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Polynomial Density Of Compact Smooth Surfaces

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Abstract

We show that any smooth closed surface has polynomial density 3 and that any connected compact smooth surface with boundary has polynomial density 2.

Keywords: Polynomial density, Whitehead triangulations, Compact smooth surfaces, Uniform algebras, Polytopes, Polynomial convexity.

Summary for Lay Audience

We assume any surfaces we consider are all in one piece. The distance between a pair of functions is defined to be the maximum over all points of the distance between their values. (When choosing functions of a particular form to approximate a given function, this distance can be thought of as an error tolerance.) We say that a specific function f can be approximated by a set of functions $\mathcal{F}$ if we can always find a function from $\mathcal{F}$ within the distance specified, no matter how small we chose the distance to be.

We say a surface has polynomial density n if there exist smooth functions $g_1, g_2, \ldots, g_n$ such that the set of polynomials in them can approximate any continuous function on the surface. We confirm that for a surface without boundary components, a closed surface, the polynomial density is three. On the other hand, for a surface with at least one boundary component, we show that the polynomial density must be two.

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Chapter 1

Introduction

The purpose of this thesis is to investigate polynomial densities of compact surfaces, both with and without boundary.

1.1 Notation and Terminology

In this chapter, M will denote a compact real manifold (possibly with boundary), K will denote a compact subset of $\mathbb{C}^n$ (for some positive integer n), and X will be a compact of either form. Denote the supremum norm on X as $\|\cdot\|_X$. The topology on X induced by it is called the **topology of uniform convergence** on $C(X)$ and the closure $\mathrm{cl}_{C(X)}(E)$ of a subset $E \subset C(X)$ in this topology is called its **uniform closure**. Note that the uniform closure of an algebra is itself an algebra since addition and multiplication are continuous in this topology.

We consider the following algebras in this introduction:

1. $C(X)$, the algebra of continous functions;

2. $P(K) = \mathrm{cl}_{C(K)}(\langle z_1, z_2, \ldots, z_n \rangle)$, the uniform closure of the unital algebra generated by $z_1, z_2, \ldots, z_n$;

3. $\mathcal{O}(K)$, the functions in $C(K)$ which extend to be holomorphic in some neighbourhood of K; and,

4. its uniform closure $\mathrm{cl}_{C(K)}(\mathcal{O}(K))$.

We will say that a function on a connected manifold with boundary is smooth if it is the restriction of a smooth function defined on some neighbourhood M_0 of M. Any manifold in the rest of this chapter is to be assumed to be connected unless otherwise specified.

1.2 Polynomial Density

The **polynomial density** of M will denote the minimum number of smooth functions $g_1, g_2, \ldots, g_n$ such that the unital algebra they generate, $\langle g_1, g_2, \ldots, g_n \rangle$, is dense in $C(M)$ in the topology of uniform convergence. This positive integer exists for any closed manifold. This can be shown by embedding the manifold M into a compact $K \subset \mathbb{R}^{2n+1}$ using the Whitney Embedding Theorem [Whi36, Thm. 1, p. 654]. Regarded as a subset of $\mathbb{C}^{2n+1}$, K is compact, polynomially convex, and totally real. By the Harvey-Wells Theorem [Sto00, Thm. 6.3.1, p. 300], $C(K) = \mathrm{cl}_{C(K)}(O(K))$ and by the Oka-Weil Theorem [Sto00, Thm. 1.5.1, p. 39], $O(K) \subset P(K)$. Consequently, $C(K) = P(K) = \mathrm{cl}_{C(K)}(\langle z_1, \ldots, z_{2n+1}\rangle)$, and the polynomial density exists and is at most $2n + 1$. Straightforward computations show that the polynomial density of a manifold is a diffeomorphism invariant.

If we have a compact manifold (with or without boundary) which is not connected, its polynomial density will be the maximum of the polynomial density of its connected components. (This can be shown by considering smooth generators $f_{k,0}, f_{k,1}, \ldots f_{k,m_j-1}$ on the connected components M_j and letting $m = \max_j m_j$. Then define f_k on M_j to be $f_{k,j}$ if $k < m_j$ and to be the constant 1 otherwise. These global smooth function f_k constitute smooth generators for M.)

Here are the 1-dimensional examples. We will see later that this pattern in which the compact manifold with boundary has a lower polynomial density than the closed manifold also holds in dimension 2.

Example 1.2.1. *An open rectifiable curve in $\mathbb{C}^n$, by diffeomorphism invariance, can be assumed to be the interval $[0, 1] \subset \mathbb{C}$. Its algebra $C([0, 1])$ has 1 generator by Weierstrass' Theorem.*

Example 1.2.2. *A closed simple curve in $\mathbb{C}^n$ can be assumed to be $S^1 \subset \mathbb{C}$. We can use Fourier series to show that $\mathrm{cl}_{C(S^1)}(\langle z, \bar{z}\rangle) = C(S^1)$, but $\mathrm{cl}_{C(S^1)}(\langle z\rangle) = P(S^1) \neq C(S^1)$, as the algebra $P(S^1)$ consists of those elements $f \in C(S^1)$ which have vanishing negative Fourier coefficients.*

1.3 Refining the Upper Bound

The work of Gupta and Shafikov in [GS20] and [GS21] provides upper bounds for the polynomial density of a compact smooth n-dimensional manifold M (possibly with boundary) whenever $n \geq 4$. They prove the existence of smooth embeddings of $2k + 1$-dimensional manifolds (without boundaries) into $\mathbb{C}^{3k}$ and of $2k$-dimensional manifolds (with or without boundaries) into $\mathbb{C}^{3k-1}$ having the following properties, where M' is the image of M under such an embedding:

1. M' is polynomially convex,
2. M' is totally real except for a finite number of isolated nondegenerate CR singularities,
3. M' is locally polynomially convex at each of its CR singularities.

The O'Farrel-Preskenis-Walsch Theorem then implies that $C(M') = \mathrm{cl}_{C(M')}(O(M'))$. (Take X to be M' and X_0 to be the set of CR singularities.)

Theorem 1.3.1. *[Sto00, Thm. 6.3.2, p. 300]*
Let X be a compact holomorphically convex set in $\mathbb{C}^n$, and let X_0 be a closed subset of X for which $X \setminus X_0$ is a totally real subset of the manifold $\mathbb{C}^n \setminus X_0$. A function $f \in C(X)$ can be approximated uniformly on X by functions holomorphic on a neighbourhood of X if and only if $f|_{X_0}$ can be approximated uniformly on X_0 by functions holomorphic on X.

Next, an application of the Oka-Weil theorem (2.6.5) shows that $\mathcal{O}(M') \subset P(M')$. Combining these results, they are able to conclude that $P(M') = C(M')$. In particular,

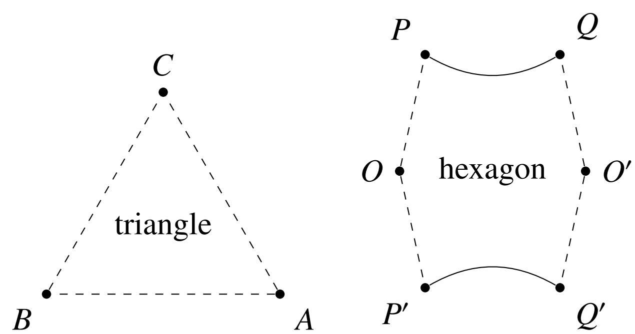
Theorem 1.3.2. *(Gupta, Shafikov)*
Any compact even-dimensional manifold (with or without boundary) or compact odd-dimensional manifold (without boundary) has polynomial density at most $\lfloor 3n/2 \rfloor - 1$*, provided that it has dimension at least* 4.

1.4 Methods and Results

Theorem 1.4.1. *(Thesis)*
Let S *be a connected compact smooth manifold. Then*

1. *its polynomial density is 3 if it is closed, and*
2. *its polynomial density is 2 if it has a boundary.*

We modify the approach of Vodovoz and Zaidenberg [VZ71], who showed that any n-dimensional simplicial polytope has a topological embedding into $\mathbb{C}^{n+1}$, to show that any closed surface has polynomial density 3 (see Chapter 3). Similar to their construction, we first build generators on the vertices, then the edges, and finally on each face. We achieve smoothness by having functions go flat near the simplex boundaries, using a Whitehead triangulation, pulling back vector fields so that we have induced normal vector fields in open neighbourhoods of the edges, and using these directions as that for which the function will be flat.



Then we consider the case of surfaces with boundary in Chapter 4. Our technique to prove that they have polynomial density 2 is analogous to that of Chapter 3 except that instead of having a decomposition into triangles we use a decomposition into 'hexagons': compact piecewise smooth polygons having six sides, two of which are opposite each other and are boundary pieces.

Chapter 2

Background

2.1 General Topology

Definition 2.1.1. *[Wil70, 11.1, p. 73]*
A set Λ is a **directed set** *if and only if there there is a relation $\leq$ on Λ satisfying:*

$\Lambda - a)$ *$\lambda \leq \lambda$ for each $\lambda \in \Lambda$,*

$\Lambda - b)$ *if $\lambda_1 \leq \lambda_2$ and $\lambda_2 \leq \lambda_3$, then $\lambda_1 \leq \lambda_3$,*

$\Lambda - c)$ *if $\lambda_1, \lambda_2 \in \Lambda$, then there is some $\lambda_3 \in \Lambda$ with $\lambda_1 \leq \lambda_3$, $\lambda_2 \leq \lambda_3$.*

Some examples of directed sets:

1. The natural numbers $\mathbb{N}$ equipped with its usual order.

2. The set of open neighbourhoods of a point x of a topological space ordered via reverse inclusion. That is, $U \geq V$ whenever $V \subseteq U$.

3. The collection $\mathcal{P}$ of all finite partitions of the closed interval $[a, b]$ into closed subintervals, ordered by the relation $A_1 \leq A_2$ if and only if A_2 refines A_1.

Definition 2.1.2. *[Wil70, 11.2, p. 74]*
A **net** *in a set X is a function $P : \Lambda \to X$, where Λ is some directed set. The point $P(\lambda)$ is usually denoted x_λ and the net denoted $(x_\lambda)_{\lambda \in \Lambda}$ or (x_λ).*

A **subnet** *of a net $P : \Lambda \to X$ is the composition $P \circ \phi$, where $\phi : M \to \Lambda$ is an increasing cofinal function from a directed set M to Λ. That is,*

1. *$\phi(\mu_1) \leq \phi(\mu_2)$ whenever $\mu_1 \leq \mu_2$ (ϕ is increasing),*

2. *for each $\lambda \in \Lambda$, there is some $\mu \in M$ such that $\lambda \leq \phi(\mu)$ (ϕ is* **cofinal** *in Λ).*

For $\mu \in M$, the point $P \circ \phi(\mu)$ is usually denoted x_{λ_μ} and the subnet denoted the subnet $\left(x_{\lambda_\mu}\right)$ of the net (x_λ).

In particular, a sequence (x_k) is a net for which the indexing set Λ is $\mathbb{N}$. A more advanced example is to consider a real-valued function $f : [a, b] \to \mathbb{R}$ and to define nets $P_L, P_U : \mathcal{P} \to \mathbb{R}$ by defining $P_L(A)$ to be the lower Riemann sum for the partition A and $P_U(A)$ to be the upper Riemann sum for the partition.

Definition 2.1.3. *[Wil70, 11.3,p. 74]*
Let (x_λ) be a net in a space X. Then (x_λ) **converges** *to a point $x \in X$ ($x_\lambda \to x$) provided for each neighbourhood U of x, there exists some $\lambda_0 \in \Lambda$ such that $\lambda \geq \lambda_0$ implies $x_\lambda \in U$. Less formally, (x_λ) converges to x provided it is* **eventually** *in every neighbourhood of x.*

The net (x_λ) has x as a **cluster point** *provided for each neighbourhood U of x and each $\lambda_0 \in \Lambda$, there is some $\lambda \geq \lambda_0$ such that $x_\lambda \in U$. Less formally, (x_λ) has x as a cluster point provided it is* **frequently** *in every neighbourhood of x.*

These definitions generalize the corresponding concepts for sequences. Consider the nets P_L and P_U for a function $f : [a, b] \to \mathbb{R}$. If they both converge to some real number c, then

$$\int_a^b f(x)\,dx = c.$$

2.2 Complex Manifolds

In this section M will be assumed to be a Hausdorff space.

Definition 2.2.1. *[GR79, IV §1, p. 154]*
An n-dimensional **complex coordinate system** *(U, ϕ) in M consists of an open set $U \subset M$ and a topological map $\phi : U \to B$ where $B \subset \mathbb{C}^n$ is open.*

A pair of complex coordinate systems (U, ϕ), (V, ψ) are said to be **holomorphically compatible** *if $U \cap V = \emptyset$ or the map $\phi \circ \psi^{-1} : \psi(U \cap V) \to \phi(U \cap V)$ is biholomorphic.*

Definition 2.2.2. *[GR79, IV §1, p. 155]*
*A covering of a Hausdorff space M with pairwise compatible n-dimensional complex coordinate systems is called an n-***dimensional complex atlas** *on M. A pair of such atlases $\mathcal{A}_1, \mathcal{A}_2$ on M are said to be equivalent if any pair of complex coordinate systems (U, ϕ), (V, ψ), taken from $\mathcal{A}_1, \mathcal{A}_2$, respectively, are holomorphically compatible. An equivalence class of complex atlases is called an n-***dimensional complex structure***.*

Definition 2.2.3. *[GR79, IV §1, p. 155]*
*An n-***dimensional complex manifold** *is a Hausdorff space M with countable basis, equipped with an n-dimensional complex structure.*

Definition 2.2.4. *[GR79, IV §1, p. 156]*
Let U be an open subset of an n-dimensional complex manifold M. Then a function $f : U \to \mathbb{C}$ is said to be **holomorphic** *if for each $p \in U$, there exists some complex coordinate system (U', ϕ) such that $f \circ \phi^{-1} : \phi(U \cap U') \to \mathbb{C}$ is holomorphic. The set of holomorphic functions on U is denoted $\mathcal{O}(U)$.*

Definition 2.2.5. *[Sto00, Def. 1.3.6, p. 25]*
A domain $\Omega \subset M$ *is a* **Runge domain** *if* $\mathcal{O}(M)|_{\Omega}$ *is dense in* $\mathcal{O}(\Omega)$.

Definition 2.2.6. *[Sto00, p. 71-72]*
Let M *be a n-dimensional complex manifold. It is said to be a* **Stein manifold** *provided it satisfies the following properties:*

1. *holomorphic functions separate points, (that is, for any points* $x \neq y$*, there exists some* $f \in \mathcal{O}(M)$ *such that* $f(x) \neq f(y)$*),*

2. *for each point* $p \in M$*, there exist* $f_1, f_2, \ldots, f_n \in \mathcal{O}(M)$ *such that* $df_1 \wedge df_2 \wedge \cdots \wedge df_n$ *does not vanish at p, (that is,* $f_1, f_2, \ldots, f_n$ *provide local holomorphic coordinates at p),*

3. *this manifold is holomorphically convex, that is, for any compact* $K \subset M$*, its holomorphically convex hull*

$$\{p \in M : |f(x)| \leq \|f\|_K \text{ for all } f \in \mathcal{O}(M)\}$$

is compact.

2.3 Foliations

Definition 2.3.1. *[Law74, Def. 1, p. 370] Let* M *be a m-dimensional smooth manifold. A p-dimensional smooth* **foliation** *on M is a decomposition of M into a union of disjoint connected subsets* $\{\mathcal{L}_\alpha\}_{\alpha \in A}$ *called the* **leaves** *of the foliation with the following property: every point in M has a neighbourhood U and a system of local, smooth coordinates* $x = (x^1, \ldots, x^n) : U \to \mathbb{R}^m$ *such that for each leaf* $\mathcal{L}_\alpha$*, the components of* $U \cap \mathcal{L}_\alpha$ *are described by the equations* $x^{p+1} = c_{p+1}, \ldots, x^m = c_m$ *for some real constants* c_k*,* $p + 1 \leq k \leq m$*. The quantity* $m - p$ *is called the* **codimension** *of the foliation.*

A foliation of dimension 1 is called a **line foliation**. One such example is to take M to be the punctured disc $\{z \in \mathbb{C} : 0 < |z| < 1\}$ and define the leaves to be the radial arcs $\left\{re^{i\theta} : 0 < r < 1\right\}$ extending between the origin and the boundary circle.

There are number of standard ways to create a foliation, including the following:

Example 2.3.2. *[Law74, Ex. A, p. 371] Let M and Q be manifolds of dimension m and* $q \leq m$*, respectively, and let* $f : M \to Q$ *be a submersion (that is,* $\mathrm{rank}(df) = q$*). Then (by the Implicit Function Theorem), f induces a codimension-q foliation where the leaves are defined to be the connected components of* $f^{-1}(x)$ *for* $x \in Q$.

Example 2.3.3. *[Law74, Ex. D, p. 372] A nonsingular system of ordinary differential equations, when reduced to first order, becomes a nonvanishing vector field. The local solutions (orbits of the local flow generated by the vector field) fit together to form a 1-dimensional foliation.*

2.4 Triangulations

Triangulations provide a way to break topological spaces into standard pieces (simplices in this case). As in Vodovoz and Zaidenberg [VZ71], we will build generators for the whole space by defining them on each simplex of the triangulation, but since we are constructing smooth ones we will require a Whitehead triangulation which additionally satisfies some smoothness conditions. We also give some examples of triangulations for the projective plane and the torus.

Definition 2.4.1. *[Ber87, Def. 2.4.3, p. 43] A set of points $v_0, v_1, \ldots, v_k$ in an affine space X is said to be* **affinely independent** *if*

$$\dim \langle v_0, v_1, \ldots, v_k \rangle = k.$$

Otherwise, they are said to be **affinely dependent**.

For $\mathbb{R}^n$, this condition is equivalent to the set of vectors $\{v_1 - v_0, v_2 - v_0, \ldots, v_k - v_0\}$ being linearly independent.

Definition 2.4.2. *[Mun68, Def. 7.1, p. 69] Let $v_0, \ldots, v_m$ be affinely independent points in $\mathbb{R}^n$. The* **simplex** *A spanned by them is the set of points x such that $x = \sum_j b_j v_j$ where $0 \leq b_j \leq 1$ and $\sum_j b_j = 1$. It is said to have* **dimension** *m, and its extreme points v_j are called its* **vertices**. *The coefficients $b_j(x)$ for a point x are called its* **barycentric coordinates**. *The special point*

$$x^* = \frac{1}{m+1} \sum_{j=0}^{m} v_j$$

is called the **barycentre** *of the simplex and it has the barycentric coordinates $(1/(m+1), \ldots, 1/(m+1))$.*

A **face** *of A is a simplex spanned by a nonempty subset of its vertices, and its* **boundary** *∂A is the union of its $m-1$-faces.*

For instance, any triangle in $\mathbb{R}^2$ is a 2-simplex whose faces are its edges and vertices. Similarly, a tetrahedron in $\mathbb{R}^3$ is a 3-simplex whose faces are its usual faces (2-faces), its edges (1-faces), and its vertices (0-faces).

Definition 2.4.3. *[Mun68, p. 69] Let $\mathcal{K}$ be a collection of simplices in $\mathbb{R}^\infty = \cup_k \mathbb{R}^k$, where $|\mathcal{K}|$ denotes their union. It is called a* **simplicial complex** *provided that*

- *every face of a simplex in $\mathcal{K}$ is in $\mathcal{K}$,*
- *the intersection of any pair of simplices of K is a face of each of them,*
- *each point of $|\mathcal{K}|$ has a neighbourhood intersecting only finitely many simplices of $\mathcal{K}$.*

If all the simplices belong to $\mathbb{R}^n$ for some finite n, we say that the simplicial complex is **finite-dimensional**.

The ***k*-skeleton** of an m-dimensional simplicial complex $\mathcal{K}$ consists of exactly those faces having dimension at most k. For instance, its 0-skeleton is its vertex set, and its m-skeleton is $\mathcal{K}$ itself.

Definition 2.4.4. *A* **triangulation** *on a topological space X is a pair $(\mathcal{K}, \phi)$ where $\mathcal{K}$ is a simplicial complex and ϕ is a homeomorphism from $|\mathcal{K}|$ to X.*

Examples of triangulations of the real projection plane and of the torus are given in figures 2.1 and 2.2, respectively. Vertices with matching labels are identified.

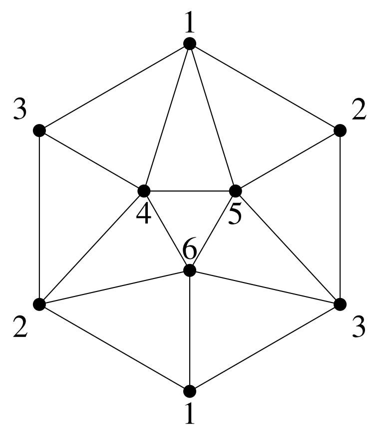


Figure 2.1: A triangulation for the real projective plane. Given in [AT07].

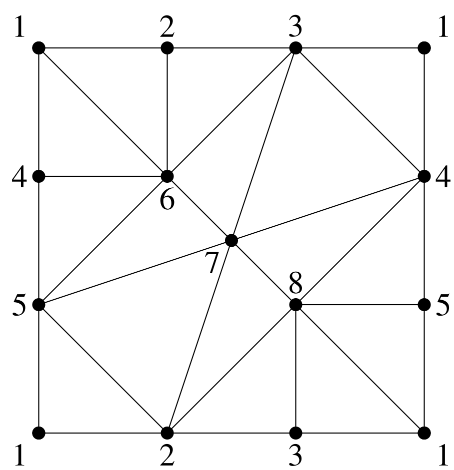


Figure 2.2: A triangulation for a torus. Given in [Law84].

For our purposes involving manifolds, we need triangulations satisfying additional properties.

Definition 2.4.5. *[Mil56, p. 14] A Whitehead triangulation $\mathcal{T}$ on a smooth m-manifold M consists of an m-dimensional simplicial complex $\mathcal{K} \subset \mathbb{R}^n$ (for some natural number n) and a homeomorphism $\phi : \mathcal{K} \to M$ such that for each m-simplex A, $\phi|_A$ is a restriction of a smooth* *nondegenerate function from some neighbourhood $V \supset A$ contained within the tangent plane to the simplex A.*

A Whitehead triangulation exists on M [Mil56, p. 3], [Whi40, p. 824], so fix some Whitehead triangulation $\mathcal{K}$ on M. Then denote the k-skeleton of $\mathcal{K}$ as X^k, and the k-simplices comprising it as X^k_j.

2.5 Homeomorphism Classes and Normal Forms

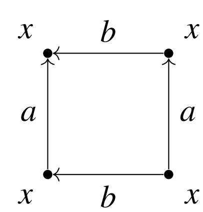
In the section, we introduce normal forms of compact surfaces (both with and without boundary) and, in order to do, we introduce cell complexes in the sense used by Ahlfors and Sario. (We adapt the notation and terminology in some ways to make it closer to the notation used elsewhere in this chapter.) Normal forms are important because of their 1:1 correspondence to homeomorphism classes of compact surfaces.

Definition 2.5.1. *[AS60, I §7 39B-C, p. 90-92, 40B, p. 94]*
A **cell complex** *K consists of a finite set $\{a, b, c, \dots, \}$ of edges, a finite set of faces $\{A, B, C, \dots, \}$, involution operators $(\cdot)^{-1}$ without fixed elements defined on these sets, and a* **boundary operator** *∂ which assigns to each face an ordered cyclic set of edges. The set of edges is permitted to be empty, but the set of faces is not. Every edge must be contained in some boundary. The boundary operator ∂ must be compatible with the involution operators: that is, for any face A with $\partial A = (a_1 a_2 \cdots a_n)$, $\partial(A^{-1}) = (a_n^{-1} \cdots a_2^{-1} a_1^{-1})$.*

If e is an arbitrary edge, an edge f is a **successor** *to e if there is some boundary ∂A in which f immediately follows e. A cyclically ordered set $(a_1 a_2 \cdots a_n)$ such that each a_i has a_{i-1}^{-1} and a_{i+1}^{-1} as successors is called an* **inner vertex***. A linearly ordered set $(a_1 a_2 \cdots a_n)$ such that the previous condition holds for each $1 < i < n$, a_1 only has a_2^{-1} as a successor, and a_n only has a_{n-1}^{-1} as a successor, is called a* **border vertex***. Any vertex $(a_1 a_2 \cdots a_n)$ is identified with $(a_n \cdots a_2 a_1)$ also. A* **contour** *is a cyclically ordered set $(a_1 a_2 \cdots a_n)$ such that each a_i belongs to the same vertex as a_{i+1}^{-1} and only appears in one boundary. Any contour $(a_1 a_2 \cdots a_n)$ is identified with $\left(a_n^{-1} \cdots a_2^{-1} a_1^{-1}\right)$.*

Example 2.5.2. *A cell complex corresponding to a torus can be given by taking the face set to be $\left\{A, A^{-1}\right\}$, the edge set to be $\left\{a, a^{-1}, b, b^{-1}\right\}$, and the boundaries $\partial A = (aba^{-1}b^{-1})$ and $\partial(A^{-1}) = (bab^{-1}a^{-1})$. Then the inner vertex is $(aba^{-1}b^{-1})$ and there are no border vertices or contours. Consider the following realization of the cell complex for the sake of clarity.*

Both A and A^{-1} represent the inside of this square, but the boundary for A comes from traversing the boundary of the square counterclockwise, while the boundary for A^{-1} comes from traversing this boundary clockwise. The vertex x here corresponds to the (abstract) inner vertex $(aba^{-1}b^{-1})$.



Example 2.5.3. *Here is a cell complex that will correspond to a torus with an excised rectangle. (While this cell complex is not the simplest possible, it does nicely illustrate multiple vertices.) Take the face set to be* $\{A, A^{-1}, B, B^{-1}, C, C^{-1}\}$ *and the edge set to be*

$$\{a_1, a_1^{-1}, a_2, a_2^{-1}, a_3, a_3^{-1}, b, b^{-1}, c, c^{-1}, d, d^{-1}, e, e^{-1}, f, f^{-1}, g, g^{-1}\},$$

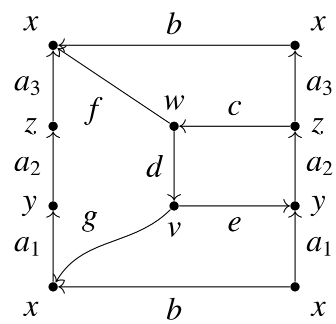
and define boundaries as follows:

$$\begin{aligned} \partial A &= a_3 b f^{-1} c^{-1}, \\ \partial B &= a_1 a_2 a_3 f^{-1} d g, \\ \partial C &= b^{-1} a_1 e^{-1} g. \end{aligned}$$

The boundaries of A^{-1}, B^{-1}, *and* C^{-1} *are uniquely determined by these in the usual way. The inner vertex of this cell complex is* $(b^{-1}a_3fbga_1^{-1})$ *and there are four border vertices:* $(ea_1a_2^{-1})$, $(a_2a_3^{-1}c^{-1})$, $(cf^{-1}d^{-1})$, *and* $(dg^{-1}e^{-1})$. *There is a single contour* $(cdea_2)$.

