetilde{f}$. *We will show that if* $g(\alpha) = \text{ess sup}_{\alpha \leq_{\mathcal{A}} \beta} f(\beta)$ *does not coincide with* $f$.

*If* $\alpha \in V$, *then* $f(\beta) = 1$ *for* $\beta \in (V \setminus \alpha) \dot{\cup} (W \setminus 0_W)$ *which has positive measure, thus* $g(\alpha) = 1$.

*If* $\alpha \in W$, *then notice that the set* $\{0_W\}$ *has the same* $\mu$*-measure as* $U \setminus (\leftarrow, \alpha]_{\mathcal{A}}$. *Hence* $g(\alpha) = f(0_W) = 0$. *Therefore* $g = \chi_V$ *which differs from* $f$ *in the set of positive measure* $W \setminus \{0_W\}$.

We finish this section with a functional description of the least core decreasing majorant, notice that the second partial order '$\preccurlyeq_{\mathcal{A}}$' is involved in this description.

**Theorem 3.2.5** *Let* $(U, \Sigma, \mu)$ *be a* $\sigma$*-finite measure space with an ordered core* $\mathcal{A}$ *and* $f, g \in L^+(\Sigma)$, *then*

$$\int_U \widetilde{f} g \, d\mu = \sup \left\{ \int_U f h \, d\mu : h \preccurlyeq_{\mathcal{A}} g \right\}.$$

We need some preparation to prove Theorem 3.2.5; the first lemma shows that the least core decreasing majorant of a simple $\Sigma$-measurable function is a simple $\sigma(\mathcal{A})$-measurable function. The second lemma is the main tool, where we 'push the mass' of $f$ to form an appropriate function $g$ to achieve the desired supremum.

**Lemma 3.2.6** *Let* $u$ *be a nonnegative, simple,* $\Sigma$*-measurable function that vanishes outside a set* $S \in \mathcal{A}$ *and let* $\{y_0, \ldots, y_n\}$ *be the values that* $u$ *takes on sets of positive measure. Then* $\widetilde{u}$ *takes its values on a subset of* $\{y_0, \ldots, y_n\}$. *Moreover,* $\widetilde{u}$ *is a* $\sigma(\mathcal{A})$*-measurable function that vanishes outside* $S$.

**Proof:**

We may assume that $\{y_0, \ldots, y_n\}$ is ordered increasingly. For $j \in \{1, \ldots, n\}$, define the sets $V_j = \{s \in U : y_{j-1} < \widetilde{u}(s) < y_j\}$ and $V_{n+1} = \{s \in U : y_n < \widetilde{u}(s)\}$. Notice that these sets belong to $\sigma(\mathcal{A})$ since $\widetilde{u}$ is core decreasing. Let $V = \cup_{j=1}^{n+1} V_j$ and define

$$h = \widetilde{u}\chi_{U\setminus V} + \sum_{j=1}^{n+1} y_{j-1}\chi_{V_j}.$$

By construction $h$ is $\sigma(\mathcal{A})$-measurable and clearly $h \leq \widetilde{u}$. Moreover, since up to measure zero, $u$ does not take any values where $\widetilde{u}$ and $h$ differ, then $h$ is also a majorant of $u$. Hence, equality will follow once we show that $h$ is core decreasing.

Let $s, t \in U$, with $s \leq_{\mathcal{A}} t$, we wish to show that $h(t) \leq h(s)$ and we do so by checking four distinct cases.

**Case 1:** $s \notin V$ **and** $t \notin V$**.** Since $\widetilde{u}$ is core decreasing, $h(t) = \widetilde{u}(t) \leq \widetilde{u}(s) = h(s)$.

**Case 2:** $s \in V$ **and** $t \in V$**.** If $s$ and $t$ belong to the same $V_j$, then $h(s) = h(t)$. Otherwise, suppose that $s \in V_j$ and $t \in V_k$. Since $\widetilde{u}$ is core decreasing, we must have that $\widetilde{u}(t) \leq \widetilde{u}(s)$, so $k < j$, hence $h(t) = y_{k-1} < y_{j-1} = h(s)$.

**Case 3:** $s \notin V$ **and** $t \in V$**.** Suppose that $t \in V_j$. Since $\widetilde{u}$ is core decreasing, and $s \notin V_j$, we get $h(t) = y_{j-1} < y_j \leq \widetilde{u}(s) = h(s)$.

**Case 4:** $s \in V$ **and** $t \notin V$**.** Suppose that $s \in V_j$. Since $\widetilde{u}$ is core decreasing and $t \notin V_j$, we get $h(t) = \widetilde{u}(t) \leq y_{j-1} \leq \widetilde{u}(s)$, we get $h(t) \leq h(s) = y_{j-1}$.

Therefore $h$ is core decreasing, and by minimality $h = \widetilde{u}$, therefore $\mu(V_j) = 0$ for all $V_j$, hence $\widetilde{u}$ takes its values in a subset of $\{y_0, \ldots, y_n\}$.

Finally, notice that $\widetilde{u}\chi_S$ is also core decreasing. Since $u$ is zero outside $S$, it is also a core decreasing majorant of $u$, hence $\widetilde{u}\chi_S = \widetilde{u}$ and completes the proof.

∎

And now, for this particular case of $u$, we construct an approximating function $g$ by 'pushing mass' on core sets.

**Lemma 3.2.7** *Let $u$ be as in the previous lemma, $f \in L^+(\Sigma)$. If $\int_U f\widetilde{u}\, d\mu = \infty$ and $n > 0$, there exists a function $g \in L^+(\Sigma)$ such that $\int_E g\, d\mu \leq \int_E f\, d\mu$ for any $E \in \mathcal{A}$ and*

$$\int_U gu\, d\mu \geq n.$$

*If $\int_U f\widetilde{u}\, d\mu < \infty$ and $\epsilon > 0$, then there exists a function $g \in L^+(\Sigma)$ such that $\int_E g\, d\mu \leq \int_E f\, d\mu$ for any $E \in \mathcal{A}$ and*

$$\int_U f\widetilde{u}\, d\mu - \epsilon < \int_U gu\, d\mu.$$

**Proof:** By the previous lemma, we may assume that $\widetilde{u} = \sum_{j=1}^m z_j\chi_{E_j}$ for a decreasing sequence $z_j$ and disjoint sets $E_j \in \sigma(\mathcal{A})$. Define the sets $B_j = \bigcup_{k=1}^j E_k$, notice that $B_j = \{s \in U : \widetilde{u}(s) \geq z_j\}$. Also notice that if $s \leq_{\mathcal{A}} t$ and $t \in B_j$, then $s \in B_j$, thus $B_j \in \mathcal{M}$. Note that $B_0 = \emptyset \in \mathcal{M}$.

Define $H = \{s \in U : \widetilde{u}(s) = u(s)\}$ and let $A \in \mathcal{A}$ satisfy $\mu(B_j \setminus A) > 0$. We claim that $\mu(H \cap (B_j \setminus A)) > 0$. To see this, suppose this is not true for some $B_j$ and some $A$. For

$s \in B_j \setminus B_{j-1}$, $\widetilde{u}(s) = z_j$ so for $\mu$-a.e. $s \in H \cap (B_j \setminus B_{j-1})$, $u(s) < \widetilde{u}(s) = z_j$ and we have $u(s) \leq z_{j+1}$ if $j < m$ and $u(s) = 0$ if $j = m$.

If $A \subseteq B_{j-1}$ define the function $h = z_{j+1}\chi_{B_j \setminus B_{j-1}} + \widetilde{u}\chi_{U \setminus (B_j \setminus B_{j-1})}$, otherwise define $h = z_{j+1}\chi_{B_j \setminus A} + \widetilde{u}\chi_{U \setminus (B_j \setminus A)}$. Clearly, $h \leq \widetilde{u}$ and by assumption $u \leq h$, by the exact same argument as the one used in the proof of the previous lemma we get that $h$ is a core decreasing majorant of $u$ strictly smaller than $\widetilde{u}$ arriving at a contradiction.

For each $j \in \{1, \ldots, m\}$ let $\alpha_j = \sup\{\mu(A) : A \in \mathcal{A}, \mu(A) < \mu(B_j)\}$. Notice that $z_m \leq \underline{u} \leq z_1$, hence

$$z_m \int\limits_{B_m} f \, d\mu \leq \int\limits_{U} \widetilde{u} f \, d\mu \leq z_1 \int\limits_{B_m} f \, d\mu.$$

Therefore, $\int\limits_U \widetilde{u} f \, d\mu$ is finite if and only if $\int\limits_{B_m} f \, d\mu$ is finite.

Suppose that $\int\limits_{B_m} f \, d\mu$ is infinite and that $\alpha_m < \mu(B_m)$. Choose $D \in \mathcal{M}$ such that $\mu(D) = \alpha_m$ and define the function

$$g = \frac{n}{z_m} \frac{\chi_{H \cap (B_m \setminus D)}}{\mu(H \cap (B_m \setminus D))}.$$

If $E \in \mathcal{A}$ then $\mu(E) \leq \mu(D)$ or $\mu(E) \geq \mu(B_m)$. In the first case we have $\int_E g \, d\mu = 0$ and in the second case we have

$$\frac{n}{z_m} = \int_E g \, d\mu < \int_E f \, d\mu = \infty.$$

Thus $\int_E g \, d\mu \leq \int_E f \, d\mu$ for each $E \in \mathcal{A}$ and

$$\int_U ug \, d\mu = \int\limits_{H \cap (B_m \setminus D)} ug \, d\mu = \int\limits_{H \cap (B_m \setminus D)} \widetilde{u} g \, d\mu = \int\limits_{H \cap (B_m \setminus D)} z_m g \, d\mu = n.$$

Above we used the fact that $\mu(B_{m-1}) \leq \mu(D) < \mu(B_m)$ so $\widetilde{u}(s) = z_m$ for $s \in B_m \setminus D$ up to a set of $\mu$-measure zero.

If $\int\limits_{B_m} f \, d\mu$ is infinite and $\alpha_m = \mu(B_m)$. Choose a set $W \in \mathcal{M}$ such that $B_{m-1} \subseteq W$, $\mu(W) < \mu(B_m)$ and $\int_W f \, d\mu \geq \frac{n}{z_m}$. Define the function

$$g = \left( \int_W f \, d\mu \right) \frac{\chi_{H \cap (B_m \setminus W)}}{\mu(H \cap (B_m \setminus W))}.$$

Let $E \in \mathcal{A}$. If $\mu(E) \leq \mu(W)$ then $\int_E g \, d\mu = 0$. If $\mu(W) \leq \mu(E)$, then

$$\int_E g \, d\mu = \int_W f \, d\mu \leq \int_E f \, d\mu.$$

Thus $\int_E g \, d\mu \leq \int_E f \, d\mu$ for each $E \in \mathcal{A}$ and

$$\int_U ug \, d\mu = \int\limits_{H \cap (B_m \setminus W)} ug \, d\mu = \int\limits_{H \cap (B_m \setminus W)} \widetilde{u} g \, d\mu = \int\limits_{H \cap (B_m \setminus W)} z_m g \, d\mu = z_m \int_W f \, d\mu \geq n.$$

For the case that $\int_{B_m} f\,d\mu$ is finite, we choose the set $C_j$ and define the function $g_j$ in two cases.

If $\alpha_j < \mu(B_j)$, choose a set $C_j \in \mathcal{M}$ such that $\mu(C_j) = \alpha_j$ and set $g_j = \left(\int_{E_j} f\,d\mu\right) \frac{\chi_{H\cap(E_j\setminus C_j)}}{\mu\left(H\cap(E_j\setminus C_j)\right)}$. The previous argument shows that the denominator is not zero.

If $\alpha_j = \mu(B_j)$, choose a set $C_j \in \mathcal{A}$ such that

$$\int_{B_j} f\,d\mu - \int_{C_j} f\,d\mu < \frac{\epsilon}{mz_j},$$

and set $g_j = \left(\int_{C_j\setminus B_{j-1}} f\,d\mu\right) \frac{\chi_{H\cap(E_j\setminus C_j)}}{\mu\left(H\cap(E_j\setminus C_j)\right)}$. Once more, notice that every set involved has finite measure.

Note than in both cases $g_j$ is supported on $E_j$, also $\int_{E_j} f\,d\mu - \int_{E_j} g_j\,d\mu < \frac{\epsilon}{mz_j}$ and $\int_{E_j} g_j\,d\mu \leq \int_{B_j\setminus B_{j-1}} f\,d\mu$.

Finally, define the function $g = \sum_{j=1}^{m} g_j$. We check that this function satisfies the requirements. Notice that since $g$ is zero outside of $H$, then $gu = g\widetilde{u}$ and

$$\begin{aligned}
\int_U f\widetilde{u}\,d\mu - \int_U gu\,d\mu &= \int_U f\widetilde{u}\,d\mu - \int_U g\widetilde{u}\,d\mu \\
&= \int_U f\widetilde{u}\,d\mu - \sum_{j=1}^{m} \int_U g_j\widetilde{u}\,d\mu \\
&= \int_U f\widetilde{u}\,d\mu - \sum_{j=1}^{m} \int_U g_j \sum_{k=1}^{m} z_k\chi_{E_k}\,d\mu \\
&= \int_U f\widetilde{u}\,d\mu - \sum_{j=1}^{m}\sum_{k=1}^{m} z_k \int_{E_k} g_j\,d\mu \\
&= \int_U f\widetilde{u}\,d\mu - \sum_{j=1}^{m} z_j \int_{E_j} g_j\,d\mu, \text{ since } g_j \text{ is supported on } E_j \\
&= \sum_{j=1}^{m} z_j \left(\int_{E_j} f\,d\mu - \int_{E_j} g_j\,d\mu\right) \\
&< \sum_{j=1}^{m} z_j \left(\frac{\epsilon}{mz_j}\right), \text{ by construction} \\
&= \epsilon.
\end{aligned}$$

This proves the statement $\int_U f\widetilde{u}\,d\mu - \epsilon < \int_U gu\,d\mu$.

To complete the proof, let $E \in \mathcal{A}$. If $\mu(E) \geq \mu(B_m)$ then

$$\int\limits_E g\,d\mu = \sum_{j=1}^{m} \int\limits_U g_j\,d\mu \leq \sum_{j=1}^{m} \int\limits_{B_j \setminus B_{j-1}} f\,d\mu = \int\limits_{B_m} f\,d\mu \leq \int\limits_E f\,d\mu.$$

If $\mu(E) < \mu(B_m)$, then there exists $k \in \{1, \ldots, m\}$ such that $\mu(B_{k-1}) \leq \mu(E) \leq \mu(B_k)$ and we have two cases.

If $\mu(E) < \mu(C_k)$ , we compute

$$\int\limits_E g\,d\mu = \sum_{j=1}^{k-1} \int\limits_U g_j\,d\mu \leq \sum_{j=1}^{k-1} \int\limits_{B_j \setminus B_{j-1}} f\,d\mu = \int\limits_{B_{k-1}} f\,d\mu \leq \int\limits_E f\,d\mu.$$

If $\mu(E) \geq \mu(C_k)$ , then $C_k \neq \alpha_k$ so $C_k$ and $g_k$ were chosen as in the case $\alpha_k = \mu(B_k)$ above. Thus,

$$\begin{aligned}\int\limits_E g\,d\mu &= \sum_{j=1}^{k-1} \int\limits_U g_j\,d\mu + \int\limits_{E\setminus C_k} g_k\,d\mu \leq \int\limits_{B_{k-1}} f\,d\mu + \int\limits_{E\setminus C_k} g_k\,d\mu \leq \int\limits_{B_{k-1}} f\,d\mu + \int\limits_{B_k\setminus C_k} g_k\,d\mu \\ &= \int\limits_{B_{k-1}} f\,d\mu + \int\limits_{C_k\setminus B_{k-1}} f\,d\mu = \int\limits_{C_k} f\,d\mu \leq \int\limits_E f\,d\mu.\end{aligned}$$

Therefore $\int_E g\,d\mu \leq \int_E f\,d\mu$ and completes the proof.

∎

We now prove the main result

**Proof of Theorem 3.2.5:** If $g$ satisfies $\int_E g\,d\mu \leq \int_E f\,d\mu$ for all $E \in \mathcal{A}$ then

$$\begin{aligned}\int_U gu\,d\mu &\leq \int_U g\widetilde{u}\,d\mu \text{ since } u \leq \widetilde{u} \\ &\leq \int_U f\widetilde{u}\,d\mu \text{ since } \widetilde{u} \text{ is core decreasing.}\end{aligned}$$

Supremum over all $g$ yields the inequality

$$\int_U f\widetilde{u}\,d\mu \geq \sup\left\{\int_U gu\,d\mu : g \preccurlyeq_{\mathcal{A}} f\right\}.$$

We now prove the converse. Since there exist a sequence $A_n \in \mathcal{A}$ increasing to $U$ and a sequence of simple $\Sigma$-measurable functions $u_n \uparrow u$, then the sequence $v_n = u_n\chi_{A_n}$ also increases to $u$. Therefore we can assume that $u_n$ is simple and vanishes outside $A_n$. If for some $n \in \mathbb{N}$ the integral $\int_U f\widetilde{u_n}\,d\mu = \infty$, then by Lemma 3.2.7 there exists a function $g$ such that $\int_E g_n\,d\mu \leq \int_E f\,d\mu$ for all $E \in \mathcal{A}$ and

$$n = \int_U u_n g\,d\mu \leq \int_U ug\,d\mu.$$

Let $n \to \infty$ to get

$$\sup\left\{\int_U gu\,d\mu : g \preccurlyeq_{\mathcal{A}} f\right\} = \infty = \int_U f\widetilde{u}\,d\mu.$$

If $\int_U \widetilde{u_n} f \, d\mu$ is finite for all $n \in \mathbb{N}$, fix $\epsilon > 0$. By Lemma 3.2.7, for each $n$ there exists $g_n$ such that $\int_E g_n \, d\mu \leq \int_E f \, d\mu$ for all $E \in \mathcal{A}$ and

$$\int_U f \widetilde{u_n} \, d\mu < \epsilon + \int_U g_n u_n \, d\mu \leq \epsilon + \int_U g_n u \, d\mu \leq \epsilon + \sup \left\{ \int_U g u \, d\mu : g \preccurlyeq_{\mathcal{A}} f \right\}.$$

Let $n \to \infty$ and then $\epsilon \to 0$ to conclude.

$$\int_U f \widetilde{u} \, d\mu \leq \sup \left\{ \int_U g u \, d\mu : g \preccurlyeq_{\mathcal{A}} f \right\},$$

and complete the proof.
∎

## 3.3 The greatest core decreasing minorant

We define an analogous projection reversing the order relation '≤'.

**Definition 3.3.1** *Let $f \in L(\Sigma)$. We say that $\underline{f}$ is a greatest core decreasing minorant of $f$ if $\underline{f} \in L^{\downarrow}(\mathcal{A})$, $\underline{f} \leq f$, and for every $g \in L^{\downarrow}(\mathcal{A})$ satisfying $g \leq |f|$ $\mu$-a.e, we have $g \leq \underline{f}$ $\mu$-a.e.*

Just as in the previous chapter, if a greatest core decreasing minorant exists, then it must be unique $\mu$-almost everywhere. The next lemma shows that a greatest core decreasing minorant always exists.

**Lemma 3.3.2** *Every $\Sigma$-measurable function $f$ has a greatest core decreasing minorant, denoted $\underline{f}$, which is unique up to $\mu$ almost everywhere equality.*

**Proof:** Suppose that $|g| \leq C$ for some $C < \infty$ and let $A_n \uparrow U$ with $A_n \in \mathcal{A}$. Let

$$\alpha_n = \sup \left\{ \int_{A_n} h \, d\mu : h \in L^{\downarrow}(\mathcal{A}) \text{ and } h \leq |g| \right\}.$$

The constant function 0 is core decreasing, so $\alpha_n \geq 0$. Since $|g|$ is bounded above, all the integrals are bounded by $C\mu(A_n)$, hence $\alpha_n \leq C\mu(A_n) < \infty$.

If $\alpha_n = 0$, define $h_n = 0$. Otherwise, choose $h_n \in L^{\downarrow}(\mathcal{A})$ such that $h_n \leq |g|$ and $\alpha_n - \frac{1}{n} < \int_{A_n} h_n \, d\mu$.

Set $f_1 = h_1$ and $f_n = \max\{f_{n-1}, h_n\}$. Since the maximum of two core decreasing functions is core decreasing, each $f_n$ is a core decreasing minorant of $|g|$. Also by construction $f_{n-1} \leq f_n$ and $\alpha_n - \frac{1}{n} < \int_{A_n} f_n \, d\mu$.

Let $h = \sup_{n \in \mathbb{N}} f_n$, which is clearly a core decreasing minorant of $|g|$ and $f_n \uparrow h$. To show that it is a greatest core decreasing minorant, let $w$ be another core decreasing minorant of $|g|$, then so is $\max\{w, h\}$, thus

$$\infty > \int_{A_n} h \, d\mu \geq \int_{A_n} h_n \, d\mu > \alpha_n - \frac{1}{n} \geq \int_{A_n} \max\{w, h\} \, d\mu - \frac{1}{n},$$

therefore

$$\frac{1}{n} \geq \int_{A_n} (\max\{w,h\} - h)\ d\mu \geq 0,$$

let $n \to \infty$ to get $h = \max\{w,h\}$ almost everywhere, so $h \geq w$ almost everywhere. This shows that $h$ is a greatest core decreasing minorant.

If $|g|$ is not bounded above, define $\varphi_n = \min(n, |g|)\chi_{A_n}$. Let $\psi_n$ be a greatest core decreasing minorant of $\varphi_n$.

Notice that since $\psi_{n-1}$ is a core decreasing minorant of $\varphi_n$ then we must have that $\psi_{n-1} \leq \psi_n$, thus $\psi_n$ is an increasing sequence of core decreasing minorants of $|g|$. Define $\psi = \sup_n \psi_n$, which is a core decreasing minorant of $|g|$.

In order to show that $\psi$ is a greatest core decreasing minorant, let $w$ be a core decreasing minorant, then $\min\{n, w\}\chi_{A_n}$ is a core decreasing minorant of $\min\{n, |g|\}\chi_{A_n} = \varphi_n$, thus $\min\{n, w\} \leq \psi_n$, hence taking limits we get $w \leq h$. This shows existence of the greatest core decreasing minorant of $g$ and completes the proof.

$\blacksquare$

Notice that, unlike the corresponding result for the least core decreasing majorant, the condition $f_n \uparrow f$ does not imply $\underline{f_n} \uparrow \underline{f}$. This is exhibited in the following example.

**Example 3.3.3** *Consider the core from Example 2.1.9, let $f = 1$ and $f_n = \chi_{\mathbb{R}^d \setminus B_{\frac{1}{n}}[0]}$. Then $\underline{f} = 1$ but $\underline{f_n} = 0$ for all $n \in \mathbb{N}$. To see the last equality, notice that $\underline{f_n}(x) = 0$ for all $x$ in $B_{\frac{1}{n}}[0]$ and since $\underline{f_n}$ is core decreasing, then $\underline{f_n}$ is identically zero.*

The next theorem is an analog of Theorem 3.2.3. The proof is analogous so we omit the details.

**Theorem 3.3.4** *Let $(U, \Sigma, \mu)$ be a $\sigma$-finite measure space with a separable ordered core $\mathcal{A}$ and $f \in L(\Sigma)$, then*

$$\underline{f}(u) = \operatorname{ess\,inf}_{t \leq_{\mathcal{A}} u} |f(t)| = \operatorname{ess\,inf}_{t \in (\leftarrow, u]_{\mathcal{A}}} |f(t)|\,.$$

**Example 3.3.5** *Let $U = \mathbb{X}$ be a metric measure space with distance function $d$, $a \in \mathbb{X}$ be any element, $\mu$ be any Borel measure and the core*

$$\mathcal{A} = \{\emptyset\} \cup \{B_{a,r} : r > 0\}$$

*where $B_{a,r} = \{x \in \mathbb{X} : d(a,x) \leq r\}$. Then*

$$\underline{g}(x) = \operatorname{ess\,inf}_{\mu} \left\{|g(t)| : t \in B_{a,|x|_a}\right\},$$

*where $|x|_a = d(a,x)$.*

The same function shown in Example 3.2.4 shows that the formula provided in Theorem 3.3.4 need not hold for non-separable ordered cores.

We finish this section with a functional description of the least core decreasing majorant. The fact that the greatest core decreasing minorant is not well behaved with increasing sequences forces us to adjust the argument and not use simple functions.

We need a technical lemma first, which will be the key to the ‘pushing mass’ argument done to the original function.

**Lemma 3.3.6** *Let $u$ be a nonnegative measurable function, $a > b \geq 0$, $A = \{s \in U : \underline{u}(s) \geq a\}$ and $B = \{s \in U : \underline{u}(s) \geq b\}$ such that $\mu(B \setminus A) > 0$. Then for $C \in \mathcal{M}$ such that $C \subseteq B$ and $\mu(C \setminus A) > 0$, the set*

$$\{s \in C \setminus A : b \leq u(s) < a\}$$

*has positive $\mu$-measure.*

**Proof:** Notice that since $\underline{u}$ is core decreasing, $A$ and $B$ belong to $\mathcal{M}$. Suppose that the statement does not hold, then there exists some $C \in \mathcal{M}$ such that $\mu(C \setminus A) > 0$, $C \subseteq B$ and $u(s) \geq a$ or $u(s) < b$ for $\mu$-almost all $s \in C \setminus A$.

