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Optimal Strategies for Introducing a New Technological Product

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Abstract

This study presents analytical frameworks to provide firms with effective strategies that can be adopted when they plan to release a new technological product (e.g., TV sets, video players, cameras, video games, etc.). Specifically, focusing on technology products, the current study recommends optimal pricing strategies for new products, introduces a multigeneration sales model developed to capture evolving sales trends across multiple product generations, outlines an optimal market entry timing strategy for a new product generation in a product line, and provides an effective marketing strategy to promote sales of a new product by offering an optimal number of free products to users.

The dissertation includes three essays. The first essay (Chapter 2) provides a modeling scheme to examine the optimal pricing strategy for a new technology product. Specifically, the first essay investigates when the widely-used price-skimming strategy should be the go-to pricing strategy. The contribution in the second essay (Chapter 3) is twofold. First, a multigeneration sales model for technological products is developed. The proposed multigeneration sales model takes into account within-generation repeat purchases as well as initial purchases and cross-generation upgrades. Next, based on the proposed sales model, a framework to derive optimal market entry timing for a new product generation is developed. The third essay (Chapter 4) examines the marketing strategy of offering free products for a new technological product. The proposed modeling framework encompasses scenarios under which: (i) the firm offers free products before product launch, (ii) the firm immediately releases the product and offers free products upon the product launch and in parallel with the sales, and (iii) the firm offers free products before product launch and simultaneously with the sales after the launch.

In summary, this study significantly contributes to the field of management science by providing comprehensive analytical frameworks developed to address the challenges of introducing a new technological product to the market. These challenges include estimating sales trends in product lines composed of successive generations, finding the optimal market entry timing strategies for new generations, finding the optimal pricing strategies, and promoting sales by offering free products to the target market.

Keywords

Pricing, product diffusion, repeat purchases, technology products, multigeneration diffusion, market entry strategy, free product offer.

Summary for Lay Audience

Firms often need to release new products to motivate sales further and stay competitive. However, a successful release of a new product requires careful planning. Focusing on technology products, the three essays presented in this dissertation help firms efficiently plan for a new product.

Essay 1 (Chapter 2) develops a modeling framework to determine the optimal pricing strategies for sales of new technology products. Specifically, Essay 1 examines the widely adopted price-skimming strategy in which firms initially set the price relatively high and later reduce it over time.

In Essay 2 (Chapter 3), an analytical model is developed to accurately capture sales trends of successive generations of a product line. Next, using the developed sales models, a modeling framework is introduced for finding the optimal market entry timing strategy for a new product generation. Two main generation transition strategies are studied in this essay: (i) phase-out transition strategy in which sales of the old generation continue after the introduction of the new generation, and (ii) total transition strategy in which by the release of the new generation, the old generation is discontinued.

Finally, Essay 3 (Chapter 4) develops an analytical modeling framework to examine the widely used marketing strategy of offering free products. In this essay, three different free product offering strategies are studied: (i) offering free products before the product launch, (ii) immediate release of the new product and offering free products upon the introduction of the new product and parallel to sales, and (iii) offering free products before and after the product launch.

In summary, the three essays of this dissertation provide analytical frameworks that help firms effectively address the challenges of introducing a new technological product to the market. These challenges encompass optimal pricing, accurate estimation of sales trends, optimal release timing of a new product generation, and promoting sales through offering free products.

Co-Authorship Statement

In Essay 1 (Chapter 2), I collaborated with Dr. Joe Naoum-Sawaya (Ivey Business School, Western University), Dr. Aslan Lotfi (Robins School of Business, University of Richmond), and Dr. Zhengrui Jiang (School of Business, Nanjing University). The specific contribution of each coauthor is detailed as follows:

Ali Lotfi: Proposing the research idea. Conducting the literature review. Conducting analysis including derivation of all the analytical and numerical results. Software implementation. Visualizations. Interpretation and presentation of the results. Preparation of the original draft. Review and edit of the drafts.

Joe Naoum-Sawaya: Project administration and supervision. Quality check of the results. Refining the drafts and providing high-level edits.

Aslan Lotfi: Assisting with the presentation and the interpretation of the results as a domain expert. Providing high-level review and edits.

Zhengrui Jiang: Offering high-level refinements and edits on the final drafts as a domain expert.

In Essay 2 (Chapter 3), I collaborated with Dr. Joe Naoum-Sawaya (Ivey Business School, Western University), Dr. Mehmet A. Begen (Ivey Business School, Western University), and Dr. Zhengrui Jiang (School of Business, Nanjing University). The specific contribution of each coauthor is detailed as follows:

Ali Lotfi: Proposing the research idea. Conducting the literature review. Conducting analysis including derivation of all the analytical and numerical results. Data curation. Software implementation. Visualizations. Interpretation and presentation of the results. Preparation of the original draft. Review and edit of the drafts.

Joe Naoum-Sawaya: Project administration and supervision. Quality check of the results. Refining the drafts and providing high-level edits.

Mehmet A. Begen: Project administration and supervision. Quality check of the results. Refining the drafts and providing high-level edits.

Zhengrui Jiang: Offering high-level refinements and edits on the final drafts as a domain expert.

In Essay 3 (Chapter 4), I collaborated with Dr. Joe Naoum-Sawaya (Ivey Business School, Western University), and Dr. Mehmet A. Begen (Ivey Business School, Western University). The specific contribution of each coauthor is detailed as follows:

Ali Lotfi: Proposing the research idea. Conducting the literature review. Conducting analysis including derivation of all the analytical and numerical results. Software implementation. Visualizations. Interpretation and presentation of the results. Preparation of the original draft. Review and edit of the drafts.

Joe Naoum-Sawaya: Project administration and supervision. Quality check of the results. Refining the drafts and providing high-level edits.

Mehmet A. Begen: Project administration and supervision. Quality check of the results. Refining the drafts and providing high-level edits.

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Chapter 1

1 Introduction

This study provides analytical models to help firms develop effective strategies for introducing new technological products such as TV sets, video players, cameras, video games, etc. Specifically, this study examines optimal pricing strategies for new products. It introduces a new multigeneration sales model developed to capture sales trends across multiple generations of a new product and examines the optimal market entry timing strategy for a new product generation within a product line. Furthermore, this study provides an effective marketing strategy to promote sales by determining the optimal number of free products to offer users.

Chapter 2 develops an analytical framework to find the optimal pricing strategy for a new technological product. Specifically, this chapter focuses on the widely-used price-skimming strategy. Price skimming is a common pricing strategy for technological products where the price is set at the highest level at the time of product release and then declines over time. Most pricing models developed in the literature are based on the assumption that sales consist only of initial purchases or adoptions. These models ignore repeat purchases, which constitute a significant portion of the sales of technological products, making them inadequate for such products. To fill this void, in Chapter 2, we develop an optimization model to find the optimal pricing strategy when sales comprise both initial purchases and repeat purchases. The results demonstrate that the effectiveness of price skimming is highly dependent on the rate of repeat purchases. Specifically, when the rate of repeat purchases is low, price skimming may not be an optimal pricing strategy, and firms may benefit more from delaying the price reduction. Moreover, when markets are highly sensitive to price changes, firms may benefit from charging a higher introductory price; however, the price must be decreased over time to motivate buyers to purchase.

Focusing on technological products, Chapter 3 formulates multigeneration sales dynamics and examines the optimal market entry timing for a new product generation in a product line. Current multigeneration diffusion models in the literature do not account for within-generation repeat purchases. Thus, these models are not suitable for capturing sales of multigeneration technological products with high rates of within-generation repeat purchases. To fill this void, we develop a multigeneration sales model that accounts for initial purchases and within-generation repeat purchases for each generation as well as cross-generation upgrades.

The model proposed in Chapter 3 includes two main generation transition strategies: (i) a phase-out transition strategy, where firms continue to sell the old generation after the release of the new generation, and (ii) a total transition strategy, where firms discontinue the old generation after introducing the new generation. Empirical results demonstrate the importance of accounting for within-generation repeat purchases. Specifically, they show that the new model provides more accurate sales estimates and forecasts than a state-of-the-art benchmark model, which does not account for within-generation repeat purchases. Furthermore, we employ the new model to examine optimal market entry timing under the two main generation transition strategies. The results highlight the critical role of the repeat purchase rate in market entry timing strategies.

Free product giveaway is a common marketing strategy to accelerate new product sales. Focusing on technology products, Chapter 4 studies three strategies a firm can employ in offering free products: (i) the firm offers free products before product launch (before strategy), (ii) the firm immediately releases the product and offers free products in parallel with sales (concurrent strategy), and (iii) the firm offers free products before product launch and simultaneously with sales after the launch (combined strategy). Considering that repeat purchases account for a large proportion of sales of technology products, we present a framework that captures sales composed of both the initial and repeat purchases. Our framework differentiates between two groups of free product users: high-valuation free users and low-valuation free users. High-valuation free users are those who are willing to buy the product if it were not offered for free, whereas those who are not willing to buy the product belong to the low-valuation group. Our results demonstrate that offering free products can be more appealing when there are high repeat purchase rates, slow product adoption among buyers, and a significant proportion of low-valuation free users. In contrast, we show that there are instances in which even a zero marginal cost and an extended market size may not justify offering free products. The results also show that targeting highly influential free users and more low-valuation free users can increase profitability in the case of a short planning horizon and this profitability can be further elongated with increase in the rate of repeat purchases. However, this strategy may result in less profitability under extensively long planning horizons. We find that if the new product release is delayed briefly, firms might benefit more from adopting the before strategy over the concurrent one. Also, in certain cases, employing a combined strategy can lead to greater profitability than just using the before strategy.

In summary, the three essays in this dissertation, presented in Chapters 2 through 4, provide comprehensive analytical frameworks that firms can use to tackle the challenges of releasing a new technological product, including finding the optimal pricing strategy, estimating and forecasting sales trends, determining the optimal market entry timing, and offering free products to promote sales.

Chapter 2

2 To Skim or not to Skim: Studying the Optimal Pricing Strategy for Technology Products[1]

2.1 Introduction

Facing rapid technological changes and ever-shortening product lifecycles, firms need a proper pricing strategy in place to recoup costs and gain profit during short product lifecycles. One popular pricing strategy for technology products is the price-skimming strategy, in which price is initially set at a relatively high level and then reduced over time (Ward, 2019). Among many technology products priced based on the skimming strategy are Apple smartphones (Holownia, 2021), video-game consoles such as the Sony PlayStation and the Nintendo Switch (Moy Media, n.d.), DVD players (Beltis, 2019), Blu-ray players (Fogel, 2023), and the latest 4K/8K television sets (Pettinger, 2019). An example reported by Statista (2014), as shown in Figure 2-1, illustrates the decline in the average price of 4K TVs between 2012 and 2017. As several examples suggest, price skimming seems to have been the *go-to* strategy for technology products.

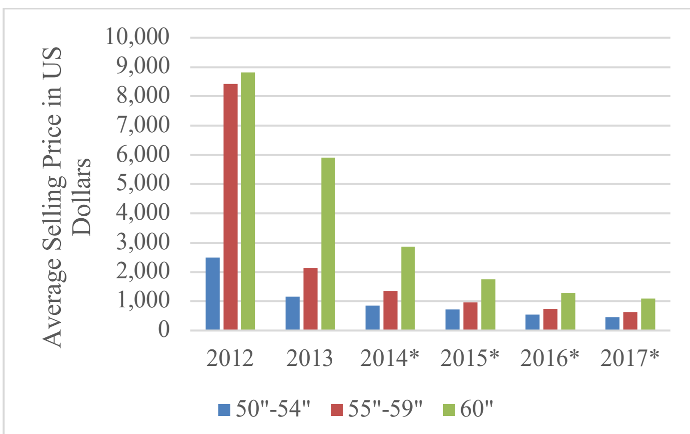

Figure 2-1: Average worldwide selling price of 4K TVs 2012-2017. NOTE: Estimated values are shown for years denoted by *s.

[1] This is an author version of the paper published as: Lotfi, A., Naoum-Sawaya, J., Lotfi, A., & Jiang, Z. (2024). To skim or not to skim: studying the optimal pricing strategy for technology products. *Omega*, *127*, 103079.

A price-skimming strategy benefits from consumer heterogeneity in terms of their willingness to pay and price sensitivity. This strategy specifically aims to “skim” off potential buyers willing to purchase the product at a higher price (Liu, 2010). For example, Apple often charges a premium price when first releasing their new products; after less price-sensitive technology-enthusiast consumers have made their purchases, the price is then reduced to attract more consumers. While being criticized for overpricing their products, Apple has fans who disagree with the criticisms and are happy to pay the premium prices (Epstein, 2017). Moreover, with significant performance improvement based on Moore’s law, components used in electronic products have been experiencing relentless price drops, driving down the cost of technology products (Rosoff, 2015), and a decline in production costs resulting from learning effects can also help firms reduce product prices over time (Besanko & Winston, 1990). All in all, the price-skimming strategy appears to be suitable for selling technology products.

However, while price-skimming strategy has been studied in the literature (e.g., Chang & Lee 2022; Yuan, et al., 2022; Du & Chen, 2017) and widely used by practitioners, our review of the existing literature reveals a gap in the study of price skimming for technology products in particular. Specifically, during the last few decades, while researchers have extensively studied optimal pricing strategies for new products (e.g., Cosguner & Seetharaman, 2022; Chutani & Sethi, 2012; Sethi et al., 2008; Debo et al., 2006; Hartl et al., 2003; Sethi & Bass, 2003; Krishnan et al., 1999; Horsky, 1990; Kalish, 1985), the majority of pricing models proposed in the literature are based on diffusion models that consider only initial product purchases while ignoring repeat purchases. Since in the market of technology products, fast-paced technology advancements lead to frequent releases of improved product versions, technology products are often characterized by a high rate of repeat purchases, sometimes starting within the earlier stages of a product lifecycle.

In this study, we examine an optimal pricing strategy for technology products that explicitly accounts for repeat purchases, thereby examining the widely-used price-skimming strategy used for marketing such products. Specifically, to derive an optimal pricing strategy for technology products, we develop an extension of the generalized diffusion model with repeat purchases (GDMR; Lotfi et al., 2023) which explicitly incorporates both initial purchases and repeat purchases. The extended sales model we propose incorporates the effects of both the baseline price and price changes on sales, while the original GDMR proposed by Lotfi et al. (2023) only incorporates the latter. The model we implement formulates sales based on a Riemann-Liouville integral of product adoption function. We name the newly-introduced sales model the *extended generalized diffusion model with repeat purchases* (*EGDMR*). Additionally, in our optimal pricing analysis based on the EGDMR, we account for both economies of scale and cost reductions due to learning effects, thus contributing to the extant literature that uses simpler cost structures.

Due to the modeling facilities brought about by applying the Riemann-Liouville integral, our findings are ample in the context of our application domain. Our findings suggest that there is a caveat to using the widely-implemented price-skimming strategy and firms should be careful when opting to use this strategy. Specifically, we find that an optimal pricing strategy for a new product is highly dependent on the volume of repeat purchases, and that the optimal strategy may deviate from skimming when repeat purchases are not ample and sales are not expected to sustain over time. Specifically, when the repeat-purchase rate is expected to be low and the sales rate is expected to drop considerably as product adoption approaches complete market penetration, to remain profitable firms are likely better off in delaying price reductions to offset drops in profit from declining sales. Particularly, our findings show that price skimming may not be optimal for fad products with low repeat-purchase rates resulting from the product losing popularity shortly after product release. Our results also demonstrate that, as opposed to common expectations, in markets highly sensitive to changes in price firms may get away with setting the introductory price at a higher level but must reduce the price over time to stimulate sales. In summary, our results suggest that overlooking repeat purchases when examining the optimal price path of a new product is likely to lead to suboptimal pricing decisions, thus underscoring the contribution of the optimization model developed in this research.

Considering that our research studies the widely used strategy of price skimming, our finding can offer valuable insights to a broad range of firms that may consider this strategy. From the methodological standpoint, our study contributes to the vast literature of fractional calculus in applied mathematics and science by introducing a novel application of Riemann-Liouville integral to solve a problem in management sciences where fractional calculus is mostly unknown. The model we implement has a clear interpretation. While fractional calculus has been extensively studied in mathematics (e.g., Kilbas et al., 2006), historically, coming by a clear interpretation of fractional calculus has proven challenging.

The remainder of the chapter is organized as follows. Following this introductory section, Section 2.2 provides a review of relevant literature. In Section 2.3, we introduce the GDMR model and then propose an extension model to incorporate the impact of pricing. Section 2.4 develops the proposed price optimization model and derives key analytical findings. Section 2.5 presents empirical analyses and discusses the results. Concluding remarks are presented in Section 2.6.

2.2 Literature Review

Examining the optimal pricing strategy for a product or service is one of the salient problems studied in the operations management and supply chain management literature (e.g., Oh & Su, 2022; Shams-Shoaaee & Hassini, 2020; Helmes et al, 2013; Cai et al., 2009; Ketzenberg & Zuidwijk, 2009; Vorasayan & Ryan 2006). There are several pricing studies on the price-skimming strategy (e.g., Chang & Lee 2022; Yuan et al., 2022; Zhang & Chiang, 2020; Mesak et al., 2020; Du & Chen, 2017; Martín-Herrán & Taboubi, 2015; Liu & Zhang, 2013; Robotis et al., 2012; Liu, 2010; Besanko & Winston, 1990). However, the aforementioned studies differ from this study in that they do not consider repeat purchases.

While there is ample research on the pricing of various technology products such as software (e.g., Brecko, 2023; Li & Kumar, 2022; Xin & Sundararajan 2020, Cheng et al., 2015; Liu et al., 2011; Cheng & Koehler, 2003; Gurnani & Karlapalem, 2001), digital music (e.g., Li et al, 2020; Ko & Lau, 2016), video game consoles (e.g., Liu, 2010), electronic products (e.g., He & Chen, 2018; Fathian et al., 2009), and digital goods in general (e.g., Taleizadeh et al., 2022; Avinadav et al., 2014; Huang & Sundararajan, 2011; Khouja & Park, 2007), research that examines optimal pricing for sales processes that include both adoptions and repeat purchases is scarce. This omission is largely due to the challenges associated with modeling repeat purchases.

Very few prior studies have accounted for repeat purchases when studying optimal price-path strategies for new products. Dolan and Jeuland (1981) develop a model for durable and non-durable products that partitions the market into users (those who have tried the product in the past) and non-users (those who have not yet tried the product), and propose a repeat purchase rate for users and a conversion rate for non-users. While the model presented accounts for repeat purchases of non-durable products, it assumes that the sales of durable products are comprised only of initial purchases.

Mesak and Berg (1995) explore the optimal price path of consumer durables while considering both adoptions and replacement purchases. Their findings suggest that accounting for replacement purchases changes the optimal price path considerably even when replacement purchases are not price-dependent. While Mesak and Berg (1995) incorporate replacement purchases, there are important differences compared to the model presented in this paper. First, the replacement purchase rate used by Mesak and Berg (1995) is unsuitable for technology products because it overestimates repeat purchases, while the model presented in this paper uses an extension of the GDMR, which is particularly suitable for technology products. In Appendix A, using data for sales of DVD players in the U.S., we demonstrate that the GDMR empirically performs well, while the sales trend by Mesak and Berg's model deviates significantly from the actual sales trend. Second, the cost function found in Mesak and Berg (1995) is comprised only of learning cost, while the cost function considered in this study incorporates both the learning effect and the economy-of-scale effect, making our results more realistic and insightful. Third, Mesak and Berg (1995) concentrate mainly on cases where future uncertainty is assumed to be negligible under non-discounted situations. For discounted situations, their analysis is limited to cases where either the word-of-mouth effect is nonexistent or where the effect of price is restricted to only replacement purchases. In contrast, our analysis considers realistic scenarios where the word-of-mouth effect is present and product price influences both initial and repeat purchases.

Debo et al. (2006) studied cases where the products offered are either new or remanufactured products and examine substitution between new and remanufactured items through repeat purchases. Their findings show that different combinations of adoption rate and repeat purchase rate may lead to optimal sales trends with either a single peak or with fluctuations. Unlike the model proposed in this study, repeat purchases in Debo et al. (2006) are limited to purchases of remanufactured products.

To the best of our knowledge, no prior research has examined the optimal price path of products while taking into consideration (i) both initial and repeat purchases, (ii) the influence of price on both initial and repeat purchases, (iii) the learning effect and the economy-of-scale effect, and (iv) future uncertainty (discounted situations). To fill this void, considering that (i)-(iv) are essential in studying sales of technology products, the model presented in this study proposes a price-optimization framework for technology products based on an extension of the generalized diffusion model with repeat purchases (GDMR; Lotfi et al., 2023). The GDMR, as detailed next, is an extension of the Bass diffusion model and has been specifically developed to model sales of technology products.

2.3 Extending GDMR to Incorporate Effect of Pricing

To derive an optimal pricing strategy for a technology product with significant repeat purchases, we need a sales model that incorporates both initial and repeat purchases, and we chose to adopt the generalized diffusion model with repeat purchases (GDMR; Lotfi et al., 2023) for this purpose. We next introduce the GDMR then extend it to incorporate pricing decisions. Following the GDMR and based on Henderson and Clark's framework (Henderson & Clark, 1990), our unit of analysis is a group of products that are only incrementally different from each other. Specifically, a new purchase is considered a repeat purchase only if the newly purchased product is the same as, or only incrementally different from, the initially purchased product.

2.3.1 The Baseline GDMR Model

The GDMR as formulated by Lotfi et al. (2023) is an extension of the Bass diffusion model that incorporates both adoptions, essentially first-time purchases, and repeat purchases, comprised of replacement purchases and multi-unit ownership purchases. The GDMR formulates sales as:

$$s(t) = I^{\beta}(y(t)) \tag{2.1}$$

where $y(t)$ denotes the noncumulative rate of adoptions, $I^{\beta}(.)$ is a fractional integral of order β ($0 \leq \beta \leq 1$), and β is the repeat purchases parameter. The GDMR captures a continuum of sales scenarios with various repeat purchase levels, ranging from a scenario in which no repeat purchase takes place (when $\beta = 0$) to a sales scenario in which all adopters on average make one repeat purchase in each unit time period ($\beta = 1$). Based on the Riemann-Liouville fractional integral formulation, Lotfi et al. (2023) reformulate sales shown in (2.1) as:

$$s(t) = \int_0^t \frac{1}{\Gamma(\beta)} (t - \tau)^{(\beta-1)} \, y(\tau) d\tau, \tag{2.2}$$

where $y(t)$ is the non-cumulative rate of adoptions specified by the Bass model (Bass, 1969). Through several empirical tests, Lotfi et al. (2023) demonstrate that the fractional integral operator accurately captures the repeat purchase rate for technological products.

The Bass model's hazard rate of adoption is formulated as:

$$\frac{f(t)}{1-F(t)} = p + qF(t), \tag{2.3}$$

where p is the coefficient of innovation that captures a potential adopters' internal motivation to adopt a new product, and q is the coefficient of imitation that captures the word-of-mouth effect on the adoption process. In (2.3), $f(t)$ is the probability density of adoption and $F(t)$ is the cumulative distribution function of adoption (i.e., $F(t) = \int_0^t f(s)ds$) showing the fraction of potential adopters who have adopted the product up to time t. Based on the closed-form solution of the Bass model (Bass, 1969), the cumulative number of adoptions by time t, $Y(t) = mF(t)$, equal to the first-order integral of $y(t) = mf(t)$, is:

$$Y(t) = \frac{m(1-e^{-(p+q)t})}{(1+\frac{q}{p}e^{-(p+q)t})}, \tag{2.4}$$

where m is the market size. Noncumulative or periodic adoption, $y(t)$, where $y(t) = Y'(t)$, takes the following form:

$$y(t) = \frac{m(p+q)^2}{p} \frac{e^{-(p+q)t}}{(1+\frac{q}{p}e^{-(p+q)t})^2}. \tag{2.5}$$

Note that a price-skimming strategy assumes that the market is comprised of customers who are heterogeneous in terms of their willingness to pay and price sensitivity. It specifically assumes that there are customers with low price sensitivity who are willing to pay higher prices shortly after the product release along with highly price-sensitive customers who can be captured by reducing price over time. Therefore, to achieve reliable results, any model used to examine price skimming needs to accommodate such customer heterogeneity. The Bass model, essentially the baseline model of the GDMR, is capable of accounting for the aforementioned market heterogeneity. Specifically, in the Bass model adopters are under the influence of both an innovation effect (p) and an imitation effect (q). From (2.3), an *innovator* who adopts a product shortly after product launch is mostly under the influence of innovation effect (p) and less influenced by the imitation effect (q), because the fraction of potential customers who have adopted the product ($F(t)$) is still small. Innovators can be characterized as being daring, venturesome (Bass, 1969), and comfortable with risk (Rogers, 1995). We expect innovators to be less price-sensitive. On the other hand, according to (2.3), an *imitator* who adopts the product later in the product lifecycle is subject to a significant imitation effect (q) because the fraction of adopters ($F(t)$) increases as market penetration progresses towards completion. An imitator's adoption timing is mostly influenced by social-system's pressure (Bass, 1969) and we expect imitators to be more price sensitive.

2.3.2 Extended GDMR Model with Pricing

The baseline GDMR shown in (2.2) does not incorporate the effects of marketing mix variables, and Lotfi et al. (2023) show that the GDMR can be extended to incorporate marketing-mix variables as follows:

$$s(t) = x(t)I^{\beta}y\big(X(t)\big), \tag{2.6}$$

where $x(t)$ and $X(t)$ represent the *current marketing effort* and the *cumulative marketing effort*, respectively, at time t. Sales at the current time are expected to be influenced by both the past marketing efforts due to these efforts' carry over effect and the current marketing efforts. Therefore, incorporating both the cumulative and current marketing efforts in sales as shown in (2.6) seems reasonable and is expected to result in a realistic sales model.

Since the goal of this study is to derive the optimal pricing path in the presence of repeat purchases, we focus on pricing as a marketing-mix variable. Following Bass et al. (1994), we can express $x(t)$ as:

$$x(t) = 1 + \alpha \frac{pr\prime(t)}{pr(t)}, \tag{2.7}$$

where $pr(t)$ is the price at time *t* and α ($\alpha < 0$) reflects the effect of price changes on sales. According to Bass et al. (1994), the cumulative marketing effort is $X(t) = \int_0^t x(s)\,ds$ and can be expressed as:

$$X(t) = t + \alpha \ln \frac{pr(t)}{pr(0)}, \tag{2.8}$$

where $pr(0)$ is the price at product launch. The current marketing effort shown in (2.7) and the cumulative marketing effort expressed in (2.8) account for the changes in price. Here, we further extend the sales model (2.6) to incorporate the absolute price level as well as price changes. Specifically, we employ the following extended cumulative marketing effort proposed by Krishnan et al. (1999):

$$X(t) = k\,t + \alpha \ln \frac{pr(t)}{pr(0)}. \tag{2.9}$$

In (2.9), parameter k ($k > 0$) captures the impact of the absolute price level and can be written as:

$$k = 1 + \gamma \ln(pr(0)), \tag{2.10}$$

where $\gamma \leq 0$ is the *baseline price coefficient*. The price level at product launch (*pr*(0)) is related to product positioning; products positioned to be more premium and benefit from more advanced technologies are often released at a higher price level. $X(t)$ in (2.9) accounts for both the absolute price level and price changes. Additionally, following Krishnan et al. (1999), we multiply the base sales by k to capture the impact of absolute price level on the market size, so the sales rate utilized in the optimal pricing model is:

$$s(t) = k\,x(t) I^{\beta} y\big(X(t)\big), \tag{2.11}$$

where, $X(t) = \int_0^t x(s)\,ds$. We refer to the sales model (2.11) as *extended generalized diffusion model with repeat purchases* (*EGDMR*). Note that when $\beta = 0$, the GDMR shown in (2.1) reduces to the original Bass adoption Model (i.e., $y(t)$), and the EGDMR presented in (2.11) reduces to the extended generalized Bass model (i.e., $k\,x(t)y\big(X(t)\big)$) proposed by Krishnan et al. (1999). The EGDMR is an extension of the GDMR in the sense that the EGDMR accounts for the effect-on-sales of the absolute price level as well as that of the price changes, while the GDMR incorporates only the effect of price changes on sales. We next discuss how the optimal price path is obtained.

2.4 Optimal Lifecycle Pricing Strategy

In this section, we examine the optimal pricing strategy within a product lifecycle. Specifically, we first present our profit function, then present a special case in which we examine the optimal price path for software products, and subsequently discuss the general price-optimization model.

2.4.1 Profit Function

This study derives a new product's optimal lifecycle pricing strategy that maximizes total profit in a demand window or planning horizon $[0,T]$. We begin with a profit function presented in the literature and then extend it to develop our own objective function. Specifically, we draw on the objective function proposed by Krishnan et al. (1999). The profit function considered by Krishnan et al. takes the following form:

$$\Pi[pr(t)] = \int_0^T [pr(t) - C(t)]\widetilde{f(t)}e^{-rt}dt, \tag{2.12}$$

where $pr(t)$ denotes product price, $\widetilde{f(t)}$ is the adoption probability density function of the modified generalized Bass model by Krishnan et al. (1999), r is the discount rate, and $C(t)$ represents cost as:

$$C(t) = \frac{c_0}{\zeta^{-l}}\left(\zeta + F(t)\right)^{-l} \tag{2.13}$$

In the cost function (2.13), c_0 is the marginal cost of making the initial batch of product, l is the learning-curve parameter, $F(t)$ is a cumulative distribution function of adoption that incorporates price effect, and ζ is a small constant. In equation (2.12), note that the profit Π is a function of price path $pr(t)$.

Next, we extend the profit function (2.12) proposed by Krishnan et al. (1999). Specifically, we replace $\widetilde{f(t)}$ in (2.12), which captures only adoption, by sales based on the EGDMR, i.e., $s(t)$ formulated by (2.11), which incorporates adoptions as well as repeat purchases. While the cost structure used in profit function (2.12), formulated in (2.13), accounts for cost reduction resulting from the learning effect, it does not take into account the effect of economies of scale that can play an important role in the price of hardware products. We extend the cost structure to account for both the effect of economies of scale on per-unit production and distribution cost as well as the learning effect on the research and development cost. Consequently, we extend the profit function by Krishnan et al. (1999), as presented in (2.12), to

$$\Pi[pr(t)] = \int_0^T [\big(pr(t) - c(t)\big)s(t) - \bar{c}(t)]e^{-rt}dt, \tag{2.14}$$

where $pr(t)$ denotes product price, $c(t)$ is per-unit production and distribution cost, $\bar{c}(t)$ is the total research and development (R&D) cost that changes with time and learning, $s(t)$ denotes sales including both adoptions and repeat purchases, and e^{-rt} represents the discount factor corresponding to rate *r*.

Due to the economies of scale, the per-unit cost is expected to decline when the number of units produced and sold increases, thus the per-unit production and distribution cost $c(t)$ can be formulated as:

$$c(t) = \frac{c_0\zeta^{l_1}}{\big(\zeta+\sigma(t)\big)^{l_1}}, \tag{2.15}$$

where c_0 is the marginal cost associated with making the initial batch of products, $\sigma(t) = s(t)/m$, making $c(t)$ dependent on dynamics of sales (s(*t*)), ζ is a positive constant, and l_1 is the *economies-of-scale* parameter. The per-unit production and distribution cost $c(t)$ in (2.15) decreases when sales increases, reflecting the effect of economies of scale. With a larger (smaller) economies-of-scale parameter (l_1), the per-unit production and distribution costs decline more (less) rapidly as the rate of sales increases.

The R&D intensity is expected to decline over time because both the product and the production process will likely become better understood through learning as time elapses. Therefore, we formulate $\bar{c}(t)$ as:

$$\bar{c}(t) = \frac{c_1\zeta^{l_2}}{(\zeta+kF(X(t)))^{l_2}}, \tag{2.16}$$

where $F(X(t)) = Y(X(t))/m$, in which *Y* is the cumulative adoption as shown in (2.4), *X*(*t*) is the cumulative marketing effort as shown in (2.9), and *m* is the market size. l_2 is the *learning rate* parameter, and c_1 represents the initial R&D investment. From (2.16), as product diffusion progresses, the learning effect leads to a reduction in R&D costs, and with a higher (lower) learning rate (l_2), the R&D cost declines more (less) quickly as the diffusion process progresses.

The discount rate (r) can be adjusted based on the level of future uncertainty. The discount rate can be set higher when future uncertainty is higher, so future profit is more heavily discounted to account for the higher risk of not realizing the projected future profit. In summary, unlike solutions from the extant literature, the profit function we consider accounts for (i) both adoptions and repeat purchases, (ii) cost reduction due to learning effect, and (iii) economies of scale.

2.4.2 The Special Case for Software Products

It is important to note that deriving analytical findings for our general case is exceptionally challenging for several reasons. By better reflecting the reality of technology products' sales, our problem formulation is significantly more complicated than the pricing models proposed in the extant literature, including the one proposed by Krishnan et al. (1999). Specifically, we incorporate repeat purchases and implement a more realistic cost structure than the prior literature has done, making our analysis more complicated. Furthermore, fractional calculus is an inherently complex tool; based on our extensive search, with the exception of the work by Lotfi et al. (2023), fractional calculus has not been applied in management science, let alone the integration of fractional calculus into a price path optimization model, further underscoring our contribution to the literature. Therefore, as expected, closed-form solutions are unattainable for problem (2.14). To demonstrate the application and utility of our price-optimization model, in this section, we analytically examine the optimal price path for specific sales scenarios involving products with key characteristics of software products.

Costs of providing software products mainly include development cost that can be considered R&D cost. The cost of distributing such products is typically low because they can be delivered electronically. For example, video game purchases increasingly are made through digital downloads (Patterson, 2021). We assume that the introductory price is predetermined (i.e. $\gamma = 0$ and $pr(0) = pr_0$), the R&D cost is constant over time (i.e. $l_2 = 0$), and there is no per-unit production and distribution cost (i.e. $c(t) = 0$). Our cost structure is compatible with that of Huang and Sundararajan (2011) that models IT-based services and products as information goods with no variable production and distribution costs. Additionally, we assume a constant introductory price because maintaining a relatively stable price at product launch is frequently observed in practice. For example, console video games, mainly due to consumer expectation, have tended to maintain the same base price of $60 for over ten years (Huang & Gilbert, 2018). Note that Section 2.4.3 considers more general cases where introductory price is not pre-determined, the R&D cost is time-varying, and the per-unit production and distribution cost is not zero.

Under the case we consider (i.e., $\gamma = 0$, $pr(0) = pr_0$, $l_2 = 0$, and $c(t) = 0$), the profit function (2.14) reduces to the following profit function:

$$\Pi[pr(t)] = \int_0^T [pr(t)I^\beta y(X(t))\left(1 + \alpha \frac{pr'(t)}{pr(t)}\right) - c_1]e^{-rt}dt. \qquad (2.17)$$

By setting $Q(t) \coloneqq \frac{pr'(t)}{pr(t)}$, the profit optimization problem is formulated as:

$$\text{Max } \Pi[Q(t)] = \int_0^T [pr(t)I^\beta y(X(t))(1 + \alpha Q(t))]e^{-rt}dt, \qquad (2.18)$$

$$s.t. \quad pr'(t) = Q(t)pr(t), \qquad (2.19)$$

$$pr(0) = pr_0, \qquad (2.20)$$

where $Q(t) \in [\underline{Q}, \overline{Q}]$, $\underline{Q} < 0$, and $\overline{Q} > 0$. Note that to guarantee the positivity of sales given the optimal price (i.e. $s(t) \geq 0$ for all $t \in [0, T]$), $\overline{Q}$ is set in such a way that $1 + \alpha\overline{Q} > 0$.

The problem presented in (2.18)-(2.20) is an optimal control-based reformulation of the initial profit optimization problem in which $Q(t)$ is the control variable. For the optimal control problem shown in (2.18)-(2.20) (see Sethi & Thompson, 2000, p. 67), the Hamiltonian is $H(t) = \Gamma_1(t) + Q(t)\Gamma_2(t)$, where $\Gamma_1(t) = pr(t)I^\beta y(X(t))$, $\Gamma_2(t) = pr(t)\alpha I^\beta y(X(t)) + \lambda(t)pr(t)$, and $\lambda(t)$ denotes the costate variable where

$$\lambda'(t) = r\lambda(t) - \frac{\partial H}{\partial pr}, \quad \lambda(T) = 0. \qquad (2.21)$$

By linearity of the Hamiltonian in $Q(t)$ and the maximum principle, the optimal $Q^*(t)$ is:

$$Q^*(t) = \begin{cases} \overline{Q} & \Gamma_2(t) > 0, \\ \underline{Q} & \Gamma_2(t) < 0. \end{cases} \qquad (2.22)$$

The characteristics of the optimal price path corresponding to the optimal control variable $Q^*(t)$ are summarized in Theorems 2.1 and 2.2 that follow. Note that the optimal price path is derived by solving the initial value problem (2.19) and (2.20) for $Q(t) = Q^*(t)$. Proofs are provided in Appendix A.

Theorem 2.1. If $-\alpha r > 1$, then $Q^*(t) = \underline{Q}$ for all $t \in [0, T]$.

Theorem 2.1 indicates that in the cases where future is highly uncertain (i.e., the discount rate *r* is large), or when sales are highly sensitive to changes in price (i.e. $-\alpha$ is large), or both, the optimal price path is declining. Therefore, Theorem 2.1 demonstrates the conditions under which a price-skimming strategy is optimal. The results of Theorem 2.1 are consistent with the frequently-observed pricing strategy for video games, because their sales price typically declines a few months after release (Patterson, 2021), and price-conscious customers often delay their purchases until the price declines. Note that the findings under Theorem 2.1 are valid regardless of the rate of repeat purchases. Therefore, price skimming can be optimal, for example, for video games even though video games have almost no repeat purchases.

With respect to situations where future profit is highly uncertain, it is more desirable to acquire revenue sooner than later, so in such environments it makes sense for firms to motivate sales by decreasing the price. The general shape of the optimal price path for products (for which sales includes both initial and repeat purchases) in the cases covered by Theorem 2.1 is declining regardless of the existence of repeat purchases, consistent with the result derived by Krishnan et al. (1999) for the optimal price path of product adoption in similar cases.

Theorem 2.2. Suppose $-\alpha r < 1$:

a. If $\Gamma_2(0) > 0$, then the optimal price path assumes a first increasing and then decreasing pattern. More specifically, there exists a unique $t^* \in [0, T]$ where

$$\begin{cases} \Gamma_2(t) > 0, & t \in [0, t^*), \\ \Gamma_2(t) < 0, & t \in (t^*, T]. \end{cases} \tag{2.23}$$

b. If $\Gamma_2(0) < 0$, then the optimal price path is monotonically declining, *i.e.* $Q^*(t) = \underline{Q}$ for all $t \in [0, T]$.

c. If $\Gamma_2(0) = 0$, then either *a* or *b* happens.

In Theorem 2.2, when future uncertainty is low (r is small) or sensitivity to changes in price is low (i.e. $-\alpha$ is small), or both, the optimal price path is either (i) first increasing, then decreasing, or (ii) monotonically declining. The general shape of the optimal price path for products with significant initial and repeat purchases in the scenarios considered in Theorem 2.2 is also consistent with that derived by Krishnan et al. (1999) for the optimal price path for initial product purchases in similar scenarios.

Theorems 2.1 and 2.2 suggest that when future is highly uncertain and the market is highly price-sensitive, it makes sense to drop the price over time to accelerate sales. Furthermore, when the market is price-indifferent, even high sales-uncertainty levels may not warrant adopting a declining pricing strategy. In fact, firms may even be able to get away with increasing the price for a while before needing to drop the price to motivate sales.

2.4.3 General Price Path Optimization Model

We next examine the optimal pricing strategy for general product categories; the profit optimization problem involves maximizing the total profit shown in (2.14). To formulate the problem, we make the following reasonable assumptions:

I. Price is always positive (i.e. $pr(t) \geq \lambda_0 > 0$, for all $t \in [0, T]$).

II. The introductory price is bounded (i.e. $\lambda_0 \leq \lambda \leq pr(0) \leq \Lambda$). We believe this is a reasonable assumption because the introductory price of a given product depends on the product segment within which the product is introduced. In other words, the price of a given product should be within the range of comparable products.

III. The rate of change in price is bounded (i.e. $\delta \leq pr'(t) \leq \Delta$, $\delta < 0$, $\Delta > 0$, $t \in [0, T]$). Naturally, in any given market there is a limit for increasing or decreasing the price from one period to the next.

IV. The price path $pr(t)$ should be admissible in the sense that the corresponding sales function should be nonnegative (i.e. $s(t) \geq 0$, for all $t \in [0, T]$).

To guarantee the positivity of k, Λ is set in such a way that $1 + \gamma \ln(\Lambda) > 0$. Similarly, to guarantee the admissibility of price, we set $x(t) \geq \omega$, where ω is a small positive constant. We thus have $X(t) \geq 0$ and $s(t) \geq 0$ for all $t \in [0, T]$. While in fact the condition $x(t) \geq 0$ is sufficient to achieve $s(t) \geq 0$ for $t \in [0, T]$, we set $x(t) \geq \omega > 0$ for small ω to guarantee the admissibility of the optimal price path solutions, $pr_N(t)$s, the derivation of which is based on the procedure introduced in Appendix A.

Given assumptions I-IV, the objective of the optimization problem is to maximize the profit Π (shown in (2.14)) over the following set:

$$M = \{pr(t) \in C^1[0, T] \mid \delta \leq pr'(t) \leq \Delta,\ pr(t) \geq \lambda_0, x(t) \geq \omega, \lambda \leq pr(0) \leq \Lambda\}, \tag{2.24}$$

where $C^1[0, T]$ denotes the space of continuously differentiable functions defined on $[0, T]$. It should be noted that the profit function (2.14) is a functional defined on the space of price functions M. Moreover, the profit (Π) maximization problem is a functional maximization problem. In Appendix A, we show that there exists a μ, $\mu < \infty$, such that $\mu = Sup\, \Pi\,|_M$. To find a profit-optimal price path in $[0, T]$, we solve the following problem:

$$Max\ \Pi\,|_{M \cap P_N[0,T]}. \tag{2.25}$$

Here, Π is maximized on $M \cap P_N[0, T]$, a subset of the space of polynomial functions of degree at most N (see Appendix A). In fact, the functional optimization problem (2.25) can be interpreted as a function optimization problem in which the coefficients of the polynomials of degree at most N should be determined. The existence of a solution for problem (2.25) is demonstrated in Appendix A. Based on formulation of (2.25), the function maximization problem can be derived as:

$$Max\ \overline{\Pi}\left[pr_0, v_0, \dots, v_{K_1}\right], \tag{2.26}$$

$$s.t. \quad R\left[pr_0, v_0, \dots, v_{K_1}, w_0, \dots, w_{K_2}, u_0, \dots, u_{K_3}, r_0, \dots, r_{K_4}\right] \leq \varepsilon, \tag{2.27}$$

$$\lambda \leq pr_0 \leq \Lambda. \tag{2.28}$$

The derivation and details of the problem represented by (2.26)-(2.28) are presented in Appendix A, where we show that, for sufficiently large $K_1, \dots, K_4$ and sufficiently small and positive ε, the solution to the problem represented by (2.26)-(2.28) corresponds to a polynomial price function $pr_N(t)$ that satisfies the desired constraints of M, with the corresponding profit $\Pi[pr_N(t)]$ close to the supremum value μ.

2.5 Numerical Examination

As explained earlier, our fractional calculus-based price-optimization model is exceptionally complex, thus deriving analytical finding for the general scenarios is infeasible. Given that we do not intend to simplify the model to produce less realistic results, we conducted numerical examination to better evaluate price optimization for general sales scenarios. Furthermore, applying a mix of analytical and numerical examinations is an approach widely used in the literature (e.g., Cosguner & Seetharaman 2022; Oh & Su, 2022; Kim & Park, 2008; Debo et al., 2006; Krankel et al., 2006; Ray et al., 2005; Krishnan et al. 1999). When an analytical examination is not feasible, it is deemed appropriate in advanced operations studies to pursue a numerical examination (e.g., Robotis et al., 2012). This section reports the results of numerical experiments conducted based on the general optimization model presented in Section 2.4.3 and examines the optimal price path for different sales scenarios.

Our results demonstrate that the optimal price path in a product's demand window exhibits one price peak, and different optimal price-path shapes can occur due to the changing position of the peak of the optimal price path under different scenarios. Briefly, a decreasing optimal price path corresponds to the case when the peak of the optimal price path occurs at product release time. The optimal price path is first increasing then decreasing when the peak occurs within the product's demand window. Finally, the optimal price path is increasing when the peak appears at the end of the demand window considered. The details of the numerical experiments are provided next.

2.5.1 Sales Scenarios

We considered different scenarios for numerical experimentations. First, we considered a base scenario developed based on a set of values for the model parameters (i.e., sales, price, and cost). Next, for the base scenario, we developed 12 variational scenarios by altering the values of the model parameters one at a time.

The parameter values for the base scenario and its corresponding variational scenarios are reported in Table 2-1. Note that (2.15) and (2.16), which represent our cost functions, are similar to the cost function proposed by Krishnan et al. (1999), thus following Krishnan et al. (1999), we set ζ=0.005, and c_0 and c_1 were set equal to 1.5 and 25000, respectively. Using different values of c_0 and c_1, our tests demonstrate that the overall behavior of the optimal price path is not sensitive to the values of c_0 and c_1. For scenarios reported in Table 2-1, we set the market size to $m = 500000$ and the planning horizon to $T = 12$. In the base and variational scenarios reported in Table 2-1, the adoption process is set to be 99% complete after 12 years ($T = 12$), making these scenarios compatible with sales of technology products with relatively short lifecycles. For example, the smartphone penetrated into its target market in about 12 years (Comscore, 2017). Our tests based on different market sizes and planning horizons lead to similar results.

Table 2-1: Base Scenario and Variations

	Sales			Price		Cost		Discount rate
	Adoption		Repeat					
	p	q	β	γ	α	l_1	l_2	r
Base	0.01	0.75	0.5	-0.3	-5.5	0	0.1	0
Changing price parameters								
Variation 1	0.01	0.75	0.5	-0.27	-5.5	0	0.1	0
Variation 2	0.01	0.75	0.5	-0.33	-5.5	0	0.1	0
Variation 3	0.01	0.75	0.5	-0.3	-5	0	0.1	0
Variation 4	0.01	0.75	0.5	-0.3	-6	0	0.1	0
Changing cost parameters								
Variation 5	0.01	0.75	0.5	-0.3	-5.5	0.1	0.1	0
Variation 6	0.01	0.75	0.5	-0.3	-5.5	0.25	0.1	0
Variation 7	0.01	0.75	0.5	-0.3	-5.5	0	0.0	0
Variation 8	0.01	0.75	0.5	-0.3	-5.5	0	0.25	0
Changing repeat purchase parameter								
Variation 9	0.01	0.75	0.2	-0.3	-5.5	0	0.1	0
Variation 10	0.01	0.75	0.8	-0.3	-5.5	0	0.1	0
Changing discount rate								
Variation 11	0.01	0.75	0.5	-0.3	-5.5	0	0.1	0.01
Variation 12	0.01	0.75	0.5	-0.3	-5.5	0	0.1	0.05

The following section presents the optimal price path and sales trend for each of the cases reported in Table 2-1. We also explored the optimal pricing strategy under several other base and variational scenarios similar to the ones presented in Table 2-1. Instances of these additional numerical analyses can be found in Appendix A. In each base scenario, we first set the discount rate equal to zero ($r = 0$) then changed it in the variational scenarios. In Section 2.5.2 we report the results corresponding to Table 2-1. The results from other base and variational scenarios considered were highly consistent with those obtained from the scenarios presented in Table 2-1.

2.5.2 Optimal Price Path and Sales Trend

Problem (2.26)-(2.28) was solved for each of the cases presented in Table 2-1. The assumptions of the optimization model require that boundary values be set for the initial price pr_0 and the price growth rate pr'. For the scenarios presented in Table 2-1, we set $\lambda = 5$, $\Lambda = 7$, $\delta = -0.5$, and $\Delta = 0.5$, and the resulting optimal price paths and their corresponding optimal sales trends for the base and variational scenarios reported in Table 2-1 are shown in Figures (2-2)-(2-7). The main observations from these experiments are outlined next.

Figure 2-2 demonstrates that, for higher repeat-purchase rates (higher β), the peak point of the optimal price path is reached earlier, leading to a longer price decline and a lower price path. When the repeat purchase rate is sufficiently large, the optimal price path can monotonically decrease for the entire product lifecycle. In other words, with ample and continuing repeat purchases, a straight price-skimming strategy is recommended. The reason behind this result is that when the repeat-purchase rate is high and sustainable sales are expected, it is reasonable for firms to begin dropping the price earlier in the product lifecycle to accelerate anticipated sales. Conversely, when the repeat-purchases rate is low, the profit declines as the diffusion process progresses, with this declining profit being a result of the drop in sales. In such a situation, to sustain profitability it is reasonable for a firm to delay decreasing the price in the product lifecycle. Furthermore, for higher repeat purchase rates, Figure 2-2 shows that the introductory price is higher as well, further making the optimal price path compatible with a skimming-based price path where the introductory price is set at a relatively high level.

Our finding corresponding to the relationship between repeat purchase rate and pricing is consistent with real-world examples. For example, repeat purchases constitute a significant proportion of DVD player sales (Lotfi et al., 2023), while DVD players having been priced based on the skimming strategy (Beltis, 2019).

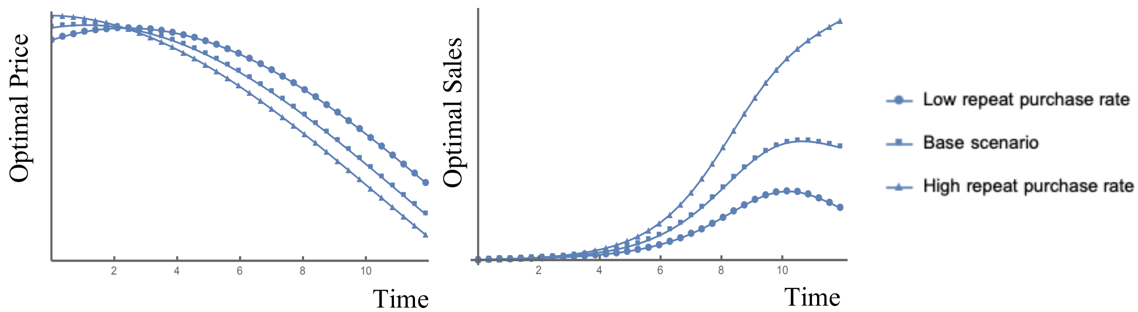

Figure 2-2: Optimal Price Path and Sales vs. Repeat Purchase Rate

This finding also has a direct implication for sales of products expected or planned as a fad with a low repeat-purchase rate. A fad is something that results in a short-lived intense enthusiasm (Steingoltz & Haslehurst, 2017). A fad product is one that at first rapidly gains popularity and then quickly loses popularity. Among prominent examples of fad technology products are 3D television sets, e-book readers (Thubron, 2017), beepers, and Google Glass (Erickson, n.d.). According to our findings, price skimming may not be the optimal pricing strategy for fad products with insignificant repeat purchases. For such products, firms may be better off with a price path that begins with a relatively lower introductory price that then peaks later in the product lifecycle.

Figure 2-3 shows that for smaller values of the baseline price parameter γ (i.e., customers are more sensitive to the baseline price), both introductory and average prices decrease. For larger values of γ (i.e., customers are less sensitive to the baseline price), both introductory and average prices increase.

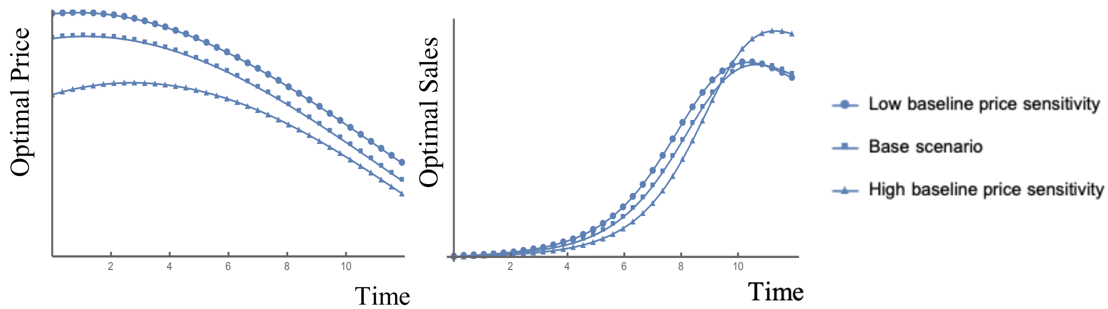

Figure 2-3: Optimal Price Path and Sales vs. Baseline Price Sensitivity (γ)

This result has implications for positioning a new product in the market, and failure to properly price a new product can be detrimental to the product's market success. For example, Google Glass, known to be an epic failure (Crothers, 2016), was priced at \$1500, despite it being a limited product (Plafke, 2013). Google Glass was marketed as an innovative product but failed to live up to its initial hype (Ward, 2019). Among other factors, Google Glass's failure has been attributed to its \$1500 price tag (Hodgkins, 2019).

Figure 2-4 demonstrates that when the value of the price-change sensitivity parameter α decreases (i.e., when customers become more sensitive to changes in price), the peak point of the optimal price path occurs earlier in the product lifecycle and the optimal introductory price increases. Figure 2-4 also demonstrates that for sufficiently high price-change sensitivity, the optimal price path monotonically decreases. This result suggests that when customers are more sensitive to price changes, contrary to common expectations, firms may get away with higher introductory prices, while increasing the price over time is only tolerated by customers for a short time before a peak point reflecting the customers' ultimate price-increase tolerance is realized. Also, when customers are highly sensitive to price changes, firms should only decrease the price over time because price increases are not tolerated. This further demonstrates that when customer sensitivity to price changes increases, the optimal pricing strategy approaches the skimming strategy. This finding is consistent with the pricing strategy of many technology products. For example, fitness trackers have become more afordable over time (Toner, 2018). One may argue that potential customers of activity trackers tend to be price sensitive due to the existance of alternative products (e.g., health and activity smartphone applications).

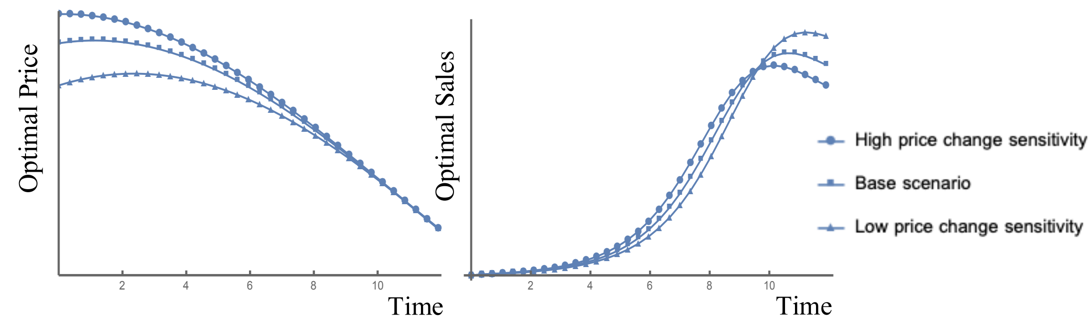

Figure 2-4: Optimal Price Path and Sales vs. Price Change Sensitivity (α)

Figure 2-5 shows that as the effect of economies of scale (l_1) increases, the optimal price path shifts downward, and vice versa. This result is expected since the greater effect of economies of scale leads to a faster drop in per-unit cost as sales increases, thus allowing a firm to achieve maximum profitability through a lower price level.

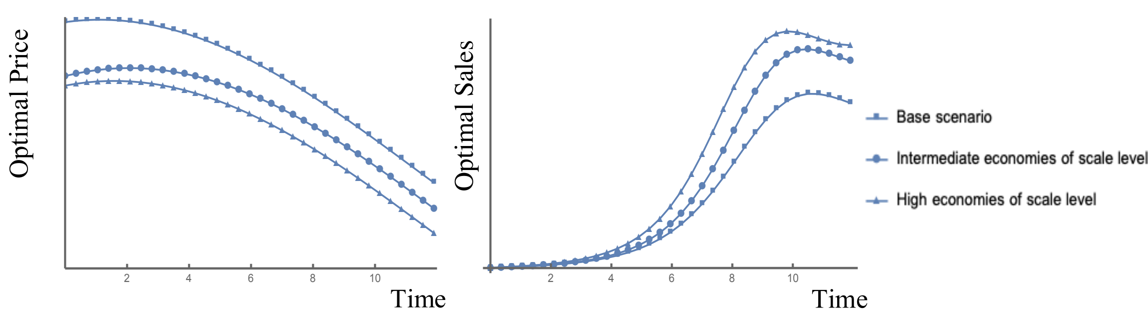

Figure 2-5: Optimal Price Path and Sales vs. Economies of Scale

Figure 2-6 demonstrates that with a more significant learning effect (l_2), the peak of the optimal price path occurs earlier. A higher learning rate allows the R&D cost to drop more rapidly, thus allowing a firm to begin dropping the product price earlier in the product lifecycle to motivate sales.

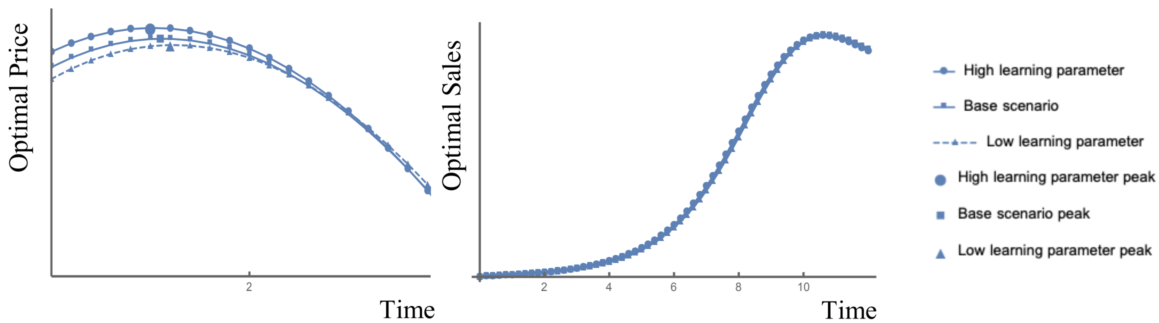

Figure 2-6: Optimal Price Path and Sales vs. R&D Learning Rate

Finally, Figure 2-7 shows that when the discount rate (r) increases, the optimal price path drops at a faster rate. This result is intuitive, because when future is uncertain, firms should drop the price at a faster rate to increase sales in the earlier stages of the product lifecycle associated with less uncertainty. This finding shows that when future is uncertain, the optimal price path assumes a declining path that resembles that of a price-skimming-based price path, resulting in more sales earlier in the product lifecycle.

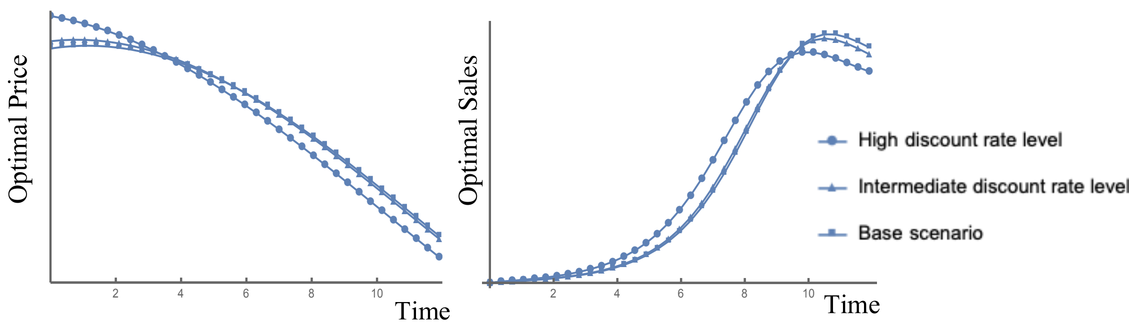

Figure 2-7: Optimal Price Path and Sales vs. Discount Rate

In summary, the most important results from our numerical examinations are as follows:

- The optimal price trend is highly dependent on the rate of repeat purchases. The optimal price path may deviate from the skimming path when the repeat purchases rate is low and sales are not expected to sustain over time.
- Price skimming may not be the optimal pricing strategy for a product expected or planned to be a fad product with a low repeat-purchase rate.

- Contrary to common expectations, customers more sensitive to changes in price may tolerate a higher introductory price. However, increasing price over time is tolerated by such customers for only a short time period, after which a peak point representing customers' maximum tolerance for price increase emerges. When market sensitivity to price changes is low, the market can accommodate an increasing price trend for a longer time, up to a point where price reaches a peak point, reflecting the maximal price tolerable by the market, after which the price should be reduced to further motivate sales.
- When the market is expected to be more (less) sensitive to the product price at the product launch time which determines the product's base-line price and its positioning in the market, the optimal product launch price tends to be lower (higher).
- When faced with highly uncertain future, maximum profit can be achieved by quickly reducing the price to motivate customers to make purchases sooner than later.

In summary, our results suggest that overlooking repeat purchases when examining the optimal price path of a new product is likely to lead to suboptimal pricing decisions, underscoring the contribution of the optimization model developed in this study.

2.6 Concluding Remarks

In this study, we examined optimal pricing strategies for technology products. Specifically, we investigated the price-skimming strategy widely used by prominent technology companies. To achieve realistic and dependable results, we adopt and extend the generalized diffusion model with repeat purchases (GDMR) by Lotfi et al. (2023), which is suitable for technology products with high rate of repeat purchases. By extending the GDMR to incorporate the effects of both baseline price and price changes, we formulate and advance a price-optimization model. The profit function we propose accounts for changes in cost due to both the learning effect and the effect of economies of scale, further contributing to the body of relevant literature that has used simpler cost structures. We examine the optimal price path under (i) a special case compatible with software sales where the introductory price is predetermined, and cost includes only a constant R&D cost with variable cost of zero, and (ii) a general case suitable for a variety of product categories when the introductory price is not predetermined and economies of scale and a dynamic R&D cost are considered.

