5-6-2011 12:00 AM

Arrangements of Submanifolds and the Tangent Bundle Complement

Priyavrat Deshpande, *The University of Western Ontario*

Supervisor: Graham Denham, *The University of Western Ontario*
Joint Supervisor: Jan Minac, *The University of Western Ontario*

A thesis submitted in partial fulfillment of the requirements for the Doctor of Philosophy degree in Mathematics

Arrangements of Submanifolds and the Tangent Bundle Complement

(Thesis Format: Monograph)

by

Priyavrat Deshpande

Department of Mathematics

A thesis submitted in partial fulfillment
of the requirements for the degree of

Doctor of Philosophy

The School of Graduate and Postdoctoral Studies
The University of Western Ontario
London, Ontario, Canada

THE UNIVERSITY OF WESTERN ONTARIO
SCHOOL OF GRADUATE AND POSTDOCTORAL STUDIES

Chief Supervisor:

Professor Graham Denham

Co-supervisor:

Professor Jan Minac

Examining Board:

Professor Mario Salvetti

Professor Rick Jardine

Professor Masoud Khalkhali

Professor Greg Reid

The thesis by
Priyavrat Deshpande

entitled:
Arrangements of Submanifolds and the Tangent Bundle Complement

is accepted in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

Date:

Chair of Examining Board
(Firstname Lastname)

Abstract

Drawing parallels with the theory of hyperplane arrangements, we develop the theory of arrangements of submanifolds. Given a smooth, finite dimensional, real manifold X we consider a finite collection $\mathcal{A}$ of locally flat codimension 1 submanifolds that intersect like hyperplanes. To such an arrangement we associate two posets: the *poset of faces* (or strata) $\mathcal{F}(\mathcal{A})$ and the *poset of intersections* $L(\mathcal{A})$. We also associate two topological spaces to $\mathcal{A}$. First, the complement of the union of submanifolds in X which we call the *set of chambers* and denote by $\mathcal{C}(\mathcal{A})$. Second, the complement of union of tangent bundles of these submanifolds inside TX which we call the *tangent bundle complement* and denote by $M(\mathcal{A})$. Our aim is to investigate the relationship between combinatorics of the posets and topology of the complements.

Using the Nerve lemma we show that $M(\mathcal{A})$ has the homotopy type of a finite simplicial complex. We generalize the classical theorem of Salvetti [73, Theorem 1] for hyperplane arrangements and show that this particular simplicial complex, called the Salvetti complex and denoted by $Sal(\mathcal{A})$, is completely determined by the face poset (Theorem 3.2.7). We also characterize all the connected covers of $Sal(\mathcal{A})$, thus generalizing the work of Delucchi [23] and Paris [68], in Section 3.4. Some general results regarding the cohomology of the tangent bundle complement are also proved.

We apply the general theory developed so far to some particular cases. First we study arrangement of spheres. Among other things we obtain a closed form formula for the homotopy type of $M(\mathcal{A})$ (Theorem 4.1.12) and the spherical version of Orlik-Solomon algebra (Theorem 4.1.17). Using the fact that a sphere is the universal cover a projective space we apply all the previous results to study arrangements of projective spaces. Then we study toric arrangements. As yet another case we consider arrangements of topologically deformed hyperplanes that arise in the study of non-realizable oriented matroids.

Finally we study the set of chambers $\mathcal{C}(\mathcal{A})$ of an arrangement $\mathcal{A}$. We obtain a formula for the number of chambers in terms of the intersection poset and the Euler characteristic of intersections. This result generalizes the classical theorem of Zaslavsky for hyperplane arrangements.

Keywords : Arrangements of hyperplanes, topological combinatorics, Salvetti complex, Orlik-Solomon algebra, toric arrangements, topological representation theorem, Euler characteristic.

To My Parents and Grand Parents...

Acknowledgements

In a matter of weeks I will defend my Ph.D. thesis and fulfill a dream that I have cherished. The awareness of how I reached this point has never left me. It makes me realize that there are many people in my life who have supported my efforts. I would like to take this opportunity to express my gratitude towards all those who have helped and encouraged me.

It is a pleasure to thank my supervisor Graham Denham. He taught me the ways of a scholar. Throughout these past five years he has been a constant source of encouragement. He has made his support available in a number of ways, notably, through fruitful discussions, by running a learning seminar on Coxeter and Artin groups and also by introducing me to the research community. I am also grateful to my co-supervisor Ján Mináč whose enthusiasm and zeal for mathematics has made a long lasting impression on me. I will always remember the teaching experiences he shared with me and the fun we had playing soccer.

This thesis would not have been possible without the discussion I had with Tadeusz Januszkiewicz at the OSU mathematics department in May 2008. The entire thesis is based on a question that he posed during that discussion. I owe him my deepest gratitude.

I profited very much from the various conferences and workshops I attended. I would like to make special mention of the 'Examples of Groups' workshop organized by Indira Chatterji. Not only did I discover my thesis problem there but I also enjoyed the exciting and rigorous mathematical environment.

The essential part of this work was done while I was in Italy in the summer of 2010. My sincere thanks to Mario Salvetti for organizing the intensive research period in Pisa and also for being on my examining committee. I learned a lot from discussions with Eric Babson, Michael Falk, Alex Suciu, Thomas Zaslavsky and Rade Živaljević.

I would also like to thank Emanuele Delucchi. I look up to him as my mentor in this field. His thesis has certainly influenced my work.

I am indebted to the Mathematics department at the University of Western Ontario. Together with higher mathematics I also enjoyed various other aspects of the graduate life. I am glad to have met friends like Richard Gonzales, Mehdi Garrousian and Enxin Wu who have made my graduate life richer.

I would like to express my gratitude towards my land-lady Ms. Lee and my friends from the Indian community of London; their love and warmth always kept me going. For the very same reasons I am grateful to all my friends from Jnana Prabodhini (India).

My wife Kavita put a lot of efforts into carefully proof-reading my thesis. She always took an active interest in my work in addition to pursuing her own Ph.D. She has been a patient listener, a critic, an English teacher and a lot more. No words are enough to express my gratitude for her love and support.

I shall never be able to thank my parents enough. I owe all my success to them. This thesis is dedicated to my father who is my intellectual hero and a life-long role model.

Priyavrat Deshpande
London, Ontario
March 24, 2011

Contents

List of Figures

Preface

An arrangement of hyperplanes is a finite set $\mathcal{A}$ consisting of linear (or affine) codimension 1 subspaces of $\mathbb{R}^l$. These hyperplanes and their intersections induce a stratification of the ambient space. The strata form a poset when ordered by inclusion and the set of all possible intersections forms a poset ordered by reverse inclusion. These posets contain important combinatorial information about the arrangement. The topological objects associated with an arrangement $\mathcal{A}$ are the *real complement* $\mathcal{C}(\mathcal{A})$ and the *complexified complement* $M(\mathcal{A})$. The real complement is the complement of the union of hyperplanes in $\mathbb{R}^l$, whereas the complexified complement is the complement of the union of the complexified hyperplanes in $\mathbb{C}^l$. Note that, unlike $\mathcal{C}(\mathcal{A})$, the topological space $M(\mathcal{A})$ is connected. There is also the *link of an arrangement* which is defined to be the union of all hyperplanes.

One of the important aspects of the theory of arrangements is to understand the interaction between the combinatorial data of an arrangement and the topology of these complements or of the link. For example, one would like to comprehend to what extent the combinatorial data of an arrangement control the topological invariants, such as (co)homology or homotopy groups etc., of these related spaces.

A fundamental problem concerning $\mathcal{C}(\mathcal{A})$ is to express the number of *chambers* (i.e. the number of its connected components) in terms of the combinatorial data. This problem has a very long history going back to the works of Jacob Steiner. We refer the reader to [41] for a comprehensive account. A pioneering result in this direction was proved by Zaslavsky in [92]. Let L denote the lattice of intersections of these hyperplanes, the characteristic polynomial of $\mathcal{A}$ is defined by

$$\chi(\mathcal{A}, t) = \sum_{X \in L} \mu(\hat{0}, X) \cdot t^{dim X}$$

where μ is the Möbius function. Zaslavsky showed that $|\mathcal{C}(\mathcal{A})| = \chi(\mathcal{A}, -1)$. Recently, Ehrenborg et al [30] have studied the arrangements of toric hyperplanes on a torus and extended Zaslavsky's result in this context.

Since the complexified complement $M(\mathcal{A})$ is connected, the natural questions that arise are about its cohomology, homotopy type and the fundamental group. The cohomology calculations of an arrangement complement are credited to Brieskorn. He computed the de Rham cohomology ring and showed that it is generated in degree 1 by certain logarithmic forms. He also showed that the cohomology groups of $M(\mathcal{A})$ have a finer grading indexed by the intersections of the hyperplanes. Orlik and Solomon found the description of the (integral) cohomology ring of the complement. They showed that it is the quotient of the exterior algebra on the set of hyperplanes by an ideal determined by the underlying combinatorics. We refer the reader to [62, Chapter 2, 5] and [91] for more on the *Orlik-Solomon algebras*.

The topological study of the complement of a complexified hyperplane arrangement can be traced back to the work of Arnold on braid groups and configuration spaces. His results led Brieskorn to conjecture that the (complexified) complement of a real reflection arrangement is a $K(\pi, 1)$ space (i.e. its universal cover is contractible). In 1973, Deligne settled this conjecture in [22]. His first step was to parametrize the universal cover of the complement by sequences of adjacent chambers (popularly known as either the positive paths or the galleries in the Coxeter complex). Using the earlier work of Garside [37] on the conjugacy problem for braid groups he showed that these positive paths can be expressed in a particular normal form. This normal form was the main ingredient in proving that the constructed universal cover was contractible. We should also note that it is still an open question whether having a $K(\pi, 1)$ complement is a combinatorial property. Deligne's work has greatly influenced research in hyperplane arrangements and geometric group theory. For example, existence of a bi-automatic structure in finite type Artin groups as proved in [12] and the discovery of Garside groups in [21] are consequences of Deligne's ideas.

The combinatorial nature of Deligne's proofs led to the search for *combinatorial models for the complement*, i.e., cell complexes built using the combinatorial data of an arrangement that are homotopy equivalent to the complement. One such model, the *Salvetti complex*, was introduced by Salvetti in [73]. He showed that the incidence relations in the face poset are sufficient to construct the Salvetti complex either as a simplicial or as a CW complex. His construction was generalized to complex arrangements in [7] where the authors also showed that the face poset determines the homeomorphism type of the complement. Combinatorial models for the connected covering spaces have also been constructed, see for example [23, 68].

As for the fundamental group, Salvetti came up with a presentation and several attempts by other mathematicians also followed. However, the task of expressing the fundamental group in terms of the combinatorial data is far from complete. Unfortunately because of the space limitations we skip the long but very exciting discussion regarding homotopy groups of the complement.

We hope that by now the reader is convinced that the theory of hyperplane arrangements provides a rich interplay between combinatorics and topology. In the present thesis we generalize hyperplane arrangements to the level of manifolds. We introduce the notion of *arrangement of codimension* 1 *submanifolds*, where the local picture is like hyperplane arrangements. We study the real complement and the tangent bundle complement, an analogue of the complexified complement. In this context we proceed along a similar line of inquiry -

do the combinatorics of an arrangement of submanifolds offer insight about the topology of corresponding complements ?

After a prelude, the intention of which is to lay the foundations and state the relevant background, we state the results obtained during our Ph.D. work that generalize results about the hyperplane arrangements.

Thesis Organization

Chapter 1: We begin this chapter by a review of basic toplogical techniques used in combinatorics. We mention topological aspects of posets and combinatorics associated with certain cell complexes. We then move on to real arrangements of hyperplanes in Section 1.2. Here we introduce basic notions and state Zaslavsky's theorem, that first appeared in his thesis and then in [92], regarding the number of connected components. Section 1.3 is about the cohomology of the complexified complement (i.e. the Orlik-Solomon algebra) and related combinatorics. We then review the construction of the Salvetti complex in Section 1.4 and recall the basics of oriented matroids in 1.5.

Chapter 2: In recent years a lot of sophisticated techniques from homotopy are being used in combinatorial applications and the theory of arrangements is no exception. In this chapter we focus on one such tool, the homotopy colimits. Inspired by the beautiful paper [89] of Welker, Ziegler and Živaljević, we have taken a concrete view point to explain homotopy colimits in Section 2.1. In Section 2.2 we present a direct derivation of the Bousfield-Kan spectral sequence to compute the cohomology groups of a homotopy colimit. Such a discussion of homotopy theoretic methods is incomplete without mentioning the role played by groupoids. This is discussed in Section 2.3. In each of these sections we survey the arrangement literature in order to present the results that were proved using these homotopy theoretic techniques.

Chapter 3: This chapter is the crux of our work. Here we introduce the submanifold arrangements and generalize some important results due to Deligne and Salvetti in this setting. In Section 3.1, we first isolate the setting to which hyperplane arrangements can be generalized. We then define *the arrangement of submanifolds of codimension* 1 and present some examples. After this we introduce the tangent bundle complement, which generalizes the complexified complement, in Section 3.2. We also prove that it has the homotopy type of a finite dimensional simplicial complex. We show that this simplicial complex is determined by the combinatorics of the incidence relations obtained by submanifold intersections. In Section 3.3 we construct a regular CW complex which also has the homotopy type of this complement. This particular cell structure helps us better understand the relationships between the combinatorics and the topology in the context of submanifold arrangements. In Section 3.3.1 we show that this CW complex has a special combinatorial structure which is very similar to that of zonotopes. We use the theory of *metrical-hemisphere complexes* (first introduced in [74]) to explain this combinatorial structure. Then we describe and characterize the connected covering spaces of the tangent bundle complement using the so called *arrangement groupoid* in Section 3.4. In Section 3.5 we describe how the relations in the fundamental group of the complement depend on the face poset of the arrangement. Section 3.6 is about the universal cover of a tangent bundle complement; here we have generalized some aspects of Deligne's original results proved in [22]. Finally, in Section 3.7 we use homotopy theoretic tools introduced in the last chapter to obtain a spectral sequence converging to cohomology of the tangent bundle complement.

Chapter 4: The chapter is divided into two sections. In Section 4.1 we look at arrangements of spheres. We start by defining what we mean by a codimension 1 subsphere. It is well known that there are infinitely many embeddings of a codimension 1 subsphere in a sphere [72, Section 2.6]. In order to avoid pathologies and subtleties we restrict our attention to the so-called *tame sub-spheres* [72, Section 1.8]. The main theorem of this section is about the homotopy type of the tangent bundle complement. We prove that the complement contains a wedge of equi-dimensional spheres and obtain a closed form formula. We then move on to arrangements in projective spaces. Using the fact that a sphere is the universal covering space of a projective space we analyze the homotopy type of the tangent bundle complement. In Section 4.2 we study arrangements of tori. Recently these arrangements have received a lot of attention.

Chapter 5: Here we concentrate on arrangements that correspond to non-realizable oriented matroids. In Section 5.1 we quickly review oriented matroids and the topological representation theorem. In order to avoid undue topological subtleties we restrict our attention to pseudosphere arrangements in the standard unit sphere. We prove that to every such pseudosphere arrangement there corresponds an arrangement of topologically deformed hyperplanes (pseudohyperplanes) in the ambient Euclidean space. In Section 5.2 we first associate a connected subset of $\mathbb{R}^{2l}$ to an arrangement of pseudohyperplanes. We then proceed to prove that the spine of this space is the Salvetti complex of the corresponding oriented matroid. Finally we apply some of the theorems proved in Chapter 3 to this setting.

Chapter 6: This chapter is about the generalization of Zaslavsky's result to arrangements of submanifolds. In Section 6.1 we revise the theory of valuations on a poset and the Euler characteristic as they are the main ingredients for our proof. Then in Section 6.2 after introducing a generalization of the characteristic polynomial we will establish a formula that combines the geometry and combinatorics of the intersections in order to count the number of chambers. We compare Zaslavsky's proof in [93] with ours. Finally in Section 6.3 we look at some particular cases of manifolds and comment about the f-vectors arising due to submanifold arrangements.

Chapter 7: In this chapter we first summarize all our important results. Finally we mention some unanswered questions and applications. Notably we outline a generalization of Coxeter arrangements. Finite Coxeter groups act on smooth manifolds in the similar way they act on Euclidean spaces (see [18, Chapter 10]). The submanifold arrangements that arise due this action are locally the Coxeter arrangements of hyperplanes. The groups that arise in this context generalize Artin groups. We hope to apply the results proved in this thesis to study these new types of groups.

Chapter 1

Arrangements of Real Hyperplanes

Hyperplane arrangements is a young branch of mathematics which has gained a lot of attention in recent years. This subject represents a beautiful blend of combinatorics, algebra and topology. It employs techniques from diverse areas such as convex geometry, homotopy theory, algebraic geometry and then answers questions in seemingly unrelated areas like geometric group theory and robotic motion planning. The book *Arrangements of Hyperplanes* [62] by Peter Orlik and Hiroaki Terao is a very good reference for understanding the modern developments of this subject.

As a simple example of hyperplane arrangements one can consider a finite collection of lines in the Euclidean plane. These lines intersect in finitely many points and the complement of the union of these lines consists of a finite number of polygonal regions. Hence to an arrangement of lines one can associate a 2-dimensional *cell complex* into which these lines decompose the plane. Such arrangements and their several generalizations have been studied by mathematicians starting from the early 19th century. For example, consider the purely combinatorial problem of counting number of connected components or partition problems in Euclidean spaces etc. Grünbaum's book [41] is an excellent reference for early work in this field. For more about the history of this subject post Grünbaum's book watch the lecture by Mike Falk [32].

This chapter reviews basic concepts and definitions that are relevant to the main work of the thesis. The main motivation is to show how the combinatorial information associated with an arrangement determines the topology of related spaces. We start this introductory chapter by recalling some essential tools that we need from Topological Combinatorics. For comprehensive accounts we direct the reader to the thesis of James Walker [86], survey by Björner [4] and (a more recent survey by) Wachs [85].

1.1 Topology of Posets

The main theme underlying our work is to understand interactions between combinatorics and topology in a particular context. Most of the tools used in our work reflect this theme. For the sake of completeness and in order to fix notation we lay out some foundation in this section.

A finite **(abstract) simplicial complex** is a finite set V (*vertex set*) together with a family Δ of nonempty subsets of V (called *simplices* or *faces*) such that if $X \in \Delta$ and $Y \subseteq X$ then $Y \in \Delta$. The *dimension* of each simplex is one less than its cardinality as a set and the dimension of Δ is the maximum of dimensions of its simplices. The complex consisting of all nonempty subsets of a $(d+1)$-element set is called the *d-simplex*. Let Δ_1, Δ_2 be two simplicial complexes. A **simplicial mapping** of Δ_1 into Δ_2 is map $f\colon V_1 \to V_2$ that takes simplices to simplices.

Now we associate a topological space to these abstract simplicial complexes. A **geometric n-simplex** is the convex hull of the set V of $d+1$ affine independent points in $\mathbb{R}^N$, for some $N \geq n$. The convex hulls of the subsets of V are called **subsimplices**. A closely related notion of **standard n-simplex** is defined as the convex hull of standard unit basis in $\mathbb{R}^{n+1}$. The following construction (and its generalizations) will be used as a path to move from combinatorics to topology.

Definition 1.1.1. Given a finite abstract simplicial complex Δ, its **standard geometric realization** is the topological space obtained by taking the union of standard k-simplex in $\mathbb{R}^{|V|}$, for all simplices of dimension k.

Any topological space that is homeomorphic to the standard realization of Δ is called the **geometric realization** of Δ, and is denoted by $|\Delta|$.

By convention a topological statement about an abstract simplicial complex is actually a statement about its geometric realization. For the sake of simplicity we will not differentiate between abstract simplicial complex and its geometric realization. We will let the context decide. A simplicial map between two simplicial complexes induces a continuous map between the corresponding geometric realizations. In some cases embedding of the geometric realization in some ambient space is important, hence the notion of a geometric simplicial complex is also defined. However in this thesis there will be no occasion to deal with this notion. Hence we will not go into technical details and just note that the realization of an abstract simplicial complex is an example of a geometric simplicial complex. On the other hand by considering the vertex set of a geometric simplex as a face, we get an abstract simplicial complex from a geometric one.

Definition 1.1.2. A *partially ordered set*, or simply *poset*, $P = (P, \leq)$ is a set together with a relation $\leq$ that satisfies the following three axioms:

1. *idempotency*: for any $x \in P$, we have $x \leq x$;

2. *antisymmetry*: for any $x, y \in P$, if $x \leq y$ and $y \leq x$ then $x = y$;

3. *transitivity*: for any $x, y, z \in P$, if $x \leq y$ and $y \leq z$ then $x \leq z$.

Unless stated otherwise P is a finite set. The subposet $P_{\leq x} := \{y \in P \mid y \leq x\}$ is called the **principal ideal** generated by x. (The notions $P_{\geq x}, P_{<x}, P_{>x}$ are defined analogously.) For $x \leq y$ define the *open interval* $(x, y) := P_{>x} \cap P_{<y}$ and the closed interval $[x, y] := P_{\geq x} \cap P_{\leq y}$. A totally ordered subset $x_0 < x_1 < \cdots < x_k$ is called a *chain of length* k. The length of $P_{\leq x}$ is called *height of* x in P. Since P is finite every element in P has finite height. Such posets are also called as *ranked posets*. The rank function $r \colon P \to \mathbb{N}$ is an order preserving function on the poset P. The dual P^* of a poset P is the poset obtained by reversing the order. A *poset map* is an order preserving (or reversing) function between two posets. Finally, the **barycentric subdivision** of a poset P which is denoted by $sd(P)$ is the poset of finite nonempty chains of P, ordered by inclusion.

Definition 1.1.3. The **order complex** $\Delta(P)$ of a poset P is the abstract simplicial complex whose vertices are all elements of P and whose k-faces are the k-chains of P. Again, we will not differentiate between the order complex and its geometric realization.

A poset map between two posets induces a simplicial (or continuous) map between the corresponding order complexes. Given a simplicial complex Δ, the set of its faces ordered by inclusion forms a poset which is called as the *face poset*. The order complex of this face poset is called the *(first) barycentric subdivision* denoted by $sd(\Delta)$. The spaces $|\Delta|$ and $|sd(\Delta)|$ are homeomorphic. Thus from a topological viewpoint simplicial complexes and posets can be considered equivalent notions.

There is one more natural way to associate a topological space to posets. The ideals of a poset P form the basic open sets of a topology on P which is called as the *ideal topology*. In most of the cases this topology is just T_0 (i.e., any two points are topologically distinguishable). This is the weakest separation condition and such topological spaces are also known as the Kolmogorov spaces. We will not be using this topology in our work and hence skip any detailed discussion about it. However, in the next chapter this topology will appear briefly when we describe sheaves over posets (see Section 2.2). The classical resources for the study of the ideal topology are [56, 83] and a more recent work which deals with general (polyhedral) complexes is [2].

The notion that encodes combinatorial as well as the topological information about a poset P is the Möbius function $\mu\colon P \times P \to \mathbb{Z}$. It is defined recursively as follows:

$$\begin{aligned}\mu(x,x) &= 1, \quad \text{for all } x \in P \\ \mu(x,y) &= -\sum_{x \leq z < y} \mu(x,z), \text{ for all } \ x < y \in P\end{aligned}$$

A poset is called *bounded* if it has a unique minimum element $\hat{0}$ and a unique maximum element $\hat{1}$. For such a poset:

$$\mu(P) := \mu(\hat{0}, \hat{1})$$

In case a poset is not bounded then one can define the *augmented poset* as $\hat{P} := P \bigcup \{\hat{0}, \hat{1}\}$, note that if P is bounded then $(P, \leq) \cong (\hat{P}, \leq)$. The Möbius function of a poset is used to obtain inversion formulas (which in some sense generalize the principal of inclusion-exclusion).

Lemma 1.1.4. *Let P be a poset and let $f, g\colon P \to \mathbb{C}$. Then*

$$g(y) = \sum_{x \leq y} f(x)$$

if and only if

$$f(y) = \sum_{x \leq y} \mu(x,y) g(x).$$

Let P be a finite poset with $\hat{0}$. The **characteristic polynomial** of P is defined as the finite sum $\sum_{x \in P} \mu(\hat{0}, x) \cdot t^{r(x)}$. The absolute value of the coefficient of t^k in the characteristic polynomial is called the *k-th Whitney number (of the second kind).*

The main reason to mention this function is its connection with the Euler characteristic. The reduced Euler characteristic of a simplicial complex Δ is defined to be

$$\tilde{\chi}(\Delta) := \sum_{i=-1}^{dim \Delta} (-1)^i f_i(\Delta)$$

where $f_i(\Delta)$ is the number of i-faces of Δ.

Lemma 1.1.5. (Hall's Theorem) *For any poset P,*

$$\mu(\hat{P}) = \tilde{\chi}(\Delta(P))$$

Since the Euler characteristic is a topological invariant the value $\mu_P(x,y)$ depends only on the topology of the open interval (x,y) of P. By (co)homology of a poset, we usually mean the reduced simplicial (co)homology of its order complex.

Next we consider the **Nerve Lemma**, which often helps simplify a given topological space for combinatorial applications. Recall that an *open covering* of a topological space is a collection of open subsets of the space such that their union is the space itself. The *nerve of an open covering* is a simplicial complex whose vertices correspond to the open sets and a set of $k+1$ vertices spans a k-simplex whenever the corresponding $k+1$ open sets have a nonempty intersection (we will refer to the geometric realization of a nerve as the nerve itself). In general the nerve need not reflect the topology of the ambient space, but the following result (famously known as the Nerve lemma) gives a useful condition when it does. The Nerve Lemma is usually attributed to Borsuk. However there are many variations of this theorem in the literature (see [4, Section 10] for history and [50, Theorem 15.21] for the proof).

Theorem 1.1.6. (Nerve Lemma)*: If $\mathcal{U}$ is a finite open cover of a topological space X such that every non empty intersection of open sets in $\mathcal{U}$ is contractible, then $X \simeq$ nerve $(\mathcal{U})$.*

We end this section by mentioning some basic facts about the cell complexes that frequently occur in Topological Combinatorics. It is often the case that given some combinatorial data one would like to synthetically construct a topological space consistent with the data. The geometric simplicial complex is one such example. Here we describe two classes of cell complexes that are sufficiently close to simplicial complexes. A *convex polytope* F is a bounded subset of $\mathbb{R}^d$ that is the solution of a finite number of linear inequalities and equalities.

Definition 1.1.7. A Hausdorff space X is called a **polyhedral (polytopal) complex** if there exists a family $\mathcal{F}$ of subsets of X called as the faces such that:

1. every element of $\mathcal{F}$ is a convex polytope;
2. $\bigcup_{F\in\mathcal{F}} F^\circ = X$;
3. if $F, F' \in \mathcal{F}$ and $F \neq F'$ then $F^\circ \cap (F')^\circ = \emptyset$ but $F \cap F' \in \mathcal{F}$

where F° is the relative interior of F.

From the definition it follows that simplicial complexes are examples of polytopal complexes. If we allow all the polytopes to be cubes of various dimensions then we have a cubical complex.

Next we consider the *regular cell complexes.* A subset e of a topological space X is called a *closed* (*open*) *k-cell* if it is homeomorphic to (interior of) the standard k-ball in $\mathbb{R}^k$.

Definition 1.1.8. A **regular cell complex** $(X, \mathcal{C})$ is a pair consisting of a Hausdorff space X and a finite collection $\mathcal{C}$ of open cells in X such that

1. $X = \bigcup_{e \in \mathcal{C}} e$,
2. the boundary $\overline{e} \setminus e$ of each cell is a union of some members of $\mathcal{C}$.

A polytopal complex is clearly a regular cell complex, whose closed cells are the participating polytopes and the whose underlying space is the union of the polytopes. Now we state a result concerning regular cell complexes which is important from the combinatorial view point.

Theorem 1.1.9. *Let $(X, \mathcal{C})$ be a regular cell complex and $\mathcal{F}(X)$ denote its face poset. Then*

$$\Delta(\mathcal{F}(X)) \cong X.$$

Furthermore, this homeomorphism can be chosen so that it restricts to a homeomorphism between $\overline{e}$ and $\Delta(\mathcal{F}_{\leq e})$, for all $e \in \mathcal{C}$.

Finally an important and relevant corollary of the above theorem.

Corollary 1.1.10. *Every d-dimensional regular cell complex can be embedded into $\mathbb{R}^{2d+1}$ so that its barycentric subdivision is a geometric simplicial complex.*

1.2 Basics of Hyperplane Arrangements

Hyperplane arrangements arise naturally in geometric, algebraic and combinatorial instances. They occur in various settings such as finite dimensional projective or affine (vector) spaces defined over field of any characteristic. In this section we will formally define hyperplane arrangements and the combinatorial data associated with it in a setting that is most relevant to our work.

Definition 1.2.1. A real, central *arrangement of hyperplanes* is a collection $\mathcal{A} = \{H_1, \ldots, H_k\}$ of finitely many codimension 1 subspaces (*hyperplanes*) in $\mathbb{R}^l$, $l \geq 1$. Here l is called as the *rank* of the arrangement.

If we allow $\mathcal{A}$ to contain affine hyperplanes (i.e., translates of codimension 1 subspaces) we call $\mathcal{A}$ an affine arrangement. However we will mostly consider central arrangements. Hence, an arrangement will always mean central, unless otherwise stated. We also assume that all our arrangements are *essential*, it means that the intersection of all the hyperplanes is the origin. For an affine subspace X of $\mathbb{R}^l$, the *contraction* of X in $\mathcal{A}$ is given by the sub-arrangement $\mathcal{A}_X := \{H \in \mathcal{A} \mid X \subseteq H\}$. The hyperplanes of $\mathcal{A}$ induce a stratification (cellular decomposition) on $\mathbb{R}^l$, components of each stratum are called *faces*.

There are two posets associated with $\mathcal{A}$, namely, the face poset and the intersection lattice which contain important combinatorial information about the arrangement.

Definition 1.2.2. The *intersection lattice* $L(\mathcal{A})$ of $\mathcal{A}$ is defined as the set of all intersections of hyperplanes ordered by reverse inclusion.

$$L(\mathcal{A}) := \{X := \bigcap_{H \in \mathcal{B}} H \mid \mathcal{B} \subseteq \mathcal{A}, X \neq \phi\}, \quad X \geq Y \Leftrightarrow X \subseteq Y$$

Note that for affine arrangements, set of all intersections only form a poset and not a lattice.

Definition 1.2.3. Let $\mathcal{A}$ be an arrangement with its intersection lattice $L(\mathcal{A})$ and let μ be the Möbius function of the lattice. Define *characteristic polynomial* of $\mathcal{A}$ as

$$\chi(\mathcal{A}, t) := \sum_{X \in L} \mu(X) \cdot t^{\dim(X)}$$

Definition 1.2.4. The *face poset* $\mathcal{F}(\mathcal{A})$ of $\mathcal{A}$ is the set of all faces ordered by inclusion: $F \leq G$ if and only if $F \subseteq \overline{G}$.

Codimension 0 faces are called *chambers*, the set of all chambers will be denoted by $\mathcal{C}(\mathcal{A})$. A chamber is bounded if and only if it is a bounded subset of $\mathbb{R}^l$. Two chambers C and D are adjacent if they have a common face. As the complement of the hyperplanes in $\mathbb{R}^l$ is disconnected, a natural question is to ask if the number of chambers depend on the intersection data. Zaslavsky in his fundamental treatise [92] studied the relationships between the intersection lattice of an arrangement and the set of chambers. He developed the enumeration theory for hyperplane arrangements by exploiting the combinatorial structure of the intersection lattice. His main result is as follows:

Theorem 1.2.5 (Theorem A [92])**.** *Let $\mathcal{A}$ be a central hyperplane arrangement in $\mathbb{R}^l$ with $L(\mathcal{A})$ as its intersection lattice and $\chi(\mathcal{A}, t)$ be the associated characteristic polynomial. Then the number chambers is given by $(-1)^l \chi(\mathcal{A}, -1)$ and the number of bounded chambers is given by $(-1)^l \chi(\mathcal{A}, 1)$.*

For the details regarding applications and more results of this kind see also [6] and [62].

An interesting space associated with a real hyperplane arrangement $\mathcal{A}$ is its *complexified complement* $M(\mathcal{A})$ which is defined as follows:

Definition 1.2.6.

$$M(\mathcal{A}) := \mathbb{C}^l \setminus (\bigcup_{H \in \mathcal{A}} H_{\mathbb{C}})$$

where $H_{\mathbb{C}}$ is the hyperplane in $\mathbb{C}^l$ with the same defining equation as $H \in \mathcal{A}$.

Note that since $M(\mathcal{A})$ is of real codimension 2 in $\mathbb{C}^l$, it is connected. In fact it is an open submanifold of $\mathbb{C}^l$ with the homotopy type of a finite dimensional CW complex [62, Section 5.1]. The study of this space was initiated in the works of Fox and Neuwirth, Arnold, Brieskorn, Deligne in the 60's and 70's. Fox, Neuwirth and Arnold were interested in some topological aspects of braid groups, Deligne was interested in reflection groups whereas Brieskorn was studying singularity theory.

1.3 Cohomology of the Complement

Let us start by defining the *Orlik-Solomon algebra* associated with an arrangement. The construction of the Orlik-Solomon algebra is completely combinatorial. This algebra is also defined for complex arrangements (where hyperplanes are defined using complex equations).

Let E_1 be the free $\mathbb{Z}$-module generated by the elements e_H for every $H \in \mathcal{A}$. Define $E(\mathcal{A})$ to be the exterior algebra on E_1. For $S = (H_1, \ldots, H_p)$ $(1 \leq p \leq n)$, call S independent if $rank(\cap S) := dim(H_1 \cap \cdots \cap H_p) = p$ and dependent if $rank(\cap S) < p$. Notice the unfortunate clash of notations, this rank is different from the one used in the intersection lattice. Geometrically independence implies that the hyperplanes of S are in general position. Let $I(\mathcal{A})$ denote the ideal of E generated by all $\partial e_S := \partial(e_{H_1} \cdots e_{H_p})$, where S is a dependent tuple and ∂ is the differential in E.

Definition 1.3.1. The *Orlik-Solomon algebra* of a complexified central arrangement $\mathcal{A}$ is the quotient algebra $A(\mathcal{A}) := E(\mathcal{A})/I(\mathcal{A})$.

The following important theorem shows how cohomology of $M(\mathcal{A})$ depends on the intersection lattice. It combines the work of Arnold, Brieskorn, Orlik and Solomon. For details and exact statements of their individual results see [62, Chapter 3, Section 5.4].

Theorem 1.3.2. *Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be a complex arrangement in $\mathbb{C}^l$. For every hyperplane $H_i \in \mathcal{A}$ choose a linear form $l_i \in (\mathbb{C}^l)^*$, such that $ker(l_i) = H_i$ $(1 \leq i \leq n)$. Then the integral cohomology algebra of the complement is generated by the classes*

$$\omega_i := \frac{1}{2\pi}\frac{dl_i}{l_i},$$

for $1 \leq i \leq n$. The map $\gamma\colon A(\mathcal{A}) \to H^*(M(\mathcal{A}), \mathbb{Z})$ defined by

$$\gamma(e_H) \mapsto \omega_H$$

induces an isomorphism of graded $\mathbb{Z}$-algebras.

This theorem asserts that a presentation of the cohomology algebra of $M(\mathcal{A})$ can be constructed from the data that are encoded by the intersection lattice. Let us state one more theorem that explicitly states the role of intersection lattice in determining the cohomology of the complement.

Theorem 1.3.3. *Let $\mathcal{A}$ be a nonempty complex arrangement and for $X \in L(\mathcal{A})$ let $M_X := M(\mathcal{A}_X)$. For $k \geq 0$ there are isomorphisms*

$$\theta_k \colon \bigoplus_{X \in L_k} H^k(M_X) \to H^k(M)$$

induced by the inclusion maps $i_X \colon M \to M_X$ (where $L_k \subset L(\mathcal{A})$ consists of elements of rank k).

We end this discussion by stating a relationship between the complexified complement and the real complement. It has been shown that the Betti numbers of the complexified complement depend only on the intersection lattice. This result can be used to prove the following theorem which is also known as the M-property:

Theorem 1.3.4. $\sum_{i \geq 0} dim(H^i(M(\mathcal{A}), \mathbb{C})) = |\mathcal{C}(\mathcal{A})|$, *i.e. the number of chambers is equal to the sum of Betti numbers of $M(\mathcal{A})$.*

1.4 The Salvetti Complex

Unlike cohomology algebra it is not known to what extent the homotopy groups of the complement can be determined combinatorially. However, there exists a construction of a regular CW-complex, introduced by Salvetti, which models the homotopy type of the complexified complement. Note that this cell complex, which we denote by $Sal(\mathcal{A})$, is defined using the face poset and not the intersection lattice.

Let us first describe its cells and how they are attached. The k-cells of $Sal(\mathcal{A})$ are in one-to-one correspondence with the pairs $[F, C]$, where F is a codimension k face of the given arrangement and C is a chamber whose closure contains F. A cell labeled $[F_1, C_1]$ is contained in the boundary of another cell $[F_2, C_2]$ if and only if $F_1 \leq F_2$ in $\mathcal{F}(\mathcal{A})$ and C_1, C_2 are contained in the same chamber of (the arrangement) $\mathcal{A}_{F_1}$ (with the attaching maps being homeomorphisms).

Theorem 1.4.1 (Salvetti [73]). *Let $\mathcal{A}$ be an arrangement of real hyperplanes and $M(\mathcal{A})$ be the complement of its complexification inside $\mathbb{C}^l$. Then there is an embedding of $Sal(\mathcal{A})$ into $M(\mathcal{A})$ moreover there is a natural map in the other direction which is a deformation retraction.*

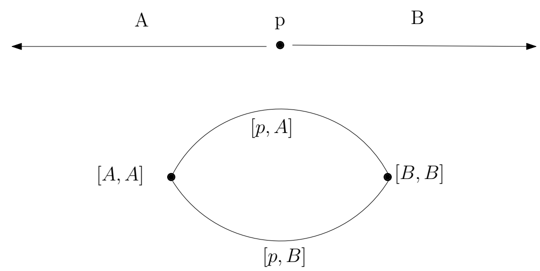


Figure 1.1: Arrangement and associated Salvetti complex.

Example 1.4.2. As an example, consider an arrangement of a point on the real line. There is one hyperplane p and two chambers A, B. The complexified complement in this case is the punctured plane which has the homotopy type of a circle. Following figure shows this arrangement and the associated Salvetti complex

As we are going to generalize this construction in Chapter 3 and extensively study its combinatorial and topological properties we skip a detailed discussion here. There is another proof of Salvetti's result by Paris in [68]. We would like to mention this result as we will adopt a similar strategy to prove our main result (Theorem 3.2.7) in Chapter 3. Consider the following poset called as the Salvetti poset (see also [69, Section 5.2]):

$$\mathcal{X} = \{(F, C) \in \mathcal{F}(\mathcal{A}) \times \mathcal{C}(\mathcal{A})\}$$

the partial order on $\mathcal{X}$ is as follows:

$$(F_1, C_1) \prec (F_2, C_2) \qquad \Longleftrightarrow \qquad F_1 \leq F_2 \text{ and } (C_1)_{F_2} = (C_2)_{F_2} \text{ as chambers of } \mathcal{A}_{F_2}$$

by C_F we mean the chamber in $\mathcal{A}_F$ which contains C.

We again denote by $Sal(\mathcal{A})$ (which is now a simplicial complex) the geometric realization of $(\mathcal{X}, \prec)$.

Theorem 1.4.3 (Paris [68])**.** *Let $\mathcal{A}$ be an arrangement of real hyperplanes and $M(\mathcal{A})$ be the complement of its complexification inside $\mathbb{C}^l$. Then for every $(F, C) \in \mathcal{X}$ there corresponds an open set of $M(\mathcal{A})$ such that the union of all these sets is an open covering of $M(\mathcal{A})$. Moreover each of these open sets are contractible and so are their intersections consequently $Sal(\mathcal{A})$ has the homotopy type of $M(\mathcal{A})$.*

As a result, every d-chain in $(\mathcal{X}, \prec)$ corresponds to a d-simplex in $Sal(\mathcal{A})$, and every simplex of $Sal(\mathcal{A})$ is of this form. Note that in general the simplicial complex $Sal(\mathcal{A})$ is the barycentric subdivision of the CW complex $Sal(\mathcal{A})$. Unlike the Orlik-Solomon algebra the Salvetti complex is defined only for the (complexified) real arrangements. See [7] for a generalization to complex arrangements.

1.5 Oriented Matroids

Oriented matroids are intimately connected to the theory of real hyperplane arrangements. As seen in the previous sections the combinatorial data associated with an arrangement is encoded in the face poset and the intersection lattice. Oriented matroids not only provide a combinatorial structure that combines the above posets but they also supply rich techniques to study arrangements. In this section we introduce oriented matroids and explain their relationship with hyperplane arrangements.

Let us first see how oriented matroids arise in the context of hyperplane arrangements. Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be an arrangement of hyperplanes in $\mathbb{R}^l$ as before. Associated with every hyperplane $H_i \in \mathcal{A}$, there are two open half-spaces bounded by the hyperplane, which will be denoted by H_i^+(plus side) and H_i^-(minus side). Accordingly, we will use H_i^0 to denote the hyperplane itself which can be called as the *zero* side. Using this we can subdivide the Euclidean space into strata of points that have the same position with respect to hyperplanes in $\mathcal{A}$. In order to achieve this we assign a sign vector $X(v) = (X_1(v), \ldots, X_n(v))$ to every point $v \in \mathbb{R}^l$ as follows:

$$X_i(v) = \begin{cases} + & \text{if } x \in H_i^+ \\ 0 & \text{if } x \in H_i^0 \\ - & \text{if } x \in H_i^- \end{cases}$$

Let $\mathcal{F}$ denote the set of all possible sign vectors that arise due to the induced stratification. It is not very difficult to verify that following properties are satisfied by $\mathcal{F}$. Obviously $|\mathcal{F}| < 2^n$. Since we are considering only central arrangements $(0, \ldots, 0) \in \mathcal{F}$. It is also clear that $-v$ realizes opposite sign configuration that of that of v. Hence, if $X \in \mathcal{F}$ then $-X \in \mathcal{F}$. Suppose that a hyperplane H separates two points v and w, but the hyperplane H' does not. Then the line segment joining v and w intersects H in a point u. It also follows that $X_H(u) = 0$, $X_{H'} \neq 0$ and if there exists a hyperplane H'' such that $X_{H''} \neq 0$ then H'' cannot contain both v and w. Finally, suppose that there are two points u and w with possibly different sign configurations and let L denote the line segment joining them. Then there exists $z \in L$ such that if $X_H(u) = 0$ then $X_H(z) \neq 0$ and for all H such that $X_H(w) \neq 0$ we have $X_H(w) = X_H(z)$.

The idea behind oriented matroids is to formalize the properties satisfied by the sign vectors of a hyperplane arrangement. Note that there are several other ways to define oriented matroids, these definitions depend only on the context in which oriented matroids arise. Essentially all the definitions are equivalent, for more details about the axioms defining oriented matroids and their equivalence see [6, Chapter 3].

Let E be a finite set and consider sign vectors $X, Y \in \{-, 0, +\}^E$. The *support* of a vector X is $\underline{X} = \{e \in E | X_e \neq 0\}$; its *zero set* is $z(X) = E \setminus \underline{X}$. The opposite of a vector X is $-X$, defined by $(-X)_e = -(X_e)$. The *zero* vector is $\mathbf{0} = (0, \ldots, 0)$. The *composition* of two sign vectors X and Y is $X \circ Y$ defined by

$$(X \circ Y)_e := \begin{cases} X_e & \text{if } X_e \neq 0 \\ Y_e & \text{otherwise} \end{cases}$$

The *separation set* of X and Y is $S(X, Y) = \{e \in E | X_e = -Y_e \neq 0\}$. With these terminologies in hand we can define oriented matroids using the *covector axioms*. These axioms generalize the geometric properties of the signed vectors (of a hyperplane arrangement) stated above.

Definition 1.5.1. A set $\mathcal{L} \subset \{-, 0, +\}^E$ (of signed vectors) is the set of covectors of an oriented matroid if and only if it satisfies:

(V0) $\mathbf{0} \in \mathcal{L}$,

(V1) $X \in \mathcal{L} \Rightarrow -X \in \mathcal{L}$,

(V2) $X, Y \in \mathcal{L} \Rightarrow X \circ Y \in \mathcal{L}$,

(V3) if $X, Y \in \mathcal{L}$ and $e \in S(X, Y)$ then there exists $Z \in \mathcal{L}$ such that $Z_e = 0$ and $Z_f = (X \circ Y)_f = (Y \circ X)_f$ for all $f \notin S(X, Y)$.

We can also put a partial ordering on the sign vectors by comparing the vectors component-wise and declaring $0 < +$, $0 < -$. It is now clear that the faces of an arrangement are nothing but the sign vectors. They satisfy the above mentioned axioms for oriented matroids and the face poset is isomorphic to the oriented matroid with the sign ordering. Hence every hyperplane arrangement gives rise to an oriented matroid and such an oriented matroid is called as *realizable*.

However it is not true that given an arbitrary oriented matroid there corresponds an arrangement of hyperplanes. Such oriented matroids are called as *non-realizable oriented matroids*. The *Folkman-Lawrence topological representation theorem* [34] states that every oriented matroid is '*almost*' realizable. There exists a generalization of hyperplane arrangements called as the pseudohyperplane arrangements and there is a one-to-one correspondence between oriented matroids and pseudohyperplane arrangements. A pseudohyperplane is a subset of the Euclidean space that is homeomorphic to a codimension 1 subspace. This homeomorphism topologically deforms the subspace in some mild way. Originally the topological representation theorem was stated in terms of *pseudo-hemisphere arrangements*. Later Arnaldo Mandel in his thesis [55] achieved much simplification using PL topology. He reproved the theorem in terms of *sphere systems* (popularly known as *arrangements of pseudospheres*). However, we end this discussion and will come back to this in Chapter (5).

Given an oriented matroid $(E, \mathcal{L})$ it can be shown that the set $L = \{z(X) | X \in \mathcal{L}\}$ forms the collection of flats of the matroid underlying $(E, \mathcal{L})$. If $(E, \mathcal{L})$ is realizable then L is the lattice of intersections of the hyperplanes. Hence an oriented matroid contains both the combinatorial data associated with an arrangement. All the results mentioned in the previous sections can now be translated in the language of matroids. For example, the Orlik-Solomon algebra can be defined for arbitrary matroids (the ideal I is defined using circuits and the basis elements of the algebra correspond to no-broken circuits).

In his thesis, Ziegler [94, Section 5.5] extended the construction of the Salvetti complex to arbitrary oriented matroids. Hence to every oriented matroid one can associate a simplicial complex and in case of a realizable oriented matroid this complex has the homotopy type of the complexified complement of the corresponding hyperplane arrangement. In their paper Gel'fand and Rybnikov [39] studied the Salvetti complex for arbitrary oriented matroids and showed that the cohomology ring of this complex is isomorphic to the Orlik-Solomon algebra of the underlying matroid (see also [7, Theorem 7.2]). For a more direct approach via discrete Morse theory see [25, Prop. 2, Lemma 5.10], but this result only proves additive isomorphism. A lot of work has been done towards studying the interaction between combinatorics of an oriented matroid and the topology of its associated Salvetti complex, see [6, chapters 2,5] for a detailed survey.

Chapter 2

Homotopy Theoretic Methods

In recent years a lot of techniques from homotopy theory have been used extensively to solve problems in topological combinatorics (see [50] for some fascinating applications and recent developments). One such technique is homotopy colimits, which is an important idea developed by Quillen, Bousfield, Kan and others in the 70's. It has not only reached remarkable extension and depth but has also proved to be a versatile tool in a lot of other areas of mathematics.

The aim of this chapter is to motivate and explain the construction of homotopy colimits and how it is used to understand the homotopy type of the arrangement complement. A lot of the literature on homotopy colimits is written in more abstract setting and uses sophisticated language of model categories. The approach we want to take here is more concrete; we would like to leave the complete generality of diagrams over arbitrary categories and focus on small, directed categories (for other concrete approaches, see [70, 84]). The main motivation for our chapter is the work of Welker, Ziegler and Živaljević [89] (before this article was published a preprint [88] was in circulation which contains more results). They developed a useful toolkit for applications of homotopy colimits in topological combinatorics and discrete mathematics.

We will start by defining homotopy colimits and then state some of the important tools. Then we will mention how they were used in the context of arrangements. In Section 2.2 we will describe how the Bousfield-Kan spectral sequence can be set up in our setting. Then we derive an explicit formula for the first page of the sequence and the first differential. We will end this chapter by a discussion on the fundamental group of homotopy colimits.

