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Characteristic Polynomial of Arrangements and Multiarrangements

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Abstract

This thesis is on algebraic and algebraic geometry aspects of complex hyperplane arrangements and multiarrangements. We start by examining the basic properties of the logarithmic modules of all orders such as their freeness, the cdga structure, the local properties and close the first chapter with a multiarrangement version of a theorem due to M. Mustaţă and H. Schenck.

In the next chapter, we obtain long exact sequences of the logarithmic modules of an arrangement and its deletion-restriction under the tame conditions. We observe how the tame conditions transfer between an arrangement and its deletion-restriction.

In chapter 3, we use some tools from the intersection theory and show that the intersection cycle of a certain projective variety has a closed answer in terms of the characteristic polynomial. This result is used to compute the leading parts of the Hilbert polynomial and Hilbert series of the logarithmic ideal. As a consequence, we recover some of the classical results of the theory such as the Solomon-Terao formula for tame arrangements. This is done by computing the Hilbert series in two different ways. We also introduce the notion of logarithmic Orlik-Terao ideal and show that the intersection lattice parametrizes a primary decomposition. The chapter is closed by a generalization of logarithmic ideals to higher orders. It is shown that these ideals detect the freeness of the corresponding logarithmic modules.

The last chapter is a generalization of the notion of logarithmic ideal to multiarrangements. Some of the basic properties of these ideals are investigated. It is shown that one obtains a natural resolution of this ideal by logarithmic modules under the tame condition. In the final section it is shown that the intersection cycle of the logarithmic ideal of a free multiarrangement is obtained from its characteristic polynomial, similar to simple arrangements.

Keywords: arrangement/multiarrangement, derivation module, logarithmic ideal, characteristic polynomial, Tutte polynomial, intersection cycle, Chow ring.

Co-Authorship

Chapter three is a joint work with my supervisor, Dr. Graham Denham, and Dr. Mathias Schulze. I made contributions to the geometric deletion-restriction formula which leads to the computation of the intersection cycle of the logarithmic variety. In particular, I gave a formulation of the main formula of Section 3.2 by giving an algebraic defining ideal for the component of the restriction. The appendix to chapter three is independent from this joint work.

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To my parents and my brothers

Table of Contents

List of Notations

Arrangements		
$(\mathcal{A}, \mathbf{m}), \mathcal{A}$	(multi)arrangement of hyperplanes	1
$|\mathcal{A}|$	the number of hyperplanes of $\mathcal{A}$	1
$\mathbb{C}[\mathcal{A}]$	complex coordinate ring of $\mathcal{A}$	1
$\exp(\mathcal{A})$	exponents of $\mathcal{A}$	11
$\mathrm{supp}(\mathbf{m})$	support of a multiplicity $\mathbf{m}$	93
Algebra		
$\mathrm{D}(\mathcal{A}), \mathrm{D}_p(\mathcal{A}, \mathbf{m})$	derivation module of $\mathcal{A}$, $(\mathcal{A}, \mathbf{m})$	9, 22
$\Omega(\mathcal{A}), \Omega^p(\mathcal{A}, \mathbf{m})$	module of logarithmic forms of $\mathcal{A}$, $(\mathcal{A}, \mathbf{m})$	10, 22
$\mathrm{Ann}(\mathcal{A})$	annihilator module of $\mathcal{A}$	13
$I(\mathcal{A}), I(\mathcal{A}, \mathbf{m})$	logarithmic ideal of $\mathcal{A}$, $(\mathcal{A}, \mathbf{m})$	57, 90
$\mathrm{depth}\,(N)$	depth of a module N	23
$(I : J)$	ideal quotient (saturation)	85
$h(N)$	Hilbert series of a graded module N	37
$p_N(x)$	Hilbert polynomial of a graded module N	74
$\widetilde{N}$	the coherent sheaf of a graded module N	41
$\mathrm{rad}(I)$	radical of an ideal I	93
$\mathrm{supp}(N)$	support of a module N	99
Combinatorics		
$\chi_{\mathcal{A}}(t)$	characteristic polynomial of $\mathcal{A}$	14
$\pi_{\mathcal{A}}(t)$	Poincaré polynomial of $\mathcal{A}$	15
$L(\mathcal{A})$	intersection lattice of $\mathcal{A}$	2
$T_{\mathcal{A}}(x, y)$	Tutte polynomial of $\mathcal{A}$	40
Geometry		
$\beta_i(X)$	i-the Betti number of a topological space X	14
$M(\mathcal{A})$	complement of $\mathcal{A}$	2
$c_i(F), c_t(F)$	i-th Chern class (Chern polynomial) of a vector bundle F	72, 65
$\mathrm{ch}(F)$	Chern character of a vector bundle F	65
$\mathrm{CH}(X)$	Chow ring of a variety X	64
$[Y]$	intersection cycle of a variety Y	66
$\Sigma(\mathcal{A})$	manifold of critical points of $\mathcal{A}$	56
$\omega_{\mathbf{a}}$	logarithmic 1-form	55
$\mathbb{P}^n$	n-dimensional complex projective space	58
$\mathfrak{X}(\mathcal{A})$	biprojective zero locus of $I(\mathcal{A})$	58

Preface

In this thesis, complex hyperplane arrangements are studied by looking at their logarithmic modules, sheaves and ideals in connection to their combinatorial structure. The content goes beyond what the title might suggest. In fact, this thesis is not directly targeted at studying characteristic polynomials. The current title is acknowledging the ubiquity of this polynomial in different contexts: For instance, the characteristic polynomial captures various data such as the number of chambers in the real picture [45], the Betti numbers of the complex complement [21], the generic number of critical points of the master function [23], Chern polynomial of the sheaf of the logarithmic modules ([20], [13]), the number of points in the complement over a finite field, the volume of the configuration space of polymers [19], the class of the union of the hyperplanes in the Grothendieck ring of varieties [5] and list goes on. Some of the above will be explained in Section 1.3 where the literature background is discussed.

The above list of contexts where the characteristic polynomial appears is extended in this thesis. The most serious contribution, which is a joint work with Graham Denham and Mathias Schulze, is the computation of the intersection cycle of the variety $\mathfrak{X}(\mathcal{A})$ in a product of projective spaces (Theorem 3.3.2). As an application, we obtain a new proof for the Solomon-Terao Formula (1.12) for tame arrangements. This formula is significant for simple arrangements and multiarrangements. It implies the celebrated Factorization Theorem (1.11) and motivates the definition of the characteristic polynomial for multiarrrangements [2]. This also provides a basis for generalizing the result of Mustaţă and Schenck from [20] to locally free multiarrangements (Theorem 1.5.10). This is interesting because except for the free case (Factorization Theorem), we do not have any results for computing the characteristic polynomial of multiarrangements.

In the rest of this section, the highlights of the content of every chapter is described. Chapter 1 is a review of the relevant material from the literature. The notion of multiarrangement has received some emphasis and some minor generalizations of the existing results are presented. In particular, we have a more general form of Saito's Criterion and a formula that relates the Poincaré polynomial of a locally free multiarrangement to a certain Chern polynomial.

In chapter 2, we impose conditions that facilitate relating the logarithmic modules of an arrangement to its deletion and restriction. As a result, we obtain long exact sequences of the logarithmic modules. See Theorems 2.1.5 and 2.2.5. These sequences are used to see how tameness and its dual notion which are homological dimension conditions are transferred between the triple (Corollaries 2.2.6 and 2.1.6).

In the last chapter as well as the last section of chapter 3, we display different generalizations of the notion of logarithmic ideal. In 3.5.1, we obtain an ideal which is something in between the Orlik-Terao ideal and the meromorphic ideal of an arrangement. The primary decomposition of this ideal is interestingly parametrized by the intersection lattice as shown in Theorem 3.5.7. Section 3.5.2 generalizes the logarithmic ideals to higher orders for all $p = 1, \ldots, \ell$, such that the ordinary logarithmic ideal corresponds to $p = 1$. The main result of the section is Theorem 3.5.14.

Chapter 4 is a generalization in the direction of multiarrangements. It is shown that the simple logarithmic ideal always defines one of the components of the primary decomposition (Corollary 4.1.10). It is conjecturally stated that the radical of the logarithmic ideal with multiplicity only depends on the support of the multiplicity (Conjecture 4.1.14). We also show similar results about the resolutions of these ideals and Hilbert series which are generalizations from the theory of simple arrangements. The very last result is an analogous version of the intersection cycle formula for free multiarrangements (Corollary 4.4.2) which is only based on the Factorization Theorem for multiarrangements. It is interesting to note that one can incorporate multiplicities in the definition of the logarithmic ideal in two ways, both of which turn to be natural and useful.

Mehdi Garrouisan
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Chapter 1
Preliminaries

This chapter serves as an introduction to the rest of the thesis. We start by recalling the standard notations and structures. The content is mostly a projection of the existing literature and provides a foundation for working on (multi)arrangements in the forthcoming chapters. We have added some proofs in cases where locating one in the literature was not easy, or a need for some face lift was felt. As general references, the reader may consult [22], [11] and [10]. The reader is advised that sections 1.2 and 1.3 are informal introductions with emphasis on motivating certain constructions and outlining some related directions of research. The rigorous study of the material starts with section 1.4.

1.1 Basic Definitions and Operations

A *complex arrangement of hyperplanes* is a finite collection of codimension one affine subspaces of a complex vector space $V \simeq \mathbb{C}^\ell$. $\mathcal{A}$ is called *central* if all of its hyperplanes pass through the origin otherwise it is an *affine* arrangement. We tend to drop the adjective complex unless emphasis is needed. A *multiarrangement of hyperplanes* is a pair $(\mathcal{A}, \mathbf{m})$, where $\mathcal{A}$ is an arrangement of hyperplanes and $\mathbf{m} : \mathcal{A} \to \mathbb{N}$ is an assignment of multiplicities. In practice, once we fix an order on the hyperplanes $H_1, \ldots, H_n$, we identify $\mathbf{m}$ with the vector $(\mathbf{m}(H_1), \ldots, \mathbf{m}(H_n))$. The size of a multiarrangement $(\mathcal{A}, \mathbf{m})$ is $|(\mathcal{A}, \mathbf{m})| = |\mathbf{m}| = \sum_{H \in \mathcal{A}} \mathbf{m}(H)$. For every hyperplane $H \in \mathcal{A}$, we fix a linear functional $f = c_1 x_1 + \cdots + c_\ell x_\ell \in V^*$, with $\ker f = H$. The *coordinate ring* of $\mathcal{A}$, denoted $\mathbb{C}[\mathcal{A}]$, is the coordinate ring of V as an affine space. The usual setup is that $V = \mathbb{C}^\ell$ and its coordinate ring is $R = \mathbb{C}[x_1, \ldots, x_\ell]$. We identify a simple arrangement $\mathcal{A}$ with the multiarrangement $(\mathcal{A}, \mathbf{1})$, where every hyperplane receives multiplicity one.

One can define an arrangement $\mathcal{A} = \{\ker f_i : 1 \leq i \leq n\}$ by its *defining polynomial* $Q = f_1 \cdots f_n$. If $(\mathcal{A}, \mathbf{m})$ is a multiarrangement, then the defining polynomial is $\widetilde{Q} = \Pi_i f_i^{\mathbf{m}(H_i)}$. This polynomial is well-defined up to a nonzero multiple. The vanishing of Q defines the union of all hyperplanes of $\mathcal{A}$.

A subarrangement $\mathcal{B} \subseteq \mathcal{A}$ is called *dependent* if $\mathrm{codim}(\cap_{H \in \mathcal{B}} H) < |\mathcal{B}|$, where absolute value denotes cardinality. This is equivalent to saying that $\{f_i : \ker f_i \in \mathcal{B}\}$ is a dependent set of vectors in V^*. A subarrangement that is not dependent is called *independent*. In fact, every arrangement has an underlying matroid which carries the combinatorial data. Matroids formalize the concept of dependence in general. See [36, Chapter 1] as a reference on matroids. A *generic* arrangement is one that has dependent subarrangements $\mathcal{B}$ only when $|\mathcal{B}| > \dim V$. The *Boolean arrangement* is defined by $Q = x_1 \cdots x_\ell$ and has the property that $|\mathcal{A}| = \dim V$. This is a direct sum of one dimensional arrangements in the sense that will be made precise later.

A central arrangement is *essential* when it has an independent subarrangement of size $\dim V$. If $\mathcal{A}$ a central arrangement, then we can projectivize it to obtain $\mathbb{P}\mathcal{A} = \{\mathbb{P}H : H \in \mathcal{A}\} \subset \mathbb{P}V$.

Knowing how the hyperplanes cut one another is extremely useful in understanding arrangements. This information is recorded in the *lattice of intersections* $(L(\mathcal{A}), \leq, \wedge, \vee)$ where

$$L(\mathcal{A}) := \{\cap_{H \in \mathcal{B}} H : \mathcal{B} \subseteq \mathcal{A}\}, \tag{1.1}$$

and $\mathcal{B}$ runs over all subarrangements of $\mathcal{A}$, including the empty one. This is a geometric lattice where the elements are ordered by reverse inclusion: $X \leq Y$ if $Y \subseteq X$, for $X, Y \in L(\mathcal{A})$. The lattice operations of *meet* and *join* are defined as follows. For all $X, Y \in L(\mathcal{A})$, we have

$$X \vee Y = X \cap Y, \quad X \wedge Y := \bigcap_{X \cup Y \subseteq Z} Z.$$

The *rank* of an element X of the lattice is $\mathrm{codim}\, X$ in V and the *rank* of a central arrangement $\mathcal{A}$ is the maximum length of the chains of $L(\mathcal{A})$, which coincides with the rank of the maximal element of $L(\mathcal{A})$, i.e. $\mathrm{codim} \cap_{H \in \mathcal{A}} H$ in V.

On the geometric side, the main source of study is the *complement* of the arrangement $M(\mathcal{A}) := V \setminus \bigcup_H H$.

Given arrangements $\mathcal{A}_1 \subset V_1$ and $\mathcal{A}_2 \subset V_2$ with coordinate rings R_1 and R_2, one can construct a few new arrangements. The first natural one is the *direct sum*

$$\mathcal{A}_1 \oplus \mathcal{A}_2 := \{H \oplus V_2 : H_1 \in \mathcal{A}_1\} \cup \{V_1 \oplus H_2 : H_2 \in \mathcal{A}_2\}.$$

The direct sum lives in $V_1 \oplus V_2$ and has $R_1 \otimes R_2$ as its coordinate ring. An arrangement $\mathcal{A}$ is called *irreducible* if it cannot be written as a direct sum of two smaller

arrangements. Also $\mathcal{A}$ fails to be essential if up to some linear change of coordinates, one can write it as $\mathcal{A}_e \oplus \Phi$, where $\mathcal{A}_e$ is the essential part and Φ is an empty arrangement.

Given a central arrangement $\mathcal{A}$ and $X \in L(\mathcal{A})$, we have the *localization* subarrangement

$$\mathcal{A}_X := \{H \in \mathcal{A} : H \leq X\}$$

and the corresponding *restriction* arrangement

$$\mathcal{A}^X := \{H \cap X : H \in \mathcal{A} \setminus \mathcal{A}_X, H \cap X \neq \emptyset\}.$$

One can also define the localization with respect to a point $x \in V$ by

$$\mathcal{A}_x := \{H \in \mathcal{A} : x \in H\}.$$

If we let $X = \cap_{x \in H \in \mathcal{A}} H$, then $\mathcal{A}_x$ will agree with $\mathcal{A}_X$.

If we distinguish a hyperplane $H \in \mathcal{A}$, we get two associated arrangements which are of special importance: The *deletion* of $\mathcal{A}$ with respect to H is defined by $\mathcal{A}' := \mathcal{A} \setminus \{H\}$ where we simply remove H. The *restriction* is obtained by cutting all hyperplanes of $\mathcal{A}'$ with the distinguished hyperplane H, that is $\mathcal{A}'' := \mathcal{A}^H = \mathcal{A}' \cap H$. The importance of this construction comes from the fact that if one succeeds in controlling a certain property or construction of an arrangement by just having information about its deletion and restriction, then one usually gets a combinatorial recipe that is governed or indexed by the lattice of intersection. In other words, deletion and restriction is a general framework for inductive arguments.

The immediate property of the triple $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ at the level of complements is listed below.

$$L(\mathcal{A}) = L(\mathcal{A}') \cup L(\mathcal{A}''),$$
$$M(A) = M(A') \setminus H,$$
$$M(A'') = M(\mathcal{A}') \cap H.$$

In general it is desirable to see how different objects that we associate to arrangements behave under deletion and restriction.

If $\mathcal{A}$ happens to be nonessential, i.e. the center $\cap \mathcal{A}$ is bigger than zero, then we can divide it by its center to essentialize it. To be precise, if $W = \cap_{H \in \mathcal{A}} H \neq 0$, then we pass to $\mathcal{A}/W$, which lives in V/W and consists of the images $\{\overline{H_1}, \ldots, \overline{H_n}\}$

of the hyperplanes of $\mathcal{A}$ under the projection $V \to V/W$. This operation leaves the lattice unchanged. The *essentialized arrangement* is denoted by $\mathcal{A}^e$.

Since we are interested in central arrangements, if $\mathcal{A}$ happens to be noncentral, we make the *cone arrangement* over $\mathcal{A}$ which is denoted by $c\mathcal{A}$. For this, homogenize the linear polynomials by introducing a new variable h. It is customary to add the hyperplane $h = 0$ to the cone to keep it essential. The coordinate ring of $c\mathcal{A}$ equals $R[h]$. Conversely, given a central arrangement $\mathcal{A}$, the *decone arrangement* of $\mathcal{A}$, denoted $d\mathcal{A}$, is obtained by intersecting the hyperplanes of $\mathcal{A}$ with the hyperplane $x_n = 1$. Note that the decone of an essential arrangement is only an affine arrangement. This is useful for depicting three dimensional real arrangements. Assume that after a change of coordinates $x_3 = 0$ is one of the hyperplanes. We first projectivize $\mathbb{R}^3$ to get $\mathbb{RP}^2$ which we identify with the upper hemisphere in S^2, where the antipodal points on the equator are identified. Now the hyperplane $x_3 = 0$ is the plane at infinity and we identify the interior of the upper hemisphere with the decone $d\mathcal{A}$ but keep in mind that parallel lines meet at infinity.

Example 1.1.1. Consider the type A_3 arrangement $\mathcal{A}(A_3) = \{\ker(x_i - x_j) : 1 \leq i < j \leq 4\}$. The maximal element of $L(\mathcal{A}(A_3))$ is the line $x_1 = x_2 = x_3 = x_4$ whose codimension in $\mathbb{R}^4$ is 3 and hence $\mathrm{rank}(\mathcal{A}) = 3$. In order to fix the difference between the rank of the ambient space and the arrangement, we slice the arrangement by $x_4 = 0$ to get $(x_1 - x_2)(x_1 - x_3)x_1(x_2 - x_3)x_2 x_3$. Decone with respect to x_3 by letting $x_3 = 1$ to get the following picture.
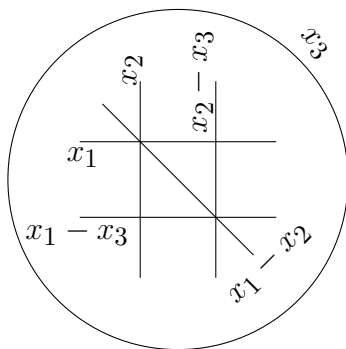
The deletion and restriction with respect to the hyperplane at infinity looks as follows.
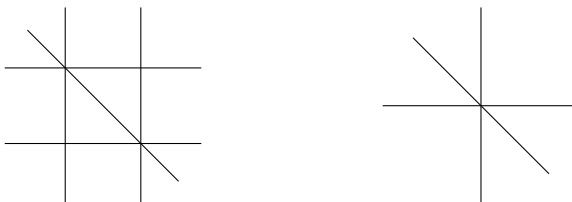
Also, their respective lattices are displayed below.
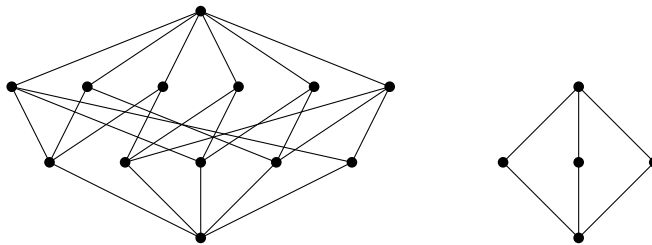
Note that two of the double points correspond to the intersection of the parallel lines at infinity. These two lattices may be combined to recover the lattice of the original arrangement which is known to be the partition lattice. The deletion-restriction process can be performed until all resulting arrangements are broken down into rank 1 pieces. The above arrangement is also known as the braid arrangement since it is the classifying space of the braids and is also the configuration space of collision-free motion of 4 particles on the complex plane.

1.2 Derivations and Kähler Forms

Modules of derivations and differential forms are going to be among the central objects of this thesis. We start with the following general definition.

Definition 1.2.1. Let A and B be commutative rings, A be a B-algebra and N be an A-module. A B-linear map $\mathrm{d} : A \to N$ is a *derivation* if it satisfies the Leibniz rule

$$\mathrm{d}(fg) = f\mathrm{d}g + g\mathrm{d}f,$$

for all $f, g \in A$. The set $\mathrm{D}_B(A, N)$ of all B-linear derivations is naturally a module over A. In case $N = A$, we use the notation $\mathrm{D}_B(A)$ as a short hand notation for $\mathrm{D}_B(A, A)$. It follows from the Leibniz property that $\mathrm{d}b = 0$ for all $b \in B$. If we let A be the coordinate ring of an algebraic variety X over a field B, then $\mathrm{D}_B(A)$ is the set of algebraic tangent vector fields to X.

Dually, the *module of Kähler forms* of A over B, denoted $\Omega_B(A)$, consists of all symbols $\mathrm{d}f$ for all $f \in A$, subject to the relations

$$\begin{aligned} \mathrm{d}(fg) &= \mathrm{d}f \cdot g + f\mathrm{d} \cdot g && \text{Leibniz} \\ \mathrm{d}(af + bg) &= a\mathrm{d}f + b\mathrm{d}g && B - \text{Linearity} \end{aligned}$$

for all $a, b \in B$ and $f, g \in A$. The map $\mathrm{d} : A \to \Omega_B(A)$ is in fact a B-linear derivation, called the *universal R-linear derivation* and satisfies the following universal property:

Given an A-module N and a B-linear derivation $e : A \to N$, there is a unique B-linear map e' that fills the following diagram.

$$
\begin{array}{ccc}
 & \xrightarrow{\ \ \mathrm{d}\ \ } & \Omega_B(A) \\
A & & \Big\downarrow \exists! \, e' \\
 & \xrightarrow[\ \ e\ \ ]{} & N
\end{array}
$$

It follows from the above diagram that

$$\mathrm{D}_B(A, N) \cong \mathrm{Hom}_B(\Omega_B(A), N).$$

In particular, we see that $\mathrm{D}_B(A)$ and $\Omega_B(A)$ are dual to one another if we let $N = A$. For more details, see [14, Chapter 16].

Now let us switch to the affine case and consider complex vectors spaces (as complex manifolds) V and W with a linear transformation $\phi : W \to V$. The map ϕ induces an algebra map in the reverse direction $\phi^* : \mathbb{C}[V] \to \mathbb{C}[W]$. If we pass to the vector bundles, we get

$$\mathrm{d}\phi : \mathrm{T}W \to \mathrm{T}V,$$

where in our linear setting $\mathrm{T}W$ really equals $W \times W$ and $\mathrm{d}\phi$ really equals $\phi \times \phi$. Dually, if we pass to the cotangent bundle, we get

$$(\mathrm{d}\phi)^* : \mathrm{T}^*V \to \mathrm{T}^*W,$$

where $(\mathrm{d}\phi)^*$ really is defined by the transpose matrix ϕ^T. For convenience, let $R = \mathbb{C}[V]$ and $S = \mathbb{C}[W]$. In general, sections of the tangent bundle $\mathrm{T}V \to V$ are called vector fields but we restrict our attention to the *polynomial vector fields* $\theta : R \to R$ which are $\mathbb{C}$-linear and satisfy the Leibniz rule. That is, $\theta(fg) = \theta(f)g + f\theta(g)$, for all $f, g \in R$. The module of polynomial vector fields is denoted by $\mathrm{D}_{\mathbb{C}}(R)$ (or alternatively $\mathrm{D}_{\mathbb{C}}[V]$).

Dually, we will be considering the *polynomial differential forms* $\Omega_{\mathbb{C}}(R) = \Omega_{\mathbb{C}}[V]$ which consist of polynomial sections of the cotangent bundle. In general, we drop the indication to the scalars if emphasis is not required. $\Omega(R)$ consists of all $\mathrm{d}f$, for

all $f \in R$, where $\mathrm{d}(fg) = \mathrm{d}f \cdot g + f \cdot \mathrm{d}g$. If $\dim V = \ell$, then for $0 \le p \le \ell$, we let $\mathrm{D}_p(R) := \wedge^p \mathrm{D}(R)$ and $\Omega^p(R) := \wedge^p \Omega(R)$.

The map ϕ may be used to send vector fields and differential forms over in a functorial way through pull-backs, denoted ϕ^*, and push-forwards, denoted ϕ_*, while one needs to be careful about the disadvantage of D in this regard: Push forward of vector fields is in general only possible when the map ϕ is either an injection or an isomorphism. D works covariantly and Ω works contravariantly:

$$\phi_* : \mathrm{D}[W] \to \mathrm{D}[V]$$
$$\phi^* : \Omega[V] \to \Omega[W]$$

The purpose of the following proposition is to explain the functorial properties. See [34] for more details.

Proposition 1.2.2. *Let $\phi : W \to V$ be a linear map between vector spaces as above.*

(1) *By pulling forms back, we get a map $\Omega[V] \to \Omega[W]$ contravariantly, i.e. Ω is a contravariant functor from the category of vector spaces to dga's.*

(2) *If ϕ is an injection or an isomorphism, then by pushing forward along ϕ, we get a natural map $\mathrm{D}[W] \to \mathrm{D}[V]$.*

Proof. If $\omega \in \Omega[V]$, then $\phi^*(\omega)$ is the composition of the following maps:

$$W \xrightarrow{\phi} V \xrightarrow{\omega} \mathrm{T}^*V \xrightarrow{\phi^T} \mathrm{T}^*W$$

On the other hand, ϕ_* works by pushing vector fields of W forward to vector fields on V:

Let $y_1, \ldots, y_k$ and $x_1, \ldots, x_\ell$ be coordinates on W and V, respectively. Given $\theta \in \mathrm{D}(S)$, we write it as

$$\sum_{j=1}^{s} g_j \frac{\partial}{\partial y_j},$$

where $g_1, \ldots, g_s \in S$. For a polynomial function f on V, i.e. an element of R, and a point $v \in \phi(V)$, we let

$$\phi_*(\theta)|_v(f) = \theta_w(\phi_*(f)) = \theta_w(f \circ \phi),$$

where $\phi(w) = v$. $\qquad \square$

The module of Kähler forms also behaves functorially in the setting of maps between algebras. Let A and A' be B and B'-algebras respectively around a commutative diagram of rings which is displayed in the following with solid arrows, i.e a morphism between algebras.

$$
\begin{array}{ccccc}
B' & \longrightarrow & A' & \xrightarrow{\ \mathrm{d}\ } & \Omega_{B'}(A') \\
\big\uparrow & & \big\uparrow & & \big\uparrow \\
B & \longrightarrow & A & \xrightarrow{\ \mathrm{d}\ } & \Omega_B(A)
\end{array}
$$

Then the universal property of Ω induces a map between the modules of Kähler differential forms as indicated by the dashed line. See [14] for some more details.

In the rest of this section, we try to clarify the geometric significance of our main objects of study, namely the derivation module and the module of logarithmic forms of an arrangements. One may bear in mind that some of these constructions are generalizable to more arbitrary divisors [25].

Fix coordinates $x_1, \ldots, x_\ell$ on the ambient space V of an arrangement $\mathcal{A}$ defined by a product Q. We are interested in the collection of vector fields that are tangent to all hyperplanes of $\mathcal{A}$. That is, for any point $p \in V(Q)$, we require that

$$
\theta\big|_p \cdot (\nabla Q)\big|_p = 0.
$$

If we let $\theta = \sum_i g_i \partial_{x_i}$, this will amount to having

$$
\Big(\sum_i g_i(p) \frac{\partial}{\partial x_i}\Big|_p\Big) \cdot \Big(\sum_i \frac{\partial Q}{\partial x_i}\Big|_p \frac{\partial}{\partial x_i}\Big|_p\Big) = \sum_i g_i(p) \frac{\partial Q}{\partial x_i}\Big|_p = \theta(Q)\big|_p = 0,
$$

for all points p, with $Q(p) = 0$. This leads us to the definition of the *derivation module* of an arrangement.

$$
\mathrm{D}(\mathcal{A}) := \{\theta \in \mathrm{D}(R) : \theta(Q) \in (Q)\} \tag{1.2}
$$

The derivation module is a submodule of

$$
\mathrm{D}(R) := R\partial_{x_1} \oplus \cdots \oplus R\partial_{x_\ell}, \tag{1.3}
$$

as an R-module.

If $Q = f_1 \cdots f_n$, then $\theta(Q) = \sum_{i=1}^n \theta(f_i) Q/f_i \in (Q)$. But this is only possible if

for each $1 \leq i \leq n$, f_i divides $\theta(f_i)$. For this, divide $\theta(f_i)$ by f_i to get $\theta(f_i) = f_i q_i + r_i$ and observe that the fact that $\theta(Q)$ is divisible by Q implies that $r_i = 0$. Conversely, if each $\theta(f_i)$ is a multiple of f_i, then we $\theta(Q)$ becomes a multiple of Q. Thus, as an alternative definition we get

$$\mathrm{D}(\mathcal{A}) := \{\theta \in \mathrm{D}(R) : \theta(f_i) \in (f_i), 1 \leq i \leq n\} \tag{1.4}$$

This form of definition allows using multiplicities as follows.

Definition 1.2.3. The *derivation module* of a multiarrangement $(\mathcal{A}, \mathbf{m})$ is defined by

$$\mathrm{D}(\mathcal{A}, \mathbf{m}) := \{\theta \in \mathrm{D}(R) : \theta(f_i) \in (f_i^{m_i}), 1 \leq i \leq n\}. \tag{1.5}$$

This defining condition is asking for tangency to the hyperplanes of higher orders according to their multiplicities.

Having derived formulations for the module of tangent vector fields, we switch to the dual situation. This time we are going to look at covector fields (differential forms) ω over V that are perpendicular to the hypersurface defined by $Q = 0$ at all of its points. Using the same coordinates, let $\omega = \sum_i g_i \mathrm{d}x_i$ and this time consider the gradient as a covector field in the cotangent space. ω is going to be normal to the surface if it is parallel to ∇Q at all points of $V(Q)$. That is, for all points p with $Q(p) = 0$, we need the matrix

$$\begin{bmatrix} g_1 & \cdots & g_\ell \\ \partial Q/\partial x_1 & \cdots & \partial Q/\partial x_\ell \end{bmatrix}$$

to have rank $= 1$. Equivalently, this requires the determinant of 2×2 minors to vanish: For all $1 \leq i < j \leq \ell$ and $p \in V(Q)$, we require

$$\det \begin{bmatrix} g_i(p) & g_j(p) \\ (\partial Q/\partial x_i)\big|_p & (\partial Q/\partial x_j)\big|_p \end{bmatrix} = 0.$$

Again, this amounts to the membership of the determinant in the ideal (Q). Equivalently, one can formulate these memberships compactly by the just asking

$$\omega \wedge \mathrm{d}Q \in Q\,\Omega^2[V],$$

simply because

$$\omega \wedge \mathrm{d}Q = \sum_{1 \leq i < j \leq \ell} \det \begin{bmatrix} g_i & g_j \\ \partial Q/\partial x_i & \partial Q/\partial x_j \end{bmatrix} \mathrm{d}x_i \wedge \mathrm{d}x_j.$$

In order to get nicer formulations later we divide these differential forms by Q. The result gives a module of differential forms with poles on the hyperplanes.

The *module of logarithmic forms* of an arrangement $\mathcal{A}$ is defined by

$$\Omega(\mathcal{A}) := \{\omega/Q : \omega \in \Omega^1[V], \omega \wedge \mathrm{d}Q \in Q\,\Omega^2[V]\}. \tag{1.6}$$

This module lives inside the localization $\Omega(R)_Q$, where

$$\Omega(R) := R\mathrm{d}x_1 \oplus \cdots \oplus R\mathrm{d}x_\ell, \tag{1.7}$$

as a module over R.

Similar to the case of derivations, we can check this condition on hyperplanes individually as follows. Equivalently,

$$\Omega(\mathcal{A}) = \{\omega : Q\omega \in \Omega^1[V], Q\omega \wedge \mathrm{d}f_i \in f_i\Omega^2[V], 1 \leq i \leq n\}. \tag{1.8}$$

Again, this definition will be more suitable for generalizing to multiarrangements.

Definition 1.2.4. The *module of logarithmic forms* of a multiarrangement $(\mathcal{A}, \mathbf{m})$ is defined by

$$\Omega(\mathcal{A}, \mathbf{m}) := \{\omega : \widetilde{Q}\omega \in \Omega^1[V], \widetilde{Q}\omega \wedge \mathrm{d}f_i \in f_i^{m_i}\,\Omega^2[V]\},$$

where $\widetilde{Q} = \Pi_i f_i^{m_i}$ is the defining polynomial of the multiarrangement.

These two logarithmic modules are graded and one can pick homogeneous minimal sets of generators for them. There are two natural choices of gradings as follows.

Definition 1.2.5. For every $j = 1, \ldots, \ell$,

- in the *polynomial grading*, $\deg \mathrm{d}x_j = \deg \partial_{x_j} = 0$ and $\deg x_j = 1$,

- and in the *total grading*, $\deg \mathrm{d}x_j = \deg x_j = 1$ and $\deg \partial_{x_j} = -1$.

Note that under the polynomial grading we have the graded isomorphism $R^\ell \cong \mathrm{D}(R)$, while under the total grading, $R^\ell[-1] \cong \mathrm{D}(R)$ and the Euler derivation $\theta_E = \sum_{j=1}^\ell x_j \partial_{x_j}$ is degree zero. Here comes the definition of a very nice class of arrangements.

Definition 1.2.6. A multiarrangement $(\mathcal{A}, \mathbf{m})$ is *free* if $\mathrm{D}(\mathcal{A}, \mathbf{m})$ is a free module over R. If $(\mathcal{A}, \mathbf{m})$ is free and $\theta_1, \ldots, \theta_\ell$ is a homogeneous basis, then $\deg \theta_1, \ldots, \deg \theta_\ell$ with respect to the polynomial grading are the *exponents* of $(\mathcal{A}, \mathbf{m})$. If exponents are listed nondecreasingly, then $\exp(\mathcal{A}, \mathbf{m}) := (\deg \theta_1, \ldots, \deg \theta_\ell)$.

It is known that the modules D and Ω are dual to one another, so one can also define freeness and the exponents by looking at $\Omega(\mathcal{A}, \mathbf{m})$.

In fact, there is a nondegenerate pairing

$$\mathrm{D}(\mathcal{A}, \mathbf{m}) \times \Omega(\mathcal{A}, \mathbf{m}) \to R,$$

which supports the duality. We will come back to this later in this chapter.

Example 1.2.7. Some examples of free and non-free arrangements:

- Every multiarrangement in rank 2 is free. In the simple case, the derivation module of an arrangement of lines in $\mathbb{C}^2$ defined by $f_1 \cdots f_n \in C[x, y]$ has a basis consisting of

$$\theta_1 = x\partial_x + y\partial_y, \quad \theta_2 = \frac{Q}{f_1}\left(\frac{\partial f_1}{\partial x}\partial_y - \frac{\partial f_1}{\partial y}\partial_x\right), \tag{1.9}$$

  implying that $\exp(\mathcal{A}) = (1, |\mathcal{A}| - 1)$. See [22] for more details. One big contrast with multiarrangements arises right here. In case of multiarrangement of rank 2, there is no general formula and the exponents are non-combinatorial. See Example 1.4.12 for an illustration.

- *Reflection arrangements* with constant multiplicities $\mathbf{m} \equiv m$ are free [32]. A reflection arrangement consists of reflecting hyperplanes of a finite irreducible subgroup G of the orthogonal group $O(V)$. The typical example is the symmetric group S_ℓ which corresponds to the $A_{\ell-1}$ root system. Each transposition (i, j) defines a reflection, namely $\ker(x_i - x_j)$, and the resulting arrangement is the so-called *braid arrangement*. Subarrangements of the braid arrangement are important examples of configurations spaces that parametrize robotic motions.

- The multiarrangement defined by $\widetilde{Q} = x_1^{m_1} x_2^{m_2} x_3^{m_3} (x_1 + x_2 + x_3)^{m_4}$ is totally non-free, meaning that $(\mathcal{A}, \mathbf{m})$ is non-free for every multiplicity $\mathbf{m}$. See [3, Example 5.6] for a direct proof. An alternative proof follows from [41, Proposition 4.1]. Yet another alternative argument may be obtained from an upgrade of Theorem 3.5.3 [43] to multiarrangements, where one sees 2-formality as an obstruction to freeness.

1.3 Background and Motivations

This section is an account of the history of the theory to the extent that is relevant to the subject of this thesis. It should be emphasized that the theory of hyperplane arrangements has had numerous interactions with different parts of mathematics and what appears here is only a brief selection.