Here the faces A, B, C are the top piece, the left piece, and the bottom piece, respectively. The right piece is the area which was excised, so it does not have a corresponding face. We orient ∂A, ∂B, ∂C *by traversing their boundaries counterclockwise, clockwise, and counterclockwise, respectively.*

The vertex x here corresponds to the inner vertex $(b^{-1}a_3fbga_1^{-1})$, *while the vertices y, z, v, w correspond to the border vertices* $(ea_1a_2^{-1})$, $(a_2a_3^{-1}c^{-1})$, $(dg^{-1}e^{-1})$, *and* $(cf^{-1}d^{-1})$, *respectively.*



Example 2.5.4. *We can represent an abstract triangulation* $\mathcal{T}$ *as a cell complex as follows. Take the edge set to consist of the symbols* a_{jk} *where* $j < k$ *and* $\{j, k\} \in \mathcal{T}$ *and let the set of faces consist of the symbols* A_{jkl} *where* $j < k < l$ *and* $\{j, k, l\} \in \mathcal{T}$. *Then define the boundaries by the rule*

$$\partial A_{jkl} = a_{jk} a_{kl} a_{jl}^{-1}.$$

Any compact surface, with or without boundary, admits a triangulation (see [AS60, I §8, p. 105–111], for instance). Consequence, any compact surface admits a cell complex also.

Definition 2.5.5. *[AS60, I §7 39D, p. 92, §4 28B, p. 59]*
An **elementary operation** *on a cell complex* $\mathcal{K}$ *is a transformation of one of the following forms:*

$a \to bc$: *replacing a pair of edges a,* a^{-1} *by bc,* $c^{-1}b^{-1}$ *in all the boundaries, or*

$A \to BC$: *replacing a face A having a boundary* $\partial A = (a_1 \cdots a_p a_{p+1} \cdots a_n)$ *with a pair of faces B, C having boundaries* $\partial B = (a_1 \cdots a_p d)$ *and* $\partial C = (d^{-1} a_{p+1} \cdots a_n)$. *(There is a related replacement of* A^{-1} *by* $B{-}1$ *and* C^{-1}.*)*

The transformed cell complex is called a **subdivision** *of* $\mathcal{K}$ *and a pair of cell complexes* $\mathcal{K}$ *and* $\mathcal{L}$ *are said to be* **equivalent** *if there exists a finite sequence* $(\mathcal{K}^1, \ldots, \mathcal{K}^n)$ *in which* $\mathcal{K}^1 = \mathcal{K}$, $\mathcal{K}^n = \mathcal{L}$, *and, for each* $1 \le k < n$, *either* $\mathcal{K}^{k+1}$ *is a subdivision of* $\mathcal{K}^k$ *or* $\mathcal{K}^k$ *is a subdivision of* $\mathcal{K}^{k+1}$.

Remark 2.5.6. *Another way to look at $\mathcal{K}^k$ being a subdivision of $\mathcal{K}^{k+1}$ is to interpret this as being either of the following inverse operations:*

$bc \to a$: *joining a pair of edges b, c which always appear in the forms bc and $c^{-1}b^{-1}$ by a new edge a, or*

$BC \to A$: *joining a pair of faces B, C along an edge d to form a new face A having the boundary $\partial A = (b_1 \cdots b_m c_1 \cdots c_n)$ where $\partial B = (b_1 \cdots b_m d)$ and $\partial C = (d^{-1} c_1 \cdots c_n)$.*

Then a pair of cell complexes $\mathcal{K}$, $\mathcal{L}$ are equivalent if there exists a sequence of operations of the forms $a \to bc$, $A \to BC$, $bc \to a$, $BC \to A$ which transform $\mathcal{K}$ into $\mathcal{L}$.

Definition 2.5.7. *[AS60, I §7 40, p. 94–95]*
A **canonical cell complex** *is a cell complex having a single pair of faces A, A^{-1} such that the boundary ∂A takes one of the following two forms:*

$$a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_p b_p a_p^{-1} b_p^{-1} \, c_1 h_1 c_1^{-1} \cdots c_q h_q c_q^{-1}, \tag{2.1}$$

for an orientable cell complex having genus p and q boundary components; or,

$$a_1^2 \cdots a_p^2 \, c_1 h_1 c_1^{-1} \cdots c_q h_q c_q^{-1}, \tag{2.2}$$

for a non-orientable cell complex having cross cut number p and q boundary components. The factors of the form $a_j b_j a_j^{-1} b_j^{-1}$ represent **handles***, those of the form a_j^2 represent* **cross-caps***, and those of the form $c_j h_j c_j^{-1}$ represent boundary components (also known as contours).*

Example 2.5.8. *Consider the cell complex of Example 2.5.3. Here we find a canonical cell complex which is equivalent to it.*

1. *First join some faces together ($A^{-1}B \to D$, $DC^{-1} \to E$) and some edges ($cd \to c'$, $c'e \to e'$) to simplify a bit. This gives the cell complex a single face E having the boundary*
$$\partial E = \left(a_1 a_2 a_3 b^{-1} a_3^{-1} e' a_1^{-1} b\right).$$
2. *The ordered cyclic set $(a_2 e')$ is a contour, so form a loop γ ($a_2 e' \to \gamma$), join the edges a_1, a_2, and a_3 ($a_2 a_3 \to a'$, $a_1 a' \to a$), so that the boundary ∂E now takes the form*
$$\partial E = \left(bab^{-1}a^{-1} \, a_1 \gamma a_1^{-1}\right).$$

The cell complex with the faces E, E^{-1} and this boundary is a canonical cell complex which is orientable, has genus equal to one, and has a single contour.

Remark 2.5.9. *The* **normal form** *(called* **canonical form** *in [AS60]) of a cell complex is the form (2.1) or (2.2) taken by the boundary of a face of any canonical cell complex equivalent to it. There is a correspondence between homeomorphism classes of realizations of cell complexes and equivalence classes of cell complexes (see [AS60, I §7 39E–F, p. 92–94]), so it well-defined.*

The normal form of a compact surface S is defined to be the normal form of a cell complex made from any triangulation. Note that the normal form of S is determined by its orientability, genus or cross cut number, and number of contours.

In particular, these observations imply the following result.

Theorem 2.5.10. *[AS60, I §7 42A, Thm, p. 98]*
The homeomorphism class of a compact surface S is determined by whether it is orientable, its number of contours, and its genus (if orientable) or its cross cut number (otherwise).

2.6 $\mathcal{F}$-convexity of a compact set

Definition 2.6.1. *Let X be a topological space, E be a compact subset of X. The convex hull of E with respect to a family $\mathcal{F}$ of continuous functions is*

$$\widehat{E}^{\mathcal{F}} = \left\{ x \in X : |f(x)| \leq \sup_{y \in E} |f(y)| \text{ for all } f \in \mathcal{F} \right\}.$$

The set E is said to be $\mathcal{F}$-convex if it equals its $\mathcal{F}$-convex hull.

Remark 2.6.2. *For some examples of convexity with respect to a family of functions, $\mathcal{F}$ can be taken to be the family of*

- *real affine functions on $\mathbb{R}^n$ or $\mathbb{C}^n$, which gives* **geometric convexity***;*
- *polynomials on $\mathbb{C}^n$, which gives* **polynomial convexity** *[Sto00, Def. 1.1.3];*
- *holomorphic functions $\mathcal{O}(\mathcal{M})$ on a Stein manifold $\mathcal{M}$, which gives* **holomorphic convexity** *[FG02, p. 251];*
- *rational functions on $\mathbb{C}^n$ not having any poles in the compact E considered, which gives* **rational convexity***;*
- *meromorphic functions on a complex manifold $\mathcal{M}$ not having any poles in the compact E considered, which gives* **meromorphic convexity**.

Another way to calculate the rationally convex hull of a compact $E \subseteq \mathbb{C}^n$ is

$$\widehat{E}^r = \{ z \in \mathbb{C}^n : p(z) \in p(E) \text{ for all polynomials } p \in \mathbb{C}[z_1, \ldots, z_n] \}.$$

Holomorphic convexity and meromorphic convexity are generalizations of polynomial convexity and rational convexity, respectively. This section will prove this in the case of holomorphic convexity.

It should also be noted that the concepts of holomorphic convexity and polynomial convexity have local versions. A set E is **locally polynomially convex** at a point p if p has a compact neighbourhood $K \subseteq E$ which is polynomially convex. The definitions of **locally holomorphically convex**, **locally rationally convex**, and **locally meromorphically convex** are analogous.

In 1-dimension, polynomial convexity and rational convexity have a straightforward characterization: any compact set $E \subset \mathbb{C}$ is rationally convex, and it is polynomially convex if and only if its complement is connected. (See Lemma 2.8.1 for a similar characterization of holomorphic convexity in a domain $U \subseteq \mathbb{C}$.)

For $\mathbb{C}^n$ where $n \geq 2$, there is no analogous characterization as these properties for a compact set are not solely determined by its topology, or that of its complement.

Proposition 2.6.3. *Let K be a compact subset in X, and $\mathcal{E}$ be a subset of the family of functions $\mathcal{F}$. Then*

$$\widehat{K}^{\mathcal{F}} \subseteq \widehat{K}^{\mathcal{E}}.$$

In particular, if K is $\mathcal{E}$-convex, then it is also $\mathcal{F}$-convex.

Here are a few immediate consequences of this:

1. any convex set is polynomially convex,
2. any polynomially convex set is rationally convex, and
3. any holomorphically convex set is meromorphically convex.

Here are some more examples for polynomial and rational convexity:

4. a disjoint union of a pair of convex sets is polynomially convex [Kal65];
5. a union of up to three disjoint balls is polynomially convex [Kal65], and
6. any finite union of disjoint balls is rationally convex [Nem08].

The least involved of these examples to demonstrate is that the disjoint union of two convex compacts E_1, E_2 is polynomially convex. Find a hyperplane between them and use it to build a holomorphic linear function $f(z_1, z_2, \ldots, z_n)$ such that $f < 0$ on E_1 and $f > 0$ on E_2. Then an application of Kallin's Lemma (see Corollary 2.8.4) gives the desired result.

On the other hand, there are some interesting counterexamples:

7. a disjoint union of three convex sets in $\mathbb{C}^2$ may fail to be polynomially convex [Ros89],
8. a disjoint union of three ellipsoids in $\mathbb{C}^3$ may fail to be polynomially convex [KK84], and
9. the image of a torus in $\mathbb{C}^2$ need not be rationally convex [Ale99].

Proposition 2.6.4. *Let K be a compact subset in X, and $\mathcal{F}_0$ be dense in the family of functions $\mathcal{F}$ (in the compact-open topology). Then the corresponding convex hulls coincide:*

$$\widehat{K}^{\mathcal{F}_0} = \widehat{K}^{\mathcal{F}}.$$

An important result about polynomially convex sets in $\mathbb{C}^n$ is the Oka-Weil Theorem. There is also a version of this theorem for rationally convex sets [Gam69, III, Ex. 4] where polynomial convexity is replaced by rational convexity and the polynomial is replaced by a rational function having no poles on K.

Theorem 2.6.5. *(Oka-Weil)*
If $K \subseteq \mathbb{C}^n$ is compact and polynomially convex, and f is holomorphic in a neighbourhood of K (that is, $f \in \mathcal{O}(K)$), then for any $\epsilon > 0$, there is a polynomial $p \in \mathbb{C}[z_1, \ldots, z_n]$ such that

$$\|f - p\|_K < \epsilon.$$

This generalizes Runge's Theorem for complex dimension 1:

Theorem 2.6.6. *[Gam69, Cor. II.1.5, p. 28]*
Let K be a compact subset of plane with connected complement. Then every function analytic in a neighbourhood of K can be uniformly approximated on K by polynomials in z.

A different result related to the Oka-Weil theorem is the following corollary:

Corollary 2.6.7. *The polynomials in $\mathbb{C}^n$ are dense in the space of entire functions on $\mathbb{C}^n$.*

Remark 2.6.8. *The notions of polynomial convexity and $\mathcal{O}(\mathbb{C}^n)$-convexity coincide since the polynomials are dense in the entire functions, and the entire functions are precisely the holomorphic functions having domain $\mathbb{C}^n$.*

Finally, the Oka-Weil Theorem has a generalization to Stein spaces, so, in particular, holds for Stein manifolds.

Theorem 2.6.9. *[For11, Thm. 2.2.5, p. 48]*
If X is a Stein space and K is a compact holomorphically convex subset of X, then every holomorphic function in an open neighbourhood of K can be approximated uniformly on K by functions in $\mathcal{O}(X)$.

2.7 Uniform algebras

In this section we introduce some fundamental concepts from uniform algebras like generalized peak sets which have the property that the corresponding restriction algebras are also uniform algebras, and the maximal antisymmetric sets whose restriction algebras are the building blocks of the algebra in a sense made precise by the Bishop-Shilov theorem. We illustrate its use by recovering the Weierstrass theorem [Apo74, Thm. 11.17, p. 322] from it.

Definition 2.7.1. *[Gam69, p. 25] A* **uniform algebra** *on a compact Hausdorff space X is a unital subalgebra $\mathcal{A}$ of the (complex-valued) continuous functions $C(X)$ which separates points and is closed under uniform limits. When equipped with the supremum norm $\|\cdot\|_\infty$, it constitutes a commutative Banach algebra.*

Definition 2.7.2. *We will call a uniform algebra $\mathcal{B}$ on X which is a subalgebra of a uniform algebra $\mathcal{A}$ on X a* **uniform subalgebra** *of $\mathcal{A}$.*

An obvious example of a uniform algebra on a compact set X is $C(X)$ itself. Many examples of uniform algebras are given by taking all uniform limits of some subalgebra $\mathcal{F}$ of $C(X)$ for some compact $X \subset \mathbb{C}^n$:

1. For the algebra of polynomials in $z_1, \ldots, z_n$, we obtain the uniform algebra $P(X)$.

2. For the algebra of rational functions in $z_1, \ldots, z_n$ not having poles in X, we obtain the uniform algebra $R(X)$.

3. The algebra generated by functions $g_1, \ldots, g_n$ in $C(X)$ which separate X. This gives a uniform algebra which we will denote as $\langle g_1, \ldots, g_n \rangle$.

A different example is given by $A(X)$ which consists of continuous functions on X which are holomorphic on its interior. It is related to some of the previous examples by the following sequence of inclusions:

$$P(E) \hookrightarrow R(E) \hookrightarrow A(E) \hookrightarrow C(E).$$

For situations where the generators can be taken to be smooth, we have the following notion:

Definition 2.7.3. *A uniform algebra $\mathcal{A}$ is n-**polynomially dense** if there exist smooth functions $g_1, \ldots, g_n$ such that $\mathcal{A} = \langle g_1, \ldots, g_n \rangle$.*

In the 1-dimensional case $X \subset \mathbb{C}$, Lavrentiev's Theorem gives a necessary and sufficient condition for z to be a generator for $C(X)$.

Theorem 2.7.4. *[Gam69, Thm 8.7, p. 48] Let $X \subset \mathbb{C}$ be a compact set. Then $C(X)$ is 1-polynomially dense (that is, $P(X) = C(X)$) if and only if X is nowhere dense and the complement of X is connected.*

Definition 2.7.5. *We will call a map $\tau : \mathcal{A} \to \mathcal{B}$ between uniform algebras a* **homomorphism** *provided that it is a unital algebra homomorphism and that it is norm-decreasing (that is, $\|\tau(f)\|_{\mathcal{B}} \leq \|f\|_{\mathcal{A}}$ for any $f \in \mathcal{A}$). If it is injective, call it a* **monomorphism***; if surjective, an* **epimorphism***; if bijective, an* **isomorphism***.*

One straightforward example of an isomorphism of a uniform algebra $\mathcal{A}$ on X is to take the map $f \mapsto f \circ \phi^{-1}$, where ϕ is a homeomorphism $X \to Y$. For future reference we note that a uniform algebra $\mathcal{A}$ (on X) which is isomorphic to $C(Y)$ for some compact space Y is equal to $C(X)$.

Any homomorphism $\phi : \mathcal{A} \to \mathbb{C}$ admits a **representing measure** (see [Gam69, Thm. 2.4]); that is, a probability measure μ such that

$$\phi(f) = \int_X f \, d\mu$$

for each $f \in \mathcal{A}$ [Gam69, II, §2].

It is sometimes useful to study a uniform algebra $\mathcal{A}$ by looking at its **maximal ideal space**. This space $\mathcal{M}$ consists of the homomorphisms $\mathcal{A} \to \mathbb{C}$ and it is equipped with the weak* topology: that is, a net ϕ_i in $\mathcal{M}$ converges to a point ϕ provided that $\lim_i \phi_i(a) = \phi(a)$ for all $a \in \mathcal{A}$. For the uniform algebras $P(K)$, $R(K)$, and $C(K)$, the maximal ideal space is homeomorphic to the polynomially convex hull of K, its rationally convex hull, and the set K itself, respectively.

Now consider the following theorem:

Theorem 2.7.6. *[Sto71, Thm. 5.8] If $\mathcal{A}$ is a commutative Banach algebra with identity and $E = (a_\lambda)_{\lambda \in \Lambda}$ is a set of generators for $\mathcal{A}$, then their joint spectrum*

$$\sigma_{\mathcal{A}}(E) = \left\{ \zeta \in \mathbb{C}^\Lambda : \text{ for some } \phi \in \mathcal{M},\ \phi(a_\lambda) = \pi_\lambda(\zeta) \text{ for all } \lambda \in \Lambda \right\}$$

is polynomially convex in $\mathbb{C}^\Lambda$, then the map $\Phi : \mathcal{M} \to \mathbb{C}^\Lambda$ determined by $\pi_\lambda(\Phi\phi) = \phi(a_\lambda)$ is a homeomorphism from the maximal ideal space $\mathcal{M}$ onto $\sigma_{\mathcal{A}}(E)$.

This has the following corollary:

Corollary 2.7.7. *A set of generators $f_1, \ldots, f_n$ for the uniform algebra $C(X)$, where X is compact, gives a polynomially convex topological embedding of X into $\mathbb{C}^n$.*

Proof. The maximal ideal space M of $C(X)$ is homeomorphic to X itself and Φ is a topological embedding of M into $\mathbb{C}^n$, so their composition comprises a topological embedding of X into $\mathbb{C}^n$. The image of this composition is the same as the image of Φ, so this topological embedding $X \hookrightarrow \mathbb{C}^n$ has a polynomially convex image also. □

Definition 2.7.8. *[Gam69, p. 52, p. 56] A set E is a* **peak set** *for a uniform algebra $\mathcal{A}$ if there exists some $g \in \mathcal{A}$ such that $g|_E = 1$ and $|g| < 1$ on $X \setminus E$. If it is a singleton, it is called a* **peak point**.

A set E is a **generalized peak set** *for $\mathcal{A}$ if it is an intersection of peak sets.*

Consider the closed disc $\overline{\Delta}$, and the algebra $C(\overline{\Delta})$. Every point is a peak point in this case. If, on the other hand, the algebra $A(\overline{\Delta})$ is considered, then the boundary points in S^1 are peak points, but the interior points are not.

Proposition 2.7.9. *[Sto71, Lem. 12.3, p. 117] Let I be an ideal in a uniform algebra $\mathcal{A}$ (on a topological space X), and let $E \subseteq X$ be a generalized peak set for $\mathcal{A}$. Then*

- *$I|_E$ is closed in $C(E)$, and*
- *$I|_E$ is isometrically isomorphic to the quotient algebra*

$$\frac{I}{I \cap \{f \in A : f|_E = 0\}},$$

equipped with the usual norm

$$\|f\| = \inf \{\|f + g\| : g \in I\}.$$

Proposition 2.7.10. *Consider a finitely generated uniform algebra*

$$\mathcal{A} = \langle g_1, g_2, \cdots, g_n \rangle$$

on X having a generalized peak set E. Then its restriction algebra takes the same general form:

$$\mathcal{A}|_E = \langle g_1|_E, g_2|_E, \cdots, g_n|_E \rangle.$$

Proof. Note first that the restriction algebra $\mathcal{A}|_E$ contains the dense subalgebra of

$$\mathcal{B} = \langle g_1|_E, g_2|_E, \cdots, g_n|_E \rangle$$

generated by $g_1|_E, g_2|_E, \ldots, g_n|_E$. By Proposition 2.7.9, $\mathcal{A}|_E$ is closed in $C(E)$; hence, it contains $\mathcal{B}$.

On the other hand, consider $a \in \mathcal{A}|_E$ and let $a = a_0|_E$, where $a_0 \in \mathcal{A}$. There exist polynomials $(P_i)_{i \in I}$ such that the net $(P_i(g_1, g_2, \ldots, g_n))_{i \in I}$ converges to a_0 in the uniform topology on X. But then the net $(P_i(g_1|_E, g_2|_E, \cdots, g_n|_E))_{i \in I}$ converges to g in the uniform topology on E. □

An important class of generalized peak sets is given by maximal antisymmetric sets.

Definition 2.7.11. *[Sto71, p. 115] A set E is an* **antisymmetric set** *for a uniform algebra $\mathcal{A}$ if whenever f is real-valued on E, then it is constant on E. An antisymmetric set E is called* **maximal** *if there does not exist an antisymmetric set F such that $E \subsetneq F$.*

Proposition 2.7.12. *[Sto71, Lem. 12.4, p. 117] Every maximal antisymmetric set K for a uniform algebra $\mathcal{A}$ is a generalized peak set.*

Corollary 2.7.13. *A consequence of Propositions 2.7.9 and 2.7.12 is that any restriction of a uniform algebra $\mathcal{A}$ to a maximal antisymmetric set A is a uniform algebra $\mathcal{A}|_A$ on A.*

Proposition 2.7.14. *Let $\mathcal{A}$ be a uniform algebra, and $\mathcal{B}$ be a uniform subalgebra ($\mathcal{B} \subseteq \mathcal{A}$). Then any antisymmetric set E for $\mathcal{A}$ is also an antisymmetric set for $\mathcal{B}$.*

Heuristically, one can say that the set of antisymmetric sets grows smaller as the uniform algebra grows larger. One consequence of this is that an antisymmetric set for $\mathcal{A}$ which is maximal for it need not be maximal in a uniform subalgebra.

Proposition 2.7.15. *Let $\mathcal{A}$ be a uniform algebra having $\mathcal{B}$ as a uniform subalgebra. If E is an antisymmetric set for $\mathcal{A}$, and is also a maximal antisymmetric set for $\mathcal{B}$, then E is a maximal antisymmetric set for $\mathcal{A}$.*

The following result for restriction algebras immediately follows from the condition for A to be an antisymmetric set.

Proposition 2.7.16. *Let $\mathcal{A}$ be a uniform algebra, let E be a generalized peak set, and A be an antisymmetric set for $\mathcal{A}$ such that $A \subseteq E$. Then A is an antisymmetric set for $\mathcal{A}|_E$ also.*

Proposition 2.7.17. *Let A be an antisymmetric set, and $f \in \mathcal{A}$ be a real-valued peak function for a set E intersecting A. Then $A \subseteq E$.*

Proposition 2.7.18. *Let $\mathcal{A}$ be a uniform algebra on the compact Hausdorff space X. Then its maximal antisymmetric sets form a disjoint cover of X.*

Proof. First consider an ascending chain A_i of antisymmetric sets indexed by a set $\mathcal{I}$. Let f be any element $f \in \mathcal{A}$ whose restriction to $\cup_i A_i$ is real. Then, in particular, its restrictions to A_i are real constants: $f|_A = c_i$. But then since $A_i \cap A_j \neq \emptyset$ for any i, j contained in the indexing set $\mathcal{I}$, $c_i = c_j$ for any $i, j \in \mathcal{I}$, so $f|_{\cup_i A_i} = c$ for the common value $c \in \mathbb{R}$ of these constants. Consequently, the union $\cup_i A_i$ is also an antisymmetric set.

From an application of Zorn's Lemma, it follows that any antisymmetric set is contained in a maximal antisymmetric set. Therefore, the maximal antisymmetric sets of $\mathcal{A}$ cover X □

The following theorem is known as the Bishop-Shilov theorem or the generalized Stone-Weierstrass theorem.

Theorem 2.7.19. *[Sto71, p. 115] Let $\mathfrak{K}$ be a cover of X by maximal antisymmetric sets of a uniform algebra $\mathcal{A}$. A continuous function $f \in C(X)$ is a member of $\mathcal{A}$ if and only if each restriction $f|_K$ where $K \in \mathfrak{K}$ is a member of the restriction algebra $\mathcal{A}|_K$.*

Corollary 2.7.20. *If $\mathcal{A}$ is a uniform algebra on X such that every maximal antisymmetric set is a point, then $\mathcal{A} = C(X)$.*

To recover the Weierstrass approximation theorem from this, consider $\mathcal{A} = \langle z \rangle \subset C([a,b])$ on $[a,b] \subset \mathbb{R} \subset \mathbb{C}$, and let A be any maximal antisymmetric set for $\mathcal{A}$. The function z is constant on A since it is real-valued there, so A consists of a single point. By the corollary, $\mathcal{A} = C([a,b])$. Now consider a real-valued continuous function $f(t) : [a,b] \to \mathbb{R}$ and some $\epsilon > 0$. There exists some complex polynomial $p(z)$ such that $|p(t) - f(t)| < \epsilon$ for any $a \leq t \leq b$. But then its real part has the same property.

2.8 A Generalization of Kallin's Lemma to Stein Manifolds

In this section, we give a generalization (Theorem 2.8.3) of Kallin's Lemma. We start by giving a characterization of holomorphic convexity in the complex plane $\mathbb{C}$.

Lemma 2.8.1. *A compact set K contained in a domain $\mathcal{U} \subseteq \mathbb{C}$ is $\mathcal{O}(\mathcal{U})$-convex if and only if no component of its relative complement $\mathcal{U} \setminus K$ is relatively compact.*

In particular, a compact set $K \subseteq \mathbb{C}$ is polynomially convex if and only if its complement is connected [Gam69, p. 68].

Proof. If K is $\mathcal{O}(\mathcal{U})$-convex and $\mathcal{U}$ has a relatively compact component E, then by the maximum principle applied to $K \cup E$, for any $z \in E$, it follows that $|f(z)| \leq \|f\|_K$ for each $f \in \mathcal{O}(\mathcal{U})$. This shows that K would have needed to contain E in the first place.

Let $z_0 \in \mathcal{U} \setminus K$. Each component of $\mathcal{U} \setminus (K \cup \{z_0\})$ is not relatively compact, so touches the boundary $\partial\mathcal{U}$ in the Riemann sphere $\overline{\mathbb{C}}$. So each connected component of $\overline{\mathbb{C}} \setminus (K \cup \{z_0\})$ contains points from $\overline{\mathbb{C}} \setminus \mathcal{U}$. Applying Runge's theorem [Gam01, p. 344] to $S = \overline{\mathbb{C}} \setminus \mathcal{U}$, the compact set $K \cup \{z_0\}$, and the holomorphic function $f : K \cup \{z_0\} \to \mathbb{C}$

$$z \mapsto \begin{cases} 0 & z \in K \\ 1 & z = z_0, \end{cases}$$

there must exist rational functions having their poles in $\overline{\mathbb{C}} \setminus \mathcal{U}$ converging uniformly on $K \cup \{z_0\}$ to f. Take some such g within $1/4$ of f. Then $g \in \mathcal{O}(\mathcal{U})$, $\|g\|_K < 1/4$, and $|g(z_0)| > 3/4$. Therefore, z_0 is not an element of the $\mathcal{O}(\mathcal{U})$-convex hull of K. As z_0 was arbitrary, K is $\mathcal{O}(\mathcal{U})$-convex. □

The following lemma is derived from [Sto00, Lemma 1.6.18] which is used in the derivation of Kallin's Lemma there, and it requires modification in this context since the complement of a holomorphically convex set K in an open set $U \subsetneq \mathbb{C}$ is not necessarily connected.

