Control of Shear Layers Using Heating Patterns

Shoyon Panday, Western University

Supervisor: Floryan, J.M., The University of Western Ontario
A thesis submitted in partial fulfillment of the requirements for the Doctor of Philosophy degree in Mechanical and Materials Engineering
© Shoyon Panday 2023

Follow this and additional works at: https://ir.lib.uwo.ca/etd

Part of the Numerical Analysis and Computation Commons

Recommended Citation
https://ir.lib.uwo.ca/etd/9214

This Dissertation/Thesis is brought to you for free and open access by Scholarship@Western. It has been accepted for inclusion in Electronic Thesis and Dissertation Repository by an authorized administrator of Scholarship@Western. For more information, please contact wlswadmin@uwo.ca.
Abstract

The presence of spatially modulated flows is universal in nature. Distributed heating and surface roughness are the most common elements to cause nonuniformity in the flows. Spatially distributed heating leads to fundamentally distinct convection, different from the classical Rayleigh-Bénard instability. Interestingly, the onset of convective motion due to horizontal temperature gradients requires no critical conditions – a forced response. At the same time, surface roughness is known to significantly influence flow behaviors and heat transfer characteristics. The current work aims to analyze modulated flows and assess their potential as a mixing technique for low Reynolds number flows. Spanwise modulations (perpendicular to the flow direction) have been considered in a three-dimensional channel. A spectrally accurate algorithm has been developed based on the Fourier expansions and Chebyshev polynomials. The immersed boundary condition method is used to overcome challenges associated with surface irregularities at the boundaries. The algorithm is gridless and provides means for analyzing numerous patterns with minimal human labor. Thermal modulations at the boundaries result in the formation of rolls/streaks in low Reynolds number flows. The strength of streaks is determined by evaluating the spanwise gradient of the streamwise (along the flow direction) velocity component and the change in kinetic energy. For all forms of nonuniform heating, an additional pressure gradient is required to maintain the same mass flowrate as the reference Poiseuille flow. Adding geometric modulation with distributed heating produces stronger streaks, and a range of wave numbers exists where pressure losses are lower than the reference flow. An optimum wave number is identified in order to generate streaks efficiently for the periodic heating of grooved surfaces. These streaks play an essential role in shear layer dynamics and are subject to instabilities, which are of interest for mixing intensification. A linear stability algorithm has been developed to study the stability characteristics of modulated flows. This algorithm avoids challenges related to the classical DNS-based approach. It is shown that a new instability mode appears due to the thermal modulations, which outstandingly reduces the critical Reynolds number.
Keywords

Convection, Galerkin, immersed boundary, incompressible flow, gridless, Navier-Stokes, spectral, Fourier expansions, Chebyshev polynomial, mixing, streaks, periodic heating, linear stability, grooves, flow control, pressure losses, laminar flows, pressure-driven flows.

Summary for Lay Audience

Mixing is essential in many industries and applications. It is a known fact that turbulent flows generally provide better mixing compared to laminar flows. The smooth, orderly movement of the fluid layers in laminar flows makes it challenging to achieve faster mixing rates. However, depending on the applications, it can be non-beneficial or impractical to always have turbulent flows in the system. Therefore, a technique that can lead to intense mixing in laminar flows will greatly interest many industries such as food & beverage, pharmaceutical, chemical processing etc. In the current work, we have assessed the potential of surface roughness and patterned heating as a mixing technique for the low Reynolds number flows. Roughness is defined by any form of irregularities at the surface, while patterned heating is identified as heating and cooling of the same surface. An efficient and highly accurate algorithm has been developed to solve systems with roughness and heating patterns. It is shown that patterned heating creates rolls which can be beneficial for mixing enhancement. Combining patterned heating with surface roughness increases the strength of these rolls. The first step towards chaotic stirring is to determine the stability condition of these rolls. In other words, it is crucial to determine when these rolls become unstable. A linear stability algorithm has been developed that can quickly determine the conditions required for the onset of instability. This algorithm introduces a small disturbance in the system and checks if the disturbance grows or decays over time. It is observed that patterned heating results in instability at a very low Reynolds number. This instability's characteristic matches with those in literature, which induced chaotic mixing in laminar flows.
Co-Authorship Statement

This dissertation is prepared in monograph format. Chapters 2 through 7 are based on the manuscripts that have been previously published, or submitted, or finalized for submission. I, Shoyon Panday, am the first author of all these manuscripts with Dr. J.M. Floryan as the co-author.
Acknowledgments

I’m grateful to Goddess Saraswati for her endless blessings. At times, it felt that She was seated next to me and steered my thoughts and actions to get through all the challenges.

I would like to express my sincere gratitude to my advisor, Dr. J.M. Floryan, for his unwavering support and guidance throughout this journey. Among many things, I greatly appreciate his ability to break down complicated ideas into simple, fundamental steps. In all those years, he immensely helped me to grow and think as a sincere researcher. I would like to extend my gratitude to the members of my advisory committee, Dr. R.E. Khayat & Dr. C. DeGroot, for their valuable suggestions and encouragement.

I appreciate the thoughtful conversation, cooperation, and proofreading from all my previous and current colleagues, including Dr. Josuel Kruppa Rogenski, Dr. Longyin Jiao, Dr. Hadi Vafadar Moradi, Dr. Mohammad Zakir Hossain, Kh. Md. Faisal, Nafisha Haq, Saajid Aman, Dwaipayan Sarkar, Sabuj Dobey and Saarah Akhand. I want to express my thankfulness to Nafisha in particular for not only the insightful research-related discussions, but also for being a wonderful friend who has provided me with motivation and mental support over the course of my doctoral program.

Finally, and more importantly, I wish to thank my parents, Ruma Panday & Goutom Panday, and my siblings, Shayontony Panday & Angona Panday, for their unconditional love, sacrifice and encouragement. This dissertation would not have been completed without their inspiration.
Table of Contents

Abstract........................................................................................................................................... ii
Summary for Lay Audience............................................................................................................. iii
Co-Authorship Statement................................................................................................................ iv
Acknowledgments........................................................................................................................... v
Table of Contents........................................................................................................................... vi
List of Tables..................................................................................................................................... x
List of Figures........................................................................................................................................ xi
List of Appendices............................................................................................................................ xxi
List of Abbreviations and Symbols.................................................................................................... xxiii
Chapter 1........................................................................................................................................... 1
1 Introduction................................................................................................................................. 1
  1.1 Objectives............................................................................................................................... 1
  1.2 Motivations............................................................................................................................. 1
  1.3 Literature Review.................................................................................................................... 4
     1.3.1 Horizontal Convection....................................................................................................... 4
     1.3.2 Surface Roughness / Grooves......................................................................................... 6
     1.3.3 Streaks / Rolls................................................................................................................ 8
     1.3.4 Modelling of Heating & Roughness Patterns............................................................... 10
     1.3.5 Algorithm for Linear Stability Analysis......................................................................... 12
     1.3.6 Flow instabilities & Stirring........................................................................................... 14
  1.4 Overview of the Present Work............................................................................................... 16
  1.5 Outline of the Dissertation................................................................................................... 19
Chapter 2........................................................................................................................................... 20
2 Algorithm for Analysis of Pressure Losses in Heated Channels................................................... 20
2.1 Introduction ............................................................... 20
2.2 Problem Formulation ................................................... 20
2.3 Analysis of the Flow Problem ........................................ 26
  2.3.1 The Spanwise Convection Problem .......................... 26
  2.3.2 The Streamwise Flow Problem ............................... 37
2.4 Extracting of the Physically Relevant Data and Post-Processing . 40
2.5 Discussion and Algorithm Testing .................................. 45
2.6 Summary ................................................................. 53

Chapter 3 ........................................................................ 54