The origins of the subject of logarithmic modules and freeness of arrangements can be traced back to [12] in 1971 where P. Deligne developed a mixed Hodge structure on the cohomology algebra $H^{\bullet}(U)$ of a complex smooth algebraic variety U. He extended his constructions in the next step to eliminate the requirement of smoothness. The main idea was the introduction of a weight filtration W on $H^{\bullet}(U)$ by means of an embedding $j : U \to X$ of U in a smooth complete variety X such that $X \setminus U = D$ is a divisor with normal crossings. He introduced the de Rham algebra $\Omega_X^{\bullet}(\log D)$ and showed that the degree one part $\Omega_X^1(\log Y)$ is locally free as a sheaf. By means of the Leray spectral sequence and the fact that there is an isomorphism between $H^{\bullet}(U, \mathbb{C})$ and the hypercohomology $\mathbb{H}^{\bullet}(X, \Omega_X^{\bullet}(\log D))$, he managed to come up with a natural Hodge filtration for the mixed Hodge structure. The rest of this section is split into two parts to show the contrast and transition from simple arrangements to multiarrangements.

1.3.1 Classical Results

In 1979, K. Saito published his paper [25] on logarithmic differential forms and vector fields in order to extend the constructions to divisors that are not necessarily complete intersection. In case of the differential forms, when X is an algebraic manifold and D is a reduced divisor on X, $\Omega_X^{\bullet}(\log D)$ has the following local picture. Let $x \in X$ be a point and let $Q_x = 0$ be a local equation for D around x, then the germ of the

logarithmic sheaf is defined by

$$\Omega_X^p(\log D)_x := \{\omega \in \Omega_{rat}(X)_x : Q_x\omega \in \Omega^p(\mathcal{O}_x),\ Q_x\mathrm{d}\omega \in \Omega^{p+1}(\mathcal{O}_x)\}.$$

The sections of the union $\cup_{x \in X}\Omega_X^p(\log D)_x$ are given a natural sheaf structure which is denoted by $\Omega_X^p(\log D)$. The dual of the sheaf of logarithmic vector fields is denoted $\mathrm{D}_p^X(\log D)$. Saito introduced the notion of freeness and found a criterion for verifying freeness once a candidate for basis is given (see Theorem 1.4.14). One can imagine that the term *logarithmic* was used in the initial definition of Deligne because he was working with normal crossing divisors and at a normal crossing of some n divisors that are defined locally by $f_1, \ldots, f_\ell$, the module $\Omega_X^1(\log D)_x$ is freely generated by $\mathrm{d}\log f_1, \ldots, \mathrm{d}\log f_\ell$ over $\mathcal{O}_x$. Similarly, the derivation module is freely generated by $f_1\frac{\partial}{\partial f_1}, \ldots, f_\ell\frac{\partial}{\partial f_\ell}$.

    H. Terao was a student of K. Saito and applied his constructions to the context of hyperplane arrangements. At the time, arrangements were gaining popularity through the works of Grünbaum and other people as an independent subject, although one could also find some hints prior to that (See [17]). Terao considered the localization of Saito's sheaf at zero. An easy but important feature of the structure of $\mathrm{D}(\mathcal{A})$ is that it admits a direct sum decomposition as

$$\mathrm{D}(\mathcal{A}) = R\theta_E \oplus \mathrm{Ann}(Q), \tag{1.10}$$

where $\mathrm{Ann}(Q) = \mathrm{Ann}(\mathcal{A}) = \{\theta \in \mathrm{D}(R) : \theta(Q) = 0\}$ and $\theta_E = \sum_j x_j\partial_{x_j}$ is the *Euler derivation*. See Proposition 1.4.9 for a slightly more general statement. Note that an arrangement $\mathcal{A}$ is free exactly when $\mathrm{Ann}(\mathcal{A})$ is free. The annihilator submodule $\mathrm{Ann}(\mathcal{A})$ has a description in terms of the Jacobian ideal of the arrangement. The *Jacobian ideal*, denoted $J(\mathcal{A})$, lives in R and is generated by all partial derivatives $\partial Q/\partial x_1, \ldots, \partial Q/\partial x_\ell$. Equivalently, one can say that the Jacobian ideal is the image of the map

$$\mathrm{D}(R) \to R.$$
$$\frac{\partial}{\partial x_j} \mapsto \frac{\partial Q}{\partial x_j}$$

A minimal free resolution of $R/J(\mathcal{A})$ starts as follows.

$$\cdots \to \mathrm{D}(R) \to R \to R/J(\mathcal{A}) \to 0$$

The annihilator module $\mathrm{Ann}(\mathcal{A})$ is naturally the kernel of the above map and makes the following sequence exact.

$$0 \to \mathrm{Ann}(\mathcal{A}) \hookrightarrow \mathrm{D}(R) \to R \to R/J(\mathcal{A}) \to 0$$

This implies that that $J(\mathcal{A})$ and $\mathrm{Ann}(\mathcal{A})$ can be recovered from each other. The next term in the free resolution must be a free module that projects to $\mathrm{Ann}(\mathcal{A}) = \mathrm{syz}_1(J(\mathcal{A})) = \ker \varphi$ and so on. So one would expect to get a Jacobian ideal interpretation of freeness from a ring theoretic point of view. The following result is exactly of this nature.

Theorem 1.3.1 (Terao, [30]). *An arrangement $\mathcal{A}$ is free if and only if its Jacobian ideal $J(\mathcal{A})$ is Cohen-Macaulay.*

The proof uses the facts that the Krull dimension of $R/J(\mathcal{A})$ and the projective dimension of $\mathrm{D}(\mathcal{A})$ are off by 2 from the rank of the arrangement, together with the Auslander-Buchsbaum formula.

A recent result of Yoshinaga and Wakefield shows that an arrangement can be recovered from the Jacobian ideal and hence derivation module. (See [33])

One of the first accomplishments of Terao's work was the so-called *Factorization Theorem* (Corollary 3.4.10) which in the free case relates the algebraic information of the derivation module in a combinatorial manner to the Betti numbers of the complement. This has its origins in the computation of the cohomology ring of an arrangement complement. The complete description of the cohomology ring with integer coefficients and multiplicative structure, known as the *Orlik-Solomon algebra*, is a result of an evolution of a chain ideas due to V.I. Arnol'd, E. Brieskorn, L. Solomon and P. Orlik. The construction of the Orlik-Solomon algebra is fully combinatorial and only uses the information of the intersection lattice $L(\mathcal{A})$. In particular, it was shown that the cohomology algebra is torsion-free and the rank of each component (the Betti numbers) may be read off from

$$\chi(M(\mathcal{A}), t) = \sum_{X \in L(\mathcal{A})} \mu(X) t^{\dim X} = \sum_{i=0}^{\ell} \beta_i(M(\mathcal{A}))(-t)^{\ell-i},$$

where $\mu : L(\mathcal{A}) \to \mathbb{Z}$ is the *Möbius inversion formula*, defined recursively by $\mu(V) = 1$ and

$$\mu(X) = - \sum_{Y < X} \mu(Y).$$

In particular, all hyperplanes receive $\mu = -1$. The *Poincaré polynomial* is related to the characteristic polynomial by $\pi(\mathcal{A}, t) = (-t)^\ell \chi(\mathcal{A}, -t^{-1})$. These formulas give combinatorial interpretation for these primarily topological invariants.

To be precise, the Poincaré polynomial version of the *Factorization Theorem* says that if $\mathcal{A}$ is a free arrangement with exponents $\exp(\mathcal{A}) = (d_1, \ldots, d_\ell)$, then

$$\pi(M(\mathcal{A})) = \Pi_{i=1}^\ell (1 + d_i t). \tag{1.11}$$

Moreover, the Betti numbers are related to the exponents by

$$\beta_i(M(\mathcal{A})) = \sum_{1 \le i_1 < \cdots < i_p \le \ell} \Pi b_{i_1} \cdots b_{i_p},$$

for every $p = 0, \ldots, \ell$.

Example 1.3.2. The lattice elements of the deleted A_3 arrangement (Example 1.1.1) are labeled with their Möbius numbers in the following picture.
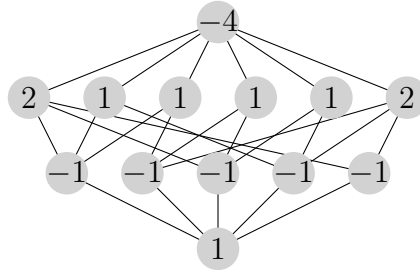
One can write down the characteristic polynomial as $\chi(t) = t^3 - 5t^2 + 8t - 4 = (t-1)(t-2)^2$ by following the above formula and looking at the labels in the picture. Also, the Poincaré polynomial equals $\pi(t) = 4t^3 + 8t^2 + 5 + 1 = (t+1)(2t+1)^2$. In fact, this arrangement is free and $\exp(\mathcal{A}) = (1, 2, 2)$.

This result gives a much bigger significance to the notion of freeness. After the factorization theorem, Terao went on to conjecture that freeness is a combinatorial property. To be precise, *Terao's conjecture* says that the information of $L(\mathcal{A})$ is enough to determine whether $\mathrm{D}(\mathcal{A})$ is a free module or not. This conjecture has been open to this day although many partial results are available in this direction. The Factorization Theorem was shown to be a specialization of a more general formula, known as the *Solomon-Terao* formula [29], which expresses the characteristic polynomial in terms of an alternating sum of certain Hilbert series as follows.

$$\chi(\mathcal{A};t) = (-1)^\ell \lim_{x \to 1} \sum_{p=0}^{\ell} h(D_p(\mathcal{A}), x)(t(x-1)-1)^p \qquad (1.12)$$

This formula holds in general regardless of the fact that Betti numbers of the derivation modules are not combinatorial in the nonfree case (see [46]). One of the main results of this thesis is to give a new proof for a weak form of the Solomon-Terao formula in Chapter 3 as a consequence of the computation of the intersection cycle of a certain variety.

The main general result in support of Terao's conjecture is the following result of S. Yuzvinsky. The idea is to fix a free arrangement $\mathcal{A}$ with intersection lattice $L(\mathcal{A})$ and consider the moduli space $\mathscr{V}$ of all arrangements with isomorphic lattices

$$\mathscr{V} = \{\mathcal{B} : L(\mathcal{B}) \simeq L(\mathcal{A})\}.$$

Every point of $\mathscr{V}$ is represented by an $n \times \ell$ matrix, in which the rows define the hyperplanes. This is a quasi-affine space in the Zariski topology, where the defining condition translates to vanishing and non-vanishing of certain minors, according to the lattice $L(\mathcal{A})$. Assuming that $\mathcal{A}$ is free to begin with, we have the following fundamental result.

Theorem 1.3.3 (Yuzvinky, [44]). *Free arrangements form an open set in $\mathscr{V}$.*

This implies that nonfree arrangements that share the same intersection lattice with some free arrangements are scarce.

1.3.2 Transition to Multiarrangements

We are going to spend the rest of this section to elaborate on the naturality and significance of multiarrangements. The notion of multiarrangement and their derivation modules came into attention through a 1989 paper of G. Ziegler [47]. He managed to upgrade the definition of Ω and D to multiarrangements and show how one naturally obtains free multiarrangements by starting from free simple arrangement. The following theorem explains this construction. His other contribution was to show that the underlying combinatorics of a free multiarrangement fails to control its exponents. Ziegler's main idea was to record the multiplicity information in the restriction process. To be precise, given a simple arrangement $\mathcal{A}$, its restriction with respect to a

distinguished hyperplane H_0 is naturally equipped with a multiarrangement structure $(\mathcal{A}'', z)$, where for each $K \in \mathcal{A}''$, we let

$$z(K) = \#\{H \in \mathcal{A}' : H \cap H_0 = K\}. \tag{1.13}$$

We will refer to this as *Ziegler's restriction multiplicity*.

Example 1.3.4. This example illustrates the Ziegler multiplicities in case of the A_3 arrangement (Example 1.1.1) when restricted to the hyperplane at infinity.
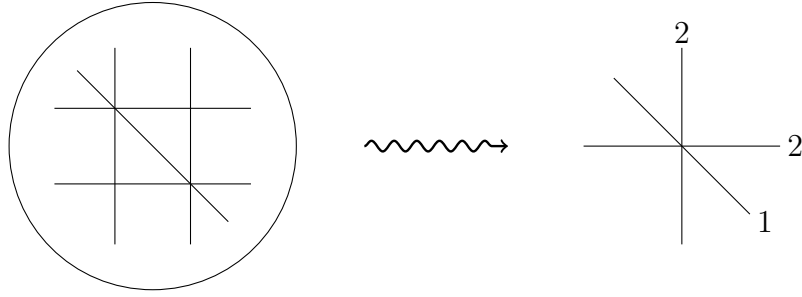
The parallel lines in the decone picture contribute to the multiple lines in the restriction.

Theorem 1.3.5 (Ziegler, [47])**.** *Let $\mathcal{A}$ be a free simple arrangement with $\exp(\mathcal{A}) = (1, d_2, \ldots, d_\ell)$, then its multi-restriction $(\mathcal{A}'', z)$ is also free with $\exp(\mathcal{A}'', z) = (d_2, \ldots, d_\ell)$.*

Proof. Suppose $\mathcal{A}$ is free and has a homogeneous basis $\theta_1 = \theta_E, \theta_2, \ldots, \theta_\ell$. Assuming that $H_0 = \ker f$, replace each such θ_i with $\theta_i - (\theta_i(f)/f)\theta_E$ to make sure that $\theta_i(f) = 0$. Now each such θ_i defines a well-defined derivation on the coordinate ring of the restriction, namely $R/(f)$.

Let θ be an element of the annihilator $\text{Ann}(H_0)$ with respect to H_0 and let f_i and f_j be defining polynomials such that $H_i \cap H_0 = H_j \cap H_0$. This is equivalent to having a dependence between f, f_i and f_j, say $f = af_i + af_j$. Since $\theta(f) = 0$, we get $a\theta(f_i) = -b\theta(f_j)$, in which one side is divisible by f_i and the other one by f_j, but since $\gcd(f_i, f_j) = 1$, it should be divisible by $f_i f_j$. After repeating this argument as many times as possible and restricting to H_0, we get $\theta(\bar{f}_i) \in (\bar{f}_i{}^{m(H \cap H_i)})$, where bar stands for reduction mod f in the ring $R/(f)$ and $m(H \cap H_i) = \#\{H_j \in \mathcal{A}' : H \cap H_j = H \cap H_i\}$. By Saito's criterion (Theorem 1.4.14), this finishes the argument. $\square$

In [2], the authors took advantage of the so called Solomon-Terao formula 1.12 to define a characteristic polynomial for multiarrangement. This will be discussed in Section 1.5.

Remark 1.3.6. The motivation of considering multiarrangements lies in their naturality in terms incorporating the multiplicity data in the restriction process that would otherwise be lost. To be more precise, one may reformulate the above theorem by saying that freeness of $\mathcal{A}$ implies the freeness of $(\mathcal{A}'', z)$ together with the formula

$$\chi(\mathcal{A}) = (t-1)\chi(\mathcal{A}'', z),$$

which fails if multiplicities are not taken into account, hence loss of information. This is reminiscent of Bezout's theorem where the number intersection number of two complex plane curves is correctly given by the product of their degrees only if we take multiplicities into account.

Apart from their naturality in the sense of Ziegler multiplicity, what makes multiarrangements worth studying lies in the fact that understanding their combinatorially wild behavior leads to a better understanding of simple arrangements. This is potentially capable of settling Terao's conjecture at least in rank three as explained here and is a current active topic of research.

The following theorem suggests that multiarrangements may be used to obtain information about simple arrangements. The idea here is to reverse Ziegler's process.

Theorem 1.3.7 (Yoshinaga, [40]). *Let $\mathcal{A}$ be a simple arrangement and H be a distinguished hyperplane, then TFAE:*

- *$\mathcal{A}$ is free;*

- *$(\mathcal{A}'', z)$ is free and the restriction map $p : \mathrm{Ann}(H) \to \mathrm{D}(\mathcal{A}'', z)$ is onto;*

- *$(\mathcal{A}'', z)$ is free, p is onto and $\chi(\mathcal{A}) = (t-1)\chi(\mathcal{A}'', z)$.*

The proof is essentially the same as Ziegler's proof followed in the reverse direction. The contribution of this theorem to rank 3 case is of special importance, since the restriction (multi)arrangements are of rank 2 and hence automatically free (See Corollary 1.4.33).

Corollary 1.3.8. *Let $\mathcal{A}$ be simple arrangement of rank 3, then TFAE:*

- *$\mathcal{A}$ is free;*

- *p is onto and $\chi(\mathcal{A}) = (t-1)\chi(\mathcal{A}'', z)$;*

- *p is onto and $\exp(\mathcal{A}) = (1, d_2, d_3)$, where $\exp(\mathcal{A}'', z) = (d_2, d_3)$.*

In view of the above result, one needs to understand the projection map p : $\mathrm{Ann}(H) \to \mathrm{D}(\mathcal{A}'', z)$, which relies on Theorem 1.3.12 and a result of M. Mustaţă and H. Schenck [20]. Their main result is a closed formula for the Chern polynomial of the sheaf associated to the derivation module after the removal of the Euler derivation, namely the annihilator $\mathrm{Ann}(H)$ but first we need the following definition.

Definition 1.3.9. A central arrangement $\mathcal{A}$ of rank ℓ is called *locally free* if for every $X \in L_{<\ell}(A)$, the localization $\mathcal{A}_X$ is free.

Theorem 1.3.10. *Let $\mathcal{A}$ be a locally free simple arrangement with a choice of a hyperplane H. Then the coherent sheaf $\mathcal{E} := \widetilde{\mathrm{Ann}(H)}$ over $\mathbb{P}^{\ell-1}$ is actually a vector bundle and its Chern polynomial, which lives in $A^*(\mathbb{P}^{\ell-1}) = \mathbb{Z}[t]/(t^\ell)$, is related to the characteristic polynomial by*

$$\chi(\mathcal{A}, t) = (1-t)t^\ell c_{-1/t}(\mathcal{E}) = (1-t)(t^\ell - c_1 t^{\ell-1} + \cdots + (-1)^\ell c_\ell). \qquad (1.14)$$

This specializes to a simpler formula in the rank 3 case where the requirement of locally freeness is automatically guaranteed (See [27]). We shall present an analogous version of this formula for locally free multiarrangements in Theorem 1.5.10 and sketch a proof.

Corollary 1.3.11. *Let $\mathcal{A}$ be a rank 3 arrangement and consider $\mathcal{E} := \widetilde{\mathrm{Ann}(H)}$ as in the last theorem. Then the Chern numbers are*

$$c_0(\mathcal{E}) = 1, \quad c_1(\mathcal{E}) = -n, \quad c_2(\mathcal{E}) = \sum_p \mu(p), \qquad (1.15)$$

where p in the sum runs over all points of intersection that are away from H_0.

Now we can state a theorem of Yoshinaga which enables us control the projection map $p : \mathrm{Ann}(H_0) \to \mathrm{D}(\mathcal{A}'', z)$.

Theorem 1.3.12 (Yoshinaga, [40])**.** *Let $\mathcal{A}$ be a rank 3 arrangement, such that $\exp(\mathcal{A}'', z) = (d_2, d_3)$, then*

$$\dim \mathrm{coker}\, p = c_2(\mathcal{E}) - d_2 d_3. \qquad (1.16)$$

In particular, if $c_2 = d_2 d_3$, then p is onto, which in turn implies $\mathcal{A}$ is free.

Multiarrangements $(\mathcal{A}'', z)$ that arise as restriction are expected to have a nicer behavior. Yoshinaga gave the following definition and tried to study them.

Definition 1.3.13. A multiarrangement $(\mathcal{A}, \mathbf{m})$ is called *extendible* if there is some simple arrangement $\mathcal{E}$ with a hyperplane H, such that $(\mathcal{A}, \mathbf{m})$ equals the restriction arrangement $\mathcal{E}^H$ with the natural restriction multiplicity z (see formula 1.13). An arrangement such as $\mathcal{E}$ will be an *extension* for $(\mathcal{A}, \mathbf{m})$.

The problem of characterization of extendible multiarrangements is discussed in [41]. It is noted that even in the rank two case, some multiarrangements are not extendible. The main result establishes links between freeness of certain multiarrangements and their extendibility.

The upshot is that the non-freeness of an arrangement $\mathcal{A}$ has to do with deviation of the characteristic polynomial $\chi(\mathcal{A}'', z)$ from $\chi(\mathcal{A})/(1-t)$. Given the fact that $\chi(\mathcal{A}'', z)$ is not entirely controlled by the underlying combinatorics, an interesting question to ask would be whether this noncombinatorial behavior can harm Terao's conjecture to the extent that it fails.

In order to exploit the combinatorics of multiarrangements, the authors in [3] crafted a suitable choice of multiplicity to relate the exponents of a multiarrangement in the free case to that of its deletion and restriction. The tricky part is the definition of the multiplicity for restriction. The main construction of [3] is the following.

Definition 1.3.14. Let $(\mathcal{A}, \mathbf{m})$ be a multiarrangement defined by $\widetilde{Q}$. The *deletion* of $(\mathcal{A}, \mathbf{m})$ with respect to a distinguished hyperplane $H_0 = \ker f_0$, denoted $(\mathcal{A}, \mathbf{m})'$, is defined by $\widetilde{Q}/f_0$. Equivalently,

$$(\mathcal{A}, \mathbf{m})' := \begin{cases} (\mathcal{A}', \mathbf{m}_{|\mathcal{A}'}) & \mathbf{m}(H_0) = 1 \\ (\mathcal{A}, \mathbf{m}') & \mathbf{m}(H_0) > 1 \end{cases}$$

where $\mathbf{m}'(H) = \mathbf{m}(H)$ if $H \neq H_0$ and $\mathbf{m}'(H_0) = \mathbf{m}(H_0) - 1$. The *restriction* $(\mathcal{A}, \mathbf{m})''$ with respect to H_0 is the pair $(\mathcal{A}'', \mathbf{m}^*)$, where the multiplicity $\mathbf{m}^*$ is defined in the following non-combinatorial process.

Given $X \in \mathcal{A}''$, choose coordinates such that $f_0 = x_1$ and $X = \{x_1 = x_2 = 0\}$ and consider the submultiarrangement $\mathcal{A}_X = \{H \in \mathcal{A} : X \subset H\}$ which is of rank two and hence free. Under these assumptions, if $\ker f \in \mathcal{A}_X$, then $f \in \mathbb{C}[x_1, x_2]$. One can pick two basis elements θ_X and Ψ_X for this multiarrangement, such that $\theta_X \notin x_1 \mathrm{D}(R)$ and $\Psi_X \in x_1 \mathrm{D}(R)$ (See Proposition 2.1 in [3]). Having chosen θ_X, define $\mathbf{m}^*(X) := \deg \theta_X$.

Theorem 1.3.15 (Deletion-Restriction, [3])**.** *Let* $(\mathcal{A}, \mathbf{m})$ *be a multiarrangement of rank* ℓ, $H_0 \in \mathcal{A}$ *and let* $(\mathcal{A}, \mathbf{m})'$ *and* $(\mathcal{A}, \mathbf{m})''$ *be the deletion and restriction with respect to* H_0*. Then any two of the following statements imply the third:*

(i) $(\mathcal{A}, m)$ *is free with* $\exp(\mathcal{A}, \mathbf{m}) = (d_1, \ldots, d_\ell)$.

(ii) $(\mathcal{A}, \mathbf{m})'$ *is free with* $\exp(\mathcal{A}, \mathbf{m})' = (d_1, \ldots, d_\ell - 1)$.

(iii) $(\mathcal{A}, \mathbf{m})''$ *is free with* $\exp(\mathcal{A}, \mathbf{m})'' = (d_1, \ldots, d_{\ell-1})$.

Note that in the above definition of deletion and restriction, if the original arrangement is simple, then the deletion and restriction will be simple as well. Simply because if $\mathbf{m} \equiv 1$, then we let θ_X just be the Euler derivation.

Theorem 1.3.16. *A multiarrangement either has infinitely many nonfree multiplicities or none.*

The case of only having free multiplicities is characterized in [4]. In fact, this can only happen in a trivial way in view of Corollary 1.4.41.

Theorem 1.3.17. *If a multiarrangement* $(\mathcal{A}, \mathbf{m})$ *has only finitely many nonfree multiplicities, then it must be a product of rank one and two arrangements, in which case it will have no nonfree multiplicity.*

Such arrangements are called *totally free*. The tool that is used in the proof is the so called *local-global mixed product formula*. Fix a number p with $1 \le p \le \ell$, together with a multiarrangement $(\mathcal{A}, \mathbf{m})$ and consider the summation

$$\mathrm{LMP}(k) = \sum_{X \in L_k} d_1^X \cdots d_k^X,$$

where L_k consists of rank k elements of the lattice and $d_1^X, \ldots, d_k^X$ is the sequence of exponents of $(\mathcal{A}, \mathbf{m})_X$ whenever it is free.

On the other hand, when $(\mathcal{A}, \mathbf{m})$ is free itself, with exponents $\exp(\mathcal{A}, \mathbf{m}) = (d_1, \ldots, d_\ell)$, let

$$\mathrm{GMP}(k) = \sum d_{i_1} \cdots d_{i_k},$$

where the sum runs over all increasing sub-k-tuples of $\exp(\mathcal{A}, \mathbf{m})$. Then, we have the following result.

Theorem 1.3.18. *If* $(\mathcal{A}, \mathbf{m})$ *is free, then the local and global mixed products agree for all* $1 \le k \le \ell$.

This is nice because it potentially provides a systematic method of showing that some multiarrangements are not free as described in [2]. There are a variety of other related results but we choose to close this section here and recall whatever that is needed as we develop the material.

1.4 Multiarrangements

This section starts with a comprehensive definition of the logarithmic modules of multiarrangements of all orders.

Definition 1.4.1. Let $(\mathcal{A}, \mathbf{m})$ be a multiarrangement with coordinate ring R, then its pth *module of logarithmic derivations* is defined by

$$\mathrm{D}_p(\mathcal{A}, \mathbf{m}) := \{\theta \in \mathrm{D}_p(R) : \theta(f, g_2, \ldots, g_p) \in f^{\mathbf{m}(H)}R, \forall g_2, \ldots, g_p \in R, \forall H = \ker f \in \mathcal{A}\}, \tag{1.17}$$

and the *module of logarithmic forms* is defined by

$$\Omega^p(\mathcal{A}, \mathbf{m}) := \{\omega/\widetilde{Q} : \omega \in \Omega^p(R), \omega \wedge \mathrm{d}f \in f^{\mathbf{m}(H)}\Omega^{p+1}(R), \forall H = \ker f \in \mathcal{A}\}. \tag{1.18}$$

Here $\mathrm{D}_p(R)$ consists of all alternating multilinear maps $\theta : R^p \to R$, which satisfy the Leibniz rule in each component. This is generated as a module over R by all partial derivatives $\partial_{i_1} \wedge \cdots \wedge \partial_{i_p}$, where each $i_j \in \{1, \ldots, \ell\}$, and if $g_1, \ldots, g_p \in R$, then

$$(\partial_{i_1} \wedge \cdots \wedge \partial_{i_p})(g_1, \ldots, g_p) = \det[\partial_{i_j}(g_k)]_{j,k} \tag{1.19}$$

We will use the compact notation ∂_I, where I is the multi-index $I = (i_1, \ldots, i_p)$. Similarly, $\Omega^p(R)$ is generated by all p-forms $\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_\ell}$ and we have a canonical pairing

$$\mathrm{D}_p(R) \times \Omega^p(R) \to R. \tag{1.20}$$

Recall the *polynomial* and *total* gradings that were introduced in Definition 1.2.5. Each of these two gradings is the appropriate grading in a different context as we shall see.

Example 1.4.2.
- The Euler derivation $\theta_E = x_1 \partial_{x_1} + \cdots + x_\ell \partial_{x_\ell}$ belongs to the derivation module of every simple arrangement $\mathcal{A}$. In fact, $\theta_E(f) = f$ for every linear form f. Moreover, for every $(\mathcal{A}, \mathbf{m})$ we have $(\widetilde{Q}/Q)\theta_E \in \mathrm{D}(\mathcal{A}, \mathbf{m})$.

- If $H = \ker f \in \mathcal{A}$, then $\mathrm{d}f/f \in \Omega(\mathcal{A})$. Also, $\mathrm{d}f/f^{\mathbf{m}(H)} \in \Omega(\mathcal{A}, \mathbf{m})$.

- Let $(\mathcal{A}, \mathbf{m})$ be the Boolean multiarrangement defined by $\widetilde{Q} = xy^2z^3$ which is free for $p = 1, 2$. Here the exponents are $\exp(\mathcal{A}, \mathbf{m}) = (1, 2, 3)$. Moreover, $x\partial_x, y^2\partial_y, z^3\partial_z$ and $\frac{\mathrm{d}x}{x}, \frac{\mathrm{d}y}{y^2}, \frac{\mathrm{d}z}{z^3}$ are bases for D and Ω, respectively. A basis for $\mathrm{D}_2(\mathcal{A}, \mathbf{m})$ consists of $xy^2\partial_x \wedge \partial_y, xz^3\partial_x \wedge \partial_z$, and $y^2z^3\partial_y \wedge \partial_z$ and similarly for Ω^2. The modules corresponding to $p = 0$ and 3 are automatically free.

1.4.1 D and Ω

For the following proposition, recall that the *depth* of a module M over a polynomial ring is the length of a maximal M-sequence in the maximal homogeneous ideal. See [7, Section 1.2] for the technicalities.

Proposition 1.4.3. *Let $(\mathcal{A}, \mathbf{m})$ be a multiarrangement, then the depth of the logarithmic modules is at least 2.*

$$\operatorname{depth} \mathrm{D}_p(\mathcal{A}, \mathbf{m}) \geq 2, \quad \operatorname{depth} \Omega^p(\mathcal{A}, \mathbf{m}) \geq 2.$$

Proof. We only prove the derivation case. The other case is treated in [20]. Let us denote $\mathrm{D}_p(\mathcal{A}, \mathbf{m})$ by D. We need two polynomials $g, h \in R$, such that $g \notin Z(D)$ and $h \notin Z(D/gD)$, where Z denotes the zero divisors. Since D is torsion-free, the first requirement is always satisfied. However, in order to make the second one work we should choose them subject to the conditions that $\gcd(g, h) = 1$ and $\gcd(\widetilde{Q}, gh) = 1$, where $\widetilde{Q}$ is the defining polynomial. The second condition would fail if we had a derivation $\theta \in D$, such that $h\theta \in gD$. This amounts to saying that there is a derivation $\eta \in D$, such that $h\theta = g\eta$. Write θ as $\sum_I h_I \partial_{x_I}$, where I runs over all p-subsets of $[\ell]$ and h_I's are polynomials in R. Now, evaluate both sides of $h\theta = g\eta$ on all choices of p-subsets of the variables to see that for each multi-index I, h_I should be divisible by g. Thus, we can divide θ by g to get $h(\theta/g) = \eta \in D$. But since $\gcd(h, \widetilde{Q}) = 1$, we should have $\theta \in gD$. $\qquad\qquad\square$

The direct sum and tensor product have the following natural behavior that one would expect.

Proposition 1.4.4. *Let $(\mathcal{A}_1, \mathbf{m}_1)$ and $(\mathcal{A}_2, \mathbf{m}_2)$ be multiarrangements. Then*

$$\mathrm{D}_p\big((\mathcal{A}_1, \mathbf{m}_1) \oplus (\mathcal{A}_2, \mathbf{m}_2)\big) = \bigoplus_{i+j=p} \mathrm{D}_i(\mathcal{A}_1, \mathbf{m}_1) \otimes_{\mathbb{C}} \mathrm{D}_j(\mathcal{A}_2, \mathbf{m}_2).$$

See [2, Lemma 1.4] for a proof. In the special case $p = 1$, if R_1, R_2 are the respective coordinate rings, then above formula reduces to the following.

Corollary 1.4.5.

$$\mathrm{D}\big((\mathcal{A}_1, \mathbf{m}_1) \oplus (\mathcal{A}_2, \mathbf{m}_2)\big) = \big(\mathrm{D}(\mathcal{A}_1, \mathbf{m}_1) \otimes R_2\big) \oplus \big(R_1 \otimes \mathrm{D}(\mathcal{A}_2, \mathbf{m}_2)\big).$$

Note that a similar formulas hold for the modules of logarithmic forms.

Definition 1.4.6. If $(\mathcal{A}, \mathbf{m})$ and $(\mathcal{B}, \mathbf{n})$ are multiarrangements in the same ambient space V, then their union $(\mathcal{A}, \mathbf{m}) \cup (\mathcal{B}, \mathbf{n})$ is based on $\mathcal{A} \cup \mathcal{B}$ and is equipped with the multiplicity $\max\{\mathbf{m}, \mathbf{n}\}$, keeping in mind that if a hyperplane is absent in an arrangement, its multiplicity is zero and that maximum is taken componentwise.

Lemma 1.4.7. *Let $(\mathcal{A}, \mathbf{m})$ and $(\mathcal{B}, \mathbf{n})$ be ℓ-multiarrangements and their* union *defined by $\widetilde{Q}$, then for all $1 \leq p \leq \ell$, we have*

(i) $\mathrm{D}_p\big((\mathcal{A}, \mathbf{m}) \cup (\mathcal{B}, \mathbf{n})\big) = \mathrm{D}_p(\mathcal{A}, \mathbf{m}) \cap \mathrm{D}_p(\mathcal{B}, \mathbf{n}),$

(ii) $\Omega^p\big((\mathcal{A}, \mathbf{m}) \cup (\mathcal{B}, \mathbf{n})\big) = \big(\Omega^p(\mathcal{A}, \mathbf{m}) : \widetilde{Q}/\widetilde{Q}_{\mathcal{A}}\big) \cap \big(\Omega^p(\mathcal{B}, \mathbf{n}) : \widetilde{Q}/\widetilde{Q}_{\mathcal{B}}\big),$

where $\widetilde{Q}_{\mathcal{A}}$ and $\widetilde{Q}_{\mathcal{B}}$ are the corresponding unreduced defining polynomials.

Proof. In the derivation case, equality follows from the definitions together with the fact that if H is the union, then $(f^{\mathbf{m}(H)}) \cap (f^{\mathbf{n}(H)}) = (f^{\max\{\mathbf{m}(H), \mathbf{n}(H)\}})$.

For the second one, let ω be a form from the left side. We have $\widetilde{Q}\omega \in \Omega^p(R)$ and $\widetilde{Q}\omega \wedge \mathrm{d}f \in f^m \Omega^{p+1}(R)$, where again m is the maximum of $\mathbf{m}(H)$ or $\mathbf{n}(H)$. But we can write $\widetilde{Q}\omega \wedge \mathrm{d}f = \widetilde{Q}_{\mathcal{A}}(\widetilde{Q}/\widetilde{Q}_{\mathcal{A}})\omega \wedge \mathrm{d}f$ and $\widetilde{Q}\omega = \widetilde{Q}_{\mathcal{B}}(\widetilde{Q}/\widetilde{Q}_{\mathcal{B}})\omega$. For the other containment, pick a hyperplane H in the union and follow the same argument backward. $\qquad\square$

Corollary 1.4.8. *Let $(\mathcal{A}, \mathbf{m})$ be a multiarrangement defined by $\widetilde{Q} = \Pi_{i=1}^{n} f_i^{m_i}$. Then*

(i) $\mathrm{D}_p(\mathcal{A}, \mathbf{m}) = \bigcap_{H_i \in \mathcal{A}} \mathrm{D}_p(H_i, m_i) = \bigcap_{i=1}^{n} \mathrm{D}_p(\mathcal{A}, \mathbf{m}_i),$

(ii) $\Omega^p(\mathcal{A}, \mathbf{m}) = \bigcap_{H_i \in \mathcal{A}} (\Omega^p(H_i, m_i) : \widetilde{Q}/f_i^{m_i}) = \bigcap_{i=1}^{n} (\Omega^p(\mathcal{A}, \mathbf{m}_i) : \widetilde{Q}/f_1 \cdots f_i^{m_i} \cdots f_n).$

where (H_i, m_i) is the multiarrangement defined by only $f_i^{m_i}$.

The derivation module of a multiarrangement $(\mathcal{A}, \mathbf{m})$ may be written as an intersection of the derivation modules of multiarrangements where every component only has one multiple hyperplane. To be precise, given a multiplicity $\mathbf{m}$ on $\mathcal{A}$ and a hyperplane $H_i \in \mathcal{A}$, define

$$\mathbf{m}_i(H) = \begin{cases} \mathbf{m}(H) & H = H_i \\ 1 & H \neq H_i \end{cases}$$

It is immediate to see that $(\mathcal{A}, \mathbf{m}) = \cup_{m_i > 1}(\mathcal{A}, \mathbf{m}_i)$ and by Lemma 1.4.7 we get

$$\mathrm{D}(\mathcal{A}, \mathbf{m}) = \cap_{m_i > 1}\mathrm{D}(\mathcal{A}, \mathbf{m}_i). \tag{1.21}$$

As mentioned earlier, multiarrangements do not admit any canonical choice of derivation such as the Euler derivation that would lead to a decomposition of the derivation module like Formula 1.10. The following Lemma is however an attempt at getting a similar splitting. This in fact will be useful in dealing with the logarithmic ideal of multiarrangements. The proof is similar to [22, Theorem 4.27] and may be considered as a special case of [3, Theorem 0.4].

Proposition 1.4.9. *Let $(\mathcal{A}, \mathbf{m})$ be a multiarrangement in $\mathbb{C}^\ell$ that is defined by*

$$\widetilde{Q} = f_1 \ldots f_{i-1} f_i^m f_{i+1} \ldots f_n, \tag{1.22}$$

then the derivation module decomposes as

$$\mathrm{D}(\mathcal{A}, \mathbf{m}) = S f_i^{m-1} \theta_E \oplus \{\theta \in \mathrm{D}(\mathcal{A}, \mathbf{m}) : \theta(f_i) = 0\}. \tag{1.23}$$

where the second component is $\mathrm{Ann}(H_i) = \{\theta \in \mathrm{D}(\mathcal{A}) : \theta(f_i) = 0\}$.