Notice that on $C \setminus A$ we have $b \leq \underline{u}(s) \leq u(s)$, therefore we may assume that $u(s) \geq a$. Define the function

$$h = \underline{u}\chi_{(U\setminus(C\setminus A))} + a\chi_{(C\setminus A)}.$$

By construction $h(s) > \underline{u}(s)$ for all $s \in C \setminus A$, which is a set of positive measure. To arrive at a contradiction it suffices to show that $h$ is a core decreasing minorant of $u$.

It was already shown that $h \leq u$ $\mu$-a.e. on $C \setminus A$, thus $h$ is a minorant of $u$. To show that it is core decreasing, notice that we have to check the two cases $s \in A$, $t \in C \setminus A$ and $s \in C \setminus A$ and $t \notin C$. In the first case:

$$h(s) = \underline{u}(s) \geq a = h(t),$$

and in the second case:

$$h(s) = a > \underline{u}(t) = h(t).$$

Therefore $h$ is a core decreasing minorant of $u$, contradicting the fact that $\underline{u}$ is the greatest core decreasing minorant of $u$ and finishing the proof.

$\blacksquare$

We now 'push the mass to the left' of $f$ to an appropriate function $g$ to achieve the desired infimum.

**Lemma 3.3.7** *Let $u$ and $f$ be nonnegative measurable functions such that $\int_U f\underline{u}\,d\mu < \infty$. Then, for any $\epsilon > 0$, there exists a measurable nonnegative function $g$ such that $\int_E g\,d\mu \geq \int_E f\,d\mu$ for any $E \in \mathcal{A}$ and*

$$\int_U gu\,d\mu - \epsilon < \int_U f\underline{u}\,d\mu.$$

**Proof:** Fix $\epsilon > 0$. Since we assume that $\int_U f\underline{u}\,d\mu < \infty$, then there exists $\alpha > 1$ such that

$$\alpha \int_U f\underline{u}\,d\mu < \int_U f\underline{u}\,d\mu + \frac{\epsilon}{2}$$

Define the sequence $\{A_n\}_{n\in\mathbb{Z}}$ as

$$A_n = \left\{s \in U : \underline{u}(s) \geq \alpha^{n+1}\right\}, \qquad \text{for each } n \in \mathbb{Z}.$$

Since $\underline{u}$ is core decreasing, the sets $A_n \in \mathcal{M}$. Define the sets $\{J_n\}_{n\in\mathbb{Z}\cup\{\pm\infty\}}$ by

$$J_\infty = \bigcap_{n\in\mathbb{Z}} A_n, \quad J_n = A_n \setminus A_{n+1}, \quad \text{for each } n \in \mathbb{Z}, \quad \text{and} \quad J_{-\infty} = U \setminus \bigcup_{n\in\mathbb{Z}\cup\{\infty\}} J_n.$$

Notice that the sets $\{J_n\}_{n\in\mathbb{Z}\cup\{\pm\infty\}}$ are disjoint and cover the whole space $U$. Also, Lemma 2.2.2 shows that $J_\infty \in \mathcal{M}$ and $J_{-\infty}$ is empty or its complement also belongs in $\mathcal{M}$. To see that the complement of $J_{-\infty}$ is empty or has finite measure, suppose that $J_{-\infty} \neq \emptyset$, then there exists $A \in \mathcal{A}$ and $x \in U$ such that

$$x \notin \bigcup_{n\in\mathbb{Z}\cup\{\infty\}} J_n = \bigcup_{n\in\mathbb{Z}} A_n,$$

by the total ordering or $\mathcal{M}$ it follows that $A_n \subseteq A$ for all $n \in \mathbb{Z}$. Therefore

$$\mu\Big(\bigcup_{n\in\mathbb{Z}\cup\{\infty\}} J_n\Big) \le \mu(A) < \infty.$$

It will be useful to consider the presentation

$$J_\infty = \Big\{s \in U : \underline{u}(s) = \infty\Big\}, \quad J_{-\infty} = \Big\{s \in U : \underline{u}(s) = 0\Big\}, \quad \text{and}$$

$$J_n = \Big\{s \in U : \alpha^n \le \underline{u}(s) < \alpha^{n+1}\Big\}, \quad \text{for each } n \in \mathbb{Z}.$$

Define the functions $f_n = f\chi_{J_n}$ for each $n \in \mathbb{Z} \cup \{\pm\infty\}$. Our goal is to build nonnegative functions $g_n$ satisfying

$$\int_E g_n \, d\mu \ge \int_E f_n \, d\mu, \quad \text{for all } E \in \mathcal{A} \quad \text{and each } n \in \mathbb{Z} \cup \{\pm\infty\}, \tag{3.1}$$

$$\int_U g_n u \, d\mu \le \alpha \int_U f_n \underline{u} \, d\mu, \quad \text{for each } n \in \mathbb{Z} \cup \{\infty\}, \text{ and} \tag{3.2}$$

$$\int_U g_{-\infty} u \, d\mu \le \frac{\epsilon}{2}. \tag{3.3}$$

Since $f = \sum_{n\in\mathbb{Z}\cup\{\pm\infty\}} f_n$ then the function $g = \sum_{n\in\mathbb{Z}\cup\{\pm\infty\}} g_n$ clearly satisfies $\int_E g \, d\mu \ge \int_E f \, d\mu$ for all $E \in \mathcal{A}$ and

$$\int_U gu \, d\mu = \sum_{n\in\mathbb{Z}\cup\{\pm\infty\}} \int_U g_n u \, d\mu \le \alpha \sum_{n\in\mathbb{Z}\cup\{\infty\}} \int_U f_n \underline{u} \, d\mu + \frac{\epsilon}{2} = \alpha \int_U f \underline{u} \, d\mu + \frac{\epsilon}{2} < \int_U f \underline{u} \, d\mu + \epsilon.$$

For any $n \in \mathbb{Z} \cup \{\infty\}$ such that $f_n = 0$ we define $g_n = 0$ and it clearly satisfies inequalities (3.1) and (3.2). For the other cases, since $\int_U f \underline{u} \, d\mu < \infty$, then we must have that $f_\infty = 0$.

Fix $n \in \mathbb{Z}$ such that that $f_n \neq 0$ $\mu$-almost everywhere. This means that $0 < \mu(J_n) = \mu(A_n) - \mu(A_{n+1})$. Since

$$\infty > \int_U f \underline{u} \, d\mu \ge \int_{J_n} f \underline{u} \, d\mu \ge \alpha^n \int_{J_n} f \, d\mu,$$

thus $\int_{J_n} f \, d\mu < \infty$.

Let

$$\beta_n = \inf\{\mu(E) : \mu(A_{n+1}) < \mu(E), E \in \mathcal{M}, \text{ and } E \subseteq A_n\}.$$

There are two cases, either $\beta_n > \mu(A_{n+1})$ or $\beta_n = \mu(A_{n+1})$. In the first case, pick $C_n \in \mathcal{M}$ such that $\mu(C_n) = \beta_n$. An application of Lemma 3.3.6 with $a = \alpha^{n+1}$, $b = \alpha^n$ $B = A_n$, $A = A_{n+1}$ and $C = C_n$ shows that the set

$$H_n = \Big\{s \in C_n \setminus A_{n+1} : \alpha^n \le \underline{u}(s) \le u(s) < \alpha^{n+1}\Big\}$$

has positive $\mu$-measure. Define

$$g_n = \Big( \int_{J_n} f \, d\mu \Big) \frac{\chi_{H_n}}{\mu(H_n)}.$$

For each $E \in \mathcal{A}$,

$$\int_E g_n \, d\mu = \begin{cases} 0, & \text{if } \mu(E) \leq \mu(A_{n+1}) \\ \int_{J_n} f \, d\mu, & \text{otherwise} \end{cases} = \int_E f_n \, d\mu,$$

therefore $g_n$ satisfies the inequality (3.1). Also

$$\int_U g_n u \, d\mu = \frac{\int_{J_n} f \, d\mu}{\mu(H_n)} \int_{H_n} u \, d\mu < \alpha^{n+1} \int_{J_n} f \, d\mu = \alpha \int_{J_n} \alpha^n f \, d\mu \leq \alpha \int_{J_n} f_n \underline{u} \, d\mu,$$

proving that $g_n$ satisfies the inequality (3.2).

The remaining case is when $\beta_n = \mu(A_{n+1})$. We prove by induction that there exists a sequence of sets $\{H_{n,m}\}_{m \in \mathbb{N}^+}$ of positive $\mu$-measure and a sequence $\{C_{n,m}\}_{m \in \mathbb{N}}$ in $\mathcal{M}$ such that $C_{n,m} \subseteq A_n$, $\mu(C_{n,m})$ is strictly decreasing to $\mu(A_{n+1})$, and

$$H_{n,m} \subseteq \Big\{ s \in C_{n,m-1} \setminus C_{n,m} : \alpha^n \leq \underline{u}(s) \leq u(s) < \alpha^{n+1} \Big\}.$$

We show the induction step first. Suppose that the sequences are constructed up to an integer $M_0 > 0$. Apply Lemma 3.3.6 with $a = \alpha^{n+1}$, $b = \alpha^n$ $B = A_n$, $A = A_{n+1}$ and $C = C_{n,M_0}$, to get that the set

$$K_{M_0} = \Big\{ s \in C_{n,M_0} \setminus A_{n+1} : \alpha^n \leq \underline{u}(s) \leq u(s) < \alpha^{n+1} \Big\}$$

has positive $\mu$-measure. Since $\beta_n = \mu(A_{n+1})$, there exists a set $C_{n,M_0+1} \in \mathcal{M}$ such that $\mu(C_{n,M_0+1} \setminus A_{n+1}) < \frac{\mu(K_{M_0})}{2}$. Another application of Lemma 3.3.6 with $a = \alpha^{n+1}$, $b = \alpha^n$ $B = A_n$, $A = A_{n+1}$ and $C = C_{n,M_0+1}$ provides a set

$$K_{M_0+1} = \Big\{ s \in (C_{n,M_0+1} \setminus A_{n+1}) : \alpha^n \leq \underline{u}(s) \leq u(s) < \alpha^{n+1} \Big\}$$

of positive $\mu$-measure. Notice that $K_{M_0+1} \subsetneq K_{M_0}$ but $\mu(K_{M_0+1}) < \mu(K_{M_0})$, therefore the difference has positive measure. Set $H_{n,M_0+1} = K_{M_0+1} \setminus K_{M_0}$ to prove the induction step. The base case follows the same argument, letting $M_0 = 0$ and $C_{n,0} = A_n$.

Define the function

$$g_n = \sum_{m=2}^{\infty} \Big( \int_{C_{n,m-2} \setminus C_{n,m-1}} f \, d\mu \Big) \frac{\chi_{H_{n,m}}}{\mu(H_{n,m})}.$$

Let $E \in \mathcal{A}$, if $\mu(E) \leq \beta_n$, then both $\int_E g_n \, d\mu$ and $\int_E f_n \, d\mu$ vanish. If $\mu(E) \geq \mu(A_n)$, then

$$\int_E g_n \, d\mu = \sum_{m=2}^{\infty} \Big( \int_{C_{n,m-2} \setminus C_{n,m-1}} f \, d\mu \Big) = \int_{J_n} f \, d\mu = \int_E f_n \, d\mu.$$

In the case that $\mu(E) \in (\beta_n, \mu(A_n))$, there exists some $M_E \in \mathbb{N}$ such that $\mu(E) \in (\mu(C_{n,M_E+1}), \mu(C_{n,M_E})]$, hence

$$\int_E g_n \, d\mu \geq \int_{C_{n,M_E+1}} g_n \, d\mu = \sum_{m=C_{n,M_E+1}}^{\infty} \Big( \int_{C_{n,m-2} \setminus C_{n,m-1}} f \, d\mu \Big) = \int_{C_{n,M_E} \setminus A_{n+1}} f \, d\mu \geq \int_{E \setminus A_{n+1}} f \, d\mu$$
$$= \int_E f_n \, d\mu.$$

Therefore $g_n$ satisfies the inequality (3.1). Also

$$\int_U g_n u \, d\mu = \sum_{m=2}^{\infty} \Big( \int_{C_{n,m-2} \setminus C_{n,m-1}} f \, d\mu \Big) \frac{\int_{H_{n,m}} u \, d\mu}{\mu(H_{n,m})} \leq \sum_{m=2}^{\infty} \Big( \int_{C_{n,m-2} \setminus C_{n,m-1}} f \, d\mu \Big) \alpha^{n+1}$$
$$= \alpha_{n+1} \int_{J_n} f \, d\mu = \alpha \int_{J_n} \alpha^n f \, d\mu \leq \alpha \int_{J_n} f_n \underline{u}.$$

proving that $g_n$ satisfies the inequality (3.2).

All that remains is defining the function $g_{-\infty}$ whenever the function $f\chi_{J_{-\infty}}$ is not $\mu$-almost everywhere zero. Let $U_0 = \cup_{n \in \mathbb{Z}} J_n$. Since $\mu(J_{-\infty}) > 0$, there exists some $E \in \mathcal{A}$ such that $U_0 \subseteq E$, therefore $\mu(U_0) < \infty$, thus $U_0 \in \mathcal{M}$. If there exists a set of positive measure $W$ such that $u(s) = 0$ for all $s \in W$ and $\mu(W \cap E) > 0$ for every $E \in \mathcal{A}$ satisfying $\mu(E) > \mu(U_0)$ then we define

$$g_{-\infty} = \infty \chi_W.$$

In this case $\int_U g_{-\infty} u \, d\mu = 0$, clearly satisfying the inequality (3.3). For any $E \in \mathcal{A}$, if $\mu(E) \leq \mu(U_0)$ then

$$0 = \int_E f_{-\infty} \leq \int_E g_{-\infty}$$

and if $\mu(E) > \mu(U_0)$, then

$$\infty = \int_E g_{-\infty} \, d\mu \geq \int_E f_{-\infty} \, d\mu,$$

thus $g_{-\infty}$ satisfies the inequality (3.1).

If such set $W$ does not exist, we will find a disjoint sequence of sets of positive measure $\{W_k\}_{k \in \mathbb{N}^+}$, such that

$$W_k \subseteq \left\{ s \in U : u(s) < \epsilon 2^{-(k+1)} \right\}, \tag{3.4}$$

also satisfying that for any $E \in \mathcal{A}$ such that $\mu(E) > \mu(U_0)$, then infinitely many sets in the sequence are subsets of $E$. The desired function will be

$$g_{-\infty} = \sum_{k=1}^{\infty} \frac{\chi_{W_k}}{\mu(W_k)}.$$

Since

$$\int_U g_{-\infty} u \, d\mu = \sum_{k=1}^{\infty} \frac{\int_{W_k} u \, d\mu}{\mu(W_k)} < \frac{\epsilon}{2} \sum_{k=1}^{\infty} 2^{-k} = \frac{\epsilon}{2},$$

$g_{-\infty}$ satisfies the inequality (3.3). For any $E \in \mathcal{A}$, if $\mu(E) \leq \mu(U_0)$ then

$$0 = \int_E f_{-\infty} \leq \int_E g_{-\infty}$$

and if $\mu(E) > \mu(U_0)$, then

$$\infty = \sum_{W_k \subseteq E} 1 = \int_E g_{-\infty}\, d\mu \geq \int_E f_{-\infty}\, d\mu,$$

thus $g_{-\infty}$ satisfies inequality (3.1). We now show that either the set $W$ exists or we build the sequence $\{W_k\}$.

Since $A_{-n}$ increases to $U_0$ whenever $n \uparrow \infty$, we have that $\mu(A_{-n}) \uparrow \mu(U_0)$. There are two possibilities, either $\mu(A_{-n}) < \mu(U_0)$ for all $n \in \mathbb{N}$ or there exists some $N_0$ such that $\mu(A_{-N_0}) = \mu(U_0)$.

In the first case, for any $j \in \mathbb{N}^+$ the set

$$G_j = \left\{ s \in U_0 : u(s) < \epsilon 2^{-(j+1)} \right\},$$

has positive $\mu$-measure. Otherwise, the function $h = \epsilon 2^{-(j+1)} \chi_{U_0}$ is a core decreasing minorant of $u$, thus $h \leq \underline{u}$. Hence, for any $n$ large enough such that $\alpha^{-n} < \epsilon 2^{-(j+1)}$ we would have $\mu(A_{-n}) = \mu(U_0)$ arriving at a contradiction. Notice that $\{G_j\}$ is a decreasing sequence, let $W = \cap G_j$. If $\mu(W) > 0$, then there is nothing left to prove. If $\mu(W) = 0$, then we may choose a subsequence $\{G_{j_k}\}_k$ such that the sequence of measures $\{\mu(G_{j_k})\}$ is strictly decreasing. Then the sequence $W_k = G_{j_k} \setminus G_{j_{k+1}}$ is disjoint and satisfies formula (3.4).

It remains to show that the set $W$ or the sequence $\{W_k\}$ exist whenever there is a positive integer $k_0$ such that $\mu(A_{-k_0}) = \mu(U_0)$, which implies that $\alpha^{-k_0} \leq \underline{u}(s) \leq u(s)$ for almost every $s \in U_0$. Let

$$\beta_{-\infty} = \inf \{\mu(E) : E \in \mathcal{M} \text{ and } \mu(E) > \mu(U_0)\}.$$

Once more, we consider the two possibilities; $\mu(U_0) < \beta_{-\infty}$ or if $\mu(U_0) = \beta_{-\infty}$.

In the first case, let $A_{-\infty} \in \mathcal{M}$ satisfy $\mu(A_{-\infty}) = \beta_{-\infty}$. Pick $r_0$ large enough, such that $\epsilon 2^{-r_0} < \alpha^{-k_0}$. For any $j > r_0$, apply Lemma 3.3.6 with $a = \epsilon 2^{-j}$, $b = 0$, $A = U_0$, $B = U$ and $C = A_{-\infty}$ to get that the set

$$G_j = \left\{ s \in (A_{-\infty} \setminus U_0) : u(s) < \epsilon 2^{-j} \right\}$$

have positive $\mu$-measure. Let $W = \cap_{k > r_0} G_j$. If $\mu(W) > 0$ there is nothing to prove, so we drop to a subsequence with strictly decreasing measures and build the sequence $\{W_k\}$ like it was done before. Note that for any $E \in \mathcal{A}$ satisfying $\mu(E) > \mu(U_0)$, then $\mu(A_{-\infty}) \leq \mu(E)$, so every set in the sequence $W_k$ is contained in $E$.

We are left with the final case, when $\mu(U_0) = \beta_{-\infty}$. Choose a sequence $\{E_j\} \subseteq \mathcal{M}$ such that $\mu(E_j) \downarrow \beta_n$, and $r_0$ large enough so $\epsilon 2^{-r_0} < \alpha^{-k_0}$. For any $j > r_0$, apply Lemma 3.3.6 with $a = \epsilon 2^{-j}$, $b = 0$, $A = U_0$, $B = U$ and $C = E_j$ to get that the set

$$G_j = \left\{ s \in (E_j \setminus U_0) : u(s) < \epsilon 2^{-j} \right\}$$

has positive $\mu$-measure. Since $\mu(G_j) \leq \mu(E_j \setminus U_0)$, we get $\mu(G_j) \downarrow 0$. Once more we can drop to a subsequence and repeat the previous process to obtain disjoint sets of positive measure $W_j$

satisfying formula (3.4) such that $W_j \subseteq E_j$. Therefore, for any $E$ such $\mu(E) > \mu(U_0)$, there are infinitely many of such sets $W_j$ contained in $E$. This finishes the proof.

∎

We now prove the main result.

**Theorem 3.3.8** *For Σ-measurable nonnegative functions $f$ and $u$, then*

$$\int_U f\underline{u}\,d\mu = \inf\left\{\int_U gu\,d\mu : \int_E g\,d\mu \geq \int_E f\,d\mu \text{ for all } E \in \mathcal{A}\right\}$$

**Proof:** If $g$ satisfies $\int_E g\,d\mu \geq \int_E f\,d\mu$ for all $E \in \mathcal{A}$ then

$$\begin{aligned}\int_U gu\,d\mu &\geq \int_U g\underline{u}\,d\mu \text{ since } u \geq \underline{u} \\ &\geq \int_U f\underline{u}\,d\mu \text{ since } \underline{u} \text{ is core decreasing.}\end{aligned}$$

If $g$ satisfies $\int_E g\,d\mu \geq \int_E f\,d\mu$ for all $E \in \mathcal{A}$ then

$$\begin{aligned}\int_U gu\,d\mu &\geq \int_U g\underline{u}\,d\mu \text{ since } u \geq \underline{u} \\ &\geq \int_U f\underline{u}\,d\mu \text{ since } \underline{u} \text{ is core decreasing.}\end{aligned}$$

Taking the infimum over all $g$ yields the inequality

$$\int_U f\underline{u}\,d\mu \leq \inf\left\{\int_U gu\,d\mu : \int_E g\,d\mu \geq \int_E f\,d\mu \text{ for all } E \in \mathcal{A}\right\}.$$

If $\infty = \int_U f\underline{u}\,d\mu$, then equality clearly follows. So we may suppose that $\int_U f\underline{u}\,d\mu < \infty$, and in this case, equality follows from Lemma 3.3.7.

∎

## 3.4 The level function

The least core decreasing majorant explored in Section 3.2 is optimal in the sense of the partial order '$\leq$' of pointwise almost everywhere inequality. We explore in this section another least core decreasing majorant but in the sense of the preorder relation '$\preccurlyeq_{\mathcal{A}}$'.

We introduce this construction via a functional description acting on the collection of core decreasing functions.

**Definition 3.4.1** *Let $f \in L(\Sigma)$. We say $f^o \in L^{\downarrow}(\mathcal{A})$ is a level function of $f$ if for all $g \in L^{\downarrow}(\mathcal{A})$,*

$$\int_U f^o g\,d\mu = \sup\left\{\int_U |f|\,h\,d\mu : h \in L^{\downarrow}(\mathcal{A}) \text{ and } h \preccurlyeq_{\mathcal{A}} g\right\}.$$

We explore the relationship between a level function and the preorder relation '$\preccurlyeq_{\mathcal{A}}$'

**Proposition 3.4.2** *Let $f \in L(\Sigma)$ and $f^o$ be a level function of $f$, then*

*(a)* $|f| \preccurlyeq_{\mathcal{A}} f^o$.

*(b) If $f \in L^{\downarrow}(\mathcal{A})$, then $f$ is a level function of $f$.*

*(c) If $v \in L^{\downarrow}(\mathcal{A})$ satisfies $|f| \preccurlyeq_{\mathcal{A}} v$, then $f^o \preccurlyeq_{\mathcal{A}} v$.*

*(d) If $f \in L^\infty_\mu$ then $f^o$ is unique.*

**Proof:** Letting $g = \chi_A$ for a core set $A \in \mathcal{A}$ yields the inequality $\int_A f^o\, d\mu \geq \int_A |f|\, d\mu$, therefore $|f| \preccurlyeq_{\mathcal{A}} f^o$ and proves (a).

To prove (b) notice that if $h \preccurlyeq_{\mathcal{A}} g$, then Proposition 3.1.4 yields $\int_U fh\, d\mu \leq \int_U fg\, d\mu$. Taking supremum yields (b).

To prove (c) notice that if $|f| \preccurlyeq_{\mathcal{A}} v$, then if $h \in L^{\downarrow}(\mathcal{A})$, Proposition 3.1.4 yields $\int_U |f|\, h\, d\mu \leq \int_U vh\, d\mu$. Taking supremum among $h$ yields

$$\begin{aligned}\int_U f^o g\, d\mu &= \sup\left\{\int_U |f|\, h\, d\mu : h \in L^{\downarrow}(\mathcal{A}) \text{ and } h \preccurlyeq_{\mathcal{A}} g\right\}\\ &\leq \sup\left\{\int_U vh\, d\mu : h \in L^{\downarrow}(\mathcal{A}) \text{ and } h \preccurlyeq_{\mathcal{A}} g\right\} = \int_U vg\, d\mu.\end{aligned}$$

Thus $f^o \preccurlyeq_{\mathcal{A}} v$.

To prove statement (d), notice that for any $A \in \mathcal{A}$

$$\begin{aligned}\int_A f^o\, d\mu &= \sup\left\{\int_U |f|\, h\, d\mu : h \in L^{\downarrow}(\mathcal{A}) \text{ and } h \preccurlyeq_{\mathcal{A}} \chi_A\right\}\\ &\leq \sup\left\{\|f\|_{L^\infty_\mu} \int_U h\, d\mu : h \in L^{\downarrow}(\mathcal{A}) \text{ and } h \preccurlyeq_{\mathcal{A}} \chi_A\right\} \leq \|f\|_{L^\infty_\mu}\mu(A) < \infty.\end{aligned}$$

Therefore any level function of $f$ belongs to $L^1_{\mathrm{Loc},\mathcal{A}}$. Statement (d) follows from the fact that a level function is $\sigma(\mathcal{A})$-measurable, so if $f^o$ and $v$ are level functions of $f$ we have $f^o \preccurlyeq_{\mathcal{A}} v$ and $v \preccurlyeq_{\mathcal{A}} f^o$ which, by Proposition 3.1.3, forces equality $\mu$-almost everywhere.

∎

The previous proposition shows that the level function is an optimal core decreasing majorant in the sense of the preorder '$\preccurlyeq_{\mathcal{A}}$'. It is worth noting, that for unbounded functions uniqueness need not hold, however, the canonical choice of level function explored below is the unique choice well behaved with increasing sequences of functions. This is done using the level function from Definition 1.6.2 and the transition maps from Section 2.4.1.