3 Time-Dependent Flows in Grooved Non-Isothermal Channels .... 54
  3.1 Introduction ............................................................... 54
  3.2 Problem Formulation ................................................... 55
  3.3 Initiation of Computations ........................................... 58
  3.4 Time Progression ........................................................ 59
    3.4.1 Convection Component ........................................ 61
    3.4.2 Streamwise Flow Component ............................... 72
  3.5 Discussion and Algorithm Performance ........................ 75
  3.6 Summary ................................................................. 81

Chapter 4 ........................................................................ 83

4 Creation of Streaks in a Smooth Channel Using Heating Patterns ... 83
  4.1 Introduction ............................................................... 83
  4.2 Problem Formulation ................................................... 84
  4.3 Rayleigh-Bénard effect ............................................... 87
  4.4 Sinusoidal Heating at the Lower Wall ........................... 90
  4.5 Combined Sinusoidal and Uniform Heating at the Lower Wall . 95
  4.6 Sinusoidal Heating at the Upper Wall ........................... 98
4.7 Sinusoidal Heating at the Both Walls ......................................................... 98
4.8 Effects of Prandtl Number ........................................................................ 105
4.9 Summary .................................................................................................... 108

Chapter 5 ......................................................................................................... 110

5 Streak Creation Using Groove and Heating Patterns ................................... 110

5.1 Introduction ............................................................................................... 110
5.2 Problem Formulation ................................................................................ 110
5.3 Uniform Heating – Isothermal Grooves ...................................................... 114
5.4 Periodic Heating and Pattern Interaction Effect .......................................... 120
5.5 Combined Periodic and Uniform Heating .................................................... 125
5.6 Summary .................................................................................................... 127

Chapter 6 ......................................................................................................... 129

6 Linear Stability of Spatially Modulated Flows .............................................. 129

6.1 Introduction ............................................................................................... 129
6.2 Problem Formulation ................................................................................ 130
6.3 Determination of Stationary States ............................................................. 133
6.4 Linear Stability Analysis ............................................................................ 134
   6.4.1 Discretization of the Modal Equation ................................................... 140
   6.4.2 Discretization of Boundary Conditions ............................................... 145
   6.4.3 The Linear Algebraic System ............................................................... 147
6.5 Method of Solution .................................................................................... 149
6.6 Testing of the Algorithm .......................................................................... 150
6.7 Summary .................................................................................................... 162

Chapter 7 ......................................................................................................... 164

7 Instabilities of the Thermally Modulated Shear Layers ............................... 164

7.1 Introduction ............................................................................................... 164
List of Tables

Table 3.1: Comparison of the computational cost of advancing solution by one time step. Cost of the 1st order method is used as the reference ........................................................................................................ 79

Table 4.1: System wavelength $\lambda$ and number of the hot and cold spots at each wall for selected values of the commensurability index CI........................................................................................................ 104

Table 6.1: Variations of the leading eigenvalue $\sigma$ as a function of the number of Fourier modes $N_D$ and the number of Chebyshev polynomials $N_K$ used in the computations for an isothermal system with $Ra_p, L = 0$, $B_L = 0.1$, $B_U = 0$, $\alpha = 1$, $\delta = 0$ and $Re_z = 3000$, $\mu = 0.4$, and $Re_z = 7000$, $\mu = 1$.................................................................................................................................................. 161

Table 6.2: Variations of the leading eigenvalue $\sigma$ as a function of the number of Fourier modes $N_D$ and the number of Chebyshev polynomials $N_K$ used in the computations for a non-isothermal system with $Re_z = 1000$, $Ra_p, L = 500$, $B_L = 0.1$, $B_U = 0$, $\alpha = 1$, $\delta = 0$, $\mu = 1$.................................................................................................................................................. 162

Table 7.1: Energy transfer from the stationary state to disturbances.............................. 189
List of Figures

Figure 1.1: Thermal map of Atlanta, Georgia. (NASA/Goddard Space Flight Center Scientific Visualization Studio)................................................................. 2

Figure 1.2: (A) Rayleigh–Bénard system (Barletta 2019) (B) Vertical natural convection (Ng et al. 2015) (C) Horizontal convection (Hossain & Floryan 2013a).................................5

Figure 1.3: Wavelength $\lambda$ of a disturbed flow system as a function of disturbance wavenumber $\delta$. Red circles and blue triangles provide wavelengths of the complete system modulated with wavenumbers $\alpha = 1$, and $\alpha = 1.1$, respectively. Vertical dotted lines identify two possible incommensurate states where the system is aperiodic................................................. 13

Figure 2.1: Schematic diagram of the flow system.................................................. 22

Figure 2.2: Structure of the coefficient matrix for the energy equation for $N_M = 2$ and $N_K = 15$. Fig 2.2A displays matrix before its re-arrangement with the data entered starting with mode -2 and ending with mode 2. Black symbols mark the non-zero elements. Fig 2.2B displays matrix after re-arrangement.................................................................................. 34

Figure 2.3: Structure of the coefficient matrix for the momentum equation for $N_M = 2$ and $N_K = 15$. Figure 2.3A displays matrix before its re-arrangement with the data entered starting with mode -2 and ending with mode 2. Black symbols mark the non-zero elements. Figure 2.3B displays matrix after re-arrangement................................................................. 36

Figure 2.4: Structure of the coefficient matrix for the streamwise flow problem for $N_M = 5$ and $N_K = 30$. The data is entered starting with mode -5 and ending with mode 5. Black symbols mark the non-zero elements. Horizontal strips correspond to boundary relation while blocks correspond to the field equation................................................................. 39

Figure 2.5: Variations of the error norms $\|Er_u\|$ (Fig 2.5A), $\|Er_w\|$ (Fig 2.5B) and $\|Er_\theta\|$ (Fig 2.5C) as functions of the number $N_M$ of Fourier modes used in the computations for $N_K = 80$. Here, $Ra_p, L = Ra_p, U = Ra_p$ ................................................................. 46
Figure 2.6: Variations of the error norms $\|E_{r_k}\|$ (Fig 2.6A), $\|E_{r_\omega}\|$ (Fig 2.6B) and $\|E_{r_\theta}\|$ (Fig 2.6C) as functions of the number $N_K$ of Chebyshev polynomials used in the computations for $N_M = 30$. Here, $Ra_{p,L} = Ra_{p,U} = Ra_p$.

Figure 2.7: Variation of the Chebyshev norms $\|u\|_\omega$ (Fig 2.7A), $\|w\|_\omega$ (Fig 2.7B) and $\|\theta\|_\omega$ (Fig 2.7C) as functions of the mode number $m$ for $N_K = 80, N_M = 30$. Here, $Ra_{p,L} = Ra_{p,U} = Ra_p$.

Figure 2.8: Variation of the force error norms $\|F\|_x$ and $\|F\|_z$ (see text for details) as functions of the number $N_M$ of Fourier modes (Figs 2.8A and 2.8B, respectively) and as functions of the number of Chebyshev polynomials $N_K$ (Figs 2.8C and 2.8D, respectively) used in the computations. $N_K = 80$ Chebyshev polynomials were used in the former case and $N_M = 30$ Fourier modes in the latter case. Here, $Ra_{p,L} = Ra_{p,U} = Ra_p$.

Figure 2.9: Variation of the pressure error norm $\|\hat{\theta}\|_z$ as a function of the number of Fourier modes $N_M$ (Fig 2.9A) and as a function of the number of Chebyshev polynomials $N_K$ (Fig 2.9B) used in the computations. See text for details. Here, $Ra_{p,L} = Ra_{p,U} = Ra_p$.

Figure 2.10: Variation of the flow rate error norm $\|Q\|_x$ as a function of the number of Fourier modes $N_M$ (Fig 2.10A) and as a function of the number of Chebyshev polynomials $N_K$ (Fig 2.10B) used in the computations. See text for details. Here, $Ra_{p,L} = Ra_{p,U} = Ra_p$.

Figure 3.1: Schematic diagram of the flow system.