2.1 Homotopy Colimit

Given a bunch of topological spaces and maps between them, taking the *colimit* of this data intuitively means that one should form a new space by gluing the domain of every map onto its image. Colimit operation is a very common mathematical construction which is usually formulated in the language of category theory. One of the drawbacks of colimits in topology is that if one were to change any of the spaces in the data up to homotopy then the homotopy type of the colimit might change. As a result one needs *homotopically smart gluing* and it turns out that homotopy colimit is the right construction. Many familiar spaces like spheres, projective spaces, simplicial complexes, toric varieties and orbit spaces can be obtained by homotopy colimit construction. Apart from being homotopy invariant, homotopy colimit is a functorial construction and there are more than one ways of constructing it. Here we will take a combinatorial viewpoint of homotopy colimits and will use results proved in [89].

A *diagram of spaces* is a covariant functor

$$\mathcal{D} \colon P \to Top$$

from a small category P to the category topological spaces and continuous maps (in this case $\mathcal{D}$ will be called a P-diagram). Since our viewpoint is combinatorial, we will concentrate only on (finite) posets and not on any other small categories. Note that poset is a small category, with objects being elements of the poset and morphisms being defined from p to q if and only if $p \geq q$ (in which case $hom(p, q)$ consists of a single element).

Before stating the formal definition of homotopy colimits let us first explain the intuitive idea behind it. Recall that the geometric realization (*the order complex*) a poset $(P, \leq)$ is a geometric simplicial complex $\Delta(P)$ whose vertices correspond to the elements of P and k-simplices correspond to k-chains in $(P, \leq)$. Now given a diagram of spaces $\mathcal{D}$ indexed over a poset P, the homotopy colimit construction is a generalization of the order complex construction as follows:

- for each object p_0 of P, take a copy of $\mathcal{D}(p_0)$;
- for each k-chain $p_0 \to \cdots \to p_k$ of composable arrows in P, take a copy of $\mathcal{D}(p_0) \times \Delta^k$ (topological k-simplex);

and make identifications as follows -

- collapse $\mathcal{D}(p_0) \times \Delta^k$ to something smaller if it arises from a chain containing an identity arrow;

- identify $\mathcal{D}(p_0) \times \partial\Delta^k \subset \mathcal{D}(p_0) \times \Delta^k$ with the appropriate subspace arising from chains of smaller lengths;
- finally identify $\mathcal{D}(p_0) \times \Delta^0$ with $\mathcal{D}(p_1) \times \Delta^0$ via the induced map $\mathcal{D}(p_0 \to p_1)$.

The space which is obtained after taking the homotopy colimit of a diagram will be called the *homotopy colimit* of the diagram and will be denoted by *hocolim*$\mathcal{D}$. Now the formal definition:

Definition 2.1.1. Consider a diagram $\mathcal{D}\colon P \to Top$ over a poset P, then the homotopy colimit of $\mathcal{D}$ is defined as

$$hocolim\mathcal{D} := \coprod_{x \in P} (\Delta(P_{\leq x}) \times \mathcal{D}(x))/\sim$$

where Δ denotes the order complex and $P_{\leq x} := \{y \in P | y \leq x\}$. The equivalence relation $\sim$ is generated by following identifications

$$\Delta(P_{\leq x}) \times \mathcal{D}(y) \hookrightarrow \Delta(P_{\leq y}) \times \mathcal{D}(y)$$

and

$$\Delta(P_{\leq x}) \times \mathcal{D}(y) \stackrel{id \times f_{yx}}{\longrightarrow} \Delta(P_{\leq x}) \times \mathcal{D}(x)$$

where $y \geq x$ and $f_{yx}\colon \mathcal{D}(y) \to \mathcal{D}(x)$.

Remark 2.1.2. Note that the above definition is not the standard definition used in homotopy theory these days. The homotopy colimit of a diagram of spaces indexed over a small category is defined to be the geometric realization of its simplicial replacement. What we have described above is this geometric realization. Also note that the according to the standard convention the above index category (the poset) is opposite. The simplicial replacements obtained are not isomorphic, although their geometric realizations are homeomorphic. In order to be more precise we should mention that homotopy colimit of a diagram is the colimit of its cofibrant replacement. The above construction is an example of one such replacement. As our scope is limited and our focus is on combinatorial applications we end this discussion and refer the interested reader to a modern textbook on homotopy theory.

A simple example of homotopy colimit is the homotopy pushout. Consider a diagram of spaces given by

$$Z \stackrel{f}{\leftarrow} X \stackrel{g}{\rightarrow} Y$$

The homotopy pushout is formed by taking

$$Z \coprod X \coprod Y \coprod (X \times \Delta^1)_f \coprod (X \times \Delta^1)_g$$

and then identifying

- $(X \times \{0\})_f \sim X$ via 1_X and $(X \times \{1\})_f \sim Z$ via f
- $(X \times \{0\})_g \sim X$ via 1_X and $(X \times \{1\})_g \sim Y$ via g.

Our main purpose is to show the reader how these homotopy theoretic methods appear in the context of arrangements. Before stating results from the literature let us mention some tools that are available to homotopy colimits. Most of the results stated here are valid in a more general setting of homotopy theory. Since our scope is limited to combinatorial applications we will not go in greater generality, proofs of all these results can be found in the references mentioned. The first lemma is about the relationship between the homotopy colimit and the usual colimit, it also gives us the condition when they are homotopy equivalent.

Lemma 2.1.3 (Projection lemma [89,96]). *Let P be a finite poset and let $\mathcal{D}\colon P \to Top$ be a diagram of spaces such that for each $p \in P$ the induced map $colim_{P_{<p}}\mathcal{D} \to \mathcal{D}(p)$ is a* closed cofibration. *Then the natural projection map*

$$hocolim_P\mathcal{D} \to colim_P\mathcal{D}$$

is a homotopy equivalence.

The next lemma says that homotopy colimit is a homotopy invariant construction.

Lemma 2.1.4 (Homotopy lemma [89]). *Let $\mathcal{D}$ and $\mathcal{E}$ be two P-diagrams such that there is a map $\alpha_p\colon \mathcal{D}(p) \to \mathcal{E}(p)$ for every $p \in P$. If α_p is a* (weak) homotopy equivalence *for every p then the induced map*

$$\hat{\alpha}\colon hocolim_P\mathcal{D} \to hocolim_P\mathcal{E}$$

is also a (weak) homotopy equivalence.

The lemma we are now going to state has proved to be very useful in arrangements and an upcoming subject called moment angle complexes. This result describes the homotopy type of homotopy colimits under certain condition.

Lemma 2.1.5 (Wedge lemma [96]). *Let $\mathcal{D}$ be a P-diagram so that there exists points $c_p \in \mathcal{D}(p)\ \forall p \in P$ such that Image $\{(\mathcal{D}(q \to p))\} = \{c_p\}$ for all $q \to p$. Then*

$$hocolim_P\mathcal{D} \simeq \Delta(P) \vee \bigvee_{p\in P} (\mathcal{D}(p) * \Delta(P_{<p}))$$

where the wedge is formed by identifying $c_p \in \mathcal{D}(p) * \Delta(P_{<p})$ with $p \in \Delta(P)$ for all p and $*$ denotes the topological join operation.

In its simplest form the next theorem provides sufficient conditions for a map between two posets to be a homotopy equivalence at the level of order complexes. The origin of this result can be traced back to a paper of McCord in 1966 [56]. Quillen in his treatise on algebraic K-theory extended this result to establish condition for which a functor between two categories induces a homotopy equivalence between the classifying spaces (or homotopy colimits in general).

Theorem 2.1.6 (Quillen's theorem A [89])**.** *Let $f\colon P \to Q$ be a poset map such that $\Delta(f^{-1}(Q_{\geq q}))$ is contractible $\forall q$. If $\mathcal{D}$ is any Q-diagram and $f^*\mathcal{D}$ the corresponding (pull back) P-diagram then,*

$$hocolim_P(f^*\mathcal{D}) \xrightarrow{\simeq} hocolim_Q\mathcal{D}$$

We will now look at an explicit connection between order complex of a poset and homotopy colimit of a diagram of spaces.

Definition 2.1.7. Let $\mathcal{P}$ be a poset. A diagram of posets is a functor $\mathcal{D}$ such that for every $p \in \mathcal{P}$ there is a poset Q_p and $\mathcal{D}(p) = \Delta(Q_p)$.

In this situation we construct a new poset, called as poset limit of the diagram and denoted by $Plim\mathcal{D}$ as follows:

$$Plim\mathcal{D} := \coprod_{p\in\mathcal{P}} \{p\} \times Q_p$$

The order relations are

$$(p, q) \geq (p', q') \Leftrightarrow \begin{cases} p \geq p' \text{ and} \\ f_{pp'}(q) \geq q' \end{cases}$$

Note that this is a special case of the Grothendieck construction, see Section 2.3 below.

Lemma 2.1.8. *(*Simplicial Model Lemma [88, Prop. 3.19]*) Let $\mathcal{D}\colon P \to Top$ be a diagram of posets. Then*

$$hocolim\mathcal{D} \simeq \Delta(Plim\mathcal{D})$$

Now let us turn to the theory of arrangements. We will mention some of the important results that were proved using the above tools. The first work in this context goes back to a paper of Ziegler and Živaljević [96] in which they studied subspace arrangements. To be more precise, using the diagram tools mentioned above they obtain some combinatorial formulas for the homotopy type of arrangement links. With the help of these formulas they derive Goresky-MacPherson results concerning homology of an arrangement link and its complement. We will start by defining subspace arrangements.

Definition 2.1.9. Arrangement of subspaces is a finite collection $\mathcal{A} = \{A_1, \ldots, A_m\}$ of affine subspaces in $\mathbb{R}^n$ s.t.

- $\mathcal{A}$ is closed under intersection, and
- for $A, B \in \mathcal{A}$ and $A \subseteq B$ the inclusion is a cofibration.

Let P be the poset of all non-empty intersections of subspaces in $\mathcal{A}$ ordered by reversed inclusion. We also define the *link* $L := \bigcup_{i=1}^{n} A_i$.

Analogously we can define arrangements of spheres and arrangements of projective spaces. We will not go into all details here and refer the reader to [96] for precise statements and proofs. Also note that an affine hyperplane arrangement with its intersections is an example of subspace arrangement. The theorem we want to state is the following:

Theorem 2.1.10 (Homotopy type of Links [96]). *If $\mathcal{A}$ is an arrangement of subspaces with L as its link, then*

$$L \simeq \Delta(P).$$

If $\hat{\mathcal{A}}$ is the compactified affine arrangement with $\hat{L}$ as its link, then

$$\hat{L} \simeq \bigvee_{p \in P} (\Delta(P_{<p}) * S^{d(p)}).$$

If $\mathcal{A}$ is an arrangement of spheres then the homotopy type of the link is

$$L \simeq \bigvee_{p \in P} (\Delta(P_{<p}) * S^{d(p)-1})$$

where $d(p)$ is the dimension of the intersection corresponding to p.

Finally we would like to mention the work of Delucchi [23, 24] which the main motivation behind our thesis work. In his thesis Delucchi introduced *Salvetti-type diagram models* [23, Chapter 4] and *Garside-type diagram models* [23, Chaper 6] to study topological covers of the complement of a complexified real arrangement. The main reason behind Garside-type models was to (re)prove an important theorem of Deligne [22] in a completely combinatorial setting. Since this is not directly relevant at this point we will concentrate only on Salvetti-type diagram models. The results proven in Delucchi's thesis are:

Theorem 2.1.11 (Salvetti-type diagram model). *Given a real arrangement of hyperplanes $\mathcal{A}$ we define a diagram of spaces*

$$\mathcal{D} \colon \mathcal{F}(\mathcal{A}) \to Top$$

with discrete spaces

$$\mathcal{D}(F) := \{C \in \mathcal{C}(\mathcal{A}) | F < C\}$$

and maps being inclusions

$$\begin{array}{rcl} \mathcal{D}(F_1 \to F_2) \colon \mathcal{D}(F_1) & \to & \mathcal{D}(F_2) \\ C & \mapsto & F_2 \circ C \end{array}$$

Then

$$hocolim_{\mathcal{F}} \mathcal{D} \cong Sal(\mathcal{A})$$

and

Theorem 2.1.12. *Given a real arrangement of hyperplanes $\mathcal{A}$, define a diagram of spaces $\mathcal{E}$ over the dual face poset $\mathcal{F}^{op}$ by*

$$\mathcal{E}(F) := Sal(\mathcal{A}_{|F|})$$

and the maps being natural inclusion of subcomplexes

$$\mathcal{E}(F_1 \to F_2) \colon Sal(\mathcal{A}_{|F_1|}) \hookrightarrow Sal(\mathcal{A}_{|F_2|})$$

Then,

$$hocolim_{\mathcal{F}^{op}} \mathcal{E} \simeq Sal(\mathcal{A})$$

Theorem 2.1.11 can be stated in a much more general context and describes all the connected covers of the Salvetti complex. As we will need to state a few more definitions in order achieve that generality and since we are going to prove this theorem in the context of manifolds, the detailed discussion of these diagram models is postponed to Section 3.4.

2.2 Cohomology of Homotopy Colimits

In this section we will apply the classical spectral sequence techniques to compute the cohomology of homotopy colimits. In particular we would like to use the Bousfield-Kan spectral sequence [9, Chapter XI]. This spectral sequence is defined for diagrams of simplicial sets and is valid for any cohomology theory. After a brief introduction we will apply elementary algebraic topology techniques to explicitly write down the terms and the differentials of this spectral sequence. We will also mention some applications of this spectral sequence to hyperplane arrangements and combinatorics. A homological version of this spectral sequence is discussed in [70], where it is introduced as homology of posets with coefficients in a local system of groups (or diagram of groups). We will follow [27] in order to define this spectral sequence.

Let $\mathcal{D}\colon P \to Top$ be a diagram of spaces, defined over an arbitrary poset P. Then there is a spectral sequence converging to the cohomology of $hocolim\mathcal{D}$, whose second page is given by,

$$E_2^{r,s} = \varprojlim{}^r H^s(\mathcal{D}, \mathbb{Z})$$

Where,

- $H^s(\mathcal{D}, \mathbb{Z})$ is the functor $P^{op} \to Ab$ obtained by composing $\mathcal{D}$ with $H^s(-, \mathbb{Z})$.
- lim is the (left exact) limit functor $Ab^{P^{op}} \to Ab$.
- $\varprojlim{}^r$ is the r-th right derived functor of lim

The right derived functors of lim can be defined as the cohomology of a certain cochain complex (cochains with coeffecient in a functor) over $\mathbb{Z}$, details of this construction can be found in [87, page 86]. In homotopy theory these derived functors are also known as the *cohomotopy* groups of the cosimplicial replacement of a diagram. If the indexing category is a poset of rank r then these derived functors are non-zero only up to step r. The simplest example of this spectral sequence is the Mayer-Vietoris sequence. A generalization of this spectral sequence to diagrams of arbitrary categories is explained in [46].

The right derived functors of a left exact functors are reminiscent of sheaf cohomology. It is interesting to note that the Bousfield-Kan spectral sequence can be realized as the sheaf cohomology of a certain topological space. In his paper [1] Backławski introduced sheaf theory for posets and then showed that for a certain locally constant sheaf defined over geometric lattices, the ranks of its cohomology groups are precisely the Whitney numbers of the lattice. He also constructed a spectral sequence (which is a particular case of the Bousfield-Kan spectral sequence) converging to the cohomology of a poset with coefficients in a sheaf [1, Corollary 4.2] (whose first page is similar to the one mentioned in Theorem 2.2.6 below).

Let us apply the above approach to our case. Note that the order relation on a poset P induces a topology in which chains of elements are open sets (more precisely subposets like $P_{\leq x}$ form a basis for the topology). Given a diagram of spaces $\mathcal{D}$ indexed over P, this diagram defines a sheaf over P whose stalk at a point $p \in P$ is $H^*(\mathcal{D}(p), \mathbb{Z})$. After sheafification one can verify that cohomology groups of P with coefficients in this sheaf are precisely the terms on the E_2 page of the Bousfield-Kan spectral sequence.

The first application of Bacławski's result in hyperplane arrangements is the Whitney homology of the intersection lattice. It is shown that the sheaf cohomology of the intersection lattice of a hyperplane arrangement is isomorphic to the associated Orlik-Solomon algebra (see [62, Section 4.5] and [91, Theorem 3.3]). One more application of this spectral sequence is due to Yuzvinsky [90]. Let $\mathcal{A}$ be an arrangement of hyperplanes in $\mathbb{C}^l$ and let S denote the symmetric algebra of $(\mathbb{C}^l)^*$. Yuzvinsky studied certain sheaves of S-modules on the intersection lattice $L(\mathcal{A})$. Using spectral sequence techniques and some other tools, he uncovered deeper connections between combinatorics and topology of an arrangement (see also [62, Section 4.6]).

We now calculate the second page of the spectral sequence. As our diagram is defined over a ranked poset, this rank structure defines a filtration on the homotopy colimit which is used to set up the spectral sequence. The arguments we will use are elementary and the overall strategy is similar to computation of cellular cohomology. This type of spectral sequence first appeared in [79] but in more abstract setting of semi-simplicial spaces. A homology version of this spectral sequence appears in [89] to compute homology of toric varieties. Our treatment is similar to [23, Chapter 5].

Henceforth $\mathcal{F}$ denotes the face poset of a regular cell complex. Let $\mathcal{D}\colon \mathcal{F} \to Top$ be a diagram of spaces and let X denote its homotopy colimit. In order to set up a spectral sequence we will construct a filtration on X using the rank structure of $\mathcal{F}$. For $0 \le r \le rk(\mathcal{F})$, let

$$X_r := \coprod_{rk(x)\le r} (\Delta(\mathcal{F}_{\le x}) \times \mathcal{D}(x))/\sim$$

The equivalence relations are same as in the definition of X. Then clearly there is an increasing filtration

$$\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset X_{rk(\mathcal{F})} = hocolim\mathcal{D}$$

As we will consider the spectral sequence of the above filtration, we will need more knowledge of the successive quotients X_r/X_{r-1}. Let Q_r denote X_r/X_{r-1} for $r \ge 0$, then we can write

$$Q_r = [\coprod_{rk(x)\le r} \Delta(\mathcal{F}_{\le x}) \times \mathcal{D}(x)/ \coprod_{rk(y)\le r-1} \Delta(\mathcal{F}_{\le y}) \times \mathcal{D}(y)]/\sim$$

It is not very hard to see that the quotient is made up of wedge sum of pieces of following type

$$\begin{aligned} K_x &:= [\Sigma(\Delta(\mathcal{F}_{<x})) \times \mathcal{D}(x)]/[\{*\} \times \mathcal{D}(x)] \\ &= [S^r \times \mathcal{D}(x)]/[\{*\} \times \mathcal{D}(x)] \end{aligned}$$

where Σ denotes the unreduced suspension and x corresponds to an open r cell, hence suspension of $\Delta(\mathcal{F}_{<x})$ has the homotopy type of S^r. The identification is done by choosing one of the suspension points, say $\{*\}$ and then collapsing $\{*\} \times \mathcal{D}(x)$. All these pieces are glued at this distinguished base point, so the quotient is

$$Q_r = \bigvee_{rk(x)=r} K_x$$

Given an increasing filtration on a space there is a spectral sequence converging to the cohomology of the space with following $E_0^{*,*}$ and the differentials $d_0^{*,*}$, see [35, page 134].

$$E_0^{r,s} = C^{r+s}(X_r, X_{r-1}) = C^{r+s}(Q_r)$$
$$d_0^{r,s} \colon C^{r+s}(Q_r) \to C^{r+(s+1)}(Q_r)$$

Let $n = r + s$. The d_0's are the usual cochain differentials and the next page looks as follows,

$$E_1^{r,s} = H^n(Q_r) = \bigoplus_{rk(x)=r} H^n(K_x)$$

The differentials $d_1^{r,s} \colon H^n(Q_r) \to H^{n+1}(Q_{r+1})$ are the connecting homomorphisms of the long exact sequence in homology of the triples (X_{r+1}, X_r, X_{r-1}) that arise from the following short exact sequence

$$0 \to C^n(X_{r+1}/X_r) \xrightarrow{j^*} C^n(X_{r+1}/X_{r-1}) \xrightarrow{i^*} C^n(X_r/X_{r-1}) \to 0 \tag{2.2.1}$$
$$0 \to C_n(X_r/X_{r-1}) \xrightarrow{i} C_n(X_{r+1}/X_{r-1}) \xrightarrow{j} C_n(X_{r+1}/X_r) \to 0 \tag{2.2.2}$$

In order to have an explicit description of the first differentials we will compute the cohomology of K_x. For notational simplicity let us write $S^r \times Z/(\{*\} \times Z)$ for K_x.

Lemma 2.2.1.

$$H^j(S^r \times Z) = \begin{cases} H^j(Z) & \text{if } j < r \\ H^j(Z) \oplus H^{j-r}(Z) & \text{if } j \geq r \end{cases}$$

Proof. This is an application of Künneth's theorem. The cochain complex $C^*(S^r \times Z)$ is the tensor product $C^*(S^r) \otimes C^*(Z)$. Since S^r has one 0-cell and one r-cell, $C^0(S^r) = C^r(S^r) = \mathbb{Z}$ and zero for all other j. Hence $C^j(S^r \times Z) = C^j(Z)$ for $j < r$ and $C^j(S^r \times Z) = C^j(Z) \oplus C^{j-r}(Z)$ for $j \geq r$, with the differentials coming from $C^*(Z)$. □

Theorem 2.2.2. *Let Z be a CW complex and $\{*\}$ denote a distinguished base point of the r sphere S^r. Then,*

$$H^k(S^r \times Z/(\{*\} \times Z)) = \begin{cases} 0 & \text{if } k < r \\ H^{k-r}(Z) & \text{if } k \geq r \end{cases}$$

Proof. Consider the long exact sequence in cohomology for the CW pair $(S^r \times Z; \{*\} \times Z)$.

For $k < r$

$$\cdots \to H^k(S^r \times Z/\{*\} \times Z) \xrightarrow{j^*} H^k(S^r \times Z) \xrightarrow{i^*} H^k(Z) \xrightarrow{\delta} H^{k+1}(S^r \times Z/\{*\} \times Z) \to \cdots$$

Now i^* is an isomorphism due to Lemma 2.2.1, hence j^* and δ are zero homomorphisms. Therefore exactness forces the cohomology groups of the quotients to be 0. For the case $k \geq r$ we have,

$$\cdots \to H^k(S^r \times Z/\{*\} \times Z) \xrightarrow{j^*} H^k(Z) \oplus H^{k-r}(Z) \xrightarrow{i^*} H^k(Z) \xrightarrow{\delta} H^{k+1}(S^r \times Z/\{*\} \times Z) \to \cdots$$

Here i^* is surjective and therefore the kernel of j^* is $H^{k-r}(Z)$ and δ is zero. □

Corollary 2.2.3. *Let $\mathcal{D}$ be a diagram of spaces over a poset $\mathcal{F}$ which is a face poset of a regular cell complex, and consider the filtration of its homotopy colimit given by the X_r as above. Then for any $0 \leq r \leq rk(\mathcal{F})$ and for any $s \geq 0$ we have*

$$H^{r+s}(Q_r) \cong \bigoplus_{rk(x)=r} H^s(\mathcal{D}(x))$$

Now for the differentials d_1, we will give a concrete expression. Note that d_1 is the connecting homomorphism in the long exact sequence coming from the short exact sequence 2.2.1. We start with a cocycle in $C^n(Q_r)$ and obtain its image in $C^{n+1}(Q_{r+1})$, (recall that $n = r + s$). Let $[\alpha] \in H^n(Q_r) = \oplus(H^n(K_x))$, without loss of generality assume that $[\alpha] \in H^n(K_x)$ for some $x \in \mathcal{F}$ of rank r. As $K_x \simeq (S^r \times \mathcal{D}(x)/\{*\} \times \mathcal{D}(x))$, cells in K_x are of the form $x \times e$ where x is the unique r cell of S^r and e is a cell in $\mathcal{D}(x)$ (by the choice of any cell structure on $\mathcal{D}(x)$). The Künneth formula translates product of cells to the tensor product of cochains. Hence we can rewrite the cochain α as $\phi \otimes \psi \in C^r(S^r) \otimes C^{n-r}(\mathcal{D}(x))$. But it is a cocycle so the differential maps it to 0 in the tensor product resolution.

$$\begin{aligned} 0 &= \delta(\phi \otimes \psi) \\ &= \delta_{\mathcal{F}}(\phi) \otimes \psi + (-1)^r \phi \otimes \delta_{\mathcal{D}(x)}(\psi) \\ \Rightarrow \delta_{\mathcal{D}(x)}(\psi) &= 0 \end{aligned}$$

Hence the necessary condition that $\phi \otimes \psi$ is a cocycle is that ψ is a cocycle in $C^{n-r}(\mathcal{D}(x))$. The map $i \colon X_r/X_{r-1} \hookrightarrow X_{r+1}/X_{r-1}$ is an inclusion so $x \times e$ is a cell in X_{r+1}/X_{r-1} too. Hence we can consider $\phi \times \psi$ as a cochain in $C^n(X_{r+1}/X_{r-1})$. Applying the coboundary map of this cochain complex we get

$$\delta(\phi \otimes \psi) = \delta\phi \otimes \psi + (-1)^r \phi \otimes \delta\psi = \delta\phi \otimes \psi$$

as $\delta\psi = 0$. Note that ϕ is not a cocycle in $C^n(X_{r+1}/X_{r-1})$. Here $\delta\phi\otimes\psi \in C^{r+1}(S^{r+1})\otimes C^{n-r}(\mathcal{D}(x))$, but $1 \otimes f^*_{yx} \colon C^{r+1}(S^{r+1}) \otimes C^{n-r}(\mathcal{D}(x)) \to C^{r+1}(S^{r+1}) \otimes C^{n-r}(\mathcal{D}(y))$ is an isomorphism induced by the identification map f_{yx} in the homotopy colimit, for any $y > x$. It is clear that $\delta\phi \otimes f^*_{yx}(\psi)$ is a cocycle in $C^{n+1}(Q_{r+1})$.

The cellular coboundary map for $\mathcal{F}$ has an explicit expression in terms of incidence numbers,

$$\delta\phi(y) = [x : y]\phi(x) \qquad \text{where rank}(y) = r + 1$$

The map $f_{yx} \colon \mathcal{D}(x) \to \mathcal{D}(y)$ induces the map $f^*_{yx} \colon H^*(\mathcal{D}(x)) \to H^*(\mathcal{D}(y))$. The cocycle now looks like

$$\delta\phi \otimes f^*_{yx}(\psi) = \sum_{rk(y)=r+1} [x : y]\phi \otimes f^*_{yx}(\psi)$$

The tensor product on the right hand side can be removed because of the Künneth isomorphism. All this discussion is summarized in the following theorem.

Theorem 2.2.4. *Let $\mathcal{D} \colon \mathcal{F} \to Top$ be a diagram of spaces and suppose that $\mathcal{F}$ is a face poset of a regular CW complex. Then there is a spectral sequence converging to the cohomology of the homotopy colimit of $\mathcal{D}$ with the E_1 page given by*

$$E_1^{r,s} = \bigoplus_{rk(x)=r} H^s(\mathcal{D}(x))$$

and the differentials

$$d_1^{r,s}(\alpha_x) = \sum_{rk(y)=r+1} [x : y] f^*_{yx}(\alpha_x),$$

where $[x : y]$ is the incidence index of x in y as cells of the CW complex, α_x is a cohomology class in $H^s(\mathcal{D}(x))$ and f^*_{yx} is induced on cohomology by $f_{yx} \colon \mathcal{D}(y) \to \mathcal{D}(x)$.

The above result can be generalized to arbitrary posets. This is done by modifying the diagram but without changing the homotopy colimit. Let P be an arbitrary poset and $sd(P)$ denote the poset of chains in P ordered by inclusion. Then $sd(P)$ is isomorphic to the face poset of $\Delta(P)$ which is a regular cell complex. We have the following lemma.

Lemma 2.2.5. *Consider the poset map $g \colon sd(P) \to P$ given by $g(\tau) = max\ \tau$, for a chain τ in P. Given a diagram $\mathcal{D} \colon P \to Top$, define a new diagram $g^*\mathcal{D}$ as follows*

$$(g^*\mathcal{D})(\tau) := \mathcal{D}(max\ \tau)$$

$$(g^*\mathcal{D})(\tau > \tau') := \mathcal{D}(max\ \tau \geq max\ \tau') \ \ \forall \tau, \tau' \in P$$

Then the map g induces a homotopy equivalence

$$hocolim(g^*\mathcal{D}) \simeq hocolim(\mathcal{D})$$

Proof. Follows from the *inverse image lemma* in [89, lemma 4.7]. □

We can thus apply Theorem 2.2.4 to $sd(P)$ and construct a spectral sequence converging the cohomology of $hocolim\mathcal{D}$.

Theorem 2.2.6. *Let $\mathcal{D}\colon P \to Top$ be a diagram of spaces over any poset P. Then there is a spectral sequence converging to $H^*(hocolim\mathcal{D})$ with the E_1 page given by,*

$$E_1^{r,s} := \bigoplus_{x_0 < \cdots < x_r} H^s(\mathcal{D}(x_r)), \qquad x_i \in P \qquad \forall i$$

and the differentials $d_1^{r,s}\colon E_1^{r,s} \to E_1^{r+1,s}$,

$$\begin{aligned}(d_1^{r,s}\alpha)(x_0 < \cdots < x_{r+1}) &= \sum_{i=0}^{r}(-1)^i\alpha(x_0 < \cdots < \hat{x}_i < \cdots < x_{r+1}) \\ &+ (-1)^{r+1} f^*_{x_{r+1}x_r}(\alpha(x_0 < \cdots < x_r))\end{aligned}$$

where $\alpha(x_0 < \cdots < x_r)$ is a cohomology class in $H^s(\mathcal{D}(x_r))$.

Proof. Follows from Lemma 2.2.5 and Theorem 2.2.4. □

2.3 Groupoids and Arrangements

Now we look at the role played by the theory of groupoids in real arrangements. Recall that a groupoid is a small category in which every morphism is invertible. A prototypical example of groupoids is a group, which is a category with a unique object and the morphisms are the group elements. The most important groupoid that appears in arrangements is the *arrangement groupoid* (or the *Deligne groupoid*), which we describe below. It was first introduced and used by Deligne in his seminal paper [22, (1.25)]. The main use of this groupoid was to prove that the universal cover of the (complexified) complement of a simplicial arrangement is contractible. In his work, Salvetti used the category of positive paths (whose groupoid completion is the arrangement groupoid) [73, Part Two] to construct a CW structure of the complexified complement of any hyperplane arrangement. In [74], Salvetti generalized his construction to metrical-hemisphere complexes and realized the arrangement groupoid as the fundamental groupoid of the associated complexes. The first explicit definition of the arrangement groupoid that works for all real arrangements appeared in the work of Paris, he called it the *oriented system* (see [65–68]). Paris not only defined it in a way that works for all arrangements but he also used it to characterize all of the connected topological covers of an arrangement complement. The approach that we want to take here first appeared in Delucchi's thesis [23, Section 2.1]. He used the systematic language from category theory to define the arrangement groupoid and from coverings of groupoid to characterize the covers of an arrangement complement (see also [24]). Apart from hyperplane arrangements the Deligne groupoid is also useful in the study of Artin groups (see [12]) and Garside groups (see [21]).

Another reason we want to consider groupoids is that the fundamental groupoid commutes with the homotopy colimit. Calculation of the fundamental group of a space could be quite complicated and complements of hyperplane arrangements are no exception. There are various techniques available but they are not as straightforward as calculations of (co)homology. Describing the fundamental group of a hyperplane complement combinatorially is even more difficult. A central part of our work is to generalize some aspects of hyperplane arrangements and their complexification. We will see in the next chapter that in this general case the analogue of the complexified complement can be seen as a homotopy colimit. Hence in this section we concentrate on the computation of the fundamental group of a homotopy colimit.

Given a diagram of spaces we will consider the corresponding diagram of fundamental groups and then compute the homotopy colimit of this diagram. In order to define such a homotopy colimit we will have to leave the realm of groups and enter the world of groupoids. If the homotopy colimit is connected then knowing the fundamental groupoid is as good as knowing the fundamental group. The main reference for this technique is [33] (see also [45]).

2.3.1 Arrangement (Deligne) groupoid

Let $\mathcal{A}$ be a real hyperplane arrangement in $\mathbb{R}^l$. To such an arrangement we associate a directed graph whose vertices correspond to chambers. For every pair of adjacent chambers we join the corresponding vertices by a pair of oppositely oriented edges. This graph is called as the *arrangement graph* and is denoted by $G(\mathcal{A})$. To every graph there corresponds a category called as the *free category* whose objects are the vertices of the graph and the morphisms are the directed paths in that graph (see [53, II.7]).

A morphism in the free category of the arrangement graph can be written as an expression

$$\alpha = (a_1, \cdots, a_n)$$

where a_i is an edge in the graph such that the terminal vertex of a_i (denoted by $t(a_i)$) is the initial vertex of a_{i+1} (denoted by $s(a_{i+1})$) for $1 \leq i \leq n$. Then for the path α, its initial vertex $s(\alpha)$ is $s(a_1)$ and its terminal vertex $t(\alpha)$ is $t(a_n)$. If α, β are two paths such that $s(\beta) = t(\alpha)$ then they can be composed into a single path $\alpha\beta$. Length of a path is the number of edges it traverses, a path is called *minimal* if its length is least among all the paths that connect its initial and terminal vertices.

We are interested in the following two quotient categories of the free category associated with an arrangement graph.

Definition 2.3.1. The **category of positive paths** (the *positive category* or the *path category*) of an arrangement $\mathcal{A}$ is obtained from the free category of the arrangement graph by identifying every two morphisms that come from positive minimal paths having the same initial and terminal vertices. This category is denoted by $\mathcal{G}^+(\mathcal{A})$.

The objects of this category correspond to the chambers of $\mathcal{A}$ and the identification implies that given two chambers C, D they determine an equivalence class of the minimal positive paths starting at C and ending at D; we denote any morphism representing this class by $(C \to D)$.

Definition 2.3.2. The **arrangement groupoid** $\mathcal{G}(\mathcal{A})$ is obtained from the positive category by groupoid completion, i.e., adding formal inverses to every morphism.

Let us see explicitly what the morphisms in this groupoid are. Each morphism can be written as

$$\alpha = (\epsilon_1 a_1, \ldots, \epsilon_n a_n)$$

where, each a_i is an edge in the graph and $\epsilon_i \in \{\pm 1\}$ denotes the direction in which a_i is traversed. The formal inverse of α is:

$$\alpha^{-1} = (-\epsilon_n a_n, \ldots, -\epsilon_1 a_1).$$

The equivalence relation $\sim$ on these paths induced by the identification in $\mathcal{G}^+(\mathcal{A})$ satisfies the following conditions:

1. if $\alpha \sim \beta$, then $s(\alpha) = s(\beta)$ and $t(\alpha) = t(\beta)$,
2. $\alpha\alpha^{-1} \sim s(\alpha)$ for any path α,
3. if $\alpha \sim \beta$ then $\alpha^{-1} \sim \beta^{-1}$,
4. if $\alpha \sim \beta$ and γ_1 is a path such that $t(\gamma_1) = s(\alpha)$, γ_2 is a path such that $s(\gamma_2) = t(\alpha)$ then $\gamma_1\alpha\gamma_2 \sim \gamma_1\beta\gamma_2$.

Remark 2.3.3. In the language of homotopy theory [36] it would be convenient to say that the groupoid $\mathcal{G}(\mathcal{A})$ can be identified with the category of fractions of $\mathcal{G}^+(\mathcal{A})$ and there is an associated canonical functor $J\colon \mathcal{G}^+(\mathcal{A}) \to \mathcal{G}(\mathcal{A})$ (see Theorem 3.4.6 for a proof).

Remark 2.3.4. The arrangement graph is the (oriented) 1-skeleton of the Salvetti complex. With this view point the arrangement groupoid can be seen as the collection of homotopy classes of paths joining the 0-cells (i.e. the fundamental groupoid of the Salvetti complex, see Definition 2.3.11).

There is a covering theory of groupoids [11, Section 10.2] similar to that of topological spaces. For simplicity we only mention the relevant facts here. We refer the reader to [24, Section 1.4] where coverings of the arrangement groupoid are well explained using examples.

Let $\mathcal{G}$ be a groupoid and $x \in \mathrm{Ob}(\mathcal{G})$ be an object. The set of all morphisms from x to x has a natural group structure, this group is denoted by $\mathcal{G}(x)$ and is called as the object group (of $\mathcal{G}$ at x). We assume that all our groupoids are *connected*, it means that for any two objects there is a morphism going from one to the other. In case of a connected groupoid $\mathcal{G}(x) \cong \mathcal{G}(y)$ hence we will denote the object group by $\pi(\mathcal{G})$. Also, so we use the language of paths when dealing with morphisms. Finally, the *star* of the object x is the set

$$St(x) := \{\alpha \in Mor(\mathcal{G}) | s(\alpha) = x\}$$

Definition 2.3.5. A *morphism of groupoids* is a functor $\rho\colon \mathcal{G}' \to \mathcal{G}$ between groupoids. The morphism ρ is called *covering* if, for every $z \in \mathrm{Ob}(\mathcal{G}')$, the induced map

$$\rho_z\colon St(z) \to St(\rho(z))$$

is bijective. Given α, a morphism of $\mathcal{G}$ and any $z \in \rho^{-1}(s(\alpha))$, the lift of α at z is the morphism $\rho_z^{-1}(\alpha)$, and will be written $\alpha^{<z>}$ when the covering ρ is understood. The object group $\pi(\mathcal{G}')$ is isomorphically mapped to a subgroup of $\pi(\mathcal{G})$ and this subgroup is called as the *characteristic group of the covering.*

One of the fundamental things about the covering space theory is the correspondence between the subgroups of the fundamental group (of the base space) and the topological covers (of the base space). This correspondence has a nice analogue in the context of groupoids (see [11, Chapter 10]).

Theorem 2.3.6. *Let $\mathcal{G}$ be a connected groupoid, H a subgroup of $\pi(\mathcal{G})$ and x be an object of the groupoid. Consider the groupoid $\mathcal{G}'$ defined by setting $Ob(\mathcal{G}') = \{H\alpha | \alpha \in St(x)\}$, the morphisms between $H\alpha_1$ and $H\alpha_2$ correspond to morphisms β from $t(\alpha_1)$ to $t(\alpha_2)$ in $\mathcal{G}$ such that $H\alpha_1\beta = H\alpha_2$.*

Then the functor $\rho\colon \mathcal{G}' \to \mathcal{G}$ such that $H\alpha \mapsto t(\alpha)$ is a covering of groupoids with the characteristic group H.

Now let us put all this in the context of arrangements. We have already introduced the Salvetti-type diagram models in the previous section. Here we will extend those diagrams in order to characterize covers of the Salvetti complex.

Definition 2.3.7. Let $\mathcal{G}(\mathcal{A})$ denote the arrangement groupoid of a hyperplane arrangement $\mathcal{A}$. Given a cover $\rho\colon \mathcal{G}_\rho \to \mathcal{G}(\mathcal{A})$, we define a diagram of posets $\mathcal{D}_\rho$ indexed over the dual face poset $(\mathcal{F}^*, \prec)$ such that

$$\mathcal{D}_\rho(F^*) := \{v \in Ob(\mathcal{G}_\rho) | \rho(v) \prec F^*\}$$

endowed with the trivial order relation defined by setting $v_1 \leq v_2$ if and only if $v_1 = v_2$, and maps being inclusions

$$\begin{array}{rcl} \mathcal{D}(F_1^* \to F_2^*)\colon \mathcal{D}(F_1^*) & \longrightarrow & \mathcal{D}(F_2^*) \\ v & \mapsto & t(\rho(v) \to F_2 \circ \rho(v))^{<v>} \end{array}$$

where $(\rho(v) \to F_2 \circ \rho(v))^{<v>}$ is the lift of the positive minimal path $(\rho(v) \to F_2 \circ \rho(v))$ in $\mathcal{G}(\mathcal{A})$ that starts at v.

Following theorem classifies the covering spaces of the Salvetti complex.

Theorem 2.3.8. *(Delucchi) For any topological cover $p\colon S \to Sal(\mathcal{A})$ of the Salvetti complex of a locally finite real arrangement $\mathcal{A}$, there exists a cover of the arrangement groupoid $\rho\colon \mathcal{G} \to \mathcal{G}(\mathcal{A})$ such that the homotopy colimit of the associated diagram of spaces $hocolim\mathcal{D}_\rho$ is isomorphic to S as a covering space of $Sal(\mathcal{A})$.*

As a corollary to the above theorem we have

Corollary 2.3.9. *Let $\hat{\rho}\colon \hat{\mathcal{G}} \to \mathcal{G}(\mathcal{A})$ be the universal cover of $\mathcal{G}(\mathcal{A})$. Then $hocolim\mathcal{D}_{\hat{\rho}}$ is the universal cover of $Sal(\mathcal{A})$.*

Remark 2.3.10. For a simplicial arrangement $\mathcal{A}$ Deligne, in [22], first proved that the morphisms of $\mathcal{G}^+(\mathcal{A})$ satisfy a certain technical property (called the *property D*) and that the canonical functor $J\colon \mathcal{G}^+(\mathcal{A}) \to \mathcal{G}(\mathcal{A})$ is faithful. Then he used this information to show that the universal cover of the complexified complement $M(\mathcal{A})$ is contractible. Several reproofs of this result have appeared since, see [15, 23, 66, 74].

2.3.2 Fundamental group of a homotopy colimit

Definition 2.3.11. Let Y be a CW complex and A be a choice of 0-skeleton for Y. Then the *fundamental groupoid of* (Y, A) denoted by $\tilde{\pi}_1(Y, A)$ is defined as the small category whose object set is A and whose morphisms are the homotopy classes of paths between any two of these zero cells.

Before stating the main theorem let us gather some tools from homotopy theory.

Definition 2.3.12. For a small category C the trivial diagram is a functor that assigns a point to every object of C. The *nerve* (the *classifying space*) of C, denoted by $|C|$, is defined as the homotopy colimit of the trivial diagram.

In general nerve of a category can be quite complicated. On the other hand, the nerve of a groupoid G is homotopy equivalent to $\coprod_\alpha BG_{x_\alpha}$, where BG_{x_α} is the classifying space of the vertex group G_{x_α} for every isomorphism class $\{x_\alpha\}$ of the objects in the groupoid [27, section 5.10]. Given a small category C, the fundamental groupoid of C, denoted by $\tilde{\pi}_1 C$, is defined to be the fundamental groupoid of $|C|$. This is equivalent to formally inverting all the morphisms in C. The important result proved in [33], which we refer to as the *general Seifert-van Kampen-Brown Theorem* is as follows:

Theorem 2.3.13. *Let $\mathcal{D}\colon \mathcal{P} \to Top$ be a diagram of spaces and let $\tilde{\pi}_1\mathcal{D}$ denote the corresponding diagram of fundamental groupoids. Then there is a natural equivalence of groupoids*

$$\tilde{\pi}_1(hocolim\ \mathcal{D}) \xrightarrow{\cong} hocolim\ \tilde{\pi}_1\mathcal{D}$$

If the homotopy colimit is a connected space, this gives an isomorphism of groups.

Proof. See Theorem 1.1 in [33] □

The only undefined notion in the above theorem is that of homotopy colimit of a diagram of groupoids. We define this homotopy colimit using the Grothendieck construction and Thomason's theorem [27, 5.15]

Let $F\colon I \to Cat$ be a functor from a small category I to the category of small categories. The *Grothendieck construction* on F, denoted by $Gr(F)$, is the category whose objects are the pairs (i, x) where i is an object of I and x is an object of $F(i)$. An arrow $(i, x) \to (j, y)$ in $Gr(F)$ is a pair (f, g) where $f \in Hom(i, j)$ in I and $g \in Hom((F(f))(x), y)$ in $F(j)$. Arrows compose according to the rule $(f, g)\cdot(f', g') = (f'', g'')$, where f'' is the composite $f \cdot f'$ and g'' is the composite of g with the image of g' under the functor $F(f)$.

Definition 2.3.14. Let $\mathcal{G}\colon I \to Gpd$ be a diagram of groupoids. The homotopy colimit of $\mathcal{G}$ is defined as follows

$$hocolim\ \mathcal{G} := \tilde{\pi}_1 Gr(\mathcal{G})$$

We will illustrate the Theorem 2.3.13 with the following example.

Example 2.3.15. As before let $\mathcal{F}$ denote the face poset of a connected, regular CW complex Y. Consider the constant diagram $\mathcal{D}\colon \mathcal{F} \to Top$ which sends every object in $\mathcal{F}$ to a point and every morphism to the identity map. Then clearly the homotopy colimit of $\mathcal{D}$ has the homotopy type of Y and $\tilde{\pi}_1(hocolim\ \mathcal{D}) \cong \tilde{\pi}_1(Y, *)$ as isomorphism of groupoids. Now the corresponding diagram of groupoids $\tilde{\pi}_1\mathcal{D}$ is a diagram of trivial groupoids with the identity morphism. Hence the Grothendieck construction is equivalent to $\mathcal{F}$ and $\tilde{\pi}_1 Gr(\tilde{\pi}_1\mathcal{D}) \cong \tilde{\pi}_1(Y, *)$, but this is same as $\pi_1(Y, *)$ since Y is connected.

Chapter 3

Arrangements of Submanifolds

This chapter contains most of the important results in the thesis. Here, we introduce a generalization of real hyperplane arrangements which we call as the *arrangements of submanifolds of codimension* 1. We consider situations in which finitely many submanifolds of a given manifold intersect in a way that the local information is same as that of a hyperplane arrangement but the global picture is quite different. Intuitively speaking it means that for every point on that manifold there exists a coordinate neighborhood homeomorphic to an arrangement of real hyperplanes. We also introduce an analogue of the complexified complement in this new setting and call it *the tangent bundle complement*. All the results proved in this chapter attempt to answer the following question: *how does the combinatorics of the intersections of these submanifolds help determine the topology of the tangent bundle complement?*

The chapter is organized as follows. In Section 3.1, we define the new object of study, *the arrangement of submanifolds* and present some examples. After these definitions and examples we introduce the tangent bundle complement in Section 3.2 and then prove that it has the homotopy type of a finite dimensional simplicial complex. We show that this simplicial complex is determined by the combinatorics of the incidence relations obtained by submanifold intersections. The proof is similar to that of Paris [68], where he reproves the theorem of Salvetti. In Section 3.3 we construct a regular CW complex which also has the homotopy type of this complement. This particular cell structure helps us better understand the relationships between the combinatorics and the topology in the context of submanifold arrangements. In Section 3.3.1 we show that this CW complex has a special combinatorial structure which is very similar to that of zonotopes. We use the theory of *metrical-hemisphere complexes* (first introduced in [74]) to explain this combinatorial structure. Then, we describe and characterize the connected covering spaces of the tangent bundle complement in 3.4. In Section 3.5 we describe how the relations in the fundamental group of the complement depend on the face poset of the arrangement. In Section 3.6 we study the higher homotopy groups of the complement. We also isolate and restate some of Deligne's original arguments in this new setting in order to achieve a mild generalization of his results. Finally, in Section 3.7 we use homotopy theoretic tools introduced in the last chapter to obtain a spectral sequence converging to cohomology of the tangent bundle complement.

This particular generalization of hyperplane arrangements and the complexified complement is due to Tduesz Januskeweisz and Richard Scott. Their work is not published but Richard Scott delivered a lecture in October 2004 at M.S.R.I. about their work and a video recording of his talk is available at [78].

3.1 Definitions

In this section we propose a generalization of arrangements of hyperplanes. In order to do this we isolate the following characteristics of a hyperplane arrangement:

1. there are finitely many codimension 1 subspaces (with nonempty intersections) separating the ambient space,

2. there is a stratification of the ambient vector space into contractible pieces,

3. (the geoemetric realization of) the face poset (of this stratification) has the homotopy type of the ambient space.

Any reasonable generalization of hyperplane arrangements should have these properties. Since smooth manifolds are locally Euclidean they are obvious candidates for the ambient space. In this setting we can study arrangements of codimension 1 submanifolds that satisfy certain nice conditions, for example, locally, we would like our submanifolds to behave like hyperplanes.

Let us make this setting precise. An l-dimensional (topological) manifold is a separable metric space in which each point has a neighborhood (*coordinate neighborhood*) homeomorphic to $\mathbb{R}^l$. An *embedding* of a topological space X into a topological space Y is a homeomorphism $f\colon X \to Y$ of X onto a subset of Y. Two such embeddings f, g are said to be *equivalent* if there is a self homeomorphism h of Y such that $hf = g$. Our focus is on the codimension 1 smooth submanifolds that are embedded as a closed subset of a finite dimensional smooth manifold. These types of submanifolds behave much like hyperplanes. The following are some of their well known properties.