Our results suggest that despite price skimming being widely used for technology products, it should not always automatically be considered as the go-to pricing strategy for such products. Whether or not price skimming should be used is highly dependent on the rate of repeat purchases. The price-skimming strategy benefits from having a group of customers who, because of their willingness to pay more for the product, can be effectively skimmed off (Liu, 2010). By incorporating such consumer heterogeneity in our analysis, we find that the optimal pricing strategy may deviate from the skimming strategy when consumers cease to demonstrate continued interest in the product through repeat purchases. In other words, our results suggest that price skimming is likely optimal when the repeat-purchases rate is high and sales are sustainable at a high level, thus underscoring the fact that accounting for repeat purchases is of prime importance when deriving an optimal pricing strategy. For example, when the rate of repeat purchases is high and sales are expected to remain sustainable over time, if firms use an optimization model that accounts only for adoption purchases, the adopted pricing strategy may deviate significantly from the optimal pricing strategy.

Our findings also have direct implications for sales of fad products with low repeat-purchase rates. Fad products experience a rapid increase in popularity followed by a quick decline in sales. Our findings indicate that price skimming may not be an optimal pricing strategy for such products. Our results instead suggest that, for fad products with low repeat-purchase rates, firms are likely better off setting their introductory price at a lower level and then increasing the price to a peak reflecting the market's maximum tolerable price before reducing it to further motivate sales. If such a pricing strategy is not followed for a product expected or planned to be a fad product with a low repeat-purchase rate, the chance of product success may be significantly lower. In particular, launching such a product with a relatively high introductory price, more compatible with a skimming strategy, can be detrimental to its success.

Other key findings are summarized as follows:

- When customers demonstrate a higher sensitivity to price changes, contrary to common expectations, a firm may get away with launching the product at a higher price. However, increasing the price over time is tolerated for a shorter time period before a peak point reflecting customers' highest tolerated price level is reached.
- When customers are more (less) sensitive to the introductory price which determines the baseline price for the product, the optimal introductory price decreases (increases).
- If a product's future is highly uncertain, a firm is better off reducing the product's price quickly so that customers will make purchases sooner rather than later.

Our results also suggest that, under conditions of significant repeat purchases, the optimal price path even in different scenarios still assumes a trend with at most one peak, so our findings are compatible with those in the price optimization research literature that consider only adoption (first-time) purchases (e.g., Krishnan et al., 1999).

The optimization model developed in this study can pave the way for further research in the price-optimization domain and in business research in general. The adoption process corresponding to the sales process examined in this study is governed by the Bass diffusion model, thus a similar modeling approach could be used to examine the optimal price path based on other diffusion models (e.g., Van den Bulte & Joshi, 2007). The optimal price path for sales of products with multiple generations can also be examined.

From a broad perspective, our research contributes to the large body of literature on fractional calculus in applied mathematics and science by introducing a novel and innovative application of Riemann-Liouville integral to solve a problem in management science where the great potentials of fractional calculus have yet to be explored. Historically, applications of fractional calculus in science have been limited by the interpretation of this type of mathematics. Tackling the problem of interest using a fractional calculus-based model with clear interpretation, our study can help encourage researchers explore and develop different interpretations of fractional calculus in different application areas.

Chapter 3

3 Modeling Sales of Multigeneration Technology Products in the Presence of Frequent Repeat Purchases: A Fractional Calculus-based Approach[2]

3.1 Introduction

Release of new product generations featuring major technological advancements takes place occasionally, while minor advancements of a current generation happen frequently. For example, digital cameras were introduced after few decades of numerous minor improvements in analogue cameras. When presented with options, some consumers may be interested in upgrading to a newer generation with a more advanced technology, while others may prefer an older generation to benefit from a more established technology. Specifically, in some cases, an older generation can have advantages over newer generations, making them favorable to some consumers. For example, film cameras have higher dynamic ranges and can capture higher resolutions than do most digital cameras, resulting in professional photographers' continued interest in them (The Dark Room, 2021). Similarly, sales of LCD TVs continued even after the introduction of LED TVs (Wang & Yu, 2010).

Repeat purchases of a product generation occur in the form of replacements and multi-unit ownership purchases. Modern technology products, encompassing both hardware and software, frequently undergo a steady stream of new releases and iterations. Many consumers exhibit a proclivity for consistently acquiring the latest product iterations. Moreover, in contrast to previous decades, it has become increasingly common for consumers to possess multiple units of the same or similar products (Lotfi et al., 2023). Given the prevalence of frequent repeat purchases, traditional multigeneration diffusion

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models no longer suffice for accurately projecting sales path for technology products. Therefore, a multigeneration sales growth model that takes into account within-generation repeat purchases as well as initial purchases and cross-generation upgrades becomes essential. In this research, we develop a comprehensive multigeneration sales model that addresses this necessity.

We name the new model the *generalized multigeneration diffusion model with repeat purchases* (GMDR). The new model accommodates two different generation transition strategies, i.e., (i) *phase-out transition strategy* and (ii) *total transition strategy* (Jiang et al., 2019). With the phase-out transition strategy, firms continue to sell an old generation after the release of a new generation to satisfy the continued demand for the old generation. The phase-out transition strategy has been widely applied. For example, as mentioned above, TV manufacturers continued to sell LCD TVs after the release of LED TVs. Similarly, although digital cameras have long been introduced to the market, some camera manufacturers such as Canon and Nikon continued to sell analog film cameras until recently. With the total transition strategy, upon the introduction of a new generation, the previous generation is immediately discontinued. For instance, Fitbit, a fitness tracker manufacturer, replaced its older products Alta, Zip, One, and Flex 2 with the new generation Fitbit Inspire (Heater, 2019).

We demonstrate that the new model can be used for both predictive and prescriptive analytics. Using real sales data and Google Trends product search data, we demonstrate that the GMDR leads to more accurate model estimations and forecasts than a state-of-the art multigeneration diffusion model that does not incorporate repeat purchases, further demonstrating the importance of incorporating repeat purchases in modeling product sales. Furthermore, we use the new model to find the optimal market entry timing of a new generation in a product line. Extensive planning goes into releasing a new product, including planning technical specifications of the product, developing a marketing strategy, and adopting a pricing plan for maximal profit. One critical and challenging aspect is to determine the best time to launch the new product to the market especially when the product is a new generation to an existing one. This is because individual products in a multigeneration product line usually have a competitive and complementary relationship. Therefore, having an effective strategy as to when to release a new product generation is of great importance. Many firms fail in transitioning from a current product generation to a new one due to improper timing of the new generation's release. One example of such failures is Kodak's release of their digital camera. While digital camera technology was first invented in the company's R&D lab in the 1970s, Kodak's leadership postponed transitioning to the digital camera technology fearing that the new technology would cannibalize the sales of the existing film cameras (Gann, 2016). Kodak ended up in a bankruptcy in 2012, a result mainly attributed to the company's slow transition from analogue technology to digital technology (Landoni, 2015).

Multigeneration diffusion models, first introduced by Norton and Bass (1987), provide a framework for studying optimal market entry timing in a product line with successive generations. However, as we discuss in the next section, there is little research in the literature studying the optimal market entry timing. In these limited few studies, the underlying multigeneration diffusion models do not consider repeat purchases within each generation, rendering them unusable for sales of technology products that include a significant proportion of such purchases. To fill this gap, we use our new model to study the optimal new-generation release timing in a product line under the phase-out transition and total transition strategies.

Our results demonstrate that the optimal entry timing is highly dependent on the rate of repeat purchases. Depending on the repeat purchases rate, under both the phase-out transition and total transition scenarios, there are cases in which firms are better off launching the new generation as early as possible and cases under which firms are advised to delay the release of the new generation as much as possible. Specifically, we find that when (i) the potential market size for the old generation is sufficiently large, (ii) the unit contribution margin for the old generation is greater than that for the new generation, and (iii) the repeat purchases rate for the old generation is at least as high as the repeat purchases rate for the new generation, firms are better off delaying the introduction of the new generation as much as possible. We also find that regardless of how large the potential market sizes for the old and new product generations are, firms are better off immediately releasing the new generation when (i) the unit contribution margin for the new generation is at least as high as that for the old generation and is declining, and (ii) the repeat purchases rate of the new generation is at least as high as the repeat purchases rate of the old generation. In other words, if the unit contribution margin for the new generation is declining, even if the repeat purchases rates and the unit contribution margins for the old and new generations are identical, an immediate release of a new generation is recommended.

In summary, the contribution of the present research to the literature is two-fold: (i) developing a new multigeneration sales model that accounts for within-generation repeat purchases and (ii) using the new model to derive the optimal market entry timing strategies under the phase-out transition and total transition scenarios.

Following this introductory section, the remainder of the chapter is organized as follows. The related literature is reviewed in Section 3.2. The development of the multigeneration sales model is detailed in Section 3.3. Empirical model testing results are provided in Section 3.4. In Section 3.5, the optimal market entry timing is modeled, and insights are provided. We present concluding remarks in Section 3.6.

3.2 Literature Review

In this section, we review the literature on multigeneration product diffusion and optimal market entry timing.

3.2.1 Multigeneration Product Diffusion

Inspired by the Bass (1969) product diffusion model, Norton and Bass (1987) introduced the first multigeneration diffusion model. Their model assumes that each generation has its own diffusion process and, with the introduction of a new generation, adopters of the old generations can switch to the new generation. Their model can be used to describe units-in-use (systems-in-use) as well as subscription purchases when all subscribers renew their subscription in each time period. Following Norton and Bass (1987), many other models have been developed to capture the number of units-in-use in a multigeneration scenario (e.g., Kim et al., 2000; Sohn & Ahn, 2003; Bass & Bass, 2004; Jiang & Jain 2012; Shi & Chumnumpan, 2019). To model sales with repeat purchases, the abovementioned models can be used only when each user on average makes one repeat purchase in each time period, rendering them unusable for sales of products with different repeat purchases rates. In contrast, the focus of this paper is to explicitly incorporate repeat purchases in a sales model and subsequently derive the optimal market entry timing for a new product generation as one possible application of the proposed model.

In the operations management literature, Qu et al. (2022) develop an Exponential-Decay proportional hazard model to predict consumers' time-to-upgrade based on their past product adoption and usage behavior. However, the model they introduce does not incorporate between-generation interactions and within generation repeat purchases.

The aforementioned studies provide continuous-time modelling frameworks. Discrete-time models have also been introduced in the literature (e.g., Danaher et al., 2001; Li et al., 2013). However, these models do not have a closed form expression, making them unsuitable for analytically studying the optimal market entry timing. Alternatively, we implement a continuous-time model with a closed form expression that enables us to analytically examine the optimal market entry timing strategy.

There are multigeneration models in the literature that combine choice and diffusion models (e.g., Jun & Park, 1999; Kim et al., 2005). However, these models do not incorporate between-generation interactions. In our model we explicitly account for between-generation interactions. Kim et al. (2001) present an econometrics model for multigeneration product diffusion capturing both initial and repeat purchases. However, as they mention, their model is operationalizable at individual consumer level and not at an aggregate level. In contrast, our model is applicable for aggregate-level multigeneration sales data.

Overall, our literature review suggests that there is a need for a comprehensive model with a closed form expression capturing multigeneration product sales including (1) initial purchases for each generation, (2) repeat purchases within each generation, and (3) cross-generation upgrades. To fill this void, we first develop such a comprehensive sales model. Next, we use our proposed model to study the optimal market entry timing in product lines with successive generations. Our proposed modelling framework examines two main generation transition scenarios, i.e., phase-out transition in which, the firm continues to sale the old generation after the introduction of a new generation, and the total transition scenario where with the release of a new generation, the old generation is discontinued.

3.2.2 Optimal Market Entry Timing

Multigeneration diffusion models provide a desirable tool to study optimal market entry timing. However, there are few research works in the diffusion literature that study the optimal market entry timing strategy. Kalish and Lilien (1986) develop a single generation diffusion model to derive the optimal market entry time for a new product. Wilson and Norton (1989) build a multigeneration diffusion model and employ it as a part of a profit optimization model to capture the optimal market entry timing. They assume the profit margin for selling one unit of a product to be constant over time, which may not be realistic. Mahajan and Muller (1996) study the optimal market entry timing in a multigeneration setting. Their model counts system-in-use for a durable technology product. Krankel et al. (2006) consider a firm's decisions on the introduction timing of successive product generations for a durable product. In their modelling framework, sales in each generation follow a diffusion process. None of the abovementioned studies incorporate within generation repeat purchases. Guo and Chen (2018) study market entry timing based on a multigeneration diffusion model they develop. However, in their analysis repeat purchases are made by strategic consumers and only in the form of upgrade from the old generation to the new generation; their model does not incorporate within generation repeat purchases. Furthermore, Guo and Chen's model does not have a closed-form expression, requiring them to study the optimal market entry timing numerically. Recently, Jiang et al. (2019) derive the optimal market entry timing based on the multigeneration model developed by Jiang and Jain (2012). Jiang et al. (2019) study both the phase-out transition scenario and total transition scenario. They incorporate repeat purchases only in the form of resubscriptions and do not take into account repeat purchases for purchase-to-own products whose life-time ownerships are transferred to buyers at the time of purchase. Moreover, they assume that the discounted profit margin for one unit of a product to be constant over time, thus leaving the door open for analyzing more complex cases.

Product upgrade timing has also been studied in the operations management literature. For example, Mehra et al. (2014) study the optimal upgrade interval in a growing market that has homogeneous customers. Kirshner et al. (2017) study product upgrade, while accounting for technology advancement, brand commitment, and product failure. Sun et al. (2022) study how a firm can implement reference-group effects in the introduction of product upgrades. These studies, however, do not account for repeat purchases made within a generation and between-generation interactions.

Our investigation in the diffusion literature reveals that there is a need for a comprehensive analysis of the optimal market entry timing strategy for product lines with successive generations in the presence of (i) initial purchases of each generation, (ii) within-generation repeat purchases and (iii) cross-generation upgrades. Therefore, we employ our proposed sales model that accounts for these three types of purchases to study the optimal market entry timing for product lines composed of successive product generations. Furthermore, we study both phase-out transition and total transition strategies for releasing a new product generation. Moreover, as opposed to Wilson and Norton (1989) and Jiang et al. (2019), we let the discounted unit profit margin of a product to vary over time.

3.3 Modeling Multigeneration Product Sales

In this section, we develop a new multigeneration product sales model. First, we define the types of purchases that we consider. Our unit of analysis is a group or line of products that fall under multiple successive generations. While the product versions belonging to a given product generation are different only incrementally, the differences from one generation to another are categorized by more significant types of innovation, as defined by Henderson and Clark (1990). We consider the following purchases:

- *Adoptions of the first generation*: Initial purchases made for the first generation.

- *Repeat purchases*: A new purchase is considered a repeat purchase only if the newly purchased product is the same as or only incrementally different from the initially purchased product. Repeat purchases happen within one generation.
- *Adoptions of a new generation*: Initial purchases made for a new generation by buyers who are specifically interested in that generation. These adopters include (i) those who have purchased one or more of the old generations but have not repeat purchased the old generations, have stopped making further repeat purchases for the old generations, or have purchased and have been repeat purchasing the old generations, and (ii) those who have never purchased the old generations. To be specific, adopters of a new generation make their adoption decision independent of the behavior they demonstrated toward previous generations.
- *Leapfrogging*: Initial purchases made for a new generation by those who would have adopted one of the old generations had there not been a new generation.
- *Switching*: Purchases of a newer generation product by (i) those who have purchased an old generation and would have made repeat purchases for the old generation had there not been a new generation, or (ii) those who have purchased an old generation and would have repeated the purchase of the old generation had it not been discontinued.

It is important to note that in multigeneration scenarios with more than two generations, leapfrogging or switching purchases skipping one or more generations can occur (e.g., a customer may leapfrog from generation one to generation three or a customer may switch from generation two to generation five).

Figure 3-1 illustrates the sales components and their interplay in a phase-out transition scenario for two generations of products. Sales components of generation 1 and generation 2 are represented in distinct shades, with generation 1 shown in blue and generation 2 in gray. Components of sales representing shifts from generation 1 to generation 2 (either through leapfrogging or switching) are depicted in white. These components would have remained in generation 1 if generation 2 did not exist.

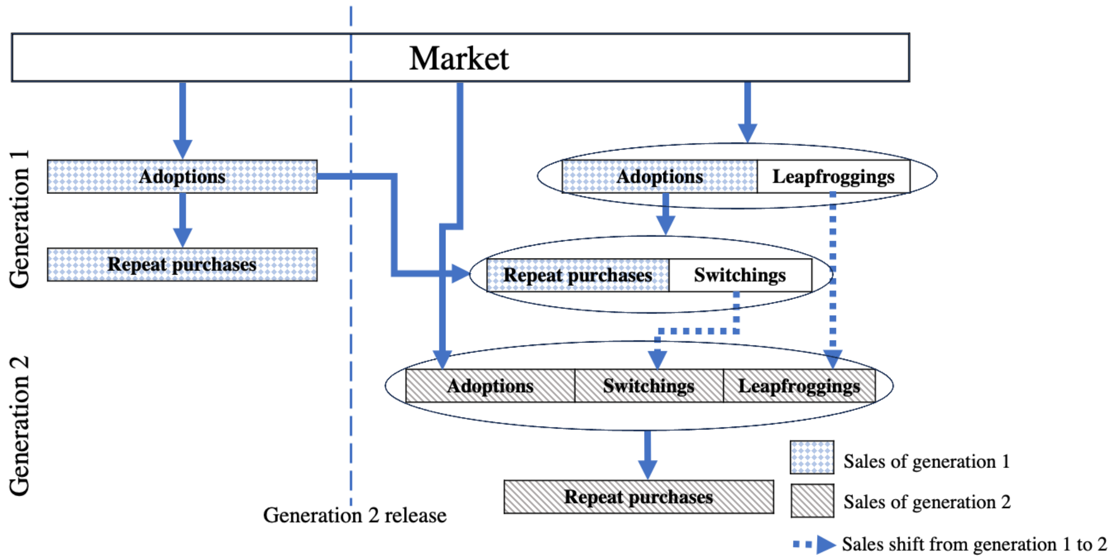

Figure 3-1: Breakdown of Sales Components in a Phase-out Transition Scenario Across Two Generations

Next, we provide the details of our multigeneration sales model. We separately model the (i) phase-out transition scenario where firms continue to sell the previous product generation along with the newly introduced generation, and (ii) the total transition scenario where the older product generation is discontinued when the new generation is introduced. Following Lotfi et al. (2023), we use a fractional calculus-based operator to incorporate repeat purchases. Lotfi et al developed a sales model for technology products that encompasses both adoptions and repeat purchases. Through extensive empirical testing, they demonstrate that the fractional integral operator accurately models the repeat purchase rate for technological products. They formulate sales as

$$S(t) = I_0^{\beta} y(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} y(s) ds, \quad t \geq 0. \tag{3.1}$$

Here, $y(t)$ represents the adoptions at time t and β ($0 \leq \beta \leq 1$) denotes the order of the fractional integral operator I_0^{β} that captures the rate of repeat purchases. A greater β value corresponds to higher repeat purchase rate. To capture the buyer's adoption, Lotfi et al use the Bass diffusion model (Bass, 1969) given by

$$y(t) = \left(p + \frac{q}{m} Y(t)\right)\left(m - Y(t)\right), \tag{3.2}$$

where $p, q,$ and m denote the coefficient of innovation, capturing potential adopters' internal motivation to adopt a new product, coefficient of imitation, capturing the word-of-mouth effect in the adoption process, and market potential, respectively. In (3.2), $y(t)$ represents adoption rate at time t and $Y(t)$, given by $Y(t) = \int_0^t y(s)\,ds$, represents the cumulative adoption at t. The closed-form solution derived from equation (3.2) can be expressed as follows:

$$y(t) = Y'(t) = m\frac{(p+q)^2}{p}\frac{e^{-(p+q)t}}{\left(1+\frac{q}{p}e^{-(p+q)t}\right)^2}, \quad t \geq 0. \tag{3.3}$$

In this research, we account for different product generations which is not considered in Lotfi et al. (2023). For further clarity, Sections 3.3.1 and 3.3.2 present a two-generation scenario and in Section 3.3.4 we present the general multiple-generation scenario.

3.3.1 Phase-out Transition Scenario with Two Product Generations

We first model the phase-out transition scenario, as also considered by Jiang and Jain (2012). Without loss of generality, we assume generation 1 is introduced at time $t = 0$ and generation 2 is released at time $t = \tau_2 > 0$. In the absence of generation 2, the adoption rate of generation 1 follows the Bass Model (Bass, 1969). Specifically, generation 1's rate of adoption before the release of generation 2 is

$$y_1(t) = m_1\, f_1(t), \tag{3.4}$$

where

$$f_1(t) = \frac{(p_1+q_1)^2}{p_1}\frac{e^{-(p_1+q_1)t}}{(1+\frac{q_1}{p_1}e^{-(p_1+q_1)t})^2}, \quad t \geq 0,$$

denotes the density function of generation 1's adoption, at time t before the release of generation 2. p_i, q_i and m_i represent the innovation effect, imitation effect, and the size of the potential market, respectively, for generation i =1, 2. We formulate the adoption rate of generation 2 starting at $t = \tau_2$ based on the Bass (1969) model as

$$\tilde{y}_2(t) = m_2 f_2(t - \tau_2), t \geq \tau_2, \tag{3.5}$$

where

$$f_2(t-\tau_2) = \frac{(p_2+q_2)^2}{p_2}\frac{e^{-(p_2+q_2)(t-\tau_2)}}{(1+\frac{q_2}{p_2}e^{-(p_2+q_2)(t-\tau_2)})^2}, \quad t \geq \tau_2.$$

The adoption rate of generation 1 is influenced by leapfroggings to generation 2 after the release of generation 2. Specifically, the rate of adoption for generation 1, $y_1(t)$, before and after the release of generation 2 can be expressed as

$$y_1(t) = \begin{cases} y_{11}(t), & 0 \le t < \tau_2, \\ y_{12}(t), & t \ge \tau_2. \end{cases} \tag{3.6}$$

In formulation (3.6), $y_{11}(t) = m_1 f_1(t)$ accounts for the adoption rate of generation 1 before the introduction of generation 2. We model leapfroggings from generation 1 to generation 2 following Jiang and Jain (2012), i.e.,

$$u_2(t) = y_{11}(t) F_2(t - \tau_2), \tag{3.7}$$

in which $F_2(t - \tau_2)$ denotes the cumulative distribution function for generation 2. Considering that $F_2(t - \tau_2)$ increases with time, the rate of leapfrogging also increases over time. Subsequently, $y_{12}(t) = y_{11}(t) - u_2(t)$ denotes the adoption rate for generation 1 after the release of generation 2 at $t = \tau_2$.

At time t, a fraction of those who have adopted generation 1 at $s \in [0, t)$, repeat purchase at t. Following Lotfi et al. (2023), the total number of adoptions and repeat purchases for generation 1 at time t, $S_1(t)$, can be captured using a β_1-order ($0 \le \beta_1 \le 1$) Riemann-Liouville integral of adoptions as

$$S_1(t) = I_0^{\beta_1} y_1(t) = \int_0^t \frac{1}{\Gamma(\beta_1)} (t - s)^{\beta_1 - 1} y_1(s) ds, \quad t > 0. \tag{3.8}$$

Here β_1 is generation 1's coefficient of repeat purchases. According to Lotfi et al. (2023), $I_0^{\beta_1} y_1(t) - y_1(t)$ captures the rate of repeat purchases taking place at time t for generation 1.

It is expected that after the release of generation 2, a lower fraction of those who have purchased generation 1 in the past, repeat purchase generation 1 at the present time and the rest of them *switch* to the new generation. Similar to leapfroggings, switchings are expected to happen at a rate proportional to generation 2's adoption rate. Specifically, the number of switching buyers from generation 1 to generation 2 at time t, $swt_2(t)$, can be written as

$$swt_2(t) = h_2 * F_2(t - \tau_2) \left(I_0^{\beta_1} y_1(t) - y_1(t) \right), \ \ t \ge \tau_2. \tag{3.9}$$

Here, parameter $0 < h_2 \le 1$ is included to capture cases in which not all buyers of generation 1 switch to generation 2. Subsequently, sales of generation 1 can be formulated as

$$S_1(t) = \begin{cases} I_0^{\beta_1} y_1(t), & 0 \leq t < \tau_2, \\ I_0^{\beta_1} y_1(t) - swt_2(t), & t \geq \tau_2. \end{cases} \quad (3.10)$$

Initial purchases of generation 2 by adopting users (i.e., $\tilde{y}_2(t)$) and leapfrogging users (i.e., $u_2(t)$) is

$$y_2(t) = \tilde{y}_2(t) + u_2(t), \;\; t \geq \tau_2. \quad (3.11)$$

At time t, repeat purchases corresponding to $y_2(t)$ can be incorporated using a Riemann-Liouville integral of order β_2 as

$$I_{\tau_2}^{\beta_2}\, y_2(t) = \int_{\tau_2}^{t} \frac{1}{\Gamma(\beta_2)} (t-s)^{\beta_2 - 1}\, y_2(s) ds, \quad t \geq \tau_2, \quad (3.12)$$

where β_2 $(0 \leq \beta_2 \leq 1)$ denotes generation 2's repeat purchases coefficient. Furthermore, incorporating switchings shown in (3.9), generation 2's sales is given by

$$S_2(t) = I_{\tau_2}^{\beta_2} y_2(t) + swt_2(t), \;\; t \geq \tau_2. \quad (3.13)$$

3.3.2 Total Transition Scenario with Two Product Generations

We next consider the total transition scenario. Again, generations 1 and 2 are introduced at $t = 0$ and $t = \tau_2 > 0$, respectively. Similar to the phase-out transition scenario, before the release of generation 2, the adoption rate of generation 1 follows the Bass Model (Bass, 1969). Specifically, generation 1's rate of adoption before the release of generation 2 is $y_1(t) = m_1\, f_1(t)$. Similar to the phase-out transition scenario, sales, including both adoptions and repeat purchases for generation 1 at t before the introduction of generation 2, $S_1(t)$, is captured by a β_1-order $(0 \leq \beta_1 \leq 1)$ integral of adoption, i.e., $I_0^{\beta_1}\, y_1(t)$.

With the release of generation 2 at $t = \tau_2$, generation 1 is discontinued. Thus, the adoption rate in generation 1 drops to zero after the release of generation 2. Thus, generation 1's adoption rate at time t, $y_1(t)$ is

$$y_1(t) = \begin{cases} y_{11}(t), & 0 \leq t < \tau_2, \\ 0, & t \geq \tau_2, \end{cases} \quad (3.14)$$

where $y_{11}(t) = m_1\, f_1(t)$. After the introduction of generation 2, repeat purchases generated by the adopters of generation 1 drops to zero, thus the entire sales, including both adoptions and repeat purchases, drops to zero after the introduction of generation 2. Consequently, the sales of generation 1 at time t, $S_1(t)$, is

$$S_1(t) = \begin{cases} I_0^{\beta_1} y_1(t), & 0 \le t < \tau_2, \\ 0, & t \ge \tau_2. \end{cases} \tag{3.15}$$

After the release of generation 2 at $t = \tau_2$, generation 2's adoption can be written as $\tilde{y}_2(t) = m_2 f_2(t - \tau_2)$. Moreover, after the release of generation 2, those who would have adopted generation 1 or leapfrogged generation 1 had it been available, make initial purchases for generation 2 instead. These initial purchases (i.e., $y_{11}(t)$) in addition to generation 2's adoptions (i.e., $\tilde{y}_2(t)$) are

$$y_2(t) = \tilde{y}_2(t) + y_{11}(t), \;\; t \ge \tau_2. \tag{3.16}$$

The initial and repeat purchases corresponding to the initial purchases shown in (3.16) can be captured by a β_2-order ($0 \le \beta_2 \le 1$) integral of $y_2(t)$ as $I_{\tau_2}^{\beta_2}\, y_2(t)$.

With the introduction of generation 2, the number of switching purchases are captured as

$$swt_2(t) = I_0^{\beta_1} y_1(t) - y_1(t), \;\; t \ge \tau_2. \tag{3.17}$$

Given that $y_1(t) = 0$, for $t \ge \tau_2$ and based on the Riemann-Liouville integral, switching from generation 1 to generation 2 is

$$swt_2(t) = I_0^{\beta_1} y_1(t) = \int_0^{\tau_2} \frac{1}{\Gamma(\beta_1)} (t-s)^{\beta_1 - 1}\, y_{11}(s) ds, \quad t \ge \tau_2. \tag{3.18}$$

Consequently, generation 2's sales are given by

$$S_2(t) = I_{\tau_2}^{\beta_2} y_2(t) + swt_2(t), \;\; t \ge \tau_2\, . \tag{3.19}$$

Assuming that all repeat purchases of generation 2 take place at the same rate β_2, then

$$\begin{gathered} S_2(t) = I_{\tau_2}^{\beta_2} y_2(t) + swt_2(t) = \\ \int_{\tau_2}^{t} \frac{1}{\Gamma(\beta_2)} (t-s)^{\beta_2 - 1} \left(\tilde{y}_2(s) + y_{11}(s)\right) ds + \int_0^{\tau_2} \frac{1}{\Gamma(\beta_2)} (t-s)^{\beta_2 - 1}\, y_{11}(s) ds = \\ I_{\tau_2}^{\beta_2} \tilde{y}_2(t) + I_0^{\beta_2} y_{11}(t), \quad t \ge \tau_2. \end{gathered} \tag{3.20}$$

3.3.3 Model Operationalization

To operationalize the Riemann-Liouville fractional integral in an empirical setting we replace the sales model shown in (3.10) and (3.13) and the sales model shown in (3.15) and (3.20) with computationally tractable approximate models that have the desirable properties of the original models as done in Lotfi et al. (2023). An approximate formulation for the phase-out transition model (3.10) and (3.13) is given by

$$S_1^n(t) = \begin{cases} \frac{y_{11}(0)t^{\beta_1}}{\Gamma(1+\beta_1)} + I_{0,n}^{1+\beta_1} y'_{11}(t), & 0 \le t < \tau_2, \\ \frac{y_{11}(0)t^{\beta_1}}{\Gamma(1+\beta_1)} + I_{0,\tau_2,n}^{1+\beta_1} y'_{11}(t) + I_{\tau_2,n}^{1+\beta_1} y'_{12}(t) - swt_2^n(t), & t \ge \tau_2, \end{cases} \tag{3.21}$$

$$S_2^n(t) = \frac{(t-\tau_2)^{\beta_2}}{\Gamma(1+\beta_2)} y_2(\tau_2) + I_{\tau_2,n}^{1+\beta_2} y'_2(t) + swt_2^n(t), \tag{3.22}$$

where

$$swt_2^n(t) = h_2 * F_2(t-\tau_2)\left(\frac{y_{11}(0)t^{\beta_1}}{\Gamma(1+\beta_1)} + I_{0,\tau_2,n}^{1+\beta_1} y'_{11}(t) + I_{\tau_2,n}^{1+\beta_1} y'_{12}(t) - y_1(t)\right), \ t \ge \tau_2. \tag{3.23}$$

Similarly, an approximate model for the total transition model (3.15) and (3.20) is formulated as

$$S_1^n(t) = \begin{cases} \frac{y_{11}(0)t^{\beta_1}}{\Gamma(1+\beta_1)} + I_{0,n}^{1+\beta_1} y'_{11}(t), & 0 \le t < \tau_2, \\ 0, & t \ge \tau_2, \end{cases} \tag{3.24}$$

$$S_2^n(t) = \frac{(t-\tau_2)^{\beta_2}}{\Gamma(1+\beta_2)} \tilde{y}_2(\tau_2) + I_{\tau_2,n}^{1+\beta_2} \tilde{y}'_2(t) + \frac{y_{11}(0)t^{\beta_2}}{\Gamma(1+\beta_2)} + I_{0,n}^{1+\beta_2} y'_{11}(t), \qquad t \ge \tau_2. \tag{3.25}$$

Here $I_{0,n}^{1+\beta}$, $I_{0,\tau_2,n}^{1+\beta}$, $I_{\tau_2,n}^{1+\beta}$, are approximate integral operators and *n* denotes the parameter of the approximate operators. Increasing the value of *n* results in approximate operators converging to the original operators. The details of the derivation of the approximate models and the convergence to the original models are provided in Appendix B. Our empirical results are based on the approximate sales models (3.21)-(3.25) whereas all the theoretical results are derived based on the original sales models (i.e., model (3.10) and (3.13) and model (3.15) and (3.20)). Figure 3-2 provides an example for each of the two scenarios, (a) the phase-out transition scenario ($\beta_1 = \beta_2 = 0.5$, $p_1 = p_2 = 0.001$, $q_1 = q_2 = 0.8$, $m_1 = m_2 = 1$, $h_2 = 0.8$, $\tau_2 = 8$) where sales of the old generation continues after the new generation's release, and (b) the total transition scenario ($\beta_1 = \beta_2 = 0.5$, $p_1 = p_2 = 0.005$, $q_1 = q_2 = 1$, $m_1 = m_2 = 1$, $\tau_2 = 15$) where the old generation is discontinued with the release of the new generation.

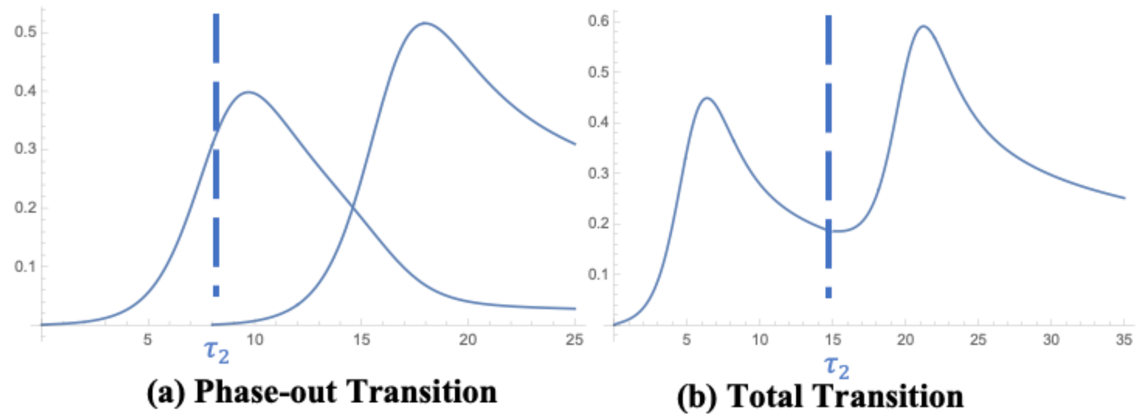

Figure 3-2: Sales for Two Product Generations with Repeat Purchases

3.3.4 N-Generations Scenarios

In this section we extend the models presented in Sections 3.3.1 and 3.3.2 to product lines including N ($N \geq 3$) successive generations. Using a procedure analogous to that outlined in Section 3.3.3, we can operationalize the extended models developed in this section.

3.3.4.1 N-Generations with Phase-out Transition Scenario

Generation 1 is introduced at $t = 0$. The adoption rate for generation 1 before and after the introduction of generation 2 at $t = \tau_2$ is formulated as

$$y_1(t) = \begin{cases} y_{11}(t), & 0 \leq t < \tau_2, \\ y_{12}(t), & t \geq \tau_2, \end{cases} \tag{3.26}$$

$$y_{11}(t) = m_1 f_1(t), \tag{3.27}$$

$$y_{12}(t) = y_{11}(t) - u_2(t), \tag{3.28}$$

where

$$u_2(t) = y_{11}(t) F_2(t - \tau_2), \tag{3.29}$$

represents leapfrogging from generation 1 to generation 2. Generation 1's sales are given by

$$S_1(t) = \begin{cases} I_0^{\beta_1} y_1(t), & 0 \leq t < \tau_2, \\ I_0^{\beta_1} y_1(t) - \ swt_2(t), & \tau_2 \leq t, \end{cases} \tag{3.30}$$

where

$$swt_2(t) = h_2 * F_2(t - \tau_2)\left(I_0^{\beta_1} y_1(t) - y_1(t)\right), \tag{3.31}$$

are the sales due to switching from generation 1 to generation 2. With the introduction of generation j $(2 \leq j \leq N-1)$ at $t = \tau_j$, the initial purchases rate for generation j, including adoptions and leapfroggings, before and after the introduction of generation $j+1$ at $t = \tau_{j+1}$ is formulated as

$$y_j(t) = \begin{cases} y_{j1}(t), & \tau_j \leq t < \tau_{j+1}, \\ y_{j2}(t), & t \geq \tau_{j+1}, \end{cases} \tag{3.32}$$

$$y_{j1}(t) = m_j f_j(t - \tau_j) + u_j(t), \tag{3.33}$$

$$y_{j2}(t) = y_{j1}(t) - u_{j+1}(t), \tag{3.34}$$

where

$$u_{j+1}(t) = y_{j1}(t) F_{j+1}(t - \tau_{j+1}), \qquad j = 2, \dots, N-1, \tag{3.35}$$

is the rate of leapfrogging to generation $j+1$. The sales for generation j are then given by

$$S_j(t) = \begin{cases} I_{\tau_j}^{\beta_j} y_j(t) + swt_j(t), & \tau_j \leq t < \tau_{j+1}, \\ I_{\tau_j}^{\beta_j} y_j(t) + swt_j(t) - swt_{j+1}(t), & \tau_{j+1} \leq t, \end{cases} \tag{3.36}$$

where the switching to generation $j+1$ $(j = 2, \dots, N-1)$ is formulated by

$$swt_{j+1}(t) = h_{j+1} * F_{j+1}(t - \tau_{j+1}) \left(I_{\tau_j}^{\beta_j} y_j(t) - y_j(t) + swt_j(t) \right). \tag{3.37}$$

Finally, the initial purchases including adoptions and leapfroggings for generation N, introduced at $t = \tau_N$, is given by

$$y_N(t) = m_N f_N(t - \tau_N) + u_N(t), \; t \geq \tau_N. \tag{3.38}$$

Thus, sales for generation N is formulated as

$$S_N(t) = I_{\tau_N}^{\beta_N} y_N(t) + swt_N(t), \quad t \geq \tau_N. \tag{3.39}$$

3.3.4.2 N-Generations with Total Transition Scenario

In this section we extend the model (3.15) and (3.20) to product lines including N $(N \geq 3)$ successive generations following a total transition strategy between generations. We assume that generation 1 is introduced at $t = 0$ while generations j $(j = 2, \dots, N)$ are introduced at τ_j. Sales for generation 1 before and after the introduction of generation 2 are given by

$$S_1(t) = \begin{cases} I_0^{\beta_1} m_1\, f_1(t), & 0 \le t < \tau_2\,, \\ 0, & t \ge \tau_2. \end{cases} \tag{3.40}$$

Similarly, sales for generation j ($j = 2, \dots, N-1$) are given by

$$S_j(t) = \begin{cases} I_0^{\beta_j} m_1\, f_1(t) + \sum_{i=2}^{j} I_{\tau_i}^{\beta_j} m_i\, f_i(t-\tau_i), & \tau_j \le t < \tau_{j+1}\,, \\ 0, & t \ge \tau_{j+1}\,. \end{cases} \tag{3.41}$$

Finally, sales for the last generation N are formulated as

$$S_N(t) = I_0^{\beta_N} m_1\, f_1(t) + \sum_{i=2}^{N} I_{\tau_i}^{\beta_N} m_i\, f_i(t-\tau_i). \tag{3.42}$$

3.4 Empirical Evaluation

In this section, we benchmark our multigeneration sales model against a state-of-the-art model from the literature using six real-world datasets that include (1) digital and analog cameras, (2) DVD and Blu-ray players, (3) CRT and flat-panel monitors, (4) three generations of Nintendo gaming console software, (5) Google Search Trends for LG Full HD and Ultra HD TVs, and (6) Google Search Trends for LED, OLED and QLED TVs. We use the model developed by Jiang et al. (2019) as our benchmark. Jiang et al. include upgrades from older generations to newer generations in their model, but they do not include repeat purchases occurring within each generation. We used Mathematica software for estimations. Our empirical tests show that the implementation of the new model is not computationally intensive compared to the benchmark.

Our fifth and sixth datasets include users' web searches as reported by Google Trends. Users' web searches can generally be used as a proxy for actual product sales. Consumers' web search data allows us to track shifts in consumers' interests (Du et al. 2015). There are several online consumer tracking services with Google Trends probably the most widely used and best known (Du et al. 2015). Recent studies have benefited from web search data for prediction purposes. For example, Choi and Varian (2012) use Google Trends data for nowcasting. Particularly, they use queries on automobile sales for a given month to predict that month's actual auto sales report that will become available several weeks later. In another study, Du and Kamakura (2012) discover consumers' common shopping interests by studying search volume patterns for 38 car makes during an 81-month time period. Given the use of web search data by the prior literature to understand the dynamics of product sales and the challenges in acquiring actual sales data, we also evaluate our model using Google Trends data.

Our empirical results demonstrate that our model is more accurate than the model proposed by Jiang et al. in estimating and forecasting multigeneration product sales. This result can be mostly attributed to the fact that Jiang et al.'s model does not incorporate repeat purchases while the model we propose does.

3.4.1 Dataset 1: Digital and Analog Cameras

The first dataset records the annual sales (approximated by unit shipments) of analog cameras from year 1951 to 2007 and the annual sales (approximated by unit shipments) of digital cameras from year 1999 to 2021 in Japan.[3] With the rise of smartphones, the digital camera market experienced an unprecedent sharp decline worldwide (Richter, 2022). Specifically, the introduction of the iPhone in 2007 has played a critical role in this disruption (Molla, 2017). Therefore, we use sales from 1951 to 2006 in our analysis to exclude the effect of smartphones on camera sales. Considering that the remaining dataset includes only few data points corresponding to sales of the new generation (i.e., digital camera), we restrict our model comparison to model estimation in which we fit both models to sales data from 1951 to 2006. Considering that very few users, if any at all, have not switched to digital cameras, we set h_2=1, indicating that all users eventually switch from analogue camera to digital camera.

The estimated parameters for GMDR are presented in Table 3-1. It can be observed that most of the estimates are statistically significant. Fitting accuracy of the GMDR and that of the benchmark are compared in Table 3-2 in terms of sum of squares error (SSE) and visually demonstrated in Figure 3-3. It can be seen both in the table and the figure that the GMDR provides a more accurate fit than does the benchmark.

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[3] The dataset is provided by Camera and Imaging Products Association and is accessible at https://www.cipa.jp/e/index.html.

Table 3-1: Parameter Estimates for Sales of Analog Cameras (1951-2006) and Digital Cameras (1999-2006) in Japan

	Estimate	Standard Error	t-Statistics	P-value
β_1	0.2829	0.2283	1.2391	0.2205
β_2	0.5697	0.1685	3.3806	0.0013
p_1	0.0039	0.0015	2.5842	0.0124
p_2	0.066	0.01812	3.6442	0.0006
q_1	0.0702	0.0073	9.5766	0.0000
q_2	0.5894	0.0925	6.3717	0.0000
m_1	88157.8	73436.2	1.2005	0.235
m_2	9911.56	2845.36	3.4834	0.001

Table 3-2: Comparison of Fit for Sales of Analog Cameras (1951-2006) and Digital Cameras (1999-2006) in Japan

	Model fit (SSE)		
	Analog	Digital	Overall
GMDR	7.41686×10^6	6.95647×10^5	8.11253×10^6
Jiang et al. (2019)	2.22954×10^7	7.23993×10^6	2.95354×10^7

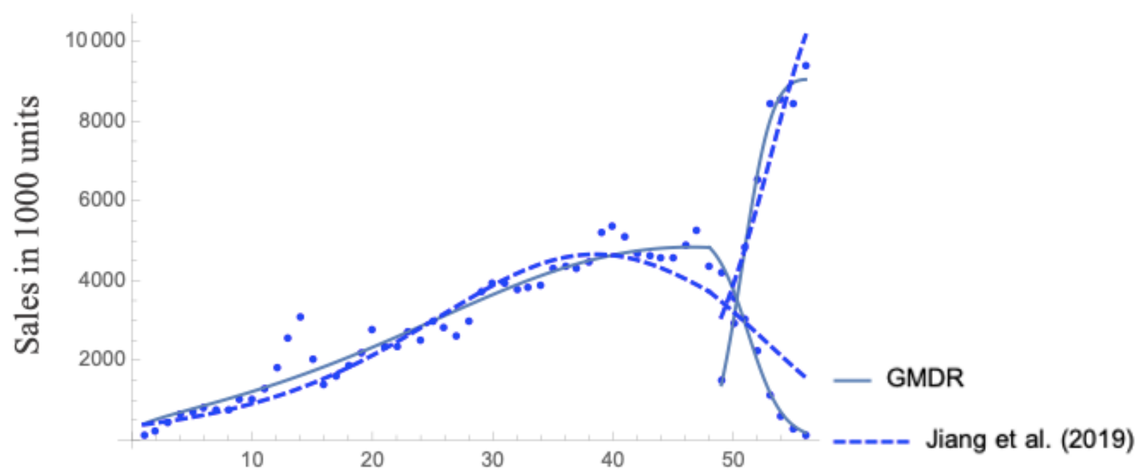

Figure 3-3: Sales of Analog Cameras (1951-2006) and Digital Cameras (1999-2006) in Japan

3.4.2 Dataset 2: DVD and Blu-ray Players

This second dataset represents annual sales (approximated by unit shipments) of DVD players from year 2000 to 2021 and annual sales (approximated by unit shipments) of Blu-ray players from year 2008 to 2021 in Japan.[4] Table 3-3 provides the parameter estimations for the sales of DVD players with Blu-ray players in Japan. It can be seen from the table that, with the exception of the repeat purchase parameter for Blu-ray players, which is close to zero, all other parameter estimates are statistically significant. The near zero value of the repeat purchase rate for Blu-ray is not surprising considering that Blu-ray technology did not achieve much market success due mostly to the emergence of streaming as a popular alternative (Vaughan-Nichols, 2019). Note that the estimated value of h_2 reflects that almost 81% of the users switch from DVD player to Blu-ray player, while 19% of users not switching and staying loyal to DVD technology. This can be attributed to the fact that Blu-ray was not a huge leap over DVD in terms of product physicality, experience, and quality (Nicholas Coker, 2020).

Table 3-3: Parameter Estimates for Sales of DVD Players (2000-2018) and Blu-ray Players (2008-2018) in Japan

	Estimate	Standard Error	t-statistics	P-value
β_1	0.5287	0.0548	9.6514	0.0000
β_2	0.0204	0.1142	0.1791	0.8596
p_1	0.0169	0.0054	3.1226	0.0051
p_2	0.0225	0.0071	3.1585	0.0047
q_1	0.9516	0.1314	7.2412	0.0000
q_2	1.1922	0.1698	7.0232	0.0000
m_1	15427.8	1991.25	7.7478	0.0000
m_2	14347.2	2861.95	5.0131	0.0001
h_2	0.8105	0.0665	12.1937	0.0000

[4] The dataset is provided by Japan Electronics and Information Technology Industries Association and is accessible at https://www.jeita.or.jp/english/.

Table 3-4 compares the model fit and the three-years-ahead prediction accuracy of the GMDR with those of the benchmark in terms of SSE and shows that the GMDR provides more accurate fit and forecast than does the benchmark. Figure 3-4 depicts the performance differences of GMDR and the model by Jiang et al. (2019) in model fitting (2000-2018) and three-years-ahead forecasting (2019-2021). It shows that the GMDR matches the real data points better than does the model by Jiang et al. (2019), thus resulting in a much smaller SSE in both model fitting and three-years-ahead forecasting.

Table 3-4: Comparison of Fit (2000-2018) and Forecast (2019-2021) for Sales of DVD Players and Blu-ray Players in Japan

	Model fit (SSE)			Three years ahead forecast (SSE)		
	DVD	Blu- ray	Overall	DVD	Blu- ray	Overall
GMDR	1.29143×10^6	2.96336×10^6	4.25479×10^6	91326.7	318143	409469
Jiang et al. (2019)	6.325×10^6	1.44819×10^7	2.08069×10^7	246993	574704	821697

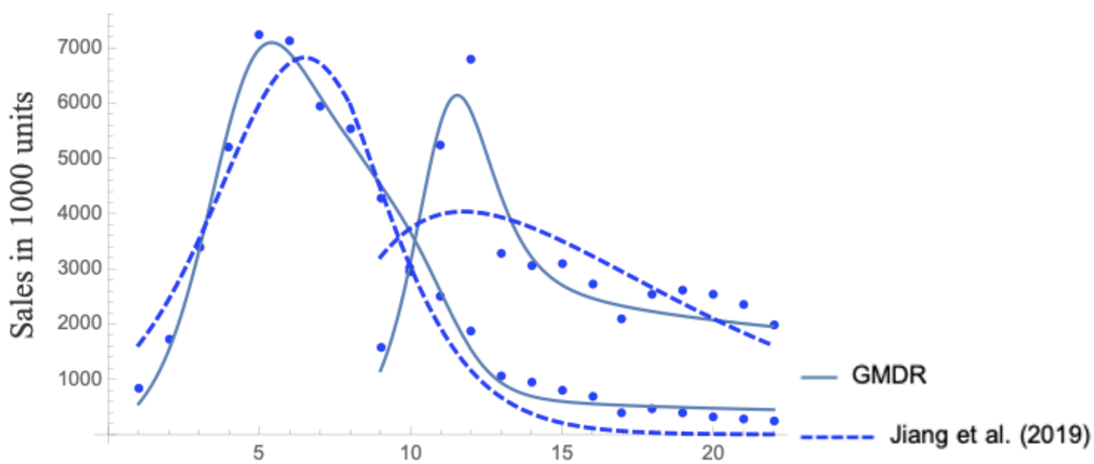

Figure 3-4: Sales of DVD Players (2000-2021) and Blu-ray Players (2008-2021) in Japan

3.4.3 Dataset 3: CRT and Flat-panel (LCD and LED) Monitors

The third dataset includes annual sales of CRT monitors from year 1980 to 2007 and annual sales of flat-panel monitors including LCD and LED from year 1999 to 2018 in the US (Althaf et al., 2021). Considering that total switching from CRT to flat-panel monitors has practically happened, we set h_2=1.

Table 3-5 reports the corresponding parameter estimates. It can be seen in the table that half of the parameter estimates are statistically significant. This can be attributed to the fact that generation 1's sales are available only until 2007 and not for the entire analysis time frame.

Table 3-5: Parameter Estimation for Sales of CRT Monitors (1980-2007) and Flat-panel Monitors (1999-2015) in the US

	Estimate	Standard Error	t-Statistics	P-value
β_1	0.111	0.1949	0.5693	0.5726
β_2	0.3569	0.1421	2.5126	0.0165
p_1	0.001	0.0003	3.0599	0.0041
p_2	0.0000	0.0000	0.42	0.6769
q_1	0.2554	0.023	11.0926	0.0000
q_2	1.5258	0.3265	4.673	0.0000
m_1	164414	85765.5	1.917	0.063
m_2	3463.73	3987.45	0.8686	0.3906

Model fit and the three-years-ahead forecasting accuracy of the GMDR and the benchmark model, measured by SSE, are summarized in Table 3-6. The results show that our model leads to considerably more accurate fit and forecast than does the benchmark. Model fit (1980-2015) and the three-years-ahead forecasting accuracy (2016-2018) of the GMDR compared to those for the benchmark model are also illustrated in Figure 3-5. The figure clearly demonstrates that the proposed model fits the data better and subsequently forecasts sales considerably more accurately than does the benchmark model.

Table 3-6: Comparison of Fit (1980-2015) and Forecast (2016-2018) for Sales of CRT Monitors and Flat-panel Monitors in the US

	Model fit (SSE)			Three years ahead forecast
	CRT	Flat panel	Overall	Flat panel
GMDR	3.85089×10^7	7.21025×10^6	4.57192×10^7	3.33512×10^6
Jiang et al. (2019)	6.73444×10^7	7.2195×10^7	1.39539×10^8	3.28688×10^7

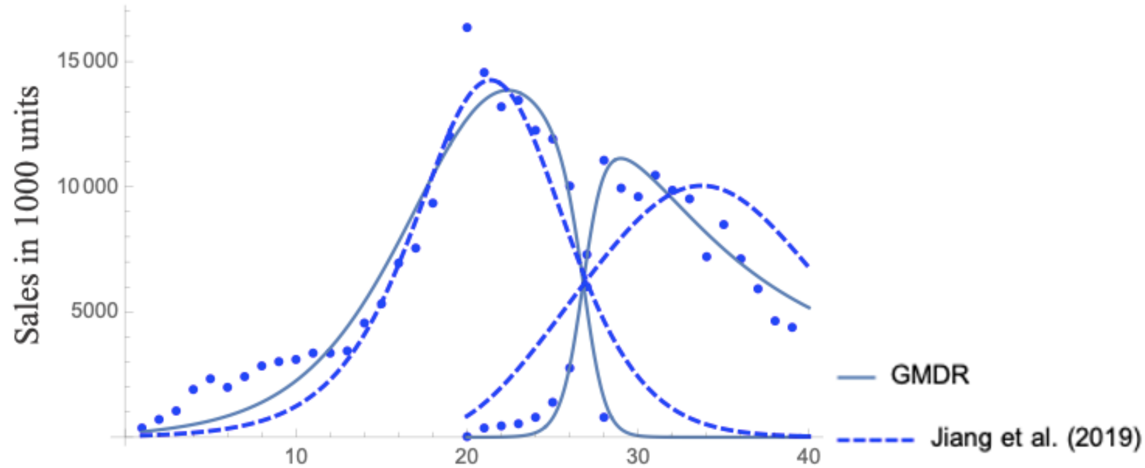

Figure 3-5: Sales of CRT Monitors (1980-2007) and Flat-panel Monitors (1999-2018) in the US

3.4.4 Dataset 4: Nintendo DS, Nintendo 3DS, and Nintendo Switch Software

The fourth dataset includes worldwide annual software sales of three generations of Nintendo gaming consoles: Nintendo DS from year 2005 to 2022,[5] Nintendo 3DS from year 2011 to 2022,[6] and Nintendo Switch from year 2017 to 2022.[7]

We set $h_2 = h_3 = 1$, which means all the users of the older generations ultimately switch to newer generations. This assumption aligns with Nintendo's strong brand loyalty (Laib, 2020). The estimated parameters for GMDR are presented in Table 3-7.[8] With one exception, all parameter estimates are statistically significant. The estimated rate of repeat purchases for the third generation is notably high. Although this estimation might

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5 From Statista and accessible at https://www.statista.com/statistics/349048/nintendo-ds-software-unit-sales/.

6 From Statista and accessible at https://www.statista.com/statistics/349072/nintendo-3ds-software-unit-sales/.

7 From Statista and accessible at https://www.statista.com/statistics/868256/nintendo-switch-software-sales/#:~:text=In%20the%20fiscal%20year%20ending,software%20amounted%20to%201.03%20billion.

8 To obtain parameter estimates within the specified range of our model, we imposed constraints on the values of m_2 and β_3, specifically requiring $m_2 > 0$ and $\beta_3 \leq 1$. As a result of these constraints, it was not possible to calculate their test statistics. Consequently, we exogenously incorporated the estimated values for m_2 and β_3 into the model and subsequently derived test statistics for the remaining parameters.

be somewhat inflated due to the limited availability of data for this particular generation, it aligns with expectations. The Nintendo Switch's remarkable success in the gaming industry (Evangelho, 2018) validates the elevated rate of repeated game purchases among console owners.

Table 3-7: Parameter Estimation for Software Sales of Nintendo DS (2005-2022), Nintendo 3DS (2011-2022), and Nintendo Switch (2017-2022) Worldwide

	Estimate	Standard Error	t-Statistics	P-value
β_1	0.3565	0.4813	7.4078	0.0000
β_2	0.1686	0.4648	0.3627	0.7198
β_3	1	--		
p_1	0.0137	0.0023	5.9986	0.0000
p_2	0.0367	0.0173	2.1226	0.0435
p_3	0.0407	0.009	4.5024	0.0001
q_1	1.1012	0.0805	13.6751	0.0000
q_2	1.2897	0.2576	5.0071	0.0000
q_3	1.0096	0.1261	8.0043	0.0000
m_1	499.438	46.3202	10.7823	0.0000
m_2	26.286	--		
m_3	211.196	10.0628	20.9877	0.0000

Table 3-8 provides a comparison of GMDR's fitting accuracy against the benchmark's using SSE, and Figure 3-6 visually presents these comparisons. Both the tabulated and graphical representations demonstrate that GMDR surpasses the benchmark in terms of fitting accuracy.

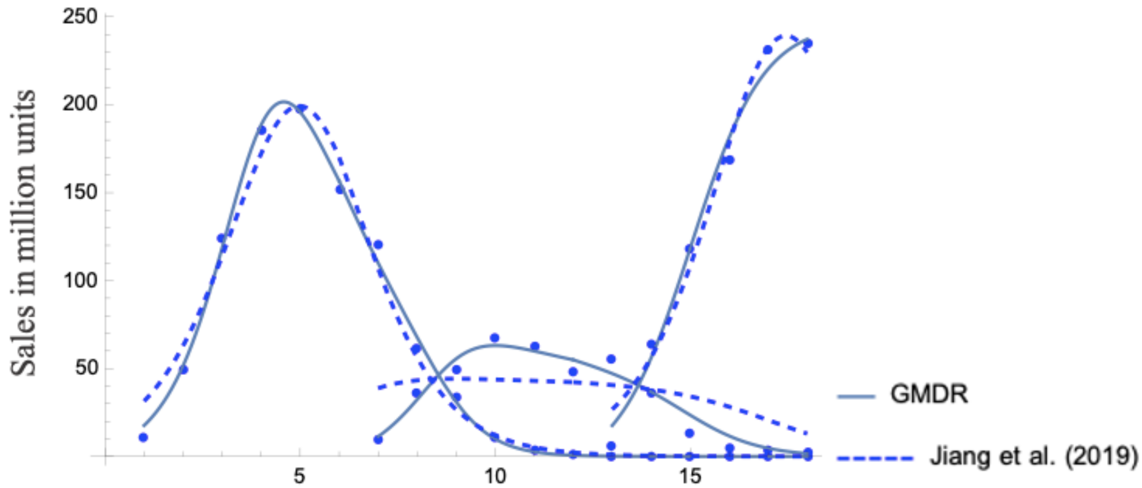

Figure 3-6: Software Sales of Nintendo DS (2005-2022), Nintendo 3DS (2011-2022), and Nintendo Switch (2017-2022) Worldwide

Table 3-8: Comparison of Fit for Software Sales of Nintendo DS (2005-2022), Nintendo 3DS (2011-2022), and Nintendo Switch (2017-2022) Worldwide

	Model fit (SSE)			
	DS	3DS	Switch	Overall
GMDR	288.753	340.509	476.238	1105.5
Jiang et al.	1502.01	3566.72	727.141	5795.86

3.4.5 Dataset 5: Google Trends Search for LG Full HD and Ultra HD TVs

The fifth dataset includes Google Trends' yearly aggregated search data for two generations of LG TVs in the UK from 2007 to 2021. The search terms we use are "LG full HD TV" and "LG Ultra HD TV". Considering that the dataset has limited data points, a more parsimonious model is expected to perform better. Therefore, we assume that repeat purchase rates for the first and the second generations are equal. Parameters estimations are reported in Table 3-9. It can be observed from the table that all parameter estimates are statistically significant, underscoring the sound performance of the parsimonious model version used.

Table 3-9: Parameter Estimation for Search Terms "LG Full HD TV" (2007-2018) and "LG Ultra HD TV" (2013-2018) in the UK

	Estimate	Standard Error	t-Statistics	P-value
β	0.385	0.0843	4.5739	0.001
p_1	0.0033	0.0014	2.2485	0.0483
p_2	0.0291	0.0084	3.4468	0.0063
q_1	1.3367	0.1645	8.1284	0.0000
q_2	0.86	0.1175	7.3189	0.0000
m_1	1079.89	161	6.7071	0.0000
m_2	965.406	252.214	3.8277	0.0033
h_2	0.7398	0.1714	4.3155	0.0015

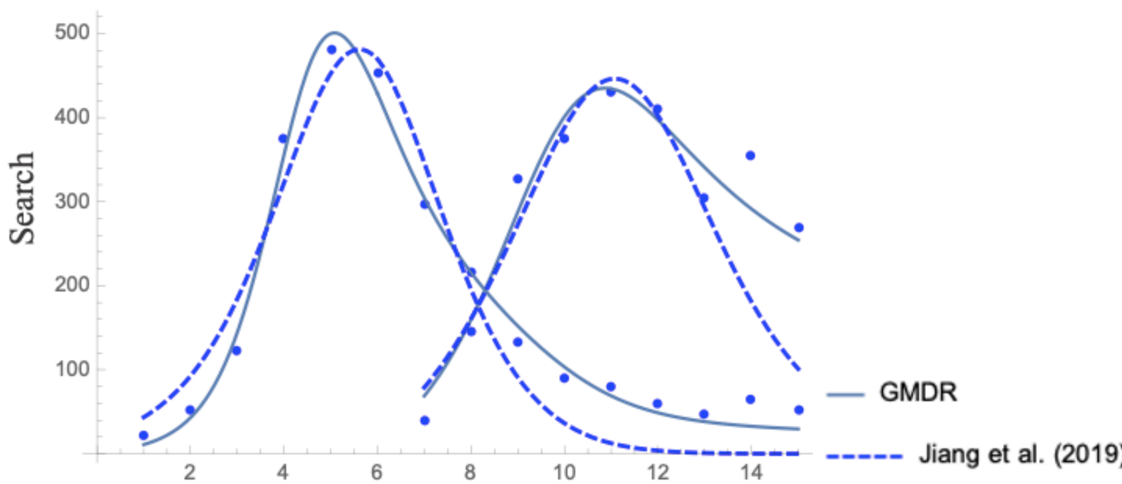

Figure 3-7: Google Trends' Search Trends for "LG Full HD TV" (2007-2021) and "LG Ultra HD TV" (2013-2021) in the UK

Table 3-10: Comparison of Fit (2007-2018) and Forecast (2019-2021) for Search of LG Full HD TVs and Ultra HD TVs in the UK

	Model fit (SSE)			Three years ahead forecast (SSE)		
	HD	UHD	Overall	HD	UHD	Overall
GMDR	2886.01	3446.07	6332.08	1680.22	5538.86	7219.09
Jiang et al.	25217.4	5178.19	30395.6	9201.8	58702.9	67904.7

Table 3-10 summarizes the fitting performance and the three-years-ahead forecasting performance of the GMDR and the benchmark model based on SSE. Similar to the prior datasets, it can be observed in the table that the proposed model provides more accurate fit and forecast than does the benchmark. The fit (2007-2018) and the forecasting results (2019-2021) are depicted in Figure 3-7. As we can see in the figure, the new model results are considerably better than those of the benchmark model.