Figure 3.2: Structure of the coefficient matrix for the energy (Fig 3.2A) and momentum (Fig 3.2B) equations ($N_M = 2, N_K = 10$). Black color identifies the non-zero elements.

Figure 3.3: Structure of the coefficient matrix for the longitudinal flow problem ($N_M = 8, N_K = 30$). Black color identifies the non-zero elements.

Figure 3.4: Variations of the error $\|E_{r_i}\|$ (see Eq. 3.5.1) in the evaluation of the pressure gradient as a function of the time step $\Delta t$. 
Figure 3.5: Variations of errors $\|E_{r_u}\|$ (Fig 3.5A) and $\|E_{r_w}\|$ (Fig 3.5B) as functions of the number of Fourier modes $N_M$ used in the computations. $N_K = 80$ Chebyshev polynomials which reduced the Chebyshev truncation error below machine accuracy………………. 77

Figure 3.6: Variations of errors $\|E_{r_u}\|$ (Fig 3.6A) and $\|E_{r_w}\|$ (Fig 3.6B) as functions of the number of Chebyshev polynomials $N_K$ used in the computations. $N_M = 30$ Fourier modes were used which reduced the Fourier truncation error below machine accuracy………………. 78

Figure 3.7: Instantaneous vectors lines for a flow in a conduit with geometry of the form $y_u(x) = 1 + 0.025 \cos(\alpha x), y_L(x) = -1 + 0.05 \cos(\alpha x)$, $\alpha = 2$, for a flow rate in the $z$-direction corresponding to $Re_z = 2$ and the zero pressure gradient in the $x$-direction ($Re_x = 0$) for a fluid with $Pr = 0.71$. The conduit is exposed to heating resulting in the lower wall temperature of the form $\theta_L(x) = Ra_{uni} + \frac{Ra_{PL}}{2} \cos(\alpha x) \sin(\omega t)$ with $Ra_{uni} = 50$, $Ra_{PL} = 800$, $\omega = 2\pi$ and the upper wall temperature of the form $\theta_U(x) = \frac{Ra_{PU}}{2} \cos(\alpha x)$ with $Ra_{PU} = 800$. $T = 1$ stands for the time-period and $t_0 = 6T$. Color coding refers to the $w$-velocity component……………………………………………………………………. 80

Figure 4.1: Schematic diagram of the flow configuration……………………………… 85

Figure 4.2: (A) Flow topology created by the Rayleigh-Bénard instability ($\alpha = 1.5585$, $Ra_{uni} = 250$, $Re_z = 15$). Golden and turquoise colors identify stream tubes corresponding to $\psi = 0.45, -0.45$, respectively, and dashed lines show particle trajectories. Colors in the $(x, y)$-plane illustrate the temperature field while black solid lines illustrate vector lines. (B) Variations of the spanwise gradient of the streamwise velocity component $dw/dx$ at $y = 0$ for $\alpha = 1.5585$, $Ra_{uni} = 200, 220, 250$, $Pr = 0.71$. (C) Variations of the spanwise gradient of the streamwise velocity component $dw/dx$ at $y = 0$ for $\alpha = 1.5585$, $Ra_{uni} = 250$, $Pr = 0.025, 0.71, 7.56$……………………………………………………………………………………………….. 88

Figure 4.3: Variations of (i) the mean pressure gradient $B = (dp/dz)/Re_z$ (Fig 4.3A), (ii) the flowrate $Q/Re_z$ (Fig 4.3B) and (iii) the average Nusselt number $Nu_{av}$ (Fig 4.3C) as functions of the uniform Rayleigh number $Ra_{uni}$ for $\alpha = 1.5585$…………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………
Figure 4.4: (A) Flow topology - golden and turquoise colors identify stream tubes corresponding to $\psi = 1.2, -1.2$; colors in the $(x, y)$-plane illustrate the temperature field; dashed lines illustrate particle trajectories. (B) Distributions of the $w$-velocity modifications $w_m = (w - w_0)$ in the $(x, y)$-plane; colors identify magnitudes of velocity modifications. (C) Distributions of the $w$-velocity in the $(x, y)$-plane; colors identify magnitudes of velocity. In all subfigures, $\alpha = 1, Ra_{P,L} = 1000, Ra_{P,U} = Ra_{uni} = 0, Re_z = 15$, black lines illustrate vector lines.

Figure 4.5: Variations of the spanwise gradient of the streamwise velocity component $dw/dx$ at $y = 0$ for $\alpha = 1$ as a function of $Ra_{P,L}$ (Fig 4.5A) and for $Ra_{P,L} = 1000$ as a function of $\alpha$ (Fig 4.5B).

Figure 4.6: Flow topologies for $\alpha = 0.2, Re_z = 15, Ra_{P,U} = Ra_{uni} = 0$ for $Ra_{P,L} = 100$ (Fig 4.6A) and for $Ra_{P,L} = 1000$ (Fig 4.6B). Yellow (ocher) and turquoise colors identify stream tubes corresponding to $\psi = 0.2, -0.2$ in Fig 4.6A, and to $\psi = 1, -1$ in Fig 4.6B. Colors in the $(x, y)$-plane illustrate the temperature field while the black solid lines illustrate vector lines.

Figure 4.7: Variations of the mean pressure gradient $B = (dp/dz)/Re_z$ (Fig 4.7A), the flowrate $Q/Re_z$ (Fig 4.7B) and the average Nusselt number $Nu_{av}$ (Fig 4.7C) as functions of the wavenumber $\alpha$ for $Ra_{P,L} = 100, 500, 1000, Ra_{P,U} = Ra_{uni} = 0$.

Figure 4.8: (A) Variations of the spanwise gradient of the streamwise velocity component $dw/dx$ at $y = 0$. Distributions of the $w$-velocity in the $(x, y)$-plane for (B) $Ra_{uni} = -200$ and (C) $Ra_{uni} = 250$. The same color scale identifying the magnitude of velocity is used in Figs 4.8 B, C. Black lines illustrate vector lines. In all subfigures, $\alpha = 1$ and $Ra_{P,L} = 500$.

Figure 4.9: Variations of the mean pressure gradient $B = (dp/dz)/Re_z$ (Fig 4.9A), the flowrate $Q/Re_z$ (Fig 4.9B) and the modulus of the average Nusselt number $|Nu_{av}|$ (Fig 4.9C) as functions of the wavenumber $\alpha$ for $Ra_{P,L} = 500$ and $Ra_{uni} = -200, -100, 0, 100, 200, 250$. Thin dotted lines in Figs 4.9 B, C illustrate the mean pressure.
gradient and the flowrate for the isothermal flow. Solid (dashed) lines in Fig 4.9C represent positive (negative) values of $Nu_{av}$.

**Figure 4.10:** Flow topologies for $\alpha = 1, Ra_{p,L} = Ra_{p,U} = 500, Re_z = 15, Ra_{uni} = 0$. Results displayed in Figs 4.10A-D correspond to phase differences $\Omega = 0, 0.5\pi, 0.999\pi, \pi$, respectively. Yellow (ocher) and green (forest) colors correspond to $\psi = 0.7, -0.7$ in Fig 4.10A, to $\psi = 0.6, -0.6$ in Fig 4.10B, to $\psi = 0.3, -0.3, -0.8$ in Fig 4.10C, and to $\psi = 0.2, -0.2$ in Fig 4.10D. Colors in the $(x, y)$-plane illustrate the temperature field while black solid lines illustrate vector lines. The same temperature color scale is used in all subfigures.

**Figure 4.11:** Variations of the spanwise gradient of the streamwise velocity component $dw/dx$ at $y = 0$ for $\alpha = 2$.

**Figure 4.12:** Variations of the flow rate $Q/Re_z$ as a function of the phase difference $\Omega$ for $\alpha = 1$ and $Ra_{p,L} = Ra_{p,U} = 500$. The insert illustrates variations of the difference between the flow rates at $\Omega = \pi$ and at any other $\Omega$ as a function of distance $(\pi - \Omega)$ expressed in the logarithmic scale. Points A, B, C, D correspond to flow conditions used in Fig 4.10 A, B, C, D.

