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Geometry in spectral triples: Immersions and fermionic fuzzy geometries

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Abstract

We investigate the metric nature of spectral triples in two ways.

Given an oriented Riemannian embedding $\imath : X \hookrightarrow Y$ of codimension 1 we construct a family of unbounded KK-cycles $\imath_!^\varepsilon$, each of which represents the shriek class of $\imath$ in KK-theory. These unbounded KK-cycles are further equipped with connections, allowing for the explicit computation of the products of $\imath_!^\varepsilon$ with the spectral triple of Y at the unbounded level. In the limit $\varepsilon \to 0$ the product of these unbounded KK-cycles with the canonical spectral triple for Y admits an asymptotic expansion. The divergent part of this expansion is known and universal, the constant term in the expansion gives the canonical spectral triple for X. Furthermore, the curvature of these unbounded KK-cycles converges to the square of the mean curvature of X in Y as $\varepsilon \to 0$.

We define a random matrix ensemble for the Dirac operator on the $(0, 1)$ fuzzy geometry incorporating both the geometric and fermionic aspects of the spectral action. This yields a unitarily invariant, single-matrix multi-trace model. We generalize Coulomb-gas techniques for finding the spectral density of single-trace models to multi-trace models and apply these to our model of a fermionic fuzzy geometry. The resulting Fredholm integral equation for the spectral density is analyzed numerically and the effect of various parameters on the spectral density is investigated.

Keywords: noncommutative geometry, spectral triples, C^*-correspondences, unbounded KK-theory, Riemannian immersions, random matrix theory, potential theory, fuzzy geometry, spectral density, Dirac ensembles.

Lay Summary

Descriptions of quantum mechanics often use matrices. These are square arrays of numbers that can be added, subtracted and multiplied similar to normal numbers with one major difference: if A and B are matrices $A \cdot B$ and $B \cdot A$ may give different results. In technical terms this is called *noncommutativity*.

On the other hand we have general relativity, in which space and time are described using geometry. The theory of *noncommutative geometry* describes geometry using matrices (or their infinite dimensional generalizations). This makes noncommutative geometry a natural language to try and unite the quantum mechanical with the world of general relativity. In this thesis we explore two aspects of noncommutative geometry.

In the first part of this thesis we contribute to one of the open questions in noncommutative geometry: How to describe maps between geometries? For these maps you can think of a map in an atlas, which describes one geometry (the world) as a part of another (the page). We construct a description in noncommutative geometry language of maps that come from *hypersurfaces*. An example of a hypersurface is the shell of a ball in 3D space.

The second part of the thesis concerns a separate project where we consider a specific class of noncommutative geometries, called the $(0, 1)$ *fuzzy geometries*. We are interested in this class of geometries since they provide a toy model of *quantum gravity*, where reality is a superposition of many different geometries of this $(0, 1)$ type.

Fuzzy geometries have many useful qualities for us. They have, for example, a minimum resolution built in. If you look very closely, details become fuzzy. Such a minimum scale, called the Planck length, is predicted by the combination of general relativity and quantum mechanics.

We focus on the fuzzy geometries of type $(0, 1)$ since for this type the situation further simplifies to allow us to find exact solutions. The main addition we make to this model of quantum gravity on fuzzy geometries is the inclusion of particles living on space time. We compute in particular how these particles affect the superposition of geometries.

Co-authorship statement

Chapter 1 is an introductory chapter outlining the basic concepts required for the remaining chapters and provides background references. I am the sole author of this chapter.

Chapter 2 is based on published work in [62] and submitted work in [63] which at the time of writing is under review at Annals of K-theory. For both these papers the research was done in close and equal collaboration between me and Dr. Walter D. van Suijlekom. The writing for these papers was done largely by me, with significant feedback from Dr. van Suijlekom. The text of Chapter 2 is an edited and extended combination of [62, 63].

Chapters 3 and 4 are based on ideas that arose during discussions within the research group of Dr. Masoud Khalkhali. The work and writing for these chapters was done by me, supplemented by useful discussions with Dr. Khalkhali and Nathan Pagliaroli.

In Chapter 4 I refer to a co-authored survey paper [33] for further background. In that paper I was the primary writer for Chapter 1 and contributed equally with the other authors in editing and finalizing the full text.

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Chapter 1

Introduction and Preliminaries

Let us start with a brief message to the reader. The goal of this thesis for me is two-fold. First of all it is intended as my Ph.D. thesis, detailing most of the work I have done over the past four years and as such form a small addition to the wide world of mathematics. Secondly I hope this can be a document that future graduate students can use to get a running start on similar projects.

The work presented in this thesis all falls under the umbrella of noncommutative geometry, but can be split into two main projects. The first project concerns the interplay between noncommutative geometry and differential geometry. We construct a representation of smooth embeddings of manifolds in the language of spectral triples. This represents a new and useful example in the search for a theory of maps in noncommutative geometry.

The second project takes on a very different flavour. We consider a Dirac ensemble based on the class of $(0, 1)$ fuzzy geometries and spectral density of the random Dirac operator of this ensemble. To accomplish this we first generalize a suite of tools for single-trace random matrix models to interacting random matrix models in Chapter 3. This is followed by Chapter 4, where we apply these techniques to a new class of Dirac ensembles incorporating the fermionic action.

More detailed introductions are given at the start of the respective chapters. The common thread in both these projects is the focus on the metric nature of spectral triples. The connection between the projects lies largely in future work that will be based on the combination between differential geometry and fuzzy Dirac ensembles [55].

First, in Sections 1.1 and 1.2, we will give very brief introductions to the parts of noncommutative geometry and random matrix theory that are relevant to our purposes. The goal of these sections is not to give a full and comprehensive introduction to the hypothetical graduate student of all the required material. The goal is to serve as a reminder of the terminology or as a check of assumed background knowledge. I have tried to include references to sufficient excellent source material that gives proper introductions to all these concepts.

1.1 Noncommutative geometry preliminaries

The distant ancestor of noncommutative geometry is one of my personally favourite theorems:

Theorem (Gelfand duality)**.** *If A is a unital commutative C^*-algebra then $A = C(\widehat{A})$ where $\widehat{A}$ is the compact Hausdorff space of characters $\phi : A \to \mathbb{C}$ with the weak-$*$ topology. Moreover, if X is a compact Hausdorff space, $C(X)$ is a unital commutative C^*-algebra and X is homeomorphic to $\widehat{C(X)}$.*

Proof. See, for example, [52, Ch. 2]. □

To paraphrase, we can study (compact Hausdorff) topological spaces by studying the associated commutative C^*-algebras.

In certain situations one comes across "degenerate" or "badly behaved" topological spaces that do, in some sense, correspond to noncommutative C^*-algebras. See [11, 16] for the example of this that formed the inspiration for Chapter 2. More generally, a lot of properties of topological spaces can be formulated for their associated commutative algebras in a way that also makes sense for noncommutative algebras. For example, compactness corresponds to unitality. These ideas give rise to what one might call noncommutative topology.

But, as we will stress again in Chapter 2, we want to do geometry, not topology. This is where spectral triples come in. The key here is Connes' reconstruction theorem [15], which provides an analogue of Gelfand duality in a geometric setting. It, very roughly, says that a particularly type of Riemannian manifolds can be reconstructed from the algebra of smooth functions, its Dirac operator, and the interaction between the Dirac operator and the smooth functions.

This gives rise to the notion of a spectral triple, consisting of an algebra A, a Dirac operator D and a Hilbert space H upon which both A and D act. Studying such spectral triples for noncommutative A (or rather not-necessarily-commutative) yields the by now widely developed area of noncommutative geometry. We will provide a very bare bones introduction here, for a more in-depth introduction see for example [12, 29] or [61] for a good didactic introduction that also explains the connections of noncommutative geometry to physics.

We assume that all C^*-algebras appearing in this section are separable.

1.1.1 Spectral triples

Spectral triples are the basic ingredient of noncommutative geometry and combine algebraic and analytic structure. Their algebraic structure will play a crucial role in the analysis of the finite dimensional fuzzy geometries in Chapter 4 while their analytic structure, primarily through unbounded KK-theory, features mostly in Chapter 2.

Definition 1.1.1. *A* spectral triple

$$(\mathcal{A}, H, D)$$

consists of a $*$-algebra $\mathcal{A}$ realized as a subalgebra of $B(H)$ for the Hilbert space H, and a possibly unbounded self-adjoint operator $D : \mathrm{Dom}(D) \subseteq H \to H$. These data must satisfy the following additional assumptions

- *$\mathcal{A}$ preserves the domain of D and $[D, a]$ extends to a bounded operator for all $a \in \mathcal{A}$,*
- *$a(D+i)^{-1}$ is compact for all $a \in A$.*

The closure $A = \overline{\mathcal{A}}$ (in operator norm) is the C^*-algebra for this spectral triple.

A graded *or* even spectral triple *has the additional datum of a $\mathbb{Z}/2\mathbb{Z}$ grading $\gamma : H \to H$ such that D is an odd operator for γ and the representation of A on H is graded as well (where $\mathcal{A}$ is possibly trivially graded).*

There is another subtype that deserves a definition of its own because of its additional complexity and central role in Chapter 4.

Definition 1.1.2. *A* real spectral triple of KO-dimension s *is a possibly graded spectral triple with additionally an antilinear isometry $J : H \to H$ such that*

$$J^2 = \varepsilon, \qquad JD = \varepsilon' DJ, \qquad J\gamma = \varepsilon'' \gamma J$$

where the signs $\varepsilon, \varepsilon', \varepsilon''$ are determined by Table 1.1. If the spectral triple is not graded the third condition is void.

For a real spectral triple we define an additional representation of the opposite algebra A^o on H by

$$a^o \psi = J a^* J^{-1} \psi = \varepsilon J a^* J \psi$$

for $a \in A$ and $\psi \in H$. This opposite representation, which we will usually call the right action of A, is required to satisfy

- *$[a, b^o] = 0$ for $a, b \in A$,*
- *$[[D, a], b^o] = 0$ for $a, b \in \mathcal{A}$ (the order one condition).*

Remark 1.1.3. *The term* order one condition *is used because in examples it requires D to be an order one differential operator. This is seen explicitly in Lemma 4.1.2 where the real structure is used to limit the space of Dirac operators. The condition is also motivated by Example 1.1.4 where the Dirac operator is, in fact, an order one differential operator.*

One of the main examples of spectral triples are the canonical spectral triples associated to spin^c Riemannian manifolds, which feature in Connes' reconstruction theorem. We will see other, finite dimensional, examples in Chapter 4. The below example is a very brief rundown of this important class of spectral triples, more background can be found in [61, Ch. 4] or [12].

Example 1.1.4. *Let M be an oriented Riemannian spin^c manifold with spinor bundle*[1] *Σ. Let $\mathbb{C}\mathrm{l}(TM)$ be the Clifford algebra bundle over M generated by the tangent bundle*

[1] We take the existence of this spinor bundle as the definition of a spin^c manifold.

$s \mod 8$	0	1	2	3	4	5	6	7
ε	+	+	−	−	−	−	+	+
ε'	+	−	+	+	+	−	+	+
ε''	+	+	−	+	+	+	−	+

Table 1.1: The usual choices of the signs associated to a KO dimension s in noncommutative geometry literature. For a real structure J the signs are $J^2 = \varepsilon$, $J\gamma_i = \varepsilon'\gamma_i J$ for γ_i generators of the Clifford algebra, and $J\gamma = \varepsilon''\gamma J$ for γ the (possibly trivial) grading operator.

and the Riemannian structure. This means that $\Sigma \to M$ is a vector bundle such that, if $\dim(M)$ is even, there is an isomorphism $c : \mathbb{C}\mathrm{l}(TM) \to \mathrm{End}(\Sigma)$. If $\dim(M)$ is odd, instead $c : \mathbb{C}\mathrm{l}^0(TM) \to \mathrm{End}(\Sigma)$ is an isomorphism.

In this setting one can always construct a lift of the Levi-Civita connection ∇^M to the spinor bundle Σ, by which we mean a metric connection ∇^Σ on Σ such that we have the Leibniz rule

$$\nabla^\Sigma_A (c(B)\psi) = c\left(\nabla^M_A(B)\right)\psi + c(B)\nabla^\Sigma_A(\psi),$$

for A, B vector fields on M. We can then define a Dirac operator by

$$D_M = ic \circ \sharp \circ \nabla^\Sigma$$

where $\sharp : \Omega^1_{dR}(M) \to TM$ is the map induced by the Riemannian metric.

Let us describe this in local coordinates over some open subset U to make it more concrete. Say we have local coordinates $\{x^\mu\}$, also choose a local orthonormal frame $\{e_i\}$. We first express the Levi-Civita connection using Christoffel symbols relative to this frame. Explicitly

$$\nabla^M_{\partial_\mu}(e_j) = \widetilde{\Gamma}^k_{\mu j} e_k.$$

Note that these are not the usual Christoffel symbols and have different symmetry properties, notably since they combine coordinate and frame indices.

Over U we can further find a basis for the spinor bundle such that $c(e_i) = \gamma_i$ for a fixed set of γ-matrices (i.e. a set of matrices satisfying $\gamma_i\gamma_j + \gamma_j\gamma_i = 2\delta_{ij}$).

It is then a cumbersome but straightforward check that

$$\nabla^\Sigma_{\partial_\mu}(\psi) = \left(\partial_\mu - \frac{1}{4}\sum_{a,b}\widetilde{\Gamma}^b_{\mu a}\gamma_a\gamma_b\right)\psi$$

defines a spin connection on Σ. The Dirac operator in these coordinates is then given by

$$D_M\psi = \sum_\mu i\gamma_\mu\left(\partial_\mu - \frac{1}{4}\sum_{a,b}\widetilde{\Gamma}^b_{\mu a}\gamma_a\gamma_b\right)\psi$$

where $\gamma_\mu = \sum_i a^i_\mu \gamma_i$ if $\partial_\mu = \sum_i a^i_\mu e_i$.

At the end we can combine all this data into the spectral triple

$$\left(C^\infty(M), L^2(\Sigma), D_M\right).$$

We call this the canonical spectral triple *of M.*

If $\dim(M)$ *is even* Σ *comes with a grading for which* $c(TM)$ *is odd, making this a graded spectral triple. In case there exists a global charge conjugation* $J : \Sigma \to \Sigma$ *the manifold* M *is called spin and this* J *makes the canonical spectral triple a real spectral triple.*

1.1.2 The external product of spectral triples

The construction in noncommutative geometry parallel to the Cartesian product in differential geometry is that of the external product (to be compared to the internal product of the next section).

We are interested in the external product for two reasons. The first is its appearance in Chapter 4, where the Dirac ensemble we are interested in is formed as an external product. Its use there is similar to the construction of almost commutative geometries, which form the basis for the noncommutative geometry approach to particle physics. The papers [13, 10] are among the foundations of this, see also [61] for a good didactic introduction as well as a more complete list of references.

The second reason, with relevance towards Chapter 2, is that the external product was the motivation for the introduction of unbounded KK-theory in [1]. As such it sets the stage for Section 1.1.3 and it provides intuition for the internal product, Section 1.1.4, as well.

Definition 1.1.5. *Given two graded spectral triples* $(\mathcal{A}_1, H_1, D_1; \gamma_1)$, $(\mathcal{A}_2, H_2, D_2; \gamma_2)$ *the external product is the spectral triple*[2]

$$(\mathcal{A}_1 \otimes \mathcal{A}_2, H_1 \otimes H_2, D_1 \otimes 1 + \gamma_1 \otimes D_2; \gamma_1 \otimes \gamma_2)\,.$$

The product of two ungraded spectral triples $(\mathcal{A}_1, H_1, D_1)$, $(\mathcal{A}_2, H_2, D_2)$ *is given by the graded spectral triple*

$$\big(\mathcal{A}_1 \otimes \mathcal{A}_2, H_1 \otimes H_2 \otimes \mathbb{C}^2, D_1 \otimes 1 \otimes \sigma_1 + D_2 \otimes 1 \otimes \sigma_2; 1 \otimes 1 \otimes \sigma_3\big)$$

where $\sigma_1, \sigma_2, \sigma_3$ *are the Pauli matrices (see Example 1.1.7).*

Taking the real structure into account complicates matters, since the obvious choice of product real structure $J = J_1 \otimes J_2$ may fail to have a well-defined KO-dimension in the sense of Table 1.1. This can be ameliorated by noting that there are choices involved in the usual sign table of KO-dimensions. In particular, if J is a real structure, so is γJ and this difference flips the ε' sign. Taking consistent graded tensor products then breaks the choices for the KO-dimension signs into two families [26, 18].

A concept closely tied to the external product and that features in both the differential geometric spectral triples of Chapter 2 and the fuzzy geometry spectral triples of Chapter 4 is that of a spinor space. We will introduce these and some associated terminology here, good further references are [61, 4].

[2] We note that there are some details to be considered when it comes to the tensor product of C^*-algebras $\mathcal{A}_1 \otimes \mathcal{A}_2$ and the choice of smooth elements. In our applications at least one of the factors will be finite dimensional which voids these concerns. In general we take the tensor product in the representation on $H_1 \otimes H_2$.

Definition 1.1.6. *The* Clifford algebra $\mathbb{C}\mathrm{l}_n$ *is the complex algebra generated by the ordered set of generator* $\{e_1, \ldots, e_n\}$ *with the relations*

$$e_i^2 = 1, \text{ for } i = 1, \ldots, n, \qquad e_i e_j = -e_j e_i \text{ for } i \neq j$$

The Clifford algebras are $\mathbb{Z}/2\mathbb{Z}$ *graded by declaring the generators to be odd.*

The element $\gamma = i^m e_1 \cdot \ldots \cdot e_n$, $m = \lfloor \frac{n}{2} \rfloor$, *is called the* grading *or* chirality element.

Example 1.1.7. *The Pauli matrices*

$$\sigma_x = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

are a representation of $\mathbb{C}\mathrm{l}_3$ *on* $\mathbb{C}^2$.

Example 1.1.8. σ_1 *and* σ_2 *are a representation of* $\mathbb{C}\mathrm{l}_2$ *on* $\mathbb{C}^2$ *and* $\sigma_3 = -i\sigma_1\sigma_2$ *is a grading operator for this representation.*

Definition 1.1.9. *We define a* spinor space *of signature* (p, q) *as a complex vector space* V *with an irreducible representation of the Clifford algebra* $\mathbb{C}\mathrm{l}_{p+q}$, *together with an antilinear map* J *called the* real structure *or* charge conjugation. *We require that for Clifford generators* $\{e_i\}$,

$$J^2 = \varepsilon, \qquad Je_i = \varepsilon' e_i J, \qquad J\gamma = \varepsilon'' \gamma J$$

with the signs $\varepsilon, \varepsilon', \varepsilon''$ *for KO-dimension* $q - p$ *in Table 1.1.*

This leads us to the final stop of this section, the doubling and halving processes. It is in many cases easier to work with graded spectral triples (and graded unbounded KK-cycles) than with ungraded ones. This preference for graded triples and cycles originates with spinor spaces, where for even n there is a unique irreducible representation while for odd n there are two irreducible representations. This leads to differences between spinor bundles over even and odd dimensional manifolds (see Example 1.1.4) and to a lot of the n even/n odd bookkeeping in Chapter 2. Ideally this could all be avoided by taking careful graded tensor products and adjusting the usual choice of KO-signs in the spirit of [26].

Definition 1.1.10. *Let* $(\mathcal{A}, H, D)$ *be an ungraded spectral triple. The doubling of* $(\mathcal{A}, H, D)$ *is the external product with* $(\mathbb{C}\mathrm{l}_1, \mathbb{C}^2, 0; \sigma_3)$, *i.e. the graded spectral triple*

$$\big(\mathcal{A} \otimes \mathbb{C}\mathrm{l}_1, H \otimes \mathbb{C}^2, D \otimes \sigma_2; 1 \otimes \sigma_3\big)$$

with C^**-algebra* $A \otimes \mathbb{C}\mathrm{l}_1$.

Conversely, if $(\mathcal{A} \otimes \mathbb{C}\mathrm{l}_1, H, D; \gamma)$ *is a graded spectral triple with* A *unital such that* $(1 \otimes e)D = -D(1 \otimes e)$, *then it is unitarily equivalent to the doubling of a spectral triple with* C^**-algebra* A.

The above definition becomes more interesting in the context of unbounded KK-cycles, where it implements unbounded representatives for the higher KK groups, see [62, App. A] and [59].

1.1.3 Hilbert bimodules and unbounded KK-cycles

We will now give a brief introduction, or rather a series of definitions, of Hilbert bimodules leading to the main definition of an unbounded KK-cycle. As the name suggests, these are representatives for classes in KK-theory. This is however not the view we will take in this thesis, instead we will interpret them as maps between spectral triples. This approach is in particular inspired by the Kasparov, or internal, product.

For an in-depth introduction to Hilbert (bi-)modules, see [44]. For an introduction to KK-theory, which we omit in this text for the reasons outlined in the introduction of Chapter 2, see [12, 8] or [34] for a more introductory text.

Definition 1.1.11. *Let B be a C^*-algebra. A* Hilbert module over B *is a vector space $\mathcal{E}$ with a right B-representation and a B-valued sesquilinear form $\langle \bullet, \bullet \rangle$. These must satisfy*

- $\langle \psi, \phi \cdot b \rangle = \langle \psi, \phi \rangle b$,
- $\langle \psi, \phi \rangle = \langle \phi, \psi \rangle^*$,
- $\langle \psi, \psi \rangle \geq 0$ *with equality only if* $\psi = 0$,

for all $\psi, \phi \in \mathcal{E}$ and $b \in B$. Finally we require that $\mathcal{E}$ is complete in the norm $\|\psi\|^2 = \|\langle \psi, \psi \rangle\|_B$.

If $\mathcal{E}$ and B are (possibly trivially) graded, all these structures must be compatible with the grading.

Example 1.1.12. *Examples of Hilbert modules are*

1. *A Hilbert space H is a Hilbert module over $\mathbb{C}$.*
2. *A vector bundle $E \to M$ with a Riemannian inner product is a Hilbert module over $C(M)$.*
3. *A C^*-algebra A is a Hilbert module over itself with inner product $\langle a, b \rangle = a^*b$.*

With the second example in mind, Hilbert modules can be thought of as vector bundles over the noncommutative space represented by the C^*-algebra B. Much of Hilbert space theory carries over the Hilbert module setting. With one major extra regularity condition for unbounded operators due to complexities with orthogonal complements.

Definition 1.1.13. *Let $\mathcal{E}$ be a Hilbert module over a C^*-algebra B. A map $T : \mathcal{E} \to \mathcal{E}$ is* adjointable *if there exists a $T^* : \mathcal{E} \to \mathcal{E}$ such that $\langle T^*\psi, \phi \rangle = \langle \psi, T\phi \rangle$ for all $\psi, \phi \in \mathcal{E}$. The* space of adjointable maps *is then predictably*

$$\mathrm{End}^*_B(\mathcal{E}) = \{T : \mathcal{E} \to \mathcal{E} \,|\, T \text{ is adjointable}\}.$$

This definition may look odd at first glance, when compared to the definition of bounded maps on a Hilbert space. In Hilbert spaces bounded linear maps are automatically adjointable, but the converse also holds, adjointable maps are automatically linear and bounded. This converse statement continues to hold for Hilbert modules, while the first one fails [44, p. 8].

Besides the space of operators we need to know what compact operators on a Hilbert module are.

Definition 1.1.14. *Let $\mathcal{E}$ be a Hilbert module over a C^*-algebra B. The space of* compact maps $K(\mathcal{E}) \subset \mathrm{End}^*_B(\mathcal{E})$ *is the (operator norm) closure of the linear space of the rank one operators*

$$\begin{aligned} |\psi\rangle\langle\phi| : \mathcal{E} &\to \mathcal{E} \\ \xi &\mapsto \psi\langle\phi, \xi\rangle. \end{aligned}$$

Example 1.1.15. *Consider A as a Hilbert module over A, then the operator $m_a : A \to A$, $b \mapsto ab$ is adjointable and compact. The adjoint map is given by $m_a^* = m_{a^*}$, indeed*

$$\langle b, m_a c\rangle = b^* a c = (a^* b)^* c = \langle m_{a^*} b, c\rangle.$$

If A is unital, $m_a = |1\rangle\langle a^*|$ and thus compact. If A is not unital observe that the operator norm of m_a is $\|a\|$ (by submultiplicativity and the C^* identity). Then if $\{\mu_n\}$ is an approximate unit for A, we have $m_a - m_{\mu_n a} = m_{a-\mu_n a}$ and $a - \mu_n a$ converges to zero in operator norm. Since $m_{\mu_n a} = |\mu_n\rangle\langle a^*|$ we get that m_a is compact in this case as well.

Now that we have the basic notions of maps on a Hilbert module we can introduce the two key components of an unbounded KK-cycle.

Definition 1.1.16. *Let A, B be C^*-algebras. A* Hilbert A-B bimodule *is a Hilbert B-module $\mathcal{E}$ with an algebra homomorphism $\rho : A \to \mathrm{End}^*_B(\mathcal{E})$. This will occasionally be written as ${}_A\mathcal{E}_B$ and the representation ρ is usually suppressed in the notation.*

If A and $\mathcal{E}$ are equipped with (possibly trivial) $\mathbb{Z}/2\mathbb{Z}$-gradings the representation of A on $\mathcal{E}$ must be compatible with these gradings.

Definition 1.1.17. *Let $\mathcal{E}$ be a Hilbert module over a C^*-algebra B. An* unbounded operator *on $\mathcal{E}$ is given by a domain $\mathrm{Dom}(D)$, dense in $\mathcal{E}$, and a map $D : \mathrm{Dom}(D) \to \mathcal{E}$ such that it has a densely defined adjoint $D^* : \mathrm{Dom}(D^*) \to \mathcal{E}$. This means that for $\psi \in \mathrm{Dom}(D^*)$ and $\phi \in \mathrm{Dom}(D)$ one has*

$$\langle D^*\psi, \phi\rangle = \langle \psi, D\phi\rangle.$$

An unbounded operator is regular *if $1 + D^*D$ has dense range.*

Other adjectives for unbounded operators, in particular *closed*, *closeable*, *symmetric*, and *self-adjoint* have the same meaning as in the Hilbert space setting. The new keyword here is *regular*, on a Hilbert space regularity is automatic but for Hilbert modules it is an essential well-behavedness property (see [44, Ch. 9] for more details).

This has, finally, put us in the position where we can define the principal object of Chapter 2, the unbounded KK-cycle. These unbounded KK-cycles were invented in [1] to simplify computations of the external product of KK-classes to, essentially, the external product we saw in Section 1.1.2.

Definition 1.1.18. *Let A, B be C^*-algebras. An* unbounded A-B KK-cycle *consists of a $\mathbb{Z}/2\mathbb{Z}$-graded Hilbert bimodule ${}_A\mathcal{E}_B$, a dense subalgebra $\mathcal{A} \subset A$, and an odd self-adjoint regular operator $D : \mathrm{Dom}(D) \to \mathcal{E}$ such that*

- *$\mathcal{A}$ preserves the domain of D and $[D,a]$ extends to an element of* $\mathrm{End}^*_B(\mathcal{E})$ *for all* $a \in \mathcal{A}$,
- $a(D+i)^{-1} \in K(\mathcal{E})$ *for all* $a \in A$.

A spectral triple $(\mathcal{A}, H, D)$ is an example of an unbounded A-$\mathbb{C}$ KK-cycle, as can easily be concluded by comparing this definition to that of a spectral triple, Definition 1.1.1. This is the crucial fact that encourages us to look at the unbounded picture of KK-theory as a source of inspiration for maps between spectral triples. In particular there is the Kasparov product, which tells us that there is a notion of composition of unbounded A-B and B-C KK-cycles to an unbounded A-C KK-cycle.

1.1.4 The Kasparov product and connections

The Kasparov, or internal, product is an operation in KK-theory that we will interpret as a composition for unbounded KK-cycles. To do this we will make use of connections and one-forms corresponding to an unbounded KK-cycle. We will make use of pre-C^*-algebras and pre-Hilbert modules in order to sidestep a lot of the analytical subtleties in this construction. We can get away with this since our application is entirely based on smooth manifolds with (sufficiently) elliptic differential operators that allow us to use the smooth functions as our pre-C^*-algebras.

For a more complete introduction to connections in noncommutative geometry one can see [12]. For the background of our particular construction see [37, 36, 49] and in particular [47]. The specifics surrounding curvature in this setting can be found in [50].

We will begin with some definitions.

Definition 1.1.19. *A* pre-C^*-algebra A *satisfies the same conditions as a C^*-algebra except that it need not be complete in the norm.*

A pre-Hilbert B module, *for a pre-C^*-algebra B, satisfies the same conditions as a Hilbert B module, except that it need not be complete in the norm. Similarly for pre-Hilbert bimodules.*

The idea behind working with pre-C^*-algebras and pre-Hilbert (bi)modules is that they correspond to the smooth functions (sections) inside the continuous functions (sections). This will allow us to talk about connections in a way that prepares us nicely for Chapter 2. Even though there we do not explicitly mention pre-C^*-algebras and pre-Hilbert bimodules, it is often in the same spirit.

We can now introduce the composition of Hilbert bimodules with the following result, masquerading as a definition. That this indeed defines a (pre-)Hilbert A-C bimodule is not obvious. In particular definiteness of the inner product requires some work, see [44, Prop. 4.5].

Definition 1.1.20. *Let A, B, C be pre-C^*-algebras and let $\mathcal{E}$ be a pre-Hilbert A-B bimodule and $\mathcal{F}$ a pre-Hilbert B-C bimodule. Then the balanced tensor product of $\mathcal{E}$ and $\mathcal{F}$, denoted $\mathcal{E} \otimes_B \mathcal{F}$ is the quotient of the (graded, if applicable) algebraic tensor product*

$$\mathcal{E} \otimes_{B,alg} \mathcal{F} = \mathcal{E} \otimes_{alg} \mathcal{F} / \sim_B$$

where $\sim_B$ is the subspace generated by elements of the form

$$\psi \cdot b \otimes \phi - \psi \otimes b \cdot \phi.$$

This balanced tensor product comes with a pre-Hilbert A-C bimodule structure with the inner product

$$\langle \psi \otimes \phi, \psi' \otimes \phi' \rangle = \langle \phi, \langle \psi, \psi' \rangle \cdot \phi' \rangle.$$

If $\mathcal{E}$ and $\mathcal{F}$ are Hilbert modules, $\mathcal{E} \otimes_B \mathcal{F}$ can be completed in the norm associated to its inner product to produce a Hilbert A-C bimodule.

The next point for composition of unbounded KK-cycles is the operator on the balanced tensor product of the Hilbert bimodules. Suppose we have two unbounded KK-cycles, $({}_A\mathcal{E}_B, D_1)$, $({}_B\mathcal{F}_C, D_2)$, the Hilbert bimodule of the composition is given by ${}_A(\mathcal{E} \otimes_B \mathcal{F})_C$ and the operator should be at least modelled on $D_1 \otimes 1 + 1 \otimes D_2$. The main issue with defining this operator is that D_2 need not commute with the B action so that $1 \otimes D_2$ is not well defined on the balanced tensor product.

To make sense of this operator we will introduce an additional piece of data on the first factor, namely a connection. To avoid talking about domains we will phrase this in terms of pre-Hilbert bimodules and pre-C^*-algebras. The operators involved are, however, still unbounded.

Definition 1.1.21. *Let B be a pre-C^*-algebra with $\mathbb{Z}/2\mathbb{Z}$-grading operator γ, possibly $\gamma = 1$, and define*

$$\begin{aligned} m : B \otimes B &\to B \\ b_1 \otimes b_2 &\mapsto b_1 \gamma(b_2). \end{aligned}$$

We define the universal space of one forms *as the B-bimodule $\Omega^1_u(B) = \ker m$, which is graded by $-\gamma \otimes \gamma$.*

The universal derivative *is the map*

$$\begin{aligned} \delta : B &\to \Omega^1_u(B) \\ b &\mapsto 1 \otimes b - \gamma(b) \otimes 1. \end{aligned}$$

The space of universal two forms *is $\Omega^2_u(B) = \Omega^1_u(B) \otimes_B \Omega^1_u(B)$ and the universal derivative extends to the map*

$$\begin{aligned} \delta : \Omega^1_u(B) &\to \Omega^2_u(B) \\ a\delta(b) &\mapsto \delta(a) \otimes \delta(b). \end{aligned}$$

Definition 1.1.22. *A* universal connection *for a pre-Hilbert B-module $\mathcal{E}$ with grading operator γ is an odd linear map*

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes_B \Omega^1_u(B)$$

such that

$$\nabla(\psi \cdot b) = \nabla(\psi) \cdot b + \gamma(\psi) \otimes \delta(b).$$

If additionally

$$\langle \nabla(\gamma(\psi)), \phi \rangle - \langle \gamma(\psi), \nabla(\phi) \rangle = \delta(\langle \psi, \phi \rangle)$$

the connection is said to be metric.

Lemma 1.1.23. *Let B be a unital pre-C^*-algebra, and $\omega \in \Omega^1_u(B)$ odd, then*

$$\begin{aligned}\nabla^B_u : B &\to B \otimes_B \Omega^1_u(B)\\ b &\mapsto 1 \otimes \delta(b) + 1 \otimes \omega b\end{aligned}$$

defines a connection.

Proof. In our setting this is a straightforward algebraic exercise,

$$\begin{aligned}\nabla^B_u(ab) &= 1 \otimes \delta(ab) + 1 \otimes \omega ab,\\ &= 1 \otimes \delta(a)b + 1 \otimes \gamma(a)\delta(b) + 1 \otimes \omega ab,\\ &= \nabla^B_u(a)b + \gamma(a) \otimes \delta(b).\end{aligned}$$

The oddness condition on ω comes from the requirement that ∇ be odd. □

With a connection we can define the product operator, but we have to leave the realm of universal forms for the represented forms.

Definition 1.1.24. *Let B, C be pre-C^*-algebras, $\mathcal{F}$ a pre-Hilbert B-C bimodule and $D_2 : \mathcal{F} \to \mathcal{F}$ an adjointable map*[3]*. Then*

$$\begin{aligned}\Omega^1_{D_2}(B) &:= \{a[D_2, b] \,|\, a, b \in B\},\\ \Omega^2_{D_2}(B) &:= \{a[D_2, b][D_2, c] \,|\, a, b, c \in B\}\end{aligned}$$

which are subspaces of the adjointable operators on $\mathcal{F}$.

Further define

$$\begin{aligned}\pi_{D_2} : \Omega^1_u(B) &\to \Omega^1_{D_2}(B)\\ a\delta(b) &\mapsto a[D_2, b].\end{aligned}$$

If $\mathcal{E}$ is a pre-Hilbert A-B bimodule with a universal connection $\nabla^{\mathcal{E}}_u$ the represented connection is defined by

$$\nabla^{\mathcal{E}}_{D_2} := (1 \otimes \pi_{D_2}) \circ \nabla^{\mathcal{E}}_u.$$

Definition 1.1.25. *Let $\mathcal{E}$, $\mathcal{F}$ be graded unbounded A-B and B-C pre-Hilbert bimodules respectively, with self-adjoint maps $D_1 : \mathcal{E} \to \mathcal{E}$ and $D_2 : \mathcal{F} \to \mathcal{F}$. Furthermore assume $\nabla^{\mathcal{E}}_u$ is a universal metric connection on $\mathcal{E}$. The* product operator *on $\mathcal{E} \otimes_B \mathcal{F}$ is defined by*

$$D_1 \times_{\nabla\varepsilon} D_2 = D_1 \otimes 1 + 1 \otimes_{\nabla\varepsilon} D_2,$$

where

$$(1 \otimes_{\nabla\varepsilon} D_2)(\psi \otimes \phi) := \gamma(\psi) \otimes D_2\phi + \nabla^{\mathcal{E}}_{D_2}(\psi)\phi.$$

This is again a proposition masquerading as a definition. One should for example check that the product operator on $\mathcal{E} \otimes_B \mathcal{F}$ is again self-adjoint. This corresponds to the (unbounded) product operator on the Hilbert module completion being symmetric on its domain. It is then a further question if this product operator has a self-adjoint extension to the Hilbert module completion. These questions are tackled on a case-by-case basis for our situation in Section 2.3 with smooth sections of spinor bundles playing the role the pre-Hilbert modules.

[3]Since $\mathcal{F}$ is only a *pre*-Hilbert bimodule this does not imply that D_2 is bounded.

Definition 1.1.26. *Let $\mathcal{E}$ be a pre-Hilbert B module for a pre-C^*-algebra B with universal connection $\nabla_u^{\mathcal{E}}$. The universal curvature of this connection is defined to be the map from $\mathcal{E}$ to $\mathcal{E} \otimes \Omega_u^2(B)$*

$$\left(\nabla^{\mathcal{E}}\right)_u^2 = \left(1 \otimes_{\nabla_u^{\mathcal{E}}} \delta\right) \circ \nabla_u^{\mathcal{E}}$$

and the represented curvature relative to a pre-Hilbert B-C bimodule $\mathcal{F}$ with $D_2 : \mathcal{F} \to \mathcal{F}$ is given by

$$\pi_{D_2}\left(\left(\nabla^{\mathcal{E}}\right)^2\right)(\psi \otimes \phi) := \left(\left((1 \otimes m) \circ (1 \otimes 1 \otimes \pi_{D_2}) \circ (1 \otimes_{\nabla_{D_2}^{\mathcal{E}}} \delta) \circ \nabla_u^{\mathcal{E}}\right)(\psi)\right)\phi.$$

where $\psi \otimes T \in \mathcal{E} \otimes \Omega_{D_2}^2(B)$ acts on $\mathcal{F}$ by $\phi \mapsto \psi \otimes T\phi$.

This definition is not terribly enlightening beyond its similarity to ∇^2 in regular differential geometry, the following lemma gives some further reason this definition is sensible by strengthening that link. A much more complete motivation for this definition can be found in [50].

Lemma 1.1.27. *For a connection ∇_u^B as in Lemma 1.1.23 the universal curvature is given by*

$$(\nabla)_u^2(a) = 1 \otimes (\delta(\omega) + \omega \otimes \omega)\, a$$

and the represented curvature relative to an operator $D_2 : \mathcal{F} \to \mathcal{F}$ on a pre-Hilbert B-C bimodule $\mathcal{F}$ is

$$\pi_{D_2}\left(\left(\nabla^A\right)^2\right)(a \otimes \phi) = \left((\pi_{D_2} \otimes \pi_{D_2})(\delta(\omega)) + \pi_{D_2}(\omega)^2\right) a\phi.$$

Proof. We compute the universal case first.

$$\begin{aligned}
(1 \otimes_{\nabla_u^B} \delta)(\nabla_u^B(b)) &= (1 \otimes_{\nabla_u^B} \delta)(1 \otimes \delta(b) + 1 \otimes \omega b), \\
&= \gamma(1) \otimes \delta^2(b) + \nabla_u^B(1) \otimes \delta(b) + 1 \otimes \delta(\omega b) + \nabla_u^B(1) \otimes \omega b, \\
&= 1 \otimes \omega \otimes \delta(b) + 1 \otimes (\delta(\omega)b + \gamma(\omega)\delta(b)) + 1 \otimes \omega \otimes \omega b, \\
&= 1 \otimes \omega \otimes \delta(b) + 1 \otimes \delta(\omega)b - 1 \otimes \omega\delta(b)) + 1 \otimes \omega \otimes \omega b, \\
&= 1 \otimes (\delta(\omega) + \omega \otimes \omega)b.
\end{aligned}$$

Note that in the fourth line we used that ω is odd.

The analogous statement for the represented curvature follows from essentially the same computation, keeping track of where the representation π_{D_2} is introduced. □

This concludes our brief and very narrow discussion of curvature for unbounded KK-cycles. The last port of call for this section is a useful result from [47] that gives us strong tools to check when the above (purely algebraic, in this incarnation) construction actually works at an analytic level.

Definition 1.1.28. *A pair of operators D_1, D_2 on a graded Hilbert B-module $\mathcal{E}$ is* weakly graded commuting *if*

- *for some C_0, C_1, C_2 and $\psi \in \mathrm{Dom}([D_1, D_2])$ we have*

$$\|[D_1, D_2]\psi\|^2 \leq C_0\|\psi\|^2 + C_1\|D_1\psi\|^2 + C_2\|D_2\psi\|^2,$$

- *There is a core E for D_2 such that $(D_1 + i\lambda)^{-1}E \subset \mathrm{Dom}([D_1, D_2])$,*

here we stress that the commutator is the graded commutator, so if both D_1 and D_2 are odd $[D_1, D_2] = D_1D_2 + D_2D_1$.

For an weakly graded commuting pair we have the following theorem.

Theorem 1.1 (Thm. 2.6 [47])**.** *If D_1, D_2 is a pair of weakly graded commuting self-adjoint regular operators on a Hilbert B-module $\mathcal{E}$, then*

- *$D_1 + D_2$ is self-adjoint and regular with domain $\mathrm{Dom}(D_1) \cap \mathrm{Dom}(D_2)$.*
- *$\|\psi\|^2 + \|(D_1 + D_2)\psi\|^2 \leq C\left(\|\psi\|^2 + \|D_1\psi\|^2 + \|D_2\psi\|^2\right)$*

This theorem is useful in the context of Kasparov products, or composition, of unbounded KK-cycles by giving a powerful criterion for the sum of $D_1 \otimes 1$ and $1 \otimes_\nabla \varepsilon\ D_2$ to be self-adjoint. The second criterion is useful in establishing that $D_1 \otimes 1 + 1 \otimes_\nabla \varepsilon\ D_2$ has compact resolvents.

We will now give two, very simple, cases as example applications of Definition 1.1.28. Both these cases can be done by other means as well.

Lemma 1.1.29. *Let $({}_A\mathcal{E}_B, D_1)$ and $({}_B\mathcal{F}_C, 0)$ be unbounded KK-cycles. Then the product cycle*

$$\left({}_A\left(\mathcal{E} \otimes_B \mathcal{F}\right)_C, D_1 \otimes 1\right)$$

is an unbounded KK-cycle.

Proof. The pair $D_1 \otimes 1$, 0 is certainly a weakly graded commuting pair of self-adjoint operators ($D_1 \otimes 1$ is self-adjoint by general theory, for example [58, Thm. VIII.33]). The operator $D_1 \otimes 1$ has compact resolvents since 0 does, as this implies that $\mathcal{F}$ is finitely generated. □

Lemma 1.1.30. *Let $({}_A(H_1)_\mathbb{C}, D_1)$ and $({}_\mathbb{C}(H_2)_\mathbb{C}, D_2)$ be unbounded KK-cycles. Then the product cycle*

$$\left({}_A\left(H_1 \otimes H_2\right)_\mathbb{C}, D_1 \otimes 1 + \gamma_1 \otimes D_2\right)$$

is an unbounded KK-cycle that moreover represents both the internal and external Kasparov product of the corresponding classes in KK-theory.

Proof. Since we are working with Hilbert spaces all operators are regular and $D_1 \otimes 1$ and $\gamma_1 \otimes D_2$ are self-adjoint by general theory of self-adjoint operators. Moreover, the algebraic tensor product $\mathrm{Dom}(D_1) \otimes_{alg} \mathrm{Dom}(D_2)$ forms a core for $\gamma_1 \otimes D_2$ and A (graded) commutes with $\gamma_1 \otimes D_2$. Finally $D_1 \otimes 1$ graded commutes with $\gamma_1 \otimes D_2$ so they form a weakly commuting pair.

This covers exactly the assumptions of [47, Thm. 7.4] which implies that the product cycle is indeed an unbounded KK-cycle that represents the Kasparov product. □

1.2 Random Matrix Preliminaries

Our introduction to random matrix theory will be as targeted as our introduction to noncommutative geometry was, skipping many interesting sub-areas and perspectives. Useful texts on random matrix theory are, for example, [25, 24, 22, 51]. The study of random matrices was originated by Wigner in [64] to study the spectra of atoms. The basic idea of random matrix theory is to choose a probability density on some space of matrices and study the statistical properties, such as the expectation value for the largest eigenvalue, of those matrices.

Our motivation for studying random matrix models can be found in more detail in Chapter 4, in particular Section 4.1. The main idea is that we would like to study a particular family of spectral triples, 0, 1-fuzzy geometries, with a random Dirac operator. This Dirac operator can be parametrized by a single Hermitian matrix, thus a model involving random Dirac operators becomes a random matrix model.

A prototypical example of random matrix models is the Gaussian Unitary Ensemble, or GUE for short. This is the random matrix model defined by the probability density

$$\frac{1}{Z_N} e^{-N \operatorname{Tr}(H^2)}\, dH$$

on the space of $N \times N$ Hermitian matrices $\mathcal{H}_N$. Here dH is the Lebesgue measure on $\mathcal{H}_N \cong \mathbb{R}^{N^2}$ and Z_N is a normalization constant. For the GUE the normalization Z_N can be computed explicitly, but usually the normalization factor is left as an abstract quantity.

Let us introduce some terminology.

Definition 1.2.1. *A* random matrix ensemble *(or* random matrix model*) is a subset $\mathcal{M}$ of $M_N(\mathbb{R})^n$, $M_N(\mathbb{C})^n$ or $M_N(\mathbb{H})^n$ together with a probability density on $\mathcal{M}$.*

If the probability density is given by a formula of the form

$$\frac{1}{Z} e^{-S(H)}\, dH,$$

where dH is the Lebesgue measure on the appropriate power of $\mathbb{R}$, the function S is called the action *of this ensemble.*

If $S(H) = \operatorname{Tr}(V(H))$ for a polynomial V this polynomial is called the potential *of this ensemble.*

The GUE is one of the simplest matrix models, in particular it has three desirable qualities.

- It is unitarily invariant, *i.e.* the density is invariant under $H \to UHU^*$ for a unitary matrix U.
- It is single matrix.
- It is single trace, meaning that the density is given by the action for a polynomial potential $V(x)$ (to be precise, for the GUE $V(x) = x^2$).

For such models many tools are available. There is even a rich theory if the potential $V(x)$ is such that $e^{-\operatorname{Tr}(V(H))}$ is not integrable, see in particular [24].

In this thesis we will be primarily concerned with a random matrix ensemble coming from a toy model for quantum gravity for which this third property fails quite definitively. First of all the action will not be single-trace, but will include products of traces. Secondly there will be a term in $S(H)$ that is not polynomial in H. Chapter 3 is entirely concerned with generalizing tools from [22] to this new situation.

The rest of this section will be an introduction to one of the main tools for unitarily invariant random matrix ensembles, Weyl's integration formula, followed by a discussion of large-N asymptotics and the spectral density. As mentioned before, this ignores a wealth of subjects that are hiding in the field of random matrix theory. For a slightly broader introduction that is still close to the subjects covered in this thesis see [33]. For a fuller introduction see the books cited at the start of this chapter.

1.2.1 Weyl's integration formula and spectral densities

One of the main tools available for single matrix, unitarily invariant random matrix ensembles over the Hermitian matrices is diagonalization. Suppose we have a random matrix ensemble given by some unitarily invariant action S over the Hermitian matrices. Then S can be written in terms of the eigenvalues of H, as those are a complete set of unitary invariants for a Hermitian matrix.

Let $O : \mathcal{H}_N \to \mathbb{R}$ be any unitarily invariant function, then the expectation value of O is given by

$$\langle O \rangle = \frac{1}{Z} \int_{\mathcal{H}_N} O(H) e^{-S(H)}\, dH = \frac{1}{Z_{ev}} \int_{\mathbb{R}^N} O_{ev}(\vec{\lambda}) e^{-S_{ev}(\vec{\lambda})} \prod_{i<j} |\lambda_i - \lambda_j|^2\, d^N\vec{\lambda} \tag{1.1}$$

where O_{ev} and S_{ev} are O and S expressed in terms of eigenvalues and Z_{ev} is a (different) normalization constant. This formula is easily proved by the change of coordinates $\mathcal{H}_N \cong U_N/(U_1^N) \times \mathbb{R}^N$ given by diagonalization (see [22, Ch. 5.3] or [25, Ch. 1.2]). The square of the Vandermonde determinant, $\prod_{i<j} |\lambda_i - \lambda_j|^2$, is the Jacobian of this change of coordinates.

This formula is specific for $\mathcal{H}_N \subset M_N(\mathbb{C})$. For the symmetric matrices in $M_N(\mathbb{R})$ the exponent of the Vandermonde determinant is 1, while for the self-adjoint matrices in $M_N(\mathbb{H})$ it is 4. This exponent is often denoted β and called the Dyson exponent in the random matrix theory literature.

The value of this exponent plays a significant role in the behaviour of the random matrix ensemble. Interestingly, for a lot of random matrix theory techniques there is no reason to restrict the Dyson exponent to $\beta = 1, 2, 4$, or even β integer (see [23] and references therein). We will see in Chapter 4 that in our toy model of quantum gravity higher (though still integer) β appear naturally.

Given a random matrix ensemble over self-adjoint real, complex or quaternionic matrices we define the *eigenvalue model* by the corresponding probability density on $\mathbb{R}^N$ for the eigenvalues of the random matrix given by the Weyl integration formula. Since for a unitarily invariant random matrix model the eigenvalues of the matrix are the only interesting observables, we choose or focus to be on the distribution of these eigenvalues.

Definition 1.2.2. *Let $(\mathcal{M}, P)$ be a unitarily invariant random matrix ensemble for $\mathcal{M} \subseteq M_N(\mathbb{C})$. Let P_{ev} be the eigenvalue density obtained by Weyl integration, then for $m = 1, \ldots, N$ the m-*point density *associated of the random matrix ensemble is given by*

$$P_m^N(x_1, ..., x_m) = \int_{\mathbb{R}^{N-m}} P_N(x_1, ..., x_N)\, dx_{m+1} ... \, dx_N.$$

The 1-point density will also be referred to as the spectral density.