It is interesting to note that the number of search queries increased significantly in 2020, deviating from the main trend. This sudden increase in consumer searches can be attributed to the COVID-19 pandemic. Reports suggest that TV watching and online streaming increased dramatically in 2020 (BBC, 2020) and TV sales in the UK experienced a steep growth during that time (Advanced Television, 2021).

3.4.6 Dataset 6: Google Trends Search for LED, OLED, and QLED TVs

The sixth dataset comprises yearly aggregated search data from Google Trends, spanning a period from 2007 to 2022, focusing on three generations of TVs in Germany. The specific search terms analyzed are "LED TV," "OLED TV," and "QLED TV." Due to the limited size of the dataset, we opt for parsimony by assuming that repeat purchase rates are consistent across all three TV generations.

Parameters estimations are reported in Table 3-11.[9] We observe that half of the parameters exhibit statistical significance. Parameters in generation 3 are estimated to be close to zero or insignificant. The underlying cause for this outcome can be primarily attributed to the limited availability of sales data for generation 3.

Table 3-12 compares the fitting accuracy of GMDR with that of the benchmark based on SSE, and Figure 3-8 visually illustrates these comparisons. Both the tabular and graphical representations indicate that GMDR outperforms the benchmark in terms of fitting accuracy.

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[9] To obtain parameter estimates within the specified range of our model, we imposed constraints on the values of q_3 and h_3, specifically requiring $q_3, h_3 > 0$. As a result of these constraints, it was not possible to calculate test statistics for these two particular parameters. Consequently, we exogenously incorporated the estimated values for q_3 and h_3 into the model and subsequently derived test statistics for the remaining parameters.

Table 3-11: Parameter Estimation for Search Terms "LED TV" (2007-2022), "OLED TV" (2010-2022), and "QLED TV" (2017-2022) in Germany

	Estimate	Standard Error	t-Statistics	P-value
β	0.5612	0.0897	6.2543	0.0000
p_1	0.0118	0.004	2.9083	0.0075
p_2	0.0031	0.0021	1.4994	0.1463
p_3	0.0397	0.2367	0.1676	0.8682
q_1	1.1134	0.1616	6.8877	0.0000
q_2	0.5795	0.1131	5.1234	0.0000
q_3	0.0000	--		
m_1	1432.74	244.598	5.8575	0.0000
m_2	647.091	555.307	1.1653	0.2549
m_3	718.472	5057.61	0.1421	0.8882
h_2	0.5335	0.134	3.9811	0.0005
h_3	0.0000	--		

Table 3-12: Comparison of Fit for Search of LED TVs (2007-2022), OLED TVs (2010-2022), and QLED TVs (2017-2022) in Germany

	Model fit (SSE)			
	LED	OLED	QLED	Overall
GMDR	32610.8	5011.36	549.368	38171.5
Jiang et al.	209891	14597.5	1311.43	225800

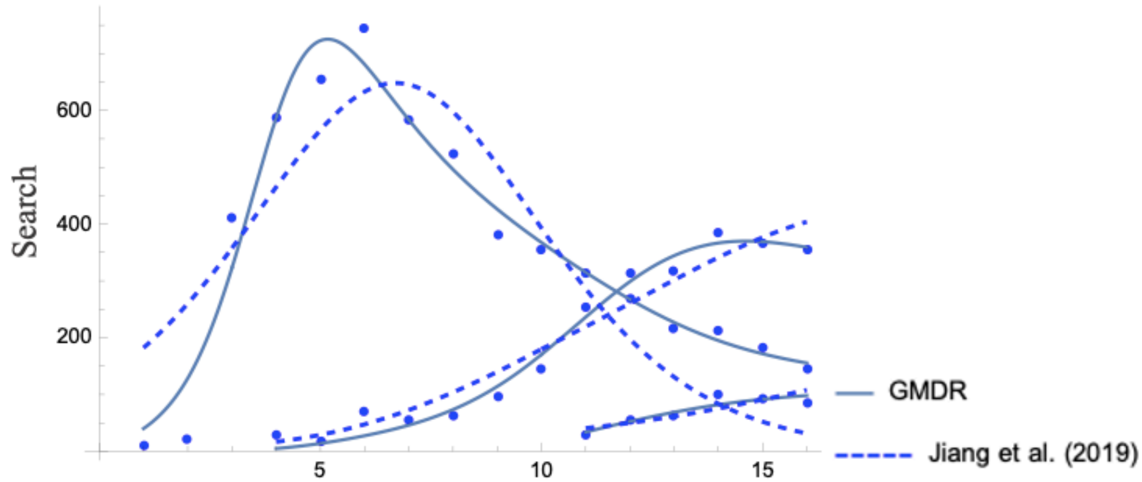

Figure 3-8: Google Trends' Search Trends for "LED TV" (2007-2022), "OLED TV" (2010-2022), and "QLED TV" (2017-2022) in Germany

3.5 Optimal Market Entry Timing Strategy

Accurately predicting and characterizing the sales growth of product generations are critically important for firms' operational and decision-making needs, thus the value of such a model can hardly be overstated. In this section, we demonstrate the utility of our model using one of the many possible applications—deciding the optimal market entry timing for a new product generation.

The multigeneration models developed in Section 3.3 provide closed form formulations for sales, which allow us to formulate the market entry optimization models analytically. Analogous to prior research (e.g., Guo & Chen, 2018; Jiang et al., 2019), we consider a scenario in which a firm tries to maximize its profit in selling two successive product generations in a finite planning horizon. We also consider a time-window within the planning horizon for the introduction of the new generation. Consequently, the profit generated by the firm from sales of two successive generations is given by

$$\Pi(\tau_2) = \int_0^T \pi_1(t)S_1(t)dt + \int_{\tau_2}^T \pi_2(t)S_2(t)dt, \tag{3.43}$$

where

$$\pi_i(t) = (pr_i(t) - c_i(t))e^{-r_i t}. \tag{3.44}$$

Here T denotes the planning horizon. The introduction of the second generation at τ_2 happens within a time-window, meaning that there exists $\lambda, \Lambda > 0$ such that $\tau_2 \in [\lambda, \Lambda] \subset [0, T]$. $S_i(t)$, denotes sales for generation $i = 1, 2$ at time *t*, given by (3.10) and (3.13) in the phase-out transition scenario, and (3.15) and (3.20) in the total transition scenario. $pr_i(t)$ and $c_i(t)$, refer to unit price and unit cost of production of the *i^th^* product generation at time *t*. $e^{-r_i t}$ denotes discount at rate r_i for generation $i = 1, 2$. $\pi_i(t), i \in \{1, 2\}$, denotes the present value of profit gained by selling one unit of the product. Following Jiang et al. (2019) we call $\pi_i(t)$ the *unit contribution margin*. We assume that profit margins are always positive (i.e., $\pi_i(t) > 0$). Given the total profit in (3.43), our goal is to find the optimal market entry timing for generation 2, τ_2, from the time-window $[\lambda, \Lambda]$. Thus, the descision making problem is formulated as

$$\max_{\tau_2 \in [\lambda, \Lambda]} \Pi(\tau_2). \tag{3.45}$$

The profit function (3.43) is similar to the one discussed in Jiang et al. (2019). However, they restrict their analysis to cases where the cost and the price of the products in both generations increase at the same rate as the discount rate. In other words, they assume that the unit contribution margins ($\pi_i(t)$) are constant over time, which can be restrictive. Moreover, they assume that the coefficient of innovation p and the coefficient of imitation q do not change across generations. In our analysis we relax these assumptions. Specifically, we consider time-varying unit contribution margins and let the diffusion parameters p and q change across generations as well.

Next, we derive analytical results regarding the optimal market entry timing. Specifically, Propositions 3.1 and 3.2 describe results under the phase-out transition scenario, and Propositions 3.3 and 3.4 depict results corresponding to the total transition scenario. In propositions 3.1 through 3.4, τ_2^* denotes the optimal market entry timing.

Proposition 3.1. Assume that the firm adopts the phase-out transition scenario. Suppose $\pi_2(t)$ is a smooth function and $T > T^*$, then

(I) If the unit contribution margin for the second generation is monotonic ($\pi_2'(t) \geq 0$ or $\pi_2'(t), \pi_2''(t) \leq 0$), $\pi_1(t) > \pi_2(t)$ for $t \in [\lambda, T]$, $\beta_1 \geq \beta_2$, and $m_1 > \gamma_1$, then $\tau_2^* = \Lambda$.

(II) If the unit contribution margin for the second generation is increasing ($\pi_2'(t) \geq 0$), $\pi_2(t) > \pi_1(t)$ for $t \in [\lambda, T]$, $\beta_2 \geq \beta_1$, and $m_1 > \gamma_2$, then $\tau_2^* = \lambda$.

(III) If the unit contribution margin for the new generation is declining ($\pi_2'(t), \pi_2''(t) \leq 0$), $\pi_2(t) \geq \pi_1(t)$ for $t \in [\lambda, T]$, and $\beta_2 \geq \beta_1$, then $\tau_2^* = \lambda$.

Proposition 3.1 provides sufficient conditions for the optimal market entry timing in a phase-out transition scenario. It is assumed that the unit contribution margin for generation 2 is monotonic over time (i.e., increasing or decreasing over time). An increasing unit contribution margin is the result of an increasing per unit profit trend $pr_i(t) - c_i(t)$ that outweighs the decreasing effect of the discount factor $e^{-r_i t}$. On the other hand, a decreasing unit contribution margin is observed in two cases; (1) when per-unit profit is decreasing over time or (2) when per-unit profit is not necessarily decreasing but is outweighed by the discount factor's decreasing trend. Thresholds T^*, γ_1, and γ_2 are provided in Lemma B.3 and the proof of Proposition 3.1 in Appendix B.

Proposition 3.1 (I) describes conditions under which the maximum delay strategy in releasing the new generation is optimal. Specifically, when the unit contribution margin for the second generation is increasing (i.e., $\pi_2'(t) \geq 0, \lambda \leq t \leq T$) or the unit contribution margin for the second generation is decreasing (i.e., $\pi_2'(t), \pi_2''(t) \leq 0, \lambda \leq t \leq T$), if the potential market size for the first generation is large enough, the unit contribution margin for the first generation is greater than that for the second generation, and the repeat purchase rate of the first generation is at least as high as that of the second generation, firms are better off delaying the introduction of the second generation as much as possible.

Proposition 3.1 (I) can be interpreted as follows. The introduction of the new generation attracts customers who would buy the old generation in the absence of the new generation through leapfrogging and switching. Considering that the unit contribution margin of the second generation is lower than that of the first generation and the repeat purchase rate of the second generation is at most as high as that of the first generation, customers shifting from the first generation to the second generation results in less profit, discouraging the earlier release of the second generation. On the other hand, the new generation creates a new market and therefore increases sales, encouraging earlier release of the second generation. However, the profit generated by the new market created by the new generation does not outweigh the first generation's substantial profit supported by a large market potential, a higher unit contribution margin, and a repeat purchase rate at least as high as that of the second generation. This finding is realistic. There are examples of firms that intentionally delay the introduction of a new generation to achieve a higher benefit. For instance, Intel deliberately delayed the release of the new generation Broadwell PC chip in 2014. The decision was attributed to the slow demand for personal computers (APH Networks, 2014).

Proposition 3.1 (II) and (III) provide sufficient conditions for the strategy of releasing the new generation as early as possible. Proposition 3.1 (II) provides a sufficient condition for the case where the unit contribution margin for the second generation is increasing (i.e., $\pi_2'(t) \geq 0, \lambda \leq t \leq T$). According to Proposition 3.1 (II), when the unit contribution margin for the second generation is greater than that for the first generation, the repeat purchase rate of the second generation is at least as high as that of the first generation, and the market size of the first generation is large enough, the firm is better off releasing the new generation as early as possible.

The connotation of Proposition 3.1 (II) is that when the market potential for the first generation is considerable, the first generation can provide a substantial number of customers for the second generation through leapfrogging and switching. These shifting customers repeat purchase at a rate at least as high as that of the first generation and generate higher unit contribution margins compared to the first generation, thus encouraging an early release of the second generation. Moreover, the release of a new generation creates new market and therefore generating more profit. On the other hand, since the unit contribution margin for the second generation is increasing, when released later, the higher sales, peak sales specifically, of the second generation coincide with higher unit contribution margins, thus encouraging a later release of the second generation. However, as long as the market potential for the first generation is sufficiently large, the profit generated by customers shifting early from the first generation to the second generation prevails over the profit generated by a delayed introduction of the second generation. Therefore, under Proposition 3.1 (II), it is recommended to release the second generation as early as possible.

According to Proposition 3.1 (III), if the unit contribution margin for the second generation is at least as high as that of the first generation and is decreasing (i.e., $\pi_2'(t), \pi_2''(t) \leq 0, \lambda \leq t \leq T$), and the repeat purchase rate in the second generation is at least as high as that of the first generation, firms are better off releasing the second generation as early as possible. This result is reasonable because when the unit contribution margin and the repeat purchase rate of the second generation is at least as high as those of the first generation, attracting customers from the first generation to the second generation through leapfrogging and switching leads to a profit at least as large as that gained under the first generation. Furthermore, the introduction of a new generation creates a new market. All of these in addition to the fact that the unit contribution margin of the second generation is decreasing over time lead to the case where introducing the new product generation as early as possible is the best strategy.

It is interesting to note that in Proposition 3.1, part (III), as opposed to parts (I) and (II), recommends the introduction of the second generation as early as possible regardless of how large the market potentials of the first and the second generations are. Moreover, the early market entry timing strategy is recommended even when the first and the second generations have equal unit contribution margins and repeat purchase rates (i.e., $\pi_1(t) = \pi_2(t), \lambda \leq t \leq T$, and $\beta_1 = \beta_2$). This interesting result can be attributed to the fact that the introduction of the second generation creates a new market, and considering that the unit contribution margin in the second generation is decreasing, then firms are better off activating this new market as early as possible.

Findings of Proposition 3.1 (II) and (III) are realistic. Many firms release the new generation early. For example, Apple released iPad 3 in March 2012 while the sales trend of iPad 2, introduced in March 2011, was strongly upward (Jiang et al., 2019).

Proposition 3.2 demonstrates a condition under which, similar to Proposition 3.1 (I), the first generation's higher unit contribution margin than that of the second generation and the repeat purchase rate at least as high as that of the second generation, motivates delaying the release of the second generation. However, unlike Proposition 3.1 (I), the maximum market entry delay is not an optimal strategy since the market size of the first generation is not big enough.

Proposition 3.2. Assume that the firm adopts the phase-out transition scenario. Suppose $\pi_2(t)$ is a smooth function and $T > T^*$. If the unit contribution margin for the second generation is monotonic ($\pi_2'(t) \geq 0$ or $\pi_2'(t), \pi_2''(t) \leq 0$), $\pi_1(t) > \pi_2(t)$ for $t \in [\lambda, T]$, $\beta_1 \geq \beta_2$, and $m_1 < \eta_1$, then $\tau_2^* \in [\lambda, \Lambda)$.

Threshold η_1 is provided in the proof of Proposition 3.2 in Appendix B. It can be observed that $\eta_1 \leq \gamma_1$. To better showcase Proposition 3.2, we provide a numerical analysis. Figure 3-9 shows changes in profit as a function of market entry timing when $\pi_1(\mathrm{t}) = 1.5$, $\pi_2(\mathrm{t}) = 1$, $p_1 = p_2 = 0.01$, $q_1 = q_2 = 0.8$, $\beta_1 = 0.3$, $\beta_2 = 0.2$, $m_1 = 400$, $m_2 = 1000$, $\mathrm{h}_2 = 1$, $\lambda = 2$, $\Lambda = 8$, and $T = 25$. It shows that the profit peaks within the introductory time window. This means that, despite the profitability of delaying the release of the new generation, the introduction of the new generation should happen before the end of the introductory time window.

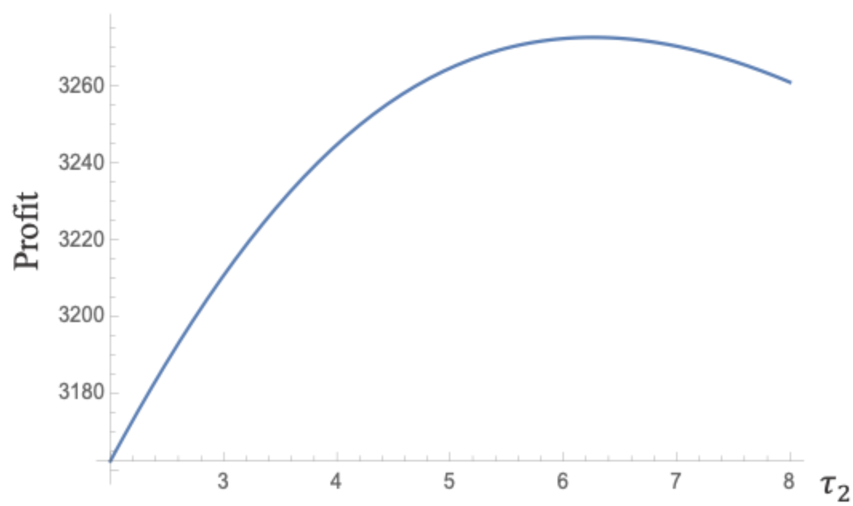

Figure 3-9: Dynamics of Profit with Respect to Market Entry Timing Under Phase-out Transition Scenario

Propositions 3.3 and 3.4 summarize findings regarding optimal market entry timing under the total transition scenario.

Proposition 3.3. Assume that the firm adopts the total transition scenario.

(I) If $\pi_1(t) \geq \pi_2(t)$ for $t \in [\lambda, T]$ and $\beta_1 \geq \beta_2$ with at least one strict inequality, and $m_1 > \tilde{\gamma}_1$, then $\tau_2^* = \Lambda$.

(II) If $\pi_2(t) \geq \pi_1(t)$ for $t \in [\lambda, T]$ and $\beta_2 \geq \beta_1$ with at least one strict inequality, and $m_1 > \tilde{\gamma}_2$, then $\tau_2^* = \lambda$.

(III) Suppose $\pi_2(t)$ declines monotonically. If $\pi_2(t) \geq \pi_1(t)$ for $t \in [\lambda, T]$ and $\beta_2 \geq \beta_1$, then $\tau_2^* = \lambda$.

Proposition 3.3 provides a set of sufficient conditions for the optimal market entry timing under the total transition scenario. Unlike the phase-out transition scenario discussed in Proposition 3.1, the results are not necessarily limited to scenarios in which the unit contribution margin for the second generation is monotonic. Thresholds $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ are provided in the proof of Proposition 3.3 in Appendix B.

Proposition 3.3 (I) provides conditions under which maximum delay in the introduction of the new generation is recommended. According to Proposition 3.3 (I), if the potential market size for the first generation is large enough, and the unit contribution margin and repeat purchase rate for the first generation are at least as large as that of the second generation, with at least one of either the unit contribution margin or the repeat purchase rate for the first generation being strictly higher than that of the second generation, firms are better off delaying the introduction of the second generation as much as possible.

The explanation for Proposition 3.3 (I) is that, with the introduction of a new generation and therefore the discontinuation of the old generation, customers who would have bought the old generation shift to the new generation. However, the unit contribution margin and the repeat purchase rate in the new generation are lower or at most as high as those of the old generation. This means that the customers shifting from the first generation to the second generation lead to a lower profit, discouraging the release of the new generation. On the other hand, the substantial profit generated by the first generation due to its large market potential, and unit contribution margin and repeat purchase rate at least as high has that of the second generation (with at least one of either the unit contribution margin or the repeat purchase rate for the first generation being strictly higher than that of the second generation) outweighs the profit generated by the new generation's created market.

The sufficient conditions provided in Proposition 3.3 (II) and (III) support the strategy of introducing the new generation as early as possible under the total transition scenario. According to Proposition 3.3 (II), if the potential market size for the first generation is large enough, and the unit contribution margin and repeat purchase rate for the second generation are at least as large as that of the first generation, with at least one of either the unit contribution margin or the repeat purchase rate for the second generation being strictly higher than that of the first generation, firms are recommended to release the new generation as early as possible.

Proposition 3.3 (II) indicates that when the market potential for the first generation is high, with the introduction of the second generation and therefore the discontinuation of the first generation under the total transition strategy, a substantial number of customers shift from the first generation to the second generation. Given that the unit contribution margin and the repeat purchase rate of the second generation are higher than or at least as high as those of the first generation, firms are recommended to introduce the second generation as early as possible to generate more profits. Moreover, in addition to the customers shifting from the first generation to the second generation, the new generation creates its own market potential and subsequently generating more profit.

Note that depending on the unit contribution margin trend and the sales trend of the second generation, delaying the introduction of the second generation may positively impact profit. For example, with a monotonically increasing unit contribution margin, a delayed introduction of the second generation can result in having the second generation's sales peaking at a point that coincides with higher unit contribution margins, thereby positively impact profit. However, this positive effect on profit does not outweigh the negative impact of a delayed release.

Based on Proposition 3.3 (III), firms are recommended to introduce the second generation as early as possible when the unit contribution margin for the second generation is at least as high as that of the first generation and is monotonically decreasing, and the repeat purchase rate of the second generation is at least as high as that of the first generation.

Proposition 3.3 (III) indicates that when the unit contribution margin and the repeat purchase rate of the second generation are at least as high as those of the first generation, the discontinuation of the first generation, causing customers to shift from the first generation to the second generation subsequently resulting in a profit at least as large as that gained under the first generation, motivates an early release of the second generation. Furthermore, the introduction of a new generation activates a new section of the market that is interested in the new generation. That along with the unit contribution margin of the second generation being decreasing over time further encourage an earlier release of the second generation.

It is important to note that in Proposition 3.3 (III), similar to the phase-out transition scenario discussed in Proposition 3.1 (III), the early market entry strategy is optimal regardless of how large the market potentials of the first and second generations are. Moreover, similar to the phase-out transition scenario, the early market entry strategy is recommended even if the unit contribution margins and the repeat purchase rates of the first and the second generation are equal (i.e., $\pi_1(t) = \pi_2(t)$, $\lambda \leq t \leq T$, and $\beta_1 = \beta_2$). Similar to the phase-out transition scenario, this can be attributed to the fact that the introduction of the second generation generates a new market. Considering that the unit contribution margin of the second generation is declining over time, attracting these second-generation customers earlier than later by releasing the second generation as early as possible leads to higher profits.

Under the total transition scenario, Proposition 3.4 provides a condition under which, similar to Proposition 3.3 (I), the first generation's unit contribution margin and repeat purchase rate at least as high as that of the second generation, with at least one of either the unit contribution margin or the repeat purchase rate for the first generation being strictly higher than that of the second generation, are in favor of a late market entry strategy; however, unlike Proposition 3.3 (I), the market size of the first generation is not big enough to support the maximum delay strategy.

Proposition 3.4. Assume that the firm adopts the total transition scenario. If $\pi_1(t) \geq \pi_2(t)$ for $t \in [\lambda, T]$ and $\beta_1 \geq \beta_2$ with at least one strict inequality, and $m_1 < \tilde{\eta}_1$, then $\tau_2^* \in [\lambda, \Lambda)$.

Threshold $\tilde{\eta}_1$ is provided in the proof of Proposition 3.4 in Appendix B. It can be observed that $\tilde{\eta}_1 \leq \tilde{\gamma}_1$. We conduct numerical analysis to demonstrate the result of Proposition 3.4. Figure 3-10 shows the profit as a function of market entry timing when $\pi_1(\mathrm{t}) = 1.2$, $\pi_2(\mathrm{t}) = 1$, $p_1 = p_2 = 0.01$, $q_1 = q_2 = 0.8$, $\beta_1 = 0.3$, $\beta_2 = 0.2$, $m_1 = 600$, $m_2 = 800$, $\lambda = 4$, $\Lambda = 15, T = 20$. It can be observed in Figure 3-10 that the profit peaks within the introductory time-window, which means that, although the delayed market entry strategy is recommended, the maximum delay strategy is not optimal.

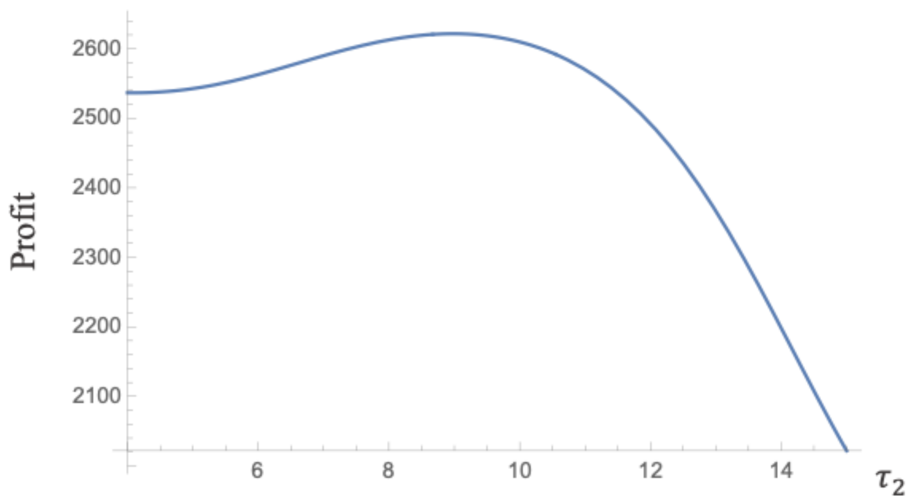

Figure 3-10: Dynamics of Profit with Respect to Market Entry Timing Under Total Transition Scenario

Note that our main model does not include the prices of the product generations. An implicit assumption is that price has already been considered in calculating the unit contribution margin. To explicitly explore the influence of pricing on the market entry timing decision, we analyze an extension scenario under phase-out transition scenario that incorporates price. In this extended analysis, we consider two key assumptions: (i) the price of the old generation is dropped (ii) the firm sets higher price for the new generation. Details of this analysis are provided in Appendix B. Our numerical results suggest that, as the price of the new generation increases, it is generally more profitable to release the new generation sooner. Given that an elevated price leads to a higher unit contribution margin, this finding is consistent with our analytical findings outlined in Propositions 1 (II) and (III) that the immediate release of the new generation is recommended when the unit contribution margin for the new generation is higher than that of the old generation.

The improved sales estimation by our model can potentially lead to better decisions on optimal market entry timing. Based on sales estimations using the DVD (2000-2018) and Blu-ray Players (2008-2018) data, we conduct numerical analysis on how a firm's profit changes with the market entry timing of a new generation under our GMDR and the benchmark model (Jiang et al., 2019). A representative instance of their difference is illustrated in Figure 3-11. The unit contribution margins for the old and new generation are assumed to follow the decreasing trends given by $\pi_1(t) = 10e^{-0.01t}$ and $\pi_2(t) = 10e^{-0.1t}$ and the planning horizon is set to $T = 22$. We can observe a stark disparity in market entry timing recommendations between GMDR and the benchmark model. The benchmark model advocates an immediate launch of the new generation, while GMDR suggests a maximum delay strategy for releasing the new generation. This divergence is further illuminated in Figure 3-11(a), where profit estimations from GMDR differ substantially between the strategies of immediate product release and maximal delay.

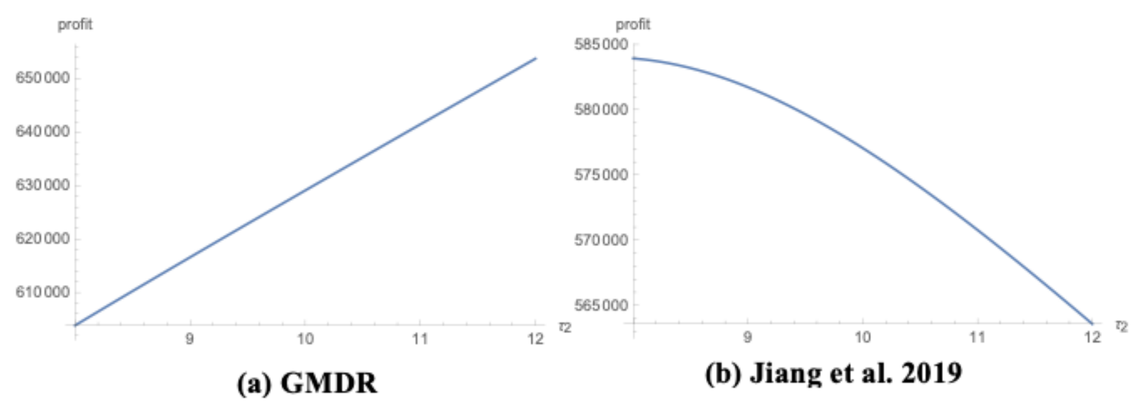

Figure 3-11: Comparison of the Profit Dynamics Under Different Market Entry Timing Models

The present research does not aim to prescribe a specific phase-out transition strategy or total transition strategy, as external factors beyond the scope of this study can impact a company's choice between these strategies. However, we also realize that an analytical comparison of the profitability outlined in Equation (3.43) between these two scenarios can shed light on the firm's decision-making process in selecting either strategy. We thus conduct such an analysis and derive some analytical findings, as summarized in Proposition 3.5. The proof of Proposition 3.5 is provided in Appendix B.

Proposition 3.5. The following hold regarding the profit under phase-out and total transition scenarios:

(I) If $\pi_1(t) = \pi_2(t)$ for $t \in [\lambda, T]$ and $\beta_1 = \beta_2$, then the profit under phase-out and total transition scenarios are equal.

(II) If $\pi_2(t) \leq \pi_1(t)$ for $t \in [\lambda, T]$ and $\beta_2 \leq \beta_1$ with at least one strict inequality, then the profit under phase-out transition scenario is higher than that under total transition scenario.

(III) If $\pi_1(t) \leq \pi_2(t)$ for $t \in [\lambda, T]$ and $\beta_1 \leq \beta_2$ with at least one strict inequality, then the profit under total transition scenario is higher than that under phase-out transition scenario.

Proposition 3.5 reveals that when the unit contribution margin and repeat purchase rates for the old and new generations are equal, the profits generated under phase-out transition and total transition scenarios are equal. However, if the unit contribution margin and repeat purchase rate for the old generation are at least as high as that of the new generation, with at least one of either the unit contribution margin or the repeat purchase rate for the old generation being strictly higher than that of the new generation, retaining the old generation in the market following the introduction of the new generation (i.e., opting for a phase-out transition scenario) yields a higher profit. Conversely, when the unit contribution margin and repeat purchase rate for the new generation are at least as high as that of the of the old generation, with at least one of either the unit contribution margin or the repeat purchase rate for the new generation being strictly higher than that of the old generation, discontinuing the old generation (i.e., employing a total transition strategy) results in a higher profit.

3.6 Concluding Remarks

In this research, we develop a new multigeneration sales model that accounts for repeat purchases. Similar to models from the extant literature, our model captures initial purchases of each generation as well as cross-generation upgrades. However, unlike the existing models that do not account for repeat purchases within each generation, our new model incorporates such purchases, making it suitable for estimating sales of technology products with high rates of within-generation repeat purchases. Furthermore, we incorporate two main generation-transition strategies: (i) a phase-out transition strategy under which firms continue to sell the old generation after the release of a new generation to fulfill the existing demand for the old generation, and (ii) a total transition strategy under which firms discontinue the old generation with the release of a new generation. The new model can be used in predictive and prescriptive analytics corresponding to sales of technology products.

We use multiple datasets to empirically evaluate the fit and forecasting accuracy of the proposed model. Furthermore, we compare our model with a benchmark model (Jiang et al., 2019) that does not incorporate repeat purchases. Our empirical results show that the new model fits and forecasts sales significantly better than does the benchmark model, demonstrating the importance of incorporating repeat purchases.

Furthermore, using our new multigeneration sales model, we derive the optimal market entry timing of a new product generation in a product line under the two generation transition strategies. We find conditions under which firms are better off releasing a new generation as early as possible or waiting as much as possible to introduce the new product generation. Specifically, under both the phase-out transition and the total transition scenarios, firms are better off adopting the maximum delay strategy in releasing the new generation when (i) the potential market size for the old generation is large enough, (ii) the unit contribution margin of the old generation is greater than that of the new generation, and (iii) the repeat purchase rate of the old generation is at least as high as that of the new generation. Moreover, independent of potential market sizes of the old and the new product generations, firms are better off releasing the new generation as early as possible when (i) the unit contribution margin for the new generation is at least as large as that of the old generation and declining, and (ii) repeat purchase rate of the new generation is at least as high as that of the old generation. This means that, even if the repeat purchase rates and the unit contribution margins for the old and new generations are identical, immediate release of the new generation is optimal if the unit contribution margin for the new generation is declining. Finally, under both generation transition scenarios, there are conditions under which neither introducing the new generation as early as possible nor delaying the release of new generation as long as possible is the optimal strategy.

In summary, the main contributions of the present research include (i) the introduction of a new model for sales of multigeneration technology products with repeat purchases and (ii) the development of a modelling framework for optimizing the market entry timing for a new product generation in a product line.

The multigeneration sales model outlined in this paper is not without limitations. Although factors such as pricing, quality and features, competition, and macro-economic conditions (e.g., recession, inflation, etc.) are implicitly considered in the model's parameters, they are not explicitly incorporated into the model. Consequently, this proposed modeling framework is best suited for situations where the primary focus of analysis does not revolve around directly assessing the influence of these factors on sales trends or the release timing of new product generation.

The new multigeneration sales model introduced in this study can serve as a foundation for future works. For instance, while the adoption rate in the current model follows the Bass Model, other diffusion models (e.g., the one by Van den Bulte and Joshi 2007) could also be considered to develop a similar multigeneration product sales model. Furthermore, by incorporating marketing mix variables such as pricing and advertising, the new multigeneration sales model can be used to find profit-maximizing optimal pricing and advertising policies in a multigenerational sales scenario when a significant portion of sales come from repeat purchases. Additionally, as a novel tool in the management science literature (Lotfi et al., 2023), we model repeat purchases using fractional integration. As a result, the fractional calculus employed in the present research contributes methodologically to the management science literature and may be used in other literatures related to business research.

Chapter 4

4 Giveaway Strategies for a New Technology Product

4.1 Introduction

Firms often offer free products to users in exchange for behaviors that positively influence product sales (e.g., relevant social media postings). To be offered a free product, one does not necessarily need to have a celebrity stature or an extensive number of social media followers. Having, for example, an Instagram account with an authentic presence and quality content may suffice (Gervin, 2021).

Firms may adopt different timings for giving away products for free. In one strategy, firms first give away their new product before launching it to the market. We call this strategy *before*. For instance, more than 5 million free copies of SpyBlocker antispyware software were distributed before officially selling the product in the market (Jiang & Sarkar, 2009). Another example is Sun Microsystems which gave away its office suite StarOffice for few years, before starting to charge new users for it (Jiang & Sarkar, 2009). Alternatively, firms may choose not to delay product release and instead offer free products upon the product launch and in parallel with sales in the market. We call this strategy *concurrent*. For instance, Google gave away Google Nest Minis for free to Spotify premium users while the product was available to purchase in the market (For the Record, 2019). In this paper, we study the before and concurrent free product offering strategies as well as the combined case in which a firm offers the free products prior to product launch in the market as well as after the release of the new product and in parallel with market sales. We call this strategy *combined*.

Free product giveaway can bring several benefits to firms. For instance, users can be offered a free product in exchange of testing a beta version of the product (Jiang et al., 2017). Moreover, offering free products can increase the network effect (Jiang et al., 2017). We focus on the role of free users in promoting the new product sales. Specifically, we present models to examine the impact of free users' word-of-mouth on the acceleration of new product sales and the corresponding profits when the before, concurrent, and combined free product offering strategies are adopted. Moreover, we assume that the free products offered are identical to those planned for market sale. This research does not consider cases where firms might give away products with fewer features.

The current literature studies free product offers for product adoption without considering repeat purchases although repeat purchases constitute a significant proportion of technology products' sales. To fill this void, focusing on technology products, we study the strategies for offering free products when sales are composed of (i) buyers' adoptions, (ii) repeat purchases made by buyers, and (iii) repeat purchases made by free users when they get additional units. Additionally, unlike the current literature which mainly studies cases where the offering of free products takes place before product launch, we study free product offering under three strategies, before, concurrent and combined. Furthermore, based on the propensity to purchase the product, we distinguish between two types of free users. We refer to free users who would buy the product if it were not offered to them for free as *high-valuation* users, whereas we refer to free users who are not willing to buy the product as *low-valuation* users. We specifically address the following research questions in this paper:

- What is the optimal number of products to offer for free under the before, concurrent, and combined strategies?
- What is the impact of the different types of free product users on the optimal number of free product offerings and the firm's profitability?
- How do market dimensions such as market size, potential buyers' internal motivation for new product adoption, buyers' and free users' word-of-mouth effect, and repeat purchases rates influence the optimal number of free product offerings and a firm's profitability?
- How do the before, concurrent, and combined strategies compare in terms of profitability?

We obtain the following managerial insights.

(i) Offering free products is highly dependent on the adoption and repeat purchases rates. Firms may benefit more from offering free products when the rate of repeat purchases for the new product is expected to be high, but the adoption is slow.

(ii) Offering free products can be more advantageous when the unit contribution margin is decreasing.

(iii) Higher ratio of low-valuation to high-valuation free users can make offering free products more appealing.

(iv) Offering free products is more favorable when the free product marginal cost is low. However, under certain conditions even a zero marginal cost may not justify the offering of free products. One such case is when the repeat purchase rate is low, the buyers' adoption process is fast, and the ratio of low-valuation to high-valuation free users is low.

(v) In the cases where offering free products is not optimal, increasing the market size does not necessarily make offering free products justifiable.

We also find that, when firms plan for a short planning horizon, in a market composed of potential buyers with effective word-of-mouth, they should consider the following for higher profit:

(i) Markets composed of potential buyers with high self-motivation to adopt the new product and/or high word-of-mouth effect should be targeted.

(ii) Highly influential users should be targeted with free products.

(iii) When the free product marginal cost is zero or relatively low compared to the unit contribution margin, higher ratio of low-valuation to high-valuation free users should be targeted with free products (i.e., corresponding to each high-valuation free product recipient, a larger number of low-valuation users should obtain free products).

(iv) When the free product marginal cost is zero or relatively low compared to the unit contribution margin, faster free product adoption by low-valuation free users after the release time should be planned.

Recommendations (i) through (iii) apply for all three strategies (before, concurrent, and combined) while recommendation (iv) applies to the concurrent and combined strategies.

The above recommendations lead to higher profit over a more extended time period when the rate of repeat purchases is high. However, intriguingly, if an extensively long planning horizon is desired, the above recommendations may lead to less profitability.

Our analysis also reveals the differences between the before, concurrent, and combined strategies. When product giveaway is profitable, firms may profit more from the before strategy than the concurrent strategy if the free product distribution time window is short and the new product is introduced without significant delay. The concurrent strategy, however, may be superior to the before strategy if the free product distribution period before the product release is long and the new product is released to the market with a significant delay. Also, the before strategy may not yield the highest profits, whereas using a combined strategy can result in the highest profits. One such condition is when there is a high rate of repeat purchases but a slow adoption of new products—a favorable condition for free product offerings.

We provide a literature review in Section 4.2. In Section 4.3, we establish our adoption and sales models for the before and concurrent scenarios. Section 4.4 presents the optimization problems related to the free product offers under the before and concurrent strategies. Section 4.5 presents a modelling framework for the combined strategy. Finally, Section 4.6 concludes the chapter.

4.2 Literature Review

Giving away products to a subset of the target population is a form of seeding in the target population. There are several publications discussing the role of seeding in product diffusion and profit optimization (e.g., Jain et al., 1995; Bakshi et al., 2007; Orbach & Fruchter, 2017). However, these papers account for new product adoptions only and ignore repeat purchases which constitute a significant proportion of the sales of technology products. Additionally, the modelling techniques used in these papers restrict the analysis to seeding only prior to product launch. We consider repeat purchases and analyze three free product offering strategies: prior to product release (before strategy), upon product release and parallel to sales (concurrent strategy), and lastly, prior to and after the product release time and in parallel with sales (combined strategy).

Lehmann and Esteban-Bravo (2006) study the effect of giving away products on a new product's diffusion acceleration. Considering three time periods for the diffusion process, Lehmann and Esteban-Bravo (2006) develop a model in which early adopters are provided with free products to accelerate product adoption. In another study, Han and Zhang (2018) examine optimal free sampling levels based on the Bass diffusion model (Bass, 1969). However, Lehmann and Esteban-Bravo (2006) and Han and Zhang (2018) do not incorporate repeat purchases in their analysis.

There are models that are specifically developed for software products. Jiang and Sarkar (2009) study the idea of offering free software in which users pay for initial subscriptions, while resubscriptions are available to users free of charge, making their model applicable only for particular cases. Jiang et al. (2017) examine giving away software for beta testing. However, they assume sales include only initial subscriptions, and exclude resubscriptions. Furthermore, they assume free users and buyers have identical word-of-mouth effect. Neither Jiang and Sarkar (2009) nor Jiang et al. (2017) examine cases in which the free offers take place after the product release and in parallel with product sales.

In a recent study Lam et al. (2020) focus on freemium games. The freemium games are free to download and only earn revenue through offering in-app purchases. Their study develops a modelling framework for finding the optimal release time for in app add-ons but does not address the problem of finding the optimal number of product giveaways when the product itself is going to be sold.

In a study, Parker and Van Alstyne (2005) examine profitability of giving away products by a firm in network markets. However, their model does not provide a temporal analysis of sales and profit. In other studies, researchers use agent-based simulation modelling to assess the effect of promotional strategies and seeding on new product diffusion. Examples of such works are Delre et al. (2007), Libai et al. (2013) and Hu et al. (2018). However, these studies do not provide mathematical models that explicitly capture product sales over time.

Our review of the literature indicates that a temporal analysis of free product offering for technology products has not been studied before when (i) sales includes both adoptions and repeat purchases and (ii) free product offers can happen before the product release time and/or after the product release time and in parallel with sales which are common in practice. We aim to fill this gap in the literature and provide valuable managerial insights.

4.3 Free Product Offer Under Before and Concurrent Strategies

We first develop novel models to capture the sales dynamics of free product offering strategies for technology products and provide new insights on how free offers affect the dynamics of sales when such sales consist of both adoptions and repeat purchases. We consider two cases: (i) free products are offered before the product launch (before strategy) and (ii) the new product is released without delay and free product offers happen following the product launch and continue in parallel with the product sales (concurrent strategy). Then we determine the optimal number of free products to offer under both strategies. Later in Section 4.5 we also formulate the combined strategy case.

A common assumption in the literature is finite-time demand window. Following Kalish (1983), Krishnan et al. (1999), Jiang and Sarkar (2009), and Jiang et al. (2017), we set time $T < \infty$ as the planning horizon, making $[0, T]$ our finite-time demand window. We use this time window to evaluate the profit based on sales. Sales are composed of adoptions and repeat purchases. Free product recipients are split into two types: *high-valuation* free users and *low-valuation* free users. High-valuation users are those who are willing to pay for the product and low-valuation users are those who would not pay for the product. Following the literature (e.g., Jiang & Sakar, 2009; Lotfi et al., 2023), we consider a potential adopters' internal motivation to adopt a new product as the *innovation effect*. Similarly, we consider the word-of-mouth effect on other potential adopters as the *imitation effect*.

We use the GDMR model developed in Lotfi et al. (2023) to model the sales process of a product after release. GDMR is a product sales model developed for technology products that captures both adoptions and repeat purchases. GDMR formulates sales as

$$S(t) = I_0^{\beta} y(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} y(s) ds, \quad t \geq 0. \tag{4.1}$$

Here, $y(t)$ denotes the adoptions at time t and β $(0 \leq \beta \leq 1)$ denotes the order of the fractional integral operator I_0^β which captures the rate of repeat purchases.[10] Through extensive empirical testing, Lotfi et al show that the fractional integral operator effectively captures the repeat purchase rate for technological product sales. Higher values of β correspond to higher repeat purchase rates. To capture the buyer's adoption, we use the Bass diffusion model (Bass, 1969) given by

$$y(t) = \left(p + \frac{q}{m} Y(t)\right)\left(m - Y(t)\right), \tag{4.2}$$

where $p, q,$ and m denote the coefficient of innovation, coefficient of imitation, and market potential respectively. In (4.2), $y(t)$ captures the adoption rate at time t and $Y(t)$, $Y(t) = \int_0^t y(s)\, ds$, is the cumulative adoption at t. One can obtain the following closed form solution from (4.2)

$$y(t) = Y'(t) = m \frac{(p+q)^2}{p} \frac{e^{-(p+q)t}}{\left(1+\frac{q}{p}e^{-(p+q)t}\right)^2}, \quad t \geq 0. \tag{4.3}$$

Following Lotfi et al. (2023), our unit of analysis is a group of products that differ only incrementally. Specifically, we consider a new purchase as a repeat purchase only if the newly purchased product is identical to the initially purchased product or differ only incrementally.

4.3.1 Before Strategy

We first consider the product sales dynamics of the before strategy, i.e., when the firm offers free products before the official release of the product to the market. Free products are offered to and adopted by users within the time-window $[0, \tau]$, $\tau > 0$ where the product release happens at τ and τ is exogenously given. Figure 4-1 illustrates the before strategy timeline.

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[10] In the numerical computations, we replace fractional integral operators with the computationally tractable approximate operators introduced in Lotfi et al. (2023).

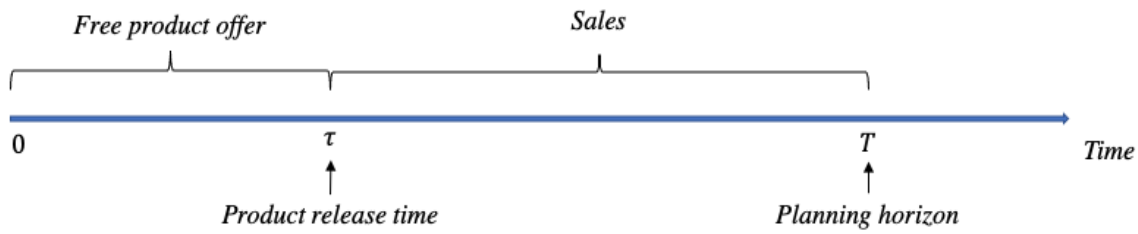

Figure 4-1: Before Strategy Timeline

Suppose n is the number of high-valuation free users. Following Jiang and Sarkar (2009) and Jiang et al. (2017), we let δn be the number of low-valuation free users where δ denotes the ratio of low-valuation to high-valuation free users. Depending on the target population where the free product is distributed, the ratio δ can be different. For instance, if the firm offers the free product to the members of a forum for the product supporters, then the ratio δ is expected to be lower than when the free product is offered to indifferent users. Since high-valuation users are those who are willing to pay for the product if they do not receive a free unit, unlike low-valuation free users, by offering free units to high-valuation users we remove them from the population of potential adopters who will purchase the product. Specifically, if n free units are offered in $[0, \tau]$ to high-valuation free users, then there will be n fewer adopters who will pay for the product after the product release and the market size will be $m - n$ which we refer to as the *actual market size*. Thus, in the presence of high-valuation and low-valuation free users after the product release and accounting for their word-of-mouth effect, the buyers' adoption at time $t \geq \tau$ follows the following diffusion process

$$y(t) = \left(p + \frac{q}{m}Y(t) + \frac{\lambda_h}{m}n + \frac{\lambda_l}{m}\delta n\right)\left(m - n - Y(t)\right), \quad t \geq \tau, \tag{4.4}$$

where, $Y(\tau) = 0$. Here, λ_h and λ_l are imitation parameters capturing high-valuation and low-valuation free users word-of-mouth effect on buyers respectively. Following the standard Bass diffusion model (Bass, 1969), the closed form solution for (4.4) can be derived as

$$y(t) = Y'(t) = (m - n)\frac{(\tilde{p}+\tilde{q})^2}{\tilde{p}}\frac{e^{-(\tilde{p}+\tilde{q})(t-\tau)}}{\left(1+\frac{\tilde{q}}{\tilde{p}}e^{-(\tilde{p}+\tilde{q})(t-\tau)}\right)^2} \quad t \geq \tau. \tag{4.5}$$

where $\tilde{p} = p + \frac{\lambda_h}{m}n + \frac{\lambda_l}{m}\delta n$ and $\tilde{q} = q\frac{m-n}{m}$. It is worth mentioning that under the special case where buyers and free users' have identical word-of-mouth effect ($q = \lambda_h = \lambda_l$), the adoption formulation (4.4) reduces to the adoption formulation presented in Jiang and Sarkar (2009) and Jiang et al. (2017). By (4.5) we have

$$Y(t) = \frac{(m-n)\left(1-e^{-(\tilde{p}+\tilde{q})(t-\tau)}\right)}{\left(1+\frac{\tilde{q}}{\tilde{p}}e^{-(\tilde{p}+\tilde{q})(t-\tau)}\right)}. \tag{4.6}$$

We observe that $Y(t)$ and $y(t)$ are proportional to the actual market size $m-n$. Given that $\frac{\partial Y}{\partial \tilde{p}}, \frac{\partial Y}{\partial \tilde{q}} > 0$, $t > 0$, and $\frac{\partial \tilde{p}}{\partial p}, \frac{\partial \tilde{p}}{\partial \lambda_h}, \frac{\partial \tilde{p}}{\partial \lambda_l}, \frac{\partial \tilde{p}}{\partial \delta}, \frac{\partial \tilde{q}}{\partial q} > 0$, we see that $Y(t)$, $t > 0$, is strictly increasing in p, q, λ_h, λ_l and δ. Additionally, according to Bass (1969), $y(t)$ has a peak if $\tilde{q} > \tilde{p}$. Thus, equivalently, $q > \frac{m}{m-n}\left(p + \frac{n}{m}(\lambda_h + \lambda_l\delta)\right)$ results in $y(t)$ having a peak. These properties are important because they offer valuable insights into the dynamics of the product adoption process. Additionally, they play a key role in the proof of subsequent analytical developments.

Incorporating the repeat purchases made by buyers, we obtain the buyers' sales at t, $t \geq \tau$ as

$$S(t) = I_\tau^\beta y(t) = \frac{1}{\Gamma(\beta)}\int_\tau^t (t-s)^{\beta-1} y(s)ds, \tag{4.7}$$

where β denotes the buyers' repeat purchases rate. If high-valuation free adopters would like to acquire additional units, then they need to pay for those units. Therefore, high-valuation free users can also be a source of sales through repeat purchases which we denote by $S_h(t)$. Following the GDMR, the high-valuation free users repeat purchases are captured by

$$S_h(t) = \frac{1}{\Gamma(\beta_h)}\int_0^\tau (t-s)^{\beta_h-1} y_h(s)ds, \tag{4.8}$$

where $y_h(t)$ denotes the free product distribution to *n* high-valuation free users within the time window $[0,\tau]$, $\tau > 0$ and β_h denotes repeat purchases rate of high-valuation free users. Thus, the total sales at t, $t \geq \tau$, is given by $S(t) + S_h(t)$.

4.3.2 Concurrent Strategy

We now examine the concurrent strategy, i.e., the new product is released without delay and free product offers take place in parallel with product sales following the product release and within the planning horizon. Figure 4-2 depicts the timeline for the concurrent strategy.

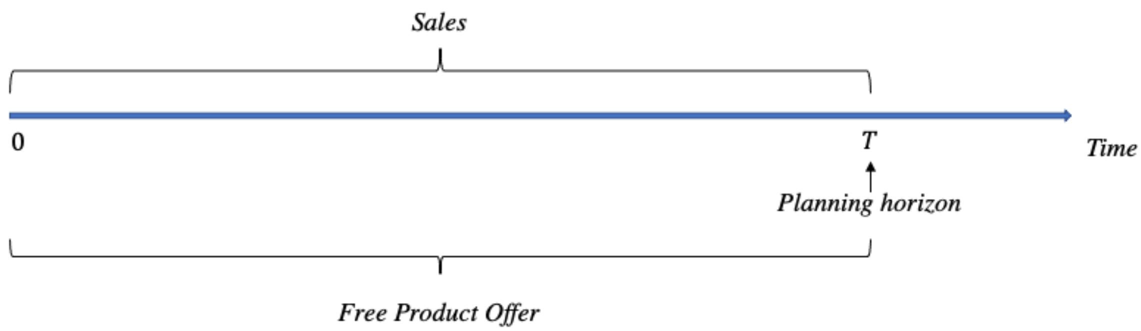

Figure 4-2: Concurrent Strategy Timeline

Let $y(t)$ be the product adoption process by buyers, and $y_h(t)$ and $y_l(t)$ be the free product adoption processes for high-valuation and low-valuation free users respectively. As before, m is the number of potential adopters, n is the number of high-valuation free users, and δn is the number of low-valuation free users with δ being the ratio of low-valuation to high-valuation free users. We consider a generic formulation for free product distributions. Specifically, we consider $Y_l(t)$, $Y_l(t) = \int_0^t y_l(s)\, ds$ and $Y_h(t)$, $Y_h(t) = \int_0^t y_h(s)\, ds$, are continuous monotonically increasing functions respectively converging to δn and n, representing the cumulative distribution of the free products to low-valuation and high-valuation free users respectively. It should be noted that by distributing all free products within the time interval $[0, T]$, we will have $Y_h(t) = n$ and $Y_l(t) = \delta n$ for $t \geq T$.

Given that the number of free products offered to high-valuation users is n, the market size should be adjusted to the actual market size $m - n$. Moreover, the buyers' adoption process $y(t)$ is influenced by $Y_h(t)$ and $Y_l(t)$ accounting for the word-of-mouth effect of the high-valuation and low-valuation free users on buyers. Thus, the buyers' adoption rate is formulated as

$$y(t) = \left(p + \frac{q}{m}Y(t) + \frac{\lambda_h}{m}Y_h(t) + \frac{\lambda_l}{m}Y_l(t)\right)\left(m - n - Y(t)\right). \qquad (4.9)$$

Here p and q are the buyers' coefficients of innovation and imitation, λ_h and λ_l denote the imitation parameters corresponding to high-valuation and low-valuation free product users' word-of-mouth, and $Y(t) = \int_0^t y(t)dt$ denotes the buyers' cumulative adoption at time t. Note that (4.9) is similar to (4.4) except that in (4.4) the free products are fully distributed to high-valuation and low-valuation free users by the product release time at $t = \tau$ ($Y_h(t) = n$ and $Y_l(t) = \delta n$ for $t \geq \tau$). Given that $Y(0) = 0$, (4.9) is an initial value problem. Although unlike (4.4), the analytical solution of (4.9) cannot be derived, the existence of a unique global solution is guaranteed by Theorem 4.1 below. Moreover, Propositions 4.1-4.3 demonstrate interesting properties of $y(t)$ and $Y(t)$, giving us insights from the modelling framework that we propose. Furthermore, they play a crucial role in the proof of subsequent analytical results. We provide all the proofs in Appendix C.

Theorem 4.1. The initial value problem (4.9) with $Y(t_0) = l$, $0 \leq t_0$, $0 \leq l < m - n$, has a unique solution $Y(t)$ defined on $[t_0, \infty)$ where $Y(t)$ is monotonically increasing to $m - n$.

Proposition 4.1 provides a sufficient condition under which the buyers' cumulative adoption rate is proportional to the actual market size $m - n$.

Proposition 4.1. If $Y_h(t)$ and $Y_l(t)$ are proportional to n, then $Y(t)$ is proportional to $m - n$.

Our next result, Proposition 4.2, demonstrates that the buyers' adoption $y(t)$ peaks if the buyers' imitation effect (q) is sufficiently high. Furthermore, the cumulative adoption rate $Y(t)$ is increasing in the innovation and the imitation parameters. It's notable that Proposition 4.2's sufficient condition for a peak in buyers' adoption is the same as the sufficient condition for the before strategy, and both reduce to the sufficient condition for the existence of a peak in the standard Bass diffusion process (4.2), which is $q > p$, when $n = 0$.

Proposition 4.2.

(i) If $q > \frac{m}{m-n}\left(p + \frac{n}{m}(\lambda_h + \lambda_l \delta)\right)$, then the adoption rate $y(t)$ peaks.

(ii) $Y(t), t > 0$ is increasing in p, q, λ_h, and λ_l.

Proposition 4.2 (i) shows that a high imitation effect increases the adoption rate stimulated by early adopters' word-of-mouth and therefore reaching a peak in adoption before a decline that is due to approaching market saturation. Proposition 4.2 (ii) demonstrates that the higher innovation effect of buyers (p) results in faster buyers' adoption process through internal motivation. Moreover, increasing the word-of-mouth effect generated by the buyers themselves (q), high-valuation free users (λ_h), and low-valuation free users (λ_l) further accelerate the buyers' adoption process. Next, Proposition 4.3 demonstrates that faster free product distribution leads to a faster buyers' adoption of the new product.

Proposition 4.3. Suppose $\bar{Y}(t)$ corresponding to $\bar{Y}_h(t)$, $\bar{Y}_h(t) > Y_h(t)$, $t \in (0, T)$, or $\bar{Y}_l(t)$, $\bar{Y}_l(t) > Y_l(t)$, $t \in (0, T)$, or both. Then, $\bar{Y}(t) \geq Y(t)$, for all $t \in [0, T]$.

Proposition 4.3 shows that having a higher number of free product users (higher $Y_h(t)$, or higher $Y_l(t)$, or both) results in higher free users' word-of-mouth effect on buyers at each time, leading to a faster buyers' adoption process.

Firms can adopt two approaches, direct or indirect, to distribute free products to potential free users. The first approach is to directly send the free products to designated users. In the second approach, the free product is distributed indirectly by making the free products available to the potential free users. In the second approach, free user's word-of-mouth plays an important role in the distribution of free products to free users. While under the direct approach $Y_h(t)$ and $Y_l(t)$ are explicitly determined by the firm, there is a need to develop formulations that capture free product distributions under the indirect approach. Next, we focus on the indirect approach and formulate the corresponding $Y_h(t)$ and $Y_l(t)$. We first consider the condition in which high-valuation and low-valuation free users have insignificant word-of-mouth effects on each other. Thus, we can formulate the distribution process for each free product user's category using the Bass diffusion model (Bass, 1969). Specifically, $Y_h(t)$ and $Y_l(t)$ are given by

$$y_h(t) = \left(p_h + \frac{q_h}{n} Y_h(t)\right)\left(n - Y_h(t)\right), \tag{4.10}$$

$$y_l(t) = \left(p_l + \frac{q_l}{\delta n} Y_l(t)\right)\left(\delta n - Y_l(t)\right). \tag{4.11}$$

Here, p_h, p_l and q_h, q_l, are the coefficients of innovation and imitation governing the free product adoption processes by high-valuation and low-valuation users. $Y_h(t) = \int_0^t y_h(t)dt$, and $Y_l(t) = \int_0^t y_l(t)dt$ capture the cumulative adoptions at time t for high-valuation and low-valuation free product users, respectively. The solution for (4.10) and (4.11) are

$$Y_h(t) = \frac{n(1-e^{-(p_h+q_h)t})}{(1+\frac{q_h}{p_h}e^{-(p_h+q_h)t})}, \tag{4.12}$$

$$Y_l(t) = \frac{\delta n(1-e^{-(p_l+q_l)t})}{(1+\frac{q_l}{p_l}e^{-(p_l+q_l)t})}. \tag{4.13}$$

From (4.12) and (4.13), we see that $Y_h(t)$ and $Y_l(t)$ are monotonically increasing to n and δn, and they are strictly increasing in p_h, q_h, and p_l, q_l, δ respectively ($\frac{\partial Y_h}{\partial p_h}, \frac{\partial Y_h}{\partial q_h}, \frac{\partial Y_l}{\partial p_l}, \frac{\partial Y_l}{\partial q_l}, \frac{\partial Y_l}{\partial \delta} > 0$). Furthermore, if $q_i > p_i$, $i = l, h$, then $y_i(t)$ peaks (Bass, 1969). The following result is derived from Proposition 4.3.

Corollary 4.1. Let $Y(t)$ be the solution for (4.9) with $Y_h(t)$ and $Y_l(t)$ as given by (4.10) and (4.11). Then, $Y(t)$, $t > 0$ is increasing in p_h, p_l, q_h, q_l, and δ.

Next, we consider a more general case in which high-valuation and low-valuation free users have word-of-mouth effects on each other. Specifically, we extend the free product adoption formulations (4.10) and (4.11) to the following

$$y_h(t) = \left(p_h + \frac{q_h}{n}Y_h(t) + \frac{q_{lh}}{n}Y_l(t)\right)\left(n - Y_h(t)\right), \tag{4.14}$$

$$y_l(t) = \left(p_l + \frac{q_l}{\delta n}Y_l(t) + \frac{q_{hl}}{\delta n}Y_h(t)\right)\left(\delta n - Y_l(t)\right). \tag{4.15}$$

Here, q_{lh} and q_{hl} are the imitation parameters capturing the word-of-mouth effect of low-valuation free users on high-valuation free users and high-valuation free users on low-valuation free users respectively. Although there is no closed-form solution for the system of (4.14) and (4.15), nevertheless, we can show that, similar to (4.10) and (4.11), the system of (4.14) and (4.15) generates monotonically increasing adoption trends for each category of free product users i.e., high-valuation and low-valuation, converging to the number of free users of each category i.e., n for high-valuation, and δn for low-valuation free users (Appendix C, Theorem C.1). We can demonstrate that similar to (4.12) and (4.13) derived from (4.10) and (4.11), $Y_h(t)$ and $Y_l(t)$, derived from (4.14) and (4.15), are proportional to n (Appendix C, Proposition C.1). The following result is obtained based on Proposition 4.1.