**Figure 4.13:** Variations of the mean pressure gradient $B = (dp/dz)/Re_z$ (Fig 4.13A), the flowrate $Q/Re_z$ (Fig 4.13B) and average Nusselt number $Nu_{av}$ (Fig 4.13C) as functions of the wavenumber $\alpha$ for $Ra_{p,L} = Ra_{p,U} = 500, Ra_{uni} = 0$ and the phase shifts $\Omega = 0, 0.25\pi, 0.5\pi, 0.75\pi, 0.99\pi, \pi$. Dotted lines correspond to the one-wall heating with $Ra_{p,L} = 500$. Solid and dashed lines are used alternatively to increase the figure readability. Thin dotted lines in Figs 4.13 A-B illustrate the mean pressure gradient and the flowrate for the isothermal flow.

**Figure 4.14:** Distributions of the upper (top) and lower (bottom) wall temperatures for the commensurability index $CI = 1.3$ (Fig 4.14A) and $CI = 0.15$ (Fig 4.14B) for $\Omega = 0$. Figure 4.14C illustrates the temperature (color) and flow (solid lines) fields for $CI = 1/3$ and $Ra_{p,L} = 500, Ra_{p,U} = 500, Ra_{uni} = 0$. 

---

97, 99, 100, 101, 102, 105

xv
**Figure 4.15:** Variations of the mean pressure gradient $B = (dp/dz)/Re_z$ (Fig 4.15A), the flowrate $Q/Re_z$ (Fig 4.15B) and the average Nusselt number $Nu_{av}$ (Fig 4.15C) as functions of the Prandtl number $Pr$ for $\alpha = 1$ for different heating conditions, i.e. periodic heating at the lower wall with $Ra_{P,L} = 500, Ra_{P,U} = 0, Ra_{uni} = 0$ (solid lines); combination of periodic and uniform heating at the lower wall with $Ra_{P,L} = 500, Ra_{P,U} = 0, Ra_{uni} = 250$ (dotted lines); periodic heating of both walls with $Ra_{P,L} = 500, Ra_{P,U} = 500, Ra_{uni} = 0, \Omega = 0$ (dashed lines)………………………………………………………………………… 106

**Figure 4.16:** (A) Variations of the spanwise gradient of the streamwise velocity component $dw/dx$ at $y = 0$ for $\alpha = 1$. (B) Variations of the streamwise velocity component $w$ at different $x$-locations for $\alpha = 1$. Green lines are multiplied by 0.9 for better readability. Black dash-dotted line corresponds to the reference velocity profile (Poiseuille). For both the subfigures, solid, dotted, and dashed lines correspond to the same heating strategies as used in Fig 4.15………………………………………………………………………………………………………………………………. 107

**Figure 4.17:** Distributions of velocity modifications $w_m = (w - w_0)$ in the $(x, y)$ – plane; colors identify magnitudes of $w_m$. First row corresponds to $Pr = 0.1$, second row corresponds to $Pr = 0.71$, and third row corresponds to $Pr = 10$………………………………………………………………………………………. 108

**Figure 5.1:** Schematic diagram of the flow configuration………………………………………………………………………………………………………………………………………… 111

**Figure 5.2:** Flow topology in an isothermal grooved channel for $\alpha = 1.57, B_L = 0.4, Ra_{uni} = 150, Ra_{P,L} = 0, Pr = 0.71$ and $Re_z = 5$. Colors in the front $(x, y)$-plane illustrate the temperature field while black solid line illustrates vector lines, colors in the rear $(x, y)$-plane represent the $w$-velocity field, and dash-dotted lines show particle trajectories…………… 115

**Figure 5.3:** (A) Variations of the change in kinetic energy $\Delta E_k$ (Eq. 5.3.1) as a function of $Ra_{uni}$ for $\alpha = 1.57, Ra_{P,L} = 0, Re_z = 5$ and selected $B_L$’s. (B) Variation of the maximum of the spanwise velocity gradient $\xi$ (Eq. 5.3.2) as a function of $\alpha$ for $B_L = 0.06, Ra_{P,L} = 0$, and selected $Ra_{uni}$’s. Dashed lines provide results for isothermal grooved channel. Red dotted line in Fig 5.3B shows results for a smooth channel with $Ra_{P,L} = 220$. Grey color identifies $\alpha$’s leading to a reduction of pressure losses in a grooved isothermal channel……………… 116
Figure 5.4: Variation of the flow rate correction $Q_c$ (Eq. 5.2.7; Fig 5.4A), the Nusselt number correction $Nu_c$ (Eq. 5.2.11; Fig 5.4B) and the pressure gradient correction $P_e$ (Eq. 5.2.9; Fig 5.4C) as functions of $\alpha$ for $B_L = 0.06$ and $Ra_{p,L} = 0$. The dashed lines in Figs 5.4A and 5.4C correspond to the isothermal grooved channel; grey color identifies $\alpha$’s leading to a reduction of pressure losses (increase of the flow rate) in such channel………………………………………………… 117

Figure 5.5: Variations of the maximum of the spanwise velocity gradient $\xi$ (Eq. 5.3.2; dashed lines) and the flow rate correction $Q_c$ (Eq. 5.2.7; solid lines) as functions of the groove amplitude $B_L$ for $\alpha = 0.6$ (green lines, the drag reduction zone) and $\alpha = 1.53$ (red lines, the RB zone) for $Ra_{uni} = 205, Ra_{p,L} = 0$. Grey color identifies conditions leading to an increase of the flow rate………………………………………………………………………………… 118

Figure 5.6: Spanwise distributions of the $w$-velocity component at different $y$-locations (A) and the transverse distributions of the $w$-velocity component at different $x$-locations (B) for $\alpha = 1.57, Ra_{uni} = 200, Ra_{p,L} = 0, B_L = 0.1$………………………………………………………………………………… 119

Figure 5.7: Variations of the change in kinetic energy $\Delta E_k$ (Eq. 5.3.1; Fig 5.7A), the maximum of the spanwise velocity gradient $\xi$ (Eq. 5.3.2; Fig 5.7B) and the flow rate correction $Q_c$ (Eq. 5.2.7; Fig 5.7C) as functions of $\Omega_{TL}$ for $\alpha = 1.57$ (the RB zone; solid blue lines) and $\alpha = 0.6$ (the drag reduction zone; solid red lines), $Ra_{uni} = 0, B_L = 0.1$, and $Ra_{p,L}$’s are specified in the figures. Reference quantities for a smooth channel exposed to the same periodic heating are illustrated using dotted lines. The reference quantities for a grooved isothermal channel are: $\alpha = 1.57$: $\Delta E_k = -0.0515, \xi = 0.0326, Q_c/Re_z = -0.0014$; $\alpha = 1$: $\Delta E_k = 0.0095, \xi = 0.0334, \frac{Q_c}{Re_z} = -0.0001$; $\alpha = 0.6$: $\Delta E_k = 0.0507, \xi = 0.0259, Q_c/Re_z = 0.0007$………………………………………………………………………………… 121

Figure 5.8: Topology of flow in an isothermal grooved channel for $\alpha = 1, B_L = 0.1, Ra_{uni} = 0, Ra_{p,L} = 200, \Omega_{TL} = \pi/2$ and $Re_z = 1$. Colors in the front $(x, y)$-plane illustrates the temperature field while the black solid line illustrates velocity vector lines, colors in the rear $(x, y)$-plane represents $w$-velocity field. Black dotted and purple solid lines inside the plotted box show particle trajectories………………………………………………………………………………… 122
Figure 5.9: Variations of the change in kinetic energy $\Delta E_k$ (Eq. 5.3.1; Fig 5.9A), the maximum of the spanwise velocity gradient $\xi$ (Eq. 5.3.2; Fig 5.9B) and the flow rate correction $Q_c$ (Eq. 5.2.7; Fig 5.9C) as functions of $\alpha$ for $Ra_{uni} = 0$, $Ra_{p,L} = 250$, $B_L = 0.1$ and selected $\Omega_{TL}$’s. Dashed and dotted lines give reference results for the grooved isothermal and smooth periodically heated channels, respectively. Grey color identifies $\alpha$’s leading to a reduction of pressure losses in a grooved isothermal channel. Green lines mark $\alpha = 1$ ................. 124