The spectral density satisfies

$$\langle \mathrm{Tr}(f(H)) \rangle = \int_{\mathbb{R}} f(x) P_1^N(x)\, dx.$$

1.2.2 Large-N limits and scaling

There are very few random matrix ensembles for which interesting observables can be explicitly computed, even for single-trace single-matrix unitarily invariant models. What *is* generally accessible is the large-N limit, or the limit where the matrix sizes tend to infinity. One of the reasons for this is that large random matrices tend to be *asymptotically free*, which is a notion of independence for noncommutative probability spaces [51].

Given a unitarily invariant random matrix model, the large-N limit is concerned with quantities of the form

$$\lim_{N\to\infty} \langle \frac{1}{N} \sum_{i=1}^{N} O(\lambda_i) \rangle_N$$

for functions $O : \mathbb{R} \to \mathbb{R}$, where $\langle \bullet \rangle_N$ is the expectation value of a random matrix model of $N \times N$ matrices[4]. To get a sensible answer out of this limit there are of course conditions on the sequence of random matrix ensembles, but requiring for example that they are "constant" is not only hard to define (since the domain of the probability density changes) it also will not generally lead to interesting results.

Suppose we have a potential V and corresponding action $S(H) = \mathrm{Tr}(V(H))$, the sequence of random matrix models to consider for a sensible large-N limit is then

$$\frac{1}{Z_N} e^{-NS(H)}\, dH$$

defined on $\mathcal{H}_N$. This is formally shown in, for example, [25, 1.4], but it can be concluded by a mostly intuitive argument as well.

Consider the Weyl integration formula, Equation 1.1, for $S(H) = \mathrm{Tr}(V(H))$. We can pull the Vandermonde determinant into the exponential to get a new, non-polynomial, potential for the eigenvalues

$$-\sum_{i=1}^{N} V(\lambda_i) + \frac{\beta}{2} \sum_{i\neq j} \log(|\lambda_i - \lambda_j|).$$

[4]This is a very informal definition, the general idea is that the size of the matrices involved grows to infinity. The specific sizes and normalization of the eigenvalue sum will depend on the specific model.

This is the potential of a gas of N particles in one dimension with $\frac{1}{r}$ repulsion inside the confining[5] potential V. As N grows, the total energy represented by the repulsive force grows like N^2. On the other hand, the total energy in the confining potential grows like N. In order to balance these energies the density of the particles must go to zero, as the particles need to be ever further apart to balance the rapidly increasing repulsion with the confining potential.

If we instead consider the sequence of models with $S_N(H) = N \operatorname{Tr}(V(H))$ both the energy represented by the confining potential and the repulsive force grow like N^2. Now they can be in balance without particles escaping to infinity (the sequence of densities is *tight*, in terms of Definition 3.1.3). Thus we conclude that the correct sequence of models for such a single-matrix ensemble defined by a potential V is $S_N(H) = N \operatorname{Tr}(V(H))$. We could also err the other direction, where we consider for example $S_N(H) = N^2 \operatorname{Tr}(V(H))$. In this case the confining potential will dominate at large N and all eigenvalues will concentrate around the minima of $V(x)$.

We will encounter higher order interactions, such as multi-trace terms, that are expressed as sums over multiple eigenvalues. These terms require a similar normalization. For example

$$\operatorname{Tr}(H)^4 = \left(\sum_{i=1}^{N} \lambda_i\right)^4 = \sum_{i,j,k,l=1}^{N} \lambda_i \lambda_j \lambda_k \lambda_l$$

should be multiplied by the power of N that makes the aggregate of order N^2, so that it is of the same order as the eigenvalue repulsion caused by the Vandermonde term. Although one could deviate from this if the action $S(H)$ has a repulsive force of a different order built in.

[5] If the potential is not confining the action does not define a probability density and should instead be treated as a divergent matrix model, see [33] or [24].

Chapter 2

Immersions between spectral triples

The goal of this chapter is to realize an immersion of manifolds in the language of spectral triples. While this will be couched in the language of unbounded KK-theory I want to stress that the goal is not to do KK-theory. In Section 2.3 we check that our construction makes sense in KK-theory (*i.e.* topologically). But everywhere else we will be thinking of the cycles themselves and not the classes they represent. The idea behind this is that KK-theory is topological in nature while we are looking for geometric data.

The idea of the construction in this chapter is based on the shriek map in noncommutative geometry [16, 11]. Given a smooth, K-oriented map $f : X \to Y$ between Riemannian manifolds, *loc. cit* gives a construction for a bounded KK-cycle $f_!$ between $C(X)$ and $C(Y)$. This class is used to prove an index theorem for foliations and, more importantly for us presently, this construction is functorial. To be precise, if $f : X \to Y$ and $g : Y \to Z$ are smooth maps, then

$$(g \circ f)_! = f_! \otimes g_! \tag{2.1}$$

where $\otimes$ denotes the Kasparov product.

Taking a closer look at the construction of this shriek class, for a spin^c manifold X, the point map $\mathrm{pt} : X \to \{*\}$ gives a class represented by the bounded transform of the canonical spectral triple of X. So if $f : X \to Y$ is a smooth, K-oriented map we get a product in KK theory

$$(\mathrm{pt}_X)_! = f_! \otimes (\mathrm{pt}_Y)_!$$

where $(\mathrm{pt}_X)_!$ and $(\mathrm{pt}_Y)_!$ are represented by the bounded transforms of the spectral triples of X and Y. Our goal is to lift this construction to the level of unbounded KK-cycles and recover geometric data of the map f, or at least in specific cases. In summary, we want to use this bounded construction as inspiration to write down a product of unbounded KK-cycles, in the sense of [49, 37, 50, 38].

In [38] this has been done for the case where $f : X \to Y$ is a submersion of manifolds, since that is one of the cases where the construction of $f_!$ already has an unbounded character. In *loc. cit* an unbounded KK-cycle $(\mathcal{E}_V, D_V)$ of "vertical spinors" and a vertical Dirac operator are constructed. The unbounded product then yields

$$D_X = D_V \otimes 1 + 1 \otimes_{\nabla^V} D_Y + \kappa$$

where κ is a bounded operator giving the curvature of the submersion. The other case where the construction of $f_!$ lends itself to this unbounded experimentation is when f is an immersion. In this chapter we study the further simplified setting of a codimension 1 embedding.

For an embedding it turns out we do not get the same type of factorization that is seen in *loc. cit.*. This is mainly due to the fact that a Riemannian submersion gives control over the Riemannian metric on the entire domain and codomain, while the metric outside the range of an immersion is unknown, see Section 2.1 for more details. This problem is sidestepped by constructing a family of unbounded KK-cycles, each corresponding to progressively smaller neighbourhoods of the embedded manifold. These unbounded KK-cycles are constructed in Section 2.2.

Next, in Section 2.3 we compute the unbounded product of our unbounded KK-cycles with the spectral triple for Y. Followed by the analysis required to verify that this product is again a spectral triple. Finally in Section 2.4 we show, as a check, that our construction gives the Kasparov product when interpreted in KK-theory. We also discuss how to recover the embedded manifold and the mean curvature of the embedded manifold from the family of products.

This chapter is based on the work also reported in the paper [62] and preprint [63].

2.1 Geometric setup

Throughout this chapter let Y be a smooth, $n+1$ dimensional, Riemannian, spin^c manifold and let $\imath : X \hookrightarrow Y$ be a fixed smooth, oriented, Riemannian codimension 1 embedding of a compact manifold X. The unit normal vector field to this embedding will be denoted ν.

Our first goal in this section is establishing the concept of Fermi coordinates, or generalized normal coordinates, around an embedded manifold. A good reference to learn more about this is [30]. Normal coordinates may be familiar from general relativity or general differential geometry, they are obtained by choosing an orthonormal frame at a point of a Riemannian manifold and defining coordinates using geodesic flow along the basis vectors of this frame. This can be adapted to an embedded manifold $X \hookrightarrow Y$ by choosing a frame on the normal bundle of X and defining coordinates around X using geodesic flow along these vectors normal to X to obtain "radial" coordinates around X. Combining this with a choice of coordinates on X itself yields Fermi coordinates. Regular normal coordinates around a point p are an example of Fermi coordinates for the embedding $\{p\} \hookrightarrow Y$.

We will make use of these Fermi coordinates to construct a family of, diffeomorphic but non-isometric, parallel hypersurfaces to X in a tubular neighbourhood of $X \subset Y$ and equip each of these copies with a spin^c structure and Dirac operator. We then find how this family of tangential Dirac operators relates to the ambient Dirac operator on Y.

Along the way several constructions will depend on which of X and Y is even dimensional. The changes between these situations are minor, so we will often do the proofs in one of these cases while recording both versions in the results.

2.1.1 Notation and Fermi coordinates

Given our embedding $\imath : X \hookrightarrow Y$ define the map $\tilde{\imath} : X \times \mathbb{R} \to Y$ by $(x, s) \mapsto \exp_{\imath(x)}(s\nu)$ where exp is the exponential map of Y. In other words, a pair (x, s) is sent to the time s flow starting at $\imath(x)$ along the vectorfield ν. Since X is compact there is some $\varepsilon_0 > 0$ such that $\tilde{\imath} : X \times (-\varepsilon_0, \varepsilon_0) \to Y$ is a diffeomorphism onto its image (this is a tubular neighbourhood theorem, see for example [30, Lemma 2.3]). For a visual guide to theses definitions see Figure 2.1.

With ε_0 we can define two Riemannian manifolds: $X \times (-\varepsilon_0, \varepsilon_0)$, with the product metric, and $W := \tilde{\imath}\,(X \times (-\varepsilon_0, \varepsilon_0))$ with the metric induced by $W \subset Y$. We now redefine $\tilde{\imath}$ to be the diffeomorphism

$$\begin{aligned} \tilde{\imath} :& X \times (-\varepsilon_0, \varepsilon_0) \to W \\ & (x, s) \mapsto \exp_{\imath(x)}(s\nu). \end{aligned}$$

We will also often use the projection

$$\begin{aligned} \pi :& X \times (-\varepsilon_0, \varepsilon_0) \to X \\ & (x, s) \mapsto x, \end{aligned}$$

and the *parallel embeddings*

$$\begin{aligned} \tilde{\imath}_s :& X \to W \\ & x \mapsto \tilde{\imath}(x, s). \end{aligned}$$

Both $X \times (-\varepsilon_0, \varepsilon_0)$ and W come with a foliation into leaves diffeomorphic to X. For W this foliation is induced by the level-sets of the *radial coordinate function*[1]

$$\begin{aligned} \mathbf{s} :& W \to (-\varepsilon_0, \varepsilon_0) \\ & \tilde{\imath}(x, s) \mapsto s. \end{aligned}$$

For $X \times (-\varepsilon_0, \varepsilon_0)$ the foliation is given by leaves having constant coordinate in $(-\varepsilon_0, \varepsilon_0)$.

We call the leaves of this foliation *parallel hypersurfaces* to X and denote them by

$$X_s = \{y \in W \,|\, \mathbf{s}(y) = s\}.$$

By the generalized Gauss Lemma (see for example [30, Cor. 2.14]) the vector field

$$\partial_{\mathbf{s}} = (\tilde{\imath}_s)_* \nu$$

is the unit normal vector field to X_s. The *tangential vector bundle* over W is then the bundle

$$\bigsqcup_{(x,s) \in X \times (-\varepsilon_0, \varepsilon_0)} T_{\tilde{\imath}(x,s)} X_s,$$

and the *normal vector bundle* over W is the line bundle generated by ∂_s.

The key point of this chapter will be to relate, or in some way intertwine, the two Riemannian manifolds $X \times (-\varepsilon_0, \varepsilon_0)$ and W.

[1] The terminology "radial" rather than normal is chosen with regard to the terminology in [30] and future generalizations to higher codimensions. For example in codimension two the tubular neighbourhood will be $X \times D$ for a disk D of radius ε_0, this gives rise to radial (away from X), tangential (along X) and spherical (circling X) directions that all likely behave differently.

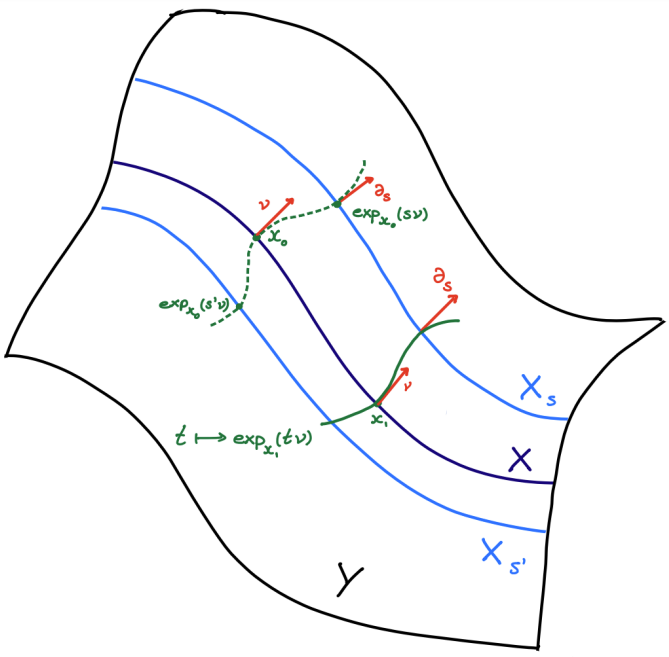


Figure 2.1: This diagram shows how X and the parallel hypersurfaces X_s sit inside Y. The green lines are the paths $\exp_{\imath(x)}(s\nu)$. The distortion of the metrics between X and X_s is illustrated by the distance along X_s between the green paths changing.

Example 2.1.1. *For the embedding of the radius 1 sphere $\imath : S^n \hookrightarrow \mathbb{R}^{n+1}$ the vector field ν is the radial vector field $\partial_r = \sum x_i \partial_i$ so that*

$$\exp_{\imath(\theta)}(s\partial_r) = (s+1)\imath(\theta)$$

for any $\theta \in S^n$ and the right-hand side uses the vector space structure on $\mathbb{R}^{n+1}$. This is a diffeomorphism for $s \in (-1, \infty)$ so we can pick any $\varepsilon_0 < 1$.

The radial coordinate function $\mathbf{s}$ *coincides with the actual radius up to an offset of 1, $\mathbf{s}(\vec{x}) = \|\vec{x}\| - 1$. For the metric g_s the parallel hypersurface $(S^n)_s$ is a sphere of radius $s+1$, while for the metric g_0 it is a sphere of radius 1.*

Using the global radial coordinate we can canonically identify bundles over W with bundles over X.

Lemma 2.1.2. *Let E be a vector bundle over W with a metric connection ∇^E. For any (s, s') there is an isomorphism of vector bundles $U_{s',s} : E|_{X_s} \to E|_{X'_s}$ and $U_{s'',s'}U_{s',s} = U_{s'',s}$.*

Proof. We will define the map $U_{s',s}$ on each fibre. Let $\gamma : (s, s') \to W$ be the path defined by $\exp_{\imath(x)}(t\nu)$. Then parallel transport along γ defines a linear isomorphism $U_{s',s,x}$ between $E_{\tilde{\imath}(x,s)}$ and $E_{\tilde{\imath}(x,s')}$. Since by definition of W these paths preserve the base point in X the linear isomorphisms $U_{s',s,x}$ assemble into an isomorphism of vector bundles $E|_{X_s} \to E|_{X_{s'}}$. □

Corollary 2.1.3. *Suppose E is a vector bundle over Y and $\psi : \imath(X) \to E$ is a section of $E|_{\imath(X)}$. Then ψ extends to a section of E over W.*

Proof. Define $\phi(\tilde{\imath}(x,s)) = U_{s,0}\psi(\imath(x))$. □

We can use Corollary 2.1.3 to construct *Fermi frames*, which have a very useful property.

Lemma 2.1.4. *Let E be a vector bundle over W with metric connection ∇^E and $\left(U, \{e_i\}_{i=1}^k\right)$ a local frame for $E|_{X_0}$. Then in the frame $\left(U \times (-\varepsilon_0, \varepsilon_0), \{U_{\bullet,0}e_i, \partial_s\}_{i=1}^k\right)$ we have*

$$\nabla^E_{\partial_s} = \frac{\partial}{\partial s}.$$

Proof. This is entirely by design. The path $s \mapsto U_{s,0}(e_i)$ is the ∇^E-parallel transport along the geodesic with tangent vector ∂_s, so by definition the frame element $Ue_i : (x,s) \mapsto U_{s,0}e_i(x)$ has

$$\nabla^E_{\partial_s}(Ue_i) = 0.$$

□

As mentioned before our main goal will be to relate or intertwine the Riemannian manifolds $X \times (-\varepsilon_0, \varepsilon_0)$ and W. For this we will employ two main tools. The first one is the *change of volume function* Λ defined by

$$\Lambda \cdot \tilde{\imath}^* \omega_Y = \pi^* \omega_X \wedge \omega_{(-\varepsilon_0,\varepsilon_0)} \tag{2.2}$$

where ω_M denotes the volume form of a Riemannian manifold M.

The change of volume function has the following local expression.

Lemma 2.1.5. *Let $(U, \vec{x})$ be a coordinate patch for X. Then for $(x,s) \in X \times (-\varepsilon_0, \varepsilon_0)$ with $x \in U$ we have*

$$\Lambda(x,s) = \sqrt{\frac{\det g_Y(x,0)}{\det g_Y(x,s)}}.$$

Here $g_Y(x,t)$ is the matrix of the Riemannian metric of Y at $\tilde{\imath}(x,t)$ in the induced Fermi coordinates.

Proof. Choose a patch of local coordinates $(U, (x_1, \ldots, x_n))$ for X, this gives coordinates $(U \times (-\varepsilon_0, \varepsilon_0), (x_1, \ldots, x_n, s))$ on $X \times (-\varepsilon_0, \varepsilon_0)$. Using $\tilde{\imath}$ it further gives coordinates $(\tilde{\imath}(U \times (-\varepsilon_0, \varepsilon_0)), (y_1, \ldots, y_n, \mathbf{s}))$ on W.

In the frame associated to these coordinates we have

$$(\tilde{\imath}^* \omega_Y)_{(x,s)} = \sqrt{\det g_Y(x,s)}\, dx_1 \wedge \ldots \wedge dx_n \wedge ds,$$

since $\tilde{\imath}$ pulls back dy_i to dx_i and $d\mathbf{s}$ to ds. On the other hand we have

$$\pi^* \omega_X \wedge \omega_{(-\varepsilon_0,\varepsilon_0)} = \left(\sqrt{\det g_X(x)}\, dx_1 \wedge \ldots \wedge dx_n\right) \wedge ds.$$

Now, Since ∂_s is the unit normal to each X_s we get $g_Y(x,s) = g_{X_s}(x) \oplus 1$, and in particular $g_Y(x,0) = g_{X_0}(x) \oplus 1 = g_X(x) \oplus 1$ since $\imath$ is a Riemannian embedding.

So in conclusion

$$\Lambda(x,s) = \frac{\sqrt{\det g_X(x)}}{\sqrt{\det g_Y(x,s)}} = \sqrt{\frac{\det g_Y(x,0)}{\det g_Y(x,s)}}.$$

□

The second object that will help us relate $X \times (-\varepsilon_0, \varepsilon_0)$ and W is the second fundamental form, or rather the family of second fundamental forms associated to the family of submanifolds X_s. The second fundamental form is the symmetric $C^\infty(X_s)$-bilinear form defined by

$$\mathrm{II}_{(x,s)}(A, B) := \langle \nabla^Y_A(B), \partial_s \rangle = -\langle B, \nabla^Y_A(\partial_s) \rangle \tag{2.3}$$

for $A, B \in T_{\tilde{\imath}(x,s)} X_s$, the Riemannian metric $\langle \bullet, \bullet \rangle$ is the metric on $X_s \subset Y$. Symmetry of II follows from the observation that $\nabla^Y_A(B) - \nabla^Y_B(A) = [A, B]$ which is tangential to X_s. For a more information on II and its role in the Gauss, Codazzi, and Ricci equations see, for example, [57].

The trace of II is defined to be the trace of the operator, also known as the shape operator,

$$A \mapsto \mathrm{II}(A, \bullet)^\sharp$$

where $\sharp$ is the musical isomorphism from $(T_{\tilde{\imath}(x,s)} X_s)^* \to T_{\tilde{\imath}(x,s)} X_s$ defined by the Riemannian metric. Alternatively

$$\mathrm{Tr}(\mathrm{II}) = \sum_{i=1}^{n} \mathrm{II}(e_i, e_i)$$

for an orthonormal basis $\{e_i\}_{i=1}^n$ of $T_{\tilde{\imath}(x,s)} X_s$.

In many references, such as [30], the shape operator S is preferred over the second fundamental form II. The shape operator is more natural in a way, since as an operator on $T_{\tilde{\imath}(x,s)} X_s$ it has a trace without needing to reference the Riemannian metric. However, we prefer the second fundamental form because it likely generalizes better to higher codimensions. II can be defined without explicit reference to the unit normal vector field by defining it as the difference of the Levi-Civita connections of X_s and Y (which we state as a consequence, in Lemma 2.1.6), in this case II becomes a bilinear form with values in the normal bundle.

Lemma 2.1.6. *Let ∇^Y and ∇^{X_s} be the Levi-Civita connections on Y and X_s respectively. Then*

$$\nabla^{X_s}_A(B) = \nabla^Y_A(B) - \mathrm{II}_{(x,s)}(A, B)\partial_s$$

for $A, B \in T_{\tilde{\imath}(x,s)} X_s$.

Proof. For an embedded submanifold $X_s \subset Y$ the Levi-Civita connection on X_s is given by projecting the Levi-Civita connection of Y to the tangent space of X_s. To be precise, let A, B as in the statement of the lemma and $\pi_{T_{\tilde{\imath}(x,s)} X_s}$ the projection from $T_{\tilde{\imath}(x,s)} Y$ to $T_{\tilde{\imath}(x,s)} X_s$. Then

$$\nabla^{X_s}_A(B) = \pi_{T_{\tilde{\imath}(x,s)} X_s} \nabla^Y_A(B).$$

Since $X_s \subset Y$ has codimension one and ∂_s is a unit normal to X_s we get

$$\nabla^{X_s}_A(B) = \nabla^Y_A(B) - \langle \nabla^Y_A(B), \partial_s \rangle \partial_s$$

which gives the desired result. □

Our two comparison tools, the second fundamental form and the change of volume function, are in fact related.

Lemma 2.1.7. *With Λ as in Equation 2.2 and* II *the second fundamental form as in Equation 2.3. Then*

$$\frac{1}{\Lambda}\left[\nabla^Y_{\partial_s}, \Lambda\right] = \mathrm{Tr}(\mathrm{II}),$$

where the trace of the bilinear form II *is defined using the metric g_Y.*

Proof. Since this is a coordinate independent statement we can verify it in a Fermi frame, i.e. a frame for the tangent bundle of W constructed from a frame for X using Corollary 2.1.3 with ∂_s added in to complete the frame.

In such a frame Lemma 2.1.4 tells us that $\nabla^Y_{\partial_s} = \frac{\partial}{\partial s}$, so we want to show that

$$\frac{1}{\Lambda}\left(\frac{\partial \Lambda}{\partial s}\right) = \mathrm{Tr}(\mathrm{II}).$$

But this is [30, Thm. 3.11], with the observation that their change of volume function θ is our $\frac{1}{\Lambda}$ which accounts for the minus sign. □

Example 2.1.8. *Consider again the embedding of the sphere $S^n \hookrightarrow \mathbb{R}^{n+1}$. For $\partial_\theta, \partial_\phi$ spherical tangent vectors we can compute* II *to be*

$$\mathrm{II}(\partial_\theta, \partial_\phi) = -\langle \partial_\theta, \nabla^{\mathbb{R}^{n+1}}_{\partial_\phi}(\partial_r)\rangle = -\langle \partial_\theta, \frac{1}{r}\partial_\phi\rangle = -\frac{1}{r}\langle \partial_\theta, \partial_\phi\rangle$$

where $\langle \bullet, \bullet\rangle$ refers to the Riemannian metric on the sphere of radius r. Since the trace is defined relative to this metric, it evaluates to $-\frac{n}{r}$ (alternatively note that the corresponding shape operator is $-\frac{1}{r}$ times the identity).

On the other hand, $\Lambda(\vec{\theta}, r) = \frac{1}{r^n}$ since Λ since the volume form on the sphere of radius r is r^n times the volume form on the sphere of radius one. So we indeed have

$$\frac{1}{\Lambda}\partial_r \Lambda = r^n(-nr^{-n-1}) = -\frac{n}{r} = \mathrm{Tr}(\mathrm{II}).$$

In this example II *is independent of the spherical coordinate $\vec{\theta}$, this will of course not be the case in general and leads to significant simplifications for the sphere.*

2.1.2 Relating the Dirac operators

We continue the geometric setup by relating the Dirac operators on the parallel hypersurfaces X_s and the Dirac operator on Y. Similar constructions are done in [2, 9] but we include the details here since we will make use of particular details of the construction. We will also make use of the Pauli matrices that are defined in Example 1.1.7.

The first step in this is to construct a spin^c structure on each X_s.

Lemma 2.1.9. *Let Σ_Y be the spinor bundle over Y. If $n = \dim(X)$ is even, define*

$$\Sigma_{X_s} = \Sigma_Y|_{X_s}$$

with Clifford multiplication

$$c_s(A) = ic_Y(A)c_Y(\partial_s).$$

Then Σ_{X_s} is a spinor bundle for X_s.

If n is odd instead define

$$\Sigma^{\pm}_{X_s} = \Sigma^{\pm}_Y|_{X_s}$$

where $\Sigma^{\pm}_Y$ refers to the $+1$ and -1 eigenspaces of the grading on Σ_Y, equipped with Clifford multiplications

$$c^{\pm}_s(A) = ic_Y(A)c_Y(\partial_s)|_{\Sigma^{\pm}_Y}.$$

In this case both $(\Sigma^{\pm}_{X_s}, c^{\pm}_s)$ define spinor bundles for X_s.

Proof. Since $T_{\tilde{\imath}(x,s)}Y = T_{\tilde{\imath}(x,s)}X_s \oplus \mathbb{R}\partial_s$ and ∂_s is globally defined, the map

$$\begin{aligned} \mathbb{C}\mathrm{l}(TX_s) &\to \mathbb{C}\mathrm{l}^0(TY)|_{X_s} \\ A &\mapsto iA\partial_s, \end{aligned}$$

defined on generators, gives an isomorphism of Clifford bundles.

If n is odd we need to establish that $\mathbb{C}\mathrm{l}^0(TX_s) \cong \mathrm{End}(\Sigma^{\pm}_{X_s})$. The inclusion $\subseteq$ is clear so let $\psi \in \mathrm{End}(\Sigma^+_{X_s}) = \mathrm{End}(\Sigma^+_Y|_{X_s})$. By Corollary 2.1.3 (note that a spinor bundle comes with a spin connection) ψ extends to an endomorphism ψ_+ of Σ^+_Y. Define $\psi_- \in \mathrm{End}(\Sigma^-_Y)$ by $c_Y(\partial_s)\psi_+c_Y(\partial_s)$ so that $\psi_+ \oplus \psi_-$ corresponds to an even endomorphism of Σ_Y.

This means $\psi_+ \oplus \psi_-$ is realized by a section of $\mathbb{C}\mathrm{l}^0(TY)$. This section commutes with $c_Y(\partial_s)$, so its restriction to X_s lies in the image of $\mathbb{C}\mathrm{l}^0(TX_s)$. Thus ψ is realized by an element of $\mathbb{C}\mathrm{l}^0(TX_s)$ and we also have the inclusion $\supseteq$. Starting from a $\psi \in \mathrm{End}(\Sigma^-_{X_s})$ the construction is the same with the correct exchanges of $+$s and $-$s.

If n is even the procedure is simpler. In this case we have $\mathbb{C}\mathrm{l}(TX_s) \cong \mathbb{C}\mathrm{l}^0(TY)|_{X_s} \cong \mathrm{End}(\Sigma_Y)|_{X_s}$, which in turn is isomorphic to $\mathrm{End}(\Sigma_{X_s})$ using Corollary 2.1.3. In this case $c_Y(\partial_s)$ becomes the grading operator for Σ_{X_s}. □

Remark 2.1.10. *If n is odd we can identify Σ^+_Y and Σ^-_Y using $c_Y(\partial_s)$, as we essentially did in the above proof. Under this identification, $c^+_s = -c^-_s$ and we can represent this in terms of Pauli matrices: $\Sigma_Y|_{X_s} \cong \Sigma_{X_s} \otimes \mathbb{C}^2$ with grading $1 \otimes \sigma_3$, $c_Y(\partial_s)$ acting by σ_1 and $\mathbb{C}\mathrm{l}(TX_s)$ acting via $c_s \otimes \sigma_3$.*

Both bundles come with their own Dirac operators, which under the isomorphism by $c_Y(\partial_s)$ differ by a minus sign. In a way, this is the first time the concept of "doubling" odd cycles shows up (see Definition 1.1.10).

In analogy with the tangential vector bundle over W and the terminology in [38] we call the bundle over W defined by

$$\bigsqcup_{s\in(-\varepsilon_0,\varepsilon_0)} \Sigma_{X_s}$$

the *tangential spin bundle* and sections of this bundle *tangential spinors*.

We have now constructed the spin bundles Σ_{X_s} for X_s, but we still need to specify the spin connections on these bundles. For this we use the spin connection version of Lemma 2.1.6 that relates the Levi-Civita connections by the second fundamental form.

Lemma 2.1.11. *Let ∇^{Σ_Y} be the spin connection on Σ_Y, A a tangential vector field over W and ψ a tangential spinor. Then, for n even,*

$$\nabla^{\Sigma_{X_s}}_{A|_{X_s}}\left(\psi|_{X_s}\right) = \left(\nabla^{\Sigma_Y}_A\psi\right)|_{X_s} - \frac{1}{2}ic_s\left(\mathrm{II}(A,\bullet)^\sharp|_{X_s}\right)\left(\psi|_{X_s}\right)$$

defines a spin connection on Σ_{X_s}. For n odd we get

$$\left(\nabla^{\Sigma^+_{X_s}}_{A|_{X_s}} \oplus \nabla^{\Sigma^-_{X_s}}_{A|_{X_s}}\right)\left(\psi|_{X_s}\right) = \left(\nabla^{\Sigma_Y}_A\psi\right)|_{X_s} - \frac{1}{2}i(c^+_s \oplus c^-_s)\left(\mathrm{II}(A,\bullet)^\sharp|_{X_s}\right)\left(\psi_{X_s}\right)$$

where $\nabla^{\Sigma^\pm_{X_s}}$ are spin connections on $\Sigma^\pm_{X_s}$.

Proof. There are several things to check here, let us start with verifying that this definition gives a well-defined spin connection on Σ_{X_s}.

While Corollary 2.1.3 guarantees that all sections of Σ_{X_s} extend to tangential spinors over W it is not a priori clear that this definition is independent of the extension. To see that the definition is in fact independent of the extension we can verify that $\nabla^{\Sigma_Y}_A$ is $C^\infty\left((-\varepsilon,\varepsilon)\right)$-linear, where $g \in C^\infty\left((-\varepsilon_0,\varepsilon_0)\right)$ acts by multiplication.

The connection property gives us

$$\nabla^{\Sigma_Y}_A(g\psi) = A(g)\psi + g\nabla^{\Sigma_Y}_A(\psi) = g\nabla^{\Sigma_Y}_A(\psi),$$

using that A is a tangential vector field and g is tangentially constant. The second term, involving II, is defined in terms of $\psi|_{X_s}$ so is certainly independent of the extension.

We then need to show that $\nabla^{\Sigma_{X_s}}$ actually defines a spin connection. In the following computations let A, B be tangential vector fields over W, and ϕ, ψ tangential spinors. Then, suppressing restrictions to X_s for legibility,

$$\begin{aligned}
\langle\nabla^{\Sigma_{X_s}}_A\psi,\phi\rangle + \langle\psi,\nabla^{\Sigma_{X_s}}_A\phi\rangle &= \langle\nabla^{\Sigma_Y}_A\psi,\phi\rangle - \langle\frac{1}{2}ic_s\left(\mathrm{II}(A,\bullet)^\sharp\right)\psi,\phi\rangle \\
&\qquad + \langle\psi,\nabla^{\Sigma_Y}_A\phi\rangle - \langle\psi,\frac{1}{2}ic_s\left(\mathrm{II}(A,\bullet)^\sharp\right)\phi\rangle, \\
&= A(\langle\psi,\phi\rangle) + \langle\psi,\frac{1}{2}ic_s\left(\mathrm{II}(A,\bullet)^\sharp\right)\phi\rangle - \langle\psi,\frac{1}{2}ic_s\left(\mathrm{II}(A,\bullet)^\sharp\right)\phi\rangle, \\
&= A(\langle\psi,\phi\rangle),
\end{aligned}$$

so $\nabla^{\Sigma_{X_s}}$ is a metric connection.

Additionally we can compute

$$\begin{aligned}
\left[\nabla_A^{\Sigma_{X_s}}, c_s(B)\right] &= \left[\nabla_A^{\Sigma_Y}, ic_Y(B)c_Y(\partial_s)\right] - \frac{1}{2}i\left[c_s\left(\mathrm{II}(A,\bullet)^\sharp\right), c_s(B)\right], \\
&= i\left[\nabla_A^{\Sigma_Y}, c_Y(B)\right]c_Y(\partial_s) + ic_Y(B)\left[\nabla_A^{\Sigma_Y}, c_Y(\partial_s)\right] - \frac{1}{2}i\left[c_s\left(\mathrm{II}(A,\bullet)^\sharp\right), c_s(B)\right], \\
&= ic_Y\left(\nabla_A^Y(B)\right)c_Y(\partial_s) + ic_Y(B)c_Y\left(\nabla_A^Y(\partial_s)\right) - \frac{1}{2}i\left[c_s\left(\mathrm{II}(A,\bullet)^\sharp\right), c_s(B)\right], \\
&= ic_Y\left(\nabla_A^{X_s}(B)\right)c_Y(\partial_s) + ic_Y\left(\mathrm{II}(A,B)\partial_s\right)c_Y(\partial_s) \\
&\qquad - ic_Y(B)c_Y\left(\mathrm{II}(A,\bullet)^\sharp\right) - \frac{1}{2}i\left[c_s\left(\mathrm{II}(A,\bullet)^\sharp\right), c_s(B)\right], \\
&= c_s\left(\nabla_A^{X_s}(B)\right) + i\,\mathrm{II}(A,B) - \frac{1}{2}i\left\{c_Y(B), c_Y\left(\mathrm{II}(A,\bullet)^\sharp\right)\right\}, \\
&= c_s\left(\nabla_A^{X_s}(B)\right) + i\,\mathrm{II}(A,B) - i\langle B, \mathrm{II}(A,\bullet)^\sharp\rangle, \\
&= c_s\left(\nabla_A^{X_s}(B)\right),
\end{aligned}$$

using Lemma 2.1.6, that $\mathrm{II}(A,\bullet)^\sharp = -\nabla_A^Y(\partial_s)$ and that $c_s(A)c_s(B) = c_Y(A)c_Y(B)$ in the third and fourth steps. So $\nabla^{\Sigma_{X_s}}$ is a Clifford connection.

The computations for n odd are identical although for well-definedness the additional observation that ∇^{Σ_Y} preserves the grading is required. This follows from the fact that the grading operator is self-adjoint, so that $\Sigma_Y^\pm$ are orthogonal, and the that the connection is metric. □

So we have now established that each X_s is a spin^c manifold by explicitly constructing a spin bundle and spin connection from the the spin structure on Y. This allows us to define Dirac operators on each X_s giving a family of *tangential Dirac operators* over W. The Dirac operator on Y can be decomposed as this family of tangential Dirac operators, a *radial Dirac operator* and a curvature term, as the next proposition shows.

Proposition 2.1.12. *On the neighbourhood W of $\imath(X)$ the Dirac operator D_Y can, if $n = \dim(X)$ is even, be written*

$$(D_Y\psi)(\tilde{\imath}(x,s)) = ic_Y(\partial_s)\left((D_{X_s}\psi|_{X_s})(x) - \frac{1}{2}\operatorname{Tr}\left(\mathrm{II}_{(x,s)}\right)\psi(\tilde{\imath}(x,s)) + \nabla_{\partial_s}^{\Sigma_Y}(\psi)(\tilde{\imath}(x,s))\right),$$

if n is odd instead

$$(D_Y\psi)(\tilde{\imath}(x,s)) = ic_Y(\partial_s)\left(\left((D_{X_s}^+ \oplus D_{X_s}^-)\psi|_{X_s}\right)(x) - \frac{1}{2}\operatorname{Tr}\left(\mathrm{II}_{(x,s)}\right)\psi(\tilde{\imath}(x,s)) + \nabla_{\partial_s}^{\Sigma_Y}(\psi)(\tilde{\imath}(x,s))\right).$$

Proof. We do the calculation for n even. Let $\{e_1, ..., e_n, \partial_s\}$ be a local orthonormal frame for TY and choose a frame for Σ_Y where the $c_Y(e_i)$, $c_Y(\partial_s)$ act by a constant matrix.

Then with summation over the repeated index j,

$$\begin{aligned}(D_Y\psi)(\tilde{\imath}(x,s)) &= ic_Y(e_j)\nabla^{\Sigma_Y}_{e_j}(\psi)(\tilde{\imath}(x,s)) + ic_Y(\partial_s)\nabla^{\Sigma_Y}_{\partial_s}(\psi)(\tilde{\imath}(x,s)),\\ &= -c_Y(\partial_s)c_s(e_j)\left(\nabla^{\Sigma_{X_s}}_{e_j}(\psi)(\tilde{\imath}(x,s)) + \frac{1}{2}ic_s\left(\mathrm{II}_{(x,s)}(e_j,\bullet)^\sharp\right)\psi(\tilde{\imath}(x,s))\right)\\ &\qquad + ic_Y(\partial_s)\nabla^{\Sigma_Y}_{\partial_s}(\psi)(\tilde{\imath}(x,s)),\\ &= ic_Y(\partial_s)\left(ic_s(e_j)\nabla^{\Sigma_{X_s}}_{e_j}(\psi|_{X_s})(\tilde{\imath}_s(x)) - \frac{1}{2}c_s(e_j)c_s\left(\mathrm{II}_{(x,s)}(e_j,\bullet)^\sharp\right)\psi(\tilde{\imath}(x,s))\right.\\ &\qquad \left. + \nabla^{\Sigma_Y}_{\partial_s}(\psi)(\tilde{\imath}(x,s))\right),\\ &= ic_Y(\partial_s)\left((D_{X_s}\psi|_{X_s})(\tilde{\imath}_s(x)) - \frac{1}{2}\operatorname{Tr}(\mathrm{II})\psi(\tilde{\imath}(x,s)) + \nabla^{\Sigma_Y}_{\partial_s}(\psi)(\tilde{\imath}(x,s))\right)\end{aligned}$$

where we use that, because $\{e_i\}$ is orthonormal and II is symmetric,

$$\begin{aligned}\sum_j c_s(e_j)c_s\left(\mathrm{II}_{(x,s)}(e_j,\bullet)^\sharp\right) &= \sum_{j,k} c_s(e_j)c_s\left(\mathrm{II}_{(x,s)}(e_j,e_k)e_k\right),\\ &= \sum_{j,k}\mathrm{II}_{(x,s)}(e_j,e_k)c_s(e_j)c_s(e_k),\\ &= \sum_j \mathrm{II}_{(x,s)}(e_j,e_j).\end{aligned}$$

The statement for n odd follows from essentially the same computation, in that case $\Sigma_Y = \Sigma^+_{X_s}\oplus\Sigma^-_{X_s}$ and on this bundle $ic_Y(e_j) = c_Y(\partial_s)(c^+_s(e_j)\oplus c^-_s(e_j))$, *c.f.* Remark 2.1.10. □

Example 2.1.13. *For the sphere $S^n\hookrightarrow\mathbb{R}^{n+1}$ this recovers the formula for the Dirac operator in spherical coordinates. For $n=1$ we get*

$$D_{\mathbb{R}^2} = ic_{\mathbb{R}^2}(\partial_r)\left(i\sigma_z\frac{1}{r}\frac{\partial}{\partial\theta} - \frac{1}{2}\operatorname{Tr}\left(\mathrm{II}_{\theta,r}\right) + \frac{\partial}{\partial r}\right) = i\frac{\sigma_\theta}{r}\frac{\partial}{\partial\theta} + i\sigma_r\frac{\partial}{\partial r} + i\sigma_r\frac{1}{2r}$$

where $\sigma_\theta = \frac{1}{r}(x\sigma_y - y\sigma_x)$ and $\sigma_r = \frac{1}{r}(x\sigma_x + y\sigma_y)$ and $\mathrm{II}_{\theta,r}$ is the bilinear form on $T_\theta S^1$ previously computed in Example 2.1.8 to be $-\frac{1}{r}\langle\bullet,\bullet\rangle$.

The appearance of the $\frac{1}{2r}$ term is due to the required frame for the spinors, namely a frame wherein σ_r and σ_θ act by constant matrices and wherein $D_{S^1_r} = \frac{1}{r}\frac{\partial}{\partial r}$ (so constant spinors are parallel along circles). Such a frame is for example given by $\psi_1 = \left(\begin{smallmatrix}\sqrt{r}(x-iy)^{-1}\\ 0\end{smallmatrix}\right)$, $\psi_2 = \left(\begin{smallmatrix}0\\ \frac{1}{\sqrt{r}}\end{smallmatrix}\right)$ relative to the spinor basis where $c(\partial_x) = \sigma_x$ and $c(\partial_y) = \sigma_y$.

Example 2.1.14. *This proposition also clarifies the doubling procedure described in Section 1.1.2. In a frame for $\Sigma_Y = \Sigma^+_Y\otimes\mathbb{C}^2$ as described in Remark 2.1.10 we get*

$$\begin{aligned}D_Y &= i\left(1\otimes\sigma_1\right)\left(D^+_{X_s}\otimes\sigma_3 - \frac{1}{2}\operatorname{Tr}\left(\mathrm{II}\right) + \nabla^{\Sigma_Y}_{\partial_s}\right),\\ &= D^+_{X_s}\otimes\sigma_2 + i\left(1\otimes\sigma_1\right)\left(\nabla^{\Sigma_Y}_{\partial_s} - \frac{1}{2}\operatorname{Tr}\left(\mathrm{II}\right)\right),\end{aligned}$$

with grading $1\otimes\sigma_3$, so we see the doubled version of D_{X_s} appear.

2.2 The unbounded KK-cycles

Let us now turn to introducing the basic building blocks of the immersion as an unbounded product. We will start by defining the spectral triples representing X and Y which are simply the (appropriately doubled) canonical spectral triples for these manifolds. Next we turn to the family of unbounded KK-cycles representing the immersion itself. We then conclude with the definition of the index class, that turns out to play a crucial role in our construction.

2.2.1 The spectral triples $[X]$ and $[Y]$

Let us start with the spectral triple for Y. The canonical spectral triple is built from the given spin structure Σ_Y and is given by

$$\left(C_0^\infty(Y), L^2(\Sigma_Y), D_Y\right).$$

If $n = \dim(X)$ is odd, Y is an even dimensional manifold and this is the representative we use.

If n is even, Y is odd dimensional so we want to work instead with the doubled version, see Definition 1.1.10,

$$\left(C_0^\infty(Y), L^2(\Sigma_Y) \otimes \mathbb{C}^2, D_Y \otimes \sigma_1; 1 \otimes \sigma_3\right).$$

This is an unbounded $C_0(Y) \otimes \mathbb{C}\mathrm{l}_1$-$\mathbb{C}$ cycle with the generator of $\mathbb{C}\mathrm{l}_1$ acting by $1 \otimes \sigma_1$.

For X the story is similar. We get a spin^c structure on X (by isometry with X_0) by Lemmas 2.1.9 and 2.1.11. If n is odd we want to use the doubled version of the canonical triple for X, which can be realized by using the full spinor bundle Σ_Y rather than just its $+1$ or -1 eigenspace, see Remark 2.1.10. This gives

$$\left(C^\infty(X), L^2(\Sigma_X^+) \otimes \mathbb{C}^2, D_X^+ \otimes \sigma_2; 1 \otimes \sigma_3\right),$$

an unbounded $C(X) \otimes \mathbb{C}\mathrm{l}_1$-$\mathbb{C}$ cycle with the generator of $\mathbb{C}\mathrm{l}_1$ acting by σ_1. For n even we get

$$\left(C^\infty(X), L^2(\Sigma_X), D_X; c(\nu)\right).$$

In all cases we use $[X]$ and $[Y]$ to refer to the (appropriately doubled) spectral triples.

2.2.2 The immersion class

Defining the unbounded KK-cycle representing the immersion, which we will call $\imath_!$, takes some more work. In part because we will be equipping it with a connection as well. Our construction is based on the simplified description of the bounded shriek class of an immersion in [16, Prop. 2.8]. In [62, Sec. 2.3] we have previously verified that the construction of this section indeed has the correct bounded transform.

Fix any $\varepsilon < \varepsilon_0$ and define the $C(X)$-$C_0(Y)$ bimodule $\mathcal{E}$ by

$$\mathcal{E}_0 = C_0\left(X \times (-\varepsilon, \varepsilon)\right) \tag{2.4}$$

with

$$(g\psi h)(x,s) = g(x)\psi(x,s)h(\tilde{\imath}(x,s))$$

for $g \in C(X)$, $h \in C_0(Y)$. Further equip it with the $C_0(Y)$ valued inner product

$$\langle \psi, \phi \rangle_{\mathcal{E}}(y) = \begin{cases} \Lambda(x,s)\overline{\psi(x,s)}\phi(x,s) & y = \tilde{\imath}(x,s),\ s \in (-\varepsilon,\varepsilon), \\ 0 & \text{elsewhere.} \end{cases}$$

Define the operator

$$\begin{aligned} S_0 : \mathrm{Dom}(S_0) &\to \mathcal{E}_0 \\ \psi &\mapsto f\psi \end{aligned} \tag{2.5}$$

where

$$f(x,s) = f(s) = -\frac{\pi}{2\varepsilon}\tan\left(\frac{\pi s}{2\varepsilon}\right) \tag{2.6}$$

and the domain $\mathrm{Dom}(S_0) = \{\psi \in \mathcal{E}_0 \,|\, f\psi \in \mathcal{E}_0\}$ is the maximal domain of S.

Lemma 2.2.1. *The pair*

$$(\mathcal{E}_0, S_0)$$

defines an unbounded $C(X)$-$C_0(Y)$ cycle.

Proof. By Definition 1.1.18 we need to verify that $\mathcal{E}_0$ is indeed a Hilbert $C(X)$-$C_0(Y)$ bimodule and that S_0 is self-adjoint, has compact resolvents and is regular. To get that $\mathcal{E}_0$ is a Hilbert bimodule we need to check the algebraic structure, which is entirely straightforward so we skip it here, and that $\mathcal{E}_0$ is complete in the induced norm.

The norm induced on $\mathcal{E}_0$ by the $C_0(Y)$-valued inner product is

$$\|\psi\|^2 = \|(\Lambda|\psi|^2) \circ \tilde{\imath}^{-1}\|_{C_0(Y)} = \sup_{(x,s)\in X\times(-\varepsilon,\varepsilon)} |\Lambda(x,s)| \cdot |\psi(x,s)|^2$$

which is equivalent to the $C_0(X \times (-\varepsilon,\varepsilon))$ norm since the change of volume Λ is bounded and bounded away from zero for $\varepsilon < \varepsilon_0$ ([30, Lem. 3.9]). This means that $\mathcal{E}_0$ is complete.

Next we need to establish that S_0 is self-adjoint and regular. As a multiplication operator this is fairly straightforward, even in the Hilbert module context. By definition,

$$\mathrm{Dom}(S_0^*) = \{\psi \in \mathcal{E}_0 \,|\, \exists \phi \text{ such that } \langle \psi, D\xi \rangle = \langle \phi, \xi \rangle \forall \xi \in \mathrm{Dom}(S_0)\}$$

So if $\psi \in \mathrm{Dom}(S_0^*)$, $\psi f\xi = \phi\xi$ for some $\phi \in \mathcal{E}_0$ and all $\xi \in \mathrm{Dom}(S_0)$ which includes all compactly supported continuous functions. This means that $\psi f = \phi$, so $\psi f \in \mathcal{E}_0$ which means $\psi \in \mathrm{Dom}(S_0)$. Since f is multiplication by a real function it is symmetric so this proves that S_0 is self-adjoint.

To find that S_0 has compact resolvents we note that $(S_0 + i)^{-1}$ is multiplication by the function $\frac{1}{f+i}$, which is a function in $C_0((-\varepsilon,\varepsilon))$ so the argument from Example 1.1.15 applies.

Finally S_0 is regular since $1 + S_0^*S_0$ is multiplication by the function $1 + f^2$, which has full range since its inverse is multiplication by the C_0-function $\frac{1}{1+f^2}$. □

Next consider the two Hilbert bimodules

$$\mathbb{C}\mathrm{l}_{1,L} = {}_{\mathbb{C}\mathrm{l}_1}(\mathbb{C}\mathrm{l}_1)_{\mathbb{C}},$$
$$\mathbb{C}\mathrm{l}_{1,R} = {}_{\mathbb{C}}(\mathbb{C}\mathrm{l}_1)_{\mathbb{C}\mathrm{l}_1},$$

where $\mathbb{C}\mathrm{l}_1$ acts by multiplication. The $\mathbb{C}$-valued inner product is given by declaring 1 and the generator e to be an orthonormal basis, the $\mathbb{C}\mathrm{l}_1$-valued inner product is the usual inner product $\langle \alpha, \beta \rangle = \alpha^* \beta$.

