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SOME REMARKS ON THE ROLE OF MONEY IN
ONE SECTOR NEOCLASSICAL GROWTH MODEL

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Tobin (1965) introduced money into Solow's growth model and showed that in the two-asset (money and capital) model (a) the equilibrium capital intensity in the money economy is lower than in the barter economy; and (b) capital intensity increases with the inflation rate. Two independent modifications of Tobin's analysis have been made by Johnson (1967), and Levhari and Patinkin (1968). Each regard the services arising from real money balances as affecting in individuals (imputed) disposable income. They show that with this modification, the direction of change described in (a) and (b) above in Tobin's model become ambiguous. Levhari and Patinkin argued that "it is this dropping of the assumption of constant 's' - rather than the different definition of disposable income" which yields qualitatively different results from those of the Tobin model.

This paper shows that it is neither the dropping of a constant saving ratio nor the different definition of disposable income but the form of the demand for money function which gives us qualitatively different results. It is also shown that dropping the assumption of constant savings ratio in Tobin's model does not yield any different result from Tobin's.

Both papers [1,2] reproduce Tobin's model incorrectly. In fact, according to their specification there does not exist a stable solution. Section I develops the model and shows under what conditions there exists a unique stable solution in Tobin's economy. Section II establishes that it is the form of the demand for money function which gives us an ambiguous result of increase in the rate of inflation on the steady state capital intensity, not the difference in the definition of disposable income. Section III shows that a variable savings ratio on its own will not yield results different from (a) and (b) in Tobin's model. Section IV deals with the problem
in an underdeveloped economy considering the Levhari-Patinkin model, where money is introduced as a consumer good and shows that though results (a) does not hold, (b) does.

I

Consider a one sector neoclassical economy producing a single homogeneous output according to the standard production function,

\[ Y = F(K, L) \quad F_K > 0, \quad F_L > 0 \]

where \( Y \) = output, \( K \) = capital and \( L \) = labor. Assume this production function to be homogeneous of degree one, so that one may write equivalently

\[ (1) \quad y = f(k), \quad y = \frac{Y}{L}, \quad k = \frac{K}{L} \]

Assume that \( f(k) \) is an increasing, strictly concave, twice differentiable continuous function and that

\[ f'(k) > 0, \quad f''(k) < 0, \]

\[ \lim_{k \to 0} f(k) = 0, \quad \lim_{k \to \infty} f(k) = \infty, \]

\[ \lim_{k \to 0} f'(k) = \infty, \quad \lim_{k \to \infty} f'(k) = 0. \]

Assume money is issued only by the state and bears zero own rate of interest legally and permanently established. The stock of money, \( M \), is exogenously determined and can be varied only by budget deficit or surplus. Money affects the working of the economy (according to Tobin) through its effect on the real disposable income. Tobin defines this disposable income as,

\[ (2) \quad Y^d = Y + \frac{d(M)}{dt}, \]

where \( P \) is the price level. An equivalent definition of \( Y^d \) is:

\[ (2') \quad Y^d = Y + \frac{M}{P} (\mu - \pi), \]
where $\mu$ is the rate of change of money supply and $\pi$ is the rate of inflation.

Saving in this economy is assumed to be a fixed proportion of disposable income, i.e.,

$$S = s[Y + \frac{M}{P} (\mu - \pi)].$$

The rate of capital accumulation is equal to the total output produced minus the amount of commodities consumed, i.e.,

$$\dot{K} = F(K, L) - (1 - s) [F(K, L) + \frac{M}{P} (\mu - \pi)].$$

Assume a demand function for real balances of the form

$$m = \lambda(i, k); \quad \lambda_i < 0 \quad \lambda_k > 0,$$

which states that the demand for per capita real balances depends on the nominal rate of interest and the capital stock per head; where the former is equal to the sum of the real rate of interest ($r$)--the marginal productivity of capital--and the rate of inflation ($\pi$). In the steady state per capita real balances are constant, and that constancy implies $(\mu - \pi - n) = 0$, i.e., $\mu - \pi = n$.