**Lemma 3.4.3** *There exists a unique map $f \mapsto f^o$ from $L(\Sigma)$ to $L^{\downarrow}(\mathcal{A})$ such that $f^o$ is a level function of $f$ and if $0 \leq f_n \uparrow f$ then $f_n^o \uparrow f^o$.*

**Proof:** Let $f \in L^+(\Sigma)$, then $Rf \in L^+(\mathcal{B})$. Its level function (see Definition 1.6.2) is a non-increasing function $(Rf)^o$. Define $f^o := Q\big((Rf)^o\big)$. Since $(Rf)^o \in L^{\downarrow}(\mathcal{B})$, $f^o \in L^{\downarrow}(\mathcal{A})$, by item (4) in Theorem 2.4.8. Fix $g \in L^{\downarrow}(\mathcal{A})$, item (5) in Theorem 2.4.8 shows that $Rg \in L^{\downarrow}(\mathcal{B})$. Proposition 1.6.3 and items (3) and (6) in Theorem 2.4.8 yields

$$\begin{aligned}\int_U f^o g\,d\mu = \int_{[0,\infty)} (Rf)^o Rg\,d\lambda &= \sup\left\{\int_{[0,\infty)} (Rf)\psi\,d\lambda : \psi \in L^{\downarrow}(\mathcal{B}) \text{ and } \psi \preccurlyeq_{\mathcal{B}} Rg\right\}\\ &= \sup\left\{\int_U f Q(\psi)\,d\mu : \psi \in L^{\downarrow}(\mathcal{B}) \text{ and } \psi \preccurlyeq_{\mathcal{B}} Rg\right\}\\ &= \sup\left\{\int_U fh\,d\mu : h \in L^{\downarrow}(\mathcal{A}) \text{ and } h \preccurlyeq_{\mathcal{A}} g\right\}.\end{aligned}$$

This shows that $f^o$ is indeed a level function of $f$. If $f \in L(\Sigma)$, then set $f^o = |f|^o$. Clearly, $f^o$ is a level function of $f$.

If $0 \leq f_n \uparrow f$, then item (3) in Proposition 2.4.6 shows that $Rf_n \uparrow Rf$. By Proposition 1.6.3 we get $(Rf_n)^o \uparrow (Rf)^o$. An application of item 3 in Proposition 2.4.7 yields $Q\big((Rf_n)^o\big) \uparrow Q\big((Rf)^o\big)$, this shows that $f_n^o \uparrow f^o$ and proves the existence of the map $f \mapsto f^o$ with the required properties.

To show uniqueness, suppose $f \mapsto f^\bullet$ is another map from $L(\Sigma) \to L^{\downarrow}(\mathcal{B})$ mapping to a level function of $f$ such that for any $0 \leq f_n \uparrow f$ it follows that $f_n^\bullet \uparrow f^\bullet$. For any $f \in L^+(\Sigma)$ we consider the sequence $f_n = \min(n, f)$. Since $f_n \in L_\mu^\infty$, $f_n^\bullet = f_n^o$ by item (d) in Proposition 3.4.2. It follows that $f^\bullet = f^o$ $\mu$-almost everywhere. This shows uniqueness for $f \in L^+(\Sigma)$.

To complete the proof fix $f \in L(\Sigma)$. It is clear from Definition 3.4.1 that $f^\bullet$ is a level function of $|f|$. Define $g_n = \min(n, |f|)$. Since $0 \leq g_n \uparrow |f|$, it follows that $g_n^\bullet \uparrow f^\bullet$, $g_n^o \uparrow f^o$ and $g_n^\bullet = f_n^o$ $\mu$-a.e. for each $n \in \mathbb{N}$. Thus $f^\bullet = f^o$ and completes the proof.

$\blacksquare$

The previous lemma is summarized in the following picture, we have the formula $f^o = Q((Rf)^o)$, where the level function on $Rf$ is with respect to the measure $\lambda$ induced by the ordered core $\mathcal{A}$ and the maps $R$,$Q$ are the transition maps from Section 2.4.1.

$$\begin{array}{ccccccccc} L(\Sigma) & \xrightarrow{\;|\;|\;} & L^+(\Sigma) & \xrightarrow{\;R\;} & L^+(\mathcal{B}) & \xrightarrow{\;o\;} & L^{\downarrow}(\mathcal{B}) & \xrightarrow{\;Q\;} & L^{\downarrow}(\mathcal{A}) \\ \\ f & \longrightarrow & |f| & \longrightarrow & R|f| & \longrightarrow & (R|f|)^o & \longrightarrow & Q(R|f|)^o \end{array}$$

With the consistent choice for level function provided by Lemma 3.4.3, for each $f \in L(\Sigma)$, we will denote

$$f^o = Q\big((R\,|f|)^o\big),$$

as *the level function* of $f$.

# Chapter 4

# Spaces defined by core decreasing functions

In this chapter, we introduce function spaces defined by the collection of core decreasing functions and study their interpolation properties. Throughout this chapter, we assume that $(U, \Sigma, \mu)$ is a $\sigma$-finite measure space with a full ordered core $\mathcal{A}$ and its enriched core $\mathcal{M}$. Let $X$ be a Banach function space over $U$ such that both $X$ and $X'$ contain all characteristic functions of sets of finite measure.

## 4.1 Down spaces and their duals

Our first function space is the down space of $X$, it is introduced as a restricted associate space over $X$.

**Definition 4.1.1** *For $f \in L(\Sigma)$, let*

$$\|f\|_{X\downarrow} = \sup\left\{\int_U |f|\, g\, d\mu : g \in X' \cap L^{\downarrow}(\mathcal{A}) \text{ and } \|g\|_{X'} \le 1\right\}.$$

*The space $X\downarrow = \{f \in L(\Sigma) : \|f\|_{X\downarrow} < \infty\}$ is called the down space of $X$.*

Since we are defining the down space by restricting the supremum, it is immediate from the definition that $\|f\|_{X\downarrow} \le \|f\|_{X''} \le \|f\|_X$. Therefore the down space $X\downarrow$ is a superspace of $X$. However, the functions have to be integrable over core sets, as the next result shows.

**Proposition 4.1.2** *Let $f \in X\downarrow$, then $f \in L^1_{Loc,\mathcal{A}}$.*

**Proof:** Fix $f \in X\downarrow$ and suppose that $f \notin L^1_{\mathrm{Loc},\mathcal{A}}$ seeking a contradiction. Then, there exists $A \in \mathcal{A}$ such that $\int_A |f|\, d\mu = \infty$. Since $\mu(A) < \infty$, then $\|\chi_A\|_{X'} < \infty$. Set $g = \frac{1}{\|\chi_A\|_{X'}}\chi_A$ and note that it is core decreasing satisfying $\|g\|_{X'} = 1$. Then

$$\|f\|_{X\downarrow} \ge \int_U |f|\, g\, d\mu = \int_A |f|\, d\mu = \infty.$$

This contradicts the hypothesis and completes the proof.

∎

The next theorem provides two fundamental examples of down spaces, which will be the key spaces in the interpolation theory at the end of this chapter. It also shows that the transition maps from Section 2.4.1 are well-behaved in those spaces.

**Theorem 4.1.3** *Let* $f \in L(\Sigma)$ *and* $\varphi \in L(\mathcal{B})$. *Then*

*(a)* $L^1_\mu\!\downarrow = L^1_\mu$ *with identical norms.*

*(b)* $\|f\|_{L^\infty_\mu\downarrow} = \sup_{A\in\mathcal{A}} \frac{1}{\mu(A)} \int_A |f|\, d\mu = \sup_{M\in\mathcal{M}} \frac{1}{\mu(M)} \int_M |f|\, d\mu.$

*(c) If* $f \in L^\infty_\mu\!\downarrow$ *then* $Rf \in L^\infty_\lambda\!\downarrow$ *and* $\|Rf\|_{L^\infty_\lambda\downarrow} \le \|f\|_{L^\infty_\mu\downarrow}$.

*(d) If* $\psi \in L^\infty_\lambda\!\downarrow$ *then* $Q\psi \in L^\infty_\mu\!\downarrow$ *and* $\|Q\psi\|_{L^\infty_\mu\downarrow} \le \|\psi\|_{L^\infty_\lambda\downarrow}$.

**Proof:** Since constant functions are core decreasing, the constant function $g = 1$ satisfies $\|g\|_{L^\infty_\mu} = 1$. By Definition 4.1.1, we get $\|f\|_{L^1_\mu} = \int_U |f|\, g\, d\mu \le \|f\|_{L^1_\mu\downarrow}$ proving (a).

To prove (b), fix $M \in \mathcal{M}$ and let $g = \frac{1}{\mu(M)}\chi_M$. Then $g \in L^\downarrow(\mathcal{A})$, $\|g\|_{L^1_\mu} = 1$ and

$$\frac{1}{\mu(M)} \int_U |f|\, d\mu = \int_U |f|\, g\, d\mu \le \|f\|_{L^\infty_\mu\downarrow}.$$

Taking the supremum over $M \in \mathcal{M}$ shows that $\sup_{M\in\mathcal{M}} \frac{1}{\mu(M)} \int_M |f|\, d\mu \le \|f\|_{L^\infty_\mu\downarrow}$. To prove the reverse inequality, suppose that $g = \sum_{k=1}^n \alpha_k \chi_{M_k}$ for some $\alpha_k \ge 0$ and $M_k \in \mathcal{M}$ such that $\|g\|_{L^1_\mu} \le 1$. That is, $\sum_{k=1}^n \alpha_k \mu(M_k) \le 1$. Then,

$$\begin{aligned}
\int_U |f|\, g\, d\mu &= \sum_{k=1}^n \alpha_k \int_{M_k} |f|\, d\mu = \sum_{k=1}^n \alpha_k \mu(M_k) \Big( \frac{1}{\mu(M_k)} \int_{M_k} |f|\, d\mu \Big) \\
&\le \Big( \sup_{M\in\mathcal{M}} \frac{1}{\mu(M)} \int_M |f|\, d\mu \Big) \Big( \sum_{k=1}^n \alpha_k \mu(M_k) \Big) \\
&\le \sup_{M\in\mathcal{M}} \frac{1}{\mu(M)} \int_M |f|\, d\mu.
\end{aligned}$$

If $g \in L^\downarrow(\mathcal{A})$ and $\|g\|_{L^1_\mu} \le 1$, then by Lemma 2.2.5 there exists a sequence $g_n \uparrow g$ of the form discussed above, therefore the monotone convergence theorem preserves this bound. Thus $\|f\|_{L^\infty_\mu\downarrow} \le \sup_{M\in\mathcal{M}} \frac{1}{\mu(M)} \int_M |f|\, d\mu$, which proves equality.

Clearly $\sup_{A\in\mathcal{A}} \frac{1}{\mu(A)} \int_A |f|\, d\mu \le \sup_{M\in\mathcal{M}} \frac{1}{\mu(M)} \int_M |f|\, d\mu$. Therefore, to prove (b) it suffices to show that the reverse inequality holds.

Fix $M \in \mathcal{M}$, by item (b) in Lemma 2.2.2, there exists a sequence $\{A_k\} \in \mathcal{A}$ such that $A_k \uparrow M$ or $A_k \downarrow M$. If $A_k \uparrow M$, then

$$\frac{1}{\mu(A_k)} \int_{A_k} |f|\, d\mu \ge \frac{1}{\mu(M)} \int_{A_k} |f|\, d\mu.$$

The monotone convergence theorem yields

$$\sup_{A\in\mathcal{A}} \frac{1}{\mu(A)} \int_A |f|\, d\mu \ge \sup_k \frac{1}{\mu(A_k)} \int_{A_k} |f|\, d\mu \ge \sup_k \frac{1}{\mu(M)} \int_{A_k} |f|\, d\mu = \frac{1}{\mu(M)} \int_M |f|\, d\mu.$$

If $A_k \downarrow M$, there are two cases. If $\int_M |f|\,d\mu = \infty$, then $\frac{1}{\mu(A_k)}\int_{A_k} |f|\,d\mu = \infty$, so both suprema are infinite. In the other case $\int_M |f|\,d\mu < \infty$. Fix $\epsilon > 0$. If $\int_{A_k} |f|\,d\mu = \infty$ for some $A_k$, then by both suprema are infinite, therefore we may assume that the sequence of functions $\{\frac{1}{\mu(A_k)}|f|\chi_{A_k}\}$ is is bounded above by the integrable function $\frac{1}{\mu(M)}|f|\chi_{A_1}$. Therefore the dominated convergence theorem applies and since $\frac{1}{\mu(A_k)}|f|\chi_{A_k} \to \frac{1}{\mu(M)}|f|\chi_M$ $\mu$-a.e we get that

$$\frac{1}{\mu(M)}\int_M |f|\,d\mu - \epsilon < \frac{1}{\mu(A_k)}\int_{A_k} |f|\,d\mu,$$

for large enough $k$. Therefore $\frac{1}{\mu(M)}\int_M |f|\,d\mu - \epsilon \le \sup_{A\in\mathcal{A}} \frac{1}{\mu(A)}\int_A |f|\,d\mu$. Letting $\epsilon \to 0$ and taking supremum over $M \in \mathcal{M}$ yields $\sup_{M\in\mathcal{M}} \frac{1}{\mu(M)}\int_M \le \sup_{A\in\mathcal{A}} \frac{1}{\mu(A)}\int_A$ and completes the proof of (b).

To prove (c), fix $x > 0$ and Theorem 2.4.3 allows us to pick $M \in \mathcal{M}$ such that $\mu(M) = \lambda([0,x])$. Then by Proposition 2.4.6 we get

$$\frac{1}{\lambda([0,x])}\int_{[0,x]} |Rf|\,d\lambda \le \frac{1}{\lambda([0,x])}\int_{[0,x]} R|f|\,d\lambda = \frac{1}{\mu(M)}\int_M |f|\,d\mu \le \|f\|_{L^\infty_\mu\downarrow}.$$

Taking supremum over all $x > 0$ completes the proof of (c).

Finally, let $M \in \mathcal{M}$. By Proposition 2.4.7 we get

$$\frac{1}{\mu(M)}\int_M |Q\psi|\,d\mu \le \frac{1}{\mu(M)}\int_M Q|\psi|\,d\mu = \frac{1}{\lambda([0,\mu(M)])}\int_{[0,\mu(M)]} |\psi|\,d\mu \le \|\psi\|_{L^\infty_\lambda\downarrow}.$$

Taking supremum over all $M \in \mathcal{M}$ proves (d).

■

Statement (b) in the previous theorem shows that $L^\infty_\mu\downarrow$ does not change when considering the ordered core $\mathcal{A}$ or its enriched core $\mathcal{M}$. The next example shows that a change in ordered cores can change the produced down space. It also exhibits a function that is not in $L^\infty_\mu$ but belongs to its down space.

**Example 4.1.4** *Let $U = [0,\infty)$ with the Lebesgue measure and consider the ordered cores $\mathcal{A}_1 = \{\emptyset\} \cup \{[0,x] : x > 0\}$ and $\mathcal{A}_2 = \{\emptyset\} \cup \{[0,n] : n \in \mathbb{N}\}$. Let $D_1$ be the down space of $L^\infty$ with respect to the core $\mathcal{A}_1$ and $D_2$ be the one with respect to $\mathcal{A}_2$. In virtue of Theorem 4.1.3 (b) it is clear that $D_1 \subseteq D_2$. We will show that equality does not hold.*

*Consider the function*

$$f(x) = \sum_{k=1}^{\infty} k^{-2}2^{k+1}\chi_{(2^{-(k+1)},2^{-k})}(x)$$

*The unbounded function $f$ belongs to $D_2$ since*

$$\frac{1}{n}\int_0^n f(x)\,dx \le \int_0^\infty f(x)\,dx = \sum_{k=1}^{\infty} k^{-2} = \pi^2/6.$$

*However, the averages are no longer uniformly bounded over intervals* $[0, x]$*, to see this note that if* $x = 2^{-j}$ *for some* $j \in \mathbb{N}$*, then*

$$2^j \int_0^{2^{-j}} f(t)\,dt = 2^j \sum_{k=j}^{\infty} k^{-2} \geq 2^j \int_{j+1}^{\infty} t^{-2}\,dt = \frac{2^j}{j+1}.$$

*Letting* $j \to \infty$ *shows that* $\|f\|_{D_1} = \infty$.

We define two more function spaces based on the constructions introduced in Chapter 3.

**Definition 4.1.5** *Let*

$$X^o = \{f \in L(\Sigma) : f^o \in X\} \quad \text{and} \quad \widetilde{X} = \left\{f \in L(\Sigma) : \widetilde{f} \in X\right\}.$$

*Also set are* $\|f\|_{X^o} = \|f^o\|_X$ *and* $\|f\|_{\widetilde{X}} = \|\widetilde{f}\|_X$.

It will be shown that the functions $f \mapsto \|f\|_{X^o}$ and $f \mapsto \|f\|_{\widetilde{X}}$ are norms. Note that the lattice property of $X$ ensures that $\|\widetilde{f}\|_X \geq \|f\|_X$, therefore $\widetilde{X}$ is a subspace of $X$.

The relationship with the down spaces is given in the next theorem. It shows that $X\downarrow$ and $\widetilde{X}$ are related via the associate space construction. For the space $X^o$, it shows that $(X'')^o \subseteq X\downarrow$. We will show that with additional conditions on the Banach function space, we have equality.

**Theorem 4.1.6** *The set* $X\downarrow$*, with its corresponding norm, is a Banach function space with the Fatou property. If* $f \in X''$*, then* $\|f\|_{X\downarrow} \leq \|f\|_{X''}$ *and, if* $f^o \in X''$*, then* $\|f\|_{X\downarrow} \leq \|f^o\|_{X''}$.

*The map* $g \to \widetilde{g}$ *is sublinear. The set* $\widetilde{X}$ *with its corresponding norm is a Banach function space. It has the Fatou property if* $X$ *does. Moreover,* $X\downarrow = (\widetilde{X'})'$ *and* $(X\downarrow)' = \widetilde{X'}$ *with identical norms.*

*The spaces* $\widetilde{L^\infty_\mu}$ *and* $L^\infty_\mu$ *are equal with identical norms.*

*If* $X$ *is a u.r.i space then* $X\downarrow = (X'')^o$ *with identical norms. If* $X$ *also has the Fatou property, then* $X\downarrow = X^o$.

**Proof:** Let $f, \{f_n\} \in L(\Sigma)$, $g \in L^{\downarrow}(\mathcal{A})$ satisfying $\|g\|_{X'} \leq 1$. Then $\int_U |f|\, g\, d\mu \geq 0$. If $f = 0$, then $\int_U |f|\, g\, d\mu = 0$. If $\alpha > 0$ then $\int_U |\alpha f|\, g\, d\mu = \alpha \int_U |f|\, g\, d\mu$. By the triangle inequality of the absolute value; $\int_U |f_1 + f_2|\, g\, d\mu \leq \int_U |f_1|\, g\, d\mu + \int_U |f_2|\, g\, d\mu$. If $|f_1| \leq |f_2|$, then $\int_U |f_1|\, g\, d\mu \leq \int_U |f_2|\, g\, d\mu$. If $|f_n| \uparrow |f|$, then by the monotone convergence theorem $\int_U |f_n|\, g\, d\mu \uparrow \int_U |f|\, g\, d\mu$.

Taking supremum over all such functions $g$ shows that the map $f \mapsto \|f\|_{X\downarrow}$ is a function semi-norm with the Fatou property. Therefore, to show that $X\downarrow$ is a Banach function space with the Fatou property, it suffices to show that $\|f\|_{X\downarrow} = 0$ implies $f = 0$ $\mu$-a.e. To see this, let $\{A_n\} \in \mathcal{A}$ satisfy $A_n \uparrow X$. Since $\mu(A_n) < \infty$, then $\|\chi_{A_n}\|_{X'} < \infty$, making the core decreasing function $g = \frac{1}{\|\chi_{A_n}\|_{X'}} \chi_{A_n}$ satisfy $\|g\|_{X'} = 1$. Thus

$$0 = \|f\|_{X\downarrow} \geq \frac{1}{\|\chi_{A_n}\|_{X'}} \int_{A_n} |f|\; d\mu.$$

It follows that $f\chi_{A_n} = 0$ $\mu$-a.e. Letting $n \to \infty$ shows that $f$ is zero $\mu$-a.e.

Since the supremum on Definition 4.1.1 is taken over a smaller collection of functions than $X'$ it is immediate that $\|f\|_{X\downarrow} \le \|f\|_{X''}$. If $f^o \in X''$, then if $g \in L^{\downarrow}(\mathcal{A})$, it follows from Definition 3.4.1 that $\int_U f^o g\,d\mu \ge \int_U |f|\,g\,d\mu$. It follows that $\|f\|_{X\downarrow} \le \|f^o\|_{X\downarrow} \le \|f^o\|_{X''}$.

We now show that the map $g \mapsto \widetilde{g}$ is sublinear. Let $g, \{g_n\} \in L(\Sigma)$ and $\alpha \ge 0$. Clearly $\alpha\widetilde{g}$ is a core decreasing majorant of $|\alpha g|$, therefore $\widetilde{\alpha g} \le \alpha\widetilde{g}$. To show equality, let $A \in \mathcal{A}$, then using Theorem 3.2.5 we get

$$\begin{aligned}\int_A \widetilde{\alpha g}\,d\mu &= \int_U \widetilde{\alpha g}\chi_A\,d\mu = \sup\left\{\int_U \alpha g h\,d\mu : h \preccurlyeq_{\mathcal{A}} \chi_A\right\} = \alpha\sup\left\{\int_U gh\,d\mu : h \preccurlyeq_{\mathcal{A}} \chi_A\right\}\\ &= \alpha\int_U \widetilde{g}\chi_A\,d\mu = \int_A \alpha\widetilde{g}\,d\mu.\end{aligned}$$

Since the above equality holds for all core sets and $\widetilde{\alpha g} \le \alpha\widetilde{g}$, then Lemma 2.3.5 shows that the equality $\widetilde{\alpha g} = \alpha\widetilde{g}$ holds.

To show that it is subadditive, if $f = g + h$, then $|f| \le |g| + |h| \le \widetilde{g} + \widetilde{h}$, which is core decreasing, so $\widetilde{f} \le \widetilde{g} + \widetilde{h}$. Thus the map is sublinear. If $|f| \le g$, then $\widetilde{f} \le \widetilde{g}$. Hence, it is immediate that $g \mapsto \|g\|_{\widetilde{X}}$ is a function semi-norm. Since $|g| \le \widetilde{g}$ then $\|g\|_{\widetilde{X}} = 0$ implies $|g| = 0$, so it is a function norm.

To show completeness, it will be shown that $\widetilde{X}$ satisfies the Riesz-Fischer property. Suppose that $0 \le g_n \in \widetilde{X}$ satisfy $\sum_{n=1}^{\infty} \|g_n\|_{\widetilde{X}} < \infty$. Let $G = \sum_{n=1}^{\infty} g_n$ and observe that $\sum_{n=1}^{\infty} \widetilde{g_n}$ is a core decreasing majorant of $G$ and hence it is a majorant of $\widetilde{G}$ . By Proposition 1.3.2, $X$ has the Riesz-Fischer property, so

$$\|G\|_{\widetilde{X}} = \|\widetilde{G}\|_X \le \left\|\sum_{n=1}^{\infty} \widetilde{g_n}\right\|_X \le \sum_{n=1}^{\infty} \|\widetilde{g_n}\|_X = \sum_{n=1}^{\infty} \|g_n\|_{\widetilde{X}} < \infty.$$

Hence, $\widetilde{X}$ is a Banach function space. Finally, if $X$ has the Fatou property, and $0 \le g_n \uparrow g$, then $\widetilde{g_n} \uparrow \widetilde{g}$, so

$$\|g_n\|_{\widetilde{X}} = \|\widetilde{g_n}\|_X \uparrow \|\widetilde{g}\|_X = \|g\|_{\widetilde{X}}.$$

This shows that $\widetilde{X}$ also has the Fatou property.

We now show that $\|f\|_{X\downarrow} = \|f\|_{(\widetilde{X'})'}$. Let $g \in L^{\downarrow}(\mathcal{A})$ such that $\|g\|_{X'} \le 1$. Since $g$ is core decreasing, then $\widetilde{g} = g$, thus $\|g\|_{\widetilde{X'}} = \|\widetilde{g}\|_{X'} = \|g\|_{X'} \le 1$, therefore

$$\int_U |f|\,g\,d\mu \le \sup\left\{\int_U |f|\,h : \|h\|_{\widetilde{X'}} \le 1\right\} = \|f\|_{(\widetilde{X'})'}.$$

Taking the supremum over all such $g$, yields $\|f\|_{X\downarrow} \le \|f\|_{(\widetilde{X'})'}$. Conversely, if $\|g\|_{\widetilde{X'}} \le 1$, then

$$\int_U |f|\,g\,d\mu \le \int_U |f|\,\widetilde{g}\,d\mu \le \sup\left\{\int_U |f|\,h : \|h\|_{X'} \le 1 \text{ and } h \in L^{\downarrow}(\mathcal{A})\right\} = \|f\|_{X\downarrow}.$$

Taking the supremum over all $g$ yields $\|f\|_{(\widetilde{X'})'} \le \|f\|_{X\downarrow}$ and proves that $X\downarrow = (\widetilde{X'})'$ with equality of norms.

If follows that $(X\downarrow)' = (\widetilde{X'})''$, but $\widetilde{X'}$ has the Fatou property. Hence $(X\downarrow)' = \widetilde{X'}$ with identical norms.

Since $|g| \le \widetilde{g}$, $\|g\|_{L^\infty_\mu} \le \|\widetilde{g}\|_{L^\infty_\mu} = \|g\|_{\widetilde{L^\infty_\mu}}$. But the constant function $\|g\|_{L^\infty_\mu}$ is a core decreasing majorant of $|g|$, hence $\widetilde{g} \le \|g\|_{L^\infty_\mu}$ and application of the $L^\infty_\mu$ norm yields $\|g\|_{\widetilde{L^\infty_\mu}} \le \|g\|_{L^\infty_\mu}$ and proves equality.