We then set

$$\mathcal{E} = \begin{cases} \mathcal{E}_0 \otimes \mathbb{C}\mathrm{l}_{1,R}, & n \text{ is even}, \\ \mathcal{E}_0 \otimes \mathbb{C}\mathrm{l}_{1,L}, & n \text{ is odd}. \end{cases} \tag{2.7}$$

and

$$S = \begin{cases} S_0 \otimes e, & n \text{ is even}, \\ S_0 \otimes ie\gamma, & n \text{ is odd}. \end{cases} \tag{2.8}$$

so that the ε immersion class defined by

$$\imath_!^\varepsilon = (\mathcal{E}, S) \tag{2.9}$$

is an unbounded KK-cycle of the right type to match up with $[X]$ and $[Y]$.

In order to compute the product of the immersion classes $\imath_!^\varepsilon$ with the spectral triple for Y we will equip $\imath_!^\varepsilon$ with a connection as described in Section 1.1.4. To make the discussion somewhat smoother we suppress the identification $\psi \in \mathcal{E}$ and $\psi \circ \tilde{\imath}^{-1} \in C_0(Y)$.

We define the universal connection using Lemma 1.1.23 on $\mathcal{E}$ by, for n even,

$$\begin{aligned} \nabla_u^{\mathcal{E}} &: \mathcal{E} \to \mathcal{E} \otimes \Omega_u^1(C_0(Y) \otimes \mathbb{C}\mathrm{l}_1), \\ &\psi \otimes \alpha \mapsto \gamma(\alpha) \otimes \delta(\psi) + \gamma(\psi\alpha) \otimes \frac{1}{2\Lambda}\delta(\Lambda). \end{aligned} \tag{2.10}$$

Here $\alpha \in \mathbb{C}\mathrm{l}_{L,R}$ and Λ is interpreted a function in $C_0(Y)$ by multiplying it by a bump function that is 1 on $(-\varepsilon, \varepsilon)$ and 0 outside $(-\varepsilon_0, \varepsilon_0)$, this bump function can then be absorbed into ψ. To get the definition for n odd the only change is to replace the codomain by $\mathcal{E} \otimes \Omega_u^1(C_0(Y))$.

Lemma 2.2.2. *The curvature of $\nabla_u^{\mathcal{E}}$ relative to $[Y]$ is*

$$\pi_{D_Y \otimes \sigma_2}\left((\nabla^{\mathcal{E}})^2\right) = 1_{\mathcal{E}} \otimes \left(\frac{1}{2\Lambda}[D_Y, \Lambda]\right)^2 \otimes 1_{\mathbb{C}^2} = 1_{\mathcal{E}} \otimes \left(\frac{1}{4\Lambda^2}[D_{X_\bullet}, \Lambda]^2 - \frac{1}{4}\operatorname{Tr}(\mathrm{II})^2\right) \otimes 1_{\mathbb{C}^2}$$

if n is even. If n is odd the curvature is almost the same, only without the $\mathbb{C}^2$ component and replacing $D_{X_\bullet}$ by $D_{X_\bullet}^+ \oplus D_{X_\bullet}^-$.

Proof. We use Lemma 1.1.27, which tells us that the represented curvature is given (for n even) by

$$\pi_{D_Y \otimes \sigma_2}\left(\left(\nabla^{\mathcal{E}}\right)^2\right) = \left(\left(\pi_{D_Y \otimes \sigma_2} \otimes \pi_{D_Y \otimes \sigma_2}\right)(\delta(\omega)) + \pi_{D_Y \otimes \sigma_2}(\omega)^2\right),$$

where in this case $\omega = \frac{1}{2\Lambda}\delta(\Lambda)$.

The part $\delta(\omega) = 0$ as it is $\frac{1}{2}\delta^2(\log(\Lambda))$ and $\delta^2 = 0$, so we compute the square of $\pi_{D_Y}(\omega)$.

$$\begin{aligned}
\pi_{D_Y\otimes\sigma_2}(\omega)^2 &= \left(\frac{1}{2\Lambda}\left[D_Y\otimes\sigma_2,\Lambda\right]\right)^2, \\
&= \frac{1}{4\Lambda^2}\left(\left[i\gamma_s D_{X_\bullet} - i\gamma_s\frac{1}{2}\operatorname{Tr}(\mathrm{II}_\bullet) + i\gamma_s\nabla^{\Sigma_Y}_{\partial_s},\Lambda\right]\right)^2\otimes\sigma_2^2, \\
&= \frac{1}{4\Lambda^2}\left(\left[i\gamma_s D_{X_\bullet},\Lambda\right] + \left[i\gamma_s\nabla^{\Sigma_Y}_{\partial_s},\Lambda\right]\right)^2\otimes 1_{\mathbb{C}^2}, \\
&= \frac{1}{4\Lambda^2}\big(-\gamma_s[D_{X_\bullet},\Lambda]\gamma_s[D_{X_\bullet},\Lambda] - \gamma_s[D_{X_\bullet},\Lambda]\gamma_s\Lambda\operatorname{Tr}(\mathrm{II}) \\
&\qquad\qquad - \gamma_s\Lambda\operatorname{Tr}(\mathrm{II})\gamma_s[D_{X_\bullet},\Lambda] - \gamma_s\Lambda\operatorname{Tr}(\mathrm{II})\gamma_s\Lambda\operatorname{Tr}(\mathrm{II})\big)\otimes 1_{\mathbb{C}^2}, \\
&= \frac{1}{4\Lambda^2}\left([D_{X_\bullet},\Lambda]^2 - \Lambda^2\operatorname{Tr}(\mathrm{II})^2\right)\otimes 1_{\mathbb{C}^2}
\end{aligned}$$

where we used that Λ commutes with all other functions as well as any Clifford matrices, but $\gamma_s = c_Y(\partial_s)$ anti-commutes with all Clifford matrices.

To get to the desired form the only step left is to note that $\mathcal{E}$ and $\pi_{D_Y\otimes\sigma_2}(\omega)^2$ commute since all Clifford matrices that appear in the curvature are even. □

2.2.3 The index class

The final unbounded KK-cycle we shall need represents a class in $KK_0(\mathbb{C},\mathbb{C})$ which we denote [1] and refer to as the index class. It will appear as a method to cancel out the radial derivative, as pairing of a dual-Dirac (or Bott) element with the radial derivative. See for example [17, Ch. 3.4] for a nice introduction to this, or any book on KK-theory.

Define the operator

$$\begin{aligned}
T : C_c^\infty\left((-\varepsilon,\varepsilon),\mathbb{C}^2\right) &\to L^2\left((-\varepsilon,\varepsilon),\mathbb{C}^2\right) \\
\psi &\mapsto i\sigma_1\frac{d\psi}{ds} - \sigma_2 f_\varepsilon\psi.
\end{aligned} \tag{2.11}$$

We shall show that T is essentially self-adjoint, has compact resolvent and has index 1. The proof of self-adjointness relies on the specific choice of f_ε defined in Equation 2.6. It may be possible to do away with this specific choice and use any function such that $f^2 \pm f'$ are bounded below. For any such f the operator T will have a self-adjoint Friedrichs extension (see for example Chapter 5 in [53]), but one needs to find a way to obtain an extension with index one.

To prove self-adjointness we will show that $T \pm \frac{\pi}{2\varepsilon}i$ have dense range by characterizing their range.

Lemma 2.2.3. *Let $\alpha = \frac{\pi}{2\varepsilon}$, then as operators on $C_c^\infty\left((-\varepsilon,\varepsilon),\mathbb{C}^2\right)$ we have for $\lambda\in\mathbb{R}$, $\lambda\neq 0$*

$$(T+\lambda i)\psi = i\sigma_1 I_\lambda^{-1}\frac{d}{ds}\left(I_\lambda\psi\right)$$

where, for $\lambda \neq \pm\alpha$,

$$I_\lambda(s) = \begin{pmatrix} -\frac{\lambda}{\sqrt{\lambda^2-\alpha^2}}\sinh\left(\sqrt{\lambda^2-\alpha^2}s\right) & \cosh\left(\sqrt{\lambda^2-\alpha^2}s\right) - \frac{1}{\sqrt{\lambda^2-\alpha^2}}f_\varepsilon(s)\sinh\left(\sqrt{\lambda^2-\alpha^2}s\right) \\ \cosh\left(\sqrt{\lambda^2-\alpha^2}s\right) & -\frac{1}{\lambda}\left(\sqrt{\lambda^2-\alpha^2}\sinh\left(\sqrt{\lambda^2-\alpha^2}s\right) - f_\varepsilon(s)\cosh\left(\sqrt{\lambda^2-\alpha^2}s\right)\right) \end{pmatrix}$$

and for $\lambda = \pm\alpha$,

$$I_\lambda = \begin{pmatrix} -\lambda s & 1 - sf_\varepsilon(s) \\ 1 & \frac{1}{\lambda}f_\varepsilon(s) \end{pmatrix}.$$

Proof. We will showcase the computation for $\lambda = \alpha$, the general computation is very similar only much lengthier.

$$\begin{aligned} i\sigma_1 I_\lambda^{-1}\frac{d}{ds}I_\lambda &= i\sigma_1 I_\lambda^{-1}\left(\frac{dI_\lambda}{ds} + I_\lambda\frac{d}{ds}\right) \\ &= i\sigma_1 I_\lambda^{-1}\frac{dI_\lambda}{ds} + i\sigma_1\frac{d}{ds}. \end{aligned}$$

so we compute

$$\begin{aligned} I_\lambda^{-1}\frac{dI_\lambda}{ds} &= -\begin{pmatrix} \frac{1}{\lambda}f(s) & -1 + sf(s) \\ -1 & -\lambda s \end{pmatrix}\begin{pmatrix} -\lambda & -f(s) - sf'(s) \\ 0 & \frac{1}{\lambda}f'(s) \end{pmatrix}, \\ &= \begin{pmatrix} f(s) & -\frac{1}{\lambda}\left(f(s)^2 + f'(s)\right) \\ \lambda & -f(s) \end{pmatrix}, \\ &= f(s)\sigma_3 - \lambda\sigma_1, \end{aligned}$$

since $\frac{1}{\lambda}(f_\varepsilon^2 + f_\varepsilon') = -\frac{\pi}{2\varepsilon}$. Multiplying by $i\sigma_1$ then gives

$$f(s)\sigma_2 - i\lambda.$$

so that in total we recover $T - \lambda i$. □

Proposition 2.2.4. *The range of* $T \pm \frac{\pi}{2\varepsilon}i$ *are dense.*

Proof. We can characterize the range of $T \pm \frac{\pi}{2\varepsilon}i$ using Lemma 2.2.3. Let $u, g \in C_c^\infty\left((-\varepsilon,\varepsilon), \mathbb{C}^2\right)$ and $\lambda = \pm\frac{\pi}{2\varepsilon}$ such that $(T + \lambda i)u = g$, then

$$I_\lambda g = i\frac{d}{ds}I_\lambda\sigma_1 u$$

so

$$\int_{-\varepsilon}^{\varepsilon} I_\lambda g(s)\, ds = i\left[I_\lambda\sigma u\right]_{-\varepsilon}^{\varepsilon} = 0.$$

Conversely, if g satisfies the above equation we can define $u \in C_c^\infty\left((-\varepsilon,\varepsilon), \mathbb{C}^2\right)$ by

$$u(s) = -i\sigma_1 I_\lambda^{-1}\int_{-\varepsilon}^{s} I_\lambda(t)g(t)\, dt.$$

To see that the set of functions $g \in C_c^\infty((-\varepsilon,\varepsilon),\mathbb{C}^2)$ satisfying $\int I_\lambda g\,ds$ is dense suppose $h \in C_c^\infty((-\varepsilon,\varepsilon),\mathbb{C}^2)$. Let

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \int_{-\varepsilon}^{\varepsilon} I_\lambda(s)h(s)\,ds.$$

Since $f(s)$ is unbounded and odd we can find an odd function h_2 of arbitrarily small L^2-norm such that $\int I_\lambda(s)\left(\begin{smallmatrix}0\\ h_2(s)\end{smallmatrix}\right)ds = \left(\begin{smallmatrix}0\\ -\beta\end{smallmatrix}\right)$. Similarly we can find an even function h_1 of similarly arbitrarily small L^2-norm such that $\int I_\lambda(s)\left(\begin{smallmatrix}0\\ h_1(s)\end{smallmatrix}\right)ds = \left(\begin{smallmatrix}-\alpha\\ 0\end{smallmatrix}\right)$.

Then $\tilde{h} := h + \left(\begin{smallmatrix}0\\ h_1+h_2\end{smallmatrix}\right)$ satisfies $\int I_\lambda \tilde{h} = 0$ and is arbitrarily close to h. □

Corollary 2.2.5. *The operator T is essentially self-adjoint with the compactly supported smooth functions as domain.*

The next point of order is to show that T has compact resolvents.

Lemma 2.2.6. *For $\lambda = \pm\frac{\pi}{2\varepsilon}$ we have*

$$\|\psi'\|^2 \leq \|(T+\lambda i)\psi\|^2.$$

Proof. We compute the right-hand side for $g \in C_c^\infty((-\varepsilon,\varepsilon),\mathbb{C}^2)$,

$$\begin{aligned}\langle (T+\lambda i)g, (T-\lambda i)g\rangle &= \langle g, \left(T^2+\lambda^2\right)g\rangle,\\ &= \langle g, \begin{pmatrix} -\frac{d^2}{ds^2}+f(s)^2-f'(s)+\lambda^2 & 0\\ 0 & -\frac{d^2}{ds^2}+f(s)^2+f'(s)+\lambda^2\end{pmatrix} g\rangle,\\ &= \|g'\|^2 + \langle g, \begin{pmatrix}-2f'(s) & 0\\ 0 & 0\end{pmatrix} g\rangle\end{aligned}$$

which gives the desired result since $-2f'(s) \geq 0$. □

Proposition 2.2.7. *For $\lambda = \pm\frac{\pi}{2\varepsilon}$ the resolvents $T+\lambda i$ are compact.*

Proof. We show that the domain of the self-adjoint extension of T is compact which implies that T has compact resolvents. Any element in the domain of the extension of T is the limit of smooth functions in the graph norm of T. Since the Sobolev norm can be bounded by the graph norm of T by Lemma 2.2.6 any such element lies in $H^1((-\varepsilon,\varepsilon),\mathbb{C}^2)$. The resolvent than factors as through $H^1((-\varepsilon,\varepsilon),\mathbb{C}^2)$ so it is compact by the Rellich embedding theorem. □

By Corollary 2.2.5 and Proposition 2.2.7 the pair

$$\left(L^2\left((-\varepsilon,\varepsilon),\mathbb{C}^2\right), T\right) \tag{2.12}$$

defines an unbounded KK-cycle for $KK_0(\mathbb{C},\mathbb{C})$ with grading σ_3.

The final property of [1] that we want to prove is that it represents the multiplicative unit in $KK_0(\mathbb{C},\mathbb{C})$. This means that we need to prove that T has (graded) index one, since the index map gives the ring isomorphism $KK_0(\mathbb{C},\mathbb{C}) \cong \mathbb{Z}$ (e.g. [8, Prop. 18.8.1]).

Proposition 2.2.8. *The operator T has index one.*

Proof. The (graded) index of T can be computed by $\dim\ker T_+ - \dim\ker T_-$ where $T_\pm = i\frac{d}{ds} \mp if$. The solution to the differential equations

$$\frac{d}{ds}g = \pm fg$$

is given by $g = e^{\pm F}$ where F is an anti-derivative for f. In our case $F(s) = \log\left(\cos\left(\frac{\pi s}{2\varepsilon}\right)\right)$, so $\ker T_+$ is spanned by $\cos\left(\frac{\pi s}{2\varepsilon}\right)$ and is therefore one dimensional. On the other hand $\ker T_-$ would be spanned by $\left(\cos\left(\frac{\pi s}{2\varepsilon}\right)\right)^{-1}$, but this is not an L^2 function so $\dim\ker T_- = 0$. So the index of T is 1. □

2.3 The product of the immersion class and X

Now that we have the unbounded cycles ready it is time to take the product of $\tilde{\imath}_!^\varepsilon$ and $[Y]$ to see how $[X]$ and [1] arise.

We will often be dealing with families of operators acting on a bundle over their base. For example with n even, we have a Dirac operator D_{X_s} for all $x \in (-\varepsilon_0, \varepsilon_0)$ that can act on $L^2(\Sigma_{X_s})$ for each s. These operators combine into the family operator on $L^2(\Sigma_Y|_W)$

$$\left(D_{X_\bullet}\psi\right)(x,s) = \left(D_{X_s}\psi|_{X_s}\right)(x).$$

We will use this $\bullet$-notation whenever we have such a family of operators.

2.3.1 Construction of the product

The first step for this is to compute the Hilbert space for the product of $\tilde{\imath}_!^\varepsilon$ and $[Y]$. This space is the balanced tensor product

$$\mathcal{E} \otimes_{C_0(Y)\otimes\mathbb{C}l_1} \left(L^2(\Sigma_Y)\otimes\mathbb{C}^2\right)$$

if n is even and

$$\mathcal{E} \otimes_{C_0(Y)} L^2(\Sigma_Y)$$

if n is odd.

Recall from Section 2.1 that $\pi : X\times(-\varepsilon_0,\varepsilon_0) \to X$ denotes the projection map. From here on we will also use π to denote the projection map to X on any $X\times(-\varepsilon,\varepsilon)$, which ε is intended will be clear from context.

Lemma 2.3.1. *The map*

$$\begin{aligned} U : \mathcal{E}_0 \otimes_{C_0(Y)} L^2(\Sigma_Y) &\to L^2(\pi^*\Sigma_Y|_{\imath(X)}) \\ \psi\otimes\phi &\mapsto \psi\cdot(\phi\circ\tilde{\imath}) \end{aligned}$$

is a unitary map that preserves the smooth sections.

Proof. We can use Lemma 2.1.2 to identify $(\Sigma_Y)_{(x,s)} \cong (\Sigma_Y)_{(x,0)}$, since this is done by parallel transport using the metric connection on Σ_Y it preserves the inner product on Σ_Y so we can calculate

$$\begin{aligned}
&\langle \psi_1 \otimes \phi_1, \psi_2 \otimes \phi_2 \rangle \\
&\quad = \langle \phi_1, \langle \psi_1, \psi_2 \rangle_{\mathcal{E}} \cdot \phi_2 \rangle, \\
&\quad = \int_Y \langle \phi_1(y), \Lambda(y) \overline{\psi_1(\tilde{\imath}^{-1}(y))} \psi_2(\tilde{\imath}^{-1}(y)) \phi_2(y) \rangle_{\Sigma_Y} \, dy, \\
&\quad = \int_{X \times (-\varepsilon,\varepsilon)} \Lambda(x,s) \langle \psi_1(x,s)\phi_1(\tilde{\imath}(x,s)), \psi_2(x,s)\phi_2(\tilde{\imath}(x,s)) \rangle_{\Sigma_Y} \sqrt{\det g_Y(x,s)} \, dx \, ds, \\
&\quad = \int_{X \times (-\varepsilon,\varepsilon)} \langle \psi_1(x,s)\phi_1(\tilde{\imath}(x,s)), \psi_2(x,s)\phi_2(\tilde{\imath}(x,s)) \rangle_{\Sigma_Y|_{\imath(X)}} \sqrt{\det g_Y(x,0)} \, dx \, ds, \\
&\quad = \langle U(\psi_1 \otimes \phi_1), U(\psi_2 \otimes \phi_2) \rangle.
\end{aligned}$$

□

With this unitary in hand we take the first step towards computing the product $\imath_!^{\varepsilon} \otimes [Y]$ by the formulas outlined in Definition 1.1.25. In the following computations of unbounded operators we do not mention their domains. These will be fixed in the next section, after we have found their formal descriptions.

Lemma 2.3.2. *Under the unitary transformations from Lemma 2.3.1, appropriately extended by the identity transformation, the product operators of $\imath_!^{\varepsilon} \otimes [Y]$ are*

If n is even:

$$U \left(S \times_{\nabla^{\varepsilon}} (D_Y \otimes \sigma_2) \right) U^* = f \otimes \sigma_1 + i c_Y(\partial_s) \left(D_{X_\bullet} + \nabla^{\Sigma_Y}_{\partial_s} + \frac{1}{2\Lambda} [D_{X_\bullet}, \Lambda] \right) \otimes \sigma_2$$

as operator on $L^2(\pi^* \Sigma_X) \otimes \mathbb{C}^2$.

If n is odd:

$$U \left(S \times_{\nabla^{\varepsilon}} D_Y \right) U^* = ie\gamma \otimes f + \gamma \otimes i c_Y(\partial_s) \left(\left(D^+_{X_\bullet} \oplus D^-_{X_\bullet} \right) + \nabla^{\Sigma_Y}_{\partial_s} + \frac{1}{2\Lambda} \left[\left(D^+_{X_\bullet} \oplus D^-_{X_\bullet} \right), \Lambda \right] \right)$$

as operator on $\mathbb{C}\mathrm{l}_L \otimes L^2 \left(\pi^* \left(\Sigma^+_X \oplus \Sigma^-_X \right) \right)$.

Proof. We start with n even and the $1 \otimes_{\nabla^{\varepsilon}} (D_Y \otimes \sigma_2)$ term. Let $\psi \otimes \phi \in \mathcal{E} \otimes_{C_0(Y) \otimes \mathbb{C}\mathrm{l}_1}$

$L^2(\Sigma_Y \otimes \mathbb{C}^2)$ and abbreviate $D_Y \otimes \sigma_2$ for now by D, then

$$\begin{aligned}
U(1 \otimes_{\nabla^{\mathcal{E}}_D} D)(\psi \otimes \phi) &= U\left(\gamma(\psi) \otimes D\phi + \nabla^{\mathcal{E}}_D(\psi)\phi\right), \\
&= U\left(\gamma(\psi) \otimes D\phi + \left(1 \otimes [D,\psi] + \gamma(\psi) \otimes \frac{1}{2\Lambda}[D,\Lambda]\right)\phi\right), \\
&= \gamma(\psi)D\phi + [D,\psi]\phi + \gamma(\psi)\frac{1}{2\Lambda}[D,\Lambda]\phi, \\
&= D\psi\phi + \frac{1}{2\Lambda}[D,\Lambda]\psi\phi, \\
&= \left(D + \frac{1}{2\Lambda}[D,\Lambda]\right) U(\psi \otimes \phi), \\
&= \left(D_Y \otimes \sigma_2 + \frac{1}{2\Lambda}[D_Y,\Lambda] \otimes \sigma_2\right) U(\psi \otimes \phi).
\end{aligned}$$

where in the third line we use that the action of $\mathcal{E}$ on $\Sigma_Y \otimes \mathbb{C}^2$ is compatible with the extension of U by identity to the tensor factors. In the fourth line we use that $[D,\psi] = D\psi - \gamma(\psi)D$ is a graded commutator and that the action of $\mathbb{C}\mathrm{l}_1$ on $\Sigma_Y \otimes \mathbb{C}^2$ graded commutes with the σ_2 hidden in D.

Using Proposition 2.1.12 to write D_Y in terms of D_X and II we can further evaluate this operator.

$$\begin{aligned}
U(1 \otimes_{\nabla^{\mathcal{E}}_D} D)U^* &= D_Y \otimes \sigma_2 + \frac{1}{2\Lambda}[D_Y,\Lambda] \otimes \sigma_2, \\
&= ic_Y(\partial_s)\left(D_{X_\bullet} - \frac{1}{2}\operatorname{Tr}(\mathrm{II}) + \nabla^{\Sigma_Y}_{\partial_s} + \frac{1}{2\Lambda}[D_{X_\bullet},\Lambda] + \frac{1}{2\Lambda}[\nabla^{\Sigma_Y}_{\partial_s},\Lambda]\right) \otimes \sigma_2, \\
&= ic_Y(\partial_s)\left(D_{X_\bullet} + \nabla^{\Sigma_Y}_{\partial_s} + \frac{1}{2\Lambda}[D_{X_\bullet},\Lambda]\right) \otimes \sigma_2
\end{aligned}$$

using Lemma 2.1.7.

The $S \otimes 1$ term is easier, there we obtain with the same techniques

$$\begin{aligned}
U(S \otimes 1)(\psi \otimes \phi) &= U(S\psi \otimes \phi), \\
&= U(fe\psi \otimes \phi), \\
&= fe\psi\phi, \\
&= (f \otimes \sigma_1)U(\psi \otimes \phi)
\end{aligned}$$

since e acts on the $\mathbb{C}^2$ component of $L^2(\Sigma_Y \otimes \mathbb{C}^2)$ by σ_1.

Now for n odd, we follow the same procedure. Let $(\psi\alpha) \otimes \phi \in \mathcal{E} \otimes_{C_0(Y)} L^2(\Sigma_Y)$ for some $\alpha \in \mathbb{C}\mathrm{l}_L$, and let the extension of U map into $\mathbb{C}\mathrm{l}_L \otimes L^2(\pi^*\Sigma_Y|_{\iota(X)})$ leaving the $\mathbb{C}\mathrm{l}_L$

component inert.

$$\begin{aligned}
U(1 \otimes_{\nabla^{\mathcal{E}}_{D}} D_Y)(\psi\alpha \otimes \phi) &= U\left(\gamma(\psi\alpha) \otimes D_Y\phi + \nabla^{\mathcal{E}}_{D_Y}(\psi)\phi\right), \\
&= U\left(\psi\gamma(\alpha) \otimes D_Y\phi + \left(\gamma(\alpha) \otimes [D_Y, \psi]\,\phi + \psi\gamma(\alpha) \otimes \frac{1}{2\Lambda}\,[D_Y, \Lambda]\,\phi\right)\right), \\
&= \gamma(\alpha) \otimes \left(\psi D_Y\phi + [D_Y, \psi]\,\phi + \frac{1}{2\Lambda}[D_Y, \Lambda]\psi\phi\right), \\
&= \gamma(\alpha) \otimes \left(D_Y\psi\phi + \frac{1}{2\Lambda}\,[D_Y, \Lambda]\,\psi\phi\right), \\
&= \left(\gamma \otimes \left(D_Y + \frac{1}{2\Lambda}\,[D_Y, \Lambda]\right)\right) U(\psi\alpha \otimes \phi),
\end{aligned}$$

so that

$$\begin{aligned}
U(1 \otimes_{\nabla^{\mathcal{E}}_{D}} D_Y)U^* &= \gamma \otimes \left(D_Y + \frac{1}{2\Lambda}\,[D_Y, \Lambda]\right), \\
&= \gamma \otimes ic_Y(\partial_s)\left(\left(D^+_{X_\bullet} \oplus D^-_{X_\bullet}\right) + \nabla^{\Sigma Y}_{\partial_s} + \frac{1}{2\Lambda}\left[\left(D^+_{X_\bullet} \oplus D^-_{X_\bullet}\right), \Lambda\right]\right).
\end{aligned}$$

And finally we get in the odd case

$$U(S \otimes 1)U^* = ie\gamma \otimes f.$$

□

The final step before the product operators reach their final form is to twist the various $\mathbb{C}\mathrm{l}_1$ and $\mathbb{C}^2$ components. This will allows us to recognize D_X (or $D_X \otimes \sigma_2$) and T in the product operator.

Lemma 2.3.3. *There exist unitary transformations $U_e, U_o : \mathbb{C}^2 \otimes \mathbb{C}^2 \to \mathbb{C}^2 \otimes \mathbb{C}^2$ such that the induced transformations on operators send*

$\mathbf{U_e}$	$\mathbf{U_o}$
$1 \otimes \sigma_1 \mapsto -\sigma_3 \otimes \sigma_2,$	$\sigma_2 \otimes 1 \mapsto -\sigma_3 \otimes \sigma_2,$
$\sigma_1 \otimes \sigma_2 \mapsto \sigma_2 \otimes 1,$	$\sigma_3 \otimes \sigma_2 \mapsto \sigma_2 \otimes 1,$
$\sigma_2 \otimes \sigma_2 \mapsto -\sigma_1 \otimes 1,$	$\sigma_3 \otimes \sigma_1 \mapsto \sigma_3 \otimes \sigma_1,$
$\sigma_3 \otimes \sigma_2 \mapsto \sigma_3 \otimes \sigma_1,$	$\sigma_3 \otimes \sigma_3 \mapsto \sigma_3 \otimes \sigma_3,$
$1 \otimes \sigma_3 \mapsto \sigma_3 \otimes \sigma_3.$	$\sigma_1 \otimes 1 \mapsto \sigma_1 \otimes 1.$

Proof. Using the isomorphism

$$A \otimes B \leftrightarrow \begin{pmatrix} Ab_{11} & Ab_{12} \\ Ab_{21} & Ab_{22} \end{pmatrix}$$

we have

$$U_e = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

as a straightforward, albeit lengthy, check shows.

For U_o the fixed points make an abstract formula reasonable,

$$U_o = \frac{1}{\sqrt{2}} 1 \otimes 1 + \frac{i}{\sqrt{2}} \sigma_1 \otimes \sigma_2$$

another straightforward but lengthy check shows that U_o does what we claim.

The formula for U_o can be derived by writing $U_o = a^{ij} \sigma_i \otimes \sigma_j$ with summation over i, j and $\sigma_0 = 1$. Then the fixed points imply that U_0 commutes with those elements, this leaves only a^{00} and a^{12} possibly non-zero and they are fixed up to a common phase by either of the remaining conditions. A similar formula exists for U_e but with the lack of fixed points the computations are easier with matrices.

It is perhaps worth noting that $U_{e,o}$ are over-determined. The product of all five operators in both cases gives a scalar multiple of the identity, which is always a fixed point. This "internal orientation" relation is the reason for the mistake noted in [63, Rem. 3.1] □

These unitaries rotate all spinor components into the right setting to recognize $[X]$ and [1]. The order of the conditions in Lemma 2.3.3 is not arbitrarily chosen, they are in the same order as they are used in the following proposition.

Proposition 2.3.4. *There are unitary transformations V so that in a Fermi frame for the respective spinor bundles*

If n is even:

$$V \left(S \times_{\nabla^{\varepsilon}} \left(D_Y \otimes \sigma_2 \right) \right) V^* = \left(D_{X_\bullet} + \frac{1}{2\Lambda} \left[D_{X_\bullet}, \Lambda \right] \right) + \gamma T$$

as operator on the Hilbert $C(X)$-$\mathbb{C}$ bimodule $L^2 \left(\pi^* \Sigma_X \otimes \mathbb{C}^2 \right)$ with grading $\gamma \otimes \sigma_3 = c_Y(\partial_s) \otimes \sigma_3$.

If n is odd:

$$V \left(S \times_{\nabla^{\varepsilon}} D_Y \right) V^* = \left(D^+_{X_\bullet} \otimes \sigma_2 + \frac{1}{2\Lambda} \left[D^+_{X_\bullet} \otimes \sigma_2, \Lambda \right] \right) + \gamma T$$

as operator on the Hilbert $C(X) \otimes \mathbb{C}\mathrm{l}_1$-$\mathbb{C}$ bimodule $L^2 \left(\Sigma^+_X \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \right)$ with grading $\gamma \otimes \sigma_3 = 1 \otimes \sigma_3 \otimes \sigma_3$.

Here, for $\Sigma = \Sigma_X$ or $\Sigma_X \otimes \mathbb{C}^2$ where appropriate,

$$T : \Gamma^\infty_c (\pi^* \Sigma \otimes \mathbb{C}^2) \to L^2 (\pi^* \Sigma \otimes \mathbb{C}^2)$$
$$\psi \otimes v \mapsto i \frac{\partial}{\partial s} \psi \otimes \sigma_1 v - f \psi \otimes \sigma_2 v.$$

Proof. The first simplification we can make is that in a Fermi frame, by Lemma 2.1.4, the covariant derivative $\nabla^{\Sigma_Y}_{\partial_s} = \frac{\partial}{\partial s}$.

For n even we are working on the space $L^2(\pi^* \Sigma_X) \otimes \mathbb{C}^2$ with grading $1 \otimes \sigma_3$, since the grading comes entirely from the $\mathbb{C}^2 = \mathbb{C}\mathrm{l}_R \otimes_{\mathbb{C}\mathrm{l}_1} \mathbb{C}^2$ component. Split $\Sigma_X = \Sigma^+_X \oplus \Sigma^-_X \cong$

$\Sigma_X^+ \otimes \mathbb{C}^2$ into the eigenspaces of $c_Y(\partial_s)$. The Clifford action of TX_s defined in Lemma 2.1.9 anticommutes with $c_Y(\partial_s)$, so in this frame $D_{X_\bullet}$ acts off-diagonal.

In this frame the operator we have at the end of Lemma 2.3.2 looks like (using Lemma 2.1.4 for $\nabla^{\Sigma_Y}_{\partial_s}$),

$$f \otimes \sigma_1 + i\sigma_3 \left(\begin{pmatrix} 0 & D_{X_\bullet}|_{\Sigma_X^-} \\ D_{X_\bullet}|_{\Sigma_X^+} & 0 \end{pmatrix} + \frac{\partial}{\partial s} + \frac{1}{2\Lambda} \left[\begin{pmatrix} 0 & D_{X_\bullet}|_{\Sigma_X^-} \\ D_{X_\bullet}|_{\Sigma_X^+} & 0 \end{pmatrix}, \Lambda \right] \right) \otimes \sigma_2,$$

with grading $1 \otimes \sigma_3$. Then the unitary U_e from Lemma 2.3.3 transforms this into

$$-\sigma_3 f \otimes \sigma_2 + \begin{pmatrix} 0 & D_{X_\bullet}|_{\Sigma_X^-} \\ D_{X_\bullet}|_{\Sigma_X^+} & 0 \end{pmatrix} \otimes 1 + \sigma_3 \frac{\partial}{\partial s} \otimes \sigma_1 + \frac{1}{2\Lambda} \left[\begin{pmatrix} 0 & D_{X_\bullet}|_{\Sigma_X^-} \\ D_{X_\bullet}|_{\Sigma_X^+} & 0 \end{pmatrix}, \Lambda \right] \otimes 1$$

with grading $\sigma_3 \otimes \sigma_3$. Rewriting $\Sigma_X^+ \oplus \Sigma_X^-$ as Σ_X gives the representation we are after.

For n odd we have the operator

$$ie\gamma \otimes f + \gamma \otimes ic_Y(\partial_s) \left(\left(D^+_{X_\bullet} \oplus D^-_{X_\bullet} \right) + \nabla^{\Sigma_Y}_{\partial_s} + \frac{1}{2\Lambda} \left[\left(D^+_{X_\bullet} \oplus D^-_{X_\bullet} \right), \Lambda \right] \right)$$

acting on $\mathbb{C}\mathrm{l}_L \otimes L^2\left(\pi^*\left(\Sigma_X^+ \oplus \Sigma_X^-\right)\right)$. Let $\mathbb{C}\mathrm{l}_1 \cong \mathbb{C}^2$ by sending 1 to $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and the generator e to $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and choose a Fermi frame in which $c_Y(\partial_s) = \sigma_1$ on $\Sigma_X^+ \oplus \Sigma_X^-$. Then, by Remark 2.1.10 (see also Example 2.1.14), we get

$$\sigma_2 \otimes f + \sigma_3 \otimes \left(D^+_{X_\bullet} \otimes \sigma_2 \right) + \sigma_3 \otimes \left(i\frac{\partial}{\partial_s} \otimes \sigma_1 \right) + \sigma_3 \otimes \left(\frac{1}{2\Lambda} \left[D^+_{X_\bullet}, \Lambda \right] \otimes \sigma_2 \right)$$

so that U_o from Lemma 2.3.3, together with

$$\begin{aligned} V : \mathbb{C}^2 \otimes L^2\left(\pi^*\Sigma_X^+ \otimes \mathbb{C}^2\right) &\to L^2(\pi^*\Sigma_X^+ \otimes \mathbb{C}^2 \otimes \mathbb{C}^2) \\ v \otimes \psi \otimes w &\mapsto \psi \otimes w \otimes v \end{aligned}$$

yields

$$\left(D^+_{X_\bullet} \otimes \sigma_2 \otimes 1 + \frac{1}{2\Lambda} \left[D^+_{X_\bullet} \otimes \sigma_2 \otimes 1, \Lambda \right] \right) + (1 \otimes \sigma_3 \otimes 1)T$$

with the Pauli matrices in T acting on the third component, as desired. □

Example 2.3.5. *For the embeddings $S^n \hookrightarrow \mathbb{R}^{n+1}$ this formula simplifies a bit further since $D_{S^n_\bullet} = \frac{1}{s+1} D_{S^n}$. If n is even we get*

$$S \times_{\nabla^\varepsilon} (D_Y \otimes \sigma_2) = D_{S^n} \otimes \frac{1}{s+1} + \gamma \otimes T$$

acting on $L^2(\Sigma_{S^n}) \otimes L^2\left((-\varepsilon, \varepsilon), \mathbb{C}^2\right)$. The term

$$\frac{1}{2\Lambda} \left[D_{S^n_\bullet}, \Lambda \right] = 0$$

since $\Lambda = \frac{1}{r^n}$ so the spherical derivatives vanish.

These effects are related: if $D_{X_s} = f(s) D_{X_0}$ we have $\Lambda_s = f(s)^n$ but the converse is not true. If, for example, we consider the metric on $\mathbb{R}^3$ with, for $z > 0$, $g(x, y, z) = \mathrm{diag}(\frac{1}{z}, z, 1)$ with $\mathbb{R}^2$ embedded at $z = 1$ the volume form of $\mathbb{R}^2_s$ is constant, but $D_{\mathbb{R}^2_s} = (s+1)\sigma_1 \partial_x + \frac{1}{s+1}\sigma_2 \partial_y$.

2.3.2 The product is a KK-cycle

The goal of this section is to establish that

$$i_!^\varepsilon \otimes [Y] = \left(L^2\left(\pi^*\Sigma_X \otimes \mathbb{C}^2\right), S \times_{\nabla^\varepsilon} D_Y\right)$$

defines an unbounded $C(X)$-$\mathbb{C}$ KK-cycle, or in other words, a spectral triple. To get this we need to establish that $S \times_{\nabla^\varepsilon} D_Y$ is self-adjoint and has compact resolvent.

The usual approach described at the end of Section 1.1.4 does not quite work. This is because the operator $1 \otimes_{\nabla^\varepsilon} D_Y$ is not self-adjoint, as it lacks appropriate boundary conditions on the boundary of $\tilde{\imath}(X \times (-\varepsilon, \varepsilon))$. We can still use much of the same theory by instead working with the unitary equivalent formulation from Proposition 2.3.4 where the two parts form a weakly graded commuting pair in the sense of Definition 1.1.28.

Let us make some notational remarks before we start. In an effort to shorten the expressions in this section we write

$$A_\bullet := \frac{1}{2\Lambda}\left[D_{X_\bullet}, \Lambda\right],$$

or with $D^+_{X_\bullet} \otimes \sigma_2 \otimes 1$ if n is odd. Additionally, almost everything in this section is essentially the same for n even and n odd. With this in mind and again to simplify the expressions we also introduce

$$D_1 = \begin{cases} D_{X_\bullet} + A_\bullet, & n \text{ is even}, \\ D^+_{X_\bullet} \otimes \sigma_2 \otimes 1 + A_\bullet, & n \text{ is odd}, \end{cases}$$
$$D_2 = \gamma T$$

and we will simply write Σ for both Σ_X and for $\Sigma^+_X \otimes \mathbb{C}^2$.

The following lemma turns out to be quite useful. It is essentially an incarnation of the local-global principle [56, 36].

Lemma 2.3.6 (Lemma 1.15 in [60])**.** *Suppose $\{D_s\}_{s\in(-\varepsilon,\varepsilon)}$ is a family of self-adjoint operators on a Hilbert space H such that there is a common core $\mathcal{H} \subset H$ for all D_s and for all $\psi \in \mathcal{H}$ the map $s \mapsto D_s\psi$ is continuous. Then the operator $D_\bullet$ on the Hilbert $C((-\varepsilon,\varepsilon))$-module $C((-\varepsilon,\varepsilon), H)$ is regular and self-adjoint.*

Lemma 2.3.7. *The operator D_1 is essentially self-adjoint on the domain* $\mathrm{Dom}(D_1) = H^1(\Sigma) \otimes_{alg} L^2((-\varepsilon,\varepsilon), \mathbb{C}^2) \subset L^2(\pi^*\Sigma \otimes \mathbb{C}^2)$.

Proof. Let us first establish that D_1 is symmetric. Assume for now that n is even, if n is

odd nothing material changes. Let $\psi, \phi \in \mathrm{Dom}(D_1)$, then

$$\begin{aligned}
\langle D_1\psi, \phi\rangle &= \int_{X\times(-\varepsilon,\varepsilon)} \langle D_{X_s}\psi(x,s) + A_s\psi(x,s), \phi(x,s)\rangle \, d\omega_X(x)\, ds,\\
&= \int_{X\times(-\varepsilon,\varepsilon)} \langle D_{X_s}\psi + \frac{1}{2\Lambda}\left[D_{X_s}, \Lambda\right]\psi, \phi\rangle\Lambda \, d\omega_{X_s}(x)\, ds,\\
&= \int_{X\times(-\varepsilon,\varepsilon)} \langle D_{X_s}\psi + \frac{1}{2\Lambda}\left[D_{X_s}, \Lambda\right]\psi, \Lambda\phi\rangle \, d\omega_{X_s}(x)\, ds,\\
&= \int_{X\times(-\varepsilon,\varepsilon)} \langle \psi, \left(D_{X_s} - \frac{1}{2\Lambda}\left[D_{X_s}, \Lambda\right]\right)\Lambda\phi\rangle \, d\omega_{X_s}(x)\, ds,\\
&= \int_{X\times(-\varepsilon,\varepsilon)} \langle \psi, \Lambda\left(D_{X_s} + \frac{1}{2\Lambda}[D_{X_s}, \Lambda]\right)\phi\rangle \, d\omega_{X_s}(x)\, ds,\\
&= \int_{X\times(-\varepsilon,\varepsilon)} \langle \psi, \left(D_{X_s} + \frac{1}{2\Lambda}[D_{X_s}, \Lambda]\right)\phi\rangle \, d\omega_X(x)\, ds,\\
&= \langle \psi, D_1\phi\rangle,
\end{aligned}$$

where the key idea is that D_{X_s} is symmetric on $L^2(\Sigma_{X_s})$ and we also use that A_s is a skew-adjoint endomorphism-valued function.

To get essential self-adjointness we use Lemma 2.3.6. For each fixed s the operator $D_{X_s} + A_s$ is a symmetric, elliptic, first order differential operator and as such self-adjoint on $H^1(\Sigma \otimes C^2)$. We get the continuity of $s \mapsto D_{X_s}\psi + A_s\psi$ by the smoothness of the coefficients of D_{X_s} and A_s. This means that the operator $D_{X_\bullet} + A_\bullet$ on $C\left((-\varepsilon,\varepsilon), L^2(\Sigma)\right)$ is regular and self-adjoint.

From there we also get that $(D_{X_\bullet} + A_\bullet) \otimes 1_2$ on $C\left((-\varepsilon,\varepsilon), L^2(\Sigma\otimes\mathbb{C}^2)\right)$ is regular and self-adjoint. The final step is to get from $C_0\left((-\varepsilon,\varepsilon), L^2\left(\Sigma\otimes\mathbb{C}^2\right)\right)$ to $L^2(\Sigma)\otimes L^2\left((-\varepsilon,\varepsilon),\mathbb{C}^2\right)$ which follows by taking the internal tensor product with the unbounded $C_0\left((-\varepsilon,\varepsilon)\right)$-$\mathbb{C}$ KK-cycle $\left(L^2\left((-\varepsilon,\varepsilon),\mathbb{C}^2\right), 0\right)$ by Lemma 1.1.29.

Then finally use that $L^2(\Sigma)\otimes L^2\left((-\varepsilon,\varepsilon),\mathbb{C}^2\right)$ is unitarily equivalent to $L^2\left(\pi^*\Sigma\otimes\mathbb{C}^2\right)$. □

The difficult in proving self-adjointness of D_1 lies in the fact that D_1 is not an elementary tensor product of operators under the isomorphism $L^2(\pi^*\Sigma\otimes\mathbb{C}^2) \cong L^2(\Sigma)\otimes L^2\left((-\varepsilon,\varepsilon),\mathbb{C}^2\right)$. In other words, it does not split into two nicely separated x and s dependent parts. In the example of $S^n \hookrightarrow \mathbb{R}^{n+1}$ this is the case, see Example 2.3.5, but in general we do not have this factorization for D_1. We do have this for D_2, so the proof for self-adjointness is much shorter.

Another way of analysing D_1 is through the concept of tangential ellipticity [41], but we use the above approach since it prepares us better for the future analyses.

Lemma 2.3.8. *The operator D_2 is essentially self-adjoint with domain* $\mathrm{Dom}(D_2) = \Gamma^\infty(\Sigma)\otimes_{alg} C_c^\infty\left((-\varepsilon,\varepsilon),\mathbb{C}^2\right)$.

Proof. Notice that under the isomorphism $L^2(\pi^*\Sigma\otimes\mathbb{C}^2) \cong L^2(\Sigma)\otimes L^2\left((-\varepsilon,\varepsilon),\mathbb{C}^2\right)$, $D_2 = \gamma\otimes T$. Here we use the symbol T for both the operator on $L^2\left((-\varepsilon,\varepsilon),\mathbb{C}^2\right)$ as well as the operator on $L^2\left(\pi^*\Sigma\otimes\mathbb{C}^2\right)$.

By Corollary 2.2.5 we have that T is essentially self-adjoint on $C_c^\infty((-\varepsilon,\varepsilon),\mathbb{C}^2)$ so $\gamma\otimes T$ is then essentially self-adjoint as well (see for example [58, Thm. VIII.33]). This then translates back to γT on $L^2(\pi^*\Sigma\otimes\mathbb{C}^2)$ with domain $\Gamma_c^\infty(\pi^*\Sigma\otimes\mathbb{C}^2)$. □

Since D_1 and D_2 are essentially self-adjoint on the domains defined in Lemmas 2.3.7 and 2.3.8 they have self-adjoint closures. From here on we use D_1 and D_2 to refer to those self-adjoint closures. With this in mind we are ready for the main workhorse of this section.

Proposition 2.3.9. *The pair (D_1,D_2) is a pair of weakly graded commuting operators.*

Proof. By Definition 1.1.28 we need to show that the graded commutator of D_1 and D_2 is relatively bounded and that there is a D_1-compatible core for D_2 or, symmetrically, a D_2-compatible core for D_1.

Let us answer this second point first. Consider $E=\Gamma_c^\infty(\pi^*\Sigma\otimes\mathbb{C}^2)$, by Lemma 2.3.7 this is a core for D_1. By Lemma 2.2.3, and smoothness of the functions I_λ appearing therein, we get that

$$(D_2-\lambda i)^{-1}\psi(x,s)=-i\sigma_1 I_\lambda(s)\int_{-\varepsilon}^{s} I_\lambda(t)\psi(x,t)\,dt$$

so that $(D_2-\lambda i)^{-1}$ preserves E.

Now let us consider the graded commutator. Since both D_1 and D_2 are odd operators this is really the anticommutator, so we compute

$$\begin{aligned} D_1D_2+D_2D_1 &= D_1(\gamma\otimes T)+(\gamma\otimes T)D_1,\\ &= -\gamma D_1(1\otimes T)+\gamma(1\otimes T)D_1,\\ &= -\gamma\left(D_1(1\otimes T)-(1\otimes T)D_1\right), \end{aligned}$$

so we get the regular commutator of D_1 with $1\otimes T=1\otimes\left(i\frac{\partial}{\partial s}\sigma_1-if\sigma_2\right)$.

Multiplication by f, which is a function depending only on s, commutes with $D_{X_\bullet}+A_\bullet$ so this commutator becomes

$$(\gamma\otimes i\sigma_1)\left(\frac{\partial D_1}{\partial s}\right).$$

We once again use the localized properties of D_1, for each fixed $s\in(-\varepsilon_0,\varepsilon_0)$ the operator $\frac{\partial D_1}{\partial s}$ is a first-order differential operator (its symbol is the s-derivative of the family of symbols for D_1), so that by Gårding's inequality there is some constant C_s such that

$$\left\|\frac{\partial D_{X_s}+A_s}{\partial s}(D_{X_s}+A_s+i)^{-1}\right\|\leq C_s$$

and we can actually choose C_s smoothly in s since the operator $D_{X_s}+A_s$ depends smoothly on s. This means that C_s attains a maximum on $[-\varepsilon,\varepsilon]\subset(-\varepsilon_0,\varepsilon_0)$ and for $C=\sup_{s\in(-\varepsilon,\varepsilon)}C_s$ this means that

$$\|[D_1,D_2]\psi\|^2\leq 2C^2\left(\|\psi\|^2+\|D_1\psi\|^2\right).$$

□

Proposition 2.3.10. *The operator $D_1 + D_2$ is self-adjoint on* $\mathrm{Dom}(D_1) \cap \mathrm{Dom}(D_2)$ *and has compact resolvents.*

Proof. Self-adjointness on the given domain follows immediately from Theorem 1.1 since (D_1, D_2) is a weakly graded commuting pair.

To get compact resolvents we can show that the domain is compact. We know for D_1, from Lemma 2.3.7, that $\mathrm{Dom}(D_1) \subset H^1(\Sigma_X) \otimes L^2((-\varepsilon, \varepsilon), \mathbb{C}^2)$. Similarly we know for D_2, from Proposition 2.2.7, that $\mathrm{Dom}(D_2) \subset L^2(\Sigma_X) \otimes H^1((-\varepsilon, \varepsilon), \mathbb{C}^2)$.