Lemma 2.8.2. *Let $\mathcal{U}$ be a domain in $\mathbb{C}$, $K \subseteq \mathcal{U}$ a compact subset which is $\mathcal{O}(\mathcal{U})$-convex, and consider the uniform algebra $Q(K)$ formed by taking the uniform closure of $\mathcal{O}(\mathcal{U})$ on K. Then each boundary point of K is a peak point for the uniform algebra $Q(K)$.*

Proof. This holds vacuously if $K = \emptyset$. If K consists of a single point w, then take $f = 1$ to be a peaking function for w on K. Otherwise, K has at least two points.

Consider some boundary point w in K. If w is in the outside boundary, choose $z_\infty = \infty$. If not, choose z_∞ to be some point of the connected component of $\mathbb{C} \setminus \mathcal{U}$ which touches the boundary component in which w resides. Also fix some point z_0 of $K \setminus \{w\}$.

Construct a path γ in $\{w\} \cup \overline{\mathbb{C}} \setminus K$ joining w to z_∞. Using γ as the branch cut, take f to be a branch of $\log(z - w)$ if $z_\infty = \infty$ or of $\log(z - w) - \log(z - z_\infty)$. Let

$$M = \max_{z \in K} |z - w|,$$

when $z_\infty = \infty$ and

$$M = \frac{\max_{z\in K} |z - w|}{\min_{z\in K} |z - z_\infty|}$$

otherwise. Then M is well-defined and strictly positive since z_∞ is outside of the closed set K and there are at least two points in K. Define

$$\psi(z) = \begin{cases} \frac{f - \log M}{f - 1 - \log M} & z \neq w \\ 1 & z = w. \end{cases}$$

This function is continuous on K and holomorphic in its interior. Denote by U^* the one-point compactification of U. Then since K is $\mathcal{O}(U)$-convex, no complement of K is relatively compact in U by Lemma 2.8.1, so $U^* \setminus K$ is connected. It is locally connected at each finite point since it is open at each of those points, so it suffices to demonstrate that this complement contains a connected neighbourhood of the infinite point ∞. By Arakelyan's Theorem [Gar95, p. 39], $f \in Q(K)$.

Also because $\mathrm{Re} f - \log M \leq 0$ on $K \setminus \{w\}$, this means that $\left|\frac{f - \log M}{f - \log M - 1}\right| < 1$ on $K \setminus \{w\}$. So $\psi(w) = 1 > |\psi(z)|$ for $z \in K \setminus \{w\}$, which shows that w is a peak point for $Q(K)$. □

Here is an extension of Kallin's Lemma to a Stein manifold where the compacts are taken to be $\mathcal{O}(\mathcal{M})$-convex sets. Its proof is based on the proof of Kallin's Lemma given in [Sto00, Thm. 1.6.19] for the polynomial convexity case.

Theorem 2.8.3. *Let $\mathcal{M}$ be a Stein manifold and X_1, X_2 be $\mathcal{O}(\mathcal{M})$-convex compact sets. Suppose that $p : \mathcal{M} \to \mathbb{C}$ is a nonconstant holomorphic function such that*

- *$p(X_1) \cup p(X_2)$ is $\mathcal{O}(p(\mathcal{M}))$-convex,*
- *$p(X_1) \cap p(X_2) \subseteq \{0\} \subseteq \partial(p(X_1) \cup p(X_2))$, and*
- *$p^{-1}(0) \cap (X_1 \cup X_2)$ is $\mathcal{O}(\mathcal{M})$-convex.*

Then $X_1 \cup X_2$ is $\mathcal{O}(\mathcal{M})$-convex.

Proof. First denote $p(X_1)$ and $p(X_2)$ as Y_1 and Y_2. Define $Q(K)$ for an arbitrary compact K in $\mathcal{M}$ to be the uniform closure of $\mathcal{O}(M)$ on K.

Note that by [Ran86, I §1 Thm 1.21], if U is a connected open set and $q : U \to \mathbb{C}$ is holomorphic and nonconstant, then $q(U)$ is a domain. This means that since $\mathcal{M}$ can be covered by connected open sets U_i with associated charts ϕ_i, that this result may then be applied to each $p \circ \phi_i$ to conclude that each of these maps is open. Each chart ϕ_i is an open map on U_i, so $\mathcal{V} = p(\mathcal{M})$ is also open.

Let x be a point of the $\mathcal{O}(\mathcal{M})$-convex hull of $X_1 \cup X_2$. Then it has a representing measure μ for $\mathcal{Q}(X_1 \cup X_2)$ supported on $X_1 \cup X_2$. Now for any $q \in \mathcal{O}(\mathcal{V})$,

$$|q(p(x))| \leq \|q \circ p\|_{X_1 \cup X_2} = \|q\|_{p(X_1 \cup X_2)} = \|q\|_{Y_1 \cup Y_2},$$

so $x \in Y_1 \cup Y_2$.

If $p(x)$ is nonzero, then take a function $g \in \mathcal{Q}(Y_1 \cup Y_2)$ such that $g(p(x)) = 1$ and its restriction $g|_{Y_2}$ is zero. This is possible by taking any $g_0 \in \mathcal{Q}(Y_1)$ such that $g_0(p(x)) \neq g_0(0)$. Then define

$$g(w) = \begin{cases} \frac{g_0(w) - g_0(0)}{g_0(p(x)) - g_0(0)} & w \in Y_1 \\ 0 & w \in Y_2. \end{cases}$$

Then for any $q \in \mathcal{O}(\mathcal{M})$ and positive integer k,

$$\left|q^k(x)\right| = \left|q^k(x) g(p(x))\right| = \left|\int_{X_1 \cup X_2} q^k(w) g(p(w))\, \mu(dw)\right|$$

but since $g|_{Y_2} = 0$, the integral just has to be taken over X_1:

$$\left|q^k(x)\right| \leq \left|\int_{X_1} q^k(w) g(p(w))\, \mu(dw)\right| \leq \|q\|_{X_1}^k \int_{X_1} |g \circ p|\, d\mu.$$

Then

$$|q(x)| \leq \|q\|_{X_1} \left(\int |g \circ p|\, d\mu\right)^{1/k}$$

for any positive integer k, which means that $|q(x)| \leq \|q\|_{X_1}$. This shows that x is in the $\mathcal{O}(\mathcal{M})$-convex hull of X_1, so $x \in X_1$.

If $p(x) = 0$ instead, then 0 is a peak point of $\mathcal{Q}(Y_1 \cup Y_2)$ since it is a boundary point (by Lemma 2.8.2) with peak function $\psi(z)$, so $\psi(0) = 1$, and $|\psi| < 1$ elsewhere. Consider, for any $q \in \mathcal{O}(\mathcal{M})$, $k \in \mathbb{N}$,

$$q(x) = q(x)\psi(p(x))^k = \int_{X_1 \cup X_2} q(w)\psi(p(w))^k\, \mu(dw).$$

Noting that $\left|\psi(p(w))^k\right|$ is dominated by 1, and that the pointwise limit of $\psi(p(w))^k$ is $\chi_{p^{-1}(0)}(w)$, use Lebesgue's dominated convergence theorem to see that

$$q(x) = \lim_{k \to \infty} \int_{X_1 \cup X_2} q(w)\psi(p(w))^k\, \mu(dw) = \int_{X_1 \cup X_2} q(w)\chi_{p^{-1}(0)}(w)\mu(dw)$$

$$q(x) = \int_{p^{-1}(0) \cap (X_1 \cup X_2)} q(w)\, \mu(dw),$$

so x is in the holomorphically convex hull of $p^{-1}(0) \cap (X_1 \cup X_2)$, so in the set itself, and consequently in $X_1 \cup X_2$.

□

Kallin's Lemma can be seen to be a special case of this.

Corollary 2.8.4. *If $X_1, X_2 \subseteq \mathbb{C}^n$ are compact and polynomially convex and there exists some polynomial $p \in \mathbb{C}[x_1, \dots, x_n]$ such that*

- $p(X_1) \cap p(X_2) \subseteq \{0\}$,
- *and the set $p^{-1}(0) \cap (X_1 \cup X_2)$ is polynomially convex,*

then $X_1 \cup X_2$ is polynomially convex.

Proof. Take $\mathcal{M} = C^n$. Now X_1, X_2 are $\mathcal{O}(\mathcal{M})$-convex and since p is entire and nonconstant, $p(\mathbb{C}^n) = \mathbb{C}$. Now $p(X_1) \cup p(X_2)$ is polynomially convex since a pair of sets in $\mathbb{C}$ having connected complements that meet in, at most, one point have a union which also has a connected component. After applying the generalization, the result follows. □

Here it can be seen that Kallin's Lemma characterizes the holomorphic convexity of a union of disjoint compact sets.

Corollary 2.8.5. *Let $X_1, X_2 \subset \mathcal{M}$ be disjoint compact sets in a Stein manifold $\mathcal{M}$. Then their union $X_1 \cup X_2$ is holomorphically convex if and only if*

- *X_1, X_2 are holomorphically convex, and*
- *there exist a holomorphic function $f : \mathcal{M} \to \mathbb{C}$ such that $f(X_1)$ and $f(X_2)$ are disjoint.*

Proof. The backward implication was just shown in Theorem 2.8.3, so it suffices to show the forward implication.

Define a function g to be -1 in a neighbourhood of X_1 and to be 1 in a neighbourhood of X_2. This function is clearly holomorphic in a neighbourhood of $X_1 \cup X_2$. Apply the Oka-Weil Theorem for Stein spaces (Theorem 2.6.9) to obtain a function $f \in \mathcal{O}(M)$ such that $\|f - g\|_{X_1 \cup X_2} < 1/10$. We have the desired function.

Now consider the fact that $\|f + 1\|_{X_1} = 1/10$ but $|1 + f(z)| \geq 1.9$ whenever $z \in X_2$; similarly, $\|1 - f\|_{X_2} = 1/10$ but $|1 - f(z)| \geq 1.9$ whenever $z \in X_1$. Consequently, $X_1 \cap \widehat{X_2} = \emptyset$ and $X_2 \cap \widehat{X_1} = \emptyset$. Now from the fact that $\widehat{X_1} \cup \widehat{X_2} = X_1 \cup X_2$,

$$\widehat{X_1} \setminus (X_1 \cup X_2) = \emptyset = \widehat{X_2} \setminus (X_1 \cup X_2)\,.$$

This simplifies to

$$\widehat{X_1} \setminus X_1 = \emptyset = \widehat{X_2} \setminus X_2.$$

Therefore, X_1 and X_2 are holomorphically convex. □

2.9 Vodovoz and Zaidenberg

This section presents Vodovoz and Zaidenberg's proof of the existence of continuous generators for $C(X)$, for any simplicial polytope X. (This simply means a space which is homeomorphic to a finite-dimensional simplicial complex). We will later adapt this method to prove the existence of smooth generators.

Theorem 2.9.1. *[VZ71, Thm. 1] Any finite n-dimensional simplicial polytope X admits a collection of $n+1$ continuous generators for its algebra $C(X)$.*

Proof. Denote as X^n the n-skeleton of X, and, for a fixed n, its faces as X_k^n, for $1 \leq k \leq m$, where m is the number of n-faces comprising X^n. Assign each face X_k^n a homeomorphism $\chi_k^n : X_k^n \to D^n$ where D^n is the closed unit ball (in $\mathbb{R}^n$). (This will be denoted as χ_k if the dimension n is fixed.) The proof will proceed by induction on the dimension n of the skeleton. The base case $n = 0$ follows by constructing a function f_0 which takes distinct positive real values on the vertex set X^0.

For the induction stage, consider functions $f_0, f_1, \ldots, f_{n-1}$ on X^{n-1} which generate $C\left(X^{n-1}\right)$, in order to construct continuous functions $\hat{f}_0, \hat{f}_1, \ldots, \hat{f}_n$ which generate $C(X^n)$. Extend each f_j, for $1 \leq j \leq n-1$, to X^n as

$$\hat{f}_j(p) = \begin{cases} |\chi_k(p)|\, f_j\left(\chi_k^{-1}\left(\frac{\chi_k(p)}{|\chi_k(p)|}\right)\right), & p \in X_k^n, \chi_k(p) \neq 0, \\ 0, & p \in X_k^n, \chi_k(p) = 0, \end{cases}$$

and define $\hat{f}_n$ as

$$\hat{f}_n(p) = \left\{ (1 - |\chi_k(p)|)\, e^{2\pi i \frac{k}{m}}, \quad p \in X_k^n; \right.$$

clearly, these functions are continuous.

Let $\mathcal{A} = \left\langle \hat{f}_0, \ldots, \hat{f}_n \right\rangle$. Now $K = f_n(X^n)$ is a union of radial line segments, so by Lavrentiev's Theorem (2.7.4), $P(K) = C(K)$. Consequently, if $\phi \in C(K)$, then $\phi \circ \hat{f}_n \in \mathcal{A}$. Any point a in K has a strictly positive peak function g_a. (That is, $g_a(a) = 1$, and $0 \leq g_a(x) < 1$ for $x \neq a \in K$.) So, in particular, any $g_a \circ \hat{f}_n \in \mathcal{A}$. Define

$$X_a = \left(g_a \circ \hat{f}_n\right)^{-1}(1) = \hat{f}_n^{-1}(a)\,.$$

These are peak sets for X^n (and have $g_a \circ \hat{f}_n$ as the corresponding peak functions). Because any peak function $g_a \circ \hat{f}_n$ is real-valued, it must be constant on any antisymmetric set A. Consequently, an antisymmetric set intersecting some X_a must be contained within it. Now, by Corollary 2.7.13, $\mathcal{A}|_{X_a}$ is closed in $C(X_a)$. We now show that it is equal to $C(X_a)$. If $a = 0$, then this is immediate since X_a is a singleton, and if $|a| = 1$, then this is the induction hypothesis. Otherwise, $0 < |a| < 1$, and X_a sits within some X_k^n, having its associated homeomorphism χ_k. Define a homeomorphism ζ from the level set X_a to the boundary ∂X_k^n of the face:

$$\zeta : p \mapsto \chi^{-1}\left(\frac{\chi(p)}{|\chi(p)|}\right).$$

The uniform algebra

$$\mathcal{A}|_{X_a} = \left\langle \hat{f}_0, \ldots, \hat{f}_{n-1}, \hat{f}_n \right\rangle\Big|_{X_a} = \langle f_0 \circ \zeta, \ldots, f_{n-1} \circ \zeta \rangle|_{X_a}$$

is then isomorphic to

$$\langle f_0, \ldots, f_{n-1} \rangle|_{\partial X_j^n}\,, \tag{2.3}$$

which is a restriction of $C(X^{n-1})$ by the induction hypothesis, so (2.3) is $C(\partial X_j^n)$. Consequently, $\mathcal{A}|_{X_a} = C(X_a)$. Since, in any of these cases, $\mathcal{A}|_{X_a} = C(X_a)$, any of its antisymmetric sets is a singleton. So any antisymmetric set in $\mathcal{A}$ must be contained within some X_a, so it would be an antisymmetric set within $\mathcal{A}|_{X_a}$. This means that any maximal antisymmetric set of $\mathcal{A}$ is a single point. By Corollary 2.7.20 (of the Bishop-Shilov Theorem), $\mathcal{A} = C(X^n)$. This completes the induction step. □

Considering Corollary 2.7.7, this further implies that any finite n-dimensional simplicial polytope X admits a polynomially convex topological embedding into $\mathbb{C}^{n+1}$.

Chapter 3

Closed surfaces are 3-polynomially dense

For this chapter, we equip the surface S with an inner product $\langle\cdot,\cdot\rangle_S$ and assign to each edge a pair $(U, \hat{\mathbf{N}})$ consisting of a neighbourhood U of its interior and a vector field defined on U which is perpendicular to the edge (in $\langle\cdot,\cdot\rangle_S$). We also equip it with a Whitehead triangulation (see §2.4 for more details and examples): a triangulation $\mathcal{K}$ and a homeomorphism $\phi : |\mathcal{K}| \to S$ such that each restriction $\phi_A : A \to \hat{A} \subset S$ has a smooth extension to a neighbourhood of A (in the smallest affine set containing it). Here we denote simplices in $\mathcal{K}$ as a uppercase letters without a hat, and their corresponding simplices in S as the same letters with a hat.

The overall plan here is that we construct a foliation on each triangle and then construct $(s, \hat{\theta})$, a piecewise smooth topological coordinate system on S. Using this, we construct functions f_j on $C(S)$, which will be shown, in the last sections, to be smooth and to constitute generators for $C(S)$.

We build our candidates for generators by specifying distinct real values on vertices, and distinct unitary complex values on the edges and faces of the triangulation $\mathcal{K}$. (The coefficients need to be distinct so that the functions f_0, f_1, f_2 will separate points later. The coefficients of the edges and faces are unitary complex so that f_1, f_2 will have images $E \subset \mathbb{C}$ such that $P(E) = C(E)$.) Then these values are interpolated smoothly using the s coordinates on faces and the linear t coordinates along the edges. We require that the interpolation be flat at all specified points (that is, the derivatives in that direction vanish to all orders).

Then the function f_0 is specified by fixing the coefficients c_u at the vertices and 0 at the centres $(p^*(T))$ of the faces T. Next, f_1 is specified by fixing the coefficients c_{uv} at the midpoints of the edges uv, 0 at the vertices, and 0 at the centres. Finally, f_2 is specified by fixing 0 along the union of all the edges, and the coefficients c_T for at the centres.

3.1 Some special smooth functions

In this section we construct a function $h(t)$ useful for joining smooth functions to each other, and a function $q(t, \epsilon)$ which is useful for smoothly approximating $\max(0, t)$, from which we build $m(a, b; \epsilon)$ which smoothly approximates $\min(a, b)$.

Proposition 3.1.1. *The function* $h : \mathbb{R} \to [0,1]$ *given by*

$$h(t) = \begin{cases} 0 & t \le 0, \\ \frac{e^{-1/t}}{e^{-1/t} + e^{-1/(1-t)}} & 0 < t < 1, \\ 1 & t \ge 1, \end{cases}$$

is smooth and has derivatives vanishing to infinite order at 0 *and at* 1*. It has the additional properties that*

$$\begin{aligned} h(t) &= 1 - h(1-t), \\ h'(t) > 0 &\ \textit{for}\ 0 < t < 1, \\ h''(t) > 0 &\ \textit{for}\ 0 < t < 1/2, \\ h''(t) < 0 &\ \textit{for}\ 1/2 < t < 1, \end{aligned}$$

$$\int_0^1 h(t)\, dt = \frac{1}{2}, \tag{3.1}$$

$$h'(t) = h'(1-t) = h(t)\, h(1-t)\, \frac{t^2 + (1-t)^2}{t^2\,(1-t)^2} = \frac{h(t)}{t^2}\frac{h(1-t)}{(1-t)^2}\left(2t^2 - 2t + 1\right), \tag{3.2}$$

and

$$h(u) + u\, h'(u) < 2. \tag{3.3}$$

Proof. Any derivative of $h(t)$ goes to 0 near $t = 0$ or $t = 1$ by routine computations, so this function extends smoothly to $\mathbb{R}$ as it has been defined. Same for the identities $h(1-t) = 1-h(t)$, (3.1), (3.2), and (3.4). An alternate form for the function is useful in some computations: using the substitution $t = (1+w)/2$, one finds that

$$h\left(\frac{1+w}{2}\right) = \frac{1}{2}\left(1 + \tanh\left(\frac{2w}{1-w^2}\right)\right) \tag{3.4}$$

whenever $-1 < w < 1$; of course, $h((1+w)/2) = 0$ for $w \le -1$, and $h((1+w)/2) = 1$ for $w \ge 1$.

From consideration of the derivatives $h'(t)$, $h''(t)$ (more easily calculated in the form involving tanh), it can be seen that $h'(t)$ is strictly positive on $(0,1)$, $h''(t) > 0$ on $(0,1/2)$ and $h''(t) < 0$ on $(1/2,1)$.

To show the inequality (3.3), first note that it follows for the interval $(0,1/2)$ immediately from the fact that $h(1/2) + (1/2)h'(1/2) = 3/2$ and that $h(u)$ and $u\,h'(u)$ are strictly increasing functions on it (since $uh''(u) + h'(u) > 0$). For an interval $[t_0, t_1]$ where $1/2 \le t_0 < t_1 \le 1$, we have the upper bound

$$h(t) + t\, h'(t) \le h(t_1) + t_1\, h'(t_0),$$

since $t, h(t)$ are increasing functions and $h'(t)$ is decreasing on $(1/2,1)$. So the inequality will be proved if we can decompose $[1/2,1]$ into intervals $[t_0,t_1]$ having the property that $h(t_1) + t_1\, h'(t_0) < 2$. That this is so can be seen in the following table.

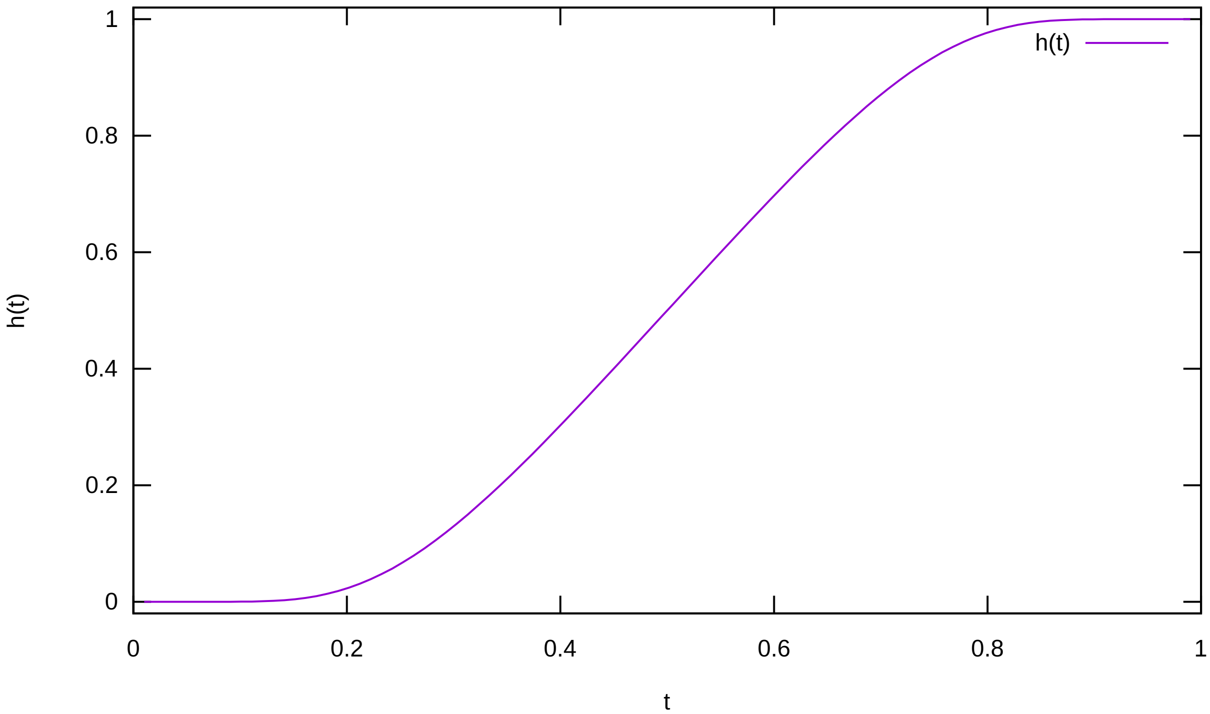


Figure 3.1: the graph of $h(t)$

$[t_0, t_1]$	$h(t_1) + t_1 h'(t_0) < 2$
$[0.500, 0.600]$	$h(0.6) + 2(0.6) \approx 1.897$
$[0.600, 0.630]$	$h(0.63) + (0.63)h'(0.60) \approx 1.954$
$[0.630, 0.650]$	$h(0.65) + (0.65)h'(0.63) \approx 1.976$
$[0.650, 0.670]$	$h(0.67) + (0.67)h'(0.65) \approx 1.998$
$[0.670, 0.685]$	$h(0.685) + (0.685)h'(0.67) \approx 1.985$
$[0.685, 0.700]$	$h(0.7) + 0.7h'(0.685) \approx 1.975$
$[0.700, 0.730]$	$h(0.73) + 0.73h'(0.7) \approx 1.994$
$[0.730, 0.800]$	$h(0.8) + 0.8h'(0.73) \approx 1.981$
$[0.800, 1.000]$	$h(1) + h'(0.8) \approx 1.596$

□

Lemma 3.1.2. *There exists a smooth function $q : \mathbb{R}^2 \setminus (0,0) \to \mathbb{R}$ such that*

1. *the function $q(t, 0) = \max(0, t)$,*
2. *the function $q(t, \epsilon) = \max(0, t)$ whenever $|t| \geq |\epsilon|$,*
3. *the function q extends continuously to $\mathbb{R}^2$,*
4. *for any fixed $t \in \mathbb{R}$, $q(t, \epsilon)$ is a nondecreasing function of ϵ,*
5. *for any fixed $\epsilon \in \mathbb{R}$, $q(t, \epsilon)$ is a nondecreasing function of t,*

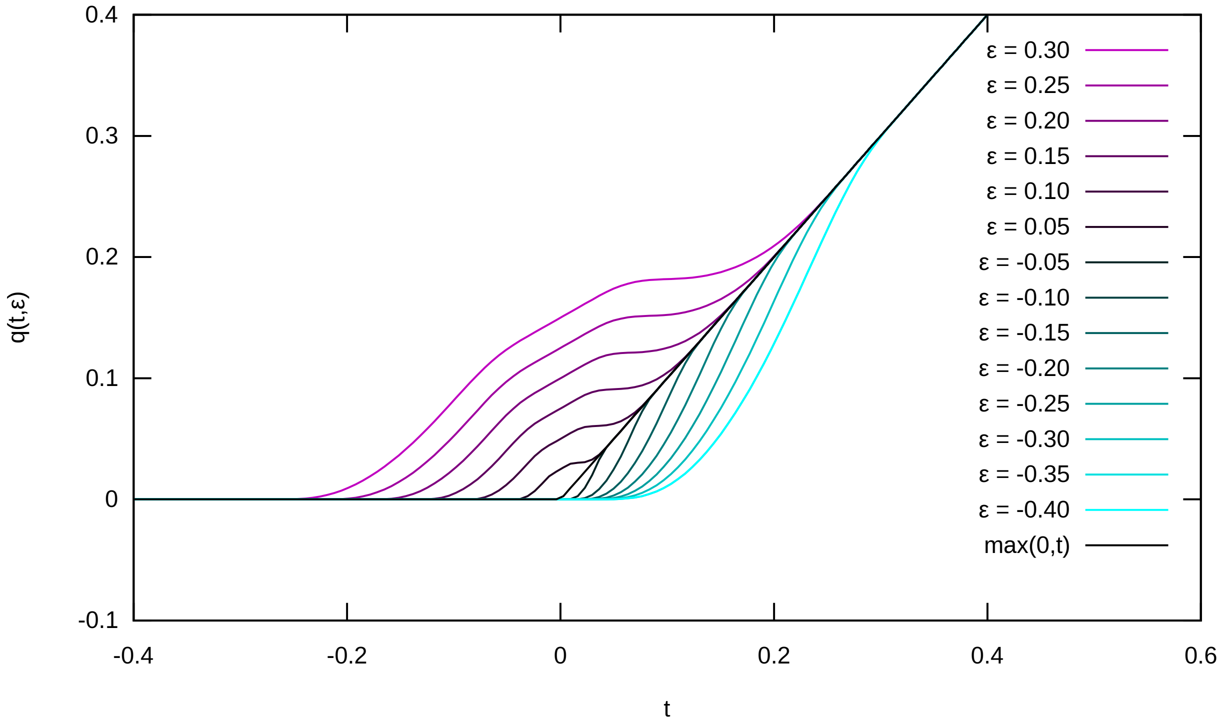


Figure 3.2: the functions $q(t, \varepsilon)$ providing smooth approximations to $\max(0, t)$

6. *for any* $\epsilon \in \mathbb{R}$,
$$|q(t, \epsilon) - \max(0, t)| \leq |\epsilon|\,,$$

7. *the derivatives of* q *near* $(0, 0)$ *have growth of, at most,* $1/\epsilon^{N-1}$ *where* N *is the total order of the derivative.*

Proof. The function is constructed as

$$q(t, \epsilon) = \begin{cases} h\left(\frac{t}{\epsilon} + 1\right) \frac{t+\epsilon}{2} + h\left(\frac{t}{\epsilon}\right) \frac{t-\epsilon}{2} & \epsilon > 0, \\ h\left(\frac{t}{-\epsilon}\right) t & \epsilon < 0, \\ \max(0, t) & \epsilon = 0. \end{cases}$$

Note that $q(t, \epsilon)$ is continuous at $(0, 0)$, even though $h(t/\epsilon)$ is not, since h is bounded and the terms $t, t \pm \epsilon$ go to 0 as $(t, \epsilon) \to (0, 0)$ which shows (3).