To prove that we have equality if $X$ is a u.r.i space, first suppose that $f, g \in L^+(\Sigma)$ are bounded and supported on some $A \in \mathcal{A}$. Then by Proposition 2.4.6 we have that $Rf, Rg$ are nonnegative, bounded, and supported on $[0, \mu(A)]$. By Proposition 1.6.4 there exist averaging operators $J_f, J_g$ such that $J_f Rf = (Rf)^o$ and $J_g Rg = (Rg)^o$ with $(Rf)^o, (Rg)^o$ also supported on $[0, \mu(A)]$. Then using Theorem 2.4.8 we get

$$\int_U f^o g\, d\mu \le \int_U f^o g^o\, d\mu = \int_{[0,\infty)} (Rf)^o (Rg)^o\, d\lambda = \int_{[0,\infty)} (J_f Rf)(J_g Rg)\, d\lambda = \int_{[0,\infty)} (Rf)(J_f J_g Rg)\, d\lambda$$
$$= \int_U f(QJ_f J_g Rg)\, d\mu = \int_U f(QJ_f (Rg)^o)\, d\mu.$$

Notice that $X'$ is also a u.r.i space and since $J_f$ and $J_g$ are admissible contractions on $L^1_\lambda + L^\infty_\lambda$, then using Propositions 2.4.6 and 2.4.7 the map $Q \circ J_f \circ J_g \circ R$ is an admissible contraction on $L^1_\mu + L^\infty_\mu$. Then, using Lemma 1.5.5 we get that $\|QJ_f J_g Rg\|_{X'} \le \|g\|_{X'}$. Since $J_f$ preserves decreasing functions, then $QJ_f(Rg)^o$ is core decreasing. Thus $QJ_f J_g Rg \in L^\downarrow(\mathcal{A})$. Hence, if $\|g\|_{X'} \le 1$ we get

$$\int_U f^o g\, d\mu \le \sup\left\{ \int_U |f|\, g\, d\mu : g \in X' \cap L^\downarrow(\mathcal{A}) \text{ and } \|g\|_{X'} \le 1 \right\} = \|f\|_{X\downarrow}.$$

Fix an arbitrary nonnegative $g \in X'$ such that $\|g\|_{X'} \le 1$. Let $g_n = \min\{n, g\}\chi_{A_n}$. Then $g_n \uparrow g$, and the lattice property of $X'$ shows that $\|g_n\|_{X'} \le 1$. The monotone convergence theorem yields

$$\int_U f^o g\, d\mu = \sup_n \int_U f^o g_n\, d\mu \le \|f\|_{X\downarrow}.$$

Taking the supremum over all such $g$ gives $\|f^o\|_{X''} \le \|f\|_{X\downarrow}$.

For a general function $f \in L(\Sigma)$, we may choose a sequence $\{f_n\}$ where $f_n$ is supported on $A_n$ and bounded such that $f_n \uparrow |f|$. The Fatou property of $X''$ and $X\downarrow$ together with Lemma 3.4.3 show that

$$\|f^o\|_{X''} = \sup_n \|f^o_n\|_{X'} \le \sup_n \|f_n\|_{X\downarrow} = \|f\|_{X\downarrow}.$$

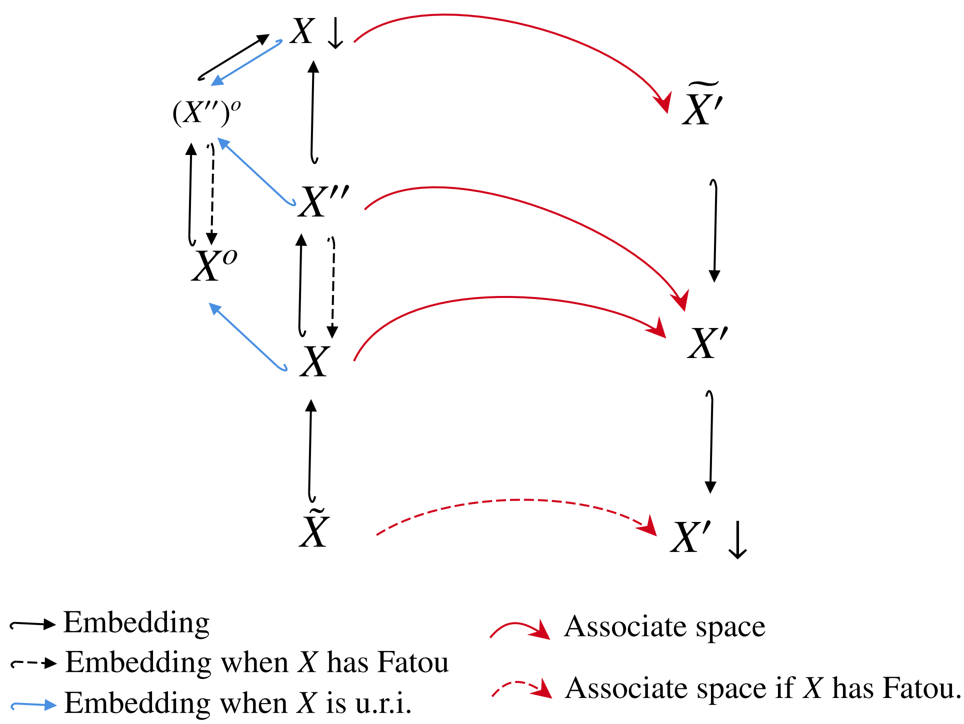
This shows the equality $\|f^o\|_{X''} = \|f\|_{X\downarrow}$ when $X$ is a u.r.i space. If in addition, $X$ has the Fatou property, then

$$\|f\|_{X^o} = \|f^o\|_X = \|f^o\|_{X''} = \|f\|_{X\downarrow}.$$

This completes the proof.

∎

The relationship among the spaces is summarized in this diagram.




This example shows that $(X'')^o$ may be strictly smaller than $X\downarrow$ whenever $X$ is not u.r.i.

**Example 4.1.7** *Consider the usual ordered core on* $[0, \infty)$ *with the Lebesgue measure, the function* $f(t) = t\chi_{(1,\infty)}(t)$ *and the weighted Lebesgue space* $X = L^1(w)$ *for the weight* $t(x) = t^{-3}$*, so the norm is*

$$\|f\|_X = \int_0^\infty |f(t)| \frac{1}{t^3}\, dt = \int_1^\infty \frac{1}{t^2}\, dt = 1.$$

*Therefore* $\|f\|_{X\downarrow} \le \|f\|_X = 1 < \infty$*, thus* $f \in X\downarrow$*. We claim that* $f^o(t) = \infty$*. To see this, notice that* $f_n(t) = t\chi_{(1,n)}$ *increases to* $f$*. Integration yields*

$$\int_0^x f_n(t)\, dt = \int_1^x t\chi_{(1,n)}\, dt = \min\left(\frac{x^2-1}{2}, \frac{n^2-1}{2}\right)\chi_{(1,n)}(x)$$

*Its least concave majorant is the function* $x \mapsto \min\left(\frac{n^2-1}{2n}x, \frac{n^2-1}{2}\right)$*, its derivative almost everywhere is*

$$f_n^o(t) = \frac{n^2-1}{2n}\chi[0,n)(t).$$

*Since* $f_n^o \uparrow \infty$*, then* $f^o = \infty$*, therefore* $f^o \notin X'' = X$*. Therefore* $f \notin (X'')^o$*. This example also shows that for this space the embeddings* $X \subseteq X^o$ *and* $X'' \subseteq (X'')^o$ *both fail.*

## 4.2 Interpolation

Since the $L^p_\mu$ spaces are u.r.i, by virtue of Theorem 4.1.6 we will not distinguish between the spaces $L^p_\mu\downarrow$ and $(L^p_\mu)^o$.

In this section we study the compatible couple $(L^1_\mu, (L^\infty_\mu)^o)$. We will show that this couple of Banach function spaces plays a fundamental role in the theory of down spaces. We will show that this couple is an exact Calderón-Mityagin couple, therefore we have a complete description of their interpolation spaces in terms of the $K$-functional. We will also show that every down space corresponds to a down space for some u.r.i space.

The first key result in this section is the computation of the $K$-functional for the couple $(L^1_\mu, (L^\infty_\mu)^o)$. We need two lemmas to prove that result. The first lemma deals with the mapping properties of the transition maps $R$, $Q$ from Subsection 2.4.1 in terms of compatible couples.

**Lemma 4.2.1** *The maps $R$ and $Q$ satisfy:*

*(a) $Q$ is an admissible contraction from $(L^1_\lambda, L^\infty_\lambda\!\downarrow)$ to $(L^1_\mu, L^\infty_\mu\!\downarrow)$.*

*(b) $Q$ is an admissible contraction from $(\widetilde{L^1_\lambda}, L^\infty_\lambda)$ to $(\widetilde{L^1_\mu}, L^\infty_\mu)$.*

*(c) $R$ is an admissible contraction from $(L^1_\mu, L^\infty_\mu\!\downarrow)$ to $(L^1_\lambda, L^\infty_\lambda\!\downarrow)$.*

*(d) $R$ is an admissible contraction from $(\widetilde{L^1_\mu}, L^\infty_\mu)$ to $(\widetilde{L^1_\lambda}, L^\infty_\lambda)$.*

**Proof:** Statement (a) is exactly item (9) in Proposition 2.4.7 and (d) in Theorem 4.1.3.

For (b), for a fixed $\psi \in \widetilde{L^1_\lambda}$ by virtue of items (2) and (7) in Proposition 2.4.7, we get that $|Q\psi| \leq Q\,|\psi| \leq Q\widetilde{\psi}$. Since $Q\widetilde{\psi}$ is core decreasing, it follows that $\widetilde{Q\psi} \leq Q\widetilde{\psi}$, hence

$$\|Q\psi\|_{\widetilde{L^1_\mu}} = \|\widetilde{Q\psi}\|_{L^1_\mu} \leq \|Q\widetilde{\psi}\|_{L^1_\mu} \leq \|\widetilde{\psi}\|_{L^1_\lambda} = \|\psi\|_{\widetilde{L^1_\lambda}}.$$

This inequality, together with item (1) in 2.4.7 prove statement (b).

Statement (c) is exactly item (9) in Proposition 2.4.6 and (c) in Theorem 4.1.3.

For (d), for a fixed $f \in \widetilde{L^1_\mu}$ by virtue of items (2) and (7) in Proposition 2.4.6, we get that $|Rf| \leq R\,|f| \leq R\widetilde{f}$. Since $R\widetilde{f}$ is decreasing, it follows that $\widetilde{Rf} \leq R\widetilde{f}$, hence

$$\|Rf\|_{\widetilde{L^1_\lambda}} = \|\widetilde{Rf}\|_{L^1_\lambda} \leq \|R\widetilde{f}\|_{L^1_\lambda} \leq \|\widetilde{f}\|_{L^1_\mu} = \|f\|_{\widetilde{L^1_\mu}}.$$

This inequality, together with item (1) in 2.4.6 prove statement (d).

∎

The next lemma builds a family of optimal decompositions of $f \in L^1_\mu + (L^\infty_\mu)^o$.

**Lemma 4.2.2** *For $0 \leq f \in L^1_\mu + (L^\infty_\mu)^o$, there is a map $D_f = [0,\infty) \to L^\downarrow(\mathcal{A})$ such that: $0 \leq D_f(t) \leq 1$ for all $t \geq 0$. If $\int_A f\,d\mu = \int_A g\,d\mu$, for all $A \in \mathcal{A}$, then $D_f = D_g$; and if $f = f_1 + f_\infty$ with $0 \leq f_1 \in L^1_\mu$ and $0 \leq f_\infty \in (L^\infty_\mu)^o$, then*

$$\|D_f(t_0)f\|_{L^1_\mu} \leq \|f_1\|_{L^1_\mu} \quad \textit{and} \quad \|(1 - D_f(t_0))f\|_{(L^\infty_\mu)^o} \leq \|f_\infty\|_{(L^\infty_\mu)^o},$$

*for $t_0 = \|f_1\|_{L^1_\mu}$.*

**Proof:** For each map $\Theta : \mathcal{M} \to [0,\infty)$ and each $x \in \Theta(\mathcal{M})$ choose an $N_x \in \mathcal{M}$ such that $\Theta(N_x) = x$. This a priori choice of $N_x$ will avoid issues with possible incompatible choices later.

Fix a nonnegative $f \in L^1_\mu + L^\infty_\mu\!\downarrow$ and suppose $0 \le f_1 \in L^1_\mu$ and $0 \le f_\infty \in L^\infty_\mu\!\downarrow$ such that $f = f_1 + f_\infty$.

By Proposition 4.1.2 each value $\int_M f\,d\mu$ is finite, thus we can set $\Theta(M) = \int_M f\,d\mu$ and let $N_x$, for $x \in \Theta(\mathcal{M})$, be those determined above. By Theorem 2.2.2, $\mathcal{M}$ is closed under countable unions and intersections, showing that $\Theta(\mathcal{M}) = \{\int_M |f|\,d\mu : M \in \mathcal{M}\}$ is a closed subset of $[0,\infty)$ containing 0. For each $\gamma \ge 0$, set

$$a_\gamma = \sup[0,\gamma] \cap \Theta(\mathcal{M}) \quad \text{and} \quad b_\gamma = \inf[\gamma,\infty) \cap \Theta(\mathcal{M}),$$

where $\inf \emptyset = \infty$. Then $a_\gamma, b_\gamma \in \Theta(\mathcal{M})$ and either $a_\gamma = \gamma = b_\gamma$ or $a_\gamma < \gamma < b_\gamma \le \infty$.

If $a_\gamma < \gamma < b_\gamma < \infty$, set

$$D_f(\gamma) = \frac{b_\gamma - \gamma}{b_\gamma - a_\gamma}\chi_{N_{a_\gamma}} + \frac{\gamma - a_\gamma}{b_\gamma - a_\gamma}\chi_{N_{b_\gamma}}.$$

Otherwise, set $D_f(\gamma) = \chi_{N_{a_\gamma}}$. Evidently, $D_f(\gamma) \in L^\downarrow(\mathcal{A})$ and $0 \le D_f(\gamma) \le 1$. If $0 \le g \in L^1_\mu + L^\infty_\mu\!\downarrow$ and $\int_M g\,d\mu = \int_M f\,d\mu$ for all $M \in \mathcal{M}$, then $f$ and $g$ give rise to the same $\Theta$, the same $N_x$ and the same $a_\gamma$ and $b_\gamma$. Therefore $D_f = D_g$.

To prove the last statement of the lemma, we let $\gamma = \|f_1\|_{L^1_\mu}$ and, for convenience, write $a = a_\gamma$ and $b = b_\gamma$. First we show that $\|D_f(\gamma)f\|_{L^1_\mu} \le \|f_1\|_{L^1_\mu}$: If $D_f(\gamma) = \chi_{N_a}$, then

$$\|D_f(\gamma)f\|_{L^1_\mu} = \int_{N_a} f\,d\mu = a \le \gamma = \|f_1\|_{L^1_\mu}.$$

Otherwise,

$$\|D_f(\gamma)f\|_{L^1_\mu} = \frac{b-\gamma}{b-a}\int_{N_a} f\,d\mu + \frac{\gamma-a}{b-a}\int_{N_b} f\,d\mu = \gamma = \|f_1\|_{L^1_\mu}.$$

On the way to proving that $\|(1-D_f(\gamma))f\|_{L^\infty_\mu\downarrow} \le \|f_\infty\|_{L^\infty_\mu\downarrow}$ we show that, for all $M \in \mathcal{M}$,

$$\int_M f_1\,d\mu \le \int_M D_f(\gamma)f\,d\mu. \tag{4.1}$$

Fix $M \in \mathcal{M}$. The definition of $\Theta(\mathcal{M})$ ensures that $\int_M f\,d\mu \le a$ or $\int_M f\,d\mu \ge b$.

Case 1. $a < \gamma < b < \infty$: If $\int_M f\,d\mu \le a$, then

$$\int_M f\,d\mu \le \int_{N_a} f\,d\mu \le \int_{N_b} f\,d\mu \;\text{ so }\; \int_M f\,d\mu = \int_{M\cap N_a} f\,d\mu = \int_{M\cap N_b} f\,d\mu,$$

because $\mathcal{M}$ is totally ordered. Thus,

$$\int_M f_1\,d\mu \le \int_M f\,d\mu = \frac{b-\gamma}{b-a}\int_{M\cap N_a} f\,d\mu + \frac{\gamma-a}{b-a}\int_{M\cap N_b} f\,d\mu = \int_M D_f(\gamma)f\,d\mu.$$

If $\int_M f\,d\mu \ge b$, then

$$\int_{N_a} f\,d\mu \le \int_{N_b} f\,d\mu \le \int_M f\,d\mu.$$

Since $\mathcal{M}$ is totally ordered,

$$a = \int_{N_a} f\,d\mu = \int_{M\cap N_a} f\,d\mu \quad \text{and} \quad b = \int_{N_b} f\,d\mu = \int_{M\cap N_b} f\,d\mu.$$

Therefore,

$$\int_M f_1\,d\mu \le \gamma = \frac{b-\gamma}{b-a}\int_{M\cap N_a} f\,d\mu + \frac{\gamma-a}{b-a}\int_{M\cap N_b} f\,d\mu = \int_M D_f(\gamma) f\,d\mu.$$

Case 2. $a < \gamma < b < \infty$ fails: If $\int_M f\,d\mu \le a$, then

$$\int_M f\,d\mu \le \int_{N_a} f\,d\mu \quad \text{so} \quad \int_M f\,d\mu = \int_{M\cap N_a} f\,d\mu,$$

because $\mathcal{M}$ is totally ordered. Thus,

$$\int_M f_1\,d\mu \le \int_M f\,d\mu = \int_{M\cap N_a} f\,d\mu = \int_M D_f(\gamma) f\,d\mu.$$

If $\int_M f\,d\mu \ge b$, then $a = \gamma = b$ so, because $\mathcal{M}$ is totally ordered,

$$\int_M f_1\,d\mu \le \gamma = \int_{N_a} f\,d\mu = \int_{M\cap N_a} f\,d\mu = \int_M D_f(\gamma) f\,d\mu.$$

This completes the proof of (4.1).

Using (4.1), we get

$$\int_M (1-D_f(\gamma)) f\,d\mu = \int_M f\,d\mu - \int_M D_f(\gamma) f\,d\mu \le \int_M f\,d\mu - \int_M f_1\,d\mu \le \int_M f_\infty\,d\mu$$

for each $M \in \mathcal{M}$ and it follows that

$$\|(1-D_f(\gamma))f\|_{L^\infty_\mu\downarrow} \le \|f_\infty\|_{L^\infty_\mu\downarrow}.$$

∎

With the aid of the previous two lemmas, we proceed to give a formula for the $K$-functional of $(L^1_\mu, (L^\infty_\mu)^o)$. Note that since $L^\infty_\mu$ is u.r.i. and has the Fatou property, $(L^\infty_\mu)^o = L^\infty_\mu \downarrow$.

**Theorem 4.2.3** *Let* $0 \le f \in L^1_\mu + (L^\infty_\mu)^o$ *and* $t > 0$*, then*

$$K(f,t,L^1_\mu,(L^\infty_\mu)^o) = K(QRf,t,L^1_\mu,(L^\infty_\mu)^o) = K(Rf,t,L^1_\lambda,(L^\infty_\lambda)^o) = \int_0^t (f^o)^*.$$

**Proof:** By Proposition 1.5.2 we may restrict ourselves to positive functions. Fix $f \in L^1_\mu + (L^\infty_\mu)^o$ nonnegative and let $0 \le f_1 \in L^1_\mu$ and $0 \le f_\infty \in (L^\infty_\mu)^o$ such that $f = f_1 + f_\infty$. By Lemma 4.2.2, we get

$$\|f_1\|_{L^1_\mu} + t\|f_\infty\|_{(L^\infty_\mu)^o} \ge \|fD_f(t_0)\|_{L^1_\mu} + t\|(1-D_f(t_0))f\|_{(L^\infty_\mu)^o},$$

for $t_0 = \|f_1\|_{L^1_\mu}$. Therefore, taking infimum over all decompositions, we get

$$K(f,t,L^1_\mu,(L^\infty_\mu)^o) = \inf\left\{\|fD_f(\gamma)\|_{L^1_\mu} + t\|(1-D_f(\gamma))f\|_{(L^\infty_\mu)^o} : \gamma \in [0,\infty)\right\}. \tag{4.2}$$

For any $M \in \mathcal{M}$, using Propositions 2.4.7 and 2.4.6 we have

$$\int_M QRf\,d\mu = \int_{[0,\mu(M)]} Rf\,d\lambda = \int_M f\,d\mu.$$

Since this holds for all $M \in \mathcal{M}$, then by Lemma 4.2.2, we have that $D_f = D_{QRf}$. Since $D_f(\gamma) \in L^\downarrow(\mathcal{A})$ for each $\gamma \geq 0$, by Propositions 2.4.6 and 2.4.7, it follows that

$$\begin{aligned}\|D_{QRf}(\gamma)QRf\|_{L^1_\mu} &= \int_U D_{QRf}(\gamma)QRf\,d\mu = \int_{[0,\infty)} RD_{QRf}(\gamma)Rf\,d\lambda = \int_U D_{QRf}(\gamma)f\,d\mu \\ &= \int_U D_f(\gamma)f\,d\mu = \|D_f(\gamma)f\|_{L^1_\mu}.\end{aligned}$$

Since the function $(1 - D_{QRf}(\gamma))\chi_M$ is nonnegative and $\sigma(\mathcal{A})$-measurable, then item (2) in Theorem 2.4.8 shows that

$$\begin{aligned}\int_M (1-D_{QRf}(\gamma))QRf\,d\mu &= \int_M QRf\,d\mu - \int_M D_{QRf}(\gamma)QRf\,d\mu = \int_M f\,d\mu - \int_M D_f(\gamma)f\,d\mu \\ &= \int_M (1-D_f(\gamma))f\,d\mu.\end{aligned}$$

Division by $\mu(M)$ and taking supremum yields

$$\|(1-D_{QRf}(\gamma))QRf\|_{(L^\infty_\mu)^o} = \|(1-D_f(\gamma))f\|_{(L^\infty_\mu)^o}.$$

Therefore, the formula (4.2) shows that $K(f,t,L^1_\mu,(L^\infty_\mu)^o) = K(QRf,t,L^1_\mu,(L^\infty_\mu)^o)$.

By Lemma 4.2.1 items (a) and (c) together with Proposition 1.5.2 we get

$$K(QRf,t,L^1_\mu,(L^\infty_\mu)^o) \leq K(Rf,t,L^1_\tau,(L^\infty_\tau)^o) \leq K(f,t,L^1_\mu,(L^\infty_\mu)^o),$$

therefore we have equality throughout. By formula (1.9) we have

$$K(Rf,t,L^1_\tau,(L^\infty_\tau)^o) = \int_0^t ((Rf)^o)^*.$$

By construction of the level function $f^o = Q((Rf)^o)$ and by item (4) in Theorem 2.4.8 we have that $((Rf)^o)^* = (Q(Rf)^o)^* = (f^o)^*$ so we get

$$K(Rf,t,L^1_\tau,(L^\infty_\tau)^o) = \int_0^t (f^o)^* = K(f,t,L^1_\mu,(L^\infty_\mu)^o) = K(QRf,t,L^1_\mu,(L^\infty_\mu)^o).$$

∎

We now compute the $K$-functional for the couple of associate spaces $(\widetilde{L^1_\mu}, L^\infty_\mu)$, which follows from the properties of the nonincreasing rearrangement.

**Theorem 4.2.4** *Let* $0 \leq g \in \widetilde{L^1_\mu} + L^\infty_\mu$ *and* $t > 0$, *then*

$$K(QRg, t, \widetilde{L^1_\mu}, L^\infty_\mu) \leq K(Rg, t, \widetilde{L^1_\lambda}, L^\infty_\lambda) \leq K(g, t, \widetilde{L^1_\mu}, L^\infty_\mu) = \int_0^t (\widetilde{g})^* .$$

*And equality throughout holds whenever* $g \in L^+(\mathcal{A})$.

**Proof:** Fix a nonnegative $g \in \widetilde{L^1_\mu} + L^\infty_\mu$. Using items (b) and (d) in Lemma 4.2.1 together with Proposition 1.5.2 we get

$$K(QRg, t, \widetilde{L^1_\mu}, L^\infty_\mu) \leq K(Rg, t, \widetilde{L^1_\lambda}, L^\infty_\lambda) \leq K(g, t, \widetilde{L^1_\mu}, L^\infty_\mu).$$

Whenever $g \in L^+(\mathcal{A})$ then $QRg = g$ and we have equality throughout. Next we show is that $K(g, t, \widetilde{L^1_\mu}, L^\infty_\mu) = \int_0^t (\widetilde{g})^*$.