Figure 5.10: Variations of the change in kinetic energy $\Delta E_k$ (Eq. 5.3.1; Fig 5.10A), the maximum of the spanwise velocity gradient $\xi$ (Eq. 5.3.2; Fig 5.10B) and the flow rate correction $Q_c$ (Eq. 5.2.7; Fig 5.10C) as functions of $Ra_{uni}$ for $B_L = 0.1$, $Ra_{p,L} = 250$, $\Omega_{TL} = \pi/2$. Blue color identifies $\alpha = 1.57$ (the RB zone), red color identifies $\alpha = 0.6$ (the drag reducing zone) and green color identifies $\alpha = 1$. Dotted and dashed lines illustrate results for smooth periodically heated channels and for isothermal grooved channels, respectively. Grey color in Fig 5.10C identifies conditions leading to a reduction of flow losses ................. 126

Figure 6.1: Schematic diagram of the flow system. Red and blue colors identify hot and cold sections of the walls ................................................................. 131

Figure 6.2: Structure of the coefficient matrix $A$ for $N_D = 2$ and $N_K = 10$ (Fig 6.2A). Green lines identify off-diagonal blocks providing coupling between different unknowns. Figure 6.2B displays the structure of a single block with green shading identifying entries corresponding to boundary relations. Black symbols mark the non-zero elements ......................... 148

Figure 6.3: Spectrum for the traveling wave instability in a plane Poiseuille flow for $B_L = 0$, $Ra_{uni} = Ra_{p,L} = Ra_{p,U} = 0$, $\delta = 0$, $\mu = 1$, $Re_z = 10000$ (plane isothermal Poiseuille flow). The results were obtained with $N_D = 1$ Fourier modes and $N_K = 120$ Chebyshev polynomials ................................................................. 151

Figure 6.4: Spectrum for flow in an isothermal channel with streamwise grooves for (A) $\alpha = 1$, $\mu = 0.4$, $\delta = 0$, $B_L = 0.1$, $B_U = 0$, $Re_z = 3000$ and (B) $\alpha = 1$, $\mu = 1$, $\delta = 0$, $B_L = 0.1$, $B_U = 0$, $Re_z = 7000$. The results were obtained with $N_D = 10$ Fourier modes and $N_K = 150$ Chebyshev polynomials ................................................................. 152
Figure 6.5: Spectrum for convection in a horizontal slot with corrugated lower wall and exposed to periodic heating with $Ra_{P,L} = Ra_{P,U} = 1600$, $Ra_{uni} = 100$, $\alpha = 3$, $\Omega_{TU} = \pi$, $B_L = 0.07$, $B_U = 0$, $Re_z = 0$, $\delta = 1.1$, $\mu = 1.1$. $N_D = 20$ Fourier modes and $N_K = 60$ Chebyshev polynomials were used in these computations............................................. 154

Figure 6.6: Spectrum for flow in a channel with longitudinal grooves exposed to spanwise periodic heating with $\alpha = 1$, $\delta = 0.8$, $\mu = 0.5$, $Ra_{P,L} = 500$, $\Omega_{TL} = \pi/2$, $B_L = 0.1$, $Re_z = 1000$. $N_D = 10$ Fourier modes and $N_K = 120$ Chebyshev polynomials were used in these computations................................................................. 155

Figure 6.7: Variations of the error $\Delta \sigma_l$ in the evaluation of the amplification rate $\sigma_l$ as a function of the number of Fourier modes $N_D$ used in the computations for (A) a two-dimensional traveling wave disturbance ($\delta = 0, \mu = 0.3$) for $\alpha = 1$, $B_L = 0.07, B_U = 0$, $Re_z = 1000$ and (B) a three-dimensional travelling wave disturbance ($\delta = 0.8, \mu = 0.5$) for $\alpha = 1$, $B_L = 0.1, B_U = 0, \Omega_{TL} = \pi/2$, $\delta = 0.8, \mu = 0.5$. The heating conditions used in the tests were $Ra_{uni} = 0$ and $Ra_{P,L} = Ra_{P,U} = 50$ and $Ra_{P,L} = Ra_{P,U} = 75$. All results were obtained with $N_K = 80$ Chebyshev polynomials................................................................. 156

Figure 6.8: Variation of $\Delta \sigma_l$ as a function of the number of Chebyshev polynomials $N_K$ for $\alpha = 1$, $B_L = 0.07, B_U = 0, Re_z = 500$ for (A) two-dimensional disturbance with $\delta = 0, \mu = 0.4$ and (B) three-dimensional disturbance with $\delta = 0.4, \mu = 0.4$. The heating conditions used in the tests were $Ra_{uni} = 0$ and $(Ra_{P,L}, Ra_{P,U}) = (100,0)$ and $(Ra_{P,L}, Ra_{P,U}) = (200,200)$. All results were obtained with $N_D = 20$ Fourier modes................................................................. 157

Figure 6.9: Distributions of $u_D$ (dashed-dotted line), $v_D$ (solid line), $w_D$ (dashed line) and $\theta_D$ (dotted line) at the lower wall for $\alpha = 1$, $B_L = 0.07$, $B_U = 0$, $Re_z = 1000$, $Ra_{P,L} = 100$, $Ra_{P,U} = 0$, $Ra_{uni} = 0$, $\mu = 0.4$ and (A) $\delta = 0$ and (B) $\delta = 0.4$. All results were obtained with $N_D = 15$ Fourier modes and $N_K = 40$ Chebyshev polynomials................................................................. 158

Figure 6.10: Variations of the boundary errors as functions of the number $N_D$ of Fourier modes used in the computations for $\alpha = 1$, $B_L = 0.07$, $Re_z = 1000$, $Ra_{P,L} = 100$, $\mu = 0.4$, $\delta = 0$ (Fig 6.10A) and $\delta = 0.4$ (Fig 6.10B). All results were obtained with $N_K = 80$ Chebyshev polynomials................................................................. 159
Figure 6.11: Variations of the Chebyshev norms as functions of the mode number for $\alpha = 1$, $B_L = 0.07, B_U = 0, Re_z = 1000, Ra_{P,L} = 100, Ra_{P,U} = Ra_{unt} = 0, \mu = 0.4$ and (A) $\delta = 0$ and (B) $\delta = 0.4$. All results were obtained with $N_D = 15$ Fourier modes and $N_K = 80$ Chebyshev polynomials.

Figure 7.1: Schematic diagram of the flow configuration.

Figure 7.2: Flow topology for $\alpha = 1, Ra_{P,L} = 1000, Re_z = 350$. Colors in the $(x,y)$- cross-sections illustrate sequentially from left to right the temperature field, the $u$-velocity field, the $v$-velocity field, and the $w$-velocity field. Black solid lines added to the temperature contour plot illustrate the velocity vector lines. The grey color illustrates stream tubes.

Figure 7.3: Variations of the spanwise (A) and transverse (B) gradients of the streamwise velocity component at different $y$-locations for $\alpha = 1, Ra_{P,L} = 1000, Re_z = 100$. Dashed lines give reference values for the isothermal channel.

Figure 7.4: Disturbance spectrum for (A) $Ra_{P,L} = 50$ and (B) $Ra_{P,L} = 200$ for $\alpha = 1, \delta = 0, \mu = 1$ and $Re_z = 7000$. Tracing of the unstable eigenvalues labeled as (I), (II), (III) is illustrated in Fig 7.5. The displayed results were obtained with $N_D = N_B = 10$ Fourier modes and $N_K = 120$ Chebyshev polynomials.