This means that any function ψ in the domain of $D_1 + D_2$ has weak derivatives tangential to X and weak derivatives in the s direction as well. So ψ is in the inverse image of $H^1(\pi^*\Sigma_X \otimes \mathbb{C}^2)$ under the unitary transformation

$$\begin{aligned} U : L^2(\Sigma_X) \otimes L^2\left((-\varepsilon, \varepsilon), \mathbb{C}^2\right) &\to L^2(\pi^*\Sigma_X \otimes \mathbb{C}^2) \\ U(\psi \otimes \phi)(x, s) &= \psi(x, s) \otimes \phi(x, s). \end{aligned}$$

This means that the resolvent $(D_1 + D_2 + i)^{-1}$ factors through $H^1(\pi^*\Sigma_X \otimes \mathbb{C}^2)$, so by the Rellich embedding theorem it is compact. □

This means that, for n even,

$$\left(C(X), L^2\left(\pi^*\Sigma_X \otimes \mathbb{C}^2\right), S \times_{\nabla^\varepsilon} (D_Y \otimes \sigma_2)\right) \tag{2.13a}$$

and, for n odd,

$$\left(C(X) \otimes \mathbb{C}\mathrm{l}_1, L^2\left(\Sigma_X^+ \otimes \mathbb{C}^2 \otimes \mathbb{C}^2\right), S \times_{\nabla^\varepsilon} D_Y\right) \tag{2.13b}$$

are spectral triples. So the construction of the unbounded product $\imath_!^\varepsilon \otimes [Y]$ is successful. But now comes the question of whether this product cycle tells us anything about the geometry of $\imath : X \hookrightarrow Y$.

2.4 Recovering the embedded manifold

We start this section by verifying that our product is sensible in the world of KK-theory. To do this we use an unbounded homotopy as in [60] to relate $[X] \otimes [1]$ and $\imath_!^\varepsilon \otimes [Y]$. Even though this homotopy is formulated at the unbounded level, it discards the geometric information contained in the immersion because the unbounded homotopies generate the regular KK-groups. To recover the geometric information we instead study the behaviour of our product as ε shrinks and explain geometrically why this preserves the geometric information contrary to the unbounded homotopy.

2.4.1 At the level of KK-theory

Let us start with the role of $[1] = (L^2((-\varepsilon, \varepsilon), \mathbb{C}^2), T)$. In Section 2.2.3 we have shown that the operator T has index 1. Since $KK_0(\mathbb{C}, \mathbb{C}) \cong \mathbb{Z}$ via the index map [8, Prop. 18.8.1] this means that [1] represents the multiplicative unit. So, in $KK_0(C(X), \mathbb{C})$, the classes $[X]$ and $[X] \otimes [1]$ represent the same element. Thus our goal is to show that our candidate representatives, Equations 2.13a and 2.13b, for $\imath_!^\varepsilon \otimes [Y]$ represent $[X] \otimes [1]$.

To show that the product spectral triples in Equation 2.13 and the unbounded cycle for the product $[X] \otimes 1$ represent the same class in $KK_0(C(X), \mathbb{C})$ we construct an unbounded homotopy in the sense of [60]. Then Theorem B in *loc. cit.* implies that they represent the same class in $KK_0(C(X), \mathbb{C})$.

An unbounded homotopy between $({}_A(\mathcal{E}_0)_B, D_0)$ and $({}_A(\mathcal{E}_1)_B, D_1)$ consists of an Hilbert A-$C([0,1], B)$ bimodule $\mathcal{E}$ and unbounded operator D, self-adjoint and regular, such that A is contained in the closure of all operators T on $\mathcal{E}$ that have both bounded commutators with D and such that D has T-locally compact resolvent. This should be compared to the usual definition of an unbounded KK-cycle, Definition 1.1.18. There we instead require a choice of dense subalgebra $\mathcal{A} \subset A$ such that $a \in \mathcal{A}$ has bounded commutators with D and D has a-locally compact resolvents.

In the following we will play fast and loose with the distinctions for n even and n odd. For the even case the precise statements can be obtained by replacing all Σ_Y by $\Sigma_Y \otimes \mathbb{C}^2$ and D_Y by $D_Y \otimes \sigma_2$. In the odd the replacements are Σ_X by $\Sigma_X^+ \otimes \mathbb{C}^2$ and $D_{X_\bullet}$ by $D^+_{X_\bullet} \otimes \sigma_2$. These adjustments lead to no changes in the actual arguments.

Lemma 2.4.1. *The Kasparov product $[X] \otimes [1]$ is represented by the unbounded KK-cycle*

$$\left(L^2(\Sigma_X) \otimes L^2\left((-\varepsilon, \varepsilon) \otimes \mathbb{C}^2\right), D_X \otimes 1 + \gamma \otimes T\right).$$

Proof. This follows from Lemma 1.1.30 since the Kasparov product is over $\mathbb{C}$. □

Proposition 2.4.2. *The Hilbert $C(X)$-$C([0,1])$ bimodule*

$$C\left([0,1], L^2(\Sigma_X) \otimes L^2\left((-\varepsilon, \varepsilon), \mathbb{C}^2\right)\right)$$

with operator

$$\widetilde{D} : C\left([0,1], \mathrm{Dom}\left(S \times_{\nabla^\varepsilon} D_Y\right)\right) \to C\left([0,1], L^2(\Sigma_X) \otimes L^2\left((-\varepsilon, \varepsilon), \mathbb{C}^2\right)\right)$$
$$\left((\widetilde{D}\psi)(t)\right)(x,s) = \left(D_{X_{st}}\psi(t)|_{X_s}\right)(x) + \left(A_{st}\psi(t)|_{X_s}\right)(x) + (\gamma \otimes T)\,\psi(t)$$

defines an unbounded homotopy between

$$[X] \otimes [1] = \left(L^2(\Sigma_X) \otimes L^2\left((-\varepsilon, \varepsilon), D_X \otimes 1 + \gamma \otimes T\right)\right)$$

and

$$\imath^\varepsilon_! \otimes [Y] = \left(L^2\left(\pi^*\Sigma_X \otimes \mathbb{C}^2\right), S \times_{\nabla^\varepsilon} D_Y\right).$$

Proof. Let us first establish that $\widetilde{D}$ indeed defines an unbounded homotopy. Note that for each fixed t the operator is given by

$$\widetilde{D}_t = D_{X_{st}} + A_{st} + \gamma \otimes T$$

acting on $\mathrm{Dom}(S \times_{\nabla^\varepsilon} D_Y) \subset L^2(\Sigma_X) \otimes L^2\left((-\varepsilon, \varepsilon), \mathbb{C}^2\right)$. So each localization at t is self-adjoint with the same domain $\mathrm{Dom}(S \times_{\nabla^\varepsilon} D_Y)$. Finally $t \mapsto \widetilde{D}_t\psi$ is continuous for ψ in the domain since $D_{X_\bullet}$ and $A_\bullet$ are smooth families of operators. Then Lemma 2.3.6 implies that $\widetilde{D}$ is regular and self-adjoint.

For the resolvents we can go one step further. The symbol of the differential operator $\widetilde{D}_t$ is, in local coordinates (x, s) on $X \times (-\varepsilon_0, \varepsilon_0)$ given by

$$\sigma(\widetilde{D}_t) = \sum_{i=1}^{n} a_i(x, st)\xi_i + b(x, st) + \gamma \otimes \sigma_1 \xi_s - \gamma \otimes f_\varepsilon(s)\sigma_2$$

for a_i, b smooth sections of $\mathrm{End}(\pi_{\varepsilon_0}^* \Sigma \otimes \mathbb{C}^2)$. Note that the symbol map $t \mapsto \sigma(\widetilde{D}_t)$ is continuous with respect to the symbol topology (see e.g. [43, Ch. 2]). This implies that the symbol of $(\widetilde{D}_t + i)^{-1}$ is also continuous in t, so that as a degree -1 pseudo-differential operator this is a norm-continuous family of operators [43, Thm. 4.1].

Next we note that the argument from Proposition 2.3.10 implies that for each fixed t $\widetilde{D}_t$ has compact resolvent. So the norm-continuity in t implies that $(\tilde{D} + i)^{-1}$ itself has compact resolvent. So for all adjointable operators T the resolvent of $\tilde{D}$ is T-locally compact. Therefore we need to establish that $C(X)$ is contained in the closure of all operators with bounded commutators with $\tilde{D}$.

Since at least the smooth functions have bounded commutator with $\widetilde{D}$ and $C(X)$ is contained in the closure of the smooth functions we get the compatibility between $C(X)$ and $\widetilde{D}$. In fact, we have shown that $\widetilde{D}$ defines an unbounded KK-cycle which is slightly stronger than a KK-homotopy as defined above.

Finally, to see that $\widetilde{D}$ defines an homotopy between $[X] \otimes [1]$ and $\imath_!^\varepsilon \otimes [Y]$ we evaluate the homotopy at the endpoints. This is done by taking the internal product with the cycles $({}_{C([0,1])}\mathbb{C}_{\mathbb{C}}, 0)$, where for the first endpoint $g \in C([0,1])$ acts by multiplication by $g(0)$ and for the other endpoint by $g(1)$. This corresponds to evaluating the function in $C([0,1], L^2(\Sigma_X) \otimes L^2((-\varepsilon, \varepsilon), \mathbb{C}^2))$ at 0 or 1 respectively.

For the first endpoint, evaluation at 0, we get

$$\widetilde{D} \otimes_{C([0,1])} 1 = D_{X_0} + A_0 + \gamma \otimes T = D_X \otimes 1 + \gamma \otimes T$$

as $A_0 = 0$. This is exactly our operator on the product $[X] \otimes [1]$.

For the second endpoint, evaluation at 1, we get

$$\widetilde{D} \otimes_{C([0,1])} 1 = D_{X_\bullet} + A_\bullet + \gamma \otimes T$$

which under the isomorphism $L^2(\pi^* \Sigma_X \otimes \mathbb{C}^2) \cong L^2(\Sigma_X) \otimes L^2((-\varepsilon, \varepsilon), \mathbb{C}^2)$ gives the operator $S \times_{\nabla^\varepsilon} D_Y$ that we found in Proposition 2.3.4. □

Corollary 2.4.3. *The unbounded KK-cycle*

$$\left(L^2(\Sigma_X) \otimes L^2\left((-\varepsilon, \varepsilon), \mathbb{C}^2\right), S \times_{\nabla^\varepsilon} D_Y\right)$$

represents the Kasparov product $\imath_!^\varepsilon \otimes [Y]$.

Proof. From the construction in [16] of the shriek classes in the bounded setting it is clear that the bounded transforms of the spectral triples for $[X]$ and $[Y]$ give the classes $(\mathrm{pt}_X)_!$ and $(\mathrm{pt}_Y)_!$ as discussed in the introduction to this chapter. By the construction of $\imath_!^\varepsilon = (\mathcal{E}, S)$ and [16, Prop. 2.8] it is clear that $\imath_!^\varepsilon$ is an unbounded representative of the $KK(C(X), C_0(Y))$ class $\imath_!$ as well.

So the functoriality of the shriek classes described in Equation 2.1 implies that, in $KK(C(X),\mathbb{C})$, $[X] = \imath_! \otimes [Y]$. Hence any unbounded KK-cycle representing $[X]$ in $KK(C(X),\mathbb{C})$ represents the Kasparov product $\imath_! \otimes [Y]$.

By Proposition 2.4.2 the unbounded KK-cycles corresponding to the spectral triples for the product, Equation 2.13, are unbounded homotopic to $[X] \otimes [1]$. So [60, Thm. B] tells us that they represent the same class in KK-theory. We get that $S \times_{\nabla^\varepsilon} D_Y$ represents the same class as $[X] \otimes [1]$, which in turn is the same class as $[X]$ since $[1]$ represents the multiplicative unit. □

In [62] we check that for $S^n \hookrightarrow \mathbb{R}^{n+1}$ the product construction represents the Kasparov product using Kucerovsky's criterion [42]. This tool does not work in this case because the positivity condition can fail, the positivity condition requires

$$\langle \psi, (D_{X_0} D_{X_\bullet} + D_{X_\bullet} D_{X_0})\, \psi \rangle \geq -k \langle \psi, \psi \rangle$$

which holds in the case of spheres since then $D_{X_s} = \frac{1}{s+1} D_{X_0}$. If D_{X_s} is not a scalar multiple of D_{X_0} the left-hand side can contain various first order differential operators that make the lower bound impossible.

2.4.2 At the geometric level

Let us spend a moment to think about what happens geometrically in the homotopy of Proposition 2.4.2. This homotopy keeps a fixed ε-neighbourhood of X but as t goes to 0 it stretches the εt neighbourhood of X out to fill the entire ε neighbourhood. Geometrically this corresponds to rescaling the radial metric so that ν becomes a length $\frac{1}{t}$ vector. Then the time s flow along the rescaled $t\nu$ corresponds to the time ts flow along ν.

This kills the geometric nature of the immersion, which essentially is about how X_s changes with s. One way to see this more clearly is by considering the second fundamental form II from Equation 2.3,

$$\mathrm{II}_{(x,0)}(A,B) = -\langle B, \nabla^Y_A(t\nu) \rangle$$

since $t\nu$ is the new unit normal. As t goes to zero, so does $\mathrm{II}_{(x,0)}$.

We do want to isolate X_0 somehow from within $\{X_s\}$. The way we can do this is by looking at progressively smaller neighbourhoods of X_0, rather than stretching the metric within a fixed neighbourhood. This is the reason we have defined the family of unbounded KK-cycles $\{\imath^\varepsilon_!\}_{\varepsilon \in (-\varepsilon_0, \varepsilon_0)}$. Throughout this section many objects will labelled with an ε where up till now they were not to save on visual clutter. For example we now write

$$\begin{aligned} \pi_\varepsilon : X \times (-\varepsilon, \varepsilon) &\to X \\ (x,s) &\mapsto x \end{aligned}$$

instead of simply π, because we will be looking at various ε at the same time.

In order to compare the operators on the neighbourhoods of progressively smaller size we do stretch them out, but this time preserving the geometric information. For this

purpose we define the maps

$$\begin{aligned} R_\varepsilon : L^2\left(\pi_\varepsilon^*\Sigma \otimes \mathbb{C}^2\right) &\to L^2\left(\pi_1^*\Sigma \otimes \mathbb{C}^2\right) \\ \left(R_\varepsilon\psi\right)(x,s) &= \sqrt{\varepsilon}\psi(x,\varepsilon s), \end{aligned} \tag{2.14}$$

where Σ can be any vector bundle. In practice it will be either $\Sigma_X \otimes \mathbb{C}^2$ or $\Sigma_X^+ \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ depending on the parity of n. These preserve the geometry by the following lemma.

Lemma 2.4.4. *The maps R_ε are unitary.*

Proof. The R_ε are clearly invertible, so we check that they preserve the inner product. Let $\psi, \phi \in L^2\left(\pi_\varepsilon^*\Sigma\right)$

$$\begin{aligned} \langle R_\varepsilon\psi, R_\varepsilon\phi\rangle &= \int_{X\times(-1,1)} \langle (R_\varepsilon\psi)(x,s), (R_\varepsilon\phi)(x,s)\rangle \, d\omega_X(x)\, ds, \\ &= \int_{X\times(-1,1)} \varepsilon\langle \psi(x,\varepsilon s), \phi(x,\varepsilon s)\rangle \, d\omega_X(x)\, ds, \\ &= \int_{X\times(-\varepsilon,\varepsilon)} \langle \psi(x,t), \phi(x,t)\rangle \, d\omega_X(x)\, dt, \\ &= \langle \psi, \phi\rangle. \end{aligned}$$

□

We want to know how the operator $S_\varepsilon \times_{\nabla\varepsilon} D_Y$ transforms under R_ε. We compute this in two steps starting with T_ε.

Lemma 2.4.5. *The map $R_\varepsilon T_\varepsilon R_\varepsilon^* : \Gamma_c^\infty\left(\pi_1^*\Sigma\right) \to L^2\left(\pi_1^*\Sigma\right)$ transforms as*

$$R_\varepsilon T_\varepsilon R_\varepsilon^* = \frac{1}{\varepsilon} T_1.$$

Proof. This is a straightforward computation, let $\psi \in \Gamma_c^\infty\left(\pi_1^*\Sigma\right)$

$$\begin{aligned} \left(R_\varepsilon T_\varepsilon R_\varepsilon^*\psi\right)(x,s) &= \sqrt{\varepsilon}(T_\varepsilon R_\varepsilon^*\psi)(x,\varepsilon s), \\ &= \sqrt{\varepsilon}\left(i\sigma_1\frac{d}{ds}(R_\varepsilon^*\psi) - \sigma_2 f_\varepsilon(\varepsilon s)(R_\varepsilon^*\psi)\right)(x,\varepsilon s), \\ &= \sqrt{\varepsilon}\left(i\sigma_1\frac{1}{\sqrt{\varepsilon}}\left.\frac{d}{ds}\right|_{(x,\varepsilon s)}\left(\psi\left(x,\frac{1}{\varepsilon}s\right)\right) - \sigma_2\frac{\pi}{2\varepsilon}\tan\left(\frac{\pi\varepsilon s}{2\varepsilon}\right)\frac{1}{\sqrt{\varepsilon}}\psi(x,s)\right), \\ &= i\sigma_1\frac{1}{\varepsilon}\psi'(x,s) - \sigma_2\frac{1}{\varepsilon}\frac{\pi}{2}\tan\left(\frac{\pi s}{2}\right)\psi(x,s), \\ &= \frac{1}{\varepsilon}\left(T_1\psi\right)(x,s). \end{aligned}$$

□

And next up is the behaviour of a family of operators.

Lemma 2.4.6. *Let $\{A_s\}_{s\in(-\varepsilon,\varepsilon)}$ be a family of operators on $L^2(\Sigma)$. Then*

$$R_\varepsilon A_\bullet R_\varepsilon^* = A_{\varepsilon\bullet}.$$

where

$$\begin{aligned} A_{\varepsilon\bullet} &: L^2(\pi_1^*\Sigma) \to L^2(\pi_1^*\Sigma) \\ &(A_{\varepsilon\bullet}\psi)(x,s) = (A_{\varepsilon s}\psi|_{X_s})(x). \end{aligned}$$

Proof. This is again a straightforward computation. Let $\psi\otimes\phi \in L^2(\pi_1^*\Sigma)$, then

$$\begin{aligned} (R_\varepsilon A_\bullet R_\varepsilon^*\psi)(x,s) &= \sqrt{\varepsilon}\,(A_\bullet(R_\varepsilon^*\psi))(x,\varepsilon s), \\ &= \sqrt{\varepsilon}\,(A_{\varepsilon s}\,(R_\varepsilon^*\psi)\,|_{X_{\varepsilon s}})(x), \\ &= (A_{\varepsilon s}\psi|_{X_s})(x). \end{aligned}$$

□

Combining Lemmas 2.4.5 and 2.4.6 and applying them to $S_\varepsilon \times_{\nabla\varepsilon} D_Y$ we get for n even

$$R_\varepsilon\,(S_\varepsilon \times_{\nabla\varepsilon} (D_Y\otimes\sigma_2))\,R_\varepsilon^* = \left(D_{X_{\varepsilon\bullet}} + \frac{1}{2\Lambda}\,[D_{X_{\varepsilon\bullet}},\Lambda]\right) + \frac{1}{\varepsilon}\gamma T_1 \tag{2.15a}$$

and for n odd

$$R_\varepsilon\,(S_\varepsilon \times_{\nabla\varepsilon} D_Y)\,R_\varepsilon^* = \left(D^+_{X_{\varepsilon\bullet}}\otimes\sigma_2 + \frac{1}{2\Lambda}\left[D^+_{X_{\varepsilon\bullet}}\otimes\sigma_2,\Lambda\right]\right) + \frac{1}{\varepsilon}\gamma T_1 \tag{2.15b}$$

So when restricting to ever smaller neighbourhoods as ε goes to zero we get a singular term $\gamma\otimes\frac{1}{\varepsilon}T$. We actually do get the useful asymptotics as ε goes to zero, by subtracting the known singular part.

Lemma 2.4.7. *For the family $\{A_s\}$ of operators on $L^2(\Sigma_X)$*

$$A_s = \frac{1}{2\Lambda|_{X_s}}\,[D_{X_s},\Lambda|_{X_s}]$$

we have that $A_{\varepsilon\bullet}$ converges to 0 in the operator norm on $L^2(\pi_1^*\Sigma_X)$ as $\varepsilon\to 0$.

Proof. Recall from Equation 2.2 that Λ is the change of volume function so that Λ is the smooth function with $\Lambda|_{X_s}\omega_{X_s} = \omega_X$. Now from this relation it is clear that $\Lambda|_{X_0}\equiv 1$, so $[D_{X_0},\Lambda|_{X_0}]\equiv 0$. By compactness of X we can find ε small enough that $[D_{X_s},\Lambda|_{X_s}]$ and $\frac{1}{\Lambda|_{X_s}}$ are arbitrarily small and close to 1, respectively, for $s\in(-\varepsilon,\varepsilon)$. □

Proposition 2.4.8. *For $\psi\in\Gamma_c^\infty(\pi_1^*\Sigma_X\otimes\mathbb{C}^2)$ and $\varepsilon<\varepsilon_1<\varepsilon_0$ we have for n even*

$$\left\|\left(R_\varepsilon(S_\varepsilon\times_{\nabla\varepsilon}(D_Y\otimes\sigma_2))R_\varepsilon^* - \frac{1}{\varepsilon}\gamma T_1 - D_X\right)\psi\right\| \le C\varepsilon(\|\psi\| + \|\psi\|_1)$$

for some $C>0$ depending only on ε_1. Here D_X acts on $\pi_1^*\Sigma_X$ as the constant family $\{D_X\}$ and $\|\psi\|_1 = \|\psi'\|_{L^2}$. For n odd we get

$$\left\|\left(R_\varepsilon(S_\varepsilon\times_{\nabla\varepsilon}D_Y)R_\varepsilon^* - \frac{1}{\varepsilon}\gamma T_1 - D_X^+\otimes\sigma_2\right)\psi\right\| \le C\varepsilon(\|\psi\| + \|\psi\|_1).$$

Proof. The proofs for the even and odd cases are essentially identical, so we will do the proof in even case. By Equation 2.15a we need to show for $\psi \in \Gamma_c^\infty(\pi_1^*\Sigma_X \otimes \mathbb{C}^2)$ that

$$\|(D_{X_{\varepsilon\bullet}}\psi + A_{\varepsilon\bullet}\psi - D_X\psi)\| \leq C\varepsilon(\|\psi\| + \|\psi\|_1).$$

Lemma 2.4.7 allows us to conclude this if we were to prove

$$\|(D_{X_{\varepsilon\bullet}}\psi - D_X\psi)\| \leq C\varepsilon(\|\psi\| + \|\psi\|_1).$$

In local coordinates $(x_1, \ldots, x_n, s)$ on $X \times (-\varepsilon, \varepsilon)$ we can write

$$\left(D_{X_{\varepsilon\bullet}}\psi\right)(\vec{x}, s) = B_i(g_{X_s}(\vec{x}))\frac{\partial\psi}{\partial x_i}(\vec{x}, s) + B(g_{X_s}(\vec{x}))\psi(\vec{x}, s)$$

for endomorphisms B, B_i of Σ_X that depend smoothly on $g_{X_s}(\vec{x})$ for $s \in (-\varepsilon_0, \varepsilon_0)$. Let C be an upper bound for $B(g_{X_s}(\vec{x})), B_i(g_{X_s}(\vec{x}))$ and their s-derivatives over the compact space $X \times [-\varepsilon_1, \varepsilon_1]$. Then, by the intermediate value theorem for operators, we get the desired $\|(D_{X_{\varepsilon\bullet}}\psi - D_X\psi)\| \leq C\varepsilon(\|\psi\| + \|\psi\|_1)$. □

We can reformulate the above proposition as the following asymptotic expansion.

Theorem 2.1. *We have the asymptotic expansion for* $\varepsilon \to 0$

$$R_\varepsilon \left(S_\varepsilon \times_{\nabla\varepsilon} D_Y\right) R_\varepsilon^* \sim \frac{1}{\varepsilon}\gamma T + D_X + O(\varepsilon)$$

in the first Sobolev norm.

In this "zooming in" scheme we do recover curvature data, as opposed to the stretching approach discussed at the start of this section.

Proposition 2.4.9. *The curvature of* $\imath_!^\varepsilon$ *converges to* $-\frac{1}{4}\operatorname{Tr}(\mathrm{II}_0)$ *as* $\varepsilon \to 0$.

Proof. By Lemma 2.2.2 the curvature of $\imath_!^\varepsilon$ is

$$\left(\nabla_{D_Y}^{\mathcal{E}}\right)^2 = \left(\frac{1}{4\Lambda}\left[D_{X_\bullet}, \Lambda\right]^2 - \frac{1}{4}\operatorname{Tr}(\mathrm{II}_\bullet)^2\right)$$

so that Lemma 2.4.7 shows that it converges to $-\frac{1}{4}\operatorname{Tr}(\mathrm{II}_\bullet)^2$ as operator on $L^2\left(\pi_1^*\Sigma_X \otimes \mathbb{C}^2\right)$ □

The conclusion we draw here is that we can recover D_X (or $D_X \otimes \sigma_2$ in the odd case) from the family of unbounded KK-cycles $\imath_!^\varepsilon$ as the constant term in the $\varepsilon \to 0$ limit. Moreover, we can recover the square of the mean curvature from this family of unbounded KK-cycles as well. Geometrically this comes down to zooming in to increasingly narrow neighbourhoods of X but without changing the metric as we did in the homotopy approach in Proposition 2.4.2. This ever smaller neighbourhood will still increasingly resemble $X \times (-\varepsilon, \varepsilon)$, but by reducing the distances over which the metric can change instead of slowing the rate of change itself.

Example 2.4.10. *For the spheres $S^n \hookrightarrow \mathbb{R}^{n+1}$ we can explicitly write down these ε products zoomed in to $\pi_1^*\Sigma_{S^n}$. Recall from Example 2.3.5 that the product with $\imath_!^\varepsilon$ is given by the operator*

$$D_{S^n} \otimes \frac{1}{s+1} + \gamma \otimes T_\varepsilon$$

on $L^2(\Sigma_{S^n}) \otimes L^2((-\varepsilon, \varepsilon), \mathbb{C}^2)$.

On $L^2(\Sigma_{S^n}) \otimes L^2((-1,1), \mathbb{C}^2)$, expanded by R_ε, this becomes

$$D_{S^n} \otimes \frac{1}{\varepsilon s + 1} + \gamma \otimes \frac{1}{\varepsilon} T_1$$

so

$$\|D_{S^n}\psi - D_{S^n_{\varepsilon\bullet}}\psi\| \leq \left|1 - \frac{1}{1-\varepsilon}\right| \|D_{S^n}\psi\| = \frac{\varepsilon}{1-\varepsilon}\|D_{S^n}\psi\| \to 0.$$

Now D_{S^n} is bounded in the Sobolev norm, so this yields the desired estimate.

This concludes our present analysis of immersions between manifolds. There are many questions left unanswered, not least among which how this works for higher codimensions. For now this is speculation, but I expect our approach to generalize. For a codimension m K-oriented immersion I believe one can construct a family of tangential Dirac operators for $s \in D^m$ with D^m an m-dimensional disk. This gives rise to tangential, radial and spherical directions, in which one should find D_{X_s}, T and some spherical operator such that T and this spherical part together will play the role of [1].

Another future project is to make the analysis in this Chapter more abstract, in the direction of [50]. The main goal of this project would be formulating the various domains more carefully in terms of the relevant operators, rather than simply relying on the smooth functions to solve the analytical problems. This would lead the way to looking for abstract criteria that can identify a family of correspondences as an immersion.

Chapter 3

Spectral Densities

This chapter is dedicated to the techniques required to analyze the model in Chapter 4. Throughout we will routinely use terminology from general random matrix theory. Much of this terminology is in Section 1.2 for unfamiliar readers. We will start in Section 3.1 with establishing some technical results and getting some definitions out of the way. These are necessary for our main results but are separated out of the main proofs of this section to help with the conceptual flow of Section 3.2.

In Section 3.2 we do the bulk of the work of this chapter. We generalize existing techniques for single-trace single-matrix models ([35], see also [22] for a more detailed explanation) to multi-trace models. We establish existence and uniqueness of a measure minimizing a specific functional and show that the spectral density converges to this measure in the large-N limit. Next, in Section 3.3, we establish an alternate characterization of this minimizing measure that allows us to actually compute this minimizing measure. Finally in Section 3.4 we present the general principles of using this alternate characterization in finding the large-N limit of the spectral density. We will apply these principles to the model defined in Chapter 4 in Section 4.3.

Remark 3.0.1. *After proving the generalized version of Theorem 3.3 it was discovered that this result had already been obtained in [21], however there it was obtained through different techniques and with slightly stronger assumptions. The main benefit of our approach is that the convexity assumptions (Assumption 3.1 point 3 and the setup before Theorem 3.3) are slightly weaker and allow for more properties of the model, such as symmetries, to be used.*

3.1 Some technical preparation

The goal of this chapter is to prove convergence of certain probability measures, for this the following notions of convergence and properties will be important. The results in this section are all classical, and will have similarities to [22, Ch. 6] since this section is a generalization of the proof therein.

Throughout this chapter let $\mathcal{P}(\mathbb{R})$ denote the space of probability measures on $\mathbb{R}$.

Definition 3.1.1. *A sequence* $\{\mu_n\}$ *in* $\mathcal{P}(\mathbb{R})$ converges weakly *to* μ*, denoted* $\mu_n \rightharpoonup \mu$*, if for all* $g \in C_b(\mathbb{R})$ *we have* $\int_{\mathbb{R}} g \, d\mu_n \to \int_{\mathbb{R}} g \, d\mu$.

Definition 3.1.2. *A sequence* $\{\mu_n\}$ *in* $\mathcal{P}(\mathbb{R})$ converges vaguely *to* μ*, denoted* $\mu_n \xrightarrow{v} \mu$*, if for all* $g \in C_c(\mathbb{R})$ *we have* $\int_{\mathbb{R}} g\, d\mu_n \to \int_{\mathbb{R}} g\, d\mu$.

Definition 3.1.3. *A sequence* $\{\mu_n\}$ *in* $\mathcal{P}(\mathbb{R})$ *is called* tight *if for all* $\varepsilon > 0$ *there is some* R *such that for all* n *we have* $\int_{|x|>R} d\mu_n < \varepsilon$.

The three concepts defined above come together in the following crucial proposition.

Proposition 3.1.4. *A tight sequence of probability measures has a weakly (thus also vaguely) convergent subsequence whose limit is a probability measure.*

Proof. Say we have a sequence of probability measures $\{\mu_n\}$ in $\mathcal{P}(\mathbb{R})$. We first want to extract a vaguely convergent subsequence. Each μ_n defines a continuous norm 1 linear functional ϕ_n on $C_c(\mathbb{R})$, thus by the Banach-Alaoglu theorem $\{\phi_n\}$ has a weak-$*$ convergent subsequence with limit ϕ (using that $C_c(\mathbb{R})$ is separable). By the Riesz-Markov theorem ϕ in turns corresponds to a measure μ on $\mathbb{R}$. Note that μ might not be a probability measure and may be the null measure.

We claim that the corresponding subsequence of measures converges vaguely to μ. Indeed, let $g \in C_c(\mathbb{R})$ then

$$\int_{\mathbb{R}} g\, d\mu_n = \phi_n(g) \to \phi(g) = \mu(g).$$

Let us forget about the original sequence and only focus on the vaguely convergent subsequence, still denoted $\{\mu_n\}$.

Now suppose the sequence of probability measures $\{\mu_n\}$ is tight (if this held for the original sequence it clearly holds for a subsequence). We wish to show two more things. First that the vague limit μ is a probability measure and second that the convergence is actually weak.

Let $\varepsilon > 0$ and R such that $\int_{|x|>R} d\mu_n < \varepsilon$ for all n. Let g be a continuous function with $g(x) = 1$ for $|x| < R$ and $g(x) = 0$ for $|x| > R + 1$, then $g \in C_c(\mathbb{R})$ and we have

$$\int_{\mathbb{R}} g(x)\, d\mu_n(x) \geq 1 - \varepsilon$$

for all n. So also

$$\int_{\mathbb{R}} g(x)\, d\mu(x) \geq 1 - \varepsilon.$$

This holds for all ε, so the total mass of μ is at least 1. Suppose the total mass of μ is $1 + \delta$ for $\delta > 0$. Then we can find a function $h \in C_c(\mathbb{R})$ of norm 1 with $\int h\, d\mu = 1 + \frac{1}{2}\delta$. This contradicts $\mu_n \xrightarrow{v} \mu$ since $\int h\, d\mu_n \leq 1$. So the total mass of μ is 1. A very similar argument shows that μ must be positive. Thus μ is a probability measure.

Moreover it has the same C_0-like behaviour as the tight sequence, *i.e.* $\int_{|x|>R} d\mu < \varepsilon$ for ε, R from the tightness of $\{\mu_n\}$.

Finally, let $f \in C_b(\mathbb{R})$. We wish to show that $\int_{\mathbb{R}} f\, d\mu_n \to \int_{\mathbb{R}} f\, d\mu$. Let $\varepsilon > 0$ and R as before. Then

$$\left| \int_{\mathbb{R}} f\, d\mu_n - \int_{\mathbb{R}} f\, d\mu \right| \leq \|f\|_\infty \varepsilon + \left| \int_{|x|<R} f\, d\mu_n - \int_{|x|<R} f\, d\mu \right|.$$

Since we are free to choose ε we can get the first term as small as desired and then we can pick n large enough such that the second term is as small as desired by vague convergence on the function $f|_{[-R,R]} \in C_c(\mathbb{R})$. □

In addition to the above notions of convergence we will need some technical results that will make sure all our steps in Section 3.2 are well defined.

Proposition 3.1.5. *If $\{\mu_n\}$ is a sequence of probability measures and $\mu_n \rightharpoonup \mu$ a probability measure μ, then $\mu_n^{\otimes k} \rightharpoonup \mu^{\otimes k}$ for $k \in \mathbb{N}$.*

Proof. Let $g : \mathbb{R}^k \to \mathbb{R}$ be a bounded continuous function. We need to establish that

$$\int_{\mathbb{R}^k} g(x_1, \ldots, x_k)\, d\mu_n(x_1) \ldots d\mu_n(x_k) \to \int_{\mathbb{R}^k} g(x_1, \ldots, x_k)\, d\mu(x_1) \ldots d\mu(x_k).$$

Let $\varepsilon > 0$ and R such that $\int_{|x|>R} d\mu < \varepsilon$ and $\int_{|x|>R} d\mu_n < \varepsilon$ for all n. This R exists since for n big enough $\int_{|x|>L} d\mu_n$ is arbitrarily close to $\int_{|x|>L} d\mu$ using weak convergence for the indicator function on $[-L, L]$.

We can decompose $\mathbb{R}^k$ into disjoint subsets by

$$\mathbb{R}^k = [-R, R]^k \cup \bigcup_{j=0}^{k-1} ([-R, R])^j \times ((-\infty, -R] \cup [R, \infty)) \times \mathbb{R}^{k-j-1}$$

and using this decomposition we get

$$\left| \int_{\mathbb{R}^k} g\, d\mu_n^{\otimes k} - \int_{\mathbb{R}^k} g\, d\mu^{\otimes k} \right| \leq \left| \int_{[-R,R]^k} g\, d\mu_n^{\otimes k} - \int_{[-R,R]^k} g\, d\mu^{\otimes k} \right| + 2k\varepsilon \|g\|_\infty$$

since on each of the parts of $\mathbb{R}^k$ with a $(-\infty, -R] \cup [R, \infty)$ component we can bound the integrand by $\|g\|_\infty$ so that both integrals can separately be bounded by $\varepsilon \|g\|_\infty$.

On the compact set $[-R, R]^k$ we can uniformly approximate g by polynomials, so we can find a polynomial p such that $\|g - p\|_\infty < \varepsilon$ this gives us

$$\left| \int_{[-R,R]^k} g\, d\mu_n^{\otimes k} - \int_{[-R,R]^k} g\, d\mu^{\otimes k} \right| \leq \left| \int_{[-R,R]^k} p\, d\mu_n^{\otimes k} - \int_{[-R,R]^k} p\, d\mu^{\otimes k} \right| + 2\varepsilon$$

since the error introduced by replacing g by p in both integrals is bounded by the ε times the integral over $[-R, R]^k$ which is less than 1. But for polynomials we clearly have weak convergence of $\mu_n^{\otimes k} \to \mu^{\otimes k}$ since the integral splits into a sum of products of one dimensional integrals.

Picking first ε small enough and then choosing n large enough we can get the right-hand side of

$$\left| \int_{\mathbb{R}^k} g\, d\mu_n^{\otimes k} - \int_{\mathbb{R}^k} g\, d\mu^{\otimes k} \right| \leq \left| \int_{[-R,R]^k} p\, d\mu_n^{\otimes k} - \int_{[-R,R]^k} p\, d\mu^{\otimes k} \right| + (2 + 2k\|g\|_\infty)\varepsilon$$

as small as we want. This establishes weak convergence $\mu_n^{\otimes k} \rightharpoonup \mu^{\otimes k}$. □

The last two results of this section are a departure from the more general discussions so far and concern the specifics of dealing with logarithmic integrals.

Proposition 3.1.6. *If ν is a compactly supported measure on $\mathbb{R}$ with $\int_{\mathbb{R}} d\nu = 0$, then*

$$-\iint \frac{1}{2}\log(m^2+(x-y)^2)\,d\nu(x)\,d\nu(y) = \int_0^\infty \frac{e^{-|m|u}}{u}\,|\hat{\nu}(u)|^2\,du$$

for $m \neq 0$.

Proof. This follows from a computation that we copy from [22]. For $m > 0$

$$\begin{aligned}
\log(m^2+s^2) &= \log(m^2) + \int_0^s \frac{2t}{t^2+m^2}\,dt,\\
&= \log(m^2) + 2\Im\int_0^s \frac{i}{t+im}\,dt,\\
&= \log(m^2) + 2\Im\int_0^s\int_0^\infty e^{i(t+im)u}\,du\,dt,\\
&= \log(m^2) + 2\Im\int_0^\infty \frac{1}{iu}e^{isu-mu} - \frac{1}{iu}e^{-mu}\,du,\\
&= \log(m^2) + 2\Im\int_0^\infty \frac{1-e^{isu}}{iu}e^{-mu}\,du.
\end{aligned}$$

For $m < 0$ one can substitute $-m$ for m starting in the second line.

So far we have creatively used the fundamental theorem of calculus to apparently complicate an expression. But now note that

$$\begin{aligned}
\iint \log\left(m^2+(x-y)^2\right)\,d\nu(x)\,d\nu(y) &= \iint \log(m^2)\,d\nu(x)\,d\nu(y)\\
&\quad + 2\Im\int_0^\infty \iint \frac{1-e^{i(x-y)u}}{iu}e^{-mu}\,d\nu(x)\,d\nu(y)\,du,\\
&= 2\Im\int_0^\infty \frac{e^{-mu}}{iu}\int e^{ixu}\,d\nu(x)\int e^{-iyu}\,d\nu(y)\,du
\end{aligned}$$

using that $\int d\nu = 0$. By definition $\int e^{ixu}\,d\nu(x) = \widehat{\nu}(u)$ is the Fourier transform of μ, so we obtain

$$\iint \log\left(m^2+(x-y)^2\right)\,d\nu(x)\,d\nu(y) = -2\int_0^\infty \frac{e^{-mu}}{u}\,|\widehat{\nu}(u)|^2\,du$$

using that ν is a real measure. □

Proposition 3.1.7. *If μ,ν are compactly supported probability measures and $-\log(|x-y|)$ is integrable with respect to $\mu\otimes\mu$ and $\nu\otimes\nu$ then it is integrable with respect to $\mu\otimes\nu$.*

Proof. Consider the measure $\rho = \mu - \nu$ with mean 0, so by Proposition 3.1.6 we get

$$-\iint \log\left(\sqrt{m^2+(x-y)^2}\right)\,d\rho(x)\,d\rho(y) = \int_0^\infty \frac{e^{-mu}}{u}\,|\widehat{\rho}(u)|^2\,du.$$

Letting m tend to zero we obtain by monotone convergence that

$$-\iint \log\left(|x-y|\right)\, d\rho(x)\, d\rho(y) = \int_0^\infty |\widehat{\rho}(u)|^2\, du \geq 0.$$

Let us consider the left-hand side of this equation, expanding ρ out again we get

$$\begin{aligned} -\iint \log\left(|x-y|\right)\, d\mu(x)\, d\mu(x) - \iint \log\left(|x-y|\right)\, d\nu(x)\, d\nu(y) \\ + \iint \log\left(|x-y|\right)\, d\mu(x)\, d\nu(x) + \iint \log\left(|x-y|\right)\, d\nu(x)\, d\mu(x) \geq 0, \end{aligned}$$

since $\log(|x-y|)$ is bounded above but not below each of these integrals may be $-\infty$. However we assume that $-\log(|x-y|)$ is integrable for $\mu \otimes \mu$ and $\nu \otimes \nu$ so the first two terms are finite. That means the mixed terms must also be finite, hence $-\log(|x-y|)$ is integrable with respect to $\mu \otimes \nu$ and $\nu \otimes \mu$. □

3.2 The equilibrium measure

In this section we will establish that the spectral density of a particular type of eigenvalue model has a large-N limit. For example those coming from a unitarily invariant multi-trace single-matrix model. We then show that this large-N limit is characterized by a variational problem. This section, and the proofs therein, are heavily based on [35, 22]. Some proofs carry over with minimal to no changes, but are included to make this text as self-contained as possible for a broader audience.

In further preparation for this section, define [1]

$$\underline{N}^k = \left\{ i \in \mathbb{N}^k \,\middle|\, 1 \leq i_j \leq N \,\forall 1 \leq j \leq k \right\}, \tag{3.1}$$

$$\Delta_N^k = \left\{ i \in \underline{N}^k \,\middle|\, i_a \neq i_b \text{ if } a \neq b \right\} \tag{3.2}$$

The situation we consider is the following. Let

$$P_N^\beta(\vec{\lambda}) = \frac{1}{Z_N} \exp\left(-N^{2-k} \sum_{i \in \underline{N}^k} U(\lambda_{i_1}, \ldots, \lambda_{i_k}) + \frac{\beta}{2} \sum_{(i,j) \in \Delta_N^2} \log|\lambda_i - \lambda_j| \right) d^N\vec{\lambda} \tag{3.3}$$

be a probability distribution on $\mathbb{R}^N$, we will call this the *eigenvalue model* or simply *model* if no confusion would arise.

This form of the interaction includes the more common form, seen for example in [21], where the potential term $N^{2-k}\sum U$ is written

$$N \sum_{i \in \underline{N}^1} V_1(\lambda_i) + \sum_{(i,j) \in \underline{N}^2} V_2(\lambda_i, \lambda_j) + \frac{1}{N} \sum_{(i,j,k) \in \underline{N}^3} V_3(\lambda_i, \lambda_j, \lambda_k) + \ldots$$

[1] In this definition Δ should be read as "different" and not "diagonal".

up to some finite interaction order k. This fits in our framework by symmetrizing, so we consider the potential U given by

$$U(\lambda_1, \ldots, \lambda_k) = \sum_{n=1}^{k} \frac{1}{k^n} \sum_{i \in \underline{k}^n} V_n(\lambda_{i_1}, \ldots, \lambda_{i_k}).$$

Working with one single function U turns out to be convenient for the proofs and not much trouble to work with in practice.

Remark 3.2.1. *The definitions and theorems in this section all assume $k \geq 2$. If $k = 1$ all results remain true, albeit with some minor changes in formulation. The $k = 1$ case is also already covered in the sources [35, 22] mentioned before. Alternatively one can use the symmetrization procedure described above to "upgrade" a non-interacting U ($k = 1$) to a trivially interacting $U(x, y) = \frac{1}{2}V(x) + \frac{1}{2}V(y)$.*

To an eigenvalue model from Equation 3.3 we associate an energy functional:

Definition 3.2.2. *The* associated energy functional *to the model in Equation 3.3 is defined on a k-tuple of measures, $(\mu_1, \ldots, \mu_k) \in \mathcal{M}(\mathbb{R})^k$, with values in $\mathbb{R} \cup \{-\infty, \infty\}$ by*

$$I(\mu_1, \ldots, \mu_k) := \int_{\mathbb{R}^k} W(x_1, \ldots, x_k) \prod_{i=1}^{k} d\mu_i(x_i).$$

Here

$$W(x_1, \ldots, x_k) := U(x_1, \ldots, x_k) - \frac{1}{k(k-1)} \sum_{(i,j) \in \Delta_k^2} \frac{\beta}{2} \log(|x_i - x_j|).$$

We will most often use the notation

$$I(\mu) := I(\mu, \ldots, \mu),$$

in which case

$$I(\mu) = \int_{\mathbb{R}^k} U(\vec{x})\, d\mu^{\otimes k}(\vec{x}) - \frac{\beta}{2} \int_{\mathbb{R}^2} \log(|x - y|)\, d\mu(x)\, d\mu(y).$$

Our first goal will be to prove the following theorem:

Theorem 3.1. *Under the assumptions on the eigenvalue model listed below in Assumption 3.1, the minimization problem*

$$\inf_{\mu \in \mathcal{P}(\mathbb{R})} I(\mu)$$

of the associated energy functional over the space of probability measures has a unique solution $\mu^E \in \mathcal{P}(\mathbb{R})$ and this measure has compact support.

Proof. This is a combination of Propositions 3.2.9, 3.2.10, (existence and uniqueness, respectively) and Lemma 3.2.6 (compact support) below. □

We will then proceed to show in Theorem 3.2, explicitly in Corollary 3.2.16, that the spectral density of the eigenvalue model in Equation 3.3 converges to this measure μ^E. We then provide an alternate characterization of the minimizer in Theorem 3.3.

Minimizing the associated energy functional is often called an equilibrium problem, since it is equivalent to finding the equilibrium distribution of a Coulomb gas. This is an infinitely divisible gas with a $\frac{1}{r}$ repulsive force, giving the log potential, in an interacting potential U. With this in mind we call the minimizing measure μ^E the *equilibrium measure*.

By the same analogy with physics we refer to k as the degree as the degree of interaction. For a gas that only interacts through a $\frac{1}{r}$ repulsive force we recover the problem in [35, 22], but in our models based on fuzzy geometries we will encounter interactions of degree 2 due to the presence of multi-trace terms, see Section 4.2.

To obtain Theorem 3.1 we assume the following properties of our eigenvalue model.

Assumption 3.1. *$U : \mathbb{R}^k \to \mathbb{R}$ is a continuous function such that:*

1. *U is invariant under permutation of its arguments.*
2. *There is a continuous function $u : \mathbb{R} \to \mathbb{R}$ such that $U(x_1, ..., x_k) \geq u(x_1)$ for all $(x_i) \in \mathbb{R}^k$ and $u(x) - \frac{\max(\beta,2)}{2} \log(1 + x^2) \to \infty$ as $|x| \to \infty$.*[2]
3. *There is a set of* candidate measures *$\mathcal{P}_{can} \subset \mathcal{P}(\mathbb{R})$ containing all minimizers of I, such that for any probability measures $\mu, \nu \in \mathcal{P}_{can}$ and any $t \in [0, 1]$*

$$\frac{d^2}{dt^2} \int_{\mathbb{R}^k} U(\vec{x})\, d(\mu + t(\nu - \mu))^{\otimes k}(\vec{x}) \geq 0.$$

Let us further discuss these assumptions, starting with the third and its set of candidate measures $\mathcal{P}_{can}$. This assumption is new compared to the single-trace case, *i.e.* the $k = 1$ case where it is automatic. It is crucial in showing that the minimizer of I is unique. The introduction of the set of candidate measures allows us to use a more relaxed convexity condition than the results for higher level interactions in the literature [21] that require the problem to be convex everywhere.

For example, the first two conditions in the above assumptions guarantee that any measure μ with $I(\mu) = +\infty$ cannot be a minimizer, by Lemma 3.2.4 below, and neither can a measure with non-compact support be a minimizer by Lemma 3.2.6. So we only need convexity between measures with $I(\mu)$ finite and compact support.

There are more universal conditions, for example

Lemma 3.2.3. *Any minimizing measure μ of I satisfies*

$$I(\mu, \ldots, \mu, \nu - \mu) \geq 0$$

for all $\nu \in \mathcal{P}(\mathbb{R})$ with compact support and $I(\nu)$ finite.

[2] The proofs in the following can be adapted to $\max(\beta, 1 + \varepsilon)$ for some $\varepsilon > 0$, but in practice u will generally have polynomial growth so we leave it at this simpler assumption.

Proof. A straightforward calculation, using Proposition 3.1.7 to make sure all integrals are well-defined, shows that

$$\left.\frac{d}{dt}\right|_{t=0} I(\mu + t(\nu - \mu)) = kI(\mu, \dots, \mu, \nu - \mu),$$

by permutation invariance of I and linearity in each argument. If this derivative is negative, $I(\mu + t(\nu - \mu)) < I(\mu)$ for t sufficiently small. □

More, and more accessible, conditions can come from the particular model under consideration. For example in the case of our fermionic fuzzy Dirac ensemble in Chapter 4 we show in Lemma 4.3.4 that any minimizer of I for that model must have mean 0. So we only need to show the convexity condition between measures of mean 0 in Proposition 4.3.5.