To show (2), note that when $t \geq |\epsilon|$, that each $h(\cdot)$ term is identically equal to 1, so $q(t, \epsilon) = t$ in this case. On the other hand, when $t \leq -|\epsilon|$, each $h(\cdot)$ term vanishes, so $q(t, \epsilon) = 0$.

To show (4), first consider $\epsilon < 0$ as then

$$\frac{\partial q}{\partial \epsilon} = h'\left(\frac{t}{-\epsilon}\right) \frac{t^2}{\epsilon^2} \geq 0.$$

On the other hand, if $\epsilon > 0$, then

$$\frac{\partial q}{\partial \epsilon} = h'\left(\frac{t}{\epsilon} + 1\right)\left(\frac{t+\epsilon}{2}\right) + \frac{1}{2} h\left(\frac{t}{\epsilon} + 1\right) \geq 0,$$

since $-\epsilon \leq t \leq 0$, and when $0 \leq t \leq \epsilon$,

$$\frac{\partial q}{\partial \epsilon} = \frac{1}{2} + h'\left(\frac{t}{\epsilon}\right)\left(\frac{-t}{\epsilon^2}\right)\left(\frac{t-\epsilon}{2}\right) - \frac{1}{2}h\left(\frac{t}{\epsilon}\right) = \frac{1}{2}\left(1 - h\left(\frac{t}{\epsilon}\right)\right) + h'\left(\frac{t}{\epsilon}\right)\frac{t(\epsilon - t)}{\epsilon^2} \geq 0.$$

To show (5), first consider $\epsilon < 0$. Then the derivative of $q(t, \epsilon)$ with ϵ fixed is

$$h\left(\frac{t}{-\epsilon}\right) + th'\left(\frac{t}{-\epsilon}\right)\left(\frac{1}{-\epsilon}\right) = h(u) + uh'(u) \geq 0,$$

where $u = -t/\epsilon$. Note that $uh'(u) \geq 0$ since if u is negative, $h'(u) = 0$. Now consider $\epsilon > 0$, and assume that $-\epsilon \leq t \leq 0$. Using the substitution $u = (t/\epsilon) + 1$, $q(t, \epsilon)$ takes the form

$$h(u)\,\frac{\epsilon(u-1)+\epsilon}{2} = \epsilon h(u)\,u,$$

which has derivative $\epsilon(h(u) + uh'(u)) \geq 0$ as before. Next assume that $0 \leq t \leq \epsilon$, and use the substitution $u = t/\epsilon$ so that q takes the form

$$\frac{\epsilon u + \epsilon}{2} + h(u)\,\frac{\epsilon u - \epsilon}{2} = \frac{\epsilon}{2}\left(u + 1 - h(u)\,(1-u)\right).$$

Comparing its derivative to 0 gives the inequality

$$\frac{\epsilon}{2}\left(1 + h(u) - (1-u)\,h'(u)\right) > 0.$$

We can disregard the factor $\epsilon/2$, use a substitution $v = 1 - u$, and rearrange and simplify to obtain the inequality

$$h(v) + v\,h'(v) < 2,$$

which is just (3.3).

To show (6), we can assume that $\epsilon \neq 0$ (as the result is otherwise trivial). Consider $\epsilon < 0$, then look at

$$\left|t - h\left(\frac{t}{-\epsilon}\right)\right| = \left|t\left(1 - h\left(\frac{t}{-\epsilon}\right)\right)\right| \leq |t| \leq |\epsilon|.$$

On the other hand, if $\epsilon > 0$ and $-\epsilon \leq t \leq 0$, consider

$$\left|h\left(\frac{t}{\epsilon}\right)\frac{t+\epsilon}{2} - 0\right| \leq |t + \epsilon|\,2 \leq \frac{\epsilon}{2}.$$

Finally, consider $\epsilon > 0$ and $0 \leq t \leq \epsilon$,

$$\left|\frac{t+\epsilon}{2} + h\left(\frac{t}{\epsilon}\right)\frac{t-\epsilon}{2} - t\right| = \left|\frac{\epsilon - t}{2} - h\left(\frac{t}{\epsilon}\right)\frac{\epsilon - t}{2}\right| \leq \left(1 - h\left(\frac{t}{\epsilon}\right)\right)\frac{\epsilon - t}{2} \leq \frac{\epsilon}{2},$$

and the results holds.

To show (7), first note (when $\epsilon > 0$) that every term of either of the partial derivatives $\frac{\partial q}{\partial t}, \frac{\partial q}{\partial \epsilon}$ is of the form

$$C\,h^{(k)}\left(\frac{t}{\epsilon} + A\right)\,\frac{p(t,\epsilon)}{\epsilon^{\deg(p)}}$$

where A, C are constants, the order $k = 0, 1$, and $p(t, \epsilon)$ is either a constant or a linear function. (To handle the case $\epsilon < 0$, use $-t/\epsilon$ instead of t/ϵ; the same computations work for this case also.) Now consider the more general form

$$C \frac{1}{\epsilon^{N-1}} h^{(k)} \left(\frac{t}{\epsilon} + A \right) \frac{p(t, \epsilon)}{\epsilon^{\deg(p)}},$$

where A, C are again constants, the order k satisfies $0 \le k \le N$, and the polynomial $p(t, \epsilon)$ is homogeneous. Computation of its partial derivatives shows that each term of those also takes the form with $N \to N+1$. By induction, any term of any higher-order partial derivative of $q(t, \epsilon)$ takes the form, and N is the total order of the derivative being computed. In addition, since the polynomial $p(t, \epsilon)$ is constant in any term of $\frac{\partial q}{\partial t}, \frac{\partial q}{\partial \epsilon}$ for which $k = 0$, any partial derivative of order at least 2 only has terms for which $k > 0$.

Next, note that when $A = 0, 1$, the $h^{(k)}(\cdot)$ factor vanishes if unless $t \in [0, \epsilon]$ or $t \in [-\epsilon, 0]$, respectively. This means that the last factor, the rational function in t, ϵ, is bounded in the region in which the second factor is non-vanishing, so the product of these factors is bounded. Therefore, the absolute value of the whole product is bounded above by some constant multiple of $1/\epsilon^{N-1}$. Consequently, any higher-order derivative of $q(t, \epsilon)$ is also. □

Theorem 3.1.3. *For any $n \ge 2$, there exists a smooth function $m(x_1, \ldots, x_n; \epsilon)$ on $\mathbb{R}^n \times (\mathbb{R} \setminus \{0\})$ having the following properties:*

1. *it extends continuously to $\mathbb{R}^{n+1}$;*
2. *it can be chosen to be within any desired distance of* $\min(\cdot)$*, specifically,*

$$|m(x_1, x_2, \ldots, x_n; \epsilon) - \min(x_1, x_2, \ldots, x_n)| \le (n-1)\,|\epsilon|\,;$$

3. *for $\epsilon < 0$, $m(x_1, x_2, \ldots, x_n; \epsilon)$ is an underestimate for* $\min(x_1, x_2, \ldots, x_n)$ *while for $\epsilon > 0$, $m(x_1, x_2, \ldots, x_n)$ is an overestimate for it;*
4. *its pointwise limit* $\lim_{\epsilon \to 0} m(x_1, x_2, \ldots, x_n; \epsilon) = \min(x_1, x_2, \ldots, x_n)$*;*
5. *interpreting $m(x; \epsilon)$ as x, if $x_n \ge x_{n-1} + \epsilon$, then*

$$m(x_1, x_2, \ldots, x_{n-1}, x_n; \epsilon) = m(x_1, x_2, \ldots, x_{n-1}; \epsilon)\,;$$

if $x_k \ge x_{k+1} + \epsilon$ for some $1 \le k < n$, then

$$m(x_1, x_2, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_n; \epsilon) = m(x_1, x_2, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n; \epsilon)\,;$$

in particular, if some element x_j is more than ϵ higher than all of the others, it can be omitted;

6. *its partial derivative*

$$\frac{\partial m}{\partial \epsilon}\Big(x_1, x_2, \ldots, x_n; \epsilon\Big) \ge 0;$$

and,

7. *its partial derivative*

$$\frac{\partial^N m}{\partial x_1^{k_1} \partial x_2^{k_2} \cdots \partial x_n^{k_n} \partial \epsilon^{k_{n+1}}} (x_1, x_2, \ldots, x_n; \epsilon)$$

has growth in $1/\epsilon$ *approaching* $(0,0)$ *of at most* $(n-1)(N-1)$.

Proof. Note that $\max(a,b) = b + \max(a-b,0)$ and that $\min(a,b) + \max(a,b) = a+b$. So

$$\min(a,b) = b + \min(a-b,0) = a - \max(a-b,0), \tag{3.5}$$

and I have a function $q(t,\epsilon)$ for approximating $\max(t,0)$ already. Note for (3) that ϵ is negated so that $\epsilon > 0$ will correspond to overestimates (since we need underestimates for $\max(t,0)$) and $\epsilon < 0$ will correspond to underestimates (in which case we use overestimates for $\max(t,0)$).

For the base case ($n = 2$) we define the function

$$m(a,b;\epsilon) = a - q(a-b,-\epsilon). \tag{3.6}$$

Then

$$\lim_{\epsilon\to 0} m(a,b;\epsilon) = a - \lim_{\epsilon\to 0} q(a-b,\epsilon) = a - \max(a-b,0) = \min(a,b),$$

showing (4).

To show (6), note that when we take this derivative of $m(a,b;\epsilon)$,

$$\frac{\partial m(a,b;\epsilon)}{\partial \epsilon} = \frac{\partial q}{\partial \epsilon}\Big(a-b,-\epsilon\Big) \geq 0,$$

since $q(t,\epsilon)$ is nondecreasing in ϵ.

Next consider the general case. Note the identity

$$\min(x_1,x_2,\ldots,x_n) = \min(x_1,\min(x_2,\ldots,x_n)) = x_1 - \max(0, x_1 - \min(x_2,\ldots,x_n)).$$

Take $m(a,b;\epsilon)$ as defined for $n = 2$ and define it for higher n as

$$m(x_1,x_2,\ldots,x_n;\epsilon) = x_1 - q(x_1 - m(x_2,\ldots,x_n;\epsilon),-\epsilon). \tag{3.7}$$

(Note that (1) holds since q satisfies the same property: Lemma 3.1.2 (3).) Assume the induction hypothesis (that (4) and (6) hold for n). For clarity, we denote the function m from the previous step as m_0. Then

$$\begin{aligned}\lim_{\epsilon\to 0} m(x_1,x_2,\ldots,x_n;\epsilon) &= x_1 - \lim_{\epsilon\to 0} q(x_1 - m_0(x_2,\ldots,x_n),-\epsilon) \\ &= x_1 - \max\Big(0, x_1 - \lim_{\epsilon\to 0} m_0(x_2,\ldots,x_n)\Big) \\ &= x_1 - \max(0, x_1 - \min(x_2,\ldots,x_n)) = \min(x_1,x_2,\ldots,x_n),\end{aligned}$$

and

$$\frac{\partial m}{\partial \epsilon} = \frac{\partial q}{\partial \epsilon} + \frac{\partial q}{\partial t}\frac{\partial m_0}{\partial \epsilon} \geq 0,$$

where I have used that q is nondecreasing in t as well.

For (2), note first that

$$\begin{aligned} |m(x_1, x_2; \epsilon) - \min(x_1, x_2)| &= |x_1 - q(x_1 - x_2, -\epsilon)) - x_1 + \min(0, x_1 - x_2)| \\ &= |\min(0, x_1 - x_2) - q(x_1 - x_2, -\epsilon)| \leq \epsilon. \end{aligned}$$

Now suppose that

$$|m(x_1, x_2, \ldots, x_{n-1}; \epsilon) - \min(x_1, x_2, \ldots, x_{n-1})| \leq (n-2)\,|\epsilon|$$

for any $x_j \in \mathbb{R}$ and $0 \neq \epsilon \in \mathbb{R}$. Then

$$\begin{aligned} & |m(x_1, x_2, \ldots, x_n; \epsilon) - \min(x_1, x_2, \ldots, x_n)| \\ = & |x_1 - q(x_1 - m(x_2, \ldots, x_n; \epsilon), -\epsilon) - x_1 + \max(0, x_1 - \min(x_2, \ldots, x_n))| \\ \leq & |q(x_1 - m(x_2, \ldots, x_n; \epsilon), -\epsilon) - \max(0, x_1 - m(x_2, \ldots, x_n; \epsilon))| \\ + & |\max(0, x_1 - m(x_2, \ldots, x_n; \epsilon)) - \max(0, x_1 - \min(x_2, \ldots, x_n))| \\ \leq & |\epsilon| + |x_1 - m(x_2, \ldots, x_n; \epsilon) - x_1 + \min(x_2, \ldots, x_n)| \\ = & |\epsilon| + |\min(x_2, \ldots, x_n) - m(x_2, \ldots, x_n; \epsilon)| \\ \leq & |\epsilon| + (n-2)\,|\epsilon| = (n-1)\,|\epsilon|\,. \end{aligned}$$

To show (5), we use Lemma 3.1.2 (2). It suffices to note that

$$m\,(x_{n-1}, x_n; \epsilon) = x_{n-1}$$

and

$$\begin{aligned} m\,(x_k, x_{k+1}, x_{k+2}, \ldots, x_n; \epsilon) &= x_k - q\,(x_k - x_{k+1} + q\,(x_{k+1} - m\,(x_{k+2}, \ldots, x_n; \epsilon)\,; -\epsilon)\,; -\epsilon) \\ &= x_{k+1} - q\,(x_{k+1} - m\,(x_{k+2}, \ldots, x_n; \epsilon)\,; -\epsilon) \\ &= m\,(x_{k+1}, x_{k+2}, \ldots, x_n; \epsilon)\,. \end{aligned}$$

For (7), consider the following:

1. the growth rate in $1/\epsilon$ of a partial derivative of $m(x_1, x_2, \ldots, x_n; \epsilon)$ is the maximum of those of its summands;

2. the growth rate in $1/\epsilon$ of a summand is the sum of the growth rates of its factors;

3. the growth rate in $1/\epsilon$ of a factor is the sum of the growth rates of the partial derivatives of q occurring in its composition.

This means that the maximum growth rate can be ascertained by considering $m(x_1, x_2, \ldots, x_n; \epsilon)$. The most factors containing $q(t, \epsilon)$ and its derivatives that can be obtained is $n-1$ and this can be done by differentiating with respect to ϵ. In any step past the first one, differentiate the factor having the most references to q and its derivatives. This raises the growth rate by $n-1$ each step. Consequently, the growth rate of a partial derivative of $m(\cdot)$ is the product of $N-1$, the number of steps that affected the growth rate, and $n-1$. □

3.2 $\mathfrak{A}$-flatness of smooth function germs

In this section, we discuss rings of germs of differential operators at a point p of some manifold M, which we call operator germ rings. (We are using a general smooth manifold M instead of a surface S in order to have greater generality; the applications in this chapter will just use surfaces.) In the specific case of $\mathbb{R}^n$ (for some positive integer n), these are partial differential operators; otherwise, finite sums of compositions $\mathbf{V}_1\mathbf{V}_2\cdots\mathbf{V}_m$ of vector fields $\mathbf{V}_j$ defined in some neighbourhood of p. The elements of these rings act on smooth function germs at the point under consideration.

This allows us to introduce notions of $\mathfrak{A}_p$-flatness, generalizing the concept of a function vanishing to infinite order, and $\mathfrak{A}_p$-matching, which generalizes the concept of two functions being equal at a point and having matching partial derivatives to all orders at it.

In this section, let $\mathbb{F}$ denote a field; for our purposes, it will either be the reals $\mathbb{R}$ or the complex numbers $\mathbb{C}$. Denote smooth $\mathbb{F}$-valued functions on an open set U as $C^\infty_{\mathbb{F}}(U)$.

Recall the definitions of a smooth function germ and an operator germ.

Definition 3.2.1. *A germ u of a smooth function or* **smooth function germ** *(at a point $p \in M$) is an equivalence class of a smooth function f defined on an open neighbourhood $U \ni p$ under the following equivalence relation $\sim$. If $f : U \to \mathbb{F}$, $g : V \to \mathbb{F}$ are smooth functions defined on open neighbourhoods of p, then $f \sim g$ provided that $f|_{U\cap V} = g|_{U\cap V}$ on $U \cap V$.*

For a smooth function germ u at p, any smooth function $f : U \to \mathbb{F}$ having u as its equivalence class is said to be a **representative** *for u. The* **evaluation** *$u(p)$ of a smooth function germ at p is defined to be $f(p)$ for any representative f.*

Remark 3.2.2. *The set of smooth function germs at a point p (denoted as $\mathfrak{F}_p$) becomes a ring when equipped with the following operations, where $f : U \to \mathbb{C}$ is a representative for f_p and $g : V \to \mathbb{C}$ is a representative for the germ g_p.*

- *The sum $f_p + g_p$ is defined to be the germ of $f + g : U \cap V \to \mathbb{C}$.*
- *The product $f_p g_p$ is defined to be the germ of $fg : U \cap V \to \mathbb{C}$.*

If, in addition, there exists an diffeomorphism $\phi : W \to U \cap V$, we can define pullbacks under it:

- *The pullback $\phi^* f_p$ is the germ at $q = \phi^{-1}(p)$ of $f \circ \phi : W \to \mathbb{C}$. It satisfies the identities $\phi^*\left(f_p + g_p\right) = \phi^* f_p + \phi^* g_p$ and $\phi^*\left(f_p g_p\right) = \phi^* f_p\ \ \phi^* g_p$ and has an inverse $\left(\phi^{-1}\right)^*$, so it is a ring isomorphism.*

Recall that a differential operator here means a finite sum of products $\mathbf{V}_1\mathbf{V}_2\cdots\mathbf{V}_m$, defined on the common domain of $\mathbf{V}_j$, as in the introduction.

Definition 3.2.3. *An* **operator germ** *is an equivalence class of a differential operator acting on smooth functions in some open neighbourhood U of $p \in M$ under the following equivalence relation $\sim$. Let $\mathbf{A}$ be a differential operator acting on smooth functions defined on an open neighbourhood $U \ni p$, and $\mathbf{B}$, similarly, be a differential operator acting on smooth functions defined on an open neighbourhood V. Then $\mathbf{A} \sim \mathbf{B}$ provided that, for any pair of smooth functions $f : U \to \mathbb{F}$, $g : V \to \mathbb{F}$ such that $f|_{U\cap V} = g|_{U\cap V}$, the smooth functions $\mathbf{A}f$ and $\mathbf{B}g$ coincide on $U \cap V$.*

Definition 3.2.4. *We define an* **operator germ ring** $\mathfrak{A}_p$ *at a point* $p \in M$ *to be a set of operator germs at* p *equipped with the following ring structure. Let* A_p *and* B_p *be arbitrary operator germs in it, having representatives* $\mathbf{A}$, $\mathbf{B}$ *defined on smooth functions on the open sets* U, V, *respectively. Let* f_p *be an arbitrary smooth function germ at* p *having a representative* $f : W \to \mathbb{F}$, *where* W *is an open neighbourhood of* p.

- *Addition of operator germs* A_p+B_p *is simply the germ of* $\mathbf{A}+\mathbf{B}$ *acting on smooth functions on* $U \cap V$.
- *Multiplication of operator germs* A_pB_p *is the germ of the composition* $\mathbf{AB}$ *acting on smooth functions on* $U \cap V$.
- *Scalar multiplication* f_pA_p *is the germ of the product* $f\mathbf{A}$ *acting on smooth functions on* $U \cap W$.
- *The set* $\mathfrak{A}_p$ *must be closed under each of these operations.*

It should be noted that an operator germ ring is additionally a unitary left $\mathfrak{F}_p$-module under the scalar multiplication just defined, and that it is a graded ring where the grade is the degree of a homogeneous operator germ.

If N is another smooth n-manifold, $q \in N$ and $\phi : N \to M$ is a diffeomorphism such that $\phi(q) = p$, then we can also define pullback ϕ^*A_p of an operator germ to be the germ of the operator ϕ^*A which acts on smooth functions on $\phi^{-1}(U)$ as

$$(\phi^*A)\, f = A\left(f \circ \phi^{-1}\right) \circ \phi.$$

It has the properties

$$\begin{aligned}
\phi^*\left(A_pf_p\right) &= \phi^*A_p\, \phi^*f_p,\\
\phi^*\left(f_pA_p\right) &= \phi^*f_p\, \phi^*A_p,\\
\phi^*\left(A_p + B_p\right) &= \phi^*A_p + \phi^*B_p,\\
\phi^*\left(A_pB_p\right) &= \phi^*A_p\, \phi^*B_p.
\end{aligned}$$

Here are some examples of operator germ rings:

1. The operator germ ring $\langle u_1, u_2, \ldots, u_m\rangle$ generated by operator germs u_j at p is the set of sums of operator germs of the form $f_pu_{j_1}\cdots u_{j_N}$, where f_p is an arbitrary smooth function germ and $j_1, j_2, \ldots, j_N$ are indices taking values in $\{1, 2, \ldots, m\}$.
2. For differential operators $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_m$, we define $\langle \mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_m\rangle_p$ to be the operator germ ring generated by their germs at p; that is, $\left\langle (\mathbf{A}_1)_p, (\mathbf{A}_2)_p, \ldots, (\mathbf{A}_m)_p\right\rangle$.
3. Define the operator germ ring

$$\mathfrak{E}_p = \left\langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}\right\rangle_p,$$

   for a point $p \in \mathbb{R}^n$, for some n. Letting $x_1, x_2, \ldots, x_n$ be a local coordinate system in a neighbourhood of p allows this definition to also work for any point p in a smooth n-manifold M.

4. Any left or right ideal of $\mathfrak{E}_p$ which is closed under multiplication by smooth function germs is an operator germ ring. To make the notation less redundant, we will denote a left ideal of the form $\mathfrak{A}_p\,(P)_p$, where P is a differential operator, as $(\mathfrak{A}P)_p$.

5. Let j be any index in $\{1,2,\dots,n\}$, where n is the dimension of the manifold. Any two-sided ideal of $\mathfrak{E}_p$ closed under multiplication by smooth function germs which contains $\frac{\partial}{\partial x_j}$ equals $\mathfrak{E}_p$ itself since

$$\frac{\partial}{\partial x_j}\left(x_j\frac{\partial}{\partial x_k}\right)-x_j\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_j}=\frac{\partial}{\partial x_k}$$

for any index $k \neq j$.

6. Given a operator germ ring $\mathfrak{A}_p$ and a diffeomorphism $\phi : W \to V$, where V is an open neighbourhood of p, then the pull-back

$$\phi^*\mathfrak{A}_p=\left\{\phi^*T\,:\,T\in\mathfrak{A}_p\right\}$$

is an operator germ ring at $\phi^{-1}(p)$. In addition, ϕ^* is a ring isomorphism between these operator germ rings. The ring $\phi^*\mathfrak{A}_p$ takes a simple form if $\mathfrak{A}_p$ is finitely generated:

$$\phi^*\left\langle u_1,u_2,\dots,u_m\right\rangle=\left\langle\phi^*u_1,\phi^*u_2,\dots,\phi^*u_m\right\rangle.$$

We are now in a position to generalize the concept of vanishing to infinite order at a point $p\in\mathbb{R}^n$.

Definition 3.2.5. *Let $\mathfrak{A}_p$ denote an operator germ ring at $p \in M$, and let f_p be a smooth function germ at p. We say that f is $\mathfrak{A}_p$-**flat** if $A_pf_p\big|_p=0$ for any $A\in\mathfrak{A}_p$, and a pair of smooth function germs f_p and g_p are said to be $\mathcal{A}_p$-**matching** if $f(p)=g(p)$ and $A_pf_p\big|_p=A_pg_p\big|_p$ for any $A_p\in\mathfrak{A}_p$.*

A smooth function f is said to be $\mathfrak{A}_p$-flat if its corresponding germ f_p is; similarly, a pair of smooth functions f,g is said to be $\mathfrak{A}_p$-matching if their corresponding germs f_p,g_p are.

Note that a pullback function germ ϕ^*f_p is $\phi^*\mathcal{A}_p$-flat if and only if f_p is $\mathcal{A}_p$-flat and that, similarly, a pair of pullback germs ϕ^*f_p and ϕ^*g_p are $\phi^*\mathcal{A}_p$-matching if and only if f_p and g_p are $\mathcal{A}_p$-matching.