Corollary 4.2. Let $Y(t)$ be derived from (4.9) with $Y_h(t)$ and $Y_l(t)$ derived either from (4.10) and (4.11) or (4.14) and (4.15). Then $Y(t)$ is proportional to $m - n$.

We can show that $Y_h(t)$ and $Y_l(t)$, derived from (4.14) and (4.15), are increasing in $p_h, q_h, q_{lh}, p_l, q_l, q_{hl}$, and δ, for $t > 0$ (Appendix C, Proposition C.2). The following is immediate from Proposition 4.3.

Corollary 4.3. Let $Y(t)$ be the solution for (4.9) with $Y_h(t)$ and $Y_l(t)$ given by (4.14) and (4.15). Then, $Y(t)$, $t > 0$ is increasing in $p_h, q_h, q_{lh}, p_l, q_l, q_{hl}$, and δ.

Following Lotfi et al. (2023), to account for repeat purchases by buyers, we model total sales generated by buyers at t, $t \geq 0$, as

$$S(t) = I_0^{\beta} y(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} y(s) ds. \tag{4.16}$$

Here, β $(0 \leq \beta \leq 1)$ is the repeat purchase parameter of buyers. If high-valuation free adopters would like to acquire additional units, then they need to pay for those units. Therefore, high-valuation free users can also be a source of sales through repeat purchases. These sales at $t \geq 0$, can be captured by

$$S_h(t) = I_0^{\beta_h} y_h(t) - y_h(t). \tag{4.17}$$

The total sales at $t \geq 0$, are then given by $S(t) + S_h(t)$. We next discuss the optimal free product offering strategies.

4.4 Optimal Free Product Offer Under Before and Concurrent Strategies

We first develop a framework to optimize the number of free products to offer under the two strategies of before and concurrent then, we discuss the combined strategy in Section 4.5.

4.4.1 Profit Optimization

Recall that, under the before strategy, free products are distributed within the time-window $[0, \tau]$, $\tau < T$, and the new product is released at τ (i.e., n free products are distributed to high-valuation free users and δn free products are distributed to low-valuation free users until time τ). We are interested in finding the optimal number of free products to offer during the time-period $[0, \tau]$, $\tau > 0$. Consider, $\pi(t)$ as the discounted unit profit. Following Jiang et al. (2019), we refer to $\pi(t)$ as the *unit contribution margin*. The unit contribution margin changes over time due to changes in the unit profit and the discount rate over time. Given that the free product offer time window under the before strategy is short compared to the entire demand window, changes in cost are not significant, hence we assume a constant free product marginal cost c_0. Furthermore, we assume that the product development cost is a sunk cost, and as such, for products with inexpensive reproduction and distribution costs, e.g., software, we consider that the marginal cost is zero. Consequently, the profit function under the before strategy is

$$\Pi(n) = \int_{\tau}^{T} \pi(t)(S(t) + S_h(t))dt - c_0(1 + \delta)n, \tag{4.18}$$

where, $S(t)$ and $S_h(t)$ are given by (4.7) and (4.8) respectively.

Under the concurrent strategy, the new product is released at $t = 0$ and all the free products are distributed within the time window $[0, T]$ (i.e., n free products are distributed to high-valuation free users and δn free products are distributed to low-valuation free users over the entire time horizon). We are interested in finding the optimal number of free products to offer during the time-period $[0, T]$. The profit function under the concurrent strategy is

$$\Pi(n) = \int_{0}^{T} \left(\pi(t)\big(S(t) + S_h(t)\big) - c(t)\big(y_h(t) + y_l(t)\big)\right) dt, \tag{4.19}$$

where, $S(t)$ and $S_h(t)$ are given by (4.16) and (4.17), respectively. Here, $c(t)$ denotes the free product marginal cost at time $t, t \in [0, T]$. Similar to the before strategy, we assume that the product development cost is sunk.

Finally, we can represent the decision-making problem of determining the optimal number of products to give away for free as

$$\max_{n \leq n_M} \Pi(n), \tag{4.20}$$

where, $n_M < m$ is an exogenous upper bound on the maximum number of free products that can be distributed. [11]

Under both the before and concurrent strategies, it is intuitive that a rise in the potential market size and the upper bound on the maximum number of free products increases the optimal number of free product users (both high-valuation and low-valuation) to effectively influence sales through word-of-mouth. Proposition 4.4 demonstrates if free product distributions are proportional to the number of free products planned to be distributed, then with an increase (decrease) in the potential market size and the maximum free product offer bound, the optimal number of free products increase (decrease) at the same rate as the market size.

Proposition 4.4. Consider profit functions (4.18) and (4.19) when $Y_h(t)$ and $Y_l(t)$ are proportional to n. Let the profit optimization problem $\overline{\Pi}^{\,*} = \overline{\Pi}\,(\bar{n}^{\,*}) = \max_{n\in[0,\bar{n}_M]} \overline{\Pi}(n)$ corresponding to $\bar{m} = rm$, and $\bar{n}_M = rn_M$, $r > 0$. Then, $\overline{\Pi}^{\,*} = r\Pi^*$, and $\bar{n}^{\,*} = rn^*$, where $\Pi^* = \Pi\,(n^*) = \max_{n\in[0,n_M]} \Pi(n)$.

When the optimal solution is not to offer any free products ($n^* = 0$), expanding the market size and the bound on the maximum number of free products ($\bar{m} = rm$, $\bar{n}_M = rn_M$, $r > 1$) does not make offering free products justifiable since if free users are not required to speed up the buyers' adoption, then the buyers' adoption is already proceeding at a reasonable rate. Through market expansion, there will be an increase in the number of early adopters of the product, preventing a decline in adoption rates through their word-of-mouth and eliminating the need for free users' word-of-mouth. We obtain the following result by Proposition 4.4, (4.12) and (4.13), and Proposition C.1.

Corollary 4.4. Consider the profit function (4.19) corresponding to (4.10) and (4.11) or (4.14) and (4.15). Then the result of Proposition 4.4 holds.

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[11] We find the optimal number of high-valuation free users n. The total free users including both high-valuation and low-valuation free users is $(1 + \delta)n$.

Our next set of Propositions (4.5-4.7) shows interesting results regarding the optimal profit under the cases when the buyers' word-of-mouth (q) is sufficiently high. Recall that when q is sufficiently high the buyers' adoption rate peaks (see the discussion provided after (4.6) and Proposition 4.2 (i)). Thresholds are given in the proofs in Appendix C.

Proposition 4.5 shows a relationship between changes in the buyers' innovation effect (p) and buyers' and free users' imitation effects (q, λ_h, λ_l) and the optimal profit.

Proposition 4.5. Let $\overline{\Pi}(n)$ and $\Pi(\mathrm{n})$ where $n \le n_M$ be the profit functions, given by (4.18) or (4.19), corresponding to $\bar{p}$, $\bar{q}$, $\bar{\lambda}_h$, $\bar{\lambda}_l$, and p, q, λ_h, λ_l respectively where, $\bar{q} > \frac{m}{m-n_M}(\bar{p} + \frac{n_M}{m}(\bar{\lambda}_h + \delta\bar{\lambda}_l))$. Suppose either one of $\bar{p} \ge p$, $\bar{q} \ge q$, $\bar{\lambda}_h \ge \lambda_h$, and $\bar{\lambda}_l \ge \lambda_l$ holds with at least one strict inequality. Then

(i) If the planning horizon is sufficiently short ($T \le \tilde{t}$), then $\overline{\Pi}(n) > \Pi(n)$ for all $0 < n \le n_M$, and $\overline{\Pi}(0) > \Pi(0)$, if either of $\bar{p} > p$ or $\bar{q} > q$ holds, otherwise $\overline{\Pi}(0) = \Pi(0)$. Thus $\overline{\Pi}^* \ge \Pi^*$.

(ii) For $\beta > 0$, the planning horizon in part (i) can be extended to T_β, $T_\beta > T$ where $y_h(t) = y_l(t) = 0$ for $t \ge T$, with the profit differences ($\overline{\Pi}(n) - \Pi(n) > 0$) being larger than those in part (i) for the planning horizon T. For $\bar{\beta}$, $\bar{\beta} > \beta$, there exists $T_{\bar{\beta}}$, $T_{\bar{\beta}} > T_\beta$.

(iii) Let the planning horizon, T, in part (i) be extended to T_L, $T_L > T_\beta$, where $y_h(t) = y_l(t) = 0$ for $t \ge T$. If $\pi(t)$ is decreasing on $[T_\beta, T_L]$ with a high declining rate, then $\overline{\Pi}(n) > \Pi(n)$ for all $0 < n \le n_M$, and $\overline{\Pi}(0) > \Pi(0)$, if either of $\bar{p} > p$ or $\bar{q} > q$ holds, otherwise $\overline{\Pi}(0) = \Pi(0)$. Thus $\overline{\Pi}^* \ge \Pi^*$.

According to Proposition 4.5 (i), under both the before and concurrent strategies, when the planning horizon is short and the buyers' word-of-mouth effect (q) is sufficiently large, having higher values for either the (1) buyers' innovation effect (p), (2) the buyers' word-of-mouth effect (q), (3) the high-valuation free users' word-of-mouth effect on buyers (λ_h), or (4) the low-valuation free users' word-of-mouth effect on buyers (λ_l), results in higher profit in the short-term. This is because the higher internal motivation to adopt the new product by buyers and higher word-of-mouth effects on buyers result in faster product adoption and therefore higher adoption purchases in the early time periods of the product lifecycle. Consequently, these higher adoptions in the early time periods lead to a higher volume of sales through adoptions and repeat purchases in the early time periods thus higher profit in the short-term. Proposition 4.5 (ii) demonstrates that having a higher buyers' innovation effect and buyers' and free users' imitation effect results in higher profit for a longer time period when the repeat purchase rate is higher. This is because a high repeat purchase rate results in substantial subsequent sales through repeat purchases. Consequently, having higher adoptions in the early time periods within the demand window and before the planning horizon leads to higher sales even after the planning horizon. Finally, we see that the conditions leading to higher short-term profitability stated in parts (i) and (ii) also result in higher long-term profitability when the unit contribution margin ($\pi(t)$) is declining over time with a sufficiently high declining rate by Proposition 4.5 (iii). This is because the unit contribution margin in the late time periods is small and therefore any changes in the profit gained in the later time periods are insignificant compared to an increase in the short-term profit listed in parts (i) and (ii).

According to Proposition 4.5, if firms plan for short-term profitability and choose to offer free products before product launch or after the product launch but within a short time interval, then they should target markets with a high level of word-of-mouth effectiveness. Higher word-of-mouth effectiveness results in higher profits in the short term. Markets composed of potential buyers with a high level of social conformity are expected to have a high level of word-of-mouth effectiveness. Moreover, markets comprised of prospective buyers who have access to various communication channels are anticipated to demonstrate strong word-of-mouth effectiveness. Furthermore, targeting highly influential free users also leads to short-term profitability. Markets composed of highly self-motivated potential buyers are also favorable for short-term profit. The recommendations also result in high profitability in the long term if the unit contribution margin is declining at a fast rate.

Contrary to the short term, markets with highly self-motivated consumers, or high level of word-of-mouth efficacy, or influential free users, may not provide higher profits in the long term. Figure 4-3 shows such a scenario under the concurrent strategy ($\beta = \beta_h = 0.35, p = 0.02, \bar{p} = 0.025, q = \lambda_h = \lambda_l = 0.5, \bar{q} = \bar{\lambda}_h = \bar{\lambda}_l = 0.6, m = 3 \times 10^8, \delta = 1, n_m = \frac{m}{5}, \pi(t) = 10 + 5t$, and $c(t) = 25$). Figure 4-3 (a) shows the profit functions when the planning horizon is set to $T = 8$ and free products are distributed in the window $[0, T = 8]$ following (4.14) and (4.15) ($p_h = p_l = 0.02, q_h = q_l = q_{hl} = q_{lh} = 0.55$). Similarly, Figure 4-3 (b) shows the profit functions when the planning horizon is set to $T = 15$ and free products are distributed in the window $[0, T = 15]$ following (4.14) and (4.15) ($p_h = p_l = 0.01, q_h = q_l = q_{hl} = q_{lh} = 0.3$).

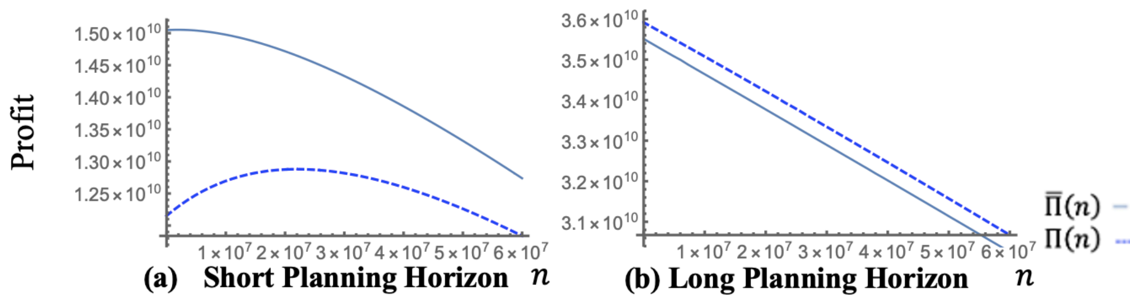

Figure 4-3: Profit Functions Under Different Innovation and Imitation Effects

The faster adoption rate derived by highly self-motivated consumers, high level of word-of-mouth efficacy, and very influential free users, causes more sales to occur earlier where the unit contribution margins are lower rather than later in time periods where the unit contribution margins are greater. In the short term (Figure 4-3 (a)), this causes higher profitability; however, when a long planning horizon is sought, it results in less profitability (Figure 4-3 (b)).

Next, we investigate the relationship between a change in the ratio of low-valuation to high-valuation free users (δ) and the optimal profit in Proposition 4.6. In Proposition 4.6 (iii), $\bar{y}_l(t)$ denotes free product distribution to low-valuation free users when the low-valuation to high-valuation ratio is set to $\bar{\delta}$.

Proposition 4.6. Let $\Pi(n)$ and $\overline{\Pi}(n)$ where $n \leq n_M$ be profit functions given by (4.18) or (4.19), corresponding to δ and $\bar{\delta}$ respectively where $\bar{\delta} > \delta > 0$ and $q > \frac{m}{m-n_M}(p + \frac{n_M}{m}(\lambda_h + \bar{\delta}\lambda_l))$.

(i) Consider the profit function (4.18) with $\pi(t)$ sufficiently high compared to c_0 or $c_0 = 0$, and the planning horizon sufficiently short ($T \leq \tilde{t}_1$). Then $\overline{\Pi}(n) > \Pi(n)$ for all $0 < n \leq n_M$, and $\overline{\Pi}(0) = \Pi(0)$, thus $\overline{\Pi}^* \geq \Pi^*$.

(ii) Consider the profit function (4.19) with $Y_l(t)$ strictly increasing in δ, $\pi(t)$ sufficiently higher than $c(t)$ on $[0, T]$ or $c(t) = 0$, and the planning horizon sufficiently short ($T \leq \tilde{t}_2$). Then $\overline{\Pi}(n) > \Pi(n)$ for all $0 < n \leq n_M$, and $\overline{\Pi}(0) = \Pi(0)$, thus $\overline{\Pi}^* \geq \Pi^*$.

(iii) Let $\beta > 0$. Then, the planning horizon in part (i) can be extended to T_β^1, $T_\beta^1 > T$ and the planning horizon in part (ii) can be extended to T_β^2, $T_\beta^2 > T$, where $y_h(t) = y_l(t) = \bar{y}_l(t) = 0$ for $t \geq T$, with the profit differences ($\overline{\Pi}(n) - \Pi(n) > 0$) being larger than those in parts (i) and (ii) for the planning horizon T. For $\bar{\beta}$, $\bar{\beta} > \beta$, there exists $T_{\bar{\beta}}^i$, $T_{\bar{\beta}}^i > T_\beta^i$, $i = 1,2$.

We see from Proposition 4.6 (i) and (ii) that if firms target a short planning horizon, then a higher ratio of low-valuation to high-valuation free users results in higher profits when the marginal cost of the free products is zero or low compared to the unit contribution margin, under both the before and concurrent strategies. A higher ratio of low-valuation to high-valuation free users means having more free users without reducing the actual market size. A large number of free users results in a faster buyers' adoption process through free users' word-of-mouth and therefore higher sales through adoptions and repeat purchases in the early time periods. The fact that the free product marginal cost is zero or at least is small compared to the unit contribution margin justifies the profitability of having a large number of low-valuation free users. Moreover, a higher ratio of low-valuation to high-valuation free users results in higher profits for a longer period of time when the repeat purchases rate is higher (Proposition 4.6 (iii)). This is because, after the planning horizon T, all the free products are already distributed therefore there is no cost for providing free products anymore. However, due to the high repeat purchase rate, having a higher adoption rate within the demand window and before the planning horizon results in higher sales even after the planning horizon due to the higher subsequent repeat purchases.

Next, we discuss the relationship between the low-valuation users' free product adoption rate and the short-term profitability when the concurrent strategy is adopted.

Proposition 4.7. Suppose $q > \frac{m}{m-n_M}\left(p + \frac{n_M}{m}(\lambda_h + \lambda_l \delta)\right)$. Let $Y_l(t) < \bar{Y}_l(t)$, $t \in (0, T)$ with $\bar{\Pi}(n)$ and $\Pi(n)$ where $n \leq n_M$ be profit functions given by (4.19) corresponding to $\bar{Y}_l(t)$ and $Y_l(t)$ respectively.

(i) If $\pi(t)$ is sufficiently higher than $c(t)$ on $[0, T]$ or $c(t) = 0$ where the planning horizon is sufficiently short ($T \leq \tilde{t}$), then $\bar{\Pi}(n) > \Pi(\mathrm{n})$, for all $0 < n \leq n_M$, and $\bar{\Pi}(0) = \Pi(0)$, thus $\bar{\Pi}^* \geq \Pi^*$.

(ii) Let $\beta > 0$ and $y_h(t) = y_l(t) = \bar{y}_l(t) = 0$ for $t \geq T$. Then, the planning horizon in part (i) can be extended to T_β, $T_\beta > T$ with the profit differences ($\bar{\Pi}(n) - \Pi(n) > 0$) being larger than those in part (i) for the planning horizon T. For $\bar{\beta}$, $\bar{\beta} > \beta$, there exists $T_{\bar{\beta}}$ such that $T_{\bar{\beta}} > T_\beta$.

We see from Proposition 4.7 (i) that if an early planning horizon is targeted and the concurrent strategy of the free product offer is adopted, then faster low-valuation users' free product adoption results in higher profit when the buyers' word-of-mouth effect is high and the marginal cost is zero or low compared to the unit contribution margin. This is because the higher number of low-valuation free users in the early stages of the product release time accelerates the buyers' adoption process. The accelerated adoption process results in higher buyers' adoption purchases in the early time periods thus higher sales including adoptions and repeat purchases. Changes in the total cost of the low-valuation free product offer due to the increased adoption rate of low-valuation free products are minimal compared to the profit generated by accelerating sales since the free product marginal cost is zero or is low relative to the unit contribution margin. The same conclusion works for a longer planning horizon if there is a higher repeat purchase rate (Proposition 4.7 (ii)) which has a similar intuition as in Proposition 4.6.

Propositions 4.6 and 4.7 provide the following managerial insight when (a) the target market enjoys a high level of buyers' word-of-mouth effect, and (b) the marginal cost is zero (e.g., software products) or the firm can charge customers a unit price that is significantly higher than the marginal cost (i.e., high unit contribution margin) in the short term after the product release.

1. If a firm's planning horizon is short, then for each high-valuation free product recipient, a high number of low-valuation free users should receive free products.
2. Firms should accelerate the free product distribution to low-valuation free users under the concurrent strategy.

These two recommendations do not necessarily lead to higher profitability in a long planning horizon. Figures 4-4 and 4-5 show examples of these situations. Figures 4-4 and 4-5 depict the results of altering the ratio δ and the low-valuation free users' adoption rate $Y_l(t)$, under a concurrent strategy in the cases of a short and a long planning horizon. We set the base profit function $\Pi(n)$ ($\beta = 0.1, \beta_h = 0.3, p = 0.03, q = \lambda_h = \lambda_l = 0.3$, $\delta = 1, m = 3 \times 10^8, n_m = \frac{m}{8}, \pi(t) = 2 + 10t$, and $c(t) = 0$) for the short and the long planning horizon ($T = 8$ and $T = 15$). Free product distributions follow (4.10) and (4.11) ($p_h = p_l = 0.03, q_h = q_l = 1$ under the short planning horizon and $p_h = p_l = 0.02$, $q_h = q_l = 0.6$ under the long planning horizon). Figure 4-4 (a) shows the base profit function and the profit function with $\bar{\delta} = 2$, under the short planning horizon. Figure 4-4 (b) shows similar profit functions for the long planning horizon. Figure 4-5 (a) shows the base profit function and the profit function corresponding to a fast free product distribution to low-valuation users ((4.11) with $\bar{p}_l = 0.04, \bar{q}_l = 1.8$), under the short planning horizon. Figure 4-5 (b) depicts the base profit function and the profit function corresponding to a fast free product distribution for low-valuation users ((4.11) with $\bar{p}_l = 0.05, \bar{q}_l = 1$) under the long planning horizon.

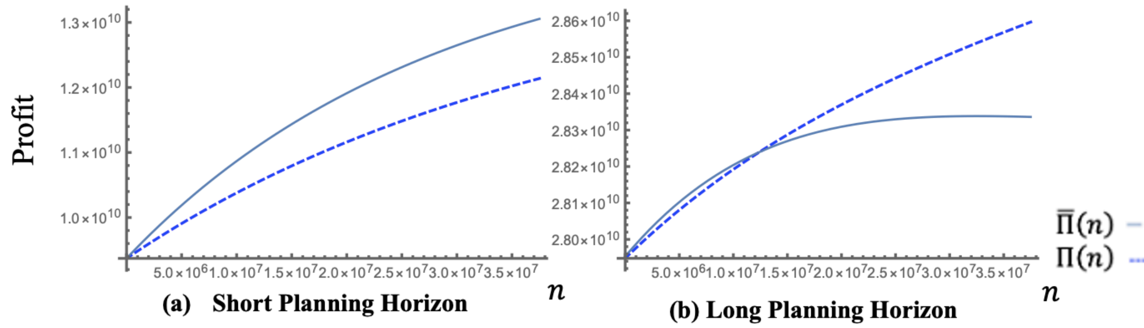

Figure 4-4: Profit Functions Under Different Ratios of Low-valuation to High-valuation Free Users

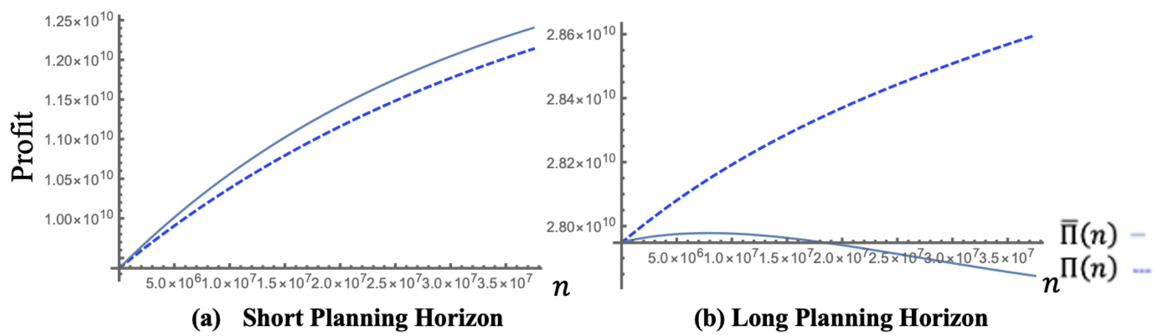

Figure 4-5: Profit Functions Under Different Free Product Distribution Rates

Similar to Figure 4-3, the results in Figures 4-4 and 4-5 are explained by the fact that a faster adoption rate enables more sales to take place sooner, when unit contribution margins are lower. As a result, when a long planning horizon is sought, profitability is lower.

4.4.2 Numerical Analysis

Optimization problems (4.18) and (4.19) are challenging to solve analytically. To gain a better understanding of the dynamics of the optimal solution we conduct a numerical analysis as typically done in the related literature (e.g., Jiang & Sarkar, 2009; Jiang et al., 2017).

For the before strategy, we numerically solve the optimization problem (4.18) with the unit contribution margin $\pi(t) = a_\pi + b_\pi t$ where parameters a_π and b_π capture the magnitude and rate of change. We consider two cases: (I) the unit contribution margin is decreasing over time ($b_\pi < 0$), and (II) the unit contribution margin is increasing over time ($b_\pi > 0$). We assume that the free products are uniformly distributed to the users before the product launch such that $Y_h(t) = \frac{tn}{\tau}, t \in [0, \tau]$. We first consider a base scenario for sales composed of adoptions and repeat purchases. Then, we create several variational scenarios by altering the values of the parameters in the base scenario one at a time. Table 4-1 provides the values of a_π, b_π and marginal cost c_0 under Cases I and II. The unit contribution margin changes from a descending trend to an ascending trend across Cases I and II, however the average unit contribution margin and the marginal cost under Cases I and II are identical within the entire demand window $[0, T]$. It is worth noting that since sales begin at time $\tau > 0$ then, the average unit contribution margin within the time interval $[\tau, T]$ under Case I is less than Case II.

Table 4-1: Optimal Solutions for Base Scenario and Variations Under Before Strategy

	Adoption				Repeat Purchase	Low to High Ratio	Case I	Case II
	p	q	λ_h	λ_l	β, β_h	δ	$a_\pi = 70, b_\pi = -5$ $c_0 = 100$	$a_\pi = 10, b_\pi = 5$ $c_0 = 100$
Base	0.037	0.656	0.656	0.656	0.278	1	$n^* = 2.8\%$	$n^* = 0\%$
Variation 1	0.037	0.656	**1**	**1**	0.278	1	$n^* = \mathbf{3.6\%}$	$n^* = 0\%$
Variation 2	0.037	0.656	0.656	0.656	**0.6**	1	$n^* = \mathbf{9.0\%}$	$n^* = \mathbf{2.2\%}$
Variation 3	**0.01**	**0.3**	0.656	0.656	0.278	1	$n^* = \mathbf{8.1\%}$	$n^* = \mathbf{4.4\%}$
Variation 4	**0.01**	**0.3**	**1**	**1**	0.278	1	$n^* = \mathbf{7.8\%}$	$n^* = \mathbf{3.6\%}$
Variation 5	0.037	0.656	0.656	0.656	0.278	**2**	$n^* = \mathbf{2.1\%}$	$n^* = 0\%$

We use the sales of the iPod as a baseline with parameter values $p = 0.037$, $q = 0.656$, $m = 2.18 \times 10^8$, and $\beta = 0.278$ as estimated in Lotfi et al. (2023). [12] Our planning horizon is set to $T = 12$. We consider equal repeat purchase rates for buyers and high-valuation free users ($\beta = \beta_h$). Our numerical experiments yield consistent results under different repeat purchases. We consider $\delta = 1$ as the base value for the ratio of low-valuation to high-valuation free users and alter this parameter in the variational scenarios. We set the new product release time at $\tau = 2$ under the base and the variational scenarios. Later in Table 4-8, we discuss the effects of changing τ. The parameter values under the

[12] Lotfi et al. (2023) estimated the parameters using iPod sales data from 2004 to 2014 obtained from Apple's quarterly summary.

base and variational scenarios and the corresponding optimal solutions for Cases I and II are exhibited in Table 4-1. The optimal solutions (n^*) are expressed as a percentage of the potential market size (m), with a 20% maximum free product offer limit ($n \leq n_M$, $n_M = \frac{m}{5}$). In Table 4-1, we highlight in bold the numbers that changed compared to the base scenario.

Table 4-2: Base Scenario and Variations Under Concurrent Strategy

	Buyers' Sales					High-valuation				Low-valuation			
	Adoption				Repeat Purchase	Adoption			Repeat Purchase	Adoption			Low to High Ratio
	p	q	λ_h	λ_l	β	p_h	q_h	q_{lh}	β_h	p_l	q_l	q_{hl}	δ
Base	0.037	0.656	0.656	0.656	0.278	0.02	0.5	0.5	0.278	0.02	0.5	0.5	1
Variation 1	0.037	0.656	**2**	**2**	0.278	0.02	0.5	0.5	0.278	0.02	0.5	0.5	1
Variation 2	0.037	0.656	0.656	0.656	**0.75**	0.02	0.5	0.5	**0.75**	0.02	0.5	0.5	1
Variation 3	**0.02**	**0.4**	0.656	0.656	0.278	0.02	0.5	0.5	0.278	0.02	0.5	0.5	1
Variation 4	**0.02**	**0.4**	**2**	**2**	0.278	0.02	0.5	0.5	0.278	0.02	0.5	0.5	1
Variation 5	**0.02**	**0.4**	0.656	0.656	0.278	0.02	0.5	0.5	0.278	0.02	0.5	0.5	**3**
Variation 6	0.037	0.656	0.656	0.656	0.278	**0.05**	**0.7**	**0.7**	0.278	0.02	0.5	0.5	1
Variation 7	0.037	0.656	0.656	0.656	0.278	0.02	0.5	0.5	0.278	**0.05**	**0.7**	**0.7**	1

Table 4-3: Optimal Solutions for Base Scenario and Variations for Different Cases Under Concurrent Strategy

	Case I	Case II	Case III	Case IV
	$a_\pi = 100, b_\pi = -6$ $a_c = 25, b_c = -2$	$a_\pi = 100, b_\pi = -6$ $a_c = 1, b_c = 2$	$a_\pi = 28, b_\pi = 6$ $a_c = 25, b_c = -2$	$a_\pi = 28, b_\pi = 6$ $a_c = 1, b_c = 2$
Base	$n^* = 0\%$	$n^* = 0\%$	$n^* = 0\%$	$n^* = 0\%$
Variation 1	$n^* = 4.2\%$	$n^* = 5.5\%$	$n^* = 0\%$	$n^* = 0\%$
Variation 2	$n^* = 4.0\%$	$n^* = 7.4\%$	$n^* = 3.3\%$	$n^* = 7.1\%$
Variation 3	$n^* = 13.6\%$	$n^* = 16.0\%$	$n^* = 5.9\%$	$n^* = 7.0\%$
Variation 4	$n^* = 12.0\%$	$n^* = 13.5\%$	$n^* = 3.7\%$	$n^* = 4.1\%$
Variation 5	$n^* = 11.0\%$	$n^* = 13.3\%$	$n^* = 4.0\%$	$n^* = 4.7\%$
Variation 6	$n^* = 0\%$	$n^* = 1.7\%$	$n^* = 0\%$	$n^* = 0\%$
Variation 7	$n^* = 0\%$	$n^* = 1.1\%$	$n^* = 0\%$	$n^* = 0\%$

For the concurrent strategy, we numerically solve the optimization problem (4.19) when free product distributions follow the indirect approach formulated by the general free product adoption formulations (4.14) and (4.15), discussed in Section 4.3.2. We consider a base scenario with (i) sales generated by product buyers, (ii) free product adoptions by high-valuation free users and their subsequent repeat purchases, and (iii) free product adoptions by low-valuation free users. Next, we generate variational scenarios by altering the base scenario parameters one at a time. Similar to the before scenario, we use the sales of iPod as the baseline with the parameters estimated in Lotfi et al. (2023) and a planning horizon set to $T = 12$. We also consider identical repeat purchase rates for buyers and high-valuation free users ($\beta = \beta_h$), and $\delta = 1$ as the base value for the ratio of low-valuation to high-valuation free users. We set the free product adoptions' innovation (p_h and p_l) and imitation parameters (q_h, q_l, q_{lh}, and q_{hl}) to values such that the free product adoptions by high-valuation and low-valuation free users reach 99 percent completion by the end of the planning horizon T. Table 4-2 shows the parameter values under the base and variational scenarios. As before, we highlight in bold the numbers that changed in each variation compared to the base scenario.

For optimization problem (4.19), we set the unit contribution margin and the free product marginal cost trends as $\pi(t) = a_\pi + b_\pi t$ and $c(t) = a_c + b_c t$ respectively with parameters a_π, b_π and a_c, b_c corresponding to the magnitude and rate of change for the unit contribution margin and the marginal cost respectively. We consider the unit contribution margin and the free product marginal cost trends under four possible cases: (I) the unit contribution margin and the marginal cost trends are both decreasing (b_π, $b_c < 0$), (II) the unit contribution margin is decreasing but the marginal cost is increasing ($b_\pi < 0$, $b_c > 0$), (III) the unit contribution margin is increasing but the marginal cost is decreasing ($b_\pi > 0$, $b_c < 0$), and (IV) both the unit contribution margin and the marginal cost are increasing (b_π, $b_c > 0$). Values of a_π, b_π, a_c, and b_c are demonstrated in Table 4-3 under Cases I through IV. To focus on the effect of the unit contribution margin and free product marginal cost trends on the optimal solution, the parameters are set to values where the average unit contribution margin and the average free product marginal cost remain identical under Cases I through IV. Subsequently, Table 4-3 shows the optimal solutions (n^*) corresponding to Cases I through IV for the base and the variational scenarios. The optimal solution is given as the percentage of the potential market size with the upper bound being 20% of the potential market ($n \leq n_M, n_M = \frac{m}{5}$).

The comparisons between Cases I and II in Table 4-1, as well as between Cases I and III and between Cases II and IV in Table 4-2, suggest that for similar free product marginal cost, offering free products is more desirable when the unit contribution margin is decreasing (Case I in Table 4-1 and Cases I and II in Table 4-2) compared to the cases with an increasing unit contribution margin (Case II in Table 4-1 and Cases III and IV in Table 4-2). The decreasing trend of the unit contribution margin further motivates employing free users' word-of-mouth to accelerate the buyer's adoption process to get a higher volume of sales in the earlier time periods which correspond to higher unit contribution margins. In contrast, the increasing trend of the unit contribution margin makes the free product offer strategy less favorable because shifting the sales peak towards earlier time periods results in less profit due to lower unit contribution margins in the early time periods.

Tables 4-1 through 4-3 Variation 1 demonstrate that the higher effectiveness of free users' word-of-mouth on the target market justifies offering a higher volume of free products under Case I in Table 4-1, and under Cases I and II in Table 4-3, since under these cases the unit contribution margin in the early time periods is higher which means a higher volume of sales in the early time periods result in higher profits. Thus, having highly influential free product users who can efficiently accelerate sales makes the cost of offering free products more justifiable. However, the conclusion is the opposite for Case II in Table 4-1, and Cases III and IV in Tables 4-2 and 4-3, because the higher word-of-mouth effect of free users results in shifting the peak for the sales towards earlier time periods where the unit contribution margin is lower, and thus, the resulting profit is lower.

Tables 4-1 through 4-3, Variation 2, demonstrate that when the repeat purchase rate is high, firms are better off giving away more products and this holds for all cases. More free users result in faster buyers' adoption due to free users' word-of-mouth. As a result, substantial sales composed of adoptions and a high volume of repeat purchases made by buyers, occur in the demand window ($[0, T]$). This high volume of sales, promoted by free users' word-of-mouth, further justifies the cost of product giveaways, thus more free product offerings are justified. Moreover, the high rate of repeat purchases results in a high volume of sales generated through high-valuation free users' repeat purchases further motivating the offering of free products.

Tables 4-1 through 4-3, Variations 3 and 4, show that a target market with weak innovation and imitation effect makes the product giveaway more appealing. This is because low innovation and imitation effects in the target market result in buyers' slow adoption process and therefore low sales volume composed of both adoptions and subsequent repeat purchases before the planning horizon. This makes the role of free users' word-of-mouth in accelerating buyers' adoption more crucial and therefore offering free products becomes more desirable in all cases. The comparison of variations 3 and 4 in Tables 4-1 through 4-3 further shows the importance of having a high number of free users to accelerate sales under scenarios in which the buyers' adoption process is slow. Specifically, under such scenarios, even less effectiveness of the free users' word-of-mouth may need to be compensated by giving away a higher number of products.

Table 4-1 variation 5 and Table 4-2 variations 3 and 5 show that increasing the ratio of low-valuation free users to the high-valuation free users favors offering more free products, however, it reduces the need to rely on high-valuation free users to accelerate buyers' adoption and sales. When the ratio of low-valuation to high-valuation free users is low, there is naturally more need for high-valuation free users to accelerate sales at the cost of lowering the actual market size. Tables 4-6 and 4-7 illustrate the situations where the ratio of low-valuation to high-valuation free users is low and offering free products, despite having no cost, is not worth reducing the actual market size. [13]

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[13] While the alteration of δ under concurrent strategy impacts adoption rates for free products (see (4.14) and (4.15)), this alteration does not ultimately affect the outcome. Our numerical tests lead to a similar outcome when using alternative formulation for free product adoption (4.10) and (4.11), where changing the value of δ does not impact the free product adoption process.

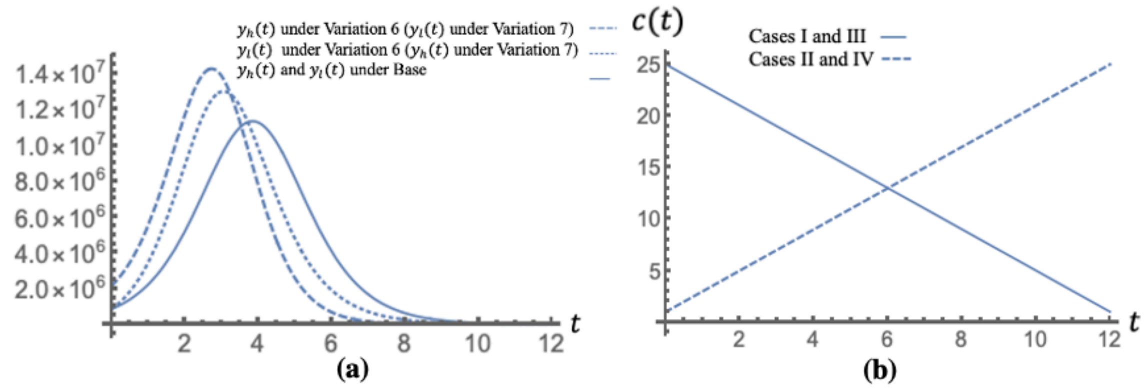

Figure 4-6: (a) Free product Distributions (b) Free Product Marginal Cost Trends

The comparisons between Cases I and II, and between Cases III and IV in Table 4-3 provide an insight regarding the dependence of the optimal free product offer on the free product marginal cost trend and the free product distribution process, under the concurrent strategy. In Table 4-3, under a similar unit contribution margin trend, Cases II and IV, where the marginal cost trends are increasing, are more favorable to offering free products than Cases I and III, where the marginal cost trends are decreasing. Figure 4-6 (a) demonstrates the free product distribution for both the high-valuation and low-valuation free users under the base scenario and the variations 6 and 7 in Table 4-2. The free product distributions in Figure 4-6 (a) are demonstrated for $n = n_M$. However, it is important to note that the form of the distribution remains identical for any value of $n > 0$ (Appendix C, Proposition C.1). Figure 4-6 (b) demonstrates the cost trends under Cases I through IV in Table 4-3. The free product distributions peak in the first half of the time horizon (before $t = 6$) (Figure 4-6 (a)). Figure 4-6 (b) demonstrates that the cost trends are symmetric around $t = 6$. This means that under Cases I and III (decreasing marginal cost), the total cost of providing free products is higher than in Cases II and IV (increasing marginal cost) making Cases I and III less favorable for product giveaway.

A faster rate of free product distribution between the high-valuation and low-valuation free users under the concurrent strategy makes it profitable to offer a higher number of free products when the unit contribution margin is decreasing and the free product unit cost is increasing (Tables 4-2 and 4-3, Case II, Variations 6 and 7). Faster adoption of free products by high-valuation and low-valuation free users results in faster adoptions and subsequent repeat purchases due to the free users' word-of-mouth. Consequently, a higher profit is gained under Case II since the unit contribution margin is higher and the free product cost is lower in the early time periods. The fact that high-valuation free users, unlike low-valuation free users, are also a source of revenue through repeat purchases can explain why more free products are offered under Case II Variation 6 compared to Variation 7.

Table 4-4: Optimal Solutions for Different Free Product Marginal Costs Under Before Strategy

	$a_\pi = 70, b_\pi = -5$ $c_0 = 30$	$a_\pi = 70, b_\pi = -5$ $c_0 = 50$	$a_\pi = 70, b_\pi = -5$ $c_0 = 100$
Base	$n^* = 9.2\%$	$n^* = 6.3\%$	$n^* = 2.8\%$

Table 4-5: Optimal Solutions for Different Free Product Marginal Costs Under Concurrent Strategy

	$a_\pi = 100, b_\pi = -6$ $a_c = 25, b_c = -2$	$a_\pi = 100, b_\pi = -6$ $a_c = 45, b_c = -2$	$a_\pi = 100, b_\pi = -6$ $a_c = 75, b_c = -2$
Variation 3	$n^* = 13.6\%$	$n^* = 9.5\%$	$n^* = 5.8\%$

Table 4-6: Optimal Solutions for Zero Free Product Marginal Cost Under Before Strategy

$a_\pi = 70, b_\pi = -5, c_0 = 0$		
Ceteris Paribus	$\beta = \beta_h = 0,\ p = 0.2,\ q = 1.5, \delta = 0$	$n^* = 0\%$
Ceteris Paribus	$\beta = \beta_h = 0,\ p = 0.2,\ q = 1.5, \delta = 10$	$n^* = 2\%$

Table 4-7: Optimal Solutions for Zero Free Product Marginal Cost Under Concurrent Strategy

$a_\pi = 100, b_\pi = -6$ $a_c = 0, b_c = 0$	
Base	$n^* = 0\%$
Ceteris Paribus $\delta = 3$	$n^* = 5.3\%$

Tables 4-4 and 4-5 demonstrate that a lower marginal cost favors giving away more free products. However, as mentioned earlier, Tables 4-6 and 4-7 provide examples of scenarios in which even zero marginal cost may not justify offering free products despite having a decreasing unit contribution margin which in general favors offering free products. The reason behind this result is that while providing free products to low-valuation free users does not affect the actual market size, offering free products to high-valuation free users results in a smaller actual market size. Therefore, when the adoption process is not slow and sales through repeat purchases made by buyers and high-valuation free users are not large, accelerating buyers' adoption process through free users' word-of-mouth is not justifiable particularly when the percentage of high-valuation free users is high and free product offers lead to lowering the actual market size.

In summary, our numerical findings suggest the following for both the before and the concurrent strategies:

- Offering free products is more appealing when the repeat purchases of the new product are anticipated to be high but adoption of the new product is taking place at a slow pace.
- Offering free products is more appealing when the unit contribution margin follows a decreasing trend. Conversely, an increasing unit contribution margin makes offering free products less favorable.
- Offering free products is more appealing when the ratio of low-valuation to high-valuation free users is high.
- Having a lower marginal cost makes offering free products more appealing. However, in certain cases where the repeat purchases rate is low, the buyers' adoption process is taking place at a high pace (high innovation and imitation effects), and the ratio of low-valuation to high-valuation free users is low, even a zero marginal cost for the free products may not justify the before or the concurrent strategies.

Our numerical experiments demonstrate the difference in the effectiveness between the before and concurrent strategies. Table 4-8 gives the optimal number of free products to offer (n^*) and the corresponding profits (Π^*) under both the before and concurrent strategies with the same unit contribution margin and free product marginal cost ($a_\pi = 100$, $b_\pi = -6$, $c_0 = a_c = 100$, and $b_c = 0$). We specifically consider the case of decreasing unit contribution margin since it favors offering free products. Furthermore, the innovation and imitation effects are considered weak (slow buyers' adoption rate) compared to the base scenarios provided in Tables 4-1 and 4-2, making the offering of free products more desirable under both strategies ($p = 0.01$ and $q = \lambda_h = \lambda_l = 0.3$).

Table 4-8: Optimal Free Product Offers and Profits Under Before and Concurrent Strategies

Before Strategy		Concurrent Strategy
$\tau = 2$	$\tau = 3$	
$n^* = 11.1\%$ $\Pi^* = 1.316 \times 10^{10}$	$n^* = 10.5\ \%$ $\Pi^* = 1.039 \times 10^{10}$	$n^* = 6.8\ \%$ $\Pi^* = 1.127 \times 10^{10}$

Despite the fact that (i) the sales period under the before strategy is shorter compared to the concurrent strategy ($[\tau, T] \subset [0, T]$) and (ii) the average unit contribution margin is lower under the before strategy (the average value of $\pi(t)$ over $[\tau, T]$ is lower compared to the average value over $[0, T]$ since $\tau > 0$ and $\pi(t)$ is decreasing), the before strategy leads to higher profit as long as the free product distribution period is not long (τ is small). When the free product offering period is short, the before strategy is more beneficial than the concurrent strategy since firms can enjoy a high level of free users' word-of-mouth instantly after the product release without sacrificing so much of the demand window. The concurrent strategy, on the other hand, leads to greater profitability when the free product offer period is lengthy (τ is large). Because the entire demand window is utilized, the sales period is longer under the concurrent strategy.

4.5 Optimal Free Product Offer Under Combined Scenario

So far, we have discussed two (before and concurrent) strategies. We now provide an extension of the before strategy under which the firm not only provides free products prior to the product launch but also continues to offer the free products after the product launch to further motivate sales. We refer to this strategy as the combined strategy which is illustrated in Figure 4-7.

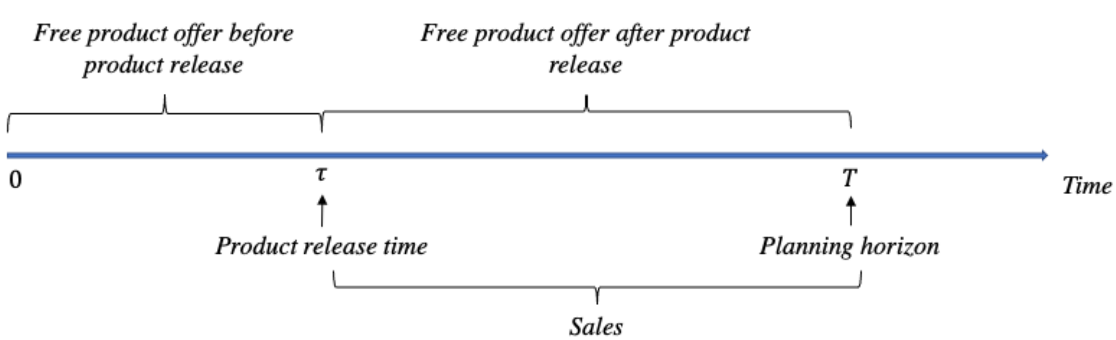

Figure 4-7: Combined Strategy Timeline

Let n_b and δn_b be the number of free products that are offered to the high-valuation free users and low-valuation free users respectively before the product launch ($t = \tau$) within $[0, \tau]$. Then analogous to the before strategy formulated by (4.4), the buyers' product adoption process after the product release follows

$$y(t) = \left(p + \frac{q}{m} Y(t) + \frac{\lambda_h}{m} n_b + \frac{\lambda_l}{m} \delta n_b\right)\left(m - n_b - Y(t)\right), \quad t \geq \tau, \tag{4.21}$$

where, $Y(\tau) = 0$. After product launch, n_a free products are offered to high-valuation users with distribution $Y_h^a(t)$, $t \in [\tau, T]$, and δn_a free products are offered to low-valuation users with distribution $Y_l^a(t)$, $t \in [\tau, T]$, where, $Y_h^a(t) = n_a$ and $Y_l^a(t) = \delta n_a$ for $t \geq T$. We also incorporate the impact of free users who received free products after the new product launch into the buyer's adoption process. Thus, the buyers' product adoption after the product launch follows

$$y(t) = \left(p + \frac{q}{m} Y(t) + \frac{\lambda_h}{m}(n_b + Y_h^a(t)) + \frac{\lambda_l}{m}(\delta n_b + Y_l^a(t))\right)\left(m - n_b - n_a - Y(t)\right), \quad t \geq \tau, \tag{4.22}$$

where, $Y(\tau) = 0$, and $n_a + n_b < m$. Similar to Theorem 4.1, we can show that the initial value problem (4.22), with $Y(\tau) = 0$ has a unique global solution $Y(t)$, $t \in [\tau, \infty)$, which is monotonically increasing to the actual market size $m - n_b - n_a$ (Appendix C, (C.10)). Moreover, like Proposition 4.1, we can demonstrate that $Y(t)$ is proportional to the actual market size $m - n_b - n_a$ if $Y_h^a(t)$ and $Y_l^a(t)$ are proportional to n_a.

Analogous to the before and concurrent strategies, we can show that the buyers' adoption peaks if the buyers' imitation effect (q) is high (Appendix C, Proposition C.3 (i)). Moreover, the buyers' cumulative adoption rate ($Y(t)$) is increasing in innovation (p) and imitation (q, λ_h, λ_l) parameters (Appendix C, Proposition C.3 (ii)). Furthermore, we can show that faster free product distribution leads to faster buyers' adoption (Appendix C, Proposition C.4). We can see that the results of Corollaries 4.1 through 4.3 are also applicable to (4.22) when $Y_h^a(t)$ and $Y_l^a(t)$ are given by (4.10) and (4.11) or (4.14) and (4.15).

Incorporating the buyers' repeat purchases, the buyers' sales is captured by

$$S(t) = I_\tau^\beta y(t) = \frac{1}{\Gamma(\beta)} \int_\tau^t (t-s)^{\beta-1} y(s) ds. \tag{4.23}$$

Similar to (4.8), the repeat purchases made by high-valuation free users who are provided with free products before the product launch is given by

$$S_h^b(t) = \frac{1}{\Gamma(\beta_h^b)} \int_0^\tau (t-s)^{\beta_h^b - 1} y_h^b(s) ds, \tag{4.24}$$

where $y_h^b(t)$, $t \in [0, \tau]$, denotes the free product distribution to n_b high-valuation free users before the product launch. Moreover, analogous to (4.17), sales through repeat purchases made by high-valuation free users who are provided with free products after the product release are captured by

$$S_h^a(t) = I_\tau^{\beta_h^a} y_h^a(t) - y_h^a(t). \tag{4.25}$$

Here, β_h^b and β_h^a denote the repeat purchase rates of high-valuation free users before and after product release, respectively. Thus, the total sales are given by $S(t) + S_h^a(t) + S_h^b(t)$. Consequently, the profit function is given by

$$\Pi(n_a, n_b) = \int_\tau^T \left(\pi(t)\left(S(t) + S_h^a(t) + S_h^b(t)\right) - c(t)\left(y_h^a(t) + y_l^a(t)\right)\right) dt - c_0(1+\delta)n_b, \tag{4.26}$$

where, c_0 and $c(t)$ denote the free product marginal costs before and after the product release respectively. Hence, the decision-making problem of determining the optimal number of free products to offer before the release (n_b) and after the release (n_a) can be represented as

$$\max_{n_a \le n_M^a, n_b \le n_M^b} \Pi(n_a, n_b), \tag{4.27}$$

where, $n_M^a + n_M^b < m$. n_M^b and n_M^a denote exogenous upper bounds on the number of free product offers before and after the product release. We note that by setting $n_a = 0$, the profit function (4.26) reduces to (4.18) which corresponds to the before strategy.

We expect that with an increase in the potential market size and the upper bound on the maximum number of free product offers, the optimal number of free product users (both high-valuation and low-valuation) should increase to effectively promote sales through word-of-mouth. Similar to the before and concurrent strategies, we observe that if free product distributions are proportional to the number of free products that are planned to be distributed, this increase in the number of optimal free users (including before and after the release low-valuation and high-valuation free product recipients) should be at the same rate as the potential market size and the maximum free product offer bound, which leads to an increase in the optimal profit. This increase in the optimal profit follows the same rate of increase as the potential market size (Appendix C, Proposition C.5). Similar to the before and concurrent strategies, we observe that expanding the market size and the bound on the maximum number of free products, however, does not make the free product offering strategy justifiable in a situation when the optimal strategy is not to offer any free products.

Analogous to the before and concurrent strategies, we find the following results for the combined strategy when the buyers' word-of-mouth (q) is sufficiently high. These findings are equally justified using the logic offered for the results of Proposition 4.5-4.7.

- When the planning horizon is short ($T \leq \tilde{t}$), having higher values for either the buyers' innovation effect (p), or the buyers' or the free users' imitation effects (q, λ_h, λ_l) increases profit (Appendix C, Proposition C.6 (i)). This higher profit is attained for a longer period of time when the repeat purchase rate is higher (Appendix C, Proposition C.6 (ii)). Furthermore, if the unit contribution margin ($\pi(t)$) is decreasing over time at a sufficiently high declining rate, this higher profitability holds in the long-term as well (Appendix C, Proposition C.6 (iii)).
- When the planning horizon is short ($T \leq \tilde{t}$), a higher ratio of low-valuation to high-valuation free users (δ) results in higher profit if the marginal cost is zero or low compared to the unit contribution margin (Proposition C.7 (i)). Higher rates of repeat purchases lead to higher profitability for a longer period of time (Proposition C.7 (ii)).
- When the planning horizon is short ($T \leq \tilde{t}$), a faster low-valuation users' free product adoption after the product release results in a higher profit if the marginal cost is zero or low compared to the unit contribution margin (Proposition C.8 (i)). This conclusion is applicable to a longer period of time if the rate of repeat purchases is higher (Proposition C.8 (ii)).

Our numerical experiments, which are presented in Tables 4-9 and 4-10, show interesting results about the efficacy of the combined strategy. Table 4-9 shows the optimal solution for (4.26) under three buyers' adoption and repeat purchase rates. In the base scenario, the values of the parameters of adoptions and repeat purchases for buyers ($y(t)$) and after-the-release free product recipients ($y_h^a(t)$ and $y_l^a(t)$) are set to those provided in the base scenario of Table 4-2. Free product distribution to high-valuation free product recipients before the product launch ($y_h^b(t)$) is identical to the one provided in Section 4.4.2 for the before strategy. Repeat purchase rates for the before and after-the-release high-valuation free product recipients are equal to the buyers' repeat purchases rate ($\beta = \beta_h^a = \beta_h^b$). The unit contribution margin ($\pi(t)$) and the free product unit cost after the product launch ($c(t)$) are set to those provided in Table 4-3 Case I. We also set the free product unit cost before product launch to $c_0 = 25$, the planning horizon (T) to 12, the product release time (τ) to 2, and $n_M^a = n_M^b = \frac{m}{10}$.

Table 4-9: Optimal Solution for Combined Strategy Under Different Scenarios

Scenarios	Ceteris Paribus $p = 0.05$, $q = 1.2$, $\beta = \beta_h^a = \beta_h^b = 0$	Base	Ceteris Paribus $p = 0.01$, $q = 0.1$, $\beta = \beta_h^a = \beta_h^b = 0.5$
Optimal Solution	$n_a^* = n_b^* = 0\%$ $\Pi^* = 1.568 \times 10^{10}$	$n_a^* = 0, n_b^* = 10\%$ $\Pi^* = 2.455 \times 10^{10}$	$n_a^* = n_b^* = 10\%$ $\Pi^* = 2.821 \times 10^{10}$

Table 4-9 demonstrates that having a high repeat purchase rate and a slow rate of buyers' adoption makes the free product offer strategy more favorable, which is consistent with the results derived earlier for the before and concurrent strategies (Section 4.4.2). In particular, we can see that when no significant repeat purchases are expected ($\beta = \beta_h^a = \beta_h^b = 0$) and buyers' product adoption is happening at a fast pace ($p = 0.05$, $q = 1.2$), offering free products is not recommended. Free product offers before the product launch are advised in the base scenario, where repeat purchase rates are higher ($\beta = \beta_h^a = \beta_h^b = 0.278$) and buyers' adoption rate is slower ($p = 0.037$, $q = 0.656$). However, offering free products after the product release is still not an optimal strategy under the base scenario, even though the unit cost ($c(t)$) is lower and declining after the product release. Finally, it is recommended to offer free products before and after the release time when the repeat purchase rate is high ($\beta = \beta_h^a = \beta_h^b = 0.5$) and the buyers' adoption rate is slow ($p = 0.01$, $q = 0.1$). This means that, while adopting the before strategy is the best course of action in the base scenario, it does not result in the highest profit when repeat purchases are high and adoption is sluggish. Instead, providing free products after the product's release time as well, leads to higher profitability.

We see that there are situations in which even zero marginal cost may not support the offering of any free products under the combined strategy. Table 4-10 provides an instance of such scenarios. Consistent with the results found under the before and concurrent strategies (Section 4.4.2), Table 4-10 shows that when adoption is occurring quickly, the repeat purchase rate is low, and the proportion of low-valuation free users is low, even a free product offer at no cost does not justify offering free products under the combined strategy ($n_a^* = n_b^* = 0\%$)[14]. Furthermore, consistent with the outcomes achieved under the before and concurrent scenarios (see Section 4.4.2), Table 4-10 shows that having a higher proportion of low-valuation free users makes offering free products justifiable ($n_b^* = 2\%$).

Table 4-10: Optimal Solution for Zero Free Product Marginal Cost Under Combined Strategy

$a_\pi = 100, b_\pi = -6 \quad a_c = b_c = c_0 = 0$	
Ceteris Paribus $\beta = \beta_h^a = \beta_h^b = 0,\ p = 0.05, q = 1.2, \delta = 0.1, p_h = 0.04, q_h = q_{lh} = 0.8$	$n_a^* = n_b^* = 0\%$
Ceteris Paribus $\beta = \beta_h^a = \beta_h^b = 0,\ p = 0.05, q = 1.2$	$n_a^* = 0, n_b^* = 2\%$

4.6 Concluding Remarks

We develop models to analyze marketing strategies of offering free products to promote sales and subsequently generate profit. In contrast to the existing literature which focuses mostly on cases where offering free products occurs prior to product launches, we examine three main product giveaway strategies: (i) free products are offered before the product release time (before), (ii) immediate product release and free product offering upon the new product release and in parallel with sales (concurrent), and (iii) free product offering before the new product release time and after the new product release and in parallel with sales (combined). The current literature also assumes that sales are composed of only adoptions whereas in this paper, sales are composed of both adoptions

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[14] By setting $p_h = 0.04$, $q_h = q_{lh} = 0.8$ we compensate for the reduction in the speed of free product adoption by high-valuation free users caused by the drop in the proportion of low-valuation users ($\delta = 0.1$).

and repeat purchases made by product buyers and high-valuation free product users. In our study, we distinguish between two types of free product users: (i) high-valuation free users who are willing to purchase the product and (ii) low-valuation free users who are not willing to pay for the product.

We assess product giveaway strategies to determine the conditions under which offering free products is advantageous. Offering free products hinges greatly on the rates of adoption and repeat purchases. Firms may gain from offering free products when high repeat purchases of the new products are expected, but adoption is sluggish. Moreover, a higher ratio of low-valuation to high-valuation free users can make offering free products more promising. Additionally, the findings indicate that offering free products can be more beneficial when the unit contribution margin follows a declining trend. Offering free products is more appealing when the free product's marginal cost is lower, but in some cases, even a zero marginal cost may not be sufficient to support any of the free product offering strategies. One such scenario occurs when the repeat purchases rate is low, the buyers' adoption process is taking place at a high pace and the ratio of low-valuation to high-valuation free users is low. Even though it is expected that the optimal number of free products to offer will rise as the potential market grows, our results demonstrate that expanding the potential market alone may not be sufficient to justify offering free products.

Our results further show that, for the three studied free product offering strategies, when firms plan for a short planning horizon in a market composed of potential customers with sufficient strong word-of-mouth effect they should target:

(i) markets composed of potential buyers with high innovation and imitation effects;
(ii) highly influential free users;
(iii) a higher ratio of low-valuation to high-valuation free users (corresponding to each high-valuation free product recipient, a larger number of low-valuation users receive the free product) when the free product marginal cost is zero or low compared to the unit contribution margin.

Furthermore, if the concurrent and combined strategies are adopted, then (iv) faster free product adoption by low-valuation free users should be planned when the free product marginal cost is zero or low in comparison to the unit contribution margin. When the rate of repeat purchases is higher, recommendations (i) through (iv) lead to higher profit over a longer period. However, recommendations (i) through (iv) may potentially have a detrimental effect on profitability if an extensively long planning horizon is chosen.

We find that when a product giveaway strategy is profitable, the before strategy can be better than the concurrent strategy if the new product release time is early. Otherwise, firms may better off immediately releasing the new product and following the concurrent strategy. Furthermore, using the before strategy may not produce the highest profit, and using a combined strategy, in which free products are also offered after the release time, may result in higher profitability. One such situation is when the adoption of the new product is sluggish, and the rate of repeat purchases is high.

Our models serve as a basis for future research. For instance, the adoption rate in the current models is based on the Bass Model, which considers the innovation and imitation effects as drivers of adoption. More extensive diffusion models can be used in place of the bass diffusion model (e.g., Van den Bulte & Joshi, 2007). Moreover, we focused on the role of free users in promoting the new product sales and firms’ profitability through word-of-mouth. Future research may consider additional possible advantages of offering free products on firms' profitability, such as enhancing the network effect, when sales are composed of both adoptions and repeat purchases.