The first assumption is automatically satisfied if this probability density originates from the Weyl integration formula. Since U only appears in a sum over all combinations of indices this can always be achieved by symmetrizing, similar to how the log-term appears in the integral over $\mathbb{R}^k$ in 3.2.2. It is therefore essentially an empty assumption that simply avoids us having to symmetrize manually and further clutter already notation-heavy proofs.

The second assumption plays a key role in showing existence of a minimizer of the associated functional I by making sure U acts like a confining potential for the eigenvalues. This is required to counteract the inherent repulsive log term.

3.2.1 Existence and uniqueness

We will start by establishing that the associated functional I has a finite minimum.

Lemma 3.2.4.

$$\inf_{\mu \in \mathcal{P}(\mathbb{R})} I(\mu) < \infty$$

Proof. Consider $\mu = \chi_{[-1,1]} dx$, then one can compute that

$$\int_{\mathbb{R}^2} \log(|x - y|)\, d\mu(x)\, d\mu(y) = \log(16) - 6 < \infty.$$

As $U : \mathbb{R}^k \to \mathbb{R}$ is continuous

$$\int_{\mathbb{R}^k} U(\vec{x})\, d\mu^{\otimes k}(\vec{x}) < \infty$$

as well, so $I(\mu)$ is finite. □

Now we move on to the real work. We start by showing that the combined potential W (consisting of U and the log interaction) is suitably confining. This then implies that the equilibrium measure will be compact, but also that any sequence of measures with finite energy is tight. Any sequence of measures with particles escaping to infinity will have increasing energy due to the confining nature of U.

Lemma 3.2.5. *The function $W : \mathbb{R}^k \to \mathbb{R}$ defined by (see also Definition 3.2.2)*

$$W(x_1, \ldots, x_k) = U(x_1, \ldots, x_k) - \frac{1}{k(k-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{k} \frac{\beta}{2} \log(|x_i - x_j|)$$

tends to infinity as $\|x\| \to \infty$.

Proof. A straightforward check by expanding both sides shows that $(x-y)^2 \leq (1+x^2)(1+y^2)$, so

$$\log(|x-y|) \leq \log\left(\sqrt{1+x^2}\right) + \log\left(\sqrt{1+y^2}\right)$$

and thus

$$\begin{aligned} W(x_1, \ldots x_k) &\geq U(x_1, \ldots, x_k) - \frac{1}{k(k-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{k} \frac{\beta}{2} \left(\log\left(\sqrt{1+x_i^2}\right) + \log\left(\sqrt{1+x_j^2}\right)\right), \\ &= U(x_1, \ldots, x_k) - \frac{1}{k} \sum_{i=1}^{k} \frac{\beta}{2} \log\left(1+x_i^2\right), \\ &\geq \frac{1}{k} \sum_{i=1}^{k} \left(u(x_i) - \frac{\beta}{2} \log\left(1+x_i^2\right)\right). \end{aligned}$$

By Assumption 3.1 each term in this sum tends to infinity as x_i tends to infinity, so W tends to infinity as $\|x\|$ tends to infinity. □

At this point we do not yet need the $\max(\beta, 2)$ from Assumption 3.1 yet. That extra condition will show up in when we need $\exp(-\frac{1}{2}u(x))$ to be integrable.

Lemma 3.2.6. *Any minimizer μ of I has compact support.*

Proof. By Lemma 3.2.5 we can find R such that $W(x_1, x_2, \ldots, x_k) > I(\mu)+1$ for $|x_1| > R$.

Let $D = \mathbb{R} \setminus [-R, R]$ and consider the probability measures

$$\begin{aligned} \mu_{+1} &= \frac{\mu + \mu|_D}{1 + \mu(D)}, \\ \mu_{-1} &= \frac{\mu - \mu|_D}{1 - \mu(D)}. \end{aligned}$$

As we assume μ is a minimizer this implies by Lemma 3.2.3 that both

$$\begin{aligned} I(\mu, \ldots, \mu, \mu_{+1}) &\geq I(\mu), \\ I(\mu, \ldots, \mu, \mu_{-1}) &\geq I(\mu). \end{aligned}$$

Taking a closer look at this first inequality, we find

$$\begin{aligned} I(\mu, \ldots, \mu, \mu + \mu|_D) &\geq (1 + \mu(D))I(\mu), \\ I(\mu, \ldots, \mu, \mu|_D) &\geq \mu(D)I(\mu). \end{aligned}$$

The second inequality gives

$$I(\mu, \dots, \mu, \mu - \mu|_D) \geq (1 - \mu(D))I(\mu),$$
$$I(\mu, \dots, \mu, -\mu|_D) \geq -\mu(D)I(\mu).$$

Together this implies

$$I(\mu, \dots, \mu, \mu|_D) = \mu(D)I(\mu).$$

But then we can compute the left hand side and find

$$\begin{aligned} I(\mu, \dots, \mu|_D) &= \int_{\mathbb{R}^k} W(\vec{x})\, d\mu|_D(x_1)\, d\mu^{\otimes k-1}(x_2, \dots, x_k), \\ &\geq \int_{\mathbb{R}^k} I(\mu) + 1\, d\mu|_D(x_1)\, d\mu^{\otimes k-1}(x_2, \dots, x_k), \\ &= (I(\mu) + 1)\mu(D), \end{aligned}$$

as for $x_1 \in D$, $W(\vec{x}) > I(\mu) + 1$. Therefore $\mu(D) = 0$. □

Lemma 3.2.7. *If $\{\mu_n\}_{n\in\mathbb{N}}$ is a sequence of probability measures such that $I(\mu_n) \leq C$ for some fixed $C \in \mathbb{R}$ and all $n \in \mathbb{N}$, then μ_n is tight.*

Proof. Let $\varepsilon > 0$. By Lemma 3.2.5, W has a lower bound b, w.o.l.o.g. $b < 0$. Find a B such that $b + B\varepsilon > C$, then, again by Lemma 3.2.5, we can find R such that if $|x_1| > R$ then $W(x_1, x_2, \dots, x_k) > B$. We get

$$C \geq I(\mu_n) \geq \int_{\mathbb{R}^{k-1}} \left(\int_{|x|<R} b\, d\mu_n(x) + \int_{|x|>R} B d\mu_n(x) \right) \prod_{i=2}^{k} d\mu_n(x_i) \geq b + B \int_{|x|>R} d\mu_n(x)$$

so $\int_{|x|>R} d\mu_n(x) < \varepsilon$, this holds independently of n so the sequence $\{\mu_n\}$ is tight. □

Having established that sequences with bounded energy are tight, we only need a simple continuity result to obtain existence of a minimizer. We then can use Assumption 3.1, in particular point 3, to establish uniqueness.

Lemma 3.2.8. *I is lower semi-continuous with respect to weak convergence on $\mathcal{P}(\mathbb{R})$.*

Proof. Suppose $\{\mu_n\}$ is a sequence in $\mathcal{P}(\mathbb{R})$ weakly converging to the probability measure μ. For any $L \in \mathbb{R}$,

$$I(\mu_n) \geq \int_{\mathbb{R}^k} \min(L, W(\vec{x}))\, d\mu_n^{\otimes k}(\vec{x}) \to \int_{\mathbb{R}^k} \min(L, W(\vec{x}))\, d\mu^{\otimes k}(\vec{x})$$

by Proposition 3.1.5. The right-hand side is monotone increasing in L and thus converges to $I(\mu)$ by the dominated convergence theorem. □

Proposition 3.2.9. *A measure μ^E minimizing I exists.*

Proof. There exists a sequence of measures μ_n such that $I(\mu_n)$ converges to $E := \inf_{\mu\in\mathcal{P}(\mathbb{R})} I(\mu)$. By Lemma 3.2.7 this sequence is tight, so by Lemma 3.1.4 it has a weakly convergent subsequence with a probability measure μ^E as weak limit. Then $E \leq I(\mu^E) \leq \liminf I(\mu_n) = E$ as by Lemma 3.2.8 I is lower semi-continuous. So $I(\mu^E) = E$. □

Proposition 3.2.10. *If μ, ν satisfy $I(\mu) = I(\nu) = \inf_{\rho\in\mathcal{P}(\mathbb{R})} I(\rho)$, then $\mu = \nu$.*

Proof. Suppose μ and ν are both minimizers of I. Consider $\rho_t = \mu + t(\nu - \mu)$, which is a probability measure on $\mathbb{R}$ for $t \in [0, 1]$. We will show that, unless $\mu = \nu$, $I(\rho_t)$ is strictly convex. This would contradict the assumption that μ, ν are both minimizers of I.

$$I(\rho_t) = \int_{\mathbb{R}^k} U(\vec{x}) d\rho_t^{\otimes k}(\vec{x}) - \int_{\mathbb{R}^2} \log(|x-y|)\, d\rho_t(x) d\rho_t(y)$$

The second derivative with respect to t is then

$$\frac{d^2}{dt^2} I(\rho_t) = \frac{d^2}{dt^2}\left[\int_{\mathbb{R}^k} U(\vec{x})\, d\rho_t^{\otimes k}(\vec{x})\right] + 2\int_{R^2} \log(|x-y|^{-1})\, d(\nu-\mu)(x)\, d(\nu-\mu)(y).$$

By Proposition 3.1.6 the integral of the log term is given by

$$-\int_{\mathbb{R}^2} \log(|x-y|)\, d(\nu-\mu)(x)\, d(\nu-\mu)(y) = \int_0^\infty \frac{1}{k}\left|\widehat{(\nu-\mu)}(k)\right|^2 dk \geq 0$$

with equality only if $\mu = \nu$ (this follows from Fourier inversion for distributions).

The set of candidate minimizers is required to contain the actual minimizers, so by Assumption 3.1 the second derivative of the U-integral is positive. Thus $\frac{d^2}{dt^2} I(\rho_t) \geq 0$, with equality only if $\mu = \nu$. But since $I(\rho_0) = I(\rho_1)$, this implies $I(\rho_t)$ attains a strict minimum for $t \in (0, 1)$ unless $\mu = \nu$. □

3.2.2 Convergence of spectral density

In this section we will show that the spectral density of the eigenvalue model converges weakly to the equilibrium measure μ^E of the associated energy functional. In fact, we will show that the n-point densities converge to $(\mu^E)^{\otimes n}$. As was the case for the previous section, this section is heavily based on [35, 22].

As a further preface to this section, we expand a little bit on the definition of the function W that defines the associated energy functional I. It is designed such that $\sum_{i\in\Delta_N^k} W(x_{i_1}, \ldots, x_{i_k})$ gives the exponent of the eigenvalue density (Equation 3.3), at least to leading order in N. A lot of the work in this section revolves around properly dealing with the lower order terms, this is especially noticeable in Proposition 3.2.13. One might hope that by changing the model to be exactly the above sum of W these proofs can be simplified considerably, but that would no longer correspond to the model we develop in Chapter 4.

The proofs in this section revolve around the sequence of sets

$$A_{N,\eta} = \left\{ x \in \mathbb{R}^N \;\middle|\; \frac{1}{N^k} \sum_{i\in\Delta_N^k} W(x_{i_1}, \ldots, x_{i_k}) \leq (E + \eta) \right\} \tag{3.4}$$

where $E = \inf_{\mu \in \mathcal{P}(\mathbb{R})} I(\mu)$.

Lemma 3.2.11. *$A_{N,\eta}$ is compact for all $N \in \mathbb{N}, \eta \in \mathbb{R}_{>0}$.*

Proof. By Lemma 3.2.5, $W(\vec{y})$ tends to infinity as $\|\vec{y}\| \to \infty$ so $A_{N,\eta}$ is bounded. Moreover, $(x, y) \mapsto \log(|x-y|)$ is continuous as a map $\mathbb{R}^2 \to \mathbb{R} \cup \{\infty\}$, hence so is the function defining $A_{N,\eta}$. Since $A_{N,\eta}$ is the inverse image of $(-\infty, E+\eta] \subset (-\infty, \infty]$ it is closed. □

The gist of the convergence proof consists of showing that, as you might expect from the definition, points $\vec{x}$ in $A_{N,\eta}$ have "low energy". So their associated counting measures will tend towards minimizers of I (Proposition 3.2.14). It also means that the probability of finding the eigenvalues of our model *outside of* $A_{N,\eta}$ is exponentially suppressed (Proposition 3.2.13). These observations combine to prove that the spectral density is a minimizer of I (Theorem 3.2). First, however, we want to be able to work with absolutely continuous measures.

Lemma 3.2.12. *Let $\mu \in \mathcal{P}(\mathbb{R})$ such that μ has compact support and $I(\mu) < \infty$. Then for $\mu_\varepsilon := \frac{1}{2\varepsilon}\chi_{[-\varepsilon,\varepsilon]} * \mu$ we have $I(\mu_\varepsilon) \to I(\mu)$.*

Proof. Since μ is assumed to have compact support and U is assumed to be continuous, Proposition 3.1.5 implies that the first integral in

$$I(\mu_\varepsilon) = \int_{\mathbb{R}^k} U(\vec{x})\, d\mu_\varepsilon^{\otimes k}(\vec{x}) - \int_{\mathbb{R}^2} \log(|x-y|)\, d\mu_\varepsilon(x)\, d\mu_\varepsilon(y)$$

converges to the corresponding integral for μ. Hence we verify convergence of the second integral.

It is a straightforward check, essentially using the adjoint formula for convolution operators, that integrating a function with respect to the "smeared" measure μ_ε is equivalent to integrating the smeared function with respect to he original measure (using that the smearing function is real and symmetric). In other words,

$$\int_{\mathbb{R}^2} \log(|x-y|)\, d\mu_\varepsilon(x)\, d\mu_\varepsilon(y) = \int_{\mathbb{R}^2} \left(\frac{1}{4\varepsilon^2} \iint_{-\varepsilon}^{\varepsilon} \log(|x-s-y+t|)\, ds\, dt \right) d\mu(x)\, d\mu(y).$$

This integral is finite by the assumption that $I(\mu) < \infty$.

We now need to investigate this ε-integral as $\varepsilon \to 0$.

$$\begin{aligned} \frac{1}{4\varepsilon^2} \iint_{-\varepsilon}^{\varepsilon} \log(|x-s-y+t|)\, ds\, dt &= \log(|x-y|) + \frac{1}{4\varepsilon^2} \iint_{-\varepsilon}^{\varepsilon} \log\left(\left| \frac{t-s}{x-y} + 1 \right| \right) ds\, dt, \\ &= \log(|x-y|) + \frac{|x-y|^2}{4\varepsilon^2} \iint_{-\frac{\varepsilon}{|x-y|}}^{\frac{\varepsilon}{|x-y|}} \log\left(|u-v+1|\right)\, du\, dv. \end{aligned}$$

If we can show that the total "error term" caused by the smearing,

$$\int_{\mathbb{R}^2} \left(\frac{|x-y|^2}{4\varepsilon^2} \iint_{-\frac{\varepsilon}{|x-y|}}^{\frac{\varepsilon}{|x-y|}} \log\left(|u-v+1|\right)\, du\, dv \right) d\mu(x)\, d\mu(y),$$

goes to 0 as $\varepsilon \to 0$, we are done.

Doing some basic, if cumbersome, calculus we get

$$\begin{aligned}\frac{1}{4a^2}\iint_{-a}^{a} \log(|u-v+1|)\,du\,dv &= \frac{1}{8a^2}\int_{-2a}^{2a}\int_{-2a+|t|}^{2a-|t|} \log(|s+1|)\,ds\,dt,\\ &= \frac{1}{4a^2}\int_{0}^{2a}\int_{1-2a+t}^{1+2a-t} \log(|s|)\,ds\,dt,\\ &= \left(\frac{(1+2a)^2}{8a^2}\log\left(|1+2a|\right) + \frac{(1-2a)^2}{8a^2}\log\left(|1-2a|\right)\right) - \frac{3}{2}.\end{aligned}$$

The limit as $a \to 0$ can be computed to be 0 using L'Hôpital's rule. As $a \to \infty$ the integral is asymptotic to $\log(1+2a) - \frac{3}{2}$. Therefore the integral is, in absolute value, bounded by $C\log(1+2a)$ for some C. Upon closer inspection one can check that in fact $C = 1$ works.

So our total error term is bounded in absolute value by

$$\int_{\mathbb{R}^2} \log\left(1 + \frac{2\varepsilon}{|x-y|}\right)\,d\mu(x)\,d\mu(y) = \int_{\mathbb{R}^2} \log(|x-y|+2\varepsilon) - \log(|x-y|)\,d\mu(x)\,d\mu(y).$$

As $\log(|x-y|)$ is integrable with respect to $\mu^{\otimes 2}$, since $I(\mu) < \infty$ and μ has compact support, this converges to 0 as $\varepsilon \to 0$. □

Proposition 3.2.13. *Let P_N be an eigenvalue model as in Equation 3.3 and $A_{N,\eta}$ as in Equation 3.4. Then, for all $\eta > 0$, there exists an N^* such that for $N \geq N^*$, $P_N(\mathbb{R}^N \setminus A_{N,\eta+a}) \leq e^{-aN^2}$.*

Proof. By Lemma 3.2.12 we can find a continuous function ϕ with compact support such that the measure $\mu = \phi(x)dx \in \mathcal{P}(\mathbb{R})$ and $I(\mu) \leq E + \frac{\eta}{4}$ (take μ to be a "smeared" version of μ^E for a sufficiently small ε). Let $D = \{x \in \mathbb{R} \,|\, \phi(x) > 0\}$.

Set further for notational convenience

$$\begin{aligned}S_N(\vec{x}) &:= N^{2-k}\sum_{i\in\underline{N}^k} U(x_{i_1},\dots,x_{i_k}) - \sum_{(i,j)\in\Delta_N^2}\frac{\beta}{2}\log(|x_i - x_j|),\\ &= N^{2-k}\sum_{i\in\underline{N}^k\setminus\Delta_N^k} U(x_{i_1},\dots,x_{i_k}) + N^{2-k}\sum_{i\in\Delta_N^k} U(x_{i_1},\dots,x_{i_k})\\ &\qquad - \frac{|\Delta_N^2|}{|\Delta_N^k|}\sum_{i\in\Delta_N^k}\frac{1}{|\Delta_k^2|}\sum_{(k,l)\in\Delta_k^2}\frac{\beta}{2}\log(|x_{i_k}-x_{i_l}|),\\ &= N^{2-k}\sum_{i\in\underline{N}^k\setminus\Delta_N^k} U(x_{i_1},\dots,x_{i_k}) + \left(N^{2-k} - \frac{|\Delta_N^2|}{|\Delta_N^k|}\right)\sum_{i\in\Delta_N^k} U(x_{i_1},\dots,x_{i_k})\\ &\qquad + \frac{|\Delta_N^2|}{|\Delta_N^k|}\sum_{i\in\Delta_N^k} W(x_{i_1},\dots,x_{i_k}),\\ &= S_N^{(1)}(\vec{x}) + S_N^{(2)}(\vec{x}) + S_N^{(3)}(\vec{x}).\end{aligned}$$

Then

$$\begin{aligned}
Z_N &= \int_{\mathbb{R}^N} \exp\left(-S_N(\vec{x})\right)\, d^N x, \\
&\geq \int_{D^N} \exp\left(-S_N(\vec{x})\right)\, d^N x, \\
&= \int_{D^N} \exp\left(-S_N(\vec{x}) - \sum_{j=1}^N \log\phi(x_j)\right) \prod_{j=1}^N \phi(x_j)\, dx_j, \\
&\geq \exp\left(\int_{D^N} -S_N(\vec{x}) - \sum_{j=1}^N \log\phi(x_j)\, d\mu^{\otimes N}(\vec{x})\right),
\end{aligned}$$

where the last step follows by Jensen's inequality. Phrasing this more in line with how we will use it, we have

$$\frac{1}{Z_N} \leq \exp\left(\int_{D^N} S_N^{(1)}(\vec{x}) + S_N^{(2)}(\vec{x}) + S_N^{(3)}(\vec{x})\, d\mu^{\otimes N} + \int_{D^N} \sum_{j=1}^N \log\phi(x_j)\, d\mu^{\otimes N}(\vec{x})\right).$$

We compute

$$\begin{aligned}
\left|\int_{D^N} S_N^{(1)}(\vec{x})\, d\mu^{\otimes N}(\vec{x})\right| &\leq \int_{D^N} N^{2-k} \sum_{i\in \underline{N}^k \setminus \Delta_N^k} |U(x_{i_1},\ldots,x_{i_k})|\, d\mu^{\otimes N}(\vec{x}), \\
&= N^{2-k} \sum_{i\in \underline{N}^k \setminus \Delta_N^k} \int_{D^N} |U(x_{i_1},\ldots,x_{i_k})|\, d\mu^{\otimes N}(\vec{x}), \\
&\leq N^{2-k}\left(N^k - |\Delta_N^k|\right) \sup_{\vec{x}\in D^k}(|U(\vec{x}|), \\
&= N^2\left(1 - \frac{|\Delta_N^k|}{N^k}\right) \sup_{\vec{x}\in D^k}(|U(\vec{x}|),
\end{aligned}$$

The supremum is finite since U is continuous and D^k is compact. Since $|\Delta_N^k| = N^k + O(N^{k-1})$, this means that $\int_{D^N} S_N^{(1)}(\vec{x}) d\mu^{\otimes N}$ is $O(N)$.

$$\begin{aligned}
\int_{D^N} S_N^{(2)}(\vec{x})\, d\mu^{\otimes N}(\vec{x}) &= \int_{D^N} \left(N^{2-k} - \frac{|\Delta_N^2|}{|\Delta_N^k|}\right) \sum_{i\in\Delta_N^k} U(x_{i_1},\ldots,x_{i_k})\, d\mu^{\otimes N}(\vec{x}), \\
&= \left(N^{2-k} - \frac{|\Delta_N^2|}{|\Delta_N^k|}\right) \sum_{i\in\Delta_N^k} \int_{D^N} U(x_{i_1},\ldots,x_{i_k})\, d\mu^{\otimes N}(\vec{x}), \\
&= \left(N^{2-k} - \frac{|\Delta_N^2|}{|\Delta_N^k|}\right) |\Delta_N^k| \int_{D^k} U(x_1,\ldots,x_k)\, d\mu^{\otimes k}(\vec{x}), \\
&= \left(N^{2-k}|\Delta_N^k| - |\Delta_N^2|\right) \int_{D^k} U(x_1,\ldots,x_k)\, d\mu^{\otimes k}(\vec{x}).
\end{aligned}$$

Both $N^{2-k}|\Delta_N^k|$ and $|\Delta_N^2|$ are $N^2 + O(N)$, so also $\int_{D^N} S_N^{(2)}(\vec{x})\, d\mu^{\otimes N}$ is $O(N)$ since the integral is again finite by the compact support of μ. In this computation we also used

that, since we are summing over Δ_N^k, all the integrals are the same. This did not hold for $S_N^{(1)}$ since there are guaranteed to be one or more double indices and this degeneracy varies between the choices of indices appearing in the sum.

Next

$$\begin{aligned}\int_{D^N} S_N^{(3)}(\vec{x})\, d\mu^{\otimes N}(\vec{x}) &= \int_{D^N} \frac{|\Delta_N^2|}{|\Delta_N^k|} \sum_{i\in\Delta_N^k} W(x_{i_1},\dots,x_{i_k})\, d\mu^{\otimes N}(\vec{x}),\\ &= \frac{|\Delta_N^2|}{|\Delta_N^k|} \sum_{i\in\Delta_N^k} \int_{D^N} \sum_{i\in\Delta_N^k} W(x_{i_1},\dots,x_{i_k})\, d\mu^{\otimes N}(\vec{x}),\\ &= |\Delta_N^2| \int_{D^k} W(x_1,\dots,x_k)\, d\mu^{\otimes k}(\vec{x}),\\ &= (N^2-N)I(\mu).\end{aligned}$$

And finally

$$\int_{D^N} \sum_{j=1}^{N} \log\phi(x_j)\, d\mu^{\otimes N}(\vec{x}) = N\int_D \log(\phi(x))\phi(x)\, dx.$$

Combining the above results, we get that

$$\frac{1}{Z_N} \le \exp\left(N^2(I(\mu)+a(N))\right)$$

with $a(N)$ some function in $O\left(\frac{1}{N}\right)$. Find N_1 such that for $N \ge N_1$ we have $a(N) \le \frac{\eta}{4}$, since also $I(\mu) \le E + \frac{\eta}{4}$ this means that for $N \ge N_1$

$$\frac{1}{Z_N} \le e^{N^2\left(E+\frac{\eta}{2}\right)}.$$

We will use this estimate on Z_N to get our estimate on P_N.

$$\begin{aligned}&P_N\left(\mathbb{R}^N \setminus A_{N,\eta+a}\right)\\ &\quad= \frac{1}{Z_N}\int_{\mathbb{R}^N\setminus A_{N,\eta+a}} \exp\left(-S_N(\vec{x})\right) d^N\vec{x},\\ &\quad= \frac{1}{Z_N}\int_{\mathbb{R}^N\setminus A_{N,\eta+a}} \exp\left(-S_N^{(1)}(\vec{x}) - S_N^{(2)}(\vec{x}) - S_N^{(3)}(\vec{x})\right) d^N\vec{x},\\ &\quad\le e^{N^2\left(E+\frac{\eta}{2}\right)}\int_{\mathbb{R}^N\setminus A_{N,\eta+a}} \exp\left(-S_N^{(1)}(\vec{x}) - S_N^{(2)}(\vec{x}) - \frac{|\Delta_N^2|}{|\Delta_N^k|}\sum_{i\in\Delta_N^k}(E+\eta+a)\right) d^N\vec{x},\\ &\quad\le e^{N^2\left(E+\frac{\eta}{2}\right)}\int_{\mathbb{R}^N\setminus A_{N,\eta+a}} \exp\left(-S_N^{(1)}(\vec{x}) - S_N^{(2)}(\vec{x}) - N(N-1)(E+\eta+a)\right) d^N\vec{x},\\ &\quad= e^{-N^2\left(\frac{\eta}{2}+a\right)}e^{N(E+\eta+a)}\int_{\mathbb{R}^N\setminus A_{N,\eta+a}} \exp\left(-S_N^{(1)}(\vec{x}) - S_N^{(2)}(\vec{x})\right) d^N\vec{x}.\end{aligned}$$

By Assumption 3.1 there exists a function u such that $\int_{\mathbb{R}} e^{-\frac{1}{2}u(x)}\,dx < \infty$ and $U(x_1,\dots,x_k) \geq u(x_1)$. In particular u is bounded below by some b. Therefore, noting that $N^{2-k} - \frac{|\Delta_N^2|}{|\Delta_N^k|} < 0$,

$$\begin{aligned}
&\int_{\mathbb{R}^N \setminus A_{N,\eta+a}} \exp\left(-S_N^{(1)}(\vec{x}) - S_N^{(2)}(\vec{x})\right) d^N\vec{x} \\
&\qquad \leq \int_{\mathbb{R}^N} \exp\left(-N^{2-k} \sum_{i \in N^k \setminus \Delta_N^k} u(x_{i_1}) - \left(N^{2-k} - \frac{|\Delta_N^2|}{|\Delta_N^k|}\right) \sum_{i \in \Delta_N^k} b\right) d^N\vec{x}, \\
&\qquad = \exp\left(\left(|\Delta_N^2| - N^{2-k}|\Delta_N^k|\right) b\right) \int_{\mathbb{R}^N} \exp\left(-\frac{\left|\underline{N}^k \setminus \Delta_N^k\right|}{N^{k-1}}(u(x_1) + \ldots + u(x_N))\right) d^N\vec{x}, \\
&\qquad = \exp\left(\left(|\Delta_N^2| - N^{2-k}|\Delta_N^k|\right) b\right) \left(\int_{\mathbb{R}} \exp\left(-\frac{\left|\underline{N}^k \setminus \Delta_N^k\right|}{N^{k-1}} u(x)\right) dx\right)^N.
\end{aligned}$$

In the second step we use that any index occurs equally often as the first element i_1 of a choice of indices i.

First consider the factor

$$\exp\left(\left(|\Delta_N^2| - N^{2-k}|\Delta_N^k|\right) b\right).$$

The combinatorial factor $|\Delta_N^2| - N^{2-k}|\Delta_N^k|$ is $O(N)$ since $|\Delta_N^n| = N^n + O(N^{n-1})$, so we can find some B such that

$$\exp\left(\left(|\Delta_N^2| - N^{2-k}|\Delta_N^k|\right) b\right) \leq e^{NB}.$$

Next we consider the integral. Since $|\underline{N}^k \setminus \Delta_N^k| = \frac{1}{2}k(k-1)N^{k-1} + O\left(N^{k-2}\right)$ as $N \to \infty$ and[3] $k \geq 2$, we can find some $N_2 \geq N_1$ such that for $N \geq N_2$ we have $\frac{|\underline{N}^k \setminus \Delta_N^k|}{N^{k-1}} \geq \frac{1}{2}$, so for $N \geq N_2$

$$\int_{\mathbb{R}} \exp\left(-\frac{\left|\underline{N}^k \setminus \Delta_N^k\right|}{N^{k-1}} u(x)\right) dx \leq C \int_{\mathbb{R}} e^{-\frac{1}{2}u(x)}\,dx$$

for some $C > 0$.

So we have established that

$$P_N\left(\mathbb{R}^N \setminus A_{N,\eta+a}\right) \leq e^{-aN^2} e^{-\frac{\eta}{2}N^2} e^{N(E+\eta+a)} e^{NB} \left(C \int_{\mathbb{R}} e^{-u(x)}\,dx\right)^N$$

for $N \geq N_2$. Now find $N^* \geq N_2$ such that for $N \geq N^*$

$$e^{-\frac{\eta}{2}N^2} e^{N(E+\eta+a)} e^{NB} \left(C \int_{\mathbb{R}} e^{-u(x)}\,dx\right)^N \leq 1$$

and we are done. □

[3] See Remark 3.2.1, if $k = 1$ the definition of S_N changes which changes this integral

Proposition 3.2.14. *Let $\eta > 0$ and $\{\vec{x}(N)\}$ a sequence of points in $A_{N,\eta}$. Then $\nu_N := \frac{1}{N}\sum \delta_{x_i(N)}$ has a weakly convergent subsequence and any weak limit point ν of $\{\nu_N\}$ has $I(\nu) \leq E + \eta$.*

Proof. Let us first establish that the sequence $\{\nu_N\}$ is tight, for which we mimic the proof of Lemma 3.2.6. Let $\varepsilon > 0$ be arbitrary and b the lower bound of W. Find B such that $\varepsilon B + (1-\varepsilon)b > E + \eta$, and R such that $W(x_1, ..., x_k) > B$ if $|x_1| > R$, using Lemma 3.2.5.

Let $f_R(N)$ be the fraction of coordinates in $\vec{x}(N)$ whose absolute value is greater than R. Then by the definition of $A_{N,\eta}$ we have that

$$\begin{aligned}(E+\eta) &\geq N^{-k}\sum_{i\in\Delta_N^k} W(x_{i_1}(N), ..., x_{i_k}(N)),\\ &> N^{-k}\sum_{\substack{i\in\Delta_N^k,\\ |x_{i_1}|>R}} B + N^{-k}\sum_{\substack{i\in\Delta_N^k,\\ |x_{i_1}|\leq R}} b,\\ &= \frac{|\Delta_N^k|}{N^k}\cdot f_R \cdot B + \frac{|\Delta_N^k|}{N^k}\cdot(1-f_R)\cdot b.\end{aligned}$$

As $\frac{|\Delta_N^k|}{N^k} \to 1$ as $N \to \infty$ this implies that $\limsup_{N\to\infty} f_R(N) < \varepsilon$, i.e. there exists an N_1 such that for $N > N_1$, $f_R = \int_{|x|>R} d\nu_N < \varepsilon$. Choosing $R' > R$ such that the same holds for $N = 1, ..., N_1$ we get that $\{\nu_N\}$ is tight.

By Proposition 3.1.4 this implies that $\{\nu_N\}$ has some probability measure ν as a weak limit point. For each ν_N we have

$$\begin{aligned}\int_{\mathbb{R}^k} \min(L, W(x_1, ..., x_k)) d\nu_N(x_1)...d\nu_N(x_k) &= N^{-k}\sum_{i\in\underline{N}^k} \min(L, W(y_{i_1}, ..., y_{i_k})),\\ &= N^{-k}\sum_{i\in\underline{N}^k} \min(L, W(y_{i_1}, ..., y_{i_k})) + \frac{|\underline{N}^k \setminus \Delta_N^k|}{N^k}L,\\ &\leq N^{-k}\sum_{i\in\Delta_N^k} W(y_{i_1}, ..., y_{i_k}) + \frac{|\underline{N}^k \setminus \Delta_N^k|}{N^k}L,\\ &\leq (E+\eta) + \frac{|\underline{N}^k \setminus \Delta_N^k|}{N^k}L\end{aligned}$$

so letting N tend to infinity we get by weak convergence and $1 - \frac{|\Delta_N^k|}{N^k} \to 0$ that

$$\int_{\mathbb{R}^k} \min(L, W(\vec{x}))\, d\nu(\vec{x}) \leq (E+\eta).$$

Letting L tend to infinity it follows that $I(\nu) \leq E + \eta$ by the monotone convergence theorem. □

Having established that the eigenvalues at finite N will tend to be found in $A_{N,\eta}$ and that this corresponds to low energy we are ready to prove the main theorem of this

section: That the spectral density converges to the minimizer of I in the large-N limit. But first a quick lemma primarily to recall some useful convergence results from Section 3.2.1.

Lemma 3.2.15. *Let $\{\nu_\eta\}$ be a sequence of probability measures such that $I(\nu_\eta) \to E$, then ν_η has a weakly convergent subsequence and any such weak limit point ν has $I(\nu) = E$.*

Proof. This follows from Lemma 3.2.7 which tells us $\{\nu_\eta\}$ is tight, so that by Proposition 3.1.4 a weak limit point exist. Finally, any limit point is a minimizer by weak lower semi-continuity of I, Lemma 3.2.8. □

Theorem 3.2. *Let $\phi : \mathbb{R}^m \to \mathbb{R}$ be a bounded continuous function and define $\Phi_N : \mathbb{R}^N \to \mathbb{R}$ by $\Phi_N(\vec{x}) = N^{-m} \sum_{i \in \underline{N}^m} \phi(x_{i_1}, \ldots, x_{i_m})$. Let μ^E be the minimizer of the associated energy functional I. Then*

$$\lim_{N\to\infty} \langle \Phi_N \rangle_{P_N} = \lim_{N\to\infty} \frac{1}{Z_N} \int_{\mathbb{R}^N} \Phi_N(\vec{x}) P_N(\vec{x})\, d^N\vec{x} = \int_{\mathbb{R}^m} \phi(\vec{y})\, d(\mu^E)^{\otimes m}(\vec{y}).$$

Proof. We will prove this by showing that

$$\limsup_{N\to\infty} \langle \Phi_N \rangle_{P_N} \leq \int_{\mathbb{R}^m} \phi(\vec{y}) d\nu_{\text{sup}}^{\otimes m}(\vec{y})$$

and

$$\liminf_{N\to\infty} \langle \Phi_N \rangle_{P_N} \geq \int_{\mathbb{R}^m} \phi(\vec{y}) d\nu_{\text{inf}}^{\otimes m}(\vec{y})$$

for, a priori different, measures ν_{sup}, ν_{inf} minimizing I. Uniqueness of the minimizer then shows that the large-N limit of $\langle \Phi_N \rangle_{P_N}$ exists and equals the integral of ϕ with respect to μ^E. We will only give the details for the lim sup, the lim inf inequality follows by similar arguments with some reversals of inequalities.

Let $\eta > 0$ be arbitrary and let $\chi_N := \chi_{A_{N,2\eta}}$ be the characteristic function of $A_{N,2\eta}$. Then

$$\int_{\mathbb{R}^N} \Phi_N(x) P_N(x)\, d^N x = \int_{\mathbb{R}^N} \Phi_N(x)\chi_N(x) P_N(x)\, d^N x + \int_{\mathbb{R}^N} \Phi_N(x)(1-\chi_N(x)) P_N(x)\, d^N x.$$

Because $\|\Phi_N\|_\infty = \|\phi\|_\infty < \infty$, the second integral can be bounded by $\|\phi\|_\infty P_N(\mathbb{R}^N \setminus A_{N,2\eta}) \leq \|\phi\|_\infty e^{-\eta N^2}$, at least for N large enough, and as such tends to zero. Hence

$$\limsup_{N\to\infty} \langle \Phi_N \rangle_N = \limsup \langle \Phi_N \chi_N \rangle_N.$$

Since $A_{N,2\eta}$ is compact, each Φ_N attains a maximum at some $\vec{y}(N) \in A_{N,2\eta}$. Set $\nu_N = \frac{1}{N} \sum_{i=1}^N \delta_{y_i(N)}$, so that

$$\langle \Phi_N \chi_N \rangle_{P_N} \leq \Phi(y_1(N), ..., y_N(N)) = \int_{\mathbb{R}^m} \phi(x_1, ..., x_m)\, d\nu_N(x_1) ... \, d\nu_N(x_m).$$

By Proposition 3.2.14 any weak limit point ν of $\{\nu_N\}$ has $I(\nu) < E + 2\eta$. Pick a subsequence of $\{\langle \Phi_N \chi_N \rangle_{P_N}\}$ converging to its own lim sup. The corresponding subsequence of $\{\nu_N\}$ still has a weakly convergent subsequence, we define ν_η as any such limit point.

So at this point we have constructed, for arbitrary $\eta > 0$, a probability measure ν_η such that

$$\limsup_{N\to\infty} \langle \Phi_N \rangle_N \leq \int_{\mathbb{R}^m} \phi(\vec{x})\, d\nu_\eta^{\otimes m}(\vec{x}) < E + 2\eta.$$

Using this, build the sequence of probability measures $\{\nu_{\frac{1}{n}}\}_{n\in\mathbb{N}}$. By Lemma 3.2.15 this sequence has a weakly converging subsequence converging to a minimizer $\nu_{\sup}$ of I and we still have

$$\limsup_{N\to\infty} \langle \Phi_N \rangle_N \leq \int_{\mathbb{R}^m} \phi(\vec{x})\, d\nu_{\sup}.$$

Repeating this for $\nu_{\inf}$ with the appropriate signs reversed, we are done. □

Corollary 3.2.16. *The m-point density of P_N converges weakly to the m-fold tensor power of the minimizer μ^E of the associated energy functional. In particular, the spectral density of P_N converges weakly to μ^E.*

Proof. By Theorem 3.2,

$$\frac{1}{N^m} \int_{\mathbb{R}^N} \sum_{i\in \underline{N}^m} \phi(x_{i_1}, \ldots, x_{i_m}) P_N(\vec{x})\, d^N x \to \int_{\mathbb{R}^m} \phi(\vec{x})\, d(\mu^E)^{\otimes m}(\vec{x})$$

for any bounded continuous $\phi : \mathbb{R}^m \to \mathbb{R}$. On the other hand

$$\begin{aligned}
\frac{1}{N^m} \int_{\mathbb{R}^N} \sum_{i\in \underline{N}^m} \phi(x_{i_1}, \ldots, x_{i_m}) P_N(\vec{x}) d^N x &= \frac{1}{N^m} \sum_{i\in \underline{N}^m} \int_{\mathbb{R}^N} \phi(x_{i_1}, \ldots, x_{i_m}) P_N(\vec{x})\, d^N x, \\
&= \frac{1}{N^m} \sum_{i\in \underline{N}^m} \int_{\mathbb{R}^m} \phi(x_1, \ldots, x_m) P_m^N(x_1, \ldots, x_m)\, d^m x, \\
&= \int_{\mathbb{R}^m} \phi(x_1, \ldots, x_m) P_m^N(x_1, \ldots, x_m)\, d^m x
\end{aligned}$$

by definition of the m-point density P_m^N. □

3.3 The variational problem

To get the variational problem for μ^E we will assume that we are in the situation of Section 3.2, so in particular that Assumption 3.1 holds. Recall that in this assumption we have a set of measures, $\mathcal{P}_{can}$, that we know the minimizer belongs to. Having established uniqueness of the minimizer we may be able to further restrict the set of measures the minimizer can be found in.

For example, the model in Chapter 4 has a reflection symmetry. Once we know that the minimizer is unique, for which we use the set of candidate measures having mean 0 (see Proposition 4.3.5), we know the minimizer must share this symmetry. Hence we can further restrict our search for the minimizing measure to the even measures.

With this example in mind, let $\mathcal{P}'_{can} \subset \mathcal{P}_{can}$ be the *reduced set of candidate measures* that we know μ^E to be in after establishing that the minimizer is unique.

For any measure ν, define the function

$$\mathcal{W}_\nu(x) := I(\bullet, \nu, \dots, \nu) = \int_{\mathbb{R}^{k-1}} W(x, x_2, ..., x_k) d\nu(x_2) ... d\nu(x_k). \tag{3.5}$$

So in particular $I(\nu) = \int_{\mathbb{R}} \mathcal{W}_\nu(x) d\nu(x)$.

In order to find the minimizing measure μ^E we will use the following characterization.

Theorem 3.3. *If $\mu \in \mathcal{P}'_{can}$ and there exists a constant $c \in \mathbb{R}$ such that*

1. *$\int_{\mathbb{R}} \mathcal{W}_\mu(x) d\nu(x) \geq c$ for all $\nu \in \mathcal{P}'_{can}$*
2. *$\mathcal{W}_\mu(x) = c$, μ-almost everywhere*

then $\mu = \mu^E$ the minimizer of I.

Conversely, if $\mu = \mu^E$ the above properties hold with $c = E = I(\mu^E)$.

Proof. We start with the converse claim. Let μ^E be the minimizer of I and let $\nu \in \mathcal{P}_0(\mathbb{R})$. Then by Lemma 3.2.3 we get

$$\int_{\mathbb{R}} \mathcal{W}_{\mu^E}(x)\, d\nu(x) = I(\mu^E, \dots, \mu^E, \nu) \geq I(\mu^E) = E.$$

For the second part we mimic the proof of compact support, Lemma 3.2.6. Let $D = \{x \,|\, \mathcal{W}_{\mu^E}(x) < E\}$ and consider

$$\mu_{+1} = \frac{\mu + \mu|_D}{1 + \mu(D)},$$
$$\mu_{-1} = \frac{\mu - \mu|_D}{1 - \mu(D)}.$$

Then by Lemma 3.2.3 again, we get

$$I(\mu, \dots, \mu_{+1}) \geq E,$$
$$I(\mu, \dots, \mu_{-1}) \geq E.$$

Combining these as in Lemma 3.2.6 we get

$$I\left(\mu, \dots, \mu|_D\right) = \mu(D)E.$$

But we can calculate the left hand side of this equation to be

$$\int \mathcal{W}_{\mu^E}(x)\, d\mu|_D(x) \leq \mu(D)E$$

but with strict inequality if $\mu|_D(D) > 0$. Hence $\mu(D) = \mu|_D(D) = 0$, and as μ^E is a probability measure with $\int_{\mathbb{R}} \mathcal{W}_{\mu^E}(x)\, d\mu^E(x) = E$ this implies equality μ^E-almost everywhere.

For the forward claim, suppose μ is as in the theorem. Then for any $\nu \in \mathcal{P}'_{can}$ and $\rho_t := \tilde{\mu} + t(\nu - \tilde{\mu})$,

$$\left.\frac{d}{dt}\right|_{t=0} I(\rho_t) = k \int_{\mathbb{R}} \mathcal{W}_\mu(x) d(\nu - \mu)(x) \geq k(c - c) = 0.$$

Thus if we choose $\nu = \mu^E$ we get, by construction of $\mathcal{P}'_{can} \subset \mathcal{P}_{can}$ and the proof of Proposition 3.2.10, that

$$\frac{d^2}{dt^2} I(\rho_t) = k(k-1) \int_{\mathbb{R}^k} W(x_1, ..., x_k) d\rho_t(x_1) ... d\rho_t(x_{k-2}) d(\mu^E - \mu)(x_{k-1}) d(\mu^E - \mu)(x_k) > 0$$

unless $\mu = \mu^E$. So if $\mu \neq \mu^E$ we get a contradiction with μ^E being the minimizer. □

Corollary 3.3.1. *If $\rho : \mathbb{R} \to \mathbb{R}$ is a continuous function with compact support such that $\rho\, dx \in \mathcal{P}'_{can}$ and for some $c \in \mathbb{R}$*

1. *$\mathcal{W}_{\rho\, dx}(x) \geq c$ for all $x \in \mathbb{R}$,*
2. *$\mathcal{W}_{\rho\, dx}(x) = c$ on $\{x \,|\, \rho(x) > 0\}$.*

Then $\rho\, dx = \mu^E$.

Conversely, if $\mu^E = \rho\, dx$ for a continuous function ρ, the above properties hold with $c = E = I(\mu^E)$.

It is this final corollary that we will use in practice, especially the fact that $\mathcal{W}_{\rho\, dx}$ should be constant on the support of ρ will turn out to be a crucial in finding ρ. There is a priori no reason to expect that μ^E is even absolutely continuous with respect to the Lebesgue measure, let alone with a continuous density function. This assumption will be justified by the ability to find ρ in practice, after which we can invoke the above Corollary 3.3.1 to confirm that $\rho = \mu^E$ is the large-N limit spectral density.

3.4 Finding the equilibrium measure

In this section we will set up much of the general strategy for finding the density function ρ in Corollary 3.3.1. We will apply these techniques in Section 4.3 to the Dirac ensemble defined in Chapter 4.

Similar to the definition of $\mathcal{W}_\nu$, define

$$\mathcal{U}_\nu(x) = \int_{\mathbb{R}^{k-1}} U(x, x_2, \ldots, x_k)\, d\nu(x_2) \ldots\, d\nu(x_k). \tag{3.6}$$

For this section we will drop the subscripts on these functions since the only measure we are concerned with is $\nu = \rho\, dx$.

The main strategy is to assume ρ exists and use various techniques from complex analysis to relate ρ to $\mathcal{U}$, or more precisely, $\mathcal{U}' = \frac{d}{dx}\mathcal{U}$. To make this work make the following assumptions.

Assumption 3.2. *We assume that*

1. *There is a continuous function ρ such that $\rho\, dx = \mu^E$, where μ^E is the equilibrium measure from Theorem 3.1.*
2. *The support of ρ is $\Sigma = \cup_{i=1}^r [a_i, b_i]$, a union of r disjoint closed intervals.*

3. *The function $\mathcal{U}(x)$ is differentiable and $\mathcal{U}'(x)$ has an analytic extension to $\mathbb{C} \setminus \Pi$ for some $\Pi \subset \mathbb{C}$ compact and $\Pi \cap \Sigma = \emptyset$.*

These assumptions will be justified by the fact that for any such support Σ we can find a corresponding function ρ_Σ and find a series of conditions that, at least in practice, guarantee that for exactly one choice of $\{a_i, b_i\}$ this ρ_Σ satisfies the conditions in Corollary 3.3.1. It should be noted that, once again, assumption 3 is the odd one out. This is really an assumption on our model. The first two assumptions on the other hand will be justified once ρ is found.

Let us now introduce the main tools from complex analysis that we will be using in this section.

Definition 3.4.1. *Let $\psi : \mathbb{R} \to \mathbb{R}$ be continuous. The* Cauchy transform *of ψ is*

$$H(\psi)(z) = \frac{1}{2\pi i} \int_{\mathrm{supp}(\psi)} \frac{\psi(x)}{x - z}\, dx$$

which defines an analytic function on $\mathbb{C} \setminus \mathrm{supp}(\psi)$.

For any continuous function f on $\mathbb{C}$ we define $f_\pm$ on $\mathbb{R}$ by

$$f_\pm(x) = \lim_{\varepsilon \to 0^+} f(x \pm i\varepsilon). \tag{3.7}$$

Lemma 3.4.2 (Sokhotski-Plemelj formula)**.** *Let $\psi : \mathbb{R} \to \mathbb{R}$ be continuous with support Σ. Then for $x \in \Sigma$,*

$$H(\psi)_\pm(x) = \frac{1}{2\pi i}\, \mathrm{PV} \int_\Sigma \frac{\psi(y)}{y - x}\, dy \pm \frac{1}{2}\psi(x),$$

where PV *denotes the Cauchy principal value.*

Proof. This follows by a computation:

$$\begin{aligned}
2\pi i \cdot H(\psi)_\pm(x) &= \lim_{\varepsilon \to 0^+} \int_\Sigma \frac{\psi(y)}{y - x \mp i\varepsilon}\, dy, \\
&= \lim_{\varepsilon \to 0^+} \int_\Sigma \frac{\psi(y)}{y - x \mp i\varepsilon} \frac{y - x \pm i\varepsilon}{y - x \pm i\varepsilon}\, dy, \\
&= \lim_{\varepsilon \to 0^+} \int_\Sigma \frac{\psi(y)(y - x)}{(y - x)^2 + \varepsilon^2}\, dy \pm i \int_\Sigma \frac{\psi(y)\varepsilon}{(y - x)^2 + \varepsilon^2}\, dy, \\
&= \mathrm{PV} \int_\Sigma \frac{\psi(y)}{y - x}\, dy \pm \pi i \psi(x).
\end{aligned}$$

Where the last step follows by $\frac{1}{\pi}\frac{\varepsilon}{(y-x)^2+\varepsilon^2}$ is a nascent δ-distribution. The first integral becomes the principal value as it is the appropriate limit of the Poisson kernel. □

Theorem 3.4 (Riemann-Hilbert problem)**.** *Suppose G is a bounded, analytic function on $\mathbb{C} \setminus \Sigma$, $\Sigma \subset \mathbb{R}$, then $G = H(v)$ for $v : \mathbb{R} \to \mathbb{C}$ defined by*

$$v(x) = G_+(x) - G_-(x).$$

Proof. By the Sokhotski-Plemelj formula (Lemma 3.4.2) the function $H(v)$ solves the scalar Riemann-Hilbert problem defined by Σ and v. To be explicit, $H(v)$ is a bounded function, analytic on $\mathbb{C} \setminus \Sigma$ such that $H(v)_+(x) - H(v)_-(x) = v(x)$ on Σ. Since this Riemann-Hilbert problem has a unique solution (the difference of two solutions would be a bounded entire function) and G is clearly also a solution, we must have $G = H(v)$. □

Since the Sokhotski-Plemelj formula (Lemma 3.4.2) features the principal value, we are interested in computing this for our specific ρ.