Example 3.2.6. *Consider the operator germ rings (in $\mathbb{R}^2$) $\mathfrak{A}_p=\left\langle\frac{\partial}{\partial x}\right\rangle$, $\mathfrak{B}_p=\left\langle\frac{\partial}{\partial x}+x\frac{\partial}{\partial y}\right\rangle$ on the vertical line $L=\{x=0\}$. Then the equations $\mathbf{A}^kf\big|_p=0$ are*

$$\left.\frac{\partial^kf}{\partial x^k}\right|_L=0,$$

so any function of the form $g(y)\mathfrak{h}(x)$, where $\mathfrak{h}$ has derivatives vanishing to infinite order at $x=0$, is $\mathfrak{A}_p$-flat. In contrast, the first three equations for $\mathbf{B}^kf\big|_p$ are

$$\left.\frac{\partial f}{\partial x}\right|_L=0,$$

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_L + \left.\frac{\partial f}{\partial y}\right|_L = 0,$$

$$\left.\frac{\partial^3 f}{\partial x^3}\right|_L + 3\left.\frac{\partial^2 f}{\partial y\,\partial x}\right|_L = 0.$$

For the function $g(y)\mathfrak{h}(x)$ to additionally be $\mathfrak{B}_p$-flat, its partial derivative (with respect to y) $g'(y)\mathfrak{h}(x)$ would have to vanish, so any $\mathfrak{B}_p$-flat function of the form $g(y)\,\mathfrak{h}(x)$ takes the form $C\mathfrak{h}(x)$. This shows that $\mathfrak{A}_p \neq \mathfrak{B}_p$ even though the tangents of the generators match.

Example 3.2.7. *Consider the left ideal $\mathfrak{I} = \left(\mathfrak{E}\,\frac{\partial}{\partial x}\right)_{(0,0)}$ of $\mathfrak{E}_{(0,0)}$ (on $\mathbb{R}^2$). Then a smooth function $f(x,y)$ is $\mathfrak{I}$-flat if and only if any partial derivative of arbitrary order having a $\frac{\partial}{\partial x}$ factor vanishes at $(0,0)$. This is a much stronger property than $\left\langle\frac{\partial}{\partial x}\right\rangle_{(0,0)}$-flatness, which only requires partial derivatives in x to vanish at $(0,0)$ to all orders.*

We say that a smooth function $f : U \to \mathbb{F}$ is **completely flat** at p if it is $\mathfrak{E}_p$-flat, and we say that a pair of smooths functions $f : U \to \mathbb{F}$, $g : V \to \mathbb{F}$ **completely match** if they are $\mathfrak{E}_p$-matching. (In other words, a function is completely flat at a point p if all partial derivatives of any order vanish at p. A pair of functions completely match at p if they are equal at that point and all partial derivatives, of any order, are equal at p also.)

Lemma 3.2.8. *Let $g_1 : U_1 \to \mathbb{F}$ and $g_2 : U_2 \to \mathbb{F}$ be smooth functions defined on connected open subsets of $\mathbb{R}^2$ such that $U_1 \cup U_2$ is also connected, and γ be a simple smooth arc in $U_1 \cap U_2$ such that $(U_1 \cup U_2) \setminus \gamma$ is disconnected and has a disconnection consisting of a pair of connected components: V_1 which is a subset of U_1, and V_2 which is a subset of U_2. Then the function*

$$g(p) = \begin{cases} g_1(p) & p \in V_1 \cup \gamma, \\ g_2(p) & p \in V_2 \cup \gamma, \end{cases}$$

is well-defined and smooth on $U_1 \cup U_2$, provided that g_1 and g_2 completely match along γ.

Proof. It is clear that $g(p)$ is well-defined since $g_1(p)$ and $g_2(p)$ agree on γ. Their derivatives (of any order) match also, so $g(p)$ has these derivatives also at each point of γ. As γ is the only set that either property needs to be checked on, $g(p)$ has the desired properties. □

3.3 A lower bound for polynomial density of a closed surface

Here we follow the proof given by Duchamp and Stout [DS81, Lem. 4.2, p. 54] that no closed n-dimensional topological manifold in $\mathbb{C}^n$ is polynomially convex. (Of course, $n = 2$ for our purposes since this chapter is concerned with closed surfaces and not closed manifolds in general.) This follows from the following observations:

1. the Čech cohomology group $H^n(U, \mathbb{Z})$ vanishes for any Runge domain U,

2. any polynomially convex compact in $\mathbb{C}^n$ is a countable intersection of Runge domains,

3. and the Čech cohomology group $H^n(K,\mathbb{C})$ is the inductive limit of the $H^n(U,\mathbb{C})$ over the set of open neighbourhoods of K ordered as $U \leq V$ (means $V \subseteq U$). (See [Car57, Prop. 3, p. 85].)

Regarding (1), Serre [Ser55] used de Rham cohomology to conclude that $H^n(U,\mathbb{C}) = 0$ for any Runge domain $U \subset \mathbb{C}^n$. This result was strengthened by Andreotti and Narasimhan [AN62, Lem. 3, Cor. (3)] who showed that $H_r(U,\mathbb{Z}) = 0$ for $r \geq n$ and $H_{n-1}(U,\mathbb{Z})$ is torsion free for any Runge domain on a contractible n-dimensional Stein space with isolated singularities. (An application of the universal coefficient theorem for cohomology [Hat02, p. 198]

$$0 \longrightarrow \mathrm{Ext}\,(H_{n-1}(U,\mathbb{Z}),\mathbb{Z}) \longrightarrow H^n(U,\mathbb{Z}) \longrightarrow \mathrm{Hom}\,(H_n(U,\mathbb{Z}),\mathbb{Z}) \longrightarrow 0$$

$$0 \longrightarrow H^n(U,\mathbb{Z}) \longrightarrow 0$$

then gives $H^n(U,\mathbb{Z}) = 0$.)

Regarding (2), this follows from the following facts:

1. any polynomially convex set in $\mathbb{C}^n$ is the zero set of a nonnegative smooth plurisubharmonic function on $\mathbb{C}^n$ (see [Sto00, p. 26]),

2. and any relatively compact sublevel set $\{\rho < c\}$ of a continuous plurisubharmonic exhaustion function ρ is a Runge domain (see [Sto00, Thm. 1.3.7]).

This means that the set of open neighbourhoods whose cohomology group $H^n(\cdot,\mathbb{Z})$ vanishes is cofinal in the set of open neighbourhoods equipped with the provided partial order. It now follows that $H^n(K,\mathbb{Z}) = 0$ by considering the commutative diagram (for any U and some $V \subseteq U$ such that $H^n(V,\mathbb{Z}) = 0$):

$$\begin{array}{ccc} H^n(U,\mathbb{Z}) & \longrightarrow & H^n(V,\mathbb{Z}) \\ & \searrow & \downarrow \\ & & H^n(K,\mathbb{Z}) \end{array} \qquad \begin{array}{ccc} H^n(U,\mathbb{Z}) & \longrightarrow & 0 \\ & \searrow & \downarrow \\ & & H^n(K,\mathbb{Z}) \end{array}$$

each of these maps $H^n(U,\mathbb{Z}) \to H^n(K,\mathbb{Z})$ is the zero map, so the colimit $H^n(K,\mathbb{Z}) = 0$.

On the other hand, any compact topological n-manifold in $\mathbb{C}^n$ has non-vanishing $H^n(K,\mathbb{Z})$; therefore, no compact n-manifold in $\mathbb{C}^n$ is polynomially convex.

(Browder [Bro61] used the weaker result that $H^n(U,\mathbb{C}) = 0$ for a Runge domain U, which means that his conclusion is limited to orientable manifolds. Vodovoz and Zaidenberg [VZ71] also showed this result by using homology with coefficients in $\mathbb{Z}_2$.)

In particular, the algebra $C(S)$ does not admit fewer than 3 continuous generators (let alone smooth ones). Consequently, its polynomial density is at least 3.

3.4 The foliation on a triangle

First of all, we assign to any edge $\hat{E}$ of the surface S a pair $(U,\hat{\mathbf{N}})$, where U is an open neighbourhood of the interior of $\hat{E}$ and $\hat{\mathbf{N}}$ is a smooth vector field on U which is normal to the edge $\hat{E}$ (in the metric $\langle\cdot,\cdot\rangle_S$).

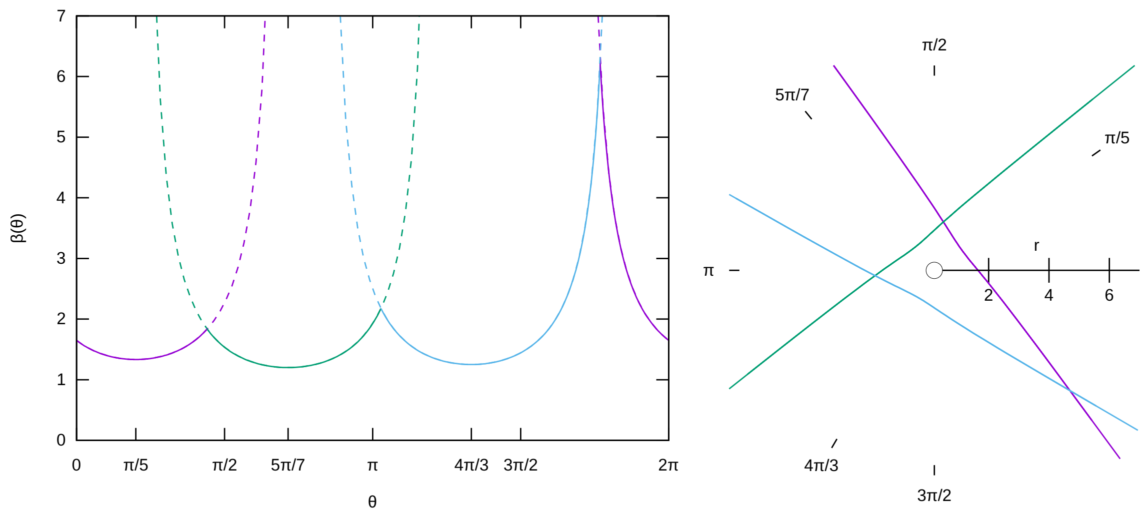


Figure 3.3: An example of $\beta(\theta)$.

For any triangle T in $\mathcal{K}$ (having a corresponding map $\phi : T \to M$) incident to E ($= \phi^{-1}(\hat{E})$), there is an induced transverse vector field $\mathbf{N} = \left(\phi^{-1}\right)_* \hat{\mathbf{N}}$ on an open neighbourhood of the interior of the edge E (after fixing a specific extension of ϕ which, by standard abuse of notation, is also denoted ϕ). That this induced vector field is transverse to the edge can be seen by considering any point p within the interior of E, a tangent vector $\frac{\partial}{\partial s}\big|_{\phi(p)}$, and a normal vector $\frac{\partial}{\partial n}\big|_{\phi(p)}$ to $\hat{E}$: the linear operator $\phi^{-1}_{*,\phi(p)}$ is nonsingular, so the tangent vector $\phi^{-1}_{*,\phi(p)} \frac{\partial}{\partial s}\big|_{\phi(p)}$ and the induced normal vector $\mathbf{N}_p$ are linearly independent.

Assign T an interior point p^* and pick some tangent vector $\mathbf{u}$ to its tangent plane Π. Now equip Π with its induced inner product $\langle \cdot, \cdot \rangle_\Pi$ and define a polar coordinate system (r, θ) as follows:

$$r(p)^2 = \langle p - p^*, p - p^* \rangle_\Pi ,$$

$$r(p) \cos \theta(p) = \langle p - p^*, \mathbf{u} \rangle_\Pi .$$

Parametrize ∂T as $r = \beta(\theta)$ where $\beta(\theta) : [0, 2\pi] \to \mathbb{R}$ is piecewise smooth and periodic. (See Figure 3.3 for a specific example.) The function $\beta(\theta)$ consists of smooth pieces taking the form $L \sec(\theta - \theta_n)$ where θ_n is the angle normal to the line segment concerned and L is the closest distance of the line segment to p^*.

Lemma 3.4.1. *There exists on any edge E of T (parametrized as $\beta(\theta) : [\theta_0, \theta_1] \to [0, \infty)$) a smooth width function $w(\theta)$ with the following properties:*

1. *this width $w(\theta)$ satisfies $0 < w(\theta) < \beta(\theta)$ for $\theta_0 < \theta < \theta_1$,*

2. *$w(\theta)$ vanishes to infinite order at the endpoints θ_0, θ_1, and*

3. *the induced normal vector field $\mathbf{N}$ is not perpendicular to the edge normal at any point of the set*

$$\{(r, \theta) : \theta_0 < \theta < \theta_1, \beta(\theta) - w(\theta) < r < \beta(\theta) + w(\theta)\} .$$

Proof. Recall the smooth increasing function h (constructed in Proposition 3.1.1) having the properties that $h(0) = 0$, $h(1) = 1$, that its derivatives vanish to all orders at these points, and that $h'(t) \leq 2$. Let

$$w_0(\theta) = A\ h'\left(\frac{\theta - \theta_0}{\theta_1 - \theta_0}\right),$$

where

$$A = \frac{1}{3}\min\left(1, \min_{[\theta_0,\theta_1]} \beta(\theta)\right);$$

this function satisfies (1) and (2).

To construct the function $w_1(\theta)$, first fix some $\theta_0 < \vartheta_0 < 1$ and choose a increasing sequence $\vartheta_0, \vartheta_1, \vartheta_2, \ldots$ converging to θ_1, as well as a decreasing sequence $\vartheta_0, \vartheta_{-1}, \vartheta_{-2}, \ldots$ converging to θ_0. This provides an increasing sequence of subintervals $[\vartheta_{-k}, \vartheta_k]$ whose union is (θ_0, θ_1). Now consider the smooth function $dx(\mathbf{N})$, the horizontal component of the induced normal vector field, on the sets

$$V_k = \{(r, \theta) : \vartheta_{-k} \leq \theta \leq \vartheta_k,\ \beta(\theta) - w_0(\theta) < r < \beta(\theta) + w_0(\theta)\},$$

for positive integers k. For each k, define Z_k to be the subset of V_k on which $dx(\mathbf{N}) \leq 0$. Now define $D_k = \min(1/2, d(E, Z_k))$, which constitutes a non-increasing sequence. Now define $w_1 : (\theta_0, \theta_1) \to \mathbb{R}^+$:

$$w_1(\theta) = \begin{cases} D_1 - \sum_{k=1}^{\infty} (D_k - D_{k+1})\ h\left(\frac{2\theta - (\vartheta_k + \vartheta_{k+1})}{\vartheta_{k+1} - \vartheta_k}\right), & \theta \geq \vartheta_0, \\ D_1 - \sum_{k=1}^{\infty} (D_k - D_{k+1})\ h\left(2\frac{\theta - \vartheta_{-k}}{\vartheta_{-k+1} - \vartheta_{-k}}\right), & \theta \leq \vartheta_0. \end{cases}$$

This is well-defined because, for any $\theta_0 < \theta < \theta_1$, the infinite sum has only a finite number of nonvanishing terms. As the sum has a finite number of nonvanishing terms, each term is a smooth function, and there is a neighbourhood of t_0 in which both sums are identical ($= D_1$), w_1 is a smooth function on $(0, 1)$.

Finally, define $w(\theta) = w_0(\theta)\, w_1(\theta)$; this is the desired width function since each of the three properties holds for any smooth function bounded above by some function satisfying it. □

Lemma 3.4.2. *Take $\mathfrak{A}$ to be the power set $\mathcal{P}V$ of some domain $V \subseteq \mathbb{R}^2$ regarded as a Boolean algebra. Let $g_1, g_2, \ldots, g_N : V \to \mathbb{R}$ be smooth functions and U be any open set in V taking the form $F(a_1, a_2, \ldots, a_N)$, where F is a Boolean function that can be expressed entirely in terms of $\cup$ and $\cap$, and the elements a_j are the superlevel sets $\left\{p \in V \middle|\ g_j(p) > 0\right\}$.*

Then there exists a smooth function $f : V \to [0, 1]$ which is non-zero on U, and is identically zero on its relative complement $V \setminus U$.

Proof. Fix a specific propositional formula for F that only uses $\cap$ and $\cup$ and define a smooth function f on V by taking

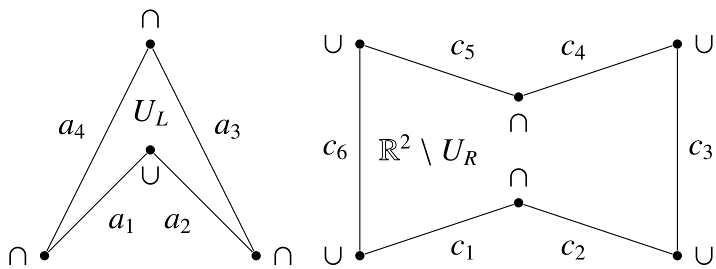
$$f = F(h(g_1), h(g_2), \ldots, h(g_N)),$$

where any term of the form $a \cap b$ (in terms of smooth functions) is understood to be ab, and any term of the form $a \cup b$ is understood to be $a + b - ab$ or $1 - (1 - a)(1 - b)$, whichever is more convenient. This satisfies the stated properties. □

Example 3.4.3. *Any open polygon can be expressed in the form just noted, whether it is bounded or unbounded. In this case, the functions g_j are taken to be the affine functions describing those lines which are incident to some side of the polygon, where the normal vectors point inside the polygon.*

The simplest examples are those in which the polygon is convex or is the complement of a convex set. Then the function $F(x_1, x_2, \ldots, x_N)$ is a conjunction $x_1 \cap x_2 \cap \cdots \cap x_N$ or disjunction $x_1 \cup x_2 \cup \cdots \cup x_N$ of its variables x_j, respectively. Otherwise, the function $F(x_1, x_2, \ldots, x_N)$ can be expressed in conjunctive normal form (a conjunction of disjunctions) or disjunctive normal form (a disjunction of conjunctions) – whichever is simpler for the chosen polygon.

Looking at a corner indicates whether it is locally of the form $a \cap b$ or $a \cup b$: if the tangents to the sides point outside of the polygon the local form is $a \cap b$; if they point inside, it is $a \cup b$. More simply, this correspond to whether the interior angle is less than π ($a \cap b$) or greater than it ($a \cup b$).



Here are two specific examples of open polygons. The one to the left is bounded (U_L is the interior) and the one to the right is unbounded (U_R is the exterior). Let $a_1, a_2, \ldots, a_4$ represent the sets $\left\{p \in \mathbb{R}^2 \middle| \, g_j(p) > 0\right\}$ for the left example, let $c_1, c_2, \ldots, c_6$ represent the sets $\left\{p \in \mathbb{R}^2 \middle| \, g_j(p) > 0\right\}$ of the right one. Then $U_L = F_{left}(a_1, a_2, a_3, a_4)$, $U_R = F_{right}(c_1, c_2, c_3, c_4, c_5, c_6)$, where

$$\begin{aligned} F_{left}(x_1, x_2, x_3, x_4) &= (x_1 \cup x_2) \cap x_3 \cap x_4, \\ F_{right}(y_1, y_2, y_3, y_4, y_5, y_6) &= (y_1 \cap y_2) \cup y_3 \cup (y_4 \cap y_5) \cup y_6. \end{aligned}$$

Proposition 3.4.4. *There exists a smooth vector field $\mathbf{V}$ on an open neighbourhood U of $T \setminus \{p^*\}$ such that:*

1. *$\mathbf{V}$ is non-vanishing on U except at vertex points of T,*
2. *$\mathbf{V}$ is parallel to $\mathbf{N}$ on the smooth part of the boundary ∂T,*
3. *the inner and outer boundaries of the set*

$$U_T = \left\{ p \in U \middle| \, \mathbf{V}_p \text{ is not parallel to } \left.\frac{\partial}{\partial r}\right|_p \right\}$$

   are tangent to ∂T at any of its incident vertices,

4. *$\mathbf{V}$ is radial sufficiently far from a boundary or sufficiently close to a vertex.*

Proof. Use Lemma 3.4.1 on each edge of T and join the resulting width functions to obtain a piecewise smooth function $w(\theta)$ defined on $[0, 2\pi]$.

First, construct an open cover of $\mathbb{R}^2 \setminus \{p^*, v_1, v_2, v_3\}$, where v_j are the vertices of T, as follows: build a domain surrounding T

$$U_T = \left\{(r,\theta) \middle| \ 1 - w(\theta) < \frac{r}{\beta(\theta)} < 1 + w(\theta)\right\},$$

an open set inside T enclosing p^*

$$U_{p^*} = \left\{(r,\theta) \middle| \ 0 < \frac{r}{\beta(\theta)} < 1 - \frac{w(\theta)}{2}\right\},$$

and an exterior domain outside T

$$U_\infty = \left\{(r,\theta) \middle| \ \frac{r}{\beta(\theta)} > 1 + \frac{w(\theta)}{2}\right\}.$$

Note that the radial width of U_T goes to 0 to infinite order as the distance from a vertex goes to 0; in particular, its inner and outer boundaries are tangent to their incident sides at any vertex, showing property (3).

In order to build smooth functions $f_T, f_{p^*}, f_\infty : \mathbb{R}^2 \to [0,1]$ which will be nonzero only on the regions U_T, U_{p^*}, and U_∞, respectively, consider the boundaries of these regions. Each boundary curve takes the general form

$$\gamma = \left\{(r,\theta) \middle| \ \frac{r}{\beta(\theta)} = 1 + C\, w(\theta)\right\} \tag{3.8}$$

where $C \in \{\pm 1, \pm 1/2\}$; it is piecewise smooth since $\beta(\theta)$ is piecewise smooth and $w(\theta)$ is smooth. The smooth segment of γ joining the vertices u and v is a segment of the smooth curve

$$\Gamma_{uv} = \left\{(r,\theta) \middle| \ \theta_n - \frac{\pi}{2} < \theta < \theta_n + \frac{\pi}{2},\ \frac{r}{\widehat{\beta_{uv}}(\theta)} = 1 + C\, \widehat{w_{uv}}(\theta)\right\}$$

where $\widehat{\beta_{uv}}$ is the smooth function $L \sec(\theta - \theta_n)$ whose restriction to $[\theta_u, \theta_v]$ is $\beta(\theta)|_{[\theta_u,\theta_v]}$, $\widehat{w_{uv}}$ is defined to be $w|_{[\theta_u,\theta_v]}$ on $[\theta_u, \theta_v]$ and to be 0 otherwise. This smooth curve is the zero set of the function $g_{uv} : \mathbb{R}^2 \setminus \{p^*\} \to \mathbb{R}$ defined as

$$g_{uv}(r,\theta) = r\cos(\theta - \theta_n) - L\left(1 + C\widehat{w_{uv}}(\theta)\right).$$

To use Lemma 3.4.2, take $V = \mathbb{R}^2 \setminus \{p^*\}$, define some elements of $\mathfrak{A}$: in the context of a fixed $C > 0$, define

$$a_{jk} = \left\{p \in V \middle| \ g_{v_j v_k}(p) > 0 \text{ using } C \text{ for the function}\right\}$$

for the area above an inner smooth curve $\Gamma_{v_j v_k}$,

$$b_{jk} = \left\{p \in V \middle| \ -g_{v_j v_k}(p) > 0 \text{ using } -C \text{ for the function}\right\}$$

for the area below an outer smooth curve. The corresponding variables to place these in will be denoted as x_{jk} and y_{jk}.

The open set U_T has 3 connected components nested between its inner and outer boundaries. Taking $C = 1$, define

$$F_T\left(x_{12}, y_{12}, x_{23}, y_{23}, x_{31}, y_{31}\right) = \left(x_{12} \cap y_{12}\right) \cup \left(x_{23} \cap y_{23}\right) \cup \left(x_{31} \cap y_{31}\right),$$

observe that $F_T\left(a_{12}, b_{12}, a_{23}, b_{23}, a_{31}, b_{31}\right) = U_T$, and apply the lemma to obtain the required function f_T.

For the open sets U_∞ and U_{p^*} there is one boundary curve of form (3.8) where $C = 1/2, -1/2$, respectively. Fixing $C = 1/2$, take

$$F_{U_\infty}\left(x_{12}, x_{23}, x_{31}\right) = x_{12} \cup x_{23} \cup x_{31}$$

and

$$F_{U_{p^*}}\left(y_{12}, y_{23}, y_{31}\right) = y_{12} \cap y_{23} \cap y_{31}$$

so that $F_{U_\infty}\left(a_{12}, a_{23}, a_{31}\right) = U_\infty$ and $F_{U_{p^*}}\left(b_{12}, b_{23}, b_{31}\right) = U_{p^*}$, and apply the lemma to F_{U_∞} and $F_{U_{p^*}}$. The required function f_{p^*} for U_{p^*} can be obtained as the product of the function provided by the lemma with the function $h(r)$; the other functions f_T, f_∞ smoothly extend to equal 0 at p^*, as they are identically zero in a neighbourhood of it in the first place.

Next, use these functions to define the vector field

$$V_p = \left(f_{p^*} + f_\infty\right) \frac{\partial}{\partial r}\bigg|_p + f_T \frac{N_p}{\left\|N_p\right\|},$$

where the last term is understood to be $\mathbf{0}$ whenever f_T vanishes. By construction, the vectors N_p and $\frac{\partial}{\partial \rho}$ are not parallel in the domain U_T. This means that V_p can only vanish if each function f_{p^*}, f_∞, and f_T vanishes. This only occurs at the vertex points v_1, v_2, v_3.

Finally, since properties (1) and (3) call for a neighbourhood of T which contains U_T, define

$$U = \{(r, \theta) |\ 0 < r < 2B\},$$

where $B = \max_\theta \beta(\theta)$. □

Consequently, $\mathbf{V}$ defines a line foliation on U. We will treat a vertex point and the leaves approaching it differently from the others: we glue them all together and call their union a **vertex leaf**. As a consequence of the previously established properties of $\mathbf{V}$ we also get:

Corollary 3.4.5. *There exists a smooth foliation $\mathcal{F}$ on $U \supset T \setminus \{p^*\}$ such that:*

- *$\mathcal{F}$ is tangent to the induced normal field $\mathbf{N}$ on the smooth part of the boundary of T,*
- *any vertex leaf is a radial line segment,*
- *any leaf is radial when it is far from the boundary.*

Corollary 3.4.6. *There exists a smooth coordinate system $(\mathfrak{b}, \hat{\theta})$ on U where the leaf angle $\hat{\theta} : U \to S^1$ carries a point p onto the direction the base of its leaf makes and the length function $\mathfrak{b}(p) : U \to \mathbb{R}^+$ gives the length along the leaf from p^* to p. This coordinate system has the property that it agrees with (r, θ) until the point is near a boundary or past it. On a vertex leaf, these coordinate systems agree all the way.*

Proof. Let $B = \max_\theta \beta(\theta)$ and take R small enough so that V_p is radial for $r < R$. Define a vector field $\mathbf{W}$ on $\mathbb{R}^2$ as

$$W_p = h\left(\frac{3B-r}{B}\right) f_\infty \left.\frac{\partial}{\partial r}\right|_p$$

when $r \geq 2B$, $W_p = V_p$ for $R \leq r \leq 2B$, and

$$W_p = h\left(\frac{r-R}{R}\right) f_{p^*} \left.\frac{\partial}{\partial r}\right|_p$$

when $r \leq R$. This vector field is compactly supported on $\mathbb{R}^2$, so it admits a global flow $\Phi : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ by [Con08, Thm. 4.1.11, p. 133].