Figure 7.5: Variations of the unstable eigenvalues labeled as (I), (II), (III) in Fig 7.4 as functions of $Ra_{P,L}$. The isothermal limit corresponds to $Ra_{P,L} = 0$.

Figure 7.6: Distributions of the real and imaginary parts of the disturbance velocity and temperature eigenfunctions (A) $g_u^{(m)}$, (B) $g_v^{(m)}$, (C) $g_w^{(m)}$, and (D) $g_\theta^{(m)}$ ($m = 0,1,2$), corresponding to the unstable eigenvalue $\sigma = 1768.648 + 11.134i$ for $Ra_{P,L} = 50$, $\alpha = 1, \delta = 0, \mu = 1$ and $Re_z = 7000$. Eigenfunctions are normalized with condition $\max |g_w^{(0)}| = 1$. The black dashed lines stand for the isothermal case.

Figure 7.7: Distributions of the real and imaginary parts of the disturbance velocity and temperature eigenfunctions (A) $g_u^{(m)}$, (B) $g_v^{(m)}$, (C) $g_w^{(m)}$, and (D) $g_\theta^{(m)}$ ($m = 0,1,2$), corresponding to the unstable eigenvalue $\sigma = 6809.698 + 19.481i$ for $Ra_{P,L} = 200$, $\alpha = \ldots$
1, \delta = 0, \mu = 1 and \text{Re}_z = 7000. Eigenfunctions are normalized with condition \( \max |g_u^{(0)}| = 1 \). The black dashed line stands for the isothermal case.

**Figure 7.8:** (A) Disturbance spectrum for the two-dimensional traveling wave for \( Ra_{p,L} = 1000, \alpha = 1, \delta = 0, \mu = 1 \) and \( \text{Re}_z = 350 \). (B) Enlargement of the red box from (A) with green and blue dots identifying isothermal Squire and OS spectra, respectively. All presented results were obtained with \( N_D = N_B = 10 \) Fourier modes and \( N_K = 120 \) Chebyshev polynomials.

**Figure 7.9:** Variations of the unstable eigenvalue from Fig 7.8 as \( Ra_p \) is reduced.

**Figure 7.10:** Topology of the disturbance velocity and temperature fields corresponding to the unstable eigenvalue \( \sigma = 297.18 + 1.163i \) for \( Ra_{p,L} = 1000, \alpha = 1, \delta = 0, \mu = 1 \) and \( \text{Re}_z = 350 \). (A) Spanwise disturbance velocity (iso-surfaces for \( u_D = 0.25, -0.25 \)). (B) normal disturbance velocity (iso-surfaces for \( v_D = 0.04, -0.04 \)). (C) streamwise disturbance velocity (iso-surfaces for \( w_D = 0.3, -0.3 \)). (D) disturbance temperature (iso-surfaces for \( \theta_D = 0.8, -0.8 \)). The red arrows show the locations of hot spots.

**Figure 7.11:** Variations of the amplification rate as a function of \( \delta \) for \( \alpha = 1, Ra_{p,L} = 1000, \) and \( \text{Re}_z = 350 \).

**Figure 7.12:** Neutral curves (A) in the \((\text{Re}_z, \mu)\) - plane and (B) in the \((\text{Re}_z, \sigma_t)\) - plane for two-dimensional traveling waves for \( \alpha = 1 \) and \( \delta = 0 \).

**Figure 7.13:** The global critical curves for \( \text{Re}_z = 350, 500 \).

**Figure 7.14:** Variations of the amplification rate \( \sigma_t \) as a function of \( \text{Re}_z \) for \( \alpha = 1, \delta = 0, \mu = 1 \).

**Figure 7.15:** Variations of the amplification rate as a function of \( Ra_{p,L} \) for \( \alpha = 1, \delta = 0, \mu = 1 \) and \( \text{Re}_z = 350 \). The solid line shows results from the complete stability analysis. The dashed line shows results from the isothermal stability analysis of the flow field established by the heating (energy equation eliminated from the stability system).
List of Appendices

Appendix A: Evaluation of the Inner Products................................................. 212

Appendix B: Evaluation of Coefficients of Fourier Expansions Describing Variations of Values of Chebyshev Polynomials Evaluated along the Walls........................................ 215

Appendix C: Pressure Normalization............................................................... 217

Appendix D: Discretization of the Fixed Flow Rate Constraint in the Streamwise Flow... 219

Appendix E: Temporal Discretization.............................................................. 221

Appendix F: Extrapolation Formulae............................................................... 226

Appendix G: Fixed Flow Rate Constraint in the Streamwise Flow (Time – Dependent Algorithm)........................................................................................................ 227

Appendix H: Orientation of Gravity Vector..................................................... 229

Appendix I: Copyright Releases...................................................................... 230
# List of Abbreviations and Symbols

## Abbreviations

<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>c.c.</td>
<td>complex conjugate</td>
</tr>
<tr>
<td>CI</td>
<td>Commensurability index</td>
</tr>
<tr>
<td>DNS</td>
<td>Direct numerical simulations</td>
</tr>
<tr>
<td>FFT</td>
<td>Fast Fourier transform</td>
</tr>
<tr>
<td>HC</td>
<td>Horizontal convection</td>
</tr>
<tr>
<td>IBC</td>
<td>Immersed boundary conditions</td>
</tr>
<tr>
<td>ODE</td>
<td>Ordinary differential equation</td>
</tr>
<tr>
<td>OS</td>
<td>Orr-Sommerfeld</td>
</tr>
<tr>
<td>PDE</td>
<td>Partial differential equation</td>
</tr>
<tr>
<td>RBC</td>
<td>Rayleigh-Bénard convection</td>
</tr>
<tr>
<td>SVD</td>
<td>Single value decomposition</td>
</tr>
<tr>
<td>TS</td>
<td>Tollmien-Schlichting</td>
</tr>
<tr>
<td>VC</td>
<td>Vertical convection</td>
</tr>
</tbody>
</table>