Lemma 3.4.3. *Suppose ρ is as in Assumption 3.2 and $\mathcal{U}$ as in Equation 3.6, then*

$$\mathrm{PV} \int_\Sigma \frac{\rho(y)}{y-x}\, dy = -\frac{k}{\beta}\frac{d}{dx}\mathcal{U}(x).$$

Proof. By Corollary 3.3.1, $\mathcal{W}(x) = E$ for $x \in \Sigma$. We want to consider the distributional derivative of this equation, for which we will use the split

$$\mathcal{W}(x) = \mathcal{U}(x) - \int_{\mathbb{R}^{k-1}} \frac{1}{k(k-1)}\frac{\beta}{2} \sum_{(i,j)\in\Delta_k^2} \log(|x_i - x_j|)\rho(x_2)\cdots\rho(x_k)\, dx_2 \ldots\, dx_k.$$

The log term becomes

$$\frac{2(k-1)}{k(k-1)}\frac{\beta}{2}\int_{\mathbb{R}} \log(|x-y|)\rho(y)\, dy + \frac{(k-1)(k-2)}{k(k-1)}\frac{\beta}{2}\int_{\mathbb{R}^2} \log(|y-z|)\rho(y)\rho(z)\, dy\, dz.$$

Since the distributional derivative of $\log(|x|)$ is $\mathrm{PV}\,\frac{1}{x}$, taking the distributional derivative of $\mathcal{W}$ we get (note the swap from $x-y$ to $y-x$),

$$\begin{aligned} \mathcal{U}'(x) - \frac{\beta}{k}\,\mathrm{PV}\int_\Sigma \frac{\rho(y)}{x-y}\, dy &= 0, \\ \frac{\beta}{k}\,\mathrm{PV}\int_\Sigma \frac{\rho(y)}{y-x}\, dy &= -\mathcal{U}'(x). \end{aligned}$$

□

Remark 3.4.4. *We continue to assume that the interaction degree k is at least 2, for the case $k = 1$ see the discussion in Remark 3.2.1.*

We are now ready for the big trick of this computation. Recall from Assumption 3.2 that the support of ρ is given by

$$\Sigma = \cup_{i=1}^r [a_i, b_i]$$

a union of disjoint closed intervals.

Lemma 3.4.5. *Let $s : \mathbb{C} \to \mathbb{C}$ be the function defined by*

$$s(z) = \prod_{i=1}^r (z-a_i)(z-b_i).$$

Then there is a function $\sqrt{s}$ that is analytic on $\mathbb{C} \setminus \Sigma$ and satisfies

1. $(\sqrt{s}(z))^2 = s(z)$,
2. $\sqrt{s}(z) \sim z^r$ *as* $z \to \infty$
3. $\sqrt{s}(\overline{z}) = \overline{\sqrt{s}(z)}$,
4. $\sqrt{s}_+(x)$ *is purely imaginary if* $x \in \Sigma$ *and real if* $x \notin \Sigma$.
5. $\sqrt{s}_+(x) = -\sqrt{s}_-(x)$ *for* $x \in \Sigma$.

For future reference we also record that, if Σ *is symmetric,*

6. $\sqrt{s}(-z) = (-1)^r \sqrt{s}(z)$,
7. $\sqrt{s}_+(-x) = (-1)^{r+1} \sqrt{s}_+(x)$ *for* $x \in \Sigma$.

Proof. We will do this construction for $r = 1$, the general case then follows by multiplying the corresponding functions for each interval $[a_i, b_i]$.

Let $\sqrt{\bullet - a_i} : \mathbb{C}\backslash(-\infty, a_i) \to \mathbb{C}$ be the square root such that $\sqrt{\bullet - a_i}(x) \geq 0$ for $x \geq 0$, and similarly $\sqrt{\bullet - b_i} : \mathbb{C}\backslash(-\infty, b_i) \to \mathbb{C}$. Then consider $\sqrt{s}_0(z) := \sqrt{\bullet - a_i}(z)\sqrt{\bullet - b_i}(z)$ defined on $\mathbb{C}\backslash(-\infty, b_i)$. It is easily seen that $\sqrt{s}_0$ extends analytically to $\sqrt{s} : \mathbb{C}\backslash[a_i, b_i] \to \mathbb{C}$ because on $(-\infty, a_i)$ the lower and upper limits match, since both factors contribute a minus sign.

The first property then follows immediately from the above construction. The second property follows from the observation that $\sqrt{s}(z) \sim \pm z^r$ and this sign has to be the same in every direction. By construction $\sqrt{s}(x) \sim x^r$ as the real number $x \to \infty$, so the sign is $+$.

For the third property a similar argument works, $\sqrt{s}(\overline{z}) = \pm\overline{\sqrt{s}(z)}$ and this sign is the same everywhere. Since $\sqrt{s}(z) \sim z^r$ the sign has to be plus. The fourth property follows immediately from $s(x) \leq 0$ for $x \in \Sigma$ and $s(x) > 0$ for $x \notin \Sigma$ and together with the third immediately implies the fifth. The sixth (if Σ is symmetric) follows from a similar argument as the third.

For the final property, if Σ is symmetric we compute

$$\begin{aligned}
\sqrt{s}_+(-x) &= \lim_{\varepsilon \to 0^+} \sqrt{s}(-x + i\varepsilon), \\
&= \lim_{\varepsilon \to 0^+} (-1)^r \sqrt{s}(x - i\varepsilon), \\
&= \lim_{\varepsilon \to 0^+} (-1)^r \overline{\sqrt{s}(x + i\varepsilon)}, \\
&= \lim_{\varepsilon \to 0^+} (-1)^{r+1} \sqrt{s}(x + i\varepsilon),
\end{aligned}$$

using, respectively, properties 6, 3 and 4. □

This function $\sqrt{s}$ will turn up frequently and is universal (in the sense that it is independent of the action). For later reference we will include the following examples.

Lemma 3.4.6. *If $\Sigma = [-a, a]$, $a < 0$, then*

$$\sqrt{s}_+(x) = i\sqrt{a^2 - x^2}$$

for $x \in \Sigma$,

$$\frac{1}{\sqrt{s}(z)} = \frac{1}{z} + \frac{a^2}{2}\frac{1}{z^3} + \frac{3a^4}{8}\frac{1}{z^5} + \frac{5a^6}{16}\frac{1}{z^7} + \dots.$$

and finally

$$\sqrt{s}(z) = z - \frac{a^2}{2}\frac{1}{z} - \frac{a^4}{8}\frac{1}{z^3} - \frac{a^6}{16}\frac{1}{z^5} - \dots.$$

Proof. The first claim follows immediately from the construction in Lemma 3.4.5, the function $\sqrt{\bullet - a}$ has $\sqrt{\bullet - a}_+(x) = i\sqrt{a - x}$ while $\sqrt{\bullet + a}$ has $\sqrt{\bullet + a}_+(x) = \sqrt{x + a}$ for $x \in \Sigma$. Multiplying these yields $\sqrt{s}_+(x) = i\sqrt{a - x}\sqrt{a + x} = i\sqrt{a^2 - x^2}$.

For the power series of $\frac{1}{\sqrt{s}(z)}$ we use fractional binomial theorem

$$\begin{aligned}
\frac{1}{\sqrt{s}(z)} &= (z^2 - a^2)^{-\frac{1}{2}}, \\
&= z^{-1}\left(1 - \frac{a^2}{z^2}\right)^{-\frac{1}{2}}, \\
&= z^{-1}\sum_{n=0}^{\infty}(-1)^n\binom{-\frac{1}{2}}{n}\frac{a^{2n}}{z^{2n}}, \\
&= \sum_{n=0}^{\infty}\frac{1}{4^n}\binom{2n}{n}\frac{a^{2n}}{z^{2n+1}},
\end{aligned}$$

using $\binom{-\frac{1}{2}}{n} = \left(-\frac{1}{4}\right)^n\binom{2n}{n}$. This selects the right square root, as the leading term for $z \to \infty$ is $z^{-1}\binom{-\frac{1}{2}}{0} = z^{-1}$. This Laurent series has inner radius of convergence a and outer radius of convergence ∞.

The power series for $\sqrt{s}(z)$ itself is found similarly

$$\begin{aligned}
\sqrt{s}(z) &= \left(z^2 - a^2\right)^{\frac{1}{2}}, \\
&= z\left(1 - \frac{a^2}{z^2}\right)^{\frac{1}{2}}, \\
&= z\sum_{n=0}^{\infty}(-1)^n\binom{\frac{1}{2}}{n}\frac{a^{2n}}{z^{2n}}.
\end{aligned}$$

Again this Laurent series converges outside the inner radius a and up to outer radius ∞, where we get the correct $\sqrt{s}$ since it is asymptotically z. □

Lemma 3.4.7. *If $\Sigma = [-b, -a] \cup [a, b]$, $0 < a < b$, then*

$$\sqrt{s}_+(x) = \begin{cases} i\sqrt{x^2 - a^2}\sqrt{b^2 - x^2} & x \in [a, b], \\ -i\sqrt{x^2 - a^2}\sqrt{b^2 - x^2} & x \in [-b, -a], \end{cases}$$

and

$$\frac{1}{\sqrt{s}(z)} = \frac{1}{z^2} + \frac{a^2+b^2}{2}\frac{1}{z^4} + \frac{3a^4+2a^2b^2+3b^4}{8}\frac{1}{z^6} + \dots,$$

and

$$\sqrt{s}(z) = z^2 - \frac{a^2+b^2}{2} - \frac{a^4-2a^2b^2+b^4}{8}\frac{1}{z^2} - \frac{a^6-a^4b^2-a^2b^4+b^6}{16}\frac{1}{z^4} - \dots.$$

Proof. The first claim follows from the same considerations as in Lemma 3.4.6 for $[a,b]$ and property 7 of Lemma 3.4.5.

For the Laurent series we also proceed similarly,

$$\begin{aligned}
\frac{1}{\sqrt{s}(z)} &= \left(z^2-a^2\right)^{-\frac{1}{2}}\left(z^2-b^2\right)^{-\frac{1}{2}}, \\
&= z^{-2}\left(\sum_{n=0}^{\infty}(-1)^n\binom{-\frac{1}{2}}{n}\frac{a^{2n}}{z^{2n}}\right)\left(\sum_{n=0}^{\infty}(-1)^n\binom{-\frac{1}{2}}{n}\frac{b^{2n}}{z^{2n}}\right), \\
&= z^{-2}\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{-\frac{1}{2}}{m}\binom{-\frac{1}{2}}{n-m}a^{2n}b^{2(n-m)}\right)(-1)^n z^{-2n}, \\
&= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{2m}{m}\binom{2(n-m)}{n-m}a^{2m}b^{2(n-m)}\right)\frac{1}{4^n}\frac{1}{z^{2n+2}}.
\end{aligned}$$

This Laurent series also has outer radius of convergence ∞ and has inner radius of convergence b.

The Laurent series near infinity for $\sqrt{s}(z)$ is also found similarly,

$$\begin{aligned}
\sqrt{s}(z) &= z^2\left(1-\frac{a^2}{z^2}\right)^{\frac{1}{2}}\left(1-\frac{b^2}{z^2}\right)^{\frac{1}{2}}, \\
&= z^2\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{\frac{1}{2}}{m}\binom{\frac{1}{2}}{n-m}a^{2n}b^{2(n-m)}\right)(-1)^n z^{-2n}.
\end{aligned}$$

□

We will now use the function $\sqrt{s}$ defined in the above Lemma 3.4.5 to take a Riemann-Hilbert problem for ρ and turn it into a Riemann-Hilbert problem for $\mathcal{U}'$. Let

$$G(z) = H(\rho)(z),$$

and define

$$\widetilde{G}(z) = \frac{G(z)}{\sqrt{s}(z)}.$$

Since $\sqrt{z} \sim z^r$ this is a bounded function, it is analytic on $\mathbb{C} \setminus \Sigma$ and we have, for $x \in \Sigma$,

$$\begin{aligned}
\widetilde{G}_+(x) - \widetilde{G}_-(x) &= \frac{G_+(x)}{\sqrt{s}_+(x)} - \frac{G_-(x)}{\sqrt{s}_-(x)}, \\
&= \frac{1}{\sqrt{s}_+(x)} \left(G_+(x) + G_-(x) \right), \\
&= \frac{1}{\sqrt{s}_+(x)} \frac{1}{\pi i} \operatorname{PV} \int_\Sigma \frac{\rho(y)}{y - x} \, dy, \\
&= \frac{1}{\sqrt{s}_+(x)} \frac{i}{\pi} \frac{k}{\beta} \mathcal{U}'(x).
\end{aligned}$$

Using first the Sokhotski-Plemelj formula (Lemma 3.4.2) and then our result about the minimizer ρ from Lemma 3.4.3. But this means $\widetilde{G}$ is the solution to the Riemann-Hilbert problem defined by Σ and $\frac{1}{\sqrt{s}_+(x)} \frac{i}{\pi} \frac{k}{\beta} \mathcal{U}'(x)$. So by Theorem 3.4,

$$\widetilde{G}(z) = \frac{1}{2\pi i} \int_\Sigma \frac{1}{\sqrt{s}_+(y)} \frac{i}{\pi} \frac{k}{\beta} \mathcal{U}'(y) \frac{dy}{y - z} = \frac{k}{2\beta\pi^2} \int_\Sigma \frac{\mathcal{U}'(y)}{\sqrt{s}_+(y)} \frac{dy}{y - z}. \tag{3.8}$$

Having established this formula for $\widetilde{G}$, we are going to consider the function

$$f(z) = \frac{\mathcal{U}'(z)}{\sqrt{s}(z)}$$

for the analytic extension of $\mathcal{U}'(x)$ to $\mathbb{C} \setminus \Pi$ assumed to exist in Assumption 3.2. The function $f(z)$ is analytic on $\mathbb{C} \setminus (\Sigma \cup \Pi)$, as both Σ and Π are assumed to be compact we can consider a contour $\gamma = \gamma_R \cup \gamma_{\Sigma,\varepsilon} \cup \gamma_{\Pi,\varepsilon}$ consisting of

γ_R: a counter-clockwise circle contour of radius R encircling Σ and Π,

$\gamma_{\Sigma,\varepsilon}$: a clockwise race-track contour around the intervals comprising Σ at a distance ε,

$\gamma_{\Pi,\varepsilon}$: a clockwise contour around Π at a distance ε.

Then by the residue theorem we get for any z_0, $|z_0| < R$ and z_0 away from Σ and Π

$$2\pi i f(z_0) = \int_{\gamma_R} f(z) \frac{dz}{z - z_0} + \int_{\gamma_{\Sigma,\varepsilon}} f(z) \frac{dz}{z - z_0} + \int_{\gamma_{\Pi,\varepsilon}} f(z) \frac{dz}{z - z_0}. \tag{3.9}$$

Let us investigate these terms separately, starting with the circle contour.

Since by Assumption 3.2 $\mathcal{U}'$ is analytic at infinity and $\sqrt{s}$ is as well,

$$\lim_{R \to \infty} \int_{\gamma_R} f(z) \frac{dz}{z - z_0} = 2\pi i \left[f(z) \right]_{-1}$$

where we use the notation $[f(z)]_n$ for the coefficient of z^n in the Laurent series for f. We can find this coefficient as follows

$$\begin{aligned}[f(z)]_{-1} &= \left[\frac{\mathcal{U}'(z)}{\sqrt{s}(z)}\frac{1}{z-z_0}\right]_{-1}, \\ &= \left[\frac{\mathcal{U}'(z)}{\sqrt{s}(z)}\left(1+\frac{z_0}{z}+\ldots\right)\right]_0, \\ &= \sum_{n=0}^{\infty}\left[\frac{\mathcal{U}'(z)}{\sqrt{s}(z)}\right]_n z_0^n\end{aligned}$$

So we can write

$$\lim_{R\to\infty}\int_{\gamma_R} f(z)\frac{dz}{z-z_0} = 2\pi i p(z_0) \tag{3.10}$$

where

$$p(z_0) = \sum_{n=0}^{\infty}\left[\frac{\mathcal{U}'(z)}{\sqrt{s}(z)}\right]_n z_0^n. \tag{3.11}$$

Remark 3.4.8. *In practice, for example in Section 4.3, p will be a polynomial (hence the choice of notation) that depends only on $S(H)$. This is due to $\mathcal{U}'(z)$ generally having polynomial growth, say of degree d, as $z\to\infty$ then the highest power of z_0 appearing will be z_0^d with coefficient simply the coefficient of z^d in the asymptotics of $\mathcal{U}'(z)$. The lower order coefficients involve the coefficients of $\sqrt{s}$ as well, but can still be computed as we do in Section 4.3.*

Next we will consider the race-track contour of radius ε around Σ, specifically in the $\varepsilon\to 0$ limit. We break this up into the semi-circles around each a_i and b_i, as well as the intervals above and below Σ and start by showing that the contribution of the semi-circles vanishes in the $\varepsilon\to 0$ limit.

Recall that

$$f(z) = \frac{\mathcal{U}'(z)}{\sqrt{s}(z)}$$

and that by Assumption 3.2 $\mathcal{U}'$ is analytic on a neighbourhood of Σ. Hence the singularities at the a_i and b_i are only due to $\frac{1}{\sqrt{s}(z)}$, which behaves like $\frac{1}{\sqrt{z}}$ near a_i, b_i. Since the length of the semi-circles is linear in ε and the function blows up at the rate $\frac{1}{\sqrt{\varepsilon}}$ the contributions of the semi-circles vanish in the limit $\varepsilon\to 0$.

Next we consider the contribution from the intervals above Σ,

$$\begin{aligned}\lim_{\varepsilon\to 0^+}\int_{\Sigma+i\varepsilon}\frac{\mathcal{U}'(z)}{\sqrt{s}(z)}\frac{dz}{z-z_0} &= \lim_{\varepsilon\to 0^+}\int_{\Sigma}\frac{\mathcal{U}'(y+i\varepsilon)}{\sqrt{s}(y+i\varepsilon)}\frac{dy}{y+i\varepsilon-z_0}, \\ &= \int_{\Sigma}\frac{\mathcal{U}'(y)}{\sqrt{s}_+(y)}\frac{dy}{y-z_0}, \\ &= \frac{2\beta\pi^2}{k}\widetilde{G}(z_0)\end{aligned}$$

The contribution from the intervals below Σ is also $\frac{2\beta\pi^2}{k}\widetilde{G}(z_0)$ by the same computation but where the minus sign due to the orientation cancels against the minus sign from $\sqrt{s}_-(y)$ v.s. $\sqrt{s}_+(y)$.

So in total we get

$$\lim_{\varepsilon\to 0^+}\int_{\gamma_{\Sigma,\varepsilon}} f(z)\frac{dz}{z-z_0} = \frac{4\beta\pi^2}{k}\widetilde{G}(z_0). \tag{3.12}$$

To work out the third contribution $\int_{\gamma_{\Pi,\varepsilon}} f(z)\frac{dz}{z-z_0}$ we need more information about the behaviour of $\mathcal{U}'$ around its singularities Π. For now we will simply give it a name

$$R(z_0) := \frac{1}{2\pi i}\int_{\gamma_{\Pi,\varepsilon}} \frac{\mathcal{U}'(z)}{\sqrt{s}(z)}\frac{dz}{z-z_0} \tag{3.13}$$

These computations leave us with the following result

Proposition 3.4.9. *For p as in Equation 3.11 and R as in Equation 3.13, we have*

$$\rho(x) = \frac{k}{\beta}\frac{\sqrt{s}_+(x)}{\pi i}p(x) + \frac{k}{\beta}\frac{\sqrt{s}_+(x)}{\pi i}R(x)$$

for $x \in \Sigma$.

Proof. From the Cauchy integral representation of f in Equation 3.9 we get using Equations 3.10, 3.12, and 3.13 that

$$2\pi i f(z_0) = 2\pi i p(z_0) + \frac{4\beta\pi^2}{k}\widetilde{G}(z_0) + 2\pi i R(z_0).$$

Isolating $\widetilde{G}$ we get

$$\frac{2\pi i\beta}{k}\widetilde{G}(z) = -\frac{\mathcal{U}'(z)}{\sqrt{s}(z)} + p(z) + R(z_0)$$

so also

$$G(z) = -\frac{k}{2\pi i\beta}\mathcal{U}'(z) + \frac{k}{2\pi i\beta}\sqrt{s}(z)p(z) + \frac{k}{2\pi i\beta}\sqrt{s}(z)R(z).$$

Now, by the Sokhotski-Plemelj formula (Lemma 3.4.2) we can recover ρ from $G = H(\rho)$ by

$$\rho(x) = G_+(x) - G_-(x) = \frac{k}{\beta}\frac{\sqrt{s}_+(x)}{\pi i}p(x) + \frac{k}{\beta}\frac{\sqrt{s}_+(x)}{\pi i}R(x)$$

for $x \in \Sigma$, using that $\mathcal{U}'(z)$, $p(z)$, and $R(z)$ are analytic near Σ and that $\sqrt{s}_+ = -\sqrt{s}_-$ on Σ (property 5 in Lemma 3.4.5). □

An important note about this result is that since p and R depend on $\mathcal{U}$ they will often contain ρ in a non-trivial way. This means Proposition 3.4.9 gives a functional equation for ρ rather than a straight up formula. An exception to this is whenever[4] and

[4] One should keep in mind that most of our formulas so far have required $k \geq 2$, however using the technique in Remark 3.2.1 $k = 1$ to transform a $k = 1$ model to a $k = 2$ model with trivial interaction still leads to a straightforward formula for ρ.

U is a polynomial. In this situation R will be 0 and $\mathcal{U}$ will only depend on ρ through its moments.

This dependence is resolvable using Proposition 3.4.10 or the approach discussed in Section 4.3.2. Then this proposition produces a formula for ρ, as seen in [22].

Also in more complicated models, such as the multi-trace models in [39], there are techniques to solve for, or at least approximate, ρ using Proposition 3.4.9.

At this point we have established an equation for ρ that, at least in practice, will allow us to uniquely determine ρ given Σ. That leaves us with the question of finding $\Sigma = \cup_{i=1}^{r}[a_i, b_i]$. This process is even more dependent on the specifics of the particular model but the general idea is that for only one choice of $r, \{a_i, b_i\}$ the corresponding ρ will be a probability density.

While the specifics are very model dependent, there are some general principles.

Proposition 3.4.10. *The support $\Sigma = \cup_{i=1}^{r}[a_i, b_i]$ is such that*

$$\int_\Sigma \frac{\mathcal{U}'(y)}{\sqrt{s}_+(y)} y^n \, dy = \begin{cases} 0, 0 \leq n < r \\ \frac{\beta\pi}{ik}, n = r, \end{cases}$$

and

$$\sum_{i=0}^{n} \left(\int_\Sigma \frac{\mathcal{U}'(y)}{\sqrt{s}_+(y)} y^{r+i} \, dy \left[\sqrt{s}(z)\right]_{r-(n-i)} \right) = \frac{\beta\pi}{ik} \mu_n^E.$$

Proof. We have established in Equation 3.8 that we can find $\widetilde{G}$ as the Cauchy transform of the function $\frac{\mathcal{U}'}{\sqrt{s}_+}$ defined on Σ. That equation implies that

$$G(z) = \frac{k\sqrt{s}(z)}{2\beta\pi^2} \int_\Sigma \frac{\mathcal{U}'(y)}{\sqrt{s}_+(y)} \frac{dy}{y - z}.$$

Moreover, for the Cauchy transform itself we know, using the geometric series for $\frac{1}{y-z}$, that

$$G(z) = -\frac{1}{2\pi i} \sum_{n=0}^{\infty} \frac{\mu_n^E}{z^{n+1}}$$

for z large enough, where μ_n^E is the n-th moment of μ^E.

So we get

$$\frac{ik}{\beta\pi} \sum_{n=0}^{\infty} \frac{\sqrt{s}(z)}{z^{n+1}} \int_\Sigma \frac{\mathcal{U}'(y)}{\sqrt{s}_+(y)} y^n \, dy = \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} \mu_n^E.$$

The right-hand side has only negative powers of z, while the left-hand side can have positive powers.

Define a_n by $a_n = \frac{ik}{\beta\pi} \int_\Sigma \frac{\mathcal{U}'(y)}{\sqrt{s}_+(y)} y^{n-1} \, dy$ for $n \geq 1$. Also define b_n by $\sqrt{s}(z) = \sum_{n=-r}^{\infty} b_n z^{-n}$. Then the above equation becomes the infinite system

$$\sum_m a_m b_{n-m} = \begin{cases} 0 & n \leq 0 \\ \mu_{n-1}^E & n \geq 1 \end{cases}$$

for all $n \in \mathbb{Z}$, where the sum is over those integers for which both a_m and b_{n-m} are defined.

This system of equations reads

$$\begin{gathered}
\vdots \\
0 = 0, \\
a_1 b_{-r} = 0, \\
a_1 b_{-r+1} + a_2 b_{-r+2} = 0, \\
\vdots \\
a_1 b_{-1} + \ldots + a_r b_{-r} = 0, \\
a_1 b_0 + \ldots + a_{r+1} b_{-r} = \mu_0^E = 1, \\
a_1 b_1 + \ldots + a_{r+1} b_{-r+1} + a_{r+2} b_{-r} = \mu_1^E, \\
\vdots
\end{gathered}$$

This implies that $a_1, \ldots, a_r = 0$ and $a_{r+1} = 1$ since $b_{-r} = 1$ by Lemma 3.4.5. We further get

$$\sum_{i=1}^{n} a_{r+i} b_{-r+(n-i)} = \mu_{n-1}^E.$$

Translating back, we obtain

$$\frac{ik}{\beta\pi} \int_\Sigma \frac{\mathcal{U}'(y)}{\sqrt{s}_+(y)} y^n \, dy = \begin{cases} 0, n < r \\ 1, n = r. \end{cases}$$

and

$$\frac{ik}{\beta\pi} \sum_{i=0}^{n} \left(\int_\Sigma \frac{\mathcal{U}'(y)}{\sqrt{s}_+(y)} y^{r+i} \, dy \left[\sqrt{s}(z) \right]_{r-(n-i)} \right) = \mu_n^E$$

as desired. □

This proposition gives us $r + 1$ conditions on the $2r$ unknowns $\{a_i, b_i\}$ and no information about r itself. There are further general conditions, called gap conditions, corresponding to the fact that not only is $\mathcal{W}$ constant on each interval (which is all we have used so far), it is the same constant on each interval. This gives us a further $r - 1$ equations, one for each gap, for a total of $2r$ equations. So, provided all these equations are independent (they turn out to be in practice) this will give us enough information to determine the support given r.

The last task is then to determine r, the number of intervals (or *cuts* in the literature) that make up Σ. This can be determined by the condition that $\rho \geq 0$ and $\Sigma = \{x \,|\, \rho(x) \geq 0\}$.

These gap conditions are once again dependent on ρ, leading to an even more complicated set of simultaneous equations for ρ, a_i, b_i when combined with Proposition 3.4.9. We will only be dealing with the $r = 1$ case, in which case $r + 1 = 2r$, or the $r = 2$ symmetric case, where one of the three moment conditions is trivial due to the symmetry but the other two moment conditions determine the two unknowns.

Chapter 4

Fuzzy Geometries with a Fermion

This chapter will cover the definition and, at least partial, solution of a very simple toy model of quantum gravity for which the techniques from Chapter 3 are applicable. As discussed in Section 1.1, in a spectral triple

$$(A, H, D)$$

the metric information is contained in the Dirac operator D. So a path integral over possible geometries corresponds to a path integral over the space of Dirac operators. A spectral triple with a random Dirac operator, in this case given by a path integral, is called a *Dirac ensemble*.

With this in mind we consider the class of *fuzzy geometries*, defined in Section 4.1, where the space of Dirac operators is a finite dimensional space parametrized by a finite set of Hermitian matrices. This makes the space of Dirac operators finite dimensional so that integration over it is well-defined. More precisely, we will consider the simplest such fuzzy geometry where the space of Dirac operators is parametrized by a single Hermitian matrix. That means that the corresponding Dirac ensemble in fact becomes a single-matrix, but multi-trace, random matrix model.

This random fuzzy geometry has already been studied analytically in [39] and more complex fuzzy geometries have been studied numerically [5, 7, 32, 28]. Both the simplest Dirac ensembles as well as the more complex ones are found to exhibit interesting behaviour, some of which we will discuss in Section 4.4 when we compare it to the behaviour of our new ensemble. Currently these models are all based entirely around a bosonic action, incorporating only the Dirac operator D.

Our goal in this project is to add a fermionic action as well. In the formulation of the standard model within noncommutative geometry [13, 10, 3] the state of the system is contained in both D, containing the metric and gauge fields, and in a vector $\psi \in H$, containing the fermionic fields. The action of such a configuration is then given by both a *bosonic* and *fermionic* part,

$$S(D, \psi) = \mathrm{Tr}(f(D)) + \langle \psi, D\psi \rangle,$$

or slight variations on this combination.

While incorporating the fermionic action $\langle \psi, D\psi \rangle$ we find we need massive fermions and therefore need to move from fuzzy geometries to *almost fuzzy geometries*. The term

almost fuzzy geometry is meant in similarity to the almost commutative spectral triples usually used in the formulation of the standard model in noncommutative geometry. So we want to consider spectral triples of the form

$$(A_{fuzzy}, H_{fuzzy}, D_{fuzzy}) \otimes (A_{finite}, H_{finite}, D_{finite}).$$

Such almost fuzzy geometries are also studied in [55, 54, 4], where they are called Yang-Mills-Higgs fuzzy geometries. This is justified since even for purely bosonic[1] actions they already show terms very reminiscent of the Yang-Mills action and the Higgs mechanism. This is a further motivation for us to consider such almost fuzzy geometries.

In Section 4.1 we will define the basic concepts that enter in to the discussion of our Dirac ensemble. In Section 4.2 we will specialize to the ensemble of interest and compute the corresponding random matrix model. We will also show this model satisfies the assumptions required for the tools from Chapter 3. The computations for finding the spectral density of our Dirac ensemble, based on the general principles of Section 3.4, are gathered in Section 4.3. Finally in Section 4.4 we discuss the effects of the fermionic sector based on the results of the preceding computations.

4.1 Almost fuzzy geometries, the fermionic action and Dirac ensembles

The terminology *fuzzy geometry* in the sense we will use it was introduced by Barrett in [4]. They are a particularly simple class of spectral triples. Examples of fuzzy geometries include the famous fuzzy sphere [48, 19, 31] and the fuzzy torus [6, 45, 46].

Definition 4.1.1. *A* fuzzy geometry of signature (p, q), *or* (p, q) fuzzy geometry, *is a spectral triple*

$$(M_N(\mathbb{C}), V \otimes M_N(\mathbb{C}), D; J, \Gamma)$$

where V is a (p, q) spinor space with charge conjugation J_V and, if $p + q$ is even grading Γ_V. The inner product for the Hilbert space $M_N(\mathbb{C})$ is the Hilbert-Schmidt inner product, and the inner product on $V \otimes M_N(\mathbb{C})$ is the induced tensor product inner product.

The real structure for the fuzzy geometry J is given by $J(v \otimes A) = J_V(v) \otimes A^*$. If $p + q$ is even the grading of the fuzzy geometry is given by $\Gamma = \Gamma_V \otimes 1$, if $p + q$ is odd the grading is omitted from the data. The Dirac operator D is any operator such that the spectral triple has KO-dimension $q - p$ (see Table 1.1).

The reason fuzzy geometries are interesting for our purposes is that the space of Dirac operators for a given signature and matrix size is given by a finite set of self-adjoint Dirac operators.

[1] By purely bosonic we mean entirely in terms of $f(D)$, calling this purely bosonic may obscure the fact that this action can also include metric information. It is motivated by the appearance of gauge fields in this term when considering inner fluctuations relative to a reference Dirac operator.

Theorem (Barrett, [4]). *The Dirac operator of a fuzzy geometry can be written*

$$D(v \otimes A) = \sum_I \gamma^I v \otimes K_I A + \varepsilon' \varepsilon_I A K_I$$

In this sum I ranges over sets of the form $\{i_1 \leq \ldots \leq i_k\}$ with $1 \leq i_j \leq p+q$, if $q-p$ is even k must be odd. By γ^I we mean the product of the gamma matrices appearing in I, in order, and K_I is a self-adjoint matrix and the sign ε_I is determined by $(\gamma^I)^* = \varepsilon_I \gamma^I$. The sign ε' is determined by the KO-dimension $s = q-p$ of the fuzzy geometry.

This theorem is a consequence of requiring D to have the correct KO-dimension. The most restrictive of the conditions is the order one condition (See Definition 1.1.2). To give an example of the proof of this theorem without needing to introduce too much additional terminology we will prove it for the simplest case, signature $(0,1)$, which also happens to be the case we are interested in.

Lemma 4.1.2. *A $(0,1)$ fuzzy geometry is of the form*

$$(M_N(\mathbb{C}), M_N(\mathbb{C}), [H, \bullet])$$

for a Hermitian matrix H, and any such triple defines a $(0,1)$ fuzzy geometry.

Proof. In the signature $(0,1)$ the spinor space V is $\mathbb{C}$ with charge conjugation given by complex conjugation. Hence $V \otimes M_N(\mathbb{C}) \cong M_N(\mathbb{C})$ with charge conjugation simply given by the adjoint.

Let $\{E_{ij}\}_{i,j=1}^N$ denote the matrix units in $M_N(\mathbb{C})$, and write $\mathbb{C}^N \otimes \mathbb{C}^N \cong M_N(\mathbb{C})$ via $v \otimes w \mapsto |v\rangle\langle w|$. Under this isomorphism the left action of $M_N(\mathbb{C})$ is given by $a \mapsto a \otimes 1$ and the right action is given by $b \mapsto 1 \otimes b^*$.

For any map $K : \mathbb{C}^N \otimes \mathbb{C}^N \to \mathbb{C}^N \otimes \mathbb{C}^N$ define

$$\pi_L K := \frac{1}{N} \sum_{i,j} (E_{i,j}^* \otimes 1) K (E_{i,j} \otimes 1),$$

note that $\pi_L K$ commutes with the left action of $M_N(\mathbb{C})$ since

$$\begin{aligned}
\left(\frac{1}{N} \sum_{i,j} (E_{i,j}^* \otimes 1) K (E_{i,j} \otimes 1) \right) (E_{k,l} \otimes 1) &= \frac{1}{N} \sum_{i,j} (E_{i,j}^* \otimes 1) K (E_{i,l} \otimes 1) \delta_{j,k}, \\
&= \frac{1}{N} \sum_{i} (E_{i,k}^* \otimes 1) K (E_{i,l} \otimes 1), \\
&= \frac{1}{N} \sum_{i,j} \delta_{j,l} (E_{i,k}^* \otimes 1) K (E_{i,j} \otimes 1), \\
&= (E_{k,l} \otimes 1) \left(\frac{1}{N} \sum_{i,j} (E_{i,j}^* \otimes 1) K (E_{i,j} \otimes 1) \right),
\end{aligned}$$

so $\pi_L K$ commutes with $a \otimes 1$ by linearity. Similarly

$$\pi_R K := \frac{1}{N} \sum_{i,j} (1 \otimes E_{i,j}^*) K (1 \otimes E_{i,j})$$

commutes with the right action of $M_N(\mathbb{C})$.

Since $\frac{1}{N}\sum_{i,j}(E_{i,j}^* \otimes 1)(E_{i,j} \otimes 1) = 1$, $\pi_L K = K$ if K already commutes with the left action. Again similarly for the right action and π_R. By the first order condition on D, $[D, a \otimes 1]$ commutes with the right action for any $a \in M_N(\mathbb{C})$. So $[D, a \otimes 1] = \pi_R([D, a \otimes 1]) = [\pi_R D, a \otimes 1]$ since $a \otimes 1$ commutes with the right action as well. That means that for any $a \in M_N(\mathbb{C})$, $[D - \pi_R D, a \otimes 1] = 0$, so that $\pi_R D - D$ commutes with the left action. So $\pi_L(\pi_R D - D) = \pi_R D - D$, or in other words

$$D = \pi_L D + \pi_R D - \pi_L \pi_R D.$$

A priori $D = \sum_i S_i \otimes T_i$ for some $S_i, T_i \in M_N(\mathbb{C})$. Then $\pi_L D = \sum_{i,j,k}(S_i)_{j,k} 1 \otimes T_i$, so we can write $\pi_L D = 1 \otimes T$ for some $T \in M_N(\mathbb{C})$. Similarly $\pi_R D = S \otimes 1$ and $\pi_L \pi_R D = c1 \otimes 1$ for some $c \in \mathbb{C}$. This shows that $D(v \otimes w) = S(v \otimes w) + (v \otimes w)T + c(v \otimes w)$.

Since D should be self-adjoint, it easily follows that $S = S^*$, $T = T^*$ and $\overline{c} = c$. Moreover, the spectral triple should have KO-dimension 1, so $JD = -DJ$ by Table 1.1. This implies that $S = -T$ and $c = -c$, so that $Da = Ha - aH = [H, a]$ for a self-adjoint matrix H.

It is a straightforward check that any self-adjoint matrix defines a valid Dirac operator by $D = [H, \bullet]$. □

Having established what a fuzzy geometry is we turn our attention to the next concept we need on our way to fermionic Dirac ensembles, the fermionic integral. This integral is also commonly referred to as the Berezin integral, after Felix Berezin credited with the invention of the concept, or Grassmann integral, since the concept is closely related to the exterior or Grassmann algebra.

Definition 4.1.3. *Let $e = \{e_i\}_{i \in I}$ be an ordered, finite set of generators for a Grassman algebra. The* fermionic, or Berezin, integral *of an element $\omega = \sum_{J \subset I} \omega_J (\wedge_{j \in J} e_j)$ in the Grassmann algebra is then defined by*

$$\int_{Ber} \omega \, de = \omega_I.$$

In other words the integral selects the coefficient of $e_1 \wedge \ldots \wedge e_n$ if $I = \{1, \ldots, n\}$.

Lemma 4.1.4. *If $e = \{e_i\}_{i=1}^n$, $f = \{f_i\}_{i=1}^n$ are two sets of generators for a finite dimensional Grassman algebra related by a matrix P, i.e. $\sum_i P_j^i e_i = f_j$, then*

$$\int_{Ber} \omega \, de = \det(P) \int_{Ber} \omega \, df.$$

Proof. It suffices to check how the elements $e_1 \wedge \ldots \wedge e_n$ and $f_1 \wedge \ldots \wedge f_n$ are related.

$$\begin{aligned} f_1 \wedge \ldots \wedge f_n &= \sum_{i_1, \ldots, i_n = 1}^{n} \left(P_1^{i_1} e_{i_1}\right) \wedge \ldots \wedge \left(P_n^{i_n} e_{i_n}\right), \\ &= \sum_{\sigma \in S_n} \left(P_1^{\sigma(1)} e_{\sigma(1)}\right) \wedge \ldots \wedge \left(P_n^{\sigma(n)} e_{\sigma(n)}\right), \\ &= \sum_{\sigma \in S_n} P_1^{\sigma(1)} \cdots P_n^{\sigma(n)} (-1)^{\mathrm{sgn}(\sigma)} e_1 \wedge \ldots \wedge e_n, \\ &= \det(P) e_1 \wedge \ldots \wedge e_n \end{aligned}$$

where we used that any choice of indices with a repeated index is 0 by the Grassmann algebra relations to obtain the sum over permutations. □

This definition applies to our situation of finite dimensional spectral triples by considering the coordinate functions of H to be Grassmann generators, rather than simply real valued functions. This is very much in line with the basic idea of noncommutative geometry, where the algebra of a spectral triple is often called the coordinate algebra.

We then interpret the inner product $\langle \psi, D\psi \rangle$ featuring in the fermionic action as an element of that Grassmann algebra. A vector ψ can then be thought of as an element in a Grassmann algebra by interpreting $\psi = \psi_1 e_1 + \ldots \psi_n e_n$ for a basis $\{e_i\}$ and Grassman generators $\{\psi_i\}$. To be precise,

Definition 4.1.5. *Let V be a finite dimensional vector space with ordered basis $\{e_i\}_{i\in I}$. Then for a real bilinear form $(\bullet,\bullet)$ on V we define the element (v,v) of the Grassmann algebra generated by $\{\varepsilon_i\}_{i=1}^n$ by*

$$(\varepsilon,\varepsilon) = \sum_{i,j\in I} (e_i, e_j)\varepsilon_i\varepsilon_j.$$

For any sesquilinear form $\langle \bullet,\bullet\rangle$ on V we define

$$\langle \varepsilon,\varepsilon\rangle = \sum_{i,j\in I} \langle e_i, e_j\rangle \overline{\varepsilon_i}\varepsilon_j$$

as an element of the Grassmann algebra generated by $\{\overline{\varepsilon_i}, \varepsilon_i\}_{i\in I}$.

We think of these Grassmann algebras as the coordinate function algebras on the (fermionic) Hilbert space.

Remark 4.1.6. *In practice we will often fail to specify an ordered set of Grassmann generators or a basis for the vector space from which one can be built, or more commonly we will fail to specify only the ordering on a basis. By Lemma 4.1.4 this leads to an ambiguity of a non-zero scalar.*

The reason for this omission on our part is that our fermionic integrals will always appear as a factor in a probability density. So any ambiguity of a non-zero scalar is removed by the normalization condition.

We will further be sloppy in notation by using the same symbol, usually ψ, for an arbitrary vector as well as for a set of generators of the Grassman algebra of coordinate functions on the vector space containing the vector.

With these definitions we want to highlight the following classical results.

Lemma 4.1.7. *Let V be a vector space with real bilinear form $(\bullet,\bullet)$. If $\{e_i\}_{i=1}^n$ is an ordered basis for V and $\{\varepsilon_i\}_{i=1}^n$ the associated fermionic coordinate functions, the fermionic integral*

$$\int_{Ber} e^{-\frac{1}{2}(\varepsilon, A\varepsilon)}\, d\varepsilon = \mathrm{Pf}(A)$$

if A is skew-adjoint and $\dim(V)$ is even. If $\langle \bullet, \bullet \rangle$ is a sesquilinear on V form then

$$\int_{Ber} e^{-\langle \varepsilon, B\varepsilon \rangle} \, d\overline{\varepsilon} \, d\varepsilon = \det(B).$$

The exponential of an element of a Grassmann algebra is defined by its power series which on a Grassmann algebra is necessarily finite.

Proof. We will prove the determinant formula, the Pfaffian case is very similar and essentially done in Lemma 4.2.4.

The first step is computing $e^{-\langle \varepsilon, B\varepsilon \rangle}$. By Definition 4.1.5 for the form $\langle \bullet, B\bullet \rangle$ we have

$$\langle \varepsilon, B\varepsilon \rangle = \sum_{i,j} B_{i,j} \overline{\varepsilon}_i \varepsilon_j.$$

The exponential is defined by its power series,

$$e^{-\langle \varepsilon, B\varepsilon \rangle} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\langle \varepsilon, B\varepsilon \rangle \right)^k,$$

where the sum is finite due to the Grassman relations.

For the fermionic integral we are after the coefficient of $\overline{\varepsilon}_1 \dots \overline{\varepsilon}_n \varepsilon_1 \dots \varepsilon_n$. This means we only need to find the $k = n$ term in the sum which is

$$\begin{aligned}
\frac{(-1)^n}{n!} \left(\sum_{i,j=1}^{n} B_{i,j} \overline{\varepsilon}_i \varepsilon_j \right)^n &= \frac{(-1)^n}{n!} \sum_{\vec{i},\vec{j} \in \underline{n}^n} \left(\prod_{k=1}^{n} B_{\vec{i}_k, \vec{j}_k} \overline{\varepsilon}_{\vec{i}_k} \varepsilon_{\vec{j}_k} \right), \\
&= \frac{(-1)^n}{n!} \sum_{\sigma, \tau \in S_n} \left(\prod_{k=1}^{n} B_{\sigma(k), \tau(k)} \overline{\varepsilon}_{\sigma(k)} \varepsilon_{\tau(k)} \right), \\
&= (-1)^n \sum_{\tau \in S_n} \left(\prod_{k=1}^{n} B_{k, \tau(k)} \overline{\varepsilon}_k \varepsilon_{\tau(k)} \right).
\end{aligned}$$

So the coefficient of $\overline{\varepsilon}_1 \dots \overline{\varepsilon}_n \varepsilon_1 \dots \varepsilon_n$ is

$$\sum_{\tau \in S_n} \prod_{k=1}^{n} (-1)^{\mathrm{sgn}(k)} B_{k, \tau(k)} = \det(B).$$

The $(-1)^n$ and $(-1)^{\mathrm{sgn}(k)}$ are due to the reordering the various ε and $\overline{\varepsilon}$ in the correct order. □

The determinant formula is, up to some factors of 2π, reminiscent of the regular Gaussian integral which is proportional to $\det(B)^{-1}$ instead. It is also worth mentioning that the above integrals *are* independent of the choice of basis for V, as opposed to the situation in Lemma 4.1.4, because the element $e^{-\langle \psi, D\psi \rangle}$ changes depending on the basis as well.

Now that we have established the necessary concepts we turn to Dirac ensembles. In general a Dirac ensemble consists of a space of Dirac operators that complete a set spectral triple, as well as a probability measure on that space. Since we are restricting ourselves to fuzzy geometries we can be a bit more precise about this space of Dirac operators and the probability measure.

Definition 4.1.8. *A* Dirac ensemble for a fuzzy geometry *of signature* (p, q) *and size* N *is a probability density of the form*

$$P(D) = \frac{1}{Z} e^{-S(D)} dD$$

where dD *is the Lebesgue measure on* $\mathcal{H}_N^k \cong \mathbb{R}^{kN^2}$ *for* k *the number of self-adjoint matrices required to parametrize the space of Dirac operators. The function* S *is called the* action *and the normalization factor* Z *is called the* partition function *when considered as a function of possible parameters (*coupling constants*) in the action.*

As discussed in the introduction the goal of this project is to include the fermionic action into these Dirac ensembles, so it is perhaps strange that the fermionic action does not feature in this definition. This is due to the fact that our goal is not, yet, the computation of fermionic observables. At the moment we are only interested in the effect that the presence of the fermionic action has on the distribution of the Dirac operator. The technical obstruction to considering mixed observables, at least analytically, is that it would induce a multi-matrix model with matrices in highly asymmetric roles, those appearing in the Dirac operator versus the matrix representing the fermionic field.

Our definition of Dirac ensembles is further motivated by the following observation. Given some Dirac ensemble with action S_b, consider a purely metric observable $A : \mathcal{D} \to \mathbb{R}$. Then the expectation value of A for this Dirac ensemble is given, in the presence of a fermionic action, by

$$\begin{aligned}
\langle A \rangle &= \frac{1}{Z} \int_{D,\psi} A(D) e^{-S_b(D) - \langle \psi, D\psi \rangle} \, d\psi \, dD, \\
&= \frac{1}{Z} \int_D A(D) e^{-S_b(D)} \int_{Ber} e^{-\langle \psi, D\psi \rangle} \, d\psi \, dD, \\
&= \frac{1}{Z} \int_D A(D) e^{-S_b(D)} F(D) \, dD, \\
&= \frac{1}{Z} \int_D A(D) e^{-S_b(D) + \log(F(D))} \, dD.
\end{aligned}$$

This leads us to make the following definition.

Definition 4.1.9. *Suppose* $((p, q), N, S_b)$ *defines a Dirac ensemble. Let*

$$F(D) = \int_{Ber} e^{-\langle \psi, D\psi \rangle} \, d\psi,$$

then the fermionic Dirac ensemble *for the bosonic action* S_b *is the Dirac ensemble with action* $S(D) = S_b(D) - \log(F(D))$.

So that for our purposes of only computing observables of the Dirac operator the fermionic action can be interpreted as a change to the bosonic action. The difference with previous models is that $\log(F(D))$ will be quite different from the trace of a polynomial in D, which is the form $S_b(D)$ usually takes.

4.2 The $(0,1)$ fuzzy geometry with one fermion

The Dirac ensemble we will be considering is based on the $(0,1)$ fuzzy geometry[2]. In Lemma 4.1.2 we have established that these geometries are of the form

$$(M_N(\mathbb{C}), M_N(\mathbb{C}), [H, \bullet]; \bullet^*)$$

for a self-adjoint $N \times N$ matrix H. We will consider the fermionic Dirac ensemble with the quartic bosonic action

$$S_b(D) = g_4 \operatorname{Tr}(D^4) + g_2 \operatorname{Tr}(D^2) \tag{4.1}$$

for $g_4 > 0$. Potentials containing higher powers of D are certainly accessible with the tools we have but will make the computations in the following sections unnecessarily complicated. The quartic Dirac ensemble already exhibits sufficiently interesting behaviour. In particular, even without fermions, it has a spectral phase transition. One of our main goals is to find the effect the fermionic action has on this phase transition. We will in the following sections also occasionally restrict to the Gaussian case given by $g_4 = 0$ and $g_2 > 0$.

However, as discussed in the introduction of this chapter, the Dirac ensemble over the $(0,1)$ fuzzy geometry is not the model we are actually interested in. We want to consider instead an almost fuzzy geometry over the $(0,1)$ space. The reason for this is the following result.