Proof. Given $\theta \in \mathrm{D}(\mathcal{A}, \mathbf{m})$, let

$$\overline{\theta} = \theta - \frac{\theta(f_i)}{f_i}\theta_E, \tag{1.24}$$

where $\theta(f_i)/f_i$ is divisible by f_i^{m-1} and $\overline{\theta}(f_i) = 0$. Also, the derivation $\overline{\theta} \in \mathrm{Ann}(H_i)$. Now if $g f_i^{m_i} \theta_E \in \mathrm{Ann}(H_i)$, then $0 = g f_i^{m_i - 1} \theta_E(f_i) = g f_i^{m_i}$, forcing $g = 0$. $\square$

Corollary 1.4.10. *Let $\mathcal{A}$ be a simple arrangement, then $\mathrm{D}(\mathcal{A}) = R\theta_E \oplus \mathrm{Ann}(H_i)$, for all $i = 1, \ldots, n$. Consequently, if $\mathcal{A}$ is free, then $\exp(\mathcal{A})$ starts with $d_1 = 1$.*

Proof. Let $m = 1$ above. $\square$

Corollary 1.4.11. *If $\mathcal{A}$ is a free simple arrangement with*

$$\exp(\mathcal{A}) = (d_1, d_2, \ldots, d_\ell), \tag{1.25}$$

then the multiarrangement $Q(\mathcal{A}, \mathbf{m}) = f_1 \ldots f_{i-1} f_i^m f_{i+1} \ldots f_n$ is also free, with exponents

$$\exp(\mathcal{A}, \mathbf{m}) = (d_2, \ldots, d_\ell) \cup \{m\}. \tag{1.26}$$

Conversely, the freeness of $(\mathcal{A}, \mathbf{m})$ as above implies the freeness of the underlying simple arrangement.

Proof. The simple arrangement $\mathcal{A}$ is free exactly when $\mathrm{Ann}(H_i)$ is a free module with a homogeneous basis $\theta_2, \ldots, \theta_\ell$, such that $\deg \theta_i = d_i$. In view of the above splitting, the same is true about $(\mathcal{A}, \mathbf{m})$. Hence, adding $f_i^{m_i} \theta_E$ (or θ_E) to a basis $\theta_2, \ldots, \theta_\ell$ of $\mathrm{Ann}(H_i)$ when it is free returns a basis of $(\mathcal{A}, \mathbf{m})$ (or $\mathcal{A}$). $\square$

Even though the freeness of multiarrangements with only one hyperplane of high multiplicity is intimately related to the underlying arrangement and its combinatorics, after taking intersection as in Formula 1.21, it can be totally distorted. The following example demonstrates that even in the free case the combinatorics is unable to control the derivation module.

Example 1.4.12 (Ziegler's pair of examples [46]). Consider the following two multiarrangement. Let $(\mathcal{A}, \mathbf{m}) = x^3 y^3 (x - y)(x + y)$ and $(\mathcal{B}, \mathbf{n}) = x^3 y^3 (x - y)(x + 2y)$. Although these two multiarrangement have isomorphic lattices with multiplicities, we have $\exp(\mathcal{A}, \mathbf{m}) = (3, 5)$ and $\exp(\mathcal{B}, \mathbf{n}) = (4, 4)$. This is an indication of the difficulty in working with multiarrangements even in the free case.

1.4.2 Saito's Criterion for Freeness

The original formulation of the Saito's Criterion is due to [25, Theorem 1.8] where it was stated in the context of logarithmic sheaves on singular divisors in complex manifolds. It provides a handy test to tell whether a set of elements freely generate a logarithmic module. The result of this section is stated in terms of the derivation modules and is slightly more general since it covers the modules of different orders. A similar dual version holds for the module of logarithmic forms.

In general, $\mathrm{D}_p(\mathcal{A}, \mathbf{m})$ is a module of rank $\binom{\ell}{p}$, simply because $\mathrm{D}_p(R)$ is of rank $\binom{\ell}{p}$ and

$$\widetilde{Q} \cdot \mathrm{D}_p(R) \subseteq \mathrm{D}_p(\mathcal{A}, \mathbf{m}) \subseteq \mathrm{D}_p(R).$$

Lemma 1.4.13. *Let $(\mathcal{A}, \mathbf{m})$ be a multiarrangement defined by $\widetilde{Q}$ and let θ_i, $1 \leq i \leq \binom{\ell}{p}$, be elements in $\mathrm{D}_p(\mathcal{A}, \mathbf{m})$, then*

$$\mathsf{M}(\theta_1, \ldots, \theta_{\binom{\ell}{p}}) := \det_{\substack{1 \leq i_1 < \cdots < i_p \leq \ell \\ i}} [\theta_i(x_{i_1}, \ldots, x_{i_p})]$$

is a multiple of $\widetilde{Q}^{\binom{\ell-1}{p-1}}$.

Proof. This is proved similar to [22, Proposition 4.12]. □

Using the same ideas as Theorem 4.19 of [22] together with the lemma above, we get the following higher order version of Saito's criterion.

Theorem 1.4.14 (Saito's Criterion). *Let $(\mathcal{A}, \mathbf{m})$ be a multiarrangement defined by $\widetilde{Q}$ and let θ_i, $1 \leq i \leq \binom{\ell}{p}$, be elements in $\mathrm{D}_p(\mathcal{A}, \mathbf{m})$. Then $\mathrm{D}_p(\mathcal{A}, \mathbf{m})$ is freely generated by these elements if and only if*

$$\det_{\substack{1 \leq i_1 < \cdots < i_p \leq \ell \\ i}} [\theta_i(x_{i_1}, \ldots, x_{i_p})] \doteq \widetilde{Q}^{\binom{\ell-1}{p-1}}.$$

Proof. Let $r := \binom{\ell}{p}$ and $s := \binom{\ell-1}{p-1}$. If the determinantal equality above holds, then the derivations $\theta_1, \ldots, \theta_r$ are linearly independent, so for the 'if' part, it remains to show that they actually generate D_p. We can use Cramer's rule to solve θ_I's for all p-indices I by considering them as unknowns in the following system of r equations.

$$\theta_i = \sum_{|I|=p} \theta_i(x_I) \partial_I, \quad 1 \leq i \leq r$$

This amounts to having $\widetilde{Q}^s \partial x_I \in \mathrm{span}_R\{\theta_i : i = 1, \ldots, r\}$. Pick an arbitrary derivation $\eta \in \mathrm{D}_p(\mathcal{A}, \mathbf{m})$ and find g_i's so that

$$\widetilde{Q}^s \eta = \sum_{i=1}^{r} g_i \theta_i.$$

We are going to show that all g_i's have to be divisible by $\widetilde{Q}^s$. This would imply that η actually lives in the span of θ_i's. By lemma 1.4.13, the determinant

$$\mathsf{M}(\theta_1, \ldots, \theta_{i-1}, \eta, \theta_{i+1}, \ldots, \theta_r)$$

is a multiple of $\widetilde{Q}^s$ and $\mathsf{M}(\theta_1, \ldots, \theta_{i-1}, \widetilde{Q}^s\eta, \theta_{i+1}, \ldots, \theta_r) \in \widetilde{Q}^{2s}R$. But the determinant remains unchanged if we replace $\widetilde{Q}^s\eta$ with $g_i\theta_i$ which is just equal to $g_i\widetilde{Q}^s$. This implies that $\widetilde{Q}^s$ divides g_i for all $i = 1, \ldots, r$.

For the other direction, pick a basis of derivations $\theta_1, \ldots, \theta_r$ and let $\mathsf{M}(\theta_1, \ldots, \theta_r) = g\widetilde{Q}^s$, where $g \in R$. Let us assume that $H_1 = \ker x_1$. Consider the following sets of derivations: First, $\eta_I = \widetilde{Q}\partial_I$, where $1 \in I$. There are s of these. Second, $\eta_I = \widetilde{Q}/x_1^{m_1}\partial_I$, where $1 \notin I$. There are $\binom{\ell-1}{p}$ of these. There is a matrix of coefficients C with respect to the basis above, such that

$$\mathsf{M}_I(\eta_I) = \mathsf{M}(\theta_1, \ldots, \theta_r)C,$$

where I varies over all p-indices. Determinant of the left hand side, up to sign, equals

$$\widetilde{Q}^s(\widetilde{Q}/x_1^{m_1})^{\binom{\ell-1}{p}},$$

which should be divisible by $g\widetilde{Q}^s$. Consequently, g should divide $(\widetilde{Q}/x_1^{m_1})^{\binom{\ell-1}{p}}$ and similarly $(\widetilde{Q}/f_i^{m_i})^{\binom{\ell-1}{p}}$ for all $1 \leq i \leq n$, but since these polynomials have no common divisor, g can only be a constant. $\qquad\square$

Corollary 1.4.15. *Let $(\mathcal{A}, \mathbf{m})$ be a multiarrangement and let θ_i, $1 \leq i \leq \binom{\ell}{p}$ be homogeneous linearly independent elements in $\mathrm{D}_p(\mathcal{A}, \mathbf{m})$. Then it is free with θ_i's as a basis, if and only if*

$$\sum_{i=1}^{\binom{\ell}{p}} \deg\theta_i = \binom{\ell-1}{p-1}\sum_{H\in\mathcal{A}} \mathbf{m}(H).$$

By using the above fact we get a slightly different proof for [2, Lemma 1.3] as follows.

Corollary 1.4.16. *If $(\mathcal{A}, \mathbf{m})$ is a free arrangement, then so is $\mathrm{D}_p(\mathcal{A}, \mathbf{m})$ for all p.*

Proof. If $\mathrm{D}(\mathcal{A}, \mathbf{m})$ is free with a basis $\theta_1, \ldots, \theta_\ell$, let $\theta_{(i_1, \ldots, i_p)} := \theta_{i_1} \wedge \cdots \wedge \theta_{i_p}$ for every p-index I and compute the determinant $\mathsf{M}_I(\theta_I)$, as I ranges over all indices. $\quad\square$

1.4.3   cdga Structure

The main theorem of this section is the following:

Theorem 1.4.17. *Let $\Omega^\bullet(\mathcal{A}, \mathbf{m}) = \bigoplus_{i=0}^{\ell} \Omega^i(\mathcal{A}, \mathbf{m})$. Then $\Omega^\bullet(\mathcal{A}, \mathbf{m})$ is a commutative differential graded $\mathbb{C}$-algebra with multiplication*

$$\Omega^p(\mathcal{A}, \mathbf{m}) \times \Omega^q(\mathcal{A}, \mathbf{m}) \to \Omega^{p+q}(\mathcal{A}, \mathbf{m}),$$

defined by exterior product and the usual exterior differentiation d.

The proof is obtained in several steps as follows. We also verify the duality between D_p and Ω^p.

Proposition 1.4.18. *Let $(\mathcal{A}, \mathbf{m})$ be a multiarrangement. A rational differential form ω belongs to $\Omega^p(\mathcal{A}, \mathbf{m})$ if and only if for every hyperplane $H = \ker f \in \mathcal{A}$, there are rational differential forms ω' and ω'', such that*

$$\omega = \frac{\mathrm{d}f}{f^{m(H)}} \wedge \omega' + \omega'',$$

where $\widetilde{Q}/f^{m(H)}\omega' \in \Omega^{p-1}[V]$ and $\widetilde{Q}/f^{m(H)}\omega'' \in \Omega^p[V]$.

Proof. If $\omega = \mathrm{d}f/f^{m(H)} \wedge \omega' + \omega''$, then $\widetilde{Q}\omega \in \Omega^p[V]$. Also, $\mathrm{d}f \wedge \widetilde{Q}\omega = \mathrm{d}f\widetilde{Q} \wedge \omega'' = f^m(\widetilde{Q}\omega''/f^m) \wedge \mathrm{d}f \in f^m\Omega^{p+1}(\mathcal{A}, \mathbf{m})$.

Conversely, let $\omega = \eta/\widetilde{Q}$ and for a hyperplane $H = \ker f \in \mathcal{A}$, introduce new coordinates if necessary such that $f = x_1$. Expand η in the standard basis of $\Omega^p(R)$ and split it as

$$\eta = \mathrm{d}x_1 \wedge \sum_{|I|=p-1} g_I \mathrm{d}x_I + \sum_{|I|=p} h_I \mathrm{d}x_I,$$

where $I \subseteq \{2, \ldots, \ell\}$ and the coefficients are polynomials in R. Here $x_1^{m(H)}$ divides $\mathrm{d}x_1 \wedge \eta$, implying that all h_I's are divisible by $x_1^{m(H)}$, for $|I| = p$. So, by letting $\omega' = f^{m(H)}/\widetilde{Q} \sum_{|I|=p-1} g_I$ and $\omega'' = 1/\widetilde{Q} \sum_{|I|=p} h_I$ we obtain the desired format of the lemma. $\square$

Proposition 1.4.19. *Wedge product defines a map*

$$\Omega^p(\mathcal{A}, \mathbf{m}) \times \Omega^q(\mathcal{A}, \mathbf{m}) \to \Omega^{p+q}(\mathcal{A}, \mathbf{m}).$$

Proof. Let $\omega_1 \in \Omega^p(\mathcal{A}, \mathbf{m})$ and $\omega_2 \in \Omega^q(\mathcal{A}, \mathbf{m})$. For each $H_i = \ker f_i \in \mathcal{A}$, we can use Proposition 1.4.18 to write

$$\omega_i = \frac{\mathrm{d}f_i}{f_i^{m_i}} \wedge \omega_i' + \omega_i'', \quad i = 1, 2$$

Calculation shows that $\widetilde{Q}/f_i^{m_i}\omega_1 \wedge \omega_2$ is of the form $\mathrm{d}f_i/f_i^{m_i}\omega' + \omega''$, with $\widetilde{Q}/f_i^{m_i}\omega' \in \Omega^{p+q-1}[V]$ and $\widetilde{Q}/f_i^{m_i}\omega'' \in \Omega^{p+q}[V]$, which again by the Proposition 1.4.18 implies that $\widetilde{Q}/f_i^{m_i}\omega_1 \wedge \omega_2 \in \Omega^{p+q}(\mathcal{A}, \mathbf{m})$. So for each $1 \leq i \leq n$, we have $\widetilde{Q}^2/f_i^{m_i}\omega_1 \wedge \omega_2 \in \Omega^{p+q}[V]$ and $\widetilde{Q}^2/f_i^{m_i}\omega_1 \wedge \omega_2 \wedge \mathrm{d}f_i \in f_i^{m_i}\Omega^{p+q+1}[V]$ but here we can drop a factor of $\widetilde{Q}$ because $\widetilde{Q}/f_1^{m_1}, \ldots, \widetilde{Q}/f_n^{m_n}$ and $\widetilde{Q}$ are relatively prime. $\qquad\square$

Proposition 1.4.20. *Let $0 \leq p \leq \ell - 1$. Then the exterior differentiation d defines a map $\Omega^p(\mathcal{A}, \mathbf{m}) \to \Omega^{p+1}(\mathcal{A}, \mathbf{m})$.*

Proof. Let $\eta \in \Omega^p(\mathcal{A}, \mathbf{m})$ be of the form $\omega/\widetilde{Q}$. By definition,

$$\omega \wedge \mathrm{d}f_i = f_i^{m_i}\alpha_i, \tag{1.27}$$

with $\alpha_i \in \Omega^{p+1}(R)$, for every $i = 1, \ldots, n$. We have

$$\mathrm{d}\eta = \frac{\mathrm{d}\omega\widetilde{Q} - \omega \wedge \mathrm{d}\widetilde{Q}}{\widetilde{Q}^2} = \frac{\mathrm{d}\omega - \omega \wedge \frac{\mathrm{d}\widetilde{Q}}{\widetilde{Q}}}{\widetilde{Q}}, \tag{1.28}$$

and we need two things. The first is that the numerator is a polynomial form. This is obvious for the first summand. For the second summand, we have

$$\omega \wedge \frac{\mathrm{d}\widetilde{Q}}{\widetilde{Q}} = \sum_{j=1}^{n} m_j \, \omega \wedge \frac{\mathrm{d}f_j}{f_j}, \tag{1.29}$$

where every time $\omega \wedge \mathrm{d}f_j$ has a factor of $f_j^{m_j}$.

The second requirement is that

$$\mathrm{d}\omega \wedge \mathrm{d}f_i - \omega \wedge \frac{\mathrm{d}\widetilde{Q}}{\widetilde{Q}} \wedge \mathrm{d}f_i \in f_i^{m_i}\Omega^{p+2}(R). \tag{1.30}$$

For the first term, by differentiating formula 1.27, we get $\mathrm{d}\omega \wedge \mathrm{d}f_i = m_i f_i^{m_i-1}\mathrm{d}f_i \wedge \alpha_i + f_i^{m_i}\mathrm{d}\alpha_i$, but since $\mathrm{d}f_i \wedge \alpha_i = 0$, we are only left with the second term, which is of the desired form. What remains to show is that $1/\widetilde{Q}\,\omega \wedge \mathrm{d}\widetilde{Q} \wedge \mathrm{d}f_i$ is in $f_i^{m_i}\Omega^{p+2}(R)$. We have

$$1/\widetilde{Q}\,\omega \wedge \mathrm{d}\widetilde{Q} \wedge \mathrm{d}f_i = \pm f_i^{m_i}\alpha_i \sum_{j=1}^{n} \mathrm{d}f_j/f_j, \tag{1.31}$$

and it is enough to show that for each j, $1 \leq j \leq \ell$, we have

$$f_i^{m_i}\alpha_i \wedge \mathrm{d}f_j/f_j \in f_i^{m_i}\Omega^{p+2}(R), \tag{1.32}$$

and for this, it is enough to observe that $\alpha_i \wedge \mathrm{d}f_j/f_j \in \Omega^{p+2}(R)$: If $j = i$, then as we have seen before, the summand vanishes. If $j \neq i$, then since $\omega \wedge \mathrm{d}f_i \wedge \mathrm{d}f_j = f_i^{m_i} \alpha_i \wedge \mathrm{d}f_j = \pm f_j^{m_j} \alpha_j \wedge \mathrm{d}f_i$, we see that $f_j^{m_j}$ divides $\alpha_i \wedge \mathrm{d}f_j$ and this finishes the argument. $\qquad\square$

Let M_1, M_2 be modules over some ring A. An A-bilinear map $\Phi : M_1 \times M_2 \to A$ is called a *nondegenerate pairing* if for every $0 \neq m_1 \in M_1$, the map $\Phi(m_1, -) : M_2 \to A$ is nonzero and similarly for every nonzero element in the second factor.

Corollary 1.4.21. *Let $(\mathcal{A}, \mathbf{m})$ be a multiarrangement of size $|\mathbf{m}| = m$, then for every $0 \leq p \leq \ell$, we have a nondegenerate pairing*

$$\Omega^p(\mathcal{A}, \mathbf{m}) \otimes_R \Omega^{\ell-p}(\mathcal{A}, \mathbf{m}) \to R[m - \ell].$$

In particular, the modules $\Omega^p(\mathcal{A}, \mathbf{m})$ and $\Omega^{\ell-p}(\mathcal{A}, \mathbf{m})$ are dual to one another. To be precise, under the total grading, we have $\mathrm{Hom}_R(\Omega^p(\mathcal{A}, \mathbf{m}), R) \simeq \Omega^{\ell-p}(\mathcal{A}, \mathbf{m})[\ell - m]$.

Proof. Use Proposition 1.4.19 together with the fact that $\Omega^\ell(\mathcal{A}, \mathbf{m}) = \widetilde{Q}^{-1}\Omega^\ell(R)$, which is isomorphic to R up to some shift of degrees. To be precise, under the polynomial (total) grading $\Omega^\ell(\mathcal{A}, \mathbf{m}) \cong R[m]$ $(R[m - \ell])$, where $m = |(\mathcal{A}, \mathbf{m})| = \sum_H \mathbf{m}(H)$. See Definition 1.2.5. For nondegeneracy, pick some $\omega \in \Omega^p(\mathcal{A}, \mathbf{m})$, expand it in the standard basis of Ω^p and assume that for all $\eta \in \Omega^{\ell-p}(\mathcal{A}, \mathbf{m})$, we have $\omega \wedge \eta = 0$. Then in particular we can let $\eta = (1/\widetilde{Q})\mathrm{d}x_{j_1} \wedge \cdots \wedge \mathrm{d}x_{j_{\ell-p}}$, for all $\ell - p$ indices $1 \leq j_1 < \cdots < j_{\ell-p} \leq \ell$, showing that all terms of ω have to be zero. $\quad\square$

Proposition 1.4.22. *The natural pairing $\langle \theta, \omega \rangle \mapsto \theta(\omega)$ extends to the following nondegenerate pairings*

$$\mathrm{D}(\mathcal{A}, \mathbf{m}) \times \Omega^p(\mathcal{A}, \mathbf{m}) \to \Omega^{p-1}(\mathcal{A}, \mathbf{m})$$
$$\mathrm{D}_p(\mathcal{A}, \mathbf{m}) \times \Omega^1(\mathcal{A}, \mathbf{m}) \to \mathrm{D}_{p-1}(\mathcal{A}, \mathbf{m})$$

which induce isomorphisms $\mathrm{D}_p(\mathcal{A}, \mathbf{m}) \simeq \Omega^p(\mathcal{A}, \mathbf{m})^*$ and $\Omega^p(\mathcal{A}, \mathbf{m}) \simeq \mathrm{D}_p(\mathcal{A}, \mathbf{m})^*$, for $p = 1, \ldots, \ell$. In particular, D_p and Ω^p are reflexive modules.

Proof. Let $\theta \in \mathrm{D}(\mathcal{A}, \mathbf{m})$ and $\omega \in \Omega^p(\mathcal{A}, \mathbf{m})$ and pick a defining form f_i and consider $\langle \theta, \omega \rangle$, for which we need to verify the two conditions above. Obviously, we have

$$\widetilde{Q}\langle \theta, \omega \rangle = \langle \theta, \widetilde{Q}\omega \rangle \in R, \tag{1.33}$$

simply because by definition $\widetilde{Q}\omega \in \Omega^p(R)$. For the other membership, we expand $\langle \theta, \widetilde{Q}\omega \wedge \mathrm{d}f_i \rangle$ as

$$\langle \theta, \widetilde{Q}\omega \rangle \wedge \mathrm{d}f_i + (-1)^p \widetilde{Q}\omega \langle \theta, \mathrm{d}f_i \rangle. \tag{1.34}$$

But by definition, $\widetilde{Q}\omega \wedge \mathrm{d}f_i \in f_i^{m_i}\Omega^{p+1}(R)$ and the R-linearity of the pairing allows us to the factor $f_i^{m_i}$ out and see that the whole expression belongs to $f_i^{m_i}\Omega^p(R)$. The same is true about the second summand in the expression above. This implies that $\langle \theta, \widetilde{Q}\omega \rangle \wedge \mathrm{d}f_i \in f_i^{m_i}\Omega^p(R)$, as desired.

In order to verify nondegeneracy, one may use the proof of [20] where they use induction and local properties of D_p and Ω^p transfers to multiarrangements but since the case $p = 1$ is of special importance, we modify the proof given in [22] to present explicit maps.

Let $\alpha : \mathrm{D}(\mathcal{A}, \mathbf{m}) \to \Omega^1(\mathcal{A}, \mathbf{m})^*$ and $\beta : \Omega^1(\mathcal{A}, \mathbf{m}) \to \mathrm{D}(\mathcal{A}, \mathbf{m})^*$ be natural maps we get from the pairing in Proposition 1.4.22. We need to show that these maps are injective and surjective.

α: If $\theta = \sum_i g_i \partial_{x_i} \in \mathrm{D}(\mathcal{A}, \mathbf{m})$ is in the kernel of α, then we pair it with $\mathrm{d}x_j \in \Omega^1(\mathcal{A}, \mathbf{m})$, $j = 1, \ldots, l$, to see that $g_j = 0$. Also, given $\eta \in \Omega^1(\mathcal{A}, \mathbf{m})^*$, define a derivation $\widetilde{\eta}$ by letting $\widetilde{\eta}(f) = \eta(\mathrm{d}f)$. In order to check that $\widetilde{\eta}$ is actually in $\mathrm{D}(\mathcal{A}, \mathbf{m})$, pick a defining form f_i and note that $\mathrm{d}f_i/f_i^{m_i} \in \Omega^1(\mathcal{A}, \mathbf{m})$, so $\eta(\mathrm{d}f_i/f_i^{m_i})$ is a polynomial and $\widetilde{\eta}(f_i) = f_i^{m_i}\eta(\mathrm{d}f_i/f_i^{m_i}) \in (f_i^{m_i})$.

β: If $\omega = 1/\widetilde{Q}\sum_i h_i \eta \mathrm{d}x_i \in \ker \beta$, then $\langle \widetilde{Q}\partial_{x_j}, \omega \rangle = h_j = 0$, so $\omega = 0$. To show that it is surjective, if $\omega \in \mathrm{D}(\mathcal{A}, \mathbf{m})^*$, then let $\widetilde{\omega} = 1/\widetilde{Q}\sum_i \omega(\widetilde{Q}\partial_{x_i})\mathrm{d}x_i$ and check that $\beta(\widetilde{\omega}) = \omega$. $\qquad \square$

Corollary 1.4.23. *Let $(\mathcal{A}, \mathbf{m})$ be a multiarrangement with $m = |\mathbf{m}|$ hyperplanes. Then*

$$\mathrm{D}_p(\mathcal{A}, \mathbf{m}) \simeq \Omega^{\ell-p}(\mathcal{A}, \mathbf{m})[\ell - m].$$

Moreover, under this isomorphism the maps of Proposition 1.4.22 fit into the diagram

$$
\begin{array}{ccc}
\Omega^{\ell-p}(\mathcal{A}, \mathbf{m})[\ell - m] & \xrightarrow{\ \omega_{\mathbf{a}}\ } & \Omega^{\ell-p+1}(\mathcal{A}, \mathbf{m})[\ell - m] \\
\downarrow & & \downarrow \\
\mathrm{D}_p(\mathcal{A}, \mathbf{m}) & \xrightarrow{\ \langle\ ,\omega_{\mathbf{a}}\rangle\ } & \mathrm{D}_{p-1}(\mathcal{A}, \mathbf{m})
\end{array}
\tag{1.35}
$$

commutes.

Proof. This is an immediate consequence of Proposition 1.4.21 and Proposition 1.4.22 where both modules are isomorphic to $\mathrm{Hom}_R(\Omega^p(\mathcal{A}, \mathbf{m}), R)$. The commutativity of the diagram follows from chasing the pertaining maps. $\qquad\square$

Remark 1.4.24. We saw in Theorem 1.4.17 that $\Omega^\bullet(\mathcal{A}, \mathbf{m})$ is a cdga, so it makes sense to consider its cohomology and in the non-simple case define

$$H^\bullet(\mathcal{A}, \mathbf{m}) := H^\bullet(\Omega^\bullet(\mathcal{A}, \mathbf{m}), \mathrm{d}).$$

This is supported by the main result of [39], in which the authors show that under the tame hypothesis (Definition 1.4.36), the de Rham cohomology chain complex of the complement of a simple arrangement is quasi-isomorphic to the algebra of logarithmic forms $\Omega^\bullet(\mathcal{A})$.

1.4.4 Homological Dimensions and Local Properties

Given a multiarrangement $(\mathcal{A}, \mathbf{m})$, we can use localization to define submultiarrangements that respect the order of $L(\mathcal{A})$ in the following way.

$$X \in L(\mathcal{A}) \mapsto (\mathcal{A}_X, \mathbf{m}|_{\mathcal{A}_X})$$

If $X \leq Y$, then $\mathcal{A}_X \subseteq \mathcal{A}_Y$ and since the multiplicities are just the restrictions of the original multiplicity, we get an inclusion of multiarrangements $(\mathcal{A}_X, \mathbf{m}|_{\mathcal{A}_X}) \hookrightarrow (\mathcal{A}_Y, \mathbf{m}|_{\mathcal{A}_Y})$. One can also compose the localization with both D_p and Ω^p in order to get functors as follows:

$$L(\mathcal{A}) \to R - \mathrm{Mod} \qquad\qquad L(\mathcal{A}) \to R - \mathrm{Mod},$$
$$X \mapsto \Omega^p(\mathcal{A}_X, \mathbf{m}|_{\mathcal{A}_X}) \qquad\qquad X \mapsto \mathrm{D}_p(\mathcal{A}_X, \mathbf{m}|_{\mathcal{A}_X})$$

Note the these two functors are covariant and contravariant, respectively.

Definition 1.4.25. For $\mathfrak{p} \in \mathrm{Spec}R$ and $X \in L(\mathcal{A})$, let

$$X(\mathfrak{p}) = \bigcap_{\substack{\ker f_i = H_i \in \mathcal{A}_X \\ f_i \in (\mathfrak{p})}} H_i.$$

A covariant/contravariant functor $\mathscr{F} : L(\mathcal{A}) \to R - \mathrm{Mod}$ is called *local* if for any prime ideal $\mathfrak{p}$ and any $X \in L(\mathcal{A})$ the localization of the map $\mathscr{F}(X(\mathfrak{p}) \to X)$ at $\mathfrak{p}$ is an isomorphism.

If the above intersection is empty, we make the convention that $X(\mathfrak{p})$ equals the ambient vector space V.

Proposition 1.4.26. *The functors $X \mapsto \Omega^p(\mathcal{A}_X, \mathbf{m}|_{\mathcal{A}_X})$ and $X \mapsto \mathrm{D}_p(\mathcal{A}_X, \mathbf{m}|_{\mathcal{A}_X})$ are local.*

Proof. For $X \in L(\mathcal{A})$, we have $X \subseteq X(\mathfrak{p})$, which implies $\mathcal{A}_{X(\mathfrak{p})} \subseteq \mathcal{A}_X$. Let $\widetilde{Q} = \widetilde{Q}(\mathcal{A}_X, \mathbf{m}|_{\mathcal{A}_X})/\widetilde{Q}(\mathcal{A}_{X(\mathfrak{p})}, \mathbf{m}|_{\mathcal{A}_{X(\mathfrak{p})}})$. Pick an element $\theta/1$ from the right hand side. By definition of $X(\mathfrak{p})$, $\widetilde{Q} \notin \mathfrak{p}$. Thus, $\theta/1$ equals $\widetilde{Q}\theta/\widetilde{Q} = \theta/1$ belongs to the left hand side too. The case of Ω is similar, except that it works covariantly. $\qquad\square$

Lemma 1.4.27. *Let $(\mathcal{A}, \mathbf{m})$ be a multiarrangement and let $\mathfrak{p}$ be the maximal ideal corresponding to a point $pt \in M(\mathcal{A})$. Then*

$$\mathrm{D}_p(\mathcal{A}, \mathbf{m})_{\mathfrak{p}} \cong \mathrm{D}_p(R)_{\mathfrak{p}} \quad and \quad \Omega^p(\mathcal{A}, \mathbf{m})_{\mathfrak{p}} \cong \Omega^p(R)_{\mathfrak{p}}.$$

Proof. Let X be the origin and observe that $X(\mathfrak{p}) = V$ and $\mathcal{A}_{X(\mathfrak{p})} = \phi$, the empty arrangement, whose derivation module is just $\mathrm{D}_p(R)$ while by our choice $(\mathcal{A}, \mathbf{m})_X = (\mathcal{A}, \mathbf{m})$. By the local property of D_p the above claim follows. The other case is similar. $\qquad\square$

Definition 1.4.28. A central ℓ-multiarrangement $(\mathcal{A}, \mathbf{m})$ is called *locally free* if for every $X \in L_{<\ell}(A)$, the localization $(\mathcal{A}, \mathbf{m})_X$ is free.

Lemma 1.4.29. *Let $(\mathcal{A}, \mathbf{m})$ be a multiarrangement. Then $\mathcal{A}$ is free iff $\mathrm{D}(\mathcal{A}, \mathbf{m})$ is locally free at all max points/prime points of $\mathrm{Spec}R$.*

Proof. Freeness is not preserved under localization in general, but in our case since we are working over either polynomial or local rings, the statement remains unchanged after switching to projective (or even flat) instead of free, which is true from the general theory. $\qquad\square$

Proposition 1.4.30. *A multiarrangement $(\mathcal{A}, \mathbf{m})$ is locally free iff $\mathrm{D}(\mathcal{A}, \mathbf{m})$ is locally free on the punctured spectrum $\mathrm{Spec}R \setminus \mathfrak{m}$.*

Proof. Let $(\mathcal{A}, \mathbf{m})$ be essential of rank ℓ. The 'only if' part only uses the local functors.

For the if part, let $X \in L_{<\ell}(\mathcal{A})$ and consider $\mathrm{D}(\mathcal{A}, \mathbf{m})_X$. By the lemma above, we need to show that $\mathrm{D}((\mathcal{A}, \mathbf{m})_X)$ is free at all max points pt of the spec. There are three cases.

If pt is outside of the complement of $\mathcal{A}_X$, then it is free. In fact it is isomorphic to $\mathrm{D}(R)|_{pt}$ in this case.

If pt is on the union of hyperplanes of $\mathcal{A}_X$ but not on X, then by the local functor business we see that it has the same localization as $(\mathcal{A}, \mathbf{m})$.

If pt is on X, then independent of where it is on X, its localization is going to have the same structure. By the proof of [42, Lemma 2.1], localization of D at $\mathfrak{m} \neq \mathfrak{p} \in X$ over $R_{\mathfrak{p}}$ is isomorphic to its localization at zero over $R_{\mathfrak{m}}$. The map comes from translating the point pt to 0 on X and this reduces the problem back to the last case. □

Corollary 1.4.31. *An arrangement $\mathcal{A}$ is free iff it is locally free and $\mathrm{D}(\mathcal{A})_{\mathfrak{m}}$ is free, where $\mathfrak{m}$ is the maximal irrelevant ideal.*

Recall that the *projective dimension* of a module, denoted pd, is the length of a minimal projective resolution. See [35, Chapter 4] for details and characterizations.

Proposition 1.4.32. *Let $(\mathcal{A}, \mathbf{m})$ be a multiarrangement of rank ℓ. Then*

$$\mathrm{pd}\, \mathrm{D}_p(\mathcal{A}, \mathbf{m}),\ \mathrm{pd}\, \Omega^p(\mathcal{A}, \mathbf{m}) \leq \ell - 2.$$

Proof. This follows from Proposition 1.4.3 and the Auslander-Buchsbaum formula $\mathrm{pd} + \mathrm{depth} = \ell$. □

Corollary 1.4.33. *Every rank 2 multiarrangement is free.*

Proof. Follows from above Proposition in view of the Quillen-Suslin Theorem [24, Theorem 4.100]. □

Proposition 1.4.34. *Let $(\mathcal{A}, \mathbf{m})$ be a multiarrangement of rank ℓ. Then the following statements are equivlanet for all $p = 1, \ldots, \ell - 1$:*

(1) $\mathrm{D}_p(\mathcal{A}, \mathbf{m})$ *is locally free.*

(2) $\Omega^p(\mathcal{A}, \mathbf{m})$ *is locally free.*

(3) $\widetilde{\mathrm{D}_p(\mathcal{A}, \mathbf{m})}$ *is a locally free sheaf (i.e. a vector bundle) on $\mathbb{P}^{\ell-1}$.*

(4) $\widetilde{\Omega^p(\mathcal{A}, \mathbf{m})}$ *is a locally free sheaf (i.e. a vector bundle) on $\mathbb{P}^{\ell-1}$.*

(5) *For every $X \in L_{<\ell}(\mathcal{A})$, $\mathrm{D}_p(\mathcal{A}, \mathbf{m})_X$ is free.*

(6) *For every $X \in L_{<\ell}(\mathcal{A})$, $\Omega^p(\mathcal{A}, \mathbf{m})_X$ is free.*

Proof. Proof is analogous to [20, Theorem 2.3]. □

Proposition 1.4.35. *If* $\widetilde{\mathrm{D}_1(\mathcal{A}, \mathbf{m})}$ *is a vector bundle (i.e.* $(\mathcal{A}, \mathbf{m})$ *is locally free), then the natural map* $\wedge^p \mathrm{D}_1(\mathcal{A}, \mathbf{m}) \to \mathrm{D}_p(\mathcal{A}, \mathbf{m})$ *induces an isomorphism* $\wedge^p \widetilde{\mathrm{D}_1(\mathcal{A}, \mathbf{m})} \simeq \widetilde{\mathrm{D}_p(\mathcal{A}, \mathbf{m})}$.

Proof. Proof is analogous to [20, Proposition 2.9] and can be formalized under Proposition 1.4.30, simply because we throw away the origion (irrelevant maximal ideal) by passing to the associated sheaf on the projective space. □

Here are two technical conditions which are useful weakening of freeness (See [39], [28] and [13]).

Definition 1.4.36. A multiarrangement $(\mathcal{A}, \mathbf{m})$ is called *tame* if $\mathrm{pd}\,\Omega^p(\mathcal{A}, \mathbf{m}) \leq p$ for $1 \leq p \leq \ell$. Also, $(\mathcal{A}, \mathbf{m})$ is called *dually tame* if $\mathrm{pd}\,\mathrm{D}_p(\mathcal{A}, \mathbf{m}) \leq p$ for all $1 \leq p \leq \ell$.

These inequalities are automatically satisfied for $p = \ell$ and $\ell - 1$ by Proposition 1.4.32.

Example 1.4.37.
- Every free arrangement is obviously tame and dually tame.

- By Proposition 1.4.32, every arrangement of rank at most 3 is both tame and dually tame.