For $r \leq R$, define $\hat{\theta}(p)$ and $\mathfrak{b}(p)$ to be $\theta(p)$ and $r(p)$, respectively. Otherwise, explicitly define $\hat{\theta}(p)$

$$\hat{\theta}(p) = \lim_{t \to -\infty} \theta(\Phi(t, p))$$

and implicitly define $\mathfrak{b}(p)$ via the following pair of equations:

$$\mathfrak{b}(p) = R + \int_{t=0}^{u} \left(\Phi(t, \sigma(\hat{\theta}))\right)^* ds,$$

$$\Phi(u, \sigma(\hat{\theta})) = p,$$

where $\sigma(\hat{\theta})$ parametrizes the circle of radius R. □

We now use $(\mathfrak{b}, \hat{\theta})$ to build a pair of parameters $(s, \hat{\theta})$ which is more convenient for building the functions f_k.

Proposition 3.4.7. *Denote as v_0, v_1, v_2 the vertices of the triangle T. There exists a pair of smooth parameters $(s, \hat{\theta})$ on $U \setminus \{v_0, v_1, v_2\}$ with $\hat{\theta} : U \to S^1$ and $s : U \to \mathbb{R}^+$ such that:*

1. *the pair $(s, \hat{\theta})$ is a smooth coordinate system on int$(T) \setminus \{p^*\}$;*

2. *s extends continuously to equal 0 at p^* and to equal 1 at any of the vertices;*

3. *s is identically 1 on the smooth part of ∂T;*

4. *the function s takes the form $s = r/\alpha(\theta)$ for some smooth $\alpha(\theta)$ in some neighbourhood of the origin p^*;*

5. *the growth rate of a partial derivative*

$$\frac{\partial^N s}{\partial^{N_1} \mathfrak{b}\, \partial^{N_2} \hat{\theta}} \left(\mathfrak{b}, \hat{\theta}\right)$$

in $\left(\left(\ell_1(\hat{\theta}) - \mathfrak{b}\right)\left(\ell_2(\hat{\theta}) - \mathfrak{b}\right)\left(\ell_3(\hat{\theta}) - \mathfrak{b}\right)\right)^{-1}$ as a nearby vertex is approached (from inside) is $N - 1$, where $\ell_k(\hat{\theta})$ (for $k = 1, 2, 3$) describe the sides of T in terms of $(\mathfrak{b}, \hat{\theta})$; and,

6. *for each vertex v, the product*

$$(1-s)^{2N-2}\,\frac{\partial^N s}{\partial^{N_1}\mathfrak{b}\,\partial^{N_2}\hat{\theta}}\left(\mathfrak{b},\hat{\theta}\right)$$

is bounded on some open subset U of int(T) having $v \in \mathrm{cl}\,(U)$.

Proof. In the coordinate system $(\mathfrak{b},\hat{\theta})$ any piecewise smooth curve in U which is transverse to the foliation at each point and intersects no leaf more than once takes the form

$$\left\{p \in U, \hat{\theta}(p) \in E : \mathfrak{b}(p) = g(\hat{\theta}(p))\right\},$$

where the function $g(\hat{\theta}) : E \to \mathbb{R}^+$ is piecewise smooth and $E \subset S^1$. For the boundary ∂T, take $E = S^1$ and define the corresponding function $\ell(\hat{\theta})$. For $k = 1, 2, 3$, define functions $\ell_k(\hat{\theta}) : S^1 \to \mathbb{R}^+$ defining curves containing the smooth sides of ∂T. Define a function $\alpha : U \to \mathbb{R}^+$:

$$\alpha(\mathfrak{b},\hat{\theta}) = m\left(\ell_1(\hat{\theta}), \ell_2(\hat{\theta}), \ell_3(\hat{\theta}); \chi\left(\mathfrak{b},\hat{\theta}\right)\right),$$

where the function $m(a, b, c; \epsilon)$ is defined in Theorem 3.1.3, and the function $\chi(\mathfrak{b},\hat{\theta})$ is defined to be

$$\chi(\mathfrak{b},\hat{\theta}) = \delta\left(2h\left(\frac{1}{2} + C(\ell_1(\hat{\theta}) - \mathfrak{b})(\ell_2(\hat{\theta}) - \mathfrak{b})(\ell_3(\hat{\theta}) - \mathfrak{b})\right) - 1\right)$$

(taking some $C > 0$ such that χ is constant on some neighbourhood of p^*, and $\delta > 0$ small enough that $m\left(\ell_1(\hat{\theta}), \ell_2(\hat{\theta}), \ell_3(\hat{\theta}); \chi\left(\mathfrak{b},\hat{\theta}\right)\right) = m\left(\ell_{k_1}(\hat{\theta}), \ell_{k_2}(\hat{\theta}); \chi\left(\mathfrak{b},\hat{\theta}\right)\right)$ for some distinct indices $k_1, k_2 \in \{1, 2, 3\}$ in some neighbourhood of $\hat{\theta}(v)$, for any vertex v). The function α is well-defined and smooth except at vertices.

We now define a function $s : U \to \mathbb{R}^+$ which will be a parameter along each leaf by taking $s(p) = \mathfrak{b}(p)/\alpha(p)$. To show that s is injective, it suffices to show that the restriction of s to any leaf $\mathcal{L}$ is strictly monotonic in $\mathfrak{b}$. Consider

$$\frac{\partial s}{\partial \mathfrak{b}} = \frac{1}{m\left(L_1, L_2, L_3; \chi\left(\mathfrak{b},\hat{\theta}_0\right)\right)^2}\left(m\left(L_1, L_2, L_3; \chi\left(\mathfrak{b},\hat{\theta}_0\right)\right) - \mathfrak{b}\,\frac{\partial m}{\partial \epsilon}\left(L_1, L_2, L_3; \chi\left(\mathfrak{b},\hat{\theta}_0\right)\right)\frac{\partial \chi}{\partial \mathfrak{b}}\left(\mathfrak{b},\hat{\theta}_0\right)\right) > 0,$$

since $\frac{\partial \chi}{\partial \mathfrak{b}} \leq 0$ within int(T): this shows that $s(\mathfrak{b})$ is strictly increasing for $\mathfrak{b} < L$. Therefore, $s|_{\mathcal{L}}$ is strictly monotonic on int(T).

That s has 0 as its infimum is a consequence of the fact that m gives overestimates of the minimum, and the fact that $\mathfrak{b}(p) \to 0^+$ as $p \to p^*$ for any leaf $\mathcal{L}$; consequently, $s(p)$ extends continuously to 0 at p^*. For a vertex v, on the other hand, the function $m(a, b; \epsilon)$ tends to $\min(a, b)$, the length of the vertex leaf containing p; consequently, $s(p)$ extends continuously to equal 1 at this point. This shows (2).

Now, consider a point $p \in \partial T$. If p is not a vertex point, then we can consider a neighbourhood in which $\alpha(p) = \mathfrak{b}(p)$, hence $s(p) = 1$. This shows (3).

Consider a neighbourhood of p^* which does not intersect U_T, on which $\chi(\mathfrak{b},\hat{\theta})$ is identically equal to δ, and for which $f_{p^*} = h(r)$. In this case, $\mathfrak{b}(p) = r(p)$ and $\hat{\theta}(p) = \theta(p)$. The function $\alpha(p)$ then is, in effect, a function of $\theta(p)$ only, so we consider $\alpha(p) = \alpha(\theta)$ in this case. Hence,

$$s(p) = \frac{r}{\alpha(\theta(p))},$$

showing (4).

To show (5), first choose a small neighbourhood $(\hat{\theta}_v - \epsilon, \hat{\theta}_v + \epsilon)$ of some $\hat{\theta}_v$, where v is a vertex of T, such that one of the $\ell_j(\hat{\theta})$ arguments of the function m drops out by Theorem 3.1.3 (5). Without loss of generality,

$$\alpha(\mathfrak{b}, \hat{\theta}) = m\left(\ell_1(\hat{\theta}), \ell_2(\hat{\theta}); \chi(\mathfrak{b}, \hat{\theta})\right).$$

The only factors in the terms of its partial derivatives to be concerned with is the derivatives of m. Using Theorem 3.1.3 (7), such a term has a growth rate in $1/\chi$ of at most $N-1$, where N is the total order of the partial derivative. (We can only get lower orders if some of the differentiation is carried out on factors other than the function m and its partial derivatives.) The growth rate of a partial derivative of the function s in $1/\chi$ has the same bounds. Now consider the restricted domain $[1-u_0, u_0]$ where $1/2 < u_0 < 1$ and do a comparison:

$$\frac{1}{h\left(\frac{1}{2} + Ct\right) - \frac{1}{2}} = \frac{1}{\int_{u=1/2}^{1/2+Ct} h'(u)\, du} \le \frac{1}{Ct\, h'(u_0)} = \frac{C'}{t},$$

whenever $0 < t < (u_0 - 1/2)/C$, where $C' = 1/(C\, h'(u_0))$. So when we consider a sector

$$V = \left\{ \left(\mathfrak{b}, \hat{\theta}\right) \middle| \, \left|\hat{\theta} - \hat{\theta}_v\right| < \epsilon,\ 0 < \left(\left(\ell_1(\hat{\theta}) - \mathfrak{b}\right)\left(\ell_2(\hat{\theta}) - \mathfrak{b}\right)\left(\ell_3(\hat{\theta}) - \mathfrak{b}\right)\right) < \frac{2u_0 - 1}{2C} \right\}$$

approaching the vertex, we find the growth rate of a partial derivative of s in

$$\left(\left(\ell_1(\hat{\theta}) - \mathfrak{b}\right)\left(\ell_2(\hat{\theta}) - \mathfrak{b}\right)\left(\ell_3(\hat{\theta}) - \mathfrak{b}\right)\right)^{-1}$$

to be bounded above by $N-1$ within this sector V, as required.

To show (6), consider the sector V just constructed and note the inequality

$$\frac{(1-s)^{2N-2}}{\left(\ell_1(\hat{\theta}) - \mathfrak{b}\right)^{2N-2}\left(\ell_2(\hat{\theta}) - \mathfrak{b}\right)^{2N-2}\left(\ell_3(\hat{\theta}) - \mathfrak{b}\right)^{2N-2}} \le C_2^{2N-2} \frac{(1-s)^{2N-2}}{\left(\ell_1(\hat{\theta}) - \mathfrak{b}\right)^{2N-2}\left(\ell_2(\hat{\theta}) - \mathfrak{b}\right)^{2N-2}},$$

where $1/C_2$ is the minimum value of $\ell_3(\hat{\theta}) - \mathfrak{b}$ on the sector. It will suffice to demonstrate boundedness of the rational functions

$$\frac{m\left(\ell_1(\hat{\theta}), \ell_2(\hat{\theta}); \chi(\mathfrak{b}, \hat{\theta})\right) - \mathfrak{b}}{\ell_j(\hat{\theta}) - \mathfrak{b}}$$

for $j = 1, 2$ in V. Without loss of generality, $\ell_1(\hat{\theta}) \le \ell_2(\hat{\theta})$ and it will suffice to show the bound for $j = 1$. An application of Theorem 3.1.3 (2) and the property $h(t + 1/2) \le t + 1/2$ for $t \ge 0$ gives

$$\frac{m\left(\ell_1(\hat{\theta}), \ell_2(\hat{\theta}); \chi(\mathfrak{b}, \hat{\theta})\right) - \mathfrak{b}}{\ell_1(\hat{\theta}) - \mathfrak{b}} \le \frac{\ell_1(\hat{\theta}) - \mathfrak{b} + \chi\left(\mathfrak{b}, \hat{\theta}\right)}{\ell_1(\hat{\theta}) - \mathfrak{b}} \le 1 + 2\delta C \frac{\left(\ell_1(\hat{\theta}) - \mathfrak{b}\right)\left(\ell_2(\hat{\theta}) - \mathfrak{b}\right)\left(\ell_3(\hat{\theta}) - \mathfrak{b}\right)}{\ell_1(\hat{\theta}) - \mathfrak{b}},$$

hence,

$$\frac{m\left(\ell_1(\hat{\theta}), \ell_2(\hat{\theta}); \chi(\mathfrak{b}, \hat{\theta})\right) - \mathfrak{b}}{\ell_1(\hat{\theta}) - \mathfrak{b}} \le 1 + 2\delta C \max_{\hat{\theta}_v - \epsilon \le \hat{\phi} \le \hat{\theta}_v + \epsilon} \left(\ell_2(\hat{\phi}) - \mathfrak{b}\right)\left(\ell_3(\hat{\phi}) - \mathfrak{b}\right).$$

□

3.5 Construction of f_j on a triangle T of $\mathcal{K}$

Assign each vertex $v \in \mathcal{K}$ a unique positive real number c_v, each edge $uv \in \mathcal{K}$ a unique unitary complex number c_{uv}, and each triangle $T \in \mathcal{K}$ a unique unitary complex number c_T.

Fix an edge uv of $\mathcal{K}$ and define functions $f_0, f_1 : uv \to \mathbb{C}$ to be

$$f_0(p) = c_u h\left(\frac{|p-v|}{|u-v|}\right) + c_v h\left(\frac{|p-u|}{|v-u|}\right),$$

$$f_1(p) = \frac{1}{2} c_{uv} h'\left(\frac{|p-u|}{|v-u|}\right).$$

These functions are well-defined because they are invariant under an exchange of the vertices u,v. Therefore, they extend to be functions $f_0, f_1 : \mathcal{K}^1 \to \mathbb{C}$ on the 1-skeleton of $\mathcal{K}$.

Next consider an arbitrary T in $\mathcal{K}$ and let $(\mathfrak{b}, \hat{\theta})$ and $(s, \hat{\theta})$ be the corresponding coordinate systems created in the previous section from a foliation on T. Let $\gamma(\hat{\theta})$ be the piecewise smooth path ∂T, which in terms of the coordinates $(\mathfrak{b}, \hat{\theta})$ takes the form $(\ell(\hat{\theta}), \hat{\theta})$. Extend the functions f_0, f_1 to its interior as

$$f_j(p) = f_j\Big(\gamma(\hat{\theta}(p))\Big)\, h\Big(s(p)\Big),$$

interpreting this expression as 0 at the centre p^*. The first factor is smooth when $\hat{\theta} = \hat{\theta}_0$ corresponds to a vertex leaf because the derivatives of $\ell(\hat{\theta})$ are bounded near $\hat{\theta}_0$ and f_j vanishes to infinite order at such points. These extensions are smooth at p^*, since the derivatives of $f_j(\gamma(\hat{\theta}))$ are bounded and $h(s)$ vanishes to infinite order at $s = 0$.

Define an additional function $f_2 : T \to \mathbb{C}$ to be

$$f_2(p) = c_T\Big(1 - h\Big(s(p)\Big)\Big)$$

on the interior of T, and to be equal to 0 on its boundary ∂T. Like the extensions, it is smooth throughout the interior of T.

3.6 Smoothness of f_j across triangles

In the previous section, functions f_0, f_1, and f_2 were defined on each triangle T of the triangulation $\mathcal{K}$. They can be used to define smooth functions (also denoted as f_j, for $j = 0, 1, 2$) on the surface S. Define

$$f_j : S \to \mathbb{C}$$

$$f_j(p) = f_j(\phi^{-1}(p))$$

whenever $\phi(p) \in T$, and $f_j : T \to \mathbb{C}$ is the function constructed on T in the previous section. The function $f_2 : S \to \mathbb{C}$ is completely flat along the image of the 1-skeleton of $\mathcal{K}$, since $f_2 : T \to \mathbb{C}$ is completely flat along ∂T whenever $T \in \mathcal{K}$. Consequently, it is smooth.

This section will show that a function $f : T \to \mathbb{C}$ which is a product $g(\hat{\theta}(p))\, h(s(p))$, where $g(\hat{\theta})$ and $\mathfrak{h}(s)$ are smooth, $\mathfrak{h}(s)$ is $\left\langle \frac{\partial}{\partial s} \right\rangle_1$-flat, and $g(\hat{\theta})$ is flat at each angle $\hat{\theta}(v)$ corresponding to a vertex leaf, is completely flat at each vertex $v \in T$, and is $\langle \mathbf{V} \rangle_p$-flat at each smooth point of the boundary. (Either of the functions f_0, f_1 constructed earlier in this chapter is of this form.)

Proposition 3.6.1. *A function $f(p)$ of the previously mentioned form is $\langle \mathbf{V} \rangle_p$-flat at any smooth point p of the boundary.*

Proof. Let p_0 be any point in the smooth part of the boundary ∂T, and U be an open neighbourhood of p_0 which only intersects the smooth part of the boundary ∂T. Consider the algebra of functions on U generated by the derivatives of $h(s(p))$ and by each function $V^k s(p)$ for each $k \in \{1, 2, 3, \ldots\}$:

$$\mathcal{F} = \left\langle h' \circ s(p), \ldots, h^{(k)} \circ s(p), \ldots \right\rangle \left\langle \mathbf{V}(s(p)), \mathbf{V}^2(s(p)), \ldots, \mathbf{V}^k(s(p)), \ldots \right\rangle.$$

This algebra $\mathcal{F}$ is closed under the derivation $\mathbf{V}$, since $\mathbf{V}\left(\mathbf{V}^k\left(s(p)\right)\right) = \mathbf{V}^{k+1}\left(s(p)\right)$ and $\mathbf{V}\left(h^{(k)} \circ s(p)\right) = h^{(k+1)} \circ s(p)\ \mathbf{V}\left(s(p)\right)$. Additionally, any function in $\mathcal{F}$ vanishes on $U \cap \partial T$ since $h^{(k)} \circ s$ does. Combined with the fact that $\mathbf{V}\left(h \circ s\right) = (h' \circ s)\mathbf{V}(s)$, it follows that $h \circ s$ is $\langle V \rangle_q$-flat for any $q \in U \cap \partial T$. To conclude that $f(p)$ is $\langle \mathbf{V} \rangle_p$-flat along $U \cap \partial T$, simply note that

$$\mathbf{V}\left(g(\hat{\theta}(p))\right) = g'(\hat{\theta}(p))\, \mathbf{V}(\hat{\theta}(p)) = 0;$$

consequently, $g(\hat{\theta}(p))$ is a constant as far as $\mathbf{V}$ is concerned. Therefore, $f(p) = g(\hat{\theta}(p))\, h(s(p))$ is $\langle V \rangle_q$-flat for any $q \in U \cap \partial T$. However, U was an arbirary neighbourhood of an arbitrary point contained in the smooth part of the boundary, so $f(p)$ is $\langle V \rangle_q$-flat at any point q of the smooth part of the boundary. □

Up to this point, we have shown that functions taking the same form as f_k on a triangle T of $\mathcal{K}$ are smooth along the smooth part of its boundary. Next, we show smoothness at vertices.

Proposition 3.6.2. *Let f be a function on the surface S such that the restriction of $f \circ \phi$ to any face $T \in \mathcal{K}$ takes the form*

$$\sum_{k=1}^{n} g_k(\hat{\theta}(p))\, \mathfrak{h}_k(s(p)) \tag{3.9}$$

where $g_k(\hat{\theta})$, $\mathfrak{h}_k(s)$ are smooth functions such that g_k is $\left\langle \frac{\partial}{\partial \hat{\theta}} \right\rangle_{\hat{\theta}(v)}$-flat (whenever v is a vertex of T) and $\mathfrak{h}_k(s)$ is $\left\langle \frac{\partial}{\partial s} \right\rangle_1$-flat. Then f is completely flat at each vertex $\hat{v}$ of S. In particular, f is smooth at each vertex.

Proof. Pick some arbitrary vertex v and a triangle T containing it. This result will be proved by induction on order n of the partial derivatives. First, consider the base case $n = 1$. The first order partial derivatives with respect to the coordinate system $(\mathfrak{b}, \hat{\theta})$ are

$$\begin{aligned} \frac{\partial}{\partial \mathfrak{b}} g(\hat{\theta})\mathfrak{h}(s) &= g(\hat{\theta})\, \mathfrak{h}'(s)\, \frac{\partial s}{\partial \mathfrak{b}}, \\ \frac{\partial}{\partial \hat{\theta}} g(\hat{\theta})\mathfrak{h}(s) &= g'(\hat{\theta})\mathfrak{h}(s) + g(\hat{\theta})\mathfrak{h}'(s) \frac{\partial s}{\partial \hat{\theta}}. \end{aligned}$$

Any term with a derivative $g^{(k)}(\hat{\theta})$ vanishes as $\hat{\theta} \to \hat{\theta}(v)$ and any term with a derivative of the form $h^{(l)}(s)$ vanishes as $s \to 1^-$. As the gradient $\nabla f(\mathfrak{b}, \hat{\theta})$ exists and tends to 0 as the vertex is approached within T, each single-sided directional derivative from inside T exists and is equal to 0. This will be true of the other triangles as well, so when we pull back to the surface, the function f has vanishing directional derivatives each way.

Next, consider the inductive step with order n, where each term of a derivative

$$\frac{\partial^n f}{\partial \mathfrak{b}^{n_1}\, \partial \hat{\theta}^{n_2}} \left(\mathfrak{b}, \hat{\theta}\right)$$

is a product of the factors $g^{(k)}(\hat{\theta})$ and $\mathfrak{h}^{(l)}(s)$, where k, l are non-negative integers such that $k + l \geq 1$, along with some (possibly empty) factors of the form

$$\frac{\partial^{k+l} s}{\partial \mathfrak{b}^{k}\, \partial \hat{\theta}^{l}} \left(\mathfrak{b}, \hat{\theta}\right),$$

where $k, l \in \mathbb{Z}^+$ and $k + l \geq 1$. It is clear that terms of the partial derivatives of these terms will take the same general form.

By Proposition 3.4.7 (6), the poles of the derivatives of s can be cancelled out using some power of $1 - s$, and we can factors any power of $1 - s$ out of $\mathfrak{h}^{(l)}(s)$ and it will still have the required properties. Any term has at least one of the derivatives $g^{(k)}(\hat{\theta})$ which vanish as $\hat{\theta} \to \hat{\theta}(v)$, or the derivatives $\mathfrak{h}^{(l)}(s)$ which vanish as $s \to 1^-$. Again, each single-sided derivative from inside T of any n-th order partial derivative exists and is equal to 0. So the $n + 1$-th order derivatives exist (and vanish) at $\hat{v}$ for the pull back.

By induction, the pull back function is completely flat at the vertex v. As this was arbitrary, the pull back function is completely flat at every vertex point, hence, smooth at every vertex point. □

Note that f_0, f_1, f_2 satisfy the conditions of this corollary, so are smooth at every vertex. They have previously been shown to be smooth everywhere else, so this completes the proof of their smoothness.

3.7 The functions f_0, f_1, f_2 generate $C(S)$

Proposition 3.7.1. *The functions f_0, f_1, f_2 are members of the uniform algebra $C(S)$ and they separate points, so they generate a uniform algebra. In addition, $f_0|_{X^1}$ and $f_1|_{X^1}$ separate the points of X^1, and $f_0|_{X^0}$ separates the points of X^0.*

Proof. First of all, the function f_0 separates points of X^0 because each vertex v has a different constant c_v associated to it.

Next we show that f_0, f_1 separate $X^1 \setminus X^0$. The argument of f_1 is unique to the edge so it suffices to show that f_0 separates the points of the edge, which follows from its monotonicity.

Finally, we show that f_0, f_1, f_2 separate $X^2 \setminus X^1$. Assume that for $p, p' \in \mathcal{K}$ that $f_0(p) = f_0(p')$, $f_1(p) = f_1(p')$, and $f_2(p) = f_2(p')$. Let the primed quantities represent those for p', while the unprimed quantities are those for p. Note that (for $j = 0, 1$)

$$\hat{f}_j(\hat{\theta})\, h(s) = \hat{f}_j(\hat{\theta}')\, h(s'),$$

$$c_T(1 - h(s)) = c_{T'}(1 - h(s')).$$

By hypothesis, $0 \le s, s' < 1$, as otherwise at least one of the points would be in X^1. The arguments of c_T and $c_{T'}$ must be the same, so p, p' belong to the same triangle T. Also, $h(s) = h(s')$, so $s = s'$ as well. For $s = 0$, p and p' must equal p^*. Otherwise,

$$\hat{f}_j(\hat{\theta}) = \hat{f}_j(\hat{\theta}'),$$

for $j = 0, 1$. The arguments of $\hat{f}_1(\hat{\theta})$ and $\hat{f}_1(\hat{\theta}')$ match, which means that the leaves of p and p' intersect the same edge. So

$$f_j(\gamma(\hat{\theta})) = f_j(\gamma(\hat{\theta}')),$$

(for $j = 0, 1$). The equality of $\hat{\theta}$ and $\hat{\theta}$ follows from the monotonicity of f_j along the edge and the injectivity of γ. Therefore the points p and p', having equal coordinates, are equal.
Consequently, f_0, f_1, f_2 separate $X^2 \setminus X^1$. □

Lemma 3.7.2. *Let X be a finite discrete set. Then any injective function $f \in C(X)$ generates it.*

Lemma 3.7.3. *Let X be a compact topological space. If the functions $f_1, f_2, \ldots, f_n, g \in C(X)$ separate X, and the image of g is nowhere dense and polynomially convex, then the uniform algebra*

$$\mathcal{A} = \langle f_1, f_2, \ldots, f_n, g \rangle = C(X)$$

if and only if the restriction algebra

$$\left\langle f_1|_E\,, f_2|_E\,, \ldots, f_n|_E \right\rangle = C(E)$$

whenever E is a non-empty level set of g.

Proof. The forward implication follows from the fact that the restriction of $C(X)$ to a compact subset E is $C(E)$.

Any point z_0 of $g(X)$ admits a non-negative peak function $g_{z_0} \in C(g(X))$: consider

$$g_{z_0}(z) = 1 - \frac{|z - z_0|}{\max_{w \in g(X)} |w - z_0|}.$$

By Lavrentiev's Theorem (2.7.4), $C(g(X)) = P(g(X))$; hence, $g_{z_0} \circ g \in \langle g \rangle \subset \mathcal{A}$. The level set $\{p \in X | \, g(p) = z_0\}$ is identically to the level set $\{p \in X \big| \, (g_{z_0} \circ g)\,(p) = 1\}$, showing that it is a peak set for $\mathcal{A}$ having a non-negative peak function $g_{z_0} \circ g$.

Let A be an arbitrary maximal antisymmetric set for $\mathcal{A}$. Pick some level set $E = \{g(z) = w_0\}$ that it intersects. By definition (see 2.7.11), the restriction of the function $g_{w_0} \circ g$ is constant on A, so $A \subseteq E$. The level set E is a peak set, so the restriction $\mathcal{A}|_E$ is a uniform algebra. By Proposition 2.7.16, it has A as an antisymmetric set.

If each restriction algebra $\mathcal{A}|_E = C(E)$, then every maximal antisymmetric set of $\mathcal{A}$ is a singleton, so $\mathcal{A} = C(X)$ by Corollary 2.7.20 of the Bishop-Shilov Theorem. The backward implication is now shown. □

Example 3.7.4. *The functions f_1 and f_2 have images in $\mathbb{C}$ taking the form of a union of radial line segments joined at the origin (see Figure 3.4); their images are nowhere dense and have connected complements.*

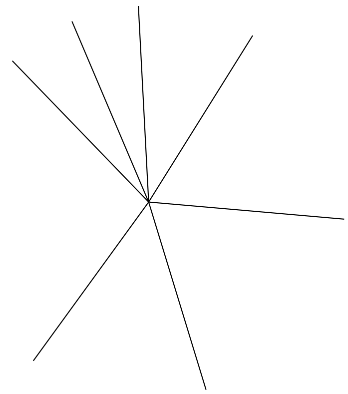


Figure 3.4: The form taken in $\mathbb{C}$ by the image of f_1 or f_2.