Lemma 3.1.1. *If X is a connected l-manifold and N is a connected $(l-1)$-manifold embedded in X as a closed subset, then $X \setminus N$ has either 1 or 2 components. If in addition $H_1(X, \mathbb{Z}_2) = 0$ then $X \setminus N$ has two components.*

Proof. Consider the following exact sequence of pairs in mod 2 homology:

$$H_1(X,\mathbb{Z}_2) \to H_1(X, X\setminus N, \mathbb{Z}_2) \to \tilde{H}_0(X\setminus N,\mathbb{Z}_2) \to \tilde{H}_0(M,\mathbb{Z}_2) = 0$$

The first statement follows from the duality $H_1(X, X\setminus N,\mathbb{Z}_2) \cong H_c^{l-1}(S,\mathbb{Z}_2) = \mathbb{Z}_2$. For the second statement $H_1(X,\mathbb{Z}_2) = 0$ implies that $\tilde{H}_0(X\setminus N,\mathbb{Z}_2) = \mathbb{Z}_2$. □

Definition 3.1.2. A connected codimension 1 submanifold N in X is *two sided* if N has a neighborhood U_N such that $U_N \setminus N$ has two connected components; otherwise N is said to be *one sided*. Generally a disconnected codimension 1 submanifold is two sided if each of its connected component. Moreover a submanifold *separates* X if its complement has 2 components.

Note that being two sided is in some sense a local condition. For example, a point in S^1 does not separate S^1, however, it is two sided. The following corollary follows from the definitions.

Corollary 3.1.3. *Every codimension* 1 *submanifold N is locally two sided in X; that is, each $x \in N$ has arbitrarily small connected neighborhoods U_x such that $U_x \setminus (U_x \cap N)$ has two components.*

The following results give an homological criterion for a closed embedded manifold to be two sided.

Lemma 3.1.4. *Let X be a connected l-manifold and let N be a connected, $(l-1)$-submanifold embedded in X as a closed subset. Then N separates X if and only if the inclusion induced homomorphism $H_c^{n-1}(X,\mathbb{Z}_2) \to H_c^{n-1}(N,\mathbb{Z}_2)$ (on the cohomology with compact supports) is trivial.*

Proof. Again the proof is a simple diagram chase:

$$\begin{tikzcd}
H_c^{n-1}(X,\mathbb{Z}_2) \arrow[r] \arrow[d, "\cong"'] & H_c^{n-1}(N,\mathbb{Z}_2) \cong \mathbb{Z}_2 \arrow[d, "\cong"] & & \\
H_1(X,\mathbb{Z}_2) \arrow[r] & H_1(X, X\setminus N,\mathbb{Z}_2) \arrow[r] & \tilde{H}_0(X\setminus N,\mathbb{Z}_2) \arrow[r] & 0.
\end{tikzcd}$$

□

Corollary 3.1.5. *If X is a l-manifold and N is a $(l-1)$ manifold embedded in X as a closed subset, where $H_1(N,\mathbb{Z}_2) \cong 0$ then N is two sided.*

From now on we assume that a submanifold is always embedded as a closed subset. An n-manifold N contained in the interior of an l-manifold X is *locally flat* at $x \in N^\circ$, if there exists a neighborhood U_x of x in X such that $(U_x, U_x \cap N) \cong (\mathbb{R}^l, \mathbb{R}^n)$. An embedding $f\colon N \to X$ such that $f(N^\circ) \subset X^\circ$ is said to be *locally flat at a point* $x \in N$ if $f(N)$ is locally flat at $f(x)$. Embeddings and submanifolds are *locally flat* if they are locally flat at every point.

It is necessary to consider this class of submanifolds otherwise there exist pathological cases. For example, the Alexander horned sphere, it is an (non flat) embedding of S^2 inside S^3 such that the connected components of its complement are not even simply connected (see [72, Page 65]). Rushing's book [72] is an excellent refernce for more examples on so called *wild* (non flat) embeddings and other important results in the field of topological emebeddings. Finally we add a technical condition so that these locally flat manifold intersect like hyperplanes.

Definition 3.1.6. Let X be a manifold of dimension l and let $\{N_1, \ldots, N_k\}, (k \geq 2)$ be codimension 1, locally flat submanifolds of X. We say that these submanifolds *intersect like hyperplanes* if and only if for every nonempty $Y \subseteq X$ which can be written as $Y = \cap_{i=1}^{j} N_i, (j \leq l)$ and for every $x \in Y$ there exists an open neighborhood V_x of x and a coordinate chart $\phi_x\colon V_x \to \mathbb{R}^l$ such that for each N_i containing x, the image $\phi_x(N_i \cap V_x)$ is a hyperplane passing through the origin.

The desired generalization of hyperplane arrangements is the following:

Definition 3.1.7. Let X be a connected, smooth, real manifold of dimension l. An **arrangement of submanifolds** is a finite collection $\mathcal{A} = \{N_1, \ldots, N_k\}$ of codimension 1 locally flat smooth submanifolds in X such that

1. The N_i's intersect like hyperplanes.
2. $X \setminus N_i$ has exactly two connected components for every i.
3. The intersections of N_i's define a regular CW decomposition of X.

(the submanifolds in an arrangement are allowed to have more than one connected component.)

Note that in the above definition we can use topological manifolds instead of smooth manifolds. The reason we have restricted our attention to smooth manifolds is because later we want to deal with the tangent bundle.

Before we convince the reader that this is the right setting, let us first look at how we can associate combinatorial data to such an arrangement and a few examples. Just like in the case of hyperplane arrangements, the intersection sets determined by these submanifolds have a combinatorial structure.

Definition 3.1.8. The *intersection poset* denoted by $L(\mathcal{A})$ is the set of connected components of all possible intersections of N_i's ordered by reverse inclusion, by convention $X \in L(\mathcal{A})$ as the smallest element. The rank of each element in $L(\mathcal{A})$ is defined to be the codimension of the corresponding intersection.

However note that (similar to the case of affine hyperplane arrangements) in general this poset need not be a geometric lattice. Also, using connected components of intersections is not a new idea, see for example [93] and more recently [30,58] (in case of toric arrangements). Now we move on to another poset associated with such an arrangement.

Definition 3.1.9. *Face poset* $\mathcal{F}(\mathcal{A})$: The intersections of these N_i's in $\mathcal{A}$ define a stratification of X as follows

$$\begin{aligned}
\mathcal{F}^0(\mathcal{A}) &:= X \setminus \cup_{i=1}^{k} N_i \\
\mathcal{F}^1(\mathcal{A}) &:= \cup_{i=1}^{k}(N_i - \cup_{j\neq i}(N_i \cap N_j)) \\
&\vdots \\
\mathcal{F}^k(\mathcal{A}) &= \cap_{i=1}^{k} N_i \\
X &= \mathcal{F}^0(\mathcal{A}) \cup \mathcal{F}^1(\mathcal{A}) \cup \cdots \cup \mathcal{F}^k(\mathcal{A})
\end{aligned}$$

The connected components in each stratum are called *faces*. Top dimensional faces are called chambers and the set of all chambers is denoted by $\mathcal{C}(\mathcal{A})$. The collection of all the faces $\mathcal{F}(\mathcal{A}) = \cup \mathcal{F}^i(\mathcal{A})$ is the *face poset* with the ordering $F \leq G \Leftrightarrow F \subseteq \overline{G}$. It is a graded poset and the rank of each face is its dimension.

Note that if there is no confusion about the arrangement, we will omit $\mathcal{A}$ and denote the two posets by $\mathcal{F}$ and L. Let us look at some examples other than hyperplane arrangements.

Example 3.1.10. Let X be the circle S^1, a smooth one dimensional manifold, the codimension 1 submanifolds are points in S^1. Consider the arrangement $\mathcal{A} = \{p, q\}$ of 2 points. For both these points there is an open neighborhood which is homeomorphic to an arrangement of a point in $\mathbb{R}$. Figure 3.1 shows this arrangement and the Hasse diagrams of the face poset and the intersection poset.

Example 3.1.11. As a 2-dimensional example consider an arrangement of 2 great circles N_1, N_2 in S^2. Figure 3.2 shows this arrangement and the related posets. The face poset has two 0-cells, four 1-cells and four 2-cells. Also note that the order complex of the face poset has the homotopy type of S^2.

Example 3.1.12. Figure 3.3 shows an arrangement of 4 circles (i.e. 1-dimensional tori) in a 2-torus and its combinatorial data. In this arrangement there are only 2 submanifolds each with exactly 2 connected components. These submanifolds correspond to kernels of the characters $r^2 = 1$ and $s^2 = 1$, where r, s are parameters describing $S^1 \times S^1$. The atoms in the intersection poset correspond to the 4 connected components of the submanifolds and the coatoms correspond to the 4 points.

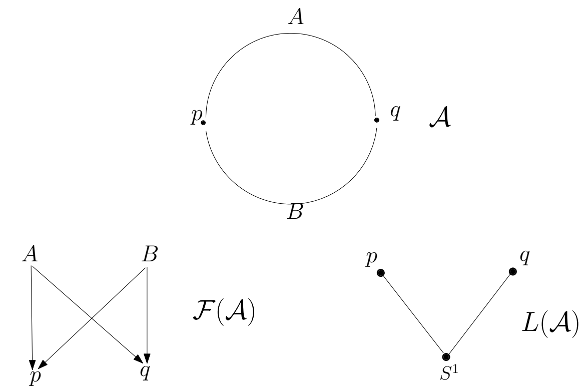


Figure 3.1: Arrangement of 2 points in a circle.

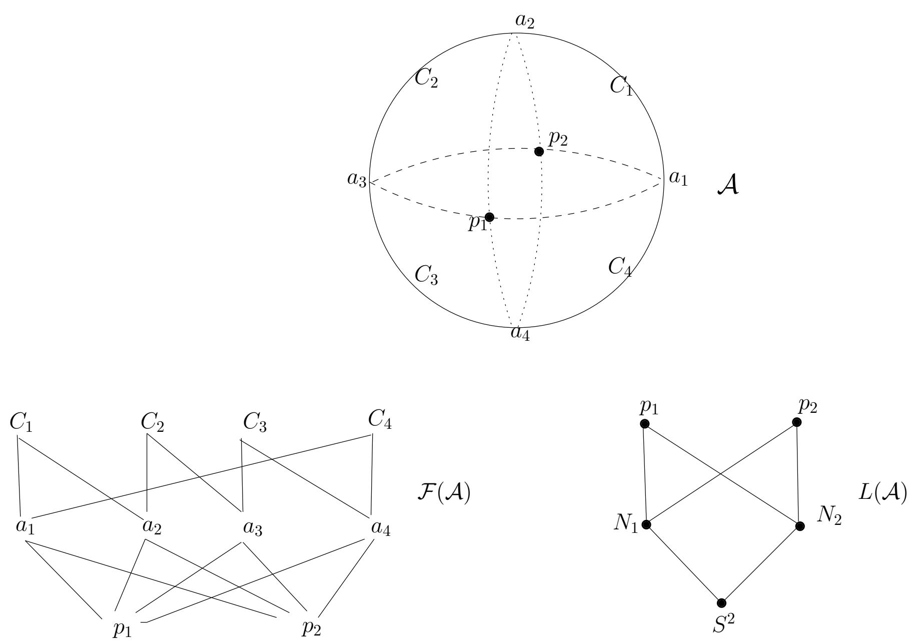


Figure 3.2: Arrangement of 2 circles in a sphere.

Recall that the face poset of a hyperplane arrangement (to be precise, its geometric realization) has the homotopy type of the ambient Euclidean space. From the definition of the face poset it is clear that an arrangement of submanifolds stratifies the manifold into open and contractible subsets. Theorem 1.1.9 implies that the face poset of an arrangement has the homotopy type of the ambient manifold. We sum up all the above arguments in Lemma 3.1.14 to show that arrangements of submanifolds

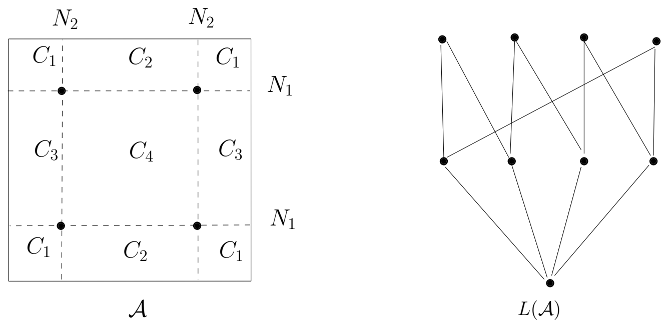


Figure 3.3: Arrangement of 4 circles in a torus.

are locally arrangements of hyperplanes. But before that a few notations.

Definition 3.1.13. Let $\mathcal{A}$ be an arrangement of submanifolds, let $\mathcal{A}_{cc}$ denote the set of all connected components of members of $\mathcal{A}$. For a point $x \in X$ define the *local arrangement* at x as follows

$$\mathcal{A}_x := \{N \in \mathcal{A}_{cc} \mid x \in N\}$$

Similarly for a face F

$$\mathcal{A}_F := \{N \in \mathcal{A}_{cc} \mid F \subset N\}$$

The *restriction* of a local arrangement to an open set $V \subseteq X$ is

$$\mathcal{A}_F|_V := \{N \cap V | N \in \mathcal{A}_F\}$$

Lemma 3.1.14. *Let F be a face of an arrangement $\mathcal{A}$ (that is $F \in \mathcal{F}(\mathcal{A})$). Then there exists an open set $V_F \subsetneq X$ containing F and a map $\phi\colon V_F \to \mathbb{R}^l$ such that*

1. *$V_F \cap N = \emptyset$ for every $N \notin \mathcal{A}_F$.*
2. *ϕ is a homeomorphism.*
3. *ϕ maps $\mathcal{A}_F|_{V_F}$ to a central arrangement of hyperplanes in $\mathbb{R}^l$.*

Proof. If C is a chamber then take $V_C = C$ and ϕ to be a coordinate chart which maps V_C to the empty arrangement. Each submanifold N has an open neighborhood in X which is homeomorphic to $N \times (-1, 1)$ (this is Brown's bicollared theorem) see [72, Theorem 1.7.5]. We call such an open neighborhood the bicollar of the submanifold. Note that every other face F is homeomorphic to an open cell of some dimension and is an intersection of finitely many codimension 1 submanifolds. Hence one can consider the open set V_F to be the intersection of all the bicollars containing F. Moreover this

V_F can be adjusted such that it intersects with only those faces which either contain $\overline{F}$ or whose closure is contained in F.

Since X is a smooth manifold, there exists a homeomorphism $\phi\colon V_F \to \mathbb{R}^l$, say a coordinate chart. If necessary ϕ can be composed with a homeomorphism of $\mathbb{R}^l$ so that for each connected component N containing F, the set $N \cap V_F$ is mapped to a hyperplane in $\mathbb{R}^l$. As ϕ is a homeomorphism it does not change the incidence relations between the faces, thus preserving the combinatorial structure of the local arrangement $\mathcal{A}_F$. The arrangement $\phi(\mathcal{A}_F|_{V_F})$ is central because of (1). □

Thus every point $x \in X$ has an open neighborhood V_x in X which is homeomorphic to a central arrangement of hyperplanes with x at its origin. The hyperplanes in that arrangement correspond to $N \cap V_x$ for every $N \in \mathcal{A}_x$. Intuitively one can think of X as a (stratified) manifold obtained by gluing central hyperplane arrangements.

3.2 The Tangent Bundle Complement

The aim of this section is to associate a connected topological space to submanifold arrangements that generalizes the construction of complexified hyperplane complement. Recall that given a hyperplane arrangement in $\mathbb{R}^l$ the complexified complement is the space obtained by removing the union of complexified hyperplanes from $\mathbb{C}^l$. If we forget the complex structure then as a topological space, $\mathbb{C}^l$ (complexification of $\mathbb{R}^l$) is homeomorphic to $\mathbb{R}^{2l}$ which also happens to be the tangent bundle of $\mathbb{R}^l$. Each hyperplane H of $\mathbb{R}^l$ is homeomorphic to $\mathbb{R}^{l-1}$ and its complexification $H_{\mathbb{C}}$ is homeomorphic to $\mathbb{C}^{l-1}$, which, as before, is the tangent bundle of $\mathbb{R}^{l-1}$. Hence the complexified complement is also a complement inside the tangent bundle. This observation suggests a different way of expressing the complexified complement (Definition (1.2.6)).

$$M(\mathcal{A}) = T\mathbb{R}^l \setminus \bigcup T\mathbb{R}^{l-1}$$

We use the above idea to define a generalization of the complexified complement in case of submanifold arrangements.

Definition 3.2.1. Let X be a l-dimensional manifold and $\mathcal{A} = \{N_1, \ldots, N_k\}$ be an arrangement of submanifolds. Let TX denote the tangent bundle of X and let $T\mathcal{A} := \bigcup_{i=1}^{k} TN_i$. The **tangent bundle complement** of the arrangement $\mathcal{A}$ is defined as

$$M(\mathcal{A}) := TX \setminus T\mathcal{A}$$

Note that $M(\mathcal{A})$ is connected as it is of codimension 2 in TX. This idea of generalizing hyperplane arrangements was explained to us by Prof. Tadeusz Januszkiewicz during a conference at Ohio State University in May 2008. He and Prof. Richard

Scott have some unpublished work on this subject (to watch the video recording of Richard Scott's talk regarding this, visit [78]).

The above definition clearly generalizes the usual notion of a complexified complement. For a point $x \in X$, let $T_x(\mathcal{A})$ denote $\bigcup T_x(N)$ for all $N \in \mathcal{A}_x$ and define the *tangent space complement* at x as

$$M(\mathcal{A}_x) := T_x(X) \setminus T_x(\mathcal{A}).$$

Then $M(\mathcal{A})$ can be rewritten as follows

$$M(\mathcal{A}) = \{(x, v) \mid x \in X, v \in M(\mathcal{A}_x)\}.$$

We start the study of $M(\mathcal{A})$ by first understanding the tangent space complement $M(\mathcal{A}_x)$.

Lemma 3.2.2. *Let $F \in \mathcal{F}(\mathcal{A})$ for a submanifold arrangement $\mathcal{A}$ in X and $\phi\colon V_F \to \mathbb{R}^l$ be a coordinate chart for an open neighborhood V_F of F. Then for every $x \in V_F$, $M(\mathcal{A}_x) \cong \phi(V_F \setminus (\mathcal{A}_x|_{V_F}))$.*

Proof. Observe that since ϕ is a homeomorphism, the linear map $(d\phi)_x\colon T_x(X) \to \phi(V_F)$ ($\cong \mathbb{R}^l$) is an isomorphism and for the same reasons $T_x(N) \cong \phi(N \cap V_F)$ ($\cong \mathbb{R}^{l-1}$) for every $N \in \mathcal{A}_x$. The map $\Phi\colon M(\mathcal{A}_x) \to \phi(V_F \setminus (\mathcal{A}_x|_{V_F}))$ defined by sending $(x, v) \mapsto (d\phi)_x((x, v))$ is a homeomorphism because it is a restriction of $(d\phi)_x$. It is also independent of the choice of coordinate charts. $\square$

Remark 3.2.3. Since $\phi(\mathcal{A}_x|_{V_F})$ is an arrangement of hyperplanes in $\phi(V_F)$ (Lemma 3.1.14), $T_x(\mathcal{A})$ is an arrangement of hyperplanes in $T_x(X)$. Moreover, the two arrangements have isomorphic face posets as well as intersection lattices.

As a consequence of the above lemma the connected components of the tangent space complement $M(\mathcal{A}_x)$ can be indexed by the chambers of $\mathcal{A}_x|_{V_F}$. Let F be a face and C be a chamber of $\mathcal{A}$ such that $F \subseteq \overline{C}$ and for every $x \in F$ define

$$C_F(x) := \{(x, v) \in M(\mathcal{A}) \mid \Phi((x, v)) \in \phi(C \cap V_F)\} \tag{3.2.1}$$

Figure 3.4 is a 2-dimensional example that intuitively illustrates Lemma 3.2.2, showing the two homeomorphisms and a component $C_F(x)$ of the tangent space complement.

Before stating the main theorem of this section let us mention one more definition. A submanifold N of $\mathcal{A}$ *separates* two chambers C and D if and only if they are contained in the distinct connected components of $X \setminus N$. For two chambers C, D the set of all the submanifolds that separate these two chambers is denoted by $R(C, D)$. The following lemma is now evident:

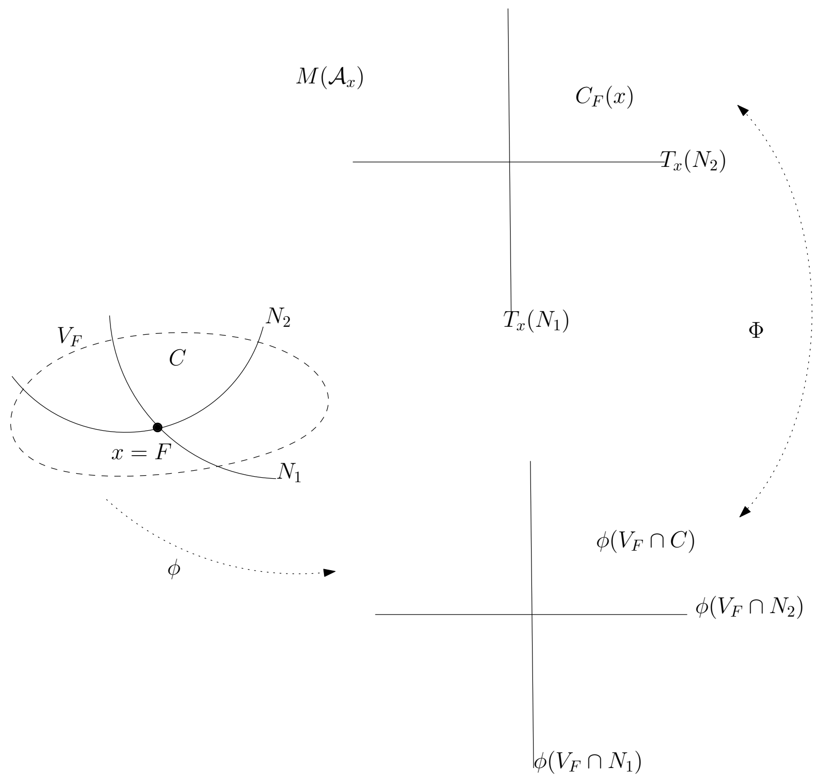


Figure 3.4: The tangent space complement

Lemma 3.2.4. *Let X be a l-manifold and $\mathcal{A}$ be an arrangement of submanifolds, let C_1, C_2, C_3 be three chambers of this arrangement. Then,*

$$R(C_1, C_3) = [R(C_1, C_2) \setminus R(C_2, C_3)] \bigcup [R(C_2, C_3) \setminus R(C_2, C_1)]$$

Proof. Let $N \in R(C_1, C_3)$ and let N^+, N^- denote the two connected components of $X \setminus N$ such that $C_1 \subset N^+$ and $C_3 \subset N^-$. The chamber C_2 is either contained in N^+, in which case $N \in R(C_2, C_3) \setminus R(C_2, C_1)$ or it is contained in N^- in which case $N \in R(C_1, C_2) \setminus R(C_2, C_3)$. The other direction follows from the definition of $R(C, D)$. See also [22, Lemma 1.2]. $\square$

The distance between two chambers is defined as the cardinality of $R(C, D)$ and denoted by $d(C, D)$. Given a face F and a chamber C of a submanifold arrangement $\mathcal{A}$, define the action of F on C as follows:

Definition 3.2.5. A face F acts on a chamber C to produce another chamber $F \circ C$ satisfying:

1. $F \subset \overline{F \circ C}$
2. $d(C, F \circ C) = \min \{d(C, C') \mid C' \in \mathcal{C}(\mathcal{A}), F \subset \overline{C'}\}$.

Lemma 3.2.6. *With the same notation as above, the chamber $F \circ C$ always exists and is unique.*

Proof. First, note that $F \circ C = F$ if F itself is a chamber. Assume that the codimension of F is greater than or equal to 1. Hence, F is given by the intersection of some submanifolds say, $\mathcal{A}' = \{N_1, \dots, N_r\}$. The collection $\mathcal{A}'$ need not be an arrangement of submanifolds. However, $\mathcal{A}'$ defines a stratification of the manifold and we will refer to the codimension 0 components of this stratification as chambers. There exists a unique chamber C' of $\mathcal{A}'$ which contains C. Then define $F \circ C$ to be the unique chamber of $\mathcal{A}$ that is contained in C' and whose closure contains F. □

An easy consequence of the definition is that for two faces F, F'

$$F' \geq F \text{ implies } F' \circ (F \circ C) = F' \circ C.$$

Also, if $F \leq C$ then $F \circ C = C$.
Now we state the main theorem of this section.

Theorem 3.2.7. *Let $\mathcal{A}$ be an arrangement of submanifolds in an l-manifold X with $\mathcal{F}$ as its poset and $M(\mathcal{A})$ as the associated tangent bundle complement. Then there exists a finite open cover of $M(\mathcal{A})$ such that each open set is indexed by a face F and a chamber C whose closure contains F. Moreover, each of these open sets is contractible and their intersections are contractible.*

Proof. First, for every face F fix an open open set V_F that completely contains F and has non-empty intersections with only those faces whose closures contain F (this is possible because of Lemma 3.1.14) . For example, if C is a chamber then $V_C = C$. Observe that $\{V_F \mid F \in \mathcal{F}\}$ is an open cover of X and the tangent bundle TX can be trivialized on each of these open sets. Let $h \colon V_F \times \mathbb{R}^l \to \pi^{-1}(V_F)$ denote the local trivialization, where $\pi \colon TX \to X$ is the projection map.

In order to construct the required open cover, arbitrarily choose a face F and fix it for rest of the proof (to avoid trivialities assume that F is not a chamber). Let C be a chamber whose closure contains F.

For every point $x \in V_F$ the map $h_x := h(x, -) \colon \mathbb{R}^l \to \pi^{-1}(x)$ is a linear isomorphism. Let y be an arbitrarily chosen point in F and $C_F(y)$ be the connected component of $M(\mathcal{A}_y)$ corresponding to $\phi(C \cap V_F)$. Denote the open subset $h_y^{-1}(C_F(y)) \subseteq \mathbb{R}^l$ by C_F and define

$$W(F, C) := h(V_F \times C_F) \subsetneq \pi^{-1}(V_F).$$

For the sake of notational simplicity we will identify $V_F \times C_F$ with its image. Clearly $W(F,C)$ is an open and contractible subset of $M(\mathcal{A})$. For a chamber C we have $W(C,C) = C \times \mathbb{R}^l$.

Now we show that these open sets cover $M(\mathcal{A})$. Let $(x,v) \in M(\mathcal{A})$ be an arbitrary point. Suppose that $x \in F$ for some face F. As $v \in M(\mathcal{A}_x)$, so $\phi^{-1}(\Phi(v)) \in C \cap V_F$, where C is some chamber whose closure contains F. Therefore $(x,v) \in W(F,C)$. The intersection of any two such open sets is given by

$$W(F,C) \cap W(F',C') = (V_F \cap V_{F'}) \times (C_F \cap C'_{F'})$$

which is clearly contractible. □

Example 3.2.8. Consider an arrangement $\mathcal{A} = \{0\}$ in $\mathbb{R}$, it divides the real line in two chambers $A = (-\infty, 0)$ and $B = (0, \infty)$. The tangent bundle complement of this arrangement is the punctured plane, since $\{0\}$ is the only point whose tangent space is disconnected.

For $p = 0$ we take $V_p = (-1, 1)$ and for the chambers we take $V_A = A$ and $V_B = B$ as the coordinate neighborhoods. With this the open cover of $M(\mathcal{A})$ is given by setting:

1. $W(p, A) = \{(x,v) \mid x \in V_p, v < 0\}$,
2. $W(p, B) = \{(x,v) \mid x \in V_p, v > 0\}$,
3. $W(A, A) = \{(x,v) \mid x \in A, v \in \mathbb{R}\}$,
4. $W(B, B) = \{(x,v) \mid x \in B, v \in \mathbb{R}\}$.

Figure 3.5 shows the open cover defined above of the tangent bundle complement.

Now going back to Theorem 3.2.7 we see that the above defined open covering $\{W(F,C) \mid F \leq C\}$ of $M(\mathcal{A})$ satisfies the hypothesis of the Theorem 1.1.6 (the Nerve Lemma), hence the nerve of that open covering has the homotopy type of $M(\mathcal{A})$. In order to identify the simplices of this nerve we need to understand how the open sets intersect.

Lemma 3.2.9. *$W(F,C) \cap W(F',C') \neq \emptyset$ if and only if $F \leq F'$ and either $C' = C$ or $F' \circ C = C'$.*

Proof. By construction of these open sets we have,

$$W(F,C) \cap W(F',C') = (V_F \cap V_{F'}) \times (C_F \cap C'_{F'})$$

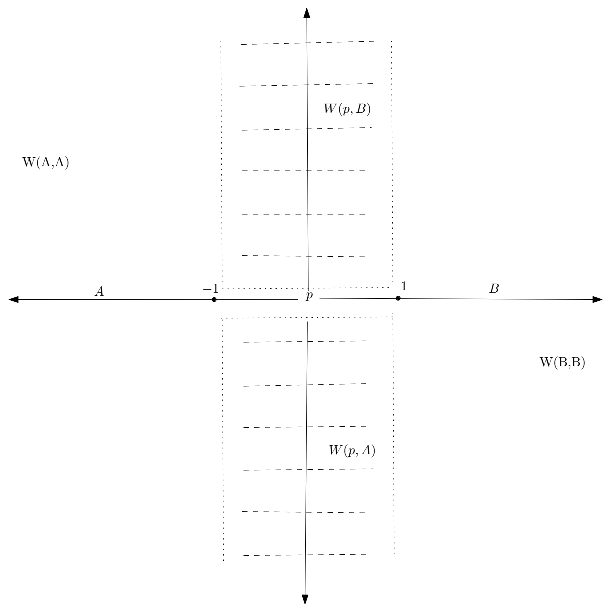


Figure 3.5: Example of an open cover of the tangent bundle complement

Clearly $V_F \cap V_{F'} \neq \emptyset$ if and only if $F \leq F'$. We also need the other intersection to be nonempty,

$$\begin{aligned} C_F \cap C'_{F'} \neq \emptyset \quad &\iff \quad (C \cap V_F) \cap (C' \cap V_{F'}) \neq \emptyset \;\; \text{due to (3.2.1)} \\ &\iff \quad (V_F \cap V_{F'}) \cap (C \cap C') \neq \emptyset \\ &\iff \quad C \cap C' \neq \emptyset \\ &\iff \quad C' = F' \circ C \text{ or } C' = C \end{aligned}$$

Which proves the theorem. □

Let $\mathcal{S}$ denote an abstract set in bijection with the open sets in the above open covering. We denote by (F, C) the element of $\mathcal{S}$ corresponding to $W(F, C)$.

Lemma 3.2.10. *The relation $(F_2, C_2) \leq_s (F_1, C_1)$ if and only if $F_1 \leq F_2$ and $F_2 \circ C_1 = C_2$. defines a partial order on $\mathcal{S}$.*

Proof. The arguments are similar to the proof of [68, Lemma 3.1]. It is obvious that the relation is reflexive and symmetric, let us check the transitivity. Pick 3 elements such that $(F_3, C_3) \leq_s (F_2, C_2)$ and $(F_2, C_2) \leq_s (F_1, C_1)$.

The first inequality implies that $F_2 \leq F_3$ and $F_3 \circ C_2 = C_3$. Similarly from the second inequality we have, $F_1 \leq F_2$ and $F_2 \circ C_1 = C_2$. Since $\mathcal{F}$ is a poset, $F_3 \leq F_1$ and $C_3 = F_3 \circ (F_2 \circ C_1) = F_3 \circ C_1$ which concludes the transitivity. □

With this partial order we can now characterize the chains in the nerve.

Lemma 3.2.11. *Let $\mathcal{A}$ be an arrangement of submanifolds, then for a chain in $\mathcal{S}$ there corresponds a chamber and a chain in $\mathcal{F}(\mathcal{A})$.*

Proof. Let $F_0 \leq \cdots \leq F_k$ be a chain in $\mathcal{F}$ and let C be a chamber. Then the Lemma 3.2.10 implies that $(F_k, F_k \circ C) \leq_s \cdots \leq_s (F_0, F_0 \circ C)$ is a chain in $\mathcal{S}$. Moreover, using the same lemma it can be shown that every chain in $\mathcal{S}$ is of this form. □

Now we have proved that $\mathcal{S}$ is a partially ordered set and it follows that the geometric realization of $(\mathcal{S}, \leq_s)$ and the nerve described by the open sets $W(F, C)$ are homeomorphic. A k-simplex in this nerve is a k-chain in $(\mathcal{S}, \leq_s)$. Let $F_0 \leq \cdots \leq F_k$ be a chain in $\mathcal{F}(\mathcal{A})$ and let C be a chamber (such that $F_k \leq C$) then both of them determine a simplex in the nerve given by

$$(F_k, C) \leq_s \cdots \leq_s (F_0, C)$$

In fact every simplex of the nerve is of this form. We summarize all this discussion in the following definition.

Definition 3.2.12 (The Salvetti Complex). Let X be a smooth l-manifold and $\mathcal{A}$ be an arrangement of codimension 1 submanifolds. Define the Salvetti poset as

$$\mathcal{S} = \{(F, C) \in \mathcal{F}(\mathcal{A}) \times \mathcal{C}(\mathcal{A}) \mid F \leq C\}$$

and the partial order as

$$(F, C) \leq_s (F', C') \text{ if and only if } F \leq F' \text{ and } F' \circ C = C'$$

The *Salvetti complex* $Sal(\mathcal{A})$ is defined as the geometric realization of $(\mathcal{S}, \leq_s)$ and it has the homotopy type of the tangent bundle complement $M(\mathcal{A})$.

Example 3.2.13. Let $\mathcal{A}$ be arrangement of 2 points in a circle. The complement $M(\mathcal{A})$ in this case is $TS^1 \setminus \{p, q\}$ (tangent bundle of a point is the point itself). As $TS^1 \cong S^1 \times \mathbb{R}$, the space $M(\mathcal{A})$ is an infinite cylinder with 2 punctures. Hence $M(\mathcal{A}) \simeq S^1 \vee S^1 \vee S^1$. Now we construct the Salvetti complex for this arrangement.

The following theorem states the connection between the manifold X and the associated tangent bundle complement.

Theorem 3.2.14. *Let X be a smooth manifold and $\mathcal{A}$ be an arrangement of submanifolds, let $M(\mathcal{A})$ denote the associated tangent bundle complement. Then the manifold X is a retract of $M(\mathcal{A})$.*

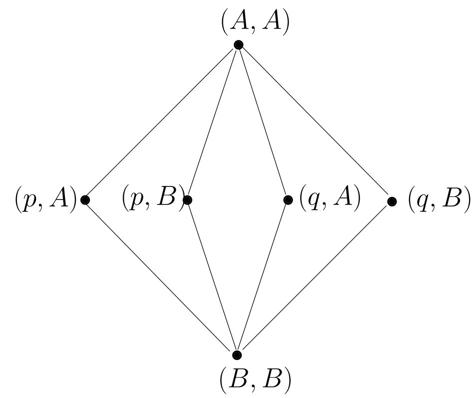


Figure 3.6: Salvetti complex

Proof. Let $r\colon \mathcal{S} \to \mathcal{F}(\mathcal{A})$ be the map defined by $(F,C) \mapsto F$. This map is order reversing and induces a simplicial map on the geometric realizations which we will again denote by r. For a chamber C define a new map $\iota_C\colon \mathcal{F}(\mathcal{A}) \hookrightarrow \mathcal{S}$ by sending F to $(F, F \circ C)$. This map is injective and order reversing, consequently it induces a continuous injective map on the geometric realizations.

Let $X(C) \subset Sal(\mathcal{A})$ denote the image of ι_C, then $X(C) \cong X$ and in fact

$$Sal(\mathcal{A}) = \bigcup_{C \in \mathcal{C}(\mathcal{A})} X(C).$$

The composition $r \circ \iota_C$ is the identity on $|\mathcal{F}(\mathcal{A})|$ which establishes the claim. $\square$

Intuitively speaking, a submanifold arrangement is made by gluing central hyperplane arrangements in a compatible way. It is natural to ask whether the associated tangent bundle complement is also made up of gluing together the Salvetti complexes of these hyperplane arrangements. This is in fact true as we now prove. We show that the complement $M(\mathcal{A})$ is a homotopy colimit of a diagram of spaces over the face poset. We make use of the fact that there is a combinatorial description of the complement.

Proposition 3.2.15. *Let X be a l-manifold and let $\mathcal{A}$ be an arrangement of codimension 1 submanifolds with $\mathcal{F}(\mathcal{A})$ as its face poset. If $F' \geq F$ are two faces then there is an inclusion $Sal(\mathcal{A}_{F'}) \hookrightarrow Sal(\mathcal{A}_F)$ as simplicial complexes.*

Proof. It suffices to prove that the Salvetti poset of $\mathcal{A}_{F'}|_{V_{F'}}$ embeds into the Salvetti poset of $\mathcal{A}_F|_{V_F}$. This follows if the face poset of $\mathcal{A}_{F'}|_{V_{F'}}$ includes into the face poset of $\mathcal{A}_F|_{V_F}$. But this is clear since $\mathcal{F}(\mathcal{A})_{\geq F'} \hookrightarrow \mathcal{F}(\mathcal{A})_{\geq F}$. $\square$

Theorem 3.2.16. *Consider the diagram of spaces $\mathcal{D}\colon \mathcal{F}(\mathcal{A}) \to Top$, given by setting $\mathcal{D}(F) = Sal(\mathcal{A}_F)$. The morphisms $\mathcal{D}(F' \to F)$ are the inclusions $Sal(\mathcal{A}_{F'}) \hookrightarrow Sal(\mathcal{A}_F)$. Then $hocolim\mathcal{D} \simeq Sal(\mathcal{A})$.*

Proof. First observe that this is indeed a diagram of posets. Hence we need to show that the poset map $\phi\colon Plim\mathcal{D} \to \mathcal{S}$ defined by $(F,(F',C)) \mapsto (F',C)$ induces a homotopy equivalence $\Delta(Plim\mathcal{D}) \simeq Sal(\mathcal{A})$. The claim follows by the application of Quillen's Theorem A. □

3.3 Combinatorics of the Cell Structure

In this section we construct a CW complex that has the homotopy type of the tangent bundle complement. Inspired by the work of Deligne [22] on simplicial arrangements, Salvetti, in his seminal paper [73], proved that the face poset of a real hyperplane arrangement defines a CW complex embedded inside the complexified complement and that the complex is a strong deformation retract of the complement. We now show that the same construction also works for tangent bundle complements.

Let $\mathcal{A}$ be an arrangement of submanifolds in a l-manifold X and let $\mathcal{F}(\mathcal{A})$ denote the associated face poset. By $(X, \mathcal{F}(\mathcal{A}))$ we will mean the regular cell structure of X induced by the arrangement. Recall that $\mathcal{F}^*(\mathcal{A})$ denotes the dual face poset, we denote by pair $(X, \mathcal{F}^*(\mathcal{A}))$ the *dual cell structure*. Every k-cell in $(X, \mathcal{F}(\mathcal{A}))$ corresponds to $(l-k)$-cell in $(X, \mathcal{F}^*(\mathcal{A}))$ for $0 \le k \le l$.

For the sake of notational simplicity, we will denote the dual, regular cell complex induced by the given submanifold arrangement $\mathcal{A}$ by $X^*(\mathcal{A})$ (X^* if the context is clear). The symbols C, D will denote vertices of X^* and the symbol F^k will denote a k-cell dual to the codimension k-face of $\mathcal{A}$. Note that a 0-cell C is a vertex of a k-cell F^k in X^* if and only if the closure $\overline{C}$ of the corresponding chamber contains the corresponding $(l-k)$-face. The action of the faces on chambers that was introduced in Definition 3.2.5 is also valid for the dual cells. The symbol $F^k \circ C$ will denote the vertex of F^k which is dual to the unique chamber 'closest' to the chamber C. The partial order on the cells of X^* will be denoted by $\prec$.

Recall that the dual cell complex of a central hyperplane arrangement has a 'special' shape called as the zonotope (see [95, Lecture 7]). The dual cell structure of a submanifold arrangement also has a special combinatorial structure which we describe in the next section. Now given $X^*(\mathcal{A})$ construct a CW complex $S(\mathcal{A})$ of dimension l as follows:

The 0-cells of $S(\mathcal{A})$ correspond to 0-cells of X^*, which we denote by the pairs $\langle C; C\rangle$. For each 1-cell $F^1 \in X^*$ with vertices C_1, C_2, assign two homeomorphic copies of F^1 denoted by $\langle F^1; C_1 \rangle, \langle F^1; C_2 \rangle$. Attach these two 1-cells in $S(\mathcal{A})_0$ so that

$$\partial \left\langle F^1; C_i \right\rangle = \{\langle C_1; C_1 \rangle, \langle C_2; C_2 \rangle\}.$$

Orient the 1-cell $\langle F^1; C_i \rangle$ so that it begins at $\langle C_i; C_i \rangle$, to obtain an oriented 1-skeleton $S(\mathcal{A})_1$.

By induction assume that we have constructed the $(k-1)$-skeleton of $S(\mathcal{A})$, $1 \leq k-1 < l$. To each k-cell $F^k \in X^*$ and to each of its vertex C assign a k-cell $\left\langle F^k; C \right\rangle$ that is isomorphic to F^k. Let $\phi(F^k, C)\colon \left\langle F^k; C \right\rangle \to S(\mathcal{A})_{k-1}$ be the same characteristic map that identifies a $(k-1)$-cell $F^{k-1} \subset \partial F^k$ with the k-cell $\left\langle F^{k-1}; F^{k-1} \circ C \right\rangle \subset \partial \left\langle F^k; C \right\rangle$. Extend the map $\phi(F^k, C)$ to whole of $\left\langle F^k; C \right\rangle$ and use it as the attaching map, hence obtaining the k-skeleton. The boundary of every k-cell in given by

$$\partial \left\langle F^k; C \right\rangle = \bigcup_{F \prec F^k} \langle F; F \circ C \rangle. \tag{3.3.1}$$

Now we prove a theorem that justifies the construction of this cell complex.

Theorem 3.3.1. *The CW complex $S(\mathcal{A})$ constructed above has the homotopy type of the tangent bundle complement $M(\mathcal{A})$.*

Proof. First, note that the above construction implies that $S(\mathcal{A})$ is a regular cell complex. Let $\mathcal{S}(\mathcal{A})$ denote the poset of cells of $S(\mathcal{A})$ ordered by inclusion. Then it is indeed isomorphic to the Salvetti poset as described in the Definition 3.2.12. Consequently the barycentric subdivision of $S(\mathcal{A})$ is isomorphic to the (simplicial) Salvetti complex $Sal(\mathcal{A})$, hence to every k-cell $\langle F; C \rangle$ in $S(\mathcal{A})$ there corresponds an ideal $\mathcal{S}(\mathcal{A})_{\leq_s [F,C]}$ (a triangulation of that k-cell). The claim now follows from the application of the Theorem 1.1.9. □

Remark 3.3.2. In order to simplify the notations and the arguments we will not distinguish between the simplicial and the CW versions of the Salvetti complex. We use the notation $[F, C]$ to denote either a cell or its barycenter and hope that the context will make it clear. More importantly, henceforth we will assume that the Salvetti complex is equipped with the above defined cell structure.

Remark 3.3.3. The original proof of Salvetti (that appeared in [73]) in the context of hyperplane arrangements is more direct. His idea is to first identify an open covering of the complexified complement (as we did in Theorem 3.2.7). Rather than considering the nerve of this covering abstractly he constructs a simplicial complex that is embedded inside the complexified complement. In order to define a CW structure on this simplicial complex, pairs of triangulated dual cells and chambers are used. The final step is to show that there is a strong deformation retraction from the complement on to the CW complex. A similar approach could be used in case of a tangent bundle complement. However, a proof involving this kind of approach is very technical since the construction of the nerve as an embedded simplicial complex is very complicated. Also, such a proof might not shed any new light and this is the reason we have taken an approach to construct the regular CW complex abstractly and then show that its barycentric subdivision is isomorphic to the nerve of the tangent bundle complement.

Example 3.3.4. As an example of this construction consider the arrangement of 2 points in a circle (Example 3.1.10). The Figure 3.7 below illustrates the dual cell structure induced by the arrangement and the associated Salvetti complex.

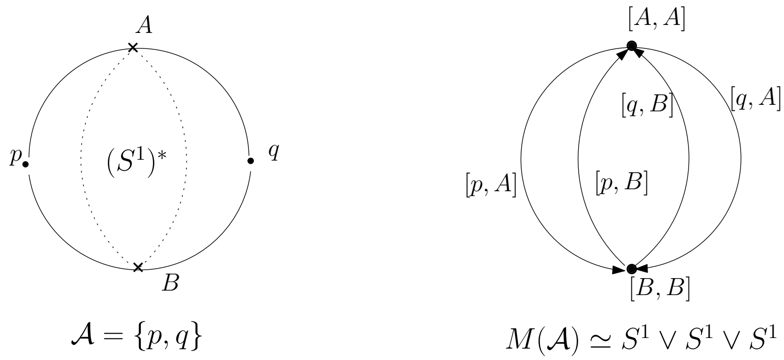


Figure 3.7: Arrangement in S^1 and the associated Salvetti complex

Example 3.3.5. Now we turn to the Salvetti complex associated with the arrangement of 2 circles in S^2 (Example 3.1.11). Note that the dual cell structure induced by the arrangement consists of two (square shaped) 2-cells with their 1-skeleton identified. According to above construction, there are 4 vertices in the Salvetti complex that correspond to 4 chambers. The eight 1-cells correspond to edge-chamber pairs in the arrangement. Figure 3.8 shows the oriented 1-skeleton of the Salvetti complex.

Finally, there are eight (square shaped) 2-cells corresponding to point-chamber pairs. It is clear, for example, that the boundaries of the two 2-cells $[p_1, C_1]$ and $[p_2, C_1]$ are attached to the central square above. Hence, these eight 2-cells form four copies of 2-spheres such that their equators are identified as shown in the figure. Note that since the attaching maps are homotopically trivial we can collapse the boundaries of the four 2-cells $\{[p_2, C_i] \mid 1 \leq i \leq 4\}$ to a point. Thus we get a wedge of a torus and four copies of 2-spheres.

We now look at some obvious properties of the above defined CW structure and also infer some more information about the tangent bundle complement.

Theorem 3.3.6. *Let $\mathcal{A}$ be an arrangement of submanifolds in a l-manifold X and let $Sal(\mathcal{A})$ denote the associated cell complex. Then*

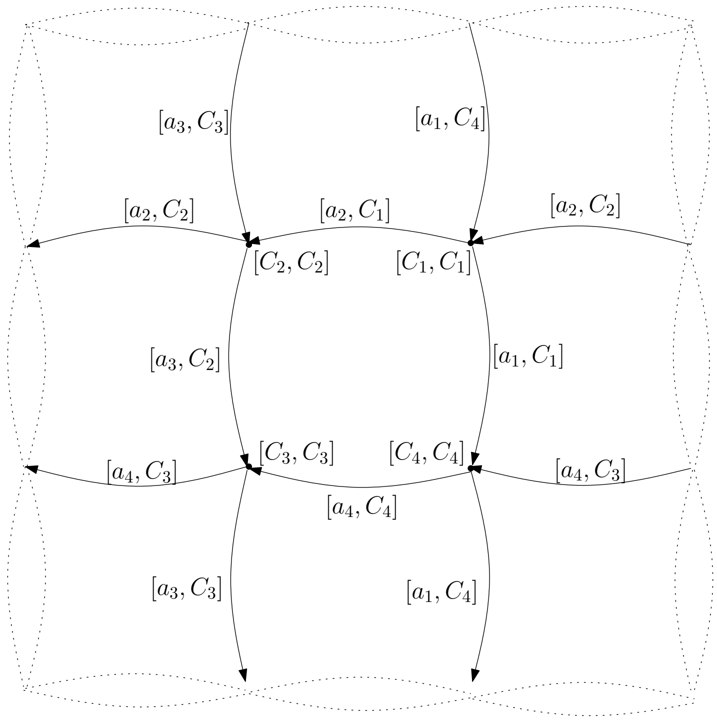


Figure 3.8: Arrangement in S^2 and the associated Salvetti complex

1. *There is a natural cellular map* $\psi\colon Sal(\mathcal{A}) \to X(\mathcal{A})^*$ *given by* $[F^k, C] \mapsto F^k$. *The restriction of* ψ *to the* 0*-skeleton is a bijection and in general*

$$\psi^{-1}(F^k) = \{C \in \mathcal{C}(\mathcal{A}) | C \prec F^k\}$$

2. *For every chamber* C *there is a cellular map* $\iota_C\colon X^*(\mathcal{A}) \to Sal(\mathcal{A})$ *taking* F^k *to* $[F^k, F^k \circ C]$ *which is an embedding of* $X^*(\mathcal{A})$ *into* $Sal(\mathcal{A})$*, and*

$$Sal(\mathcal{A}) = \bigcup_{C \in \mathcal{C}(\mathcal{A})} \iota_C(X^*).$$

3. *The absolute value of the Euler characteristic of the complement is the number of bounded chambers.*

4. *Let the link of* $\mathcal{A}$ *(denoted by* $\bigcup \mathcal{A}$*) be the union of submanifolds in* $\mathcal{A}$ *then*

$$\bigcup \mathcal{A} \simeq \bigvee_r S^{l-1}$$

where $r = |\chi(M(\mathcal{A}))|$. *When* $r = 0$*, the link is contractible.*

Proof. Claims (1) and (2) follow from the construction and the Theorem 3.2.14.