Let $g = g_1 + g_\infty$ with $0 \leq g_1 \in \widetilde{L^1_\mu}$ and $0 \leq g_\infty \in L^\infty_\mu$. By Theorem 4.1.6 the map $g \to \widetilde{g}$ is sublinear, thus $\widetilde{g} \leq \widetilde{g_1} + \widetilde{g_\infty}$. By Proposition 1.3.9, for each $\epsilon \in (0,1)$ we have $(\widetilde{g_1} + \widetilde{g_\infty})^*(s) \leq (\widetilde{g_1})^*((1-\epsilon)s) + (\widetilde{g_\infty})^*(\epsilon s)$. Integration yields

$$\int_0^t (\widetilde{g})^*(s)\, ds \leq \int_0^t (\widetilde{g_1} + \widetilde{g_\infty})^*(s)\, ds \leq \int_0^t (\widetilde{g_1})^*((1-\epsilon)s)\, ds + \int_0^t (\widetilde{g_\infty})^*(\epsilon s)\, ds. \tag{4.3}$$

But

$$\begin{aligned}\int_0^t (\widetilde{g_1})^*((1-\epsilon)s)\, ds &= \frac{1}{1-\epsilon}\int_0^{(1-\epsilon)t} (\widetilde{g_1})^*(s)\, ds \leq \frac{1}{1-\epsilon}\int_0^\infty (\widetilde{g_1})^*(s)\, ds \\ &= \frac{1}{1-\epsilon}\int_U \widetilde{g_1}\, d\mu = \frac{1}{1-\epsilon}\|g_1\|_{\widetilde{L^1_\mu}},\end{aligned}$$

and

$$\int_0^t (\widetilde{g_\infty})^*(\epsilon s)\, ds \leq \int_0^t \|g_\infty\|_{L^\infty_\mu}\, ds = t\|g_\infty\|_{L^\infty_\mu}.$$

We may let $\epsilon \to 0$ in (4.3) to get

$$\int_0^t (\widetilde{g})^*(s)\, ds \leq \|g_1\|_{\widetilde{L^1_\mu}} + t\|g_\infty\|_{L^\infty_\mu}.$$

Taking the infimum over all such decompositions of $g$ yields the inequality

$$\int_0^t (\widetilde{g})^*(s)\, ds \leq K(g, t, \widetilde{L^1_\mu}, L^\infty_\mu).$$

To get the reverse inequality, fix $t > 0$ and set $y = \widetilde{g}(t)$. Let $g_1 = \max(0, g - y)$. Since $g \leq \widetilde{g} \in L^\downarrow(\mathcal{A})$, then $g_1 \leq \max(0, \widetilde{g} - y)$. Since the maximum of core decreasing functions is core decreasing, then $\max(0, \widetilde{g} - y)$ is a core decreasing majorant of $g_1$, thus $\widetilde{g_1} \leq \max(0, \widetilde{g} - y)$. If $h \in L^\downarrow(\mathcal{A})$ such that $g_1 \leq h$, then $g \leq h + y \in L^\downarrow(\mathcal{A})$, so $\widetilde{g} \leq h + y$. It follows that $\max(0, \widetilde{g} - y) \leq h$, this shows that $\max(0, \widetilde{g} - y) = \widetilde{g_1}$. Since $(\widetilde{g})^*$ is decreasing, $(\widetilde{g})^* \geq y$ on $[0, t]$ and $(\widetilde{g})^* \leq y$ on $[t, \infty)$, we can apply Proposition 1.3.9, to get

$$\|g_1\|_{\widetilde{L^1_\mu}} = \int_U \max(0, \widetilde{g} - y)\, d\mu = \int_0^\infty \max(0, \widetilde{g} - y)^*(s)\, ds = \int_0^t (\widetilde{g} - y)^* = \int_0^t (\widetilde{g})^*\, ds - ty.$$

Evidently, $g_\infty = g - g_1 = \min(g, y) \le y$, so $\|g - g_1\|_{L^\infty_\mu} \le y$ and it follows that

$$\|g_1\|_{\widetilde{L^1_\mu}} + t\|g_\infty\|_{L^\infty_\mu} \le \|g_1\|_{\widetilde{L^1_\mu}} + ty \le \int_0^t (\widetilde{g})^*\, ds.$$

We get

$$\int_0^t (\widetilde{g})^*(s)\, ds \ge K(g, t, \widetilde{L^1_\mu}, L^\infty_\mu)$$

and prove equality.

To finish the proof, notice that $\widetilde{g} \in L^+\mathcal{A}$ and using the fact that the K-functional is monotone, together with the bound $|g| \le \widetilde{g}$, Lemma 2.4.6, and $\widetilde{\widetilde{g}} = \widetilde{g}$, we get

$$K(g, t, \widetilde{L^1_\mu}, L^\infty_\mu) \le K(\widetilde{g}, t, \widetilde{L^1_\mu}, L^\infty_\mu) = K(R\widetilde{g}, t, \widetilde{L^1_\lambda}, L^\infty_\lambda) = \int_0^t (\widetilde{g})^* = K(g, t, \widetilde{L^1_\mu}, L^\infty_\mu).$$

This completes the proof.

$\blacksquare$

The next theorems show that both $(L^1_\mu, (L^\infty_\mu)^o)$ and $(\widetilde{L^1_\mu}, L^\infty_\mu)$ are exact Calderón couples.

**Theorem 4.2.5** *The couple $(L^1_\mu, (L^\infty_\mu)^o)$ is an exact Calderón couple.*

**Proof:** Suppose $f, g \in L^1_\mu + (L^\infty_\mu)^o$ satisfy $K(f, t, L^1_\mu, (L^\infty_\mu)^o) \le K(g, t, L^1_\mu, (L^\infty_\mu)^o)$ for all $t > 0$. To show that $(L^1_\mu, (L^\infty_\mu)^o)$ is an exact Calderón couple we need to find an admissible contraction from $(L^1_\mu, (L^\infty_\mu)^o)$ to itself that sends $g$ to $f$.

Consider the map $h \mapsto \operatorname{sgn}(g)h$, clearly it maps $g \to \operatorname{sgn}(g)g = |g|$ and it satisfies the pointwise bound $\left|\operatorname{sgn}(g)h\right| \le |h|$. It follows that it is an admissible contraction from $(L^1_\mu, (L^\infty_\mu)^o)$ to itself. Similarly the map $h \mapsto \operatorname{sgn}(f)h$ is also an admissible contraction and it satisfies $|f| \mapsto \operatorname{sgn}(f)\,|f| = f$.

Therefore we may assume that the functions $f$ and $g$ are nonnegative. We will build the desired operator as a composition of three admissible contractions, as shown in the following diagram.

$$\begin{array}{ccccccc} L^1_\mu + \left(L^\infty_\mu\right)^o & \xrightarrow{R} & L^1_\lambda + \left(L^\infty_\lambda\right)^o & \xrightarrow{H} & L^1_\lambda + \left(L^\infty_\lambda\right)^o & \xrightarrow{W} & L^1_\mu + \left(L^\infty_\mu\right)^o \\ g & \xrightarrow{R} & Rg & \xrightarrow{H} & Rf & \xrightarrow{W} & f \end{array}$$

The operator $R$ is given by Lemma 2.4.6 and is shown to be an admissible contraction in Proposition 4.2.1.

By virtue of Theorem 4.2.3, the inequality

$$K(Rf, t, L^1_\lambda, (L^\infty_\lambda)^o) \le K(Rg, t, L^1_\lambda, (L^\infty_\lambda)^o)$$

holds for all $t > 0$, then (3) in Theorem 1.6.11 provides the operator $H$.

Since $f$ is nonnegative, so is $Rf$. Let $E = \{x \in [0, \infty) : Rf(x) > 0\}$ and $w(x) = 1/Rf(x)\chi_E(x)$. For nonnegative functions $\psi \in L^1_{\mathrm{loc},\mathcal{B}}$ define $W(\psi) = fQ(\psi w)$, which is clearly an additive mapping. For each $M \in \mathcal{M}$,

$$\int_M W\psi\, d\mu = \int_M fQ(\psi w)\, d\mu = \int_{[0,\mu(M)]} Rf\psi w\, d\lambda \le \int_{[0,\mu(M)]} \psi\, d\lambda. \tag{4.4}$$

Therefore $W(\psi) \in L^1_{\mathrm{Loc},\mathcal{A}}$, and by Theorem 1.2.4 the map $W$ extends uniquely to a positive linear mapping defined on $L^1_{\mathrm{Loc},\mathcal{B}}$. The monotone convergence theorem, together with equation (4.4) yields $\|W\psi\|_{L^1_\mu} \le \|\psi\|_{L^1_\lambda}$. Once more, division by $\mu(M)$ on equation (4.4) and taking supremum over $M \in \mathcal{M}$ yields $\|W\psi\|_{(L^\infty_\mu)^o} \le \|\psi\|_{(L^\infty_\lambda)^o}$. Thus, $W$ is an admissible contraction from $L^1_\lambda + (L^\infty_\lambda)^o$ to $L^1_\mu + (L^\infty_\mu)^o$.

Therefore, the operator $W \circ H \circ R$ is the desired from $L^1_\lambda + (L^\infty_\lambda)^o$ to itself taking $g \to f$ and completes the proof.

∎

**Theorem 4.2.6** *The couple $(\widetilde{L^1_\mu}, L^\infty_\mu)$ is an exact Calderón couple.*

**Proof:** Suppose $f, g \in \widetilde{L^1_\mu} + L^\infty_\mu$ satisfy $K(f, t, \widetilde{L^1_\mu}, L^\infty_\mu) \le K(g, t, \widetilde{L^1_\mu}, L^\infty_\mu)$ for all $t > 0$. To show that $(\widetilde{L^1_\mu}, L^\infty_\mu)$ is an exact Caldeón couple we need to find an admissible contraction from $(\widetilde{L^1_\mu}, L^\infty_\mu)$ to itself that sends $g$ to $f$.

Just as the proof of Theorem 4.2.5 we may assume that $f$ and $g$ are nonnegative and we will build the operator as a composition of admissible contractions, as shown in the following diagram.

$$\begin{array}{ccccccccccc}
\widetilde{L^1_\mu} + L^\infty_\mu & \xrightarrow{W_1} & \widetilde{L^1_\mu} + L^\infty_\mu & \xrightarrow{R} & \widetilde{L^1_\lambda} + L^\infty_\lambda & \xrightarrow{H} & \widetilde{L^1_\lambda} + L^\infty_\lambda & \xrightarrow{Q} & \widetilde{L^1_\mu} + L^\infty_\mu & \xrightarrow{W_2} & \widetilde{L^1_\mu} + L^\infty_\mu \\
g & \xrightarrow{W_1} & \tilde{g} & \xrightarrow{R} & R\tilde{g} & \xrightarrow{H} & R\tilde{f} & \xrightarrow{Q} & \tilde{f} & \xrightarrow{W_2} & f
\end{array}$$

Notice $\widetilde{L^1_\mu} + L^\infty_\mu$ is a subspace of $L^1_{\mathrm{Loc},\mathcal{A}}$: Since if $h = h_1 + h_\infty$ with $h_1 \in \widetilde{L^1_\mu}$ and $h_\infty \in L^\infty_\mu$, then for any $M \in \mathcal{M}$,

$$\begin{aligned}
\int_M |h|\, d\mu &\le \int_M |h_1|\, d\mu + \int_M |h_\infty|\, d\mu \le \int_M |h_1|\, d\mu + \mu(M)\|h\|_{L^\infty_\mu} \\
&\le \int_U \widetilde{h_1}\, d\mu + \mu(M)\|h\|_{L^\infty_\mu} = \|h_1\|_{\widetilde{L^1_\mu}} + \mu(M)\|h\|_{L^\infty_\mu} < \infty.
\end{aligned}$$

The maps $W_1$ and $W_2$ are constructed in the same way as in the proof of [17, Theorem 2.3], for completeness we repeat the argument. Define the map $W_1$ on the one dimensional space $\mathbb{R}g$ of $L^1_{\mathrm{Loc},\mathcal{A}}$ by the formula $W_1(\alpha g) = \alpha\widetilde{g}$. The map is trivially linear and satisfies $W_1(h) \le \widetilde{h}$ for each $h \in \mathbb{R}g$. Theorem 4.1.6 shows that $h \mapsto \widetilde{h}$ is sublinear and it maps $L^1_{\mathrm{Loc},\mathcal{A}}$ which is a Dedekind complete vector lattice, by Proposition 2.3.4. So we may apply Theorem 1.2.6 to extend $W_1$ to a linear map preserving the bound $W_1(h) \le \widetilde{h}$. At $-h$ it gives $-W_1 h \le \widetilde{-h} = \widetilde{h}$ so $|W_1 h| \le \widetilde{h}$. It follows that

$$\|W_1 h\|_{L^\infty_\mu} \le \|\widetilde{h}\|_{L^\infty_\mu} = \|h\|_{L^\infty_\mu}$$

and

$$\|W_1 h\|_{\widetilde{L^1_\mu}} \le \|\widetilde{h}\|_{\widetilde{L^1_\mu}} = \|\widetilde{\widetilde{h}}\|_{L^1_\mu} = \|\widetilde{h}\|_{L^1_\mu} = \|h\|_{\widetilde{L^1_\mu}}.$$

Therefore $W_1$ is an admissible contraction from $(\widetilde{L^1_\mu}, L^\infty_\mu)$ to itself that maps $g \to \widetilde{g}$.

The operators $R$ and $Q$ are provided by Propositions 2.4.6 and 2.4.7 and are shown to be admissible contractions in Lemma 4.2.1.

Since $\widetilde{f} \in \sigma(\mathcal{A})$, Theorem 4.2.4 shows that

$$K(R\widetilde{f}, t, \widetilde{L^1_\mu}, L^\infty_\mu) = K(f, t, \widetilde{L^1_\mu}, L^\infty_\mu) \le K(g, t, \widetilde{L^1_\mu}, L^\infty_\mu) = K(R\widetilde{g}, t, \widetilde{L^1_\mu}, L^\infty_\mu).$$

By (4) in Theorem 1.6.11 we have an admissible contraction $H$ mapping $R\widetilde{g}$ to $R\widetilde{f}$.

It remains to prove the existence of the operator $W_2$. To define it, consider $\theta(s) = f(s)/\widetilde{f}(s)$ when $\widetilde{f}(s) \ne 0$ and $\theta(s) = 0$ otherwise. Then let $W_2 h = \theta h$ and note that $W_2 \widetilde{f} = f$. Since $|\theta| \le 1$, $W_2$ is an admissible contraction from $(\widetilde{L^1_\mu}, L^\infty_\mu)$ to itself.

Therefore, the operator; $W_2 \circ Q \circ H \circ R \circ W_1$ is an admissible contraction from $(\widetilde{L^1_\mu}, L^\infty_\mu)$ to itself mapping $g$ to $f$ and completes the proof.

∎

To exploit the previous results we compute the divisibility constants for both compatible couples.

**Lemma 4.2.7** *The couple $(L^1_\mu, (L^\infty_\mu)^o)$ has divisibility constant* 1.

**Proof:** Fix $f \in L^1_\mu + (L^\infty_\mu)^o$ and let $\{\omega_j\}$ be nonnegative concave functions satisfying $\sum_{j=1}^\infty \omega_j(1) < \infty$ and $K(f, t; L^1_\mu, (L^\infty_\mu)^o) \le \sum_{j=1}^\infty \omega_j(t)$, for each $t > 0$. Suppose that $f$ is nonnegative, then $Rf \ge 0$ by Lemma 2.4.6. And by Theorem 4.2.3 we get

$$K(Rf, t; L^1_\lambda, (L^\infty_\lambda)^o) = K(f, t; L^1_\mu, (L^\infty_\mu)^o) \le \sum_{j=1}^{\infty} \omega_j(t),$$

for each $t > 0$. By (3) in Theorem 1.6.11, the couple $(L^1_\lambda, (L^\infty_\lambda)^o)$ has divisibility constant 1. So, there exist functions $\varphi_j \in L^1_\lambda + (L^\infty_\lambda)^o$ such that $K(\varphi_j, t; L^1_\lambda, (L^\infty_\lambda)^o) \le \omega_j(t)$, for all $j$ and $t$, and $\sum_{j=1}^\infty \varphi_j$ converges to $Rf$ in $L^1_\lambda + (L^\infty_\lambda)^o$. Because $Rf$ is nonnegative, we may assume that $\varphi_j \ge 0$ for all $j$, since otherwise we can replace them with $\psi_j$, defined by $\psi_1 = \min(|\varphi_1|, Rf)$ and $\psi_{n+1} = \min(|\varphi_{n+1}|, Rf - (\psi_1 + \cdots + \psi_n))$ for $n = 1, 2, \ldots$.

Since $QRf$ is $\sigma(\mathcal{A})$-measurable, the set $E = \{u \in U : QRf(u) = 0\}$ is $\sigma(\mathcal{A})$-measurable, hence $\chi_E$ is $\sigma(\mathcal{A})$-measurable, hence by item (11) in Proposition 2.4.6,

$$0 = \int_U (QRf)\chi_E \, d\mu = \int_U (Rf)R\chi_E \, d\mu = \int_U f\chi_E,$$

therefore $\{u \in U : QRf(u) = 0\}$ is $\mu$-almost contained in $\{u \in U : f(u) = 0\}$.

Set $f_j = (fQ\varphi_j)/QRf$ when $QRf$ is nonzero and $f_j = 0$ otherwise. Then by item (3) in Proposition 2.4.7 we get

$$\sum_{j=1}^{\infty} f_j = \sum_{j=1}^{\infty} \frac{f}{QRf} Q(\varphi_j)\chi_E = \frac{f}{QRf} Q\Big(\sum_{j=1}^{\infty} \varphi_j\Big)\chi_E = \frac{f}{QRf} Q\big(Rf\big)\chi_E = f \tag{4.5}$$

$\mu$-a.e. In particular, $0 \le f_j \le f$ $\mu$-a.e for all $j$. Now fix $j$. The definition of $f_j$ implies that $fQ\varphi_j = (QRf)f_j$ $\mu$-a.e. and we get $QRf, Q\varphi_j \in L^+(\mathcal{A})$ from item (6) in Proposition 2.4.7, so we may use item (11) of Proposition 2.4.6 and Theorem 2.4.8 to get

$$(Rf)RQ\varphi_j = R(fQ\varphi_j) = R((QRf)f_j) = (RQRf)Rf_j = (Rf)Rf_j.$$

Since $f - f_j \geq 0$, $Rf - Rf_j \geq 0$ so $Rf_j = 0$ whenever $Rf = 0$. Therefore, we may cancel $Rf$ above to get $RQ\varphi_j \geq Rf_j$. This inequality, Lemma 4.2.1 and Proposition 1.5.2, give

$$K(Rf_j, t; L^1_\lambda, (L^\infty_\lambda)^o) \leq K(RQ\varphi_j, t; (L^1_\lambda, (L^\infty_\lambda)^o) \leq K(\varphi_j, t; L^1_\lambda, (L^1_\lambda, (L^\infty_\lambda)^o) \leq \omega_j(t).$$

By Theorem 4.2.3, this gives $K(f_j, t; L^1_\mu, (L^\infty_\mu)^o) \leq \omega_j(t)$. This estimate, along with (4.5) and the completeness of $L^1_\mu + (L^\infty_\lambda)^o$, shows that for each $n$,

$$\Big\| f - \sum_{j=1}^{n-1} f_j \Big\|_{L^1_\mu + (L^\infty_\lambda)^o} = \Big\| \sum_{j=n}^{\infty} f_j \Big\|_{L^1_\mu + (L^\infty_\lambda)^o} \leq \sum_{j=n}^{\infty} \|f_j\|_{L^1_\mu + (L^\infty_\lambda)^o} \leq \sum_{j=n}^{\infty} \omega_j(1),$$

where we have used the identity $\|f_j\|_{L^1_\mu + (L^\infty_\lambda)^o} = K(f_j, 1; L^1_\mu, (L^\infty_\lambda)^o)$. The right-hand side above is the tail of a convergent series, so it goes to zero as $n \to \infty$. We conclude that $\sum_{j=1}^{\infty} f_j$ converges to $f$ in $L^1_\mu + (L^\infty_\lambda)^o$.

To drop the nonnegativity assumption on $f$, construct the $f_j$ for $|f|$, the functions $\mathrm{sgn}(f) f_j$ give the required decomposition of $f$.

∎

**Lemma 4.2.8** *The couple* $(\widetilde{L^1_\mu}, L^\infty_\mu)$ *has divisibility constant* 1.

**Proof:** Let $0 \leq g \in \widetilde{L^1_\mu} + L^\infty_\mu$ and $\omega_j$ be nonnegative, concave functions on $[0, \infty)$ such that $\sum_{j=1}^{\infty} \omega_j(1) < \infty$ and $K(g, t, \widetilde{L^1_\mu}, L^\infty_\mu) \leq \sum_{j=1}^{\infty} \omega_j(t)$ for all $t$. Our object is to find $g_j \in \widetilde{L^1_\mu} + L^\infty_\mu$ such that $K(g_j, t, \widetilde{L^1_\mu}, L^\infty_\mu) \leq \omega_j(t)$ for all $t$ and $\sum_{j=1}^{\infty} g_j$ converges to $g$ in $\widetilde{L^1_\mu} + L^\infty_\mu$.

Define $v_j(t) = \min\left(\omega_j(t), \int_0^t (\widetilde{g})^*\right)$. Since $v_j$ is defined as the minimum of nonnegative concave functions, it is nonnegative and concave. Also $v_j(0^+) = 0$, thus there exists $h_j \in L^\downarrow$ such that $v_j(t) = \int_0^t h_j$. Then, for each $t > 0$:

$$\int_0^t (\widetilde{g})^* = \min\left( \int_0^t (\widetilde{g})^*, \sum_{j=1}^{\infty} \omega_j(t) \right) \leq \sum_{j=1}^{\infty} \min\left( \int_0^t (\widetilde{g})^*, \omega_j(t) \right) = \sum_{j=1}^{\infty} \int_0^t h_j \leq \sum_{j=1}^{\infty} \omega_j(t).$$

Set $h = \sum_{j=1}^{\infty} h_j$. Since the partial sums $\sum_{j=1}^{n} h_j$ are decreasing and nonnegative, $h$ is decreasing and nonnegative. By the monotone convergence theorem, we have

$$K((\widetilde{g})^*, t; L^1, L^\infty) = \int_0^t (\widetilde{g})^* \leq \int_0^t \sum_{j=1}^{\infty} h_j = \int_0^t h = K(h, t; L^1, L^\infty),$$

for each $t > 0$. By Theorem 1.5.8, there exists an admissible contraction $T$ from $L^1 + L^\infty$ to itself such that $Th = (\widetilde{g})^*$ with $Th_j \in L^\downarrow$. Notice that

$$\Big\| h - \sum_{j=1}^{n} h_j \Big\|_{L^1 + L^\infty} = \Big\| \sum_{j=n+1}^{\infty} h_j \Big\|_{L^1 + L^\infty} = \int_0^1 \sum_{j=n+1}^{\infty} h_j = \sum_{j=n+1}^{\infty} \int_0^1 h_j \leq \sum_{j=n+1}^{\infty} \omega_j(1).$$

The right-hand side is the tail of a convergent series, therefore $\sum_{j=1}^{n} h_j \to h$ in $L^1 + L^\infty$. By continuity, $T(\sum_{j=1}^{n} h_j) \to Th = (\widetilde{g})^*$. Let $\kappa_j = Th_j$, then by linearity, $\sum_{j=1}^{\infty} \kappa_j = (\widetilde{g})^*$ with convergence in $L^1 + L^\infty$. Notice that the sequence of partial sums $\sum_{j=1}^n h_j$ is increasing, thus $\sum_{j=1}^n Th_j \uparrow (\widetilde{g})^*$ $\lambda$-a.e.

Since $\widetilde{g} \in L^{\downarrow}(\mathcal{A})$, Propositions 2.4.6 and Lemma 2.4.5 yield

$$(R\widetilde{g}) \circ b = (R\widetilde{g})^* = (\widetilde{g})^*,$$

hence $Th = (R\widetilde{g}) \circ b$, where $b$ is the function from Theorem 2.4.3. Recall that $b \circ b(x) = b(x)$ and $x \leq b(x)$ for all $x > 0$. Since $\kappa_j$ is nonincreasing, we have $\kappa_j(x) \geq \kappa_j \circ b(x)$, then

$$0 = (R\widetilde{g}) \circ b(x) - (R\widetilde{g}) \circ b \circ b(x) = \sum_{j=1}^{\infty} \kappa_j(x) - \sum_{j=1}^{\infty} \kappa_j \circ b(x) = \sum_{j=1}^{\infty} \left(\kappa_j(x) - \kappa_j \circ b(x)\right).$$

Since $\kappa_j(x) - \kappa_j \circ b(x) \geq 0$, we obtain $\kappa_j(x) - \kappa_j \circ b(x) = 0$ and so $\kappa_j = \kappa_j \circ b$ for all $j$. This shows that $\kappa_j \in L(\mathcal{B})$, hence $\kappa_j \in L^{\downarrow}(\mathcal{B})$ and its rearrangement with respect to $\lambda$ is $\kappa_j$. By Proposition 2.4.7, we have $Q\kappa_j \in L^{\downarrow}(\mathcal{A})$ and $(Q\kappa_j)^* = \kappa_j$.

Set $g_j = (g/\widetilde{g})Q\kappa_j$. Since $g \leq \widetilde{g}$, we get $g_j \leq Q\kappa_j$ and

$$(\widetilde{g_j})^* \leq (Q\kappa_j)^* = \kappa_j.$$

Thus, for all $j$ and $t$, Theorem 4.2.4 and Proposition 1.5.2 show

$$K(g_j, t; \widetilde{L^1_\mu}, L^\infty_\mu) = \int_0^t (\widetilde{g_j})^* \leq \int_0^t \kappa_j = K(Th_j, t; L^1, L^\infty) \leq K(h_j, t; L^1, L^\infty) = \int_0^t h_j \leq \omega_j(t).$$

Since $x = b(x)$ $\lambda - a.e.$, we get $(R\widetilde{g}) \circ b = R\widetilde{g}$. Then, item (3) in Proposition 2.4.7, item (3) in Theorem 2.4.8 yield

$$\sum_{j=1}^{\infty} g_j = (g/\widetilde{g})Q\left(\sum_{j=1}^{\infty} \kappa_j\right) = (g/\widetilde{g})Q\left((\widetilde{g})^*\right) = (g/\widetilde{g})Q\left((R\widetilde{g}) \circ b\right) = (g/\widetilde{g})QR\widetilde{g} = \widetilde{g},$$

$\mu$-a.e. This, and the completeness of $\widetilde{L^1_\mu} + L^\infty_\mu$, shows that for each $n$,

$$\left\| g - \sum_{j=1}^{n-1} g_j \right\|_{\widetilde{L^1_\mu}+L^\infty_\mu} = \left\| \sum_{j=n}^{\infty} g_j \right\|_{\widetilde{L^1_\mu}+L^\infty_\mu} \leq \sum_{j=n}^{\infty} \|g_j\|_{\widetilde{L^1_\mu}+L^\infty_\mu} = \sum_{j=n}^{\infty} K(g_j, 1; \widetilde{L^1_\mu}, L^\infty_\mu) \leq \sum_{j=n}^{\infty} \omega_j(1).$$

The right-hand side is the tail of a convergent series so it goes to zero as $n \to \infty$. We conclude that $\sum_{j=1}^{\infty} g_j$ converges to $g$ in $\widetilde{L^1_\mu} + L^\infty_\mu$.