Chapter 5

5 Conclusion

Focusing on technological products, this research provides comprehensive analytical models that firms can use when planning to release a new product to the market. The problems addressed in the three essays of this dissertation include: finding the optimal pricing strategy for a new product, estimating and forecasting sales trends in a product line composed of multiple generations, determining the optimal market entry timing strategy for a new product generation in a product line, and identifying the optimal number of free products that should be offered to promote sales and maximize profit.

Focusing on the price skimming strategy, Chapter 2 examines optimal pricing strategies for technological products. In this research, we first extend the generalized diffusion model proposed by Lotfi et al. (2023), which captures initial purchases or adoptions and repeat purchases, to incorporate the effects of introductory price and price changes on sales. Next, based on the extended sales model, we formulate a profit optimization problem.

Our results in Chapter 2 reveal that the effectiveness of a price-skimming strategy is highly dependent on the rate of repeat purchases. Specifically, the results show that price skimming may not be an optimal pricing strategy when sales through repeat purchases are weak. Conversely, our results suggest that price skimming is likely optimal when the repeat purchase rate is high.

Chapter 2 also provides the following important findings:

- When the market's sensitivity to price changes is high, firms may be better off setting a higher introductory price; however, the price should start to decline earlier.
- When customers are more (less) sensitive to the baseline price, the optimal introductory price decreases (increases).

Chapter 3 introduces a new model that captures the sales trends of a product line composed of multiple generations of a technological product. The new model accounts for initial purchases of each generation, within-generation repeat purchases, and cross-generation upgrades. It considers two main generation transition strategies: (i) a phase-out transition strategy, under which firms continue to sell the old generation after the release of a new generation, and (ii) a total transition strategy, under which firms discontinue the old generation with the release of a new generation.

Our empirical results in Chapter 3 demonstrate that the new model leads to significantly more accurate fits and forecasts than a benchmark model that does not account for within-generation repeat purchases. Therefore, our results underscore the importance of incorporating repeat purchases.

In Chapter 3, by employing our new multigeneration sales model, we develop a framework to determine the optimal market entry timing strategy for a new product generation in a product line. Depending on the rate of repeat purchases, under both generation transition strategies, we identify conditions under which the new generation should be released as early as possible, or its introduction should be delayed as much as possible. Furthermore, under both strategies, we find conditions where neither the immediate release strategy nor the maximum release delay strategy is optimal.

Chapter 4 introduces a new model to examine marketing strategy of offering free products to promote sales and maximize profit for a new technological product. The new model accounts for three main product giveaway strategies: (i) free products are offered before the product release (before), (ii) the product is released immediately, and free products are offered in parallel with sales (concurrent), and (iii) free products are offered both before and after the product release (combined). In the new modeling framework, sales comprise initial purchases or adoptions and repeat purchases. This study considers two types of free product recipients: (i) high-valuation free users who are willing to purchase the product and (ii) low-valuation free users who are not willing to pay for the product.

We find that offering free products is highly dependent on the rate of adoptions and repeat purchases. Specifically, offering free products is more appealing when adoptions occur at a slow pace, but a high rate of repeat purchases is expected. Additionally, a higher ratio of low-valuation to high-valuation free users can make the free product offering strategy more favorable. Moreover, we find that when the rate of repeat purchases is low, buyers' product adoption is fast, and the ratio of low-valuation to high-valuation free users is low, even zero-cost free product offers may not justify offering free products. Furthermore, our results show that when offering free products is not an optimal strategy, even expanding the market size may not make offering free products justifiable.

In Chapter 4, we further find that when a short planning horizon is considered in a market composed of potential customers with a sufficiently strong word-of-mouth effect, the following strategies should be adopted by firms under the three free product offer timing strategies:

(i) Markets composed of potential buyers with high innovation and imitation effects should be targeted.
(ii) Highly influential free users should be targeted.
(iii) When the free product marginal cost is zero or low compared to the unit contribution margin, for each high-valuation free product recipient, a larger number of low-valuation users should receive the free product.

We find that when the rate of repeat purchases is higher, recommendations (i) through (iii) lead to higher profit over a longer period. However, these recommendations may potentially lead to lower profitability over an extensively long planning horizon.

Additionally, the results in Chapter 4 indicate that if the new product is released without a significant delay, the before strategy may lead to higher profit than the concurrent strategy. Moreover, when the buyers' product adoption rate is slow, but the rate of repeat purchases is high, the combined strategy may lead to higher profit than the before strategy.

Methodologically, the three essays presented in Chapters 2 through 4 of this dissertation contribute to the management science literature by implementing fractional calculus as a novel tool. Fractional calculus, a well-established field in applied mathematics, may find more applications in different areas of management science research.

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Appendices

Appendix A: To Skim or not to Skim: Studying the Optimal Pricing Strategy for Technology Products

A.1 Software Sales

Before proceeding to prove Theorems 2.1 and 2.2 we state the following theorems from fractional calculus needed for the subsequent developments.

Theorem A.1 For f continuous on $[0, T]$ and $\beta_1, \beta_2 > 0$, $I^{\beta_1+\beta_2} f(t) = I^{\beta_1} I^{\beta_2} f(t)$.
Proof (Kilbas et al., 2006).

Theorem A.2 For f continuous on $[0, T]$ and $\beta > 1$, $\frac{d}{dt} I^{\beta} f(t) = I^{\beta-1} f(t)$.
Proof (Kilbas et al., 2006).

Proof of Theorem 2.1 Since $\lambda(T) = 0$, $\Gamma_2(T) = pr(T)\alpha I^{\beta} y(X(T))$. Because $\alpha < 0$, $\Gamma_2(T) < 0$. In the remainder of the proof, it is shown that $\Gamma_2(t) \leq 0$ for $t \in [0, T)$. Suppose there exists $t_1 \in [0, T)$ such that $\Gamma_2(t_1) > 0$. Then there exists $t_2 \in (t_1, T)$ such that $\Gamma_2(t_2) = 0$ and $\Gamma_2'(t_2) \leq 0$. Clearly

$$\Gamma_2'(t) = \alpha \left(\frac{d}{dt} I^{\beta} y(X(t)) pr(t) + I^{\beta} y(X(t)) pr'(t) \right) + \lambda(t) pr'(t) + pr(t) \lambda'(t), \tag{A.1}$$

where

$$\lambda'(t) = r\lambda(t) - \frac{\partial H}{\partial pr}(t), \tag{A.2}$$

$$\frac{\partial H}{\partial pr}(t) = (1 + \alpha Q(t))(I^{\beta} y(X(t)) + pr(t) \frac{\partial I^{\beta} y(X)}{\partial pr}(t)) + \lambda(t) Q(t). \tag{A.3}$$

By Theorem A.1

$$I^{\beta} y(X(t)) = \frac{X(t)^{\beta}}{\Gamma(\beta+1)} y(0) + I^{\beta+1} y'(X(t)). \tag{A.4}$$

Then, by Theorem A.2

$$\frac{\partial I^{\beta}y(X)}{\partial pr}(t) = \frac{\alpha}{pr(t)}\left(\frac{X(t)^{\beta-1}}{\Gamma(\beta)}y(0) + I^{\beta}y'\big(X(t)\big)\right), \quad t \in (0,T]. \tag{A.5}$$

Similarly, by Theorem A.2 it can be observed that

$$\frac{d}{dt}I^{\beta}y\big(X(t)\big) = \big(1+\alpha Q(t)\big)\left(\frac{X(t)^{\beta-1}}{\Gamma(\beta)}y(0) + I^{\beta}y'\big(X(t)\big)\right), \quad t \in (0,T]. \tag{A.6}$$

Substituting (A.2), (A.3), (A.5), and (A.6) in (A.1) and noting (2.19),

$$\Gamma_2'(t) = pr(t)(r\lambda(t) - I^{\beta}y\big(X(t)\big)), \qquad t \in (0,T]. \tag{A.7}$$

On the other hand, $\Gamma_2(t_2) = 0$ implies that, $-\alpha I^{\beta}y\big(X(t_2)\big) = \lambda(t_2)$. So, by (A.7)

$$\Gamma_2'(t_2) = pr(t_2)I^{\beta}y\big(X(t_2)\big)(-\alpha r - 1). \tag{A.8}$$

The assumption $-\alpha r > 1$ implies that $\Gamma_2'(t_2) > 0$ which is a contradiction. □

Proof of Theorem 2.2a By assuming $\Gamma_2(0) > 0$ and the fact that $\Gamma_2(T) < 0$ (see the proof of Theorem 4.1), there exists $t^* \in (0,T)$ such that $\Gamma_2(t^*) = 0$ which means that $-\alpha I^{\beta}y\big(X(t^*)\big) = \lambda(t^*)$. On the other hand, by (A.7)

$$\Gamma_2'(t^*) = pr(t^*)I^{\beta}y\big(X(t^*)\big)(-\alpha r - 1) < 0. \tag{A.9}$$

This means that at any point in $(0,T)$, if $\Gamma_2 = 0$, then $\Gamma_2' < 0$. Now let $t_1 \neq t^*$, such that $\Gamma_2(t_1) = 0$ and without loss of generality $t_1 \in (t^*,T)$. Then by continuity of $\Gamma_2(t)$ and the fact that $\Gamma_2'(t^*) < 0$, we must have $\Gamma_2'(t_1) \geq 0$ which is a contradiction. □

Proof of Theorem 2.2b If there exists $t^* \in (0,T)$ such that $\Gamma_2(t^*) > 0$, then by continuity of $\Gamma_2(t)$ and the fact that $\Gamma_2(0), \Gamma_2(T) < 0$, there must exist $t_1 \in (0,t^*)$ and $t_2 \in (t^*,T)$ such that $\Gamma_2(t_1) = \Gamma_2(t_2) = 0$ and $\Gamma_2'(t_1) \geq 0$ and $\Gamma_2'(t_2) \leq 0$. Since $\Gamma_2(t_1) = 0$, then $\Gamma_2'(t_1) < 0$ (see proof of Theorem 2.2a) which is a contradiction. □

Proof of Theorem 2.2c By assuming $\Gamma_2(0) = 0$ and the fact that $\Gamma_2(T) < 0$ two cases can happen. First, there exists $\bar{t} \in (0,T)$ such that $\Gamma_2(\bar{t}) > 0$. Then, there exists $t^* \in (\bar{t},T)$ such that $\Gamma_2(t^*) = 0$ which means Theorem 2.2a happens (see proof of Theorem 2.2a). Second, $\Gamma_2(t) \leq 0$ for $t \in (0,T]$ which means Theorem 2.2b happens. □

A.2 General Optimization Model

It should first be demonstrated that the functional Π has an upper bound and therefore supremum.

According to Assumption III, the inequality

$$\delta \leq pr'(t) \leq \Delta, \tag{A.10}$$

holds on $[0, T]$. Integrating inequality (A.10) over $[0, T]$ along with Assumptions I and II leads to

$$\lambda_0 \leq pr(t) \leq \Lambda + \Delta T, \ t \in [0, T]. \tag{A.11}$$

It can be observed that

$$0 < \omega \leq x(t) = \left(k + \alpha \frac{pr'(t)}{pr(t)}\right) \leq 1 + \gamma \ln(\lambda) + \alpha \frac{\delta}{\lambda_0}, \ t \in [0, T]. \tag{A.12}$$

Simply integrating (A.12) leads to

$$0 \leq X(t) \leq \left(1 + \gamma \ln(\lambda) + \alpha \frac{\delta}{\lambda_0}\right) T, t \in [0, T]. \tag{A.13}$$

Since $y(t)$ is a continuous function, it can be observed that $I^{\beta} y(t)$ defined in (2.1) is a continuous function on any closed interval $[0, L]$. Therefore, the sales function $s(t)$ defined in (2.11) is clearly continuous with respect to arguments k, $x(t)$ and $X(t)$ and therefore, by inequalities (A.12), (A.13), and the boundedness of pr_0, $s(t)$ is bounded on $[0, T]$. In the same way it can be observed that the cost functions $c(t)$ and $\bar{c}(t)$ are also bounded on $[0, T]$. Therefore, it can be deduced that the functional Π is bounded from above on M and therefore there exists $\mu < \infty$ such that $\mu = Sup\ \Pi \mid_M$.

In the remainder of this section, the objective is to find polynomial functions of degree at most N, such that each $pr_N(t)$ satisfies assumptions I-IV and also that $\Pi[pr_N(t)]$ be sufficiently close to μ for large enough values of N. To do so, the focus is placed on the optimization problem (2.25). The existence of a solution for the problem expressed by (2.25) is proven in Theorem A.4. The continuity of functional Π is needed for the subsequent development, and this is demonstrated in Lemma A.1 using the following theorem.

Theorem A.3 Let g be a continuous mapping of a compact metric space X into a metric space Y, then g is uniformly continuous.
Proof (Royden, 1988).

Note that $C^1[0,T]$ denotes the Banach space of continuously-differentiable functions defined on $[0,T]$ equipped with the uniform norm, $\|.\|_1$where, $\|f\|_1 = \|f\|_\infty + \|f'\|_\infty$, $\|f\|_\infty = Max\,\{|f(t)||0 \le t \le T\}$, $f \in C^1[0,T]$.

Lemma A.1 For any $\rho > 0$, the functional Π is continuous on G_ρ where

$$G_\rho := \{pr(t) \in C^1[0,T]\ |pr(t) \ge \rho, x(t) > 0, k > 0\}. \tag{A.14}$$

Proof Let $pr^* \in G_\rho$. Consider

$$I = [0,T] \times [\rho, pr^*(0) + 1] \times [\rho, \|pr^*\|_\infty + 1] \times [-\|pr^{*\prime}\|_\infty - 1, \|pr^{*\prime}\|_\infty + 1].$$

Obviously for $t \in [0,T]$ we have $P^*(t) := \left(t, pr^*(0), pr^*(t), pr^{*'}(t)\right) \in I$. Let $pr(t) \in G_\rho$, $\eta > 0$ and $\|pr - pr^*\|_1 < \eta$. Then for $\eta < 1$, $P(t) = (t, pr(0), pr(t), pr'(t)) \in I$ for $t \in [0,T]$. Define

$$H_1(t, pr(0), pr(t), pr'(t)) := k + \alpha \frac{pr'(t)}{pr(t)}, \tag{A.15}$$

$$H_2(t, pr(0), pr(t), pr'(t)) := kt + \alpha \ln \frac{pr(t)}{pr(0)}. \tag{A.16}$$

Because H_1 and H_2 are continuous on I, where I is a compact subset of the metric space $\mathbb{R}^4$, then, according to Theorem A.3, H_1 and H_2 are uniformly continuous on I. Now suppose $\varepsilon > 0$ is given. Clearly for $0 < \eta < 1$ sufficiently small, $|H_1(P(t)) - H_1(P^*(t))|$, $|H_2(P(t)) - H_2(P^*(t))| < \varepsilon$ for $t \in [0,T]$, and therefore $\|H_1(P) - H_1(P^*)\|_\infty$, $\|H_2(P) - H_2(P^*)\|_\infty < \varepsilon$. It can thus be concluded that $x \to x^*$, $X \to X^*$ with respect to $\|.\|_\infty$ as $pr \to pr^*$ with respect to $\|.\|_1$ where $x^*(t) := k^* + \alpha \frac{pr(t)^{*\prime}}{pr(t)^*}$, $X^*(t) := k^* t + \alpha \ln \frac{pr(t)^*}{pr(0)^*}$ and, $k^* = 1 + \gamma \ln(pr(0)^*)$. Let m_{x^*}, M_{x^*} be the respective minimum and maximum values of $x^*(t)$ on $[0,T]$. Then for $\tau > 0$ small enough such that $m_{x^*} - \tau > 0$, $x(t), x^*(t) \in [m_{x^*} - \tau, M_{x^*} + \tau]$ and $X(t), X^*(t) \in [0, (M_{x^*} + \tau)T]$ for pr sufficiently close to pr^*. Therefore, by Theorem A.3 it can be concluded that $(1 + \gamma \ln(pr(0)))F\big(X(t)\big) \to (1 + \gamma \ln(pr^*(0)))F\big(X^*(t)\big)$ and $(1 +$

$\gamma \ln(pr(0)))x(t)I^{\beta}y(X(t)) \to (1+\gamma \ln(pr^*(0)))x^*(t)I^{\beta}y(X^*(t))$ with respect to $\|.\|_{\infty}$ as $pr \to pr^*$ with respect to $\|.\|_1$.

On the other hand, for $\tau > 0$, $(1+\gamma \ln(pr(0)))F(X(t))$, $(1+\gamma \ln(pr^*(0)))F(X^*(t))$, $(1+\gamma \ln(pr(0)))x(t)I^{\beta}y(X(t))$ and $(1+\gamma \ln(pr^*(0)))x^*(t)I^{\beta}y(X^*(t)) \in [0, M^*+\tau]$ for pr sufficiently close to pr^* with respect to $\|.\|_1$, where

$$M^* = Max\left\{Max\{(1+\gamma \ln(pr^*(0)))F(X^*(t)) | t \in [0,T]\}, Max\{(1+\gamma \ln(pr^*(0)))x^*(t)I^{\beta}y(X^*(t)) | t \in [0,T]\}\right\}. \tag{A.17}$$

Again, by Theorem A.3 it can be observed that $c(t) \to c^*(t)$, and $\bar{c}(t) \to \bar{c}^*(t)$, with respect to $\|.\|_{\infty}$ as $pr \to pr^*$ with respect to $\|.\|_1$ where

$$c^*(t) = \frac{C_0\zeta^{l_1}}{(\zeta+\sigma^*(t))^{l_1}}, \quad \bar{c}^*(t) = \frac{c_1\zeta^{l_2}}{(\zeta+k^*F(X^*(t)))^{l_2}}, \tag{A.18}$$

$$\sigma^*(t) = \frac{(1+\gamma \ln(pr^*(0)))x^*(t)I^{\beta}y(X^*(t))}{m}. \tag{A.19}$$

This completes the proof. □

Theorem A.4 For all $N \in \mathbb{N}$, the optimization problem expressed in (2.25) has a solution.

Proof First it can be shown that M is a closed subset of $C^1[0,T]$. Consider $\{pr_n(t)\}_{n\in\mathbb{N}} \subset M$ such that $pr_n \longrightarrow p$ with respect to $\|.\|_1$. Clearly $\delta \le pr_n{}'(t) \le \Delta$ implies that $\delta \le p'(t) \le \Delta$ on $[0,T]$. $\lambda \le pr_n(0) \le \Lambda$ and $pr_n(t) \ge \lambda_0$ on $[0,T]$ also result in $\lambda \le p(0) \le \Lambda$ and $pr(t) \ge \lambda_0$ on $[0,T]$, respectively. Similar to the proof of Lemma A.1, it can be demonstrated that $x_n \longrightarrow x$ with respect to $\|.\|_{\infty}$ where $x_n(t) = k_n + \alpha \frac{pr_n{}'(t)}{pr_n(t)}$, $x(t) = k + \alpha \frac{p'(t)}{p(t)}$, $k_n = 1 + \gamma \ln(pr_n(0))$, and $k = 1 + \gamma \ln(p(0))$ which means that $x(t) \ge \omega$. So, $p(t) \in M$, which means that M is closed. Since $P_N[0,T]$ is a finite dimensional subspace of $C^1[0,T]$ it is closed and therefore $M \cap P_N[0,T]$is a closed subset of $P_N[0,T]$. On the other hand, by inequalities (A.10) and (A.11), $M \cap P_N[0,T]$ is also bounded. Therefore, by the Heine-Borel theorem $M \cap P_N[0,T]$ is compact. Because Π is continuous by Lemma A.1 and $M \cap P_N[0,T]$ is nonempty, the existence of a solution for (2.25) is guaranteed. □

So far, Theorem A.4 demonstrates that problem expressed in (2.25) has a solution. Although the functional optimization problem of (2.25) can be interpreted as a function optimization problem in which coefficients of polynomials of degree at most N should be determined to get the optimal solution, the existence of inequality constraints make it impossible to easily deal with (2.25) as a function optimization problem. The objective is therefore to derive a sequence of function optimization problems and show that solutions for these problems provide us with the desired sequence of polynomial price functions $\{pr_N\}_{N\in\mathbb{N}}$ discussed earlier in Section 2.4.3.

By using the transformation

$$pr'(t) = v^2(t) + \delta, \tag{A.20}$$

(2.14) and (2.24) are transformed into (A.21) and (A.22) respectively

$$\widetilde{\Pi}[pr_0, v(t)] = \Pi\left[pr_0 + \delta t + \int_0^t v^2(s)ds\right], \tag{A.21}$$

$$\widetilde{M} = \Big\{(pr_0, v(t), w(t), u(t), r(t)) \in [\lambda, \Lambda] \times \prod_{i=1}^{4} C[0,T] \mid \int_0^T [\big(v^2(t) + w^2(t) - (\Delta - \delta)\big)^2 + \left(pr_0 + \delta t + \int_0^t v^2(s)ds - u^2(t) - \lambda_0\right)^2 + (x_v(t) - r^2(t) - \omega)^2] = 0\Big\}, \tag{A.22}$$

where

$$x_v(t) = (1 + \gamma \ln(pr_0)) + \alpha \frac{v^2(t)+\delta}{pr_0+\delta t+\int_0^t v^2(s)ds}. \tag{A.23}$$

Lemma A.2 demonstrates the equivalence of the optimization problem described by (2.14), (2.24) with (A.21), (A.22).

Lemma A.2 $Sup\ \widetilde{\Pi}|_{\widetilde{M}} = \mu$.

Proof Let $(pr_0, v(t), w(t), u(t), r(t)) \in \widetilde{M}$. Define

$$pr(t) := pr_0 + \delta t + \int_0^t v^2(s)ds. \tag{A.24}$$

Clearly $pr(t) \in C^1[0,T]$. Also $pr'(t) = \delta + v^2(t)$, which means that $pr'(t) \geq \delta$ on $[0,T]$. On the other hand, $\int_0^T \big(v^2(t) + w^2(t) - (\Delta - \delta)\big)^2 dt = 0$ which implies that $pr'(t) \leq \Delta$ on $[0,T]$. It is clear that $\lambda \leq pr(0) \leq \Lambda$. It can also be observed that

$\int_0^T (pr(t) - u^2(t) - \lambda_0)^2 dt = 0$, implying that $pr(t) \geq \lambda_0$ on $[0, T]$. Also $\int_0^T (x_v(t) - r^2(t) - \omega)^2 dt = 0$ implies that $x(t) \geq \omega$. Hence $pr(t) \in M$ and $\widetilde{\Pi}[pr_0, v(t)] = \Pi[pr(t)]$, which means that $Sup\ \widetilde{\Pi}|_{\widetilde{M}} \leq \mu$. Now consider an arbitrarily small $\eta > 0$ and $pr_\eta(t) \in M$ such that $\Pi\left[pr_\eta(t)\right] > \mu - \eta$. Define $pr_0{}^\eta := pr_\eta(0)$, $v_\eta(t) := \sqrt{pr_\eta{}'(t) - \delta}$, $w_\eta(t) := \sqrt{\Delta - pr_\eta{}'(t)}$, $u_\eta(t) := \sqrt{pr_\eta(t) - \lambda_0}$ and $r_\eta(t) := \sqrt{x_\eta(t) - \omega}$ where $x_\eta(t) := (1 + \gamma \ln(pr_0{}^\eta)) + \alpha \frac{pr_\eta{}'(t)}{pr_\eta(t)}$. Clearly $v_\eta{}^2(t) + w_\eta{}^2(t) = \Delta - \delta$, $u_\eta{}^2(t) + \lambda_0 = pr_\eta(t)$ and $r_\eta{}^2(t) + \omega = x_\eta(t)$. On the other hand, $pr_\eta(t) = pr_\eta(0) + \int_0^t pr_\eta{}'(s) ds = pr_0{}^\eta + \delta t + \int_0^t v_\eta{}^2(s) ds$. Hence $(pr_0{}^\eta, v_\eta(t), w_\eta(t), u_\eta(t), r_\eta(t)) \in \widetilde{M}$ and $\widetilde{\Pi}(pr_0{}^\eta, v_\eta(t)) = \Pi\left[pr_\eta(t)\right] > \mu - \eta$, meaning that $Sup\ \widetilde{\Pi}|_{\widetilde{M}} > \mu - \eta$; this completes the proof. □

Consider vectors

$$\phi_K(t) = \begin{bmatrix} \varphi_0(t) \\ \varphi_1(t) \\ \vdots \\ \varphi_K(t) \end{bmatrix}, \quad V_K = \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_K \end{bmatrix}, \quad W_K = \begin{bmatrix} w_0 \\ w_1 \\ \vdots \\ w_K \end{bmatrix}, \quad U_K = \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_K \end{bmatrix}, \quad R_K = \begin{bmatrix} r_0 \\ r_1 \\ \vdots \\ r_K \end{bmatrix}, \tag{A.25}$$

and expansions

$$v_{K_1}(t) = V_{K_1}{}^T . \phi_{K_1}(t), \tag{A.26}$$

$$w_{K_2}(t) = W_{K_2}{}^T . \phi_{K_2}(t), \tag{A.27}$$

$$u_{K_3}(t) = U_{K_3}{}^T . \phi_{K_3}(t), \tag{A.28}$$

$$r_{K_4}(t) = R_{K_4}{}^T . \phi_{K_4}(t), \tag{A.29}$$

where φ_is are polynomial functions of degree i, defined on $[0, T]$.

Now for given $\varepsilon > 0$, the constrained function optimization problem represented by (2.26)-(2.28) is defined as follows

$$\overline{\Pi}\left[pr_0, v_0, \dots, v_{K_1}\right] := \widetilde{\Pi}\left[pr_0, v_{K_1}(t)\right], \tag{A.30}$$

$$R\left[pr_0, v_0, \dots, v_{K_1}, w_0, \dots, w_{K_2}, u_0, \dots, u_{K_3}, r_0, \dots, r_{K_4}\right] := \int_0^T [(v_{K_1}{}^2(t) + w_{K_2}{}^2(t) - (\Delta - \delta))^2 + \left(pr_0 + \delta t + \int_0^t v_{K_1}{}^2(s) ds - u_{K_3}{}^2(t) - \lambda_0\right)^2 + \left(x_{v_{K_1}}(t) - r_{K_4}{}^2(t) - \omega\right)^2] \, dt. \tag{A.31}$$

Theorem A.5 For $\varepsilon > 0$ sufficiently small, the optimization problem (2.26)-(2.28) has a solution.

Proof Define

$$\widetilde{M}^{\epsilon}{}_{K_1,K_2,K_3,K_4} = \Big\{(pr_0, v(t), w(t), u(t), r(t)) \in [\lambda, \Lambda] \times \textstyle\prod_{l=1}^{4} P_{K_l}[0,T] \mid \int_0^T [\big(v^2(t) + w^2(t) \; - (\Delta - \delta)\big)^2 + \Big(pr_0 + \delta t + \int_0^t v^2(s)ds - u^2(t) - \lambda_0\Big)^2 + (x_v(t) - r^2(t) - \omega)^2]\, dt \le \varepsilon\Big\}. \quad \text{(A.32)}$$

It can be easily observed that $\widetilde{M}^{\epsilon}{}_{K_1,K_2,K_3,K_4}$ is nonempty (e.g. set $v(t) = \sqrt{-\delta}$, $w(t) = \sqrt{\Delta}$, $u(t) = \sqrt{pr_0 - \lambda_0}$ and $r(t) = \sqrt{k - \omega}$). Let $(pr_0, v(t), w(t), u(t), r(t)) \in \widetilde{M}^{\epsilon}{}_{K_1,K_2,K_3,K_4}$. It can be observed that

$$\|v^2 + w^2 \; - (\Delta - \delta)\|_{L^2} \le \sqrt{\varepsilon}, \quad \text{(A.33)}$$

$$\Big\|pr_0 + \delta t + \int_0^t v^2(s)ds - u^2(t) - \lambda_0\Big\|_{L^2} \le \sqrt{\varepsilon}, \quad \text{(A.34)}$$

$$\|x_v - r^2 - \omega\|_{L^2} \le \sqrt{\varepsilon}. \quad \text{(A.35)}$$

By (A.33) and the equivalence of norms on finite-dimensional spaces (Folland, 1999) it can be observed that for $0 < \varepsilon < 1$, $v(t)$ and $w(t)$ are both bounded on $[0, T]$. Inequality expressed in (A.34) indicates that $u(t)$ is also bounded on $[0, T]$. Furthermore, by (A.34) and the equivalence of norms it can be observed that for $0 < \varepsilon < 1$ sufficiently small there exists $\tau_\varepsilon > 0$ such that $pr_0 + \delta t + \int_0^t v^2(s)ds \ge \lambda_0 - \tau_\varepsilon > 0$ on $[0, T]$. This means that $x_v(t)$ is bounded and therefore $r(t)$ is bounded by (A.35) for $0 < \varepsilon < 1$ sufficiently small. Hence, $\widetilde{M}^{\epsilon}{}_{K_1,K_2,K_3,K_4}$ is bounded. (A.35) also implies that $x_v(t) > 0$ for small enough $0 < \varepsilon < 1$. To show that $\widetilde{M}^{\epsilon}{}_{K_1,K_2,K_3,K_4}$ is also closed, suppose $\Big\{\Big(pr_{0_i}, v_i(t), w_i(t), u_i(t), r_i(t)\Big)\Big\}_{i\in\mathbb{N}} \subset \widetilde{M}^{\epsilon}{}_{K_1,K_2,K_3,K_4}$ and $\Big(pr_{0_i}, v_i(t), w_i(t), u_i(t), r_i(t)\Big) \longrightarrow (pr_0, v(t), w(t), u(t), r(t))$. By the equivalence of norms in finite-dimensional spaces we obtain $|pr_{0_i} - pr_0|$, $\|v_i - v\|_\infty$, $\|w_i - w\|_\infty$, and $\|r_i - r\|_\infty \to 0$ as $i \to \infty$. Similar to the proof of Lemma A.1, it is easy to observe that $x_{v_i} \longrightarrow x_v$ with respect to $\|.\|_\infty$, and therefore

$$\int_0^T [(v_i^2(t) + w_i^2(t) - (\Delta - \delta))^2 + \left(pr_{0_i} + \delta t + \int_0^t v_i^2(s)ds - u_i^2(t) - \lambda_0\right)^2 + \left(x_{v_i}(t) - r_i^2(t) - \omega\right)^2]\, dt \longrightarrow \int_0^T [\left(v^2(t) + w^2(t) - (\Delta - \delta)\right)^2 + \left(pr_0 + \delta t + \int_0^t v^2(s)ds - u^2(t) - \lambda_0\right)^2 + (x_v(t) - r^2(t) - \omega)^2]dt, \tag{A.36}$$

meaning that $\int_0^T [\left(v^2(t) + w^2(t) - (\Delta - \delta)\right)^2 + \left(pr_0 + \delta t + \int_0^t v^2(s)ds - u^2(t) - \lambda_0\right)^2 + (x_v(t) - r^2(t) - \omega)^2]dt \leq \varepsilon$ and also, $pr_0 \in [\lambda, \Lambda]$. Hence, $\widetilde{M}^{\epsilon}{}_{K_1,K_2,K_3,K_4}$ is closed. $\widetilde{M}^{\epsilon}{}_{K_1,K_2,K_3,K_4}$ is a closed and bounded subset of $\mathbb{R} \times \prod_{l=1}^{4} P_{K_l}[0,T]$, so by the Heine-Borel theorem $\widetilde{M}^{\epsilon}{}_{K_1,K_2,K_3,K_4}$ is a compact set. Since Π is continuous by Lemma A.1, $\widetilde{\Pi}$ is continuous on $\widetilde{M}^{\epsilon}{}_{K_1,K_2,K_3,K_4}$, implying that it attains a maximum on $\widetilde{M}^{\epsilon}{}_{K_1,K_2,K_3,K_4}$. □

By solving the problem expressed in (2.26)-(2.28) for small values of $\varepsilon > 0$, corresponding optimal values, say $pr_0{}^{\varepsilon}$, $v_0{}^{\varepsilon}, \ldots, v_{K_1}{}^{\varepsilon}$, are determined. By substituting these values in (A.26), $v_{K_1}{}^{\varepsilon}(t)$ and subsequently the desired $pr_N{}^{\varepsilon}(t)$ are given by

$$pr_N{}^{\varepsilon}(t) = pr_0{}^{\varepsilon} + \delta t + \int_0^t v_{K_1}{}^{\varepsilon 2}(s)ds, \tag{A.37}$$

where $N = 2K_1 + 1$. Note that $\widetilde{\Pi}\left[pr_0{}^{\varepsilon}, v_{K_1}{}^{\varepsilon}(t)\right] = \Pi[pr_N{}^{\varepsilon}(t)]$. Theorem A.6 demonstrates the desired convergence.

Theorem A.6 For given $\varepsilon, \eta > 0$, there exists $K_1, \ldots, K_4 \in \mathbb{N}$ such that $\widetilde{\Pi}\left[pr_0{}^{\varepsilon}, v_{K_1}{}^{\varepsilon}(t)\right] > \mu - \eta$.

Proof Referring to Lemma A.2, it can be observed that there exists $(pr_0{}^{\eta}, v_\eta(t), w_\eta(t), u_\eta(t), r_\eta(t)) \in \widetilde{M}$ such that $\widetilde{\Pi}\left[pr_0{}^{\eta}, v_\eta(t)\right] > \mu - \frac{\eta}{2}$. By the Weierstrass theorem there exist sequences of polynomial functions $\left\{v_{K_1}(t)\right\}_{K_1 \in \mathbb{N}}$, $\left\{w_{K_2}(t)\right\}_{K_2 \in \mathbb{N}}$, $\left\{u_{K_3}(t)\right\}_{K_3 \in \mathbb{N}}$ and, $\left\{r_{K_4}(t)\right\}_{K_4 \in \mathbb{N}}$ such that $v_{K_1} \to v_\eta$, $w_{K_2} \to w_\eta$, $u_{K_3} \to u_\eta$ and $r_{K_4} \to r_\eta$with respect to $\|.\|_\infty$. From the fact that

$$\int_0^T [\left(v_\eta{}^2(t) + w_\eta{}^2(t) - (\Delta - \delta)\right)^2 + \left(pr_0{}^{\eta} + \delta t + \int_0^t v_\eta{}^2(s)ds - u_\eta{}^2(t) - \lambda_0\right)^2 + \left(x_{v_\eta}(t) - r_\eta{}^2(t) - \omega\right)^2]dt = 0, \tag{A.38}$$

it can be easily observed that for $K_1, \dots, K_4 \in \mathbb{N}$ sufficiently large

$$\int_0^T [(v_{K_1}{}^2(t) + w_{K_2}{}^2(t) - (\Delta - \delta))^2 + \left(pr_0{}^\eta + \delta t + \int_0^t v_{K_1}{}^2(s)ds - u_{K_3}{}^2(t) - \lambda_0\right)^2 + \left(x_{v_{K_1}}(t) - r_{K_4}{}^2(t) - \omega\right)^2] \, dt \leq \varepsilon. \quad \text{(A.39)}$$

Thus, for $K_1, \dots, K_4 \in \mathbb{N}$ sufficiently large $(pr_0{}^\eta, v_{K_1}(t), w_{K_2}(t), u_{K_3}(t), r_{K_4}(t)) \in \widetilde{M}^\epsilon{}_{K_1,K_2,K_3,K_4}$. Hence

$$\widetilde{\Pi}\left[pr_0{}^\varepsilon, v_{K_1}{}^\varepsilon(t)\right] \geq \widetilde{\Pi}\left[pr_0{}^\eta, v_{K_1}(t)\right]. \quad \text{(A.40)}$$

On the other hand, Lemma A.1 implies that $\widetilde{\Pi}\left[pr_0{}^\eta, v_{K_1}(t)\right] \to \widetilde{\Pi}\left[pr_0{}^\eta, v_\eta(t)\right]$ as $v_{K_1} \to v_\eta$ with respect to $\|.\|_\infty$. This means for $K_1, \dots, K_4 \in \mathbb{N}$ sufficiently large

$$\widetilde{\Pi}\left[pr_0{}^\eta, v_{K_1}(t)\right] \geq \widetilde{\Pi}\left[pr_0{}^\eta, v_\eta(t)\right] - \frac{\eta}{2} > \mu - \eta. \quad \text{(A.41)}$$

Now, by (A.40) and (A.41) $\widetilde{\Pi}\left[pr_0{}^\varepsilon, v_{K_1}{}^\varepsilon(t)\right] > \mu - \eta$. □

A.3 Empirical Testing

In this appendix, we demonstrate the empirical superiority of the GDMR over the sales model proposed by Mesak and Berg (1995). Mesak and Berg's sales model can be expressed as:

$$S(t) = y(t) + \lambda * y(t), \quad \text{(A.42)}$$

in which $y(t)$ is noncumulative adoption at t and λ represents the repeat purchases rate.

We empirically test both the GDMR and Mesak and Berg's model on the sales data of DVD players in the U.S. Data sources are provided in Lotfi et al. (2023). Considering that the adoption trend corresponding to the sales trend is available, we first estimate the adoption parameters by fitting a Bass curve to the adoption trend. Next, we fit the GDMR and Mesak and Berg's model to the DVD player sales data, while inputting the estimated adoption parameters into both models. We use the first 13 sales data points for model-fitting and the remaining data points for examining the models' forecasting performance. It can be seen in Figure A-1 that the GDMR results in considerably more accurate forecasts than does the model by Mesak and Berg (1995).

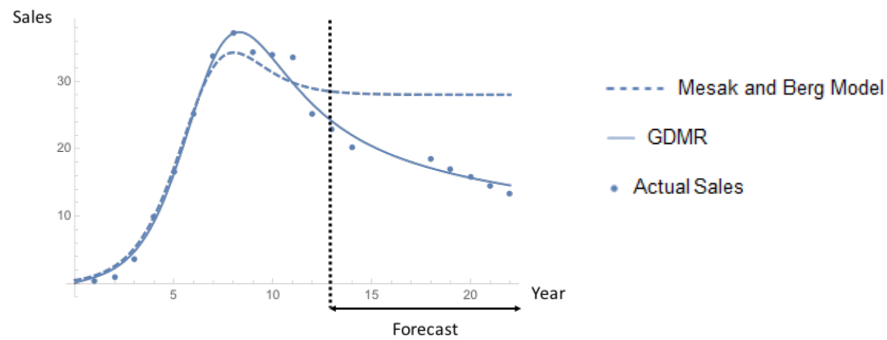


Figure A-1: Model Fitting and Forecasting on DVD Player Sales in the U.S

A.4 Further Numerical Examinations

In this appendix we conduct more numerical examinations in addition to those reported in Section 2.5. We consider two more base scenarios based on the two planning horizons of $T = 10$ and $T = 14$. The adoption processes corresponding to each base scenario is set to be 99% complete when the process reaches the planning horizon. The market sizes for these scenarios are set at $m = 500000$. Additionally, we set $\lambda = 5$ and $\Lambda = 7$ for the introductory price, and $\delta = -0.5$ and $\Delta = 0.5$ for the price change rate. Moreover, we set ζ=0.005, $c_0 = 1.5$, and $c_1 = 25000$ in cost functions (2.15) and (2.16). For each base scenario, we consider 12 variation scenarios by changing the parameters one at a time.

Tables A-1 and A-2 present the base and variation scenarios corresponding to short and long planning horizons (i.e., $T = 10$ and $T = 14$, respectively), while the corresponding optimal price paths are presented in Figures A-2 and A-3, respectively. It can be observed in Figures A-2 and A-3 that the results from scenarios presented in Tables A-1 and A-2 are consistent with those from the scenarios shown in Table 2-1.

Table A-1 Base and Variation Scenarios in Short Planning Horizon Case ($T = 10$)

	Sales			Price		Cost		Discount rate
	Adoption		Repeat					
	p	q	β	γ	α	l_1	l_2	r
Base	0.01	0.95	0.4	-0.25	-3	0	0.1	0
Changing price parameters								
Variation 1	0.01	0.95	0.4	-0.23	-3	0	0.1	0
Variation 2	0.01	0.95	0.4	-0.27	-3	0	0.1	0
Variation 3	0.01	0.95	0.4	-0.25	-2.8	0	0.1	0
Variation 4	0.01	0.95	0.4	-0.25	-3.2	0	0.1	0
Changing cost parameters								
Variation 5	0.01	0.95	0.4	-0.25	-3	0.01	0.1	0
Variation 6	0.01	0.95	0.4	-0.25	-3	0.02	0.1	0
Variation 7	0.01	0.95	0.4	-0.25	-3	0	0.0	0
Variation 8	0.01	0.95	0.4	-0.25	-3	0	0.25	0
Changing repeat purchase parameter								
Variation 9	0.01	0.95	0.2	-0.25	-3	0	0.1	0
Variation 10	0.01	0.95	0.6	-0.25	-3	0	0.1	0
Changing discount rate								
Variation 11	0.01	0.95	0.4	-0.25	-3	0	0.1	0.01
Variation 12	0.01	0.95	0.4	-0.25	-3	0	0.1	0.05

Table A-2 Base and Variation Scenarios in Long Planning Horizon Case ($T = 14$)

	Sales			Price		Cost		Discount rate
	Adoption		Repeat					
	p	q	β	γ	α	l_1	l_2	r
Base	0.01	0.65	0.65	-0.3	-6	0	0.1	0
Changing price parameters								
Variation 1	0.01	0.65	0.65	-0.27	-6	0	0.1	0
Variation 2	0.01	0.65	0.65	-0.33	-6	0	0.1	0
Variation 3	0.01	0.65	0.65	-0.3	-5.5	0	0.1	0
Variation 4	0.01	0.65	0.65	-0.3	-6.5	0	0.1	0
Changing cost parameters								
Variation 5	0.01	0.65	0.65	-0.3	-6	0.05	0.1	0
Variation 6	0.01	0.65	0.65	-0.3	-6	0.1	0.1	0
Variation 7	0.01	0.65	0.65	-0.3	-6	0	0.0	0
Variation 8	0.01	0.65	0.65	-0.3	-6	0	0.25	0
Changing repeat purchase parameter								
Variation 9	0.01	0.65	0.5	-0.3	-6	0	0.1	0
Variation 10	0.01	0.65	0.8	-0.3	-6	0	0.1	0
Changing discount rate								
Variation 11	0.01	0.65	0.65	-0.3	-6	0	0.1	0.01
Variation 12	0.01	0.65	0.65	-0.3	-6	0	0.1	0.05

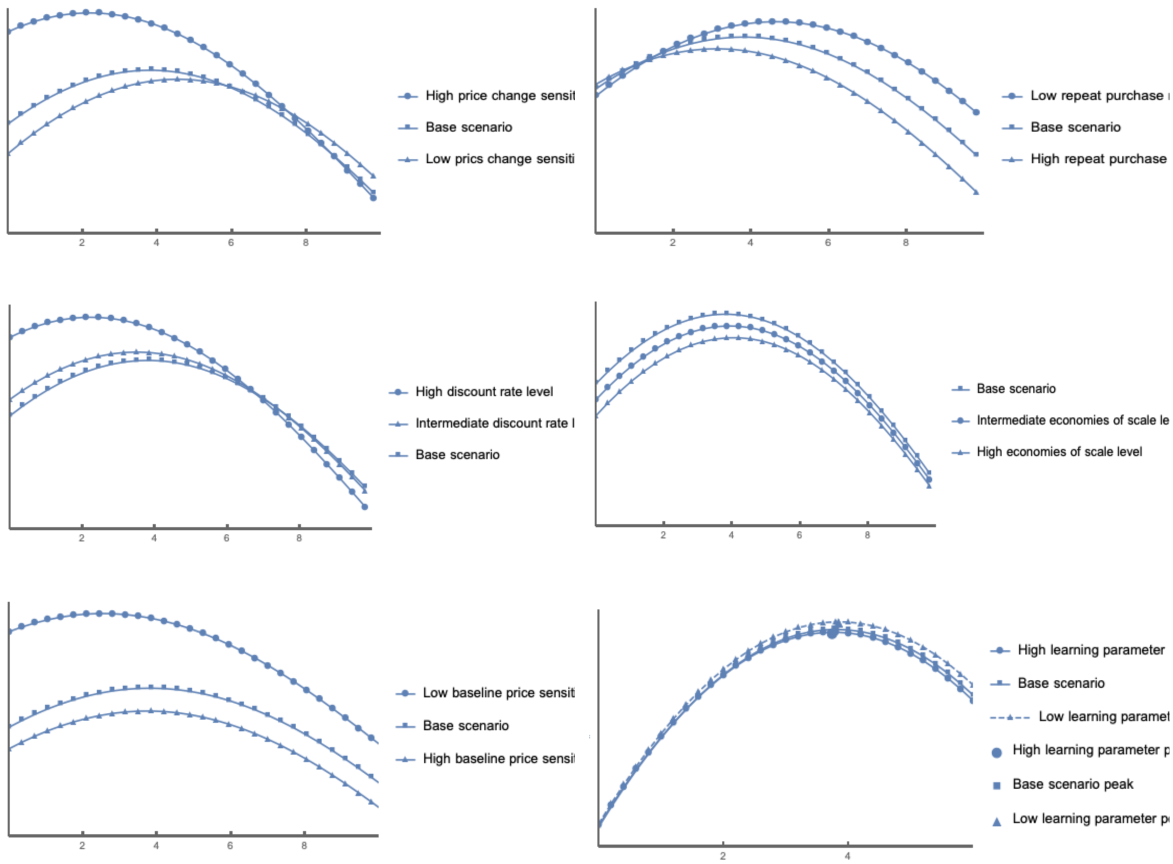


Figure A-2: Optimal Prices for Short Planning Horizon Case ($T = 10$) Under Base and Variation Scenarios

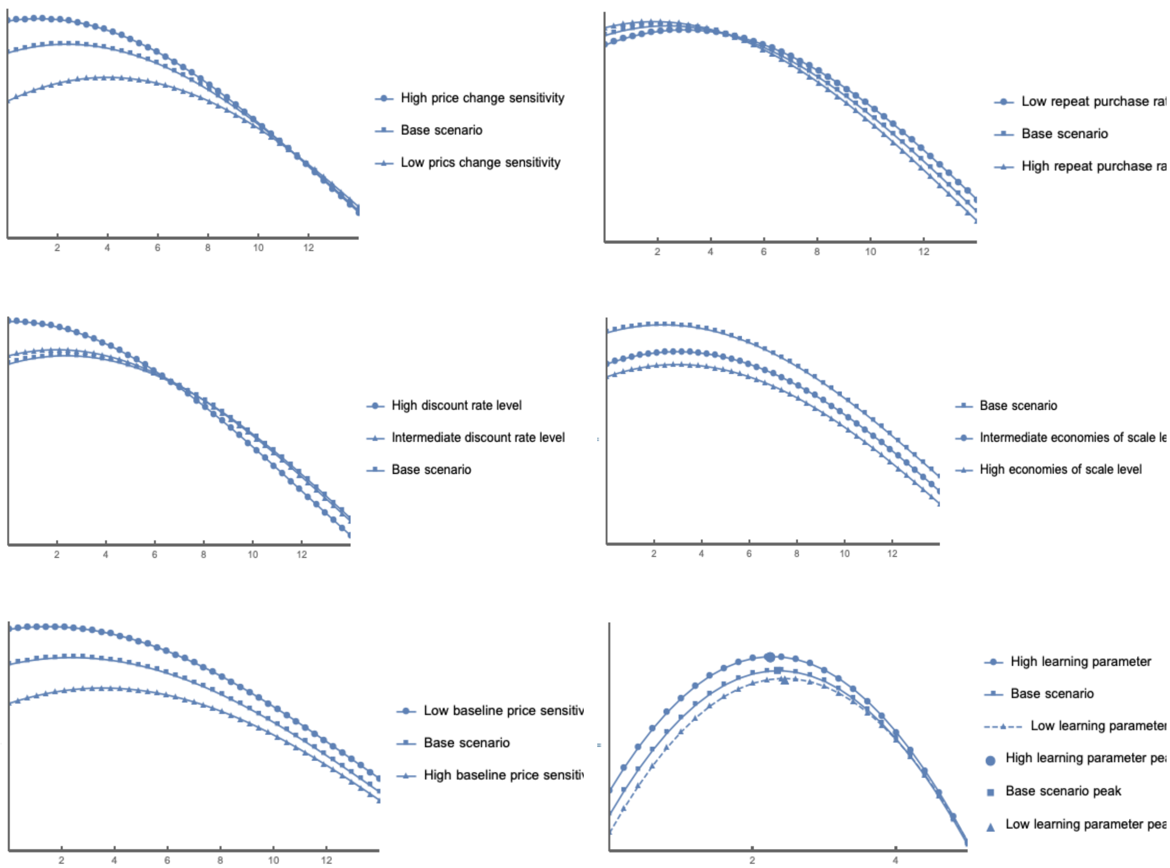


Figure A-3: Optimal Prices for Long Planning Horizon Case ($T = 14$) Under Base and Variation Scenarios

Appendix B: Modeling Sales of Multigeneration Technology Products in the Presence of Frequent Repeat Purchases: A Fractional Calculus-based Approach

B.1 Approximate Operators

We start with introducing substitute operators. After introducing the new operators, by showing that the approximate operators, I_n^{β}, converge to I^{β} with respect to the operator norm, we demonstrate that I_n^{β} has the same properties as those of I^{β}. We discuss the phase-out transition scenario formulas (3.21)-(3.23). The total transition scenario formulas (3.24) and (3.25) are derived similarly.

Theorem B.1. (Semigroup property) For f continuous on $[t_0, T]$ and $\beta_1, \beta_2 > 0$, $I_{t_0}^{\beta_1+\beta_2} f(t) = I_{t_0}^{\beta_1} I_{t_0}^{\beta_2} f(t)$.

Proof (Kilbas et al., 2006).

Applying the semigroup property of fractional integral operator, for $t \leq \tau_2$ we have

$$\begin{aligned} I_0^{\beta_1} y_1(t) &= I_0^{\beta_1}[y_{11}(0) + I_0^1 y'_{11}(t)] \\ &= I_0^{\beta_1} y_{11}(0) + I_0^{1+\beta_1} y'_{11}(t) \\ &= \frac{y_{11}(0) t^{\beta_1}}{\Gamma(1+\beta_1)} + I_0^{1+\beta_1} y'_{11}(t). \end{aligned} \tag{B.1}$$

On the other hand, for $t \geq \tau_2$

$$I_0^{\beta_1} y_1(t) = \int_0^{\tau_2} \frac{(t-s)^{\beta_1-1}}{\Gamma(\beta_1)} y_{11}(s) ds + \int_{\tau_2}^{t} \frac{(t-s)^{\beta_1-1}}{\Gamma(\beta_1)} y_{12}(s) ds. \tag{B.2}$$

By integration by parts

$$\int_0^{\tau_2} \frac{(t-s)^{\beta_1-1}}{\Gamma(\beta_1)} y_{11}(s) ds = -\frac{(t-\tau_2)^{\beta_1}}{\Gamma(1+\beta_1)} y_{11}(\tau_2) + \frac{t^{\beta_1}}{\Gamma(1+\beta_1)} y_{11}(0) + \overbrace{\int_0^{\tau_2} \frac{(t-s)^{\beta_1}}{\Gamma(1+\beta_1)} y'_{11}(s) ds}^{I_{0,\tau_2}^{1+\beta_1} y'_{11}(t)}. \tag{B.3}$$

Given that

$$y_{12}(t) = y_{12}(\tau_2) + \int_{\tau_2}^{t} y'_{12}(s) ds, \tag{B.4}$$

by (B.4) and Theorem B.1

$$\int_{\tau_2}^{t} \frac{(t-s)^{\beta_1-1}}{\Gamma(\beta_1)} y_{12}(s)ds = \frac{(t-\tau_2)^{\beta_1}}{\Gamma(1+\beta_1)} y_{12}(\tau_2) + \int_{\tau_2}^{t} \frac{(t-s)^{\beta_1}}{\Gamma(1+\beta_1)} y_{12}'(s)ds. \tag{B.5}$$

Finally, by (B.2)-(B.5) for $t \geq \tau_2$

$$I_0^{\beta_1} y_1(t) = \frac{t^{\beta_1}}{\Gamma(1+\beta_1)} y_{11}(0) + I_{0,\tau_2}^{1+\beta_1} y_{11}'(t) + I_{\tau_2}^{1+\beta_1} y_{12}'(t). \tag{B.6}$$

Similarly, by Theorem B.1 we get

$$I_{\tau_2}^{\beta_2} y_2(t) = \frac{(t-\tau_2)^{\beta_2}}{\Gamma(1+\beta_2)} y_2(\tau_2) + I_{\tau_2}^{1+\beta_2} y_2'(t). \tag{B.7}$$

We approximate fractional integration operators by applying the well-known n-point Gauss quadrature formula for integrals knowing that w_i and x_i denote the quadrature nodes and weights (DeVore & Scott, 1984). For $t \leq \tau_2$

$$I_0^{\beta_1} y_1(t) \approx \frac{y_{11}(0) t^{\beta_1}}{\Gamma(1+\beta_1)} + I_{0,n}^{1+\beta_1} y_{11}'(t) \tag{B.8}$$

where

$$I_{0,n}^{1+\beta_1} y_{11}'(t) := \frac{1}{\Gamma(1+\beta_1)} \frac{t}{2} \sum_{i=1}^{n} w_i \left(t - \frac{t}{2}(1+x_i)\right)^{\beta_1} y_{11}' \left(\frac{t}{2}(1+x_i)\right).$$

For $t > \tau_2$

$$I_0^{\beta_1} y_1(t) \approx \frac{t^{\beta_1}}{\Gamma(1+\beta_1)} y_{11}(0) + I_{0,\tau_2,n}^{1+\beta_1} y_{11}'(t) + I_{\tau_2,n}^{1+\beta_1} y_{12}'(t). \tag{B.9}$$

where

$$I_{0,\tau_2,n}^{1+\beta_1} y_{11}'(t) := \frac{1}{\Gamma(1+\beta_1)} \frac{\tau_2}{2} \sum_{i=1}^{n} w_i \left(t - \frac{\tau_2}{2}(1+x_i)\right)^{\beta_1} y_{11}' \left(\frac{\tau_2}{2}(1+x_i)\right),$$

$$I_{\tau_2,n}^{1+\beta_1} {y_{12}}'(t) :=$$

$$\frac{1}{\Gamma(1+\beta_1)} \frac{(t-\tau_2)}{2} \sum_{i=1}^{n} w_i \left(t - (\frac{t-\tau_2}{2}(1+x_i) + \tau_2)\right)^{\beta_1} y_{12}' \left((\frac{t-\tau_2}{2}(1+x_i) + \tau_2)\right).$$

Similarly

$$I_{\tau_2}^{\beta_2} y_2(t) \approx \frac{(t-\tau_2)^{\beta_2}}{\Gamma(1+\beta_2)} y_2(\tau_2) + I_{\tau_2,n}^{1+\beta_2} y_2'(t). \tag{B.10}$$

Now we need to show that the computationally implementable approximate operators given in (B.8)-(B.10) maintain the desired characteristics of the main operators. Specifically, we will show that, with increase in value of n, approximate operators converge to the main operator. We begin with some definitions and theorems from functional analysis (Kreyszig, 1978) that are needed in this argument.

The normed space $(C^1[t_0,t_1], \|.\|_1)$ is defined as follows:

$$C^1[t_0,t_1] = \{f(t)|\, f'(t) \in C[t_0,t_1]\},$$
$$\|f\|_1 = \|f\|_\infty + \|f'\|_\infty,$$

where $C[t_0,t_1]$ denotes the space of continuous functions on the interval $[t_0,t_1]$ equipped with the uniform norm,

$$\|f\|_\infty = \sup\{|f(t)||\, t \in [t_0,t_1]\}$$

Definition B.1 (Bounded linear operator). Let $L\colon C^1[t_0,t_1] \to C[\overline{t_0},\overline{t_1}]$ be a linear operator. The operator L is said to be bounded if there exists a real number c in such a way that for all $f \in C^1[t_0,t_1]$,

$$\|L(f)\|_\infty \leq c\|f\|_1.$$

Definition B.2 (Operator Norm). Let L be a bounded linear operator as defined in Definition B.1. $\|L\|$ is called the norm of the operator L and is defined as

$$\|L\| = \inf\{c : \|L(f)\|_\infty \leq c\|f\|_1, for\ all\ f \in C^1[t_0,t_1]\,\}.$$

Theorem B.2 presents an error upper bound for the Gauss quadrature formula.

Theorem B.2. Let $E_n(f)$ denote the error in n-point Gaussian quadrature applied to function f on the interval $[t_0,t_1]$. If $|f'(s)|\sqrt{1-(\frac{2(s-t_0)}{t_1-t_0}-1)^2}$ be integrable, then

$$|E_n(f)| \leq \frac{3\pi(t_1-t_0)}{n}\int_{t_0}^{t_1}|f'(s)|\sqrt{1-(\frac{2(s-t_0)}{t_1-t_0}-1)^2}ds.$$

Proof (DeVore & Scott, 1984).

Theorem B.3. Consider $I_0^{1+\beta}, I_{0,n}^{1+\beta} : C^1[0,\tau_2] \to C[0,\tau_2]$, $I_{0,\tau_2}^{1+\beta}, I_{0,\tau_2,n}^{1+\beta} : C^1[0,\tau_2] \to C[\tau_2,T]$, $I_{\tau_2}^{1+\beta}, I_{\tau_2,n}^{1+\beta}\colon C^1[\tau_2,T] \to C[\tau_2,T]$, then as the value of n increases, $\left\|I_0^{1+\beta} - I_{0,n}^{1+\beta}\right\|$, $\left\|I_{0,\tau_2}^{1+\beta} - I_{0,\tau_2,n}^{1+\beta}\right\|$ and, $\left\|I_{\tau_2}^{1+\beta} - I_{\tau_2,n}^{1+\beta}\right\|$ converge to zero.

Proof

(I) We show that $\left\| I_0^{1+\beta} - I_{0,n}^{1+\beta} \right\|$ converges to zero. From Theorem B.2 we get

$$\left| \int_0^t \frac{(t-s)^\beta}{\Gamma(1+\beta)} f(s)ds - \frac{1}{\Gamma(1+\beta)} \frac{t}{2} \sum_{i=1}^n w_i \left(\frac{t}{2}(1-x_i)\right)^\beta f\left(\frac{t}{2}(1+x_i)\right) \right| \leq$$

$$\frac{3\pi t}{n} \int_0^t \left[\left| \frac{(t-s)^\beta}{\Gamma(1+\beta)} f'(\mathrm{s}) - \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f(s) \right| \sqrt{1 - \left(\frac{2}{t}s - 1\right)^2} \right] ds \leq$$

$$\frac{3\pi t}{n} \left(\frac{t^{\beta+1}}{\Gamma(2+\beta)} \|f'\|_\infty + \frac{t^\beta}{\Gamma(\beta+1)} \|f\|_\infty \right) \leq \frac{3\pi}{n} \left(\frac{\tau_2^{\beta+2}}{\Gamma(2+\beta)} + \frac{\tau_2^{\beta+1}}{\Gamma(\beta+1)} \right) \|f\|_1.$$

According to Definition B.2, $\left\| I_0^{1+\beta} - I_{0,n}^{1+\beta} \right\| \leq \frac{3\pi}{n} \left(\frac{\tau_2^{\beta+2}}{\Gamma(2+\beta)} + \frac{\tau_2^{\beta+1}}{\Gamma(\beta+1)} \right)$ which means $\left\| I_0^{1+\beta} - I_{0,n}^{1+\beta} \right\|$ converges to zero.

(II) We show that $\left\| I_{0,\tau_2}^{1+\beta} - I_{0,\tau_2,n}^{1+\beta} \right\|$ converges to zero. By Theorem B.2 we get

$$\left| \int_0^{\tau_2} \frac{(t-s)^\beta}{\Gamma(1+\beta)} f(s)ds - \frac{1}{\Gamma(1+\beta)} \frac{\tau_2}{2} \sum_{i=1}^n w_i \left(t - \frac{\tau_2}{2}(1+x_i)\right)^\beta f\left(\frac{\tau_2}{2}(1+x_i)\right) \right| \leq$$

$$\frac{3\pi\tau_2}{n} \int_0^{\tau_2} [\left| \frac{(t-s)^\beta}{\Gamma(1+\beta)} f'(s) - \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f(s) \right| \sqrt{1 - (\frac{2}{\tau_2}s - 1)^2}] ds \leq$$

$$\frac{3\pi\tau_2}{n} \left(\frac{t^{\beta+1} + (t-\tau_2)^{\beta+1}}{\Gamma(2+\beta)} \|f'\|_\infty + \frac{t^\beta + (t-\tau_2)^\beta}{\Gamma(\beta+1)} \|f\|_\infty \right) \leq \frac{3\pi\tau_2}{n} \left(\frac{T^{\beta+1} + (T-\tau_2)^{\beta+1}}{\Gamma(2+\beta)} + \frac{T^\beta + (T-\tau_2)^\beta}{\Gamma(\beta+1)} \right) \|f\|_1.$$

According to Definition B.2, $\left\| I_{0,\tau_2}^{1+\beta} - I_{0,\tau_2,n}^{1+\beta} \right\| \leq \frac{3\pi\tau_2}{n} \left(\frac{T^{\beta+1} + (T-\tau_2)^{\beta+1}}{\Gamma(2+\beta)} + \frac{T^\beta + (T-\tau_2)^\beta}{\Gamma(\beta+1)} \right)$ which implies that $\left\| I_{0,\tau_2}^{1+\beta} - I_{0,\tau_2,n}^{1+\beta} \right\|$ converges to zero.