We prove (3) by explicitly counting cells in the Salvetti complex. The Euler characteristic of a CW complex K is equal to the alternating sum of number of cells of each dimension. Given a k-dimensional dual cell F^k there are as many as $|\{C \in \mathcal{C}(\mathcal{A}) | F \leq C\}|$ k-dimensional cells in $Sal(\mathcal{A})$. Hence for a vertex $[C, C] \in Sal(\mathcal{A})$ the number of k-dimensional cells that have this particular 0-cell as a vertex is equal to the number of k-faces of $\mathcal{F}(\mathcal{A})^*$ that contain C. The alternating sum of number of cells that contain a particular vertex C of $\mathcal{F}^*(\mathcal{A})$ is equal to $1 - \chi(Lk(C))$, where $Lk(C)$ is the link of C in $X^*(\mathcal{A})$. Applying this we get,

$$\chi(Sal(\mathcal{A})) = \sum_{C \in \mathcal{C}(\mathcal{A})} (1 - \chi(Lk(C)))$$

If a chamber is unbounded then $Lk(C) \simeq \mathbb{B}^l$ and on the other hand if it is bounded then $Lk(C) \simeq S^{l-1}$. Hence we have,

$$\begin{aligned}
\chi(Sal(\mathcal{A})) &= \sum_{C \in \mathcal{C}(\mathcal{A})} (1 - \chi(Lk(C))) \\
&= \sum_{C \ unbounded} (1 - 1) + \sum_{C \ bounded} (1 - [1 + (-1)^{l-1}]) \\
&= (-1)^l \sum_{C \ bounded} 1
\end{aligned}$$

Hence,

$$\chi(M(\mathcal{A})) = (-1)^l (\text{number of bounded chambers}).$$

Proof of (4) is now clear. □

3.3.1 Metrical-hemisphere complexes

We now take a closer look at the above defined CW structure on Salvetti complexes in order study its combinatorial properties. Our aim is to understand how the combinatorial properties of the Salvetti complex associated to an arrangement of hyperplanes generalize in the context of submanifold arrangements. In particular, we show that even in the general context the combinatorial properties of the CW structure are similar to that of a zonotope. Recall that a central arrangement of hyperplanes decomposes the ambient Euclidean space into open polyhedral cones. As a matter of fact every hyperplane arrangement is a normal fan of a very special polytope known as the zonotope. Zonotopes can be defined in various ways: for example, projections of cubes, Minkowski sums of line segments, dual (polar) of hyperplane arrangements etc. Being a zonotope is a strictly geometric property and not combinatorial. For example, in $\mathbb{R}^2$ a parallelogram is a zonotope whereas an arbitrary quadrilateral need not be, even though they have isomorphic face posets. For more on the relationship between zonotopes and hyperplane arrangements see [95, Lecture 7] and [6, Section 2.2].

Definition 3.3.7. A *zonotope* is a polytope all of whose faces are centrally symmetric (equivalently every 2-face is centrally symmetric). A *zonotopal cell* is a (closed) k-cell such that its face poset is isomorphic to the face poset of a k-zonotope for some k.

The face poset of a zonotope has some special combinatorial properties, the most important of which is the product structure. This product is basically the one on the face poset of a hyperplane arrangement or on the set of covectors of an oriented matroid. We show that the dual cells of a submanifold arrangement and the cells of its associated Salvetti complex enjoy similar combinatorial structure. In order to do this we do not use the language of zonotopes (since they are exclusive to Euclidean settings) but use the language of *metrical-hemisphere complexes*. These cell complexes possess all the essential combinatorial properties of a zonotope (and also of zonotopal tilings). The metrical-hemisphere complexes (MH-complexes for short) were first introduced in [74] where Salvetti generalized his construction and proved an analogue of Deligne's theorem for oriented matroids.

Let Q denote a connected, regular, CW complex (and $|Q|$ be the underlying space). The 1-skeleton of such a complex Q is a graph $G(Q)$ with no loops (abbreviated to G if the context is clear). The vertex set of this graph will be denoted by VG and the edge set by EG. An edge-path in $G(Q)$ is a sequence $\alpha = (l_1, \ldots, l_n)$ of edges that correspond to a connected path in $|Q|$. The inverse of a path is again a path $\alpha^{-1} = (l_n, \ldots, l_1)$. Two paths are composed by concatenation if ending vertex of one of the paths is the starting vertex of another. The distance $d(v, v')$ between two vertices will be the least of the lengths of paths joining v to v'. Given an i-cell $e^i \in Q$, $Q(e^i) := \{e^j \in Q : |e^j| \subset |e^i|\}$ and let $V(e^i) = VG \cap Q(e^i)$.

Definition 3.3.8. A regular CW complex Q is a QMH-complex (quasi-metrical-hemisphere complex) if and only if there exist two maps

$$\underline{\omega}, \overline{\omega} \colon VG \times Q \to VG$$

such that for all $v \in VG, e^i \in Q$ following properties are satisfied.

1. $\underline{\omega}(v, e^i) \in V(e^i)$ and $d(v, \underline{\omega}(v, e^i)) = \text{minimum}\{d(v, u) | u \in V(e^i)\}$.

2. $\overline{\omega}(v, e^i) \in V(e^i)$ and $d(v, \overline{\omega}(v, e^i)) = \text{maximum}\{d(v, u) | u \in V(e^i)\}$.

3. $d(v, \overline{\omega}(v, e^i)) = d(v, u) + d(u, \overline{\omega}(v, e^i))$ for all $u \in V(e^i)$.

This definition imposes a strong restriction on the 1-skeletons of such complexes (see [74, Proposition 1]).

Lemma 3.3.9. *If Q is a QMH-complex then each circuit in G has an even number of edges.*

Proof. Assume that Q is a QMH-complex and let $\alpha = (l_1, \dots, l_n)$ be a circuit such that edge l_1 starts at a vertex v which is also the ending vertex of the edge l_n. If n is odd then the two paths $\alpha_1 = (l_1, \dots, l_{(n-1)/2})$ and $\alpha_2 = (l_n, \dots, l_{(n+3)/2})$ join the vertex v respectively to the vertices v_1, v_2 of the edge $l_{(n+1)/2}$, also they are of the same length $(n-1)/2$. Without loss of generality let $v_1 = \underline{\omega}(v, l_{(n+1)/2})$. Then there must exist another path β joining v to v_1 whose length is strictly less than $(n-1)/2$. Then $\alpha = (\alpha_1 \beta^{-1})(\beta l_{(n+1)/2} \alpha_2^{-1})$ is a decomposition of α in circuits of shortest length. Hence, n cannot be odd. $\square$

The next corollary follows from the above lemma and the definition of a zonotope (see [6, Proposition 2.2.14]).

Corollary 3.3.10. *Let Q be a closed k-cell which also is a QMH-complex. Then Q is a zonotopal cell.*

For any $e^i \in Q$, indicate by $G(e^i) \subset G(Q)$ the subgraph corresponding to the 1-skeleton of e^i and by $d_{G(e^i)}$ the distance computed using $G(e^i)$.

Definition 3.3.11. A regular CW complex will be called a LMH-complex (local-metrical-hemisphere complex) if and only if each $Q(e^i)$ is a QMH-complex with respect to $d_{G(e^i)}$. Moreover, the following compatibility condition also holds: if $e^k \in Q(e^i) \cap Q(e^j), v \in V(e^i) \cap V(e^j)$ then

$$\underline{\omega}_{(e^j)}(v, e^k) = \underline{\omega}_{(e^i)}(v, e^k), \quad \overline{\omega}_{(e^j)}(v, e^k) = \overline{\omega}_{(e^i)}(v, e^k).$$

Here, $\underline{\omega}_{(e^j)}, \overline{\omega}_{(e^j)}$ are defined similar to $\underline{\omega}, \overline{\omega}$ but using $d_{G(e^j)}$.

Finally, Q will be called a MH-complex if Q is both a QMH-complex and a LMH-complex and for all $e^i \in Q, e^j \in Q(e^i), v \in V(e^i)$

$$\underline{\omega}(v, e^j) = \underline{\omega}_{(e^i)}(v, e^j) \quad \overline{\omega}(v, e^j) = \overline{\omega}_{(e^i)}(v, e^j)$$

Remark 3.3.12. Note that the 1-skeleton of a MH-complex has very special properties with respect to the distance. It is not enough to have a cell complex all of whose cells are zonotopal. Here are three examples that illustrate the special nature of MH-complexes. The first example describes a cell complex (Figure 3.9) which is made up of three square shaped 2-cells and one hexagonal 2-cell. The boundaries of the three square shaped cells are glued to form a hexagon. Finally the boundaries of

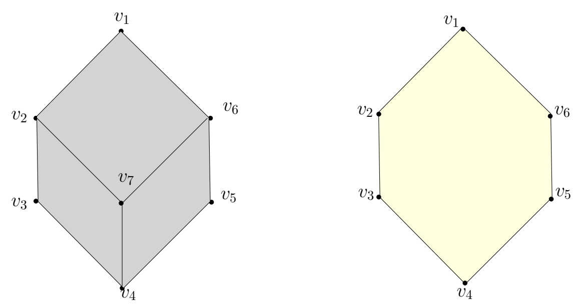


Figure 3.9: Zonotopal complex which is not QMH.

these hexagons is identified. One can see that even though each cell in this complex is zonotopal the distance maps $\overline{\omega}$ and $\underline{\omega}$ are not well defined.

Next we consider a cell complex made up of two 1-cells attached to an octagonal 2-cell. There is a one more trapezoidal 2-cell whose three 1-cells (in the boundary) are attached to three 1-cells in the boundary of the octagonal cell as shown below in Figure 3.10. The resulting complex is QMH but not LMH. Consider the 1-cell labeled by e in the figure. There are two vertices, namely v_1, v_2, in its boundary. Considering e as a member of the trapizoidal cell we see that the vertex v_1 is closest to the vertex v_4. On the other hand as a member of the octagonal 2-cell vertex v_2 is closest to v_4.

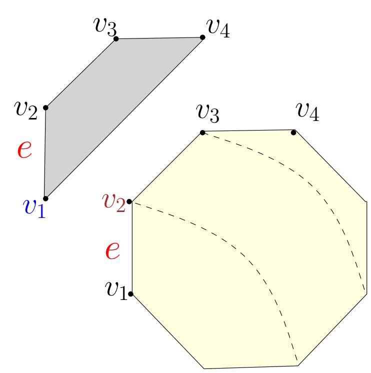


Figure 3.10: A QMH complex without LMH structure.

The final example shows a cell complex (Figure 3.11) obtained by removing the trapezoidal 2-cell from the first example. The resulting cell complex is both QMH and LMH but not a MH-complex. Consider the 1-cell labeled by e, there are two boundary vertices v_1, v_2. Considering e as a member of the octagonal cell the vertex v_2 is closest to v_3. But in the whole complex the boundary vertex of e closest to v_3 is v_1.

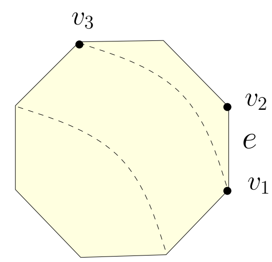


Figure 3.11: A QMH and LMH-complex which is not a MH-complex

The following lemma establishes the combinatorial connection between zonotopes (zonotopal tilings to be specific) and MH-complexes. It states that the distance between any two vertices is the same no matter how it is measured, either locally or globally (see [74, Proposition 5]).

Lemma 3.3.13. *Let Q be a MH-complex, $e^i \in Q, v, v' \in V(e^i)$. Then,*

$$d(v, v') = d_{G(e^i)}(v, v').$$

Proof. Let $\alpha = (l_1, \ldots, l_n)$ be a minimal path of $G(e^i)$ between v and v' (so $d_{G(e^i)}(v, v') = n$). Let v_{j-1}, v_j be the vertices of l_j ordered according to the orientation of α from v to v'. Since α is minimal in $G(e^i)$ and Q is a MH-complex we have,

$$\underline{\omega}_{(e^i)}(v, l_j) = v_{j-1} = \underline{\omega}(v, l_j)$$

Hence, $d_{G(e^i)}(v, v_j) = d_{G(e^i)}(v, v_{j-1}) + 1$ and $d(v, v_j) = d(v, v_{j-1}) + 1$ for $j = 1, \ldots, n$ which proves the lemma. □

We now state the theorem that generalizes the relationship between hyperplane arrangements and zonotopes.

Theorem 3.3.14. *Let X be a smooth manifold of dimension l, $\mathcal{A}$ denote an arrangement of submanifolds. The (dual) cell complex $X^* = (X, \mathfrak{F}^*)$ is a MH-complex.*

Proof. First, we need to define the two maps $\underline{\omega}, \overline{\omega}$ and then show that they are well defined. Let F^i be an i-cell and C be a vertex of X^* then,

$$\underline{\omega}(C, F^i) := F^i \circ C.$$

Using the same strategy as in the proof of Lemma 3.2.6 we can identify a unique chamber (of $\mathcal{A}$) whose closure contains (dual of) F^i and is farthest from C, denote this chamber by $F^i * C$ and

$$\overline{\omega}(C, F^i) := F^i * C.$$

This shows that the maps $\underline{\omega}, \overline{\omega}$ are well defined. Second, note that a path between two vertices C, C' has minimal length (among all paths from C to C') if and only if it crosses the faces that separate C from C' exactly once and does not cross any other face. The distance between any two vertices of X^* is thus

$$d(C, C') := |R(C, C')|.$$

Observe that if $F^i = N_1 \cap \cdots \cap N_r$ then $R(F^i \circ C, F^i * C) = \{N_1, \ldots, N_r\}$ and if $D \in V(F^i)$ then

$$R(F^i \circ C, D) \bigcup R(F^i * C, D) = R(F^i \circ C, F^i * C) \text{ and } R(F^i * C, D) \cap R(F^i \circ C, D) = \emptyset$$

(by Lemma 3.2.4). Moreover,

$$R(C, D) = R(C, F^i \circ C) \bigcup R(F^i \circ C, D), \quad R(C, F^i \circ C) \cap R(F \circ C, D) = \emptyset$$

and

$$R(C, D) = R(C, F * C) \bigcup R(F * C, D), \quad R(C, F * C) \cap R(F * C, D) = \emptyset.$$

Using the last equality we see that X^* is a QMH-complex. The other compatibility conditions also follow easily. □

Similar combinatorial properties are enjoyed by the cell structure of a tangent bundle complement that was described at the beginning of the previous section. In case of hyperplane arrangements, each cell of the associated Salvetti complex is zonotopal (see [7, Proposition 5.7]). We generalize this claim and also show that in general the Salvetti complex is a MH-complex.

Theorem 3.3.15. *Let $\mathcal{A}$ be an arrangement of submanifolds in a manifold X and $Sal(\mathcal{A})$ denote the associated Salvetti complex. Then $Sal(\mathcal{A})$, with the CW structure described above, is a MH-complex.*

Proof. The 1-skeleton of $Sal(\mathcal{A})$ is obtained by 'doubling' the edges in the 1-skeleton of X^*. Hence the distance between any two vertices of $Sal(\mathcal{A})$ is same as the distance between the corresponding vertices of X^*. Also, by construction, there is a one-to-one correspondence between vertices of $[F^i, C]$ and the vertices of F^i for all $F^i \in X^*$. □

Remark 3.3.16. Note that in the construction of the Salvetti complex at the beginning of this section we used the $\underline{\omega}(F, C) = F \circ C$ map. It is possible to repeat the same construction using the $\overline{\omega}$ map. However one can show that the two complexes are isomorphic.

Remark 3.3.17. The oriented 1-skeleton of the Salvetti complex along with the circuits satisfies the *circuit axioms* [6, Definition 3.2.1] to become an oriented matroid. However, we will not prove it here.

As a consequence of the Corollary 3.3.10, every 2-cell of the Salvetti complex (of an arrangement) is combinatorially equivalent to an oriented polygon with an even number of edges. This observation together with the condition (3) in Definition 3.3.8 (QMH-complex) proves the following.

Corollary 3.3.18. *In the Salvetti complex $Sal(\mathcal{A})$, the (oriented) 1-skeleton of a k-cell $[F^k, C]$ is composed of edge-paths going from $[C, C]$ to $[F^k * C, F^k * C]$. All these paths have same lengths and are directed away from $[C, C]$.*

In particular, the boundary of each 2-cell $[F^2, C]$ is composed of two minimal positive paths of the same length from $[C, C]$ to $[F^2 * C, F^2 * C]$.

Proof. Assume that there is an edge $[F^1, C']$ contained in the boundary of $[F^k, C]$. Therefore, $C' = F^1 \circ C$ and the result follows because of the condition (3) defining QMH-complex. $\square$

Remark 3.3.19. The reason we have defined the Salvetti complex is because it has the homotopy type of the tangent bundle complement associated to a submanifold arrangement. In light of the results proved in this section it is now clear that one can associate the Salvetti complex construction to any regular cell complex which is a MH-complex (either LMH or QMH-complex). This generalization was first studied in [74]. Although the dual cell complex induced by a submanifold arrangement has the structure of a MH-complex it is not true that an arbitrary MH-complexes corresponds to some submanifold arrangement. In fact, by a theorem of Salvetti [74, Proposition 6] it follows that even the dual of a pseudosphere arrangement is a MH-complex. Hence it is natural to ask whether there is a *topological representation theorem* for MH-complexes. By that we mean, is there a generalization of submanifold and pseudosphere arrangements so that the induced dual cell complex is a MH-complex?

We would like to state a conjecture that might prove one direction of such a topological representation theorem.

Conjecture 3.3.20. Let X be a finite dimensional manifold and $\mathcal{A} = \{N_1, \ldots, N_k\}$ be a collection of finitely many submanifolds such that:

1. each N_i is an embedded, locally flat, closed submanifold of codimension 1,

2. the intersections of these submanifolds give a stratification of X into a regular cell complex.

Then the cell complex obtained by considering the dual cells is a MH-complex.

Remark 3.3.21. Since the Salvetti complex $Sal(\mathcal{A})$ is a MH-complex we can repeat its construction to obtain a new complex. We denote it by $Sal(\mathcal{A})^{(2)}$ and call it the *second associated Salvetti complex*. In fact in light of Theorem 3.3.15 one can define $Sal(\mathcal{A})^{(n)}$ for every $n \geq 1$. A k-cell of such a complex can be written as $[F^k; C_1, \ldots, C_n]$ for every $C_i \prec F^k$. As in Theorem 3.3.1 there is a sequence of maps $\psi_n \colon Sal(\mathcal{A})^{(n)} \to X$. Setting $Sal(\mathcal{A})^{(0)} = X$ we have

$$\chi(Sal(\mathcal{A})^{(n)}) = \sum_{F^i \in X^*} (-1)^i \psi_n^{-1}(F^i)$$

By induction on n we also have an embedding $\iota_{C_1,\ldots,C_n} \colon X^* \to Sal(\mathcal{A})^{(n)}$ given by $F^i \mapsto [F^i; F^i \circ C_1, \ldots, F^i \circ C_n]$.

We end this section with homotopy colimits. The next result follows from the observation that the barycentric subdivision of any regular cell complex is a poset limit (Definition 2.1.7).

Theorem 3.3.22. *Given a l-manifold X and a submanifold arrangement $\mathcal{A}$, let $\mathcal{F}^*$ denote the dual face poset. Define a diagram of spaces*

$$\mathcal{D} \colon \mathcal{F}^* \to Top$$

with discrete spaces

$$\mathcal{D}(F^i) := \{C \in X_0^* | C \prec F^i\}$$

and maps being inclusions

$$\begin{array}{rcll} \mathcal{D}(F^i \to F^j) \colon \mathcal{D}(F^i) & \to & \mathcal{D}(F^j) & (i > j) \\ C & \mapsto & F^j \circ C & \end{array}$$

Then

$$hocolim_{\mathcal{F}^*} \mathcal{D} \cong Sal(\mathcal{A})$$

Proof. Since all the space in the above diagram are in fact finite sets of points, we can consider these sets as posets with trivial order relation ($C \leq C' \Leftrightarrow C = C'$). In this situation we can apply the *simplicial model Lemma* 2.1.8. Hence the homotopy colimit of $\mathcal{D}$ has the homotopy type of $\Delta(Plim\mathcal{D})$.

The vertex set of $\Delta(Plim\mathcal{D})$ is clearly

$$\{(F, C) \in \mathcal{F}(\mathcal{A}) \times \mathcal{C}(\mathcal{A}) | F \subset \overline{C}\}.$$

In order to determine higher dimensional simplices we look at the chains in $Plim\mathcal{D}$. According to the Definition 2.1.7, the order relation on $Plim\mathcal{D}$ is the following

$$(F, C) \leq (F', C') \Leftrightarrow F \prec F' \text{ in } \mathcal{F}^* \text{ and } F \circ C' = C.$$

Then it follows that the poset $Plim\mathcal{D}$ is isomorphic to the Salvetti poset, which establishes the claim. $\square$

3.4 Covers of the Tangent Bundle Complement

We now look at the covering spaces of the tangent bundle complement. Our aim is to show that the covering spaces of the tangent bundle complement are determined by the arrangement graph. Recall that in case of a hyperplane arrangement the associated arrangement graph is the oriented 1-skeleton of the Salvetti complex. Our plan is to generalize the theory of *Salvetti-type diagram models* introduced by Delucchi in [23] (building on the work of Paris [66]) to submanifold arrangements. In this section we start with a short discussion of arrangement groupoids. Then corresponding to a cover of the arrangement groupoid we construct covering spaces of the Salvetti complex. Constructing a covering a space of the tangent bundle complement itself is fairly technical and does not provide any new insight. For this reason we have decided to work with the Salvetti complex instead. We discuss this point at length towards the end of this section. The classification of the covering spaces will be proved in the next section (after a discussion of fundamental groups).

Recall the definition of *arrangement graph* associated to a hyperplane arrangement (Section 2.3.1). Note that this notion is also valid for the arrangement of submanifolds and it is isomorphic to the oriented 1-skeleton of the associated Salvetti complex. For the sake of completeness we define it here again.

Definition 3.4.1. Given an arrangement of submanifolds $\mathcal{A}$ in a manifold X, the arrangement graph, denoted by $G(\mathcal{A})$, is the directed graph whose vertex set corresponds to the set of chambers of $\mathcal{A}$. Add is a pair of oppositely oriented edges between two vertices whenever there is a codimension 1 face common to the corresponding chambers.

Once we have the arrangement graph we can also define the arrangement groupoid and the category of positive paths for an arrangement of submanifolds. Before introducing these notions we would like to recall a relevant terminology.

A *path* in the arrangement graph is a sequence of edges traversed not necessarily according to their orientations. The length of a path is equal to the number of edges traversed. An edge is a path of length 1 and vertex is path of length 0. Initial vertex of a path α is denoted by $s(\alpha)$ and the terminal vertex by $t(\alpha)$. Such directed paths can be represented by words whose letters correspond to edges. If α is an undirected path then $\alpha = (\epsilon_1 a_1, \dots, \epsilon_n a_n)$ where $\epsilon_i = \pm 1$ depending on whether a_i is traversed along or opposite to its orientation, the length of α is equal to 1 and $s(\epsilon_{i+1}a_{i+1}) = t(\epsilon_i a_i)$ for every i. If α and β are two paths such that $s(\beta) = t(\alpha)$ then their concatenation gives a new path denoted by $\alpha\beta$. Inverse of a path $\alpha = (\epsilon_1 a_1, \dots, \epsilon_n a_n)$ is the path $\alpha^{-1} = (-\epsilon_n a_n, \dots, -\epsilon_1 a_1)$.

A *directed path* is a sequence of consecutive edges traversed according to their orientations. For example, $\alpha = (+a_1, \dots, +a_n)$ is directed path of length n. A *minimal positive path* is a path having shortest length among all the positive paths that join its endpoints. Unless otherwise stated by a *path* we mean an *undirected path* and by a *positive path* we mean a directed path.

Recall that every 2-cell in the Salvetti complex is a polygon with an even number of (oriented) edges and that the boundary of such a 2-cell is composed of two minimal positive paths of same length (Corollary 3.3.18).

We define two types of '*moves*' on the paths in the arrangement graph:

1. If α is a path such that $\alpha = \alpha_1\gamma\alpha_2$ and if γ, γ' are two minimal positive paths joining the opposite vertices of a 2-cell, then substitute γ' for γ in α.

2. If α is a path such that $\alpha = \alpha_1\gamma\alpha_2$, where γ is a sub-path of a boundary path of a 2-cell of $Sal(\mathcal{A})$, substitute for γ the path γ' having same endpoints (i.e. $s(\gamma') = s(\gamma), t(\gamma') = t(\gamma)$) and containing all the remaining edges of that 2-cell.

Definition 3.4.2. Given a submanifold arrangement $\mathcal{A}$, the associated *category of positive paths*, denoted by $\mathcal{G}^+(\mathcal{A})$, is defined as follows. The objects of this category are the vertices of the arrangement graph. The morphisms in this category are equivalence classes of positive paths. Two positive paths α and β from a vertex C to a vertex D are equivalent if and only if α is obtained from β by a finite sequence of moves of type (1). It means that the two paths are connected by a sequence of substitutions of minimal positive paths. This equivalence relation will be called the *positive equivalence*, the equivalence class of a path α will be denoted by $[\alpha]_+$ and if need the relation will be denoted by $\stackrel{+}{\sim}$.

The main difference between the positive category associated to a hyperplane arrangement and the above definition is the equivalence relation. In case of hyperplane arrangements the equivalence relation is generated by declaring two minimal positive paths to be equivalent. Since two such paths are homotopic in the associated Salvetti complex. However this is not true for submanifold arrangements. For example, consider the Salvetti complex (Figure 3.7) associated to the arrangement of 2 points in S^1 (Example 3.1.10). The two edges labeled $[p, A]$ and $[q, A]$ are minimal positive paths with the same end points however they are positive equivalent.

Lemma 3.4.3. *Let F be a face of a submanifold arrangement and C be a chamber such that $C \prec F$. Then any two minimal positive paths from C to $F * C$ are positive equivalent.*

Proof. The statement follows from the application of Corollary 3.3.18. Any two minimal positive paths from C to $F * C$ are in fact contained in the boundary of the cell $[F, C]$. □

Hence we can rephrase the positive equivalence as follows. Let α, α' be two positive paths from C to C'. Set $\alpha \overset{+}{\sim} \alpha'$ if and only if $\alpha = \alpha_1 \beta \alpha_2$, $\alpha' = \alpha_1 \beta' \alpha_2$ such that

1. α_1 is a positive path from C to C'',
2. β and β' are minimal positive paths from C'' to $F * C''$ (for some $C'' \prec C$),
3. α_2 is a positive path from $F * C''$ to C'.

More generally, set $\alpha \overset{+}{\sim} \alpha'$ if and only if α is obtained from α' by a finite sequence of positive paths related as above. We now define the arrangement groupoid.

Definition 3.4.4. The *arrangement groupoid* of a submanifold arrangement is a category whose objects are the vertices of the arrangement graph. The morphisms in this category are equivalence classes of paths. Two paths are equivalent if and only if one is obtained from the other by a finite sequence of moves of type (2). The category will be denoted by $\mathcal{G}(\mathcal{A})$ and the morphisms will be denoted by $[\alpha]$, where α is a representative path.

Before moving on we would like to explicitly state the relationship between the above defined categories.

Lemma 3.4.5. *Moves of type* (2) *are composed of moves of type* (1).

Proof. Follows from the description of a 2-cell. $\square$

As a result, if $[\alpha]_+ = [\beta]_+$ then $[\alpha] = [\beta]$. So there is a functor

$$J \colon \mathcal{G}^+ \to \mathcal{G}$$

given by sending $[\alpha]_+$ to $[\alpha]$.

Theorem 3.4.6. *For a submanifold arrangement $\mathcal{A}$ the category $\mathcal{G}(\mathcal{A})$ can be identified with the category of fractions of $\mathcal{G}^+(\mathcal{A})$ and J with the associated canonical functor.*

Proof. Recall from [36, Chapter 1] that to each category $\mathcal{C}$ and to each subset $\Sigma \subset Mor(\mathcal{C})$, of morphisms in $\mathcal{C}$, there is an associated category of fractions $\mathcal{C}[\Sigma^{-1}]$ and a functor $P_\Sigma \colon \mathcal{C} \to \mathcal{C}[\Sigma^{-1}]$. The functor P_Σ makes all morphisms in Σ invertible and it is universal in this sense.

Given $(+a_1, \ldots, +a_n)$ a representative of a morphism in $\mathcal{G}^+$, its inverse is the morphism given by the equivalence class of $(-a_n, \ldots, -a_1)$. Hence J makes all morphisms of $\mathcal{G}^+$ invertible.

The universality of this functor is easy to check. Note that given a morphism $[\alpha]$ in $\mathcal{G}(\mathcal{A})$ it can be written as $[\prod_{i=1}^{n}(\alpha_i)^{\epsilon_i}]$ where each α_i is a positive path, $\epsilon_i = \pm 1$ and the product means the concatenation. Let $J' \colon \mathcal{G}^+ \to \mathcal{E}$ be another functor making all morphisms of $\mathcal{G}^+$ invertible. Then define $K \colon \mathcal{G} \to \mathcal{E}$ as

$$K([\prod_{i=1}^{n}(\alpha_i)^{\epsilon_i}]) := \prod_{i=1}^{n} J'([\alpha_i]_+)^{\epsilon_i}$$

If the path $\prod_{i=1}^{n}(\alpha_i)^{\epsilon_i}$ is equivalent to another path $\prod_{j=1}^{m}(\beta_j)^{\nu_j}$ via moves of type (1) or (2) then

$$\prod_{i=1}^{n} J'([\alpha_i]_+)^{\epsilon_i} = \prod_{j=1}^{m} J'([\beta_j]_+)^{\nu_j}$$

Since the functor K is well defined we have $J' = K \circ J$ which completes the proof. □

The category of fractions is a categorical analogue of the ring of fractions. The connection between the arrangement groupoid and the Salvetti complex is the following:

Lemma 3.4.7. *Let $\mathcal{A}$ be an arrangement of submanifolds in a manifold X. The arrangement groupoid $\mathcal{G}(\mathcal{A})$ is equivalent to the fundamental groupoid $\tilde{\pi}_1(Sal(\mathcal{A}))$ of the associated Salvetti complex.*

Proof. The cell structure we have for the Salvetti complex is special in the sense that its 0-cells correspond to chambers of $\mathcal{A}$, which happen to be the objects of the arrangement groupoid. Let α, β be two paths with same end points. If they are equivalent in $\mathcal{G}(\mathcal{A})$ then the moves of type (2) determine a homotopy between the two. Conversely if α, β are homotopic then type (2) moves provide a homotopy that connects them. □

Consequently, the fundamental group of the Salvetti complex is isomorphic to the object group $\pi(\mathcal{G})$. Hence the connected topological covers of the Salvetti complex are indexed by the subgroups of $\pi(\mathcal{G})$. According to Theorem 2.3.6, connected covers of the arrangement groupoid $\mathcal{G}(\mathcal{A})$ are also indexed by the subgroups of $\pi(\mathcal{G})$. This correspondence was first exploited by Paris in the context of hyperplane arrangements (see [66]). He showed that the covers of the complexified complement are indexed by the covers of the oriented systems. Our treatment is based on Delucchi's reformulation [23, 24] of this idea in terms of diagram of spaces. We show that a cover of the arrangement groupoid can be used to construct a simplicial complex which is a covering space of the Salvetti complex.

3.4.1 Salvetti-type diagram models

In his thesis [23], Delucchi characterized all the connected covers of an arrangement complement[1] using diagrams of spaces and homotopy colimits. To be precise, using covers of the arrangement groupoid he defined a family of diagrams of spaces such that their homotopy colimits have the homotopy type of covers of arrangement complement. One of the benefits of his elegant treatment is that it is easy to generalize to the tangent bundle complement. This is what we plan to do now. We will define the Salvetti-type diagrams and show that their homotopy colimits are covers of the associated Salvetti complex. We closely follow Delucchi's arguments as well as the notation. Let $\mathcal{A}$ denote a submanifold arrangement in some manifold X. Let $Sal(\mathcal{A})$ be the associated Salvetti complex, the context will decide whether we view it as a CW-complex or as simplicial complex. Finally, let $\mathcal{G}$ denote the arrangement groupoid. When dealing with morphisms in $\mathcal{G}$ we will use the path-terminology. For example, domain of a morphism $[\alpha]$ will be denoted by $s(\alpha)$. Hence whenever we make statement about a path it is actually about its equivalence class in $\mathcal{G}$.

For the convenience of the reader we quickly recall relevant facts about the covers of the arrangement groupoid. Given $z \in Ob(\mathcal{G})$ its *star* is the following set of morphisms-

$$St(z) = \{[\alpha] \in Mor(\mathcal{G}) \mid s(\alpha) = z\}.$$

A *covering* of $\mathcal{G}$ is a functor $\rho\colon \mathcal{G}_\rho \to \mathcal{G}$ such that for every $z \in Ob(\mathcal{G}_\rho)$, the induced map

$$\rho_z\colon St(z) \to St(\rho(z))$$

is bijective. Given $[\alpha]$, a morphism in $\mathcal{G}$ and any $z \in \rho^{-1}(s(\alpha))$, the lift of α at z is the morphism $\rho_z^{-1}(\alpha)$, denoted by $\alpha^{<z>}$ when the covering ρ is understood. Recall that according to Theorem 2.3.6 each object of $\mathcal{G}_\rho$ is a right coset of homotopy class of paths in the arrangement graph. Let $\mu(C \to D, F)$ denote a minimal positive path starting at a vertex $[C, C]$ ending at another vertex $[D, D]$ and it traverses an edge contained in the boundary of $[F, C]$. It is assumed that $C \prec F$.

Lemma 3.4.8. *With the same notation as above, $\mu(C \to D, F) \overset{+}{\sim} \mu(C \to D, F')$ if there exists $G \in X^*$ such that on of the following is satisfied:*

1. *$F, F' \prec G$ and $\mu(C \to D, F), \mu(C \to D, G)$ are in the boundary of $[G, C]$;*
2. *$D = G * C$.*

Proof. Follows from Corollary 3.3.18. $\square$

[1]'Arrangement complement' is an often used abbreviation for complexified complement of a real hyperplane arrangement.

Definition 3.4.9. Given a cover of the arrangement groupoid $\rho\colon \mathcal{G}_\rho \to \mathcal{G}(\mathcal{A})$, we define a *diagram of posets* $\mathcal{D}_\rho$ on the dual face poset $\mathcal{F}^*$ as

$$\mathcal{D}_\rho(F^i) := \{v \in Ob(\mathcal{G}_\rho) \mid \rho(v) \prec F^i\}.$$

Each poset is endowed with the trivial order relation. The maps between these spaces are the following inclusions

$$\begin{array}{rcl} \mathcal{D}_\rho(F^i \to F^j)\colon \mathcal{D}(F^i) & \to & \mathcal{D}(F^j) \quad (i > j) \\ v & \mapsto & t(\mu(\rho(v) \to F^j \circ \rho(v), F^i)^{<v>}) \end{array}$$

where $\mu(\rho(v) \to F^j \circ \rho(v), F^i)^{<v>}$ is the lift of the minimal positive path in $\mathcal{G}$ that starts at $[\rho(v), \rho(v)]$ and traverses an edge contained in $[F^i, \rho(v)]$.

According to the *simplicial model lemma* (Lemma 2.1.8), the above homotopy colimit is in fact an order complex. We will denote this order complex by $\mathcal{S}_\rho$, its vertex set is

$$\{(F, v) \in \mathcal{F}^* \times Ob(\mathcal{G}_\rho) \mid \rho(v) \prec F\}$$

the simplices of $\mathcal{S}_\rho$ are chains with respect to the following partial order

$$(F_2, v_2) \prec_\rho (F_1, v_1) \Leftrightarrow \begin{cases} F_2 \prec F_1 \text{ and} \\ v_2 = t(\mu(\rho(v_1) \to F_2 \circ \rho(v_1), F^1)^{<v_1>}) \end{cases}$$

Remark 3.4.10. If ρ is the identity map then $\mathcal{S}_{id}$ is precisely the Salvetti complex, for the proof see Theorem 3.3.22.

Remark 3.4.11. Using the same arguments as in the proof of Lemma 3.2.11 it follows that there is a one-to-one correspondence between the chains in $\mathcal{S}_\rho$ and the chains in $\mathcal{F}^*$. If ϕ denotes a chain in $\mathcal{F}^*$ and $v \in Ob(\mathcal{G}_\rho)$ such that $\rho(v) \prec max(\phi)$, then the pair (ϕ, v) corresponds to a simplex of $\mathcal{S}_\rho$ and every simplex is of this form. Moreover, the simplicial complex $\mathcal{S}_\rho$ is the barycentric subdivision of a CW-complex $\mathcal{S}_\rho^{CW}$; the barycentric subdivision of a (closed) k-cell of this CW-complex, which we denote by a pair $[F^k, v]$, is given by

$$[F^k, v] := \bigcup_{max(\phi) = F^k} (\phi, v).$$

Vertices of $\mathcal{S}_\rho^{CW}$ are of the form $[\rho(v), v]$, hence are in bijection with the objects of $\mathcal{G}_\rho$. For a k-cell $[F^k, v]$ a 0-cell $[\rho(v'), v']$ is its vertex if and only if

$$v' = t(\mu(\rho(v) \to F' \circ \rho(v'), F^k)^{<v>})$$

for some $F' \prec F^k$.

Our aim is to show that the simplicial complex $\mathcal{S}_\rho$ is a covering space of the (simplicial) Salvetti complex. The strategy we employ here is motivated by the proof of a well known fact in elementary algebraic topology. A simplicial and fixed point free action of a discrete group on a simplicial complex is a covering space action (see for example [44, Section 1.B]). First, we prove that there is a natural, simplicial map between these two complexes and them proceed on to prove that it is indeed a covering map.

The functor $\rho\colon \mathcal{G}_\rho \to \mathcal{G}$ naturally induces a morphism of diagrams $\lambda\colon \mathcal{D}_\rho \to \mathcal{D}_{id}$. On the level of spaces it is given by -

$$\begin{array}{rcll} \lambda_F\colon \mathcal{D}_\rho(F) & \to & \mathcal{D}_{id}(F) & \forall F \in \mathcal{F}^* \\ v & \mapsto & \rho(v) & \end{array}$$

In order to check the compatibility of this morphism with the maps between the spaces, consider $F_2 \prec F_1$ and $v \in \mathcal{D}_\rho(F_2)$. Then,

$$\begin{array}{rcl} \lambda_{F_2}(\mathcal{D}_\rho(F_1 \to F_2)(v)) & = & \rho(t(\mu(\rho(v) \to F_2 \circ \rho(v), F_1)^{<v>})) \\ & = & F_2 \circ \rho(v) \\ & = & \mathcal{D}_{id}(F_1 \to F_2)(\rho(v)) \\ & = & \mathcal{D}_{id}(F_1 \to F_2)(\lambda_{F_1}(v)) \end{array}$$

The morphism λ induces a map between homotopy colimits by functoriality, hence a map of simplicial complexes

$$\Lambda_\rho\colon \mathcal{S}_\rho \to Sal(\mathcal{A}).$$

Λ_ρ maps the simplex (ϕ, v) of $\mathcal{S}_\rho$ to the simplex $(\phi, \rho(v))$ of $Sal(\mathcal{A})$. Moreover, a morphism $\eta\colon \mathcal{G}_{\rho_1} \to \mathcal{G}_{\rho_2}$ between two covers of the arrangement groupoids induces a map $\Lambda_\eta\colon \mathcal{S}_{\rho_1} \to \mathcal{S}_{\rho_2}$.

Theorem 3.4.12. *The map $\Lambda_\rho\colon \mathcal{S}_\rho \to Sal(\mathcal{A})$, induced by a cover $\rho\colon \mathcal{G}_\rho \to \mathcal{G}$ of the arrangement groupoid, is a covering map.*

Proof. First note that $Sal(\mathcal{A})$ is a finite dimensional and locally finite simplicial complex implying that it is connected and locally path connected. Hence all its connected covering spaces exist.

Let $p \in Sal(\mathcal{A})$ be some point, σ be the smallest dimensional simplex containing p, let U denote the star of σ (set of all simplices of $Sal(\mathcal{A})$ that contain σ). In order to show that Λ_ρ is a covering map it is enough to show that $\Lambda_\rho^{-1}(U)$ is an even covering of U.

Let us first characterize U and its inverse image in terms of chains and chambers. As σ is a simplex of $Sal(\mathcal{A})$ there exists a chain $\tilde{\phi}$ in $\mathcal{F}^*$ and (a chamber) $\tilde{C} \prec$

$max(\tilde{\phi}) =: \tilde{F}$ such that $\sigma = (\tilde{\phi}, \tilde{C})$. Thus star of σ takes the following form:

$$U = \bigcup_{\substack{\phi \supseteq \tilde{\phi} \\ C \in R(\phi, \tilde{C})}} (\phi, C)$$

where $R(\phi, \tilde{C}) := \{C \prec max(\phi) \mid \tilde{F} \circ C = \tilde{C}\}$. As a set, the pre-image of U is:

$$\Lambda_\rho^{-1}(U) = \bigcup_{\substack{\phi \supseteq \tilde{\phi} \\ \{v \mid \rho(v) \in R(\phi, \tilde{C})\}}} (\phi, v).$$

For a fixed object $w \in \rho^{-1}(\tilde{C})$ define

$$W_w := \bigcup_{\substack{\phi \supseteq \tilde{\phi} \\ C \in R(\phi, \tilde{C})}} (\phi, v(C, w))$$

where $v(C, w)$ is the (unique) object, of $\mathcal{G}_\rho$, in $\rho^{-1}(C)$ from which w can be reached by the lift of a minimal positive path.

We now show that these sets W_w are disjoint, their union is $\Lambda_\rho^{-1}(U)$ and each of them is homeomorphic to U.

Claim 1: For any $w_1 \neq w_2 \in \rho^{-1}(\tilde{C})$, $W_{w_1} \cap W_{w_2} = \emptyset$.

Assume that the intersection is non-empty and there is a simplex

$$(F, u) \subset W_{w_1} \cap W_{w_2}.$$

By definition of W_w,

$$u = v(C, w_1) = v(C, w_2)$$

and $\rho(u) = C$. This is same as

$$\begin{aligned} w_1 &= t(\mu(C \to \rho(w_1), F)^{<u>}) \\ w_2 &= t(\mu(C \to \rho(w_2), F)^{<u>}) \end{aligned}$$

which implies $w_1 = w_2$ (since $\rho(w_1) = \rho(w_2) = \tilde{C}$), a contradiction.

Claim 2:

$$\coprod_{w \in \rho^{-1}(\tilde{C})} W_w = \Lambda_\rho^{-1}(U).$$

To see that the left hand side is contained in the right hand side, consider a simplex $(\phi, v(C, w))$. Its image $\Lambda_\rho(\phi, v(C, w)) = (\phi, C)$ is a simplex of the right hand side.

For the converse, consider (ϕ, v) a simplex of the left hand side (such that $\tilde{\phi} \subseteq \phi$ and $\rho(v) \in R(\phi, \tilde{C})$). Define $w := t(\mu(\rho(c) \to \tilde{C}, max(\phi))^{<v>})$. By construction, $\rho(w) = \tilde{C}$, hence (ϕ, v) is a simplex of W_w.

Claim 3: Fix $w \in \rho^{-1}(\tilde{C})$, then $\Lambda_\rho \colon W_w \to U$ is a homeomorphism.

Clearly Λ_ρ is a simplicial map and also it is surjective, since given (ϕ, C) a simplex of U, $(\phi, v(C, w))$ is a simplex of W_w which lies in its pre-image. The injectivity of this map follows from [23, Lemma 4.3.3] and [73, Lemma 3]. □

Example 3.4.13. Consider the arrangement $\mathcal{A}_1$ of 2 points in S^1 (Example 3.1.10), its associated Salvetti complex $Sal(\mathcal{A}_1)$ was described in Figure 3.7. Figure 3.12 depicts a portion of its universal cover. The objects of the covering groupoid are denoted by $\tilde{A}, \tilde{B}$ (indices are suppressed for notational simplicity). Note that there

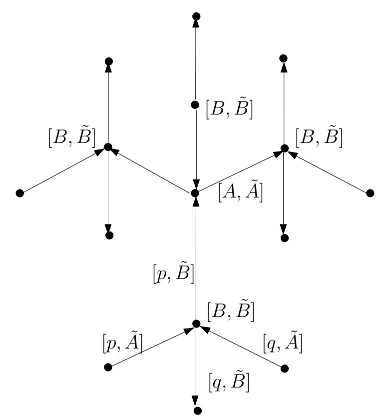


Figure 3.12: The Universal cover of $Sal(\mathcal{A}_1)$

are two minimal positive paths, in $Sal(\mathcal{A})$, from $[B, B]$ to $[A, A]$. One of them is the (oriented) edge $[p, B]$ and the other is the edge $[q, B]$. Both of these paths are represent distinct equivalence classes in $\mathcal{G}^+(A)$ as well as $\mathcal{G}(A)$. In the covering space, end points of the lifts of these two paths are distinct and are contained in $\rho^{-1}([A, A])$. Also note that the functor $J \colon \mathcal{G}^+(A) \to \mathcal{G}(A)$ is faithful in this example.

Example 3.4.14. Consider the arrangement $\mathcal{A}_2$ of 2 circles in S^2 (Example 3.1.11), its associated Salvetti complex $Sal(\mathcal{A}_2)$ was described in Figure 3.8. Figure 3.13 shows a portion of the 1-skeleton of its universal cover. The objects of the covering groupoid are denoted by $\tilde{C}_i, 1 \leq i \leq 4$ (indices are suppressed for notational simplicity). Note that each square is a boundary of a 2-sphere, whose two 2-cells are indexed by $[p_1, \tilde{C}_i]$ and $[p_2, \tilde{C}_i]$.

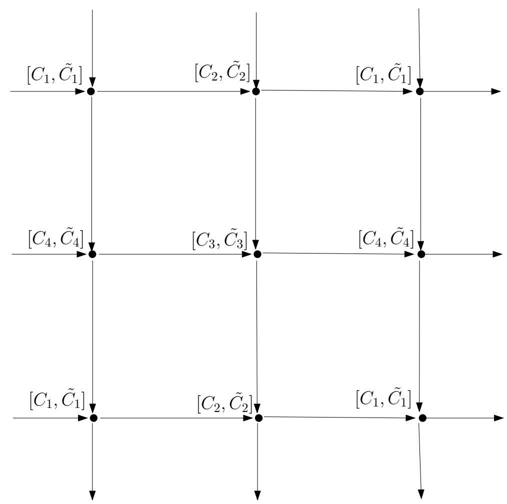


Figure 3.13: 1-skeleton of the Universal cover of $Sal(\mathcal{A}_2)$

3.5 The Fundamental Group

Since the Salvetti complex carries the homotopy type of the tangent bundle complement both have isomorphic fundamental groups. By studying the 2-skeleton of $Sal(\mathcal{A})$ we can deduce a presentation for the fundamental group. The aim of this section is to understand $\pi_1(Sal(\mathcal{A}))$ (as much as possible) using the arrangement groupoid. We will also identify a class of arrangements for which the word problem for the fundamental group is solvable. Finally, we will prove a result regarding fundamental groups of covering spaces constructed in the previous section. This result will lead to the classification of the covering spaces of $Sal(\mathcal{A})$.

In the previous section we proved that the arrangement groupoid $\mathcal{G}(\mathcal{A})$ is the category of fractions of $\mathcal{G}^+(\mathcal{A})$. The study of the fundamental group requires more information regarding the relationship between these two categories. Existence of calculus of fractions is one such property that is useful. We proceed by recalling definitions and some results.