To drop the nonnegativity assumption on $g$, repeat the same construction of $\{g_j\}$ for $|g|$. Then the functions $\mathrm{sgn}(g)g_j$ provide the required decomposition of $g$.

∎

The following theorem summarizes the results

**Theorem 4.2.9** *Let $X$, $Y$, and $Z$ be Banach function spaces of $\mu$-measurable functions. Then*

1. $X \in Int_1(L^1_\mu, L^\infty_\mu)$ *if and only if $X$ is u.r.i.*
2. $Y \in Int_1(L^1_\mu, (L^\infty_\mu)^o)$ *if and only if $Y = X^o$ for some u.r.i space $X$.*

3. *$Z \in Int_1(\widetilde{L^1_\mu}, L^\infty_\mu)$ if and only if $Z = \widetilde{X}$ for some u.r.i space $X$.*

**Proof:** Consider the collection of all triples

$$((L^1_\mu, L^\infty_\mu)_\Phi, (L^1_\mu, L^\infty_\mu\downarrow)_\Phi, (\widetilde{L^1_\mu}, L^\infty_\mu)_\Phi)$$

as $\Phi$ runs through all parameters of the $K$-method. By Proposition 1.5.4; the first entry of each triple is in $\mathrm{Int}_1(L^1_\mu, L^\infty_\mu)$, the second entry is in $\mathrm{Int}_1(L^1_\mu, L^\infty_\mu\downarrow)$ and the third entry is in $\mathrm{Int}_1(\widetilde{L^1_\mu}, L^\infty_\mu)$.

Item (1), above, is Theorem 1.5.5 and was proved in [6]. It shows that the collection of first entries in these triples is exactly the set of u.r.i. spaces over $U$.

Suppose $Y \in \mathrm{Int}_1(L^1_\mu, (L^\infty_\mu)^o)$; by Theorem 4.2.5 and Lemma 4.2.7, there exists a parameter of the $K$-method $\Phi$ such that $Y = (L^1_\mu, (L^\infty_\mu)^o)_\Phi$ with identical norms. For any $f \in L^1_\mu + (L^\infty_\mu)^o$, Theorem 4.2.3 shows that $K(f, t; L^1_\mu, (L^\infty_\mu)^o) = K(f^o, t; L^1_\mu, L^\infty_\mu)$, therefore

$$\|f\|_{(L^1_\mu, (L^\infty_\mu)^o)_\Phi} = \|f^o\|_{(L^1_\mu, L^\infty_\mu)_\Phi} = \|f\|_{\left((L^1_\mu, L^\infty_\mu)_\Phi\right)^o}.$$

Hence, $Y = X^o$ where $X$ is the u.r.i space $(L^1_\mu, L^\infty_\mu)_\Phi$, this proves item (2).

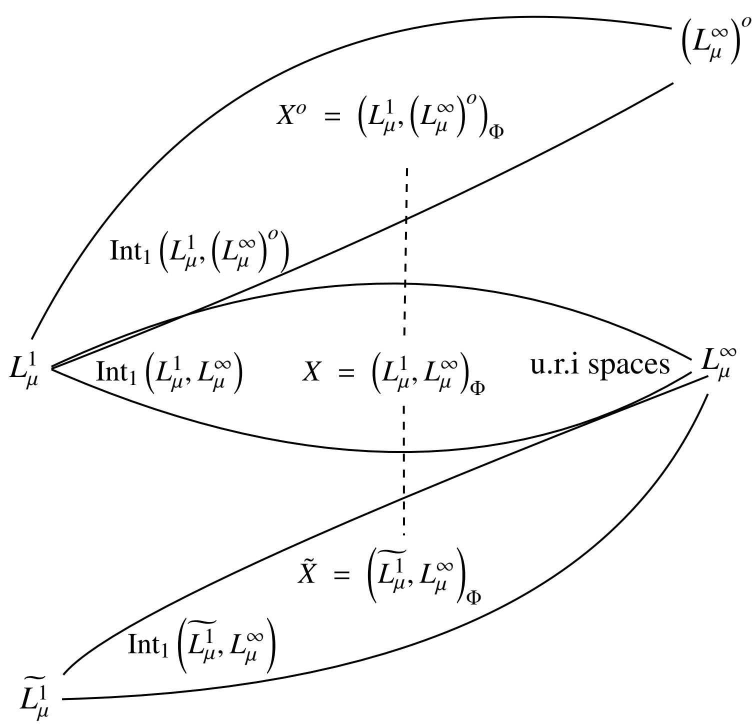
Similarly, if $Y \in \mathrm{Int}_1(\widetilde{L^1_\mu}, L^\infty_\mu)$, by Theorem 4.2.6 and Lemma 4.2.8, there exists a parameter of the $K$-method $\Phi$ such that $Y = (\widetilde{L^1_\mu}, L^\infty_\mu)_\Phi$ with identical norms. For any $g \in \widetilde{L^1_\mu} + L^\infty_\mu$, Theorem 4.2.4 shows that $K(f, t; \widetilde{L^1_\mu}, L^\infty_\mu) = K(\widetilde{g}, t; L^1_\mu, L^\infty_\mu)$, therefore

$$\|g\|_{(L^1_\mu, (L^\infty_\mu)^o)_\Phi} = \|\widetilde{g}\|_{(L^1_\mu, L^\infty_\mu)_\Phi} = \|g\|_{\widetilde{(L^1_\mu, L^\infty_\mu)_\Phi}}.$$

Hence, $Y = \widetilde{X}$ where $X$ is the u.r.i space $(L^1_\mu, L^\infty_\mu)_\Phi$, this proves item (3).

∎

The following diagram may help to clarify the interpolation results summarized from the last theorem.



# Chapter 5

# Kernel operators

In this section, we turn our attention to linear operators of the form

$$Tf(y) = \int_U k(y,t) f(t)\, d\mu(t),$$

where $k : Y \times U \to [0,\infty)$ is a $\tau \otimes \mu$-measurable function. We call the function $k$ a kernel and $T$ its related kernel operator.

We will focus on the kernel operators that have the following compatibility condition on their level sets.

**Definition 5.0.1** *We say that the kernel is consonant if*

$$\mathcal{K} = \{\emptyset\} \cup \{\{u \in U : k(y,u) \geq t\} : y \in Y, t > 0\}$$

*is a totally ordered collection of sets of finite $\mu$-measure, that is, if $\mathcal{K}$ is an ordered core of $(U, \Sigma, \mu)$.*

## 5.1 Transferring monotonicity

We will focus on the kernels satisfying Definition 5.0.1. We begin by noting that, the functions $u \mapsto k(y,u)$ are core decreasing with respect to the ordered core $\mathcal{K}$, for all $y \in Y$.

**Proposition 5.1.1** *Let $\mathcal{K}$ be the ordered core defined above and let $y \in Y$, then the function $u \mapsto k(y,u)$ is core decreasing. Moreover, if $f, g \in L^+(\Sigma)$ and $f \preccurlyeq_{\mathcal{K}} g$, then $Tf \leq Tg$ $\tau$-almost everywhere.*

**Proof:** Suppose $u \leq_{\mathcal{K}} v$ but $k(y,u) < k(y,v)$ seeking a contradiction. Then there exists $t \in (k(y,u), k(y,v))$. Hence $v \in \{s \in U : k(y,s) \geq t\}$ but $u \notin \{s \in U : k(y,s) \geq t\}$. Thus $u \leq_{\mathcal{K}} v$ fails. This contradicts our original assumption and proves that $u \mapsto k(y,u)$ is core decreasing.

To complete the proof, suppose that $f, g \in L^+(\Sigma)$ and $f \preccurlyeq_{\mathcal{K}} g$. Then by Proposition 3.1.4 and the fact that $u \mapsto k(y,u)$ is core decreasing for each $y \in Y$, we get

$$Tf(y) = \int_U k(y,u)\, f(u)\, d\mu(u) \leq \int_U k(y,u)\, g(u)\, d\mu(u) = Tg(y).$$

∎

For all the following results we suppose that $X$ is a quasi Banach function space of $\tau$-measurable functions.

**Theorem 5.1.2** *If $k$ is a kernel that satisfies Definition 5.0.1 and $T$ is its associated kernel operator. Then, the least constant $C$, infinite or finite, for which the inequality*

$$\|Tf\|_X \leq C \int_U fw\, d\mu, \quad f \in L^+_\mu$$

*holds is unchanged when $w$ is replaced with $\underline{w}$. That is,*

$$\sup_{f\geq 0} \frac{\|Tf\|_X}{\int_U fw\, d\mu} = \sup_{f\geq 0} \frac{\|Tf\|_X}{\int_U f\underline{w}\, d\mu}.$$

**Proof:** Since $\underline{w} \leq w$ $\mu$-almost everywhere, the inequality '$\leq$' is clear. Conversely, we use Theorem 3.3.8 to get

$$\frac{\|Tf\|_X}{\int_U f\underline{w}\, d\mu} = \frac{\|Tf\|_X}{\inf\left\{\int_U gw\, d\mu : f \preccurlyeq_{\mathcal{K}} g\right\}} = \sup\left\{\frac{\|Tf\|_X}{\int_U gw\, d\mu} : f \preccurlyeq_{\mathcal{K}} g\right\} \leq \sup\left\{\frac{\|Tg\|_X}{\int_U gw\, d\mu} : f \preccurlyeq_{\mathcal{K}} g\right\}$$
$$\leq \sup\left\{\sup_{g\geq 0} \frac{\|Tg\|_X}{\int_U gw\, d\mu} : f \preccurlyeq_{\mathcal{K}} g\right\} = \sup_{g\geq 0} \frac{\|Tg\|_X}{\int_U gw\, d\mu}.$$

Above, we have used Proposition 5.1.1 and the fact that $Tf \leq Tg$ implies $\|Tf\|_X \leq \|Tg\|_X$. To complete the proof, we take the supremum over all $f \geq 0$.

∎

We now look at the reversed inequality.

**Theorem 5.1.3** *If $k$ is a kernel that satisfies Definition 5.0.1 and $T$ its associated kernel integral operator. Then, the least constant $C$, infinite or finite, for which the inequality*

$$\int_U fw\, d\mu \leq C\|Tf\|_X, \quad f \in L^+_\mu,$$

*holds is unchanged when $w$ is replaced with $\widetilde{w}$. That is,*

$$\sup_{f\geq 0} \frac{\int_U fw\, d\mu}{\|Tf\|_X} = \sup_{f\geq 0} \frac{\int_U f\widetilde{w}\, d\mu}{\|Tf\|_X}.$$

**Proof:** Since $\widetilde{w} \geq w$ $\mu$-almost everywhere. Then, the inequality '$\leq$' is clear. Conversely, we use Theorem 3.2.5 to get

$$\frac{\int_U f\widetilde{w}\, d\mu}{\|Tf\|_X} = \frac{\sup\left\{\int_U gw\, d\mu : g \preccurlyeq_{\mathcal{K}} f\right\}}{\|Tf\|_X} = \sup\left\{\frac{\int_U gw\, d\mu}{\|Tf\|_X} : g \preccurlyeq_{\mathcal{K}} f\right\} \leq \sup\left\{\frac{\int_U gw\, d\mu}{\|Tg\|_X} : g \preccurlyeq_{\mathcal{K}} f\right\}$$
$$\leq \sup\left\{\sup_{g\geq 0} \frac{\int_U gw\, d\mu}{\|Tg\|_X} : g \preccurlyeq_{\mathcal{K}} f\right\} = \sup_{g\geq 0} \frac{\int_U gw\, d\mu}{\|Tg\|_X}.$$

Above, we have used Proposition 5.1.1 and the fact that $Tg \le Tf$ implies $\|Tg\|_X \le \|Tf\|_X$. To complete the proof, we take the supremum over all $f \ge 0$.

∎

And a corresponding result for core decreasing functions.

**Theorem 5.1.4** *If k is a kernel that satisfies Definition 5.0.1 and T its associated kernel integral operator. Then, the least constant C, infinite or finite, for which the inequality*

$$\int_U fw\,d\mu \le C\|Tf\|_X, \quad f \in L^{\downarrow}(\mathcal{K}).$$

*holds is unchanged when w is replaced with $w^o$. That is,*

$$\sup_{f\ge 0} \frac{\int_U fw\,d\mu}{\|Tf\|_X} = \sup_{f\ge 0} \frac{\int_U fw^o\,d\mu}{\|Tf\|_X}.$$

**Proof:** Let $f \in L^{\downarrow}(\mathcal{K})$. Item (a) in Proposition 3.4.2, together with Proposition 3.1.4 shows that $\int_U fw\,d\mu \le \int_U fw^o\,d\mu$. Therefore, the inequality '$\le$' holds. Conversely, we get

$$\begin{aligned}
\frac{\int_U fw^o\,d\mu}{\|Tf\|_X} &= \frac{\sup\left\{\int_U gw\,d\mu : g \preccurlyeq_{\mathcal{K}} f, g \in L^{\downarrow}(\mathcal{K})\right\}}{\|Tf\|_X} = \sup\left\{\frac{\int_U gw\,d\mu}{\|Tf\|_X} : g \preccurlyeq_{\mathcal{K}} f, g \in L^{\downarrow}(\mathcal{K})\right\} \\
&\le \sup\left\{\frac{\int_U gw\,d\mu}{\|Tg\|_X} : g \preccurlyeq_{\mathcal{K}} f, g \in L^{\downarrow}(\mathcal{K})\right\} \\
&\le \sup\left\{\sup_{g\in L^{\downarrow}(\mathcal{K})} \frac{\int_U gw\,d\mu}{\|Tg\|_X} : g \preccurlyeq_{\mathcal{K}} f, g \in L^{\downarrow}(\mathcal{K})\right\} = \sup_{g\in L^{\downarrow}(\mathcal{K})} \frac{\int_U gw\,d\mu}{\|Tg\|_X}.
\end{aligned}$$

Again, we have used Proposition 5.1.1 and the fact that $Tg \le Tf$ implies $\|Tg\|_X \le \|Tf\|_X$. To complete the proof, we take the supremum over all $f \in L^{\downarrow}(\mathcal{K})$.

∎

## 5.2 The case $p = 1$ for abstract Hardy inequalities

Recall the abstract Hardy inequality (1.14). The conditions on the core map guarantee that $\mathcal{A} = \{B(y)\}_{y\in Y} \cup \{\emptyset\}$ is a $\sigma$-bounded ordered core. That same ordered core can also be induced by considering the kernel $k : Y \times U \to [0, \infty)$ by $k(y, u) = \chi_{B(y)}(u)$ and following the construction in Section 5.1.

From this point forward we will assume that $\mathcal{K} = \mathcal{A}$, that the kernel is $\chi_{B(y)}(u)$.

$$Tf(y) = \int_{B(y)} f\,d\mu = \int_U k(y,u) f(u)\,d\mu(u),$$

Our approach to finding necessary and sufficient conditions on the measures for inequality (1.14) is to find an equivalent inequality involving only two measures and a weight function, then to use Theorem 5.1.2 to replace the weight function with a core decreasing function. Finally, we find an equivalent Hardy inequality on the half line.

**Proposition 5.2.1** *Fix $q > 0$, let $\eta$ and $\mu$ be $\sigma$-finite measures over $(U, \Sigma)$, let $\tau$ be a $\sigma$-finite measure over $(Y, \tau)$ and let $B : Y \to \Sigma$ be a core map. Then there exists a positive $\Sigma$-measurable function $u$ such that the best constant in the inequality*

$$\left( \int_Y \left( \int_{B(y)} f \, d\mu \right)^q d\tau(y) \right)^{1/q} \leq C \int_U f \, d\eta, \tag{5.1}$$

*is the same as the best constant in the inequality*

$$\left( \int_Y \left( \int_{B(y)} f \, d\mu \right)^q d\tau(y) \right)^{1/q} \leq C \int_U f \underline{u} \, d\mu, \ \forall f \in L^+_\mu. \tag{5.2}$$

**Proof:** First, we reduce the problem to the case $U = \cup_{y \in Y} B(y)$. Fix $f \in L^+(\Sigma)$, set $U_0 = \cup_{y \in Y} B(y)$ and $g = f \chi_{U_0}$. Then

$$\frac{\left( \int_Y \left( \int_{B(y)} g \, d\mu \right)^q d\tau(y) \right)^{1/q}}{\int_{U_0} g \, d\eta} = \frac{\left( \int_Y \left( \int_{B(y)} f \, d\mu \right)^q d\tau(y) \right)^{1/q}}{\int_U g \, d\eta} \geq \frac{\left( \int_Y \left( \int_{B(y)} f \, d\mu \right)^q d\tau(y) \right)^{1/q}}{\int_U f \, d\eta}.$$

Taking the supremum over all $f \in L^+(\Sigma)$ shows that

$$\sup_{f \in L^+(\Sigma)} \frac{\left( \int_Y \left( \int_{B(y)} f \, d\mu \right)^q d\tau(y) \right)^{1/q}}{\int_U f \, d\eta} \leq \sup_{f \in L^+(\Sigma)} \frac{\left( \int_Y \left( \int_{B(y)} f \, d\mu \right)^q d\tau(y) \right)^{1/q}}{\int_{U_0} f \, d\eta}.$$

Conversely,

$$\sup_{f \in L^+(\Sigma)} \frac{\left( \int_Y \left( \int_{B(y)} f \, d\mu \right)^q d\tau(y) \right)^{1/q}}{\int_{U_0} f \, d\eta} = \sup_{f \chi_{U_0} \in L^+(\Sigma)} \frac{\left( \int_Y \left( \int_{B(y)} f \, d\mu \right)^q d\tau(y) \right)^{1/q}}{\int_U f \, d\eta}$$

$$\leq \sup_{f \in L^+(\Sigma)} \frac{\left( \int_Y \left( \int_{B(y)} f \, d\mu \right)^q d\tau(y) \right)^{1/q}}{\int_U f \, d\eta}.$$

Therefore, we may replace $U$ with $U_0$ in (5.1). The same argument shows that we may replace $U$ with $U_0$ in (5.2). Hence, we may suppose that $U = U_0$.

An application of the Lesbesgue decomposition theorem shows that $\mu = \mu_1 + \mu_2$, with $\mu_2 << \eta$ and $\mu_1 \perp \eta$. Also $U = U_1 \cup U_2$ with $U_1 \cap U_2 = \emptyset$ and $\mu_2(U_1) = 0 = \eta(U_2)$. The Radon-Nikodym theorem provides a $\Sigma$-measurable nonnegative function $h$ such that $d\mu_2 = h \, d\eta$. If $E = \{s \in U : h(s) = 0\}$ we can define the function $g = h \chi_{(U \setminus E)}$ and the sets $V_1 = U_1 \setminus E$ and

$V_2 = U_2 \cup E$ to get a decomposition $d\mu = g\,d\eta + d\mu_1$ supported on $V_1$ and $V_2$ respectively, moreover $g$ is never zero on $V_1$. Thus the inequality (5.1) becomes

$$\left(\int\limits_Y \left(\int\limits_{B(y)} fg\,d\eta + \int\limits_{B(y)} f\,d\mu_1\right)^q d\tau(y)\right)^{\frac{1}{q}} \le C \int_U f\,d\eta,\ \forall f \in L^+_\mu.$$

Fix $z \in Y$ and set $f = \chi_{(B(z)\cap V_2)}$, then if $C$ is finite, we have

$$\left(\int\limits_Y \left(\mu_1\left(B(y)\cap B(z)\right)\right)^q d\tau(y)\right)^{\frac{1}{q}} = \left(\int\limits_Y \left(\int\limits_{B(y)\cap B(z)} d\mu_1\right)^q d\tau(y)\right)^{\frac{1}{q}} \le C\eta\left(B(z)\cap V_2\right) = 0.$$

Therefore $\mu_1\left(B(y)\cap B(z)\right) = 0$ for $\tau$-almost every $y$. Since this holds for all $z \in Y$, letting $B(z) \uparrow U$ yields $\mu_1(U) = 0$.

Hence, the inequality becomes

$$\left(\int\limits_Y \left(\int\limits_{B(y)} fg\,d\eta\right)^q d\tau(y)\right)^{\frac{1}{q}} \le C \int_U f\,d\eta,\ \forall f \in L^+_\mu.$$

Since $g$ is non-zero $\eta$-almost everywhere, we can define $u = \frac{1}{g}$, so $d\eta = u\,d\mu$. Notice that the sets $L^+_\mu$ and $L^+_\eta$ are only dependant on $\Sigma$, thus the the substitution $f \mapsto fu$ is a bijection from $L^+_\eta \to L^+_\mu$ and yields the inequality

$$\left(\int\limits_Y \left(\int\limits_{B(y)} f\,d\mu\right)^q d\tau(y)\right)^{1/q} \le C \int\limits_U fu\,d\mu,\ \forall f \in L^+_\mu. \tag{5.3}$$

This shows that if the best constant in the inequality (5.1) is finite, then it is also the best constant in the inequality (5.3). For the remaining case, notice that we can decompose $d\eta = ud\mu + d\eta_2$ for some measure $\eta_2$ satisfying $\eta \perp \eta_2$. Therefore

$$\sup_{f\in L^+_\mu} \frac{\left(\int\limits_Y \left(\int\limits_{B(y)} f\,d\mu\right)^q d\tau(y)\right)^{\frac{1}{q}}}{\int_U f\,d\eta} \le \sup_{f\in L^+_\mu} \frac{\left(\int\limits_Y \left(\int\limits_{B(y)} f\,d\mu\right)^q d\tau(y)\right)^{\frac{1}{q}}}{\int_U fu\,d\mu},$$

thus if the best constant in inequality (5.1) is infinite, then it is also the best constant in the inequality (5.3).

To finish the proof, apply Theorem 5.1.2 with the kernel $k(y,u) = \chi_{B(y)}(u)$ and the core $\mathcal{K} = \mathcal{A}$ to replace $u$ with $\underline{u}$.

$\blacksquare$

We now reduce the problem to a Hardy inequality with measures over the half line.

**Lemma 5.2.2** *Given $B, \tau, \mu$ as in the previous propositions, then there exist Borel measures $\nu, \lambda$ on $[0,\infty)$ and a nonincreasing function $w$ finite $\lambda$-almost everywhere, such that the best constant in inequality (5.2) is the best constant in*

$$\left(\int\limits_{[0,\infty)} \left(\int\limits_{[0,x]} f\,d\lambda\right)^q d\nu(x)\right)^{1/q} \le C \int\limits_{[0,\infty)} fw\,d\lambda,\ \forall f \in L^+_\lambda \tag{5.4}$$

**Proof:** Since $B$ is a core map, the function $\varphi : Y \to [0,\infty)$ defined by $\varphi(y) = \mu(B(y))$ is measurable. Let $\nu$ be the push-forward Borel measure associated to $\varphi$ from Theorem 1.1.5. Let $\lambda$ be the Borel measure induced by ordered core $\mathcal{A}$ with enriched core $\mathcal{M}$, and $R, Q$ the transition maps from Theorems 2.4.6 and 2.4.7.

Fix a positive $\Sigma$-measurable function $f \in L^1_{\mathrm{Loc},\mathcal{A}}$ and define

$$Hf(x) = \int_{[0,x]} R(f)\,d\lambda, \quad \text{and} \quad Tf(y) = \int_{B(y)} f\,d\lambda.$$

We will show that $Hf$ and $Tf$ are equimeasurable with respect to the measures $\nu$ and $\tau$, respectively, by computing their distribution functions. First notice that for all $y \in Y$ we have

$$(Hf)\circ\varphi(y) = Hf\big(\mu(B(y))\big) = \int_{[0,\mu(B(y))]} R(f)\,d\mu = \int_{B(y)} f\,d\lambda = Tf(y).$$

Fix $\alpha > 0$ and define the sets

$$E_\alpha = \{x \in [0,\infty) : Hf(x) > \alpha\} \text{ and } F_\alpha = \{y \in Y : T(y) > \alpha\}.$$

Let

$$\gamma = \sup\left\{x \in [0,\infty) : \int_{[0,x]} Rf\,d\lambda \le \alpha\right\}.$$

Notice that by the monotone convergence theorem $Hf(\gamma) \le \alpha$. We claim that $E_\alpha = (\gamma,\infty)$ and that $F_\alpha = \varphi^{-1}(E_\alpha)$.

Let $x \in E_\alpha$, then since $Hf$ is increasing, we must have that $x > \gamma$, thus $E_\alpha \subseteq (\gamma,\infty)$. Conversely, let $x > \gamma$, then $Hf(x) > \alpha$, thus $x \in E_\alpha$, this shows the first equation.