- A small non-example (which should be of rank at least 4) is defined by

$$\widetilde{Q} = x_1 x_2 x_3 x_4 (x_1 + x_2)(x_2 + x_3)(x_3 + x_1)(x_1 + 2x_2 + 3x_3 + 4x_4). \quad (1.36)$$

  Computation by the computer algebra system [16, Macaulay 2] shows that

$$\mathrm{pd}\,\mathrm{D}(\mathcal{A}) = 2,$$

  therefore $\mathcal{A}$ is not dually tame, although it is tame. Also, the multiarrangement defined by Q^2 serves as non-simple example.

Proposition 1.4.38. *Let* $(\mathcal{A}, \mathbf{m})$ *be a multiarrangement and let* $X \in L(\mathcal{A})$, *then*

$$\mathrm{pd}\,\mathrm{D}_p(\mathcal{A}, \mathbf{m})_X \leq \mathrm{pd}\,\mathrm{D}_p(\mathcal{A}, \mathbf{m}),$$
$$\mathrm{pd}\,\Omega^p(\mathcal{A}, \mathbf{m})_X \leq \mathrm{pd}\,\Omega^p(\mathcal{A}, \mathbf{m}).$$

Proof. See the proof of [42, Lemma 2.1] where this is proven in the derivation case for $p = 1$ and note that the proof works in general. □

Corollary 1.4.39. *If $(\mathcal{A}, \mathbf{m})$ is free (tame, dually tame), then all of its localizations are free (tame, dually tame).*

Proposition 1.4.40. *Let $(\mathcal{A}_i, \mathbf{m}_i)$, $i = 1, 2$, be two multiarrangements. The projective dimension of the direct sum is given by*

$$\mathrm{pd}\, \mathrm{D}\big((\mathcal{A}_1, \mathbf{m}_1) \oplus (\mathcal{A}_2, \mathbf{m}_2)\big) \quad = \quad \max_{i=1,2} \mathrm{pd}\, \mathrm{D}(\mathcal{A}_i, \mathbf{m}_i).$$

Proof. Let $R_i = \mathbb{C}[\mathcal{A}_i]$, for $i = 1, 2$ and $R = R_1 \otimes R_2$. By Corollary 1.4.5

$$\mathrm{D}\big((\mathcal{A}_1, \mathbf{m}_1) \oplus (\mathcal{A}_2, \mathbf{m}_2)\big) \cong \big(\mathrm{D}(\mathcal{A}_1, \mathbf{m}_1) \otimes_{\mathbb{C}} R_2\big) \oplus \big(\mathrm{D}(\mathcal{A}_2, \mathbf{m}_2) \otimes_{\mathbb{C}} R_1\big).$$

The projective dimension of a direct sum is the maximum of the projective dimensions and

$$\mathrm{pd}_R \big(\mathrm{D}(\mathcal{A}_i, \mathbf{m}_i) \otimes_{\mathbb{C}} R_j\big) = \mathrm{pd}_{R_i} \mathrm{D}(\mathcal{A}_i, \mathbf{m}_i)$$

for $i \neq j \in \{1, 2\}$. $\qquad\square$

Corollary 1.4.41. *The direct sum is free if and only if its components are free.*

1.5 Characteristic and Poincaré Polynomials

In this section we discuss the characteristic and Poincaré polynomials of arrangements and multiarrangements in connection to the Solomon-Terao formula and the Tutte polynomial. The main theorem of this section is a generalization of the Mustaţă-Schenck formula to multiarrangements.

Definition 1.5.1. Let $R = \oplus_{d \in D} R_d$ be graded algebra over a field F that is graded by a monoid D. Also let M be a graded module or algebra of finite type over R, i.e. each component M_d for each degree d is a finite dimensional vector space. Then the *Hilbert function* is defined by

$$f(d) := \dim M_d.$$

The *Hilbert Series* is defined by

$$h(M) := \sum_{d \in D} f(d) t^d.$$

Here we are mostly interested in gradings over either $\mathbb{N}$ or $\mathbb{N}^2$.

Note 1.5.2. All modules in this section are polynomially graded (Definition 1.2.5).

Definition 1.5.3. Define $\tilde{\Psi}$ and $\bar{\Psi}$ for $(\mathcal{A}, \mathbf{m})$ by

$$\tilde{\Psi}(\mathcal{A}, \mathbf{m}; x, y) := \sum_{p=0}^{\ell} h(\mathrm{D}_p(\mathcal{A}, \mathbf{m}), x)(y(x-1)-1)^p \tag{1.37}$$

$$\bar{\Psi}(\mathcal{A}, \mathbf{m}; x, y) := \sum_{p=0}^{\ell} h(\Omega^p(\mathcal{A}, \mathbf{m}), x)(y(1-x)-1)^p \tag{1.38}$$

which are a priori formal power series. However, it turns out that $\tilde{\Psi}$ is in fact a polynomial in x, i.e. it does not have poles at $x = 1$. See [2, Theorem 2.5]. Also, see [22, Proposition 4.133] where it is shown that $\bar{\Psi}$ is a polynomial in x for simple arrangements.

Definition 1.5.4 (Abe, Terao, Wakefield, [2])**.** The *characteristic* and *Poincaré polynomials* of a multiarrangement $(\mathcal{A}, \mathbf{m})$ are defined by

$$\chi\big((\mathcal{A}, \mathbf{m}), t\big) = (-1)^\ell \tilde{\Psi}(\mathcal{A}, \mathbf{m}; 1, t) = (-1)^\ell \lim_{x \to 1} \sum_{p=0}^{\ell} h(\mathrm{D}_p(\mathcal{A}, \mathbf{m}), x)(t(x-1)-1)^p \tag{1.39}$$

$$\pi\big((\mathcal{A}, \mathbf{m}), t\big) = (-t)^\ell \tilde{\Psi}(\mathcal{A}, \mathbf{m}; 1, -t^{-1}) = t^\ell \lim_{x \to 1} \sum_{p=0}^{\ell} h(\mathrm{D}_p(\mathcal{A}, \mathbf{m}), x)(\frac{-1}{t}(x-1)-1)^p \tag{1.40}$$

respectively. These two polynomials are related by the formula

$$\pi\big((\mathcal{A}, \mathbf{m}), t\big) = (-t)^\ell \chi\big((\mathcal{A}, \mathbf{m}), -t^{-1}\big).$$

The characteristic polynomial may alternatively be defined by

$$\chi\big((\mathcal{A}, \mathbf{m}), t\big) := \bar{\Psi}(\mathcal{A}, \mathbf{m}; 1, t) = \lim_{x \to 1} \sum_{p=0}^{\ell} h(\Omega^p(\mathcal{A}, \mathbf{m}), x)(t(1-x)-1)^p$$

which implies an alternative formulation for the Poincaré polynomial as

$$\pi\big((\mathcal{A}, \mathbf{m}), t\big) = (-t)^\ell \bar{\Psi}(\mathcal{A}, \mathbf{m}; 1, -t^{-1}) = (-t)^\ell \lim_{x \to 1} \sum_{p=0}^{\ell} h(\Omega^p(\mathcal{A}, \mathbf{m}), x)(\frac{-1}{t}(1-x)-1)^p$$

The equivalence of the definitions follows from the following lemma.

Lemma 1.5.5. *Let $(\mathcal{A}, \mathbf{m})$ be an arrangement of $m = |\mathbf{m}|$ hyperplanes. Then for every $d \in \mathbb{N}$, we have*

$$(-x^{-d})^{\ell} \tilde{\Psi}(\mathcal{A}, \mathbf{m}; x, \frac{x^d - 1}{1 - x}) = x^{m-\ell} \bar{\Psi}(\mathcal{A}, \mathbf{m}; x, \frac{1 - x^{-d}}{1 - x}).$$

As a consequence, $(-1)^{\ell} \tilde{\Psi}(\mathcal{A}, \mathbf{m}; 1, t) = \bar{\Psi}(\mathcal{A}, \mathbf{m}; 1, t)$.

Proof. Let d be an arbitrary number. Expand the left hand side as

$$(-x^{-d})^{\ell} \sum_{p=0}^{\ell} h(\mathrm{D}_p(\mathcal{A}, \mathbf{m}), x)(-x^d)^p$$

in which we use Corollary 1.4.23 to switch to the logarithmic forms

$$\sum_{p=0}^{\ell} h(\Omega^{\ell-p}(\mathcal{A}, \mathbf{m}), x) x^{m-\ell}(-x^{-d})^{\ell-p}$$

By [2, Theorem 2.5], the left hand side has no poles at $x = 1$ which will also be true for the right hand side. So, we may evaluate both sides at $x = 1$ to get

$$(-1)^{\ell} \tilde{\Psi}(\mathcal{A}, \mathbf{m}; 1, -d) = \bar{\Psi}(\mathcal{A}, \mathbf{m}; 1, -d).$$

Now since the left hand side is a polynomial and this identity holds for infinitely many values of d, it hold in general. $\square$

As noted earlier, the characteristic polynomial of simple arrangements has a combinatorial definition which only makes use of the lattice $L(\mathcal{A})$. The following pair of examples illustrate the fact the converse of the Factorization Theorem (Corollary 3.4.10) is false even in the case of simple arrangements.

Example 1.5.6 (Nonfree with split characteristic polynomial)**.** Let $\mathcal{A}$ be the arrangement $Q = yz(x-y)(x-2y)(x+y)(x+2y)(x+z)$ which is illustrated below [22, Figure

4.5].
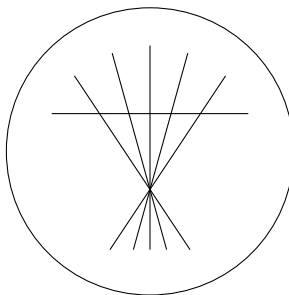
An easy computation gives

$$\chi(\mathcal{A}; t) = (t-1)(t-3)^2,$$

although $\mathcal{A}$ is not free. A multiarrangement example with the same property is supported by $\mathcal{A}$ with multiplicity $\mathbf{m} = (1, 1, 2, 2, 2, 1, 1)$. Computation shows that $(\mathcal{A}, \mathbf{m})$ is not free, but

$$\chi((\mathcal{A}, \mathbf{m}); t) = (t-4)(t-3)^2.$$

These examples show that the converse of the Factorization Theorem (Corollary 3.4.10) is not valid. However, if we are lucky, we might be able to tell if an arrangement is non-free by just computing its characteristic polynomial.

Definition 1.5.7. Let $\mathcal{A}$ be an arrangement, then its *Tutte polynomial* is defined by

$$T_\mathcal{A}(x, y) = \sum_{\mathcal{B} \subseteq \mathcal{A}} (x-1)^{r-\mathrm{rank}\mathcal{B}} (y-1)^{|\mathcal{B}|-\mathrm{rank}\mathcal{B}},$$

where r denotes the rank of the original arrangement $\mathcal{A}$, and $\mathcal{B}$ runs over all subarrangements, including the empty one.

Tutte polynomial has an alternative recursive definition which will be discussed in Section 3.3. It is worth noting that the Tutte polynomial is a stronger invariant than the characteristic polynomial as the following theorem and example show.

Theorem 1.5.8. *Let $\mathcal{A}$ be an essential ℓ-arrangement, then*
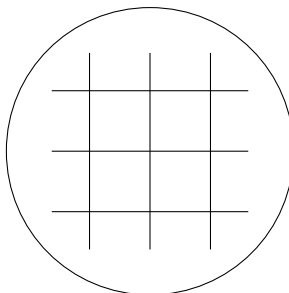
$$\chi(\mathcal{A}; t) = (-1)^\ell T_\mathcal{A}(1-t, 0).$$

See [11, Theorem 2.33] for a proof.

Example 1.5.9. Let us compare the arrangement $\mathcal{A}$ of Example 1.5.6, defined by

$$Q_\mathcal{A} = xyz(x-z)(x+z)(y+z)(y-z),$$

with the arrangement displayed below

which is defined by

$$Q_{\mathcal{B}} = xyz(x \pm z)(y \pm z).$$

The interesting fact in the comparison is that these two arrangements share the same characteristic polynomial, namely $(t - 1)(t - 3)^2$, while $\mathcal{A}$ is non-free in contrast to $\mathcal{B}$ which is free.

Computation shows that

$$T_{\mathcal{A}}(x, y) = y^4 + 2y^3 + 3y^2 + 4y + 4x + 4x^2 + x^3 + 3xy + 2xy^2 + xy^3$$
$$T_{\mathcal{B}}(x, y) = y^4 + 3y^3 + 4y^2 + 4y + 4x + 4x^2 + x^3 + 4xy + 2xy^2$$

So the Tutte polynomial is actually more sensitive than the characteristic polynomial which fails to detect the difference between $\mathcal{A}$ and $\mathcal{B}$ in terms of the combinatorial type and freeness.

The following theorem is the multiarrangement version of the main result of [41] by Mustaţă and Schenck. It seems that in general one can only compute the Poincaré polynomial of a multiarrangement when it is free (Factorization Theorem). This theorem presents an answer in terms of the Chern polynomial which might potentially be useful in computations.

Theorem 1.5.10. *Let $(\mathcal{A}, \mathbf{m})$ be a locally free multiarrangement. Then*

$$\overline{\pi}\big((\mathcal{A}, \mathbf{m}), t\big) = c_t(\widetilde{\Omega^1(\mathcal{A}, \mathbf{m})}),$$

where $\overline{\pi}$ is the class of the Poincaré polynomial in the Chow ring $\mathbb{Z}[t]/(t^\ell)$.

Proof. We only give a sketch of the proof because it goes parallel to the original one

given in [20]. If $\mathscr{E}$ is a vector bundle of rank r on $\mathbb{P}^{\ell-1}$, then define

$$R(\mathscr{E}; t, x) = (-t)^r (1-x)^{\ell-r} \sum_{i=0}^{r} h(H_\bullet^0(\wedge^i \mathscr{E}); x) \cdot (\frac{x-1}{t} - 1)^i, \qquad (1.41)$$

where $H_\bullet^0(\wedge^i \mathscr{E})$ is the finitely generated graded module $\bigoplus_{d \in \mathbb{Z}} H^0(\mathbb{P}^{\ell-1}, \wedge^i \mathscr{E}(d))$. The main ingredient of the proof is [20, Theorem 3.1] which states that the limit

$$\lim_{x \to 1} R(\mathcal{E}; t, x) \qquad (1.42)$$

exists and equals $= c_t(\mathcal{E})$. The proof is straight forward in case of split vector bundles. The idea in general case is to express the requirements of this identity in terms of some rational coefficient polynomial identities in the Chern classes of $\mathscr{E}$. The fact that these relations hold needs Hirzebruch-Riemann-Roch while comparing the Hilbert series of the global sections with the Hilbert polynomial.

Next step is to use Proposition 1.4.35 to observe that under the locally free condition, $\widetilde{D_p(\mathcal{A}, \mathbf{m})}$ is a vector bundle for all p, and that $\widetilde{D_p(\mathcal{A}, \mathbf{m})} = \wedge^p \widetilde{D_1(\mathcal{A}, \mathbf{m})}$. Here if we let $\mathscr{E} = \widetilde{D_1(\mathcal{A}, \mathbf{m})}$, then from formula 1.41, we get

$$\lim_{x \to 1} R(\widetilde{D_1(\mathcal{A}, \mathbf{m})}; -t, x) = c_{-t}(\widetilde{D_1(\mathcal{A}, \mathbf{m})})$$
$$= c_t(\widetilde{\Omega^1(\mathcal{A}, \mathbf{m})})$$

Final observation is that one is allowed to replace $H_\bullet^0(\wedge^i \widetilde{D_1})$ with $D_p(\mathcal{A}, \mathbf{m})$ and expand the limit as follows.

$$\lim_{x \to 1} R(\widetilde{D_1(\mathcal{A}, \mathbf{m})}; -t, x) = t^\ell \lim_{x \to 1} \sum_{p=0}^{\ell} h(D_p(\mathcal{A}, \mathbf{m}); x)(-\frac{1}{t}(x-1) - 1)^p$$
$$= \pi((\mathcal{A}, \mathbf{m}), t).$$

The Solomon-Terao Formula is used in the last equality. $\qquad \square$

Corollary 1.5.11. *The formula above is valid for all multiarrangements $(\mathcal{A}, \mathbf{m})$ of rank 3. In particular, it holds for the multirestriction of any rank 4 simple arrangement.*

Proof. If $\mathrm{rank}(\mathcal{A}, \mathbf{m}) = 3$, then for every $X \in L_{<\ell}(\mathcal{A})$, the localization $(\mathcal{A}, \mathbf{m})_X$ is of rank 2, which makes it automatically free by Corollary 1.4.33. $\qquad \square$

In the absence of an analogue for the Euler derivation for multiarrangements, it is not possible to recover the Poincaré polynomial after it is truncated in the ring $\mathbb{Z}[t]/t^\ell$. The following approach is an attempt to fix this problem.

Recall that extendible arrangements are introduced in Definition 1.3.13.

Proposition 1.5.12. *Let $(\mathcal{A}, \mathbf{m})$ be an extendible multiarrangement, such that it has an extension $\mathcal{E}$ which is (by definition a simple arrangement and) locally free, then $(\mathcal{A}, \mathbf{m})$ is a locally free multiarrangement.*

Proof. Assume that H is the distinguished hyperplane of $\mathcal{E}$ with $\mathcal{E}^H = (\mathcal{A}, \mathbf{m})$, where we think of $\mathcal{E}^H$ as natural restriction multiarrangement that is obtained by restriction at H. One can check that if $X \in L(\mathcal{A}) \subseteq L(\mathcal{E})$, then $(\mathcal{A}, \mathbf{m})_X = (\mathcal{E}^H)_X = (\mathcal{E}_X)^H$. Now since by assumption, $\mathcal{E}_X$ is free, we can apply Theorem 1.3.5 to see that $(\mathcal{E}_X)^H$ is a free multiarrangement. $\square$

Proposition 1.5.13. *Let $(\mathcal{A}, \mathbf{m})$ be an extendible multiarrangement which has a free extension $\mathcal{E}$, then*

$$\pi((\mathcal{A}, \mathbf{m}), t) = \frac{\pi(\mathcal{E}, t)}{1 + t}.$$

Proof. Let

$$\pi(\mathcal{E}, t) = (1 + t)\Pi_{i=2}^{\ell+1}(1 + d_i t),$$

be the Poincaré polynomial of $\mathcal{E}$, where $d_1(= 1), d_2, \ldots, d_{\ell+1}$ are the exponents of $\mathcal{E}$. Then by Ziegler's theorem and Theorem 4.1 of [2], the Poincaré polynomial of $(\mathcal{A}, \mathbf{m})$ is just $\Pi_{i=2}^{\ell+1}(1 + d_i t)$. $\square$

If we dualize the fomula $\mathrm{D}(\mathcal{A}) = R[-1] \oplus \mathrm{D}_0(\mathcal{A})$, we get $\Omega(\mathcal{A}) = R[1] \oplus \Omega_0(\mathcal{A})$ which is used in the following corollary.

Corollary 1.5.14. *Let $(\mathcal{A}, \mathbf{m})$ be an extendible multiarrangement which has a locally free extension $\mathcal{E}$, then for every $X \in L_{<\ell}(\mathcal{A})$, we have*

$$\pi((\mathcal{A}, \mathbf{m})_X, t) = c_t(\Omega_0^1(\mathcal{E}_X)).$$

Proof. Using the last Proposition together with Corollary 4.3 of [20], we have

$$\begin{aligned}
\pi((\mathcal{A}, \mathbf{m})_X, t) &= \frac{\pi(\mathcal{E})}{1 + t} \\
&= \frac{(1 + t)c_t(\Omega_0^1(\mathcal{E}_X))}{1 + t} = c_t(\Omega_0^1(\mathcal{E}_X)).
\end{aligned}$$

$\square$

It is very difficult to strengthen the statement of this corollary as the following example illustrates.

Remark 1.5.15. One might expect that under the assumptions of the last corollary we get

$$\pi(\mathcal{A}, \mathbf{m}) = c_t(\Omega_0^1(\mathcal{E})).$$

This is however not true. As a counter example, consider a simple 3-arrangement $\mathcal{E}$ with a non-split characteristic polynomial of the form $(t-1)(t^2 + bt + c)$. Note that in rank 3 all arrangements are locally free. By assumption, if we drop the $t-1$ factor, we get a polynomial that is irreducible over $\mathbb{Z}$, while by the Factorization Theorem for multiarrangements, $\pi(\mathcal{E}^{H_0})$ must split.

This failure is there even when the characteristic polynomial splits.

Example 1.5.16. Let $\mathcal{A}$ be arrangement defined in Example 1.5.6 and let $H = \ker(x - y)$. The natural multiarrangement obtained by restricting $\mathcal{A}$ to H is defined by

$$\widetilde{Q} = y^4(y + z)z$$

in $\mathbb{C}^2$, which has characteristic polynomial $(t-2)(t-4)$, while the characteristic polynomial of $\mathcal{A}$ is $(t-1)(t-3)^2$.

Chapter 2
Long Exact Sequences

The aim of this chapter is to obtain long exact sequences of the modules of derivations and forms under deletion and restriction when certain conditions are satisfied. As a result, we will get some insight into how the properties of tameness and dual tameness behave under deletion and restriction when certain conditions are satisfied.

2.1 LES for D_p

The following left-exact sequence relates the derivation module of a multiarrangement to that of its deletion and restriction. Here we are using the Euler multiplicity (Definition 1.3.14) for the restriction. Note that this sequence is in general not exact.

Proposition 2.1.1. *Let $(\mathcal{A}, \mathbf{m})$ be a multiarrangement, and let $(\mathcal{A}, \mathbf{m})'$ and $(\mathcal{A}, \mathbf{m})''$ be its deletion and restriction with respect to some hyperplane $H = \ker f \in \mathcal{A}$, then the sequence*

$$0 \to \mathrm{D}_p(\mathcal{A}, \mathbf{m})'[-1] \xrightarrow{\cdot f} \mathrm{D}_p(\mathcal{A}, \mathbf{m}) \xrightarrow{\rho} \mathrm{D}_p(\mathcal{A}, \mathbf{m})'' \tag{2.1}$$

is exact over R, where the first map is multiplication by f and ρ is restriction to H. In particular, this holds for simple arrangements.

Proof. Derivation modules live inside free modules and are hence torsion-free, so multiplication by f is injective and sends elements of $\mathrm{D}_p(\mathcal{A}, \mathbf{m})'$ injectively to derivations in $\mathrm{D}_p(\mathcal{A}, \mathbf{m})$. Looking at the second map, we need to check that it is well-defined and that it really lands where it should. The latter claim follows from applying the argument in the proof of [3, Proposition 2.2] to each component. To be precise, let $\theta \in \mathrm{D}_p(\mathcal{A}, \mathbf{m})$ and $g_1, \ldots, g_p \in R$, then

$$\overline{\theta}(\overline{g_1}, \ldots, \overline{g_p}) := \overline{\theta(g_1, \ldots, g_p)},$$

where bar denotes reduction mod f. If $(\overline{g_1}, \ldots, \overline{g_p}) = (\overline{h_1}, \ldots, \overline{h_p})$, then for each $1 \leq i \leq p$, $g_i - h_i \in fR$. Since $\theta \in \mathrm{D}_p(\mathcal{A}, \mathbf{m})$, and $g_1 - f_1$ is a multiple of f,

$\theta(g_1 - h_1, g_2, \ldots, g_p)$ is dividable by f and hence

$$\theta(g_1, \ldots, g_p) = \theta(h_1, g_2, \ldots, g_p) \pmod{f}.$$

Similarly,

$$\theta(h_1, g_2, \ldots, g_p) = \theta(h_1, h_2, g_3 \ldots, g_p) \pmod{f},$$

and we inductively see that we can replace all g_i's on the right hand side with h_i's to eventually get

$$\theta(\overline{g_1}, \ldots, \overline{g_p}) = \theta(\overline{h_1}, \ldots, \overline{h_p}) \pmod{f}.$$

For exactness in the middle, note that if $\eta \in D_p(\mathcal{A}, \mathbf{m})'$, then $\overline{f\eta} = 0$. Also, if $\theta \in D_p(\mathcal{A}, \mathbf{m})$ and $\overline{\theta} = 0$, then we express θ as

$$\theta = \sum_I h_I \partial_I,$$

where I runs over all increasing p-indices (notation of Definition 1.4.1) and proceed by evaluating it at each $x_{I_0} = (x_{i_1}, \ldots, x_{i_p})$, once at a time. Note that values of a p-derivation at p-tuples consisting of only the coordinate functions determine the p-derivation. We have

$$\theta(x_{I_0}) = \sum_I h_I \partial_I(x_{I_0}) = h_{I_0} \in (f),$$

since in the determinant definition of $\partial_I(x_{I_0})$, we get 1 exactly when $I = I_0$, and zero otherwise. This computation implies $\theta/f_n = \sum_I (h_I/f)\partial_I$ is still a valid polynomial p-derivation which by comparison to the definition of $(\mathcal{A}, \mathbf{m})'$ is seen to live in $D(\mathcal{A}, \mathbf{m})'$. $\square$

Definition 2.1.2. A triple $(\mathcal{A}, \mathbf{m}, H)$ is called D_p-exact if the sequence 2.1 is short exact, i.e. the last map is onto.

Example 2.1.3. Here are a few examples of D_p-exact pairs.

- If $\mathcal{A}$ is rank 2 and $H \in \mathcal{A}$, then $(\mathcal{A}, H)$ is D_1-exact: Use x, y as coordinates and let $H = \ker x$. Then $\mathcal{A}''$ is defined by just y and $D(\mathcal{A}'')$ is freely generated by $y\partial_y$, which is the restriction of the Euler derivation $x\partial_x + y\partial_y$ to H.

- Direct sum of a D_p-exact pairs is again D_p-exact. To be precise, if $(\mathcal{A}_1, H)$ is

D$_p$-exact and $\mathcal{A}_2$ is any other arrangement, then

$$D_p(\mathcal{A}_1 \oplus \mathcal{A}_2) \to D_p(\mathcal{A}_1 \oplus \mathcal{A}_2)''$$

is also surjective for the following reason. Since $H \in \mathcal{A}_1$, by following the definitions we see that $(\mathcal{A}_1 \oplus \mathcal{A}_2)'' = (\mathcal{A}_1'' \oplus \mathcal{A}_2)$. By Corollary 1.4.5 D$_p(\mathcal{A}_1 \oplus \mathcal{A}_2) \simeq [D_p(\mathcal{A}_1) \otimes R_2] \oplus [R_1 \oplus D_p(\mathcal{A}_2)]$ and similarly for the restriction. Therefore the above map is identical to the natural map

$$[D_p(\mathcal{A}_1) \otimes R_2] \oplus [R_1 \oplus D_p(\mathcal{A}_2)] \to [D_p(\mathcal{A}_1'') \otimes R_2] \oplus [R_1'' \otimes D_p(\mathcal{A}_2)]$$

which is clearly surjective.

- As a consequence, totally free arrangements are D$_p$-exact for all p. (See Theorem 1.3.17)

- By [38, Theorem 3.4], if $\mathcal{A}$ is a non-Boolean generic arrangement and $H \in \mathcal{A}$, then $(\mathcal{A}, H)$ is D_1-exact.

Lemma 2.1.4. *Suppose f is a nonzerodivisor in a commutative ring R. If M is a R/f-module, then*

$$\operatorname{Ext}_{R/f}^q(M, R/f)[1] \cong \operatorname{Ext}_R^{q+1}(M, R)$$

for $q \geq 0$, and $\operatorname{Hom}_R(M, R) = 0$.

Proof. Since f is a non-zerodivisor, R/f has a free resolution

$$0 \to R \xrightarrow{f} R \to R/f \to 0.$$

So $\operatorname{Ext}_R^q(R/f, R) = 0$ unless $q = 1$, in which case we get $(R/f)[1]$. The change of rings spectral sequence

$$E_2^{pq} = \operatorname{Ext}_{R/f}^p(M, \operatorname{Ext}_R^q(R/f, R)) \Rightarrow \operatorname{Ext}_R^{p+q}(M, R),$$

has only one nonzero row by $q = 1$. $\qquad\square$

Theorem 2.1.5. *If $(\mathcal{A}, \mathbf{m}, H)$ is D$_p$-exact, then there is a long exact sequence of*

R-modules as follows.

$$
\begin{array}{cccc}
\xrightarrow{} & \mathrm{Ext}_R^1(\mathrm{D}_p(\mathcal{A},\mathbf{m}),R)[-1] \longrightarrow & \mathrm{Ext}_R^1(\mathrm{D}_p(\mathcal{A},\mathbf{m})',R) \longrightarrow & \mathrm{Ext}_{R''}^1(\mathrm{D}_p(\mathcal{A},\mathbf{m})'',R'') \\
0 \xrightarrow{} & \Omega^p(\mathcal{A},\mathbf{m})[-1] \xrightarrow{\ \cdot f\ } & \Omega^p(\mathcal{A},\mathbf{m})' \xrightarrow{\ \varrho'\ } & \Omega^p(\mathcal{A},\mathbf{m})''
\end{array}
$$

$$(2.2)$$

Here R'' *denotes the coordinate ring of* $\mathcal{A}''$, *namely* R/f. *In particular, this holds for simple arrangements.*

Proof. Start with the usual long exact sequence of the short exact sequence above

$$
\begin{array}{cccc}
\xrightarrow{} & \mathrm{Ext}_R^1(\mathrm{D}_p(\mathcal{A},\mathbf{m})'',R) \longrightarrow & \mathrm{Ext}_R^1(\mathrm{D}_p(\mathcal{A},\mathbf{m}),R) \longrightarrow & \mathrm{Ext}_R^1(\mathrm{D}_p(\mathcal{A},\mathbf{m})',R)[+1] \\
0 \to & \mathrm{Hom}_R(\mathrm{D}_p(\mathcal{A},\mathbf{m})'',R) \xrightarrow{} & \Omega^p(\mathcal{A},\mathbf{m}) \xrightarrow{} & \Omega^p(\mathcal{A},\mathbf{m})'[+1]
\end{array}
$$

and use Lemma 2.1.4 to see the first term dies, i.e $\mathrm{Hom}_R(\mathrm{D}_p(\mathcal{A},\mathbf{m})'',R) = 0$, since $\mathrm{D}_p(\mathcal{A},\mathbf{m})''$ is an R/f-module. Again, by the same lemma, we replace $\mathrm{Ext}_R^1(\mathrm{D}_p(\mathcal{A},\mathbf{m})'',R)$ with $\Omega^p(\mathcal{A},\mathbf{m})''$. Similarly, replace every $\mathrm{Ext}_R^i(\mathrm{D}_p(\mathcal{A},\mathbf{m})'',R)$ with $\mathrm{Ext}_R^{i-1}\mathrm{D}_p(\mathcal{A},\mathbf{m})'',R)[1]$.

Finally, adjust the shift of degrees by shifting the degrees of terms involving $(\mathcal{A},\mathbf{m})$ by 1 instead of shifting the degrees of the other two terms by -1. □

Corollary 2.1.6. *Let* $(\mathcal{A},H)$ *be* D_p-*exact for* $1 \le p \le \ell - 1$, *then* $\mathcal{A}'$ *is dually tame if both* $\mathcal{A}$ *and* $\mathcal{A}''$ *are dually tame.*

Proof. Let M be an arbitrary R-module. One needs to show that $\mathrm{Ext}_R^{p+1}(\mathrm{D}_p(\mathcal{A}),M)$ vanishes. Apply $\mathrm{Ext}_R^*(-,M)$ to the short exact sequence

$$0 \to \mathrm{D}_p(\mathcal{A},\mathbf{m})'[-1] \xrightarrow{\cdot f} \mathrm{D}_p(\mathcal{A},\mathbf{m}) \xrightarrow{\rho} \mathrm{D}_p(\mathcal{A},\mathbf{m})'' \to 0$$

and consider the part

$$\mathrm{Ext}_R^{p+1}(\mathrm{D}_p(\mathcal{A}),M) \to \mathrm{Ext}_R^{p+1}(\mathrm{D}_p(\mathcal{A}'),M)[1] \to \mathrm{Ext}_R^{p+2}(\mathrm{D}_p(\mathcal{A}''),M),$$

for all each p.

The first term is zero since $\mathcal{A}'$ is tame. In order to show that the last term is zero, use the following change of base spectral sequence

$$E_2^{r,s} = \mathrm{Ext}_{R''}^r(\mathrm{D}_p(A''),\mathrm{Ext}_R^s(R'',M)) \Rightarrow \mathrm{Ext}_R^{r+s}(\mathrm{D}_p(\mathcal{A}''),M)$$

and observe that $E_2^{r,s}$ is zero when $r + s = p + 2$. To be more precise, using $0 \to R \xrightarrow{f} R \to R'' \to 0$ as an R-resolution for R'', we get

$$\operatorname{Ext}_R^s(R'', M) = \begin{cases} (0 :_M f) & s = 0 \\ M/fM & s = 1 \\ 0 & s > 1 \end{cases}$$

On the other hand, since $\mathcal{A}''$ was assumed to be tame, after the pth row, all the terms are going to be zero. So, our spectral sequence only has the columns corresponding to $q = 0, 1$. As a consequence, this forces the last term of the exact sequence above and hence the middle term to vanish. $\qquad \square$

Proposition 2.1.7. *Let $(\mathcal{A}, H)$ be an D_1-exact pair such that $\mathcal{A}''$ is tame whose non-free locus is of codimension k in H. Then $(\mathcal{A}, H)$ is D_p-exact for all $p \leq k - 1$. In particular, in rank 4, every D_1-exact pair is also D_2-exact without any further assumptions.*

Proof. Similar to Proposition 2.2.7. $\qquad \square$

2.2 LES for Ω^p

Proposition 2.2.1. *Let $\mathcal{A}$ be a central arrangement of hyperplanes and let $\mathcal{A}'$ and $\mathcal{A}''$ be its deletion and restriction with respect to a hyperplane $\ker f = H \in \mathcal{A}$, then the sequence*

$$0 \to \Omega^p(\mathcal{A})[-1] \xrightarrow{\cdot f} \Omega^p(\mathcal{A}') \xrightarrow{\varrho} \Omega^p(\mathcal{A}'') \tag{2.3}$$

is exact, where the first map is multiplication by f and ϱ is restriction to H.

Proof. An element of $\Omega^p(\mathcal{A})$ is of the form ω/Q which gets sent to $f\omega/Q = \omega/Q' \in \Omega^p(\mathcal{A}')$. By lemma 1.4.18, $\omega/Q = (\mathrm{d}f/f) \wedge \omega' + \omega''$, where ω' and ω'' do not have poles on H. If we multiply this by f, we get $\mathrm{d}f \wedge \omega' + f\omega''$, which after restriction to $\ker f$ vanishes. The second map uses [46, Theorem 4.4], which says that Ω^p is a covariant functor from the category of arrangements to vector spaces. For exactness in the middle, note that if $\omega/Q' \in \Omega^p(\mathcal{A}')$ vanishes after restriction to H, then ω/Q becomes an element of $\Omega^p(\mathcal{A})$. $\qquad \square$

Definition 2.2.2. A pair $(\mathcal{A}, H)$ is called Ω_p-*exact* if sequence 2.3 is short exact, i.e. the last map is onto.

Similar to the derivation case, the rank 2 arrangements, direct sums and totally free arrangements provide examples for Ω^p-exactness property. Additionally, the functorial property of Ω^p allows using the following process to produce more examples.

Example 2.2.3. A hyperplane $H \in \mathcal{A}$ is called a *bridge* if $\text{rank}(\mathcal{A}) > \text{rank}(\mathcal{A}')$. This is motivated by the graphic terminology where a bridge is the last edge to connect two parts of a graph that would otherwise become two disjoint connected components.

Example 2.2.4. If $H \in \mathcal{A}$ is a bridge, then $(\mathcal{A}, H)$ is an Ω^p-exact pair for all p.

This fact was proven in [46] for $p = 1$ but its proof works in general. The fact that rank drops by pulling H out means that after a change of coordinates, there is a variable that does not appear in every other equation. So we may assume that $H = \ker x_\ell$ and $\partial_{x_\ell} f_i = 0$ for any other hyperplane $\ker f \in \mathcal{A}$. Now, the projection map $\pi : V \to H$ supports a hyperplane morphism $\Pi : \mathcal{A}'' \to \mathcal{A}'$. On the other hand, we have the usual morphism $I : \mathcal{A}' \to \mathcal{A}''$, which is supported by the inclusion $i : H \to V$. As a result, the composition $i \circ \pi = id_H$, which by functoriality implies that $i^* : \Omega^*(\mathcal{A}') \to \Omega^*(\mathcal{A}'')$ must be onto.

Theorem 2.2.5. *If $(\mathcal{A}, H)$ is Ω^p-exact, then there is a long exact sequence as follows.*

$$
\begin{array}{l}
\cdots \\
\hookrightarrow \text{Ext}_R^1(\Omega^p(\mathcal{A}'), R)[-1] \longrightarrow \text{Ext}_R^1(\Omega^p(\mathcal{A}), R) \longrightarrow \text{Ext}_{R''}^1(\Omega^p(\mathcal{A}''), R'') \\
0 \longrightarrow D_p(\mathcal{A}')[-1] \xrightarrow{\quad \cdot f \quad} D_p(\mathcal{A}) \xrightarrow{\quad \rho' \quad} D_p(\mathcal{A}'')
\end{array}
$$

$$(2.4)$$

Proof. Similar to Theorem 2.1.5. □

Corollary 2.2.6. *Let $(\mathcal{A}, H)$ be Ω^p-exact for $1 \le p \le \ell - 1$, then $\mathcal{A}$ is tame if both $\mathcal{A}'$ and $\mathcal{A}''$ are tame.*

Proof. Similar to Corollary 2.1.6. □

One may utilize the following proposition to make the requirements of the above corollary more manageable.