(III) We show that $\left\| I_{\tau_2}^{1+\beta} - I_{\tau_2,n}^{1+\beta} \right\|$ converges to zero. From Theorem B.2, we get

$$\left| \int_{\tau_2}^t \frac{(t-s)^\beta}{\Gamma(1+\beta)} f(s)ds - \frac{1}{\Gamma(1+\beta_1)} \frac{(t-\tau_2)}{2} \sum_{i=1}^n w_i \left(t - (\frac{t-\tau_2}{2}(1+x_i) + \tau_2)\right)^{\beta_1} f\left((\frac{t-\tau_2}{2}(1+x_i) + \tau_2)\right) \right| \leq \frac{3\pi(t-\tau_2)}{n} \int_{\tau_2}^t [\left| \frac{(t-s)^\beta}{\Gamma(1+\beta)} f'(s) - \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f(s) \right| \sqrt{1 - (\frac{2(s-\tau_2)}{t-\tau_2} - 1)^2}] ds \leq$$

$$\frac{3\pi(t-\tau_2)}{n} \left(\frac{(t-\tau_2)^{\beta+1}}{\Gamma(2+\beta)} \|f'\|_\infty + \frac{(t-\tau_2)^\beta}{\Gamma(\beta+1)} \|f\|_\infty \right) \leq \frac{3\pi}{n} \left(\frac{(T-\tau_2)^{\beta+2}}{\Gamma(2+\beta)} + \frac{(T-\tau_2)^{\beta+1}}{\Gamma(\beta+1)} \right) \|f\|_1.$$

According to Definition B.2, $\left\| I_{\tau_2}^{1+\beta} - I_{\tau_2,n}^{1+\beta} \right\| \leq \frac{3\pi}{n} \left(\frac{(T-\tau_2)^{\beta+2}}{\Gamma(2+\beta)} + \frac{(T-\tau_2)^{\beta+1}}{\Gamma(\beta+1)} \right)$ and therefore $\left\| I_{\tau_2}^{1+\beta} - I_{\tau_2,n}^{1+\beta} \right\|$ converges to zero. □

B.2 Optimal Market Entry Timing Strategy

Before proving Propositions 3.1 through 3.5 we provide several theorems and lemmas needed for the subsequent developments.

Theorem B.4. For f continuous on $[0,T]$ and $\beta > 1$, $\frac{d}{dt} I^{\beta} f(t) = I^{\beta-1} f(t)$.

Proof (Kilbas et al., 2006).

Theorem B.5. Let $f: [a,b] \to [0,\infty)$ be a monotonic function, whereas $g: [a,b] \to \mathbb{R}$ be a Lebesgue integrable function.

(I) If the function f is non-decreasing, then there exists $\xi \in [a,b]$ such that

$$\int_a^b f(x)g(x)dx = f(b-)\int_\xi^b g(x)dx.$$

(II) If the function f is non-increasing, then there exists $\eta \in [a,b]$ such that

$$\int_a^b f(x)g(x)dx = f(a+)\int_a^\eta g(x)dx.$$

Proof (Witula et al., 2012).

In Theorem B.6 we derive a version of the Leibnitz rule applicable in the subsequent development.

Theorem B.6. (Leibnitz rule) Suppose $g(t,\tau)$ continuous for $\tau \le t \le T$ and $\frac{\partial}{\partial \tau} g(t,\tau)$ uniformly bounded for $\tau < t \le T$, $\tau \in [\lambda, \Lambda] \subset [0,T]$. Then

$$\frac{d}{d\tau}\int_\tau^T g(t,\tau)dt = -g(\tau,\tau) + \int_\tau^T \frac{d}{d\tau} g(t,\tau)dt.$$

Proof Define $\phi(\tau) = \int_\tau^T g(t,\tau)dt$ and let $\Delta_n > 0$. Then

$$\frac{\Delta\phi}{\Delta_n} = \frac{\phi(\tau+\Delta_n)-\phi(\tau)}{\Delta_n} = \frac{1}{\Delta_n}\left(\int_{\tau+\Delta_n}^T g(t,\tau+\Delta_n)dt - \int_\tau^T g(t,\tau)dt\right) =$$

$$\frac{1}{\Delta_n}\left(\int_{\tau+\Delta_n}^T [g(t,\tau+\Delta_n) - g(t,\tau)]dt - \int_\tau^{\tau+\Delta_n} g(t,\tau)dt\right) = \quad \text{(B.11)}$$

$$\int_{\tau+\Delta_n}^T \frac{g(t,\tau+\Delta_n)-g(t,\tau)}{\Delta_n}dt - g(\zeta,\tau),$$

where $\zeta \in [\tau, \tau + \Delta_n]$. Note that the last term is derived by mean value theorem for integrals. By continuity of g, $g(\zeta, \tau) \to g(\tau, \tau)$ as $\Delta_n \to 0$. On the other hand,

$$\left|\int_{\tau+\Delta_n}^{T} \frac{g(t,\tau+\Delta_n)-g(t,\tau)}{\Delta_n} dt - \int_{\tau}^{T} \frac{\partial}{\partial\tau} g(t,\tau) dt\right| \leq$$

$$\int_{\tau+\Delta_n}^{T} \left|\frac{g(t,\tau+\Delta_n)-g(t,\tau)}{\Delta_n} - \frac{\partial}{\partial\tau} g(t,\tau)\right| dt + \int_{\tau}^{\tau+\Delta_n} \left|\frac{\partial}{\partial\tau} g(t,\tau)\right| dt \,. \tag{B.12}$$

By boundedness of $\frac{\partial}{\partial\tau} g(t,\tau)$ there exists $M_1 > 0$ such that $\int_{\tau}^{\tau+\Delta_n} \left|\frac{\partial}{\partial\tau} g(t,\tau)\right| dt \leq \Delta_n M_1$. Now let $\epsilon > 0$ arbitrarily small and $\Delta_n < \epsilon$, then we have

$$\int_{\tau+\Delta_n}^{T} \left|\frac{g(t,\tau+\Delta_n)-g(t,\tau)}{\Delta_n} - \frac{\partial}{\partial\tau} g(t,\tau)\right| dt \leq$$

$$\int_{\tau+\epsilon}^{T} \left|\frac{g(t,\tau+\Delta_n)-g(t,\tau)}{\Delta_n} - \frac{\partial}{\partial\tau} g(t,\tau)\right| dt + \int_{\tau+\Delta_n}^{\tau+\epsilon} \left|\frac{g(t,\tau+\Delta_n)-g(t,\tau)}{\Delta_n} - \frac{\partial}{\partial\tau} g(t,\tau)\right| dt.$$

Clearly for $t \in [\tau + \epsilon, T]$, $\left|\frac{g(t,\tau+\Delta_n)-g(t,\tau)}{\Delta_n} - \frac{\partial}{\partial\tau} g(t,\tau)\right| \to 0$ as $\Delta_n \to 0$. By mean value theorem, for $t \in [\tau + \Delta_n, T]$ there exists $\zeta_n \in (\tau, \tau + \Delta_n)$ such that $\frac{g(t,\tau+\Delta_n)-g(t,\tau)}{\Delta_n} = \frac{\partial}{\partial\tau} g(t, \zeta_n)$. So it can be concluded that the sequence $\{\frac{g(t,\tau+\Delta_n)-g(t,\tau)}{\Delta_n}\}$ is uniformly bounded on $[\tau + \Delta_n, T]$, which means there exists $M > 0$ such that $\left|\frac{g(t,\tau+\Delta_n)-g(t,\tau)}{\Delta_n} - \frac{\partial}{\partial\tau} g(t,\tau)\right| \leq M$ for all $t \in [\tau + \Delta_n, T]$, $\Delta_n > 0$. So, by the dominated convergence theorem (Folland, 1999) $\int_{\tau+\epsilon}^{T} \left|\frac{g(t,\tau+\Delta_n)-g(t,\tau)}{\Delta_n} - \frac{\partial}{\partial\tau} g(t,\tau)\right| dt \to 0$ as $\Delta_n \to 0$ and therefore

$$\limsup_{\Delta_n\to 0} \int_{\tau+\Delta_n}^{T} \left|\frac{g(t,\tau+\Delta_n)-g(t,\tau)}{\Delta_n} - \frac{\partial}{\partial\tau} g(t,\tau)\right| dt \leq M\epsilon.$$

Since $\epsilon > 0$ is arbitrary we get

$$\lim_{\Delta_n\to 0} \int_{\tau+\Delta_n}^{T} \left|\frac{g(t,\tau+\Delta_n)-g(t,\tau)}{\Delta_n} - \frac{\partial}{\partial\tau} g(t,\tau)\right| dt = 0.$$

Now by (B.11) and (B.12) we can conclude that

$$\lim_{\Delta_n\to 0} \frac{\Delta\phi}{\Delta_n} = -\, g(\tau,\tau) + \int_{\tau}^{T} \frac{d}{d\tau} g(t,\tau) dt,$$

let $\Delta_n < 0$. Then

$$\frac{\Delta\phi}{\Delta_n} = \frac{\phi(\tau+\Delta_n)-\phi(\tau)}{\Delta_n} = \frac{1}{\Delta_n}\left(\int_{\tau+\Delta_n}^{T} g(t,\tau+\Delta_n)dt - \int_{\tau}^{T} g(t,\tau)dt\right) =$$

$$\frac{1}{\Delta_n}\left(\int_{\tau}^{T} [g(t,\tau+\Delta_n) - g(t,\tau)]dt + \int_{\tau+\Delta_n}^{\tau} g(t,\tau+\Delta_n)dt\right) =$$

$$\int_{\tau}^{T}\frac{g(t,\tau+\Delta_n)-g(t,\tau)}{\Delta_n}dt-g(\zeta,\tau+\Delta_n),$$

where $\zeta\in[\tau+\Delta_n,\tau]$. Again, by continuity of g, $g(\zeta,\tau+\Delta_n)\to g(\tau,\tau)$ as $\Delta_n\to 0$ and similar to the case of $\Delta_n>0$, by the dominated convergence theorem we have

$$\lim_{\Delta_n\to 0}\int_{\tau}^{T}\left|\frac{g(t,\tau+\Delta_n)-g(t,\tau)}{\Delta_n}-\frac{\partial}{\partial\tau}g(t,\tau)\right|dt=0.$$

and this completes the proof. □

Note that in the case of fixed integral bounds Theorem B.6 reduces to the standard form of the Leibnitz rule (i.e., $\frac{d}{d\tau}\int_{\tau^*}^{T}g(t,\tau)dt=\int_{\tau^*}^{T}\frac{d}{d\tau}g(t,\tau)dt$).

Lemma B.1. Let $g(s,\tau)$ be a smooth function, $\mu>0$ and, $t>\tau\geq 0$, then

$$\frac{d}{d\tau}\frac{1}{\Gamma(\mu)}\int_{\tau}^{t}(t-s)^{\mu-1}g(s,\tau)ds=-\frac{(t-\tau)^{\mu-1}}{\Gamma(\mu)}g(\tau,\tau)+\frac{1}{\Gamma(\mu)}\int_{\tau}^{t}(t-s)^{\mu-1}\frac{d}{d\tau}g(s,\tau)ds.$$

Proof By using Theorem B.1 we have

$$\frac{1}{\Gamma(\mu)}\int_{\tau}^{t}(t-s)^{\mu-1}g(s,\tau)d\tau=g(\tau,\tau)\frac{(t-\tau)^{\mu}}{\Gamma(\mu+1)}+\frac{1}{\Gamma(\mu+1)}\int_{\tau}^{t}(t-s)^{\mu}\frac{d}{ds}g(s,\tau)ds.$$

Now using Leibnitz rule, it can be observed that

$$\frac{d}{d\tau}\frac{1}{\Gamma(\mu)}\int_{\tau}^{t}(t-s)^{\mu-1}g(s,\tau)d\tau=\partial_1 g(\tau,\tau)\frac{(t-\tau)^{\mu}}{\Gamma(\mu+1)}+\partial_2 g(\tau,\tau)\frac{(t-\tau)^{\mu}}{\Gamma(\mu+1)}-g(\tau,\tau)\frac{(t-\tau)^{\mu-1}}{\Gamma(\mu)}-$$

$$\partial_1 g(\tau,\tau)\frac{(t-\tau)^{\mu}}{\Gamma(\mu+1)}+\frac{1}{\Gamma(\mu+1)}\int_{\tau}^{t}(t-s)^{\mu}\frac{d}{d\tau}\partial_1 g(s,\tau)ds=\partial_2 g(\tau,\tau)\frac{(t-\tau)^{\mu}}{\Gamma(\mu+1)}-$$

$$g(\tau,\tau)\frac{(t-\tau)^{\mu-1}}{\Gamma(\mu)}+\frac{1}{\Gamma(\mu+1)}\int_{\tau}^{t}(t-s)^{\mu}\partial_1\frac{d}{d\tau}g(s,\tau)ds=-g(\tau,\tau)\frac{(t-\tau)^{\mu-1}}{\Gamma(\mu)}+\frac{1}{\Gamma(\mu)}\int_{\tau}^{t}(t-$$

$$s)^{\mu-1}\frac{d}{d\tau}g(s,\tau)ds.$$

□

Lemma B.2 Let $\beta>0$ and $T>1$, then

(I) $I_0^{\beta}y_1(t)$, $I_{\tau_2}^{\beta}u_2(t)$, and $I_{\tau_2}^{\beta}\tilde{y}_2(t)$ are continuous for $\tau_2\leq t\leq T$, $\tau_2\in[\lambda,\Lambda]\subset[0,T]$,

(II) $\frac{\partial}{\partial\tau_2}I_0^{\beta}y_1(t)$, $\frac{\partial}{\partial\tau_2}I_{\tau_2}^{\beta}u_2(t)$, and $\frac{\partial}{\partial\tau_2}I_{\tau_2}^{\beta}\tilde{y}_2(t)$ are uniformly bounded for $\tau_2<t\leq T$, $\tau_2\in[\lambda,\Lambda]\subset[0,T]$,

(III) $I_{\tau_2}^{\beta+1} f_1(t) f_2(t-\tau_2)$ is continuous for $\tau_2 \le t \le T$, $\tau_2 \in [\lambda, \Lambda] \subset [0, T]$, and $\frac{d}{d\beta} I_{\tau_2}^{\beta+1} f_1(t) f_2(t-\tau_2)$ is continuous for $\tau_2 < t \le T$ and has a removable discontinuity at $t = \tau_2$. $\frac{d}{d\beta} I_{\tau_2}^{\beta+1} f_1(t) f_2(t-\tau_2)$ is uniformly bounded for all $t \in [\tau_2, T]$, $\tau_2 \in [\lambda, \Lambda] \subset [0, T]$, $\beta \le 2$.

Proof

(I) We show the result for $I_0^{\beta} y_1(t)$. For the others the proof is similar. Without loss of generality assume $t^* > t$, $\tau_2^* > \tau_2$, and $t^* > \tau_2^*$, $t > \tau_2$. Then for (t, τ_2) sufficiently close to (t^*, τ_2^*) we have

$$\left|I_0^{\beta} y_1(t^*) - I_0^{\beta} y_1(t)\right| = \frac{1}{\Gamma(\beta)}\Big(\Big|\int_0^{\tau_2^*}(t^*-s)^{\beta-1} m_1 f_1(s) ds + \int_{\tau_2^*}^{t^*}(t^*-s)^{\beta-1} m_1 f_1(s)(1 - F_2(s-\tau_2^*))ds - \int_0^{\tau_2}(t-s)^{\beta-1} m_1 f_1(s) ds - \int_{\tau_2}^{t}(t-s)^{\beta-1} m_1 f_1(s)(1 - F_2(s - \tau_2))ds\Big|\Big) \le$$

$$\frac{1}{\Gamma(\beta)}\Big(\Big|\int_0^{\tau_2^*}(t^*-s)^{\beta-1} m_1 f_1(s) ds - \int_0^{\tau_2}(t-s)^{\beta-1} m_1 f_1(s) ds\Big| + \Big|\int_{\tau_2^*}^{t^*}(t^*-s)^{\beta-1} m_1 f_1(s)(1 - F_2(s-\tau_2^*))ds - \int_{\tau_2}^{t}(t-s)^{\beta-1} m_1 f_1(s)(1 - F_2(s-\tau_2))ds\Big|\Big) \le$$

$$\frac{1}{\Gamma(\beta)}\Big(\Big|\int_0^{\tau_2}((t^*-s)^{\beta-1} - (t-s)^{\beta-1}) m_1 f_1(s) ds + \int_{\tau_2}^{\tau_2^*}(t^*-s)^{\beta-1} m_1 f_1(s) ds\Big|\Big) + \Big|\int_{\tau_2^*}^{t^*}(t^*-s)^{\beta-1} m_1 f_1(s)\big(1 - F_2(s-\tau_2^*)\big)ds - \int_{\tau_2^*}^{t^*}(t^*-s)^{\beta-1} m_1 f_1(s)\big(1 - F_2(s-\tau_2)\big)ds + \int_{\tau_2^*}^{t^*}(t^*-s)^{\beta-1} m_1 f_1(s)\big(1 - F_2(s-\tau_2)\big)ds - \int_{\tau_2}^{t}(t-s)^{\beta-1} m_1 f_1(s)(1 - F_2(s-\tau_2))ds\Big| \le$$

$$\frac{1}{\Gamma(\beta)}\Big(\Big|\int_0^{\tau_2}((t^*-s)^{\beta-1} - (t-s)^{\beta-1}) m_1 f_1(s) ds + \int_{\tau_2}^{\tau_2^*}(t^*-s)^{\beta-1} m_1 f_1(s) ds\Big|\Big) + \Big|\int_{\tau_2^*}^{t^*}(t^*-s)^{\beta-1}\Big(m_1 f_1(s)\big(1 - F_2(s-\tau_2^*)\big) - m_1 f_1(s)\big(1 - F_2(s-\tau_2)\big)\Big) ds + \int_{\tau_2^*}^{t}((t^*-s)^{\beta-1} - (t-s)^{\beta-1}) m_1 f_1(s)\big(1 - F_2(s-\tau_2)\big)ds - \int_{\tau_2}^{\tau_2^*}(t-s)^{\beta-1} m_1 f_1(s)\big(1 - F_2(s-\tau_2)\big)ds + \int_{t}^{t^*}(t^*-s)^{\beta-1} m_1 f_1(s)\big(1 - F_2(s-\tau_2)\big)ds\Big| \le$$

$$\frac{m_1}{\Gamma(\beta+1)}\Big(\big|(t-\tau_2)^{\beta} - (t^*-\tau_2)^{\beta}\big| + \big|t^{\beta} - t^{*\beta}\big| + \big|(t^*-\tau_2)^{\beta} - (t^*-\tau_2^*)^{\beta}\big| + \big|(t-\tau_2^*)^{\beta} - (t^*-\tau_2^*)^{\beta}\big| + \big|(t-\tau_2)^{\beta} - (t-\tau_2^*)^{\beta}\big| + 2\big|(t^*-t)^{\beta}\big|\Big) +$$

$$\frac{\epsilon(t^*-\tau^*)^\beta}{\Gamma(\beta+1)}. \tag{B.13}$$

Note that in (B.13) we have used the fact that, by continuity of F_2, for any given $\epsilon > 0$, if $|\tau_2 - \tau_2^*|$ be sufficiently small then,

$$\left|m_1 f_1(s)\big(1 - F_2(s-\tau_2^*)\big) - m_1 f_1(s)\big(1 - F_2(s-\tau_2)\big)\right| < \epsilon.$$

Since $\epsilon > 0$ is arbitrary, it can be observed that the inequality (B.13) tends to zero as $(t, \tau_2) \to (t^*, \tau_2^*)$ and this shows the continuity.

(II) We show the result for $\frac{\partial}{\partial \tau_2} I_0^\beta y_1(t)$. For the others the proof is similar. By employing Lemma B.1. for $t > \tau_2$ we have

$$\frac{d}{d\tau_2} I_0^\beta y_1(t) = \frac{d}{d\tau_2}\frac{1}{\Gamma(\beta)}\left(\int_0^{\tau_2}(t-s)^{\beta-1}\, y_{11}(s)ds + \int_{\tau_2}^t (t-s)^{\beta-1}\, y_{12}(s)ds\right) =$$

$$\left(\frac{(t-\tau_2)^{\beta-1}}{\Gamma(\beta)} y_{11}(\tau_2) - \frac{(t-\tau_2)^{\beta-1}}{\Gamma(\beta)} y_{12}(\tau_2) + \frac{1}{\Gamma(\beta)}\int_{\tau_2}^t (t-s)^{\beta-1}\frac{d}{d\tau_2} y_{12}(s)ds\right) =$$

$$I_{\tau_2}^\beta y_{11}(t) f_2(t-\tau_2).$$

It can be observed that

$$\left|\frac{d}{d\tau_2} I_0^\beta y_1(t)\right| = \left|I_{\tau_2}^\beta y_{11}(t) f_2(t-\tau_2)\right| \le \frac{m_1 T^\beta}{\Gamma(\beta+1)}.$$

(III) The continuity of $I_{\tau_2}^{\beta+1} f_1(t) f_2(t-\tau_2)$ can be demonstrated similar to the part (I).

Next, we show the continuity of $\frac{d}{d\beta} I_{\tau_2}^{\beta+1} f_1(t) f_2(t-\tau_2)$.

Define $f: [\tau_2, t] \times [\epsilon, 2] \to \mathbb{R}$ where $f(s,\beta) := (t-s)^\beta f_1(s) f_2(s-\tau_2)$, $s \in [\tau_2, t]$ and, $\beta \in [\epsilon, 2]$, $\epsilon > 0$. Then, $f_\beta(s,\beta) = (t-s)^\beta \ln(t-s)\, f_1(s) f_2(s-\tau_2)$. Clearly $f_\beta(s,\beta)$ is continuous on $[\tau_2, t] \times [\epsilon, 2]$. (Note that f_β has a removable discontinuity at t). Then by standard Leibnitz rule we have

$$\frac{d}{d\beta} I_{\tau_2}^{\beta+1} f_1(t) f_2(t-\tau_2) = \frac{d}{d\beta}\frac{1}{\Gamma(\beta+1)}\int_{\tau_2}^t (t-s)^\beta\, f_1(s) f_2(s-\tau_2)ds =$$

$$= -\frac{\Gamma'(\beta+1)}{\Gamma^2(\beta+1)}\int_{\tau_2}^t (t-s)^\beta\, f_1(s) f_2(s-\tau_2)ds +$$

$$\frac{1}{\Gamma(\beta+1)}\int_{\tau_2}^t (t-s)^\beta \ln(t-s)\, f_1(s) f_2(s-\tau_2)ds.$$

We focus on showing the continuity of $\int_{\tau_2}^t (t-s)^\beta \ln(t-s)\, f_1(s) f_2(s-\tau_2)ds$. In a similar fashion we can show that $\int_{\tau_2}^t (t-s)^\beta\, f_1(s) f_2(s-\tau_2)ds$ is also continuous. Let

$\tau_2 < t^* \leq T$ and without loss of generality assume $t > t^*$. Suppose $|t - t^*| < \delta < 1$. Then

$$\left|\int_{\tau_2}^{t} (t-s)^\beta \ln(t-s)\, f_1(s) f_2(s-\tau_2) ds - \int_{\tau_2}^{t^*} (t^*-s)^\beta \ln(t^*-s)\, f_1(s) f_2(s-\tau_2) ds\right| \leq$$

$$\int_{\tau_2}^{t^*} \left|(t-s)^\beta \ln(t-s) - (t^*-s)^\beta \ln(t^*-s)\right| ds + \int_{t^*}^{t} \left|(t-s)^\beta \ln(t-s)\right| ds. \quad \text{(B.14)}$$

On the other hand

$$\int_{\tau_2}^{t^*} \left|(t-s)^\beta \ln(t-s) - (t^*-s)^\beta \ln(t^*-s)\right| ds \leq$$

$$\int_{\tau_2}^{t^*} \left|(t-s)^\beta \ln(t-s) - (t^*-s)^\beta \ln(t-s)\right| ds + \int_{\tau_2}^{t^*} \left|(t^*-s)^\beta \ln(t-s) - (t^*-s)^\beta \ln(t^*-s)\right| ds. \quad \text{(B.15)}$$

We also have

$$\left|(t-s)^\beta - (t^*-s)^\beta\right| \leq \begin{cases} (t-t^*)^\beta, & \beta \leq 1 \\ (t-\tau_2)^\beta + (t^*-\tau_2)^\beta, & \beta > 1 \end{cases}, \; s \in [\tau_2, t^*].$$

So, for any $\eta > 0$ there exists $\delta < 1$ sufficiently small such that

$$\int_{\tau_2}^{t^*} \left|(t-s)^\beta \ln(t-s) - (t^*-s)^\beta \ln(t-s)\right| ds \leq \eta \int_{\tau_2}^{t^*} |\ln(t-s)|\, ds, \quad \text{(B.16)}$$

where

$$\int_{\tau_2}^{t^*} |\ln(t-s)|\, ds =$$

$$\begin{cases} 2 + t^* - 2t + \tau_2 + (t-t^*)\ln(t-t^*) + (t-\tau_2)\ln(t-\tau_2), & t - \tau_2 \geq 1, \\ t^* - \tau_2 + (t-t^*)\ln(t-t^*) - (t-\tau_2)\ln(t-\tau_2), & t - \tau_2 < 1\,. \end{cases} \quad \text{(B.17)}$$

On the other hand

$$\int_{\tau_2}^{t^*} \left|(t^*-s)^\beta \ln(t-s) - (t^*-s)^\beta \ln(t^*-s)\right| ds \leq (t^*-\tau_2)^\beta \int_{\tau_2}^{t^*} |\ln(t-s) - \ln(t^*-s)|\, ds =$$

$$(t^*-\tau_2)^\beta \left(\left((\tau_2 - t^*) - (t-t^*)\ln(t-t^*) + (t-\tau_2)\ln(t-\tau_2)\right) - \left((\tau_2 - t^*) + (t^*-\tau_2)\ln(t^*-\tau_2)\right)\right). \quad \text{(B.18)}$$

It can be observed that

$$\int_{t^*}^{t} \left|(t-s)^\beta \ln(t-s)\right| ds < \delta^\beta \int_{t^*}^{t} |\ln(t-s)|\, ds, \quad \text{(B.19)}$$

where

$$\int_{t^*}^{t} |\ln(t-s)|\, ds = (t-t^*) - (t-t^*)\ln(t-t^*). \tag{B.20}$$

Now by (B.14)-(B.20) we can conclude that $\left|\int_{\tau_2}^{t}(t-s)^{\beta}\ln(t-s)\, f_1(s) f_2(s-\tau_2)ds - \int_{\tau_2}^{t^*}(t^*-s)^{\beta}\ln(t^*-s)\, f_1(s) f_2(s-\tau_2)ds\right|$ tends to zero as $t,\ t > t^*$ tends to t^* which demonstrates the continuity of $\int_{\tau_2}^{t}(t-s)^{\beta}\ln(t-s)\, f_1(s) f_2(s-\tau_2)ds$.

It can be observed that for $(t-\tau_2) < 1$

$$\left|\int_{\tau_2}^{t}(t-s)^{\beta}\ln(t-s)\, f_1(s) f_2(s-\tau)ds\right| \le \int_{\tau_2}^{t} |\ln(t-s)|\, ds =$$

$$((t-\tau_2) - (t-\tau_2)\ln(t-\tau_2)),$$

which means $\int_{\tau_2}^{t}(t-s)^{\beta}\ln(t-s)\, f_1(s) f_2(s-\tau_2)ds$ converges to zero as t tends to τ_2. We can easily show a similar result for $\int_{\tau_2}^{t}(t-s)^{\beta}\, f_1(s) f_2(s-\tau_2)ds$. Hence $\frac{d}{d\beta} I_{\tau_2}^{\beta+1} f_1(t) f_2(t-\tau_2)$ has a removable discontinuity at $t = \tau_2$.

To show the uniform boundedness of $\frac{d}{d\beta} I_{\tau_2}^{\beta+1} f_1(t) f_2(t-\tau_2)$ it is enough to observe that

$$\frac{d}{d\beta} I_{\tau_2}^{\beta+1} f_1(t) f_2(t-\tau_2)$$

$$= -\frac{\Gamma'(\beta+1)}{\Gamma^2(\beta+1)}\int_{\tau_2}^{t}(t-s)^{\beta}\, f_1(s) f_2(s-\tau_2)ds +$$

$$\frac{1}{\Gamma(\beta+1)}\int_{\tau_2}^{t}(t-s)^{\beta}\ln(t-s)\, f_1(s) f_2(s-\tau_2)ds \le$$

$$\int_{\tau_2}^{t}(t-s)^{\beta}\, ds + 2\int_{\tau_2}^{t}(t-s)^{\beta}|\ln(t-s)|\, ds \le$$

$$\int_{0}^{t}(t-s)^{\beta}\, ds + 2T^2\int_{0}^{t}|\ln(t-s)|\, ds \le T^3 + 2T^2(3 + T + T\ln T).$$

□

Lemma B.3 Let $0 < \beta \le 1$, $T > T^*$, where $T^* = \max\{\theta_1, \theta_2\} + e^2 + e$, and θ_1 and θ_2 are time to peak for $f_1(t)$ and $f_2(t-\Lambda)$ respectively. Suppose $\pi(t)$ a smooth monotonic function and $\tau \in [\lambda, \Lambda]$.

(I) Assume $\pi'(t) \ge 0$, then $\frac{d}{d\beta}\int_{\tau}^{T}\pi(t) I_{\tau}^{\beta} f_1(t) f_2(t-\tau)dt \ge 0$.

(II) Assume $\pi'(t), \pi''(t) \le 0$, then $\frac{d}{d\beta}\int_{\tau}^{T}\pi(t) I_{\tau}^{\beta} f_1(t) f_2(t-\tau)dt \ge 0$.

Proof Let $\epsilon > 0$. For any $\beta \in [\epsilon, 2]$ we have

$$\frac{d}{d\beta} I_\tau^{\beta+1} f_1(t) f_2(t-\tau) = \frac{d}{d\beta} \frac{1}{\Gamma(\beta+1)} \int_\tau^t (t-s)^\beta f_1(s) f_2(s-\tau) ds =$$

$$\frac{1}{\Gamma(\beta+1)} \int_\tau^t (t-s)^\beta (\ln(t-s) - \delta) f_1(s) f_2(s-\tau) ds, \quad \text{(B.21)}$$

where $\delta = \frac{\Gamma'(\beta+1)}{\Gamma(\beta+1)}$. Note that for $\beta \in [0,2]$, $-0.6 < \delta < 1$. Also let $r > 0$ such that $\ln r = \delta$. Let $\alpha = \max\{\bar{\theta}_1, \bar{\theta}_2\}$ and $\bar{\theta}_1$ and $\bar{\theta}_2$ are time to peak for $f_1(t)$ and $f_2(t-\tau)$ respectively. Since $T > T^*$, $f_1(t) f_2(t-\tau)$ is decreasing on $[\alpha, T]$.

First, we show that $\frac{d}{d\beta} I_\tau^{\beta+1} f_1(T) f_2(T-\tau) \geq 0$ for $\beta \in [\epsilon, 2]$, $\epsilon > 0$. It can be observed that

$$\int_\tau^T (T-s)^\beta (\ln(T-s) - \delta) f_1(s) f_2(s-\tau) ds = \int_\tau^{T-e^2} (T-s)^\beta (\ln(T-s) - \delta) f_1(s) f_2(s-\tau) ds + \int_{T-e^2}^T (T-s)^\beta (\ln(T-s) - \delta) f_1(s) f_2(s-\tau) ds. \quad \text{(B.22)}$$

In the equation (B.22), the first integral in right hand side is clearly nonnegative. On the other hand, by Theorem B.5 (II), there exists $\zeta \in [T-e^2, T]$ such that

$$\int_{T-e^2}^T (T-s)^\beta (\ln(T-s) - \delta) f_1(s) f_2(s-\tau) ds = f_1(T-e^2) f_2((T-e^2) - \tau) \int_{T-e^2}^\zeta (T-s)^\beta (\ln(T-s) - \delta)\, ds$$

$$= f_1(T-e^2) f_2(T-e^2-\tau) \Big(\frac{e^{2(\beta+1)}(-1-(1+\beta)\delta + (1+\beta)\ln(T-(T-e^2)))}{(1+\beta)^2} - \frac{(T-\zeta)^{\beta+1}(-1-(1+\beta)\delta + (1+\beta)\ln(T-\zeta))}{(1+\beta)^2} \Big) > 0. \quad \text{(B.23)}$$

By (B.21), (B.22) and (B.23), $\frac{d}{d\beta} I_\tau^{\beta+1} f_1(T) f_2(T-\tau) > 0$, for $\beta \in [\epsilon, 2]$.

Next we show that $\frac{d}{d\beta} I_\tau^{\beta+1} f_1(T) f_2(T-\tau) - \frac{d}{d\beta} I_\tau^{\beta+1} f_1(\eta) f_2(\eta-\tau) \geq 0$ for $\beta \in [\epsilon, 2]$, $\epsilon > 0$, and different possible values that $\eta \in [\tau, T]$ can take.

(i) $\eta \leq \alpha$. Then

$$\int_\tau^T (T-s)^\beta (\ln(T-s) - \delta) f_1(s) f_2(s-\tau) ds - \int_\tau^\eta (\eta-s)^\beta (\ln(\eta-s) - \delta) f_1(s) f_2(s-\tau) ds$$

$$= \int_\tau^\eta ((T-s)^\beta (\ln(T-s) - \delta) - (\eta-s)^\beta (\ln(\eta-s) - \delta)) f_1(s) f_2(s-\tau) + \int_\eta^T (T-s)^\beta (\ln(T-s) - \delta) f_1(s) f_2(s-\tau) ds. \quad \text{(B.24)}$$

Clearly the first integral in (B.24) right hand side is nonnegative. On the other hand

$$\int_\eta^T (T-s)^\beta\,(\ln(T-s)-\delta)\,f_1(s)f_2(s-\tau)ds =$$
$$\int_\eta^\alpha (T-s)^\beta\,(\ln(T-s)-\delta)\,f_1(s)f_2(s-\tau)ds +$$
$$\int_\alpha^T (T-s)^\beta\,(\ln(T-s)-\delta)\,f_1(s)f_2(s-\tau)ds. \tag{B.25}$$

The first integral in (B.25) right hand side is nonnegative. By Theorem B.5 (II), there exists $\zeta \in [\alpha, T]$ such that

$$\int_\alpha^T (T-s)^\beta\,(\ln(T-s)-\delta)\,f_1(s)f_2(s-\tau)ds = f_1(\alpha)f_2(\alpha-\tau)\int_\alpha^\zeta (T-s)^\beta\,(\ln(T-s)-\delta)\,ds$$
$$= f_1(\alpha)f_2(\alpha-\tau)\Big(\frac{(T-\alpha)^{\beta+1}(-1-(1+\beta)\delta+(1+\beta)\ln(T-\alpha))}{(1+\beta)^2} -$$
$$\frac{(T-\zeta)^{\beta+1}\,(-1-(1+\beta)\delta+(1+\beta)\ln(T-\zeta))}{(1+\beta)^2}\Big) > 0. \tag{B.26}$$

By (B.21) and (B.24)-(B.26), $\frac{d}{d\beta} I_\tau^{\beta+1} f_1(T)f_2(T-\tau) - \frac{d}{d\beta} I_\tau^{\beta+1} f_1(\eta)f_2(\eta-\tau) > 0$ for $\beta \in [\epsilon, 2]$, $\epsilon > 0$ and $\eta \le \alpha$.

(ii) $\alpha < \eta \le T - e^2$. Then

$$\int_\tau^T (T-s)^\beta\,(\ln(T-s)-\delta)\,f_1(s)f_2(s-\tau)ds - \int_\tau^\eta (\eta-s)^\beta\,(\ln(\eta-s)-\delta)\,f_1(s)f_2(s-\tau)ds$$
$$= \int_\tau^\eta ((T-s)^\beta\,(\ln(T-s)-\delta) - (\eta-s)^\beta\,(\ln(\eta-s)-\delta))f_1(s)f_2(s-\tau) +$$
$$\int_\eta^T (T-s)^\beta\,(\ln(T-s)-\delta)\,f_1(s)f_2(s-\tau)ds. \tag{B.27}$$

Obviously the first integral in (B.27) right hand side is nonnegative. On the other hand

$$\int_\eta^T (T-s)^\beta\,(\ln(T-s)-\delta)\,f_1(s)f_2(s-\tau)ds =$$
$$\int_\eta^{T-e^2} (T-s)^\beta\,(\ln(T-s)-\delta)\,f_1(s)f_2(s-\tau)ds + \tag{B.28}$$
$$\int_{T-e^2}^T (T-s)^\beta\,(\ln(T-s)-\delta)\,f_1(s)f_2(s-\tau)ds.$$

The first integral in (B.28) right hand side is obviously nonnegative. For the second integral by Theorem B.5 (II), there exists $\zeta \in [T-e^2, T]$ such that

$$\int_{T-e^2}^T (T-s)^\beta\,(\ln(T-s)-\delta)\,f_1(s)f_2(s-\tau)ds =$$
$$f_1(T-e^2)f_2(T-e^2-\tau)\int_{T-e^2}^\zeta (T-s)^\beta\,(\ln(T-s)-\delta)\,ds = \tag{B.29}$$

$$f_1(T-e^2)f_2(T-e^2-\tau)(\frac{e^{2(\beta+1)}(-1-(1+\beta)\delta+(1+\beta)\ln(T-(T-e^2)))}{(1+\beta)^2}-$$

$$\frac{(T-\zeta)^{\beta+1}\,(-1-(1+\beta)\delta+(1+\beta)\ln(T-\zeta))}{(1+\beta)^2})>0.$$

By (B.21) and (B.27)-(B.29), $\frac{d}{d\beta}I_\tau^{\beta+1}f_1(T)f_2(T-\tau)-\frac{d}{d\beta}I_\tau^{\beta+1}f_1(\eta)f_2(\eta-\tau)>0$ for $\beta\in[\epsilon,2]$, $\epsilon>0$ and $\alpha<\eta\leq T-e^2$.

(iii) $T-e^2<\eta<T$. Then

$$\int_\tau^T(T-s)^\beta\,(\ln(T-s)-\delta)\,f_1(s)f_2(s-\tau)ds-\int_\tau^\eta(\eta-s)^\beta\,(\ln(\eta-s)-\delta)\,f_1(s)f_2(s-\tau)ds$$

$$=\int_\tau^{T-r}(T-s)^\beta\,(\ln(T-s)-\delta)\,f_1(s)f_2(s-\tau)ds-\int_\tau^{\eta-r}(\eta-s)^\beta\,(\ln(\eta-s)-\delta)\,f_1(s)f_2(s-\tau)ds$$

$$+\int_{T-r}^T(T-s)^\beta\,(\ln(T-s)-\delta)\,f_1(s)f_2(s-\tau)ds-\int_{\eta-r}^\eta(\eta-s)^\beta\,(\ln(\eta-s)-\delta)\,f_1(s)f_2(s-\tau)ds. \tag{B.30}$$

We have

$$\int_\tau^{T-r}(T-s)^\beta\,(\ln(T-s)-\delta)\,f_1(s)f_2(s-\tau)ds-\int_\tau^{\eta-r}(\eta-s)^\beta\,(\ln(\eta-s)-\delta)\,f_1(s)f_2(s-\tau)ds=$$

$$\int_\tau^{\eta-r}((T-s)^\beta\,(\ln(T-s)-\delta)-(\eta-s)^\beta(\ln(\eta-s)-\delta))\,f_1(s)f_2(s-\tau)ds \tag{B.31}$$

$$+\int_{\eta-r}^{T-r}(T-s)^\beta\,(\ln(T-s)-\delta)\,f_1(s)f_2(s-\tau)ds.$$

The first and the second integrals in (B.31) right hand side are both nonnegative. On the other hand

$$\int_{\eta-r}^\eta(\eta-s)^\beta\,(\ln(\eta-s)-\delta)\,f_1(s)f_2(s-\tau)ds=\int_0^r u^\beta\,(\ln u-\delta)\,f_1(\eta-u)f_2\big((\eta-u)-\tau\big)du, \tag{B.32}$$

$$\int_{T-r}^T(T-s)^\beta\,(\ln(T-s)-\delta)\,f_1(s)f_2(s-\tau)ds=\int_0^r u^\beta\,(\ln u-\delta)\,f_1(T-u)f_2\big((T-u)-\tau\big)du. \tag{B.33}$$

But for $0<u\leq r$, $u^\beta(\ln u-\delta)\,f_1(T-u)f_2((T-u)-\tau)\geq u^\beta(\ln u-\delta)\,f_1(\eta-u)f_2((\eta-u)-\tau)$ which by (B.32) and (B.33) implies that

$$\int_{T-r}^T(T-s)^\beta\,(\ln(T-s)-\delta)\,f_1(s)f_2(s-\tau)ds-\int_{\eta-r}^\eta(\eta-s)^\beta\,(\ln(\eta-s)-\delta)\,f_1(s)f_2(s-\tau)ds\geq\;0. \tag{B.34}$$

Hence by (B.21) and (B.30)-(B.34), $\frac{d}{d\beta} I_\tau^{\beta+1} f_1(T) f_2(T-\tau) - \frac{d}{d\beta} I_\tau^{\beta+1} f_1(\eta) f_2(\eta-\tau) \geq 0$ for $\beta \in [\epsilon, 2]$, $\epsilon > 0$ and $T - e^2 < \eta < T$.

Proof of part (I).

Employing integration by part we get

$$\int_\tau^T \pi(t) I_\tau^\beta f_1(t) f_2(t-\tau) dt = \pi(T) I_\tau^{\beta+1} f_1(T) f_2(T-\tau) - \int_\tau^T \pi'(t) I_\tau^{\beta+1} f_1(t) f_2(t-\tau) dt. \quad \text{(B.35)}$$

It can be verified by Lemma B.2 that $I_\tau^{\beta+1} f_1(t) f_2(t-\tau)$ is continuous and $\frac{d}{d\beta} I_\tau^{\beta+1} f_1(t) f_2(t-\tau)$ is continuous and uniformly bounded on $[\tau, T] \times [\epsilon, 1]$. So, by Leibnitz rule (Folland, 1999, p. 56, Theorem 2.27) we get

$$\frac{d}{d\beta} \int_\tau^T \pi(t) I_\tau^\beta f_1(t) f_2(t-\tau) dt = \pi(T) \frac{d}{d\beta} I_\tau^{\beta+1} f_1(T) f_2(T-\tau) - \int_\tau^T \pi'(t) \frac{d}{d\beta} I_\tau^{\beta+1} f_1(t) f_2(t-\tau) dt.$$

Since $\pi'(t)$ does not change sign and by Lemma B.2 $\frac{d}{d\beta} I_\tau^{\beta+1} f_1(t) f_2(t-\tau)$ is continuous, according to the mean value theorem for integrals there exists $\eta \in [\tau, T]$ such that

$$\frac{d}{d\beta} \int_\tau^T \pi(t) I_\tau^\beta f_1(t) f_2(t-\tau) dt = \pi(T) \frac{d}{d\beta} I_\tau^{\beta+1} f_1(T) f_2(T-\tau) - \frac{d}{d\beta} I_\tau^{\beta+1} f_1(\eta) f_2(\eta-\tau)\, (\pi(T) - \pi(\tau)).$$

Since $\frac{d}{d\beta} I_\tau^{\beta+1} f_1(T) f_2(T-\tau) \geq \frac{d}{d\beta} I_\tau^{\beta+1} f_1(\eta) f_2(\eta-\tau)$, by the fact that $\pi(t)$ is increasing it can be observed that

$$\frac{d}{d\beta} \int_\tau^T \pi(t) I_\tau^\beta f_1(t) f_2(t-\tau) dt = \pi(T) \frac{d}{d\beta} I_\tau^{\beta+1} f_1(T) f_2(T-\tau) - \frac{d}{d\beta} I_\tau^{\beta+1} f_1(\eta) f_2(\eta-\tau) \left(\pi(T) - \pi(\tau)\right) \geq 0.$$

and this completes the proof for part (I).

Proof of part (II).

Employing integration by part we get

$$\int_\tau^T \pi(t) I_\tau^\beta f_1(t) f_2(t-\tau) dt = \pi(T) I_\tau^{\beta+1} f_1(T) f_2(T-\tau) - \int_\tau^T \pi'(t) I_\tau^{\beta+1} f_1(t) f_2(t-\tau) dt,$$

and

$$\frac{d}{d\beta}\int_{\tau}^{T}\pi(t)I_{\tau}^{\beta}f_1(t)f_2(t-\tau)dt = \pi(T)\frac{d}{d\beta}I_{\tau}^{\beta+1}f_1(T)f_2(T-\tau) - \frac{d}{d\beta}\int_{\tau}^{T}\pi'(t)I_{\tau}^{\beta+1}f_1(t)f_2(t-\tau)dt. \quad \text{(B.36)}$$

Employing integration by part we get

$$\int_{\tau}^{T}-\pi'(t)I_{\tau}^{\beta+1}f_1(t)f_2(t-\tau)dt = -\pi'(T)I_{\tau}^{\beta+2}f_1(T)f_2(T-\tau) + \int_{\tau}^{T}\pi''(t)I_{\tau}^{\beta+2}f_1(t)f_2(t-\tau)dt.$$

It can be verified by Lemma B.2 that $I_{\tau}^{\beta+2}f_1(t)f_2(t-\tau)$ is continuous and $\frac{d}{d\beta}I_{\tau}^{\beta+2}f_1(t)f_2(t-\tau)$ is continuous and uniformly bounded on $[\tau,T]\times[\epsilon,1]$. So, by Leibnitz rule (Folland, 1999, p. 56, Theorem 2.27) we get

$$\frac{d}{d\beta}\int_{\tau}^{T}-\pi'(t)I_{\tau}^{\beta+1}f_1(t)f_2(t-\tau)dt = -\pi'(T)\frac{d}{d\beta}I_{\tau}^{\beta+2}f_1(T)f_2(T-\tau) + \int_{\tau}^{T}\pi''(t)\frac{d}{d\beta}I_{\tau}^{\beta+2}f_1(t)f_2(t-\tau)dt. \quad \text{(B.37)}$$

Since $\pi''(t)\le 0$ and by Lemma B.2 $\frac{d}{d\beta}I_{\tau}^{\beta+2}f_1(t)f_2(t-\tau)$ is continuous, according to the mean value theorem for integrals there exists $\eta\in[\tau,T]$ such that

$$\frac{d}{d\beta}\int_{\tau}^{T}-\pi'(t)I_{\tau}^{\beta+1}f_1(t)f_2(t-\tau)dt = -\pi'(T)\frac{d}{d\beta}I_{\tau}^{\beta+2}f_1(T)f_2(T-\tau) - \frac{d}{d\beta}I_{\tau}^{\beta+2}f_1(\eta)f_2(\eta-\tau)\,(-\pi'(T)+\pi'(\tau)).$$

Since $\frac{d}{d\beta}I_{\tau}^{\beta+2}f_1(T)f_2(T-\tau)\ge\frac{d}{d\beta}I_{\tau}^{\beta+2}f_1(\eta)f_2(\eta-\tau)$ and $\frac{d}{d\beta}I_{\tau}^{\beta+2}f_1(T)f_2(T-\tau)\ge 0$, by (B.37) and the fact that $\pi'(t)\le 0,\ \pi''(t)\le 0$ it can be observed that $\frac{d}{d\beta}\int_{\tau}^{T}-\pi'(t)I_{\tau}^{\beta+1}f_1(t)f_2(t-\tau)dt\ge 0$. Hence by (B.36) and the fact that $\frac{d}{d\beta}I_{\tau}^{\beta+1}f_1(T)f_2(T-\tau)\ge 0$ it can be concluded that $\frac{d}{d\beta}\int_{\tau}^{T}\pi(t)I_{\tau}^{\beta+1}f_1(t)f_2(t-\tau)dt\ge 0$, and this completes the proof for part (II). □

Proof of Proposition 3.1.

Profit function can be written as

$$\Pi(\tau_2) = \int_0^T\pi_1(t)I_0^{\beta_1}y_1(t)dt - \int_{\tau_2}^T\pi_1(t)swt_2(t)dt + \int_{\tau_2}^T\pi_2(t)I_{\tau_2}^{\beta_2}\tilde{y}_2(t)dt + \int_{\tau_2}^T\pi_2(t)I_{\tau_2}^{\beta_2}u_2(t)dt + \int_{\tau_2}^T\pi_2(t)swt_2(t)dt. \quad \text{(B.38)}$$

By Lemmas B.1 and B.2 and Theorem B.6 we get

$$\frac{d}{d\tau_2}\int_0^T \pi_1(t) I_0^{\beta_1} y_1(t)dt = \int_{\tau_2}^T \pi_1(t) I_{\tau_2}^{\beta_1} y_{11}(t) f_2(t-\tau_2)dt, \quad \text{(B.39)}$$

$$\frac{d}{d\tau_2}\int_{\tau_2}^T \pi_2(t) I_{\tau_2}^{\beta_2} u_2(t)dt = -\int_{\tau_2}^T \pi_2(t) I_{\tau_2}^{\beta_2} y_{11}(t) f_2(t-\tau_2)dt, \quad \text{(B.40)}$$

and for $i \in \{1,2\}$,

$$\frac{d}{d\tau_2}\int_{\tau_2}^T \pi_i(t) swt_2(t)dt = -\pi_i(\tau_2) swt_2(\tau_2) +$$
$$\int_{\tau_2}^T [-\pi_i(t) h_2 f_2(t-\tau_2)\left(I_0^{\beta_1} y_1(t) - y_1(t)\right) +$$
$$\pi_i(t) h_2 F_2(t-\tau_2)\frac{d}{d\tau_2}\left(I_0^{\beta_1} y_1(t) - y_1(t)\right)]dt, \quad \text{(B.41)}$$

where

$$\frac{d}{d\tau_2}\left(I_0^{\beta_1} y_1(t) - y_1(t)\right) = I_{\tau_2}^{\beta_1} y_{11}(t) f_2(t-\tau_2) - y_{11}(t) f_2(t-\tau_2). \quad \text{(B.42)}$$

Now by (B.38)-(B.41) we get

$$\frac{d}{d\tau_2}\Pi(\tau_2) = \int_{\tau_2}^T \pi_1(t) I_{\tau_2}^{\beta_1} y_{11}(t) f_2(t-\tau_2)dt +$$
$$\frac{d}{d\tau_2}\int_{\tau_2}^T \pi_2(t) I_{\tau_2}^{\beta_2} \tilde{y}_2(t)dt - \int_{\tau_2}^T \pi_2(t) I_{\tau_2}^{\beta_2} y_{11}(t) f_2(t-\tau_2)dt +$$
$$\int_{\tau_2}^T [-(\pi_2(t) - \pi_1(t)) h_2 f_2(t-\tau_2)(I_0^{\beta_1} y_1(t) - y_1(t)) +$$
$$(\pi_2(t) - \pi_1(t)) h_2 F_2(t-\tau_2)\frac{d}{d\tau_2}\left(I_0^{\beta_1} y_1(t) - y_1(t)\right)]dt. \quad \text{(B.43)}$$

Proof of part (I).

Let $\pi_2'(t) \geq 0$ or $\pi_2'(t), \pi_2''(t) \leq 0$, $\pi_1(t) > \pi_2(t)$ for $\lambda \leq t \leq T$, and $\beta_2 \leq \beta_1$.
By Lemma B.2 and Theorem B.6 we get

$$\frac{d}{d\tau_2}\int_{\tau_2}^T \pi_2(t) I_{\tau_2}^{\beta_2} \tilde{y}_2(t)dt = \int_{\tau_2}^T \pi_2(t)\frac{d}{d\tau_2} I_{\tau_2}^{\beta_2} \tilde{y}_2(t)dt, \quad \text{(B.44)}$$

where, by Lemma B.1

$$\frac{d}{d\tau_2} I_{\tau_2}^{\beta_2} \tilde{y}_2(t) = -\frac{(t-\tau_2)^{\beta_2 - 1}}{\Gamma(\beta_2)}\tilde{y}_2(\tau_2) - I_{\tau_2}^{\beta_2} \tilde{y}_2'(t). \quad \text{(B.45)}$$

By (B.42)-(B.44) we get

$$\frac{d}{d\tau_2}\Pi(\tau_2) = \int_{\tau_2}^T \pi_1(t) I_{\tau_2}^{\beta_1} y_{11}(t) f_2(t-\tau_2) - \pi_2(t) I_{\tau_2}^{\beta_2} y_{11}(t) f_2(t-\tau_2)dt +$$
$$\int_{\tau_2}^T \pi_2(t)\frac{d}{d\tau_2} I_{\tau_2}^{\beta_2} \tilde{y}_2(t)dt +$$
$$\int_{\tau_2}^T [-(\pi_2(t) - \pi_1(t)) h_2 f_2(t-\tau_2)\left(I_0^{\beta_1} y_1(t) - y_1(t)\right) + (\pi_2(t) - \pi_1(t)) h_2 F_2(t -$$
$$\tau_2)\frac{d}{d\tau_2}\left(I_0^{\beta_1} y_1(t) - y_1(t)\right)]dt \geq$$

$$\int_{\tau_2}^{T} \pi_1(t) I_{\tau_2}^{\beta_1} y_{11}(t) f_2(t-\tau_2) - \pi_2(t) I_{\tau_2}^{\beta_2} y_{11}(t) f_2(t-\tau_2) dt + \int_{\tau_2}^{T} \pi_2(t) \frac{d}{d\tau_2} I_{\tau_2}^{\beta_2} \tilde{y}_2(t) dt$$

$$+ \int_{\tau_2}^{T} -(\pi_2(t) - \pi_1(t)) h_2 f_2(t-\tau_2) \left(I_0^{\beta_1} y_1(t) - y_1(t) \right) dt + \tag{B.46}$$

$$\int_{\tau_2}^{T} (\pi_2(t) - \pi_1(t)) I_{\tau_2}^{\beta_1} y_{11}(t) f_2(t-\tau_2) dt - \int_{\tau_2}^{T} (\pi_2(t) - \pi_1(t)) h_2 F_2(t-$$

$$\tau_2) y_{11}(t) f_2(t-\tau_2) dt =$$

$$\int_{\tau_2}^{T} \pi_2(t) (I_{\tau_2}^{\beta_1} y_{11}(t) f_2(t-\tau_2) - I_{\tau_2}^{\beta_2} y_{11}(t) f_2(t-\tau_2)) dt + \int_{\tau_2}^{T} \pi_2(t) \frac{d}{d\tau_2} I_{\tau_2}^{\beta_2} \tilde{y}_2(t) dt -$$

$$\int_{\tau_2}^{T} (\pi_2(t) - \pi_1(t))\, h_2 f_2(t-\tau_2) \left(I_0^{\beta_1} y_1(t) - y_1(t) \right) dt -$$

$$\int_{\tau_2}^{T} (\pi_2(t) - \pi_1(t)) h_2 F_2(t-\tau_2) y_{11}(t) f_2(t-\tau_2) dt.$$

By Lemma B.3 and the fact that $\beta_2 \le \beta_1$ we have

$$\int_{\tau_2}^{T} \pi_2(t) (I_{\tau_2}^{\beta_1} f_1(t) f_2(t-\tau_2) - I_{\tau_2}^{\beta_2} f_1(t) f_2(t-\tau_2)) dt \ge 0. \tag{B.47}$$

On the other hand, for $t \in [\tau_2, T]$

$$m_1 (I_0^{\beta_1} \Phi_1(t) - f_1(t)\big(1 - F_2(t-\tau_2)\big)) = I_0^{\beta_1} y_1(t) - y_1(t) \ge 0, \tag{B.48}$$

where

$$\Phi_1(t) = \begin{cases} f_1(t) & 0 < t \le \tau_2 \\ f_1(t)(1 - F_2(t-\tau_2)) & t > \tau_2 \end{cases},$$

So, by (B.45), (B.47) and (B.48), for m_1 sufficiently large, $m_1 r(\tau_2) + \int_{\tau_2}^{T} \pi_2(t) \frac{d}{d\tau_2} I_{\tau_2}^{\beta_2} \tilde{y}_2(t) dt \ge 0$, for all $\tau_2 \in [\lambda, \Lambda]$ where

$$r(\tau_2) = \int_{\tau_2}^{T} \pi_2(t) (I_{\tau_2}^{\beta_1} f_1(t) f_2(t-\tau_2) - I_{\tau_2}^{\beta_2} f_1(t) f_2(t-\tau_2)) dt +$$

$$\int_{\tau_2}^{T} (\pi_1(t) - \pi_2(t))\, h_2 f_2(t-\tau_2) \left(I_0^{\beta_1} \Phi_1(t) - f_1(t)\big(1 - F_2(t-\tau_2)\big) \right) dt + \tag{B.49}$$

$$\int_{\tau_2}^{T} (\pi_1(t) - \pi_2(t)) h_2 F_2(t-\tau_2) f_1(t) f_2(t-\tau_2) dt.$$

Let $\gamma_1 = \inf\{m_1 : m_1 r(\tau_2) + \int_{\tau_2}^{T} \pi_2(t) \frac{d}{d\tau_2} I_{\tau_2}^{\beta_2} \tilde{y}_2(t) dt \ge 0 \text{ for all } \tau_2 \in [\lambda, \Lambda]\}$. Then by (B.46) and (B.49) it can be observed that, for $m_1 > \gamma_1$ we get $\frac{d}{d\tau_2} \Pi(\tau_2) \ge 0$ which implies that $\tau_2^* = \Lambda$.

Proof of part (II).

Let $\pi_2'(t) \ge 0$, $\pi_2(t) > \pi_1(t)$ for $\lambda \le t \le T$, and $\beta_1 \le \beta_2$.

By (B.42)-(B.44) we get

$$\frac{d}{d\tau_2}\Pi(\tau_2) \leq$$

$$\int_{\tau_2}^{T} \pi_2(t)(I_{\tau_2}^{\beta_1} y_{11}(t) f_2(t-\tau_2) - I_{\tau_2}^{\beta_2} y_{11}(t) f_2(t-\tau_2))dt + \int_{\tau_2}^{T} \pi_2(t) \frac{d}{d\tau_2} I_{\tau_2}^{\beta_2} \tilde{y}_2(t)dt -$$

$$\int_{\tau_2}^{T} (\pi_2(t) - \pi_1(t))\, h_2 f_2(t-\tau_2)\left(I_0^{\beta_1} y_1(t) - y_1(t)\right)dt - \quad \text{(B.50)}$$

$$\int_{\tau_2}^{T} (\pi_2(t) - \pi_1(t)) h_2 F_2(t-\tau_2) y_{11}(t) f_2(t-\tau_2)dt.$$

By Lemma B.3 (I) and the fact that $\beta_1 \leq \beta_2$ we have

$$\int_{\tau_2}^{T} \pi_2(t)(I_{\tau_2}^{\beta_1} y_{11}(t) f_2(t-\tau_2) - I_{\tau_2}^{\beta_2} y_{11}(t) f_2(t-\tau_2))dt \leq 0. \quad \text{(B.51)}$$

So, by (B.45), (B.48) and (B.51), for m_1 sufficiently large, $m_1 r(\tau_2) + \int_{\tau_2}^{T} \pi_2(t) \frac{d}{d\tau_2} I_{\tau_2}^{\beta_2} \tilde{y}_2(t)dt \leq 0$, for all $\tau_2 \in [\lambda, \Lambda]$. Let $\gamma_2 = \inf\{m_1 : m_1 r(\tau_2) + \int_{\tau_2}^{T} \pi_2(t) \frac{d}{d\tau_2} I_{\tau_2}^{\beta_2} \tilde{y}_2(t)dt \leq 0$ for all $\tau_2 \in [\lambda, \Lambda]\}$. Then by (B.49) and (B.50) it can be observed that for $m_1 > \gamma_2$ we get $\frac{d}{d\tau_2}\Pi(\tau_2) \leq 0$ which implies that $\tau_2^* = \lambda$.

Proof of part (III).

Let $\pi_2'(t), \pi_2''(t) \leq 0, \pi_2(t) \geq \pi_1(t)$ for $\lambda \leq t \leq T$ and, $\beta_1 \leq \beta_2$.
By Lemma B.3 (II) we have

$$\int_{\tau_2}^{T} \pi_2(t)(I_{\tau_2}^{\beta_1} y_{11}(t) f_2(t-\tau_2) - I_{\tau_2}^{\beta_2} y_{11}(t) f_2(t-\tau_2))dt \leq 0. \quad \text{(B.52)}$$

By Lemma B.2, Theorem B.5 (II) and Theorem B.6 we get

$$\frac{d}{d\tau_2}\int_{\tau_2}^{T} \pi_2(t) I_{\tau_2}^{\beta_2} \tilde{y}_2(t)dt = \int_{\tau_2}^{T} \pi_2(t) \frac{d}{d\tau_2} I_{\tau_2}^{\beta_2} \tilde{y}_2(t)dt = \pi_2(\tau_2)\int_{\tau_2}^{\eta} \frac{d}{d\tau_2} I_{\tau_2}^{\beta_2} \tilde{y}_2(t)dt, \quad \text{(B.53)}$$

where $\eta \in [\tau_2, T]$. By (B.45) and Theorem B.1 we get

$$\int_{\tau_2}^{\eta} \frac{d}{d\tau_2} I_{\tau_2}^{\beta_2} \tilde{y}_2(t)dt = -\int_{\tau_2}^{\eta}\left(\frac{(t-\tau_2)^{\beta_2-1}}{\Gamma(\beta_2)}\tilde{y}_2(\tau_2) + I_{\tau_2}^{\beta_2}\tilde{y}_2{}'(t)\right)dt = -I_{\tau_2}^{\beta_2}\tilde{y}_2(\eta). \quad \text{(B.54)}$$

Therefore by (B.53) and (B.54)

$$\frac{d}{d\tau_2}\int_{\tau_2}^{T} \pi_2(t) I_{\tau_2}^{\beta_2} \tilde{y}_2(t)dt = -\pi_2(\tau_2) I_{\tau_2}^{\beta_2} \tilde{y}_2(\eta). \quad \text{(B.55)}$$

By (B.42), (B.43), and (B.55) we get

$$\frac{d}{d\tau_2}\Pi(\tau_2) \leq \int_{\tau_2}^{T} \pi_2(t)(I_{\tau_2}^{\beta_1} y_{11}(t) f_2(t-\tau_2) - I_{\tau_2}^{\beta_2} y_{11}(t) f_2(t-\tau_2))dt -$$

$$\pi_2(\tau_2) I_{\tau_2}^{\beta_2} \tilde{y}_2(\eta)$$

$$-\int_{\tau_2}^{T} (\pi_2(t) - \pi_1(t))\, h_2 f_2(t-\tau_2)\left(I_0^{\beta_1} y_1(t) - y_1(t)\right)dt \quad \text{(B.56)}$$

$$-\int_{\tau_2}^{T}(\pi_2(t)-\pi_1(t))h_2F_2(t-\tau_2)y_{11}(t)f_2(t-\tau_2)dt.$$

Noting (B.52), (B.56) implies that $\frac{d}{d\tau_2}\Pi(\tau_2)\le 0$ and therefore $\tau_2^*=\lambda$. □

Proof of Proposition 3.2.