Definition 3.5.1. The category $\mathcal{G}^+$ admits a (strong) calculus of left fractions if and only if the following conditions are satisfied ([36, Section 2.2]):

1. If $[\alpha_1], [\alpha_2]$ are such that $s(\alpha_1) = s(\alpha_2)$ then there exist $[\beta_1], [\beta_2]$, with $s(\beta_i) = t(\alpha_i)(i = 1, 2)$, such that $[\alpha_1\beta_1]_+ = [\alpha_2\beta_2]_+$.

2. (*left cancellation law*) If $[\alpha], [\alpha_1], [\alpha_2]$ are such that $t(\alpha) = s(\alpha_i)(i = 1, 2)$, and $[\alpha\alpha_1]_+ = [\alpha\alpha_2]_+$ then $[\alpha_1]_+ = [\alpha_2]_+$.

Remark 3.5.2. Usually there are two more conditions mentioned in the definition of calculus of fractions. Since we are inverting all the morphisms in $\mathcal{G}^+$ those conditions are always satisfied, hence skipped in the above definition.

Remark 3.5.3. The conditions for a calculus of right fractions are dual to the ones specified above. If a morphism $[\alpha]$ in $\mathcal{G}^+$ is represented by a sequence of edges as $(+a_1, \ldots, +a_n)$, then define the *opposite* of $[\alpha]$ as the morphism represented by $(+a'_n, \ldots, +a'_1)$ where a'_i is the edge opposite to a_i. By passing to the opposite paths it is clear that $\mathcal{G}^+$ admits a calculus of left fractions if and only if it admits a calculus of right fractions. Hence when it is the case, we will just say that $\mathcal{G}^+$ admits a *calculus of fractions*.

Recall that the elements of a ring of fractions can be written as $s^{-1}r$. The same is true when a category admits a calculus of fractions. Assume that $\mathcal{G}^+$ admits a calculus of fraction, then every morphism $[\alpha] \in Mor(\mathcal{G})$ (i.e. a homotopy class of paths) can be written in the form

$$[\alpha] = [\alpha_1][\alpha_2]^{-1},$$

where α_1, α_2 are positive paths.

Theorem 3.5.4. *The category $\mathcal{G}^+$ admits a calculus of fractions if and only if the canonical functor $J\colon \mathcal{G}^+ \to \mathcal{G}$ is faithful.*

Proof. It is clear that if J is faithful then $\mathcal{G}^+$ admits a calculus of fractions.
Conversely, assume that there two positive paths α, β such that they have same end points and they are homotopic. Then $[\alpha][1]^{-1} = [\beta][1]^{-1}$. So there exist two more positive paths γ_1, γ_2 such that

$$[\alpha\gamma_1]_+ = [\beta\gamma_2]_+, [\gamma_1]_+ = [\gamma_2]_+.$$

By the cancellation law $[\alpha]_+ = [\beta]_+$. □

We start by characterizing arrangements for which the restriction of J to the class of minimal positive paths is always faithful.

Theorem 3.5.5. *Let X be a l-manifold and $\mathcal{A}$ be an arrangement of submanifolds with $Sal(\mathcal{A})$ being the associated Salvetti complex. If X is simply connected then two minimal positive paths in the 1-skeleton of $Sal(\mathcal{A})$ that have same initial as well as terminal vertex are homotopic relative to $(0,1)$.*

Proof. Let α, β be two positive paths such that $s(\alpha) = s(\beta) = [C, C]$, $t(\alpha) = t(\beta) = [D, D]$ and both of them have same lengths $(= d(C, D))$. Clearly, $\psi(\alpha), \psi(\beta) \subset X_1^*$ (recall that $\psi\colon Sal(\mathcal{A}) \to X^*$ was defined in Theorem 3.3.6) are minimal paths of length $d(C, D)$, so no two edges of these two paths are sent (by ψ) to the same edge of X^*. Without loss of generality, let $a = [F^1, C']$ be an edge of the path α, consider (a) as a sub-path of α. Let $s(a) = [C', C']$ then, $F^1 \circ C = C'$ since α is minimal. Repeating same argument for every edge of β we get that the paths α and β are contained in $\iota_C(X^*)$. Hence by Theorem 3.3.6 and the fact that $\pi_1(X^*) = \{1\}$, it follows that α and β are homotopic. $\square$

An immediate consequence of this theorem is the following:

Corollary 3.5.6. *If X is a l-manifold, $\mathcal{A}$ is an arrangement of submanifolds and $M(\mathcal{A})$ is the tangent bundle complement then,*

$$\pi_1(M(\mathcal{A})) \cong \pi_1(G(\mathcal{A}))/\{\alpha\beta^{-1} \mid \alpha, \beta \ \ \textit{are minimal positive paths with same end points}\}.$$

Proof. A loop in the arrangement graph $G(\mathcal{A})$ is contractible if and only if it is the boundary of a 2-cell (of $Sal(\mathcal{A})$). According to Corollary 3.3.18 the boundary of every 2-cell is composed of two minimal positive paths with same ends. $\square$

In particular Theorem 3.5.5 applies to hyperplane arrangements and arrangements in a sphere (dimension ≥ 2). Minimal positive paths with same end points are homotopic in the Salvetti complex associated to such an arrangement. Hence such paths represent the same morphism in the corresponding arrangement groupoid. More is true for (central) hyperplane arrangements; minimal positive paths with same end points represent same morphism in the positive category (see [22, 1.12] for the original proof). This means that the restriction of functor J to the class of minimal positive paths is faithful (this holds true even for arrangements of pseudospheres [74, Theorem 20]).

Definition 3.5.7. A submanifold arrangement $\mathcal{A}$ in a l-manifold X will be called *flat* if and only if it satisfies the following conditions:

1. X is a simply connected manifold,
2. restriction of $J\colon \mathcal{G}^+(\mathcal{A}) \to \mathcal{G}(\mathcal{A})$ to the class of minimal positive paths is faithful.

The dual complex $(X, \mathcal{F}^*)$ in this case will be called *flat MH-complex*.

For a flat arrangement $\mathcal{A}$, we indicate by $[\mu(C \to D)]$ the unique equivalence class (in $\mathcal{G}^+(\mathcal{A})$ or $\mathcal{G}(\mathcal{A})$) determined by a minimal positive path from a chamber C to another chamber D.

For every chamber C of a central hyperplane arrangement $\mathcal{A}$, there exists a unique chamber $-C$ such that $R(C, -C) = \mathcal{A}$ (in our notation $-C = \{\mathbf{0}\} * C$). This property induces an involution on the associated arrangement graph. Also in case of pseudosphere arrangements the antipodal action on the sphere induces an involution on the associated arrangement graph. We formalize this property in the following definition:

Definition 3.5.8. An arrangement of submanifolds $\mathcal{A}$ in a l-manifold X is said to have *the involution property* if and only if there exists a graph automorphism $\phi\colon X_1^* \to X_1^*$ of the dual 1-skeleton (considered as a graph) satisfying:

1. ϕ is an involution (which induces involution on the vertices as well as the edges);
2. for every vertex C, $d(C, \phi(C)) = max_{D \in X_0^*} d(C, D)$;
3. $d(C, \phi(C)) = d(C, D) + d(D, \phi(C))$ for every vertex C and D.

The dual complex X^* of such an arrangement will be called as the *MH*-complex*.

The image of either a vertex or an edge under ϕ will be denoted by writing # on its top, for example, $C^\# := \phi(C)$.

Lemma 3.5.9. *If $\mathcal{A}$ is a submanifold arrangement in X with the involution property then:*

1. *$d(C, C^\#) = |\mathcal{A}|$ (number of submanifolds in $\mathcal{A}$) for all C.*
2. *$d(C, D) = d(C^\#, D^\#)$ for all C, D.*

Proof. Using the property (3) in Definition 3.5.8 we have:

$$\begin{aligned} d(C, C^\#) &= d(C, D) + d(D, C^\#) && (3.5.1)\\ \text{also } d(C, C^\#) &= d(C, D^\#) + d(D^\#, C^\#) && (3.5.2) \end{aligned}$$

Adding equations (3.5.1) and (3.5.2) we get

$$2d(C, C^\#) = 2d(D, D^\#)$$

Without loss of generality assume that $d(C, C^\#) = |\mathcal{A}| - 1$. Hence there is $N \in \mathcal{A}$ such that $N \notin R(C, C^\#)$. Choose C' such that $N \in R(C, C')$. Using the condition (3) defining QMH-complexes we get that $d(C, C') > d(C, C^\#)$, a contradiction. Therefore no such N exists, which proves (1). We call the number $d(C, C^\#) = |\mathcal{A}|$, the *diameter of X* (with respect to $\mathcal{A}$).

Now subtracting $d(D, D^\#) = d(D, C^\#) + d(C^\#, D^\#)$ from (3.5.1) we get

$$0 = d(C, D) - d(C^\#, D^\#)$$

which proves (2). □

Now we show that $[C, C]$ and $[C^\#, C^\#]$ are vertices of a cell in $Sal(\mathcal{A})$.

Lemma 3.5.10. *For a chamber C there exists a face F such that $C^\# = F * C$.*

Proof. Choose F such that $C \prec F$. Again using Definition 3.3.8,

$$d(C, F * C) = d(C, C^{\#}) + d(C^{\#}, F * C)$$

Since $d(C^{\#}, F * C) = 0$, the statement follows. $\square$

Remark 3.5.11. Given a chamber C, let $Opp(C, 0)$ denote the set of all codimension l faces that are contained in the closure of $C^{\#}$. For $F \in Opp(C, 0)$ the cell $[F, F \circ C]$ in $Sal(\mathcal{A})$ is of dimension l. Moreover, $[F \circ C, F \circ C]$ and $[C^{\#}, C^{\#}]$ are its (opposite) vertices. Therefore any minimal positive path from $[C, C]$ to $[C^{\#}, C^{\#}]$ is composed of a minimal positive path from $[C, C]$ to $[F \circ C, F \circ C]$ and a minimal positive path from $[F \circ C, F \circ C]$ to $[F * C, F * C]$. The latter path is contained in the boundary of of the l-cell $[F, F \circ C]$. As an application of Corollary 3.3.18 it follows that any two such paths are not only homotopic but they are also positive equivalent. However two minimal positive paths from $[C, C]$ to $[F \circ C, F \circ C]$ need not be positive equivalent in general.

Before we move on to explore more properties of the positive category let us mention one more combinatorial structure on chambers. Using the combinatorial distance (Lemma 3.2.4) we can equip the chambers with an ordering relation. This was first done for hyperplane arrangements by Edelman in [29].

Fix a chamber C now define a partial order $\prec_C$ on the set of all chambers as follows:

$$D \prec_C D' \iff R(C, D) \subseteq R(C, D')$$

Let us denote by $\mathcal{P}_C(\mathcal{A})$ the set of all chambers together with the above ordering then we have the following.

Lemma 3.5.12. *For an arrangement $\mathcal{A}$ with the involution property, $\mathcal{P}_C(\mathcal{A})$ is a graded and self-dual poset for every chamber C. The rank of $D \in \mathcal{P}_C(\mathcal{A})$ is equal to $|R(C, D)|$ and the self-duality is given by $D \mapsto D^{\#}$. Moreover for $C, D \in \mathcal{C}(\mathcal{A})$ there is a natural isomorphism $\mathcal{P}_C(\mathcal{A}) \xrightarrow{\cong} \mathcal{P}_D(\mathcal{A})$.*

We refer to $\mathcal{P}_C(\mathcal{A})$ as the *poset of chambers* with a base chamber C. The natural question now is to find a criterion when this poset becomes a lattice. For hyperplane arrangements this was done in [5]. The authors showed that poset of chambers is a lattice for every choice of base chamber if and only if each chamber is a cone over a simplex.

Now we come back to the positive category. Note that if the involution exists then it also induces an involution (which we again denote by ϕ) on the arrangement graph. Moreover this involution preserves the positive equivalence on paths as proved in the next lemma.

Lemma 3.5.13. *If $\mathcal{A}$ is an arrangement of submanifolds with the involution property then the involution ϕ induces a functor on $\mathcal{G}^+$ which is also an involution.*

Proof. We start by showing that there is a bijection between the set of edge-paths of X_1^* and the set of all positive paths in $Sal(\mathcal{A})_1$. In particular this bijection is given by $[F^1, C] \mapsto F^1$. Extend the given involution to $Sal(\mathcal{A})_1$ by sending $[F^1, C]$ to $[(F^1)^\#, C^\#]$. Under this involution a positive path $\alpha = (a_1, \ldots, a_n)$ goes to a positive path

$$\alpha^\# := (a_1^\#, \ldots, a_n^\#)$$

If γ_1, γ_2 are two minimal positive boundary paths of a 2-cell in $Sal(\mathcal{A})$ then so are $\gamma_1^\#, \gamma_2^\#$. Therefore $[\gamma_1]_+ = [\gamma_2]_+ \Rightarrow [\gamma_1^\#]_+ = [\gamma_2^\#]_+$. □

For the next few results we consider only the flat arrangements with the involution property. One of the important examples of these arrangements is the simplicial arrangement of hyperplanes. Recall that an arrangement of hyperplanes is said to be *simplicial* when its chambers are cones over simplices. All of the following results were proved in this context by Deligne in [22]. Our proofs are along the same lines.

As the functor J is faithful on the class of minimal positive paths there is only one positive equivalence class of such paths. We denote this unique equivalence class by the symbol $[\mu(C \to C^\#)]$. Further define a *'positive'* loop, in $G(\mathcal{A})$, based at $[C, C]$ as

$$\delta(C) := \mu(C \to C^\#)\mu(C^\# \to C)$$

By $\delta^k(C)$ we mean that the loop is traversed k times in the same direction if $k > 0$ and in the reverse direction if $k < 0$. We will say that a positive path α begins (or ends) with a positive path α' if and only if $\alpha = \alpha'\beta (= \beta\alpha')$ for some positive path β.

Lemma 3.5.14. *Let $\mathcal{A}$ be a flat arrangement with the involution property and α be a positive path from C to D. Then:*

1. *$[\alpha][\mu(D \to D^\#)] = [\mu(C \to C^\#)][\alpha^\#]$;*
2. *if for a chamber D', β is some positive path from C to D' then $\alpha\delta^n(D)$ begins with β;*
3. *if $[\gamma] \in \mathcal{G}(C, D)$ then there exists $n \in \mathbb{N}$ and a positive path γ' such that*

$$[\gamma] = [\delta^{-n}(C)][\gamma'].$$

Proof. For (1) we use induction on the length of α. In fact, it is enough to assume that $\alpha = \mu(C \to C_1)$ such that $d(C, C_1) = 1$. Thus:

$$\begin{array}{rcl} \alpha\mu(C_1 \to C_1^{\#}) & = & \mu(C \to C_1)\mu(C_1 \to C_1^{\#}) \\ & \overset{+}{\sim} & \mu(C \to C_1)\mu(C_1 \to C^{\#})\mu(C^{\#} \to C_1^{\#}) \\ & \overset{+}{\sim} & \mu(C \to C^{\#})\mu(C^{\#} \to C_1^{\#}) \\ & \overset{+}{\sim} & \mu(C \to C^{\#})\alpha^{\#} \end{array}$$

By the same arguments, the following stronger statement is true:

$$[\alpha][\delta^k(D)] = [\delta^k(C)][\alpha], \quad k \geq 1 \tag{3.5.3}$$

For (2), let $\beta = (b_1, \ldots, b_n)$ where b_i is an edge from B_{i-1} to B_i ($B_0 = C, B_n = D'$). Observe that $\beta\mu(B_n \to B_{n-1}^{\#}) = (b_1, \ldots, b_{n-1})\mu(B_{n-1} \to B_{n-1}^{\#})$. By induction on n assume that there exists a positive path η from B_{n-1} to D such that -

$$(b_1, \ldots, b_{n-1})\eta = \alpha\delta^{n-1}(D).$$

Using (1), we get

$$\beta\mu(B_n \to B_{n-1}^{\#})\eta^{\#} = (b_1, \ldots, b_{n-1})\eta\delta(D) = \alpha\delta^n(D)$$

which proves (2).

For γ an arbitrary path from C to D assume $\gamma = (\epsilon_1 a_1, \ldots, \epsilon_n a_n), \epsilon_i \in \{\pm 1\}$. Let $A_i = t(\epsilon_1 a_1, \ldots, \epsilon_i a_i)$. Set $k = |\{1 \leq i \leq n | \epsilon_i = -1\}|$, we prove (3) by induction on k. The case $k = 0$ is clear since it means that γ is a positive path. Assume that the statement is true for $k - 1$. Now the general case; there exists an index j such that $\epsilon_1 = \cdots = \epsilon_{j-1} = 1$ and $\epsilon_j = -1$. We have

$$\begin{array}{rcl} \delta(C)\gamma & = & \mu(C \to C^{\#})\mu(C^{\#} \to C)(a_1, \ldots, a_{j-1}, -a_j, \epsilon_{j+1}a_{j+1}, \ldots, \epsilon_n a_n) \\ & \overset{+}{\sim} & \mu(C \to C^{\#})a_1^{\#}\mu(A_1^{\#} \to A_1)(a_2, \ldots, a_{j-1}, -a_j, \epsilon_{j+1}a_{j+1}, \ldots, \epsilon_n a_n)(\text{ from1}) \\ & \overset{+}{\sim} & \mu(C \to C^{\#})a_1^{\#} \cdots a_{j-1}^{\#}\mu(A_{j-1}^{\#} \to A_{j-1})(-a_j, \epsilon_{j+1}a_{j+1}, \ldots, \epsilon_n a_n) \\ & \overset{+}{\sim} & \mu(C \to C^{\#})a_1^{\#} \cdots a_{j-1}^{\#}\mu(A_{j-1}^{\#} \to A_j)(\epsilon_{j+1}a_{j+1}, \ldots, \epsilon_n a_n) \\ & \overset{+}{\sim} & \delta^{1-n}(C)\gamma' \text{ (by induction hypothesis,)} \end{array}$$

where γ' is a positive path. Hence $[\gamma] = [\delta^{-n}(C)]\gamma'$. □

One of the immediate consequence of the above result is the following:

Corollary 3.5.15. *Let $\mathcal{A}$ be a flat arrangement with the involution property. The axiom (1) and its dual which define a calculus of fractions (Definition 3.5.1) are satisfied. So $J\colon \mathcal{G}^+ \to \mathcal{G}$ is faithful if and only if the cancellation laws hold in $\mathcal{G}^+$.*

Proof. Use the equality (2) proved above in the Lemma 3.5.14. □

Recall that [82, Section 0.5.7] the *word problem* for a group G is the problem of deciding whether or not an arbitrary word w in G is the identity of G. The word problem for G is *solvable* if and only if there exists an algorithm to determine whether $w = 1_G$ or equivalently, if there exists an algorithm to determine when two arbitrary words represent the same element of G.

Theorem 3.5.16. *Let X be a simply connected l-manifold and $\mathcal{A}$ be a submanifold arrangement with the involution property. Then if $J\colon \mathcal{G}^+(\mathcal{A}) \to \mathcal{G}(\mathcal{A})$ is faithful the word problem for $\pi_1(M(\mathcal{A}))$ is solvable.*

Proof. Let $[\alpha], [\beta]$ be two loops in $\pi_1(Sal(\mathcal{A}))$ based at a vertex $[C, C]$. Then according to Lemma 3.5.14 there is a finite algorithm to write -

$$[\beta] = [\delta^{-k}(C)][\beta'], \quad [\alpha] = [\delta^{-k}(C)][\alpha']$$

where β', α' are positive loops based at $[C, C]$. Hence, $[\alpha] = [\beta]$ if and only if $[\alpha']_+ = [\beta']_+$. The theorem follows because there are only finitely many positive paths of given length to choose from. □

Remark 3.5.17. It follows from the work of Deligne [22], Paris [66], Salvetti [74] and recently Delucchi [23] that the functor J is faithful for simplicial arrangements of hyperplanes. Since these arrangements also have the involution property it follows that the fundamental group of the complexified complement has solvable word problem (pure Artin groups also occur as these fundamental groups). In fact, more is known about these groups. It has been shown by Charney [12] that these groups are biautomatic. One more class of examples that satisfy the above property is the simplicial arrangements of pseudospheres (see [74]). In the next chapter we will present a new class of arrangements that satisfy above property.

Finally, we mention the diagram of spaces introduced in the Theorem 3.2.16. Using the theory fundamental groupoid of a homotopy colimit discussed in Section 2.3.2 we get one more tool to compute the fundamental group of the tangent bundle complement. The spaces in that diagram are connected and also the homotopy colimit of the diagram is connected. Hence the *Seifert-van Kampen-Brown Theorem* (see Theorem 2.3.13) can be restated as -

Corollary 3.5.18. *Let $\mathcal{A}$ be an arrangement of submanifolds in X and let $\mathcal{D}\colon \mathcal{F}(\mathcal{A}) \to$ Top be the diagram of spaces whose homotopy colimit has the homotopy type of $M(\mathcal{A})$. For the diagram $\pi_1(\mathcal{D})\colon \mathcal{F}(\mathcal{A}) \to$ Grps of fundamental groups, there is a pushout diagram of groups*

$$\begin{array}{ccc} \pi_1(X) & \longrightarrow & \pi_1(M(\mathcal{A})) \\ \Big\downarrow & & \Big\downarrow \\ 1 & \longrightarrow & colim\ \pi_1(\mathcal{D}) \end{array}$$

Example 3.5.19. For the arrangement of 2 points in a circle we get that $\pi_1(M(\mathcal{A})) \cong \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$.

Example 3.5.20. For the arrangement of 2 circles on S^2 as $\pi_1(S^2) = 1$, using the previous corollary we have

$$\pi_1(M(\mathcal{A})) \cong \text{colim } \pi_1(\mathcal{D}) \cong \mathbb{Z}^2$$

3.5.1 Classification of Covering Spaces

In this section we return to the covering spaces of the Salvetti complex. The aim is to classify these covering spaces according to subgroups of the fundamental group. In Section 3.4.1, using the Salvetti-type diagram models, we constructed covering spaces of the Salvetti complex. Now we prove that any connected topological covering space of the Salvetti complex corresponds to a Salvetti-type diagram model. The strategy we employ is a standard result in algebraic topology establishing correspondence between isomorphism classes of covering spaces and conjugacy classes of subgroups (see for example, [44, Theorem 1.38]).

Recall that in previous section we saw that given a cover $\rho\colon \mathcal{G}_\rho \to \mathcal{G}$ of the arrangement groupoid $\mathcal{G}$, one can construct a simplicial complex $\mathcal{S}_\rho$ which is a covering space of the associated Salvetti complex $Sal(\mathcal{A})$. In Remark 3.4.11 it was mentioned that the simplicial complex $\mathcal{S}_\rho$ is indeed the barycentric subdivision of a CW-complex. For notational simplicity we denote the CW-complex by $\mathcal{S}_\rho$ and start by analyzing its 2-skeleton. The plan is to show that the fundamental group of $\mathcal{S}_\rho$ is isomorphic to the object group of $\mathcal{G}_\rho$.

0-**skeleton**: The 0-cells of $\mathcal{S}_\rho$ correspond bijectively to the objects of $\mathcal{G}_\rho$ and can be written as $[\rho(v), v]$ where $v \in Ob(\mathcal{G}_\rho)$ and $\rho(v) \in \mathcal{C}(\mathcal{A})$.

1-**skeleton**: There is a 1-cell between two vertices $[\rho(v_1), v_1], [\rho(v_2), v_2]$ if and only if the corresponding chambers $\rho(v_1), \rho(v_2)$ share a codimension 1 face F^1. In fact, there is one 1-cell $[F^1, v_1]$ directed from $[\rho(v_1), v_1]$ to $[\rho(v_2), v_2]$ and another 1-cell $[F^1, v_2]$ directed from $[\rho(v_2), v_2]$ to $[\rho(v_1), v_1]$.

2-skeleton: For a codimension 2 face F^2 and an object $v \in Ob(\mathcal{G}_\rho)$ (such that $\rho(v) \prec F^2$) there is one 2-cell $[F^2, v]$ in $\mathcal{S}_\rho$. The vertices of $[F^2, v]$ are indexed by the chambers whose closures contain F^2. To such a chamber C, there corresponds an object in the covering groupoid given by $t(\rho(v) \to C, F^2)^{<v>}$. Hence the vertex set of $[F^2, v]$ is given by $\{[\rho(v_i), v_i] \mid \rho(v_i) \prec F^2, i = 1, \ldots, 2k\}$. A 1-cell $[F^1, v_i]$ in the boundary of $[F^2, v]$ corresponds to codimension 1 face F^1 such that $F^1 \prec F^2$. The vertex v_1 is such that the chamber $\rho(v_i)$ is on the same side as $\rho(v)$ with respect to F^1. Moreover, the cell is oriented to have v_i as its start vertex.

Remark 3.5.21. Attachment of higher dimensional cells is analogous to the construction of Salvetti complex. Hence it can be shown that these covering complexes $\mathcal{S}_\rho$ have the structure of a MH-complex. However we will not prove it here.

The next step in the classification is to show that the fundamental group of $\mathcal{S}_\rho$ is isomorphic to the object group of $\mathcal{G}_\rho$. We compare the relations given by 2-cells of $\mathcal{S}_\rho$ and the relations that define $\mathcal{G}_\rho$.

Recall that the equivalence relation in the arrangement groupoid $\mathcal{G}$ is generated by identifying two paths (in the arrangement graph) with same end points and form the boundary of a 2-cell (of $Sal(\mathcal{A})$). As a result, the identity in an object group is an equivalence class of loops based at that object that can be written as $\gamma\gamma'^{-1}$, where γ, γ' are paths bounding a 2-cell. For a given groupoid cover $\rho\colon \mathcal{G}_\rho \to \mathcal{G}$ let G_ρ denote the graph underlying $\mathcal{G}_\rho$. The morphisms of $\mathcal{G}_\rho$ are equivalence classes of paths in the graph G_ρ. This equivalence relation is the one induced by ρ. To be precise, two paths $\tilde{\alpha}, \tilde{\beta}$ in G_ρ are identified if and only if they have same end points and $\rho(\tilde{\alpha}) \sim \rho(\tilde{\beta})$ (where $\tilde{\alpha}, \tilde{\beta}$ are lifts of α, β respectively). Hence the relations in $\mathcal{G}_\rho$ are generated by the loops of the form $\tilde{\gamma_1}\tilde{\gamma_2}^{-1}$ such that $[\gamma_1] = [\gamma_2]$ (in $\mathcal{G}_\rho$), denote this set by Σ_ρ. In a nutshell we have the following:

$$\pi(\mathcal{G}_\rho) \cong \pi_1(G_\rho)/\Sigma_\rho$$

On the other hand let σ_ρ denote the smallest normal subgroup of $\pi_1((\mathcal{S}_\rho)_1)$ (1-skeleton) generated by the relations imposed by the boundary of every 2-cell. Then:

$$\pi_1(\mathcal{S}_\rho) \cong \pi_1((\mathcal{S}_\rho)_1)/\sigma_\rho$$

Theorem 3.5.22. *With the notation as above we have following isomorphism*

$$\pi(\mathcal{G}_\rho) \cong \pi_1(\mathcal{S}_\rho)$$

Proof. The proof is clear once we use the above discussion, the 'moves' defining the arrangement groupoid and the Corollary 3.5.6. □

Remark 3.5.23. We note that the above theorem was first proved in the context of covers of the hyperplane complement, in Delucchi's thesis [23, Lemma 4.2.2, Proposition 4.2.3], the proof is more direct and transparent.

Now we are in the position to state the classification of covering spaces of the Salvetti complex. The proofs of following statements essentially involve diagram chasing and are fairly straightforward (see [23, Section 4.5] for the original arguments). Finally, we recall that the covering spaces $\mathcal{S}_\rho$ are homotopy colimits of diagrams $\mathcal{D}_\rho$ (defined over $\mathcal{F}^*$) of posets, see Definition 3.4.9.

Theorem 3.5.24. *Let $\mathcal{A}$ be a submanifold arrangement in a l-manifold X, let $Sal(\mathcal{A})$ denote the associated Salvetti complex and $\mathcal{G}$ denote the arrangement groupoid. For any topological cover $p\colon S \to Sal(\mathcal{A})$, there exists a cover of the arrangement groupoid $\rho\colon \mathcal{G}_\rho \to \mathcal{G}$ such that the homotopy colimit of the associated diagram of spaces $\mathcal{D}_\rho$ is isomorphic to S as a covering space of $Sal(\mathcal{A})$*

Proof. Let H denote the isomorphic image of $p_*(\pi_1(S))$ inside $\pi(\mathcal{G}(\mathcal{A}))$ (under the isomorphism in Theorem 3.5.22). Applying Theorem 2.3.6 we see that there is a covering groupoid $\rho\colon \mathcal{G}_\rho \to \mathcal{G}(A)$ such that $\pi(\mathcal{G}_\rho) \cong H$. Let S_ρ denote the homotopy colimit of the diagram of spaces defined using $\mathcal{G}_\rho$. Again appealing to Theorem 3.5.22 it implies that $\pi_1(S_\rho) \cong H$. Let ι_ρ denote the inclusion of G_ρ (the graph underlying $\mathcal{G}_\rho$) into S_ρ as its 1-skeleton. Since $\mathcal{G}_\rho$ is equivalent to the fundamental groupoid of S_ρ the following diagram commutes.

$$\begin{tikzcd}
\pi(\mathcal{G}_\rho) \arrow[r, "\cong"] \arrow[d, "\rho_*"'] & \pi_1(S_\rho) \arrow[d, "(\Lambda_\rho)_*"] \\
\pi(\mathcal{G}) \arrow[r, "\cong"] & \pi_1(Sal(\mathcal{A}))
\end{tikzcd}$$

Since $(\Lambda_\rho)_*(S_\rho) \cong H$ we have that S and S_ρ are isomorphic as covering spaces. $\square$

Following corollaries are immediate.

Corollary 3.5.25. *Any cover $p\colon S \to Sal(\mathcal{A})$ of the Salvetti complex can be written as the order complex of a poset.*

Corollary 3.5.26. *Let $\hat{\rho}\colon \mathcal{G}_{\hat{\rho}} \to \mathcal{G}$ denote the universal cover of the arrangement groupoid. Then $\mathcal{S}_{\hat{\rho}}$, the homotopy colimit of the associated diagram of spaces (posets) is the universal cover of $Sal(\mathcal{A})$.*

3.6 Higher Homotopy Groups

As mentioned before Deligne's work on simplicial arrangements [22] brought the subject of hyperplane arrangements to the forefront. Using combinatorics of positive paths he showed that the universal cover of the complexified complement is contractible. As a result simplicial arrangements are called $K(\pi, 1)$. Since Deligne's work one of the main open problems in the theory of arrangements is to find out whether being $K(\pi, 1)$ is a combinatorial property. Deligne's work also had profound effect on the field of geometric group theory. For example, Deligne's use of Garside's work (on the word problem for braid groups) [37] gave rise to study of Garside groups (a generalization of finite-type Artin groups), see [21]. Also see [12] for new examples of automatic groups.

We end this section by taking a closer look at the universal cover of the tangent bundle complement. Two essential ingredients in Deligne's proof were faithfulness of the functor J and the existence of a canonical form for morphisms in $\mathcal{G}(\mathcal{A})$ (also known as the *Deligne normal form* [12], *property D* [66]). We generalize Deligne's original arguments in order to derive conditions for an arrangement $\mathcal{A}$ such that the functor J is faithful. We will also show that for a certain class of flat arrangements the universal cover of its associated tangent bundle complement is contractible. Before that let us identify a class of arrangements that are not $K(\pi, 1)$.

Lemma 3.6.1. *Let $\mathcal{A}$ be an arrangement of submanifolds in a l-manifold X. If $\pi_n(X) \neq 0$ for some $n \geq 2$ then $\pi_n(M(\mathcal{A})) \neq 0$.*

Proof. Recall that in Theorem 3.2.14 we showed that there is a retraction

$$r\colon Sal(\mathcal{A}) \to X$$

Hence $r_*\colon \pi_n(M(\mathcal{A})) \to \pi_n(X)$ is surjective. □

Hence the only hope to find $K(\pi, 1)$ arrangements is to consider arrangements in $K(\pi, 1)$ manifolds. We now generalize Deligne's arguments. Let $\Lambda\colon \tilde{\mathcal{S}} \to Sal(\mathcal{A})$ denote the universal cover complex of the Salvetti complex. Recall that the cells of this complex correspond to pairs $[F, v]$, F being a face of the arrangement and v an object in the universal cover groupoid $\tilde{\mathcal{G}}$. Let G and $\tilde{G}$ denote the 1-skeletons of $Sal(\mathcal{A})$ and $\tilde{\mathcal{S}}$ respectively.

Lemma 3.6.2. *If α is a positive path in $G(\mathcal{A})$ then the pull-back $\Lambda^{-1}(\alpha)$ is a minimal positive path in $\tilde{G}$. The canonical functor $J\colon \mathcal{G}^+(\mathcal{A}) \to \mathcal{G}(\mathcal{A})$ is faithful if and only if $\tilde{\mathcal{S}}$ is a flat MH-complex.*

Proof. Recall that every point in the universal cover corresponds to a relative homotopy class of a path in the base space [44, Page 64]. Let $[C, C]$ be a fixed base point of $Sal(\mathcal{A})$. Therefore each vertex of $\tilde{G}$ is given by a relative homotopy class of a path in $G(\mathcal{A})$ that starts at $[C, C]$. Let $u_0 \in \Lambda^{-1}([C, C])$ be a fixed base point of $\tilde{\mathcal{S}}$. Let α be a positive path in $G(\mathcal{A})$ from $[C, C]$ to $[D, D]$ and $\tilde{\alpha}$ be its lift in $\tilde{G}$ that starts at u_1 and ends at v. The end points u_1 and v correspond to homotopy classes of paths starting at u_0. Let $\tilde{\beta}$ denote the path that represents the homotopy class corresponding to u_1 and $\tilde{\gamma}$ be the path representing v. Note that $\tilde{\beta}$ projects to a loop based at $[C, C]$ and $\tilde{\gamma}$ projects to another path from $[C, C]$ to $[D, D]$. If $l(\beta)$ denotes the edge length of a path then the distance between u_1 and v is given by

$$min\{l(\eta) \mid [\eta] = [\beta^{-1}\gamma]\}$$

Since the length of the path α attains the above minimum the first statement follows.

Two positive paths α, β are relative homotopic if and only if their lifts have the same end points. By the above arguments their lifts $\tilde{\alpha}, \tilde{\beta}$ are minimal positive paths in $\tilde{G}$. According to the Definition 3.5.7, a MH-complex is *flat* if and only if the complex is simply connected and any two minimal positive paths are relative homotopic. □

Rest of the results concerning contractiblity of the universal cover require that there is only one equivalence class of minimal positive paths between any two chambers. A property that is satisfied by central hyperplane arrangements. For this reason we restrict ourselves to flat arrangements. As this proves to be a mild generalization all of the arguments made by Deligne carry through. Consequently we have omitted the proofs.

Let $\tilde{\mathcal{S}}^+$ denote the sub-complex of $\tilde{\mathcal{S}}$ generated by the cells of the form $[F, v]$ where v corresponds to a homotopy class of a positive path starting at $[C, C]$.

Lemma 3.6.3. *Let $\mathcal{A}$ be a flat arrangement of submanifolds with the involution property in a l-manifold X. If the canonical functor $J\colon \mathcal{G}^+ \to \mathcal{G}$ is faithful then there exists a sequence of embeddings*

$$j_n\colon \tilde{\mathcal{S}}^+ \to \tilde{\mathcal{S}},\ n \geq 0$$

such that

$$j_0(\tilde{\mathcal{S}}^+) \subset j_1(\tilde{\mathcal{S}}^+) \subset \cdots \subset \bigcup_n j_n(\tilde{\mathcal{S}}^+) = \tilde{\mathcal{S}}.$$

Proof. Since J is faithful, the positive category $\mathcal{G}^+$ admits a calculus of fractions. Consequently, for every homotopy class $[\alpha] \in \mathcal{G}$ that starts from the base point $[C, C]$ we have an expression

$$[\alpha] = [\delta^{-n}(C)\alpha'],\ \alpha' \text{ a positive path}$$

(due to Lemma 3.5.14). For a fixed n, $[\delta^{-n}(C)\alpha'] = [\delta^{-n}(C)\beta']$ if and only if $[\alpha]_+ = [\beta]_+$. Hence the functor $J_n\colon \mathcal{G}^+ \to \mathcal{G}$ defined by $J_n([\alpha]_+) := [\delta^{-n}(C)\alpha]$ is faithful. This functor induces a map $j_n\colon \tilde{\mathcal{S}}^+ \to \mathcal{S}$ as follows

$$j_n([F, v]) = [F, J_n(v)]$$

It is clear that $j_n(\tilde{\mathcal{S}}^+) \subset j_{n+1}(\tilde{\mathcal{S}}^+)$ and the fact that $\lim J_n(\mathcal{G}^+) = \mathcal{G}$ establishes the claim. □

All of the above arguments can be put together in a theorem that gives a criterion for the tangent bundle complement to be a $K(\pi, 1)$ space.

Theorem 3.6.4. *Let X be a $K(\pi, 1)$ l-manifold and $\mathcal{A}$ be an arrangement of submanifolds with the involution property. If:*

1. *$\mathcal{A}$ is a flat arrangement;*
2. *the canonical functor $J\colon \mathcal{G}^+ \to \mathcal{G}$ is faithful;*
3. *the complex $\tilde{\mathcal{S}}^+$ is contractible;*

then the universal cover of the tangent bundle complement is contractible.

Proof. If J is faithful then the universal cover $\tilde{\mathcal{S}}$ is a colimit of its sub-complexes as proved in Lemma 3.6.3.

Now for a fixed base point $v \in \tilde{\mathcal{S}}$, $j_n(\tilde{\mathcal{S}}^+)$ is the sub-complex whose cells are indexed by the positive paths starting from $J_n(v)$. Since the action of π_1 induces an isomorphism of the universal cover, the covers $\tilde{\mathcal{S}}^+$ and $j_n(\tilde{\mathcal{S}}^+)$ are isomorphic. □

We now identify the combinatorial conditions for flat arrangements so that the above theorem holds true. We start with the faithfulness of J and the Deligne normal form. Recall that an arrangement is *simplicial* if and only if each chamber is a simplex (or cone over an open simplex). Set a partial order on the morphisms of $\mathcal{G}^+(\mathcal{A})$ as follows

$$[\alpha]_+ \sqsubseteq [\beta]_+ \iff [\beta]_+ = [\gamma]_+[\alpha]_+ \text{ for some } [\gamma]_+ \in \mathcal{G}^+(s(\beta), s(\alpha))$$

Definition 3.6.5. Let $\mathcal{A}$ be a flat arrangement of submanifolds. The associated positive category $\mathcal{G}^+(\mathcal{A})$ admits the *Deligne normal form* if and only if for every morphism $[\beta]_+ \in \mathcal{G}^+(C, D)$ there exists $A \in \mathcal{C}(\mathcal{A})$ such that

$$[\mu(B \to D)]_+ \sqsubseteq [\beta]_+ \iff B \prec_A D.$$

Note that the above condition is equivalent to the existence of a minimal positive path $\mu(A \to D)$ such that

$$[\mu(B \to D)]_+ \sqsubseteq [\beta]_+ \iff [\mu(B \to D)]_+ \sqsubseteq [\mu(A \to D)]_+$$

All the (classes of) minimal paths that are smaller, with respect to $\sqsubseteq$, than a given morphism $[\beta]_+$ form a poset. Then the above definition states that this poset has a unique maximal element. Let $[\mu_0]_+$ denote this maximal element, hence $[\beta]_+ = [\beta_1]_+[\mu_0]_+$. Since $\mathcal{G}^+$ admits the Deligne normal form we can write $[\beta]_+ = [\beta_2]_+[\mu_1]_+[\mu_0]_+$. We can repeat this process for only a finitely many times since these minimal paths have strictly decreasing lengths. The result is that the morphism $[\beta]_+$ can be written as a unique product of classes of minimal paths as follows:

$$[\beta]_+ = [\mu_k]_+ \cdots [\mu_0]_+.$$

Such an expression is in fact the Deligne normal form of a positive morphism (some times the terms *canonical minimal decomposition* or *right greedy canonical form* are also used).

Lemma 3.6.6. *If $\mathcal{A}$ is a flat and simplicial arrangement in X then*

1. *the functor $J\colon \mathcal{G}^+(\mathcal{A}) \to \mathcal{G}(\mathcal{A})$ is faithful;*
2. *$\mathcal{G}^+(\mathcal{A})$ admits the Deligne normal form.*

Proof. We have already shown that the condition (1) defining the strong calculus of fractions is satisfied (lemma). the cancellation laws (Definition 3.5.1, condition (2)) hold true in $\mathcal{G}^+(\mathcal{A})$. In light of Remark 3.5.3 it is enough to prove that right cancellation laws hold. The proof that the cancellation laws hold and that the normal form exists is the same as Deligne's original proof for simplicial hyperplane arrangements. We refer the reader to following [22, Theorem 1.19], [23, Theorem 6.4.6], [66, Theorem 4.1], [74, Theorem 31]. □

The final piece of the puzzle is to show that the positive complex $\tilde{\mathcal{S}}^+$ is contractible. This depends on one more property of simplicial hyperplane arrangements which we now state. But before that some notations. For $A, B \in \mathcal{C}(\mathcal{A})$, set -

$$\mathcal{E}(A, B) := \{F \in \mathcal{F}^*(\mathcal{A}) \mid \text{if } C \prec F \text{ then } A \prec_A C \prec_A B\}$$

$$\mathcal{E}_1(A, B) := \{F \in \mathcal{E}(A, B) \mid B \not\prec F\}$$

Definition 3.6.7. The faces of a flat arrangement $\mathcal{A}$ have the *strong contraction property* if and only if for every $A, B \in \mathcal{C}(\mathcal{A}), A \neq B$ the set $\mathcal{E}(A, B)$ is simply connected and contracts over $\mathcal{E}_1(A, B)$ by leaving it point wise fixed.

We can now the state necessity of above condition.

Lemma 3.6.8. *Let $\mathcal{A}$ be a simplicial and flat arrangement with the involution. Then if the faces of $\mathcal{A}$ have the strong contraction property then the positive complex $\tilde{\mathbb{S}}^+$ is contractible. Consequently the arrangement is $K(\pi, 1)$*

Proof. The proof is same as that of [74, Theorem 33] or [66, Lemma 4.11]. □

Remark 3.6.9. The reason we have explicitly mentioned this contraction property is the following. In the next chapter we will show that there are examples of simplicial and flat arrangements that do not satisfy the contraction property. Hence the universal cover is not contractible even though the associated positive category admits the Deligne normal form. For arrangements of hyperplanes the contraction property is always satisfied (see [66, Lemma 4.8]). In Chapter 5 we will show that the simplicial arrangements of pseudo-hyperplanes also satisfy all these properties, hence are $K(\pi, 1)$. It would be interesting to find new examples of manifolds such that the simplicial arrangements in that manifold are $K(\pi, 1)$. We leave this as an open question.

Remark 3.6.10. An important aspect that we did not discuss in detail is regarding the strong lattice property. For simplicial arrangements of hyperplanes the poset of chambers is a lattice for every choice of base chamber. Delucchi proves that the existence of the Deligne normal form for hyperplane arrangements is equivalent to strong the lattice property [23, Theorem 6.5.3] (see also [65]).

Remark 3.6.11. There are examples of submanifold arrangements in torus that are neither flat nor simplicial but their associated Salvetti complex is aspherical. Our future plan is to generalize the results proved in this section to these new settings.

3.7 Cohomology Calculations

The final section is devoted to the cohomology calculations. The aim is to find a connection between the cohomology algebra of the tangent bundle complement and the intersections of the submanifolds. We will make use of the fact that the tangent bundle complement is a homotopy colimit as proved in Theorem 3.2.16. We will apply the Bousfield-Kan spectral sequence to this data (refer Theorem 2.2.4). Before that a result about the ring structure.

Let $\mathcal{A}$ be an arrangement of submanifolds in X and let $L(\mathcal{A})$ be its intersection poset (Definition 3.1.8). For every $Y \in L(\mathcal{A})$ define

$$\mathcal{F}_Y := \{F \in \mathcal{F}(\mathcal{A}) \mid F \subset Y\}$$

Since Y is a closed subset of X, $(Y, \mathcal{F}_Y)$ is a regular cell structure for Y. Let Y^* denote Y with the 'dual' cell structure given by $\mathcal{F}_Y^* \bigcup \{C \in X_0^* \mid C \prec F, \forall F \in \mathcal{F}_Y^*\}$. Then Y^* is a regular cell complex homeomorphic to Y.

Lemma 3.7.1. *For every $Y \in L(\mathcal{A})$, the cell complex Y^* is a retract of $Sal(\mathcal{A})$.*

Proof. The map ι_C defined in Theorem 3.3.6, by sending $F \mapsto [F, F \circ C]$, restricted to Y^* is still an embedding. Also for the same reasons the map $\psi\colon Sal(\mathcal{A}) \to Y^*$ defined by $[F, C] \mapsto F$ is a retraction. □

In the context of cohomology the above result takes the following form.

Theorem 3.7.2. *Let $\mathcal{A}$ be an arrangement of submanifolds in an l-manifold X. If $M(\mathcal{A})$ is the associated tangent bundle complement then $H^*(M(\mathcal{A}), \mathbb{Z})$ is a finitely generated $H^*(Y, \mathbb{Z})$ module for every $Y \in L(\mathcal{A})$.*

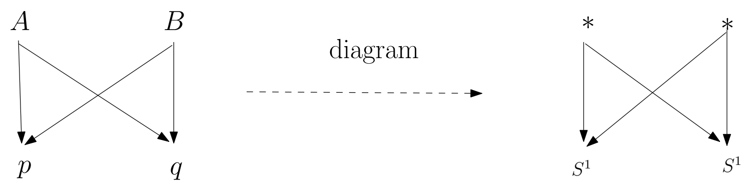
Proof. Since each Y is a retract of $M(\mathcal{A})$ the claim follows from the fact that the map induced in cohomology by the retraction is a injective ring homomorphism. In particular it means that $H^*(X, \mathbb{Z})$ is a subring of $H^*(M(\mathcal{A}), \mathcal{A})$ and that it need not be generated in degree 1. □

We would like to mention that we do not yet have an analogue of the Brieskorn-Orlik-Solomon theorem (see Theorem 1.3.2) in this general context, it is work in progress. However, in all our examples we observe that there is a finer grading of the cohomology groups indexed by the Whitney numbers of the intersection poset. This observation, which might serve as a generalization of Theorem 1.3.3, is stated as a conjecture at the end of this section. In all our examples the spectral sequence collapses (for dimension reasons) on E_2 page (we do not have a proof yet). Also the 0-th row of this page contains cohomology groups of the ambient manifold as suggested by above Theorem 3.7.2.

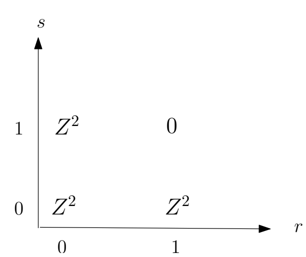
Recall that in Theorem 3.2.16 it was proved that the Salvetti complex associated to a tangent bundle complement is made up by gluing Salvetti complexes of local (hyperplane) arrangements. Since the cohomology groups of a complexified real arrangement are well known they will serve as an input to the Bousfield-Kan spectral sequence. As we still lack a general theorem, in the rest of this section we explicitly describe the spectral sequence calculations with the help of some examples. Finally recall that for a poset P, its *characteristic polynomial* is defined as the finite sum $\sum_{x \in P} \mu(\hat{0}, x) \cdot t^{r(x)}$ and the absolute value of the coefficient of t^k is called the *k-th Whitney number (of the second kind).*

We start our examples with an arrangement in S^1.