Proposition 2.2.7. *Let $(\mathcal{A}, H)$ be an Ω^1-exact pair such that $\mathcal{A}''$ is tame whose codimension of non-free locus is k in H. Then $(\mathcal{A}, H)$ is Ω^p-exact for all $p \le k - 1$. In particular, in rank 4, every Ω^1-exact pair is also Ω^2-exact without any further assumptions.*

Proof. For every p, we have a map $\wedge^p \Omega \to \Omega^p$ which a monomorphism for all arrangements. Under the above assumptions, when $p < k$, [13, Proposition 2.9] states that $\wedge^p \Omega^1(\mathcal{A}'') \to \Omega^p(\mathcal{A}'')$ is an isomorphism, as indicated in the following commutative diagram.

$$
\begin{array}{ccccc}
0 & \longrightarrow & \Omega^p(\mathcal{A})[-1] & \longrightarrow & \Omega^p(\mathcal{A}') & \overset{?}{\longrightarrow\!\!\!\!\longrightarrow} & \Omega^p(\mathcal{A}'') \\
& & & & \big\uparrow & & \big\uparrow \\
& & & & \wedge^p \Omega(\mathcal{A}') & \longrightarrow\!\!\!\!\longrightarrow & \wedge^p \Omega(\mathcal{A}'')
\end{array}
$$

$$(2.5)$$

The bottom map is onto because $(\mathcal{A}, H)$ is Ω^1-exact and we passed to the p-fold exterior product. The fact that the above map is onto follows from an easy chase of diagram. $\qquad\square$

2.3 D$_p$-exactness vs. Ω^p-exactness

The following lemma allows us to compare the D-exact and Ω-exact properties by providing alternative descriptions of the first connecting homomorphisms in the above long exact sequences. We state and prove it for $p = 1$ although it works for higher indices as well. Note that we are using simple arrangements, i.e. $\mathbf{m} \equiv 1$.

Lemma 2.3.1. *Let $\mathcal{A}$ be simple arrangement. The maps ϱ (2.3) and ϱ' (2.2) agree when $(\mathcal{A}, H)$ is D_1-exact. Similarly, the maps ρ (2.1) and ρ' (2.4) agree when $(\mathcal{A}, H)$ is Ω^1-exact.*

Proof. We show the proof for the first case. Let $(\mathcal{A}, H)$ be D_1-exact and consider the map ϱ' in (2.2) for $p = 1$ and $\mathbf{m} \equiv 1$. We have the following explicit description. Let $D = D(\mathcal{A})$. Given $\varphi \in \mathrm{Hom}_R(D(\mathcal{A}'), R) \simeq \Omega^1(\mathcal{A})$, the following diagram gets naturally filled with the restriction $\varphi_{|D}$ of ρ to $D(\mathcal{A}) \subset D(\mathcal{A}')$ and it commutes since ρ is R-linear. This forces the last map to agree with the reduction of φ modulo f on the right hand side.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & D(\mathcal{A}') & \overset{\cdot f}{\longrightarrow} & D(\mathcal{A}) & \overset{\rho}{\longrightarrow} & D(\mathcal{A}'') & \longrightarrow & 0 \\
& & \big\downarrow{\scriptstyle \varphi} & & \big\downarrow{\scriptstyle \varphi_{|D}} & & \big\downarrow{\scriptstyle \overline{\varphi}} & & \\
0 & \longrightarrow & R & \overset{\cdot f}{\longrightarrow} & R & \longrightarrow & R'' & \longrightarrow & 0
\end{array}
$$

$$(2.6)$$

Therefore the map $\varrho' : \mathrm{Hom}_R(\mathrm{D}(\mathcal{A}'), R) \to \mathrm{Hom}_{R''}(\mathrm{D}(\mathcal{A}''), R'')$ sends each φ to $\overline{\varphi}$, just as ϱ does. The proof in the other case is similar. $\qquad\square$

As a consequence we get the following theorem which shows how these two properties interact.

Theorem 2.3.2. *Let $\mathcal{A}$ be an arrangement.*

i. *If $(\mathcal{A}, H)$ is D_p-exact and $\mathrm{D}_p(\mathcal{A})$ is free (more generally, $\mathrm{Ext}^1_R(\mathrm{D}_p(\mathcal{A}), R) = 0$), then $(\mathcal{A}, H)$ is Ω^p-exact.*

ii. *If $(\mathcal{A}, H)$ is Ω^p-exact and $\Omega^p(\mathcal{A}')$ free (more generally, $\mathrm{Ext}^1_R(\Omega^p(\mathcal{A}'), R) = 0$), then $(\mathcal{A}, H)$ is D_p-exact.*

iii. *If $\mathrm{rank}(\mathcal{A}) = 3$ and $(\mathcal{A}, H)$ is Ω^1-exact, then $\mathcal{A}'$ is free if $\mathcal{A}$ is free.*

iv. *If $\mathrm{rank}(\mathcal{A}) = 3$ and $(\mathcal{A}, H)$ is D_1-exact, then $\mathcal{A}$ is free if $\mathcal{A}'$ is free.*

v. *In particular, if $\mathcal{A}$ has rank 3 and exactly one of $\mathcal{A}$ and $\mathcal{A}'$ is free, then $(\mathcal{A}, H)$ cannot be both D_1-exact and Ω^1-exact for any H.*

Proof. The first claim follows from the consequence

$$0 \to \Omega^p(\mathcal{A})[-1] \xrightarrow{\cdot f} \Omega^p(\mathcal{A}') \xrightarrow{\varrho = \varrho'} \Omega^p(\mathcal{A}'') \to \mathrm{Ext}^1_R(\mathrm{D}_p(\mathcal{A}), R),$$

which is a segment of the sequence 2.1.5, together with the assumption that the last module is zero. The second claim is similar and follows from 2.2.5.

The next two statements follow from the above sequences and the fact that when $\mathcal{A}$ is rank 3, then $\mathcal{A}''$ is of rank 2 and as a consequence automatically free. Last statement is similar.

$\qquad\square$

2.4 Examples

Ziegler gave a pair of examples of arrangements in [46] to illustrate the fact that the degree of generators of the derivation modules are not controlled by the combinatorics of the intersection lattice. This pair of examples appeared in [43] in a different guise to demonstrate the non-combinatorial behaviour of the 2-formality property. See 3.5.2. In fact, Ziegler originally prepared these examples to illustrate the fact the Betti

numbers of the logarithmic modules are not combinatorial in general. The following arrangements are from [43].

Example 2.4.1. Define the arrangements $\mathcal{A}_1$ and A_2 by

$$Q_1 = xyz(x+y+z)(2x+y+z)(2x+3y+z)(2x+3y+4z)(3x+5z)(3x+4y+5z)$$
$$Q_2 = xyz(x+y+z)(2x+y+z)(2x+3y+z)(2x+3y+4z)(x+3z)(x+2y+3z)$$
$$(2.7)$$

and note that only the last two hyperplanes are different. It is worth mentioning that the triple points of $\mathcal{A}_1$ are on a conic while the triple points of $\mathcal{A}_2$ are not. Consider the restrictions with respect to the first hyperplane $H = \ker x$. Direct computation shows that $\mathcal{A}_1''$ and $\mathcal{A}_2''$ are both free with $\exp = (1, 5)$, however in $D(\mathcal{A}_1)$ is minimally generated by one generator of degree 1 and eight generators of degree 8, while $D(\mathcal{A}_2)$ has one generator in degree 1, one in degree 5, five in degree 6 and two in degree 7. In summary, we have

$$\begin{aligned} D(\mathcal{A}_1): \quad & \beta_{0,1} = 1, \qquad\qquad\qquad\qquad\qquad\qquad \beta_{0,8} = 8 \\ D(\mathcal{A}_2): \quad & \beta_{0,1} = 1, \ \beta_{0,5} = 1, \ \beta_{0,6} = 5, \ \beta_{0,7} = 2 \end{aligned}$$

It is verified that the projection map $p_2 : D(\mathcal{A}_2) \to D(\mathcal{A}_2'')$ is onto, where as $p_1 : D(\mathcal{A}_1) \to D(\mathcal{A}_1'')$ fails to be onto. The reason is that the degree 5 generator of the target is not covered under p_1, since $D(\mathcal{A}_1)$ does not have a generator in degree 5. Therefore, $(\mathcal{A}_1, H)$ is not D_1-exact, while $(\mathcal{A}_2, H)$ is, although they have isomorphic lattices, $L(\mathcal{A}_1) \cong L(\mathcal{A}_2)$.

Chapter 3
Intersection Cycle, Recurrence and the Characteristic Polynomial

In this chapter, we switch to an ideal theoretic analog of the logarithmic modules to study simple arrangements. As seen in the last chapter, working for example with D has the disadvantage that the natural sequence

$$0 \to \mathrm{D}(\mathcal{A}')[-1] \xrightarrow{\cdot f} \mathrm{D}(\mathcal{A}) \xrightarrow{\rho} \mathrm{D}(\mathcal{A}'') \tag{3.1}$$

leaks information, as the last map fails to be surjective in general. This problem will be fixed in the new approach. The present chapter contains a new proof for the Solomon-Terao formula under the tame hypothesis. This is achieved by calculating the intersection cycle of the variety of an ideal related to the derivation module and comparison with the Grothendieck group of coherent sheaves.

3.1 Critical Points

Definition 3.1.1. Given an arrangement $\mathcal{A}$ with defining equation $Q = f_1 \cdots f_n$ and a choice of *weight vector* $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$, the corresponding *master function* is defined by

$$\Phi_\lambda = f_1^{\lambda_1} \cdots f_n^{\lambda_n}.$$

A point $x \in \mathbb{C}^\ell$ is called a *critical point* with respect to weight λ if it is a critical point of the function Φ_λ, i.e.

$$\frac{\partial \Phi_\lambda}{\partial x_i}\big|_x = 0,$$

for $i = 1, \ldots, \ell$.

Note that regardless of the terminology, a master function is not actually a function, as it is multi-valued. However, by the following lemma, if we restrict to points where the $\mathbb{C}^\ell$ factor is in the complement of the hyperplanes, we get a description as

the vanishing of the differential of a logarithmic function, which is a (well-defined) meromorphic 1-form.

Lemma 3.1.2. *A point $x \in M(\mathcal{A})$ is critical with respect to a weight λ if and only if it is a root of the differential 1-from*

$$\omega_\lambda = \sum_{i=1}^n \lambda_i \mathrm{d} f_i / f_i = \mathrm{d} \log \Phi_\lambda. \tag{3.2}$$

Proof. A point $x \in M(\mathcal{A})$ is critical if it is a root of

$$\partial \Phi / \partial x_j = \Phi \sum_{i=1}^n \lambda_i (\partial f_i / \partial x_j) f_i^{-1}$$

for all $j = 1, \ldots, \ell$. Since the value of Φ is nonzero, these equations can be compactly written as the vanishing of the single formula

$$\sum_{j=1}^\ell \sum_{i=1}^n \lambda_i (\partial f_i / \partial x_j) f_i^{-1} \mathrm{d} x_j = 0$$

which recovers ω_λ by just flipping the summations. See [23] for a slightly more general treatment. □

Definition 3.1.3. Let C denote the coordinate ring of the space of weights $\mathbb{C}^n$, namely $\mathbb{C}[a_1, \ldots, a_n]$. Define the *logarithmic 1-form* by

$$\omega_{\mathbf{a}} = \sum_{i=1}^n a_i \frac{\mathrm{d} f_i}{f_i}.$$

It is immediate to see from the definitions that $\omega_{\mathbf{a}} \in \Omega^1(\mathcal{A}) \otimes_\mathbb{C} C =: \Omega_C^1(\mathcal{A})$. The vanishing of this one form is defined by an ideal, called the *meromorphic ideal I_{mer}*, which is generated by ℓ rational polynomials, namely $\langle \partial_{x_j}, \omega_{\mathbf{a}} \rangle$, for $j = 1, \ldots, \ell$.

Let us save the letter S for the ring $R \otimes_\mathbb{C} C = \mathbb{C}[x_1, \ldots, x_\ell; a_1, \ldots, a_n]$. Note that $\Omega_C^1(\mathcal{A})$ is an S-module. Note that the meromorphic ideal I_{mer} lives in the local ring S_Q.

Definition 3.1.4. Given an arrangement $\mathcal{A} = \{H_1, \ldots, H_n\}$, together with a surjective map $\mu : \{a_1, \ldots, a_m\} \to \mathcal{A}$ for some $m \geq n$, one naturally gets a multiarrangement $(\mathcal{A}, \mathbf{m})$ by letting $\mathbf{m}(H) := |\mu^{-1}(H)|$, for all $H \in \mathcal{A}$. For every $\mathcal{A}$ and μ as

above, let

$$\omega_\mu = \sum_{i=1}^{n} \sum_{j \in \mu^{-1}(H_i)} a_j \frac{\mathrm{d}f_i}{f_i}.$$

In simple case, $n = m$ and $\mu : a_i \mapsto H_i$, for $i = 1, \ldots, n$.

Multirestriction in the sense of Ziegler provides a natural example of the above construction as follows. Let $\mathcal{A}$ be a simple arrangement of size n with a distinguished hyperplane H_0. Then define a map

$$z : \{a_1, \ldots, a_{n-1}\} \to \mathcal{A}''$$

by assigning $a_i \mapsto K \in \mathcal{A}''$, if $H_i \cap H_0 = K$. Note that $|(\mathcal{A}'', z)| = n - 1$, where $z : \mathcal{A}'' \to \mathbb{N}$ as a multiplicity is defined in Formula 1.13. Under the notation just introduced, we have the following Proposition, which is a modification of [23, Proposition 4.1].

Proposition 3.1.5. *Let $(\mathcal{A}, \mathbf{m})$ be a multiarrangement, then the zero locus of ω_μ is a nonsingular quasi-affine variety, denoted $\Sigma_\mu(\mathcal{A})$, whose codimension in $\mathbb{C}^\ell \times \mathbb{C}^m$ is* rank $(\mathcal{A})$.

Proof. Without loss of generality, we may assume that $\mathcal{A}$ is full rank. Let $f_i(x) = \sum_{j=1}^{\ell} c_{ji} x_j$. The solution space of $\omega_\mu = 0$ is the common zero set of

$$d_{\mu,i}(x) = \sum_{i=1}^{n} \Big(\sum_{j \in \mu^{-1}(H_i)} a_j \Big) \frac{c_{ji}}{f_i(x)},$$

for $j = 1, \ldots, \ell$, with $x \in \mathbb{C}^\ell$ and $\mathbf{a} \in \mathbb{C}^m$, where $m = |\mathbf{m}|$ and $n = |\mathcal{A}|$. In matrix notation, this is the same as the solution space of

$$\Big[c_{ji}/f_i(x) \Big] \begin{bmatrix} \sum_{j \in \mu^{-1}(H_1)} a_j \\ \vdots \\ \sum_{j \in \mu^{-1}(H_n)} a_j \end{bmatrix} = 0,$$

where we think of x as a parameter that runs over all points of the complement, a space of dimension ℓ. Having picked an $x \in M(\mathcal{A})$, the linear system above admits a solution space of dimension $n - \ell$. Again after fixing every solution, we get a space of dimension $\mathbf{m}(H_i) - 1$, for the ith component. Adding up the dimensions returns $m - \ell$. Therefore the solution space in $\mathbb{C}^\ell \times \mathbb{C}^n$ is of dimension m after the contribution of

$x \in M(\mathcal{A})$ is added, hence of codimension ℓ. Smoothness is implicitly verified in the above argument, as the dimension of the tangent space is the same at all points. The above argument also verifies the fact the natural projection $\Sigma_\mu \to M(\mathcal{A})$ is a trivial vector bundle. $\qquad\square$

Definition 3.1.6. Let $\mathcal{A}$ be a simple arrangement. The *logarithmic ideal* $I(\mathcal{A})$ (as introduced in [9]) is the image of the module of C-linear derivations $\mathrm{D}_C(\mathcal{A}) := \mathrm{D}(\mathcal{A}) \otimes_{\mathbb{C}} C$ (as S-module) and $\omega_{\mathbf{a}}$ under the pairing of Proposition 1.4.22 for $p = 1$.

$$I(\mathcal{A}) := \langle \mathrm{D}_C(\mathcal{A}), \omega_{\mathbf{a}} \rangle \tag{3.3}$$

Proposition 3.1.7. *Let $(\mathcal{A}, \mathbf{m})$ be a multiarrangement. The closure of Σ_μ is defined by the ideal $I_\mu(\mathcal{A}) = \langle \mathrm{D}_C(\mathcal{A}), \omega_\mu \rangle$, which is bi-homogeneous with respect to the grading of R and C.*

Proof. Similar to [9, Theorem 2.9]. $\qquad\square$

Proposition 3.1.8. *Under the above notations, the intersection $V(I_\mu(\mathcal{A})) \cap (M(\mathcal{A}) \times \mathbb{C}^m)$ equals Σ_μ.*

Proof. This follows from Lemma 1.4.27. See [23] for a proof in the simple case. $\qquad\square$

3.1.1 A Concrete Example

One of the key results of this chapter is Theorem 3.2.2 which relates the logarithmic ideal of an arrangement to that of its deletion and restriction. Here is an example to illustrate the answer.

Consider the rank 2 braid arrangement $\mathcal{A}$, defined by

$$Q = xy(x - y)$$

The logarithmic ideal $I(\mathcal{A})$ lives in the polynomial ring $R \otimes_{\mathbb{C}} C$, where $R = \mathbb{C}[x, y]$ and $C = [a_1, a_2, a_3]$. In an algebraic language, the main result of this chapter is centered around understanding the ideal $I(\mathcal{A}) + (a_i)$, for any variable a_i. It turns out that the combinatorics of $\mathcal{A}$ emerges in the associated primes of this ideal.

We know that $\mathcal{A}$ is free with a basis consisting of the Euler derivation $\theta_E = x\partial_x + y\partial_y$ and $\theta = xy(\partial_x + \partial_y)$ [use Saito's criterion, Theorem 1.4.14]. Pairing these derivations with the meromorphic one form

$$\omega_{\mathbf{a}} = a_1 \frac{\mathrm{d}x}{x} + a_2 \frac{\mathrm{d}y}{y} + a_3 \frac{\mathrm{d}(x - y)}{x - y}$$

returns the ideal $I(\mathcal{A}) = (a_1 + a_2 + a_3, a_1 y + a_2 x)$. One may notice that the exponents of $\mathcal{A}$ are realized as the degrees of these generators.

Consider $I(\mathcal{A}) + (a_3)$. Computation by hand or a computer ([16]) gives the prime decomposition

$$I(\mathcal{A}) + (a_3) = \big(a_1, a_2, a_3\big) \cap \big(a_1 + a_2, a_3, x - y\big)$$

One careful look at the decomposition suggests that the first ideal in the intersection is $I(\mathcal{A}') + (a_3)$ and the second one comes from the restriction multiarrangement A'' which consists of one hyperplane of multiplicity 2 in the following way.

Let ω_z be the meromorphic 1-form obtained by restricting $\omega_{\mathbf{a}'}$ to the last hyperplane, namely

$$\omega_z = (a_1 + a_2)\frac{\mathrm{d}\bar{x}}{\bar{x}},$$

where bar denotes the variable in the coordinate ring of restriction $R/(x - y)$. The variables a_1 and a_2 appear together because the first and second hyperplane have the same intersection with the last one.

One can now see that the second ideal equals the pull back of $\langle \mathrm{D}(\mathcal{A}''), \omega_z \rangle$ under the natural map

$$R \otimes C \to R \otimes C/(a_3) \to R/(x - y) \otimes C/(a_3).$$

One main result of this chapter is to show that this holds at a coarser geometric level in general.

3.2 A Geometric Deletion-Restriction Formula

Let $\mathcal{A}$ a simple arrangement. By Proposition 3.1.7, the ideal $I(\mathcal{A})$ is bihomogeneous in $R \otimes C[a_0]$ and defines a variety in $\mathbb{P}^{\ell-1} \times \mathbb{P}^n$ which we denote by $\mathfrak{X}(\mathcal{A})$. We are using the extra variable a_0 to avoid irrelevant ideals. Similarly, if $(\mathcal{A}, \mu)$ is a multiarrangement (see Definition 3.1.4), denote the projective zero locus of $I_\mu(\mathcal{A})$ in $\mathbb{P}^{\ell-1} \times \mathbb{P}^m$ by $\mathfrak{X}_\mu(\mathcal{A})$.

Starting with a simple arrangement $\mathcal{A}$, the following calculation gives a decomposition of $\mathfrak{X}(\mathcal{A}) \cap K$, where $K = \mathbb{P}^{\ell-1} \times \mathbb{P}(\ker a_n)$.

We are going to consider the deletion and restriction of $\mathcal{A}$ with respect to the last hyperplane H_n. The following two inclusion maps are going to be used to adjust

the ambient spaces of the components that appear.

$$\mathbb{P}(H_n) \times \mathbb{P}^{n-1} \xrightarrow{j_a} \mathbb{P}^{\ell-1} \times \mathbb{P}^{n-1} \xrightarrow{i_a} \mathbb{P}^{\ell-1} \times \mathbb{P}^n$$

Here j_a and i_a are the maps associated to the natural projection maps

$$R \otimes C[a_0] \xrightarrow{i} R \otimes C'[a_0] \xrightarrow{j} R'' \otimes C'[a_0],$$

where $C' = \mathbb{C}[a_1, \ldots, a_{n-1}] = C/(a_n)$ and $R'' = R/(f_n)$.

First use the map j_a to pull the meromorphic 1-form $\omega_{\mathbf{a}'} = \sum_{i=1}^{n-1} a_i \mathrm{d}f_i/f_i$ back to a form on $\mathbb{P}(H_n) \times \mathbb{P}^{n-1}$. This happens to be the same as ω_z, in the notation of Definition 3.1.4, where $z : \{a_1, \ldots, a_{n-1}\} \to \mathcal{A}''$ is Ziegler's natural restriction multiplicity (see formula 1.13). The vanishing of this form defines a manifold $\mathbb{P}\Sigma_z$. It follows from 3.1.7 that the closure of this manifold is defined by the ideal

$$(\theta, \langle j_a^*(\omega_{\mathbf{a}'}) \rangle : \theta \in \mathrm{D}(\mathcal{A}''))$$

of the ring $R'' \otimes_{\mathbb{C}} C'[a_0]$, which we denote by $I_z(\mathcal{A}'')$.

Definition 3.2.1. Let $\mathfrak{X}' := i_{a*}\mathfrak{X}(\mathcal{A}')$ and $\mathfrak{X}'' = (i_a \circ j_a)_* \mathfrak{X}_z(\mathcal{A}'')$. Algebraically, $\mathfrak{X}'$ and $\mathfrak{X}''$ are defined by $i^{-1}(I(\mathcal{A}'))$ and $(j \circ i)^{-1}(I_z(\mathcal{A}''))$, respectively. Note that $a_n \in i^{-1}(I(\mathcal{A}'))$ and $a_n, f_n \in (j \circ i)^{-1}(I_z(\mathcal{A}''))$.

One of the main theorems of this chapter is the following. Recall that $K = \mathbb{P}(H_n)$.

Theorem 3.2.2. *If H_n is a bridge, then $\mathfrak{X} \cap K = \mathfrak{X}'$, otherwise $\mathfrak{X} \cap K = \mathfrak{X}' \cup \mathfrak{X}''$.*

Algebraically, this comes down to showing that $I(\mathcal{A}) + (a_n)$ has two minimal ideals that define the components of $\mathfrak{X}(\mathcal{A}) \cap K$. As mentioned above, in our notation $I_z(\mathcal{A}'') = \langle \mathrm{D}(\mathcal{A}''), \omega_z \rangle$. The minimal ideals are

$$\min(I(\mathcal{A}) + (a_n)) = \{I(\mathcal{A}') + (a_n), I_z(\mathcal{A}'') + (a_n, f_n)\}.$$

In general, there are embedded primes. Proof will follow from a collection of partial steps as follows.

Lemma 3.2.3. $\mathfrak{X}', \mathfrak{X}'' \subseteq \mathfrak{X}(\mathcal{A}) \cap K$

Proof. Algebraically, these inclusions translate to the following inclusion of ideals:

$$I(\mathcal{A}) + (a_n) \subseteq i^{-1}(I(\mathcal{A}')) \quad \text{and} \quad (j \circ i)^{-1}(I_z(\mathcal{A}''))$$

Given an element $\langle \theta, \omega_{\mathbf{a}} \rangle \in I(\mathcal{A})$, with $\theta \in \mathrm{D}(\mathcal{A})$, since $\mathrm{D}(\mathcal{A}) \subseteq \mathrm{D}(\mathcal{A}')$, we have

$$\langle \theta, \omega_{\mathbf{a}} \rangle = \langle \theta, \omega_{\mathbf{a}'} \rangle + a_n \frac{\theta(f_n)}{f_n} \in i^{-1}(I(A')).$$

The second inclusion follows from the same observation, together with the fact that restriction of a derivation $\theta \in \mathrm{D}(\mathcal{A})$ to H_n (formula 3.1), returns $\theta|_{H_n} \in \mathrm{D}(\mathcal{A}'')$. To be more precise, we have $\langle \theta, \omega_{\mathbf{a}'} \rangle \in (j \circ i)^{-1}(I(\mathcal{A}''))$ because

$$\langle \theta, \omega_{\mathbf{a}'} \rangle|_{H_n} = \langle \theta|_{H_n}, j^*(\omega_{\mathbf{a}'}) \rangle.$$

$\square$

Proposition 3.2.4. *The connected components of the underlying matroid of an arrangement $\mathcal{A}$ contribute linear elements in $I(\mathcal{A}) \cap C$. More precisely, $\mathcal{B} \subseteq \mathcal{A}$ is a union of connected components if and only if $\sum_{H_i \in \mathcal{B}} a_i$ is an element of $I(\mathcal{A})$. Moreover, each connected component of the matroid corresponds to a sum of the variables $a_1, \ldots, a_n$ with minimal terms.*

Proof. After a change of coordinates, decompose $\mathcal{A}$ as $\mathcal{A}_1 \oplus \cdots \oplus \mathcal{A}_r$. Use formula $\mathrm{D}(\oplus_{i=1}^r \mathcal{A}_i) = \oplus_{i=1}^r R\,\mathrm{D}(\mathcal{A}_i)$ of Corollary 1.4.5 and apply Euler derivations $\theta_{E_1}, \cdots, \theta_{E_r}$ to the logarithmic form $\omega_{\mathbf{a}}$. For each index i we get

$$\langle \theta_{E_i}, \omega_{\mathbf{a}} \rangle = \sum_{H_j \in \mathcal{A}_i} a_j.$$

Conversely, if $\langle \theta, \omega_{\mathbf{a}} \rangle = \sum_{H_k \in \mathcal{B}} a_k$ for some subarrangement $\mathcal{B}$, then $\theta(f_i) = 0$ for $i \notin K$ and $\theta(f_k) = f_k$ for all $k \in K$. As a consequence, $\mathcal{B}$ will be forced to be a union of the irreducible components as above and θ has be equal to tensor product of the corresponding Euler derivations. A slightly weaker form of this is implicit in [9, Proposition 2.8]. $\square$

Corollary 3.2.5. *The following statements are equivalent:*

  i) H_n is a bridge;

  ii) $a_n \in I(\mathcal{A})$;

iii) $a_n \in \mathrm{rad}(I(\mathcal{A}))$.

Proof. By the above proposition, it is clear that the second statement follows from the first one and the last one is obviously a consequence of the second one. It remains to see that the last statement implies the first one. Let $H_n = \ker x_\ell$ and note that every element of $I(\mathcal{A})$ is of the form $\langle \theta, \omega_{\mathbf{a}} \rangle$, for some $\theta \in \mathrm{D}_C(\mathcal{A})$. If for some $r \in \mathbb{N}$, we get $a_n^r = \langle \theta, \omega_{\mathbf{a}} \rangle = \langle \theta, \omega_{\mathbf{a}'} \rangle + \langle \theta, a_n \mathrm{d}x_\ell/x_\ell \rangle$. It is clear that the first summand involves other a_i's with $i < \ell$ and can not contribute a_n^r. In fact, a_n^r can only be achieved by $a_n^{r-1} x_\ell \partial_{x_\ell}$ which implies that $x_{ell} \partial_{x_\ell}$ is a derivation on $\mathcal{A}$. This is only possible when H_n is disconnected from the other hyperplanes. $\square$

Proposition 3.2.6. *Let H_n be a bridge. Then $\mathfrak{X} = \mathfrak{X} \cap K = \mathfrak{X}'$.*

Proof. The fact that H_n is a bridge implies that $a_n \in I(\mathcal{A})$ and this verifies the first equality. For the second one, we need to show that $I(\mathcal{A}') + (a_n) \subseteq I(\mathcal{A}) + (a_n)$. Without loss of generality, assume that $f_n = x_\ell$ and that the other equations are free of x_ℓ, so $\mathcal{A}$ is the direct sum of $\mathcal{A}'$ and the hyperplane $H_n = \ker x_\ell$. Similar to the proof of the first lemma, we see that

$$\mathrm{D}(\mathcal{A}) = \mathbb{C}[x_\ell] \cdot \mathrm{D}(\mathcal{A}') \oplus \mathbb{C}[x_1, \ldots, x_{\ell-1}] \cdot x_\ell \partial_{x_\ell},$$

so in particular, $\mathrm{D}(\mathcal{A}') \subseteq \mathrm{D}(\mathcal{A})$, which implies the desired inclusion. The second formula follows from intersecting the first one with K. $\square$

The following three cases cover the proof of Theorem 3.2.2 in the non-bridge case.

3.2.1   case 1: Localization

Lemma 3.2.7. *If $H = \ker f_n$, then there is an isomorphism of R_{f_n}-modules*

$$\mathrm{D}(\mathcal{A})_{f_n} \cong \mathrm{D}(\mathcal{A}')_{f_n}.$$

Proof. After passing to localization, the multiplication map

$$\mathrm{D}(\mathcal{A}') \xrightarrow{\cdot f_n} \mathrm{D}(\mathcal{A})$$

becomes an inverse to the localization of the inclusion $\mathrm{D}(\mathcal{A}) \hookrightarrow \mathrm{D}(\mathcal{A}')$. Therefore, the localized modules are isomorphic. $\square$

Proposition 3.2.8. *The restriction of $\mathfrak{X}(\mathcal{A})$ to $\{a_n = 0\}$ agrees with $\mathfrak{X}(\mathcal{A}')$, away from the hyperplane defined by f_n. To be more precise, we have*

$$\mathfrak{X} \cap V(a_n) \cap D(f_n) = i_*(\mathfrak{X}(\mathcal{A}')) \cap D(f_n).$$

Proof. The inclusion $i_*(\mathfrak{X}(\mathcal{A}')) \hookrightarrow \mathbb{P}^{\ell-1} \times \mathbb{P}^n$ is defined by $I(\mathcal{A}') + (a_n)$. Intersection with $D(f_n)$ corresponds to passing to the localization modulo f_n, so we have

$$
\begin{aligned}
(I(\mathcal{A}) + (a_n))_{f_n} &= (\langle \mathrm{D}(\mathcal{A}), \omega_{\mathbf{a}} \rangle, a_n)_{f_n} \\
&= (\langle \mathrm{D}(\mathcal{A})_{f_n}, \omega_{\mathbf{a}} \rangle, a_n/1) \\
&= (\langle \mathrm{D}(\mathcal{A}')_{f_n}, \omega_{\mathbf{a}} \rangle, a_n/1) \quad \text{by Lemma 3.2.7} \\
&= (\langle \mathrm{D}(\mathcal{A}'), \omega_{\mathbf{a}} \rangle, a_n)_{f_n} \\
&= (I(\mathcal{A}') + (a_n))_{f_n}
\end{aligned}
$$

$\square$

3.2.2 case 2: Complement of Restriction

Proposition 3.2.9. *Consider $(x, \lambda) \in \mathfrak{X} \cap K$, where $x \in M(\mathcal{A}'')$, i.e. $f_n(x) = 0$ and $f_i(x) \neq 0$, for all $i = 1, \ldots, n-1$. Then $(x, \lambda) \in \mathfrak{X}''$.*

Proof. Since $\mathfrak{X} = \overline{\Sigma}$, we may think of (x, λ) as the limit point of a sequence in Σ and under the above assumptions modify the sequence in order to get a new sequence with the same limit point, such that the new sequence avoids the unwanted area.

For convenience assume that $f_n = x_\ell$ and note that modulo f_n, the vanishing of the 1-form along the sequence converging to (x, λ), becomes the vanishing of following $\ell - 1$ equations

$$\sum_{i=1}^{n-1} \lambda_i \frac{c_{j,i}}{f_i} = 0, \quad j = 1, \ldots, \ell - 1,$$

which is what it takes for the point to be in $\mathfrak{X}''$, except that the x-coordinates might not lie on the complement M''. In order to fix this, let us modify the sequences to get a new one for which this problem is fixed.

Assume that $f_1, \ldots, f_{\ell-1}$ are independent. Choose a new sequence $\{y_k\}$ in M'' with the same limit as $\{x_k\}$, namely $x \in M''$. Having made this choice, define a new

sequence $\{\gamma_k = (\gamma_{k,1}, \ldots, \gamma_{k,n})\}_k$, by setting

$$\frac{\gamma_{k,i}}{f_i(y_k)} = \frac{\lambda_{k,i}}{f_i(x_k)},$$

for $\ell \leq i \leq n-1$ and $\gamma_{k,n} = 0$. This ensures that $\gamma_{k,i} \to \lambda_i$ for $i > \ell$. In order to determine the first $\ell-1$ coordinates of γ_k's use the following system of $\ell-1$ equations and $\ell-1$ unknowns for each k.

$$\sum_{i=1}^{\ell-1} \gamma_{k,i} \frac{c_{j,i}}{f_i(y_k)} = -\sum_{i=\ell}^{n-1} \gamma_{k,i} \frac{c_{j,i}}{f_i(y_k)}, \quad j = 1, \ldots, \ell-1$$

It remains to verify that $\lim_k \gamma_{k,i} = \lim \lambda_{k,i} = \lambda_i$, for $i = 1, \ldots, \ell-1$. For this use Cramer's Formula to find the unique answer to each $\gamma_{k,i}$'s, for $i < \ell$. For instance $\gamma_{k,i} =$

$$\det \begin{bmatrix} -\sum_{i=\ell}^{n-1} \gamma_{k,i} \frac{c_{1,i}}{f_i(y_k)} & \frac{c_{1,2}}{f_2(y_k)} & \cdots & \frac{c_{1,\ell-1}}{f_{\ell-1}(y_k)} \\ \vdots & \vdots & & \vdots \\ -\sum_{i=\ell}^{n-1} \gamma_{k,i} \frac{c_{\ell-1,i}}{f_i(y_k)} & \frac{c_{\ell-1,2}}{f_2(y_k)} & \cdots & \frac{c_{\ell-1,\ell-1}}{f_{\ell-1}(y_k)} \end{bmatrix} / \det \begin{bmatrix} \frac{c_{1,1}}{f_1(y_k)} & \frac{c_{1,2}}{f_2(y_k)} & \cdots & \frac{c_{1,\ell-1}}{f_{\ell-1}(y_k)} \\ \vdots & \vdots & & \vdots \\ \frac{c_{\ell-1,1}}{f_1(y_k)} & \frac{c_{\ell-1,2}}{f_2(y_k)} & \cdots & \frac{c_{\ell-1,\ell-1}}{f_{\ell-1}(y_k)} \end{bmatrix}$$

Note that the denominator is still nonsingular because it is the coefficient matrix of the (independent) linear forms $f_1, \ldots, f_{\ell-1}$ in which the columns are divided by nonzero numbers. Similarly, if we switch to x_k and λ_k in the Cramer's formula above, we get $\lambda_{k,1}$ and since the matrices have the same limit, their solutions must have the same limit as well. This shows that the new sequence (y_k, γ_k) lives in $\mathbb{P}\Sigma_z$ and converges to (x, λ). $\qquad \square$

3.2.3 case 3: Blow Up

For the last case, consider $f_n \omega_{\mathbf{a}}$, which has the same roots as $\omega_{\mathbf{a}}$ on the complement M.

Proposition 3.2.10. *Consider* $(x, \lambda) \in \mathfrak{X} \cap K$, *such that* $f_n(x) = 0$ *but* $x \notin M''$, *i.e.* $f_n(x) = 0$ *and* $f_i(x) = 0$ *too, for some* $i = 1, \ldots, n-1$. *Then* $(x, \lambda) \in \mathfrak{X}' \cup \mathfrak{X}''$.

Proof. Let $X = \cap_{x \in H \in \mathcal{A}} H$, which is to say that X is the biggest element of $(L(\mathcal{A}), <)$ that contains x. The proof is an induction on the codimension of X. The base case of codim $X = 1$ is already covered in the last two cases: If $X = H_n$, then the point lies in $\mathfrak{X}''$ and if $X = H_i$, for $i \neq n$, then it goes to $\mathfrak{X}'$. Without loss of generality, let

us assume that $f_r, \ldots, f_n$ vanish at x, among which $f_s, \ldots, f_n$ forms a basis, $r \leq s$. Blow up $\mathbb{P}V$ along $\mathbb{P}X$ to obtain Y. The resulting space is defined by $f_i t_j = f_j t_i$, $s \leq i < j \leq n$, and lives inside $\mathbb{P}^{\ell-1} \times \mathbb{P}^{n-s+1}$, where the t_j's are coordinates on the second factor. Since the projection from the blow up is a bijection away from the proper transform, we can lift our convergent sequence and the lift will have a convergent subsequence, as $Y \times \mathbb{P}^n$ is a compact space.