Levhari and Patinkin assume that individuals hold a certain proportion, $\lambda$, of income, $Y$, in the form of real money balances, i.e.,

$$\frac{M}{P} = \lambda Y,$$

where $\lambda$ is the function of money rate of interest $i$, that is,

$$\lambda = \lambda(i), \quad \lambda_i < 0,$$

where $i = f'(k) + \pi$, and this means that money rate of interest is the sum of real return on capital and rate of inflation.

Assuming equilibrium always prevails in the money market (by instantaneous adjustment in the price level), for over all equilibrium in the economy we need equilibrium in capital goods market and labor market (as by Walras Law...
consumption goods market then automatically will be in equilibrium.

Equation (4) guarantees the equilibrium in capital market. And the labor market equilibrium condition will be satisfied by the relation,

\[ \frac{\dot{L}}{L} = n. \]  

(7)

Capital market equilibrium is given (in per capita form) by

\[ \frac{\dot{K}}{L} = sf(k) - (1-s)(\mu - \eta) \lambda(i) f(k). \]  

(8)

Combining (7) and (8) we get a fundamental differential equation for overall equilibrium condition of the economy as

\[ \frac{\dot{k}}{k} = \phi(k) - n, \]  

(9)

where

\[ \phi(k) = \frac{sf(k)}{k} - (1-s)(\mu - \eta) \lambda(i) \frac{f(k)}{k}. \]

And a stable solution will exist if \( \phi'(k) < 0 \) in the neighborhood of equilibrium. Differentiating \( \phi(k) \) with respect to \( k \) at the steady state, yields,

\[ \phi'(k^*) = s[k^* f'(k^*) - f(k^*)]/k^{*2} - (1-s)n \frac{d(m)}{dk^*}. \]

Since marginal productivity of labor is positive, the first term of the right-hand side is negative. Thus,

\[ \phi'(k^*) < 0, \]

if

\[ \frac{d(m)}{dk^*} \equiv 0. \]

Clearly this will depend on the form of the demand for money function.

The demand for money function, specified above is:

\[ m = \lambda(i, k), \]

and differentiate with respect to \( k \) and multiply both sides by \( \frac{k^*}{m} \) to obtain

\[ \frac{dm}{dk^*} \frac{k^*}{m} = \eta_i \frac{di}{dk^*} k^* + \frac{\partial \lambda}{\partial k^*} \frac{k^*}{\lambda}. \]
Now, the stability condition:

\[ \frac{\frac{d(m)}{k^*}}{dk^*} \approx 0 , \]

implies \[ \frac{dm}{dk^*} \frac{k^*}{m} \approx 1 . \]

Tobin assumes a proportionality relationship between demand for per capita real balances and per capita capital stock, hence,

\[ \frac{\partial \lambda}{\partial k^*} \frac{k^*}{\lambda} = 1 , \]

which ensures that \[ \frac{dm}{dk^*} \frac{k^*}{m} \approx 1 . \]

But in allegedly reproducing the Tobin model, both Johnson and Levhari-Patinkin make different assumptions about the demand for money function, and later assert that this change in the form of money demand function does not affect the steady state results. Following their assumption, let us consider a special case of a general demand for money function as,

(5') \[ m = \lambda(i) y . \]

Substituting this (5') in equation (9) we get,

\[ \frac{k}{k^*} = \frac{sf(k^*)}{k^*} - (1 - s)n \lambda(i) \frac{f(k^*)}{k^*} - n , \]

and

\[ \phi'(k^*) = s[k^* f'(k^*) - f(k^*)]/k^* \]

\[ - (1 - s)n \left[ \lambda(i) \frac{k^* f'(k^*) - f(k^*)}{k^*} + \frac{f(k^*)}{k^*} \lambda_1^* f''(k^*) \right] . \]

Since the sign of the second term is ambiguous, therefore

\[ \phi'(k^*) \approx 0 , \]
and hence Johnson and Levhari-Patinkin reproduce Tobin's model incorrectly.

We now examine the implication of their demand for money function.