Example 3.7.3. Consider the arrangement of two points in a circle (Example 3.1.10). At each point of the arrangement the local picture is like an arrangement of a point in a line. Therefore the local Salvetti complex is homeomorphic to S^1. The corresponding diagram of spaces takes the following form



The face poset above realizes a regular CW complex. We apply Theorem 2.2.4 to set up the first page of the spectral sequence. The 0th column on the E_1 page contains the cohomology of 2 disjoint copies of S^1 and the next column contains the cohomology of 2 disjoint points, with possibly only one non zero differential $d_1^{0,0}\colon E_1^{0,0} \to E_1^{1,0}$. The page is shown below



Let α_p be the generator of $H^0(S^1, \mathbb{Z})$ (the circle assigned to p). Note that the inclusion $* \hookrightarrow S^1$ induces isomorphism on H^0. Then using the formula for the differential we see that

$$\alpha_p \mapsto [p : A] f^*_{Ap}(\alpha_p) + [p : B] f^*_{Bp}(\alpha_p)$$

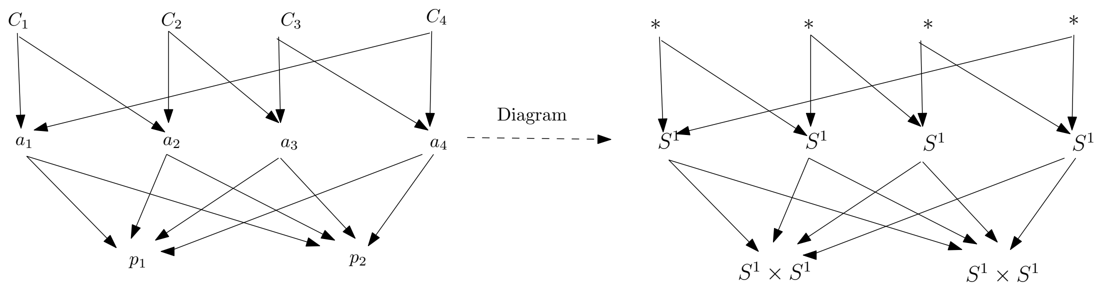
and there is a similar expression for $d_1(\alpha_q)$. Hence kernel and the image of $d_1^{0,0}$ are isomorphic to $\mathbb{Z}$. Consequently, $E_2^{0,0} \cong \mathbb{Z}$ and $E_2^{1,0} \cong \mathbb{Z}$ and all of the differentials are zero. The spectral sequence collapses at E_2 and the cohomology ring of the complement is

$$H^*(M(\mathcal{A}), \mathbb{Z}) \cong \frac{\mathbb{Z}[x_1, x_2, x_3]}{< x_i^2 = 0 >},\ |x_i| = 1 \text{ for } i = 1, 2, 3$$

Also observe that the ranks of the cohomology groups in the first column (and on the diagonal) of E_2 are the Whitney numbers $\{1, 2\}$ of the intersection poset.

We now move on to arrangements in S^2.

Example 3.7.4. Consider the arrangement of two circles in a sphere as in Example 3.1.11. At each of the two intersection points the local picture is of two lines in a plane. Hence the local Salvetti complex is homeomorphic to $S^1 \times S^1$. Similarly the local Salvetti complex at each of the 4 arcs is homeomorphic to S^1. The diagram of spaces is



Now the first page of the spectral sequence is easy to setup, without going into details we can see that it looks like

$$\begin{array}{c|ccc}
s & & & \\
2 & Z^2 & 0 & 0 \\
1 & Z^4 & Z^4 & 0 \\
0 & Z^2 & Z^4 & Z^4 \\
\hline
 & 0 & 1 & 2 \quad r
\end{array}$$

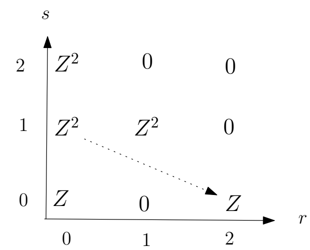
We will leave it to the reader to workout the differentials of the 0th row. We will calculate $d_1^{0,1}$. Let α_{13}, α_{24} generate $H^1(\mathcal{D}(p_1))$ and β_{13}, β_{24} generate $H^1(\mathcal{D}(p_2))$. Let ψ_j generate $H^1(\mathcal{D}(a_j))$ for $1 \leq j \leq 4$. Recall that the inclusion $S^1 \hookrightarrow S^1 \times S^1$ induces projection $H^1(S^1 \times S^1) \to S^1$. Using this we have

$$\begin{aligned}
d_1^{0,1}(\alpha_{13}) &= \sum_{j=1}^{4} [p_1 : a_j] f^*_{a_j p_1}(\alpha_{13}) \\
&= f^*_{a_1 p_1}(\alpha_{13}) + f^*_{a_3 p_1}(\alpha_{13}) \\
&= \psi_1 + \psi_3 \\
d_1^{0,1}(\alpha_{24}) &= \psi_2 + \psi_4
\end{aligned}$$

There will be similar expressions for the other two generators, consequently we have

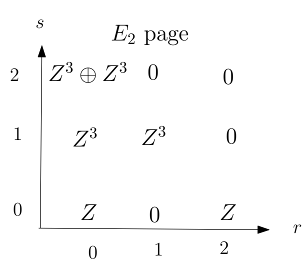
$$E_2^{0,1} = ker d_1^{0,1} = < \alpha_{13} - \beta_{13}, \alpha_{24} - \beta_{24} > \cong \mathbb{Z}^2$$

The second page of the spectral sequence is shown below and it is not very difficult to verify that the differential $d_2^{0,1}$ is zero. Hence the spectral sequence collapses at E_2.

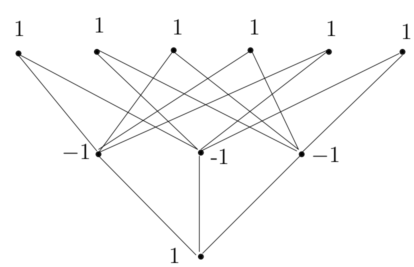


Recall that in Example 3.3.5 we have seen that the tangent bundle complement has the homotopy type of $T^2 \vee S^2 \vee S^2 \vee S^2 \vee S^2$. The Whitney numbers of the intersection poset (Figure 3.2) are $\{1, 2, 2\}$. As observed in the previous example these numbers can be seen as the ranks of the cohomology groups in the first column as well as the groups on the diagonal. The 0th row of E_2 contains the cohomology of S^2 which is consistent with Theorem 3.7.2.

Example 3.7.5. As a next example consider 3 circles arranged on S^2 in general position. The intersection poset and E_2 page of the cohomology spectral sequence are shown below.



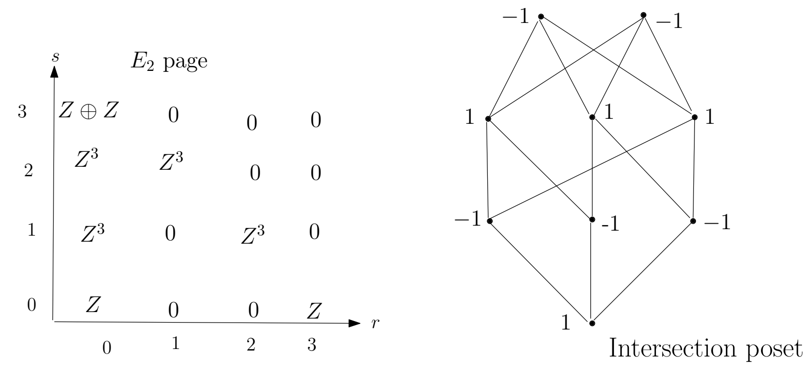
$H^i(M(\mathcal{A})) = \{Z, Z^3, Z^{10}\}$



Intersection poset

Even in this case the Whitney numbers $\{1, 3, 6\}$ can be seen in the first column and the diagonal of the E_2 page.

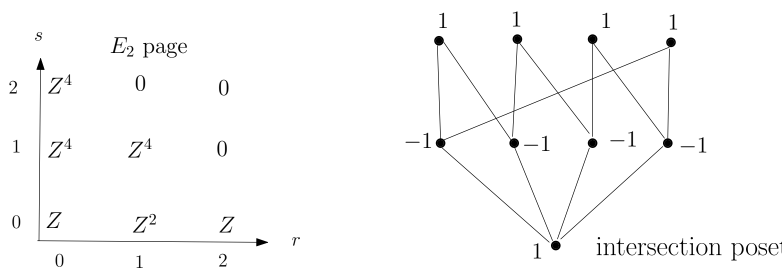
Example 3.7.6. In this example we will look at the 3-sphere. Consider the arrangement of 3 S^2's that intersect twice like coordinate hyperplanes in $\mathbb{R}^3$. The intersection poset and the E_2 page of the cohomology spectral sequence are shown below.



$H^*(M(\mathcal{A})) \cong H^*(T^3 \bigvee (\vee_8 S^3))\}$

Here $M(\mathcal{A}) \cong T^3 \bigvee (\vee_8 S^3)$ and the Whitney numbers appear on the E_2 page.

Example 3.7.7. Finally let us look at the arrangement of circles on a 2-torus introduced in Example 3.1.12. The E_2 page of the cohomology spectral sequence is shown below with the intersection poset.



Without going into of details we mention that the homotopy type of the tangent bundle complement is $(S^1 \vee S^1 \vee S^1) \times (S^1 \vee S^1 \vee S^1)$. In fact, this is not very hard to see. Observe that this arrangement is a product of two arrangements in S^1. Both are the arrangements of 2 points in S^1 (Example 3.1.10). As the tangent bundle of a torus is the product of two infinite cylinders ($\cong TS^1$) the conclusion follows. Finally, the ranks of the groups in the first column and on the diagonal are the Whitney numbers $\{1, 4, 4\}$. The groups in the 0th row are the cohomology groups of the torus.

Based on the calculations presented in this section, a theorem proved in the next section and a result in [19, Theorem 4.2] we propose the following conjecture.

Conjecture 3.7.8. Let $\mathcal{A}$ be a submanifold arrangement in an orientable l-manifold X and let $M(\mathcal{A})$ be the associated tangent bundle complement. If $L(\mathcal{A})$ is the intersection poset then

$$rank(H^i(M(\mathcal{A}), \mathbb{Z})) = \sum_{\substack{Y \in L(\mathcal{A}) \\ 0 \le rk(Y) \le i}} rank(H^{i-rk(Y)}(\coprod_{|\mu(X,Y)|} Y, \mathbb{Z})) \quad 0 \le i \le l$$

where μ is the Möbius function of $L(\mathcal{A})$ and $rk(Y)$ is the codimension of the corresponding intersection.

Lemma 3.7.9. *Let $\mathcal{A}$ be an arrangement of real hyperplanes in $\mathbb{R}^l$ and $M(\mathcal{A})$ be the associated complexified complement. Then the Conjecture 3.7.8 is satisfied for $H^*(M(\mathcal{A}), \mathbb{Z})$.*

Proof. Follows from Theorem 3.72 and Proposition 3.75 of [62]. $\square$

Chapter 4

Arrangements of Spheres and Tori

In this chapter we consider some specific examples of submanifold arrangements. We examine arrangements in spheres, projective spaces and tori. Our aim, as before, is to study the relationship between the combinatorics of the arrangement, geometry of the ambient manifold and the topology of the tangent bundle complement. We will apply the theory developed in Chapter 3 to arrangements in these manifolds. We obtain some specific information about the tangent bundle complement.

The chapter is divided into two sections. In Section 4.1 we look at arrangements of spheres. We start by defining a codimension 1 sub-sphere. It is well known that there are infinitely many embeddings of a codimension 1 sphere in a sphere [72, Section 2.6]. In order to avoid pathologies and subtleties we restrict our attention to so-called *tame sub-spheres* [72, Section 1.8]. Our first theorem is to characterize *flat arrangements* (Definition 3.5.7). The main theorem of this section is about the homotopy type of the tangent bundle complement. We prove that the complement contains a wedge of equi-dimensional spheres and obtain a closed form formula for its homotopy type. We then move on to arrangements in projective spaces. Using the fact that a sphere is the universal covering space of a projective space we analyze the homotopy type of the tangent bundle complement.

In Section 4.2 we study arrangements in tori. Recently these arrangements have received a lot of attention. Usually these arrangements are defined in a complex (or an algebraic) torus. The members of such an arrangement are the kernels of some homomorphisms, which are of real codimension 2 in the given complex torus. We start by defining arrangements in a real (or compact) torus. Then we show that the tangent bundle complement is homeomorphic to the complement of a complex toric arrangement. This situation is analogous to hyperplane arrangements in the sense that the complexified complement and the tangent bundle complement coincide. Consequently, we have the Salvetti complex construction for toric arrangements (see also [61] and [17] for an alternate treatment).

4.1 Arrangements of Spheres

As stated in the introduction, the codimension 1 sub-spheres in a sphere could be very difficult to deal with. Hence we restrict our selves to a nice class of spheres.

Definition 4.1.1. Let S^l denote the unit sphere in $\mathbb{R}^{l+1}$, a subset S of the unit sphere is called a *hypersphere* if and only if it is neither empty nor singleton and $S = H \cap S^l$ for some (affine) hyperplane H in $\mathbb{R}^{l+1}$.

An important property, that will be relevant to us, is that for $S^l (l \geq 2)$ the complement of a hypersphere contains exactly two connected components homeomorphic to an open ball (this follows from Lemma 3.1.1).

Definition 4.1.2. An *arrangement of spheres* in the unit sphere S^l is a finite collection $\mathcal{A} = \{S_1, \ldots, S_k\}$ of hyperspheres satisfying the following conditions:

1. $\mathcal{A}_I := \cap_{i \in I} S_i$ is a sphere of some dimension, for all $I \subseteq \{1, \ldots, k\}$.
2. If $\mathcal{A}_I \nsubseteq S_i$, for some I and $i \in \{1, \ldots, k\}$, then $\mathcal{A}_I \cap S_i$ is a hypersphere in $\mathcal{A}_I$.
3. The hyperspheres in $\mathcal{A}$ decompose S^l into a regular cell complex.

If all the hyperspheres are obtained by intersecting with the linear hyperplanes then we call such an arrangement a *centrally symmetric arrangement of spheres.*

We assume that the empty set is the unit sphere of dimension -1 and that S^0 consists of two points. For $S \in \mathcal{A}$ let H_S denote the hyperplane in $\mathbb{R}^{l+1}$ such that $S = H_S \cap S^l$. Also because of the above definition all the sphere arrangements we consider satisfy the following non-degeneracy condition

$$dim(\mathcal{A}_I) < dim(\cap_{S \in \mathcal{A}_I} H_S) \text{ for every non-empty subset } I.$$

Lemma 4.1.3. *An arrangement spheres is an arrangement of submanifolds, i.e., it satisfies the 3 conditions of Definition 3.1.7.*

Proof. Observe that whenever these hyperspheres intersect they intersect like hyperplanes. This is because each intersection is an intersection of some hyperplanes with the unit sphere. In dimension 1 the hypersphere S^0 contains 2 points and its complement has exactly 2 connected complements. In higher dimensions every hypersphere is separating. Finally the regular cell complex condition is part of the definition. $\square$

First we look at arrangements in S^1. An arrangement in S^1 consists of n copies of S^0, i.e. $2n$ points. The tangent bundle complement of such an arrangement is homeomorphic to the infinite cylinder with $2n$ punctures. Thus we have the following theorem.

Theorem 4.1.4. *Let $\mathcal{A}$ be an arrangement of* 0*-spheres in S^1. If $|\mathcal{A}| = n$ then*

$$M(\mathcal{A}) \simeq \bigvee_{2n+1} S^1.$$

From now on we assume that all our spheres are simply connected. Let $\mathcal{A}$ denote an arrangement of spheres in S^l. Then S^l is not a $K(\pi, 1)$ space and consequently the associated tangent bundle complement $M(\mathcal{A})$ is also not $K(\pi, 1)$ (by Lemma 3.6.1). However our aim is to understand more about the topology of $M(\mathcal{A})$ by studying the associated Salvetti complex $Sal(\mathcal{A})$. Since S^l is simply connected it follows from Theorem 3.5.5 that the minimal positive paths with same end points in the associated arrangement graph $G(\mathcal{A})$ are relative homotopic in $Sal(\mathcal{A})$. If we can show that two such paths are the same under the positive equivalence (given by Definition 3.4.2). It follows that the hypersphere arrangements are *flat*. Recall that by Definition 3.5.7 for flat arrangements the restriction of the canonical functor $J: \mathcal{G}^+(\mathcal{A}) \to \mathcal{G}(A)$ to the class of minimal positive paths is faithful.

Theorem 4.1.5. *Let $\mathcal{A}$ be a centrally symmetric arrangement of spheres in $S^l (l \geq 2)$ then it is a flat arrangement.*

Proof. We already know that if α, β are two minimal positive paths with the same end points then $[\alpha] = [\beta]$ in $\mathcal{G}(A)$. Hence it is enough to show that $[\alpha]_+ = [\beta]_+$. We argue on the lines of the proof of [74, Theorem 20]. Since each $S \in \mathcal{A}$ is centrally symmetric around the origin the antipodal map induces a fixed point free cellular action on the faces of $\mathcal{A}$.

Suppose $\alpha = (a_1, \ldots, a_n)$ and $\beta = (b_1, \ldots, b_n)$ are two minimal positive paths in $G(\mathcal{A})$ that start at C and end at D. We proceed by induction on n, cases $n = 0, 1$ being trivial. Assume that the statement is true for all minimal positive paths with same end points and of length strictly less than n. If $a_1 = b_1$ then we are done by induction. Hence assume that a_1, b_1 are distinct and are dual to the hyperspheres S_a, S_b respectively.

We have that $S_a, S_b \in R(C, D)$ (the set of hyperspheres separating C and D) and $S_a \cap S_b \cong S^{l-2}$, since both these hyperspheres are equatorial. For every $S \in \mathcal{A}$ let $X_S(C, D)$ denote the closures of the connected components of $S^l \setminus S$ that contain either C or D (or both) and let $\mathcal{H}(C, D) = \cap_{S \in \mathcal{A}} X_S(C, D)$. Note that if $S \in R(C, D)$ then $X_S(C, D)$ contains closures of both the components of $S^l \setminus S$. On the other hand if $S \notin R(C, D)$ then $X_S(C, D)$ is homeomorphic to the l-disk $\mathbb{D}^l$.

Claim 1: The set $\mathcal{H}(C, D)$ is either connected or empty.

$$\begin{aligned}\mathcal{H}(C,D) &= \cap_{S\in\mathcal{A}} X_S(C,D)\\ &= (\cap_{S\in R(C,D)} X_S(C,D)) \cap (\cap_{S\notin R(C,D)} X_S(C,D))\\ &= (\cap_{S\in R(C,D)} S) \cap (\cap_{S\notin R(C,D)} \mathbb{D}^l)\end{aligned}$$

The definition of sphere arrangements implies that the first intersection is either a sphere of some finite dimension or it is empty. Hence the set $\mathcal{H}(C,D)$ is either homeomorphic to some closed disk (of possibly lower dimension) or it is empty.

Claim 2: If $\mathcal{H}(C,D) \neq \emptyset$ then $\mathcal{H}(C,D) \cap S_a \cap S_b \neq \emptyset$.
Let S_a^+, S_a^- (respectively S_b^+, S_b^-) denote the (closures of the) connected components of $S_a \setminus (S_a \cap S_b)$ (respectively $S_b \setminus (S_a \cap S_b)$). Without loss of generality assume that S_a^+ and S_b^+ intersect $\overline{C}$. This implies

$$\mathcal{H}(C,D) \cap S_a^+ \neq \emptyset \neq \mathcal{H}(C,D) \cap S_b^+.$$

A similar argument using $\overline{D}$ establishes the claim.

Hence the set $\mathcal{H}(C,D) \cap S_a \cap S_b$ contains a codimension 2 face say F^2. Let C' denote $F^2 * C$. Since $\mathcal{A}$ is an arrangement of submanifolds there exists a minimal positive path γ_0 from C' to D. Also, there exist two minimal positive paths γ_1, γ_2 such that γ_1 starts at $a_1 \circ C$ and γ_2 starts at $b_1 \circ C$ such that both of them end at C'. Using this we can construct two new minimal positive paths $\eta = a_1\gamma_1\gamma_0$ and $\eta' = b_1\gamma_2\gamma_0$. The paths α, η are minimal positive with the same end points and share the same first edge. Hence by induction, $[\alpha]_+ = [\eta]_+$. For the same reasons $[\beta]_+ = [\eta']_+$. If $C' \neq D$ then the path γ_0 is of nonzero length and again by induction $[a_1\gamma_1]_+ = [b_1\gamma_2]_+$ implying $[\eta]_+ = [\eta']_+$. Now the transitivity of the equivalence relation establishes the claim.

The cases in which either $C' = D$ or $\mathcal{H}(C,D) = \emptyset$ can be treated similarly. □

One more obvious property of a centrally symmetric sphere arrangement is the following.

Lemma 4.1.6. *Let $\mathcal{A}$ be a centrally symmetric sphere arrangement in S^l then $\mathcal{A}$ has the involution property.*

Proof. The antipodal action on S^l provides the required graph automorphism on the 1-skeleton of the associated Salvetti complex. □

Now onwards we focus only on centrally symmetric arrangements in spheres of dimension ≥ 2.

Theorem 4.1.7. *If $\mathcal{A}$ is a centrally symmetric, simplicial arrangement of spheres in S^l then the associated canonical functor J is faithful and the positive category $\mathcal{G}^+$ admits the Deligne normal form. Also, the word problem for $\pi_1(M(\mathcal{A}))$ is solvable.*

Proof. All these claims follow from the results proved in previous chapter. In particular Corollary 3.5.15 and Lemma 3.6.6 imply that the functor J is faithful and the positive category admits the Deligne normal form. The word problem for the fundamental group is solvable because the hypothesis of Theorem 3.5.16 is satisfied. □

Remark 4.1.8. We have already noted that arrangements in a sphere are never $K(\pi, 1)$. This can also be seen by observing that the strong contraction property (Definition 3.5.16) is not satisfied by such arrangements.

We now proceed to analyze the homotopy type of the tangent bundle complement associated with a centrally symmetric arrangement of spheres. We say that two arrangements are combinatorially isomorphic if their corresponding face posets and intersection posets are isomorphic.

Lemma 4.1.9. *Given a centrally symmetric sphere arrangement $\mathcal{A}$ in S^l there exists a generic hypersphere S_0 such that S_0 intersects every member of $\mathcal{A}$ in general position. Let S_0^+, S_0^- denote the connected components of $S^l \setminus S_0$ and $\mathcal{A}^+ := \mathcal{A}|S_0^+$, $\mathcal{A}^- := \mathcal{A}|S_0^-$ be the restricted arrangements. Then $\mathcal{A}^+$ and $\mathcal{A}^-$ are combinatorially isomorphic arrangements of hyperplanes in S_0^+ and S_0^- (both $\cong \mathbb{R}^l$) respectively.*

Proof. Since the arrangement is centrally symmetric each individual hypersphere in $\mathcal{A}$ is invariant under the antipodal mapping $x \mapsto -x$ of S^l. For every $S \in \mathcal{A}$ let $a(S) := S/(x \sim -x) \cong \mathbb{P}^{l-1}$, let S_0 be the equator with respect to this action and let S_0^+, S_0^- denote the hemispheres whose boundary is S_0. This equator S_0 generically intersects with every S and $a(S) \setminus (S_0 \cap a(S))$ is a hyperplane contained in $S_0^+ \cong \mathbb{R}^l$. Under this correspondence an intersection of hyperspheres is mapped to the intersection of hyperplanes. The restrictions of the arrangement to the hemispheres S_0^+ and S_0^- gives us two hyperplane arrangements in $\mathbb{R}^l$ which are combinatorially isomorphic. □

Here are two well known facts that we will use later.

Lemma 4.1.10. *If (Y, A) is a CW pair such that the inclusion $A \hookrightarrow Y$ is null homotopic then $Y/A \simeq Y \vee SA$, where SA is the suspension of A.*

Proof. See [44, Chapter 0]. □

Lemma 4.1.11. *Let $\mathcal{B}$ be an essential and affine arrangement of hyperplanes in $\mathbb{R}^l$. Then the cell complex which is dual to the induced stratification is regular and homeomorphic to a closed ball of dimension l.*

Proof. See [73, Lemma 9]. In particular this ball is a MH-complex. □

Now the main theorem of this section.

Theorem 4.1.12. *Let $\mathcal{A}$ be a centrally symmetric arrangement of spheres in S^l. Let $\mathcal{A}^+$ and $\mathcal{A}^-$ be the hyperplane arrangements as in Lemma 4.1.9 and $\mathcal{C}(\mathcal{A}^+)$ be the set of its chambers. Then the tangent bundle complement*

$$M(\mathcal{A}) \simeq Sal(\mathcal{A}^-) \vee \bigvee_{|\mathcal{C}(\mathcal{A}^+)|} S^l.$$

Proof. Let $C \in \mathcal{C}(\mathcal{A}^+)$ and let Q denote the dual cell complex $(S_0^+, \mathcal{F}^*(\mathcal{A}^+))$. Define the map ι_C^+ as follows:

$$\begin{array}{rcl} \iota_C^+ \colon Q & \hookrightarrow & Sal(\mathcal{A}) \\ F & \mapsto & [F, F \circ C] \end{array}$$

Claim 1: The image of ι_C^+, in $Sal(\mathcal{A})$, is homeomorphic to Q (which is a closed ball of dimension l).

Observe that ι_C^+ is just the restriction of the map ι_C, defined in Theorem 3.3.6, which is an embedding of S^l into $M(\mathcal{A})$. Hence ι_C^+ maps Q homeomorphically onto its image.

Hence ι_C^+ is the characteristic map which attaches the boundary ∂Q to the $(l-1)$-skeleton of $Sal(\mathcal{A}^-)$. For notational simplicity let j_C denote the restriction of ι_C^+ to ∂Q.

Claim 2: The image of j_C is also the boundary of an l-cell in $Sal(\mathcal{A}^-)$.
Consider the subcomplex of $Sal(\mathcal{A}^-)$ given by the cells $\{[F, F \circ C] \mid F \in \mathcal{F}^*(\mathcal{A}^-)\}$. By Lemma 4.1.10 above this subcomplex is homeomorphic to the closed l-ball. The boundary of this closed ball is precisely the image of j_C.

Therefore the characteristic map ι_C^+ is the extension of j_C to the cone over ∂Q (which is Q). Hence j_C is null homotopic. Applying the above arguments to every chamber of $\mathcal{A}^+$ establishes the theorem. □

We state the following obvious corollary for the sake of completeness.

Corollary 4.1.13. *Let $\mathcal{A}$ be a centrally symmetric arrangement of spheres in S^l. With the notation as in Lemma 4.1.9 we have:*

$$\pi_1(M(\mathcal{A})) \cong \pi_1(M(\mathcal{A}^-)).$$

Example 4.1.14. Consider the arrangement of 2 circles in S^2 introduced in Example 3.1.11. It is clear that the arrangement $\mathcal{A}^-$ in this case is the arrangement of two lines in $\mathbb{R}^2$ that intersect in a single point. Hence

$$M(\mathcal{A}) \simeq T^2 \vee S^2 \vee S^2 \vee S^2 \vee S^2.$$

The Salvetti complex consists of four 0-cells, eight 1-cells and eight 2-cells, see Figure 3.8. The T^2 component in the above formula corresponds to $M(\mathcal{A}^-)$ and the spheres correspond to chambers of this arrangement.

Example 4.1.15. Consider the arrangement of three circles in S^2 that intersect in general position. This arrangement arises as the intersection of S^2 with the coordinate hyperplanes in $\mathbb{R}^3$. In this case $\mathcal{A}^-$ is the arrangement of three lines in general position. Thus

$$M(\mathcal{A}) \simeq Sal(\mathcal{A}^+) \bigvee \vee_7 S^2.$$

Let us compare this with the cohomology calculations done in Example 3.7.5. The first column of the E_2 page contains the cohomology (ring) of of $M(\mathcal{A}^-)$. To be precise, the cohomology groups of $Sal(\mathcal{A}^-)$ are $\{\mathbb{Z}, \mathbb{Z}^3, \mathbb{Z}^3\}$ the seven extra cohomology classes in degree 2 correspond to seven copies of S^2.

Example 4.1.16. Finally, consider the arrangement of three S^2s in S^3 that intersect like co-ordinate hyperplanes in $\mathbb{R}^3$ (Example 3.7.6). The $\mathcal{A}^-$ in this case is the arrangement of co-ordinate hyperplanes hence $Sal(\mathcal{A}^-) \simeq T^3$, the 3-torus. This arrangement has 8 chambers. So we have the following

$$M(\mathcal{A}) \simeq T^3 \bigvee \vee_8 S^2.$$

We now establish a relationship between the cohomology of the tangent bundle complement and the intersection poset. Let $\mathcal{A}$ be a centrally symmetric arrangement of spheres in S^l, let $\mathcal{A}^+$ be the affine hyperplane arrangement in the positive hemisphere. Let L and L^+ denote the corresponding intersection posets. Observe that the map from L to L^+ that sends $Y \in L$ to $Y|S_0^+ =: Y^+$ is one-to-one up to rank $l-1$. If L_{l-1} and L_{l-1}^+ denote the sub-posets consisting of elements of rank less than or equal to $l-1$ then the previous map is a poset isomorphism. For notational simplicity we use M^- for $M(\mathcal{A}^-)$.

Theorem 4.1.17. *With the notation above, we have the following*

$$rank(H^i(M,\mathbb{Z})) = \begin{cases} \sum_{\substack{Y \in L \\ rk(Y)=i}} |\mu(S^l, Y)| \text{ for } 0 \le i < l \\ \sum_{Y \in L} |\mu(S^l, Y)| \text{ for } i = l \end{cases}$$

Where μ is the Möbius function of the intersection poset.

Proof. We use Theorem 4.1.12 above and Lemma 3.7.9 in order to prove the assertion by considering two cases.
Case 1: $i < l$

$$\begin{aligned}
rank(H^i(M)) &= rank(H^i(M^-)) + \sum_{|\mathcal{C}(\mathcal{A}^+)|} rank(H^i(S^l)) \\
&= rank(H^i(M^+)) + 0 \\
&= \sum_{rk(Y^-)=i} rank(H^0(\coprod_{|\mu(Y^-)|} Y^-)) \\
&= \sum_{rk(Y)=i} rank(H^0(\coprod_{|\mu(Y)|} Y)) \\
&= \sum_{rk(Y)=i} |\mu(S^l, Y)|
\end{aligned}$$

The last equality follows from the fact that each Y is a sphere of dimension $l - i$.
Case 2: $i = l$

$$\begin{aligned}
rank(H^l(M)) &= rank(H^l(M^-)) + \sum_{|\mathcal{C}(\mathcal{A}^+)|} rank(H^l(S^l)) \\
&= \sum_{rk(Y^-)=l} |\mu(Y^-)| + |\mathcal{C}(\mathcal{A}^+)| \\
&= \sum_{rk(Y^-)=l} |\mu(Y^-)| + \sum_{Y^+ \in L^+} |\mu(Y^+)| \\
&= \sum_{Y \in L} |\mu(Y)|
\end{aligned}$$

The third equality follows from the expression for the number of chambers of an affine hyperplane arrangement. The last equality is true because the number of rank l elements in L are twice the corresponding number in L^-. □

In particular the above theorem verifies the Conjecture 3.7.8 for sphere arrangements. The coholomogy ring of the tangent bundle complement in this case can be expressed as a direct sum of an Orlik-Solomon algebra and some top dimensional classes. The number of these top dimensional classes is equal to the number of graded pieces in the Orlik-Solomon algebra. If $\mathcal{A}$ is a centrally symmetric arrangement of spheres then one might call the cohomology algebra $H^*(M(\mathcal{A}), \mathbb{Z})$ the *spherical Orlik-Solomon algebra*.

Next we consider the arrangements in projective spaces. Given a finite dimensional real projective space $\mathbb{P}^l$ we consider a finite collection of subspaces that are homeomorphic to $\mathbb{P}^{l-1}$. We define the projective arrangements as follows.

Definition 4.1.18. Let $\mathbb{P}^l$ denote the l-dimensional projective space and $a\colon S^l \to \mathbb{P}^l$ be the antipodal map. A finite collection $\mathcal{A} = \{H_1, \dots, H_n\}$ of codimension 1 projective spaces is called an *arrangement of projective spaces* (or a *projective arrangement*) if and only if $\tilde{\mathcal{A}} = \{a^{-1}(H) \mid H \in \mathcal{A}\}$ is a centrally symmetric arrangement of spheres in S^l.

It is not hard to see that the above defined arrangements are indeed arrangements of submanifolds. The homotopy type of the tangent bundle complement associated to a projective arrangement is easier to understand because of the antipodal action.

Theorem 4.1.19. *Let $\mathcal{A}$ be a projective arrangement in $\mathbb{P}^l$ and $\tilde{\mathcal{A}}$ be the corresponding centrally symmetric sphere arrangement in S^l. Then the antipodal map on the sphere extends to its tangent bundle and*

$$M(\mathcal{A}) \cong M(\tilde{\mathcal{A}})/((x,v) \sim a(x,v)).$$

Proof. If (x,v) is a point in the tangent bundle of S^l extend the antipodal map in the obvious way, $a(x,v) = (-x,v)$. We now prove that the space $M(\tilde{\mathcal{A}})$ is a covering space of $M(\mathcal{A})$. This follows from the fact that $a\colon TS^l \to T\mathbb{P}^l$ is a covering map for every l. Note that the antipodal map is also cellular on the faces of the arrangement.

Consequently it induces a cellular map on $Sal(\tilde{\mathcal{A}})$ by sending a cell $[F, C]$ to $[a(F), a(C)]$. Hence we get a cell structure for $Sal(\mathcal{A})$. In particular the tangent bundle complement associated to a projective arrangement contains a wedge of projective spaces. Hence there is a torsion in the homology as well as the fundamental group of the tangent bundle complement. Moreover $\pi_1(M(\tilde{A}))$ is an index 2 subgroup of $\pi_1(M(\mathcal{A}))$. □

Example 4.1.20. Consider a projective arrangement $\mathcal{A}$ in $\mathbb{P}^2$ corresponding to the arrangement of 2 circles in S^2 (Example 3.1.11). In this projective arrangement we have two $\mathbb{P}^1$'s intersecting in a point and there are two chambers. Taking the quotient as above of the space obtained in Example 4.1.14 we get the following

$$M(\mathcal{A}) \simeq K \vee \mathbb{P}^2 \vee \mathbb{P}^2$$

where K denotes the Klein bottle. Recall that the 2-torus is a two-fold cover of the Klein bottle.

Given a projective arrangement $\mathcal{A}$ let $J\colon \mathcal{G}^+ \to \mathcal{G}$ denote the canonical functor between the positive category and the arrangement groupoid. For the corresponding (centrally symmetric) sphere arrangement $\tilde{\mathcal{A}}$ let $\tilde{J}\colon \tilde{\mathcal{G}}^+ \to \tilde{\mathcal{G}}$ be the associated canonical functor. Because of the antipodal action on S^n the arrangement $\tilde{\mathcal{A}}$ has the involution property (Definition 3.5.8). It follows from Lemma 3.5.13 that this action induces an 'antipodal' functor on $\mathcal{G}^+$. Recall that under this functor an object C (which is a chamber) is mapped to its antipodal (chamber) $C^{\#}$ and a morphism $[\alpha]$ is mapped to $[\alpha^{\#}]$.

Lemma 4.1.21. *With the notation as above the following diagram commutes:*

$$\begin{array}{ccc} \tilde{\mathcal{G}}^+ & \xrightarrow{\tilde{J}} & \tilde{\mathcal{G}} \\ {\scriptstyle \Phi^+}\downarrow & & \downarrow{\scriptstyle \Phi} \\ \mathcal{G}^+ & \xrightarrow{J} & \mathcal{G} \end{array}$$

where the functor Φ^+ identifies antipodal objects and morphisms and Φ is the covering functor.

Proof. Follows from a simple diagram chase and the fact that S^l is the universal cover of $\mathbb{R}P^l$. □

An immediate consequence of the lemma is -

Corollary 4.1.22. *The restriction of J to the class of minimal positive paths is faithful and the word problem for $\pi_1(M(\mathcal{A}))$ is solvable. Moreover if $\mathcal{A}$ is a simplicial arrangement then J is faithful.*

Proof. The first statement follows from the commutativity of the diagram in the previous lemma. If $[\alpha]_+$ is a class of minimal positive path in $\mathcal{G}^+$ then the class representing either of α's lift is also minimal positive in $\tilde{\mathcal{G}}^+$. If there are two distinct classes of minimal positive paths then first applying $\tilde{J}$ to their lifts in $\tilde{\mathcal{G}}^+$ and then applying applying Φ results in producing two distinct classes of minimal positive paths in $\mathcal{G}$. By the same argument if $\tilde{J}$ is faithful then J is also faithful.

Let us see why the word problem is solvable. Let $[\alpha]$ be a loop based at a vertex C in $\mathcal{G}$. Let $[\tilde{\alpha}]$ be the class representing a loop based at $\tilde{C}$ (a vertex in the fiber over C). Then by statement 3 in Lemma 3.5.14 we have the following

$$[\tilde{\alpha}] = [\delta^{-n}(\tilde{C})][\tilde{\alpha}']$$

where $\delta^{-n}(\tilde{C}) = \mu(\tilde{C} \to \tilde{C}^{\#})\mu(\tilde{C}^{\#} \to \tilde{C})$ and $\tilde{\alpha}'$ is a positive loop based at $\tilde{C}$. Since Φ is the covering functor, $\Phi([\delta^{-n}(\tilde{C})]) = [\delta^{-2n}(C)]$ here $\delta(C)$ is a positive loop based at C which traverses every vertex twice. Let $[\alpha']$ be the image $\Phi([\tilde{\alpha}'])$, it represents a class positive loop based at C. Note that choosing another lift of α that is based at the antipodal point $\tilde{C}^{\#}$ does not make any difference. Hence we have proved that any loop in $Sal(\mathcal{A})$ can be expressed as a composition of a 'special loop' (which traverses each vertex a fixed number of times) and a positive loop. Now the same argument as in the proof of Theorem 3.5.16 shows that the word problem for $\pi_1(M(\mathcal{A}))$ is solvable. $\square$

Remark 4.1.23. From the commutativity of the above diagram it follows that $\mathcal{G}^+$ admits the Deligne normal form whenever $\tilde{\mathcal{G}}^+$ does. It also suggests that the 'simple connectedness' requirement in the definition of a flat arrangement might be unnecessary.

4.2 Toric Arrangements

In this section we study the arrangements in a torus. These type of arrangements are of interest because of their deep connections to diverse fields like algebraic groups, integral polytopes, approximation theory etc. To the best of our knowledge the idea of toric arrangements can be traced back to a paper of Lehrer [51]. However an explicit study of toric arrangements started with the work of Douglas in [26]. Analogous to hyperplane arrangements he defined arrangements in an algebraic torus. His main aim was to study the associated derivations of the co-ordinate ring. He characterized the arrangements for which the derivations localized at the identity form a free module over the (localized) co-ordinate ring. His student Sawyer [75] extended Brieskorn's theorem using the earlier work of Jozsa and Rice [47]. In particular she showed that the (complex) cohomology algebra of the complement of a toric arrangement is generated by certain degree 1 logarithmic forms (see [75, Theorem 5.2]). Extending the work of Douglas, Macmeiken derived conditions under which the module of derivations associated to a toric arrangement is a free module over the co-ordinate ring [54].

Probably the most influential work on this subject has appeared in the paper [19] of De Concini and Procesi. They computed cohomology groups of (the complement of) toric arrangements using the theory of D-modules and no-broken circuits. They used these results to derive counting formulas for the number of integer points in a parametrized polytope. An extensive account of their work and its applications to approximation theory and box splines can be found in their book [20]. Moci has generalized the wonderful compactification for complements of toric arrangements in [60]. In [59] Moci describes a new polynomial, called the multiplicity Tutte polynomial, for toric arrangements. Similar to the case of hyperplane arrangements, this polynomial counts the number of chambers, gives Poincaré polynomial of the complex complement and the characteristic polynomial of the intersection poset. Some other applications to approximation theory and zonotopes are also discussed. He also studies the toric arrangements defined by root systems in [58]. Recently Moci and Settepanella have generalized the construction of Salvetti complex for toric arrangements in [61]. Building up on their work in [80], Settepanella proves that the integral cohomology groups of the complement of the toric arrangement defined by an affine Weyl group are torsion free.

However arrangements in a real (or compact) torus have received very little attention. The paper [30], by Ehrenborg, Readdy and Slone does an extensive study of real toric arrangements from a combinatorial point of view. For example, they generalize Zaslavsky's theorem (see [30, Theorem 3.6]) and obtain a toric version of Bayer-Sturmfels result for hyperplane arrangements. They also compute the **ab**-index of the associated face poset. We now end the literature survey and move on to our work. We start by defining arrangements in a real torus. Then we find an obvious relation between the tangent bundle complement of a real toric arrangement and the complement of a complex toric arrangements. Using this relationship we construct the Salvetti complex for toric arrangements.

Let $S^1 \hookrightarrow \mathbb{C}^*$ denote the circle group whose every element is given by the expression $e^{2\pi i\theta}, \theta \in \mathbb{R}$. The l-dimensional torus denoted by T^l is the l-fold product of S^1 with itself. An element of the jth component of $S^1 \hookrightarrow T^l$ will be denoted by r_j. There is a natural projection map (in fact a covering map) $p\colon \mathbb{R}^l \to T^l$ given by $(x_1, \ldots, x_l) \mapsto (e^{2\pi i x_1}, \ldots, e^{2\pi i x_l})$. This map is equivalent to the covering space action of $\mathbb{Z}^l$ on $\mathbb{R}^l$ by translation. The fundamental domain with respect to this action is the (open) l-cube $[0,1)^l$. Under this covering map every codimension 1 (rational) subspace of $\mathbb{R}^l$ is mapped to a codimension 1 subtorus (which we sometimes refer to as a toric hyperplane) of T^l. The inverse image of a toric hyperplane consists of (rational) translates of a codimension 1 subspaces of $\mathbb{R}^l$. As a matter of fact the image of a (rational) k-subspace of $\mathbb{R}^l$ under the covering map is homeomorphic to a k-torus in T^l. Now we define what we mean by a toric arrangement.

Definition 4.2.1. Let T^l be the l-torus, a *toric arrangement* is a collection $\mathcal{A} = \{N_1, \ldots, N_n\}$ of finitely many codimension 1 subtori (toric hyperplanes). Moreover the collection $\mathcal{A}$ satisfies the axioms of a submanifold arrangement.

A toric hyperplane also corresponds to the kernel of a group homomorphism $\phi\colon T^l \to S^1$. For example, given a 2-dimensional torus T^2 the kernel of the homomorphism $r_1^2 = 1$ is a toric hyperplane having 2 connected components. Also, when two toric hyperplanes intersect the intersection need not be connected. In general for two subspaces $V, W \subset \mathbb{R}^l$, we have that $p(V \cap W) \subseteq p(V) \cap p(W)$ and this containment could be strict. Without loss of generality assume that $dim(V) = dim(W) = k$. The inverse image $p^{-1}(p(V))$ consists of V and its (rational) translates. The intersection of $p^{-1}(p(V)) \cap p^{-1}(p(W))$ with the fundamental region $[0,1)^l$ is disconnected and consists of finitely many translates of $V \cap W$, say n. As the subspace $V \cap W$ is $(k-1)$-dimensional the image $p(V \cap W)$ is homeomorphic to disjoint copies of T^{k-1}. Moreover we have the following:

$$p(V) \cap p(W) = \bigcup_{i=1}^{n} p(a_i + (V \cap W))$$

where each $a_i \in [0,1)^l$ (we may assume that $a_1 = (0,\dots,0)$) and each connected component is homeomorphic to T^{k-1}. In group theoretic terms the intersection can be expressed as $\mathbb{Z}_n \times T^{k-1}$.

Now we define the toric arrangements from a complex perspective. Let $(\mathbb{C}^*)^l$ denote the complex l-torus, which is an algebraic group under multiplication. A *rational character* is an algebraic group homomorphism $\chi\colon (\mathbb{C}^*)^l \to \mathbb{C}^*$. The kernel of such a character is a hypersurface in $(\mathbb{C}^*)^l$. A *complex toric arrangement* in $(\mathbb{C}^*)^l$ is a finite collection $\mathcal{A}_{\mathbb{C}} = \{ker(\chi_1),\dots,ker(\chi_n)\}$ of kernels of rational characters. The complement of such an arrangement is defined as

$$\mathcal{R}(\mathcal{A}_{\mathbb{C}}) := (\mathbb{C}^*)^l \setminus \bigcup_{i=1}^{n} ker(\chi_i).$$

The co-ordinate ring of the complex torus is given by $\mathbb{C}[z_1^{\pm 1},\dots,z_l^{\pm 1}]$ (which is a unique factorization domain). Then it is well known that a character χ is of the form $\chi = z_1^{n_1}\cdots z_l^{n_l}$ where each $n_i \in \mathbb{Z}$. As before let $p\colon \mathbb{C}^l \to (\mathbb{C}^*)^l$ be the universal cover. The inverse image of the kernel of a rational character is a family of infinite, parallel hyperplanes in $\mathbb{C}^l$. The defining equations of these hyperplanes are:

$$\{n_1 z_1 + \cdots + n_l z_l = m \mid m \in \mathbb{Z}\}$$

Since the ring of (complex) Laurent polynomials is a UFD each character can be uniquely expressed as a product of irreducible factors. For example, the irreducible factors of a character $\chi - 1$ are of the form $(\chi'_i - \mu_i)$ where χ'_i is again a character with a connected kernel and μ_i is an n-th root of unity for some n. Hence given a complex toric arrangement $\mathcal{A}_{\mathbb{C}}$ consider a new arrangement $\mathcal{A}'_{\mathbb{C}}$ which consists of distinct irreducible characters that come from factors of the characters in $\mathcal{A}_{\mathbb{C}}$. From the combinatorial and topological viewpoint nothing has changed.

From now on, without loss of generality, we assume that all our complex toric arrangements consist of irreducible characters. Let $\mathcal{A}_{\mathbb{C}} = \{ker(\chi'_1),\dots,ker(\chi'_n)\}$ be such a complex toric arrangement in $(\mathbb{C}^*)^l$ (here $ker(\chi'_i) = \{z \in (\mathbb{C}^*)^l \mid \chi'_i(z) = \mu_i\}$). Since there is a natural (topological) embedding $T^l \hookrightarrow (\mathbb{C}^*)^l$ for every complex toric arrangement there is a unique real toric arrangement in T^l. This arrangement is given by $\mathcal{A} = \{ker(\chi'_1) \cap T^l, \dots, ker(\chi'_n) \cap T^l\}$. Moreover if we denote $ker(\chi'_j) \cap T^l$ by N_j then

$$p^{-1}(N_j) = \{n_1x_1 + \cdots + n_1x_l = \theta_j + m \mid m \in \mathbb{Z}\}$$

where $\theta_j \in [0, 1)$ and $e^{i\theta_j} = \mu_j$. We refer to $\mathcal{A}$ as the associated real (toric) arrangement. Conversely given a real toric arrangement in T^l we can define the associated complex (toric) arrangement in $(\mathbb{C}^*)^l$ by extension of scalers.

Lemma 4.2.2. *Let $\mathcal{A}_{\mathbb{C}}$ be a complex toric arrangement in $(\mathbb{C}^*)^l$ and let $\mathcal{A}$ be the associated real arrangement in T^l. If $M(\mathcal{A})$ denotes the tangent bundle complement then*

$$\mathcal{R}(\mathcal{A}_{\mathbb{C}}) \cong M(\mathcal{A})$$

Proof. We start by analyzing the tangent bundle of a real torus. In dimension 1, the tangent bundle of S^1 (is trivial) is the infinite cylinder $S^1 \times \mathbb{R}$. The infinite cylinder is homeomorphic (biholomorphic, to be precise) to $\mathbb{C}^*$. If every element of the tangent bundle is given by $(e^{2\pi i\theta}, v)$ then the homeomorphism can be given by

$$(e^{2\pi i\theta}, v) \mapsto e^v(\cos(2\pi\theta) + i\sin(2\pi\theta)).$$

Extending this homeomorphism to the tangent bundle of T^l we get that it is homeomorphic to $(\mathbb{C}^*)^l$, the complex l-torus. We denote this homeomorphism by

$$h_l \colon T^l \times \mathbb{R}^l \xrightarrow{\cong} (\mathbb{C}^*)^l.$$

Since $h_l(\bigcup_{i=1}^n TN_i) \cong \bigcup_{i=1}^n ker(\chi_i)$ the desired homeomorphism between $M(\mathcal{A})$ and $\mathcal{R}(\mathcal{A}_{\mathbb{C}})$ is given by $(x, v) \mapsto h_l(x, v)$. $\square$

Remark 4.2.3. Given a real toric arrangement $\mathcal{A}$ in T^l there corresponds an arrangement of periodic hyperplanes $\tilde{\mathcal{A}}$ in $\mathbb{R}^l$. The hyperplanes in $\tilde{\mathcal{A}}$ can be grouped in to a finite family of parallel hyperplanes. The complexification of these hyperplanes gives us an arrangement of periodic hyperplanes in $\mathbb{C}^l$. Under the covering map these hyperplanes are mapped to finitely many codimension 1 subtori in $(\mathbb{C}^*)^l$. Intersection of each of these subtori with the compact torus gives the arrangement in T^l that we started with.

As a consequence of the above Lemma 4.2.2 we have the following theorem.