Along the new sequences that is obtained as above, each quotient $f_n/f_i = t_n/t_i$, for $i = s, \ldots, n$ and $f_n/f_i = t_n/\sum t$ for $i = r, \ldots, s-1$, where $\sum t$ is a linear combination of $t_s, \ldots, t_n$. At limit, at least one of the t_i's must be nonzero. As a result, the codimension of the space over which divsion by zero occurs goes down by at least one and depending on whether at limit $t_n \neq 0$ or $t_i \neq 0$, for some $s \leq i \leq n-1$. In the former case the point goes to $\mathfrak{X}'$ or otherwise to $\mathfrak{X}''$. $\qquad\square$

3.2.4 Intersection Classes and Multiplicities

Using the geometric formula we obtained in the last section, we are going to use intersection theory to first compute the intersection cycle of the logarithmic ideal and then reformulate the answer in terms of Chern classes. Our general reference on intersection theory is [15]. We are going to quickly recall some basic facts.

A cycle of codimension r on a variety X of dimension d is an element of the free abelian group generated by closed irreducible varieties of X of codimension r. It follows from the Moving Lemma that the *Chow ring* $\mathrm{CH}(X) = \oplus_{r=0}^{d} A^r(X)$ is a graded ring. If Y and Z are subvarieties which intersect properly (i.e. the codimension of their intersection is the sum of their codimensions), then the multiplication in the Chow ring is defined by

$$Y \cdot Z = \sum i(Y, Z; W_j) W_j, \tag{3.4}$$

where the sum runs over all irreducible components W_j of the intersection and the coefficient is the local intersection multiplicity of Y and Z along W_j.

The space we are considering here is $\mathbb{P}^{\ell-1} \times \mathbb{P}^n$, whose Chow ring is

$$\mathrm{CH}(\mathbb{P}^{\ell-1} \times \mathbb{P}^n) = \frac{\mathbb{Z}[h, k]}{(h^\ell, k^{n+1})}. \tag{3.5}$$

See [18, Example 2.0.1.] and [15, Example 8.3.7.] for a proof.

Next items in our toolkit are Chern classes and Chern characters. These are initially defined for vector bundles via construction that use connections and certain symmetric functions but one may use resolutions to extend the constructions

to sheaves over nonsingular varieties and that is exactly what we are going to do in case of S/I. This is supported by the fact that at the level of schemes over a given space, there is a canonical map from the *Grothendieck group of vector bundles* to the *Grothendieck group of coherent sheaves*

$$K^\circ \to K_\circ \qquad (3.6)$$

that sends a vector bundle to its sheaf of sections. This homomorphism upgrades to an isomorphism in case of a nonsingular scheme, such as $\mathbb{P}^{\ell-1} \times \mathbb{P}^{n-1}$ in our case. The map in the reverse direction uses the fact that over a nonsingular scheme, every coherent sheaf $\mathcal{F}$ admits a finite resolution $0 \to E_q \to \cdots \to E_0 \to \mathcal{F}$ by bundles. So one may simply send $[\mathcal{F}] \in K_\circ \mapsto \sum_i (-1)^i [E_i] \in K_\circ$. Since we are in the nonsingular case, we do not distinguish between the two variations of K-theory and just use K to denote the common object. The following lemma shows how different numerical invariants of coherent sheaves (graded modules) are related.

Lemma 3.2.11. *Consider graded modules over a polynomial ring $R = \mathbb{C}[x_1, \ldots, x_d]$. The Hilbert series is characterized by the following two properties:*

- $h(R[-a], t) = t^a/(1-t)^d$;

- h is additive on short exact sequences.

The Chern polynomial c_t is characterized by

- $c_t(\widetilde{R[-a]}) = 1 - at$;

- c_t is multiplicative on short exact sequences.

And finally, the Chern character is characterized by

- $\mathrm{ch}(\widetilde{R[-a]}) = e^{-at}$;

- ch *is additive on short exact sequences.*

See [14, Exercise 19.18] as a reference for the first two items. The fact about the Chern character follows from definition in comparison with Chern character.

Having talked about the general intersection theory machinery, we now state the main theorem of this section. Brackets are used to denote the intersection class of a variety in the Chow ring.

Theorem 3.2.12. *Let $\mathcal{A}$ be an arrangement of rank ℓ. If H_n is a bridge, then*

$$[\mathfrak{X}(\mathcal{A})] = [\mathfrak{X}(\mathcal{A}')]k, \tag{3.7}$$

otherwise,

$$[\mathfrak{X}(\mathcal{A})]k = [\mathfrak{X}(\mathcal{A}')]k + [\mathfrak{X}(\mathcal{A}'')]hk. \tag{3.8}$$

The proof follows from the following lemmas and propositions. Recall that $K = \mathbb{P}^{\ell-1} \times \mathbb{P}(\ker a_n)$.

Lemma 3.2.13. *The intersection $\mathfrak{X}(\mathcal{A}) \cap K$ is proper exactly when H_n is not a bridge.*

Proof. Since K is a hyperplane, the intersection is not proper iff $\mathfrak{X}(\mathcal{A}) \cap K = \mathfrak{X}(\mathcal{A})$. The intersection is defined by the ideal $I(\mathcal{A}) + (a_n)$ and by Corollary 3.2.5, $I(\mathcal{A}) + (a_n) = \mathrm{rad}(I(\mathcal{A}))$ exactly when H_n is a bridge. $\qquad\square$

For the following lemma, we embed Σ' and Σ^z in the ambient space of Σ by letting the last coordinate be zero.

Lemma 3.2.14. *When H_n is not a bridge, $\mathbb{P}\Sigma$ and K intersect properly.*

Proof. Since K is a hyperplane, the intersection is not proper only if $\mathbb{P}\Sigma \subseteq K$, which implies $\mathbb{P}\Sigma \subseteq \mathbb{P}\Sigma'$ which is impossible, since $\dim \Sigma = n = \dim \Sigma' + 1$.

An alternative proof is to recall that the closure $\overline{\mathbb{P}\Sigma}$ equals $\mathfrak{X}(\mathcal{A})$. Again, if Σ was inside K, then so would be its closure, which cannot be the case by Lemma 3.2.13. $\qquad\square$

Lemma 3.2.15. *The intersection in the last lemma is generically nonsingular and transversal.*

Proof. We projectivize all spaces in $\mathbb{P}^{\ell-1} \times \mathbb{P}^n$ and let $K = \{a_n = 0\}$ as before. It is straight forward to verify that $\Sigma \cap K \subseteq \Sigma' \cap D(f_n) \subseteq \Sigma \cap K$. Thus $\Sigma' \cap D(f_n) = \Sigma \cap V(a_n)$ in which the left hand side is an open and hence dense subset of $\mathfrak{X}'$.

For the other component, recall that $\mathfrak{X}'' = \overline{\Sigma^z}$. We also have $\Sigma^z \subset \overline{\Sigma^z} \subset \mathfrak{X} \cap K \subset \mathfrak{X}$.

For transversality, write $\mathrm{D}(\mathcal{A}) = R\theta_E \oplus \mathrm{Ann}(H_n)$ and consider a generating set consisting of the Euler derivation θ_E together with some generators $\theta_1, \ldots, \theta_r$ for $\mathrm{Ann}(H_n)$. The normal to the hypersurface defined by $\langle \theta_E, \omega_{\mathbf{a}} \rangle = \sum_i a_i$ is $(0, \ldots, 0; 1, \ldots, 1)$. For every $\theta \in \mathrm{Ann}(H_n)$ we have $\partial_{a_n} \langle \theta, \omega_{\mathbf{a}} \rangle = 0$. Therefore the kernel of the matrix $J_{I(\mathcal{A})}$ with respect to these generators has some nonzero component in the a_n coordinate.

$\qquad\square$

Lemma 3.2.16 (Proposition 8.2 and Remark 8.2, [15])**.** *Let V and W be varieties in some nonsingular variety Y and let Z be a proper component of the intersection $V \cap W$. The intersection multiplicity $i(Z, V \cdot W, Y) = 1$ iff the maximal ideal of $\mathcal{O}_{Z,Y}$ is the sum of the prime ideals of V and W. When $\dim Z > 0$, this translates to the following geometric condition: V and W are generically nonsingular along Z and meet transversally along Z.*

Proposition 3.2.17. *Let H be a non-bridge hyperplane in an arrangement $\mathcal{A}$, then*

$$[\mathfrak{X}] \cdot [K] = [\mathfrak{X}'] + [\mathfrak{X}''].\tag{3.9}$$

Proof. This follows from Theorem 3.2.2, Lemma 3.2.15 and the above lemma that ensures getting intersection multiplicity 1. $\qquad\square$

In the rest of this section, we restate the right hand side of the formula above with explicit reference to the intersection classes of the original varieties $\mathfrak{X}(\mathcal{A}')$ and $\mathfrak{X}(\mathcal{A}')$.

Lemma 3.2.18. *Let V be a variety in $\mathbb{P}^d$ and T an invertible linear transformation on $\mathbb{C}^{d+1}$. Then V is rationally equivalent to its image TV. Moreover, this is true for a product of linear transformations on a product of projective spaces.*

Proof. By functoriality, pull back along T provides an isomorphism from the Chow ring of $\mathbb{P}^d$, $\mathbb{Z}[t]/(t^{d+1})$, to itself. But there are only two choices of isomorphisms, namely the one defined by $t \mapsto -t$ and the identity. To see why the first one is ruled out, note that the class of a hyperplane H is t and its pullback is again a hyperplane for which we have $T^*([H]) = t = T^*(t)$. As a result, we have

$$[TV] = T^*[TV] = [T^{-1}TV] = [V].$$

Equivalently, one could argue that given any variety V, there is a linear subspace U of the same dimension such that $[TV] = d[U]$, where d is the degree. Now, since taking pullback does not affect the constant c and the dimension of U, we again see that $[V] = c[U]$, as well. $\qquad\square$

Lemma 3.2.19. *Let $b < d$ and let R_1 and R_2 be the polynomial rings $\mathbb{C}[x_0, \ldots, x_b]$ and $\mathbb{C}[x_0, \ldots, x_d]$, respectively. For a homogeneous ideal I of R_1, let V_1 and V_2 be the varieties defined by I in $\mathbb{P}^b$ and $\mathbb{P}^d$, respectively. Then $[V_1] = i^*[V_2] = [V_2]$, where $i : \mathbb{P}^b \hookrightarrow \mathbb{P}^d$ is the natural inclusion.*

Proof. The inclusion map i induces

$$\mathrm{CH}(\mathbb{P}^d) \to \mathrm{CH}(\mathbb{P}^b)$$
$$[V] \mapsto [i^{-1}(V)]$$

between the Chow rings. If $\mathbb{P}^b$ is embedded in $\mathbb{P}^d$, then $i^{-1}(V)$ is simply $V \cap \mathbb{P}^b$.

The map between the chow rings is determined by the image of $[\mathbb{P}^d] = 1$ and $[H] = t$, where H is a generic hyperplane on the first side. Intersect with $\mathbb{P}^b$ to get $1 \mapsto 1$ and $t \mapsto t$. Also, since V_1 and V_2 have the same codimension which should be $\leq b$, $[V_1]$ stays away from the kernel and the equality follows. $\square$

For the following Proposition, recall that $|(\mathcal{A}, \mathbf{m})| = m$ and $|\mathcal{A}| = n$.

Proposition 3.2.20. *Let $(\mathcal{A}, \mathbf{m})$ be a multiarrangement, then $\mathfrak{X}(\mathcal{A})$ and $\mathfrak{X}_\mu(\mathcal{A})$ have the same intersection class in $\mathbb{P}^{\ell-1} \times \mathbb{P}^m$. In particular, for a simple arrangement $\mathcal{A}$, we have $[\mathfrak{X}(\mathcal{A}'')] = [\mathfrak{X}_z(\mathcal{A}'')]$.*

Proof. By the lemma above, one only needs to give an invertible linear transformation

$$T : \mathbb{C}^\ell \times \mathbb{C}^{m+1} \to \mathbb{C}^\ell \times \mathbb{C}^{m+1},$$

which after passing to the projective spaces, maps one variety to the other one.

Define a linear map T by letting it be identity on $\mathbb{C}^\ell$ and on the second factor, define it by

$$(a_0, a_1, \ldots, a_m) \mapsto (a_0, \sum_{i \in \mu^{-1}(H_1)} a_i, \ldots, \sum_{i \in \mu^{-1}(H_n)} a_i, a_{n+1}, \ldots, a_m).$$

By Lemma 3.2.18, this gives $[\mathfrak{X}_\mu(\mathcal{A})] = [V(I(\mathcal{A}))]$, where on the right hand side we consider $I(\mathcal{A})$ as an ideal of the ring $R \otimes \mathbb{C}[a_0, \ldots, a_m]$. Also by the above lemma, we know that the answer to the intersection class only depends on the defining ideal and not the ring in which it is being realized. Therefore, $I(\mathcal{A}'')$ gives the equal intersection cycles $[\mathfrak{X}(\mathcal{A})] = [V(I(\mathcal{A}))]$ which belong to the Chow rings of $\mathbb{P}^{\ell-2} \times \mathbb{P}^{|\mathcal{A}|}$ and $\mathbb{P}^{\ell-2} \times \mathbb{P}^m$, respectively. $\square$

Lemma 3.2.21. *The maps i_a and j_a are proper and their push forwards induce additive maps on the Chow rings.*

$$\frac{\mathbb{Z}[h, k]}{(h^{\ell-1}, k^n)} \xrightarrow{\cdot h} \frac{\mathbb{Z}[h, k]}{(h^\ell, k^n)} \xrightarrow{\cdot k} \frac{\mathbb{Z}[h, k]}{(h^\ell, k^{n+1})}$$

In particular, $[\mathfrak{X}'] = [i_{a*}\mathfrak{X}(\mathcal{A}')] = k[\mathfrak{X}(\mathcal{A}')]$ and $[\mathfrak{X}''] = [(i_a \circ j_a)_* \mathfrak{X}_z(\mathcal{A}'')] = hk[\mathfrak{X}_z(\mathcal{A}'')]$.

Proof. The first map is induced from inclusion $\mathbb{P}(H_n) \times \mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^{\ell-1} \times \mathbb{P}^{n-1}$ and sends an intersection class $[Y] \mapsto [Y \cap V(f_n)] = [Y] \cdot h$. Similarly, the second map is induced from $\mathbb{P}^{\ell-1} \times \mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^{\ell-1} \times \mathbb{P}^n$ and sends an intersection class $[Z] \mapsto [Z \cap V(a_n)] = [Z] \cdot k$. □

Proof of Theorem 3.2.12

We are now in the position to combine the above partial results to prove the main theorem of the section.

Proof. If H_n is a bridge, then

$$
\begin{aligned}
[\mathfrak{X}(\mathcal{A})] &= [\mathfrak{X}'] && \text{Proposition 3.2.6} \\
&= [\mathfrak{X}(\mathcal{A}')] \cdot k && \text{Lemma 3.2.21.}
\end{aligned}
$$

And when H_n is not a bridge, then

$$
\begin{aligned}
[\mathfrak{X}(\mathcal{A})] \cdot k &= [\mathfrak{X}(\mathcal{A}) \cap K] && \text{intersection is proper by Proposition 3.2.5} \\
&= [\mathfrak{X}'] + [\mathfrak{X}''] && \text{Proposition 3.2.17} \\
&= [\mathfrak{X}(\mathcal{A}')] \cdot k + [\mathfrak{X}_z(\mathcal{A}'')] \cdot hk && \text{Lemma 3.2.21} \\
&= [\mathfrak{X}(\mathcal{A}')] \cdot k + [\mathfrak{X}(\mathcal{A}'')] \cdot hk && \text{Lemma 3.2.20.}
\end{aligned}
$$

□

3.3 Tutte Polynomial and Recursion

Recall that a hyperplane H is a *bridge* if $\text{rank}(\mathcal{A}') = \text{rank}(\mathcal{A}) - 1$, otherwise $\text{rank}(\mathcal{A}') = \text{rank}(\mathcal{A})$. In case of restriction, $\text{rank}(\mathcal{A}'') = \text{rank}(\mathcal{A}) - 1$ regardless of whether H is a bridge or not. The Tutte polynomial (Definition 1.5.7) may also be defined recursively for loopless matroids by $T_{\mathcal{A}}(x, y) = 1$ for the empty matroid, and

$$
T_{\mathcal{A}}(x, y) = \begin{cases} x T_{\mathcal{A}'}(x, y) & \text{if } H \text{ is a bridge} \\ T_{\mathcal{A}'}(x, y) + T_{\mathcal{A}''}(x, y) & \text{otherwise} \end{cases} . \tag{3.10}
$$

As a reference, see [37, Theorem 6.2.2] by Brylawski for a strong form of this fact.

Example 3.3.1. An easy computation using the above recursion formula shows that the Tutte polynomial of $Q = xy(x - y)$ is $T_{\mathcal{A}}(x, y) = x^2 + x + y$.

Substitute k/h and 0 for x and y respectively and multiply both sides of by k^ℓ. To be precise, note that for an ℓ-arrangement $\mathcal{A}$, the polynomial $T_{\mathcal{A}}(x, 0)$ is of the form $\sum_{i=1}^{\ell} t_{i,0} x^i$ and what we mean by $h^\ell T_{\mathcal{A}}(h, k)$ is simply $\sum_{i=1}^{\ell} t_{i,0} k^i h^{\ell-i}$. Also, the fact that $T_{\mathcal{A}}(x, 0)$ is divisible by x is a consequence of the Euler derivation. Recall that $\mathrm{D}(\mathcal{A}) = R\theta_E \oplus \mathrm{Ann}(Q)$ which implies that the characteristic polynomial $\chi(t)$ is divisible by $t - 1$. By comparing with Theorem 1.5.8, one immediately sees that $T_{\mathcal{A}}(x, 0)$ is divisible by x. The recurrence is as follows.

$$h^\ell T_{\mathcal{A}}(k/h, 0) = \begin{cases} kh^{\ell-1} T_{\mathcal{A}}(k/h, 0) & \text{if } H \text{ is a bridge} \\ h^\ell T_{\mathcal{A}'}(k/h, 0) + hh^{\ell-1} T_{\mathcal{A}''}(k/h, 0) & \text{otherwise} \end{cases} \tag{3.11}$$

Note that the rank of each arrangement appears as the exponent of h everywhere in this formula. For the arrangement of only one hyperplane, we have $T(x, y) = x$ and $[\mathfrak{X}] = k$, because its logarithmic ideal has only one generator which defines a hyperplane in the second factor. In this case $[\mathfrak{X}] = hT(k/h, 0)$ is immediate, verifying the base case. For the general step, compare the above formula with the formulas of Theorem 3.2.12 to see that both recursions have the same pattern with

$$[\mathfrak{X}(\mathcal{A})] = h^\ell T_{\mathcal{A}}(k/h, 0),$$

where $\ell = \mathrm{rank}(\mathcal{A})$. This answer maybe reformulated in terms of either the Poincaré or characteristic polynomial as follows.

Theorem 3.3.2. *Let $\mathcal{A}$ be an arrangement of rank ℓ, then the intersection class is given by*

$$[\mathfrak{X}(\mathcal{A})] = h^\ell T_{\mathcal{A}}(k/h, 0) = (-h)^\ell \chi(\mathcal{A}, (h - k)/h) = (h - k)^\ell \pi(\mathcal{A}, h/(k - h)).$$

Proof. Use the above argument together with the formulas $T_{\mathcal{A}}(t, 0) = (-1)^\ell \chi(\mathcal{A}; 1 - t)$ and $\pi(\mathcal{A}, t) = (-t)^\ell \chi(\mathcal{A}; -t^{-1})$. $\qquad\square$

All denominators in the above formulas are formal. In every case we homogenize the expression after multiplying by the denominator to the power ℓ in order to get a bivariate formula of total degree ℓ.

Example 3.3.3. The following calculations demonstrate the deletion-restriction formula for computing the intersection cycle.

- Let $\mathcal{A}$ be the Boolean arrangement $Q = x_1 \cdots x_\ell$. The logarithmic ideal $I(\mathcal{A}) = (a_1, \ldots, a_\ell)$ defines an intersection of ℓ hyperplanes and the cycle $[\mathfrak{X}(\mathcal{A})] \in \mathbb{Z}[h, k]/(h^\ell, k^{\ell+1})$ equals k^ℓ.

- As the first example consider the arrangement the A_2 arrangement $Q = xy(x - y)$ of Section 3.1.1. Let us use Formula 3.11 to compute the intersection class $[\mathfrak{X}]$. Since $\mathfrak{X}$ has codimension 2, the answer will be a polynomial of degree 2 of the form $ak^2 + bkh$ in $\mathbb{Z}[h, k]/(h^2, k^3)$, however in order to avoid getting irrelevant ideals, let us include $\mathbb{P}^1 \times \mathbb{P}^2$ in $\mathbb{P}^1 \times \mathbb{P}^3$ and work in the Chow ring $\mathbb{Z}[h, k]/(h^2, k^4)$. Start off by deleting the hyperplane $x = y$. By Formula 3.11, we have $k[\mathfrak{X}(\mathcal{A})] = k[\mathfrak{X}(\mathcal{A}')] + kh[\mathfrak{X}(\mathcal{A}'')]$. On the other hand $\mathcal{A}'$ and $\mathcal{A}''$ are the boolean arrangements of rank 2 and 1, and $I(\mathcal{A}')$ and $I(\mathcal{A}'')$ are defined by (a_1, a_2) and (a_1) respectively. Therefore $[\mathfrak{X}(\mathcal{A}')] = k^2$ and $[\mathfrak{X}(\mathcal{A}'')] = k$ and we obtain the relation

$$k \cdot (ak^2 + bkh) = k \cdot k^2 + k \cdot kh.$$

  By comparing the coefficients of k^3 and k^2h on both sides we see that $a = b = 1$, as predicted by above theorem. So, the final answer is $[\mathfrak{X}(\mathcal{A})] = k^2 + kh$.

- The second example is the arrangement $\mathcal{A}$ defined by $Q = xyz(x + y + z)$, which contrary to the first example is nonfree (last item in Example 1.2.7). The Tutte polynomial is given by $T_\mathcal{A}(x, y) = x^3 + x^2 + x + y$. Since the codimension of $\mathfrak{X}(\mathcal{A})$ is 3, we expect a polynomial of the form $ak^3 + bk^2h + ckh^2$ as the answer to $[\mathfrak{X}(\mathcal{A})] \in \text{CH}(\mathbb{P}^2 \times \mathbb{P}^4)$. Let us again use the reduction formula $k[\mathfrak{X}(\mathcal{A})] = k[\mathfrak{X}(\mathcal{A}')] + kh[\mathfrak{X}(\mathcal{A}'')]$ with respect to the last hyperplane. The deletion $\mathcal{A}'$ is the rank 3 boolean arrangement and the deletion $\mathcal{A}''$ is isomorphic to the A_2 arrangement for which we know the answer from the above calculation. If we substitute in the formula, we get

$$k \cdot (ak^3 + bk^2h + ckh^2) = k \cdot k^3 + kh \cdot (k^2 + kh)$$

  which implies that $a = b = c = 1$. This verifies the answer of above theorem by showing that $[\mathfrak{X}(\mathcal{A})] = k^3 + k^2h + kh^2$.

  Every reflection group G has a ring of invariants $\mathbb{C}[V]^G = \{f \in R : g \cdot f = f, \text{for all } g \in G\}$. The Chevalley-Shephard-Todd theorem states that the ring of invariants is always polynomial, meaning that there are polynomials $p_1, \ldots, p_\ell$ such that $\mathbb{C}[V]^G = \mathbb{C}[p_1, \ldots, p_\ell]$. The list of degrees of these polynomials is called the *basic invariants* of G.

Corollary 3.3.4. *Let G be a finite reflection group with basic invariants $b_1, \ldots, b_\ell$ and let $\mathcal{A}(G)$ be the reflection arrangement consisting of the reflecting hyperplanes of G, then*

$$[\mathfrak{X}(\mathcal{A}(G))] = \Pi_{i=1}^\ell ((b_i - 2)h + k).$$

Example 3.3.5. Let G be the symmetric group S_3. Then $\mathcal{A}(G)$ is the type A arrangement of rank 2 in $\mathbb{C}^3$. The ring of invariants is

$$\mathbb{C}[x_1, x_2, x_3]^{S_3} = \mathbb{C}[x_1 + x_2 + x_3, x_1 x_2 + x_2 x_3 + x_3 x_1, x_1 x_2 x_3]$$

and the list of basic invariants are $(1, 2, 3)$. One may divide by this ring by the sum of the variables to avoid negative coefficients. By the above corollary we get $h(h + k)$ which is the correct answer to the intersection cycle $[\mathfrak{X}]$ for the arrangement $Q = xy(x - y)$.

Lemma 3.3.6. *Let $\mathcal{F}$ be a sheaf of codimension s over a nonsingular scheme, then for every $0 < i < s$, $c_i(\mathcal{F}) = 0$.*

Proof. See [15, Example 15.3.6]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Theorem 3.3.7. *For any $\mathcal{A}$, $c_i([\mathcal{O}_\mathfrak{X}]) = 0$, for $i < \ell$, and the Chern character has order ℓ:*

$$\mathrm{ch}([\mathcal{O}_\mathfrak{X}]) = \frac{(-1)^{\ell-1}}{(\ell - 1)!} c_\ell([\mathcal{O}_\mathfrak{X}]) + h.o.t.$$

where the higher order terms involve Chern classes of order higher than ℓ, i.e. terms in h and k with total degree greater than ℓ. Moreover, $c_\ell([\mathcal{O}_\mathfrak{X}]) = (-1)^{\ell-1}(\ell-1)![\mathfrak{X}]$ in $A^\ell(\mathbb{P}^{\ell-1} \times \mathbb{P}^{n-1})$.

Proof. Let $\mathcal{F}_i$ be a resolution of $\mathcal{O}_\mathfrak{X}$ by locally free sheaves (bundles) and let $d = n - \ell - 1$. The Chern class of $\mathcal{F}_i$ is an alternating sum of the Chern classes of these bundles. Consider an arbitrary term $\mathcal{F}$ in the resolution. Using the Newton formulas to relate the power-sums to the symmetric functions to recover the Chern character in terms of the Chern classes. To be precise we have

$$c_t(\mathcal{F}) = \sum_{i=0}^d c_i(\mathcal{F}) t^i = \Pi_{i=1}^d (1 + a_i t),$$

$$\mathrm{ch}(\mathcal{F}) = \sum_{i=1}^d e^{a_i} = \sum_{i=1}^d \sum_{k=0}^\infty \frac{(a_i)^k}{k!} = \sum_{k=0}^\infty \frac{p_k(a_1, \ldots, a_d)}{k!},$$

where $p_k(a_1, \ldots, a_d) = \sum_{i=1}^{d} a_i^k$ and $c_k(\mathcal{F}) = e_k(a_1, \ldots, a_d) = \sum a_{i_1} \cdots a_{i_k}$, where the sum runs over all subsets of $a_1, \ldots, a_d$ of size k. The power-sums and symmetric functions are related by the formula

$$ke_k(a_1, \ldots, a_d) = \sum_{i=1}^{k} (-1)^{i-1} e_{k-i}(a_1, \ldots, a_d) p_i(a_1, \ldots, a_d), \quad k \geq 1.$$

By the last lemma, $e_i = 0$, for $i = 1, \ldots, \ell - 1$, which implies that $p_i(a_1, \ldots, a_d) = 0$. Also, for $k = \ell$, we get $p_\ell = (-1)^{\ell-1} \ell c_\ell(\mathcal{F})$. Substituting in the above formula for $k = \ell$ returns

$$\frac{p_\ell}{\ell!} = \frac{(-1)^{\ell-1} \ell c_\ell(\mathcal{F})}{\ell!} = \frac{(-1)^{\ell-1}}{(\ell-1)!} c_\ell(\mathcal{F}). \tag{3.12}$$

Moreover, by [15, Theorem 18.3(5)] and the fact that we are working over a nonsingular space, the Chern character $\mathrm{ch} : K(X) \to \mathrm{CH}_{\mathbb{Q}}$ sends $[\mathcal{O}_{\mathfrak{X}}]$ to

$$\mathrm{ch}([\mathcal{O}_{\mathfrak{X}}]) = [\mathfrak{X}] + \text{terms of codimensions} > \ell. \tag{3.13}$$

By comparison, we see that $\frac{(-1)^{\ell-1}}{(\ell-1)!} c_\ell([\mathcal{O}_{\mathfrak{X}}]) = [\mathfrak{X}]$. $\qquad \square$

Corollary 3.3.8. *For any $\mathcal{A}$, $c_i(\mathcal{O}_{\mathfrak{X}}) = 0$, for $i < \ell$ and*

$$\frac{(-1)^{\ell-1}}{(\ell-1)!} c_\ell([\mathcal{O}_{\mathfrak{X}}]) = h^\ell T_{\mathcal{A}}(k/h, 0) = (h-k)^\ell \pi(\mathcal{A}, \frac{h}{k-h}),$$

which also equals $(-h)^\ell \chi(\mathcal{A}; (h-k)/h)$.

Proof. This is an immediate consequence of 3.3.2 and 3.3.7. $\qquad \square$

Corollary 3.3.9. *Let $\mathcal{A}$ be a free arrangement with $\exp(\mathcal{A}) = (d_1, \ldots, d_\ell)$. Then*

$$[\mathfrak{X}(\mathcal{A})] = \Pi_{i=1}^{\ell}((d_i - 1)h + k).$$

Proof. This follows from the Factorization Theorem (Formula 1.11) which will also be a consequence of the results of this chapter. $\qquad \square$

See Corollary 4.4.2 for a generalization.

3.4 Applications

In this section we investigate the consequences of the intersection cycle formula of Theorem 3.3.2 on the Hilbert polynomial and Hilbert series of the logarithmic ideal and we shall see that we get a new proof for the Solomon-Terao formula for tame arrangements by computing their Hilbert series in two different ways.

Recall that if F is a coherent sheaf on a projective scheme over a field, then the *Euler characteristic* of F is the alternating sum

$$\chi(X, F) = \sum (-1)^i \dim_k H^i(X, F).$$

The Hirzebruch-Riemann-Roch Formula relates this cohomological information to the more topological information in the following way.

Lemma 3.4.1 (HRR, Corollary 15.2.1, [15]). *Let E be a vector bundle on a nonsingular complete variety X. Then*

$$\chi(X, E) = \int_X \mathrm{ch}(E) \mathrm{td}(T_X).$$

In practice integration comes down to collecting the leading term of the integrand.

Let M be a homogeneous module over a ring $k[x_0, \ldots, x_r]$ where k is a field. The *Hilbert function* of M is defined by $f_M : n \mapsto \dim_k M_n$ and the *Hilbert polynomial* is the unique polynomial p_M that agrees with f_M for large values of t.

$$t \gg 0 \qquad f_M(t) = p_M(t)$$

Every homogeneous module M gives rise to a sheaf $\widetilde{M}$ over $\mathbb{P}_k^r = \mathrm{Proj}(k[x_0, \ldots, x_r])$ with $\Gamma_\bullet(\widetilde{M}) = M$. Also, given a sheaf F over the projective space, the global sections $\Gamma_\bullet(F)$ are a graded module over the coordinate ring. The following lemma relates the Euler characteristic to the Hilbert polynomial.

Lemma 3.4.2 (Exercise 5.2, [18]). *Let $X = \mathbb{P}_k^r$ and let $M = \Gamma_\bullet(F)$. Then*

$$\chi(X, F(t)) = p_M(t).$$

This prepares us to state the following result about the Hilbert function of the logarithmic ideal.

Proposition 3.4.3. *If $\mathcal{A}$ is a rank ℓ arrangement of $n \geq 2$ hyperplanes, then*

$$p_{S/I}(x,y) = \Big[\frac{1}{(n-1)!}\frac{1}{tx}T_{\mathcal{A}}(tx,0)(1+ty)^{n-1}\Big]_{t^{n-1}} + \Omega(\{x,y\}^{n-1}), \qquad (3.14)$$

where $[\ldots]_{t^{n-1}}$ stands for the coefficient of t^{n-1} and $\Omega(\{x,y\}^{n-1})$ denotes a polynomial in x and y of total degree strictly less than $n-1$

Proof. By Lemma 3.4.1, the Hilbert polynomial is computed by the Hirzebruch-Riemann-Roch formula as follows.

$$p_{S/I}(x,y) = \chi(\mathbb{P}^{\ell-1} \times \mathbb{P}^n, \mathcal{O}_{\mathfrak{X}}(x,y)) = \int\limits_{\mathbb{P}^{\ell-1}\times\mathbb{P}^n} \mathrm{ch}(\mathcal{O}_{\mathfrak{X}}(x,y))\mathrm{td}(\mathcal{T}_{\mathbb{P}^{\ell-1}\times\mathbb{P}^n})$$

$$= \big[\mathrm{ch}(\mathcal{O}_{\mathfrak{X}}(x,y))\mathrm{td}(\mathcal{T}_{\mathbb{P}^{\ell-1}\times\mathbb{P}^n})\big]_{h^{\ell-1}k^n}$$

$$= \big[\mathrm{ch}(\mathcal{O}_{\mathfrak{X}})e^{xh+yk}\mathrm{td}(\mathcal{T}_{\mathbb{P}^{\ell-1}\times\mathbb{P}^n})\big]_{h^{\ell-1}k^n}$$

Here we are interested in the leading term of the polynomial in x and y. In order to pick up the highest powers of x and y, we use the constant term of the Todd class, namely 1. So there is no loss in restricting to

$$\big[\mathrm{ch}(\mathcal{O}_{\mathfrak{X}})e^{xh+yk}\big].$$

Moreover, the largest powers of these variables come from the terms of Chern character which have the lowest total degree in h and k, namely $h^{\ell}T_{\mathcal{A}}(k/h,0) = \sum_{i=1}^{\ell}t_{i0}h^{\ell-i}k^i$. Therefore the leading term in x and y is in

$$\sum_{i=1}^{\ell}t_{i0}h^{\ell-i}k^i\sum_{r\geq 0}\frac{(hx)^r}{r!}\sum_{s\geq 0}\frac{(ky)^s}{s!}$$

which actually equals

$$\sum_{i=1}^{\ell}t_{i0}h^{\ell-i}k^i\frac{(hx)^{i-1}}{(i-1)!}\frac{(ky)^{n-i}}{(n-i)!} = \frac{1}{(n-1)!}\sum_{i=1}^{\ell}t_{i0}\binom{n-1}{i-1}x^{i-1}y^{n-i} + \Omega(\{x,y\}^{n-1})$$

$$(3.15)$$

where the tail consists of terms of total degree less than $n-1$ in x and y. $\qquad\square$

Note that the denominator $1/(n-1)!$ in the leading term of the Hilbert polynomial indicates the dimension of $\mathfrak{X}$, as expected.

Example 3.4.4. The Tutte polynomial of the arrangement $Q = xy(x - y)$ equals $T(x, y) = x^2 + x + y$ and by the above theorem, the leading term of the Hilbert polynomial is

$$x + y,$$

as $n = 3$, $\ell = 2$ and $t_{1,0} = t_{2,0} = 1$. Let us verify this answer by direct computation. Use the resolution in Proposition 4.3.6 to compute the Hilbert series of S/I as follows. We have $\exp(\mathcal{A}) = (1, 2)$, $n = 3$, $\ell = 2$, $\Omega^0(\mathcal{A}) = R$, $\Omega^1(\mathcal{A}) = R[1] \oplus R[2]$ and $\Omega^2(\mathcal{A}) = R[3]$ by Corollary 1.4.16.

$$
\begin{aligned}
h(S/I) &= \frac{tu^2}{(1-u)^4}\Big[\frac{1}{(1-t)^2} + \Big(\frac{-t}{u}\Big)\frac{t^{-1} + t^{-2}}{(1-t)^2} + \Big(\frac{-t}{u}\Big)^2\frac{t^{-3}}{(1-t)^2}\Big] \\
&= \frac{1 - u - tu + tu^2}{(1-t)^2(1-u)^4} \\
&= (1 - u - tu + tu^2)\Big(\sum_{x \geq 0} t^x\Big)^2\Big(\sum_{y \geq 0} u^y\Big)^4 \\
&= (1 - u - tu + tu^2)\Big(\sum_{x \geq 0}(x+1)t^x\Big)\Big(\sum_{y \geq 0}\frac{(y+1)(y+2)(y+3)}{3!}u^y\Big)
\end{aligned}
$$

Now if we multiply out the entire expression and collect the coefficients of monomials in t and u as in $\sum \dim(S/I)_{(x,y)}t^x u^y$, then we see that after the first few terms the dimension stabilizes as a polynomial in x and y. For every $x \geq 1, y \geq 2$, $\dim(S/I)_{(x,y)}$ is given by the Hilbert polynomial

$$p_{S/I}(x, y) = \frac{1}{2}y^2 + xy + \frac{3}{2}y + x + 1,$$

where the leading part was predicted by the last theorem:

$$\frac{1}{2}y^2 + xy = \frac{1}{(3-1)!}\Big[\binom{2}{0}y^2 + \binom{2}{1}xy\Big]$$

Moreover, the Hilbert series admits the following partial fraction presentation.

$$h(S/I; t, u) = \frac{1}{(1-u)^4}T_{\mathcal{A}}\Big(\frac{1-u}{1-t}, 0\Big) - \frac{1}{(1-t)(1-u)^2}.$$

The significance is due to the fact that the total order of poles in the error part is strictly less than the number of hyperplanes, i.e. 3 in this case.

The following theorem recovers the leading part of the Hilbert series in its partial fraction decomposition in terms of the coefficients of the Tutte polynomial.