## Symbols

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>Pressure gradient in $x$-direction</td>
</tr>
<tr>
<td>$A_L^{(n)}, A_U^{(n)}$</td>
<td>Coefficients of the Fourier expansions describing shape of corrugations at the lower and upper walls in computational domain</td>
</tr>
<tr>
<td>$B$</td>
<td>Pressure gradient in $z$-direction</td>
</tr>
</tbody>
</table>
\begin{itemize}
  \item $B_L, B_U$ Amplitude of lower and upper wall corrugation
  \item $c$ Specific heat
  \item $d_L, d_U$ Coefficients of Fourier expansions describing first derivative of Chebyshev polynomials at the lower and upper walls
  \item $E$ Eigenvector
  \item $E_L, E_U$ Coefficients of Fourier expansions describing Chebyshev polynomials at the lower and upper walls
  \item $E_k$ Kinetic energy
  \item $\Delta E_k$ Change in kinetic energy
  \item $f^{<m>}$ Fourier coefficient for base flow quantities
  \item $F_x, F_z$ Force in the $x$ and $z$-direction
  \item $g^{<m>}$ Fourier coefficient for disturbance flow quantities
  \item $g$ Gravitational acceleration
  \item $g$ Wave number
  \item $G\psi, G\theta$ Chebyshev coefficients for stream function and relative temperature
  \item $h$ Half of the channel height
  \item $\mathcal{H}$ Wave number
  \item $H_L^{(n)}, H_U^{(n)}$ Coefficients of the Fourier expansions describing shape of corrugations at the lower and upper walls in physical domain
  \item $I_k$ Integrals of Chebyshev polynomials
  \item $k$ Thermal conductivity
  \item $\vec{n}_L$ Normal unit vector
  \item $N_G, N_T, N_M, N_B, N_D$ Number of Fourier modes
  \item $N_K, N_c$ Number of Chebyshev polynomials
\end{itemize}
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_{uu}$, $N_{u\theta}$</td>
<td>Nonlinear terms of momentum and energy equations</td>
</tr>
<tr>
<td>$N_{vv}$, $N_{v\theta}$</td>
<td>Nonlinear terms of momentum and energy equations</td>
</tr>
<tr>
<td>$Nu_{loc}$, $Nu_{av}$</td>
<td>Local and mean Nusselt number</td>
</tr>
<tr>
<td>$Nu_e$</td>
<td>Nusselt number correction</td>
</tr>
<tr>
<td>$Nu_{cond}$</td>
<td>Nusselt number associated with conduction</td>
</tr>
<tr>
<td>$Pr$</td>
<td>Prandtl Number</td>
</tr>
<tr>
<td>$\phi_x$, $\phi_z$</td>
<td>Mean pressure gradients in the $x$- and $z$-directions</td>
</tr>
<tr>
<td>$Q_c$</td>
<td>Flow rate correction</td>
</tr>
<tr>
<td>$Q_x$, $Q_z$</td>
<td>Flow rates in the $x$- and $z$-directions</td>
</tr>
<tr>
<td>$Ra_c$</td>
<td>Critical Rayleigh number</td>
</tr>
<tr>
<td>$Ra_{uni}$</td>
<td>Uniform Rayleigh number</td>
</tr>
<tr>
<td>$Ra_{p,L}$, $Ra_{p,U}$</td>
<td>Lower and upper periodic Rayleigh number</td>
</tr>
<tr>
<td>$t$</td>
<td>Time</td>
</tr>
<tr>
<td>$t_{init}$</td>
<td>Initial time</td>
</tr>
<tr>
<td>$\Delta t$</td>
<td>Time step</td>
</tr>
<tr>
<td>$T$</td>
<td>Time period</td>
</tr>
<tr>
<td>$T_k$, $T_r$</td>
<td>Chebyshev polynomial</td>
</tr>
<tr>
<td>$T_L$, $T_U$</td>
<td>Lower and upper wall temperature</td>
</tr>
<tr>
<td>$T_{L,mean}$, $T_{U,mean}$</td>
<td>The mean temperatures of the lower and upper walls</td>
</tr>
<tr>
<td>$\tilde{T}<em>{p,L}^{(n)}$, $\tilde{T}</em>{p,U}^{(n)}$</td>
<td>Coefficients of Fourier expansions describing periodic components of temperatures at the lower and upper walls</td>
</tr>
<tr>
<td>$\vec{u}$</td>
<td>Velocity vector</td>
</tr>
<tr>
<td>$U_{max}$</td>
<td>Maximum velocity in $x$-direction</td>
</tr>
<tr>
<td>$U_v$</td>
<td>Velocity scale</td>
</tr>
</tbody>
</table>
Velocity vector

Angular velocity

Streamwise velocity modification

Maximum velocity in $z$-direction

Y coordinate in computational domain

Lower and upper wall geometry

Locations of extremities of the upper and lower walls

Greek Symbols

Wave number

Critical wavenumber

Thermal expansion coefficient

Angles associated with gravity vector

Real disturbance wavenumbers in the $x$-direction

Thermal diffusivity

Wavelength

Real disturbance wavenumbers in the $z$-direction

Kinematic viscosity

Relative temperature

Difference between the maximum and minimum of the lower and upper periodic temperature components

Density

Complex frequency
\( \sigma_i \) The rate of growth of disturbances
\( \sigma_r \) Disturbance frequency
\( \bar{\sigma} \) Stress vector
\( \omega \) Weigh function
\( \vec{\omega} = (\xi, \eta, \phi) \) Vorticity vector
\( \Omega \) Phase shift between the upper and lower temperature patterns
\( \Omega_G \) Phase shift between both corrugation systems
\( \Omega_{T,L} \) Phase shift between the lower corrugation and the lower temperature pattern
\( \Omega_{T,U} \) Phase shift between the lower corrugation and the upper temperature pattern
\( \xi \) Maximum of the spanwise gradient of the longitudinal velocity component at the middle of the channel
\( \psi \) Stream function

Subscripts

0 Reference and isothermal quantities
\( L, U \) Refer to the lower and upper walls
\( B, D \) Stationary and disturbance quantities

Superscripts

\( m, n, s \) Index for Fourier modes
\( * \) Denotes dimensional quantity
Chapter 1

1 Introduction

1.1 Objective

The main objective of this dissertation is to analyze the spatially modulated flow and to develop suitable algorithms for the analysis. Heating patterns and surface roughness are used to achieve spatial modulation, which can be implemented for intentional flow actuation. The analysis includes the effect of modulations on the flow structure, flow resistance, heat transfer, and transition to the secondary state. A fundamental understanding of these effects provides a starting point for the development of mixing enhancement techniques for laminar flows.

The specific objectives of this work are as follows –

(i) To develop a spectrally accurate numerical method for solving fluid flow and heat transfer problems with boundary irregularities.
(ii) To create streaks/rolls using heating patterns, which is of interest for chaotic mixing.
(iii) To use longitudinal grooves for intensification of streaks.
(iv) To develop a specialized algorithm for analyzing the stability characteristics of modulated flows.
(v) To identify critical conditions when streaks/rolls jump to the secondary state.

1.2 Motivations

The use of spatial modulations for the purpose of flow control is a very common practice. Over the years, modulations in terms of topography patterns, wall transpiration, distributed heating, and surface vibrations have been extensively studied for drag reduction,
propulsion augmentation, and energy-efficient chaotic stirring (Yadav, Gepner & Szumbarski 2021; Fukunishi & Ebina 2001; Jiao & Floryan 2021). Amongst other forms of modulation, existing literature suggests that the use of distributed heating has attracted less attention from the scientific community (Hughes & Griffiths 2008). Unlike the more commonly explored Rayleigh-Bénard configuration, distributed or patterned heating initiates convective motion due to the horizontal temperature gradient. Numerous examples of this fundamentally different form of convection can be spotted in the atmosphere, ocean, earth’s mantle, as well as in industrial processes (Rossby 1965; Lenardic et al. 2005; Vila et al. 2016). The temperature gradient between zonal and meridional surfaces determines large-scale atmospheric circulation patterns (Karamperidou, Cioffi & Lall 2012). Dissimilar thermal properties of roofs, streets, and parks in the metropolitan city (see Fig 1.1) cause local circulation in the atmosphere, while the same effect may occur in the countryside steered by variations in the heating rates of forests, fields, and lakes. System of fractures, leads and polynyas in sea ice can trigger horizontal convection in the ocean and atmosphere (Marcq & Weiss 2012). Distributed heating also plays a relevant role in the dynamics of localized fires and their propagations’ prediction (Finney et al. 2012).

**Figure 1.1** Thermal map of Atlanta, Georgia. (NASA/Goddard Space Flight Center Scientific Visualization Studio)

2
Besides a long list of geophysical events, recent research shows that heating patterns reduce frictional losses in laminar flows (Floryan & Floryan 2015; Hossain, Floryan & Floryan 2012; Hossain & Floryan 2015). A combination of groove and heating patterns may lead either to a significant reduction or a significant increase of losses depending on the relative position of both patterns (Hossain & Floryan 2020) and may activate the pattern interaction effect (Floryan & Inasawa 2021), which, in turn, generates thermal drift (Abtahi & Floryan 2017; Inasawa, Hara & Floryan 2021). Streamwise differential heating has been used for all these analyses. Past literature does not report the effect of spanwise heating patterns on flow behavior.

Again, the ability to mix effectively is imperative in many applications. Turbulence is suitable for mixing augmentation but not a viable solution in all circumstances. For example, delicate fluids with long molecular chains (used in medical fields) can be damaged by turbulent stresses. Besides, pressure losses along the conduit rise in turbulent flows. Consequently, a technique to enhance mixing in laminar flows is desired. Streaks can facilitate mixing intensification. But it naturally occurs only in high Reynolds number flows, implying that external forcing is necessary to create streaks in laminar flows. Spanwise heating patterns can be a suitable option in this regard.