Lemma 4.2.1. *Let $D = [H, \bullet]$ be the Dirac operator of a $(0,1)$ fuzzy geometry. Then the fermionic contribution*

$$F(D) = \int_{Ber} e^{-\langle \psi, D\psi \rangle} \, d\psi = 0.$$

Proof. Let $\{e_i\}_{i=1}^N$ be an orthonormal (relative to the usual $\mathbb{C}^N$ inner product) basis of eigenvectors for H with $He_i = \lambda_i e_i$ and let $E_{i,j}$ be the matrix units relative to $\{e_i\}$. Then

$$\begin{aligned} D(E_{i,j})e_k &= [H, E_{i,j}]e_k, \\ &= \delta_{j,k} H e_i - \lambda_k E_{i,j} e_k, \\ &= \lambda_i \delta_{j,k} e_i - \lambda_k \delta_{j,k} e_i, \\ &= (\lambda_i - \lambda_j)\delta_{j,k} e_i, \\ &= (\lambda_i - \lambda_j) E_{i,j} e_k. \end{aligned}$$

[2] One might ask why we consider the signature $(0,1)$ geometry only and not the signature $(1,0)$ as well. This is because our main techniques are not available for signature $(1,0)$, see Remark 4.3.2.

So the eigenvalues of D are differences of eigenvalues of H. Moreover $E_{i,j}$ are orthonormal with respect to the Hilbert-Schmidt inner product so that by Lemma 4.1.7

$$F(D) = \int_{Ber} e^{-\langle \psi, D\psi \rangle} \, d\psi = \det(D) = \prod_{i,j} (\lambda_i - \lambda_j) = 0^N (-1)^N \prod_{i<j} (\lambda_i - \lambda_j)^2.$$

□

This means that we cannot, directly, define a probability density on the space of Dirac operators $\mathcal{D} = \mathcal{H}_N$ by the recipe for a fermionic fuzzy Dirac ensemble (Definition 4.1.9). However, we could naively attempt to cancel the 0^N in $F(D)$ against the corresponding 0^N appearing in Z. One way we can make this precise is by considering $\langle \psi, (D + m)\psi \rangle$ as fermionic action instead and letting m tend to zero.

This trick of adding a mass m (by analogy with physics, e.g. [27, Ch. 4.1]) is technically sufficient, but not satisfactory. In particular $D + m$ is not a Dirac operator, as Lemma 4.1.2 shows that any $(0, 1)$ Dirac operator is purely a commutator. Moreover both the analogy with physics, where m appears with a Clifford-type matrix, as well as the standard model in noncommutative geometry, where fermion masses are added through the external product with a finite space, run against the grain of simply considering $D + m$.

Instead we will add the mass term by building an almost fuzzy geometry. As our finite space we consider the spectral triple

$$(\mathbb{C}, \mathbb{C}, m; \overline{\bullet}) \tag{4.2}$$

of KO-dimension 7. The external product with a $(0, 1)$ fuzzy geometry is then given by

$$\big(M_N(\mathbb{C}), M_N(\mathbb{C}) \otimes \mathbb{C}^2, D = [H, \bullet] \otimes \sigma_1 + m \otimes \sigma_2; \bullet^* \otimes J, 1 \otimes \sigma_3\big) \tag{4.3}$$

where the σ_i are the Pauli matrices and $J\left(\begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} \overline{v_1} \\ -\overline{v_2} \end{smallmatrix}\right)$. This is a spectral triple of KO-dimension 0, see Sections 1.1.1 and 1.1.2 for more details on external products and KO-dimension.

Our tools are capable of handling more complicated finite spaces, and this is something we will definitely continue to explore in the future. It is also important to note that we treat m differently from H, while it is, essentially, a Dirac operator we consider it as a fixed physical input to the model. It would be extremely interesting to consider m as a random variable and let the model make predictions for the mass, but the tools developed in Chapter 3 do not apply to the asymmetric situation of having the spectrum of H and the variable m. However, since H and m (or a more general D_{finite}) are not only simultaneously but *independently* diagonalizable due to the external product structure. We believe that, with significant extra work, the tools from Chapter 3 can be adapted to this situation.

4.2.1 The bosonic and fermionic actions

The rest of this section will be spent on getting the Dirac ensemble to a point where the technology developed in Chapter 3 applies. The first step in this process is to compute

the bosonic action from Equation 4.1 for our new product Dirac operator in terms of H, *i.e.* translate the Dirac ensemble into random matrix model.

Lemma 4.2.2. *For the Dirac operator $D = D(H, m)$ as defined in Equation 4.3 we have*

$$\begin{aligned}\operatorname{Tr}(D^2) &= 4N \operatorname{Tr}(H^2) - 4 \operatorname{Tr}(H)^2 + 2N^2 m^2,\\ \operatorname{Tr}(D^4) &= 4N \operatorname{Tr}(H^4) + 12 \operatorname{Tr}(H^2)^2 - 16 \operatorname{Tr}(H) \operatorname{Tr}(H^3)\\ &\qquad + 2m^2 \left(4N \operatorname{Tr}(H^2) - 4 \operatorname{Tr}(H)^2\right) + 2N^2 m^4.\end{aligned}$$

Proof. This is a fairly straightforward computation. We start by computing D^2.

$$\begin{aligned}([H, \bullet] \otimes \sigma_1 + m \otimes \sigma_2)^2 &= [H, \bullet]^2 \otimes 1 + ([H, \bullet] \circ m) \otimes \sigma_1\sigma_2 + (m \circ [H, \bullet]) \otimes \sigma_2\sigma_1 + m^2 \otimes 1,\\ &= [H, \bullet]^2 \otimes 1 + m^2 \otimes 1,\\ &= \left([H, \bullet]^2 + m^2\right) \otimes 1.\end{aligned}$$

To get the trace of the commutator we use the isomorphism of vector spaces $\mathbb{C}^N \otimes \mathbb{C}^N \to M_N(\mathbb{C})$ given by $v \otimes w \mapsto |v\rangle\langle w|$. Under this isomorphism, taking the commutator with the self-adjoint H becomes $H \otimes 1_N - 1_N \otimes H$. The subscript on 1_N denotes that it is the $N \times N$ identity matrix. Then

$$\begin{aligned}D^2 &= \left((H \otimes 1_N - 1_N \otimes H)^2 + m^2\right) \otimes 1_2,\\ &= \left(H^2 \otimes 1_N - 2H \otimes H + 1_N \otimes H^2 + m^2 1_N \otimes 1_N\right) \otimes 1_2\end{aligned}$$

from which the trace is easily obtained.

We square this again to get

$$\begin{aligned}D^4 = &\left(H^4 \otimes 1_N + 6H^2 \otimes H^2 + 1_N \otimes H^4 - 4H^3 \otimes H - 4H \otimes H^3\right) \otimes 1_2\\ &+ 2m^2 \left(H^2 \otimes 1_N - 2H \otimes H + 1_N \otimes H^2\right) \otimes 1_2 + m^4 1_N \otimes 1_N \otimes 1_2.\end{aligned}$$

□

The next step is to compute the fermionic part of the action, $F(D)$. Since this is related to the determinant, or Pfaffian, of D we are interested in the spectrum of D in terms of the spectrum of the parametrizing matrix H. This will also be relevant when we want to find the spectral density of D from the spectral density of H.

Lemma 4.2.3. *The spectrum of $D = D(H, m)$ as in Equation 4.3 is*

$$\left\{ -\sqrt{(\lambda_i - \lambda_j)^2 + m^2}, \sqrt{(\lambda_i - \lambda_j)^2 + m^2} \right\}_{i,j=1,\ldots,N}$$

where $\{\lambda_i\}_{i=1}^N$ is the spectrum of H.

Proof. Let $\{e_i\}_{i=1}^N$ be a basis of orthonormal eigenvectors for H with eigenvalues $\{\lambda_i\}_{i=1}^N$ and $E_{i,j}$ the matrix units relative to that basis. Then

$$D(E_{i,j} \otimes v) = (\lambda_i - \lambda_j) E_{i,j} \otimes \sigma_1 v + m E_{i,j} \otimes \sigma_2 v$$

so in the basis $\{E_{i,j} \otimes e_k\}$ with $i, j = 1, \ldots, N$ and $k = 1, 2$ we see that D acts like a direct sum of N^2 block matrices

$$\begin{pmatrix} 0 & (\lambda_i - \lambda_j) - im \\ (\lambda_i - \lambda_j) + im & 0 \end{pmatrix}$$

so that it has spectrum

$$\left\{ -\sqrt{(\lambda_i - \lambda_j)^2 + m^2}, \sqrt{(\lambda_i - \lambda_j)^2 + m^2} \right\}_{i,j=1,\ldots,N}.$$

□

Lemma 4.2.4. *For $D = D(H, m)$ as in Equation 4.3 we have*

$$\int_{Ber} e^{-\langle \psi, D\psi \rangle} \, d\psi = (-1)^{N^2} \prod_{i,j} \left((\lambda_i - \lambda_j)^2 + m^2 \right),$$

$$\int_{Ber} e^{-\frac{1}{2}\langle J\psi, D\psi \rangle} \, d\psi = (im)^N (-1)^{\frac{N^2 - N}{2}} \prod_{i<j} \left((\lambda_i - \lambda_j)^2 + m^2 \right)$$

where i, j run over $1, \ldots, N$.

Proof. We will prove these two statements by different means, even though both methods work in both cases, for illustration purposes.

The first claim easily follows from the spectrum of D, Lemma 4.2.3, and Lemma 4.1.7 which tells us this integral is the determinant of D.

The second claim we will prove from the definition of the fermionic integral. Let as before $\{e_i\}_{i=1}^N$ be an orthonormal basis of eigenvectors for H with eigenvalues $\{\lambda_i\}$ and let $\{E_{i,j}\}$ be the matrix units relative to this basis. Further, let $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ be the standard basis for $\mathbb{C}^2$ so that $\{E_{i,j} \otimes v_a\}$ forms a basis for $M_N(\mathbb{C}) \otimes \mathbb{C}^2$. To start, by Definition 4.1.5, we wish to know the matrix elements

$$\begin{aligned} \langle J(E_{i,j} \otimes v_a), D(E_{k,l} \otimes v_b) \rangle &= \langle E_{j,i} \otimes v_a, (\lambda_k - \lambda_l) E_{k,l} \otimes \sigma_1 v_b \rangle + \langle E_{j,i} \otimes v_a, m E_{k,l} \otimes \sigma_2 v_b \rangle, \\ &= \left((\lambda_j - \lambda_i) \langle v_a, \sigma_1 v_b \rangle + m \langle v_a, \sigma_2 v_b \rangle \right) \delta_{j,k} \delta_{i,l}, \\ &= \left((\lambda_j - \lambda_i) + (-1)^a im \right) \delta_{j,k} \delta_{i,l} (1 - \delta_{a,b}). \end{aligned}$$

Let $\psi = \{\psi_{(i,j),a}\}$ be the coordinate functions corresponding to the basis $\{E_{i,j} \otimes v_a\}$, then

$$\begin{aligned} \langle J\psi, D\psi \rangle &= \sum_{(i,j)} \left((\lambda_j - \lambda_i) - im \right) \psi_{(i,j),1} \psi_{(j,i),2} + \sum_{(i,j)} \left((\lambda_j - \lambda_i) + im \right) \psi_{(i,j),2} \psi_{(j,i),1}, \\ &= \sum_{(i,j)} \left((\lambda_j - \lambda_i) - im \right) \psi_{(i,j),1} \psi_{(j,i),2} + \sum_{(i,j)} \left((\lambda_i - \lambda_j) - im \right) \psi_{(j,i),1} \psi_{(i,j),2}, \\ &= \sum_{(i,j)} \left((\lambda_j - \lambda_i) - im \right) \psi_{(i,j),1} \psi_{(j,i),2} + \sum_{(j,i)} \left((\lambda_j - \lambda_i) - im \right) \psi_{(i,j),1} \psi_{(j,i),2}, \\ &= 2 \sum_{(i,j)} \left((\lambda_j - \lambda_i) - im \right) \psi_{(i,j),1} \psi_{(j,i),2} \end{aligned}$$

where we first use the matrix elements, then swap the Grassmann generators in the second sum, re-index the second sum, and realize for the final step that summing (i,j) or (j,i) for $i,j = 1,\ldots,N$ is the same.

Further note that for $(i,j) \neq (k,l)$ the products $\psi_{(i,j),1}\psi_{(j,i),2}$ and $\psi_{(k,l),1}\psi_{(l,k),2}$ commute so that we can compute

$$e^{-\frac{1}{2}\langle J\psi, D\psi\rangle}$$

using the $e^{a+b} = e^a e^b$ formula for the exponential (the exponential on the Grassmann algebra is defined by its, necessarily finite, power series).

For each individual term in the sum we have

$$e^{-((\lambda_j-\lambda_i)-im)\psi_{(i,j),1}\psi_{(j,i),2}} = 1 - ((\lambda_j - \lambda_i) - im)\,\psi_{(i,j),1}\psi_{(j,i),2}$$

as the square of $\psi_{(i,j),1}\psi_{(j,i),2}$ is zero. Then we get for the full exponential

$$e^{-\frac{1}{2}\langle J\psi, D\psi\rangle} = \prod_{i,j}\left(1 - ((\lambda_j - \lambda_i) - im)\,\psi_{(i,j),1}\psi_{(j,i),2}\right)$$

so that

$$\int_{Ber} e^{-\frac{1}{2}\langle J\psi, D\psi\rangle}\, d\psi = \prod_{i,j}((\lambda_i - \lambda_j) + im) = (im)^N(-1)^{\frac{N^2-N}{2}}\prod_{i<j}\left((\lambda_i-\lambda_j)^2 + m^2\right)$$

□

Observe that the proof of this lemma shows that the difference between the determinant and the Pfaffian is really due to halving the number of generators from $\{\overline{\psi}_i, \psi_i\}$ to $\{\psi_i\}$. This is possible with the real bilinear form $\langle J\psi, D\psi\rangle$ and is unrelated to the $\frac{1}{2}$ appearing in the exponent. Indeed it is easy to check that removing that $\frac{1}{2}$ simply leads to a factor 2^{N^2} in the final answer.

But perhaps a more obvious question is why we compute the fermionic action for both $\langle\psi, D\psi\rangle$ and $\langle J\psi, D\psi\rangle$, while so far we have said that the fermionic action is given by $\langle\psi, D\psi\rangle$. This is related to the issue of fermion doubling that happens in the almost commutative framework for the standard model, and that also happens here for the almost fuzzy framework. While our goal in Equation 4.2 was to add one fermion to the Dirac ensemble, the nature of the odd-odd product of spectral triples caused us to gain two fermions (one right-handed and one left-handed).

This is similar to, although not quite the same as, the issue of fermion doubling that was encountered when formulating the standard model of particle physics in almost commutative noncommutative geometry. There both the base space and the finite space have a left- and right-handed structure as well as a particle anti-particle structure, leading to a situation with too many fermions, some of which could be called unphysical (*e.g.* those that have a left-handed spin according to the base space and a right-handed spin according to the finite space).

Multiple solutions have been offered for the fermion doubling problem. One approach that works for the standard model but not for us is to change the finite space, see for example [3]. Another solution is to (if applicable) restrict the fermionic integral to the

+1 eigenspace of Γ and consider the fermionic action $\frac{1}{2}\langle J\psi, D\psi\rangle$ instead, see for example [14].

Looking at the spectrum of $D = D(H, m)$ in Lemma 4.2.3 we see that the "doubling factor" $\mathbb{C}^2$ in the spectral triple causes a duplication of the eigenvalues. Indeed if $m = 0$ the spectrum of D is exactly two copies of the spectrum of $[H, \bullet] : M_N(\mathbb{C}) \to M_N(\mathbb{C})$. This duplication of eigenvalues can be countered by considering the Pfaffian instead of the determinant. The classical results on the fermionic integral, Lemma 4.1.7, then show that this can be accomplished by introducing the J in the fermionic action.

We will take the easy way out of this problem for now and leave it ambiguous which fermionic action we pick. Instead we say that the fermionic contribution is given by

$$F(D) = \prod_{i<j} \left((\lambda_i - \lambda_j)^2 + m^2\right)^{\frac{\beta_2}{2}}. \tag{4.4}$$

If $\beta_2 = 2$ this corresponds to the action $\langle J\psi, D\psi\rangle$ and if $\beta_2 = 4$ this corresponds to the action $\langle \psi, D\psi\rangle$. Note that in both cases what we call the fermionic contribution is only proportional to the actual fermionic integral computed in Lemma 4.2.4. The proportionality constant is independent of H and can thus be absorbed into Z.

We choose the notation β_2, and the factor $\frac{1}{2}$, by analogy with the Dyson exponent β from random matrix theory. The parameter β_2 can be thought of as a coupling constant for the fermionic action, as we will see in the following, but in the current setup can only take even integer values (where values higher than 4 are obtained by considering a higher dimensional finite space).

This means that our fermionic Dirac ensemble is given by the density

$$\frac{1}{Z_{Dirac}} \exp\left(-g_4 \operatorname{Tr}(D^4) - g_2 \operatorname{Tr}(D^2) + \log(F(D))\right).$$

with $F(D)$ from Equation 4.4.

4.2.2 The eigenvalue model for the fermionic fuzzy model

There is one final problem when transforming this into a matrix model, and eventually eigenvalue model, for H through $D = D(H)$. Observe that the map $\mathcal{H}_N \to \mathcal{D}$ is not injective, as any scalar multiple of the identity matrix is mapped to the zero operator. This means that the induced density on $\mathcal{H}_N$ will not be finite, since it is constant along the paths $t \mapsto H + t1_N$. Alternatively we can take a peek forward to the application of the tools from Section 3.2 and observe that $U(x + t, y + t) = U(x, y)$. This implies that the associated energy functional $I(\mu)$ is invariant under translations of the measure. In particular there cannot be a unique minimizer to I, as any translated measure will have the same energy.

The resolution to this is to manually put a probability density on the fibres of $\mathcal{H}_N \to \mathcal{D}$. In our case we opt for the probability density proportional to $e^{-a \operatorname{Tr}(H)^2}$ for some $a > 0$, the proportionality constant will be absorbed into the global normalization constant Z. Since this probability density varies exclusively along the fibres of $\mathcal{H}_N \to \mathcal{D}$ it does not affect the probability of finding any particular Dirac operator. But it does eliminate the translation invariance, as shown in Lemma 4.3.4.

It might appear that a more natural fix to this problem is to consider the subspace of traceless self-adjoint matrices, on which the map $\mathcal{H}_N \to \mathcal{D}$ is an isomorphism of vector spaces. This is equivalent to choosing the probability density $\delta_0(\mathrm{Tr}(H))$ on the fibres, which in terms of the eigenvalues of H is $\delta_0(\sum \lambda_i)$ and as such is a degree N interaction (borrowing terminology from Section 3.2). On the other hand, the Gaussian density we use becomes $(\sum \lambda_i)^2 = \sum \lambda_i \lambda_j$ which is a degree 2 interaction. The technology developed in Chapter 3 is only equipped to deal with interactions of fixed degree, so restricting to the traceless matrices would invalidate those tools.

With the above observations in mind, the Weyl integration formula (see Section 1.2.1) gives us the following eigenvalue density on $\mathbb{R}^N$:

$$\frac{1}{Z_{eigen}} \exp\left(\begin{array}{c} -g_4 \left(4N \sum \lambda_i^4 + 12 \left(\sum \lambda_i^2\right)^2 - 16 \left(\sum \lambda_i\right)\left(\sum \lambda_i^3\right)\right) \\ -g_2' \left(4N \sum \lambda_i^2 - 4 \left(\sum \lambda_i\right)^2\right) - a \left(\sum \lambda_i\right)^2 \\ + \frac{\beta_2}{4} \sum_{i,j} \log\left((\lambda_i - \lambda_j)^2 + m^2\right) + \frac{\beta}{2} \sum_{i \neq j} \log\left(|\lambda_i - \lambda_j|\right) \end{array} \right) \tag{4.5}$$

where $g_2' = g_2 + 2m^2 g_4$ and the unlabelled sums are over i, in all cases i, j run from 1 to N.

We will briefly recap where each term comes from.

- The first line is the quartic part of the quartic term in the bosonic action.
- The first term in the second line consists of the quadratic term of the action, together with the quadratic part of the quartic term accounting for the change in coupling constant from g_2 to g_2'.
- The second term in the second line accounts for the kernel of the map $\mathcal{H}_N \to \mathcal{D}$, as discussed below Lemma 4.2.2. We use this term in Lemma 4.3.4, but the value of $a > 0$ will turn out not to affect the spectral density.
- The first logarithmic term in the third line is the fermionic action, β_2 is either 2 or 4. The factor β_2 here justifies us calling it the fermionic coupling constant. The observant reader might note the discrepancy in the sum including $i = j$ terms and Equation 4.4. The difference is a scalar independent of λ so this is absorbed into Z_{eigen}.
- The second logarithmic term in the third line is the Jacobian that we pick up in Weyl's integration formula. In our case $\beta = 2$ since we are working with complex matrices, but we keep β as a variable to more easily deal with the $m = 0$ case, in which case the two logarithmic terms combine to one with coefficient $\frac{\beta+\beta_2}{2}$.

4.3 The spectral density: computations

In this section we will first prove that the tools from Chapter 3 apply to our fuzzy fermionic Dirac ensemble and then do the associated computations for this specific ensemble. To do this we first need to give the function U defining our eigenvalue model and then verify that this function U satisfies the three parts of Assumption 3.1.

Comparing the form of the eigenvalue model in Equation 3.3 and the eigenvalue density for our fuzzy fermionic Dirac ensemble in Equation 4.5, we see that the degree of our interaction is $k = 2$ and that we should define the potential function U, for $m \neq 0$, by

$$\begin{aligned} U(x,y) = g_4\left(2(x^4+y^4)+12x^2y^2-8(x^3y+xy^3)\right) \\ +g_2'\left(2(x^2+y^2)-4xy\right)+axy-\tfrac{\beta_2}{4}\log\left((x-y)^2+m^2\right). \end{aligned} \tag{4.6}$$

For $m = 0$ we define U by this formula without the logarithmic term.

Recall from Section 3.2 that to this U we associate an energy functional

$$I(\mu_1,\mu_2) = \int_{\mathbb{R}^2} W(x,y)\, d\mu_1(x)\, d\mu_2(y)$$

with

$$W(x,y) = U(x,y) - \frac{\beta}{2}\log(|x-y|),$$

where β is replaced by $\beta+\beta_2$ if $m = 0$. This is the main reason we keep track of β and β_2 throughout all computations, it makes it easy to get $m = 0$ results without extra work. It is also interesting to keep track of them as they function like coupling constants for the strength of the native eigenvalue repulsion and the fermionic action allowing us to artificially play with the "strength" of the fermionic action.

4.3.1 Check of the assumptions

We will establish the three parts of Assumption 3.1 for this function U, repeated here for convenience and specialized to $k = 2$. $U : \mathbb{R}^2 \to \mathbb{R}$ is a continuous function such that

1. U is invariant under permutation of its arguments.
2. There is a continuous function $u : \mathbb{R} \to \mathbb{R}$ such that $U(x,y) \geq u(x)$ for all $y \in \mathbb{R}$ and $u(x) - \frac{\max(\beta,2)}{2}\log(1+x^2) \to \infty$ as $|x| \to \infty$.
3. There is a set of *candidate measures* $\mathcal{P}_{can} \subset \mathcal{P}(\mathbb{R})$ containing the minimizer of I, such that for any probability measures $\mu, \nu \in \mathcal{P}_{can}$ and any $t \in [0,1]$

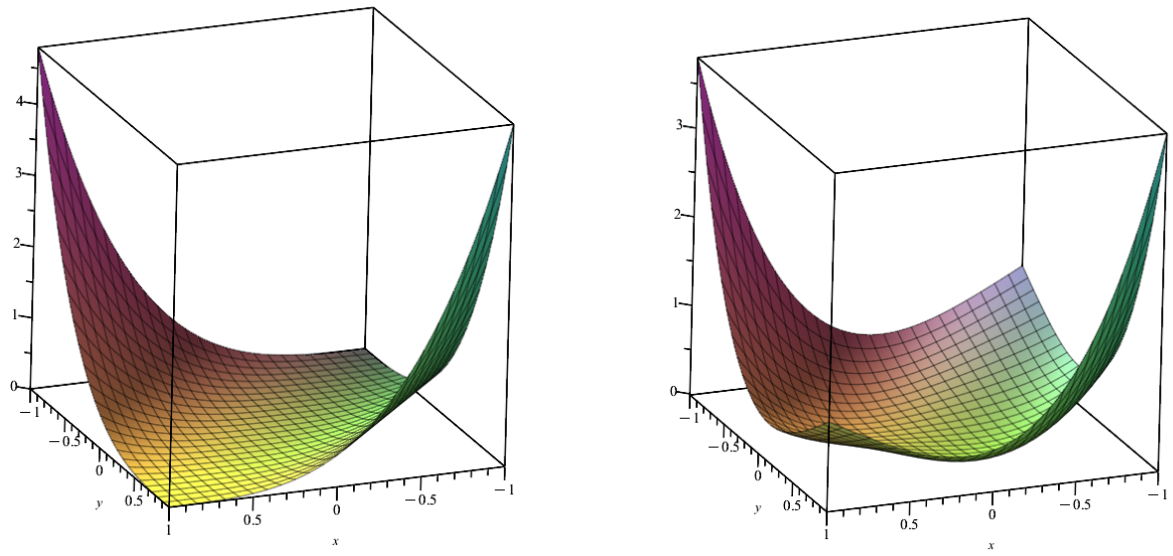
$$\frac{d^2}{dt^2}\int_{\mathbb{R}^2} U(x,y)\, d(\mu+t(\nu-\mu))^{\otimes 2}(x,y) \geq 0.$$

Certainly U defined in Equation 4.6 is continuous and well-defined on all of $\mathbb{R}$, and is invariant under the swap $(x,y) \mapsto (y,x)$. To establish the second and third property the following lemma turns out to be very useful and essential, respectively. This highlights the importance of the $a\,\mathrm{Tr}(H)^2$ term added in Section 4.2.2 and is visualized in Figure 4.1.

Lemma 4.3.1. *For U defined in Equation 4.6 we have*

$$U(x+t,y+t) = U(x,y) - axy + a(x+t)(y+t),$$

and the same holds for the corresponding function W, as defined in Definition 3.2.2.



(a) $a = 0$ (b) $a = 1$

Figure 4.1: The function $U(x, y)$ plotted over $[-1, 1]^2$ for $g_4 = g_2 = 0.1$, $m = 1$, $\beta_2 = 2$. The second part of Assumption 3.1 requires U to be increasing as $\|(x, y)\| \to \infty$. As discussed in Section 4.2.2, this is not the case if U is based only on the (combined fermionic and bosonic) action of D since that would leave a shift invariance $U(x+t, y+t) = U(x, y)$ corresponding to the kernel of the map $\mathcal{H}_N \to \mathcal{D}$ (see Lemma 4.3.1). This shift invariance can be seen in plot (a) and is broken by the $a \operatorname{Tr}(H)^2$ term, as plot (b) shows.

Proof. The logarithmic parts of U and W are clearly invariant under this shift so we consider U_{pol} the polynomial part of U.

Consider the matrix

$$H = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix},$$

and $D = [H, \bullet]$ as operator on $M_2(\mathbb{C})$. Then by Lemma 4.2.2 we have

$$\begin{aligned} \operatorname{Tr}(D^2) &= 8(x^2 + y^2) - 4(x + y)^2 + 2N^2m^2, \\ &= 4x^2 + 4y^2 - 8xy + 8m^2, \end{aligned}$$

and

$$\begin{aligned} \operatorname{Tr}(D^4) &= 8(x^4 + y^4) + 12(x^2 + y^2)^2 - 16(x + y)(x^3 + y^3) + 2m^2 \operatorname{Tr}(D^2) + 2N^2m^4, \\ &= 4x^4 + 4y^4 + 24x^2y^2 - 16xy^3 - 16x^3y + 2m^2 \operatorname{Tr}(D^2) + 8m^4. \end{aligned}$$

So

$$\begin{aligned} g_4 \operatorname{Tr}(D^4) + g_2 \operatorname{Tr}(D^2) &= g_4 \left(4x^4 + 4y^4 + 24x^2y^2 - 16xy^3 - 16x^3y + 8m^4\right) \\ &\quad + g_2' \left(4x^2 + 4y^2 - 8xy + 8m^2\right), \\ &= 2\left(U_{pol}(x, y) - axy\right) + 8m^2(g_2' + g_4m^2). \end{aligned}$$

Since H and $H + t1_2$ clearly induce the same D, the left-hand side of this equation is invariant under the shift $(x, y) \mapsto (x + t, y + t)$ so the right-hand side must be as well. Thus

$$U_{pol}(x + t, y + t) - a(x + t)(y + t) = U_{pol}(x, y) - axy.$$

□

Remark 4.3.2. *We should finally explain why the other single-matrix model that comes up in the context of fuzzy geometries is absent from this discussion. This is the Dirac ensemble with signature* $(1, 0)$ *and Dirac operator* $D = \{H, \bullet\}$*. In this ensemble the map from the Hermitian matrices to Dirac operators is injective and defines a convergent random matrix model on* $\mathcal{H}_N$ *by itself.*

The associated function U *is very similar to the one we obtained for the* $(0, 1)$ *model, simply change the* $-$ *signs in front of the* $\operatorname{Tr}(H)\operatorname{Tr}(H^3)$ *and* $\operatorname{Tr}(H)^2$ *terms to* $+$*. Because of this similarity we get a similar plot of* U *as Figure 4.1a, but instead of* U *being constant along lines* $x - y = c$ *it is constant along lines with* $x + y = c$*. So this ensemble does not immediately fit into the theory from Chapter 3.*

For signature $(1, 0)$ *there is no clear way to adjust* U *as we did for for signature* $(0, 1)$ *in Section 4.2.2. This means we cannot apply the same techniques to the signature* $(1, 0)$ *case blindly. It is likely that some relatively minor modification of the theory in Section 3.2 applies, but this requires further work.*

Additionally there is strong numerical evidence in [20] that the spectral density for the $(1, 0)$ *is not symmetric, and thus in particular not unique. This means that the convexity assumption will not hold for this model, even if the assumptions on the confining nature of* U*, or* U *itself, are suitably changed.*

Lemma 4.3.3. *There is some* $R \in \mathbb{R}$ *such that*

$$u(x) = \inf_{y \in \mathbb{R}} U(x, y) \geq \frac{1}{2} a x^2$$

for $|x| > R$*.*

Proof. We compute, using Lemma 4.3.1,

$$\begin{aligned}
u(x) &= \inf_{y \in \mathbb{R}} U(x, y), \\
&= \inf_{y \in \mathbb{R}} \left(U(0, y - x) + axy \right), \\
&= \inf_{y \in \mathbb{R}} \left(U(0, y) + ax(y + x) \right), \\
&= ax^2 + \inf_{y \in \mathbb{R}} \left(2g_4 y^4 + 2g_2' y^2 - \frac{\beta_2}{4} \log(m^2 + y^2) + axy \right), \\
&\geq ax^2 + \inf_{y \in \mathbb{R}} \left(g_4 y^4 + 2g_2' y^2 - \frac{\beta_2}{4} \log(m^2 + y^2) \right) + \inf_{y \in \mathbb{R}} \left(g_4 y^4 + axy \right).
\end{aligned}$$

Using that $g_4 > 0$, some basic calculus shows that the third term is proportional to $x^{\frac{4}{3}}$ and the second term is finite. Thus we can find some R such that for $|x| > R$ this lower bound is bigger than $\frac{1}{2} a x^2$. □

Lemma 4.3.4. *Any measure that minimizes the associated energy functional of the fuzzy fermionic eigenvalue model, Equation 4.6, has mean 0.*

Proof. Suppose μ is any probability measure on $\mathbb{R}$ and let $\tau_t : \mathbb{R} \to \mathbb{R}$, $\tau_t(x) = x + t$ so that $\{\tau_t^* \mu\}_{t\in\mathbb{R}}$ is the family of translates of μ. Then

$$\begin{aligned} I(\tau_t^*\mu) &= \int_{\mathbb{R}^2} W(x,y)\, d\tau_t^*\mu(x)\, d\tau_t^*\mu(y), \\ &= \int_{\mathbb{R}^2} W(x,y)\, d\mu(x-t)\, d\mu(y-t), \\ &= \int_{\mathbb{R}^2} W(x+t,y+t)\, d\mu(x)\, d\mu(y), \\ &= \int_{\mathbb{R}^2} W(x,y) - axy + a(x+t)(y+t)\, d\mu(x)\, d\mu(y), \\ &= I(\mu) + 2a\mu_1 t + at^2 \end{aligned}$$

using Lemma 4.3.1 and where μ_1 is the mean of μ.

This means $I(\tau_t^*\mu)$ is minimized when $t = -\mu_1$, *i.e.* exactly when $\tau_t^*\mu$ is centered. □

Proposition 4.3.5. *The function U defined in Equation 4.6 satisfies Assumption 3.1 with the set of centered probability measures of compact support as candidate measures, $\mathcal{P}_{can} = \{\mu \in \mathcal{P}_c(\mathbb{R}) \,|\, \mu_1 = 0\}$.*

Proof. The function U is clearly invariant under $x \leftrightarrow y$ so the first property is satisfied and Lemma 4.3.3 gives the second property.

For the third property Lemma 4.3.4 shows that a minimizer of I must have mean zero, and the first two properties already guarantee that a minimizer must have compact support (see the discussion below Assumption 3.1). So any minimizer lies in the set of candidate measures.

We again split the potential U into its logarithmic and polynomial parts

$$U(x,y) = U_{pol}(x,y) - \log(m^2 + (x-y)^2).$$

By Proposition 3.1.6 we have

$$-\int_{\mathbb{R}^2} \log(m^2 + (x-y)^2)\, d(\nu-\mu)(x)\, d(\nu-\mu)(y) = 2\int_0^\infty e^{-m^2 k} \frac{1}{k} |\widehat{(\nu-\mu)}(k)|^2\, dk \geq 0.$$

To find the contribution of U_{pol} first observe that

$$\begin{aligned} \int_{\mathbb{R}^2} x^a y^b\, d(\nu-\mu)(x)\, d(\nu-\mu)(y) &= \nu_a\nu_b - \nu_a\mu_b - \mu_a\nu_b + \mu_a\mu_b, \\ &= (\nu_a - \mu_a)(\nu_b - \mu_b), \end{aligned}$$

where as usual μ_k denotes the k-th moment of a measure μ. Using this we get

$$\begin{aligned} \int_{\mathbb{R}^2} U_{pol}(x,y)\, d(\nu-\mu)(x)\, d(\nu-\mu)(y) = 12g_4(\nu_2-\mu_2)^2 - 16g_2'(\nu_1-\mu_1)(\nu_3-\mu_3) \\ - 4g_2'(\nu_1-\mu_1)^2 + a(\nu_1-\mu_1)^2 \end{aligned}$$

As μ, ν are candidate measures we have $\mu_1 = 0 = \nu_1$ and we find

$$\int_{\mathbb{R}^2} U_{pol}(x,y)\, d(\nu-\mu)(x)\, d(\nu-\mu)(y) = 12g_4(\nu_2-\mu_2)^2 \geq 0,$$

so we are done. □

Remark 4.3.6. *Proposition 4.3.5 still holds for the Gaussian model where $g_4 = 0$, $g_2 > 0$. In fact the only proof that needs to be adapted slightly is the proof of Lemma 4.3.3. The results also hold for the case $m = 0$.*

4.3.2 Computation of the spectral density

In this section we will do the computations required to obtain systems of equations for ρ and the support, being either $[-a,a]$ or $[-b,-a]\cup[a,b]$ (see the discussion at the end of this section for why these cases are exhaustive). These systems will generally not be exactly solvable, at least not as far as we have found, but they are sufficient for numerical approximations that we will show in Section 4.4.

Since we established in Proposition 4.3.5 that the eigenvalue model for our fermionic fuzzy $(0,1)$ Dirac ensemble satisfies the assumptions for Section 3.2, we know that Theorems 3.1, 3.2, and 3.3 hold for the associated energy functional I and its equilibrium measure μ^E. In particular the equilibrium measure is unique and can be found using the tools from Section 3.4. We can take the set of even measures as our space of reduced candidate measures since $U(-x,-y) = U(x,y)$.

To start this process we make the assumptions in Assumption 3.2, so we assume the equilibrium measure is given by

$$\mu^E = \rho(x)\, dx$$

and

$$\operatorname{supp}\left(\mu^E\right) = \Sigma = \cup_{i=1}^{r}[a_i,b_i].$$

We further know that $\rho(-x) = \rho(x)$ and that Σ is symmetric, since we have the set of even measures as our reduced candidate measures.

The first stage of these computations is finding $\mathcal{U}(x)$ and the set Π such that $\mathcal{U}'(x)$ can be extended to $\mathbb{C}\setminus\Pi$.

$$\begin{aligned}
\mathcal{U}(x) &= \int_{\mathbb{R}} U(x,y)\, d\mu^E(y),\\
&= \int_{\mathbb{R}} g_4\left(2(x^4+y^4)+12x^2y^2-8(x^3y+xy^3)\right)\\
&\qquad + g_2'\left(2(x^2+y^2)-4xy\right)+axy-\frac{\beta_2}{4}\log\left((x-y)^2+m^2\right)\, d\mu^E(y),\\
&= 2g_4x^4+2g_4\mu_4^E+12g_4x^2\mu_2^E-8x^3\mu_1^E-8x\mu_3^E\\
&\qquad + 2g_2'x^2+2g_2'\mu_2^E-4g_2'x\mu_1^E-\frac{\beta_2}{4}\int_{\mathbb{R}}\log\left((x-y)^2+m^2\right)\, d\mu^E(y),\\
&= 2g_4x^4+2g_4\mu_4^E+12g_4x^2\mu_2^E+2g_2'x^2+2g_2'\mu_2^E-\frac{\beta_2}{4}\int_{\Sigma}\log\left((x-y)^2+m^2\right)\, d\mu^E(y),
\end{aligned}$$

so that

$$\mathcal{U}'(x) = 8g_4x^3 + \left(24g_4\mu_2^E + 4g_2'\right)x - \frac{\beta_2}{2}\int_\Sigma \frac{x-y}{(x-y)^2+m^2}\,d\mu^E(y)$$

which has an analytic extension

$$\mathcal{U}'(z) = 8g_4z^3 + \left(24g_4\mu_2^E + 4g_2'\right)z - \frac{\beta_2}{2}\int_\Sigma \frac{z-y}{(z-y)^2+m^2}\,d\mu^E(y) \tag{4.7}$$

to $\mathbb{C}\setminus\Pi$, $\Pi = (\Sigma + im)\cup(\Sigma - im)$. If instead $m = 0$ we instead get

$$\mathcal{U}'(z) = 8g_4z^3 + \left(24g_4\mu_2^E + 4g_2'\right)z$$

on all of $\mathbb{C}$. The second moment of $\mu^E = \rho(x)\,dx$ still appears in this expression, but this can be expressed in terms of the other functions involved through Proposition 3.4.10. We keep μ_2^E in the expressions to simplify the notation.

Let us start with two computational lemmas that we will use throughout this section.

Lemma 4.3.7. *Let $\gamma_{\Pi,\varepsilon}$ be the clockwise racetrack of radius ε around $\Pi = \Sigma \pm im$, then for any function f analytic on a neighbourhood of Π and satisfying $f(\overline{z}) = \overline{f(z)}$, we have*

$$\lim_{\varepsilon\to 0}\int_{\gamma_{\Pi,\varepsilon}} f(z)\frac{\mathcal{U}'(z)}{\sqrt{s}(z)}\,dz = \pi i\beta_2\int_\Sigma \Re\left(\frac{f(y+im)}{\sqrt{s}(y+im)}\right)\rho(y)\,dy.$$

Proof. We simply compute this, using that only the fractional part of $\mathcal{U}'$ causes singularities around Π,

$$\begin{aligned}
\lim_{\varepsilon\to 0}\int_{\gamma_{\Pi,\varepsilon}} \frac{\mathcal{U}'(z)}{\sqrt{s}(z)}f(z)\,dz &= \lim_{\varepsilon\to 0}\int_{\gamma_{\Pi,\varepsilon}}\int_\Sigma -\frac{\beta_2}{2}\frac{z-y}{(z-y)^2+m^2}\rho(y)\,dy\frac{f(z)}{\sqrt{s}(z)}\,dz,\\
&= -\frac{\beta_2}{2}\lim_{\varepsilon\to 0}\int_\Sigma\int_{\gamma_{\Pi,\varepsilon}} \frac{f(z)}{\sqrt{s}(z)}\frac{z-y}{(z-y)^2+m^2}\,dz\,\rho(y)\,dy,\\
&= -\frac{\beta_2}{2}\int_\Sigma\left(-2\pi i\frac{1}{2}\frac{f(y+im)}{\sqrt{s}(y+im)} - 2\pi i\frac{1}{2}\frac{f(y-im)}{\sqrt{s}(y-im)}\right)\rho(y)\,dy,\\
&= \frac{\beta_2\pi i}{2}\int_\Sigma\left(\frac{f(y+im)}{\sqrt{s}(y+im)} + \frac{f(y-im)}{\sqrt{s}(y-im)}\right)\rho(y)\,dy,
\end{aligned}$$

using that

$$\operatorname{res}_{z=y\pm im}\left(\frac{z-y}{(z-y)^2+m^2}\right) = \operatorname{res}_{z=0}\left(\frac{z\pm im}{z^2\pm 2zim}\right) = \operatorname{res}_{z=0}\left(\frac{1}{z+2im} + \frac{1}{z}\frac{\pm im}{z\pm 2im}\right) = \frac{1}{2}.$$

Finally, using the conjugacy property assumed for f and proved for $\sqrt{s}$ in Lemma 3.4.5 we get the lemma. □

Lemma 4.3.8.

$$\int_\Sigma y^n\frac{\mathcal{U}'(y)}{\sqrt{s}_+(y)}\,dy = -\frac{\pi i\beta_2}{2}\int_\Sigma R^n_{\Sigma,m}(y)\rho(y)\,dy - \pi i\left[\frac{\mathcal{U}'(z)}{\sqrt{s}(z)}\right]_{-(n+1)}$$

where

$$R^n_{\Sigma,m}(y) = \Re\left(\frac{(y+im)^n}{\sqrt{s}(y+im)}\right).$$

Proof. Using the same contours that were used in the proof of Proposition 3.4.9, consisting of (clockwise) racetracks around Σ and Π and a (counter-clockwise) circle of large radius applied to the function $\frac{\mathcal{U}'(z)}{\sqrt{s}(z)}$, which has no singularities outside of Σ and Π, we get

$$\begin{aligned} 2\int_\Sigma y^n \frac{\mathcal{U}'(y)}{\sqrt{s}_+(y)}\,dy &= \lim_{\varepsilon\to 0}\int_{\gamma_{\Sigma,\varepsilon}} z^n \frac{\mathcal{U}'(z)}{\sqrt{s}(z)}\,dz, \\ &= -\lim_{\varepsilon\to 0}\int_{\gamma_{\Pi,\varepsilon}} z^n \frac{\mathcal{U}'(z)}{\sqrt{s}(z)}\,dz - \lim_{R\to\infty}\int_{\gamma_R} z^n \frac{\mathcal{U}'(z)}{\sqrt{s}(z)}\,dz, \\ &= -\pi i\beta_2 \int_\Sigma \Re\left(\frac{(y+im)^n}{\sqrt{s}(y+im)}\right)\rho(y)\,dy - 2\pi i\left[z^n\frac{\mathcal{U}'(z)}{\sqrt{s}(z)}\right]_{-1}, \end{aligned}$$

Here we used Lemma 4.3.7 for the Π contour. □

With these lemmas out of the way, we are ready to find the equation for ρ. By Proposition 3.4.9 we get that

$$\rho(x) = \frac{2}{\beta}\frac{\sqrt{s}_+(x)}{\pi i}p(x) + \frac{2}{\beta}\frac{\sqrt{s}_+(x)}{\pi i}R(x)$$

where

$$p(z) = \sum_{n=0}^{\infty}\left[\frac{\mathcal{U}'(z)}{\sqrt{s}(z)}\right]_k z_0^k$$

and

$$R(x) = \frac{1}{2\pi i}\int_{\gamma_{\Pi,\varepsilon}} \frac{\mathcal{U}'(z)}{\sqrt{s}(z)}\frac{dz}{z-x}$$

where $\gamma_{\Pi,\varepsilon}$ is a clockwise contour around Π at radius ε.

To find $R(x)$ we can use Lemma 4.3.7 with $f(z) = \frac{1}{z-x}$. We get

$$R(x) = \frac{\beta_2}{2}\int_\Sigma \Re\left(\frac{1}{\sqrt{s}(y+im)}\frac{1}{y+im-x}\right)\rho(y)\,dy,$$

and define

$$K_{\Sigma,m}(x,y) = \frac{\sqrt{s}_+(x)}{\pi i}\Re\left(\frac{1}{\sqrt{s}(y+im)}\frac{1}{y+im-x}\right) \tag{4.8}$$

so that

$$\frac{\sqrt{s}_+(x)}{\pi i}R(x) = \frac{\beta_2}{2}\int_\Sigma K_{\Sigma,m}(x,y)\rho(y)\,dy.$$

This gives us the following Fredholm integral equation

$$\rho(x) = \frac{2}{\beta}\frac{\sqrt{s}_+(x)}{\pi i}p(x) + \frac{\beta_2}{\beta}\int_\Sigma K_{\Sigma,m}(x,y)\rho(y)\,dy. \tag{4.9}$$

The remaining computations, $p(x)$, the support conditions and the expression for μ_2^E, depend more explicitly on the number of intervals of the support so we do those separately below.

One interval

If $\Sigma = [-a, a]$ we get from Equation 4.7 and Lemma 3.4.6 that

$$\frac{\mathcal{U}'(z)}{\sqrt{s}(z)} = \left(8g_4z^3 + \left(24g_4\mu_2^E + 4g_2'\right)z - \frac{\beta_2}{2}\int_\Sigma \frac{z-y}{(z-y)^2+m^2}\rho(y)\,dy\right)\left(\frac{1}{z} + \frac{a^2}{2}\frac{1}{z^3} + \frac{3a^4}{8}\frac{1}{z^5} + \dots\right).$$

The polynomial p appearing in the equation for ρ is the positive part of the Laurent series for $\frac{\mathcal{U}'(z)}{\sqrt{s}(z)}$ that converges around infinity, which is

$$p(z) = 8g_4z^2 + 4g_4a^2 + 24g_4\mu_2^E + 4g_2', \tag{4.10}$$

where we use that the integral term of $\mathcal{U}'$ is asymptotically $\frac{1}{z}$ so that it does not contribute.

This completes the equation for ρ in terms of a. To complete the problem we compute the support conditions and expression for μ_2^E using Proposition 3.4.10.

The first condition is

$$0 = \int_{-a}^{a} \frac{\mathcal{U}'(x)}{\sqrt{s}_+(x)}\,dx$$

but this is automatically satisfied as $\mathcal{U}'(x)$ is odd and $\sqrt{s}_+(x)$ is even.

The normalization condition is

$$\frac{\beta\pi}{2i} = \int_{-a}^{a} x\frac{\mathcal{U}'(x)}{\sqrt{s}_+(x)}\,dx.$$

which by Lemma 4.3.8 means

$$\frac{\beta\pi}{2i} = -\frac{\pi i\beta_2}{2}\int_{-a}^{a} R^1_{\Sigma,m}(y)\rho(y)\,dy - \pi i\left[\frac{\mathcal{U}'(z)}{\sqrt{s}(z)}\right]_{-2}.$$

The term in the Laurent expansion at infinity can be computed using our work for the polynomial $p(z)$ above Equation 4.10, but this time we are looking for the coefficient of z^{-2} instead of the positive part, so we get

$$\left[\frac{\mathcal{U}'(z)}{\sqrt{s}_+(z)}\right]_{-2} = 3g_4a^4 + \left(12g_4\mu_2^E + 2g_2'\right)a^2 - \frac{\beta_2}{2}.$$

Gathering everything up we get the normalization condition

$$\beta + \beta_2\left(1 - \int_\Sigma R^1_{\Sigma,m}(y)\rho(y)\,dy\right) = 6g_4a^4 + \left(24g_4\mu_2^E + 4g_2'\right)a^2 \tag{4.11}$$

Then we conclude the one interval analysis by relating μ_2^E to a and ρ, by Proposition 3.4.10 we have

$$\frac{\beta\pi}{2i}\mu_2^E = \int_{-a}^{a}\frac{\mathcal{U}'(x)}{\sqrt{s}_+(x)}x\,dx\left[\sqrt{s}(z)\right]_{-1} + \int_{-a}^{a}\frac{\mathcal{U}'(x)}{\sqrt{s}_+(x)}x^2\,dx\left[\sqrt{s}(z)\right]_{0} + \int_{-a}^{a}\frac{\mathcal{U}'(x)}{\sqrt{s}_+(x)}x^3\,dx\left[\sqrt{s}(z)\right]_{1}.$$

The middle term is zero, both by symmetry of the integrand and because $\sqrt{s}(z)$ has no constant term in its Laurent expansion at infinity. Using the normalization condition, the Laurent expansion for $\sqrt{s}(z)$ and Lemma 4.3.8 we get

$$\mu_2^E = -\frac{a^2}{2} + \frac{\beta_2}{\beta}\int_{-a}^{a} R^3_{\Sigma,m}(y)\rho(y)\,dy + \frac{2}{\beta}\left[\frac{\mathcal{U}'(z)}{\sqrt{s}(z)}\right]_{-4}$$

We can calculate

$$\left[\frac{\mathcal{U}'(z)}{\sqrt{s}(z)}\right]_{-4} = \frac{5}{2}a^6 g_4 + 3a^4\left(3g_4\mu_2^E + \frac{1}{2}g_2'\right) - \frac{\beta_2}{2}\left(\frac{a^2}{2} + \mu_2^E - m^2\right)$$

by expanding

$$\frac{z-y}{(z-y)^2+m^2} = \frac{1}{z}\left(1 + \frac{y}{z} + \frac{y^2}{z^2} + \dots\right)\left(1 - \frac{m^2}{z^2}\frac{1}{1-\frac{y^2}{z^2}} + \dots\right).$$

This gives

$$(\beta+\beta_2)\left(\mu_2^E + \frac{a^2}{2}\right) = 5a^6 g_4 + 3a^4\left(6g_4\mu_2^E + g_2'\right) + \beta_2\left(m^2 + \int_{-a}^{a} R^3_{\Sigma,m}(y)\rho(y)\,dy\right)$$

or solving for μ_2^E

$$\mu_2^E = \frac{5a^6 g_4 + 3a^4 g_2' - \frac{1}{2}(\beta+\beta_2)a^2 + \beta_2\left(m^2 + \int_{-a}^{a} R^3_{\Sigma,m}(y)\rho(y)\,dy\right)}{\beta + \beta_2 - 18a^4 g_4}.$$

This expression is not terribly useful since it expresses μ_2^E in terms of a and ρ which can also be done much easier by

$$\mu_2^E = \int_{-a}^{a} x^2\rho(x)\,dx.$$

At first glance, the first expression might be useful if $m = 0$. Since then it gives us μ_2^E purely in terms of a and the coupling constants. This has the beneficial effect that instead of searching for a pair (a, μ_2^E) that is self-consistent (so μ_2^E is indeed the second moment of ρ) and satisfies the normalization condition we can let the computer search for an a that satisfies the normalization condition.