Theorem 4.2.4. *Let $\mathcal{A}_{\mathbb{C}}$ be a complex tori arrangement in $(\mathbb{C}^*)^l$ and let $\mathcal{A}$ denote the corresponding real arrangement in T^l. Also assume that $(T^l, \mathcal{F}(\mathcal{A}))$ is a regular CW-complex. If $Sal(\mathcal{A})$ is the associated Salvetti complex then*

$$M(\mathcal{A}_{\mathbb{C}}) \simeq Sal(\mathcal{A}).$$

Remark 4.2.5. The requirement that $(T^l, \mathcal{F}(\mathcal{A}))$ is a regular CW-complex is not very restrictive. In fact the quotient map $p\colon (\mathbb{R}^l, \mathcal{F}(\tilde{\mathcal{A}})) \to (T^l, \mathcal{F}(\mathcal{A}))$ is injective on the faces if and only if $(T^l, \mathcal{F}(\mathcal{A}))$ is a regular CW-complex. Recently d'Antonio and Delucchi have dealt the 'non-regular' case using nerves of acyclic categories in [17].

Remark 4.2.6. By Theorem 3.7.2 we know that the cohomology ring of the tangent bundle complement contains $H^*(T^k, \mathbb{Z})$ as a subring for every $0 \leq k \leq l$. Also by Corollary 3.5.18 it is clear that $\pi_1(M(\mathcal{A}))$ is isomorphic to the free product of $\mathbb{Z}^l$ with another subgroup.

Chapter 5

Arrangements of Pseudohyperplanes

In this chapter we study arrangements of pseudohyperplanes (hyperplanes that are topologically deformed in some mild way). As explained in Section 1.5 oriented matroids are intimately connected to hyperplane arrangements. The faces of a hyperplane arrangement satisfy covector axioms of an oriented matroid. The oriented matroids which correspond to faces of a hyperplane arrangement are known as the *realizable oriented matroids*. There are oriented matroids that do not correspond to hyperplane arrangements (e.g. non-Pappus configuration). Hence for a long time an important question in this field was to come up with the right topological model for oriented matroids. This was settled by Folkman and Lawrence in [34]. The *Folkman-Lawrence Topological Representation Theorem* states that oriented matroids are completely realizable in terms of geometric topology: they may not correspond to real hyperplane arrangements, but they correspond to certain collections of topological spheres and balls (i.e. *arrangements of pseudo-hemispheres*). These pseudo arrangements not only create oriented matroids in the same way that $\mathbb{R}^l$ and collections of half spaces create an obvious combinatorial structure but there is a one-to-one correspondence between such arrangements and the oriented matroids. In his thesis Mandel [55] introduced "*sphere systems*" that simplified some aspects of the pseudo-hemisphere arrangements and also proved the stronger piecewise linear version of the representation theorem.

In his thesis Ziegler [94, Section 5.5] extended the definition of the Salvetti complex in the context of arbitrary oriented matroids. To every oriented matroid one can associate a simplicial complex and in case of a realizable oriented matroid this complex has the homotopy type of the complexified complement of the corresponding hyperplane arrangement. In their paper Gel'fand and Rybnikov [39] studied the Salvetti complex for arbitrary oriented matroids and showed that the cohomology ring of this complex is isomorphic to the Orlik-Solomon algebra of the associated lattice of flats (see also [7]). This result not only extends the classical theorem of Brieskorn and Orlik-Solomon but also gives a completely combinatorial proof. For a more direct approach via discrete Morse theory see [25, Prop. 2, Lemma 5.10], but this result only proves additive isomorphism.

An important thing missing in this study is a $2l$-dimensional space naturally associated with the pseudo arrangements that has the homotopy type of the associated Salvetti complex, i.e. a generalization of the complexified complement. The aim of this chapter is to introduce such a space.

In Section 5.1 we quickly review the oriented matroids and the topological representation theorem. In order to avoid undue topological subtleties we restrict our attention to pseudosphere arrangements in the standard unit sphere. We prove that to every such pseudosphere arrangement there corresponds an arrangement of topologically deformed hyperplanes (pseudohyperplanes) in the ambient Euclidean space.

In Section 5.2 we first associate a connected subset of $\mathbb{R}^{2l}$ to an arrangement of pseudohyperplanes. We then proceed to prove that the spine of this space is the Salvetti complex of the corresponding oriented matroid. Finally we apply some of the theorems proved in Chapter 3 to this setting.

5.1 Topological Representation Theorem

Let $E = \{1, \dots, n\}$ be the finite ground set for some $n > 0$. A *sign vector* is a function $X\colon E \to \{+, 0, -\}$, i.e., an assignment of signs to each element of E. The set of all possible possible sign vectors is denoted by $\{+, 0, -\}^E$ and X_e stands for $X(e)$ for all $e \in E$.

Definition 5.1.1. A set $\mathcal{L} \subset \{-, 0, +\}^E$ is the set of *covectors* of an *oriented matroid* if and only if it satisfies:

(V0) $\mathbf{0} \in \mathcal{L}$,

(V1) $X \in \mathcal{L} \Rightarrow -X \in \mathcal{L}$,

(V2) $X, Y \in \mathcal{L} \Rightarrow X \circ Y \in \mathcal{L}$,

(V3) if $X, Y \in \mathcal{L}$ and $e \in S(X, Y)$ then there exists $Z \in \mathcal{L}$ such that $Z_e = 0$ and $Z_f = (X \circ Y)_f = (Y \circ X)_f$ for all $f \notin S(X, Y)$.

Here $\mathbf{0} = (0, \dots, 0)$, $-X$ is the opposite sign vector defined by $(-X)_e = -(X_e)$ and $S(X, Y) = \{e \in E \mid X_e = -Y_e \neq 0\}$ is called as the *separation set* of X and Y.

The *support* of a vector X is $\underline{X} = \{e \in E | X_e \neq 0\}$; its *zero set* is $z(X) = E \setminus \underline{X}$. Finally, the *composition* of two sign vectors X and Y is $X \circ Y$ defined by

$$(X \circ Y)_e := \begin{cases} X_e & \text{if } X_e \neq 0 \\ Y_e & \text{otherwise} \end{cases}$$

There is a partial order on the sign vectors defined as follows:

$$Y \leq X \iff Y_e \in \{0, X_e\} \quad \forall\, e \in E$$

If $\mathcal{L} \subset \{+, 0, -\}^E$ is a set of covectors of an oriented matroid then it inherits the above defined partial ordering to become a poset with the bottom element $\mathbf{0}$. The poset $\hat{\mathcal{L}} := (\mathcal{L} \bigcup \{\hat{1}\}, \leq)$ is a lattice. The join in $\hat{\mathcal{L}}$ of X and Y is $X \circ Y = Y \circ X$ if $S(X, Y) = \emptyset$, and equals $\hat{1}$ otherwise.

Definition 5.1.2. The lattice $\mathcal{F}(\mathcal{L}) = (\hat{\mathcal{L}}, \leq)$ is called the *face lattice* of the oriented matroid $\mathcal{L}$. The maximal elements of $\mathcal{L}$ are called *topes* (or *regions*). Let $\mathcal{T}(\mathcal{L})$ denote the set of topes. The *rank* of $\mathcal{L}$ is the length of a maximal chain in $(\mathcal{L}, \leq)$.

Remark 5.1.3. If $\mathcal{L}$ is a (linear) oriented matroid coming from a central hyperplane arrangement $\mathcal{A}$ in $\mathbb{R}^l$ (see Section 1.5 for details), then $\mathcal{F}(\mathcal{L})$ is isomorphic to the face poset $\mathcal{F}(\mathcal{A})$. In particular the topes of $\mathcal{L}$ correspond to the chambers of $\mathcal{A}$.

We now turn to the topological side of the representation theorem. We first recall standard terminologies from PL topology. Let K and L denote two geometric simplicial complexes. A map (between the underlying spaces) $f\colon ||K|| \to ||L||$ is said to be *piecewise linear (PL)* if it is linear with respect to some simplicial subdivision of K. A *PL homeomorphism* is a PL map which is also a homeomorphism of underlying spaces, a *PL embedding* is defined analogously. A *PL n-sphere* is a (geometric) simplicial complex which is PL homeomorphic to the boundary of a $(n+1)$-simplex, analogously a *PL n-ball* is PL homeomorphic to standard topological n-simplex. Following are some (relevant) well known facts in this field (we refer the reader to [72]). Recall that an embedding of a submanifold is locally flat if every point in the image has a neighborhood in which the submanifold is homeomorphic to a Euclidean subspace.

Theorem 5.1.4. *If $f\colon M \to N$ is a PL embedding of the PL m-manifold M into the PL n-manifold N and $m - n \neq 2$, then f is locally flat.*

Theorem 5.1.5. *Let S^l denote the standard unit sphere in $\mathbb{R}^{l+1}$. If $f\colon S^l \to \mathbb{R}^n$, $n - l \neq 2$ is a locally flat embedding, then there exists a homeomorphism $h\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $h \circ f$ is the inclusion map. The same conclusion holds for an embedding of $\mathbb{R}^l$ into $\mathbb{R}^n$.*

A subset of the standard unit sphere is called a subsphere if it is homeomorphic to some lower dimensional sphere. We single out a class of subspheres that play an important role in defining more general types of arrangements.

Lemma 5.1.6. *For a $(l-1)$-subsphere S of S^l the following conditions are equivalent:*

1. *embedding of S is equivalent to the inclusion map,*
2. *embedding of S is equivalent to some PL $(l-1)$-subsphere of S^l,*
3. *the closure of each connected component of $S^l \setminus S$ is homeomorphic to the l-ball.*

The equivalence class of these subspheres is known as *tame*, all other embeddings are called *wild*. It is known that all embeddings of S^1 into S^2 are tame (the Schönflies theorem). However, there are wild 2-spheres in S^3, for example, the Alexander horned sphere.

Definition 5.1.7. A $(l-1)$-subsphere S in S^l satisfying any of the equivalent conditions in Lemma 5.1.6 is called a *pseudosphere* in S^l. The two connected components of $S^l \setminus S$ are its *sides*, denoted by S^+ and S^-. The closures of the sides are called the *closed sides* (or *pseudohemispheres*)

We can now present the generalization of hyperplane arrangements that was used to prove the representation theorem.

Definition 5.1.8. A *signed arrangement of pseudospheres* in the standard unit sphere $S^l \subseteq \mathbb{R}^{l+1}$ is a finite collection $\mathcal{A} = \{(S_i^+, S_i^0, S_i^-) \mid i \in E\}$ where $E = \{1, \ldots, n\}$ such that

1. Each S_i^0 is a pseudosphere in S^l with sides S_i^+ and S_i^-.
2. $S_I := \cap_{i \in I} S_i^0$ is a sphere, for all $I \subseteq E$ ($\emptyset$ is the (-1)-sphere).
3. If $S_I \nsubseteq S_j$, for some subset I, an index j, then $S_I \cap S_j$ is a pseudosphere in S_I with sides $S_I \cap S_j^+$ and $S_I \cap S_j^-$.

For the sake of notational simplicity we assume that both the sides of each pseudosphere are equipped with a sign and we will not explicitly mention it every time. Since each side has a sign attached to it one can define a sign function similar to that for hyperplane arrangements. Equivalently the position of each point $x \in S^l$ with respect to each pseudosphere in the arrangement $\mathcal{A}$ is given by a sign vector $\sigma(x) \in \{+, 0, -\}^E$, defined by

$$\sigma(x)_i = \begin{cases} + & \text{if } x \in S_i^+ \\ 0 & \text{if } x \in S_i^0 \\ - & \text{if } x \in S_i^- \end{cases}$$

The arrangement defines a stratification of the ambient sphere, and each strata is indexed by the sign vectors in $\sigma(S^l)$. One of the reasons why this type of generalization is necessary is the following:

Theorem 5.1.9. *Let $\mathcal{A}$ be a signed, essential arrangement of pseudospheres in S^l. Then*

$$\mathcal{L}(\mathcal{A}) := \{\sigma(x) \mid x \in S^l\} \bigcup \{\boldsymbol{0}\} \subseteq \{+, 0, -\}^E$$

is the set of covectors of an oriented matroid and the rank of $\mathcal{L}(\mathcal{A}) = l + 1$.

Some of the topological properties of hyperplane arrangements also hold.

Lemma 5.1.10 ([55] Lemma 3, page 201)**.** *Let $\mathcal{A}$ be a signed and essential arrangement of pseudospheres in S^l. For every $X \in \mathcal{L}(\mathcal{A}) \setminus \{\boldsymbol{0}\}$ the strata $\sigma^{-1}(X)$ is an open cell of a regular cell decomposition $\Delta(\mathcal{A})$ of S^l. The boundary of $\overline{\sigma^{-1}(X)}$ is the union of all those $\sigma^{-1}(Y)$ such that Y is properly covered by X. Furthermore, the mapping $X \mapsto \{y \in S^l \mid \sigma(y) \leq X\}$ gives an isomorphism*

$$\hat{\mathcal{L}}(\mathcal{A}) \cong \hat{\mathcal{F}}(\Delta(\mathcal{A}))$$

of the face lattice of $\mathcal{L}(\mathcal{A})$ and face lattice of the regular cell complex $\Delta(\mathcal{A})$.

Two signed arrangements $\mathcal{A} = \{S_1, \ldots, S_n\}$ and $\mathcal{A}' = \{S'_1, \ldots, S'_n\}$ of pseudospheres in S^l are *topologically equivalent* ($\mathcal{A} \sim \mathcal{A}'$) if there exists some homeomorphism $h\colon S^l \to S^l$ such that $h(S_i) = S'_i$ and $h(S_i^+) = (S'_i)^+$ for all $1 \leq i \leq n$. This topological equivalence is combinatorially determined.

Theorem 5.1.11. *Two signed arrangements $\mathcal{A}$ and $\mathcal{A}'$ in S^l are topologically equivalent if and only if $\mathcal{L}(\mathcal{A}) \cong \mathcal{L}(\mathcal{A}')$.*

If the oriented matroid obtained from such an arrangement is realizable then we can retrieve hyperplane arrangements.

Corollary 5.1.12. *Let $\mathcal{A} = \{S_1, \ldots, S_n\}$ be a signed arrangement of pseudospheres in S^l. The oriented matroid $\mathcal{L}(\mathcal{A})$ is realizable if and only if there exists a homeomorphism $h\colon S^l \to S^l$ such that $h(S_i) = S^l \cap H_i$, where H_i is a codimension 1 subspace of $\mathbb{R}^{l+1}$, for every i.*

The topological representation theorem [34, Theorem 20], [55] and [6, Theorem 5.2.1], however, includes a converse to all of the above.

Theorem 5.1.13 (**Topological Representation Theorem**). *Let $\mathcal{L} \subseteq \{+, 0, -\}^E$. Then following conditions are equivalent:*

1. *$\mathcal{L}$ is the set of covectors of a (simple) oriented matroid of rank l.*
2. *$\mathcal{L} = \mathcal{L}(\mathcal{A})$ for some signed arrangement $\mathcal{A} = \{S_1, \dots, S_n\}$ of pseudospheres in S^{l-1}, which is essential and centrally symmetric and whose induced cell complex $\Delta(\mathcal{A})$ is regular.*

An arrangement is said to be *centrally symmetric* if each pseudosphere $S \in \mathcal{A}$ is invariant under the antipodal mapping of S^l (and so are the sides, i.e. $S_i^+ \mapsto S_i^-$ for every i).

Let $\mathcal{L}$ be the set of covectors of a rank l oriented matroid. According to Theorem 5.1.13 there corresponds a signed arrangement $\mathcal{A} = \{S_1, \dots, S_n\}$ of pseudospheres in S^{l-1}, the unit sphere in $\mathbb{R}^l$. Since each pseudosphere S is centrally symmetric any pair of antipodal points $x, -x \in S$ generates a line through the origin in $\mathbb{R}^l$. For $S \in \mathcal{A}$ let H_S be the set of all rays from the origin passing through S. Specifically this set can be expressed as the cone over S as follows:

$$H_S = S \times [0, \infty)/\{S \times 0\}$$

The next result is now immediate and follows from Lemma 5.1.6.

Lemma 5.1.14. *Let S be a pseudosphere in the unit sphere S^{l-1} and H_S be the cone. Then there exists a homeomorphism of $\mathbb{R}^l$ such that it maps H_S to a hyperplane passing through the origin.*

Definition 5.1.15. A *pseudohyperplane* in $\mathbb{R}^l$ is defined as the cone over some pseudosphere in S^{l-1}. An *arrangement of pseudohyperplanes* is a finite collection $\mathcal{A}$ of pseudohyperplanes in $\mathbb{R}^l$ such that $\{H \cap S^{l-1} \mid H \in \mathcal{A}\}$ is an arrangement of pseudospheres in S^{l-1}.

Given an arrangement $\mathcal{A}$ of pseudospheres in S^{l-1} we denote by $c\mathcal{A}$ the corresponding arrangement of pseudohyperplanes. A face of $c\mathcal{A}$ is the cone over some face of $\mathcal{A}$ and hence homeomorphic to an open polyhedral cone of 1 dimension higher.

Example 5.1.16. Consider the arrangement of circles in S^2 as shown in Figure 5.1. It corresponds to the non-Pappus oriented matroid of rank 3. We first construct an arrangement of 8 circles in S^2 such that points a, b, c are collinear and other three points a', b', c' are also collinear. According Pappus theorem the points d, e, f are also

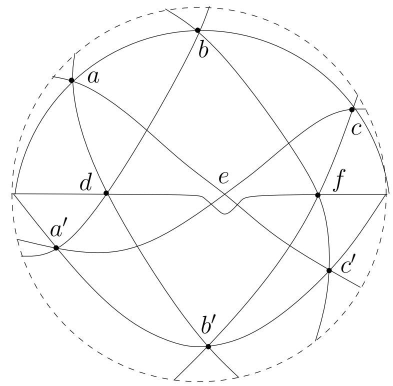


Figure 5.1: Non-Pappus arrangement

collinear. However we add the 9th circle which passes through the points d and f but not e. The resulting pseudosphere arrangement represents a non-oriented matroid. The corresponding pseudo-plane arrangement in $\mathbb{R}^3$ is obtained by letting rays from the origin pass through each of these 9 circles.

We state the following corollary for the sake of completeness.

Corollary 5.1.17. *Let $\mathcal{L} \subseteq \{+, 0, -\}^E$ be the set of covectors of a oriented matroid of rank l. Then there exists an arrangement of pseudohyperplanes $c\mathcal{A}$ such that*

$$\mathcal{F}(c\mathcal{A}) \cong (\mathcal{L}, \leq).$$

Remark 5.1.18. In the literature related to topological representation theorem the word *pseudohyperplane* is used for the (codimension 1) projective space obtained by applying the antipodal map. However here we have used this term for a tame embedding of a hyperplane. Miller has also used this word for topologically deformed hyperplanes in [57] where he describes a slightly different topological representation for a certain class of oriented matroids.

5.2 Complexification of Pseudohyperplanes

Throughout this section we fix an arbitrary simple oriented matroid $\mathcal{L}$ of rank $l-1$, let $\mathcal{A}$ and $c\mathcal{A}$ denote the corresponding arrangements of pseudospheres (in S^{l-1}) and pseudohyperplanes (in $\mathbb{R}^l$) respectively. Our aim is to construct a connected subspace of $\mathbb{R}^{2l}$ and then show that it has the homotopy type of a simplicial complex that is determined by the oriented matroid.

Lemma 5.2.1. *If $c\mathcal{A}$ is the pseudohyperplane arrangement corresponding to an oriented matroid then it is an arrangement with the involution property. Let $\mathcal{F}(c\mathcal{A})$ denote the associated face poset and $\mathcal{F}^*$ be the dual face poset then the dual cell complex $(\mathbb{R}^l, \mathcal{F}^*)$ is a MH*-complex.*

Proof. The involution on the dual 1-skeleton is induced by the antipodal map. Observe that the distance between two dual vertices is equal to the number of pseudohyperplanes that separate corresponding chambers. The action of faces on chambers is given by composition of corresponding covectors. The proof that $(\mathbb{R}^l, \mathcal{F}^*)$ is a MH-complex is exactly same as that of Theorem 3.3.14. □

Let $c\mathcal{A} = \{H_1, \ldots, H_n\}$ be an arrangement of pseudohyperplanes in $\mathbb{R}^l$. For every $x \in \mathbb{R}^l$ the *arrangement restricted* at x is

$$c\mathcal{A}_x := \{H \in c\mathcal{A} \mid x \in H\}.$$

Define the *local complement* at x as:

$$M(c\mathcal{A}_x) := \mathbb{R}^l \setminus c\mathcal{A}_x.$$

Finally, define the *complexified complement* of $c\mathcal{A}$ as:

$$M(c\mathcal{A}) := \coprod_{x \in \mathbb{R}^l} \mathbb{R}^l \setminus c\mathcal{A}_x = \{(x, v) \mid x \in \mathbb{R}^l, v \in M(c\mathcal{A}_x)\} \subseteq \mathbb{R}^{2l}$$

Lemma 5.2.2. *The space $M(c\mathcal{A})$ is connected.*

Proof. For any two points (x_1, v_1) and (x_2, v_2) we show that there is path in $M(c\mathcal{A})$ connecting these two points. Let $\{\alpha(t) \mid t \in [0, 1]\}$ be a continuous path starting from x_1 and ending at x_2 in $\mathbb{R}^l$. Let F be the face containing x_1. The local complement $M(c\mathcal{A}_{x_1})$ is disconnected and its components correspond to chambers of the arrangement $c\mathcal{A}_x$. Let C be the chamber of $c\mathcal{A}$ containing v_1 and and C_{x_1} be the chamber of $c\mathcal{A}_{x_1}$ containing C. Now for every $y \in F$ the local complement $M(c\mathcal{A}_y)$ contains connected component C_y such that $C \subseteq C_y$. Therefore $v_1 \in C_y$ for every $y \in F$ and $\{(\alpha(t) \cap F, v_1) \mid t \in [0, 1]\}$ is a continuous path in $M(c\mathcal{A})$, call it β_F. Now we have two cases to deal with.

Case 1: Let G be a face such that F covers G and $Im(\alpha) \cap G \neq \emptyset$. As $G \leq C$ in the face poset we have that for every $y \in G$ there is a connected component C_y of $M(c\mathcal{A}_y)$ that contains C. Hence $\{(\alpha(t) \cap G, v_1) \mid t \in [0, 1]\}$ is also a continuous path, denote it by β_G.

Case 2: Let G be a face such that F is covered by G and $Im(\alpha) \cap G \neq \emptyset$. Hence for every $y \in G$ the local complement $M(c\mathcal{A}_y)$ has a component C_y that contains C and $G \circ C$. Let $z \in G \circ C$ and γ_G be a continuous path in C_y joining z and v_1. Let β_G denote the path which is made up of concatenating γ_G with $\{(\alpha(t) \cap G, z) \mid t \in [0,1]\}$ (appropriately).

Continuing this process one can construct a path β, by concatenating the paths β_G (for every face G that intersects with the path α) which joins any two points. Hence $M(c\mathcal{A})$ is path connected. $\square$

We now want to construct an open covering of the space $M(c\mathcal{A})$. First we state some more terminology and results from topological embeddings that we need. A connected codimension 1 submanifold N of a manifold M is *two-sided* if there is a connected open neighborhood of U of N in M such that $U \setminus N$ has exactly two components each of which is open in M. Further, N is said to be *bicollared* in M if it has an open neighborhood homeomorphic to $N \times (-1, 1)$ with N itself corresponding to $N \times \{0\}$. We will use the following theorem originally due to M. Brown in 1964.

Theorem 5.2.3. *Let N be a locally flat, connected, two-sided, codimension 1 submanifold of M. Then N is bicollared in M.*

For every $F \in \mathcal{F}(c\mathcal{A}) \setminus \{\mathbf{0}\}$ let $\tilde{F}$ be the face of $\mathcal{A}$ such that $\tilde{F} = F \cap S^{l-1}$. If σ is the function assigning signs to every face then

$$\begin{aligned} \sigma(F) &= \sigma(\tilde{F}) \qquad \forall F \in \mathcal{F}(c\mathcal{A}) \setminus \{\mathbf{0}\} \\ \sigma(\mathbf{0}) &= (0, \ldots, 0) \end{aligned}$$

As stated before, such a face F is just the cone over $\tilde{F}$, hence homeomorphic to an open polyhedral cone in $\mathbb{R}^l$. For a tope T let V_T denote the corresponding chamber in $\mathbb{R}^l$. For the sign vector $\mathbf{0}$ let $V_{\mathbf{0}}$ be the open unit ball. Let F be a face which is neither a chamber nor $\mathbf{0}$. Let H_F be the support of F and $B(H_F)$ be its bicollar (pseudohyperplanes satisfy the hypothesis of Theorem 5.2.3). Let V_F be the portion of $B(H_F)$ that contains F and intersects only those faces whose closures contain F.

From the above construction it is easy to prove the following lemma which explains properties of these open sets.

Lemma 5.2.4. *With the notation as above, the following statements are true:*

1. *For every $F \in \mathcal{F}(c\mathcal{A})$ the open set V_F contains F and is homeomorphic to $\mathbb{R}^l$.*

2. *If $F \leq F'$ in $\mathcal{F}(\mathcal{A})$ then $V_F \cap V_{F'} \neq \emptyset$ and $F \nsubseteq V_{F'}$.*

3. If F and F' are not comparable in $\mathcal{F}(c\mathcal{A})$ then $V_F \cap V_{F'} = \emptyset$.

For every pair (F, T), where F is a face and T is a tope covering it, define a subset of $M(c\mathcal{A})$ as follows:

$$W(F, T) := V_F \times V_T$$

Theorem 5.2.5. *The collection $\{W(F, T) \mid (F, T) \in \mathcal{F} \times \mathcal{T}, F \leq T\}$ forms an open covering of $M(c\mathcal{A})$ and whenever these open sets intersect the intersection is contractible.*

Proof. Let $(x, v) \in M(c\mathcal{A})$ be any point. Therefore there is some face F such that $x \in F \subseteq V_F$ and some chamber C such that $v \in C \subset M(c\mathcal{A}_x)$. These sets are open and contractible because they are products of open and contractible subsets. The intersections are contractible for the same reasons. □

As the hypothesis for the Nerve Lemma 1.1.6 is satisfied, the nerve of this open covering has the homotopy type of $M(c\mathcal{A})$. We can also deduce the criterion for their intersections to be non-empty as it is needed to identify the simplices.

Lemma 5.2.6.

$$W(F_1, T_1) \cap W(F_2, T_2) \neq \emptyset \iff F_1 \leq F_2 \text{ and } T_2 = F_2 \circ T_1$$

Proof. By construction of these open sets we have,

$$W(F_1, T_1) \cap W(F_2, T_2) = (V_{F_1} \cap V_{F_2}) \times (V_{T_1} \cap V_{T_2})$$

Clearly $V_{F_1} \cap V_{F_2} \neq \emptyset$ if and only if $F_1 \leq F_2$. We also need the other intersection to be nonempty,

$$\begin{aligned} V_{T_1} \cap V_{T_2} \neq \emptyset &\iff T_1 \cap T_2 \neq \emptyset \\ &\iff T_2 = F_2 \circ T_1 \text{ or } T_2 = T_1. \quad \square \end{aligned}$$

Let us first construct the nerve as an abstract simplicial complex.

Definition 5.2.7. Let $\mathcal{L}$ be the set of covectors of an oriented matroid and let $\mathcal{T}$ be the set of all topes. Define a partial order on the set of all pairs (X, T) for which $X \in \mathcal{L}$, and $T \in \mathcal{T}$, by the following rule:

$$(X_2, T_2) \leq_S (X_1, T_1) \iff X_1 \leq X_2 \text{ and } X_2 \circ T_1 = T_2$$

The **Salvetti complex** $Sal(\mathcal{L})$ is the regular cell complex having this poset as its face poset.

The next result is now immediate.

Theorem 5.2.8. *Let $\mathcal{L}$ denote the set of covectors of an oriented matroid and $c\mathcal{A}$ be the associated arrangement of pseudohyperplanes. If $M(c\mathcal{A})$ is the associated space then*

$$M(c\mathcal{A}) \simeq Sal(\mathcal{L})$$

Here are some obvious properties of the Salvetti complex.

Corollary 5.2.9. *With the notation as above we have the following:*

1. *The complex $Sal(\mathcal{L})$ is a MH-complex.*

2. *The number of* 0*-cells of $Sal(\mathcal{L})$ is equal to the number of its l-cells which is also equal to the number of topes of $\mathcal{L}$.*

3. *Every chain in $(\{(X,T) \in \mathcal{L} \times \mathcal{T}\}, \leq_S)$ corresponds to pair consisting of a chain in $(\mathcal{L}, \leq)$ and a tope.*

4. *The geometric realization of $(\mathcal{L}, \leq)$ is a retract of $Sal(\mathcal{L})$.*

5. $\chi(M(c\mathcal{A})) = 0$.

6. *The homeomorphism type of $M(c\mathcal{A})$ is completely determined by the oriented matroid $\mathcal{L}$.*

Proof. The first statement follows from the Theorem 3.3.15. Second statement follows from the construction of the Salvetti complex as described in Section 3.3. The proof of the third statement is same as that of the Lemma 3.2.11. For statements 4 and 5 see Theorem 3.3.6. The last statement is proved in [7]. □

As before we can now define the positive category and the arrangement groupoid. We have already seen that for hyperplane arrangements any two minimal positive paths with same end points are positive equivalent. This is also true for pseudohyperplane arrangements.

Lemma 5.2.10. *An arrangement of pseudohyperplanes is flat.*

Proof. Recall that according to Definition 3.5.7 we need to show that if two minimal positive paths are relative homotopic then they are positive equivalent. Since the ambient space is simply connected we can apply Theorem 3.5.5 to conclude that any two minimal positive paths with same end points are relative homotopic.

For a chamber C let $C^{\#} = \{\mathbf{0}\} * C$, the chamber opposite to C. Let α, β be two minimal positive paths from C to another chamber D. Since $[C, C]$ and $[D, D]$ are vertices of the l-cell $[\mathbf{0}, C]$ the paths α and β are contained in its boundary. By the definition of MH-complex these two paths can be extended to two minimal positive paths from $[C, C]$ to $[C^{\#}, C^{\#}]$ which are certainly positve equivalent. For a proof that only uses oriented matroid arguments, see [16, Theorem 2.4]. $\square$

Let $\mathcal{G}^+$ denote the positive category associated with a pseudohyperplane arrangement. The objects in this category are the chambers and morphisms are equivalence classes of positive paths (the equivalence is generated by declaring two minimal positive paths with same end points to be same). Let $\mathcal{G}$ denote the arrangement (fundamental) groupoid of the associated Salvetti complex. The theory of Salvetti-type diagram models also works in case of oriented matroids (see [23]).

Definition 5.2.11. Given a cover $\rho\colon \mathcal{G}_\rho \to \mathcal{G}$, define a diagram of posets $\mathcal{D}_\rho$ indexed over the dual face poset $(\mathcal{F}^*, \prec)$ such that

$$\mathcal{D}_\rho(F^*) := \{v \in Ob(\mathcal{G}_\rho) | \rho(v) \prec F^*\}$$

endowed with the trivial order relation defined by setting $v_1 \leq v_2$ if and only if $v_1 = v_2$, and maps being inclusions

$$\begin{array}{rcl} \mathcal{D}(F_1^* \to F_2^*)\colon \mathcal{D}(F_1^*) & \longrightarrow & \mathcal{D}(F_2^*) \\ v & \mapsto & t(\rho(v) \to F_2 \circ \rho(v))^{<v>} \end{array}$$

where $(\rho(v) \to F_2 \circ \rho(v))^{<v>}$ is the lift of the minimal positive path $(\rho(v) \to F_2 \circ \rho(v))$ in $\mathcal{G}$ that starts at v.

Following theorem classifies the covering spaces of the Salvetti complex.

Theorem 5.2.12. *(Delucchi) For any topological cover $p\colon S \to Sal(\mathcal{L})$ of the Salvetti complex of a locally finite pseudohyperplane arrangement $c\mathcal{A}$, there exists a cover of the arrangement groupoid $\rho\colon \mathcal{G}_\rho \to \mathcal{G}$ such that the homotopy colimit of the associated diagram of spaces hocolim$\mathcal{D}_\rho$ is isomorphic to S as a covering space of $Sal(\mathcal{L})$.*

As a corollary to the above theorem we have

Corollary 5.2.13. *Let $\hat{\rho}\colon \hat{\mathcal{G}} \to \mathcal{G}$ be the universal cover of $\mathcal{G}$. Then hocolim$\mathcal{D}_{\hat{\rho}}$ is the universal cover of $Sal(\mathcal{L})$.*

We now turn to simplicial arrangements, that is, arrangements in which every chamber is a cone over an open simplex. Alternately, an oriented matroid is simplicial if $\mathcal{L} \setminus \mathbf{0}$ is isomorphic to the face poset of a simplicial decomposition of the sphere (or for every tope T the interval $[\mathbf{0}, T]$ is Boolean). Such an arrangement is an example of a flat, simplicial arrangement with the involution property. Hence we can apply the results proved in Section 3.6 (those are Deligne's original arguments). In this setting Lemma 3.6.6, Lemma 3.6.8 and Theorem 3.6.4 imply the following.

Theorem 5.2.14. *Let $\mathcal{L}$ be an oriented matroid and $c\mathcal{A}$ be the associated pseudohyperplane arrangement. Then the following are equivalent*

1. *$\mathcal{L}$ is simplicial.*
2. *The positive category admits the Deligne normal form.*
3. *The tope poset $\mathcal{P}_T(\mathcal{L})$ is a lattice for every tope T.*

All of the above conditions imply that the space $M(c\mathcal{A})$ is $K(\pi, 1)$.

Proof. Here the tope poset is defined similarly as the poset of chambers was defined before Lemma 3.5.12. $1 \Rightarrow 2$ is originally due to Deligne [22] (and a reproof by Paris [66]); both these proofs are for realizable oriented matroids. For non-realizable oriented matroids the proof was given by Cordovil [15, Theorem 4.1] and by Salvetti [74, Theorem 33]. $2 \Rightarrow 1$ is due to Paris [67]. For $1 \iff 3$ see [5] and the proof of $3 \iff 1$ is in Delucchi's thesis [23, Theorem 6.4.6, Lemma 6.5.2]. $\square$

Obvious examples of arrangements that were not covered by Deligne's theorem are the simplicial arrangements of pseudolines. A simplicial arrangement of pseudolines in $\mathbb{R}P^2$ consists of a finite family of simple closed curves such that every two curves have precisely one point in common and every 2-face is isomorphic to a triangle. By applying the coning process we get an arrangement of (non-stretchable) pseudoplanes in $\mathbb{R}^3$ whose face poset correspond to a rank 3 non-realizable oriented matroid. The Salvetti complex associated to such oriented matroids is a $K(\pi, 1)$ space. In fact there are at least seven infinite families of non-stretchable simplicial arrangement of pseudolines are known, see [40, Chapter 3] for details and examples of such arrangements.

Chapter 6

The Topological Dissection Problem

The aim of this chapter is to study the classical problem of determining the number of pieces into which a certain geometric set is divided by a given collection of subsets. In our context this problem boils down to counting the number of faces of a submanifold arrangement. This problem has a long history in combinatorial geometry. In 1826, Steiner considered the problem of counting the pieces of a plane cut by a finite collection of lines, circle etc. In 1901, Schläfli obtained a formula for counting the number of regions in a Euclidean space when it is cut by hyperplanes in general position. Subsequently many mathematicians studied various aspects and generalizations of this problem. We refer to [40, Chapter 18] and [41] for more history related to this problem.

As pointed out in Chapter 1, for real hyperplane arrangements Zaslavsky [92] discovered a face counting formula involving the Möbius function of the intersection lattice. Our aim is to generalize his formula to the case of submanifold arrangements. Our result is motivated by the techniques used in [31], where the authors generalize Zaslavsky's formula for toric arrangements. During a discussion with Thomas Zaslavsky about this generalization, he directed us to his paper [93] in which he proves a much more general result. The name of this chapter is borrowed from the title of that paper. We would like to point out that though we have proved similar results, the techniques used are different.

In Section 6.1 we revise the theory of valuations on a poset and the Euler characteristic, since they are the main ingredients of our proof. Then, in Section 6.2 after introducing a generalization of the characteristic polynomial, we will establish a formula that combines the geometry and combinatorics of the intersections in order to count the number of chambers. We compare Zaslavsky's proof in [93] with ours. Finally in Section 6.3 we look at some particular cases of manifolds and comment about the f-vectors arising due to submanifold arrangements.

6.1 Valuations on a Lattice

In his attempt to classify the convex polyhedra in 3-space, Leonhard Euler discovered that the number of vertices of a polyhedron minus the number of its edges plus the number of its faces is an invariant. He published a proof of this result in 1758 and also conjectured that the result is true for higher dimensional polytopes. The invariant popularly known as the *Euler characteristic* not only appears in many branches of mathematics but plays an important role in those areas. A century later Schläfli proved the *Euler relation* for polytopes of higher dimensions and Poincaré, using homology theory, extended the result to manifolds. For more on the history of the Euler characteristic see [40, Chapter 8] and [28].

Our focus here is the combinatorial nature of this invariant. In 1955 Hadwiger [43] characterized the Euler characteristic as the unique translation-invariant, finitely additive set function defined on finite unions of compact convex subsets of $\mathbb{R}^n$. Inspired by Hadwiger's work, Victor Klee gave a relatively elementary proof of the Euler formula for polytopes in [49]. Motivated by these results Rota [71] established a combinatorial connection between the Euler characteristic and underlying order-theoretic structure. His work revealed that the Euler characteristic can be thought of as a fundamental dimension-less invariant, associated with any mathematical structure, that can be defined in much more general context. For example, in case of a finite set the Euler characteristic is its cardinality. Generalizing this basic idea, Schanuel [76] showed that the Euler characteristic of certain polyhedra is determined by a simple universal property. A further generalization is achieved by Leinster in [52] by defining the Euler characteristic for finite categories.

The unique universal property of the Euler characteristic will be used to generalize Zaslavsky's result. We will devote this section to the introduction of the theory of valuations on lattices and will also explain how the Euler characteristic is defined combinatorially. The main references for this introductory material are [38], [71] and [48, Chapter 2]. We will assume that the reader is familiar with the basic facts about posets and lattices ([81, Chapter 3] is a good reference).

Let D be a family of subsets of a set S such that D is closed under finite unions and finite intersections. Such a family is a distributive lattice in which the partial ordering is given by the inclusion of subsets while the meet and join are defined by intersection and union of subsets, respectively. All of the following theory holds true for arbitrary distributive lattices but we will state it in the context that best suits our purpose. Let R be a commutative ring with 1.

Definition 6.1.1. An *R-valuation* on D is a function $\nu\colon D \to R$, satisfying

$$\nu(A \bigcup B) = \nu(A) + \nu(B) - \nu(A \cap B) \quad (6.1.1)$$
$$\nu(\emptyset) = 0 \quad (6.1.2)$$

By iterating the identity (6.1.1) we get the *inclusion-exclusion principle* for ν, namely

$$\nu(A_1 \bigcup \cdots \bigcup A_n) = \sum_i \nu(A_i) - \sum_{i<j} \nu(A_i \cap A_j) + \sum_{i<j<k} \nu(A_i \cap A_j \cap A_k) + \cdots \quad (6.1.3)$$

The above definition is clearly similar to that of a measure on a Boolean algebra. But the theory of valuations is in some sense richer, for example, the Euler characteristic (which we will show is a valuation) has no counterpart in measure theory. In functional analysis a measure is regarded either as a linear functional or as an abstract integral on a function space. Drawing parallels with this aspect, Rota [71] defined a ring for distributive lattices called the *valuation ring* (see Definition 6.2.9 below), denoted by $V(D, R)$ and identified R-valuations on D with R-valued functionals on $V(D, R)$. Moreover when the distributive lattice is finite any valuation can be uniquely determined by the following theorem.

Theorem 6.1.2 (Rota [71])**.** *A valuation on a finite distributive lattice D is uniquely determined by the values it takes on the set of join-irreducible elements of D, and these values can be arbitrarily assigned.*

With this theorem we are now in a position to define the Euler characteristic.

Definition 6.1.3. The *Euler characteristic* of a finite distributive lattice D is the unique valuation χ such that $\chi(x) = 1$ for all join-irreducible elements x and $\chi(\hat{0}) = 0$.

Before moving on let us look at some examples.

Example 6.1.4. Let S be a finite set and D be the lattice consisting of all abstract simplicial complexes defined on the power set of S. In this case the the Euler characteristic defined above coincides with the classical definition. Also note that the join-irreducible elements of D are the simplices (simplicial complex with exactly one maximal element) and their Euler characteristic is exactly 1. Moreover, for $A \in D$ if f_i denotes the number of cardinality i subsets, then

$$\chi(A) = \sum_{i \geq 0} (-1)^i f_i.$$

The above example justifies the name Euler characteristic and the next examples show how it is defined in a different context and for infinite lattices.

Example 6.1.5. Let D denote the lattice of positive integers, ordered by divisibility. Then, using elementary facts about the Möbius function, one can show that for a positive integer n its Euler characteristic $\chi(n)$ is equal to the number of distinct prime divisors of n.

Example 6.1.6. Let D be the lattice generated by n-dimensional polytopes in $\mathbb{R}^n$. In this lattice the join-irreducible elements are the compact convex polytopes and the unique $\mathbb{R}$-valuation that takes value 1 on them is the (classical) Euler characteristic. Moreover if we consider the lattice generated by all relative interiors of convex n-polytopes then the definition extends to polytopal complexes and non-closed convex sets. For the proof and other details see [48, Chapter 5] and [77, Chapter 3].

Just like measure, a valuation on a lattice can be used to construct abstract integrals. Since integrals with respect to a valuation will be used to generalize the characteristic polynomial we will briefly sketch some important facts. From now on let D be a finite distributive lattice consisting of subsets of a finite set S.

A D-*simple* function f is a finite linear combination

$$f = \sum_{i=1}^{k} r_i I_{A_i} \tag{6.1.4}$$

where $r_i \in R$ and $I_{A_i} \colon S \to \{0, 1\}$ are the *indicator (or characteristic) functions* with $A_i \in D$ for $1 \leq i \leq k$. The set of all D-simple functions forms a ring under point wise addition and multiplication. Note that unlike the valuations, these simple functions are defined on S.

A subset L of D is called a *generating set* if it is closed under finite intersections and if every element of D can be expressed as a finite union of members of L. Using the inclusion-exclusion formula for the indicator functions it can be shown that every D-simple function can be rewritten as a linear combination

$$f = \sum_{i=1}^{m} s_i I_{B_i} \tag{6.1.5}$$

where each $B_i \in L$. An R-valued function ν on L is called a *valuation* on L provided that ν satisfies the identities (6.1.1) and (6.1.2) for all sets $A, B \in L$ such that $A \cup B \in L$. For a D-simple function f, define the *integral* of f with respect to ν as

$$\int f d\nu = \sum_{i=1}^{m} s_i \nu(B_i) \tag{6.1.6}$$

The existence of the extension of ν to D and also the existence of the integral are equivalent properties of ν. This nontrivial fact is stated as the following -

Theorem 6.1.7. ***(Groemer's Integral Theorem)*** *Let L be a generating set for a lattice D, and let ν be a valuation on L. Then the following are equivalent :*

1. *ν extends uniquely to a valuation on D.*

2. *ν satisfies the inclusion-exclusion identities*

$$\nu(B_1 \bigcup \cdots \bigcup B_n) = \sum_i \nu(B_i) - \sum_{i<j} \nu(B_i \cap B_j) + \sum_{i<j<k} \nu(B_i \cap B_j \cap B_k) + \cdots \tag{6.1.7}$$

   whenever $B_i \in L \forall i$ and $B_1 \bigcup \cdots \bigcup B_n \in L$, for all $n \geq 2$.

3. *ν defines an integral on the R-algebra of D-simple functions.*

Proof. See [48, Theorem 2.2.7] □

6.2 The Chamber Counting Formula

Let $\mathcal{A}$ be an arrangement of submanifolds of a smooth l- manifold X. The problem at hand is to count the number of connected components of the complement $X \setminus \bigcup_{N \in \mathcal{A}} N$, which will be denoted by $|\mathcal{C}(\mathcal{A})|$. Let D be the lattice of sets generated by the intersection poset $L(\mathcal{A})$ and the members of $\mathcal{C}(\mathcal{A})$ through finite unions and finite intersections. Recall that to count the number of chambers of a hyperplane arrangement, we use the *characteristic polynomial* of the intersection lattice (Theorem 1.2.5). We will start by generalizing this polynomial.

Define the *Poincaré polynomial with compact support* of a topological space A as

$$Poin_c(A, t) := \sum_{i \geq 0} rank(H_c^i(A, \mathbb{Z}))t^i$$

where H_c^i is the cohomology with compact supports.

Lemma 6.2.1. *The function $\nu \colon D \to \mathbb{Z}[t]$ defined by $\nu(A) = Poin_c(A, t)$, $\forall A \in D$ is a valuation on D.*

Proof. The first step is to find a generating set for D. Let G denote the union of all faces of $\mathcal{A}$ and the empty set $\emptyset$. Clearly the collection G is closed under intersections. Each member of $L(\mathcal{A})$ is a union of finitely many faces and as all the chambers are faces any finite union (or intersection) of members of $L(\mathcal{A})$ and $\mathcal{C}(\mathcal{A})$ can be expressed as a finite union of faces. This observation proves that G is a generating set for D.

Now we have to show that ν defines a valuation on G. This is clear because $\nu(\emptyset) = 0$ and all the non-empty faces are disjoint, open topological cells. For the same reasons the inclusion-exclusion identities (6.1.7) are satisfied by ν. Hence as a consequence of *Groemer's integral theorem* 6.1.7, ν extends to whole of D. □

For each $Y \in L$ $(= L(\mathcal{A}))$, define

$$f(Y) = Y \setminus \bigcup_{\substack{Y<Z \\ Z \in L}} Z$$

Then $\{f(Y) \mid Y \in L\}$ is the set of all faces of the arrangement, in particular it is a disjoint collection. Hence

$$\begin{aligned} X &= \coprod_{\substack{X \leq Y \\ Y \in L}} f(Y) \\ \text{so} \quad I_X &= \sum_{X \leq Y} I_{f(Y)} \\ \text{and} \quad I_{f(X)} &= \sum_{X \leq Y} \mu(X,Y) I_Y \quad \text{(by Möbius inversion)} \end{aligned} \tag{6.2.1}$$

Note that $f(X)$ is in fact the union of all the chambers, therefore $I_{f(X)}$ is a D-simple function. As ν extends uniquely to a valuation on D, it defines an integral on the algebra of D-simple functions. Integrating $I_{\mathcal{C}(\mathcal{A})}$ with respect to ν, we get

$$\begin{aligned} \int I_{\mathcal{C}(\mathcal{A})} d\nu &= \sum_{Y \in L} \mu(X,Y) \nu(Y) \\ &= \sum_{Y \in L} \mu(X,Y) Poin_c(Y,t) \end{aligned} \tag{6.2.2}$$

Definition 6.2.2. Let $\mathcal{A}$ be an arrangement of submanifolds of a l-manifold X and L be the associated intersection poset. The *generalized characteristic polynomial* of $\mathcal{A}$ is

$$\chi(\mathcal{A},t) := \sum_{Y \in L} \mu(X,Y) Poin_c(Y,t)$$

Note the unfortunate clash of notations.

Definition 6.2.3. The *combinatorial Euler characteristic* κ of a finite CW complex P is defined as

$$\kappa(P) = \begin{cases} \chi(\hat{P}) - 1 & \text{if } P \text{ is not compact} \\ \chi(P) & \text{if } P \text{ is compact} \end{cases}$$

where $\chi(\hat{P})$ is the Euler characteristic of the one-point compactification of P.

Note that this is not a new notion, this is just a topological description of Definition 6.1.3. To give an intrinsic topological description of the combinatorial Euler characteristic for arbitrary spaces is not an easy job. The theory of o-minimal structures has to be used in order to define valuations and integrals on arbitrary spaces, which is beyond the scope of this chapter. However, the above notion is a topological invariant and it satisfies the Euler relation, that is the number of even dimensional cells minus the number of odd dimensional cells is equal to the κ value. The following lemma is now clear.

Lemma 6.2.4. *The combinatorial Euler characteristic κ defines a $\mathbb{R}$-valuation on D.*

Using all of the theory developed so far we can now generalize Zaslavsky's theorem.