From their demand for money function, we observe

\[
\frac{dm}{dk} \frac{k}{m} = \eta_i \frac{di}{dk} \frac{k}{i} + \frac{kf'(k)}{f(k)}.
\]

Assume without loss of generality, the rate of inflation to be zero. Then, the above relation may be written in alternative form as

\[
\frac{dm}{dk} \frac{k}{m} = (-\eta_i) \frac{\Theta}{\theta} + (1-\rho),
\]

where \(\rho\) and \(1-\rho\) are respectively the share of labor and the share of capital in total output and \(\theta\) is the elasticity of substitution. In this case for a stable equilibrium solution, we need very high value of interest elasticity and/or a very low value of elasticity of substitution. Levhari and Patinkin in accordance with the findings of most empirical studies,\(^2\) assume that \(\eta_i\) is less than unity in absolute value. Hence for \(\frac{dm}{dk} \frac{k}{m} \geq 1\), the elasticity of substitution should have to be very close to zero. Notice however in that case the more familiar problem of Harrodian instability arises, an instability which among other things the neoclassical model was designed to overcome.

With this kind of money demand function, one can avoid this instability problem only by assuming infinitely (or rather highly) elastic demand for money balance with respect to nominal rate of interest. By accepting an extreme Keynesian position one can get rid of this situation. If we consider the proportionality to capital stock version of the demand for money function, stability is ensured for any value of the partial elasticity of the demand function for real balances with respect to the money rate of interest and elasticity of substitution.
II

We will consider in this section money as a consumer good following Johnson, Levhari and Patinkin and will show that again, it is the difference in the form of the demand function for money which makes a difference in the direction of change of steady state capital intensity with respect to change in the rate of inflation and not the definition of disposable income.

Johnson, Levhari and Patinkin criticized Tobin saying that money is neither consumer's good nor producer's good in his model. They argued that when money is treated as a consumer's good, it enters the utility function and its imputed services are included in disposable income; and these services must clearly be valued at the alternative cost at the margin of holding money balances which is the money rate of interest. Accordingly their disposable income,

$$Y^d = Y + \frac{M}{P} (\mu - \pi) + \frac{M}{P} (r + \pi), \quad r: \text{real rate of interest}$$

$$= Y + \frac{M}{P} (\mu + r).$$

In this case, accumulation of capital,

$$\dot{K} = F(K, L) - \{(1 - s)[F(K, L) + \frac{M}{P} (\mu + r) - \frac{M}{P} (r + \pi) \}.$$

Alternatively at steady state,

$$(10) \quad \frac{\dot{k}}{k^*} = \frac{sf(k^*)}{k^*} - (1 - s)n \frac{m}{k^*} + si \frac{m}{k^*} - n,$$

where,

$$\phi(k^*) = \frac{sf(k^*)}{k^*} - (1 - s)n \frac{m}{k^*} + si \frac{m}{k^*},$$

and

$$\phi'(k^*) = sk^2 \frac{f'(k^*) - f(k^*)}{k^*} - (1 - s)n \frac{d(m)}{dk^*} + s \left[ i \frac{d(m)}{dk^*} + \frac{m}{k^*} \frac{di}{dk^*} \right].$$

Following Levhari-Patinkin, first consider the transaction version of the demand for money function, i.e., \(m = \lambda(i)f(k)\).
Dividing both sides by $k$ and then differentiating with respect to $k$ to obtain,

$$\frac{d\left(\frac{m}{k}\right)}{dk} = \lambda(i) \frac{d\left(\frac{f(k)}{k}\right)}{dk} + \frac{f(k)}{k} \lambda_i f''(k),$$

and therefore $\phi'(k^*) > 0$, hence there is no unique stable steady state solution at $k^*$.