However, it turns out that it is still easier to use the equation defining the second moment even in the zero mass case. Since then we get

$$\begin{aligned}
\mu_2^E &= \int_{-a}^{a} x^2 \frac{2}{\beta+\beta_2}\frac{\sqrt{s}_+(x)}{\pi i}p(x)\,dx, \\
&= \frac{2}{(\beta+\beta_2)}\lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_{\gamma_{\Sigma,\varepsilon}} z^2\sqrt{s}(z)p(z)\,dz, \\
&= \frac{2}{(\beta+\beta_2)}\lim_{R\to\infty}\frac{1}{2\pi i}\int_{\gamma_{R'}} z^2\sqrt{s}(z)p(z)\,dz, \\
&= -\frac{2}{\beta+\beta_2}\left[z^2\sqrt{s}(z)p(z)\right]_{-1},
\end{aligned}$$

where $\gamma_{R'}$ it the clockwise circle contour of radius R. This gives

$$\mu_2^E = -\frac{2}{\beta+\beta_2}\left(-\frac{1}{2}g_4a^6 - a^4\left(\frac{1}{2}g_4a^2 + 3g_4\mu_2^E + \frac{1}{2}g_2\right)\right)$$

or, when solved for μ_2^E,

$$\mu_2^E = \frac{2g_4a^6 + g_2a^4}{\beta+\beta_2 - 6g_4a^4}. \tag{4.12}$$

Two intervals

We proceed similarly to the single interval case above. For the polynomial p we get, now using Lemma 3.4.7,

$$\frac{\mathcal{U}'(z)}{\sqrt{s}(z)} = \left(8g_4z^3 + (24g_4\mu_2^E + 4g_2')z - \frac{\beta_2}{2}\int_\Sigma \frac{z-y}{(z-y)^2+m^2}\rho(y)\,dy\right) \times \left(\frac{1}{z^2} + \frac{a^2+b^2}{2}\frac{1}{z^4} + \frac{3a^4+2a^2b^2+3b^4}{8}\frac{1}{z^6} + \dots\right).$$

So p, the positive part of this Laurent series, is

$$p(z) = 8g_4z. \tag{4.13}$$

This time the conditions from Proposition 3.4.10 that are not automatic by symmetry are the order zero condition and the order two, or normalization, condition.

$$0 = \int_\Sigma \frac{\mathcal{U}'(x)}{\sqrt{s}_+(x)}\,dx,$$
$$\frac{\beta\pi}{2i} = \int_\Sigma \frac{\mathcal{U}'(x)}{\sqrt{s}_+(x)}x^2\,dx.$$

Using again Lemma 4.3.8, the order zero condition becomes

$$0 = \beta_2\int_\Sigma R^0_{\Sigma,m}(y)\rho(y)\,dy + 2\left[\frac{\mathcal{U}'(z)}{\sqrt{s}(z)}\right]_{-1}.$$

The Laurent coefficient can be read off again from the computation for $p(z)$ and gives

$$0 = \beta_2\int_\Sigma R^0_{\Sigma,m}(y)\rho(y)\,dy + 8g_4(a^2+b^2) + (48g_4\mu_2^E + 8g_2').$$

The normalization condition is similarly given by

$$\beta = \beta_2\int_\Sigma R^2_{\Sigma,m}(y)\rho(y)\,dy + 2\left[\frac{\mathcal{U}'(z)}{\sqrt{s}(z)}\right]_{-3}$$

and

$$2\left[\frac{\mathcal{U}'(z)}{\sqrt{s}(z)}\right]_{-3} = 2g_4\left(3a^4+2a^2b^2+3b^4\right) + (24g_4\mu_2^E + 4g_2')(a^2+b^2) - \beta_2.$$

Gathering everything up again we get the conditions

$$0 = \beta_2 \int_\Sigma R^0_{\Sigma,m}(y)\rho(y)\,dy + 4g_4(a^2+b^2) + (24g_4\mu_2^E + 4g_2'), \tag{4.14a}$$

$$\beta + \beta_2\left(1 - \int_\Sigma R^2_{\Sigma,m}(y)\rho(y)\,dy\right) = 2g_4\left(3a^4 + 2a^2b^2 + 3b^4\right) + (24g_4\mu_2^E + 4g_2')(a^2+b^2). \tag{4.14b}$$

We can again express μ_2^E in terms of a, b and ρ using Proposition 3.4.10, but it turns out we do not need to do this. Since in the two interval case $p(x)$, and thus ρ, does not depend on μ_2^E we can simply compute ρ from a choice of a and b and then compute the corresponding μ_2^E by

$$\mu_2^E = \int_\Sigma x^2\rho(x)\,dx.$$

We do however for future reference compute this equation in the $m = 0$ case, where we get the same simplification as for the single interval case.

$$\begin{aligned}\mu_2^E &= -\frac{2}{\beta+\beta_2}\left[z^2\sqrt{s}(z)p(z)\right]_{-1},\\ &= -\frac{16g_4}{\beta+\beta_2}\left[\sqrt{s}(z)\right]_{-4},\\ &= \frac{g_4}{\beta+\beta_2}\left(a^6 - a^4b^2 - a^2b^4 + b^6\right).\end{aligned}$$

Which, rewritten slightly with future work in mind, is

$$\mu_2^E = \frac{g_4}{\beta+\beta_2}\left(a^2+b^2\right)\left(a^2-b^2\right)^2. \tag{4.15}$$

Three or more intervals

For the quartic action the support of ρ cannot consist of three or more intervals, at least in the $m = 0$ case. There are two ways to conclude this.

The first, more intuitive, explanation is that a quartic polynomial with positive leading coefficient has at most two minima. The eigenvalue model will be in the single interval phase if the polynomial $g_4x^4 + g_2x^2$ has one minimum, or when the barrier separating the two minima is small enough compared to the eigenvalue repulsion. It will transition to the two interval phase when the barrier separating the minima becomes too large.

The second explanation relies on counting the degrees of the various polynomials involved in ρ. If the potential of D has degree $2n$, $\mathcal{U}'$ will be degree $2n-1$. So the degree of p will be $2n-1-r$. On the other hand, the sign of $\sqrt{s}_+(x)$ changes across each gap between intervals, so in order for $\rho(x) \propto \sqrt{s}_+(x)p(x)$ to be positive we need $p(x)$ to change sign $r-1$ times as well. So we need $2n-1-r \geq r-1$, or $n \geq r$.

For an eigenvalue model with a fermion of finite mass the same should hold, but we do not yet have a formal proof. The intuitive explanation carries over, although we expect the phase transition to occur for a lower barrier (so less negative g_2') since the eigenvalue repulsion caused by the fermion is relatively weaker in the presence of a mass. The more

precise argument fails for m small, since now $\rho(x) \propto (1-K)\sqrt{s}_+(x)p(x)$ where K is the integral operator with kernel $K_{\Sigma,m}(x,y)$. For m large enough (depending on Σ) the operator K will have norm less than 1, so that the degree counting argument does carry over.

All tools we used for the quartic model can also be applied to higher degree models, but the quartic already exhibits a phase transition and thus suffices for the goals of this project.

4.4 Results of the spectral density

We will start this section with some general observations about Equation 4.9, followed by an analysis of the $m = 0$ models and then numerical results for finite mass. In particular we find the phase transition in the $m = 0$ quartic Dirac ensemble using the same techniques as in [39, 20], to verify our results.

Before we start let us gather the results from Section 4.3.2 in one convenient spot.

Single interval problem:

$$\rho(x) = \frac{2}{\beta}\frac{\sqrt{a^2-x^2}}{\pi}\left(8g_4x^2 + \left(4g_4a^2 + 24g_4\mu_2^E + 4g_2'\right)\right) + \frac{\beta_2}{\beta}\int_\Sigma K_{\Sigma,m}(x,y)\rho(y)\,dy, \tag{4.16a}$$

$$\beta + \beta_2\left(1 - \int_\Sigma R^1_{\Sigma,m}(y)\rho(y)\,dy\right) = 6g_4a^4 + \left(24g_4\mu_2^E + 4g_2'\right)a^2, \tag{4.16b}$$

$$\mu_2^E = \int_\Sigma x^2\rho(x)\,dx, \tag{4.16c}$$

Two interval problem:

$$\rho(x) = \frac{2}{\beta}\frac{\sqrt{(x^2-a^2)(b^2-x^2)}}{\pi}8g_4|x| + \frac{\beta_2}{\beta}\int_\Sigma K_{\Sigma,m}(x,y)\rho(y)\,dy, \tag{4.17a}$$

$$0 = \beta_2\int_\Sigma R^0_{\Sigma,m}(y)\rho(y)\,dy + 8g_4\left(a^2+b^2\right) + \left(48g_4\mu_2^E + 8g_2'\right), \tag{4.17b}$$

$$\beta + \beta_2\left(1 - \int_\Sigma R^2_{\Sigma,m}(y)\rho(y)\,dy\right) = 2g_4\left(3a^4 + 2a^2b^2 + 3b^4\right) + \left(24g_4\mu_2^E + 4g_2'\right)\left(a^2+b^2\right), \tag{4.17c}$$

$$\mu_2^E = \int_\Sigma x^2\rho(x)\,dx, \tag{4.17d}$$

where

$$K_{\Sigma,m}(x,y) = \frac{\sqrt{s}_+(x)}{\pi i}\Re\left(\frac{1}{\sqrt{s}(y+im)}\frac{1}{y+im-x}\right)$$

and

$$R^n_{\Sigma,m}(y) = \Re\left(\frac{(y+im)^n}{\sqrt{s}(y+im)}\right).$$

Here we have also used Lemmas 3.4.6, 3.4.7 to evaluate $\sqrt{s}_+(x)$ in the equations for ρ.

Next we can see that this equation is consistent with an even solution. By the properties of $\sqrt{s}$ from Lemma 3.4.5 we get

$$\begin{aligned}
K_{\Sigma,m}(-x,-y) &= \frac{\sqrt{s}_+(-x)}{\pi i}\Re\left(\frac{1}{\sqrt{s}(-y+im)}\frac{1}{-y+im+x}\right),\\
&= (-1)^{r+1}\frac{\sqrt{s}_+(x)}{\pi i}\Re\left(\frac{(-1)^r}{\sqrt{s}(y-im)}\frac{-1}{y-im-x}\right),\\
&= \frac{\sqrt{s}_+(x)}{\pi i}\Re\left(\frac{1}{\sqrt{s}\left(\overline{y+im}\right)}\frac{1}{\overline{y+im-x}}\right),\\
&= \frac{\sqrt{s}_+(x)}{\pi i}\Re\left(\frac{1}{\overline{\sqrt{s}(y+im)}}\frac{1}{\overline{y+im-x}}\right),\\
&= \frac{\sqrt{s}_+(x)}{\pi i}\Re\left(\frac{1}{\sqrt{s}(y+im)}\frac{1}{y+im-x}\right) = K_{\Sigma,m}(x,y).
\end{aligned}$$

Moreover, if $r=1$ (or odd in general) $p(x)$ and $\sqrt{s}_+(x)$ are both even, while if $r=2$ (or even in general) both are odd, so that their product is still even.

We also note that the $m=0$ case agrees with the $m\to 0$ limit, as for $m\to 0$ we get

$$\begin{aligned}
\lim_{m\to 0^+} K_{\Sigma,m}(x,y) &= \lim_{m\to 0^+}\frac{\sqrt{s}_+(x)}{\pi i}\Re\left(\frac{1}{\sqrt{s}(y+im)}\frac{1}{y+im-x}\right),\\
&= \lim_{m\to 0^+}\frac{1}{\pi}\Im\left(\frac{\sqrt{s}_+(x)}{\sqrt{s}(y+im)}\frac{1}{y+im-x}\right),\\
&= \frac{1}{\pi}\lim_{m\to 0^+}\Im\left(\frac{1}{y+im-x}\right),\\
&= \frac{1}{\pi}\lim_{m\to 0^+}\Im\left(\frac{y-im-x}{(y-x)^2+m^2}\right),\\
&= -\delta(y-x).
\end{aligned}$$

So the $K_{\Sigma,m}$ integral becomes $\frac{\beta_2}{\beta}\rho(x)$ which has the effect of changing β to $\beta+\beta_2$. The functions $R^n_{\Sigma,m}(y)$ converge to 0 as m goes to zero, since $\sqrt{s}(y+im)$ converges to the purely imaginary $\sqrt{s}_+(y)$ while $(y+im)^n$ converges to y^n, so the support conditions also behave nicely in the limit.

In the limit $m\to\infty$ the kernel $K_{\Sigma,m}$ converges to 0 while the functions $R^n_{\Sigma,m}$ converge to 1 by the asymptotic properties of $\sqrt{s}$ from Lemma 3.4.5. So this limit appears to have the effect of simply setting $\beta_2=0$, however one should note that $g_2' = g_2+2g_4m^2$ so that the potential for H becomes infinitely confining. By Lemma 4.2.3 this means that the spectrum of D would become two δ peaks at $\pm m$. If one instead adjusts g_2 such that g_2' remains constant, the $m\to\infty$ limit reproduces the purely bosonic $\beta_2=0$ models.

4.4.1 The $m = 0$ case

The zero mass case equations can be obtained from Equations 4.16, 4.17 by first setting $\beta_2 = 0$ and then replacing β by $\beta + \beta_2$. In the Gaussian case, $g_4 = 0$, $g_2 > 0$, the model is well-known to give the Wigner semi-circle law for ρ [40], and this indeed follows for us as well with the system becoming

$$\begin{aligned} \rho(x) &= \frac{8g_2}{\beta+\beta_2}\frac{\sqrt{a^2-x^2}}{\pi}, \\ \beta+\beta_2 &= 4g_2a^2. \end{aligned}$$

This system is easily checked to indeed be normalized and has a valid solution for all $g_2 > 0$, so as expected there is no phase transition in the Gaussian model.

The quartic problem for $m = 0$ is given by

$$\begin{aligned} \rho(x) &= \frac{2}{\beta+\beta_2}\frac{\sqrt{a^2-x^2}}{\pi}\left(8g_4x^2 + \left(4g_4a^2 + 24g_4\mu_2^E + 4g_2\right)\right), \\ \beta+\beta_2 &= 6g_4a^4 + \left(24g_4\mu_2^E + 4g_2\right)a^2, \\ \mu_2^E &= \frac{2g_4a^6 + g_2a^4}{\beta+\beta_2 - 6g_4a^4}, \end{aligned}$$

for the solution with $\Sigma = [-a, a]$ and

$$\begin{aligned} \rho(x) &= \frac{2}{\beta+\beta_2}\frac{\sqrt{(x^2-a^2)(b^2-x^2)}}{\pi}8g_4|x|, \\ 0 &= 8g_4\left(a^2+b^2\right) + \left(48g_4\mu_2^E + 8g_2\right), \\ \beta+\beta_2 &= 2g_4\left(3a^4+2a^2b^2+3b^4\right) + \left(24g_4\mu_2^E + 4g_2\right)\left(a^2+b^2\right), \\ \mu_2^E &= \frac{g_4}{\beta+\beta_2}\left(a^2+b^2\right)\left(a^2-b^2\right)^2, \end{aligned}$$

for $\Sigma = [-b, -a] \cup [a, b]$, using Equations 4.12, 4.15 for the consistency conditions on μ_2^E.

Our main goal with this section is to locate the transition between these two phases. With this in mind let ρ_1 denote solutions to the single interval problem and ρ_2 solutions to two interval problems. There are several ways to find this, the first is to identify when $\rho_1(x)$ starts to fail to be a positive function. This is done by finding the coupling constants for which $\rho_1(0) = 0$. A second way is by solving the support conditions for $\Sigma = [-b, -a] \cup [a, b]$ and finding for which values of the coupling constants this solution is valid, in the sense that $0 < a < b$.

This second way turns out to be the more convenient, although it is not a priori clear that it finds the precise phase transition. We can simply check this afterwards. To simplify notation while solving the system of support and consistency equations, let us introduce some short hand:

$$x = a^2 + b^2, \qquad y = a^2 - b^2, \qquad z = \mu_2^E, \qquad G_{2,4} = \frac{g_{2,4}}{\beta+\beta_2}.$$

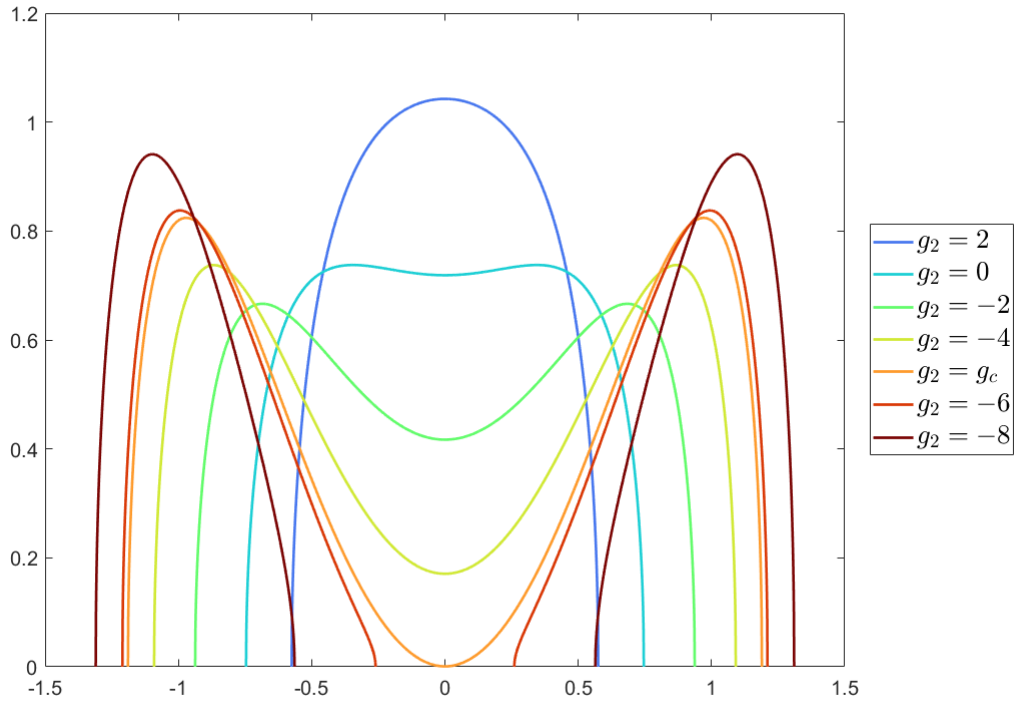


Figure 4.2: The spectral density of the matrix H in the model with $m = 0$, $\beta = 2$, $\beta_2 = 2$, $g_4 = 1$ for varying values of g_2, here $g_c = -4\sqrt{2}$ is the phase transition for the model with the given values for the other coupling constants.

Thus we want to solve

$$\begin{cases} 0 = 4G_4x + 24G_4z + 4G_2, \\ 1 = 2G_4\left(2x^2 + y^2\right) + 24G_4xz + 4G_2x, \\ z = G_4xy^2. \end{cases}$$

This system is solvable through some straightforward algebra that we will not repeat here, and one finds that the unique solution is given by

$$x = -\frac{1}{4}\frac{G_2}{G_4}, \qquad y = -\frac{1}{\sqrt{2G_4}}, \qquad z = -\frac{1}{8}\frac{G_2}{G_4}.$$

Returning to our actual variables, this means

$$a^2 = -\frac{1}{8}\frac{g_2}{g_4} - \frac{1}{2}\sqrt{\frac{\beta+\beta_2}{2g_4}}, \qquad b^2 = -\frac{1}{8}\frac{g_2}{g_4} + \frac{1}{2}\sqrt{\frac{\beta+\beta_2}{2g_4}}, \qquad \mu_2^E = -\frac{1}{8}\frac{g_2}{g_4}.$$

which is a valid solution if

$$g_2 \leq -2\sqrt{2}\sqrt{g_4(\beta+\beta_2)}. \tag{4.18}$$

This suspected location for the phase transition is in agreement with the phase transition found analytically and through Monte-Carlo simulations in [20], with the observation

that they have $\beta = 2$, $\beta_2 = 0$ and that their coupling constants $g_{i,b}$ are $g_{i,b} = 2g_{i,f}$ in terms of our coupling constants $g_{i,f}$. The difference in coupling constants is caused by the dimension increase from the addition of fermions. Note that this is not in agreement with [39] due to some minor calculation errors in *loc. cit.*.

We can check this location for the phase transition with our first approach by verifying that if $g_2 = -2\sqrt{2g_4(\beta+\beta_2)}$, $a^2 = \sqrt{\frac{\beta+\beta_2}{g_4}}$ and $\mu_2^E = -\frac{1}{8}\frac{g_2}{g_4}$ solves the system for the one-interval density ρ_1 and has $\rho_1(0) = 0$. This is again a straightforward bit of algebra that we will not repeat here. Finally observing that $\rho_1(0)$ is decreasing in g_2 confirms the location of the phase transition.

We can find the spectral density of H for any set of coupling constants β, β_2, g_2, g_4 if $m = 0$ by solving the above system. While it is possible to solve the corresponding systems exactly the graphs in this section have been generated using numerical approximations to test the systems necessary for the $m \neq 0$ case.

Let us discuss the effect of the various coupling constants, this discussion is based on two effects that together describe the main phenomena visible in the spectral density. The first major coupling constant to vary is g_2, as g_2 becomes negative the potential $g_4x^4 + g_2x^2$ takes on the shape of a double well. The eigenvalues will tend to be found

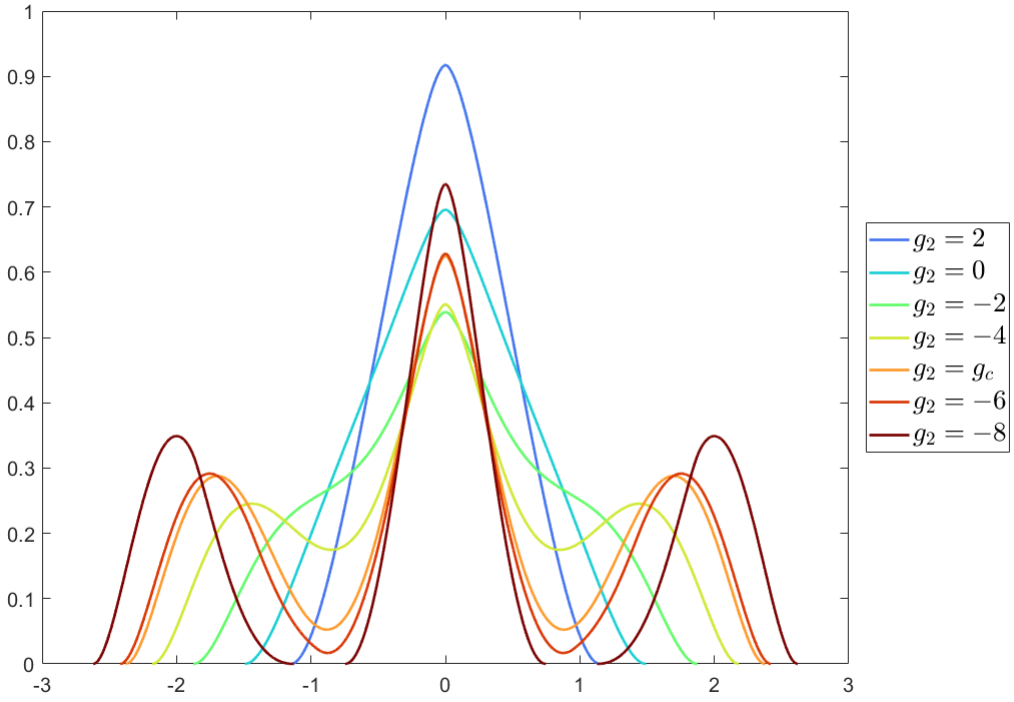


Figure 4.3: The spectral density of D in the model with $m = 0$, $\beta = 2$, $\beta_2 = 2$, $g_4 = 1$ for varying values of g_2, here $g_c = -4\sqrt{2}$ is the phase transition of H for the model with the given values for the other coupling constants. The colours correspond to those in Figure 4.2.

near the minima of these wells, this is not obvious since the potential for H is more complicated than the potential for D, but the symmetric solution causes the potential for H to be approximately quartic as well.

This effect is countered by the eigenvalue repulsion inherent in unitarily invariant matrix ensembles. The strength of this is governed by the Dyson exponent β, and in our model strengthened by the fermionic action, governed by the coupling constant β_2. The eigenvalue repulsion prevents all eigenvalues from sitting at the minima of the potential. The phase transition computed above corresponds to the balancing point where the repulsive force between the eigenvalues becomes strong enough to push eigenvalues above the barrier separating the wells of the $g_4x^4 + g_2x^2$ potential.

These effects can be seen in Figures 4.2, 4.4 and 4.5. In Figure 4.2 we vary g_2. Initially, for $g_2 = 2$ there is no double well and all eigenvalues are clustered around zero. For $g_2 = 0$ we see the inaccuracy caused by the difference between the purely quartic potential for D and the effective potential for H as there is already a hint of separation for the eigenvalues. As g_2 decreases the double well nature becomes more obvious until for $g_2 < -4\sqrt{2}$ the wells are too deep for the repulsive force.

Given the spectral density of H we can also find the spectral density of D using Lemma 4.2.3. The spectral densities of D for various values of the coupling constants can be found in Figure 4.3.

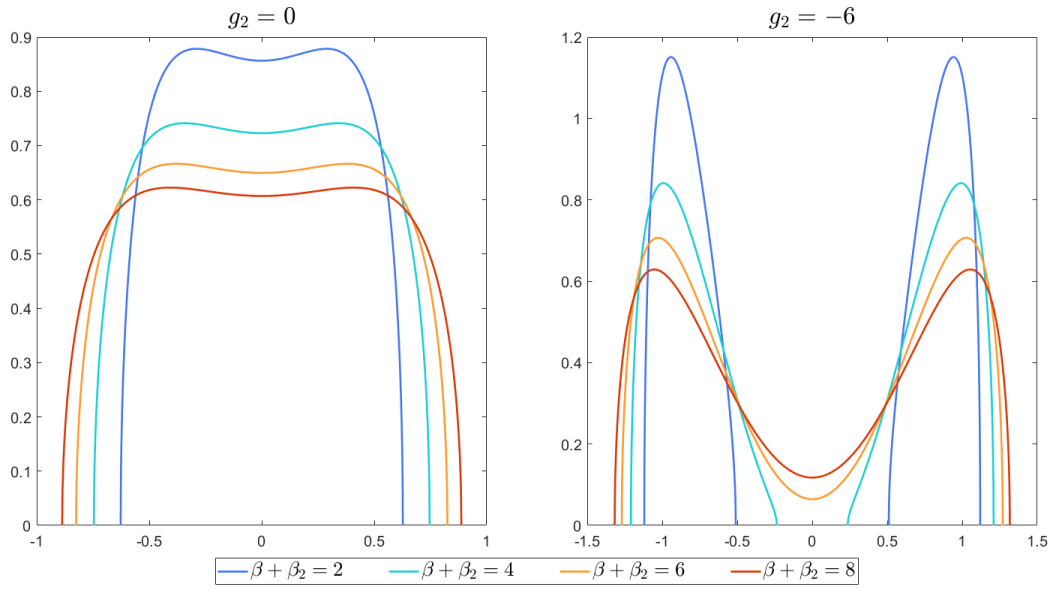


Figure 4.4: The spectral density of the matrix H in the model with varying $\beta+\beta_2$, $m = 0$, $g_4 = 1$ and in the left plot $g_2 = 0$ and $g_2 = -6$ in the right plot. The effect of raising $\beta + \beta_2$ is to strengthen the repulsive force in the spectrum, so it will tend to spread out the spectral density. The phase transition in terms of β_2 for the $g_2 = -6$ model happens at $\beta_2 = \frac{5}{2}$, so when $\beta + \beta_2 = \frac{9}{2}$ and can be seen in the right graph.

There are some observations that can be made based on Figure 4.3 for the spectral density of D. The spectral density for H generally consists of two peaks, they may overlap almost completely ($g_2 = 2$), overlap partially ($g_2 = 0, -2, -4$) or be disjoint

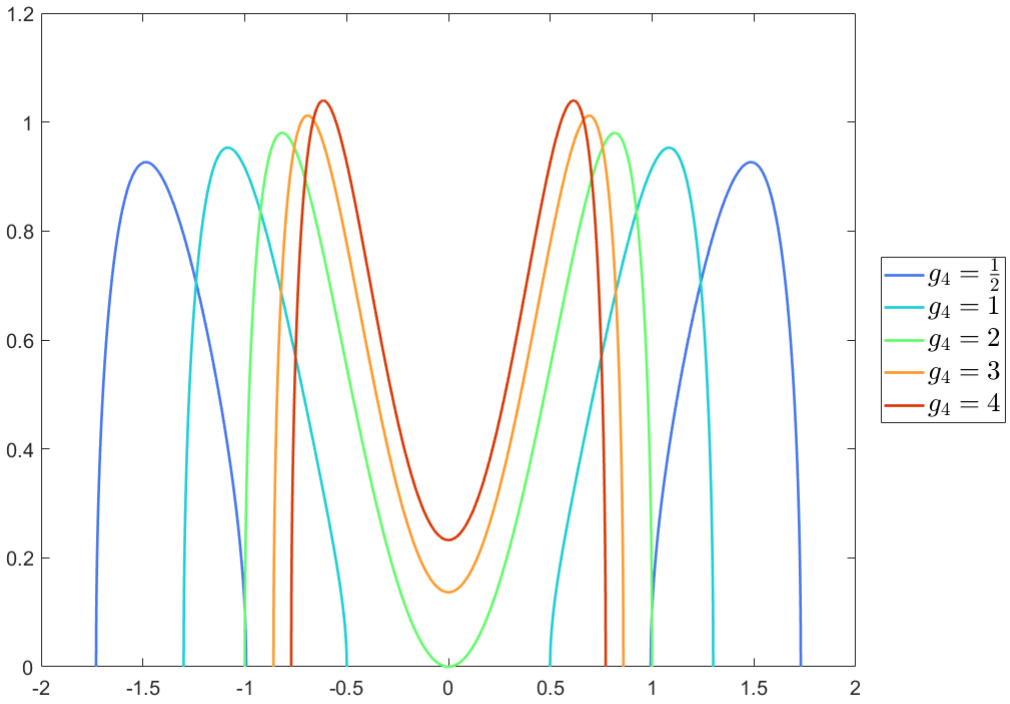


Figure 4.5: The spectral density of the matrix H in the model with varying g_4 and $m = 0$, $\beta = 2$, $\beta_2 = 2$, $g_2 = -8$. The effect of increasing g_4 is, as expected, to confine the spectral density. In this case the phase transition occurs at $g_4 = 2$, with the connected phase for larger values of g_4.

($g_2 < g_c$). Correspondingly, the spectrum of D consists of three peaks, with the center peak corresponding to differences of eigenvalues within one peak of H and the two outer peaks consisting of inter-H-peak differences. These peaks also show various degrees of overlap, from complete overlap ($g_2 = 2, 0$) to less overlap ($g_2 = -2, -4, g_c, -6$) to three disjoint peaks ($g_2 = -8$). It is interesting to note that these changes, which might also warrant the name phase transition, occur at different places than the phase transition for the spectral density of H. These "phase transitions" for D have not yet been located analytically.

In Figure 4.4 we investigate the opposite effect where we adjust the repulsive force rather than the depth of the wells. For $g_2 = 0$ we see that strengthening the repulsive force, increasing $\beta + \beta_2$, spreads out the eigenvalues as expected. For $g_2 = -6$ we see that the eigenvalues are pushed apart and that for $\beta + \beta_2 = 6$ this effect is strong enough to once again overcome the barrier between the wells.

Figure 4.5 shows the effect of varying g_4. As expected a stronger g_4 is more confining for the eigenvalues. This confining effect can, as Equation 4.18 shows, cause a phase transition by reducing the size of the wells to the point where the repulsive force can once again overpower the barrier.

4.4.2 Numerical analysis for finite mass

We will now present the results of a series of numerical approximations of the spectral densities of H and D for finite masses. Before we get to the results let us briefly describe the process by which these results were obtained.

We use a Riemann-sum-esque approximation for our functions, we choose some resolution Δx and consider a vector $\vec{x} = (x_1, \ldots, x_M)$ of points in Σ separated by Δx. A function f is then replaced by the vector $\vec{f} = (f(x_1), \ldots, f(x_M))$ and the integral kernel $K_{\Sigma,m}(x, y)$ becomes the matrix $K = (K(x_i, x_j))_{i,j=1}^{M}$. This reduces the Fredholm integral Equation 4.9 to a linear equation

$$\left(1 - \frac{\beta_2}{\beta} K \Delta x\right) \vec{\rho} = \frac{2}{\beta} \frac{\vec{\sqrt{s}}_+}{\pi i} \odot \vec{\rho} \tag{4.19}$$

where $\odot$ is component-wise multiplication of vectors. The support conditions for the single interval problem become

$$\beta + \beta_2 \left(1 - \vec{R}^1_{\Sigma,m} \cdot \vec{\rho} \Delta x\right) = 6 g_4 a^4 + \left(24 g_4 \mu_2^E + 4 g_2'\right) a^2, \tag{4.20a}$$

$$\mu_2^E = (\vec{x} \odot \vec{x}) \cdot \vec{\rho} \Delta x, \tag{4.20b}$$

where $\cdot$ is the dot product of vectors. The conditions for the two interval problem can be encoded similarly.

Given a choice of a (and b, in the two interval case) and μ_2^E a programming language like MATLAB has little trouble computing $\vec{\rho}$ from Equation 4.19 as well as both sides of the system in Equation 4.20. We were unfortunately unable to reliably have the computer find values of a, b and/or μ_2^E that make the support conditions close to correct.

Instead we have landed on the following procedure to establish numerics, differing slightly between the one and two interval cases. In the two interval case we have ostensibly three parameters we need to guess, a, b and μ_2^E. However, looking at Equation 4.17a we note that ρ is unaffected by μ_2^E so that the second moment only features in the support conditions. Hence we can, given a choice of a and b, compute ρ, compute μ_2^E and check how far the support conditions are from being true. This gives us a two parameter search space.

Starting from a manually estimated range for a and b we generate the graphs seen in Figure 4.6. We use two normalization conditions, one coming from the general theory as in Equation 4.17c and one corresponding to $\int \rho(x)\, dx = 1$ since the latter turns out to be useful and is easily computed, and the "moment zero" condition corresponding to Equation 4.17b. In the fourth graph of Figure 4.6 we see the three, approximate, contours where each individual support condition is true. The actual values of a and b for the given coupling constants then correspond to the intersection of these contours.

These values can be read off from the graph and then manually further optimized. The goal that turned out to be feasible in general is to get the difference between the sides of the support conditions to be on the order of 0.01 to 0.001 while both sides of the equations tend to be on the order of 1 to 10.

For the single interval case the procedure is very similar, but with a and μ_2^E as parameters instead of a and b. The place of the third graph in Figure 4.6 is then taken

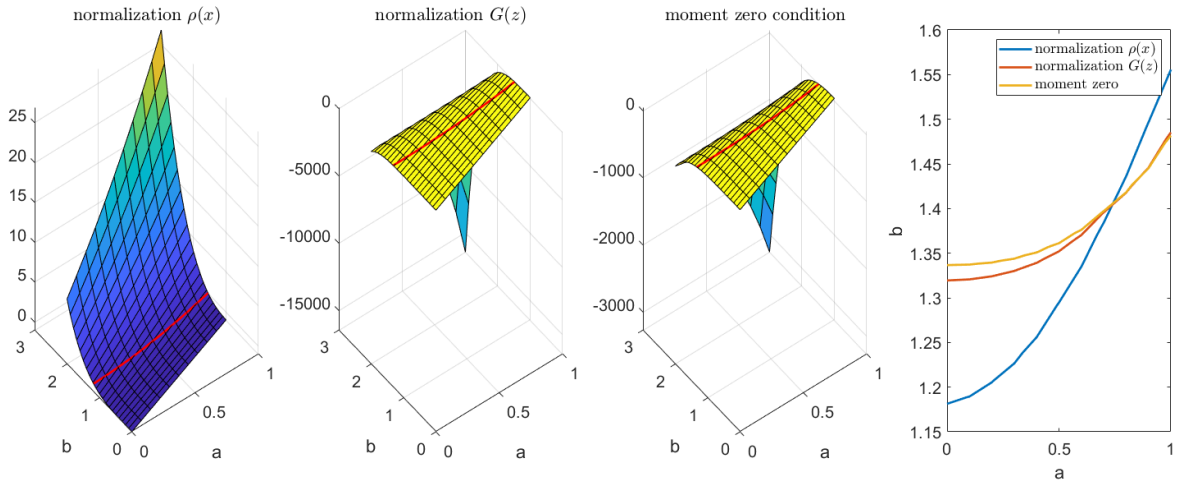


Figure 4.6: Output of the MATLAB program used to estimate the correct values of a and b for a particular model. The first graph shows the value of $\Sigma(\vec{\rho}\Delta x) - 1$, the second graph shows the value of the left-hand side minus the right-hand side of the normalization condition in Equation 4.17c. The third graph shows the left-hand side minus the right-hand side of the condition in Equation 4.17b. The red line in the first three graphs is the contour of the surface at 0. These contours are also plotted in the fourth graph, the correct value of (a, b) corresponds to the (approximate) intersection of the three contours.

by the self-consistency condition corresponding to $\mu_2^E = \int x^2 \rho(x)\, dx$. Since in the single interval case μ_2^E affects ρ there is feedback[3] between the chosen parameter value μ_2^E and the actual second moment of ρ. It is worth mentioning that in the single interval case the two normalization conditions coincide almost perfectly in all trials.

Let us now explain the effect the mass has on the spectral densities. There are two main effects that play a role for the spectral density of H. The first effect is the changing of the coupling constant g_2 to $g_2' = g_2 + 2g_4 m^2$. This makes the potential increasingly confining as m increases, so that in the $m \to \infty$ limit the spectral density of H becomes point-like. The second effect is that the mass term lowers the repulsive force due to the fermions. As m grows the eigenvalues remain confined in the same interval (really a slightly smaller interval due to the first effect) so $\lambda_i - \lambda_j$ remains of the same order. Therefore if m grows the term

$$-\frac{\beta_2}{4} \log\left((\lambda_i - \lambda_j)^2 + m^2\right)$$

becomes less and less sensitive to the eigenvalues and more dominated by the mass term.

These effects and their balance can be seen in Figure 4.7. For $m = 0$ the model is exactly at its phase transition, as m starts to increase both effects start to play a role. The increasing g_2' shrinks the total support of the spectral density, but the reducing eigenvalue repulsion starts out dominant and pushes the model to a two-interval phase

[3]This feedback can be computed given a, since changing μ_2^E changes ρ by adding a multiple of $\sqrt{s}_+$. We chose not to do these computations in the interest of efficiency since we could simply reuse the code developed for the two interval case.

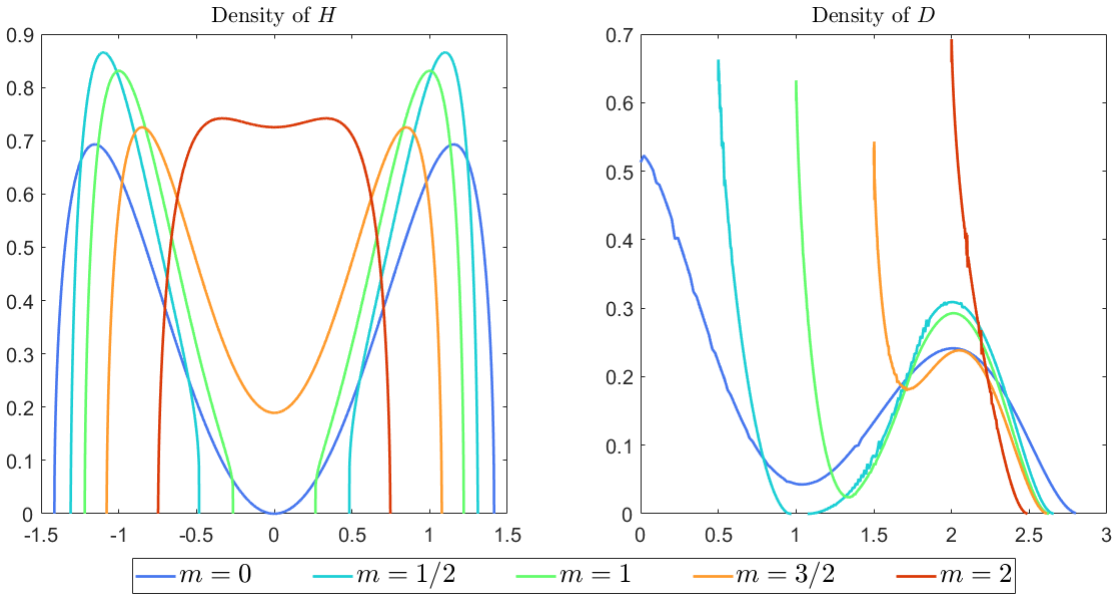


Figure 4.7: The spectral densities of H and D for the model with $\beta = 2$, $\beta_2 = 2$, $g_2 = -4$, $g_4 = \frac{1}{2}$ and varying masses. The spectral density is only shown for $x > 0$ since it is symmetric. The irregularities in the spectral density of D are numerical artefacts.

for $m = \frac{1}{2}$ and $m = 1$. As m continues to increase the effect of g_2' becomes ever stronger while the relative effect of the reduced eigenvalue repulsion becomes less, so the model transitions back to a single interval phase. The exact point at which this occurs might be interesting to determine, and from trial and error appears to be just below $m = 1.1$.

This shifting of the phase transition due to reduced eigenvalue repulsion is further shown in Figure 4.8. The phase transition would be expected at $g_2 = -4$ in the massless case, but for $m = 1$ the model is still in the two interval phase at $g_2 = -4$. Besides this there is very little evident impact of the mass when compared to Figures 4.2 and 4.3 besided the shifting of the Dirac spectrum.

In Figure 4.9 we look at the effect of the mass if g_2' is kept constant, so g_2 is adjusted with the mass. As expected the behaviour can be explained entirely by the reduction of eigenvalue repulsion, with the spectral density shrinking into the wells of the potential, one can also observe that the effect of the mass on the eigenvalue repulsion diminishes quickly as m grows. If m would continue to become larger beyond $m = 5$ the system would not reach a phase transition since $g_2' = -3 > -4$ which is the location of the phase transition for the model with $g_2 = -3$, $g_4 = 1$, $\beta = 2$ and $\beta_2 = 0$. As m increases only the eigenvalue repulsion due to β_2 shrinks, so the base repulsion of β remains.

From both Figures 4.7 and 4.9 the effect of the mass on the spectral density of D can be seen. By Lemma 4.2.3 the spectral density is only non-zero outside of $[-m, m]$. Outside of that the spectral density is essentially squished as the differences of the eigenvalues of H only shrink and the m term in $\sqrt{(\lambda_i - \lambda_j)^2 + m^2}$ becomes dominant.

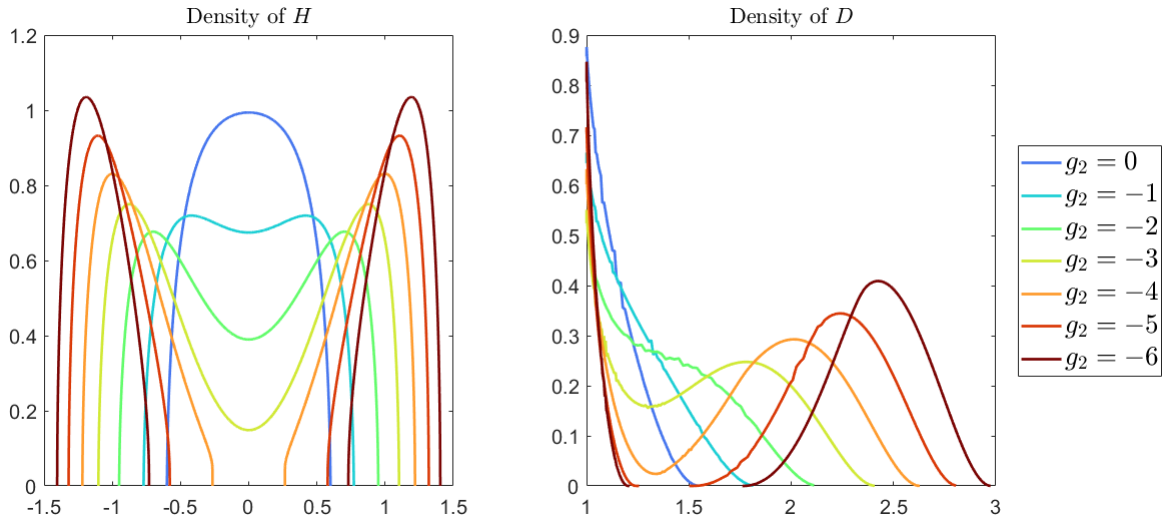


Figure 4.8: The spectral densities for H and D for the model with $m = 1$, $\beta = 2$, $\beta_2 = 2$, $g_4 = \frac{1}{2}$ and varying g_2. For the spectral density of D only the part greater than $m = 1$ is plotted, the density is zero on $[-1, 1]$ and symmetric. In the massless model the phase transition for H should occur at $g_2 = -4$. The minor irregulirities in the densities of D are numerical artefacts.

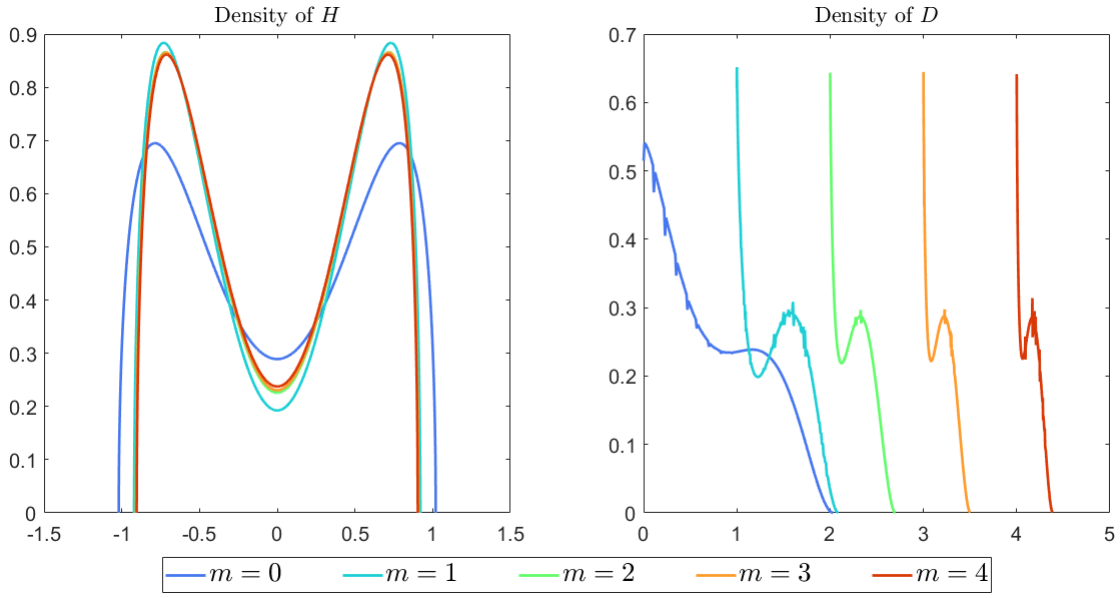


Figure 4.9: The spectral densities of H and D for the model with $\beta = 2$, $\beta_2 = 2$, $g_4 = 1$ and $g_2' = -3$ for varying masses m. Note that we are keeping $g_2' = g_2 + 2g_4m^2$ constant. The spectral density of D is only shown for $x > 0$ since it is symmetric. The density is 0 on the interval $[-m, m]$ by Lemma 4.2.3.

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