Theorem 6.2.5. *Let $\mathcal{A}$ be an arrangement of submanifolds in an l-manifold X (that is $\mathcal{A}$ subdivides the manifold into chambers homeomorphic to open l-dimensional balls). Then the number of chambers is given by $(-1)^l \sum_{Y \in L} \mu(X,Y)\kappa(Y)$, where μ is the Möbius function on $L \times L$ and κ is the combinatorial Euler characteristic.*

Proof. First note that κ and $\nu|_{t=-1}$ agree on every element of G. Consequently, they also agree on every member of D. Hence, we have that

$$\kappa(\mathcal{C}(\mathcal{A})) = \int I_{\mathcal{C}(\mathcal{A})} d\kappa = \int I_{\mathcal{C}(\mathcal{A})} d\nu|_{t=-1} \tag{6.2.3}$$

The set $\mathcal{C}(\mathcal{A})$ is a disjoint union of chambers, each of which is homeomorphic to an open ball of dimension l. Consequently, the combinatorial Euler characteristic of a chamber is $(-1)^l$, substituting this in the equation 6.2.3, we get

$$\begin{aligned}
|\mathcal{C}(\mathcal{A})| &= (-1)^l \kappa(\mathcal{C}(\mathcal{A})) \\
&= (-1)^l \int I_{\mathcal{C}(\mathcal{A})} d\nu|_{t=-1} \\
&= (-1)^l \sum_{Y \in L} \mu(X,Y) Poin_c(Y,-1) \\
&= (-1)^l \sum_{Y \in L} \mu(X,Y)\kappa(Y) \qquad (6.2.4)
\end{aligned}$$

□

Corollary 6.2.6. *(**Zaslavsky** [92]) Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{R}^n$. The number of chambers of this arrangement is equal to $(-1)^n \chi(\mathcal{A}, -1)$.*

Proof. Every member of the intersection poset in this case is homeomorphic to an open ball, hence $Poin_c(Y,t) = t^{\dim Y}$ for every $Y \in L(\mathcal{A})$. The result follows from the observation that $\int I_{\mathcal{C}(\mathcal{A})} d\nu = \chi(\mathcal{A}, t)$. □

Corollary 6.2.7. ***(Ehrenborg et al.*** *[30]) For a toric hyperplane arrangement $\mathcal{A}$ in a torus T^n that subdivides the torus into open n-dimensional balls, the number of chambers is given by*

$$(-1)^n \sum_{dim\ Y=0} \mu(T^n, Y)$$

Proof. Note that toric hyperplanes are homeomorphic to T^{n-1} and they intersect into lower dimensional subtori. Hence the elements of the intersection poset L are tori of some finite dimension, except for the coatoms that are points. If $Y \in L$ is k-dimensional torus then $Poin_c(Y,t) = (1+t)^k$.

$$\begin{aligned}
\int I_{\mathcal{C}(\mathcal{A})} d\nu &= \sum_{1 \leq \text{ dim } Y \leq n} \mu(T^n, Y)(1+t)^{\text{dim } Y} + \sum_{\text{dim } Y=0} \mu(T^n, Y) \\
\Rightarrow |\mathcal{C}(\mathcal{A})| &= (-1)^n \sum_{\text{dim } Y=0} \mu(T^n, Y)
\end{aligned}$$

□

Though the Theorem 6.2.5 is stated for submanifold arrangements it is valid in a more general context. The only thing we have used in the proof is that the combinatorial Euler characteristic is a valuation. Now consider a more general situation where X is a topological space and $\mathcal{A}$ is a finite collection of subspaces that are removed from X. Let L denote the poset consisting of X and connected components of the all possible finite intersections of members of $\mathcal{A}$ ordered by reverse inclusion. The topological dissection problem asks whether it is possible to express the number of connected components of the complement in terms of L (dissection of X intuitively means that X is expressed as a union of pairwise disjoint subspaces). Let $\{C_1, \dots, C_m\}$ denote the connected components of the complement of union of members of $\mathcal{A}$, in this context Theorem 6.2.5 takes the following form

Theorem 6.2.8. *If the combinatorial Euler characteristic κ is a valuation on the lattice of sets generated by $L \bigcup \{C_1, \dots, C_m\}$ then,*

$$\sum_{j=1}^{m} \kappa(C_j) = \sum_{Y \in L} \mu(X, Y) \kappa(Y)$$

The above statement is referred to as the *fundamental theorem of dissection theory* in [93, Theorem 1.2]. In a nutshell, the number of connected components of the complement (combinatorial Euler characteristic of the complement, to be precise) depend on a condition on the intersections and that condition turns out to be just the Euler relation. Moreover the combinatorial Euler characteristic is a valuation if and only if each face of the dissection is a finite, disjoint union of open topological cells (see [93, Lemma 1.1]). Some authors have also considered more general types of arrangements. For example, see [63, 64] where arrangements of topological spheres having homologically trivial chambers are studied.

The approach we took to prove Theorem 6.2.5 can be traced back to a paper of Blass and Sagan [8, Theorem 2.1] where they show how to evaluate the characteristic polynomial of subarrangements of the type B braid arrangement. Ehrenborg and Readdy [31] generalized this work and used it to determine the characteristic polynomial of any subspace arrangement defined over an infinite field. They explicitly used the Gromer's integral theorem to prove that the characteristic polynomial is a valuation. While studying toric arrangements with M. Slone [30] they generalized the previous result and proved the above mentioned Corollary 6.2.7. The idea of looking at the Euler characteristic as an integral of indicator functions is due to Chen [13], see also [14].

Finally we compare our strategy with that of Zaslavsky's, used to prove the fundamental theorem of dissection theory in [93]. In order to do this we will briefly sketch the outline of his proof. At the very foundation of both strategies lies the idea of using the combinatorial Euler characteristic as a valuation. In order to implement this idea we have used Möbius inversion whereas Zaslavsky has used a technical property of valuations. But before that a few definitions.

Definition 6.2.9. Let D be a distributive lattice and R be a unitary ring. Let $M(D,R)$ denote the free R-algebra whose basis is the elements of D and the multiplication on the basis elements is defined by setting $xy = x \wedge y$ and then extended by linearity. In this algebra the set $N(D,R)$ of all linear combinations of elements of the form $x \vee y + x \wedge y - x - y$ is an ideal. The *valuation ring* of D over R is defined to be the quotient $M(D,R)/N(D,R)$ and denoted by $V(D,R)$.

Definition 6.2.10. Let P be a finite poset and R be a unitary ring. The *Möbius algebra* $M(P,R)$ of P is the free R-module whose basis is the elements of P, with a product defined by

$$xy := \sum_{t|\ t\leq x,y} e_t(P), \quad \forall x,y \in P$$

and extended to $M(P,R)$ by linearity, where we define

$$e_t(P) := \sum_{s\in P} \mu_P(s,t)s, \quad \forall t \in P.$$

Note that the elements of the type $e_t(P)$ are orthogonal idempotents in $M(P,R)$ and they also form a basis of this algebra. Moreover for a distributive lattice the two definitions of $M(D,R)$ agree. Zaslavsky proves a theorem that establishes a relationship between the valuation ring of D and the canonical idempotents $e_t(P)$ of a subset of D. Instead of stating this theorem we will state an important and relevant consequence.

Theorem 6.2.11. *Let ϕ be a valuation of the finite distributive lattice D, and let P be a subset of D containing $\hat{0}$ and every join-irreducible element. Then, for any $t \in P$ which is not $\hat{0}$ or a join-irreducible element of D,*

$$\sum_{s \in P; s \leq t} \mu_P(s,t)\phi(s) = 0.$$

In order to apply this theorem to dissection theory note that D is the lattice of sets generated by the intersection poset and all the chambers. Theorem 6.2.8 then follows once we use the valuation κ.

6.3 Faces of an Arrangement

In this section we will use formula (6.2.4) to find the number of various dimensional faces of an arrangement. We start with a definition.

Definition 6.3.1. The *f-vector* of an arrangement of submanifolds in a l-dimensional manifold is the vector

$$f = (f_0, f_1, \ldots, f_l) \in \mathbb{N}^{l+1}$$

where f_k denotes the number of k-dimensional faces of the arrangement.

Theorem 6.2.5 can be used to count the number of faces (of all dimensions) of an arrangement. Note that these faces are the chambers of the restricted arrangements defined as follows. For $Y \in L(\mathcal{A})$, the arrangement restricted to Y is

$$\mathcal{A}^Y := \{N \cap Y | N \in \mathcal{A} \text{ and } \emptyset \neq N \cap Y \neq Y\}$$

Theorem 6.3.2. *Let X be a smooth, real manifold of dimension l and $\mathcal{A}$ be an arrangement of submanifolds. Then the numbers f_k, are given by*

$$f_k = \sum_{dim\ Y=k} (-1)^k (\sum_{\substack{Z \in L \\ Y \leq Z}} \mu(Y,Z)\kappa(Z)) \tag{6.3.1}$$

Proof. As $\mathcal{F}(\mathcal{A}) = \{\mathcal{C}(\mathcal{A}^Y) \mid Y \in L\}$ the number of k-faces of $\mathcal{A}$ is given by

$$f_k = \sum_{\substack{Y \in L(\mathcal{A}) \\ \dim\ Y=k}} |\mathcal{C}(\mathcal{A}^Y)|$$

Use (6.2.4) to substitute for $|\mathcal{C}(\mathcal{A}^Y)|$ in the above formula and note that

$$L(\mathcal{A}^Y) = \{Z \in L(\mathcal{A}) \mid Y \leq Z\}$$

By convention, these f_k's are also expressed as the coefficients of the following *generating polynomial*

$$\sum_{k=0}^{l} f_k x^{l-k} = (-1)^l \sum_{Z \in L(\mathcal{A})} \kappa(Z) \sum_{Y \leq Z} \mu(Y,Z)(-x)^{l-dim(Y)}$$

□

We have already seen in the Lemma 3.1.14 that the local picture at each face of the arrangement of submanifolds is that of a hyperplane arrangement. The following lemma is a detailed proof of this fact from a combinatorial viewpoint.

Lemma 6.3.3. *Let $\mathcal{A}$ be an arrangement of submanifolds in a l-manifold X. Then every interval of the intersection poset $L(\mathcal{A})$ is a geometric lattice.*

Proof. Consider an interval $[Y,Z]$ in $L(\mathcal{A})$, such that $\dim Y = i$ and $\dim Z = j$. There exists an open set V in X and a coordinate chart ϕ such that $\phi(V \cap Y)$ is homeomorphic to an i-dimensional subspace of $\mathbb{R}^l$. Moreover, $\{\phi(N \cap V) | N \in \mathcal{A}^Y\}$ is a central arrangement of hyperplanes in $\mathbb{R}^i$. For any $W \in [Y,Z]$ the subspace $\phi(W \cap V)$ is homeomorphic to a subspace of $\mathbb{R}^i$ that contains $\phi(Z \cap V)$. In particular, the $(i-1)$-dimensional subspaces in $[Y,Z]$ map to hyperplanes in $\mathbb{R}^i$ that contain $\phi(Z \cap V)$. This correspondence gives us an essential central arrangement of hyperplanes in $\mathbb{R}^{i-j}$ when we quotient out $\phi(Z \cap V)$. This correspondence is also a poset isomorphism and hence $[Y,Z]$ is a geometric lattice. Moreover for geometric lattices the the Möbius function alternates in sign and consequently $(-1)^{dim\ Y - dim\ Z}\mu(Y,Z) > 0$ (see [81, Proposition 3.10.1]). □

We will say that the submanifolds X are in general position if intersection of any i of the submanifolds, $i \geq 1$, is either empty or $(l-i)$-dimensional. In the case of general position arrangements, every interval of $L(\mathcal{A})$ is a Boolean algebra hence the rank generating function for an interval $[Y,Z]$ is $(x+1)^{l-dimZ}$ and also $\mu(Y,Z) = (-1)^{dimY - dimZ}$ (see [81, Example 3.8.3]). Now we will compute f-vectors for some particular examples of arrangements. In each of the following examples we substitute the appropriate values of κ in the equation 6.3.1 and also use the above stated facts regarding the Möbius function. For similar calculations also see [30, 93].

Example 6.3.4. Let $\mathcal{A}$ be an arrangement of submanifolds in l-manifold X. Let $L(\mathcal{A})$ denote the intersection poset of this arrangement let a_k denote the number of k-dimensional members of this poset.

1. If $X \cong \mathbb{R}^l$ and $\mathcal{A}$ is an arrangement of hyperplanes then

$$f_k = \sum_{dim\ Y=k} \sum_{Y \leq Z} |\mu(Y,Z)|$$

If the hyperplanes are in general position then

$$f_k = \sum_{j=0}^{k} a_j \binom{l-j}{l-k}.$$

2. For an arrangement of codimension 1 subtori on a torus of dimension l,

$$f_k = \sum_{\substack{dim\ Y=k \\ dim\ Z=0 \\ Y \leq Z}} |\mu(Y,Z)|$$

If the subtori are in general position then the number f_l takes the following simpler form

$$f_k = a_0 \binom{l}{l-k}.$$

3. If $X \cong S^l$ an l-dimensional sphere and all the intersections are lower dimensional spheres then

$$f_k = 2 \sum_{dim\ Y=k} \sum_{\substack{Y \leq Z \\ dim\ Z \geq 2 \\ even}} |\mu(Y,Z)| + \sum_{dim\ Y=k} \sum_{\substack{Y \leq Z \\ dim Z=0}} |\mu(Y,Z)|$$

If all of the hyperspheres are in general position

$$f_k = 2 \sum_{j=2,even}^{k} a_j \binom{l-j}{l-k} + a_0 \binom{l}{k}.$$

4. Let X be the l-dimensional projective space and $\mathcal{A}$ be an arrangement of $l-1$-dimensional projective space

$$f_k = \sum_{dim Y=k} \sum_{\substack{Y \leq Z \\ dim Z \geq 0 \\ dim Z even}} |\mu(Y,Z)|$$

If all the $l-1$ projective spaces are in general position

$$f_k = \sum_{j=0,even}^{k} a_j \binom{l-j}{l-k}$$

We end this section by describing a relationship between the intersection poset and the face poset, which extends a result due to Bayer and Sturmfels [3] for hyperplane arrangements. Let $\mathcal{F}^*$ be the dual of the face poset, define the map $\psi\colon \mathcal{F}^* \to L(\mathcal{A})$ by sending each face to the smallest dimensional subspace in the intersection poset that contains the face. For oriented matroids this is the support map going to its underlying matroid. The map ψ is order and rank preserving, as well as surjective hence we will look at it as a map from the set of chains of $\mathcal{F}^*$ to the set of chains of $L(\mathcal{A})$.

Theorem 6.3.5. *Let $c = \{Y_1 \leq Y_2 \leq \cdots \leq Y_k\}$, $k \geq 2$, be a chain in the intersection poset $L(\mathcal{A})$ of an arrangement of submanifolds $\mathcal{A}$. Then the cardinality of the inverse image of the chain c under the map ψ is given by the following formula*

$$|\psi^{-1}(c)| = \prod_{i=1}^{k-1}\Big(\sum_{Y_i \leq Z \leq Y_{i+1}} (-1)^{l-dimZ}\mu(Y_i, Z)\Big) \cdot |\mathcal{C}(\mathcal{A}^{Y_k})|$$

Proof. The arguments are similar to the proof of [30, Theorem 3.13]. The number of ways of selecting a face F_k such that $\psi(F_k) = Y_k$ is equal to the number of chambers of $\mathcal{A}^{Y_k}$. A face F_{k-1} is in $\psi^{-1}(Y_{k-1})$ if it is a chamber in the arrangement $\mathcal{A}^{Y_{k-1}}$ and also contains the face F_k. The number of such faces is equal to the number of chambers in the central hyperplane arrangement whose intersection lattice is isomorphic to $[Y_{k-1}, Y_k]$. By repeating this process for all the subspaces up to Y_1 we get the desired formula. □

Chapter 7

Conclusions and Future Work

In the final chapter we summarize our work and also mention some open questions that outline future directions.

7.1 Conclusions and Summary

The main motivation behind this research work is to generalize some aspects of the theory of hyperplane arrangements. In particular we study arrangements of codimension 1 submanifolds. The aim is to obtain a better understanding of the interaction between combinatorics and topology in this new context. We show that, under these new conditions, most of the well known constructions (and results) in Hyperplane Arrangements can be generalized to submanifold arrangements.

The most general case in which one considers arbitrary submanifolds and allows all possible intersections is indeed very difficult to deal with. Hence we introduce some restrictions that are described in the definition below (see Section 3.1 for details).

Definition 7.1.1. Let X be a smooth, real manifold of dimension l. An *arrangement of submanifolds* is a finite collection $\mathcal{A} = \{N_1, \dots, N_k\}$ of codimension 1 submanifolds of X such that

1. The N_i's intersect transversally.
2. $X \setminus N_i$ has exactly two connected components for every i.
3. The intersections of N_i's define a regular CW decomposition of X.

Intuitively it means that for every point on the manifold there is a coordinate neighborhood homeomorphic to a hyperplane arrangement. Moreover, the arrangement stratifies the manifold into open, contractible pieces. The complement of this arrangement inside the manifold is disconnected. For examples other than hyperplane arrangements, consider arrangement of points on a circle or arrangement of circles on a 2-sphere.

In order to generalize the notion of the complexified complement, let us first forget the complex structure on $\mathbb{C}^l$, then as a topological space it is homeomorphic to $\mathbb{R}^{2l}$ which also happens to be the tangent bundle of $\mathbb{R}^l$. Hence the complexified complement is also a complement inside the tangent bundle. This point of view has a potential to generalize, as the following definition suggests.

Definition 7.1.2. Let X be a l-dimensional manifold and $\mathcal{A} = \{N_1, \ldots, N_k\}$ be an arrangement of submanifolds. Let TX denote the tangent bundle of X and $T\mathcal{A} := \bigcup_{i=1}^{k} TN_i$. The *tangent bundle complement* of the arrangement $\mathcal{A}$ is defined as

$$M(\mathcal{A}) := TX \setminus T\mathcal{A}.$$

Similar to hyperplane arrangements one can define the face poset (denoted by $\mathcal{F}(\mathcal{A})$) and the intersection poset (denoted by $L(\mathcal{A})$) in this new setting (see Definitions 3.1.9 and 3.1.8 respectively). Elements of the face poset are called faces, codimension 0 faces are called chambers. The main theme of investigation is:

To what extent do the face poset and intersection poset determine the topology of the tangent bundle complement?

Here is a summary of results obtained while investigating this question.

Generalization of the Salvetti complex

Let us first look at the tangent bundle complement. We prove that it has the homotopy type of a finite dimensional simplicial complex (see Theorem 3.2.7). This can be considered as the main theorem of this thesis. The key is to identify an open covering of this space such that it is finite, each open set is contractible and also their intersections are contractible. Moreover, construction of each of these open sets depends only on the incidence relations in the face poset. As a result, using Nerve lemma (Theorem 1.1.6) we can identify a simplicial complex whose simplices are indexed by face-chamber pairs and that has the homotopy type of the tangent bundle complement. The scheme of this proof is analogous to the work of Paris in [68], where he gives a different proof of Salvetti's seminal result and this is the reason the simplicial complex is called the Salvetti complex. Following Salvetti [73], we also describe a regular cell structure of the tangent bundle complement whose barycentric subdivision is the above stated simplicial complex. This is done using metrical-hemisphere complexes (see Section 3.3 and Subsection 3.3.1 in particular). The combinatorial structure of the Salvetti complex is also studied in detail. We also show that the Salvetti complex is a homotopy colimit of certain diagrams of spaces (Theorems 3.2.16 and 3.3.22).

Covering spaces of the tangent bundle complement

Building on the work of Salvetti and Paris, Delucchi studied the Salvetti complex (in the context of real hyperplane arrangements) and its connected covers using methods from homotopy theory in his thesis, see [23, Chapter 4] and [24]. To be precise Delucchi described diagrams of spaces indexed over the face poset of an arrangement such that the homotopy colimit of a diagram is some connected cover of the Salvetti complex. This result is generalized for the tangent bundle complements (Theorem 3.4.12). The homotopy-theoretic tools for combinatorial applications developed in the remarkable paper of Welker, Ziegler and Živaljević [89] are extensively used to prove these results (see Chapter 2 for details).

Homotopy Groups

A presentation for the fundamental group of the tangent bundle complement is obtained in Corollary 3.5.6. A class of submanifold arrangements for which the word problem (for the fundamental group) is solvable is identified in Theorem 3.5.16. These results are proved using the path category and the fundamental groupoid of the Salvetti complex. Inspired by the seminal work of Deligne we also obtain a mild generalization of his main result in Theorem 3.6.4.

Cohomology groups of the tangent bundle complement

One of the benefits of using the language of homotopy colimits is that some machinery from homotopy theory can be applied. For example, if a topological space is expressed as a homotopy colimit then (in principle) its cohomology can be computed using the Bousfield-Kan spectral sequence. Since the Salvetti complex is expressed as a homotopy colimit, we can apply the Bousfield-Kan spectral sequence. An explicit description of the differentials on the E_1 page is obtained, with the help of which terms on the E_2 page are calculated. It is observed that in most of the examples the spectral sequence collapses on the second page (see Section 3.7).

Arrangements of spheres

As a particular example of the theory developed so far, we study the arrangements of hyperspheres and obtain an explicit description for the homotopy type of the tangent bundle complement. These arrangements are defined as follows.

Definition 7.1.3. Let S^l denote the unit sphere in $\mathbb{R}^{l+1}$. An arrangement of spheres in S^l is a finite collection $\mathcal{A}$ of codimension 1 sub-spheres obtained by intersecting with a central and essential hyperplane arrangement in $\mathbb{R}^{l+1}$.

The result that describes the homotopy type of the tangent bundle complement for these types of arrangements is (see Lemma 4.1.9 and Theorem 4.1.12):

Theorem 7.1.4. *Let $\mathcal{A}$ be an arrangement of hyperspheres in S^l. Then there exists a generic equator which cuts the sphere into two open disks D^l_+ and D^l_- such that $\mathcal{A}_+ := \mathcal{A}|D^l_+$ and $\mathcal{A}_- := \mathcal{A}|D^l_-$ are (combinatorially isomorphic) hyperplane arrangements in $\mathbb{R}^l$. Without loss of generality, let $m = |\mathcal{C}(\mathcal{A}_-)|$. Then,*

$$M(\mathcal{A}) \simeq (\bigvee_m S^l) \bigvee Sal(\mathcal{A}_+)$$

where $|\mathcal{C}(\mathcal{A})|$ denotes the number of chambers of an arrangement.

All the above results are also extended to arrangements of projective spaces that arise due to antipodal action.

Arrangements of pseudohyperplanes

Finally we also study arrangements of pseudohyperplanes (topologically tame hyperplanes). Oriented matroids are intimately connected to hyperplane arrangements. The faces of a hyperplane arrangement satisfy covector axioms of an oriented matroid. An important landmark in the theory of oriented matroids is the *Folkman-Lawrence Topological Representation Theorem* [34]. The theorem states that there is a one-to-one correspondence between oriented matroids of rank l and arrangements of pseudospheres in S^{l+1}. An arrangement of pseudospheres is obtained by intersecting an arrangement of (central) pseudohyperplanes with the unit sphere (see Corollary 5.1.17). Under this correspondence realizable oriented matroids correspond to arrangements of central hyperplanes.

A generalization of the complexification of a (non-stretchable) pseudohyperplane arrangement was missing, as it is not always possible to put a smooth structure on these pseudohyperplanes. In Section 5.2 we introduce a (connected, $2l$-dimensional) topological space naturally associated with a pseudohyperplane arrangement that has the homotopy type of the corresponding Salvetti complex (Theorem 5.2.8). When the arrangement is stretchable this space is homeomorphic to the complexified complement.

Generalization of Zaslavsky's theorem

Let us now consider the complement of the arrangement inside the manifold. This is a disconnected topological space and we are interested in counting the number of connected components (chambers). This counting is done using the Möbius function of the intersection poset and the compactly supported cohomology of the intersections. This result generalizes the classical theorem of Zaslavsky. The theorem we prove is the following (Lemma 6.2.1 and Theorem 6.2.5).

Theorem 7.1.5. *Let $\mathcal{A}$ be an arrangement of submanifolds with the intersection poset $L(\mathcal{A})$ and let $Poin_c(Y,t)$ denote the Poincaré polynomial with compact supports of a topological space Y. The function $\nu\colon L \to \mathbb{Z}[t]$ defined by $\nu(A) = Poin_c(A,t)$, $\forall A \in L$ extends uniquely to a measure on the distributive lattice generated by L. Moreover if $I_{\mathcal{C}(\mathcal{A})}$ denotes the characteristic function on the set chambers then*

$$\int I_{\mathcal{C}(\mathcal{A})} d\nu = \sum_{Y\in L} \mu(X,Y) Poin_c(Y,t)$$

and the number of chambers is given by evaluating the above integral at $t=-1$ i.e.

$$|\mathcal{C}(\mathcal{A})| = (-1)^l \sum_{Y\in L} \mu(X,Y) Poin_c(Y,-1)$$

The main tool used in the proof is the *Groemer's integral theorem* and the fact that the Euler characteristic serves as a measure on certain posets. The idea of using this theory in the context of arrangements goes back to Blass and Sagan [8] (see also [14, 30, 31]). We would like to mention here that this generalization is not completely new. In fact it appears in the work of Zaslavsky [93] but our proof is different from his and makes it possible to introduce a generalization of the characteristic polynomial (suggested in the work of Chen [14]).

7.2 Future Work

Cohomology algebra of the tangent bundle complement

Though we have established some information about the cohomology groups of the tangent bundle complement in Section 3.7, understanding of the cup product is far from complete. Based on these calculations, the result about sphere arrangements (Theorem 4.1.17) and ongoing work on toric arrangements we propose the following conjecture. This conjecture states that there is a finer grading of cohomology groups indexed by the intersection data.

Conjecture 7.2.1. Let $\mathcal{A}$ be a submanifold arrangement in an orientable l-manifold X and let $M(\mathcal{A})$ be the associated tangent bundle complement. If $L(\mathcal{A})$ is the intersection poset then

$$rank(H^i(M(\mathcal{A}),\mathbb{Z})) = \sum_{\substack{Y\in L(\mathcal{A})\\ 0\le rk(Y)\le i}} rank(H^{i-rk(Y)}(\coprod_{|\mu(X,Y)|} Y, \mathbb{Z})) \qquad 0\le i\le l$$

where μ is the Möbius function of $L(\mathcal{A})$ and $rk(Y)$ is the codimension of the corresponding intersection.

Note that the above conjecture is true for hyperplane arrangements (known as the Brieskorn lemma [62, Proposition 3.75]) and sphere arrangements (Theorem 4.1.17). One more relevant step in this direction would be to see if it is possible to obtain a combinatorial description of homology cells of the Salvetti complex. That is to figure out whether the cells of the minimal complex for the tangent bundle complement are determined combinatorially. For hyperplane arrangements, this kind of explicit description is obtained in [25] using discrete Morse theory.

Toric arrangements

These types of arrangements are studied extensively and in various contexts (see Section 4.2). Let us start by defining complex toric arrangements.

Definition 7.2.2. Let $(\mathbb{C}^*)^l$ be the complex l-torus, a (complex) toric arrangement is a collection $\mathcal{A} = \{H_1, \dots, H_n\}$ of finitely many codimension 1 complex subtori.

Since $\mathbb{C}^*$ is biholomorphic to an infinite cylinder, a complex torus is homeomorphic to the tangent bundle of its underlying compact (or the real) torus. If $\mathcal{A}$ is a (complex) toric arrangement in $(\mathbb{C}^*)^l$ then $(\mathbb{C}^*)^l \cong (S^1)^l \times \mathbb{R}^l$ and each of the subtori in $\mathcal{A}$ are homeomorphic to $(S^1)^{l-1} \times \mathbb{R}^{l-1}$. So given a real toric arrangement (in the sense of submanifold arrangement) the tangent bundle complement is homeomorphic to the complement of a complex toric arrangement. Hence for a complex toric arrangement its associated real toric arrangement can also tell us a lot about the complement $M(\mathcal{A})$, similar to the hyperplane arrangement case. Though a lot of results developed so far for the submanifold arrangements apply in this context. Here is a list of some immediate questions that shape our future work.

1. To remove the regular cell division condition from the definition of the submanifold arrangements.

2. To find a relationship between the generalized characteristic polynomial and the multiplicity Tutte polynomial.

3. To show that the integral cohomology algebra of the complement of a (complex) toric arrangement is an Orlik-Solomon algebra.

4. To show that the toric arrangements that correspond to affine Coxeter groups are $K(\pi, 1)$.

Reflections on Manifolds

The most important work that initiated the study of the complexified complement of a real hyperplane arrangement is due to Deligne [22]. He proved a conjecture due to Brieskorn [10] that the complexified complement of a reflection arrangement is aspherical. Deligne proved this conjecture for a larger class of arrangements called as *simplicial arrangements*. An arrangement is called simplicial if its chambers are cones over simplices. Deligne proved that the complexified complement of such an arrangement is $K(\pi, 1)$ (i.e. the complement is an aspherical space). His proof has the following main ideas -

1. If $\mathcal{A}$ is a simplicial arrangement, then $\mathcal{A}$ has property D ([22, Prop. 1.9], [68, Theorem 3.1]).

2. If a real arrangement $\mathcal{A}$ of hyperplanes has property D, then the complexified complement $M(\mathcal{A})$ is a $K(\pi, 1)$ space ([22], [68, Theorem 3.6]).

Property D is another name, in the literature, for the *Deligne normal form* of the associated positive category (see Definition 3.6.5). Deligne uses this property to show that the universal cover of $M(\mathcal{A})$ is contractible (see Section 3.6 for a sketch of the proof). The arrangements that arise due to (faithful linear) representations of finite reflection groups (called Coxeter arrangements) are examples of simplicial arrangements (see [6, 62]), hence are $K(\pi, 1)$. Our interest is to generalize the theory of Coxeter arrangements in the context of real, smooth manifolds.

Now, with this canvas in mind let us look at the problem we want to investigate. First a definition -

Definition 7.2.3. Let X be a real, connected, smooth manifold of finite dimension l. A *reflection of* X is a self-diffeomorphism s such that

1. $s^2 = 1$,

2. the set X_s of fixed points of s is a smooth, codimension 1 submanifold,

3. $X \setminus X_s$ has 2 different components.

Let $W(X, n)$ denote the group generated by n distinct reflections of X and let N_i denote the submanifold fixed by the i-th reflection. Then it is not hard to see that N_i's intersect like hyperplanes (however it is not guaranteed that their intersections will always describe a regular CW structure of X). If we assume that the intersections of these submanifolds give us a regular CW decomposition then these fixed submanifolds form a submanifold arrangement $\mathcal{A}$.

We refer to [18, Chapter 10] and [42] for more information on these reflections and a presentation of the corresponding finite reflection group. According to [18, Theorem 10.1.5] the group $W(X, n)$ is actually a Coxeter group and the chambers of the corresponding arrangement are labeled by the elements of $W(X, n)$. Since each isotropy subgroup acts linearly on the tangent space the whole group has a proper action on the tangent bundle. The tangent bundles of these fixed submanifolds are also fixed under this action. As a result the reflection group $W(X, n)$ has a fixed point free, proper action on the tangent bundle complement $M(\mathcal{A})$. Let $Y(\mathcal{A})$ denote the qoutient of $M(\mathcal{A})$ under this action and let $PA(X, \mathcal{A})$ (respectively $A(X, \mathcal{A})$) denote the fundamental group of $M(\mathcal{A})$ (respectively of $Y(\mathcal{A})$). Note that if $X \cong \mathbb{R}^l$ then $PA(X, \mathcal{A})$ is the pure Artin group and $A(X, \mathcal{A})$ is the Artin group. Hence this setting generalizes not only Coxeter arrangements but also Artin groups. Now, the following ideas form natural problems in this context-

1. To find a labeling of the cells of the associated Salvetti complex that is based on the reflection group data.
2. To naturally extend the action of $G(X, n)$ to the tangent bundle complement.
3. To generalize the theory of Garside type models [23, Chapter 6] in this context.
4. To characterize the cases in which the tangent bundle complement is a $K(\pi, 1)$ space.
5. To understand the geometric group theoretic properties of the groups $PA(X, \mathcal{A})$ and $A(X, \mathcal{A})$.

We hope that this project will generalize the study initiated by Deligne and hopefully add an interesting aspect to the $K(\pi, 1)$ problem.

References

[1] K. Bacławski. Whitney numbers of geometric lattices. *Advances in Math.*, 16:125–138, 1975. 21

[2] T. Basak. Combinatorial cell complexes and Poincaré duality. *Geom. Dedicata*, 147:357–387, 2010. 3

[3] M. Bayer and B. Sturmfels. Lawrence polytopes. *Canad. J. Math.*, 42(1):62–79, 1990. 130

[4] A. Björner. Topological methods. In *Handbook of combinatorics, Vol. 1, 2*, pages 1819–1872. Elsevier, Amsterdam, 1995. 1, 5

[5] A. Björner, P. H. Edelman, and G. M. Ziegler. Hyperplane arrangements with a lattice of regions. *Discrete Comput. Geom.*, 5(3):263–288, 1990. 72, 117

[6] A. Björner, M. Vergnas, B. Strumfels, N. White, and G. M. Ziegler. *Oriented matroids*. Cambridge University Press, 2 edition, 1999. 7, 11, 13, 52, 53, 57, 110, 137

[7] A. Björner and G. M. Ziegler. Combinatorial stratification of complex arrangements. *J. Amer. Math. Soc.*, 5(1):105–149, 1992. xi, 10, 13, 56, 106, 115

[8] A. Blass and B. E. Sagan. Characteristic and Ehrhart polynomials. *J. Algebraic Combin.*, 7(2):115–126, 1998. 126, 135

[9] A. K. Bousfield and D. M. Kan. *Homotopy limits, completions and localizations*. Lecture Notes in Mathematics, Vol. 304. Springer-Verlag, Berlin, 1972. 20

[10] E. Brieskorn. Sur les groupes de tresses. In *Séminaire Bourbaki, (1971/1972), Exp. No. 401*, pages 21–44. Lecture Notes in Math., Vol. 317. Springer, Berlin, 1973. 137

[11] R. Brown. *Topology and groupoids*. BookSurge, LLC, Charleston, SC, 2006. 29

[12] R. Charney. Artin groups of finite type are biautomatic. *Math. Ann.*, 292(4):671–683, 1992. xi, 27, 75, 79

[13] B. Chen. On the euler characteristic of finite unions of convex sets. *Discrete & Computational Geometry. An International Journal of Mathematics and Computer Science*, 10(1):79–93, 1993. 126

[14] B. Chen. On characteristic polynomials of subspace arrangements. *Journal of Combinatorial Theory. Series A*, 90(2):347–352, 2000. 126, 135

[15] R. Cordovil. On the homotopy type of the Salvetti complexes determined by simplicial arrangements. *European J. Combin.*, 15(3):207–215, 1994. 30, 117

[16] R. Cordovil and M. L. Moreira. A homotopy theorem on oriented matroids. *Discrete Math.*, 111(1-3):131–136, 1993. Graph theory and combinatorics (Marseille-Luminy, 1990). 116

[17] G. d'Antonio and E. Delucchi. A salvetti complex for toric arrangements and its fundamental group. *Arxiv 1101.4111*, 2011. 90, 104

[18] M. W. Davis. *The geometry and topology of Coxeter groups*, volume 32 of *London Mathematical Society Monographs Series*. Princeton University Press, Princeton, NJ, 2008. xiv, 138

[19] C. De Concini and C. Procesi. On the geometry of toric arrangements. *Transform. Groups*, 10(3-4):387–422, 2005. 88, 100

[20] C. De Concini and C. Procesi. *Topics in hyperplane arrangements, polytopes and box-splines*. Universitext. Springer, New York, 2011. 100

[21] P. Dehornoy and L. Paris. Gaussian groups and Garside groups, two generalisations of Artin groups. *Proc. London Math. Soc. (3)*, 79(3):569–604, 1999. xi, 27, 79

[22] P. Deligne. Les immeubles des groupes de tresses généralisés. *Invent. Math.*, 17:273–302, 1972. xi, xiii, 19, 26, 30, 41, 47, 70, 73, 75, 79, 82, 117, 137

[23] E. Delucchi. *Topology and combinatorics of arrangement covers and of nested set complexes*. PhD thesis, ETH Zürich, 2006. iii, xi, 19, 22, 27, 30, 59, 62, 63, 67, 75, 78, 82, 83, 116, 117, 133, 138

[24] E. Delucchi. Combinatorics of covers of complexified hyperplane arrangements. *Progress in Mathematics*, 283:138, 2007. 19, 27, 29, 62, 133

[25] E. Delucchi. Shelling-type orderings of regular CW-complexes and acyclic matchings of the Salvetti complex. *Int. Math. Res. Not. IMRN*, 2008(6):Art. ID rnm167, 39, 2008. 13, 106, 136

[26] J. M. Douglass. Toral arrangements and hyperplane arrangements. *Rocky Mountain J. Math.*, 28(3):939–956, 1998. 100

[27] W. G. Dwyer and H. Henn. *Homotopy theoretic methods in group cohomology*. Advanced Courses in Mathematics. CRM Barcelona. Birkhäuser Verlag, Basel, 2001. 20, 31

[28] B. Eckmann. The euler characteristic a few highlights in its long history. In *Mathematical Survey Lectures*, pages 177–188. Springer Berlin Heidelberg, 2006. 119

[29] P. H. Edelman. A partial order on the regions of $\mathbf{R}^n$ dissected by hyperplanes. *Trans. Amer. Math. Soc.*, 283(2):617–631, 1984. 72

[30] R. Ehrenborg, M. Readdy, and M. Slone. Affine and toric hyperplane arrangements. *Discrete & Computational Geometry. An International Journal of Mathematics and Computer Science*, 41(4):481–512, 2009. x, 36, 101, 125, 126, 128, 130, 135

[31] R. Ehrenborg and M. A. Readdy. On valuations, the characteristic polynomial, and complex subspace arrangements. *Advances in Mathematics*, 134(1):32–42, 1998. 118, 126, 135

[32] M. Falk. Introduction: The combinatorics and topology of hyperplane arrangements, 2004. visit `http://www.msri.org/communications/vmath/VMathVideos/VideoInfo/2185/show_video`. 1

[33] E. D. Farjoun. Fundamental group of homotopy colimits. *Advances in Mathematics*, 182(1):127, 2004. 27, 31

[34] J. Folkman and J. Lawrence. oreinted matroids. *Journal of Combinatorial Theory. Series B*, 25(2):199–236, 1978. 12, 105, 110, 134

[35] A. T. Fomenko, D. B. Fuchs, and V. L. Gutenmacher. *Homotopic topology*. Akadémiai Kiadó (Publishing House of the Hungarian Academy of Sciences), Budapest, 1986. 23

[36] P. Gabriel and M. Zisman. *Calculus of fractions and homotopy theory*. Springer-Verlag, 1967. 28, 61, 68

[37] F. A. Garside. The braid group and other groups. *Quart. J. Math. Oxford Ser. (2)*, 20:235–254, 1969. xi, 79

[38] L. Geissinger. Valuations on distributive lattices. I. *Arch. Math. (Basel)*, 24:230–239, 1973. 119

[39] I. M. Gel'fand and G. L. Rybnikov. Algebraic and topological invariants of oriented matroids. *Dokl. Akad. Nauk SSSR*, 307(4):791–795, 1989. 13, 105

[40] B. Grünbaum. *Convex polytopes*. Pure and Applied Mathematics, Vol. 16. Interscience Publishers John Wiley & Sons, Inc., New York, 1967. 117, 118, 119

[41] B. Grünbaum. *Arrangements and spreads*. American Mathematical Society Providence, R.I., 1972. Conference Board of the Mathematical Sciences Regional Conference Series in Mathematics, No. 10. x, 1, 118

[42] E. Gutkin. Geometry and combinatorics of groups generated by reflections. *Enseign. Math. (2)*, 32(1-2):95–110, 1986. 138

[43] H. Hadwiger. Eulers Charakteristik und kombinatorische Geometrie. *J. Reine Angew. Math.*, 194:101–110, 1955. 119

[44] A. Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002. 65, 76, 80, 94

[45] P. J. Higgins. *Notes on categories and groupoids*. Van Nostrand Reinhold Co., London, 1971. Van Nostrand Rienhold Mathematical Studies, No. 32. 27

[46] A. A. Husainov. Homological dimension theory of small categories. *Journal of Mathematical Sciences (New York)*, 110(1):22732321, 2002. Algebra, 17. 21

[47] R. Jozsa and J. Rice. On the cohomology ring of hyperplane complements. *Proc. Amer. Math. Soc.*, 113(4):973–981, 1991. 100

[48] D. A. Klain and G. Rota. *Introduction to geometric probability*. Lezioni Lincee. [Lincei Lectures]. Cambridge University Press, Cambridge, 1997. 119, 121, 122

[49] V. Klee. The Euler characteristic in combinatorial geometry. *Amer. Math. Monthly*, 70:119–127, 1963. 119

[50] D. Kozlov. *Combinatorial algebraic topology*, volume 21 of *Algorithms and Computation in Mathematics*. Springer, Berlin, 2008. 5, 14

[51] G. I. Lehrer. A toral configuration space and regular semisimple conjugacy classes. *Math. Proc. Cambridge Philos. Soc.*, 118(1):105–113, 1995. 100

[52] T. Leinster. The Euler characteristic of a category. *Doc. Math.*, 13:21–49, 2008. 119

[53] S. Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998. 27

[54] C. Macmeikan. Modules of derivations for toral arrangements. *Indag. Math. (N.S.)*, 15(2):257–267, 2004. 100

[55] A. Mandel. *Topology of Oriented Matroids*. PhD thesis, Thesis (Ph.D.)–University of Waterloo (Canada), 1982. 12, 105, 109, 110

[56] M. C. McCord. Singular homology groups and homotopy groups of finite topological spaces. *Duke Math. J.*, 33:465–474, 1966. 3, 18

[57] D. A. Miller. Oriented matroids from smooth manifolds. *J. Combin. Theory Ser. B*, 43(2):173–186, 1987. 111

[58] L. Moci. Combinatorics and topology of toric arrangements defined by root systems. *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl.*, 19(4):293–308, 2008. 36, 101

[59] L. Moci. A tutte polynomial for toric arrangements. *Arxiv 0911.4823*, Nov. 2009. 100

[60] L. Moci. Wonderful models for toric arrangements. *Arxiv 0912.5461*, Dec. 2009. 100

[61] L. Moci and S. Settepanella. The homotopy type of toric arrangements. *Journal of Pure and Applied Algebra*, In Press:–, 2010. 90, 101

[62] P. Orlik and H. Terao. *Arrangements of hyperplanes*, volume 300 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin, 1992. xi, 1, 7, 8, 21, 22, 89, 136, 137

[63] L. Pakula. Regions cut by arrangements of topological spheres. *Canad. Math. Bull.*, 36(2):241–244, 1993. 126

[64] L. Pakula. Pseudosphere arrangements with simple complements. *Rocky Mountain J. Math.*, 33(4):1465–1477, 2003. 126

[65] L. Paris. Arrangements of hyperplanes with property D. *Geom. Dedicata*, 45(2):171–176, 1993. 26, 83

[66] L. Paris. The covers of a complexified real arrangement of hyperplanes and their fundamental groups. *Topology Appl.*, 53(1):75–103, 1993. 26, 30, 59, 62, 75, 79, 82, 83, 117

[67] L. Paris. The Deligne complex of a real arrangement of hyperplanes. *Nagoya Math. J.*, 131:39–65, 1993. 26, 117

[68] L. Paris. Universal cover of salvetti's complex and topology of simplicial arrangements of hyperplanes. *Transactions of the American Mathematical Society*, 340(1):149–178, Nov. 1993. iii, xi, 10, 26, 32, 44, 132, 137

[69] L. Paris. Braid groups and Artin groups. In *Handbook of Teichmüller theory. Vol. II*, volume 13 of *IRMA Lect. Math. Theor. Phys.*, pages 389–451. Eur. Math. Soc., Zürich, 2009. 10

[70] D. Quillen. Homotopy properties of the poset of nontrivial p-subgroups of a group. *Advances in Mathematics*, 28(2):101–128, 1978. 14, 20

[71] G.-C. Rota. On the combinatorics of the Euler characteristic. In *Studies in Pure Mathematics (Presented to Richard Rado)*, pages 221–233. Academic Press, London, 1971. 119, 120

[72] T. B. Rushing. *Topological embeddings*. Academic Press, New York, 1973. Pure and Applied Mathematics, Vol. 52. xiii, 35, 38, 90, 107

[73] M. Salvetti. Topology of the complement of real hyperplanes in N. *Inventiones Mathematicae*, 88(3):603–618, Oct. 1987. iii, xi, 9, 26, 47, 48, 67, 95, 132

[74] M. Salvetti. On the homotopy theory of complexes associated to metrical-hemisphere complexes. *Discrete Math.*, 113(1-3):155–177, 1993. xiii, 26, 30, 32, 52, 53, 55, 57, 70, 75, 82, 83, 92, 117

[75] C. C. Sawyer. *On the cohomology of the complement of a toral arrangement*. PhD thesis, University of North Texas, TX, 1999. 100

[76] S. H. Schanuel. Negative sets have Euler characteristic and dimension. In *Category theory (Como, 1990)*, volume 1488 of *Lecture Notes in Math.*, pages 379–385. Springer, Berlin, 1991. 119

[77] R. Schneider. *Convex bodies: the Brunn-Minkowski theory*, volume 44 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1993. 121

[78] R. Scott. Blow-ups and complements in zonotopal complexes, Oct. 2004. Available at `http://www.msri.org/communications/vmath/VMathVideos/VideoInfo/2184/show_video`. 33, 40

[79] G. Segal. Classifying spaces and spectral sequences. *Publ. Math. des Inst. des H. É. S.*, 34:105–112, 1968. 22

[80] S. Settepanella. The integer cohomology of toric weyl arrangements. *Arxiv preprint 10080631*, pages 1–14, 2010. 101

[81] R. P. Stanley. *Enumerative combinatorics. Vol. 1*, volume 49 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1997. 119, 128

[82] J. Stillwell. *Classical topology and combinatorial group theory*, volume 72 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1993. 75

[83] R. E. Stong. Finite topological spaces. *Trans. Amer. Math. Soc.*, 123:325–340, 1966. 3

[84] R. M. Vogt. Homotopy limits and colimits. *Math. Z.*, 134:11–52, 1973. 14

[85] M. L. Wachs. Poset topology: tools and applications. In *Geometric combinatorics*, volume 13 of *IAS/Park City Math. Ser.*, pages 497–615. Amer. Math. Soc., Providence, RI, 2007. 1

[86] J. W. Walker. *Topology and combinatorics of Ordered Sets*. PhD thesis, MIT, 1981. Available at `http://hdl.handle.net/1721.1/16153`. 1

[87] C. A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994. 21

[88] V. Welker, G. M. Ziegler, and R. T. Živaljević. Comparison lemmas and applications for diagrams of spaces. *preprint*, pages 1–35, 1997. 14, 18

[89] V. Welker, G. M. Ziegler, and R. T. Živaljević. Homotopy colimits - comparison lemmas for combinatorial applications. *Journal für die Reine und Angewandte Mathematik*, 509:117–149, 1999. xii, 14, 15, 17, 18, 22, 26, 133

[90] S. Yuzvinsky. Cohomology of local sheaves on arrangement lattices. *Proc. Amer. Math. Soc.*, 112(4):1207–1217, 1991. 21

[91] S. Yuzvinsky. Orlik-Solomon algebras in algebra and topology. *Uspekhi Mat. Nauk*, 56(2(338)):87–166, 2001. xi, 21

[92] T. Zaslavsky. Facing up to arrangements: face-count formulas for partitions of space by hyperplanes. *Memoirs of the American Mathematical Society*, 1(issue 1, 154):vii+102, 1975. x, xii, 7, 118, 124

[93] T. Zaslavsky. A combinatorial analysis of topological dissections. *Advances in Mathematics*, 25(3):267–285, 1977. xiii, 36, 118, 125, 126, 128, 135

[94] G. M. Ziegler. *Algebraic Combinatorics of hyperplane arrangements*. PhD thesis, MIT Cambridge MA, 1987. 13, 105

[95] G. M. Ziegler. *Lectures on polytopes*, volume 152 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. 47, 52

[96] G. M. Ziegler and R. T. Živaljević. Homotopy types of subspace arrangements via diagrams of spaces. *Mathematische Annalen*, 295(3):527–548, 1993. 17, 18, 19

Curriculum Vitae

Priyavrat C. Deshpande

Education

- **Ph.D. Student (Mathematics)**, University of Western Ontario *Present.*
- **Master of Science (Mathematics)**, University of Western Ontario *2007.*
- **Master of Science (Mathematics)**, University of Pune *June 2002.*
- **Bachelor of Science (Mathematics)**, University of Pune *June 2000.*

Research Publications

- *Arrangements of submanifolds and the tangent bundle complement*, preprint.
- *On a generalization of Zaslavsky's theorem*, preprint.
- *Arrangements of spheres and projective spaces*, preprint.
- *On a complexification of pseudohyperplane arrangements*, preprint.

Teaching

- **Teaching Assistant**, University of Western Ontario *Fall 2006-present*
- **Lecturer**, S.P. College, Pune, India *2004-06*

Scholarships and Awards

- Ruth and Robert Lumsden Fellowship, Faculty of Science, UWO *2010*
- Graduate Thesis Research Award, Faculty of Science, UWO *2009, 2011*
- Faculty of Science Graduate Student Teaching Award, UWO *2008*
- Western Graduate Research Scholarship *2006 - present*