Alternatively consider another special case of a general demand function for real balances as suggested by Tobin,

$$m = \lambda(i) k,$$

and dividing both sides by $k$ and differentiating with respect to $k$ we obtain,

$$\frac{d\left(\frac{m}{k}\right)}{dk} = \lambda'(i) f''(k),$$

which is positive. Thus

$$\phi'(k^*) = sk^* \frac{f'(k^*) - f(k^*)}{k^*^2} - (1-s)n \lambda'(i) f''(k^*)$$

$$+ s \lambda \left[ i \frac{\lambda'}{\lambda} + 1 \right] f''(k^*).$$

The first term and second term of the right-hand side are clearly negative. Since the partial elasticity of the demand for money with respect to nominal interest rate is less then unity in absolute value, therefore $[i \frac{\lambda'}{\lambda} + 1]$ is positive and third term is negative. Hence,

$$\phi'(k^*) < 0,$$

unambiguously which is sufficient for unique stable steady state solution at $k = k^*$. Since, $\phi'(k^*) < 0$ therefore,

$$\frac{\partial\left(\frac{m}{k}\right)}{\partial k^*} < 0.$$
Now to show that $\frac{dk^*}{d\pi^*} > 0$ (assuming proportionality to stock of capital version of the demand for money function), differentiate (10) with respect to rate of inflation, i.e.,

$$\frac{\partial (\frac{k}{k^*})}{\partial \pi^*} = -(1-s) \lambda'(i) + s[i \lambda' + \lambda]$$

$$= -(1-s) \lambda'(i) + s \lambda \left[ i \frac{\lambda'}{\lambda} + 1 \right].$$

Since $\lambda$ is less than unity in absolute value and $\lambda'(i) < 0$, then

$$\frac{\partial (\frac{k}{k^*})}{\partial \pi^*} > 0.$$

Hence by the implicit function theorem,

$$\frac{dk^*}{d\pi^*} = - \frac{\frac{\partial (\frac{k}{k^*})}{\partial \pi^*}}{\frac{\partial (\frac{k}{k^*})}{\partial k^*}} > 0.$$

Thus we observe that it is not the definition of disposable income that causes the ambiguity of the effect of rate of inflation on steady state capital intensity, but the form of the demand for money function makes a qualitative different result. To explain the ambiguity of the sign of $\frac{dk^*}{d\pi^*}$, Levhari and Patinkin asserted that it is the constancy of the saving ratio $s$ in the Tobin model that makes the result unambiguous. In the next section it will be shown that this claim is wrong.
II

This section will show that with a variable \( s \) there exists a unique stable solution to Tobin's model and that his results (a) and (b) are correct. To examine the validity of their observation we will assume (following Johnson and Levhari-Patinkin) in the rest of the paper a generalized form of the demand for money function,

\[
(11) \quad m = h(i, y) \quad h_i < 0 \quad h_y < 0
\]

Tobin's model assumes that there is a single aggregate savings ratio in the economy. It is more usual, and more interesting, to assume that capital and labour have different savings ratios, capital's savings ratio being higher than labour's. Accordingly total savings in the economy are

\[
S = s_p \frac{Y_d}{Y} + s_w \frac{Y_d}{Y}, \quad 0 \leq s_w < s_p < 1,
\]

where \( Y_d^p \) and \( Y_d^w \) are the disposable income of the profit earners and wage earners respectively. Assume that the government distributes additions to the money supply in proportion to the share of income of two classes. In that case savings ratio

\[
(12) \quad s = \frac{S}{Y_d} = \frac{s_p r K + s_w wL}{Y}
\]

and a change in the distribution of income will change \( s \). Since real return on capital,

\[ r = f'(k), \]

and wage rate,

\[ w = f(k) - k f'(k), \]

alternatively we can write (12) as,

\[
(13) \quad s = \frac{(s_p - s_w) k f'(k) + s_w f(k)}{f(k)}.
\]
As $k$ increases the rate of return on capital and wage rate will change and make $s$ vary. Notice that, since $0 \leq s_w < s_p < 1$, therefore this variable $s$ lies between 0 and 1 and hence from relation (4) Tobin's result that equilibrium capital intensity in the money economy is lower than in the barter economy holds.