By (B.46)

$$\frac{d}{d\tau_2}\Pi(\tau_2)=m_1\mu_1(\tau_2)+m_2\mu_2(\tau_2),$$

where

$$\mu_1(\tau_2)=\int_{\tau_2}^{T}\pi_1(t)I_{\tau_2}^{\beta_1}f_1(t)f_2(t-\tau_2)-\pi_2(t)I_{\tau_2}^{\beta_2}f_1(t)f_2(t-\tau_2)dt+$$

$$\int_{\tau_2}^{T}[-(\pi_2(t)-\pi_1(t))h_2f_2(t-\tau_2)(I_0^{\beta_1}\Phi_1(t)-f_1(t)(1-F_2(t-\tau_2)))+$$

$$(\pi_2(t)-\pi_1(t))h_2F_2(t-\tau_2)(I_{\tau_2}^{\beta_1}f_1(t)f_2(t-\tau_2)-f_1(t)f_2(t-\tau_2))]dt,$$

$$\mu_2(\tau_2)=\int_{\tau_2}^{T}\pi_2(t)\frac{d}{d\tau_2}I_{\tau_2}^{\beta_2}f_2(t-\tau_2)dt.$$

Note that $\mu_1(\tau_2)\ge r(\tau_2)>0$, $\tau_2\in[\lambda,\Lambda]$. It can be observed that if $m_1<\underbrace{\frac{-\mathrm{m}_2\mu_2(\Lambda)}{\mu_1(\Lambda)}}_{\eta_1}$

then $\frac{d}{d\tau_2}\Pi(\Lambda)<0$ which means $\tau_2^*\in[\lambda,\Lambda)$. □

Proof of Proposition 3.3.

Profit function can be written as

$$\Pi(\tau_2)=\int_0^{\tau_2}\pi_1(t)I_0^{\beta_1}y_1(t)dt+\int_{\tau_2}^{T}\pi_2(t)I_{\tau_2}^{\beta_2}\tilde{y}_2(t)dt+\int_{\tau_2}^{T}\pi_2(t)I_0^{\beta_2}y_{11}(t)dt. \quad \text{(B.57)}$$

It can be easily observed that

$$\frac{d}{d\tau_2}\int_0^{\tau_2}\pi_1(t)I_0^{\beta_1}y_1(t)dt=\pi_1(\tau_2)I_0^{\beta_1}y_{11}(\tau_2), \quad \text{(B.58)}$$

$$\frac{d}{d\tau_2}\int_{\tau_2}^{T}\pi_2(t)I_0^{\beta_2}y_{11}(t)dt=-\pi_2(\tau_2)I_0^{\beta_2}y_{11}(\tau_2). \quad \text{(B.59)}$$

Now by (B.57)-(B.59) we get

$$\frac{d}{d\tau_2}\Pi(\tau_2)=\pi_1(\tau_2)I_0^{\beta_1}y_{11}(\tau_2)-\pi_2(\tau_2)I_0^{\beta_2}y_{11}(\tau_2)+\frac{d}{d\tau_2}\int_{\tau_2}^{T}\pi_2(t)I_{\tau_2}^{\beta_2}\tilde{y}_2(t)dt. \quad \text{(B.60)}$$

Proof of part (I).

Let $\pi_1(t)\ge\pi_2(t)$ for $\lambda\le t\le T$, $\beta_2<\beta_1$ (or $\pi_1(t)>\pi_2(t)$ for $\lambda\le t\le T$, $\beta_2\le\beta_1$). Then

$$\pi_1(\tau_2)I_0^{\beta_1}y_{11}(\tau_2) - \pi_2(\tau_2)I_0^{\beta_2}y_{11}(\tau_2) > 0. \quad \text{(B.61)}$$

By (B.44), (B.45), and (B.61), for m_1 sufficiently large, $m_1(\pi_1(\tau_2)I_0^{\beta_1}f_1(\tau_2) - \pi_2(\tau_2)I_0^{\beta_2}f_1(\tau_2)) + \int_{\tau_2}^{T}\pi_2(t)\frac{d}{d\tau_2}I_{\tau_2}^{\beta_2}\tilde{y}_2(t)dt \geq 0$, for all $\tau_2 \in [\lambda, \Lambda]$. Let $\tilde{\gamma}_1 = \inf\left\{m_1: m_1(\pi_1(\tau_2)I_0^{\beta_1}f_1(\tau_2) - \pi_2(\tau_2)I_0^{\beta_2}f_1(\tau_2)) + \int_{\tau_2}^{T}\pi_2(t)\frac{d}{d\tau_2}I_{\tau_2}^{\beta_2}\tilde{y}_2(t)dt \geq 0 \text{ for all } \tau_2 \in [\lambda, \Lambda]\right\}$. Then by (B.60) it can be observed that for $m_1 > \tilde{\gamma}_1$ we get $\frac{d}{d\tau_2}\Pi(\tau_2) \geq 0$ for $\tau_2 \in [\lambda, \Lambda]$ which implies that $\tau_2^* = \Lambda$.

Proof of part (II).

Let $\pi_2(t) \geq \pi_1(t)$ for $\lambda \leq t \leq T$, $\beta_1 < \beta_2$ (or $\pi_2(t) > \pi_1(t)$ for $\lambda \leq t \leq T$, $\beta_1 \leq \beta_2$). Then

$$\pi_1(\tau_2)I_0^{\beta_1}y_{11}(\tau_2) - \pi_2(\tau_2)I_0^{\beta_2}y_{11}(\tau_2) < 0. \quad \text{(B.62)}$$

By (B.44), (B.45), and (B.62), for m_1 sufficiently large, $m_1(\pi_1(\tau_2)I_0^{\beta_1}f_1(\tau_2) - \pi_2(\tau_2)I_0^{\beta_2}f_1(\tau_2)) + \int_{\tau_2}^{T}\pi_2(t)\frac{d}{d\tau_2}I_{\tau_2}^{\beta_2}\tilde{y}_2(t)dt \leq 0$, for all $\tau_2 \in [\lambda, \Lambda]$. Let $\tilde{\gamma}_2 = \inf\left\{m_1: m_1(\pi_1(\tau_2)I_0^{\beta_1}f_1(\tau_2) - \pi_2(\tau_2)I_0^{\beta_2}f_1(\tau_2)) + \int_{\tau_2}^{T}\pi_2(t)\frac{d}{d\tau_2}I_{\tau_2}^{\beta_2}\tilde{y}_2(t)dt \leq 0 \text{ for all } \tau_2 \in [\lambda, \Lambda]\right\}$. Then by (B.60), for $m_1 > \tilde{\gamma}_2$ we get $\frac{d}{d\tau_2}\Pi(\tau_2) \leq 0$ which implies that $\tau_2^* = \lambda$.

Proof of part (III).

Suppose $\pi_2(t)$ be monotonically declining. Let $\pi_2(t) \geq \pi_1(t)$ for $\lambda \leq t \leq T$ and, $\beta_1 \leq \beta_2$. Then

$$\pi_1(\tau_2)I_0^{\beta_1}y_{11}(\tau_2) - \pi_2(\tau_2)I_0^{\beta_2}y_{11}(\tau_2) \leq 0. \quad \text{(B.63)}$$

By (B.55), (B.60) and (B.63) it can be concluded that $\frac{d}{d\tau_2}\Pi(\tau_2) \leq 0$ for $\tau_2 \in [\lambda, \Lambda]$ which means $\tau_2^* = \lambda$. □

Proof of Proposition 3.4.

By (B.60)

$$\frac{d}{d\tau_2}\Pi(\tau_2) = m_1(\pi_1(\tau_2)I_0^{\beta_1}f_1(\tau_2) - \pi_2(\tau_2)I_0^{\beta_2}f_1(\tau_2)) + m_2 \underbrace{\frac{d}{d\tau_2}\int_{\tau_2}^{T}\pi_2(t)I_{\tau_2}^{\beta_2}f_2(t-\tau_2)dt}_{\tilde{\mu}(\tau_2)}.$$

It can be observed that if

$$m_1 < \underbrace{\frac{-\mathrm{m}_2\tilde{\mu}(\Lambda)}{\left(\pi_1(\Lambda)I_0^{\beta_1}f_1(\Lambda)-\pi_2(\Lambda)I_0^{\beta_2}f_1(\Lambda)\right)}}_{\tilde{\eta}_1},$$

then $\frac{d}{d\tau_2}\Pi(\Lambda) < 0$ which means $\tau_2^* \in [\lambda, \Lambda)$. □

Proof of Proposition 3.5.

We refer to the profit function (3.43) under phase-out transition and total transition scenarios as Π^{ph} and Π^{T} respectively. By (3.10) and (3.13) profit function (3.43) under phase-out transition scenario is given by

$$\Pi^{ph}(\tau_2) = \int_0^T \pi_1(t)I_0^{\beta_1}y_1(t)dt + \int_{\tau_2}^T \pi_2(t)I_{\tau_2}^{\beta_2}\tilde{y}_2(t)dt + \int_{\tau_2}^T \pi_2(t)I_{\tau_2}^{\beta_2}u_2(t)dt$$
$$- \int_{\tau_2}^T\big(\pi_1(t) - \pi_2(t)\big)swt_2(t)dt. \quad \text{(B.64)}$$

By (3.15) and (3.20), profit function (3.43) under total transition scenario is given by

$$\Pi^{\mathrm{T}}(\tau_2) = \int_0^{\tau_2} \pi_1(t)I_0^{\beta_1}y_{11}(t)dt + \int_{\tau_2}^T \pi_2(t)I_{\tau_2}^{\beta_2}\tilde{y}_2(t)dt + \int_{\tau_2}^T \pi_2(t)I_0^{\beta_2}y_{11}(t)dt. \quad \text{(B.65)}$$

Proof of part (I).

Suppose $\pi_1(t) = \pi_2(t)$ and $\beta_1 = \beta_2$. By (B.64) and (B.65)

$$\Pi^{ph}(\tau_2) = \int_0^T \pi_1(t)I_0^{\beta_1}y_1(t)dt + \int_{\tau_2}^T \pi_2(t)I_{\tau_2}^{\beta_2}\tilde{y}_2(t)dt + \int_{\tau_2}^T \pi_2(t)I_{\tau_2}^{\beta_2}u_2(t)dt =$$
$$\int_0^{\tau_2} \pi_1(t)I_0^{\beta_1}y_{11}(t)dt + \int_{\tau_2}^T \pi_1(t)I_0^{\beta_1}y_{11}(t)dt - \int_{\tau_2}^T \pi_1(t)I_{\tau_2}^{\beta_1}u_2(t)dt +$$
$$\int_{\tau_2}^T \pi_2(t)I_{\tau_2}^{\beta_2}\tilde{y}_2(t)dt + \int_{\tau_2}^T \pi_2(t)I_{\tau_2}^{\beta_2}u_2(t)dt =$$
$$\int_0^{\tau_2} \pi_1(t)I_0^{\beta_1}y_{11}(t)dt + \int_{\tau_2}^T \pi_2(t)I_0^{\beta_2}y_{11}(t)dt + \int_{\tau_2}^T \pi_2(t)I_{\tau_2}^{\beta_2}\tilde{y}_2(t)dt = \Pi^{\mathrm{T}}(\tau_2).$$

Proof of part (II).

Suppose $\pi_2(t) < \pi_1(t)$ and $\beta_2 \leq \beta_1$. By (B.64) and (B.65) we get

$$\Pi^{ph}(\tau_2) = \int_0^T \pi_1(t)I_0^{\beta_1}y_1(t)dt + \int_{\tau_2}^T \pi_2(t)I_{\tau_2}^{\beta_2}\tilde{y}_2(t)dt + \int_{\tau_2}^T \pi_2(t)I_{\tau_2}^{\beta_2}u_2(t)dt -$$
$$\int_{\tau_2}^T\big(\pi_1(t) - \pi_2(t)\big)h_2F_2(t-\tau_2)\left(I_0^{\beta_1}y_1(t) - y_1(t)\right)dt >$$

$$\int_0^T \pi_1(t) I_0^{\beta_1} y_1(t)dt + \int_{\tau_2}^T \pi_2(t) I_{\tau_2}^{\beta_2} \tilde{y}_2(t)dt + \int_{\tau_2}^T \pi_2(t) I_{\tau_2}^{\beta_2} u_2(t)dt - \int_{\tau_2}^T \big(\pi_1(t) - \pi_2(t)\big)\left(I_0^{\beta_1} y_1(t) - y_1(t)\right)dt =$$

$$\int_0^{\tau_2} \pi_1(t) I_0^{\beta_1} y_{11}(t)dt + \int_{\tau_2}^T \pi_1(t) I_0^{\beta_1} y_{11}(t)dt - \int_{\tau_2}^T \pi_1(t) I_{\tau_2}^{\beta_1} u_2(t)dt + \int_{\tau_2}^T \pi_2(t) I_{\tau_2}^{\beta_2} \tilde{y}_2(t)dt + \int_{\tau_2}^T \pi_2(t) I_{\tau_2}^{\beta_2} u_2(t)dt - \int_{\tau_2}^T \pi_1(t)\left(I_0^{\beta_1} y_{11}(t) - I_{\tau_2}^{\beta_1} u_2(t) - y_1(t)\right)dt + \int_{\tau_2}^T \pi_2(t)\left(I_0^{\beta_1} y_1(t) - y_1(t)\right)dt =$$

$$\int_0^{\tau_2} \pi_1(t) I_0^{\beta_1} y_{11}(t)dt + \int_{\tau_2}^T \pi_2(t) I_{\tau_2}^{\beta_2} \tilde{y}_2(t)dt + \int_{\tau_2}^T \pi_2(t) I_{\tau_2}^{\beta_2} u_2(t)dt + \int_{\tau_2}^T \pi_1(t) y_1(t)dt + \int_{\tau_2}^T \pi_2(t)\left(I_0^{\beta_1} y_1(t) - y_1(t)\right)dt \geq$$

$$\int_0^{\tau_2} \pi_1(t) I_0^{\beta_1} y_{11}(t)dt + \int_{\tau_2}^T \pi_2(t) I_{\tau_2}^{\beta_2} \tilde{y}_2(t)dt + \int_{\tau_2}^T \pi_2(t) I_{\tau_2}^{\beta_2} u_2(t)dt + \int_{\tau_2}^T \pi_1(t) y_1(t)dt + \int_{\tau_2}^T \pi_2(t)\left(I_0^{\beta_2} y_1(t) - y_1(t)\right)dt =$$

$$\int_0^{\tau_2} \pi_1(t) I_0^{\beta_1} y_{11}(t)dt + \int_{\tau_2}^T \pi_2(t) I_{\tau_2}^{\beta_2} \tilde{y}_2(t)dt + \int_{\tau_2}^T \pi_2(t) I_{\tau_2}^{\beta_2} u_2(t)dt + \int_{\tau_2}^T \pi_1(t) y_1(t)dt + \int_{\tau_2}^T \pi_2(t)\left(I_0^{\beta_2} y_{11}(t) - I_{\tau_2}^{\beta_2} u_2(t) - y_1(t)\right)dt =$$

$$\int_0^{\tau_2} \pi_1(t) I_0^{\beta_1} y_{11}(t)dt + \int_{\tau_2}^T \pi_2(t) I_{\tau_2}^{\beta_2} \tilde{y}_2(t)dt + \int_{\tau_2}^T (\pi_1(t) - \pi_2(t)) y_1(t)dt + \int_{\tau_2}^T \pi_2(t) I_0^{\beta_2} y_{11}(t)dt > \Pi^{\mathrm{T}}(\tau_2).$$

Suppose $\pi_1(t) = \pi_2(t)$ and $\beta_2 < \beta_1$. By (B.64) and (B.65) we get

$$\Pi^{ph}(\tau_2) = \int_0^T \pi_1(t) I_0^{\beta_1} y_1(t)dt + \int_{\tau_2}^T \pi_2(t) I_{\tau_2}^{\beta_2} \tilde{y}_2(t)dt + \int_{\tau_2}^T \pi_2(t) I_{\tau_2}^{\beta_2} u_2(t)dt =$$

$$\int_0^{\tau_2} \pi_1(t) I_0^{\beta_1} y_{11}(t)dt + \int_{\tau_2}^T \pi_1(t) I_0^{\beta_1} y_1(t)dt + \int_{\tau_2}^T \pi_2(t) I_{\tau_2}^{\beta_2} \tilde{y}_2(t)dt + \int_{\tau_2}^T \pi_2(t) I_{\tau_2}^{\beta_2} u_2(t)dt >$$

$$\int_0^{\tau_2} \pi_1(t) I_0^{\beta_1} y_{11}(t)dt + \int_{\tau_2}^T \pi_1(t) I_0^{\beta_2} y_1(t)dt + \int_{\tau_2}^T \pi_2(t) I_{\tau_2}^{\beta_2} \tilde{y}_2(t)dt + \int_{\tau_2}^T \pi_2(t) I_{\tau_2}^{\beta_2} u_2(t)dt =$$

$$\int_0^{\tau_2} \pi_1(t) I_0^{\beta_1} y_{11}(t)dt + \int_{\tau_2}^T \pi_1(t) I_0^{\beta_2} y_{11}(t)dt - \int_{\tau_2}^T \pi_1(t) I_{\tau_2}^{\beta_2} u_2(t)dt + \int_{\tau_2}^T \pi_2(t) I_{\tau_2}^{\beta_2} \tilde{y}_2(t)dt + \int_{\tau_2}^T \pi_2(t) I_{\tau_2}^{\beta_2} u_2(t)dt =$$

$$\int_0^{\tau_2} \pi_1(t) I_0^{\beta_1} y_{11}(t)dt + \int_{\tau_2}^T \pi_2(t) I_0^{\beta_2} y_{11}(t)dt + \int_{\tau_2}^T \pi_2(t) I_{\tau_2}^{\beta_2} \tilde{y}_2(t)dt = \Pi^{\mathrm{T}}(\tau_2).$$

Proof of part (III).

Similar to the proof of part (II) but with the direction of inequalities reversed. □

B.3 Impact of Price on Optimal Market Entry Timing

Our main model does not take into consideration the impact of pricing. In an extension scenario, we conduct numerical analysis to investigate the impact of pricing on the optimal market entry timing strategy under the phase-out transition scenario.

We consider a declining price trend for the old generation, $pr_1(t)$, as follows:

$$pr_1(t) = d_1^p e^{-r_p * t} \ , t \geq 0.$$

Here, d_1^p represents the baseline price and $r_p > 0$ denotes the declining rate of price. We let the price for the new generation, denoted as pr_2, be higher than the old generation (i.e., $pr_2 > pr_1(t)$, $t \geq \tau_2$). Our goal is to examine how the price differential between the old and new generations impacts the optimal market entry timing. In pursuit of this goal, we fix the price trend of the old generation while varying the price trend of the new generation.

We extend the sales model outlined in equations (3.10) and (3.13) to account for variations in the sales of the new generation resulting from alterations in the price trend (pr_2) of the new generation. To this end, following Mesak and Berg (1995), we incorporate price in the new generation's market size m_2 and adoption parameters p_2 and q_2. Specifically, we set

$$p_2(pr_2) = p_2 e^{-r_d (pr_2 - l_2)},$$

$$q_2(pr_2) = q_2 e^{-r_d (pr_2 - l_2)}$$

$$m_2(pr_2) = m_2 e^{-r_d (pr_2 - l_2)},$$

where p_2, q_2, and m_2 represent the baseline values for the innovation parameter, imitation parameter, and market size for the new generation respectively. l_2 denotes the baseline price for the new generation, and $r_d > 0$ denotes the declining rate. We consider the following monotone cost functions for the old ($c_1(t)$) and the new generation ($c_2(t)$):

$$c_1(t) = d_1^c e^{r_c * t} \ ,$$

$$c_2(t) = d_2^c e^{r_c * t},$$

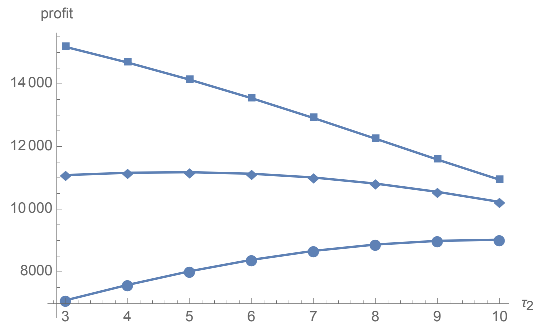
where, d_1^c and d_2^c represent the baseline cost for generations 1 and 2, respectively and r_c denotes the rate of change in cost.

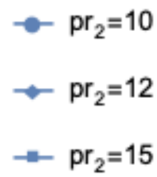
In our numerical analysis we consider a base sales scenario with parameters, $p_1 = p_2 = 0.01$, $q_1 = q_2 = 0.5$, $\beta_1 = \beta_2 = 0.3$, $m_1 = 800$, $m_2 = 600$, $r_d = 0.05$, $l_2 = 10$, and $h_2 = 1$. Price, cost, and discount rate parameters are set to $d_1^p = 10$, $r_p = 0.01$, $d_1^c = 5$, $d_2^c = 10$, $r_c = -0.01$, and $r_1 = r_2 = 0$. The Planning horizon is set to $T = 20$. Under the base scenario, we set the price levels for the new generation to be $pr_2 = 10$, 12, and 15. Next, we change the base scenario to variational scenarios by changing the sales parameters one at the time. Table B-1 demonstrates the profit dynamics with respect to the market entry timing across three different pricing levels for the new generation ($pr_2 = 10$, 12, and 15), under the base and variational scenarios. We repeat the previously described process, taking into account increasing cost trends with $r_c = 0.01$ and $d_2^c = 8$. The results are demonstrated in Table B-2.

It is evident that across all scenarios presented in Tables B-1 and B-2, as the price of the new generation (pr_2) rises, it is more profitable to release the new generation earlier. Furthermore, it is noteworthy that when the price is set to the highest level $pr_2 = 15$, where the unit contribution margin for the new generation surpasses the old generation (i.e., $\pi_2(t) > \pi_1(t)$), the immediate release of the new generation is the optimal strategy. This result is consistent with our earlier analytical findings presented in Proposition 3.1 (II) and (III).

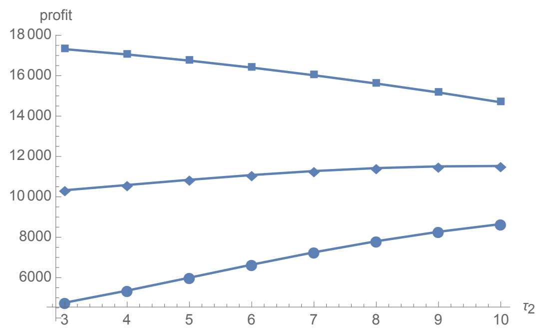
Table B-1: Dynamics of Profit with Respect to Market Entry Timing with Decreasing Cost Trends

Base

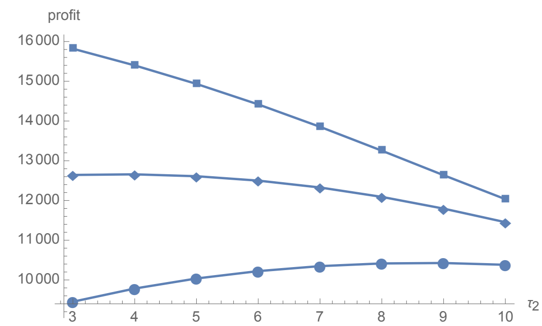




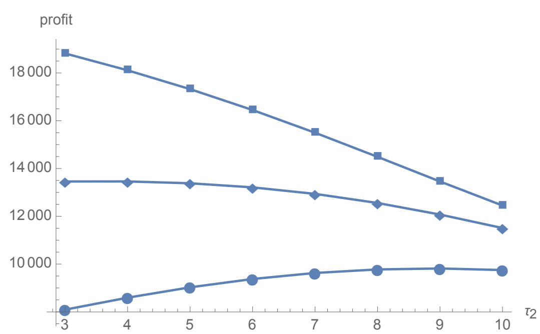
New generation fast adoption ($p_2 = 0.05, q_2 = 0.8$)



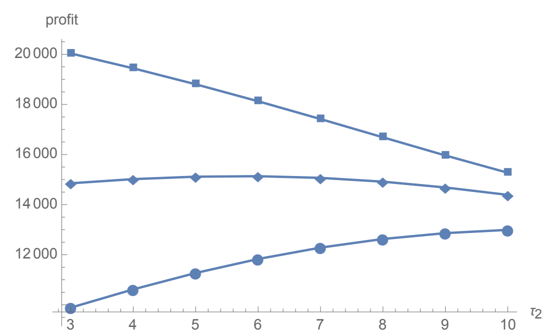
Old generation fast adoption ($p_1 = 0.05, q_1 = 0.8$)



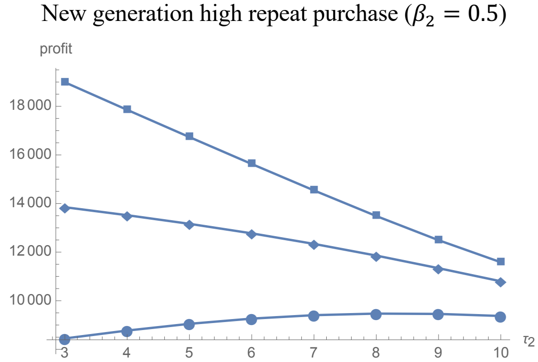
New generation big market size ($m_2 = 1000$)



Old generation big market size ($m_1 = 1200$)



New generation high repeat purchase ($\beta_2 = 0.5$)



Old generation high repeat purchase rate ($\beta_1 = 0.5$)

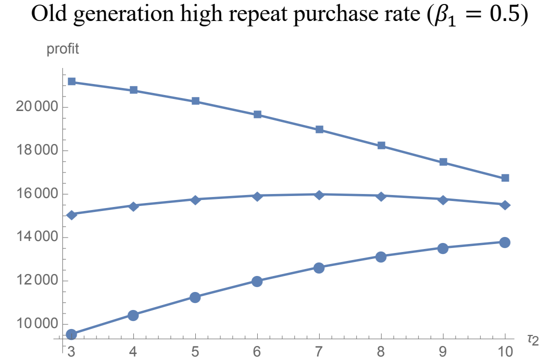
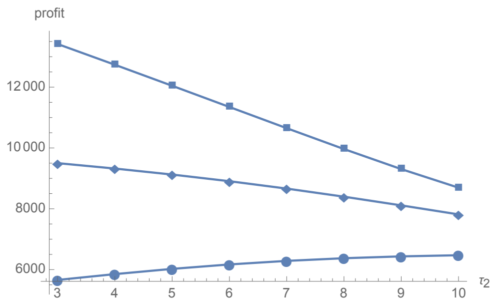
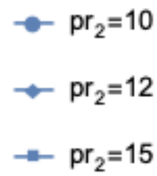


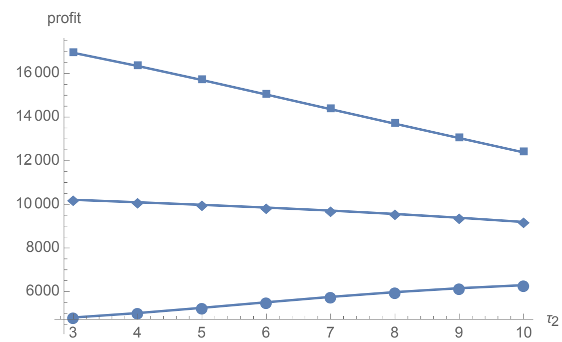
Table B-2: Dynamics of Profit with Respect to Market Entry Timing with Increasing Cost Trends

Base

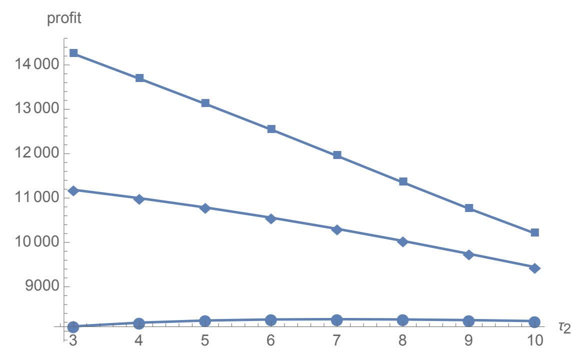




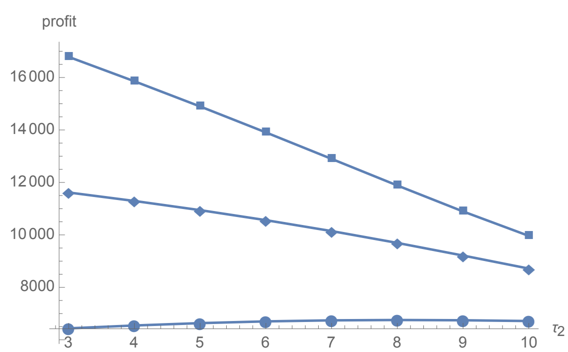
New generation fast adoption ($p_2 = 0.05, q_2 = 0.8$)



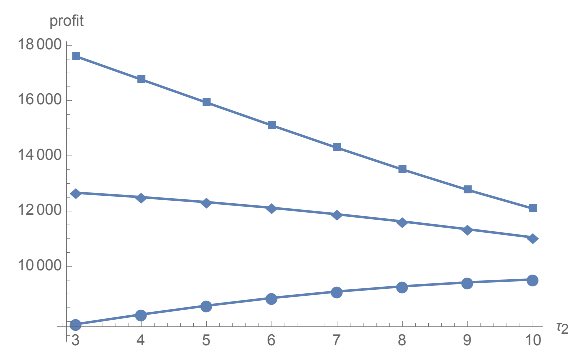
Old generation fast adoption ($p_1 = 0.05, q_1 = 0.8$)



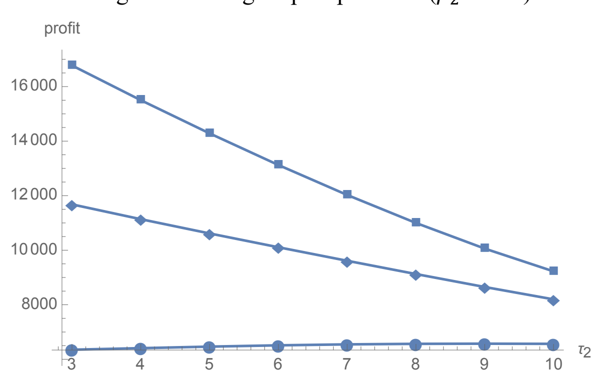
New generation big market size ($m_2 = 1000$)



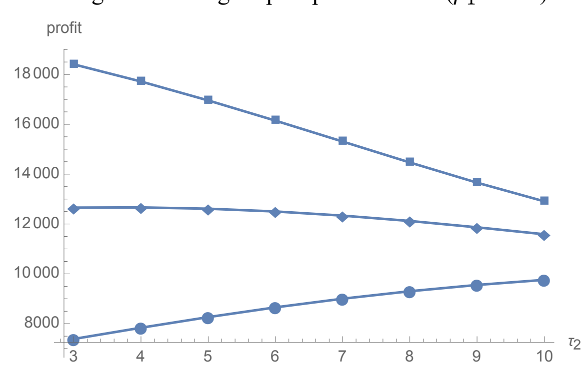
Old generation big market size ($m_1 = 1200$)



New generation high repeat purchase ($\beta_2 = 0.5$)



Old generation high repeat purchase rate ($\beta_1 = 0.5$)



Appendix C: Giveaway Strategies for a New Technology Product

C.1 Supplementary Analysis

We provide all the proofs in Section C.2.

Theorem C.1. The initial value problem (4.14) and (4.15) with $Y_l(t_0) = l_1$, $0 \leq l_1 < \delta n$, $Y_h(t_0) = l_2$, $0 \leq l_2 < n$, $t_0 \geq 0$, has a unique solution $Y_h(t)$, $Y_l(t)$ defined on $[t_0, \infty)$ where $Y_h(t)$ and $Y_l(t)$ are monotonically increasing to n and δn respectively.

Proposition C.1. $Y_h(t)$ and $Y_l(t)$ (derived from (4.14) and (4.15)) are proportional to n.

Proposition C.2. $Y_h(t)$ and $Y_l(t)$, $t > 0$ (derived from (4.14) and (4.15)) are increasing in $p_h, q_h, q_{lh}, p_l, q_l, q_{hl}$, and δ.

Proposition C.3.

(i) If $q > \frac{m}{m-(n_a+n_b)}\left(p + \frac{n_a+n_b}{m}(\lambda_h + \lambda_l \delta)\right)$, then the adoption rate $y(t)$ peaks.

(ii) $Y(t)$, $t > \tau$ is increasing in p, q, λ_h, and λ_l.

Proposition C.4. Suppose $\bar{Y}(t)$ corresponding to $\bar{Y}_h^a(t)$, $\bar{Y}_h^a(t) > Y_h^a(t)$, $t \in (\tau, T)$, or $\bar{Y}_l^a(t)$, $\bar{Y}_l^a(t) > Y_l^a(t)$, $t \in (\tau, T)$, or both. Then, $\bar{Y}(t) \geq Y(t)$, for all $t \in [\tau, T]$.

Proposition C.5. Consider profit function (4.26) when $Y_h^a(t)$ and $Y_l^a(t)$ are proportional to n_a and $Y_h^b(t)$ is proportional to n_b. Let the profit optimization problem $\bar{\Pi}^* = \bar{\Pi}(\bar{n}_a^*, \bar{n}_b^*) = \max_{n_a \in [0, \bar{n}_M^a], n_b \in [0, \bar{n}_M^b]} \bar{\Pi}(n_a, n_b)$ corresponding to $\bar{m} = rm$, $\bar{n}_M^a = rn_M^a$, and $\bar{n}_M^b = rn_M^b$, $r > 0$. Then, $\bar{\Pi}^* = r\Pi^*$, with $\bar{n}_a^* = rn_a^*$ and $\bar{n}_b^* = rn_b^*$, where $\Pi^* = \Pi(n_a^*, n_b^*) = \max_{n_a \in [0, n_M^a], n_b \in [0, n_M^b]} \Pi(n_a, n_b)$.

Proposition C.6. Let $\bar{\Pi}(n_a, n_b)$ and $\Pi(n_a, n_b)$ where $n_a \leq n_M^a$, $n_b \leq n_M^b$, be profit functions, given by (4.26), corresponding to $\bar{p}$, $\bar{q}$, $\bar{\lambda}_h$, $\bar{\lambda}_l$, and p, q, λ_h, λ_l respectively where, $\bar{q} > \frac{m}{m-n_M^a-n_M^b}\left(\bar{p} + \frac{n_M^a+n_M^b}{m}\left(\bar{\lambda}_h + \delta\bar{\lambda}_l\right)\right)$. Suppose either one of $\bar{p} \geq p, \bar{q} \geq q$, $\bar{\lambda}_h \geq \lambda_h$, and $\bar{\lambda}_l \geq \lambda_l$ holds with at least one strict inequality. Then

(i) If the planning horizon is sufficiently short ($T \leq \tilde{t}$), then $\overline{\Pi}(n_a, n_b) > \Pi(n_a, n_b)$ for all $n_a \leq n_M^a$, $n_b \leq n_M^b$, $(n_a, n_b) \neq (0,0)$, and $\overline{\Pi}(0,0) > \Pi(0,0)$ if either of $\bar{p} > p$ or $\bar{q} > q$ holds, otherwise $\overline{\Pi}(0,0) = \Pi(0,0)$. Thus, $\overline{\Pi}^* \geq \Pi^*$.

(ii) For $\beta > 0$, the planning horizon in part (i) can be extended to T_β, $T_\beta > T$ where $y_h^a(t) = y_l^a(t) = 0$ for $t \geq T$ with the profit differences ($\overline{\Pi}(n_a, n_b) - \Pi(n_a, n_b) > 0$) being larger than those in part (i) for the planning horizon T. For $\bar{\beta}$, $\bar{\beta} > \beta$, there exists $T_{\bar{\beta}}$ such that $T_{\bar{\beta}} > T_\beta$.

(iii) Let the planning horizon in part (i) be extended to T_L, $T_L > T_\beta$, where $y_h^a(t) = y_l^a(t) = 0$ for $t \geq T$. If $\pi(t)$ be decreasing on $[T_\beta, T_L]$ with high declining rate, then $\overline{\Pi}(n_a, n_b) > \Pi(n_a, n_b)$ for all $n_a \leq n_M^a$, $n_b \leq n_M^b$, $(n_a, n_b) \neq (0,0)$, and $\overline{\Pi}(0,0) > \Pi(0,0)$ if either of $\bar{p} > p$ or $\bar{q} > q$ holds, otherwise $\overline{\Pi}(0,0) = \Pi(0,0)$. Thus, $\overline{\Pi}^* \geq \Pi^*$.

Proposition C.7. Suppose $\overline{\Pi}(n_a, n_b)$ and $\Pi(n_a, n_b)$ where $n_a \leq n_M^a$, $n_b \leq n_M^b$ be profit functions given by (4.26), corresponding to $\bar{\delta}$ and δ respectively where $\bar{\delta} > \delta > 0$ and $q > \frac{m}{m-n_M^a-n_M^b}\left(p + \frac{n_M^a+n_M^b}{m}\left(\lambda_h + \bar{\delta}\lambda_l\right)\right)$.

(i) Let $Y_l(t)$ be strictly increasing in δ, $\pi(t)$ sufficiently high compared to c_0 and $c(t)$ on $[\tau, T]$ or $c(t) = c_0 = 0$, and the planning horizon be sufficiently short ($T \leq \tilde{t}$). Then $\overline{\Pi}(n_a, n_b) > \Pi(n_a, n_b)$ for all $n_a \leq n_M^a$, $n_b \leq n_M^b$, $(n_a, n_b) \neq (0,0)$ and $\overline{\Pi}(0,0) = \Pi(0,0)$, thus $\overline{\Pi}^* \geq \Pi^*$.

(ii) For $\beta > 0$, the planning horizon in part (i) can be extended to T_β, $T_\beta > T$ where $y_h^a(t) = y_l^a(t) = \bar{y}_l^a(t) = 0$ for $t \geq T$ with the profit differences ($\overline{\Pi}(n_a, n_b) - \Pi(n_a, n_b) > 0$) being larger than those in part (i) for the planning horizon T. For $\bar{\beta}$, $\bar{\beta} > \beta$, there exists $T_{\bar{\beta}}$ such that $T_{\bar{\beta}} > T_\beta$.

In Proposition C.7 (ii), $\bar{y}_l^a(t)$ denotes free product distribution to low-valuation free users after product launch when the low-valuation to high-valuation ratio is set to $\bar{\delta}$.

Proposition C.8. Suppose $q > \frac{m}{m-n_M^a-n_M^b}\left(p + \frac{n_M^a+n_M^b}{m}(\lambda_h + \delta\lambda_l)\right)$. Let $Y_l^a(t) < \bar{Y}_l^a(t)$, $t \in (\tau, T)$, with $\bar{\Pi}(n_a, n_b)$ and $\Pi(n_a, n_b)$ where $n_a \leq n_M^a$, $n_b \leq n_M^b$ be profit functions given by (4.26) corresponding to $\bar{Y}_l^a(t)$ and $Y_l^a(t)$ respectively.

(i) If $\pi(t)$ is sufficiently higher than $c(t)$ on $[\tau, T]$ or $c(t) = 0$ where the planning horizon is sufficiently short ($T \leq \tilde{t}$), then $\bar{\Pi}(n_a, n_b) > \Pi(n_a, n_b)$ for all $0 < n_a \leq n_M^a$, $n_b \leq n_M^b$, and $\bar{\Pi}(0, n_b) = \Pi(0, n_b)$, $n_b \leq n_M^b$, thus $\bar{\Pi}^* \geq \Pi^*$.

(ii) Let $\beta > 0$ and $y_h^a(t) = y_l^a(t) = \bar{y}_l^a(t) = 0$ for $t \geq T$. Then, the planning horizon in part (i) can be extended to T_β, $T_\beta > T$ with the profit differences ($\bar{\Pi}(n_a, n_b) - \Pi(n_a, n_b) > 0$) being larger than those in part (i) for the planning horizon T. For $\bar{\beta}$, $\bar{\beta} > \beta$, there exists $T_{\bar{\beta}}$ such that $T_{\bar{\beta}} > T_\beta$.

C.2 Proofs

We begin by presenting a theorem and a lemma that will be used in the subsequent proofs.

Theorem C.2. Let f be a vector function defined on $D \subseteq \mathbb{R}^{n+1}$ open. Let f and $\frac{\partial f}{\partial y_i}$, $i = 1, \dots, n$, be continuous in D. Then given any point $(t_0, \eta) \in D$, there exists a unique solution ϕ of the system

$$y' = f(t, y),$$

satisfying the initial condition $\phi(t_0) = \eta$ on an interval I containing t_0.

Proof (Brauer & Nohel, 1989).

Lemma C.1. Let $[t_0, t_L)$ be the maximal interval of existence of solution of the initial value problem (4.9) with $Y(t_0) = l$, $0 \leq t_0$, $0 \leq l < m - n$. If $t_L < \infty$, then given any compact set K, there exists a $t \in [t_0, t_L)$ such that $Y(t) \notin K$.

Proof Suppose $t_L < \infty$. Let compact set K be such that $Y(t) \in K$, for all $t \in [t_0, t_L)$. Define

$$\psi(Y, Y_h, Y_l) = \left(p + \frac{q}{m}Y + \frac{\lambda_h}{m}Y_h + \frac{\lambda_l}{m}Y_l\right)(m - n - Y).$$

Given that ψ is continuous with respect to its arguments and by the fact that $0 \leq Y_h \leq n$ and $0 \leq Y_l \leq \delta n$ there exists $M > 0$ such that $\psi(Y, Y_h, Y_l) \leq M$ for all $Y \in K$. We show that there exists Y_L such that $\lim_{t \to t_L^-} Y(t) = Y_L$. Let $t_0 < t_1 < t_2 < t_L$, then

$$|Y(t_1) - Y(t_2)| \leq \int_{t_1}^{t_2} |\psi(Y(s), Y_h(s), Y_l(s))| \, ds \leq M|t_2 - t_1|.$$

We observe that $\{Y(t_n)\}$, $t_n \to t_L^-$, is a Cauchy sequence. Thus, by completeness of $\mathbb{R}$ the existence of Y_L is guaranteed. Moreover, since K is compact we see $Y_L \in K$. Now define $X(t)$ as

$$X(t) = \begin{cases} Y(t) & t \in [t_0, t_L), \\ Y_L & t = t_L. \end{cases}$$

Then,

$$X(t) = l + \int_{t_0}^{t} \psi(X(s), Y_h(s), Y_l(s)) ds,$$

which means $\frac{dX}{dt}(t_L) = \psi(X(t_L), Y_h(t_L), Y_l(t_L))$. This means $X(t)$ is a solution for the problem (4.9) on $[t_0, t_L]$. By Theorem C.2 the equation $\frac{dY}{dt} = \psi(Y(t), Y_h(t), Y_l(t))$ with the initial condition $Y(t_L) = Y_L$ has a unique solution $\bar{Y}(t)$ on $(t_L - \delta, t_L + \delta)$ for some $\delta > 0$. But $\bar{Y}(t) = X(t)$ on $(t_L - \delta, t_L)$ and $\bar{Y}(t_L) = X(t_L) = Y_L$. Now let

$$U(t) = \begin{cases} X(t) & t \in [t_0, t_L], \\ \bar{Y}(t) & t \in [t_L, t_L + \delta). \end{cases}$$

Then $U(t)$ is a solution for (4.9) on $[t_0, t_L + \delta)$ but this contradicts the fact that $[t_0, t_L)$ is the maximal interval of existence. □

Proof of Theorem 4.1

Consider the initial value problem (4.9) with $Y(t_0) = l$, $0 \leq t_0$, $0 \leq l < m - n$, on maximal interval of existence of solution ($t \in [t_0, t_L)$). We see that $\frac{dY}{dt}(t_0) = (p + \frac{q}{m} l + \frac{\lambda_h}{m} Y_h(t_0) + \frac{\lambda_l}{m} Y_l(t_0))(m - n - l) > 0$. We show that there exists $t_1 > t_0$ such that $Y(t) > l$, for $t \in (t_0, t_1]$. Suppose for all $t_1^k > t_0$, $k \in \mathbb{N}$, there exists $t_k \in (t_0, t_1^k]$ such that $Y(t_k) \leq l$. Then

$$\frac{Y(t_k) - Y(t_0)}{t_k - t_0} \leq 0.$$

Consider t_1^k is a monotonically decreasing sequence approaching t_0. Then, it can be observed that

$$\lim_{k\to\infty} \frac{Y(t_k)-Y(t_0)}{t_k-t_0} = \frac{d}{dt}Y(t_0),$$

and therefore $\frac{d}{dt}Y(t_0) \leq 0$ which is a contradiction. Now suppose there exists $\hat{t} \in [t_0, t_L)$, $\hat{t} > t_1$ such that $Y(\hat{t}) < l$. Then by continuity of Y and the fact that $Y(t_1) > l$ there exists $t^* \in (t_1, \hat{t})$ such that $Y(t^*) = l$. We consider $\hat{t}$ small enough to have $t^* \in (t_1, \hat{t})$ unique. Therefore, $Y(t) < l$, $t \in (t^*, \hat{t}]$. On the other hand $\frac{dY}{dt}(t^*) = (p + \frac{q}{m}Y(t^*) + \frac{\lambda_h}{m}Y_h(t^*) + \frac{\lambda_l}{m}Y_l(t^*))(m - n - Y(t^*)) > 0$. Similar to the above discussion we can show that there exists $\eta > 0$ such that $Y(t) > l$, $t \in (t^*, t^* + \eta]$ which is a contradiction. This means $Y(t) \geq l$, $t \in [t_0, t_L)$ and $\frac{dY}{dt}(t) > 0$ as long as $Y(t) < m - n$. This means $Y(t)$ is increasing on $[t_0, t_L)$ as long as $Y(t) < m - n$. However, with increase in the value of $Y(t)$ and approaching $m - n$, the increasing rate $\frac{dY}{dt}(t)$ approaches to zero which means $Y(t)$ is bounded from above to the asymptote $m - n$. Hence, $Y(t) \in [l, (m - n)]$, for $t \in [t_0, t_L)$, which by Lemma C.1 means $t_L = \infty$. □

Lemma C.2. Let $Y(t, t_0, l)$ be the solution for (4.9) where $Y(t_0, t_0, l) = l$, $t_0, l \geq 0$. Then for $t > t_0$, $Y(t, t_0, l) \to Y(t, 0,0)$, when $t_0, l \to 0$.

Proof Consider $\psi(Y, Y_h, Y_l)$ as defined in the proof of Lemma C.1. We have

$$Y(t, t_0, l) = l + \int_{t_0}^{t} \psi(Y(s, t_0, l), Y_h(s), Y_l(s))ds,$$

$$Y(t, 0,0) = \int_{0}^{t} \psi(Y(s, 0,0), Y_h(s), Y_l(s))ds.$$

By the fact that $0 \leq Y_h \leq n$, $0 \leq Y_l \leq \delta n$, and $0 \leq Y \leq m - n$, there exists $M, K > 0$ such that $\psi(Y, Y_h, Y_l) \leq M$ and $\frac{\partial}{\partial Y}\psi(Y, Y_h, Y_l) \leq K$. Therefore

$$|Y(t, t_0, l) - Y(t, 0,0)| \leq l + \int_{0}^{t_0} |\psi(Y(s, 0,0), Y_h(s), Y_l(s))|\, ds +$$

$$\int_{t_0}^{t} \left|\psi\big(Y(s, t_0, l), Y_h(s), Y_l(s)\big) - \psi\big(Y(s, 0,0), Y_h(s), Y_l(s)\big)\right| ds \leq$$

$$l + t_0 M + K \int_{t_0}^{t} |Y(s, t_0, l) - Y(s, 0,0)|\, ds.$$

Using Gronwall inequality (Brauer & Nohel, 1989, p.31, Theorem 1.4) we get

$$|Y(t, t_0, l) - Y(t, 0,0)| \leq (l + t_0 M)e^{K(t-t_0)}.$$

Thus, we can observe that $|Y(t,t_0,l) - Y(t,0,0)| \to 0$, when $t_0, l \to 0$. □

Proof of Proposition 4.1

Let $\bar{Y}_h, \bar{Y}_l$ and $\bar{Y}$ corresponding to $\bar{m} = rm$ and $\bar{n} = rn$. Then $\bar{Y}_h = rY_h$ and $\bar{Y}_l = rY_l$ where Y_h and Y_l are corresponding to n. It can be observed that

$$ry(t) = \left(p + \frac{q}{rm} rY(t) + \frac{\lambda_h}{rm} rY_h(t) + \frac{\lambda_l}{rm} rY_l(t))\right)\left(r(m-n) - rY(t)\right).$$

By Theorem 4.1 we conclude that $\bar{Y}(t) = rY(t)$ and this completes the proof. □

Proof of Proposition 4.2

For $n = 0$, (4.9) reduces to the standard Bass equation (4.2) for which both (i) and (ii) hold. We show the results for $n > 0$.

(i) Consider $\psi(Y, Y_h, Y_l)$ as defined in the proof of Lemma C.1. Then

$$\frac{\partial \psi}{\partial Y}(Y, Y_h, Y_l) = q\frac{m-n}{m} - 2\frac{q}{m}Y - \left(p + \frac{\lambda_h}{m}Y_h + \frac{\lambda_l}{m}Y_l\right).$$

We see that for

$$Y \le \frac{m}{2q}\left(q\left(\frac{m-n}{m}\right) - \left(p + \frac{\lambda_h}{m}Y_h + \frac{\lambda_l}{m}Y_l\right)\right), \tag{C.1}$$

we have $\frac{\partial \psi}{\partial Y}(Y, Y_h, Y_l) \ge 0$. On the other hand, $q > \frac{m}{m-n}(p + \frac{n}{m}(\lambda_h + \lambda_l \delta))$ implies that there exists a time period $[0, \theta]$ for which the inequality (C.1) is satisfied. On the other hand, $\frac{\partial \psi}{\partial Y_h}(Y, Y_h, Y_l) = \frac{\lambda_h}{m}\left((m-n) - Y\right) > 0$, $\frac{\partial \psi}{\partial Y_l}(Y, Y_h, Y_l) = \frac{\lambda_l}{m}\left((m-n) - Y\right) > 0$. By Theorem 4.1, $Y(t)$ converges to the asymptote $(m-n)$ which means $\frac{dY}{dt}(t)$ approaches zero. Thus, we conclude that $y(t) = \frac{dY}{dt}(t)$ first increases in the time period $[0, \theta]$ and reaches a peak before approaching zero.

(ii) Suppose $\bar{Y}(t, l, t_0)$ and $Y(t, l, t_0)$ derived from (4.9) corresponding to $\bar{p}$, $\bar{q}$, $\bar{\lambda}_h$, $\bar{\lambda}_l$ and p, q, λ_h, λ_l, respectively, with initial conditions $\bar{Y}(t_0, l, t_0) = Y(t_0, l, t_0) = l$, $0 < l < m - n$, $0 < t_0 < 1$, where $\bar{p} \ge p$, $\bar{q} \ge q$, $\bar{\lambda}_h \ge \lambda_h$, $\bar{\lambda}_l \ge \lambda_l$, and at least one of the inequalities is strict. Theorem 4.1 guarantees the existence of such solutions on $[t_0, \infty)$. First, we show that $\bar{Y}(t, l, t_0) \ge Y(t, l, t_0)$ for $t \ge t_0$. It can be observed that

$$\left(\bar{p} + \frac{\bar{q}}{m} l + \frac{\bar{\lambda}_h}{m} Y_h(t_0) + \frac{\bar{\lambda}_l}{m} Y_l(t_0)\right)\left((m-n) - l\right) >$$

$$\left(p + \frac{q}{m} l + \frac{\lambda_h}{m} Y_h(t_0) + \frac{\lambda_l}{m} Y_l(t_0)\right) \left((m-n) - l\right), \tag{C.2}$$

which means $\frac{d}{dt}\bar{Y}(t_0, l, t_0) > \frac{d}{dt}Y(t_0, l, t_0)$. Now we show that there exists $t_1 > t_0$ such that for all $t \in (t_0, t_1]$, $\bar{Y}(t, l, t_0) > Y(t, l, t_0)$. Suppose for all $t_1^k > t_0$, $k \in \mathbb{N}$, there exists $t_k \in (t_0, t_1^k]$ such that $\bar{Y}(t_k, l, t_0) \leq Y(t_k, l, t_0)$. Then

$$\frac{\bar{Y}(t_k,l,t_0) - \bar{Y}(t_0,l,t_0)}{t_k - t_0} \leq \frac{Y(t_k,l,t_0) - Y(t_0,l,t_0)}{t_k - t_0}.$$

Consider t_1^k is a monotonically decreasing sequence approaching to t_0. Then, it can be observed that

$$\lim_{k\to\infty} \frac{\bar{Y}(t_k,l,t_0) - \bar{Y}(t_0,l,t_0)}{t_k - t_0} = \frac{d}{dt}\bar{Y}(t_0, l, t_0),$$

$$\lim_{k\to\infty} \frac{Y(t_k,l,t_0) - Y(t_0,l,t_0)}{t_k - t_0} = \frac{d}{dt}Y(t_0, l, t_0),$$

and therefore $\frac{d}{dt}\bar{Y}(t_0, l, t_0) \leq \frac{d}{dt}Y(t_0, l, t_0)$ which is a contradiction. Now suppose there exists $\hat{t} > t_1$ such that $\bar{Y}(\hat{t}, l, t_0) < Y(\hat{t}, l, t_0)$. Then by continuity of $\bar{Y}$ and Y and the fact that $\bar{Y}(t_1, l, t_0) > Y(t_1, l, t_0)$ there exists $t^* \in (t_1, \hat{t})$ such that $\bar{Y}(t^*, l, t_0) = Y(t^*, l, t_0)$. We consider $\hat{t}$ small enough to have $t^* \in (t_1, \hat{t})$ unique. Therefore, $\bar{Y}(t, l, t_0) < Y(t, l, t_0)$ for all $t \in (t^*, \hat{t}]$. It can be seen that

$$\left(\bar{p} + \frac{\bar{q}}{m}\bar{Y}(t^*, l, t_0) + \frac{\bar{\lambda}_h}{m} Y_h(t^*) + \frac{\bar{\lambda}_l}{m} Y_l(t^*)\right)\right) \left(m - n - \bar{Y}(t^*, l, t_0)\right) >$$

$$\left(p + \frac{q}{m}Y(t^*, l, t_0) + \frac{\lambda_h}{m} Y_h(t^*) + \frac{\lambda_l}{m} Y_l(t^*)\right)\right) \left(m - n - Y(t^*, l, t_0)\right), \tag{C.3}$$

which means $\frac{d}{dt}\bar{Y}(t^*, l, t_0) > \frac{d}{dt}Y(t^*, l, t_0)$. Similar to the above argument we can demonstrate that there exists an $\eta > 0$ such that $\bar{Y}(t, l, t_0) > Y(t, l, t_0)$ for all $t \in (t^*, t^* + \eta]$ which is a contradiction.

So far, we have shown that $\bar{Y}(t, l, t_0) \geq Y(t, l, t_0)$ for $t \geq t_0$. Now let $t > 0$ and set t_0 such that $t > t_0$. By Lemma C.2, $\bar{Y}(t, l, t_0)$ and $Y(t, l, t_0)$ tend to $\bar{Y}(t, 0,0)$ and $Y(t, 0,0)$ respectively as l and t_0 tend to zero. Since $\bar{Y}(t, l, t_0) \geq Y(t, l, t_0)$ it can be concluded that $\bar{Y}(t, 0,0) \geq Y(t, 0,0)$ for $t \geq 0$ and this completes the proof. □

Proof of Proposition 4.3

Suppose $\bar{Y}(t,l,t_0)$ and $Y(t,l,t_0)$ derived from (4.9), with initial conditions $\bar{Y}(t_0,l,t_0) = Y(t_0,l,t_0) = l$, $0 < l < 1$, $0 < t_0 < 1$, corresponding to $\bar{Y}_h(t)$, $\bar{Y}_l(t)$ and $Y_h(t)$, $Y_l(t)$ respectively. Theorem 4.1 guarantees the existence of such solutions on $[t_0,\infty)$. The rest of the proof is similar to the proof of Proposition 4.2 (ii) in which for (C.2) we have

$$\left(p + \frac{q}{m}l + \frac{\lambda_h}{m}\bar{Y}_h(t_0) + \frac{\lambda_l}{m}\bar{Y}_l(t_0)\right)\big((m-n)-l\big) >$$

$$\left(p + \frac{q}{m}l + \frac{\lambda_h}{m}Y_h(t_0) + \frac{\lambda_l}{m}Y_l(t_0)\right)\big((m-n)-l\big).$$

Similar to Proposition 4.2 (ii), we suppose there exists $\hat{t} \in (t_1,T]$ such that $\bar{Y}(t,l,t_0) < Y(t,l,t_0)$ and we find $t^* \in (t_1,\hat{t})$ such that $\bar{Y}(t^*,l,t_0) = Y(t^*,l,t_0)$. We proceed similar to the Proposition 4.2 (ii) in which for (C.3) we have

$$\left(p + \frac{q}{m}\bar{Y}(t^*,l,t_0) + \frac{\lambda_h}{m}\bar{Y}_h(t^*) + \frac{\lambda_l}{m}\bar{Y}_l(t^*))\right)\big(m-n-\bar{Y}(t^*,l,t_0)\big) >$$

$$\left(p + \frac{q}{m}Y(t^*,l,t_0) + \frac{\lambda_h}{m}Y_h(t^*) + \frac{\lambda_l}{m}Y_l(t^*))\right)\big(m-n-Y(t^*,l,t_0)\big).$$

□

Lemma C.3 is needed in the proof of Theorem C.1.

Lemma C.3. Let $[t_0,t_L)$ be the maximal interval of existence of solution of the initial value problem (4.14) and (4.15) with $Y_l(t_0) = l_1$, $0 \le l_1 < \delta n$, $Y_h(t_0) = l_2$, $0 \le l_2 < \delta n$, $t_0 \ge 0$. If a compact set K be such that $\big(Y_l(t), Y_h(t)\big) \in K$, $t \in [t_0,t_L)$, then $t_L = \infty$.
Proof (Perko, 2001, p. 91, Corollary 2).

Proof of Theorem C.1

With a procedure similar to the one presented in the proof of Theorem 4.1 we show that there exists t_1^h and t_1^l, $t_1^h, t_1^l > t_0$ such that $Y_l(t) > l_1$, for $t \in (t_0,t_1^l]$ and $Y_h(t) > l_2$, for $t \in (t_0,t_1^h]$. Now suppose there exists $\hat{t} \in [t_0,t_L)$, $\hat{t} > t_1^h$ such that $Y_h(\hat{t}) < l_2$. Then by continuity of Y_h and the fact that $Y_h\big(t_1^h\big) > l_2$ there exists $t_h^* \in (t_1^h,\hat{t})$ such that $Y_h(t_h^*) = l_2$. We consider $\hat{t}$ small enough to have $t_h^* \in (t_1^h,\hat{t})$ unique. Suppose $Y_l(t_h^*) < l_1$. Then by continuity of Y_l and the fact that $Y_l\big(t_1^l\big) > l_1$ there exists $t_l^* \in (t_1^l,t_h^*)$ such that $Y_l(t_l^*) = l_1$. Let t_l^*, be the smallest value where $Y_l(t_l^*) = l_1$. But $Y_h(t_l^*) > l_2$. Therefore, $\frac{dY_l}{dt}(t_l^*) > 0$. Similar to the proof of Theorem 4.1 we can show that there exists $\eta >$

0 such that $Y_l(t) > l_1$, $t \in (t_l^*, t_l^* + \eta]$ which is a contradiction. Suppose $Y_l(t_h^*) \geq l_1$. Then, $\frac{dY_h}{dt}(t_h^*) > 0$ and again similar to the proof of Theorem 4.1 we can show that there exists $\eta > 0$ such that $Y_h(t) > l_2$, $t \in (t_h^*, t_h^* + \eta]$ which is a contradiction. In a similar fashion we can show that there is no $\hat{t} \in [t_0, t_L)$, $\hat{t} > t_1^l$ such that $Y_l(\hat{t}) < l_1$. Thus, $Y_l(t) \geq l_1$ and $Y_h(t) \geq l_2$, for $t \in [t_0, t_L)$. This means $\frac{dY_h}{dt}(t), \frac{dY_l}{dt}(t) > 0$, $t \in [t_0, t_L)$, as long as $Y_h(t) < n$ and $Y_l(t) < \delta n$. Thus $Y_h(t)$ and $Y_l(t)$ are increasing on $[t_0, t_L)$ as long as $Y_h(t) < n$ and $Y_l(t) < \delta n$. However, as $Y_h(t)$ approaches n and $Y_l(t)$ approaches δn, the increasing rates $\frac{dY_h}{dt}(t)$ and $\frac{dY_l}{dt}(t)$ approach zero which means $Y_h(t)$ and $Y_l(t)$ are bounded from above to asymptotes n and δn respectively. Hence, $\left(Y_l(t), Y_h(t)\right) \in [l_1, \delta n] \times [l_2, n]$, for $t \in [t_0, t_L)$, which by Lemma C.3 means $t_L = \infty$. ☐

Proof of Proposition C.1

Similar to the proof of Proposition 4.1. ☐

Lemma C.4 is needed in the proof of Proposition C.2.