Lemma 3.7.5. *Let X be either the 1-skeleton X^1 or boundary ∂T of some triangle in $\mathcal{K}$. Then the functions $f_0|_X$, $f_1|_X$ generate $C(X)$.*

Proof. Consider any non-empty level set $E = \{p \in X | f_1(p) = k\}$ of $f_1|_X$. If $k = 0$ then $E = X^0 \cap X$. As f_0 takes on distinct values on X^0, $f_0|_E$ is certainly injective in this case. For the case $|k| = 1$, E is a singleton, so there is nothing to show then. Finally, for the case $0 < |k| < 1$, focus on the edge singled out by the phase of k, and notice that (in terms of a parameter along the edge) $E = \{t, 1 - t\}$ for some $0 < t < 1/2$. (See Figure 3.5 to see what the functions $h(t)$, $h'(t)/2$ look like side by side. The second function is, of course, $|f_1|$.) The function f_0 is monotonic along the edge, so it is injective on E also.

In each case, f_0 is injective on the level set E, so $C(E) = \langle f_0|_E \rangle$ by Lemma 3.7.2. As the level set E is arbitrary, an appeal to Lemma 3.7.3 yields the result $C(X) = \langle f_0|_X , f_1|_X \rangle$. □

Theorem 3.7.6. *The functions f_0, f_1, f_2 generate $C(S)$.*

Proof. Let $E = \left\{p \in X^2 \middle| f_2(p) = k\right\}$ be an arbitrary non-empty level set of f_2. For $|k| = 1$, E is a singleton. If $k = 0$, then $E = X^1$ and Lemma 3.7.5 tells us that $\mathcal{A}|_E = C(E)$. Finally, if $0 < |k| < 1$, then E is a level set $\{s = s_0\}$ of the function s inside the triangle T. So

$$\mathcal{A}|_E = \langle f_0|_E , f_1|_E \rangle = \left\langle C\hat{f}_0(\hat{\theta}), C\hat{f}_1(\hat{\theta}) \right\rangle = \left\langle \hat{f}_0(\hat{\theta}), \hat{f}_1(\hat{\theta}) \right\rangle ,$$

where $C = h(s_0)$, so $\mathcal{A}|_E$ is isomorphic to $\mathcal{A}|_{\partial T}$, which is known to equal $C(\partial T)$ by Lemma 3.7.5. Consequently, $\mathcal{A}|_E = C(E)$.

As the level set E was arbitrary, an appeal to Lemma 3.7.3 gives

$$\mathcal{A} = \langle f_0, f_1, f_2 \rangle = C(X^2).$$

Now X^2 is homeomorphic to S via ϕ^{-1}, hence,

$$\left\langle f_0 \circ \phi^{-1}, f_1 \circ \phi^{-1}, f_2 \circ \phi^{-1} \right\rangle = C(S).$$

□

In combination with the fact that S can have no fewer than 3 continuous generators, this gives us the desired conclusion that the polynomial density of S is 3.

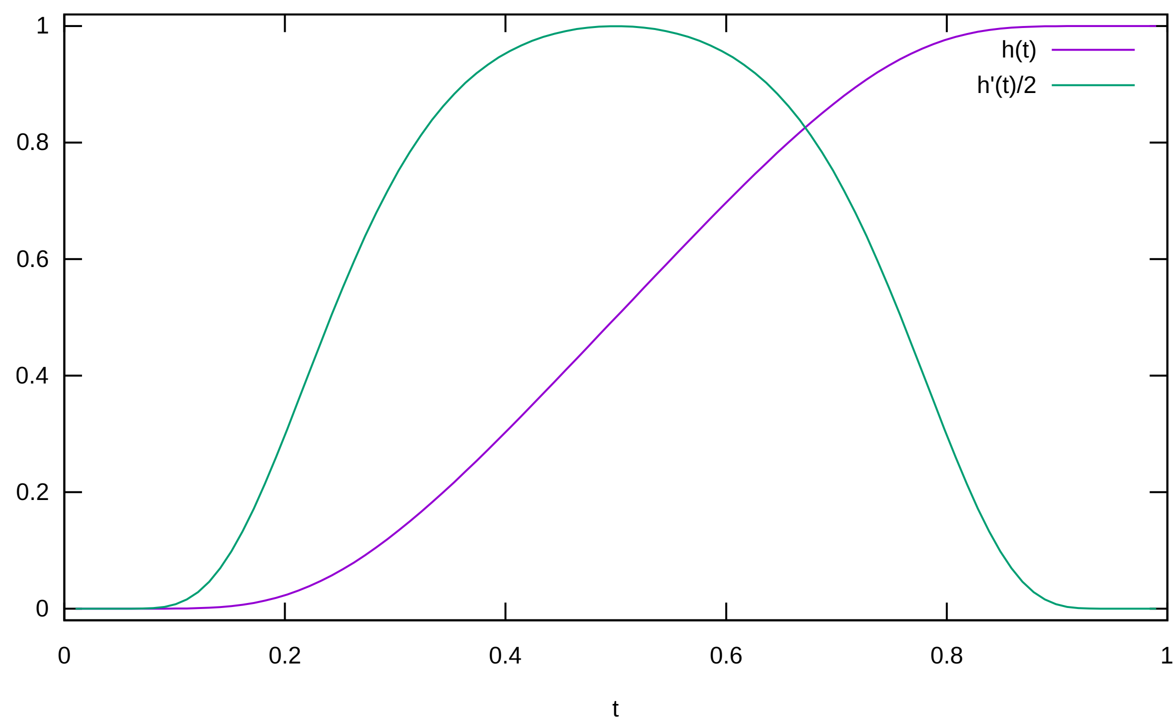


Figure 3.5: The form of $h(t), h'(t)/2$.

Chapter 4

Compact surfaces with boundary are 2-polynomially dense

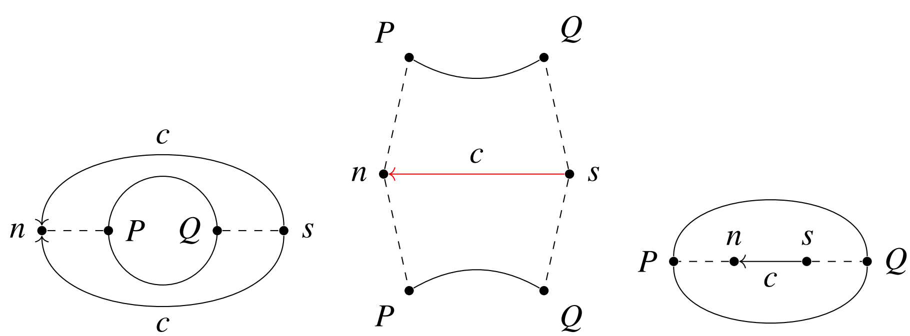
In this chapter, we show how to construct a pair of smooth generators for a compact surface S with boundary. Let S' be an open surface containing S. We will call a continuous function $f : S \to \mathbb{C}$ smooth if there exists a function $g : U \to \mathbb{C}$ such that $f = g|_S$, where U is an open neighbourhood of S in S'.

The first section gives examples of what this type of decomposition looks like, the next pair of sections define the hexagonal decomposition and prove that every compact surface with boundary admits one. The final sections build a coordinate system closely related to the foliation and develop a piecewise construction for smooth generators f_1, f_2 generating $C(S)$, proving that the polynomial density of such a surface is at most 2.

The polynomial density of a surface must be greater than 1. Otherwise, there would exist a smooth function $f_1 : S \to \mathbb{C}$ which generated $C(S)$; that is f_1 would be injective. The uniform algebra $C(f_1(S))$ would be generated by z, so $f_1(S)$ would have to be nowhere dense by Lavrentiev's Theorem (see Theorem 2.7.4) which is inconsistent with it being the smooth image of S under an injective function.

4.1 Example Decompositions

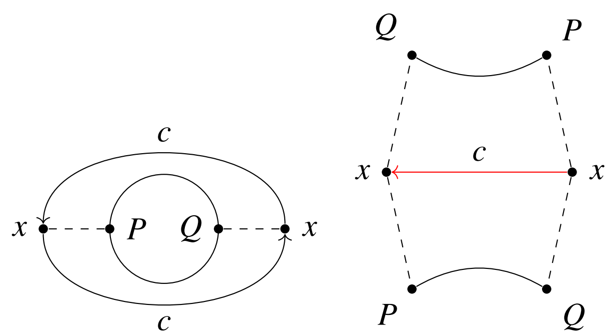
Example 4.1.1. *The sphere with excised disc (equivalently, the closed disc) has a hexagonal structure consisting of a single hexagon having two interior vertices n, s (north pole and south pole, for instance) and two boundary vertices P, Q.*



The set

$$\mathcal{G} = \Big\{\big\{\{P, n\}, \{Q, s\}, \{P, Q\}\big\}\Big\}.$$

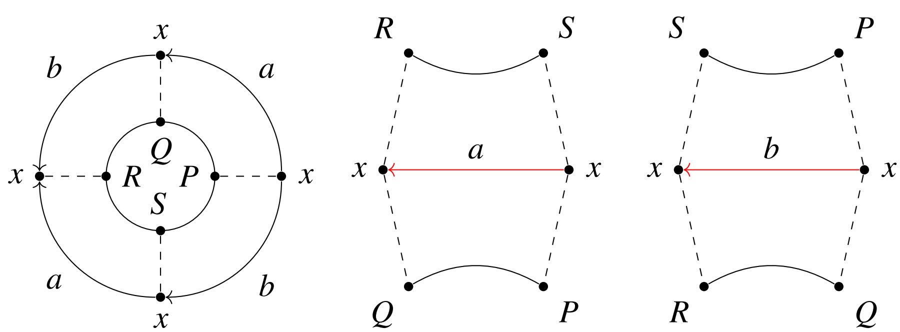
Example 4.1.2. *The projective plane with excised disc has a hexagonal structure consisting of a single hexagon, which has one interior vertex x and two boundary vertices P, Q.*



As a set,

$$\mathcal{G} = \Big\{\big\{\{P, x, Q\}, \{P, Q\}\big\}\Big\}.$$

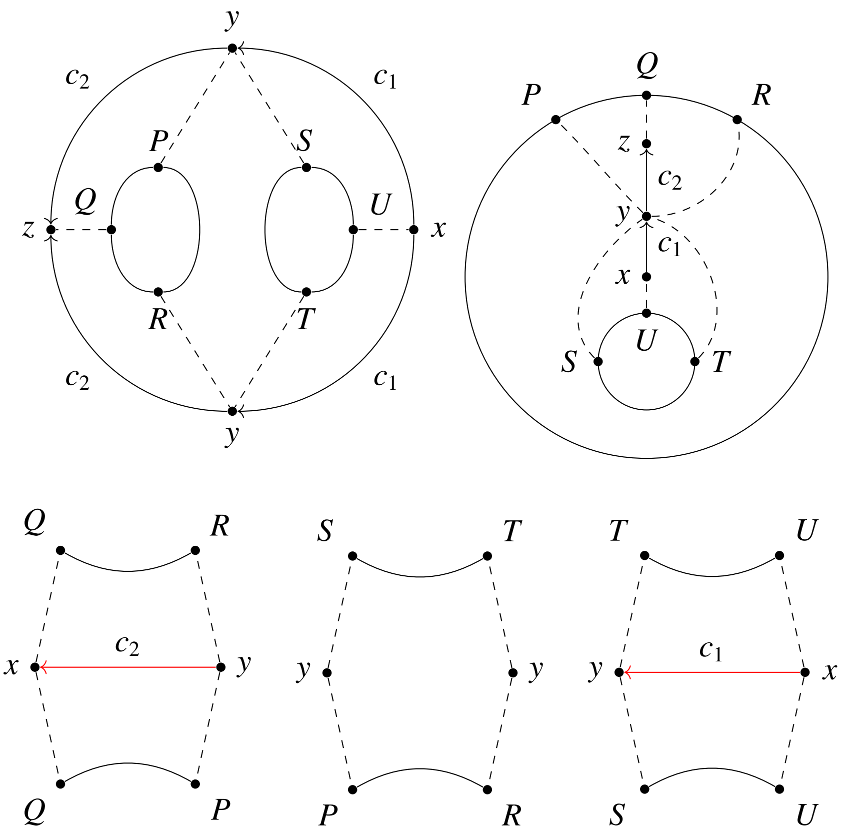
Example 4.1.3. *The torus with excised disc has a hexagonal structure consisting of a pair of hexagons. There is one interior vertex x and four boundary vertices P, Q, R, S.*



As a set,

$$\mathcal{G} \quad = \quad \Big\{\big\{\{Q, x, R\}, \{P, x, S\}, \{R, S\}, \{P, Q\}\big\},\ \big\{\{R, x, S\}, \{P, x, Q\}, \{P, S\}, \{Q, R\}\big\}\Big\}.$$

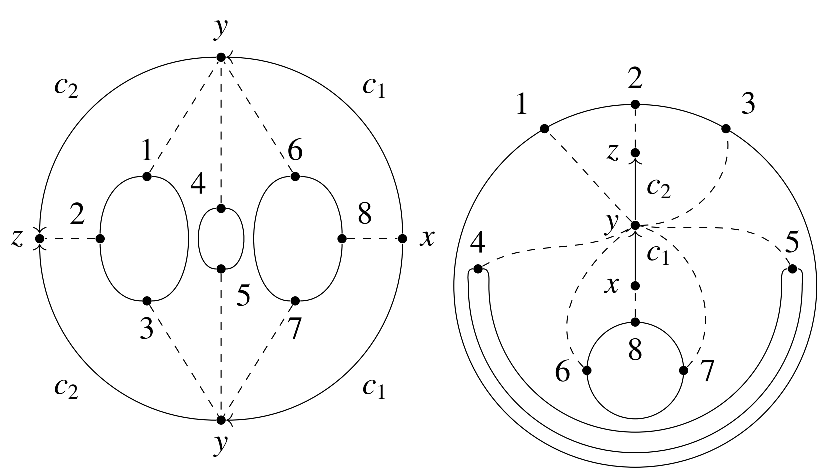
Example 4.1.4. *The closed annulus (equivalently, the cylinder, or the sphere with two excised discs) can be equipped with a hexagonal structure consisting of three hexagons, having two interior vertices x,y and six boundary vertices P,Q,R,S,T,U.*

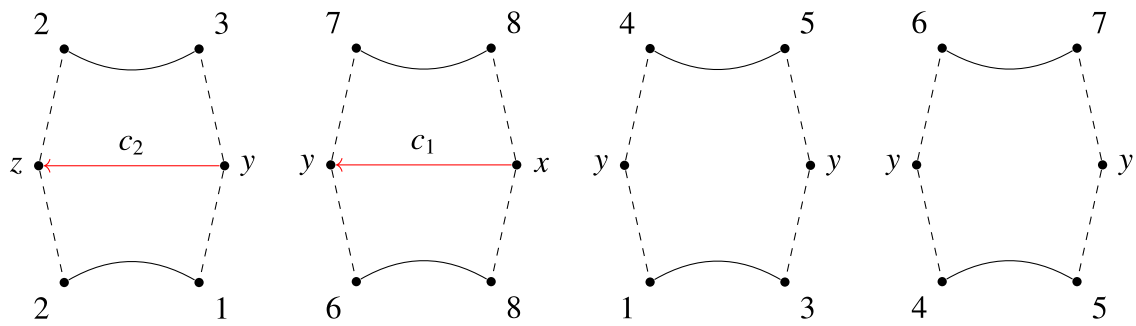


As a set,

$$\mathcal{G} = \Big\{ \big\{ \{x, Q\}, \{P, y, R\}, \{P, Q\}, \{Q, R\} \big\},\ \big\{ \{P, y, S\}, \{R, y, T\}, \{P, R\}, \{S, T\} \big\},$$
$$\big\{ \{S, y, T\}, \{x, U\}, \{S, U\}, \{T, U\} \big\} \Big\}.$$

Example 4.1.5. *The pair of pants (equivalently, the sphere with three excised disks) can be equipped with a hexagonal structure consisting of four hexagons having three interior vertices x, y, z and eight boundary vertices numbered* 1 *through* 8.



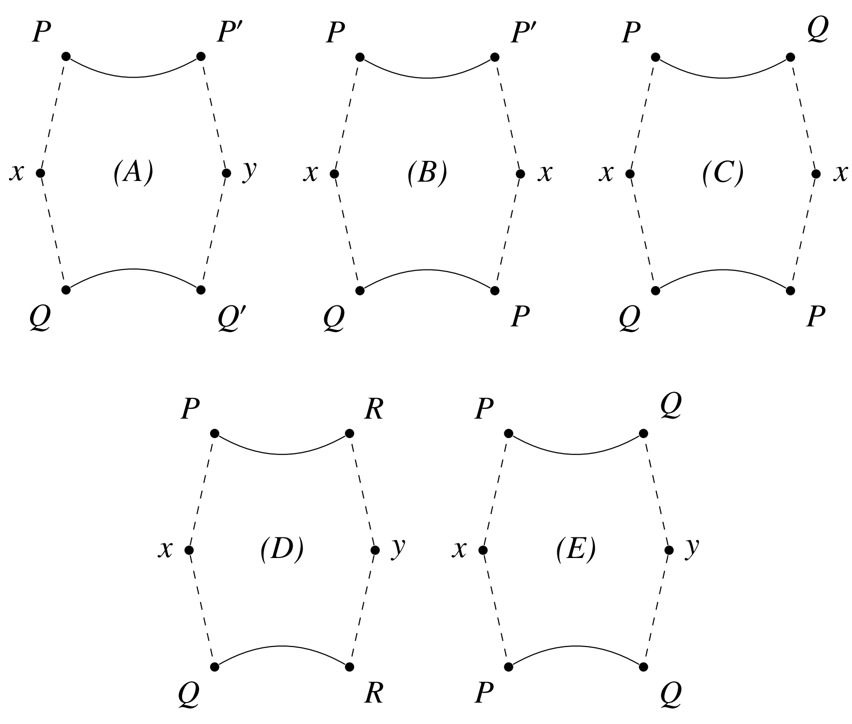


As a set,

$$\mathcal{G} = \Big\{\big\{\{2,z\}, \{1,3,y\}, \{1,2\}, \{2,3\}\big\},\ \big\{\{6,y,7\}, \{8,x\}, \{6,8\}, \{7,8\}\big\},$$
$$\big\{\{1,y,4\}, \{3,y,5\}, \{1,3\}, \{4,5\}\big\},\ \big\{\{4,y,6\}, \{5,y,7\}, \{4,5\}, \{6,7\}\big\}\Big\}.$$

4.2 Hexagonal decompositions

Definition 4.2.1. *Let S be a closed surface with boundary. A* **hexagon** *on S will be a pair (H, ϕ) where ϕ is a homeomorphism from the interior of H to an open set of S and H is a compact curvilinear polygon taking one of the following forms. If ϕ is a diffeomorphism then (H, ϕ) will be called a* **smooth hexagon***.*



The solid curves represents the boundary arcs (in S) while the dashed line segments represent the part of the boundary of the hexagon which maps onto the interior of S.

The uppercase letters P, Q, and R represent distinct boundary vertices and the lowercase letters x and y represent interior vertices. Note that $x = y$ is permissible for form (A) but not for forms (D) and (E).

The images (under ϕ) of the line segments in S are called the **cut lines** *of the hexagon.*

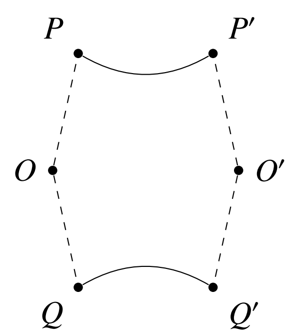
The **multiplicity** *of a vertex is the number of times it occurs in the form for H.*

Remark 4.2.2. *The forms (C) and (E) represent the surface with boundary obtained by excising a disc from real projective plane or the sphere, respectively. Any other compact surface with boundary can only have the forms (A), (B), or (D) in its hexagonal decomposition.*

Definition 4.2.3. *An* **abstract hexagon** *on a pair $(\mathcal{B}, \mathcal{I})$, where $\mathcal{B}$ will be denoted as* **boundary vertices** *and $\mathcal{I}$ as* **interior vertices**, *respectively, is a set H of the form*

$$H = \{\{P, O, Q\}, \{P', O', Q'\}, \{P, P'\}, \{Q, Q'\}\},$$

where P, Q, P', Q' are boundary vertices, O and O' are interior vertices, and any boundary vertex E in $\mathcal{B}$ can be equal to at most 2 of the vertices P, Q, P', Q'. (This number is called its **multiplicity** *in H.) The following diagram is its visual representation.*



We call OP, OQ, $O'P'$, $O'Q'$ its **cut lines**, *and PP', QQ' its* **boundary arcs**.[1] *If the boundary vertices are equipped with an equivalent relation $\cong$, then H is said to be* **compatible** *with it if $P \cong Q$ and $P' \cong Q'$.*

Definition 4.2.4. *An* **abstract hexagonal decomposition** *is a quadruple $(\mathcal{I}, \mathcal{B}, C, \mathcal{G})$ where*

- *$\mathcal{I}$ is a set of interior vertices;*
- *$\mathcal{B}$ is a set of boundary vertices;*
- *C is a set of cut lines (pairs (O, P) with $O \in \mathcal{I}$, $P \in \mathcal{B}$) such that each boundary vertex belongs to exactly one cut line;*
- *and a set $\mathcal{G}$ of abstract hexagons on $(\mathcal{B}, \mathcal{I})$ compatible with the equivalence relation in which a pair of boundary vertices E, F satisfy $E \cong F$ if their corresponding cut lines share an interior vertex, such that the sum of the multiplicity of any boundary vertex over all hexagons is 2.*

[1] In the absence of additional information, these will just be notation for the corresponding sets $\{O, P\}$ etc. Most boundary arcs will be uniquely determined by the pair of boundary points; at worst, one will need to specify the direction taken between them. One could stipulate that the cut lines would be segments of geodesics; then, they would be unique determined once a metric has been prescribed on the surface.

Each boundary vertex is joined to exactly one interior vertex by a cut line, but an interior vertex may be joined to many boundary vertices by cut lines. Consequently, the graph $(\mathcal{B}\cup\mathcal{I}, C)$ defined by taking the vertex set equipped with the set of cut lines constitutes a disjoint union of stars.

Remark 4.2.5. *The sphere and real projective plane (each with excised disc) are the only compact surfaces with boundary which admit a hexagonal structure consisting of a single hexagon.*

4.3 Decomposing a compact surface with boundary

This section will show that any compact surface with boundary admits a smooth hexagonal decomposition.

First consider Munkres' Approximation Theorem for C^r-surfaces which shows that compact smooth surfaces (with or without boundary) in the same homeomorphism class are in the same diffeomorphism class also.

Theorem 4.3.1. *[Mun56, Cor. 5.24, p. 113]*
Let M and N be differentiable surfaces of class C^r ($r \leq \infty$); let f be a homeomorphism mapping M onto N. Then there is a non-degenerate homeomorphism $\overline{f}$ of class C^r mapping M onto N. Indeed, if $\phi(x)$ is a positive continuous function defined on M, then $\overline{f}$ may be chosen so that it is a ϕ-approximation to f.

Corollary 4.3.2. *Any pair of differentiable surfaces of class C^r which are homeomorphic are also C^r-diffeomorphic.*

Proof. Consider the function $\overline{f}$ provided by the theorem and let p be an arbitrary point of M.

Choosing local charts (U, ϕ) at p and (V, ψ) on N at $\overline{f}(p)$, the the Inverse Function Theorem [KP14, Thm. 3.3.2, p. 43] applied to $\psi \circ \overline{f} \circ \phi^{-1}$ gives a local C^r inverse $g : \psi(V) \to \phi(U)$. This gives $\overline{f}$ a local C^r-smooth inverse $\phi^{-1} \circ g \circ \psi : V \to U$.

Note p is arbitrary, so $\overline{f}(p)$ is also. Additionally, any pair of local inverses g_1, g_2 defined on domains V_1, V_2 which intersect must be equal on the intersection $V_1 \cap V_2$ of their domains. Therefore, a global C^r-smooth function $G : N \to M$ may be defined as $G(q) = g(q)$ where g is any local inverse to $\overline{f}$ defined on a neighbourhood V of q. This constitutes a global inverse to $\overline{f}$ which is C^r-smooth, so $\overline{f} : M \to N$ is a C^r-diffeomorphism. □

Next consider a characterization of the homeomorphism classes of a compact surface (with or without boundary).

Lemma 4.3.3. *Any compact surface S with boundary is diffeomorphic to a smooth model constructed by taking a closed disc $\overline{\mathbb{D}}$, excising a finite set of disjoint open discs D_1, D_2, ..., D_H, where H is the number of contours, subdividing its boundary S^1 into paths γ_1, γ_2, ..., γ_{2N}, where N is either the cross cut number (if non-orientable) or twice the genus (if orientable), and identifying each path with exactly one other.*

In addition, provided that the excised discs do not intersect S^1 and are pairwise disjoint, their positions and sizes do not affect the diffeomorphism class. Similarly, the lengths of the paths γ_j do not affect it either.

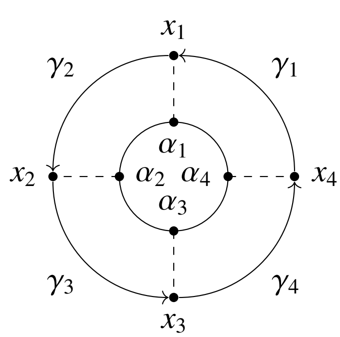
Proof. First assign S the normal form form (2.1) or (2.2) corresponding to its triangulation whose existence is shown in [AS60, I §8, p. 105–111]. Then define a smooth model corresponding to the normal form by taking a closed disc $\overline{\mathbb{D}}$, excising a finite set of disjoint open discs – one disc D_j for each boundary component $c_j h_j c_j^{-1}$, subdividing the boundary circle S^1 into $4p = 4g$ equal-sized arcs if the normal form is orientable (genus g), subdividing S^1 into $2p = 2K$ equal-sized paths otherwise (cross cut number K), and identifying these paths with the sides a_j and b_j (if applicable) of the normal form. (The orientation of this identification will, of course, depend on the orientations of the sides.)

The locations and sizes of the excised discs can be arbitrarily chosen (subject to the condition of not intersecting each other or S^1) since this does not affect the homeomorphism class by Theorem 2.5.10; hence, this does not affect the diffeomorphism class either by Corollary 4.3.2. The same reasoning applies to the length of the arcs since this clearly does not alter the normal form. □

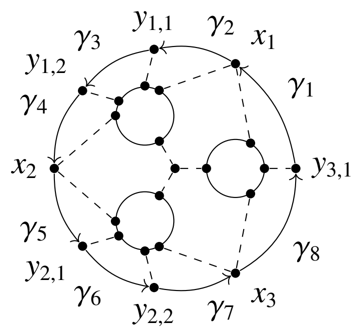
Remark 4.3.4. *Any compact surface S of this form has an inclusion into an open surface S' having nearly the same form. This can be done by excising closed discs of half the original radii at the same locations. When S is not of this form, fix a smooth model of this form for S and interpret S' according to it.*

Theorem 4.3.5. *Any compact surface S with boundary admits a hexagonal decomposition and a family of diffeomorphisms $\{\phi_H : U_H \to \mathbb{C}\}$, where U_H is an open neighbourhood of H in S', such that the sides of any image $\phi_H(H)$ consist of line segments and circular arcs.*

Proof. Without loss of generality, S is a smooth model of the form described in the previous corollary (with O representing the origin). Additionally, since a family of diffeomorphisms $\{\phi_H\}$ will need to be built later, assume without loss of generality that no successive pair of points x_k, x_{k+1} are equal. (Subdividing some of the arcs γ_k can be used to impose this condition.)