Theorem 3.4.5. *The leading term of $h(S/I; t, u)$ is combinatorial:*

$$h(S/I; t, u) = (1 - u)^{-(n+1)} T_{\mathcal{A}}(\frac{1 - u}{1 - t}, 0) + \Omega(\{(1 - t)^{-1}, (1 - u)^{-1}\}^{n+1}) \quad (3.16)$$

Proof. In the rational function presentation of the Hilbert series

$$h(S/I; t, u) = \frac{Q(x, y)}{(1 - t)^{\ell}(1 - u)^{n+1}},$$

replace the numerator by its Taylor series expansion

$$Q(t, u) = \sum_{i,j} e_{i,j}(t - 1)^i(u - 1)^j,$$

and rewrite the Hilbert series as

$$h(S/I; t, u) = \sum_{i,j}(-1)^{i+j}e_{i,j}(1 - t)^{i-\ell}(1 - u)^{j-n-1}.$$

The Hilbert function $p_{S/I}(x, y)$ is the coefficient of $t^x u^y$ in the above formal power series for large values of x and y. The stable part of the Hilbert series equals

$$\sum_{i,j}(-1)^{i+j}e_{i,j}\sum_{x \geq 0}\binom{x + \ell - i - 1}{x}t^x\sum_{x \geq 0}\binom{y + n - j}{y}u^y$$

$$= \sum_{x,y \geq 0}\sum_{i,j}(-1)^{i+j}e_{i,j}\binom{x + \ell - i - 1}{x}\binom{y + n - j}{y}t^x u^y$$

therefore the Hilbert polynomial equals

$$p_{S/I}(x, y) = \sum_{i,j}(-1)^{i+j}e_{i,j}\binom{x + \ell - i - 1}{x}\binom{y + n - j}{y}$$

and since we know that $\mathfrak{X}$ is of dimension $n - 1$, the coefficients $e_{i,j}$ can only be nonzero for $i + j \geq \ell$. In particular, the leading term of the Hilbert polynomial is to be looked for among the terms of

$$\sum_{i+j=\ell}(-1)^{i+j}e_{i,j}\binom{x + \ell - i - 1}{x}\binom{y + n - j - 1}{y}.$$

Eliminate i and extract the leading terms to obtain

$$(-1)^\ell \sum_{j=1}^{\ell} e_{\ell-j,j} \frac{x^{j-1}}{(j-1)!} \frac{y^{n-j}}{(n-j)!}.$$

Comparing this with formula 3.15 implies that $(-1)^\ell e_{\ell-j,j} = t_{j,0}$, where $t_{j,0}$ is the coefficient of x^i in the Tutte polynomial. So, we get to rewrite the Hilbert series as

$$h(S/I; t, u) = \sum_{i+j=\ell} (-1)^\ell e_{i,j}(1-t)^{i-\ell}(1-u)^{j-n-1} + \sum_{i+j>\ell} (-1)^{i+j} e_{i,j}(1-t)^{i-\ell}(1-u)^{j-n-1}$$

plus possibly some other terms with smaller order of poles. The first sum simplifies to

$$\sum_{j=1}^{\ell} (-1)^\ell e_{\ell-j,j}(1-t)^j(1-u)^{j-n-1} = \frac{1}{(1-u)^{n+1}} \sum_{j=1}^{\ell} t_{j,0}\left(\frac{1-u}{1-t}\right)^j.$$

Also, the total oder of poles in the second sum will be at most n, as $i+j > \ell$. $\quad\square$

Corollary 3.4.6. *The formal power series*

$$(1 - t + st(1-t))^{n+1} h(S/I; t, t-st(1-t))$$

is in fact a polynomial in s and t, and its specialization to $t = 1$ equals $T_{\mathcal{A}}(1+s, 0) = (-1)^\ell \chi(\mathcal{A}, -s)$.

Proof. In the last theorem, denote the error term by $q(t, u)$ and make the substitution $t \mapsto t$ and $u \mapsto t - st(1-t)$. Note that under this change of variable $1 - u \mapsto (1-t)(1+st)$. By the theorem,

$$(1-t+st(1-t))^{n+1} h(S/I; t, t-st(1-t)) = T_{\mathcal{A}}(1+st, 0) + (1-t)^{n+1}(1+st)^{n+1} q(t, t-st(1-t)),$$

where the right hand side is a polynomial. This only needs a verification for the error part. The error term is of the form

$$q(t, u) = \frac{g(t, u)}{(1-t)^b (1-u)^c}$$

with $b + c < n$. After making the substitution, we get

$$(1-t)^{n+1}(1+st)^{n+1} q(t, t-st(1-t)) = (1-t)^{n+1-(b+c)}(1+st)^{n+1-c} g(t, t-st(1-t)),$$

78

which is a polynomial and since $n + 1 - (b + c) \geq 1$, it vanishes at $t = 1$. The last claim follows from subbing in $t = 1$ everywhere. □

Definition 3.4.7. Let $D_\bullet(\mathcal{A})$ be totally bigraded, i.e. $\deg \partial_{x_j} = (-1, 0)$, $\deg x_j = (1, 0)$ and $\deg a_i = (0, 1)$, for all $1 \leq i \leq n$ and $1 \leq j \leq \ell$. Define two rational functions by

$$\Psi_\mathcal{A}(t, u) = \sum_{p=0}^{\ell} h(D_p(\mathcal{A}, \mathbf{m}), t) t^p (u(1 - t) - 1)^p$$

$$P_\mathcal{A}(t, u) = \sum_{p=0}^{\ell} h(D_p(\mathcal{A}, \mathbf{m}), t) u^p$$

Note that $\Psi_\mathcal{A}(t, u) = P_\mathcal{A}(t, tu(1 - t) - t)$.

In the tame case, we have a resolution of S/I by logarithmic differential modules. The proof of this fact is given in the multiarrangement case in the next chapter. (See Theorem 4.3.1)

In the following proposition we use C-linear p-derivations

$$D_p^{S/C}(\mathcal{A}) := D_p(\mathcal{A}) \otimes_{\mathbb{C}} C[a_0].$$

The following proposition provides a resolution of S/I by logarithmic derivation modules and computes the full answer to the Hilbert series under the tame condition. See Theorem 4.3.1 for a multiarrangement version and [9] where this fact was originally proven.

Proposition 3.4.8. *Let $\mathcal{A}$ be a tame arrangement. Then we have a resolution*

$$0 \to D_\ell^{S/C}(\mathcal{A})[(0, -\ell)] \to \cdots \to D_1^{S/C}(\mathcal{A})[(0, -1)] \to D_0^{S/C}(\mathcal{A}) \to S/I \to 0$$

where the differentials are defined by contraction along $\omega_\mathbf{a}$:

$$\theta \mapsto \langle \theta, \omega_\mathbf{a} \rangle$$

Moreover, the Hilbert series of $S/I(\mathcal{A})$ equals

$$h(S/I; t, u) = \frac{P_\mathcal{A}(t, -u)}{(1 - u)^{n+1}}.$$

Proof. The first statement is a direct consequence of Theorem 4.3.1 and Corollary 1.4.23. The Hilbert series is calculated in the following process.

$$h(S/I; t, u) = \sum_{p=0}^{\ell} (-1)^p h(D_p^{S/C}(\mathcal{A})[(0, -p)]; t, u)$$

$$= \sum_{p=0}^{\ell} (-1)^p u^p h(D_p(\mathcal{A}), t) h(C)$$

$$= \frac{1}{(1-u)^{n+1}} \sum_{p=0}^{\ell} (-u)^p h(D_p(\mathcal{A}), t) \qquad \square$$

Theorem 3.4.9 (Solomon-Terao Formula). *If $\mathcal{A}$ is tame, then*

$$\sum_{p=0}^{\ell} h(D_p(\mathcal{A}, \mathbf{m}), t) t^p (s(1-t) - 1)^p \Big|_{t=1} = (-1)^\ell \chi(\mathcal{A}, -s).$$

Proof. This follows from computing the specialization of $(1-t+st(1-t))^{n+1} h(S/I; t, t-st(1-t))$ to $t = 1$ in two different ways via Corollary 3.4.6 and Proposition 3.4.8.

$$(1 - t + st(1-t))^{n+1} h(S/I; t, t - st(1-t)) = P_{\mathcal{A}}(t, st(1-t) - t) \qquad \text{Proposition 3.4.8}$$

$$= \Psi_{\mathcal{A}}(t, s) \qquad \text{Definition 3.4.7}$$

We get the left hand side of the above formula by evaluating $\Psi_{\mathcal{A}}(t, s)$ at $t = 1$. On the other hand, by Corollary 3.4.6, we have

$$(1 - t + st(1-t))^{n+1} h(S/I; t, t - st(1-t)) \Big|_{t=1} = T_{\mathcal{A}}(1 + s, 0) = (-1)^\ell \chi(\mathcal{A}, -s).$$

$$\square$$

Corollary 3.4.10 (Factorization Theorem). *If $\mathcal{A}$ is free with $\exp(\mathcal{A}) = (d_1, \ldots, d_\ell)$, then*

$$\pi(\mathcal{A}, t) = (1 + t)(1 + d_2 t) \cdots (1 + d_\ell t).$$

Proof. Compute the left hand side of the Solomon-Terao formula in terms of the exponents. See [22, Theorem 4.137]. $\square$

It follows from Proposition 1.4.9 that

$$D(\mathcal{A}) = R\theta_E \oplus \text{Ann}(Q),$$

where $\mathrm{Ann}(Q) = \{\theta \in \mathrm{D}(R) : \theta(Q) = 0\}$. If we pass to the logarithmic ideals we get

$$I(\mathcal{A}) = (\langle \theta_E, \omega_{\mathbf{a}} \rangle) + I_0(\mathcal{A}),$$

where $I_0(\mathcal{A}) = \langle \mathrm{Ann}, \omega_{\mathbf{a}} \rangle$ and $\langle \theta_E, \omega_{\mathbf{a}} \rangle = \sum_i a_i$. Let $\mathfrak{X}_0$ be the variety defined by $I_0(\mathcal{A})$ in $\mathbb{P}^{\ell-1} \times \mathbb{P}^{n-1}$. The addition of ideal turns into a proper intersection intersection between the variety $\mathfrak{X}_0$ and $V(\sum_i a_i)$ which is a hyperplane in the second factor.

$$\mathfrak{X} = V(a_1 + \cdots + a_n) \cap \mathfrak{X}_0$$

The properness of the intersection is discussed in [13] and is revisited in the next chapter. It is implicit in the last formula that $[\mathfrak{X}]$ is divisible by k. Moreover, it allows us to compute the intersection cycle of $\mathfrak{X}_0$ as follows.

Lemma 3.4.11. *The variety $\mathfrak{X}_0$ has codimension $\ell - 1$ and its intersection cycle is given by*

$$\frac{h^\ell}{k} T_{\mathcal{A}}(k/h, 0).$$

Proof. The intersection class of $V(\sum_i a_i)$ simply equals k as it is a hyperplane in $\mathbb{P}^{n-1}$. Therefore

$$[\mathfrak{X}(\mathcal{A})] = k[\mathfrak{X}_0(\mathcal{A})],$$

from which the formula follows via Theorem 3.3.2. $\square$

Theorem 3.4.12 (Orlik-Terao)**.** *For a generic weight $\lambda \in \mathbb{C}^n$, the number of projective critical points is given by the coefficient of x in the Tutte Polynomial.*

Proof. A weight vector $\lambda \in \mathbb{C}^n$ spans a line L_λ in $\mathbb{C}^n$ which is also a point of codimension $n - 1$ in $\mathbb{P}^{n-1}$. In general, the codimension of an intersection is bounded above by the sum of the codimensions. In this situation, we are interested in the set $\mathfrak{X}_0 \cap L_\lambda$.

$$\mathrm{codim}\,(L_\lambda \cap \mathfrak{X}_0) \leq \mathrm{codim}\, L_\lambda + \mathrm{codim}\, \mathfrak{X}_0 = (n-1) + (\ell-1)$$

This implies that the intersection is proper and in fact forces it to be zero dimensional, i.e. a finite number of points. In order to compute the exact number we do the following calculation which is subject to the relations $k^n = 0 = h^\ell$ and $T_{\mathcal{A}}(x, y) =$

$\sum_{i,j} t_{i,j} x^i y^j$.

$$[L_\lambda \cap \mathfrak{X}_0] = [L_\lambda] \cdot [\mathfrak{X}_0] = k^{n-1} \cdot \frac{h^\ell}{k} T_{\mathcal{A}}(k/h, 0)$$

$$= k^{n-1} \cdot \frac{h^\ell}{k} \sum_{i=1}^{\ell} t_{j,0}(k/h)^i = \sum_{i=1}^{\ell} t_{i,0} \, h^{\ell-i} k^{n+i-2}$$

$$= t_{1,0} \, h^{\ell-1} k^{n-1}$$

According to our notation $t_{1,0}$ is the coefficient of x in the Tutte polynomial. $\qquad \square$

Example 3.4.13. In the example of three lines in $\mathbb{C}^2$ defined by $Q = xy(x - y)$, the Tutte polynomial is $T(x, y) = x^2 + x + y$ and there is generically exactly one critical point.

See [8, Theorem 28] for a related generalization.

3.5 Appendix to Chapter 3

3.5.1 Logarithmic OT Ideals

This section is about an ideal which is in essence a blend of the meromorphic ideal and the Orlik-Terao ideal in the following sense.

Recall that the meromorphic ideal of $\mathcal{A}$ is the ideal defined by

$$\sum_{i=1}^{n} c_{ij} \frac{a_i}{f_i}$$

for $j = 1, \ldots, \ell$. When $\mathcal{A}$ is essential, I_{mer} defines a manifold Σ of codimension ℓ. The idea here is to mix I_{mer} with the *Orlik-Terao ideal* $I_{OT}(\mathcal{A})$ which is the kernel of the following map.

$$\mathbb{C}[y_1, \ldots, y_n] \to \mathbb{C}[1/f_1, \ldots, 1/f_n]$$
$$y_i \mapsto 1/f_i$$

The right hand side is called the *Orlik-Terao algebra*, which serves as a commutative analog for the Orlik-Solomon algebra. (See [26] and [31].)

One can list a set of generators for the OT ideal that is indexed by the lattice of intersections $L(\mathcal{A})$ as follows.

Theorem 3.5.1 (Schenck-Tohaneanu, [26])**.** *The kernel of the above map, i.e. the Orlik-Terao ideal, is generated by elements of the form*

$$\sum_{j} b_j y_{i_1} \cdots \widehat{y_{i_j}} \cdots y_{i_t},$$

for every dependence $\sum_{j=1}^{t} b_j f_{i_j} = 0$ among the defining linear functionals.

We will refer to these generating elements as the OT relations. The significance of the OT algebra lies in the fact that it provides a commutative model for the anti-commutative Orlik-Solomon algebra and that it detects a non-combinatorial property that is an obstruction to freeness: 2-formality.

Definition 3.5.2. An arrangement $\mathcal{A}$ is 2-*formal* if every dependence between the linear functionals $f_1, \ldots, f_n$ is a linear combination of dependences between subsets of size three.

Now, we can state the following theorem, due to Yuzvinsky [43].

Theorem 3.5.3. *If an arrangement $\mathcal{A}$ if free, then $\mathcal{A}$ is a 2-formal arrangement.*

It is expected that the result remains true for multiarrangements as well.

Definition 3.5.4. Let $\mathcal{A}$ be a central hyperplane arrangement. The ideal generated by all OT relations together with the relations obtained from the generators of I_{mer}, with $1/f_i$ replaced with y_i will be called the *logarithmic OT ideal*. We denote this ideal by $J(\mathcal{A})$. To be precise, we have

$$I_{\ell ot}(\mathcal{A}) := I_{OT}(\mathcal{A}) + (\sum_{i=1}^{n} c_{ij} a_i y_i : 1 \leq j \leq n).$$

Example 3.5.5. Consider the first arrangement in Example 1.2.7, defined by $Q = xy(x-y)$. The logarithmic form is defined by

$$\omega_{\mathrm{a}} = a_1 \frac{\mathrm{d}x}{x} + a_2 \frac{\mathrm{d}y}{y} + a_3 \frac{\mathrm{d}(x-y)}{x-y},$$

which contributes two relation, namely $a_1 y_1 + a_3 y_3$ and $a_2 y_2 - a_3 y_3$. On the OT side, since there is only one dependence, $-f_1 + f_2 + f_3$, we get the relation $-y_2 y_3 + y_1 y_3 + y_1 y_2$. Therefore,

$$I_{\ell ot}(\mathcal{A}) = (-y_2 y_3 + y_1 y_3 + y_1 y_2, a_1 y_1 + a_3 y_3, a_2 y_2 - a_3 y_3).$$

We will give a presentation of the minimal ideals of such ideals in the following section.

3.5.1.1 Minimal Primes of Log OT Ideals

Lemma 3.5.6. *Let $\mathcal{A}$ be an arrangement in $\mathbb{C}^{\ell}$ and let (y, a) be a point on $V(I_{\ell ot}(\mathcal{A}))$ with $y_i = 0$ for some $1 \leq i \leq n$, then there is a flat X of rank $r - 1$ such that*

$$\{j : H_j \notin \mathcal{A}_X\} \subseteq \{j : y_j = 0\}.$$

Proof. We use induction on the number of hyperplanes of $\mathcal{A}$. Let $y_i = 0$ and consider $\mathcal{A}' = \mathcal{A} \setminus \{H_i\}$. If $\mathcal{A}'$ is essential, then by induction hypothesis, there is a flat $X \in L_{r-1}(\mathcal{A}')$ such that $\{j : H_j \notin \mathcal{A}'_X\} \subseteq \{j : j \neq i, y_j = 0\}$. This choice of X works for $\mathcal{A}$ too. We might happen to have $H_i \notin \mathcal{A}_X$, which is ok because of our assumption $y_i = 0$.

If $\mathcal{A}'$ is not essential, then its center $\cap \mathcal{A}'$ is a line, i.e. of rank $\ell - 1$, and $\{j : H_j \notin \mathcal{A}_X\}$ is the singleton $\{i\}$. The inclusion holds in this case too. The base of induction where we only have one hyperplane is easy. $\square$

In general, one can obtain all associated primes of an ideal J but looking among the ideals $(J : z)$, by letting z vary among all elements of the ring. The drawback is that depending on where z comes from, $(J : z)$ might only be primary. The following lemma shows that under the tame condition, saturation at the product of variables $y_1 \cdots y_n$ returns a prime ideal. (see [6])

Theorem 3.5.7. *Let $\mathcal{A}$ be a central hyperplane arrangement, then the logarithmic Orlik-Terao ideal decomposes as follows.*

$$\mathrm{rad}(I_{\ell ot}(\mathcal{A})) = \bigcap_{X \in L(\mathcal{A})} \mathrm{rad}(I_{\ell ot}(\mathcal{A}_X) : \Pi_{H_i \in \mathcal{A}_X} y_i) + (y_j)_{H_j \notin \mathcal{A}_X}$$

In particular, the components in the intersection above are the minimal primes of the logarithmic Orlik-Terao ideal.

Proof. The proof works by induction on the number of hyperplanes in $\mathcal{A}$. The base case where there is only one hyperplane is easy to verify. In order to use the induction hypothesis, we need to establish the formula

$$V(I_{\ell ot}(\mathcal{A})) = V(J(\mathcal{A}) : y_1 \cdots y_n) \cup (\cup_{X \in L_{r-1}(\mathcal{A})} V(I_{\ell ot}(\mathcal{A}_X) + (y_j)_{j \notin \mathcal{A}_X})), \quad (3.17)$$

where r is the rank of our arrangement.

Once this proven, we turn to the defining ideals to get

$$\mathrm{rad}(I_{\ell ot}(\mathcal{A})) = \mathrm{rad}(I_{\ell ot}(\mathcal{A}) : y_1 \cdots y_n) \cap (\cap_{X \in L_{r-1}(\mathcal{A})} \mathrm{rad}(I_{\ell ot}(\mathcal{A}_X)) + (y_j)_{H_j \notin \mathcal{A}_X})),$$

and use the induction hypothesis for each $\mathcal{A}_X$ to get replace $\mathrm{rad}(I_{\ell ot}(\mathcal{A}_X))$ with

$$\bigcap_{Y \in L(\mathcal{A}_X)} \mathrm{rad}(I_{\ell ot}((\mathcal{A}_X)_Y) : \Pi_{H_i \in (\mathcal{A}_X)_Y} y_i) + (y_j)_{H_j \in \mathcal{A}_X \setminus (\mathcal{A}_X)_Y}.$$

But since $(\mathcal{A}_X)_Y = \mathcal{A}_Y$ for $Y \in L(\mathcal{A}_X)$, after plugging these for all $X \in L_{r-1}(\mathcal{A})$ back into the formula above, the desired formula emerges.

Now, let us go back to the formula 3.17 above. If $(\mathbf{y}, \mathbf{a}) \in V(I_{\ell ot}(\mathcal{A}))$ is a point with all y_i's nonzero. This point will be in $V(I_{\ell ot}(\mathcal{A}) : y_1 \cdots y_n)$, because the saturated ideal defines the closure of $V(I_{\ell ot}(\mathcal{A})) - V(y_1 \cdots y_n)$. Otherwise, if

some of y_i coordinates are zero, then by lemma 3.5.6, there is a flat $X \in L(\mathcal{A})$ that accommodates our point in one of the other components of the formula 3.17. $\qquad\square$

Example 3.5.8. Associated primes of the ideal in Example 3.5.5 are listed as follows.

$$(y_1, y_2, y_3)$$
$$(a_1, y_2, y_3) \quad (y_1, a_2, y_3) \quad (y_1, y_2, a_3)$$
$$(a_1 + a_2 + a_3, y_1 y_2 + y_1 y_3 - y_2 y_3, y_2 a_2 + y_3 a_1 + y_3 a_2, y_1 a_1 + y_2 a_2)$$

As it is evident, these ideals provides labels for the lattice of intersections of the A_2 arrangement. See the deletion in Example 1.1.1.

3.5.2 Higher Order Logarithmic Ideals

One would like to have an analogous definition for the logarithmic ideal in higher orders to capture the properties of higher order logarithmic modules. The following definition seems to be a natural one:

Definition 3.5.9. Let $\mathcal{A}$ be an ℓ-arrangement and for every $p = 1, \ldots, \ell$, let

$$C^p = \mathbb{C}[a_{i_1, \ldots, i_p} : 1 \le i_1 < i_2 < \cdots < i_p \le n],$$

with variables that are independent. Now, define the pth logarithmic form by

$$\omega_{\mathbf{a}}^p = \sum_{1 \le i_1 < i_2 < \cdots < i_p \le n} a_{i_1, \ldots, i_p} \frac{\mathrm{d}f_{i_1} \wedge \cdots \wedge \mathrm{d}f_{i_p}}{f_{i_1} \cdots f_{i_p}}.$$

It is immediate to verify that $\omega_{\mathbf{a}}^p \in \Omega^p(\mathcal{A}) \otimes_{\mathbb{C}} C^p$. We extend the scalars of $\mathrm{D}_p(\mathcal{A})$ to C^p and define

$$I_p(\mathcal{A}) = \langle \mathrm{D}_p(\mathcal{A}) \otimes_{\mathbb{C}} C^p, \omega_{\mathbf{a}}^p \rangle.$$

Note that again the p-derivations clear the denominators and we get an ideal which actually lives in the polynomial ring $R \otimes_{\mathbb{C}} C^p$. We also have the following analogous definition.

Definition 3.5.10. Under the above setting, let $\Sigma^p \subset M(\mathcal{A}) \times \mathbb{C}^{\binom{n}{p}}$ be the vanishing of the pth logarithmic form $\omega_{\mathbf{a}}^p$.

$$\Sigma^p(\mathcal{A}) = \{(\mathrm{x}, \mathbf{a}) : \sum_{1 \le i_1 < i_2 < \cdots < i_p \le n} a_{i_1, \ldots, i_p} \frac{\mathrm{d}f_{i_1} \wedge \cdots \wedge \mathrm{d}f_{i_p}}{f_{i_1} \cdots f_{i_p}(\mathrm{x})} = 0\}$$

The vanishing of this meromorphic p-form on $M \times \mathbb{C}^{\binom{n}{2}}$ is equivalent to the vanishing of $\binom{\ell}{2}$ forms as follows. Recall that $f_i = \sum_{j=1}^{\ell} c_{j,i} x_j$ which implies $\mathrm{d}f_i = \sum_{j=1}^{\ell} c_{j,i} \mathrm{d}x_j$. We collect all of the coefficients in the matrix

$$
\begin{pmatrix}
c_{1,1} & \cdots & c_{1,n} \\
\vdots & & \vdots \\
c_{\ell,1} & \cdots & c_{\ell,n}
\end{pmatrix}
$$

where the ith column stores the coefficients of H_i. Consider the vanishing of the following sum.

$$
\sum_{1 \leq i_1 < \cdots < i_p \leq n} a_{i_1,\ldots,i_p} \frac{\mathrm{d}f_{i_1} \wedge \cdots \wedge \mathrm{d}f_{i_p}}{f_{i_1} \cdots f_{i_p}(\mathrm{x})} =
$$

$$
\sum_{1 \leq i_1 < \cdots < i_p \leq n} a_{i_1,\ldots,i_p} \frac{\sum_{j_1=1}^{\ell} c_{j_1,i_1} \mathrm{d}x_{j_1} \wedge \cdots \wedge \sum_{j_p=1}^{\ell} c_{j_p,i_p} \mathrm{d}x_{j_p}}{f_{i_1} \cdots f_{i_p}(\mathrm{x})} =
$$

$$
\sum_{1 \leq i_1 < \cdots < i_p \leq n} a_{i_1,\ldots,i_p} \frac{\sum_{1 \leq j_1 < \cdots < j_p \leq \ell} \det[\text{minor}] \, \mathrm{d}x_{j_1} \wedge \cdots \wedge \mathrm{d}x_{j_p}}{f_{i_1} \cdots f_{i_p}(\mathrm{x})}
$$

The minor corresponds to rows $(j_1, \ldots, j_p)$ and columns $(i_1, \ldots, i_p)$ for all increasing index lists. Now we can flip the double sum as follows.

$$
\sum_{1 \leq j_1 < \cdots < j_p \leq \ell} \left[\sum_{1 \leq i_1 < \cdots < i_p \leq n} \det[\text{minor}] \frac{a_{i_1,\ldots,i_p}}{f_{i_1} \cdots f_{i_p}(\mathrm{x})} \right] \mathrm{d}x_{j_1} \wedge \cdots \wedge \mathrm{d}x_{j_p}
$$

So vanishing translates to the vanishing of all inner sums of which we have $\binom{\ell}{p}$. We expect to get an analogous result to Proposition 3.1.5 which requires the following linear algebra fact. In the following lemma, let A_k denote the matrix of $k \times k$ minors of a matrix A.

Lemma 3.5.11. *Let A be an $n \times n$ matrix. Then the matrix of $k \times k$ minors A_k is invertible if A is invertible. More generally,*

$$
\det(A_k) = \det(A)^{\binom{n-1}{k-1}}.
$$

Note that this is obvious for $k = 1, n$, and for $k = n-1$ follows from $A \cdot \mathrm{Adj}A = \det(A) \cdot I$.

Proof. Without loss of generality one may assume that A is a diagonal matrix because otherwise one may apply row operations (adding a multiple of a row to another) to transform A into a diagonal matrix with the same determinant. Also, because determinants are multilinear, the row operations do not change the determinant of the matrix of $k \times k$ minors. Now we prove the statement about the determinant by induction on n. The claim is obvious for $n = 1, 2$. In general, since A is diagonal, we only get contributions from the minors which are diagonal, i.e. if the intersection of any of the k chosen row and columns at any time is off the diagonal, then the minor will have an entirely zero row. Let $n > 2$ be arbitrary and assume that the claim is true for all diagonal matrices of size less than n and all $k \leq n$ for each n. Let A be an arbitrary diagonal matrix of dimension n and let $k < n$. The matrix of k minors is again a diagonal matrix where each diagonal entry is a product of a k-subset of the original diagonal entries. Let A' be the matrix that is obtained by removing the first row and column of A. The determinant of A_k is

$$a_{1,1}^{\binom{n-1}{k-1}} \det(A')^{\binom{n-2}{k-2}} \cdot \det(A')^{\binom{n-2}{k-1}}$$

where the first factor corresponds to the products that contain $a_{1,1}$ and the second one to the ones that do not. Therefore the final answer is

$$a_{1,1}^{\binom{n-1}{k-1}} \det(A')^{\binom{n-2}{k-2}+\binom{n-2}{k-1}} = a_{1,1}^{\binom{n-1}{k-1}} \det(A')^{\binom{n-1}{k-1}} = \det(A)^{\binom{n-1}{k-1}}.$$

$\square$

Note that the above formula is similar to the general form of the Saito's Criterion (Theorem 1.4.14).

Proposition 3.5.12. *Let $\mathcal{A}$ be an ℓ-arrangement. The vanishing of the pth logarithmic form defines a quasi-affine variety in $M \times \mathbb{C}^{\binom{n}{2}}$ which is of codimension $\binom{\ell}{p}$.*

Proof. Modify the argument of Proposition 3.1.5 with the above lemma and the argument preceding it. $\square$

The following statement is an extension of Proposition 3.1.7.

Proposition 3.5.13. *Let $\mathcal{A}$ be an ℓ-arrangement. Then $V(I_p(\mathcal{A})) = \overline{\Sigma^P}$.*

As a consequence, we get a similar description for freeness of logarithmic modules of all order.

Theorem 3.5.14. *Let $\mathcal{A}$ be an ℓ-arrangement. Then for every p, the module $\mathrm{D}_p(\mathcal{A})$ is free if and only if the logarithmic ideal $I^p(\mathcal{A})$ is a complete intersection.*

Proof. Proof is again analogous to the proof of the $p = 1$ case. See [9, Theorem 2.10]. $\qquad\square$

Chapter 4
Logarithmic Ideals of Multiarrangements

In this chapter we are going to extend the notion of logarithmic ideal to multiarrangements and study its properties. The goal is to get an ideal-theoretic equivalent description of freeness. Such a description is already known for simple arrangements in at least two ways: Theorem 1.3.1 about the Jacobian ideal; and Theorem 3.5.14 about the logarithmic ideal. As it was noted in chapter 1, it does not make sense to talk about a multiarrangement analog of the Jacobian ideal, however the approach of logarithmic ideal does produce meaningful consequences.

We keep our notations consistent and let $D_C(\mathcal{A}, \mathbf{m}) := D_\mathbb{C}(\mathcal{A}, \mathbf{m}) \otimes_\mathbb{C} C$, which again lives in $D_C(S) = D(R) \otimes C$. The following definition proves to be natural.

Definition 4.0.15. Let $(\mathcal{A}, \mathbf{m})$ be a multiarrangement, then its *logarithmic ideal* is defined by

$$I(\mathcal{A}, \mathbf{m}) = \langle D_C(\mathcal{A}, \mathbf{m}), \omega_\mathbf{a} \rangle.$$

Remark 4.0.16. The logarithmic ideal of a multiarrangement $(\mathcal{A}, \mathbf{m})$ actually lives in the ring S. As noted before $\omega_\mathbf{a} \in \Omega^1_C(\mathcal{A})$. We have the following pairings which come from Proposition 1.4.22 after tensoring with C.

$$
\begin{array}{ccccc}
D_C(\mathcal{A}) & \times & \Omega^1_C(\mathcal{A}) & & \\
\cup| & & \cap| & \searrow & S \\
D_C(\mathcal{A}, \mathbf{m}) & \times & \Omega^1_C(\mathcal{A}, \mathbf{m}) & \nearrow &
\end{array}
$$

So the image is an ideal in the polynomial ring. Also, because of the first inclusion, we have $I(\mathcal{A}, \mathbf{m}) \subseteq I(\mathcal{A})$.

Equivalently, if $\theta \in D_C(\mathcal{A}, \mathbf{m})$, then

$$\langle \theta, \omega_\mathbf{a} \rangle = \theta\left(\sum_{i=1}^{n} a_i \frac{\mathrm{d}f_i}{f_i}\right) = \sum_{i=1}^{n} a_i \frac{\theta(f_i)}{f_i} \tag{4.1}$$

which is a polynomial since $\theta(f_i)$ is divisible by f_i.

Example 4.0.17. Let $(\mathcal{A}, \mathbf{m})$ be a Boolean multiarrangement defined by $\widetilde{Q} = x_1^{m_1} \ldots x_\ell^{m_\ell}$, then the module of derivations $D(\mathcal{A}, \mathbf{m})$ is free and has the following elements as a basis.

$$x_1^{m_1} \partial_{x_1}, \ldots, x_\ell^{m_\ell} \partial_{x_\ell},$$

Thus the logarithmic ideal is

$$I(\mathcal{A}, \mathbf{m}) = (a_1 x_1^{m_1 - 1}, \ldots, a_\ell m_\ell^{m_\ell - 1}).$$

Primary decomposition components of this ideal are of the form

$$(a_1, \ldots, a_{i-1}, x_i^{m_i - 1}, a_{i+1}, \ldots, a_\ell), \tag{4.2}$$

for every i with $m_i > 1$, together with the ideal $(a_1, \ldots, a_n)$ which comes from the underlying simple arrangement.

A dual description of the logarithmic ideal is as follows. Consider the image of the map

$$\Omega^{\ell - 1}(\mathcal{A}, \mathbf{m}) \xrightarrow{\omega_{\mathbf{a}}} \Omega^\ell(\mathcal{A}, \mathbf{m}) \xrightarrow{\cdot \widetilde{Q}} R, \tag{4.3}$$

where the first map is multiplication by $\omega_{\mathbf{a}}$ and second map drops the basis $dx_1 \wedge \cdots \wedge dx_\ell$ and multiplies by the defining polynomial. This is supported by Corollary 1.4.23, which allows us to replace pairing with exterior multiplication and derivation module $D(\mathcal{A}, \mathbf{m})$ with $\Omega^{\ell - 1}(\mathcal{A}, \mathbf{m})$.

Since the derivation modules are homogeneous with the usual grading of R, the logarithmic ideal inherits a graded structure too. Moreover, $I(\mathcal{A}, \mathbf{m})$ has a bigrading given by $\deg x_j = (1, 0)$ and $\deg a_i = (0, 1)$, for $1 \leq j \leq \ell$ and $1 \leq i \leq n$.

Proposition 4.0.18. *If $(\mathcal{A}, \mathbf{m})$ is a free multiarrangement, then the ideal $I(\mathcal{A}, \mathbf{m})$ has generators in the polynomial degrees of $(\mathcal{A}, \mathbf{m})$. More precisely, if $D(\mathcal{A}, \mathbf{m})$ is generated in polynomial degrees $d_1, \ldots, d_t$ (with $t \geq \ell$), then $I(\mathcal{A}, \mathbf{m})$ is generated in degrees $(d_1 - 1, 1), \ldots, (d_t - 1, 1)$.*

Proof. Applying a homogeneous element $\theta \in D(\mathcal{A}, \mathbf{m})$ of polynomial degree d to $\omega_{\mathbf{a}}$, we get

$$\sum_{i=1}^{n} a_i \frac{\theta(f_i)}{f_i}, \tag{4.4}$$

where $\deg \theta(f_i)/f_i = d - 1$ and $\deg a_i = 1$, hence $\deg = (d - 1, 1)$. The total degree is just the addition of the two components. $\qquad\square$

4.1 Freeness via Log Ideals

As the title suggests, this section is targeted at detecting freeness of multiarrangements by looking at their logarithmic ideals. This is done through understanding the geometry of the logarithmic ideal $I(\mathcal{A}, \mathbf{m})$. The main result, Theorem 4.1.13, is an extension of [9, Theorem 2.13] from simple arrangements to multiarrangements. The knowledge of the simple case is useful in understanding the multiarrangement case, although the general case is considerably more complicated.

The following lemma is analogous to [9, Lemma 3.10] and relates the logarithmic module of forms/ideal of ideal of a multiarrangement to those of its irreducible components.

Lemma 4.1.1. *If* $(\mathcal{A}, \mathbf{m}) = (\mathcal{A}_1, \mathbf{m}_1) \oplus (\mathcal{A}_2, \mathbf{m}_2)$, *then* $I(\mathcal{A}, \mathbf{m}) = S_2 I(\mathcal{A}_1, \mathbf{m}_1) + S_1 I(\mathcal{A}_2, \mathbf{m}_2)$.

Proof. This simply follows from the corresponding formula for the logarithmic modules. See Lemma 1.4.5. $\square$

The following fact is analogous to [23, Proposition 2.7]. This has to do with the local nature of the derivation module. See Lemma 1.4.27.

Proposition 4.1.2. *Let* $(\mathcal{A}, \mathbf{m})$ *be a multiarrangement, then*

$$V(I(\mathcal{A}, \mathbf{m})) \cap (M(\mathcal{A}) \times \mathbb{C}^n) = \Sigma(\mathcal{A}). \tag{4.5}$$

Proof. Pick a point $\mathrm{x} \in M$ and assume that for some $\lambda \in \mathbb{C}^n$, (x, λ) is in the zero locus of $I(\mathcal{A}, \mathbf{m})$. Consider $\widetilde{Q} \partial_{x_i}$ which is a derivation on $(\mathcal{A}, \mathbf{m})$, for all $1 \leq i \leq \ell$. Since x is in the complement, $\widetilde{Q}(\mathrm{x}) \neq 0$ and we get $\langle \partial_{x_i}, \omega_\lambda \rangle = 0$, implying that the point is critical. $\square$

We first treat the case of multiarrangements with only one multiple hyperplane, i.e. there is some i, such that $\mathbf{m}(H_i) > 1$ and for all $H \neq H_i$, we have $\mathbf{m}(H) = 1$ (See Proposition 1.4.9).