Now substituting equations (13) and (10) in equation (9) and after a little manipulation, the fundamental difference equation of the system is obtained as

\begin{equation}
\dot{k} = \varphi(k) - n,
\end{equation}

where,

\[ \varphi(k) = (s_p - s_w)f'(k) + s_w \frac{f(k)}{k} + n[(s_p - s_w) \frac{f'(k)}{f(k)} h(i,y) - (1 - s_w) \frac{h(i,y)}{k}] \]

and the sufficient condition of this unique stable solution at this steady state depends upon the sign of $\varphi'(k)$. Differentiate $\varphi(k)$ with respect to $k$ at steady state and obtain

\[ \varphi'(k) = \left[ (s_p - s_w)f''(k) + s_w \frac{k f'(k) - f(k)}{k^2} \right] \]

\[ + \left[ (s_p - s_w) n \left( \frac{h(i,y)}{f(k)} f''(k) + k' (k) \frac{d(h/f(k))}{dk} \right) \right] \]

\[ - \left[ (1 - s_w) n \frac{k^2 h(k)}{k^2} - h \right]. \]

Consider the first term of the right hand side. Since $f''(k) < 0$ and marginal productivity of labour is positive [i.e., $(f(k) - k'f(k)) > 0$], this term is clearly negative. The second term will be negative if

\[ \frac{d}{dk} \left[ \frac{h(i,y)}{f(k)} \right] < 0 \]
Differentiating,
\[
\frac{d}{dk^*} \left( \frac{h}{f} \right) = \frac{f h^* - h f'}{f^2} = h \frac{f'(h^*) - f}{f''(h^*)} - 1.
\]

As the income elasticity of the demand for money,
\[
\eta_y = \frac{\partial h}{\partial f} \frac{f}{h} = \frac{\partial h}{\partial k} \frac{\partial k}{\partial f} \frac{f}{h} = \left( \frac{h}{f} \right) \frac{f'}{f^2}.
\]

and is equal to unity from the nature of the money demand function assumed in the model. Therefore, \( \frac{d}{dk} (h/f(k^*)) = 0 \) and the second term is negative. Since \( \frac{dm}{dk} m \geq 1 \), the third term is non-negative and hence \( \phi'(k) < 0 \). The economic interpretation of \( \phi'(k) < 0 \) is straightforward. An increase in the capital stock lowers the average product of capital and as a result capital accumulation declines. Further it reduces the marginal product of capital which is the real rate of interest. This in turn will increase the per capita demand for real money if the increase in the stock of real money produces an increase in private disposable real income, consumption is stimulated and the rate of capital accumulation is further reduced.

In order to analyze the stability of the equilibrium growth path the effect on the growth rate of changes in the endogenous variable \( \pi \) as well as \( k \) must be determined. An increase in the expected rate of inflation lowers private disposable income in two accounts: (1) it increases expected capital losses on the stock of money balances, and (2) it lowers the amount of real money that the individual (and the community) desires to hold. Differentiating equation (14) with respect to \( \pi \), we obtain,
\[
\frac{\partial (k/k^*)}{\partial \pi^*} = (s_p - s_w)n \cdot \frac{f'(k*)}{f(k*)} h_1 - (1 - s_w)n \cdot \frac{h_1}{k^*} \\
= \left[ (s_p \cdot \frac{f'(k*)}{f(k*)} - \frac{1}{k^*} \right] + s_w \left( \frac{f'(k*)}{f(k*)} - \frac{1}{k^*} \right) n h_1.
\]

Since \( h_1 < 0 \) and marginal productivity of labour is positive, therefore,

\[
\frac{\partial (k/k^*)}{\partial \pi^*} > 0
\]

Now using implicit function theorem,

\[
\frac{dk^*}{d\pi^*} = -\frac{\frac{\partial (k^*)}{\partial \pi^*}}{\frac{\partial (k^*)}{\partial k^*}} > 0.
\]