For the second equation, notice that

$$F_\alpha = \{y \in Y : T(y) > \alpha\} = \{y \in Y : (Hf)\circ\varphi(y) > \alpha\}.$$

So if $y \in F_\alpha$, then $\varphi(y) \in E_\alpha$, this shows $F_\alpha \subseteq \varphi^{-1}(E_\alpha)$. Conversely, if $y \in \varphi^{-1}(E_\alpha)$, then $T(y) > \alpha$, hence $y \in F_\alpha$.

Computation of the distribution functions yields

$$\nu(E_\alpha) = \tau(\varphi^{-1}(E_\alpha)) = \tau(F_\alpha).$$

Therefore $Hf$ and $Tf$ are equimeasurable, hence

$$\left(\int_{[0,\infty)}\left(\int_{[0,x]} R(f)\,d\lambda\right)^q d\nu\right)^{\frac{1}{q}} = \left(\int_{[0,\infty)}\Big(Hf\Big)^q d\nu\right)^{\frac{1}{q}} = \left(\int_Y \Big(Tf\Big)^q d\tau\right)^{\frac{1}{q}}$$

$$= \left(\int_Y\left(\int_{B(y)} f\,d\mu\right)^q d\tau\right)^{\frac{1}{q}}.$$

Since $\underline{u}$ is core decreasing, Theorem 2.4.6, items (8) and (10) show that

$$\int_U f\underline{u}\, d\mu = \int_{[0,\infty)} Rf R\underline{u}\, d\lambda.$$

Therefore, if inequality (5.2) holds, so does

$$\left(\int_{[0,\infty)} \left(\int_{[0,x]} Rf\, d\lambda\right)^q d\nu(x)\right)^{\frac{1}{q}} \leq C \int_{[0,\infty)} Rf R\underline{u}\, d\lambda,\ \forall f \in L^+_\mu.$$

Notice that $R\underline{u}$ must be finite almost everywhere, otherwise, the original measures are not $\sigma$-finite. The result follows by letting $w = R\underline{u}$ and noting that $R$ maps $L^+_\mu$ onto $L^+_\lambda$, which follows from part 8 of Theorem 2.4.1 and the monotone convergence theorem.

∎

We are ready to prove the main result.

**Theorem 5.2.3** *Let $(Y, \mathcal{T}, \tau), (U, \Sigma, \mu), (U, \Sigma, \nu)$ be $\sigma$-finite measure spaces and $B : Y \to \Sigma$ a core map. Set $\eta = \eta_a + \eta_s$, where $d\eta_a = u d\mu$ and $\eta_s \perp \mu$. Then the best constant $C$ in the inequality*

$$\left(\int_Y \left(\int_{B(y)} f\, d\mu\right)^q d\tau(y)\right)^{1/q} \leq C \int_U f\, d\eta, \tag{5.5}$$

*satisfies*

$$C \approx \left[\int_Y \left(\int_{\mu(B(z)) \leq \mu(B(y))} R\left(\frac{1}{\underline{u}}\right) \circ \mu \circ B(y)\, d\tau(y)\right)^{\frac{q}{1-q}} d\tau(z)\right]^{\frac{1-q}{q}},\ \textit{for } q \in (0,1),$$

*and*

$$C = \sup_{s \in U} \left(\frac{1}{\underline{u}}(s)\right) \tau(\{y \in Y : s \in B(y)\})^{1/q},\ \textit{for } q \in [1,\infty).$$

*Where the greatest core decreasing minorant is taken with respect to the core $\mathcal{A} = \{\emptyset\} \cup \{B(y) : y \in Y\}$ and $R$ is the map from Proposition 2.4.6.*

**Proof:** Suppose that $q \in (0,1)$, then by Lemma 5.2.2 and Theorem 1.7.2 (Theorem 3.1 of [11]) the best constant is equivalent to

$$\left(\int_{[0,\infty)} \left(\int_{[0,x]} \frac{1}{\underline{w}}\, d\nu\right)^{\frac{q}{1-q}} d\nu(x)\right)^{1/q},$$

where $w = R(\underline{u})$ and $\nu$ is the push-forward measure for the map $\varphi(y) = \mu \circ B(y)$. Notice that $\underline{w} = w$, and it follows from item (11) of Proposition 2.4.6, that

$$1 = R(1) = R\left(\frac{\underline{u}}{\underline{u}}\right) = R(\underline{u}) R\left(\frac{1}{\underline{u}}\right)$$

Hence,

$$\int\limits_{[0,x]} \frac{1}{\underline{w}}\, d\nu = \int_{[0,\infty)} R\left(\frac{1}{\underline{u}}\right)\chi_{[0,x]}\, d\nu = \int_Y R\left(\frac{1}{\underline{u}}\right) \circ \varphi(y)\chi_{[0,x]} \circ \varphi(y)\, d\tau(y)$$
$$= \int\limits_{\varphi(y)\le x} R\left(\frac{1}{\underline{u}}\right) \circ \varphi(y)\, d\tau(y).$$

Thus

$$\int\limits_{[0,\infty)} \Bigg( \int\limits_{[0,x]} \frac{1}{\underline{w}}\, d\nu\Bigg)^{\frac{q}{1-q}} d\nu(x) = \int_Y \Bigg( \int\limits_{\varphi(y)\le\varphi(z)} R\left(\frac{1}{\underline{u}}\right) \circ \varphi(y)\, d\tau(y)\Bigg)^{\frac{q}{1-q}} d\tau(z)$$

and completes the proof for the case $q \in (0, 1)$.

The case $q \in [1, \infty)$ follows directly from duality and is included for completeness.

By Proposition 5.2.1 the best constant in inequality (5.5) is the norm of the integral operator $Kf(y) = \int_U k(y, s)f(s)\, d\theta(s)$ acting from $L^1_\theta \to L^q_\tau$ where $d\theta = \underline{u}d\mu$ and $k(y, s) = \frac{1}{\underline{u}(s)}\chi_{B(y)}(s)$. By duality, it is the best constant in the inequality

$$\left\| \int\limits_Y k(y, \cdot)h(y)\, d\tau(y) \right\|_{L^\infty_\theta} \le C\left(\int_Y h^{q'}\, d\tau\right)^{\frac{1}{q'}}, \forall h \in L^+_\tau.$$

Define $\psi_s(y) = 1$ if $s \in B(y)$ and $\psi_s(y) = 0$ otherwise. Divide both sides of the equation by $\|h\|_{L^{q'}_\tau}$ to get

$$\sup \left\{ \frac{1}{\underline{u}(s)} \int\limits_Y \psi_s(y)\frac{h(y)}{\|h\|_{L^{q'}_\tau}}\, d\tau(y) : s \in U \right\} \le C.$$

Taking supremum over non-zero positive functions $h$ yields

$$\sup_{s\in U} \frac{1}{\underline{u}(s)}\|\psi_s\|_{L^q_\tau} \le C,$$

which is the same as

$$C \ge \sup_{s\in U}\left(\frac{1}{\underline{u}}(s)\right)\tau\left(\{y \in Y : s \in B(y)\}\right)^{1/q}.$$

For the reverse inequality, an application of Minkowski's integral inequality yields

$$\left(\int\limits_Y \left(\int_U k(s, y)f(s)\, d\theta(s)\right)^q d\tau(y)\right)^{1/q} \le \left(\int\limits_U \left(\int_Y \psi_s(y)\, d\tau(y)\right)^{1/q} \frac{f(s)}{\underline{u}(s)}\, d\theta(s)\right)$$
$$\le \sup_{s\in U}\left(\frac{1}{\underline{u}}(s)\right)\tau\left(\{y \in Y : s \in B(y)\}\right)^{1/q} \int_U f(s)\, d\theta(s)$$

hence $C \le \sup_{s\in U}\left(\frac{1}{\underline{u}}(s)\right)\tau\left(\{y \in Y : s \in B(y)\}\right)^{1/q}$ and proves the statement for $q \in (1, \infty)$.

∎

## 5.3 Hardy inequalities in metric measure spaces

In this section, we show that the framework of abstract Hardy inequalities can be used to give different proofs to [21, Theorem 2.1 Condition $\mathcal{D}_1$], [22, Theorem 2.1] and [20, Theorem 3.1]. These theorems give necessary and sufficient conditions for Hardy inequalities to hold in metric measure spaces; they cover three cases depending on the indices $p$ and $q$, provided the existence of a locally integrable function $\lambda \in L^1_{\text{loc}}$ such that for all $f \in L^1(\mathbb{X})$ the following *polar decomposition* at $a \in \mathbb{X}$ holds:

$$\int_{\mathbb{X}} f d\mu = \int_0^\infty \int_{\Sigma_r} f(r,\omega)\lambda(r,\omega)\, d\omega_r dr,$$

for a family of measures $d\omega_r$, where $\Sigma_r = \{x \in \mathbb{X} : d(x,a) = r\}$.

Our new proofs show that the polar decomposition hypothesis is not required so the results hold in all metric measure spaces.

We also give the corresponding results regarding the conjugate Hardy inequality discussed in [21, Theorem 2.2 Condition $\mathcal{D}_1^*$] and [20, Theorem 3.2].

We begin with the case $p > 1$, extending [21, Theorem 2.1 Condition $\mathcal{D}_1$], [22, Theorem 2.1] to all metric measure spaces.

**Theorem 5.3.1** *Let $\mu$ be a $\sigma$-finite measure on a metric measure space $\mathbb{X}$. Fix $a \in \mathbb{X}$ and let $p \in (1,\infty)$, $q > 0$, $q \neq 1$ and $\omega, v$ be measurable functions, positive $\mu$-almost everywhere such that $\omega$ is integrable over $\mathbb{X} \setminus B[a;|x|_a]$ and $v^{1-p'}$ is integrable over $B[a;|x|_a]$ for all $x \in \mathbb{X}$. Then the Hardy inequality*

$$\left(\int_{\mathbb{X}}\Big(\int_{B[a;|x|_a]} f(y)\, d\mu(y)\Big)^q \omega(x)\, d\mu(x)\right)^{\frac{1}{q}} \le C\Big(\int_{\mathbb{X}} f(x)^p v(x)\, d\mu(x)\Big)^{\frac{1}{p}},\ \forall f \in L^+_\mu \tag{5.6}$$

*holds if and only if $p \le q$ and:*

$$\sup_{x\neq a}\left\{\Big(\int_{\mathbb{X}\setminus B[a;|x|_a]} \omega\, d\mu\Big)^{\frac{1}{q}}\Big(\int_{B[a;|x|_a]} v^{1-p'}\, d\mu\Big)^{\frac{1}{p'}}\right\} < \infty,$$

*$0 < q < 1 < p$ and*

$$\int_{\mathbb{X}}\Big(\int_{\mathbb{X}\setminus B[a;|x|_a]} \omega\, d\mu\Big)^{\frac{r}{p}}\Big(\int_{B[a;|x|_a]} v^{1-p'}\, d\mu\Big)^{\frac{r}{p'}} u(s)\, d\mu(s) < \infty,$$

*or $1 < q < p$ and*

$$\int_{\mathbb{X}}\Big(\int_{\mathbb{X}\setminus B[a;|x|_a]} \omega\, d\mu\Big)^{\frac{r}{q}}\Big(\int_{B[a;|x|_a]} v^{1-p'}\, d\mu\Big)^{\frac{r}{q'}} v^{1-p'}(s)\, d\mu(s) < \infty.$$

*Here $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$.*

**Proof:** By hypothesis $v > 0$ and $v < \infty$ $\mu$-almost everywhere, then the mapping $f \mapsto v^{1-p'} f$ is a bijection on $L^+_\mu$. Then, the inequality (5.6) is equivalent to

$$\left(\int_{\mathbb{X}}\Big(\int_{B[a;|x|_a]} f(y)v^{1-p'}(y)\,d\mu(y)\Big)^q \omega(x)\,d\mu(x)\right)^{\frac{1}{q}} \le C\Big(\int_{\mathbb{X}} f(x)^p v^{1-p'}(x)\,d\mu(x)\Big)^{\frac{1}{p}},\ \forall f \in L^+_\mu \quad (5.7)$$

Above we have used the identity $v^{p(1-p')}v = v^{1-p'}$. Let $d\tau = v^{1-p'}d\mu$ and define the map $B : \mathbb{X} \to \Sigma$ by

$$B(x) = B[a; |y|_a].$$

The image of $B$ is a totally ordered set. By hypothesis $\tau(B(x)) < \infty$ for each $x \in \mathbb{X}$. Therefore, it is an ordered core with respect to the measure $\tau$. We get the equivalent abstract Hardy inequality

$$\left(\int_{\mathbb{X}}\Big(\int_{B[a;|y|_a]} f\,d\tau\Big)^q \omega(y)\,d\mu(y)\right)^{\frac{1}{q}} \le C\Big(\int_{\mathbb{X}} f^p\,d\tau\Big)^{\frac{1}{p}},\ \forall f \in L^+_\mu.$$

Let $\lambda$ be the measure on $[0,\infty)$ induced by the core, so that for every $M$ in the core

$$\int_{[0,x]} Rf\,d\lambda = \int_M f v^{1-p'}\,d\mu,\ \text{where } x = \int_M v^{1-p'}\,d\mu.$$

We claim that inequality (5.7) is equivalent to the Hardy inequality

$$\left(\int_{[0,\infty)}\Big(\int_{[0,y]} g\,d\lambda\Big)^q R\Big(\frac{\omega}{v^{1-p'}}\Big)\,d\lambda(y)\right)^{\frac{1}{q}} \le C\Big(\int_{[0,\infty)} g^p d\lambda\Big)^{\frac{1}{p}}, \quad \forall g \in L^+_\lambda. \quad (5.8)$$

By Theorem 1.7.5, to complete the proof, it suffices to show that the maps

$$b_1(s) = \int_{B[a;|s|_a]} v^{1-p'}\,d\mu \quad \text{and} \quad b_2(x) = \lambda([0,x])$$

have the same distribution functions with respect to the measures $\omega\,d\mu$ and $R\left(\frac{\omega}{v^{1-p'}}\right)d\lambda$ respectively.

Fix $t > 0$ and consider the sets $E_1 = b_1^{-1}(t,\infty)$ and $E_2 = b_2^{-1}(t,\infty)$, we give a characterization for these sets.

Define the set $W$ as follows

$$W = \bigcup\left\{B[a;|s|_a] : \int_{B[a;|s|_a]} v^{1-p'}\,d\mu \le t\right\}.$$

If $z \in E_1$, then $b_1(z) > t$, thus $z \notin W$, conversely if $z \in W$ then $b_1(z) \le t$, therefore $z \notin E_1$. Hence $W^c = E_1$. Since $W$ is a union of closed balls centered at $a$, then there exists a sequence $s_n$ such that $B[a;|s_n|_a] \uparrow W$. Let $t_n = \int_{B[a;|s_n|_a]} v^{1-p'}\,d\mu$.

Let $\widetilde{t}$ be defined as

$$\widetilde{t} = \sup\left\{ z \le t : z = \int_{B[a;|s|_a]} v^{1-p'}\, d\mu \text{ for some } s \in \mathbb{X} \right\},$$

hence $\widetilde{t} = \lambda[0, t]$.

Therefore,

$$\begin{aligned}
\int_{E_1^c} \omega\, d\mu &= \sup_{n\in\mathbb{N}} \int_{B[a;|s_n|_a]} \omega\, d\mu \text{ by the monotone convergence theorem} \\
&= \sup_{n\in\mathbb{N}} \int_{[0,t_n]} R\left(\frac{\omega}{v^{1-p'}}\right) d\lambda \text{ by the action of } R \\
&= \int_{[0,t]} R\left(\frac{\omega}{v^{1-p'}}\right) d\lambda \text{ by monotone convergence theorem} \\
&= \int_{E_2^c} R\left(\frac{\omega}{v^{1-p'}}\right) d\lambda.
\end{aligned}$$

By hypothesis $\int_{E_1^c} \omega\, d\mu < \infty$, we have that

$$\int_{b_1^{-1}(t,\infty)} \omega\, d\mu = \int_{b_2^{-1}(t,\infty)} R\left(\frac{\omega}{v^{1-p'}}\right) d\lambda.$$

It follows that the distribution functions coincide which proves that the Hardy inequalities (5.7) and (5.8) have the same normal form parameter, therefore, they are equivalent by Theorem 1.7.5.

For all the index cases, we can apply Theorem 1.7.1. In the case $1 < p \le q < \infty$, the inequality (5.8) holds if and only if

$$\sup_x \left( \int_{[x,\infty)} R\left(\frac{\omega}{v^{1-p'}}\right) d\lambda(t) \right)^{\frac{1}{q}} \left( \int_{[0,x]} d\lambda \right)^{\frac{1}{p'}} < \infty$$

which is equivalent to

$$\sup_{s\neq a} \left( \int_{\mathbb{X}\setminus B[a;|s|_a]} \omega\, d\mu \right)^{\frac{1}{q}} \left( \int_{B[a;|s|_a]} v^{1-p'}\, d\mu \right)^{\frac{1}{p'}} < \infty.$$

In the case $0 < q < 1 < p < \infty$, the inequality (5.8) holds if and only if

$$\int_{[0,\infty)} \left( \int_{[x,\infty)} R\left(\frac{\omega}{v^{1-p'}}\right) d\lambda \right)^{\frac{r}{p}} \left( \int_{[0,x]} d\lambda \right)^{\frac{r}{p'}} R\left(\frac{\omega}{v^{1-p'}}\right) d\lambda(x) < \infty$$

which is equivalent to

$$\int\limits_{\mathbb{X}} \Big( \int\limits_{\mathbb{X}\setminus B[a;|s|_a]} \omega \, d\mu \Big)^{\frac{r}{p}} \Big( \int\limits_{B[a;|s|_a]} v^{1-p'} \, d\mu \Big)^{\frac{r}{p'}} \omega(s) \, d\mu(s) < \infty.$$

In the case $1 < q < p$ we have that inequality (5.8) holds if and only if

$$\int\limits_{[0,\infty)} \Big( \int\limits_{[x,\infty)} R\Big(\frac{\omega}{v^{1-p'}}\Big) \, d\lambda \Big)^{\frac{r}{q}} \Big( \int\limits_{[0,x]} d\lambda \Big)^{\frac{r}{q'}} d\lambda(x) < \infty$$

which is equivalent to

$$\int\limits_{\mathbb{X}} \Big( \int\limits_{\mathbb{X}\setminus B[a;|s|_a]} \omega \, d\mu \Big)^{\frac{r}{q}} \Big( \int\limits_{B[a;|s|_a]} v^{1-p'} \, d\mu \Big)^{\frac{r}{q'}} v^{1-p'} \, d\mu(s) < \infty$$

completing the proof.

$\blacksquare$

We also have a corresponding result to the conjugate Hardy inequality, extending [21, Theorem 2.1 Condition $\mathcal{D}_1$].

**Theorem 5.3.2** *Let $\mu$ be a $\sigma$-finite measure on a metric measure space $\mathbb{X}$. Fix $a \in \mathbb{X}$ and let $p \in (1,\infty)$, $q > 0$, $q \neq 1$ and $\omega, v$ be measurable functions, positive $\mu$-almost everywhere satisfying that $v^{1-p'}$ is integrable over $\mathbb{X} \setminus B[a;|x|_a]$ for each $x \in \mathbb{X}$ and $\omega \in L^1_{Loc}(\mathbb{X})$. Then, the Hardy inequality*

$$\Bigg( \int\limits_{\mathbb{X}} \Big( \int\limits_{\mathbb{X}\setminus B[a;|x|_a]} f(y) \, d\mu(y) \Big)^{q} \omega(x) \, d\mu(x) \Bigg)^{\frac{1}{q}} \leq C \Big( \int\limits_{\mathbb{X}} f(x)^p v(x) \, d\mu(x) \Big)^{\frac{1}{p}}, \ \forall f \in L^+_\mu$$

*holds if and only if $p \leq q$ and:*

$$\sup_{x \neq a} \left\{ \Big( \int\limits_{B[a;|x|_a]} \omega \, d\mu \Big)^{\frac{1}{q}} \Big( \int\limits_{\mathbb{X}\setminus B[a;|x|_a]} v^{1-p'} \, d\mu \Big)^{\frac{1}{p'}} \right\} < \infty,$$

*$0 < q < 1 < p$ and*

$$\int\limits_{\mathbb{X}} \Big( \int\limits_{B[a;|x|_a]} \omega \, d\mu \Big)^{\frac{r}{p}} \Big( \int\limits_{\mathbb{X}\setminus B[a;|x|_a]} v^{1-p'} \, d\mu \Big)^{\frac{r}{p'}} u(s) \, d\mu(s) < \infty,$$

*or $1 < q < p$ and*

$$\int\limits_{\mathbb{X}} \Big( \int\limits_{B[a;|x|_a]} \omega \, d\mu \Big)^{\frac{r}{q}} \Big( \int\limits_{\mathbb{X}\setminus B[a;|x|_a]} v^{1-p'} \, d\mu \Big)^{\frac{r}{q'}} v^{1-p'}(s) \, d\mu(s) < \infty.$$

*Here $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$.*

**Proof:** We only sketch the proof as most details follow the same argument as Theorem 5.3.1.

Let $d\tau = v^{1-p'}\,d\mu$. Observe that the hypothesis on $v$ guarantees that, for each $y \in \mathbb{X}$, the sets $\mathbb{X} \setminus B[a; |y|_a]$ have finite $\tau$ measure. Thus the map $B(y) = \mathbb{X} \setminus B[a; |y|_a]$ is a core map.

Then the Lebesgue decomposition theorem and the substitution $f \mapsto v^{1-p'}$ provides the equivalent abstract Hardy inequality

$$\Bigg(\int_{\mathbb{X}}\Bigg(\int_{\mathbb{X}\setminus B[a;|y|_a]} f\,d\tau\Bigg)^q \omega(y)\,d\mu(y)\Bigg)^{\frac{1}{q}} \le C\Bigg(\int_{\mathbb{X}} f^p\,d\tau\Bigg)^{\frac{1}{p}},\ \forall f \in L^+_\mu.$$

By definition of $\tau$ this is equivalent to

$$\Bigg(\int_{\mathbb{X}}\Bigg(\int_{\mathbb{X}\setminus B[a;|y|_a]} f v^{1-p'}\,d\mu\Bigg)^q \omega(y)\,d\mu(y)\Bigg)^{\frac{1}{q}} \le C\Bigg(\int_{\mathbb{X}} f^p v^{1-p'}\,d\mu\Bigg)^{\frac{1}{p}},\ \forall f \in L^+_\mu. \tag{5.9}$$

Let $\lambda$ be the measure on $[0,\infty)$ induced by the core, so that for every $y \in \mathbb{X}$:

$$\int_{[0,x]} Rf\,d\lambda = \int_{\mathbb{X}\setminus B[a;|y|_a]} f v^{1-p'}\,d\mu,\ \text{where } x = \int_{\mathbb{X}\setminus B[a;|y|_a]} v^{1-p'}\,d\mu.$$

The maps

$$b_1(s) = \int_{\mathbb{X}\setminus B[a;|s|_a]} v^{1-p'}\,d\mu \quad \text{and} \quad b_2(x) = \lambda([0,x])$$

have the same distribution functions with respect to the measures $\omega\,d\mu$ and $R\left(\frac{\omega}{v^{1-p'}}\right)d\lambda$ respectively. Then, by Theorem 1.7.5, we get that the inequality (5.9) is equivalent to the Hardy inequality

$$\Bigg(\int_{[0,\infty)}\Bigg(\int_{[0,y]} g\,d\lambda\Bigg)^q R\left(\frac{\omega}{v^{1-p'}}\right) d\lambda(y)\Bigg)^{\frac{1}{q}} \le C\Bigg(\int_{[0,\infty)} g^p d\lambda\Bigg)^{\frac{1}{p}}, \quad \forall g \in L^+_\lambda. \tag{5.10}$$

For all the index cases, we can apply Theorem 1.7.1. In the case $1 < p \le q < \infty$, the inequality (5.10) holds if and only if

$$\sup_x \Bigg(\int_{[x,\infty)} R\left(\frac{\omega}{v^{1-p'}}\right) d\lambda(t)\Bigg)^{\frac{1}{q}}\Bigg(\int_{[0,x]} d\lambda\Bigg)^{\frac{1}{p'}} < \infty$$

which is equivalent to

$$\sup_{s\neq a}\Bigg(\int_{B[a;|s|_a]} \omega\,d\mu\Bigg)^{\frac{1}{q}}\Bigg(\int_{\mathbb{X}\setminus B[a;|s|_a]} v^{1-p'}\,d\mu\Bigg)^{\frac{1}{p'}} < \infty.$$

In the case $0 < q < 1 < p < \infty$, the inequality (5.10) holds if and only if

$$\int_{[0,\infty)}\Bigg(\int_{[x,\infty)} R\left(\frac{\omega}{v^{1-p'}}\right) d\lambda\Bigg)^{\frac{r}{p}}\Bigg(\int_{[0,x]} d\lambda\Bigg)^{\frac{r}{p'}} R\left(\frac{\omega}{v^{1-p'}}\right) d\lambda(x) < \infty$$

which is equivalent to

$$\int_{\mathbb{X}}\Big(\int_{B[a;|s|_a]}\omega\,d\mu\Big)^{\frac{r}{p}}\Big(\int_{\mathbb{X}\setminus B[a;|s|_a]}v^{1-p'}\,d\mu\Big)^{\frac{r}{p'}}\omega(s)\,d\mu(s)<\infty.$$

In the case $1<q<p$ we have that inequality (5.10) holds if and only if

$$\int_{[0,\infty)}\Big(\int_{[x,\infty)}R\Big(\frac{\omega}{v^{1-p'}}\Big)\,d\lambda\Big)^{\frac{r}{q}}\Big(\int_{[0,x]}d\lambda\Big)^{\frac{r}{q'}}\,d\lambda(x)<\infty$$

which is equivalent to

$$\int_{\mathbb{X}}\Big(\int_{B[a;|s|_a]}\omega\,d\mu\Big)^{\frac{r}{q}}\Big(\int_{\mathbb{X}\setminus B[a;|s|_a]}v^{1-p'}\,d\mu\Big)^{\frac{r}{q'}}v^{1-p'}\,d\mu(s)<\infty$$

completing the proof.