Proposition 4.1.3. *If* $(\mathcal{A}, \mathbf{m})$ *is defined by* $\widetilde{Q} = f_1 \ldots f_{i-1} f_i^m f_{i+1} \ldots f_n$, *then the radical of* $I(\mathcal{A}, \mathbf{m})$ *is independent of m and the zero locus of the logarithmic ideal is given by*

$$V(I(\mathcal{A}, \mathbf{m})) = \overline{\Sigma} \cup V((f_i) + \langle \mathrm{Ann}(H_i), \omega_{\mathbf{a}} \rangle).$$

In particular, the codimension of $I(\mathcal{A}, \mathbf{m})$ *equals its rank.*

Proof. By Proposition 1.4.9, which provides a decomposition of the derivation module, we compute the following factorization.

$$
\begin{aligned}
I(\mathcal{A}, \mathbf{m}) &= \left(\langle f_i^{m-1} \theta_E, \omega_\mathbf{a} \rangle \right) + \left(\langle \theta, \omega_\mathbf{a} \rangle : \theta \in \mathrm{D}(\mathcal{A}, \mathbf{m}), \theta(f_i) = 0 \right) \\
&= \left(f_i^{m-1}(a_1 + \cdots + a_n) \right) + \left(\langle \theta, \omega_\mathbf{a} \rangle : \theta \in \mathrm{Ann}(H_i) \right) \\
&= \left[(f_i^{m-1}) \cap (a_1 + \cdots + a_n) \right] + \langle \mathrm{Ann}(H_i), \omega_\mathbf{a} \rangle \\
&= \left[(f_i^{m-1}) + \langle \mathrm{Ann}(H_i), \omega_\mathbf{a} \rangle \right] \cap \left[(a_1 + \cdots + a_n) + \langle \mathrm{Ann}(H_i), \omega_\mathbf{a} \rangle \right] \\
&= \left[(f_i^{m-1}) + \langle \mathrm{Ann}(H_i), \omega_\mathbf{a} \rangle \right] \cap I(\mathcal{A}) \qquad \text{(by Corollary 1.4.10)}
\end{aligned}
$$

This directly verifies that $I(\mathcal{A})$ defines one of the components of $V(I(\mathcal{A}, \mathbf{m}))$ (See Corollary 4.1.10). In the last Lemma 4.1.4, letting $\mathfrak{a} := (f_i^{m-1})$ and $\mathfrak{c} := I(\mathrm{Ann}(H_i))$, implies that the radical of $(f_i^{m-1}) + I(\mathrm{Ann}(H_i))$ is independent of m, as long as $m > 1$. As a result, one sees that the minimal primes of this ideal only depend on whether $m = 1$ or 2. $\qquad\square$

Lemma 4.1.4 (Exercise 1.13 in [6])**.** *Let $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ and $\mathfrak{d}$ be ideals in some commutative ring and suppose that $\mathrm{rad}(\mathfrak{a}) = \mathrm{rad}(\mathfrak{c})$ and $\mathrm{rad}(\mathfrak{b}) = \mathrm{rad}(\mathfrak{d})$, then*

$$
\mathrm{rad}(\mathfrak{a} + \mathfrak{b}) = \mathrm{rad}(\mathfrak{c} + \mathfrak{d}). \tag{4.6}
$$

The proof is easy and follows from the definitions.

Corollary 4.1.5. *Let $(\mathcal{A}, \mathbf{m})$ be a multiarrangement with only one multiple hyperplane H as in the above proposition. Then the radical of $I(\mathcal{A}, \mathbf{m})$ does not depend on the support of $\mathbf{m}$, i.e. whether $\mathbf{m}(H) > 1$ or $\mathbf{m}(H) = 1$.*

Proof. By the above proposition, the radical of $I(\mathcal{A}, \mathbf{m})$ equals

$$
\mathrm{rad}\left[\left((f_i^{m-1}) + \langle \mathrm{Ann}(H_i), \omega_\mathbf{a} \rangle \right) \cap I(\mathcal{A}) \right] = \mathrm{rad}\left((f_i^{m-1}) + \langle \mathrm{Ann}(H_i), \omega_\mathbf{a} \rangle \right) \cap \mathrm{rad} I(\mathcal{A})
$$

where by the above lemma, radical of $(f_i^{m-1}) + \langle \mathrm{Ann}(H_i), \omega_\mathbf{a} \rangle$ is independent of $m > 1$. $\qquad\square$

Definition 4.1.6. If $\mathbf{m}$ is a multiplicity on an arrangement $\mathcal{A}$, then the support of $\mathbf{m}$ is the subarrangement

$$
\mathrm{supp}(\mathbf{m}) := \{ H \in \mathcal{A} : \mathbf{m}(H) > 1 \}. \tag{4.7}
$$

Corollary 4.1.7. *The zero locus $V(I(\mathcal{A}, \mathbf{m}))$ contains*

$$\overline{\Sigma} \cup [\cup_{H_i \in \mathrm{supp}(\mathbf{m})} V((f_i) + \langle \mathrm{Ann}(H_i), \omega_{\mathbf{a}} \rangle)]. \tag{4.8}$$

Proof. For every $H_i \in \mathrm{supp}$, we have the inclusion $\mathrm{D}(\mathcal{A}, \mathbf{m}) \subseteq \mathrm{D}(\mathcal{A}, \mathbf{m}_i)$. Therefore,

$$V(I(\mathcal{A}, \mathbf{m}_i)) \subseteq V(I(\mathcal{A}, \mathbf{m})),$$

and the left hand side splits as in Proposition 4.1.3. $\square$

Remark 4.1.8. It should be emphasized that the inclusion of the above Corollary is in general far from equality. If D_1 and D_2 are two derivation modules on some arrangement $\mathcal{A}$, then $\langle D_1 \cap D_2, \omega_{\mathbf{a}} \rangle \subseteq \langle D_1, \omega_{\mathbf{a}} \rangle \cap \langle D_2, \omega_{\mathbf{a}} \rangle$, where as computing the right hand side is completely nontrivial: If $\theta_1 \in D_1$ and $\theta_2 \in D_2$ and $\langle \theta_1, \omega_{\mathbf{a}} \rangle = \langle \theta_2, \omega_{\mathbf{a}} \rangle$, then $\langle \theta_1 - \theta_2, \omega_{\mathbf{a}} \rangle$, implying that $\theta_1 - \theta_2$ is in the kernel of the map $\theta \mapsto \langle \theta, \omega_{\mathbf{a}} \rangle$. This kernel is generated by syzygies of the form $\langle \theta_1, \omega_{\mathbf{a}} \rangle \theta_2 - \langle \theta_2, \omega_{\mathbf{a}} \rangle \theta_1$, for all $\theta_1 \in D_1$ and $\theta_2 \in D_2$, which is almost always bigger than $D_1 \cap D_2$.

Theorem 4.1.9. *Let $\mathcal{A}$ be a rank ℓ multiarrangement. Then*

$$\mathrm{codim}\, I(\mathcal{A}, \mathbf{m}) = \ell.$$

Proof. It is enough to verify this locally because (co)dimension is determined locally. By Proposition 4.1.2, $V(I(\mathcal{A}, \mathbf{m}))$ agrees with the ℓ-codimensional Σ away from the hyperplanes. It remains to verify the claim on the hyperplanes. For this, it suffices to consider irreducible multiarrangements. The reason is that up to a linear change of coordinates we can break $(\mathcal{A}, \mathbf{m})$ into its connected components and by Lemma 4.1.1 write the ideal as the sum of the components.

$$\mathrm{codim}\, V(I(\mathcal{A}, \mathbf{m})) = \mathrm{codim}\, V(\sum_i I(\mathcal{A}_i, \mathbf{m}_i)) = \mathrm{codim}\, \cap_i V(I(\mathcal{A}_i, \mathbf{m}_i))$$

$$= \sum_i \mathrm{codim}\, V(I(\mathcal{A}_i, \mathbf{m}_i)) \overset{?}{=} \sum_i \mathrm{rank}(\mathcal{A}_i) = \mathrm{rank}\mathcal{A} = \ell$$

So without loss of generality we let $(\mathcal{A}, \mathbf{m})$ be irreducible and essential and use induction on $|\mathcal{A}|$. As a consequence all deletions of $\mathcal{A}$ will be essential too, because otherwise the deleted hyperplane would be a bridge and hence $\mathcal{A}$ reducible. Now, pick a point $\mathfrak{p} \neq \mathfrak{m}$ on one of the hyperplanes that is different from the origin. For

every hyperplane $H = \ker f \in \mathcal{A}$ with $\mathbf{m}(H) = m$ we have the sequence

$$\mathrm{D}(\mathcal{A}, \mathbf{m}) \hookrightarrow \mathrm{D}(\mathcal{A}', \mathbf{m}|_{\mathcal{A}'}) \xrightarrow{f^m} \mathrm{D}(\mathcal{A}, \mathbf{m})$$

which turns to equality after localizing at f as f^m is invertible in R_f. We have $\mathrm{D}(\mathcal{A}, \mathbf{m})_f \simeq \mathrm{D}(\mathcal{A}', \mathbf{m}|_{\mathcal{A}'})_f$ which implies $I(\mathcal{A}, \mathbf{m})_f + (a_n)_f = I(\mathcal{A}', \mathbf{m}|_{\mathcal{A}'}) + (a_n)_f$ and this ideal defines the zero locus of $I(\mathcal{A}, \mathbf{m})$ away from the hyperplane $H \times \mathbb{C}^n$. Since this works for every hyperplane, let H be a hyperplane that does not contain $\mathfrak{p}$. This is possible because $\mathcal{A}$ was assumed to be essential. The local codimension of $V(I(\mathcal{A}, \mathbf{m}))$ around $\mathfrak{p}$ is at most $\ell + 1$ by the induction hypothesis but by the above corollary every such component of $V(I(\mathcal{A}, \mathbf{m}))$ lies in some hyperplanes of the form $H_i \times \mathbb{C}^n$ with $\mathfrak{p} \in H_i$. So we intersect with this hyperplane and this brings the codimension down to ℓ.

By essentiality, we have codimension ℓ everywhere on $\cup_{i=1}^n D(f_i) = \mathbb{C}^\ell \times \mathbb{C}^n \backslash 0 \times \mathbb{C}^n$. What remains is a component that is contained in $0 \times \mathbb{C}^n$ which has codimension at least ℓ. This will not harm the overall codimension of $I(\mathcal{A}, \mathbf{m})$. $\qquad\square$

Corollary 4.1.10. *Let* $\mathbf{m}$ *be a multiplicity on an arrangement* $\mathcal{A}$, *then* $V(I)$ *is contained in* $V(I(\mathcal{A}, \mathbf{m}))$ *and moreover defines one of the irreducible components.*

Proof. In the inclusion $\Sigma \subset V(I(\mathcal{A}, \mathbf{m}))$ take closure and use [9, Theorem 2.9] to replace $\overline{\Sigma}$ with $V(I(\mathcal{A}))$. The fact that $V(I(\mathcal{A}))$ defines one of the irreducible components is an immediate by dimension reasons. $\qquad\square$

Remark 4.1.11.   • Apart from $I(\mathcal{A})$, the rest of the components of $I(\mathcal{A}, \mathbf{m})$ are to be found among the components of the ideal quotient $(I(\mathcal{A}, \mathbf{m}) : I(\mathcal{A}))$.

• It is worth remembering from [9] that $I(\mathcal{A})$ is prime when $\mathcal{A}$ is tame. In this case, $I(\mathcal{A})$ appears as one the associated primes of $I(\mathcal{A}, \mathbf{m})$. This requirement will be satisfied for free in rank 3.

Definition 4.1.12. An ideal I is called a *complete intersection* if it can be generated by as many generators as its codimension. By the Krull's principal ideal theorem, the codimension $\mathrm{codim}\,(I)$ is less than or equal to the minimal number of generators $\mu(I)$.

The following Theorem is analogous to Theorem 2.13 in [9].

Theorem 4.1.13. *A multiarrangement* $(\mathcal{A}, \mathbf{m})$ *is free if and only if the ideal* $I(\mathcal{A}, \mathbf{m})$ *is a complete intersection.*

Proof. If $(\mathcal{A}, \mathbf{m})$ is free, then we take a basis $\theta_1, \ldots, \theta_\ell$ and apply them to the critical 1-form $\omega_\mathbf{a}$ to get ℓ generators for the $I(\mathcal{A}, \mathbf{m})$. On the other hand, as seen in Theorem 4.1.9, the logarithmic ideal $I(\mathcal{A}, \mathbf{m})$ is of height ℓ, showing that it is a complete intersection.

Conversely, if $I(\mathcal{A}, \mathbf{m})$ is a complete intersection, then it has some ℓ generating elements, say $\langle \theta_1, \omega_\mathbf{a} \rangle, \ldots, \langle \theta_\ell, \omega_\mathbf{a} \rangle$, where $\theta_1, \ldots, \theta_\ell \in D_C(\mathcal{A}, \mathbf{m})$. Comparison with $I(\mathcal{A}, \mathbf{m})$ suggests that these elements form a set of generators for $D_C(\mathcal{A}, \mathbf{m})$ over S. As a result, $D_C(\mathcal{A}, \mathbf{m})$ must be free over S, because S is a domain. It particular, it is flat over S. The next step is to refine this by showing that $D(\mathcal{A}, \mathbf{m})$ is flat over R. Let M be an arbitrary R module and consider it as a S-module by letting the variables of C act trivially. Let $n > 0$ and use the flat base change formula for Tor to get the following.

$$\operatorname{Tor}_n^R(D(\mathcal{A}, \mathbf{m}), M) \cong \operatorname{Tor}_n^S(D(\mathcal{A}, \mathbf{m}) \otimes_R S, M) \tag{4.9}$$

But here $D(\mathcal{A}, \mathbf{m}) \otimes_R S$ is the same as $D_C(\mathcal{A}, \mathbf{m})$ and by flatness we see that the right hand side vanishes. This shows that $D(\mathcal{A}, \mathbf{m})$ is flat and hence free over R. $\square$

The following conjecture indicates the invariance of the radical logarithmic ideal under the support of the multiplicity. It is proved in case of multiarrangements with only one multiple hyperplane and in general is supported by symbolic calculations.

Conjecture 4.1.14. The radical ideal $\operatorname{rad}(I(\mathcal{A}, \mathbf{m}))$ only depends on $\operatorname{supp}(\mathbf{m})$. To be precise, if $\mathbf{m}_1$ and $\mathbf{m}_2$ are multiplicities on an arrangement $\mathcal{A}$, then

$$\operatorname{rad}(I(\mathcal{A}, \mathbf{m}_1)) = \operatorname{rad}(I(\mathcal{A}, \mathbf{m}_2))$$

if and only if $\operatorname{supp}(\mathbf{m}_1) = \operatorname{supp}(\mathbf{m}_2)$.

The author expects to find a proof for this by understanding the ideal quotient $(I(\mathcal{A}, \mathbf{m}) : I(\mathcal{A}))$ and relating it to radical of the principal ideal generated by $\widetilde{Q}/Q$.

4.2 Examples

Example 4.2.1 (Multipencils). The following computation gives the minimal associated primes of a multiarrangement $(\mathcal{A}, \mathbf{m})$ of lines in $\mathbb{C}^2$.

Let us consider the arrangement defined by $\widetilde{Q} = f_1^2 f_2 \dots f_n \in \mathbb{C}[x,y]$. The module of derivations of this multiarrangement of lines is free and is generated by

$$\theta_1 = f_1 \theta_E \,, \theta_2 = \frac{Q}{f_1}\left(\frac{\partial f_1}{\partial x}\frac{\partial}{\partial y} - \frac{\partial f_1}{\partial y}\frac{\partial}{\partial x}\right). \tag{4.10}$$

Assuming that $f_i = c_i x + d_i y$, we get

$$I(\mathcal{A}, \mathbf{m}) = \left(f_1(a_1 + \dots + a_n), \sum_{i=2}^{n} a_i(c_1 d_i - c_i d_1)f_2 \cdots \widehat{f_i} \cdots f_n\right)$$
$$= (f_1, g) \cap (a_1 + \dots + a_n, g),$$

where g denotes the second generator on the fist line above. Also note that

$$(f_1, g) = \left(f_1, (a_2 + \dots + a_n)y^{n-2}\right), \tag{4.11}$$

simply because each of g and $y^{n-2}(a_2 + \dots + a_n)$ is a linear combination of f_1 and the other one. Separating the generators takes us to the list of primary components that include $(f_i, a_2 + \dots + \hat{a}_i + \dots + a_n)$. In genesral, if the size of supp is at least 2, then every line $H_i = \ker f_i$ of multiplicity greater than one, contributes one component to the intersection, namely the ideal generated by $(f_i, a_2 + \dots + \hat{a}_i + \dots + a_n)$. In this easy case, the primary decomposition is given by

$$I(\mathcal{A}, \mathbf{m}) = I(\mathcal{A}) \cap (x,y) \cap \left(\bigcap_{m_i > 1} (f_i^{m_i - 1}, a_+ \dots + \hat{a}_i + \dots + a_n)\right).$$

The associated primes in this case just come from flattening the exponents $m_i - 1$ to 1, for all $m_i > 1$.

The problem of finding the primary decomposition is not expected to have a clean answer because the ideals $\langle \mathrm{Ann}(H_i), \omega_{\mathbf{a}} \rangle$ might have embedded primes which are already not easy to describe.

Example 4.2.2 (Deleted A_3 Arrangement). Let $\mathcal{A}$ be the arrangement defined by $Q = (y-z)y(x-y)x(x-z)$ and consider multiplicities $\mathbf{m}_1 \equiv 2$ and $\mathbf{m}_2 = (2,2,3,2,2)$. Resolving the ideals $I_1 = I(\mathcal{A}, \mathbf{m}_1)$ and $I_2 = I(\mathcal{A}, \mathbf{m}_2)$ in Macaulay 2 [16] returns

$$0 \to S^2 \to S^5 \to S^4 \to S^1 \to S/I_1 \to 0$$
$$0 \to S^1 \to S^3 \to S^3 \to S^1 \to S/I_2 \to 0$$

as minimal free resolutions, where the first Betti number measures the minimal number of generators. Therefore, $\mathbf{m}_1$ is non-free multiplicity where the ideal I_2 that has codimension 3 needs 4 generators, where as $\mathbf{m}_2$ is a free multiplicity with $\beta_1 = 3 = \operatorname{codim} I_2$. It is also worth noting that by Proposition 4.1.9, they have the same ideal and share the same zero locus. The non/freeness of these two examples were predicted by a result of T. Abe. See [1].

The Theorem in the following section explains the occurrence of binomial Betti numbers in the resolution of the free multiplicity in the Example above.

4.3 Tameness, Resolutions and Hilbert Series

By Proposition 1.4.19, together with the fact that $\omega_{\mathbf{a}} \in \Omega^1_C(\mathcal{A}) \subseteq \Omega^1_C(\mathcal{A}, \mathbf{m})$ we get a well-defined complex as in the following theorem where the differential is multiplication by the logarithmic 1-from $\omega_{\mathbf{a}}$. The statement and proof is analogous to [9, Theorem 3.5].

Theorem 4.3.1. *Let $(\mathcal{A}, \mathbf{m})$ be a tame multiarrangement, then the complex*

$$0 \to \Omega^0(\mathcal{A}, \mathbf{m}) \xrightarrow{\omega_{\mathbf{a}}} \Omega^1(\mathcal{A}, \mathbf{m}) \xrightarrow{\omega_{\mathbf{a}}} \cdots \xrightarrow{\omega_{\mathbf{a}}} \Omega^{\ell-1}(\mathcal{A}, \mathbf{m}) \xrightarrow{\omega_{\mathbf{a}}} \Omega^{\ell}(\mathcal{A}, \mathbf{m}) \to S/I(\mathcal{A}, \mathbf{m}) \to 0$$
(4.12)

is exact.

Proof. The idea is to show that all localizations of the complex above at maximal points of R are exact. Again without loss of generality, we can assume that the arrangement is full rank. In this case, the union of the complements $\cup_{i=1}^n D(f_i)$ covers everything except the origin in the first factor. The first step is remove all multiple hyperplanes one by one and show that the localization of the complex on each complement agrees with that of the deletion. Then it remains to investigate exactness at the origin for which the tameness hypothesis will be used.

Case I) *Local exactness everywhere except at the origin.* We use induction on the size of the underlying arrangement. With only one hyperplane the problem is empty. Let $|\mathcal{A}| > 1$. For every $H \in \mathcal{A}$ of multiplicity m we have the sequence

$$\Omega^{\bullet}_R(\mathcal{A}', \mathbf{m}|_{\mathcal{A}'}) \hookrightarrow \Omega^{\bullet}_R(\mathcal{A}, \mathbf{m}) \xrightarrow{f^m} \Omega^{\bullet}_R(\mathcal{A}', \mathbf{m}|_{\mathcal{A}'})$$

as a complex over R. Next we tensor this with $C = \mathbb{C}[a_1, \ldots, a_{n-1}] \otimes \mathbb{C}[a_n]$.

$$\Omega^{\bullet}_{C'}(\mathcal{A}', \mathbf{m}|_{\mathcal{A}'}) \otimes \mathbb{C}[a_n] \hookrightarrow \Omega^{\bullet}_C(\mathcal{A}, \mathbf{m}) \xrightarrow{f^m} \Omega^{\bullet}_{C'}(\mathcal{A}', \mathbf{m}|_{\mathcal{A}'}) \otimes \mathbb{C}[a_n]$$

If we localize at f, the inclusion upgrades to an isomorphism between dga's since over S_f, multiplication by f^m will define a surjective map as f^m will be invertible. Therefore we get the following isomorphism of chain complex cohomologies

$$H^\bullet(\Omega_{C'}^\bullet(\mathcal{A}', \mathbf{m}|_{\mathcal{A}'}) \otimes \mathbb{C}[a_n])_f \simeq H^\bullet(\Omega_C^\bullet(\mathcal{A}, \mathbf{m}))_f$$

where $(\Omega_{C'}^\bullet(\mathcal{A}', \mathbf{m}|_{\mathcal{A}'}) \otimes \mathbb{C}[a_n], \mathrm{d}) \simeq (\Omega_{C'}^\bullet(\mathcal{A}', \mathbf{m}|_{\mathcal{A}'}), \mathrm{d}) \otimes (\mathbb{C}[a_n], 0)$. As a result, $H^\bullet(\Omega_{C'}^\bullet(\mathcal{A}', \mathbf{m}|_{\mathcal{A}'}) \otimes \mathbb{C}[a_n])_f$ will be isomorphic to $H^\bullet(\Omega_{C'}^\bullet(\mathcal{A}', \mathbf{m}|_{\mathcal{A}'})_f) \otimes \mathbb{C}[a_n]_f$ where the first factor is zero by induction hypothesis.

Case II) *Local exactness at the origin.* This uses the tame hypothesis and follows from the following Lemma 4.3.2. □

Let us denote the chain cochain complex $(\Omega_{S/C}^\bullet(\mathcal{A}, \mathbf{m}), \omega_{\mathbf{a}})$ by $\Omega^\bullet$ and its cohomology $H^q(\Omega^\bullet)$ by H^q, for $q = 1, \ldots, \ell - 1$. It follows from the lemma above that for each q in the range, we have

$$\operatorname{supp} H^q \subseteq \{\mathfrak{m}\},$$

and we want to show that the left hand side is actually empty. Comparison with the formula

$$\operatorname{supp} H^q = V(\operatorname{Ann} H^q),$$

in view of the Nullstellensatz correspondence implies that the annihilator is of the form $\mathfrak{m}^r$, for some $r \geq 0$. This turns the problem to showing that $r = 0$. The possibility of having $r > 0$ is ruled out by the following lemma, where we show that

$$H_\mathfrak{m}^0(H^q) = \cup_{i=1}^\infty (0 :_{H^q} \mathfrak{m}^i) = 0,$$

for $q = 1, \ldots, \ell - 1$.

Lemma 4.3.2. *Let $(\mathcal{A}, \mathbf{m})$ be a tame multiarrangement and let $\mathfrak{m} = SR_+$. If $1 \leq q < \ell$ is the first spot where $\Omega^\bullet$ is non-exact, then the zeroth local cohomology module $H_\mathfrak{m}^0(H^q)$ vanishes. In particular, the localization of the complex 4.3 at $\mathfrak{m}$ is exact.*

Proof. We use the following two hypercohomology local spectral sequences.

$$'E_2^{p,q} = H^p(H_\mathfrak{m}^q(\Omega^\bullet)) \Rightarrow \mathbb{H}_\mathfrak{m}^{p+q}(\Omega^\bullet) \tag{4.13}$$

$$''E_2^{p,q} = H_\mathfrak{m}^p(H^q(\Omega^\bullet)) \Rightarrow \mathbb{H}_\mathfrak{m}^{p+q}(\Omega^\bullet) \tag{4.14}$$

The fact the first non-vanishing of local cohomology characterizes the depth, together with the Auslander-Buchsbaum theorem, under the tame hypothesis, imply that

$$H_{\mathfrak{m}}^q(\Omega^\bullet) = 0,$$

for $0 \leq q < \ell - p$. As a consequence of the first spectral sequence, we have $\mathbb{H}_{\mathfrak{m}}^k(\Omega^\bullet)$, for $k < \ell$.

Considering the second spectral sequence, choose the smallest q with $H^q \neq 0$. The map $''E_2^{0,q} \to ''E_2^{1,q-1} = 0$ is stable, thus

$$''E_\infty^{0q} = ''E_2^{0q}.$$

This, combined with the fact that $H_{\mathfrak{m}}^q(\Omega^\bullet) = 0$, for this specific choice of q, shows that $''E_2^{0q} = H_{\mathfrak{m}}^0(\Omega^\bullet) = 0$. By argument preceding the lemma, implies that the sequence is exact, contradicting the choice of q. $\qquad\square$

Here is another implication of tameness, similar to Theorem 3.7 in [9]. Contrary to the simple case, this does not imply that the logarithmic ideal is prime. In fact, the logarithmic ideal of nonsimple arrangements is never prime and has at least two isolated associated primes.

Theorem 4.3.3. *Let $(\mathcal{A}, \mathbf{m})$ be a tame multiarrangement, then the logarithmic ideal $I(\mathcal{A}, \mathbf{m})$ is Cohen-Macaulay.*

Proof. This is similar to the proof of [9, Theorem 3.7] which uses the above resolution together with Auslander-Buchsbaum Formula. $\qquad\square$

The following is a corollary to Theorem 4.3.1.

Corollary 4.3.4. *A multiarrangement $(\mathcal{A}, \mathbf{m})$ is free if and only if the logarithmic ideal $I(\mathcal{A}, \mathbf{m})$ admits binomial Betti number, i.e.*

$$\beta_i(S/I(\mathcal{A}, \mathbf{m})) = \binom{\ell}{i},$$

for $i = 1, \ldots, \ell$.

Proof. The only if part is a simple consequence of the last theorem together with Corollary 1.4.16 and Proposition 1.4.22. Conversely, if the Betti numbers are binomial, then just because $\beta_1 = \ell$, the ideal $I(\mathcal{A}, \mathbf{m})$ is generated by ℓ elements and freeness follows from the proof of Theorem 4.1.13. $\qquad\square$

The following two proposition compute the Hilbert series of $S/I(\mathcal{A}, \mathbf{m})$ in different gradings.

Definition 4.3.5. Let $\Omega^\bullet(\mathcal{A}, \mathbf{m})$ be graded polynomially, i.e. $\deg x_j = 1$ and $\deg \mathrm{d}x_j = 0$, for $1 \leq j \leq \ell$. Then define

$$\mathfrak{P}_{(\mathcal{A}, \mathbf{m})}(t, u) := \sum_{p=0}^{\ell} h(\Omega_{\mathbb{C}}^p(\mathcal{A}, \mathbf{m}), t)u^p. \tag{4.15}$$

For the following Proposition, we are going to bigrade the resolution by letting $\deg \mathrm{d}x_j = \deg x_j = (1, 0)$ and $\deg a_i = (0, 1)$, for all $1 \leq i \leq n$ and $1 \leq j \leq \ell$. As a result, multiplication by $\omega_{\mathbf{a}}$ will be a map of degree $(0, 1)$.

Proposition 4.3.6. *If $(\mathcal{A}, \mathbf{m})$ is tame, then*

$$\frac{t^{m-\ell}(-u)^\ell}{(1-u)^n} \mathfrak{P}_{(\mathcal{A}, \mathbf{m})}(t, -t/u)$$

is the bigraded Hilbert Series of $S/I(\mathcal{A}, \mathbf{m})$ under the total grading.

Proof. By Theorem 4.3.1, for a tame multiarrangement $(\mathcal{A}, \mathbf{m})$, the complex

$$0 \to \Omega_{S/C}^0[(0, -\ell)] \xrightarrow{\omega_{\mathbf{a}}} \cdots \xrightarrow{\omega_{\mathbf{a}}} \Omega_{S/C}^{\ell-1}[(0, -1)] \xrightarrow{\omega_{\mathbf{a}}} \Omega_{S/C}^\ell \to (S/I)[(m-\ell, 0)] \to 0$$

is exact where the reference to $(\mathcal{A}, \mathbf{m})$ is dropped to keep the formula short. In our grading, $\deg \omega_{\mathbf{a}} = (0, 1)$, so the term $\Omega_{S/C}^p(\mathcal{A}, \mathbf{m})$ in the sequences should be shifted to $\Omega_{S/C}^p[(0, p - \ell)]$ in order to keep the sequence homogeneous. Let t and u record the degrees in the R and C variables respectively.

$$h(S/I)[(m-\ell, 0)] = \sum_{p=\ell}^{0} (-1)^{\ell-p} h(\Omega_{S/C}^p(\mathcal{A}, \mathbf{m})[-(0, \ell-p)]; t, u), \quad \deg \mathrm{d}x_j = 1$$

$$= (-1)^\ell \sum_{p=0}^{\ell} (-1)^p u^{\ell-p} t^p h(\Omega_{\mathbb{C}}^p(\mathcal{A}, \mathbf{m}), t) h(C, u), \quad \deg \mathrm{d}x_j = 0$$

$$= \frac{(-u)^\ell}{(1-u)^n} \sum_{p=0}^{\ell} (-t/u)^p h(\Omega_{\mathbb{C}}^p(\mathcal{A}, \mathbf{m}), t)$$

Therefore the final answer is

$$h(S/I) = \frac{t^{m-\ell}(-u)^\ell}{(1-u)^n} \sum_{p=0}^{\ell}(-t/u)^p h(\Omega_{\mathbb{C}}^p(\mathcal{A}, \mathbf{m}), t), \quad \deg \mathrm{d}x_j = 0.$$

□

In the next calculation, we let the algebra $\Omega_{S/C}^\bullet(\mathcal{A}, \mathbf{m})$ be single graded by letting $\deg \mathrm{d}x_i = 0$ and

$$\deg x_j = \deg a_i = 1,$$

for $1 \le j \le \ell$ and $1 \le i \le n$. Under this setup, the 1-form $\omega_{\mathbf{a}} = \sum_{i=1}^n a_i \mathrm{d}f_i/f_i$ will be homogeneous of degree 0. In this grading, the complex

$$0 \to \Omega_{S/C}^0(\mathcal{A}, \mathbf{m}) \cdots \xrightarrow{\omega_{\mathbf{a}}} \Omega_{S/C}^{\ell-p}(\mathcal{A}, \mathbf{m}) \xrightarrow{\omega_{\mathbf{a}}} \cdots \Omega_{S/C}^\ell(\mathcal{A}, \mathbf{m}) \to (S/I)[m] \to 0, \quad (4.16)$$

is exact, where $m = \deg \widetilde{Q} = |\mathbf{m}|$ is the size of the multiarrangement.

Theorem 4.3.7. *Let $(\mathcal{A}, \mathbf{m})$ be an free multiarrangement of rank ℓ with $|\mathcal{A}| = n$. Then the Hilbert series of $S/I(\mathcal{A}, \mathbf{m})$ is given by*

$$h(S/I) = \frac{\Pi_{i=1}^\ell(1 - t^{d_i})}{(1-t)^{n+\ell}}, \tag{4.17}$$

where $(d_1, \ldots, d_\ell) = \exp(\mathcal{A}, \mathbf{m})$.

Proof. By exactness of the complex above, we have

$$h(S/I)[m - \ell] = \sum_{p=0}^{\ell}(-1)^{\ell-p} h(\Omega_{S/C}^p(\mathcal{A}, \mathbf{m}))$$

$$= (-1)^\ell \sum_{p=0}^{\ell}(-1)^p h(\Omega_{\mathbb{C}}^p(\mathcal{A}, \mathbf{m})) \cdot h(C)$$

$$= \frac{(-1)^\ell}{(1-t)^{n+\ell}} \sum_{p=0}^{\ell}(-1)^p \sum_{1 \le i_1 < \cdots < i_p \le \ell} t^{-(d_{i_1} + \cdots + d_{i_p})}$$

$$= \frac{(-1)^\ell}{(1-t)^{n+\ell}} \sum_{p=0}^{\ell} \sum_{1 \le i_1 < \cdots < i_p \le \ell} (-t)^{-(d_{i_1} + \cdots + d_{i_p})}$$

$$= \frac{(-1)^\ell}{(1-t)^{n+\ell}} \Pi_{p=0}^\ell(1 - t^{-d_i}) = \frac{\Pi_{p=0}^\ell(t^{-d_i} - 1)}{(1-t)^{n+\ell}}$$

Therefore

$$h(S/I) = \frac{t^m \Pi_{p=0}^{\ell}(t^{-d_i} - 1)}{(1-t)^{n+\ell}} = \frac{\Pi_{p=0}^{\ell} t^{d_i}(t^{-d_i} - 1)}{(1-t)^{n+\ell}}, \tag{4.18}$$

where the fact $m = \sum_{i=1}^{\ell} d_i$ is used. $\qquad\square$

Example 4.3.8. The Hilbert series of $S/I(\mathcal{A}, \mathbf{m})$ is not combinatorial. For the multiarrangements of Example 1.4.12, we have

$$h(S/I(\mathcal{A}_1, \mathbf{m}_1)) = \frac{1 - t^3 - t^5 + t^8}{(1-t)^6}, \tag{4.19}$$

while

$$h(S/I(\mathcal{A}_2, \mathbf{m}_2)) = \frac{1 - 2t^4 + t^8}{(1-t)^6}. \tag{4.20}$$

This is in sharp contrast to the simple case where the Hilbert series of locally free arrangements is combinatorially determined. See [44].

4.4 Intersection Class with Multiplicity

Proposition 4.4.1. *Let I be a bihomogeneous complete intersection ideal in*

$$\mathbb{C}[x_0, \ldots, x_m; y_0, \ldots, y_n]$$

of codimension c, which is generated by $g_1, \ldots, g_c$, such that $\deg g_i = (b_i, d_i)$, for $i = 1, \ldots, c$, then the intersection class $[V(I)] \in \mathrm{CH}(\mathbb{P}^m \times \mathbb{P}^n) = \mathrm{Z}[h, k]/(h^{m+1}, k^{n+1})$ is given by

$$[V(I)] = \Pi_{i=1}^{c}(b_i h + d_i k).$$

Proof. One can use induction on the number of generators of I, since if we pull the first generator g_1 out, we get the two ideals (g_1) and $I' = (g_2, \ldots, g_c)$, where the first one defines a hypersurface and the second one is again a complete intersection. This is because subsequences of regular sequences are again regular sequences. Since

$$V(I) = V((g_1) + I') = V(g_1) \cap V(I),$$

we get $[V(I)] = [V(g_1)] \cdot [V(I')]$ and the answer follows by using the induction hypothesis.

For the base case, consider a bihomogeneous polynomial g of degree (b, d) and look at the hypersurface it defines in $\mathbb{P}^m \times \mathbb{P}^n$. Since its codimension is one and the

degree one part of the cohomology ring is generated by linear combinations of h and k, the answer should be of the form $\alpha h + \beta k$, for some numbers α, β. In order to determine these, multiply it by $h^{m-1}k^n$, which is the cocycle of $L \times pt$, where L is a line in $\mathbb{P}^m$ and pt is a point in $\mathbb{P}^n$. By $g|_{pt}$ denote the polynomial which is obtained by partially evaluating g at the point pt. This will be a homogeneous polynomial in x's of degree b. We have

$$V(g) \cap (L \times pt) = (V(g|_{pt}) \cap L) \times pt,$$

and $[V(g|_{pt}) \cap L] = bh^m$ and $[pt] = k^n$, and by Kuenneth formula we get $[V(g) \cap (L \times pt)] = bh^m k^n$, which by comparison equals $\alpha h^m k^n$ and we get the first one. For the other coefficient, one uses $pt \times L$, where this time the point is the first factor and L is a line in the second factor. $\qquad\square$

Corollary 4.4.2. *Let $(\mathcal{A}, \mathbf{m})$ be a free multiarrangement with* $\exp(\mathcal{A}, \mathbf{m}) = (d_1, \dots, d_\ell)$, *then*

$$[V(I(\mathcal{A}, \mathbf{m}))] = \Pi_{i=1}^{\ell}((d_i-1)h+k) = (k-h)^{\ell}\pi(\mathcal{A}, \mathbf{m}; h/(k-h)) = (-h)^{\ell}\chi(\mathcal{A}, \mathbf{m}; (h-k)/h).$$

Proof. When $(\mathcal{A}, \mathbf{m})$ is free with a basis $\theta_1, \dots, \theta_\ell$, then $I(\mathcal{A}, \mathbf{m})$ becomes a complete intersection, generated by $\langle \theta_i, \omega_{\mathbf{a}} \rangle$, for $i = 1, \dots, \ell$, which is of bidegree $(d_i - 1, 1)$. $\quad\square$

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