Lemma C.4. Let $Y_h(t, t_0, l)$, $Y_l(t, t_0, l)$ be the solution for (4.14), (4.15) where $Y_h(t_0, t_0, l) = Y_l(t_0, t_0, l) = l$, $t_0, l \geq 0$. Then for $t > t_0$, $Y_h(t, t_0, l) \to Y_h(t, 0,0)$, and $Y_l(t, t_0, l) \to Y_l(t, 0,0)$, when $t_0, l \to 0$.

Proof Similar to the proof of Lemma C.2. ☐

Proof of Proposition C.2

Suppose $\bar{Y}_h(t, l, t_0), \bar{Y}_l(t, l, t_0)$ and $Y_h(t, l, t_0), Y_l(t, l, t_0)$ derived from (4.14) and (4.15) with initial conditions $\bar{Y}_h(t_0, l, t_0) = Y_h(t_0, l, t_0) = l$, $\bar{Y}_l(t_0, l, t_0) = Y_l(t_0, l, t_0) = l$, $0 < l < \min\{n, \delta n\}$, $0 < t_0 < 1$, corresponding to $\bar{p}_h$, $\bar{q}_h$, $\bar{p}_l$, $\bar{q}_l$, $\bar{\lambda}_h$, $\bar{\lambda}_l$, $\bar{\delta}$ and p_h, q_h, p_l, q_l, λ_h, λ_l, δ where $\bar{p}_h \geq p_h$, $\bar{q}_h \geq q_h$, $\bar{p}_l \geq p_l$, $\bar{q}_l \geq q_l$, $\bar{q}_{lh} \geq q_{lh}$, $\bar{q}_{hl} \geq q_{hl}$, $\bar{\delta} \geq \delta$ and at least one of the inequalities are strict. Theorem C.1 guarantees the existence of such solutions on $[t_0, \infty)$. We show that $\bar{Y}_h(t, l, t_0) \geq Y_h(t, l, t_0)$ and $\bar{Y}_l(t, l, t_0) \geq Y_l(t, l, t_0)$ for $t \geq t_0$. It can be observed that

$$\left(\bar{p}_h + \frac{\bar{q}_h}{n} l + \frac{\bar{q}_{lh}}{n} l\right)(n - l) \geq \left(p_h + \frac{q_h}{n} l + \frac{q_{lh}}{n} l\right)(n - l),$$

$$\left(\bar{p}_l + \frac{\bar{q}_l}{\bar{\delta}n} l + \frac{\bar{q}_{hl}}{\bar{\delta}n} l\right)\left(\bar{\delta}n - l\right) \geq \left(p_l + \frac{q_l}{\delta n} l + \frac{\bar{q}_{hl}}{\delta n} l\right)(\delta n - l),$$

which means $\frac{d}{dt}\bar{Y}_h(t_0, l, t_0) \geq \frac{d}{dt}Y_h(t_0, l, t_0)$ and $\frac{d}{dt}\bar{Y}_l(t_0, l, t_0) \geq \frac{d}{dt}Y_l(t_0, l, t_0)$, with at least one of the inequalities being strict. Without loss of generality, we assume the first inequality is strict ($\bar{p}_l = p_l$, $\bar{q}_l = q_l$, $\bar{q}_{hl} = q_{hl}$, $\bar{\delta} = \delta$). Similar to the proof of Proposition 4.2 (ii) we can show that there exists $t_1^h > t_0$ such that for all $t \in (t_0, t_1^h]$, $\bar{Y}_h(t, l, t_0) > Y_h(t, l, t_0)$. Let $\bar{Y}_l(t, l, t_1)$ and $Y_l(t, l, t_1)$ be the solution for (4.15) corresponding to $\bar{Y}_h(t, l, t_0)$, and $Y_h(t, l, t_0)$, $t \in (t_0, t_1^h]$, with t_1, $t_0 < t_1 < t_1^h$ and $\bar{Y}_l(t_1, l, t_1) = Y_l(t_1, l, t_1) = l$. Similar to Theorem 4.1 we can show that solutions $\bar{Y}_l(t, l, t_1)$ and $Y_l(t, l, t_1)$ are increasing to δn. Then

$$\left(p_l + \frac{q_l}{\delta n} l + \frac{q_{hl}}{\delta n}\bar{Y}_h(t_1, l, t_0)\right)(\delta n - l) > \left(p_l + \frac{q_l}{\delta n} l + \frac{q_{hl}}{\delta n} Y_h(t_1, l, t_0)\right)(\delta n - l),$$

which means, $\frac{d}{dt}\bar{Y}_l(t_1, l, t_1) > \frac{d}{dt}Y_l(t_1, l, t_1)$. Similar to the proof of Proposition 4.2 (ii) we can show that there exists $t_1^l > t_1$ such that for all $t \in (t_1, t_1^l]$, $\bar{Y}_l(t, l, t_1) > Y_l(t, l, t_1)$. Suppose there exists $\hat{t}_l$, $t_1 < \hat{t}_l \leq t_1^h$, such that $\bar{Y}_l(\hat{t}_l, l, t_1) < Y_l(\hat{t}_l, l, t_1)$. Then by continuity of $\bar{Y}_l$ and Y_l and the fact that for all $t \in (t_1, t_1^l]$, $\bar{Y}_l(t, l, t_1) > Y_l(t, l, t_1)$, there exists $t_l^* < \hat{t}_l$ such that $\bar{Y}_l(t_l^*, l, t_1) = Y_l(t_l^*, l, t_1)$. Let t_l^*, be the smallest value where $\bar{Y}_l(t_l^*, l, t_1) = Y_l(t_l^*, l, t_1)$. Then $\frac{d}{dt}\bar{Y}_l(t_l^*, l, t_1) > \frac{d}{dt}Y_l(t_l^*, l, t_1)$ and similar to the proof of Proposition 4.2 (ii) we can show that there exists an $\eta > 0$ such that $\bar{Y}_l(t, l, t_1) > Y_l(t, l, t_1)$, for $t \in (t_l^*, t_l^* + \eta]$ which is a contradiction. Thus, $\bar{Y}_l(t, l, t_1) \geq Y_l(t, l, t_1)$ for all $t \in (t_1, t_1^h]$. For t, $t_0 < t \leq t_1^h$, let t_1, $t_0 < t_1 < t$. Then, we have $\bar{Y}_l(t, l, t_1) \geq Y_l(t, l, t_1)$. Similar to Lemma C.2, we can show that $\bar{Y}_l(t, l, t_1)$ and $Y_l(t, l, t_1)$ tend to $\bar{Y}_l(t, l, t_0)$ and $Y_l(t, l, t_0)$ respectively, as t_1 tends to t_0. Thus, $\bar{Y}_l(t, l, t_0) \geq Y_l(t, l, t_0)$, $t \in (t_0, t_1^h]$.

Suppose there exists $\hat{t}_h$, $\hat{t}_h > t_1^h$, and $\hat{t}_l$, $\hat{t}_l > t_1^h$ such that $\bar{Y}_h(\hat{t}_h, l, t_0) < Y_h(\hat{t}_h, l, t_0)$ and $\bar{Y}_l(\hat{t}_l, l, t_0) < Y_l(\hat{t}_l, l, t_0)$. Then by continuity of $\bar{Y}_h$, $\bar{Y}_l$, Y_h, and Y_l there exist $t_h^* \in (t_1^h, \hat{t}_h)$ and $t_l^* \in [t_1^h, \hat{t}_l)$ such that $\bar{Y}_h(t_h^*, l, t_0) = Y_h(t_h^*, l, t_0)$ and $\bar{Y}_l(t_l^*, l, t_0) = Y_l(t_l^*, l, t_0)$. We consider $\hat{t}_h$ and $\hat{t}_l$ small enough to have $t_h^* \in (t_1^h, \hat{t}_h)$ and $t_l^* \in [t_1^h, \hat{t}_l)$ unique. If $t_h^* \leq t_l^*$, then $\bar{Y}_l(t_h^*, l, t_0) \geq Y_l(t_h^*, l, t_0)$ and therefore

$$\left(\bar{p}_h + \frac{\bar{q}_h}{n}\bar{Y}_h(t_h^*, l, t_0) + \frac{\bar{q}_{lh}}{n}\bar{Y}_l(t_h^*, l, t_0)\right)\left(n - \bar{Y}_h(t_h^*, l, t_0)\right) > \left(p_h + \frac{q_h}{n}Y_h(t_h^*, l, t_0) + \frac{q_{lh}}{n}Y_l(t_h^*, l, t_0)\right)\left(n - Y_h(t_h^*, l, t_0)\right).$$

Thus, $\frac{d}{dt}\bar{Y}_h(t_h^*, l, t_0) > \frac{d}{dt}Y_h(t_h^*, l, t_0)$ and similar to the proof of Proposition 4.2 (ii) we can show that there exists an $\eta > 0$ such that $\bar{Y}_h(t, l, t_0) > Y_h(t, l, t_0)$, for $t \in (t_h^*, t_h^* + \eta]$ which is a contradiction. If $t_h^* > t_l^*$, then $\bar{Y}_h(t_l^*, l, t_0) > Y_h(t_l^*, l, t_0)$ and therefore

$$\left(p_l + \frac{q_l}{\delta n}\bar{Y}_l(t_l^*, l, t_0) + \frac{q_{hl}}{\delta n}\bar{Y}_h(t_l^*, l, t_0)\right)\left(\delta n - \bar{Y}_l(t_l^*, l, t_0)\right) > \left(p_l + \frac{q_l}{\delta n}Y_l(t_l^*, l, t_0) + \frac{q_{hl}}{\delta n}Y_h(t_l^*, l, t_0)\right)\left(\delta n - Y_l(t_l^*, l, t_0)\right).$$

Thus, $\frac{d}{dt}\bar{Y}_l(t_l^*, l, t_0) > \frac{d}{dt}Y_l(t_l^*, l, t_0)$ and similar to the proof of Proposition 4.2 (ii) it can be shown that there exists an $\eta > 0$ such that $\bar{Y}_l(t, l, t_0) > Y_l(t, l, t_0)$, for $t \in (t_l^*, t_l^* + \eta]$ which is a contradiction.

Suppose $\bar{Y}_h(t, l, t_0) \geq Y_h(t, l, t_0)$ for $t \geq t_0$ but there exists $\hat{t}_l$, $\hat{t}_l > t_1^h$ such that $\bar{Y}_l(\hat{t}_l, l, t_0) < Y_l(\hat{t}_l, l, t_0)$. Then by continuity of $\bar{Y}_l$ and Y_l there exist $t_l^* \in [t_1^h, \hat{t}_l)$ such that $\bar{Y}_l(t_l^*, l, t_0) = Y_l(t_l^*, l, t_0)$. We consider $\hat{t}_l$ small enough to have $t_l^* \in [t_1^h, \hat{t}_l)$ unique. If $\bar{Y}_h(t_l^*, l, t_0) = Y_h(t_l^*, l, t_0)$, then $\frac{d}{dt}\bar{Y}_h(t_l^*, l, t_0) = \frac{d}{dt}Y_h(t_l^*, l, t_0)$, since $\bar{Y}_h(t, l, t_0) \geq Y_h(t, l, t_0)$ for $t \geq t_0$. On the other hand

$$\left(\bar{p}_h + \frac{\bar{q}_h}{n}\bar{Y}_h(t_l^*, l, t_0) + \frac{\bar{q}_{lh}}{n}\bar{Y}_l(t_l^*, l, t_0)\right)\left(n - \bar{Y}_h(t_l^*, l, t_0)\right)$$
$$> \left(p_h + \frac{q_h}{n}Y_h(t_l^*, l, t_0) + \frac{q_{lh}}{n}Y_l(t_l^*, l, t_0)\right)\left(n - Y_h(t_l^*, l, t_0)\right).$$

Thus, $\frac{d}{dt}\bar{Y}_h(t_l^*, l, t_0) > \frac{d}{dt}Y_h(t_l^*, l, t_0)$ which is a contradiction. If $\bar{Y}_h(t_l^*, l, t_0) > Y_h(t_l^*, l, t_0)$, then

$$\left(p_l + \frac{q_l}{\delta n}\bar{Y}_l(t_l^*, l, t_0) + \frac{q_{hl}}{\delta n}\bar{Y}_h(t_l^*, l, t_0)\right)\left(\delta n - \bar{Y}_l(t_l^*, l, t_0)\right)$$
$$> \left(p_l + \frac{q_l}{\delta n}Y_l(t_l^*, l, t_0) + \frac{q_{hl}}{\delta n}Y_h(t_l^*, l, t_0)\right)\left(\delta n - Y_l(t_l^*, l, t_0)\right).$$

Thus, $\frac{d}{dt}\bar{Y}_l(t_l^*, l, t_0) > \frac{d}{dt}Y_l(t_l^*, l, t_0)$ and similar to the proof of Proposition 4.2 (ii) it can be shown that there exists an $\eta > 0$ such that $\bar{Y}_l(t, l, t_0) > Y_l(t, l, t_0)$, for $t \in (t_l^*, t_l^* + \eta]$ which is a contradiction.

Suppose $\bar{Y}_l(t, l, t_0) \geq Y_l(t, l, t_0)$ for $t \geq t_0$ but there exists $\hat{t}_h$, $\hat{t}_h > t_1^h$, such that $\bar{Y}_h(\hat{t}_h, l, t_0) < Y_h(\hat{t}_h, l, t_0)$. Then by continuity of $\bar{Y}_h$ and Y_h, there exist $t_h^* \in (t_1^h, \hat{t}_h)$ such that $\bar{Y}_h(t_h^*, l, t_0) = Y_h(t_h^*, l, t_0)$. We consider $\hat{t}_h$ small enough to have $t_h^* \in (t_1^h, \hat{t}_h)$ unique. Therefore

$$\left(\bar{p}_h + \frac{\bar{q}_h}{n}\bar{Y}_h(t_h^*, l, t_0) + \frac{\bar{q}_{lh}}{n}\bar{Y}_l(t_h^*, l, t_0)\right)\left(n - \bar{Y}_h(t_h^*, l, t_0)\right)$$
$$> \left(p_h + \frac{q_h}{n}Y_h(t_h^*, l, t_0) + \frac{q_{lh}}{n}Y_l(t_h^*, l, t_0)\right)\left(n - Y_h(t_h^*, l, t_0)\right).$$

Thus, $\frac{d}{dt}\bar{Y}_h(t_h^*, l, t_0) > \frac{d}{dt}Y_h(t_h^*, l, t_0)$ and similar to the proof of Proposition 4.2 (ii) we can show that there exists an $\eta > 0$ such that $\bar{Y}_h(t, l, t_0) > Y_h(t, l, t_0)$, for $t \in (t_h^*, t_h^* + \eta]$ which is a contradiction.

So far, we have shown that $\bar{Y}_h(t, l, t_0) \geq Y_h(t, l, t_0)$ and $\bar{Y}_l(t, l, t_0) \geq Y_l(t, l, t_0)$ for $t \geq t_0$. Now let $t > 0$ and set t_0 such that $t > t_0$. By Lemma C.4, we observe that

$$\bar{Y}_h(t, l, t_0) \to \bar{Y}_h(t, 0,0),$$
$$\bar{Y}_l(t, l, t_0) \to \bar{Y}_l(t, 0,0),$$
$$Y_h(t, l, t_0) \to Y_h(t, 0,0),$$
$$Y_l(t, l, t_0) \to Y_l(t, 0,0),$$

as l and t_0 tend to zero. Thus, $\bar{Y}_h(t, 0,0) \geq Y_h(t, 0,0)$ and $\bar{Y}_l(t, 0,0) \geq Y_l(t, 0,0)$ for $t \geq 0$, and this completes the proof. □

Proof of Proposition 4.4

We prove the proposition under the concurrent strategy. The proof under the before strategy is similar. Let $\hat{y}_h^*(t)$, $\hat{y}_l^*(t)$, and $\hat{y}^*(t)$ corresponding to $\bar{m}$ and rn^*, and $y_h^*(t)$, $y_l^*(t)$, and $y^*(t)$ corresponding to m and n^*. Then by Proposition 4.1, $\hat{y}^*(t) = ry^*(t)$. Moreover, $\hat{y}_h^*(t) = ry_h^*(t)$ and $\hat{y}_l^*(t) = ry_l^*(t)$. Hence,

$$\bar{\Pi}(rn^*) = \int_0^T (\pi(t)(I_0^{\beta}\hat{y}^*(t) + I_0^{\beta_h}\hat{y}_h^*(t) - \hat{y}_h^*(t)) - c(t)(\hat{y}_h^*(t) + \hat{y}_l^*(t)))dt =$$
$$r\int_0^T (\pi(t)(I_0^{\beta}y^*(t) + I_0^{\beta_h}y_h^*(t) - y_h^*(t)) - c(t)(y_h^*(t) + y_l^*(t)))dt =$$

$$r\Pi(n^*) = r\Pi^*, \tag{C.4}$$

which means $\bar{\Pi}^* \geq r\Pi^*$. Now let $\tilde{y}_h^*(t)$, $\tilde{y}_l^*(t)$, and $\tilde{y}^*(t)$ corresponding to m and $\frac{\bar{n}^*}{r}$ and similarly $\bar{y}_h^*(t)$, $\bar{y}_l^*(t)$, and $\bar{y}^*(t)$ corresponding to $\bar{m}$ and $\bar{n}^*$. Then $\tilde{y}_h^*(t) = \frac{1}{r}\bar{y}_h^*(t)$, $\tilde{y}_l^*(t) = \frac{1}{r}\bar{y}_l^*(t)$, and by Proposition 4.1, $\tilde{y}^*(t) = \frac{1}{r}\bar{y}^*(t)$. Therefore,

$$\Pi\left(\frac{\bar{n}^*}{r}\right) = \int_0^T (\pi(t)(I_0^{\beta}\tilde{y}^*(t) + I_0^{\beta_h}\tilde{y}_h^*(t) - \tilde{y}_h^*(t)) - c(t)\big(\tilde{y}_h^*(t) + \tilde{y}_l^*(t)\big))dt =$$

$$\frac{1}{r}\int_0^T (\pi(t)(I_0^{\beta}\bar{y}^*(t) + I_0^{\beta_h}\bar{y}_h^*(t) - \bar{y}_h^*(t)) - c(t)\big(\bar{y}_h^*(t) + \bar{y}_l^*(t)\big))dt = \frac{1}{r}\bar{\Pi}\,(\bar{n}^*) = \frac{1}{r}\bar{\Pi}^*,$$

which means $\Pi^* \geq \frac{1}{r}\bar{\Pi}^*$. Hence, $r\Pi^* = \bar{\Pi}^*$ and therefore by (C.4) we conclude $\bar{n}^* = rn^*$ and this completes the proof. □

Theorem C.3 from Kilbas et al. (2006) demonstrates an important property of fractional integral operators we use in the subsequent proofs.

Theorem C.3. (Semigroup property) For f continuous on $[t_0, t_1]$, and $\beta_1, \beta_2 > 0$, $I_{t_0}^{\beta_1+\beta_2} f(t) = I_{t_0}^{\beta_1} I_{t_0}^{\beta_2} f(t)$.

Proof (Kilbas et al., 2006).

Proof of Proposition 4.5

We prove the proposition under the concurrent strategy. The before strategy proof is similar in which we set $Y_h(t) = n$ and $Y_l(t) = \delta n$, buyers' adoption $Y(t)$ is formulated by (4.4) with closed form solution given by (4.5) and (4.6), sales are given by (4.7) and (4.8), and the profit is given by (4.18).

(i) Suppose $Y(t)$, $\bar{Y}(t)$ derived from (4.9) corresponding to $\bar{p}, \bar{q}, \bar{\lambda}_h, \bar{\lambda}_l$ and $p, q, \lambda_h, \lambda_l$, respectively for $0 < n \leq n_M$ and initial conditions $\bar{Y}(0) = Y(0) = 0$. Let

$$\phi := \frac{m}{2\bar{q}}\left(\bar{q}\,\frac{m - n_M}{m} - \left(\bar{p} + \frac{\bar{\lambda}_h}{m} n_M + \frac{\bar{\lambda}_l}{m}\delta n_M\right)\right).$$

It can be observed that for all $n \leq n_M$

$$0 < \phi \leq \frac{m}{2\bar{q}}\left(\bar{q}\,\frac{m-n}{m} - \left(\bar{p} + \frac{\bar{\lambda}_h}{m} Y_h(t) + \frac{\bar{\lambda}_l}{m} Y_l(t)\right)\right),\ t \geq 0. \tag{C.5}$$

Let $\bar{\hat{Y}}(t)$ be the solution for (4.9) corresponding to $\bar{p}, \bar{q}, \bar{\lambda}_h, \bar{\lambda}_l$, $Y_h(t) = n$, and $Y_l(t) = \delta n$. For $n \in \{1, \dots, n_M\}$, let $\tilde{t}_n$ be such that $\bar{\hat{Y}}(\tilde{t}_n) = \phi$. Then for $t \in [0, \hat{t}]$, $\bar{\hat{Y}}(t) \leq \phi$, for all $0 < n \leq n_M$ where $\hat{t} = \min_{n \in \{1, \dots, n_M\}} \tilde{t}_n$. According to Proposition 4.3, for any $Y_h(t)$, $Y_l(t)$, and $T > 0$, $\bar{Y}(t) \leq \bar{\hat{Y}}(t)$ for all $t \in [0, T]$ and $0 < n \leq n_M$. Moreover, by Proposition 4.2 (ii), $Y(t) \leq \bar{Y}(t)$ for $t \geq 0$ and $0 < n \leq n_M$. Consider $T > 0$ such that $T \leq \hat{t}$. By the fact that $Y(t) \leq \bar{Y}(t) \leq \phi$ for $t \in [0, T]$ and $0 < n \leq n_M$, and considering (C.5), it can be observed that for $t \in (0, T]$

$$\bar{y}(t) = \left(\bar{p} + \frac{\bar{q}}{m}\bar{Y}(t) + \frac{\bar{\lambda}_h}{m}Y_h(t) + \frac{\bar{\lambda}_l}{m}Y_l(t)\right)\left((m-n) - \bar{Y}(t)\right) \geq \left(\bar{p} + \frac{\bar{q}}{m}Y(t) + \frac{\bar{\lambda}_h}{m}Y_h(t) + \frac{\bar{\lambda}_l}{m}Y_l(t)\right)\left((m-n) - Y(t)\right) > \left(p + \frac{q}{m}Y(t) + \frac{\lambda_h}{m}Y_h(t) + \frac{\lambda_l}{m}Y_l(t)\right)\left((m-n) - Y(t)\right) = y(t).$$

Let the sales processes $\bar{S}(t)$ and $S(t)$ corresponding to diffusion processes $\bar{y}(t)$ and $y(t)$ respectively for $0 < n \leq n_M$. Then $\bar{S}(t) > S(t)$ for $t \in (0, T]$. It can be observed that $\int_0^T \pi(t)\bar{S}(t)dt > \int_0^T \pi(t)S(t)dt$ and therefore $\overline{\Pi}(n) > \Pi(\text{n})$ for $0 < n \leq n_M$. Let $n = 0$. If neither of $\bar{p} > p$ or $\bar{q} > q$ holds, then $\overline{\Pi}(0) = \Pi(0)$. Otherwise, let $\tilde{t}_0$ be such that $\bar{Y}(\tilde{t}_0) = \phi$. By Proposition 4.2 (ii), $Y(t) \leq \bar{Y}(t) \leq \phi$ for $t \in [0, \tilde{t}_0]$. Considering (C.5), it can be observed that for $t \in (0, T]$, $T \leq \tilde{t}_0$

$$\bar{y}(t) = \left(\bar{p} + \frac{\bar{q}}{m}\bar{Y}(t)\right)\left(m - \bar{Y}(t)\right) \geq \left(\bar{p} + \frac{\bar{q}}{m}Y(t)\right)\left(m - Y(t)\right) > \left(p + \frac{q}{m}Y(t)\right)\left(m - Y(t)\right) = y(t).$$

Thus, for sales processes $\bar{S}(t)$ and $S(t)$ corresponding to diffusion processes $\bar{y}(t)$ and $y(t)$ respectively we have $\bar{S}(t) > S(t)$ for $t \in (0, T]$. Hence, $\int_0^T \pi(t)\bar{S}(t)dt > \int_0^T \pi(t)S(t)dt$ and therefore $\overline{\Pi}(0) > \Pi(0)$. Hence, for $T \leq \tilde{t}$, $\tilde{t} = \min\{\tilde{t}_0, \hat{t}\}$ we have $\overline{\Pi}(n) > \Pi(\text{n})$ for $0 < n \leq n_M$, $\overline{\Pi}(0) > \Pi(0)$, if either of $\bar{p} > p$ or $\bar{q} > q$ holds, otherwise $\overline{\Pi}(0) = \Pi(0)$. Thus $\overline{\Pi}^* \geq \Pi^*$.

(ii) By continuity of $\bar{S}(t) - S(t)$ and the fact that $\bar{S}(T) - S(T) > 0$, for each n, $0 < n \leq n_M$ there exists $T_\beta^n > T$ such that $\bar{S}(t) \geq S(t)$ for $t \in [0, T_\beta^n]$. Let $\hat{T}_\beta = \min_{0 < n \leq n_m} T_\beta^n$. Then, $\hat{T}_\beta > T$ and $\bar{S}(t) \geq S(t)$ for $t \in [0, \hat{T}_\beta]$. Hence, $\int_0^{\hat{T}_\beta} \pi(t)\bar{S}(t)dt > \int_0^{\hat{T}_\beta} \pi(t)S(t)dt$, therefore $\overline{\Pi}(n) > \Pi(n)$ for $0 < n \le n_M$. On the other hand, for $n = 0$, when either of $\bar{p} > p$ or $\bar{q} > q$ holds we have $\bar{S}(T) - S(T) > 0$. So, by continuity of $\bar{S}(t) - S(t)$, there exists $T_\beta^0 > T$ such that $\bar{S}(t) \ge S(t)$ for $t \in [0, T_\beta^0]$. Hence, $\int_0^{T_\beta^0} \pi(t)\bar{S}(t)dt > \int_0^{T_\beta^0} \pi(t)S(t)dt$, therefore $\overline{\Pi}(0) > \Pi(0)$. If $n = 0$, and neither of $\bar{p} > p$ or $\bar{q} > q$ holds we have $\overline{\Pi}(0) = \Pi(0)$. Let, $T_\beta = \min\{T_\beta^0, \hat{T}_\beta\}$, then $T_\beta > T$ and under the planning horizon T_β we have $\overline{\Pi}(n) > \Pi(n)$ for $0 < n \le n_M$, and $\overline{\Pi}(0) > \Pi(0)$, when either of $\bar{p} > p$ or $\bar{q} > q$ holds, otherwise $\overline{\Pi}(0) = \Pi(0)$. Thus, $\overline{\Pi}^* \ge \Pi^*$. Moreover, by Theorem C.3 it can be observed that

$$I_0^{\bar{\beta}}(\bar{y} - y)(t) = I_0^{\bar{\beta}-\beta} I_0^{\beta}(\bar{y} - y)(t) = I_0^{\bar{\beta}-\beta}\left(I_0^{\beta}\bar{y}(t) - I_0^{\beta}y(t)\right).$$

Thus, $I_0^{\bar{\beta}}(\bar{y} - y)(t) > 0$ for $t \in (0, \hat{T}_\beta]$ and $0 < n \le n_M$. Continuity of $I_0^{\bar{\beta}}(\bar{y} - y)(t)$ implies the existence of $T_{\bar{\beta}}^n > \hat{T}_\beta$ for each n, $0 < n \le n_M$ such that $I_0^{\bar{\beta}}(\bar{y} - y)(t) \ge 0$ for $t \in [0, T_{\bar{\beta}}^n]$. Set $\hat{T}_{\bar{\beta}} = \min_{0<n\le n_M} T_{\bar{\beta}}^n$. Then, $\hat{T}_{\bar{\beta}} > \hat{T}_\beta$, and $\bar{S}(t) = I_0^{\bar{\beta}}\bar{y}(t) \ge I_0^{\bar{\beta}}y(t) = S(t)$, for $t \in [0, \hat{T}_{\bar{\beta}}]$, and $0 < n \le n_M$. Similarly, for $n = 0$, when either of $\bar{p} > p$ or $\bar{q} > q$ holds we have $T_{\bar{\beta}}^0 > T_\beta^0$ such that $\bar{S}(t) = I_0^{\bar{\beta}}\bar{y}(t) \ge I_0^{\bar{\beta}}y(t) = S(t)$, for $t \in [0, T_{\bar{\beta}}^0]$. Set $T_{\bar{\beta}} = \min\{T_{\bar{\beta}}^0, \hat{T}_{\bar{\beta}}\}$. Then, $T_{\bar{\beta}} > T_\beta$, and $\bar{S}(t) = I_0^{\bar{\beta}}\bar{y}(t) \ge I_0^{\bar{\beta}}y(t) = S(t)$, $t \in [0, T_{\bar{\beta}}]$ for $0 < n \le n_M$, and for $n = 0$ when either of $\bar{p} > p$ or $\bar{q} > q$ holds. This completes the proof for part (ii).

(iii) We have

$$\int_0^{T_L} \pi(t)\bar{S}(t)dt - \int_0^{T_L} \pi(t)S(t)dt =$$
$$\int_0^{T_\beta}(\bar{S}(t) - S(t))\pi(t)dt + \int_{T_\beta}^{T_L}(\bar{S}(t) - S(t))\pi(t)dt.$$

By mean value theorem there exists $\zeta_n^1 \in (0, T_\beta)$ and $\zeta_n^2 \in (T_\beta, T_L)$ such that

$$\int_0^{T_\beta}(\bar{S}(t) - S(t))\pi(t)dt = \left(\bar{S}(\zeta_n^1) - S(\zeta_n^1)\right)\int_0^{T_\beta}\pi(t)dt,$$
$$\int_{T_\beta}^{T_L}(\bar{S}(t) - S(t))\pi(t)dt = \left(\bar{S}(\zeta_n^2) - S(\zeta_n^2)\right)\int_{T_\beta}^{T_L}\pi(t)dt.$$

Let $\Omega = \min\ \{\bar{S}(\zeta_n^1) - S(\zeta_n^1): 0 < n \le n_M\}$ and $\Gamma = \min\ \{\bar{S}(\zeta_n^2) - S(\zeta_n^2): 0 < n \le n_M\}$ when neither of $\bar{p} > p$ or $\bar{q} > q$ holds. Also let $\Omega = \min\ \{\bar{S}(\zeta_n^1) - S(\zeta_n^1): 0 \le n \le n_M\}$ and $\Gamma = \min\ \{\bar{S}(\zeta_n^2) - S(\zeta_n^2): 0 \le n \le n_M\}$ when either of $\bar{p} > p$ or $\bar{q} > q$ holds. Then

$$\int_0^{T_L} \pi(t)\bar{S}(t)dt - \int_0^{T_L} \pi(t)S(t)dt \geq \Omega \int_0^{T_\beta} \pi(t)dt + \Gamma \int_{T_\beta}^{T_L} \pi(t)dt.$$

It can be observed that if $\Gamma \geq 0$ then $\int_0^{T_L} \pi(t)\bar{S}(t)dt - \int_0^{T_L} \pi(t)S(t)dt > 0$ which implies that $\bar{\Pi}(n) > \Pi(n)$ for all $0 < n \leq n_M$ and $\bar{\Pi}(0) > \Pi(0)$ when either of $\bar{p} > p$ or $\bar{q} > q$ holds. Otherwise, if the following inequality is satisfied

$$\frac{\int_{T_\beta}^{T_L} \pi(t)dt}{\int_0^{T_\beta} \pi(t)dt} < \frac{\Omega}{-\Gamma}, \tag{C.6}$$

then we conclude that $\int_0^{T_L} \pi(t)\bar{S}(t)dt > \int_0^{T_L} \pi(t)S(t)dt$ and therefore $\bar{\Pi}(n) > \Pi(\text{n})$ for all $0 < n \leq n_M$, and $\bar{\Pi}(0) > \Pi(0)$ when either of $\bar{p} > p$ or $\bar{q} > q$ holds. We see that if $\pi(t)$ is decreasing on $[T_\beta, T_L]$ with high declining rate, then the inequality (C.6) is satisfied. When $n = 0$ and neither of $\bar{p} > p$ or $\bar{q} > q$ holds, then $\int_0^{T_L} \pi(t)\bar{S}(t)dt = \int_0^{T_L} \pi(t)S(t)dt$ and this completes the proof for part (iii). □

Proof of Proposition 4.6

We prove part (ii) and the result of part (iii) corresponding to part (ii). For part (i) and part (iii) corresponding to part (i) proof is similar in which we set $Y_h(t) = n$, $Y_l(t) = \delta n$, buyers' adoption $Y(t)$ is formulated by (4.4) with closed form solution given by (4.5) and (4.6), sales are given by (4.7) and (4.8), and the profit is given by (4.18).

(ii) Let $\bar{Y}_l(t)$ and $Y_l(t)$ corresponding to $\bar{\delta}$ and δ respectively. Suppose $\bar{Y}(t)$ and $Y(t)$ derived from (4.9) corresponding to $\bar{Y}_l(t)$ and $Y_l(t)$ respectively where, $0 < n \leq n_M$, and $\bar{Y}(0) = Y(0) = 0$. Define

$$\phi := \frac{m}{2q}\left(q\frac{m-n_M}{m} - \left(p + \frac{\lambda_h}{m}n_M + \frac{\lambda_l}{m}\bar{\delta}n_M\right)\right).$$

We can see that for all $n \leq n_M$

$$0 < \phi \leq \frac{m}{2q}\left(q\frac{m-n}{m} - \left(p + \frac{\lambda_h}{m}Y_h(t) + \frac{\lambda_l}{m}\bar{Y}_l(t)\right)\right), \; t \geq 0. \tag{C.7}$$

Let $\hat{\bar{Y}}(t)$ be the solution for (4.9) corresponding to $Y_h(t) = n$ and $\bar{Y}_l(t) = \bar{\delta}n$. For $n \in \{1, \dots, n_M\}$, let $\tilde{t}_n$ be such that $\hat{\bar{Y}}(\tilde{t}_n) = \phi$. Then for $t \in [0, \tilde{t}_2]$, $\hat{\bar{Y}}(t) \leq \phi$, for all $0 < n \leq n_M$ where $\tilde{t}_2 = \min_{n \in \{1,\dots,n_M\}} \tilde{t}_n$. According to Proposition 4.3, for any $Y_h(t)$, $\bar{Y}_l(t)$, and $T > 0$, $\bar{Y}(t) \leq \hat{\bar{Y}}(t)$ for $t \in [0, T]$ and $0 < n \leq n_M$. Moreover, by Proposition 4.3, $Y(t) \leq$

$\bar{Y}(t)$ for $t \geq 0$ and $0 < n \leq n_M$. Consider $T > 0$ such that $T \leq \tilde{t}_2$. By the fact that $Y(t) \leq \bar{Y}(t) \leq \phi$ and $\bar{Y}_l(t) > Y_l(t)$ for $t \in (0, T]$ and $0 < n \leq n_M$, and considering (C.7), we see that for $t \in (0, T]$

$$\begin{aligned} \bar{y}(t) = \left(p + \tfrac{q}{m}\bar{Y}(t) + \tfrac{\lambda_h}{m}Y_h(t) + \tfrac{\lambda_l}{m}\bar{Y}_l(t)\right)\left((m-n) - \bar{Y}(t)\right) \geq \\ \left(p + \tfrac{q}{m}Y(t) + \tfrac{\lambda_h}{m}Y_h(t) + \tfrac{\lambda_l}{m}\bar{Y}_l(t)\right)\left((m-n) - Y(t)\right) > \\ \left(p + \tfrac{q}{m}Y(t) + \tfrac{\lambda_h}{m}Y_h(t) + \tfrac{\lambda_l}{m}Y_l(t)\right)\left((m-n) - Y(t)\right) = y(t). \end{aligned} \tag{C.8}$$

Let the sales processes $\bar{S}(t)$ and $S(t)$ corresponding to diffusion processes $\bar{y}(t)$ and $y(t)$ respectively for $0 < n \leq n_M$. Then $\bar{S}(t) > S(t)$ for $t \in (0, T]$, therefore $\int_0^T \pi(t)\bar{S}(t)dt > \int_0^T \pi(t)S(t)dt$ for $0 < n \leq n_M$. If $\pi(t)$ is sufficiently higher than $c(t)$ on $[0, T]$ or $c(t) = 0$, then for $0 < n \leq n_M$

$$\int_0^T \pi(t)(\bar{S}(t) - S(t))dt > \int_0^T c(t)\big(\bar{y}_l(t) - y_l(t)\big)dt, \tag{C.9}$$

which implies that $\bar{\Pi}(n) > \Pi(\mathrm{n})$ for $0 < n \leq n_M$. On the other hand, $\bar{\Pi}(0) = \Pi(0)$. Thus, $\bar{\Pi}^* \geq \Pi^*$.

(iii) By continuity of $\bar{S}(t) - S(t)$ and the fact that $\bar{S}(T) - S(T) > 0$, for each n, $0 < n \leq n_M$ there exists $T_\beta^n > T$ such that $\bar{S}(t) \geq S(t)$ for $t \in [0, T_\beta^n]$. Let $T_\beta^2 = \min_{0<n\leq n_m} T_\beta^n$. Then, $T_\beta^2 > T$ and $\bar{S}(t) \geq S(t)$ for $t \in [0, T_\beta^2]$ and $0 < n \leq n_m$. Thus

$$\int_0^{T_\beta^2} \pi(t)\bar{S}(t)dt > \int_0^{T_\beta^2} \pi(t)S(t)dt.$$

On the other hand, for $0 < n \leq n_M$

$$\int_0^{T_\beta^2} \pi(t)(\bar{S}(t) - S(t))dt > \int_0^T \pi(t)(\bar{S}(t) - S(t))dt,$$

$$\int_0^{T_\beta^2} c(t)(\bar{y}_l(t) - y_l(t))dt = \int_0^T c(t)(\bar{y}_l(t) - y_l(t))dt.$$

Therefore by (C.9), under the planning horizon T_β^2, $\bar{\Pi}(n) > \Pi(\mathrm{n})$ for $0 < n \leq n_M$. On the other hand, $\bar{\Pi}(0) = \Pi(0)$. Thus, $\bar{\Pi}^* \geq \Pi^*$. Moreover, by Theorem C.3 we observe that

$$I_0^{\bar{\beta}}(\bar{y} - y)(t) = I_0^{\bar{\beta}-\beta} I_0^{\beta}(\bar{y} - y)(t) = I_0^{\bar{\beta}-\beta}\left(I_0^{\beta}\bar{y}(t) - I_0^{\beta}y(t)\right).$$

Thus, $I_0^{\bar{\beta}}(\bar{y} - y)(t) > 0$ for $t \in (0, T_\beta^2]$. The continuity of $I_0^{\bar{\beta}}(\bar{y} - y)(t)$ implies the existence of $T_{\bar{\beta}}^n > T_\beta^2$ for each n, $0 < n \leq n_M$ such that $I_0^{\bar{\beta}}(\bar{y} - y)(t) \geq 0$ for $t \in$

$[0, T_{\overline{\beta}}^{n}]$. Set $T_{\overline{\beta}}^{2} = \min_{0<n\leq n_M} T_{\overline{\beta}}^{n}$. Then, $T_{\overline{\beta}}^{2} > T_{\beta}^{2}$, and $\bar{S}(t) = I_0^{\overline{\beta}} \bar{y}(t) \geq I_0^{\overline{\beta}} y(t) = S(t)$, for $t \in [0, T_{\overline{\beta}}^{2}]$ and this completes the proof. □

Proof of Proposition 4.7

(i) Similar to the proof of Proposition 4.6 (ii).

(ii) Similar to the proof of Proposition 4.6 (iii). □

We rewrite (4.22) as follows

$$y(u) = \left(\tilde{p} + \frac{\tilde{q}}{\tilde{m}} Y(u) + \frac{\tilde{\lambda}_h}{\tilde{m}} Y_h^a(u) + \frac{\tilde{\lambda}_l}{\tilde{m}} Y_l^a(u))\right)\left(\tilde{m} - n_a - Y(u)\right) \quad u \geq 0, \quad \text{(C.10)}$$

where $\tilde{p} = p + \frac{n_b}{m}(\lambda_h + \delta\lambda_l)$, $\tilde{q} = q\frac{\tilde{m}}{m}$, $\tilde{\lambda}_h = \lambda_h \frac{\tilde{m}}{m}$, $\tilde{\lambda}_l = \lambda_l \frac{\tilde{m}}{m}$, $\tilde{m} = m - n_b$, and $u = t - \tau$, $t \geq \tau$. We can observe that (C.10) with the initial condition $Y(0) = 0$ is identical to (4.9) with p, q, λ_h, λ_l, n, m, Y_h, and Y_l given as $\tilde{p}$, $\tilde{q}$, $\tilde{\lambda}_h$, $\tilde{\lambda}_l$, n_a, $\tilde{m}$, Y_h^a, and Y_l^a respectively and the initial condition $Y(0) = 0$. Accordingly, we rewrite buyers' sales (4.23) as

$$S(u) = I_0^{\beta} y(u) = \frac{1}{\Gamma(\beta)} \int_0^u (u - s)^{\beta-1} y(s) ds, \quad \text{(C.11)}$$

which is identical to (4.16). We also rewrite after the release high-valuation free users' sales (4.25) as

$$S_h^a(u) = I_0^{\beta_h^a} y_h^a(u) - y_h^a(u). \quad \text{(C.12)}$$

We can see that

$$\Pi(n_a, n_b) = \int_0^{T-\tau} \left(\pi(u)\big(S(u) + S_h^a(u)\big) - c(u)\big(y_h^a(u) + y_l^a(u)\big)\right) du + \int_\tau^T \pi(t) S_h^b(t) dt - c_0(1 + \delta) n_b, \quad \text{(C.13)}$$

where, the first term in (C.13) right hand side is similar to (4.19) with the planning horizon given as $T - \tau$. We employ (C.10)-(C.13) in the following proofs.

Proof of Proposition C.3

(i) Since $q > \frac{m}{m-(n_a+n_b)}\left(p + \frac{n_a+n_b}{m}(\lambda_h + \lambda_l\delta)\right)$, then $\tilde{q} > \frac{\tilde{m}}{\tilde{m}-n_a}(\tilde{p} + \frac{n_a}{\tilde{m}}(\tilde{\lambda}_h + \tilde{\lambda}_l\delta))$. Thus, considering (C.10) we obtain the result by Proposition 4.2 (i).

(ii) Given that increasing p, q, λ_h, and λ_l result in increasing $\tilde{p}$, $\tilde{q}$, $\tilde{\lambda}_h$, and $\tilde{\lambda}_l$, we obtain the result considering (C.10) and Proposition 4.2 (ii). □

Proof of Proposition C.4

We derive the result considering (C.10) and using Proposition 4.3 with the planning horizon $T - \tau$. □

Proof of Proposition C.5

Analogous to the proof of Proposition 4.4. □

Proof of Proposition C.6

(i) We can see that either one of $\tilde{\bar{p}} \geq \tilde{p}$ and $\tilde{\bar{q}} \geq \tilde{q}$, $\tilde{\bar{\lambda}}_h \geq \tilde{\lambda}_h$, and $\tilde{\bar{\lambda}}_l \geq \tilde{\lambda}_l$ holds with at least one strict inequality where $\tilde{\bar{p}} = \bar{p} + \frac{n_b}{m}(\bar{\lambda}_h + \delta\bar{\lambda}_l)$, $\tilde{\bar{q}} = \bar{q}\frac{\tilde{m}}{m}$, $\tilde{\bar{\lambda}}_h = \bar{\lambda}_h\frac{\tilde{m}}{m}$, and $\tilde{\bar{\lambda}}_l = \bar{\lambda}_l\frac{\tilde{m}}{m}$. Since $\bar{q} > \frac{m}{m-n_M^a-n_M^b}\left(\bar{p} + \frac{n_M^a+n_M^b}{m}(\bar{\lambda}_h + \delta\bar{\lambda}_l)\right)$, we see that $\tilde{\bar{q}} > \frac{\tilde{m}}{\tilde{m}-n_M^a}(\tilde{\bar{p}} + \frac{n_M^a}{\tilde{m}}(\tilde{\bar{\lambda}}_h + \delta\tilde{\bar{\lambda}}_l))$ for all $n_b \leq n_M^b$. Thus, considering (C.10)-(C.13), and $n_b \leq n_M^b$, by Proposition 4.5 (i), there exists a threshold $\tilde{t}_{n_b}$ such that for $T \leq \tilde{t}_{n_b}$, $\bar{\Pi}(n_a, n_b) > \Pi(n_a, n_b)$ for all $n_a \leq n_M^a$, if $n_b > 0$, $\bar{\Pi}(n_a, 0) > \Pi(n_a, 0)$, $0 < n_a \leq n_M^a$, and $\bar{\Pi}(0,0) > \Pi(0,0)$ if either of $\bar{p} > p$ or $\bar{q} > q$ holds, otherwise $\bar{\Pi}(0,0) = \Pi(0,0)$. Let $\tilde{t} = \min_{n_b \leq n_M^b} \tilde{t}_{n_b}$. Then, $\bar{\Pi}(n_a, n_b) > \Pi(n_a, n_b)$ for all $n_a \leq n_M^a$, $n_b \leq n_M^b$, $(n_a, n_b) \neq (0,0)$, and $\bar{\Pi}(0,0) > \Pi(0,0)$ if either of $\bar{p} > p$ or $\bar{q} > q$ holds, otherwise $\bar{\Pi}(0,0) = \Pi(0,0)$.

(ii) For $n_b \leq n_M^b$, by Proposition 4.5 (ii), there exists $T_\beta^{n_b}$, $T_\beta^{n_b} > T$, $T \leq \tilde{t}$, where $y_h^a(u) = y_l^a(u) = 0$ for $u \geq T - \tau$, and (1) if $n_b > 0$, $\bar{\Pi}(n_a, n_b) > \Pi(n_a, n_b)$ for all $n_a \leq n_M^a$, (2) $\bar{\Pi}(n_a, 0) > \Pi(n_a, 0)$, $0 < n_a \leq n_M^a$, and (3) $\bar{\Pi}(0,0) > \Pi(0,0)$ if either of $\bar{p} > p$ or $\bar{q} > q$ holds, otherwise $\bar{\Pi}(0,0) = \Pi(0,0)$. Let, $T_\beta = \min_{n_b \leq n_M^b} T_\beta^{n_b}$. Then, $T_\beta > T$ and under the planning horizon $T_\beta - \tau$ we have $\bar{\Pi}(n_a, n_b) > \Pi(n_a, n_b)$ for all $n_a \leq n_M^a$, $n_b \leq n_M^b$, $(n_a, n_b) \neq (0,0)$, and $\bar{\Pi}(0,0) > \Pi(0,0)$ if either of $\bar{p} > p$ or $\bar{q} > q$ holds, otherwise $\overline{\Pi}(0,0) = \Pi(0,0)$. In a similar fashion, for $n_b \leq n_M^b$, by Proposition 4.5 (ii), there exists $T_{\overline{\beta}}^{n_b}$ such that $T_{\overline{\beta}}^{n_b} > T_{\beta}^{n_b}$. Let, $T_{\overline{\beta}} = \min_{n_b \leq n_M^b} T_{\overline{\beta}}^{n_b}$. Then, $T_{\overline{\beta}} > T_{\beta}$.

(iii) Let $\bar{S}(t)$ and $S(t)$, corresponding to $\bar{p}$, $\bar{q}$, $\bar{\lambda}_h$, $\bar{\lambda}_l$, and p, q, λ_h, λ_l respectively. We have

$$\int_{\tau}^{T_L} \pi(t)\bar{S}(t)dt - \int_{\tau}^{T_L} \pi(t)S(t)dt =$$

$$\int_{\tau}^{T_\beta} (\bar{S}(t) - S(t))\pi(t)dt + \int_{T_\beta}^{T_L} (\bar{S}(t) - S(t))\pi(t)dt.$$

By mean value theorem there exists $\zeta_{(n_a,n_b)}^1 \in (\tau, T_\beta)$ and $\zeta_{(n_a,n_b)}^2 \in (T_\beta, T_L)$ such that

$$\int_{\tau}^{T_\beta} (\bar{S}(t) - S(t))\pi(t)dt = \left(\bar{S}\left(\zeta_{(n_a,n_b)}^1\right) - S\left(\zeta_{(n_a,n_b)}^1\right)\right) \int_{\tau}^{T_\beta} \pi(t)dt,$$

$$\int_{T_\beta}^{T_L} (\bar{S}(t) - S(t))\pi(t)dt = \left(\bar{S}\left(\zeta_{(n_a,n_b)}^2\right) - S\left(\zeta_{(n_a,n_b)}^2\right)\right) \int_{T_\beta}^{T_L} \pi(t)dt.$$

Let

$$\Omega = \min \left\{\bar{S}\left(\zeta_{(n_a,n_b)}^1\right) - S\left(\zeta_{(n_a,n_b)}^1\right): n_a \leq n_M^a, n_b \leq n_M^b, (n_a, n_b) \neq (0,0)\right\},$$

$$\Gamma = \min \left\{\bar{S}\left(\zeta_{(n_a,n_b)}^2\right) - S\left(\zeta_{(n_a,n_b)}^2\right): n_a \leq n_M^a, n_b \leq n_M^b, (n_a, n_b) \neq (0,0)\right\},$$

when neither of $\bar{p} > p$ or $\bar{q} > q$ holds. Also

$$\Omega = \min \left\{\bar{S}\left(\zeta_{(n_a,n_b)}^1\right) - S\left(\zeta_{(n_a,n_b)}^1\right): n_a \leq n_M^a, n_b \leq n_M^b\right\},$$

$$\Gamma = \min \left\{\bar{S}\left(\zeta_{(n_a,n_b)}^2\right) - S\left(\zeta_{(n_a,n_b)}^2\right): n_a \leq n_M^a, n_b \leq n_M^b\right\},$$

when either of $\bar{p} > p$ or $\bar{q} > q$ holds. Then

$$\int_{\tau}^{T_L} \pi(t)\bar{S}(t)dt - \int_{\tau}^{T_L} \pi(t)S(t)dt \geq \Omega \int_{\tau}^{T_\beta} \pi(t)dt + \Gamma \int_{T_\beta}^{T_L} \pi(t)dt.$$

It can be observed that if $\Gamma \geq 0$ then $\int_{\tau}^{T_L} \pi(t)\bar{S}(t)dt - \int_{\tau}^{T_L} \pi(t)S(t)dt > 0$ which implies that $\overline{\Pi}(n_a, n_b) > \Pi(n_a, n_b)$ for all $n_a \leq n_M^a$, $n_b \leq n_M^b$, $(n_a, n_b) \neq (0,0)$, and $\overline{\Pi}(0,0) > \Pi(0,0)$ when either of $\bar{p} > p$ or $\bar{q} > q$ holds. Otherwise, if the following inequality is satisfied

$$\frac{\int_{T_\beta}^{T_L} \pi(t)dt}{\int_{\tau}^{T_\beta} \pi(t)dt} < \frac{\Omega}{-\Gamma}, \tag{C.14}$$

then we conclude that $\int_{\tau}^{T_L} \pi(t)\bar{S}(t)dt > \int_{\tau}^{T_L} \pi(t)S(t)dt$ and therefore $\overline{\Pi}(n_a, n_b) > \Pi(n_a, n_b)$ for all $n_a \leq n_M^a$, $n_b \leq n_M^b$, $(n_a, n_b) \neq (0,0)$, and $\overline{\Pi}(0,0) > \Pi(0,0)$ when either of $\bar{p} > p$ or $\bar{q} > q$ holds. We see that if $\pi(t)$ is decreasing on $[T_\beta, T_L]$ with high declining rate, then the inequality (C.14) is satisfied. When $n_a = n_b = 0$ and neither of

$\bar{p} > p$ or $\bar{q} > q$ holds, then $\int_{\tau}^{T_L} \pi(t)\bar{S}(t)dt = \int_{\tau}^{T_L} \pi(t)S(t)dt$ and this completes the proof for part (iii). □

Proof of Proposition C.7

(i) We have $\tilde{q} > \frac{\tilde{m}}{\tilde{m}-n_M^a}(\tilde{p} + \frac{n_M^a}{\tilde{m}}(\tilde{\lambda}_h + \bar{\delta}\tilde{\lambda}_l))$ for all $n_b \le n_M^b$, because $q > \frac{m}{m-n_M^a-n_M^b}\left(p + \frac{n_M^a+n_M^b}{m}(\lambda_h + \bar{\delta}\lambda_l)\right)$. Let $\bar{y}_l^a(u)$ corresponding to $\bar{\delta}$ and $\bar{S}(u)$ derived from (C.10) and (C.11) corresponding to $\bar{y}_l^a(u)$. Let

$$\widehat{\Pi}(n_a, n_b) = \int_0^{T-\tau} \left(\pi(u)\left(\bar{S}(u) + S_h^a(u)\right) - c(u)(y_h^a(u) + \bar{y}_l^a(u))\right) du + \int_{\tau}^{T} \pi(t)S_h^b(t)dt - c_0\left(1+\bar{\delta}\right)n_b.$$

Using Proposition 4.6 (ii) we can show that if $\pi(u)$ be sufficiently high compared to c_0 and $c(u)$ on $[0, T-\tau]$ or $c_0 = c(u) = 0$ then for $n_b \le n_M^b$ there exists a threshold $\tilde{t}_{n_b}^1$ such that for $T \le \tilde{t}_{n_b}^1$, $\widehat{\Pi}(n_a, n_b) > \Pi(n_a, n_b)$ for all $0 < n_a \le n_M^a$. Let $\bar{\tilde{p}} = p + \frac{n_b}{m}(\lambda_h + \bar{\delta}\lambda_l)$. Then, $\bar{\tilde{p}} > \tilde{p}$ for $n_b > 0$ and $\bar{\tilde{p}} = \tilde{p} = p$ when $n_b = 0$. Since $q > \frac{m}{m-n_M^a-n_M^b}\left(p + \frac{n_M^a+n_M^b}{m}(\lambda_h + \bar{\delta}\lambda_l)\right)$, we have $\tilde{q} > \frac{\tilde{m}}{\tilde{m}-n_M^a}(\bar{\tilde{p}} + \frac{n_M^a}{\tilde{m}}(\tilde{\lambda}_h + \bar{\delta}\tilde{\lambda}_l))$ for $n_b \le n_M^b$. Considering (C.10)-(C.13), by Proposition 4.5 (i), there exists a threshold $\tilde{t}_{n_b}^2$ such that for $T \le \tilde{t}_{n_b}^2$, $\overline{\Pi}(n_a, n_b) > \widehat{\Pi}(n_a, n_b)$ for all $n_a \le n_M^a$ if $n_b > 0$ and $\overline{\Pi}(n_a, 0) = \widehat{\Pi}(n_a, 0)$. Let, $\tilde{t}_{n_b} = \min\{\tilde{t}_{n_b}^1, \tilde{t}_{n_b}^2\}$. Then, for $T \le \tilde{t}_{n_b}$, $\overline{\Pi}(n_a, n_b) > \Pi(n_a, n_b)$ for all $0 < n_a \le n_M^a$, if $\pi(u)$ be sufficiently high compared to c_0 and $c(u)$ on $[0, T-\tau]$ or $c(u) = c_0 = 0$. Let, $\tilde{t} = \min_{n_b \le n_M^b} \tilde{t}_{n_b}$. Then for $T \le \tilde{t}$, $\overline{\Pi}(n_a, n_b) > \Pi(n_a, n_b)$ for $0 < n_a \le n_M^a$ and $n_b \le n_M^b$, if $\pi(u)$ be sufficiently high compared to c_0 and $c(u)$ on $[0, T-\tau]$ or $c(u) = c_0 = 0$. For $T \le \tilde{t}$, $\overline{\Pi}(0, n_b) > \Pi(0, n_b)$ for $0 < n_b \le n_M^b$, if $\pi(u)$ be sufficiently high compared to c_0 and $c(u)$ on $[0, T-\tau]$ or $c(u) = c_0 = 0$. Moreover, $\overline{\Pi}(0,0) = \Pi(0,0)$.

(ii) For $n_b \le n_M^b$, by Proposition 4.6 (iii), there exists $T_\beta^{n_b 1}, T_\beta^{n_b 1} > T, T \le \tilde{t}$, where $y_h^a(u) = y_l^a(u) = \bar{y}_l^a(u) = 0$ for $u \ge T - \tau$ and $\widehat{\Pi}(n_a, n_b) > \Pi(n_a, n_b)$ for all $0 < n_a \le n_M^a$. Moreover, by Proposition 4.5 (ii), there exists $T_\beta^{n_b 2}, T_\beta^{n_b 2} > T, T \le \tilde{t}$, where

$y_h^a(u) = \bar{y}_l^a(u) = 0$ for $u \geq T - \tau$ and $\overline{\Pi}(n_a, n_b) > \widehat{\Pi}(n_a, n_b)$ for all $n_a \leq n_M^a$ if $n_b > 0$ and $\overline{\Pi}(n_a, 0) = \widehat{\Pi}(n_a, 0)$. Let $T_\beta^{n_b} = \min\{T_\beta^{n_b 1}, T_\beta^{n_b 2}\}$. Then for the planning horizon $T_\beta^{n_b}$, where $y_h^a(u) = y_l^a(u) = \bar{y}_l^a(u) = 0$ for $u \geq T - \tau, T \leq \tilde{t}$, we have $\overline{\Pi}(n_a, n_b) > \Pi(n_a, n_b)$ for all $0 < n_a \leq n_M^a$. Let, $T_\beta = \min_{n_b \leq n_M^b} T_\beta^{n_b}$. Then, $T_\beta > T, T \leq \tilde{t}$. Under the planning horizon $T_\beta - \tau$, where $y_h^a(u) = y_l^a(u) = \bar{y}_l^a(u) = 0$ for $u \geq T - \tau$, we have $\overline{\Pi}(n_a, n_b) > \Pi(n_a, n_b)$ for $0 < n_a \leq n_M^a$ and $n_b \leq n_M^b$. Under the planning horizon $T_\beta - \tau$, where $y_h^a(u) = y_l^a(u) = \bar{y}_l^a(u) = 0$ for $u \geq T - \tau$, we have $\overline{\Pi}(0, n_b) > \Pi(0, n_b)$ for $0 < n_b \leq n_M^b$. Furthermore, for $\bar{\beta}$, $\bar{\beta} > \beta$, by Proposition 4.6 (iii) we will have $T_{\bar{\beta}}^{n_b 1} > T_\beta^{n_b 1}$ and by Proposition 4.5 (ii) we will have $T_{\bar{\beta}}^{n_b 2} > T_\beta^{n_b 2}$. In a similar fashion we find $T_{\bar{\beta}}^{n_b} = \min\{T_{\bar{\beta}}^{n_b 1}, T_{\bar{\beta}}^{n_b 2}\}$ and $T_{\bar{\beta}} = \min_{n_b \leq n_M} T_{\bar{\beta}}^{n_b}$ where, $T_{\bar{\beta}} > T_\beta$. □

Proof of Proposition C.8

(i) Since $q > \frac{m}{m - n_M^a - n_M^b}\left(p + \frac{n_M^a + n_M^b}{m}(\lambda_h + \delta\lambda_l)\right)$ we have $\tilde{q} > \frac{\tilde{m}}{\tilde{m} - n_M^a}(\tilde{p} + \frac{n_M^a}{\tilde{m}}(\tilde{\lambda}_h + \delta\tilde{\lambda}_l))$ for $n_b \leq n_M^b$. Thus, considering (C.10)-(C.13) and $n_b \leq n_M^b$, by Proposition 4.7 (i), there exists a threshold $\tilde{t}_{n_b}$ such that for $T \leq \tilde{t}_{n_b}$, $\overline{\Pi}(n_a, n_b) \geq \Pi(n_a, n_b)$ for all $0 < n_a \leq n_M^a$, if $\pi(u)$ be sufficiently high compared to $c(u)$ on $[0, T - \tau]$ or $c(u) = 0$. Let, $\tilde{t} = \min_{n_b \leq n_M^b} \tilde{t}_{n_b}$. Then for $T \leq \tilde{t}$, we have $\overline{\Pi}(n_a, n_b) > \Pi(n_a, n_b)$ for all $0 < n_a \leq n_M^a$, $n_b \leq n_M^b$ if $\pi(u)$ be sufficiently high compared to $c(u)$ on $[0, T - \tau]$ or $c(u) = 0$ and $\overline{\Pi}(0, n_b) = \Pi(0, n_b)$, $n_b \leq n_M^b$.

(ii) For $n_b \leq n_M^b$, by Proposition 4.7 (ii), there exists $T_\beta^{n_b}, T_\beta^{n_b} > T, T \leq \tilde{t}$, where $y_h^a(u) = y_l^a(u) = \bar{y}_l^a(u) = 0$ for $u \geq T - \tau$, and $\overline{\Pi}(n_a, n_b) > \Pi(n_a, n_b)$ for all $0 < n_a \leq n_M^a$. Let, $T_\beta = \min_{n_b \leq n_M^b} T_\beta^{n_b}$. Then, $T_\beta > T$ and under the planning horizon $T_\beta - \tau$ we have $\overline{\Pi}(n_a, n_b) > \Pi(n_a, n_b)$ for all $0 < n_a \leq n_M^a$, $n_b \leq n_M^b$, and $\overline{\Pi}(0, n_b) = \Pi(0, n_b)$, $n_b \leq n_M^b$. Furthermore, for $\bar{\beta}$, $\bar{\beta} > \beta$, by Proposition 4.7 (ii) we will have $T_{\bar{\beta}}^{n_b} > T_\beta^{n_b}$, and $T_{\bar{\beta}} = \min_{n_b \leq n_M^b} T_{\bar{\beta}}^{n_b}$ where, $T_{\bar{\beta}} > T_\beta$. □

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