∎

For the case $p=1$, Theorem 5.2.3 implies the following characterization.

**Corollary 5.3.3** *Let $\mu$ be a $\sigma$-finite measure on a metric measure space $\mathbb{X}$. Fix $a\in\mathbb{X}$, let $q\in(0,\infty)$ and $\omega, v$ be measurable functions, positive $\mu$-almost everywhere satisfying that $\omega$ is integrable over $\mathbb{X}\setminus B[a;|x|_a]$ for each $x\in\mathbb{X}$ and $v^{1-p'}\in L^1_{Loc}(\mathbb{X})$. Then the best constant in the Hardy inequality*

$$\left(\int_{\mathbb{X}}\Big(\int_{B[a;|x|_a]}f(y)\,d\mu(y)\Big)^{q}\omega(x)\,d\mu(x)\right)^{\frac{1}{q}}\leq C\int_{\mathbb{X}}f(x)v(x)\,d\mu(x),\ \forall f\in L^+_\mu$$

*satisfies*

$$C\approx\left(\int_{\mathbb{X}}\Big(\int_{z\leq_{\mathcal{A}}x}\frac{1}{\underline{v}}(x)\omega(x)\,d\mu(x)\Big)^{\frac{q}{1-q}}\omega(z)\,d\mu(z)\right)^{\frac{1-q}{q}},\ \textit{for } q\in(0,1),$$

*and*

$$C=\sup_{x\in X}\left(\frac{1}{\underline{v}}(x)\right)\Big(\int_{x\leq_{\mathcal{A}}t}\omega(t)\,d\mu(t)\Big)^{1/q},\ \textit{for } q\in[1,\infty).$$

*Where* $\underline{v}(x)=\operatorname{ess\,inf}_\mu\{v(t):t\in B[a;|x|_a]\}$, *$x\leq_{\mathcal{A}}t$ means $B[a;|x|_a]\subseteq B(a,|t|)$ and $B[a;|x|_a]=\{z\in\mathbb{X}: dist(a,z)\leq dist(a,x)\}$.*

**Proof:** Let $\mathcal{A}=\{\emptyset\}\cup\{B[a;|x|_a]\}_{x\in\mathbb{X}}$, it is the full ordered core induced by the core map $x\to B[a;|x|_a]$. Let $d\tau=\omega d\mu$, $d\eta=v d\mu$ and $\lambda$ be the measure on $[0,\infty)$ induced by the ordered core.

Consider the function $\varphi:\mathbb{X}\to[0,\infty)$ defined by $\varphi(x)=\mu(B[a;|x|_a])$ and let $\nu$ be the pushforward measure. Then, if $y=\varphi(x)$ we have

$$\nu([0,y])=\mu(\varphi^{-1}([0,y]))=\int_{\varphi(t)\leq y}d\mu(t)=\int_{B[a;|x|_a]}d\mu=\lambda([0,\varphi(x)])=\lambda([0,y]).$$

It follows that the Borel measures $\nu$ and $\lambda$ coincide and are finite over $[0, y]$ for all $y > 0$, therefore $\lambda$ is the pushforward measure of $\varphi$.

We now show that $R\left(\frac{1}{\underline{v}}\right) = \frac{1}{\underline{v}} \circ \varphi$ up to a set of $\mu$-measure zero.

Indeed

$$\int\limits_{B[a;|x|_a]} \frac{1}{\underline{v}}\, d\mu = \int\limits_{\varphi(t)\le\varphi(x)} \frac{1}{\underline{v}}(t)\, d\mu(t) = \int\limits_{[0,\varphi(x)]} R\left(\frac{1}{\underline{v}}\right)(t)\, d\lambda(t) = \int\limits_{[0,\infty)} R\left(\frac{1}{\underline{v}}\right)(t)\chi_{[0,\varphi(x)]}(t)\, d\lambda(t)$$

$$= \int\limits_{\mathbb{X}} R\left(\frac{1}{\underline{v}}\right) \circ \varphi(t)\chi_{[0,\varphi(x)]} \circ \varphi(t)\, d\mu(t) = \int\limits_{\varphi(t)\le\varphi(x)} R\left(\frac{1}{\underline{v}}\right) \circ \varphi(t)\, d\mu(t)$$

$$= \int\limits_{B[a;|x|_a]} R\left(\frac{1}{\underline{v}}\right) \circ \varphi(t)\, d\mu(t).$$

Since the equality holds for all core sets, then $R\left(\frac{1}{\underline{v}}\right) = \frac{1}{\underline{v}} \circ \varphi$ almost everywhere.

Then for $q \in (0, 1)$, Theorem 5.2.3 yields

$$C \approx \left(\int\limits_{\mathbb{X}} \left(\int\limits_{\varphi(z)\le\varphi(x)} R\left(\frac{1}{\underline{v}}\right) \circ \varphi(x)\omega(x)\, d\mu\right)^{\frac{q}{1-q}} \omega(z)\, d\mu(z)\right)^{\frac{1-q}{q}}$$

$$\approx \left(\int\limits_{\mathbb{X}} \left(\int\limits_{z\le_{\mathcal{A}} x} \frac{1}{\underline{v}}(x)\omega(x)\, d\mu\right)^{\frac{q}{1-q}} \omega(z)\, d\mu(z)\right)^{\frac{1-q}{q}}.$$

The statement for $q \in [1, \infty)$ follows directly from Theorem 5.2.3. The description of $\underline{v}$ was given in Example 3.3.5. This completes the proof.

■

Our result regarding the conjugate Hardy inequality to Corollary 5.3.1 needs an adjustment. Since for a metric measure space $\mathbb{X}$, the sets $(\mathbb{X} \setminus B[a; |x|_a])$ may have infinite measure, the collection $\{\mathbb{X} \setminus B[a; |x|_a]\}_{x\in\mathbb{X}}$ may fail to be an ordered core. This obstruction is addressed in the following lemma.

**Lemma 5.3.4** *Let $\mu$ be a $\sigma$-finite measure on a metric measure space $\mathbb{X}$. Fix $a \in \mathbb{X}$. Let $\{\mathbb{X}_n\}$ be a sequence of sets of finite $\mu$-measure such that $a \in \mathbb{X}_n \uparrow \mathbb{X}$, $q \in (0, \infty)$ and $\omega, v$ be measurable functions, positive $\mu$-almost everywhere satisfying that $v^{1-p'}$ is integrable over $\mathbb{X} \setminus B[a; |x|_a]$ for each $x \in \mathbb{X}$ and $\omega \in L^1_{Loc}(\mathbb{X})$.*

*For each $n \in \mathbb{N}$. Let $C_n$ be the best constant in the inequality*

$$\left(\int\limits_{\mathbb{X}_n} \left(\int\limits_{\mathbb{X}_n\setminus B[a;|x|_a]} f(y)\, d\mu(y)\right)^q \omega(x)\, d\mu(x)\right)^{\frac{1}{q}} \le C_n \int\limits_{\mathbb{X}_n} f(x)v(x)\, d\mu(x),\ \forall f \in L^+_\mu \tag{5.11}$$

*and $C$ be the best constant in the inequality*

$$\left(\int\limits_{\mathbb{X}} \left(\int\limits_{\mathbb{X}\setminus B[a;|x|_a]} f(y)\, d\mu(y)\right)^q \omega(x)\, d\mu(x)\right)^{\frac{1}{q}} \le C \int\limits_{\mathbb{X}} f(x)v(x)\, d\mu(x),\ \forall f \in L^+_\mu, \tag{5.12}$$

*where* $B[a;|x|_a] = \{z \in \mathbb{X} : dist(a,z) \leq dist(a,x)\}$.

*Then,*

$$C = \sup_{n\in\mathbb{N}} C_n.$$

**Proof:**

Fix $f \in L^+_\mu$, then an application of inequality (5.12) yields

$$\Bigg(\int\limits_{\mathbb{X}_n}\Big(\int\limits_{\mathbb{X}_n\setminus B[a;|x|_a]} f(y)\,d\mu(y)\Big)^q \omega(x)\,d\mu(x)\Bigg)^{\frac{1}{q}} \leq \Bigg(\int\limits_{\mathbb{X}}\Big(\int\limits_{\mathbb{X}\setminus B[a;|x|_a]} f(y)\chi_{\mathbb{X}_n}(y)\,d\mu(y)\Big)^q \omega(x)\,d\mu(x)\Bigg)^{\frac{1}{q}}$$
$$\leq C\int\limits_{\mathbb{X}} f(y)\chi_{\mathbb{X}_n}(y)\,v(y)d\mu(y) = C\int\limits_{\mathbb{X}_n} f(y)\,v(y)d\mu(y).$$

Thus $\sup_{n\in\mathbb{N}} C_n \leq C$.

Conversely, the monotone convergence theorem together with equation (5.11) yields

$$\Bigg(\int\limits_{\mathbb{X}}\Big(\int\limits_{\mathbb{X}\setminus B[a;|x|_a]} f(y)\,d\mu(y)\Big)^q \omega(x)\,d\mu(x)\Bigg)^{\frac{1}{q}} = \sup_n\Bigg(\int\limits_{\mathbb{X}_n}\Big(\int\limits_{\mathbb{X}\setminus B[a;|x|_a]} f(y)\chi_{\mathbb{X}_n}(y)\,d\mu(y)\Big)^q \omega(x)\,d\mu(x)\Bigg)^{\frac{1}{q}}$$
$$= \sup_n\Bigg(\int\limits_{\mathbb{X}_n}\Big(\int\limits_{\mathbb{X}_n\setminus B[a;|x|_a]} f(y)\,d\mu(y)\Big)^q \omega(x)\,d\mu(x)\Bigg)^{\frac{1}{q}}$$
$$\leq \sup_n C_n\int\limits_{\mathbb{X}_n} f(x)v(x)\,d\mu(x) \leq (\sup_n C_n)\int\limits_{\mathbb{X}} f(x)v(x)\,d\mu(x).$$

Division by $\int\limits_{\mathbb{X}} f(x)v(x)\,d\mu(x)$ and taking supremum over $f$ yields $C \leq \sup_{n\in\mathbb{N}} C_n$ and completes the proof.

■

We are ready to state our result for the conjugate Hardy inequality with $p = 1$.

**Corollary 5.3.5** *Let $\mu$ be a $\sigma$-finite measure on a metric measure space $\mathbb{X}$. Fix $a \in \mathbb{X}$, let $q \in (0,\infty)$ and $\omega, v$ be measurable functions, positive $\mu$-almost everywhere satisfying $v^{1-p'}$ is integrable over $\mathbb{X} \setminus B[a;|x|_a]$ for each $x \in \mathbb{X}$ and $\omega \in L^1_{Loc}(\mathbb{X})$. Then the best constant in the Hardy inequality*

$$\Bigg(\int\limits_{\mathbb{X}}\Big(\int\limits_{\mathbb{X}\setminus B[a;|x|_a]} f(y)\,d\mu(y)\Big)^q \omega(x)\,d\mu(x)\Bigg)^{\frac{1}{q}} \leq C\int\limits_{\mathbb{X}} f(x)v(x)\,d\mu(x),\ \forall f \in L^+_\mu$$

*satisfies*

$$C \approx \Bigg(\int\limits_{\mathbb{X}}\Big(\int\limits_{x\leq_{\mathcal{A}} z} \frac{1}{\underline{v}}(x)\omega(x)\,d\mu(x)\Big)^{\frac{q}{1-q}} \omega(z)\,d\mu(z)\Bigg)^{\frac{1-q}{q}},\ \textit{for}\ q \in (0,1),$$

*and*

$$C = \sup_{x\in X}\left(\frac{1}{\underline{v}}(x)\right)\Big(\int_{t\leq_{\mathcal{A}}x} \omega(t)\,d\mu(t)\Big)^{1/q}, \textit{ for } q\in[1,\infty).$$

*Where* $\underline{v}(x) = \operatorname{ess\,inf}_\mu\{v(t) : t\notin B[a;|x|_a]\}$, $x\leq_{\mathcal{A}} t$ *means* $B[a;|x|_a]\subseteq B_{a,|t|}$ *and* $B[a;|x|_a] = \{z\in\mathbb{X} : dist(a,z)\leq dist(a,x)\}$.

**Proof:** For each $n\in\mathbb{N}^+$ define $\mathbb{X}_n = \{x\in\mathbb{X} : \operatorname{dist}(a,x)\leq n\}$. Let $C_n$ be the best constant in the inequality (5.11). Let $\mathcal{A}_n = \{\emptyset\}\cup\{\mathbb{X}_n B[a;|x|_a]\}_{x\in\mathbb{X}}$, it is the full ordered core over $\mathbb{X}_n$ induced by the core map $x\to(\mathbb{X}_n\setminus B[a;|x|_a])$. Let $d\tau=\omega d\mu$, $d\eta = vd\mu$ and $\lambda_n$ be the measure on $[0,\infty)$ induced by the ordered core. Notice that $\lambda_n$ is supported on the compact interval $[0,\mu(\mathbb{X}_n)]$.

Consider the function $\varphi_n:\mathbb{X}\to[0,\infty)$ defined by $\varphi_n(x) = \mu(\mathbb{X}_n\setminus B[a;|x|_a])$ and let $\nu_n$ be the pushforward measure.

Then, if $y=\varphi_n(x)$ we have

$$\nu_n([0,y]) = \mu(\varphi_n^{-1}([0,y])) = \int_{\varphi_n(t)\leq y} d\mu(t) = \int_{\mathbb{X}_n\setminus B[a;|x|_a]} d\mu = \lambda([0,\varphi(x)]) = \lambda([0,y]).$$

It follows that the Borel measures $\nu_n$ and $\lambda_n$ coincide and are finite over $[0,y]$ for all $y>0$, therefore $\lambda_n$ is the pushforward measure of $\varphi_n$.

We now show that $R_n\Big(\frac{1}{\underline{v_n}}\Big) = \frac{1}{\underline{v_n}}\circ\varphi$ up to a set of $\mu$-measure zero, here $R_n$ is the transition map between $\mu$ and $\lambda_n$ and $\underline{v_n}$ is the greatest core decreasing minorant of $v$ relative to the core $\mathcal{A}_n$.

Indeed

$$\begin{aligned}\int_{\mathbb{X}_n\setminus B[a;|x|_a]} \frac{1}{\underline{v_n}}\,d\mu &= \int_{\varphi_n(t)\leq\varphi_n(x)} \frac{1}{\underline{v_n}}(t)\,d\mu(t) = \int_{[0,\varphi_n(x)]} R_n\Big(\frac{1}{\underline{v_n}}\Big)(t)\,d\lambda(t)\\ &= \int_{[0,\infty)} R_n\Big(\frac{1}{\underline{v_n}}\Big)(t)\chi_{[0,\varphi_n(x)]}(t)\,d\lambda(t)\\ &= \int_{\mathbb{X}} R_n\Big(\frac{1}{\underline{v_n}}\Big)\circ\varphi_n(t)\chi_{[0,\varphi(x)]}\circ\varphi_n(t)\,d\mu(t) = \int_{\varphi_n(t)\leq\varphi_n(x)} R_n\Big(\frac{1}{\underline{v_n}}\Big)\circ\varphi_n(t)\,d\mu(t)\\ &= \int_{\mathbb{X}_n\setminus B[a;|x|_a]} R_n\Big(\frac{1}{\underline{v_n}}\Big)\circ\varphi_n(t)\,d\mu(t).\end{aligned}$$

Since the equality holds for all core sets, then $R_n\Big(\frac{1}{\underline{v_n}}\Big) = \frac{1}{\underline{v_n}}\circ\varphi_n$ almost everywhere.

Then for $q\in(0,1)$, Theorem 5.2.3 yields

$$\begin{aligned}C_n &\approx \Bigg(\int_{\mathbb{X}_n}\Big(\int_{\varphi_n(z)\leq\varphi_n(x)} R_n\left(\frac{1}{\underline{v_n}}\right)\circ\varphi_n(x)\omega(x)\,d\mu\Big)^{\frac{q}{1-q}}\omega(z)\,d\mu(z)\Bigg)^{\frac{1-q}{q}}\\ &\approx \Bigg(\int_{\mathbb{X}_n}\Big(\int_{x\leq_{\mathcal{A}}z} \frac{1}{\underline{v_n}}(x)\omega(x)\,d\mu\Big)^{\frac{q}{1-q}}\omega(z)\,d\mu(z)\Bigg)^{\frac{1-q}{q}}.\end{aligned}$$

Notice that

$$\frac{1}{\underline{v_n}}(x) = \frac{1}{\text{ess inf}_\mu \{v(t) : t \in \mathbb{X}_n \setminus B[a; |x|_a]\}} = \text{ess sup}_\mu \left\{ \frac{1}{v(t)} : t \in \mathbb{X}_n \setminus B[a; |x|_a] \right\},$$

therefore

$$\sup_n \frac{1}{\underline{v_n}}(x) = \text{ess sup}_\mu \left\{ \frac{1}{v(t)} : t \in \mathbb{X} \setminus B[a; |x|_a] \right\} = \frac{1}{\underline{v}}(x).$$

An application of Lemma 5.3.4 and the monotone convergence theorem yields

$$\begin{aligned} C &\approx \sup_{n\in\mathbb{N}} \left( \int_{\mathbb{X}_n} \left( \int_{x \leq_{\mathcal{A}} z} \frac{1}{\underline{v}}(x)\omega(x)\, d\mu \right)^{\frac{q}{1-q}} \omega(z)\, d\mu(z) \right)^{\frac{1-q}{q}} \\ &= \left( \int_{\mathbb{X}} \left( \int_{x \leq_{\mathcal{A}} z} \frac{1}{\underline{v}}(x)\omega(x)\, d\mu \right)^{\frac{q}{1-q}} \omega(z)\, d\mu(z) \right)^{\frac{1-q}{q}}. \end{aligned}$$

For $q \geq 1$ we get

$$C_n = \sup_{x\in\mathbb{X}_n} \left( \frac{1}{\underline{v_n}}(x) \right) \left( \int_{t \leq_{\mathcal{A}} x} \omega(t)\, d\mu(t) \right)^{1/q} = \sup_{x\in\mathbb{X}} \left( \frac{1}{\underline{v_n}}(x) \right) \left( \int_{t \leq_{\mathcal{A}} x} \omega(t)\, d\mu(t) \right)^{1/q} \chi_{\mathbb{X}_n}(x).$$

By Lemma 5.3.4 we get

$$\begin{aligned} C &= \sup_n C_n = \sup_n \sup_{x\in\mathbb{X}} \left( \frac{1}{\underline{v_n}}(x) \right) \left( \int_{t \leq_{\mathcal{A}} x} \omega(t)\, d\mu(t) \right)^{1/q} \chi_{\mathbb{X}_n}(x) \\ &= \sup_{x\in\mathbb{X}} \left( \frac{1}{\underline{v}}(x) \right) \left( \int_{t \leq_{\mathcal{A}} x} \omega(t)\, d\mu(t) \right)^{1/q}. \end{aligned}$$

This completes the proof.

∎

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# Appendix A

# The category of ordered cores

We define a category of ordered cores and show the functoriality properties derived from Theorem 2.3.6.
**Objects:** An object in this category consists of a 4-tuple $(U, \Sigma, \mu, \mathcal{A})$ where $U$ is a set, $\Sigma$ a $\sigma-$algebra on $U$, $\mu$ a $\sigma-$finite measure defined on $\Sigma$ and $\mathcal{A}$ a $\sigma-$bounded full ordered core.
**Morphisms:** There is a morphism $r : (U, \Sigma, \mu, \mathcal{A}) \to (T, \mathcal{T}, \tau, \mathcal{A}')$ if $r : \mathcal{A} \to \mathcal{A}'$ is a surjective order-preserving function and there exists a constant $c$ such that

$$\tau(r(B) \setminus r(A)) \leq c\mu(B \setminus A), \quad \text{for all } B, A \in \mathcal{A}.$$

The identity morphism is the identity map of ordered cores $r$. Surjectivity and monotonicity of functions are preserved by compositions, moreover, if $r_1 : (U, \Sigma, \mu, \mathcal{A}) \to (T, \mathcal{T}, \tau, \mathcal{A}')$ and $r_2 : (T, \mathcal{T}, \tau, \mathcal{A}') \to (T_2, \mathcal{T}_2, \tau_2, \mathcal{A}'')$ are morphisms with constants $c_1, c_2$ respectively, then

$$\tau_2(r_2 \circ r_1(B) \setminus r_2 \circ r_1(A)) \leq c_2\tau_1(r_1(B) \setminus r_1(A)) \leq c_2c_1\mu(B \setminus A), \quad \forall A, B \in \mathcal{A}.$$

The following proposition shows the existence of a semifunctor mapping to the category of $\mathbb{R}-$vector spaces.

**Proposition A.0.1** *There is a contravariant semifunctor $\mathcal{F}_V$ mapping to the category of $\mathbb{R}-$vector spaces, mapping $\mathcal{F}_V(U, \Sigma, \mu, \mathcal{A}) = L^1_{Loc,\mathcal{A}}$ and $r : \mathcal{A} \to \mathcal{A}'$ gets mapped to $\mathcal{F}_V(r) = R$, where $R$ is the linear operator described in Theorem 2.3.6.*

**Proof:** Existence and linearity were shown in Theorem 2.3.6. The only property remaining is the preservation of composition, that is, if $r_1 : (U, \Sigma, \mu, \mathcal{A}) \to (T, \mathcal{T}, \tau, \mathcal{A}')$ and $r_2 : (T, \mathcal{T}, \tau, \mathcal{A}') \to (T_2, \mathcal{T}_2, \tau_2, \mathcal{A}'')$ are morphisms then $\mathcal{F}_V(r_2 \circ r_1) = \mathcal{F}_V(r_1) \circ \mathcal{F}_V(r_2)$.
Indeed; Let $f \in L^1_{\text{Loc},\mathcal{A}''}$. By item (6) in 2.3.6, both $\mathcal{F}_V(r_1) \circ \mathcal{F}_V(r_2)f$ and $\mathcal{F}_V(r_2 \circ r_1)f$ are $\sigma(\mathcal{A})-$measurable functions. By Lemma 2.3.5, to show equality, it suffices to show that their integrals match on each core set.
For each $A \in \mathcal{A}$:

$$\int\limits_A \mathcal{F}_V(r_1) \circ \mathcal{F}_V(r_2)f \, d\mu = \int\limits_{r_1(A)} \mathcal{F}_V(r_2)f \, d\tau = \int\limits_{r_2 \circ r_1(A)} f \, d\tau_2 = \int\limits_A \mathcal{F}_V(r_2 \circ r_1)f \, d\mu.$$

Hence, $\mathcal{F}_V(r_2 \circ r_1)f = \mathcal{F}_V(r_1) \circ \mathcal{F}_V(r_2)f$ $\mu-$a.e. and completes the proof.
■
The following diagram expresses the semifunctoriality of $\mathcal{F}_V$.

$$\begin{array}{ccc} (U,\Sigma,\mu,\mathcal{A}) & \xrightarrow{\mathcal{F}_V} & L^1_{\mathrm{Loc},\mathcal{A}} \\ \Big\downarrow r & & \Big\uparrow R \\ (T,\mathcal{T},\tau,\mathcal{A}') & \xrightarrow{\mathcal{F}_V} & L^1_{\mathrm{Loc},\mathcal{A}'} \end{array}$$

Similarly, we can define a functor $\mathcal{F}_1$ mapping to the category of compatible couples. We map $\mathcal{F}_1(U,\Sigma,\mu,\mathcal{A}) = L^1_\mu + L^\infty_\mu$ and $r : \mathcal{A} \to \mathcal{A}'$ gets mapped to $\mathcal{F}_1(r) = R$, where $R$ is the linear operator described in Theorem 2.3.6. Notice that items (1) and (9) show that the linear map $\mathcal{F}_1(r)$ is admissible. The diagram becomes

$$\begin{array}{ccc} (U,\Sigma,\mu,\mathcal{A}) & \xrightarrow{\mathcal{F}_1} & L^1_\mu + L^\infty_\mu \\ \Big\downarrow r & & \Big\uparrow R \\ (T,\mathcal{T},\tau,\mathcal{A}') & \xrightarrow{\mathcal{F}_1} & L^1_\tau + L^\infty_\tau \end{array}$$

Also, we have a corresponding semifunctor $\mathcal{F}_2$ when considering down spaces.

$$\begin{array}{ccc} (U,\Sigma,\mu,\mathcal{A}) & \xrightarrow{\mathcal{F}_2} & L^1_\mu + \left(L^\infty_\mu\right)^o \\ \Big\downarrow r & & \Big\uparrow R \\ (T,\mathcal{T},\tau,\mathcal{A}') & \xrightarrow{\mathcal{F}_2} & L^1_\tau + (L^\infty_\tau)^o \end{array}$$

And also a semifunctor $\mathcal{F}_3$ when considering the dual couple.

$$\begin{array}{ccc} (U,\Sigma,\mu,\mathcal{A}) & \xrightarrow{\mathcal{F}_3} & \widetilde{L^1_\mu} + L^\infty_\mu \\ \Big\downarrow r & & \Big\uparrow R \\ (T,\mathcal{T},\tau,\mathcal{A}') & \xrightarrow{\mathcal{F}_3} & \widetilde{L^1_\tau} + L^\infty_\tau \end{array}$$

# Curriculum Vitae

| | |
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