For the case of one contour ($H = 1$), let x_k be the terminal point of the path γ_k, for $1 \le k \le 2N$, centre the excised disc D at the origin, cut along the radial line segments $x_k O$ (denoting each intersection with ∂D as α_k), and, given any pair of paths γ_k and $\gamma_{k'}$, glue the curved polygons $x_{k-1}\alpha_{k-1}\alpha_k x_k \gamma_k^{-1}$ and $x_{k'-1}\alpha_{k'-1}\alpha_{k'} x_{k'} \gamma_{k'}^{-1}$ together along γ_k to form a hexagon.



Consider the case of multiple contours ($H \geq 2$). If $H > 2N$, subdivide some of the paths γ_k. Each time a path is split into p pieces, it raises N by $p - 1$. Consider this process to be complete once $H \leq 2N$. Choose the model to satisfy these additional properties:

1. each excised disc D_k has a centre located at $\zeta^k/2$, where $\zeta = e^{2\pi i/H}$, and a common radius $\rho < 1/4$ small enough that the discs D_k are pairwise disjoint and do not intersect either the origin or the unit circle S^1;

2. denote as n_k the nearest point of ∂D_k to O and those points which are $2\pi/3$ radian clockwise or counterclockwise of n_k along ∂D_k as l_k, r_k, respectively; and,

3. the points $x_k = \zeta^k \sqrt{\zeta}$ are endpoints of a subset of the paths γ_j (where $\sqrt{\zeta}$ is taken to be $e^{2\pi i/2H}$), and the other endpoints between a pair x_k, x_{k+1} are denoted as $y_{k,q}$ (the range of q, of course, depends on k).

Then carry out the following procedure to build the hexagons.

1. If $H = 2$, build the link hexagon $x_1 r_1 l_1 x_2 r_2 l_2 x_1$; otherwise, build the link hexagons $On_{k-1} l_{k-1} x_k r_k n_k O$, where the curves incident with O or some x_j are line segments, and curves joining points on ∂D_j are the arcs of shortest length.

2. Subdivide an arc $r_k l_k$ of D_k into $n + 1$ subarcs whenever there are n endpoints between x_k and x_{k+1} and denote the points between the subarcs as $P_{k,1}, \ldots, P_{k,n}$.

3. Make a hexagon piece taking one of the forms $x_k r_k P_{k,1} y_{k,1} \gamma_j^{-1}$, $y_{k,q} P_{k,q} P_{k,q+1} y_{k,q+1} \gamma_j^{-1}$, or $y_{k,n} P_{k,n} l_k x_{k+1} \gamma_j^{-1}$, for some $1 \leq j \leq 2N$, whenever there are points $P_{k,q}$ between the pair of points x_k, x_{k+1}; otherwise, make a hexagon piece taking the form $x_k r_k l_k x_{k+1} \gamma_k^{-1}$.

4. Build the side hexagons by gluing together the pieces that reference γ_k and $\gamma_{k'}$ along their common edge γ_k (which is identified with $\gamma_{k'}$).

The link hexagons along with the side hexagons comprise a hexagonal decomposition covering S. Next, the open neighbourhoods U_H and diffeomorphisms ϕ_H need to be constructed.

1. For a link hexagon H, let U_H be an open neighbourhood (in the related open surface S') small enough that no pair of points in U_H are identified with each other. Then ϕ_H can be taken to be the inclusion map $U_H \hookrightarrow \mathbb{C}$. Clearly, in this case, $\phi_H(H)$ has the claimed property.

2. For a side hexagon H, there are a couple of pieces H_1 and H_2, each containing a boundary arc, a pair of cut lines, and a curve γ (used for the gluing the pieces together). Equip H_1 and H_2 with disjoint open neighbourhoods U_1 and U_2 in S' having the properties that $U_1 \cap \partial\mathbb{D}$ and $U_2 \cap \partial\mathbb{D}$ are connected subsets of $\partial\mathbb{D}$ having no identified points on their own, such that each point of $U_1 \cap \partial\mathbb{D}$ is in the same equivalence class of precisely one point of $U_2 \cap \partial\mathbb{D}$, and neither set contains the point 0. Then a suitably chosen transformation $\tau(z)$ of the form $z \mapsto \zeta/z$ or $z \mapsto \zeta/\bar{z}$ (where $|\zeta| = 1$) will carry $U_1 \cap \partial D$ onto $U_2 \cap \partial D$, so the piecewise defined map ϕ_H equal to the identity on $U_2 \cap \partial\mathbb{D}$ and equal to $\tau(z)$ on $U_1 \cap \partial\mathbb{D}$ has the required properties. (The second condition comes from the circle-preserving property of $1/z$ and $\bar{z}$.)

□

4.4 Defining a foliation and coordinate system on a hexagon

For this section, fix a concrete hexagon H (and its neighbourhood U in S') having a piecewise smooth boundary ∂H consisting of line segments and circular arcs. Without loss of generality, assume that the line segment OO' is a horizontal line passing through $\mathbf{0}$.

Proposition 4.4.1. *There exists a smooth vector field* $\mathbf{V}$ *on* U *with the following properties.*

1. *It only vanishes at vertex points.*
2. *It is normal to any interior point of a cut line and is parallel to both boundary arcs.*
3. *The inner and outer boundaries of the set on which* $\mathbf{V}$ *is not horizontal are tangent to the boundary* ∂H *at any vertex point.*
4. *It is horizontal sufficiently far from* ∂H *or sufficiently close to a vertex point.*

Proof. Build a polar coordinate system (r, θ) using the midpoint p^* of the line segment OO' as the pole.

For a line segment, the line properly containing it takes the form

$$r = a \sec(\theta - \theta_n),$$

where a is the distance between p^* and the line L and θ_n is the angle at which this distance is achieved. In this case, define the interval (θ_0, θ_1) to be $(\theta_n - \pi/2, \theta_n + \pi/2)$.

For a circular arc, the circle properly containing it takes the form

$$r^2 - 2ar\cos(\theta - \theta_c) = a^2,$$

where a is the distance between p^* and the circle and θ_c is the angle at which this achieved (eqivalently, the angle of the centre of the circle). In this case, define the interval to be (θ_0, θ_1) where θ_0 and θ_1 are the angles of the rays tangent to the circle.

Let $\mathfrak{A}$ be the algebra of subsets of U. Define elements a_i and b_i (representing the area above or below a given curve, respectfully) where i is indexed by the finite set $\{LL, L, LR, UL, U, UR\}$

where LL, L, LR, UL, U, UR represent the lower left cut line, the lower boundary arc, the lower right cut line, the upper left cut line, the upper boundary arc, and the upper right cut line, respectively. Define $w_i(\theta)$ to be a width function on the segment or arc flat at both ends of it, and bounded above by half the distance between the arc and p^*. Now extend it to the rest of the interval (θ_0, θ_1) by defining it to be identically zero there. Define

$$g_i(r, \theta; C) = r\cos(\theta - \theta_n) - a(1 + Cw(\theta))$$

when considering a line segment; otherwise, for a circular arc, define

$$g_i(r, \theta; C) = r^2 - 2ar\cos(\theta - \theta_c) - a^2(1 + Cw(\theta))^2 .$$

The elements a_i and b_i (for a fixed C such that $0 < C < 1$) are defined to be

$$a_i = \{(r, \theta) \in U |\ \theta_0 < \theta < \theta_1,\ g_i(r, \theta; -C) > 0\},$$

$$b_i = \{(r, \theta) \in U |\ \theta \text{ not contained in } (\theta_0, \theta_1) \text{ or } g_i(r, \theta; C) < 0\}.$$

Let x_i, y_i be variables valued in $\mathfrak{A}$. For the sake of clarity, a variable x_i will be evaluated at a_i (for a region being constructed) and y_i will be evaluated at b_i.

Define Boolean functions

$$F_{\text{in}}(x_i, y_i) = y_{UL} \cap y_U \cap y_{UR} \cap x_{LL} \cap x_L \cap x_{LR} \text{ where } C = 2/3,$$

$$F_{\text{out}}(x_i, y_i) = x_{UL} \cup x_U \cup x_{UR} \cup y_{LL} \cup y_L \cup y_{LR} \text{ where } C = 2/3,$$

$$F_{\text{cut}}(x_i, y_i) = (x_{LL} \cap y_{LL}) \cup (x_{LR} \cap y_{LR}) \cup (x_{UL} \cap y_{UL}) \cup (x_{UR} \cap y_{UR}) \text{ where } C = 1/3,$$

and

$$F_{\text{bdry}}(x_i, y_i) = (x_L \cap y_L) \cup (x_R \cap y_R) \text{ where } C = 1/3.$$

Use Lemma 3.4.2 to obtain functions f_{in}, f_{out}, f_{cut}, and f_{bdry}. Define vector fields $\mathbf{T}_p$ and $\mathbf{N}_p$ to be the tangent vector field on the boundary arcs extended above and below and the normal vector field on the cut lines extended to the sides. Define

$$\mathbf{V}_p = \left(f_{\text{in}}(p) + f_{\text{out}}(p)\right) \frac{\partial}{\partial x}\bigg|_p + f_{\text{cut}}(p)\,\mathbf{N}_p + f_{\text{bdry}}(p)\,\mathbf{T}_p.$$

Condition (1) is satisfied since the functions only vanish simultaneously on vertex points. Condition (2) is satisfied because $\mathbf{V}_p$ is a scalar multiple of $\mathbf{T}_p$ at each interior point of a boundary arc and a scalar multiple of $\mathbf{N}_p$ at each interior point of a cut line. Conditions (3) and (4) hold by construction. □

As in the case of closed surfaces, the vector field $\mathbf{V}$ constructed for this case defines a line foliation on U, the open neighbourhood constructed for the hexagon H in this case.

Corollary 4.4.2. *There exists a smooth foliation $\mathcal{F}$ on U such that*

1. *$\mathcal{F}$ is tangent to the normal vector field* $\mathbf{V}$ *on the interior of any cut line,*

2. *$\mathcal{F}$ is tangent to the boundary arcs, and*

3. *any leaf is horizontal when it is far from the cut lines and the boundary arcs of* H.

Corollary 4.4.3. *There exists a smooth coordinate system* $(\mathfrak{b}, \hat{y})$ *on* U *in which* $\hat{y}$ *parametrizes the leaves of* $\mathcal{F}$ *with the interval* $[-1, 1]$ *where* $\hat{y} = -1$ *corresponds to the lower boundary arc,* $\hat{y} = +1$ *corresponds to the upper boundary arc, and* $\mathfrak{b}(p)$ *measure the signed distance along the leaf from L, the line segment bisecting both boundary arcs.*

Proof. Construct vectors field $\mathbf{W}_1$ and $\mathbf{W}_2$ from $\mathbf{V}$ by multiplying by terms which vanishing on the right or left side of L, respectively and use them to construct flows Φ_1 and Φ_2. Define $\hat{y}(p)$ as

$$\lim_{t\to\infty} \frac{2y(\Phi_1(t, p)) - (y_0 + y_1)}{y_1 - y_0}$$

for points p which at L or left of it, and, otherwise, as

$$\lim_{t\to-\infty} \frac{2y(\Phi_2(t, p)) - (y_0 + y_1)}{y_1 - y_0},$$

where y_0 and y_1 are the lowest and highest y values occurring in the line segment L.

Parametrize the line segment L as $\gamma(\hat{y})$. To define $\mathfrak{b}(p)$ along the left side of L, consider the pair of equations

$$\mathfrak{b}(p) = -\int_0^u \Phi_1(t, \gamma(\hat{y}))^* ds,$$

$$\Phi_1(-u, \gamma(\hat{y})) = p;$$

for the right side of L, consider the pair

$$\mathfrak{b}(p) = \int_0^u \Phi_2(t, \gamma(\hat{y}))^* ds,$$

$$\Phi_2(u, \gamma(\hat{y})) = p.$$

□

Proposition 4.4.4. *Let* K *denote the union of those points to the left of* O *and to the right of* O'. *There exists a smooth function* s *on* $U \setminus (K \cup \{O, O'\})$ *such that*

1. $s(p)$ *is identically 0 on the left cut lines and identically 1 on the right cut lines, and*

2. $s(p)$ *extends continuously to 0 at* O *and to 1 at* O'.

Using the continuous extension of s *to* $U \setminus K$*, we obtain a couple of further properties:*

3. *the preimage of* $[0, 1] \times [-1, 1]$ *under* $(s, \hat{y})$ *is exactly* H*, and*

4. *the pair* $(s, \hat{y})$ *constitutes an injective map* $(U \setminus K) \cup \{O, O'\} \to \mathbb{R}^2$.

Proof. Define smooth functions $\ell_{LL}(\hat{y})$, $\ell_{UL}(\hat{y})$, $\ell_{LR}(\hat{y})$, $\ell_{UR}(\hat{y})$ to parametrize the cut lines (and their containing line segments and circular arcs) in the form $\{\mathfrak{b} = \ell_{\square}(\hat{y})\}$. Let ℓ_L and ℓ_R be the piecewise smooth functions $\max(\ell_{LL}, \ell_{UL})$ and $\min(\ell_{LR}, \ell_{UR})$, respectively. Define a function $\alpha(p) : U \setminus K \to \mathbb{R}$:

$$\alpha(\mathfrak{b}, \hat{y}) = \begin{cases} \mathfrak{b}\ h\left(\frac{-\mathfrak{b}}{\mathfrak{b}_0}\right)\ m\Big(-\ell_{LL}(\hat{y}), -\ell_{UL}(\hat{y}); \epsilon\ \chi_-(\mathfrak{b}, \hat{y})\Big)^{-1} & \mathfrak{b} \leq 0, \\ \mathfrak{b}\ h\left(\frac{\mathfrak{b}}{\mathfrak{b}_0}\right)\ m\Big(\ell_{LR}(\hat{y}), \ell_{UR}(\hat{y}); \epsilon\ \chi_+(\mathfrak{b}, \hat{y})\Big)^{-1} & \mathfrak{b} \geq 0, \end{cases}$$

where

$$\chi_-(\mathfrak{b}, \hat{y}) = 2h\left(\frac{1}{2}\Big(1 + (\mathfrak{b} - \ell_{LL}(\hat{y}))\,(\mathfrak{b} - \ell_{UL}(\hat{y}))\Big)\right) - 1,$$

and

$$\chi_+(\mathfrak{b}, \hat{y}) = 2h\left(\frac{1}{2}\Big(1 + (\ell_{LR}(\hat{y}) - \mathfrak{b})\,(\ell_{UR}(\hat{y}) - \mathfrak{b})\Big)\right) - 1.$$

Then define $s : U \setminus K \to \mathbb{R}$ as

$$s(p) = \frac{1}{2}\Big(1 + \alpha(p)\Big).$$

To show (1) and (2), note that $\alpha(p)$ simplifies to $\ell_L(\hat{y}(p))$ (so that $s = 0$) for the left cut lines and that it simplifies to $\ell_R(\hat{y}(p))$ (giving $s = 1$) for the right cut lines. The difference between these conditions is a consequence of the fact that the functions $-m(-\ell_{UL}, -\ell_{LL}; 0) = \ell_L$ and $m(\ell_{UR}, \ell_{LR}; 0) = \ell_R$ are only continuous at O or O', but are smooth elsewhere.

To show (4), the injectivity of $(s, \hat{y})$, it is sufficient to show that $\alpha(\mathfrak{b}, \hat{y})$ is injective on each leaf (that is, for each fixed $\hat{y}$). For the intervals $(-\mathfrak{b}_0, 0)$ or $(0, \mathfrak{b}_0)$, $\alpha(\mathfrak{b}, \hat{y})$ takes the form

$$\mathfrak{b}\ h\left(\frac{\pm\mathfrak{b}}{\mathfrak{b}_0}\right)\ m_0(\hat{y})^{-1},$$

with $\pm$ corresponding to the sign of $\mathfrak{b}$ and $m_0(\hat{y})$ being a smooth function of $\hat{y}$ coming from the appropriate $m(\ldots)$ term in $\alpha(\mathfrak{b}, \hat{y})$. An easy computation shows that the partial derivative is $\frac{\partial\alpha}{\partial\mathfrak{b}} > 0$ for either interval. For other intervals, $\alpha(\mathfrak{b}, \hat{y})$ takes the form

$$\mathfrak{b}\ m\Big(\ell_1(\hat{y}), \ell_2(\hat{y}); \epsilon\ \chi_\pm\big(\mathfrak{b}, \hat{y}\big)\Big)^{-1},$$

where $\ell_1(\hat{y})$ and $\ell_2(\hat{y})$ are smooth functions and $\pm$ depends on the sign of $\mathfrak{b}$ in the interval. Its derivative $\frac{\partial\alpha}{\partial\mathfrak{b}}$ then takes the general form

$$\frac{1}{m\,(\cdots)} - \frac{\mathfrak{b}\epsilon}{m^2\,(\cdots)}\ \frac{\partial m}{\partial\epsilon}(\cdots)\ \frac{\partial\chi_\pm}{\partial\mathfrak{b}}(\cdots) > 0,$$

so $\alpha(\mathfrak{b}, \hat{y})$ is monotonic in those regions also.

To show (3), it suffices to show that the preimage of the set $\{0, 1\} \times [-1, 1] \cup [0, 1] \times \{-1, 1\}$ is ∂H. That the preimage of $[0, 1] \times \{-1, 1\}$ is the union of the boundary arcs is a result from the previous corollary. That the preimage of $\{0, 1\} \times [-1, 1]$ is the union of the cut lines follows from conditions (1) and (2). $\square$

4.5 Defining generators f_1, f_2

In this section, we construct smooth functions f_1, f_2 which generate the algebra $C(S)$ of continuous functions on S.

Equip S with a smooth hexagonal decomposition $\mathcal{H}$, denoting the interior and boundary vertices as $\mathcal{I}$, $\mathcal{B}$, respectively, and each cut line or boundary arc with the smooth assignment $p \mapsto \mathfrak{n}_p$ of operator germs given by the normals at each point.

Suppose there is a collection $\{f_H : H \to \mathbb{C}\}$ of continuous functions having the following properties. For each smooth hexagon H in the hexagonal composition:

1. its associated function f_H is smooth at any interior point of H,
2. the function f_H is $\mathfrak{En}$-flat along all of the cut lines or boundary arcs incident to H, and
3. the function f_H is completely flat at each vertex point of the hexagon H.

For any pair of smooth hexagons H, H' whose intersection is non-empty,

4. the associated functions $f_H, f_{H'}$ are identically equal on $H \cap H'$.

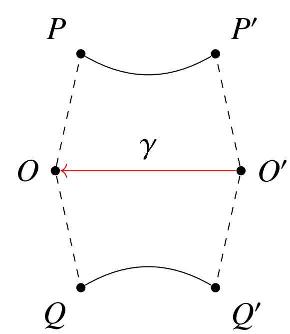
This collection of functions then defines a smooth function f on the surface with boundary S via the obvious construction $f(p) = f_H(p)$ whenever $p \in H$. This shows us how to construct smooth functions on S piecewise. Additionally, this function f is a restriction of a smooth function defined on an open neighbourhood of S in S' because of its $\mathfrak{En}$-flatness along the boundary arcs and its complete flatness at boundary vertices.

Now an application of the Bishop-Shilov theorem tells us that the following properties suffice for a pair of functions f_1 and f_2 to generate $C(S)$:

1. they separate points on the surface,
2. the uniform algebras $C(f_2(S))$ and $P(f_2(S))$ coincide,
3. f_2 has a polynomially convex image in $\mathbb{C}$, and
4. the image of any level set of f_2 under f_1 is polynomially convex.

Assign a distinct complex unitary c_H to each hexagon which will be used to define f_2. Additionally assign a unique element of the lattice $\mathbb{Z} + i\mathbb{Z}$ (in $\mathbb{C}$) to each interior vertex and equip each boundary vertex with the unitary corresponding to its direction in the polygonal model. These will be used to define f_1.

Consider a hexagon in the decomposition of S. Denote the lattice points assigned to O, O' as o, o', respectively. Then denote the unitaries corresponding to the vertex points P, P', Q, Q' as u, u', v, v'. The points $o + 1/6u$, $o + 1/6v$, $o' + 1/6u'$, $o' + 1/6v'$ will be denoted as p,q, p', q'. When $O = O'$ the unitaries u,u' are adjacent to each other as are v,v', since their locations in the models used to construct the hexagonal decomposition are themselves adjacent.



We build the functions $f_1(p)$ and $f_2(p)$ using the pair of smooth functions $s(p)$ and $\hat{y}(p)$ constructed in Proposition 4.4.4. To define the function $f_2(p) = f_2(s(p), \hat{y}(p))$ on an individual hexagon, let

$$f_2(s, \hat{y}) = c_H \, h(s) \, h(1 - s),$$

where c_H is its associated unitary. Its image $f_2(S)$ is a union of radial line segments joined at a single common point, so it is polynomially convex in $\mathbb{C}$ and it is nowhere dense; consequently, $C(f_2(S)) = P(f_2(S))$ and each condition only involving f_2 is satisfied.

We will construct f_1 by smoothly joining three pieces defined on the intervals $I_A = [0, 1/3]$, $I_B = [1/3, 2/3]$ and $I_C = [2/3, 1]$: an initial piece extending outwards from the convex region poq, a middle piece which keeps its distance from the points o and o', and a final piece extending inward to the convex region $p'o'q'$. Specifically, define $f_1(p) = f_1(s(p), \hat{y}(p))$ from

$$f_1(s, \hat{y}) = \begin{cases} f_A\left(3h(s), 2h\left(\frac{\hat{y}+1}{2}\right) - 1\right) & 0 \leq x \leq h^{-1}(1/3), \\ f_B\left(3h(s) - 1, 2h\left(\frac{\hat{y}+1}{2}\right) - 1\right) & h^{-1}(1/3) \leq x \leq h^{-1}(2/3), \\ f_C\left(3h(1 - s), 2h\left(\frac{\hat{y}+1}{2}\right) - 1\right) & h^{-1}(2/3) \leq x \leq 1, \end{cases}$$

where

- the functions $f_A(x, y)$, $f_B(x, y)$, and $f_C(x, y)$ on $[0, 1] \times [-1, 1]$ are smooth;
- the set $f_A([0] \times [-1, 1])$ is the union of the line segments op and oq, and the set $f_C([0] \times [-1, 1]$ is the union of the line segments $o'p'$ and $o'q'$;
- for each $y \in [-1, 1]$, the functions $f_A(x, y)$ and $f_C(x, y)$ are $\left(\mathfrak{E} \frac{\partial}{\partial x}\right)_{(0,y)}$-flat, $f_A(x, y)$ and $f_B(x + 1, y)$ are $\left(\mathfrak{E} \frac{\partial}{\partial x}\right)_{(0,y)}$-matching, and $f_B(x, y)$ and $f_C(2 - x, y)$ are $\left(\mathfrak{E} \frac{\partial}{\partial x}\right)_{(1,y)}$-matching;
- the images of $(0, 1) \times [-1, 1]$ under f_A and f_C are disjoint;
- the image of $(0, 1) \times [-1, 1]$ under f_B does not intersect either of the discs $B(o, 1/6)$ and $B(o', 1/6)$; and
- there exists a $0 < x_1 < 1/2$ such that the images of $(0, x_1) \times [-1, 1]$ and $(1 - x_1, 1) \times [-1, 1]$ under f_B are disjoint, and the restriction of f_B to $[x_1, 1 - x_1] \times [-1, 1]$ is bijective.

In order to define either $f_A(x, y)$ and $f_C(x, y)$ let ζ, ζ' be the unitaries which bisect the triangle poq or $p'o'q'$, respectively, and let α, α' denote the angles between u and v and between u' and v', respectively. Let

$$f_A(x, y) = o + \frac{1}{6}\zeta\left(iy + 2\,(x - h(x)) + \sqrt{4h(x)^2 + y^2 \,\tan^2\left(\frac{\alpha}{2}\right)}\right),$$

$$f_C(x, y) = o' + \frac{1}{6}\zeta'\left(iy + 2\,(x - h(x)) + \sqrt{4h(x)^2 + y^2 \,\tan^2\left(\frac{\alpha'}{2}\right)}\right).$$

To construct $f_B(x, y)$, define a pair of paths (taken from f_A and f_C)

$$\gamma_A(t) = \frac{1}{6}\zeta\left(it + \sqrt{4 + t^2 \tan^2\left(\frac{\alpha}{2}\right)} - 2\right)$$

and

$$\gamma_C(t) = \frac{1}{6}\zeta' \left(it + \sqrt{4 + t^2 \tan^2\left(\frac{\alpha'}{2}\right)} - 2 \right),$$

and create a bijective smooth homotopy $H(s,t)$ between $\gamma_A(t)$ and $\gamma_C(t)$ with the property that $|H(s,t)| \leq 1/6$ for any $0 \leq s \leq 1$, $-1 \leq t \leq 1$, along with a smooth path $\gamma(t)$ such that

- $\gamma(t) = o + 1/3\zeta(1+t)$ near $t = 0$,
- $\gamma(t) = o' + 1/3\zeta'(2-t)$ near $t = 1$,
- there exists a neighbourhood $[t_0, 1-t_0]$ of $t = 1/2$ in which $\gamma(t)$ is linear and $|\gamma(1-t_0) - \gamma(t_0)| \geq 1/3$,
- there exists some $s_1 \in [t_0, 1-t_0]$ such that the image of $H(s_1, t)$ is not parallel to the tangent vector of γ at s_1 for any $-1 \leq y \leq 1$,
- this path never gets closer than $1/3$ to either of the lattice points o,o'.

Finally, we construct the function $f_B(x,y)$ as

$$f_B(x,y) = \gamma(x) + H(u(x), y),$$

where $u(t) : [0,1] \to [0,1]$ is a non-decreasing surjective function such that u is constant in a neighbourhood of $t = 0$, in a neighbourhood of $t = 1$, and the restriction of u to the neighbourhood $[t_0, t_1]$ is equal to s_1. (This is to force f_1 to be injective in the region where we can not rely on the distance between the components of the level sets of f_2 to make sure that points are separated by f_1 and f_2.)

Chapter 5

Conclusion

In this thesis, we have shown that any connected closed surface has polynomial density 3 and any connected compact surface with boundary has polynomial density 2. It would be interesting to learn whether any closed manifold of dimension n has polynomial density $n + 1$ and whether any connected compact manifold with boundary has polynomial density n.

Some other interesting questions are whether there is a similar bound for a connected compact surface with boundary if the boundary is permitted to be non-smooth, or whether a n-dimensional topological manifold not admitting a triangulation still admits $n + 1$ continuous generators. On the other hand, a related problem involving rational density instead of polynomial density is that of determining the rational density of the sphere and the real projective plane.

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