This means that an increasing rate of inflation increases the steady state capital intensity in the economy. Hence, dropping the assumption of constant \( s \) cannot yield qualitatively different results. Levhari-Patinkin said that it is the variable physical savings ratio, \( \sigma \), which makes a qualitatively different result, not the way they define disposable income. But how does this physical saving ratio become variable? Precisely the way they define disposable income. Their variable \( \sigma \) is a function of capital intensity and the rate of inflation. This can be true in the short run, but there is no rationale to believe it in the long run. From the studies of Friedman \(^3\) and Ando-Brumberg \(^4\), we know that \( s \) being dependent on the capital intensity (actually they consider rate of growth) is constant in the long run. We are interested in the long-run equilibrium growth, and thus long run nature of the 's' function, which cannot depend on the short run rates of return. This \( s \) may be different over time in an economy where means of production are widely separated from the majority of the masses and the banking system
is underdeveloped, and thus redistribution of income over time causes a change in s in the long run. Levhari-Patinkin obtain a different result because they analyze a long-run problem with short-run apparatus - and the result created an impasse problem.

IV

Here it will be shown that in a very special situations there exists a unique stable steady state solution to the money in one sector neoclassical growth model where money is introduced as a consumer good as suggested by Johnson, and Levhari and Patinkin. Let us consider a situation where aggregate savings ratio is very low and the savings ratio of workers is very close to zero. Assume $s_w = 0$, and thus our aggregate savings ratio is

$$s = s_p k \frac{f'(k)}{f(k)} .$$

Now substitute (15) into (14), and obtain

$$k = s_p kf'(k) + s_p (\mu + \pi)k \frac{f'(k)}{f(k)} h(i,y) - (\mu - \pi)h(i,y) - nk .$$

And in the steady state,

$$\frac{k}{k^*} = s_p [f'(k^*) + (n+1) \frac{f'(k^*)}{f(k^*)} h(i,y)] - n \frac{h(i,y)}{k^*} - n .$$

Define once again,

$$\phi(k^*) = s_p [f'(k^*) + (n+1) \frac{f'(k^*)}{f(k^*)} h(i,y)] - n \frac{h(i,y)}{k^*} ,$$

and,
\[ \varphi'(k^*) = s_p [f''(k^*) + (n+1)(f'(k^*) \frac{d(h/f)}{dk} + \frac{h}{f} f''(k^*)) + \frac{f'(k^*)}{f(k^*)} h(i,y) f''(k^*)] - n \frac{k^* h k^* - h}{k^*^2}, \]

which is clearly negative. Hence at the steady state

\[ \frac{\partial (k/k^*)}{\partial k^*} < 0, \]

which guarantees that there exists a unique stable solution. To show that in this economy the steady state capital intensity will increase with an increase in the rate of inflation, we have to examine the sign of \( \frac{\partial (k/k^*)}{\partial \pi^*} \).

We have,

\[ \frac{k}{k^*} = s_p [f'(k^*) + (n+1) \frac{f'(k^*)}{f(k^*)} h(i,y)] - n \frac{h(i,y)}{k^*} - n, \]

and thus,

\[ \frac{\partial (k/k^*)}{\partial \pi^*} = s_p [(n+1) \frac{f'(k^*)}{f(k^*)} h_i + \frac{f'(k^*)}{f(k^*)} h(i,y)] - \frac{n}{k^*} h_i \]

\[ = [s_p \frac{f'(k^*)}{f(k^*)} - \frac{1}{k^*}] h_i + [s_p \frac{h_i}{h} + 1] \frac{f'(k^*)}{f(k^*)} h. \]

Consider the first term of the right hand side. Since bracketed term is negative and \( h_i \) is negative, the first term is positive. Following Levhari-Patinkin, assume that the partial elasticity of real money balances with respect to the rate of interest and is less than unity in absolute value, then the second
term is also positive and thus,

$$\frac{\partial (k/k^*)}{\partial \pi^*} > 0.$$ 

Hence using the implicit function theorem,

$$\frac{dk^*}{d\pi^*} > 0,$$

that is, capital intensity increases with the inflation rate, or equivalently the real interest rate falls with the inflation rate—the major result in the Tobin model.
Footnotes


This special situation can nicely be fitted in an underdeveloped country.

In this case for \( \eta_k \geq 1 \) elasticity of substitution, \( \theta \), is to be very low; and since we are in an underdeveloped economy we can safely assume that.
References