Understanding a system equipped with spanwise patterned heating and surface roughness is still incomplete. Simultaneously, the formation of streaks in laminar flow represents a significant challenge. The current work focuses on the creation of streaks and their stability conditions due to spatially modulated flows. Determination of the most effective spatial patterns to produce desired flow characteristics will require analysis of thousands of patterns that cannot be handled using the standard grid-based methods and justifies the development of suitable algorithms. These algorithms are not limited to these particular applications but can be applied to solve a wide range of fluid flow and heat transfer problems. On a final note, the present work is fundamental research which aims to expand the knowledge about patterned heating and will provide the foundation for future applications such as mixing intensification.
1.3 Literature Review

The existing literature on convective flows is vast and diverse. Discussion of the following section is organized in a way that it focuses on the literature related to the objectives of the present work.

1.3.1 Horizontal Convection

Analysis of thermally driven flows is commonly performed based on the three classical systems (Shishkina 2017), i.e., Rayleigh-Bénard convection (RBC) (Bénard 1900; Rayleigh 1916), vertical convection (VC) (Ng et al. 2015), and horizontal convection (HC) (Hughes & Griffiths 2008). RBC is the most extensively studied form of convection, where a fluid layer is heated from the bottom and cooled from the top. In VC, heating and cooling are applied over the opposite vertical walls. For both configurations, the addition and removal of heat occur through two different parallel surfaces (Fig 1.2 A, B). On the contrary, heating and cooling are applied to different locations on the same horizontal surface in HC (Fig 1.2 C). The slightest temperature difference in the horizontal direction sets the fluid into motion; thereby, the critical Rayleigh number for HC is zero (Paparella & Young 2002). This type of convection is vital in geophysical flows, especially in ocean circulation (Stern 1975). However, for an extended period of time, the geophysical community has disregarded its importance in large-scale overturning flows (Scotti & White 2011) due to a century-old experiment by Sandström (1908). According to Sandström, sustained circulation was possible only when the heating source was situated at a level lower than the cooling source. This is known as Sandström's theorem, and it was widely cited until Rossby demonstrated a steady large-scale circulation due to differential heating in 1965 (Rossby 1965). Modern experiments also support Rossby's observation (Mullarney, Griffiths & Hughes 2004; Wang & Huang 2005). Lately, Mullarney et al. (2006) utilized the concept of HC to study ocean overturning circulation, where differential heating was imposed at the top surface and additional heat flux at the bottom.
Horizontal convection has also been studied as a method for flow control in channels. Such convection reduces pressure losses in the horizontal channel (Insawa, Taneda & Floryan 2019) as well as the driving force in case of the relative motion between two parallel plates (Floryan, Shadman & Hossain 2018). Rolls due to the horizontal convection act as separation bubbles, responsible for the reduction of losses at low Reynolds numbers. High
flow velocity washes away these separation bubbles. It is known that HC in a slot exhibits a strong dependence on the Prandtl number $Pr$, where critical conditions for the frequently studied RBC do not depend on $Pr$. Differential heating of the lower surface produces symmetric convection patterns to that of the upper surface (Hossain and Floryan 2014). For low-intensity HC, the flow structure is locked in with the heating pattern, while high-intensity HC results in a transition to secondary states that may have different structures (Hossain & Floryan 2013). The form of secondary convection changes significantly with the heating wave numbers and may lead to frustrated systems and responses in the form of soliton lattices (Hossain & Floryan 2013; Nixon et al. 2013). Analysis of heat transfer resulting from the HC in an infinite horizontal slot can be found in Hossain & Floryan (2013a).

Furthermore, HC is relevant in astrophysical systems (Hanasoge, Gizon & Sreenivasan 2016) and can drive engineering processes such as glass melting (Gramberg, Howell & Ockendon 2007; Chiu-Webster, Hinch & Liter 2008).

1.3.2 Surface Roughness / Grooves

The effect of surface roughness is one of the classical topics in fluid dynamics. Due to the large variety of possible roughness forms, it is difficult to draw any general conclusion about their effects. Mainly, the analysis of roughness is focused on determining the changes of resistance experienced by a flow. Traditionally, it is believed that roughness always increases the surface area and, therefore, increases the flow resistance (Hagen 1854; Darcy 1857). Nikuradse (1933) and Moody (1944) quantified the overall drag in terms of friction factor and demonstrated that roughness does not affect drag in laminar flow or the effect is too small to be measured. Walsh (1979) first contradicted this longstanding belief in turbulent flow and showed that certain surface geometries could reduce drag, changing the near-wall structure of the turbulent boundary layer. The use of roughness in laminar flow attracted less attention, and the verdict regarding their effect on laminar drag was amended by current research focused on microchannels (Gamrat et al. 2008).
Roughness has been studied significantly with a view to delaying/promoting laminar-turbulent transition. The general view is that a hydraulically rough wall always promotes transition (Reynolds 1883), while a hydraulically smooth wall does not affect the transition. Recent work demonstrated that roughness could be stabilizing and, thereby, could delay the onset of transition (Sarric, Carrillo & Reibert 1998). Separating roughness elements from each other eliminates interaction between them, where they behave as individual roughness elements and have hardly any effect on the flow (Floryan & Asai 2011). However, a system of such roughness may result in significant changes in flow stability. It is known that two-dimensional distributed roughness can destabilize travelling wave disturbances (Floryan 2006), and the same class of roughness amplifies disturbances in the form of streamwise vortices (Floryan, 2007). Experimental study confirms these theoretical predictions (Asai & Floryan 2006).

Grooves are also known for influencing heat transfer characteristics (Lee et al. 2003). A lot of research has been conducted to optimize surface geometries with a view to enhancing the heat transfer rate. Rosaguti et al. (2007) demonstrated that the use of sinusoidal geometry results in high rates of heat transfer in comparison with straight pipe. The wavy channel is also a common choice for compact heat exchangers (Zhang, Kundu & Manglik 2004) due to their superior thermo-hydraulic performance (Manglik, Zhang & Muley 2005).

Rib arrays inside the channel are frequently used to enhance the heat transfer rate. Park et al. (1992) compared different rectangular channels with parallel angled ribs and found effective angles for narrow and wide aspect ratio channels. Tanda (2004) quantified the local heat transfer coefficient for transverse and V-shaped broken ribs for different Reynolds numbers in turbulent regimes. It showed that heat transfer augmentation is higher in transverse broken ribs relative to the reference smooth channel. Optimization of rib-roughened shape to enhance turbulent heat transfer was done by Kim & Kim (2002). However, the heat transfer enhancement of the scale roughened surface (Chang, Liou & Lu 2005) is surprisingly good compared to rib roughened and dimpled surfaces (Ventola et al. 2014).
1.3.3 Streaks / Rolls

Streaks are characterized by the alternating zones of high and low speed fluid in the near-wall region due to the wall normal motions. The streak velocity profile has inflection points and disturbances of such velocity profile grow much faster. It is known that rolls and streaks play a crucial role in the dynamics of shear layers, including instabilities, the laminar-turbulent transition and turbulence itself (Buttler & Farrel 1992; Waleffe 1997, 2003; Chernyshenko & Baig 2005; Jiménez 2013). Rolls are naturally occurring vortices in the cross-plane of the flow. They transport high-speed fluid toward the wall (downwash) and low-speed fluid away from the wall (upwash), thereby creating streaks in the main flow velocity field (Floryan 1991). The natural formation of streaks generally takes place in high Reynolds number flows, indicating that an external forcing may be required to generate streaks in low Reynolds number laminar flows. These streaks are themselves subject to instabilities (Park, Hwang & Cossu 2011) and can play a central role in the self-sustained process of wall-bounded turbulence (Waleffe 1997, Hwang & Cossu 2010). In the instability of laminar flows, either the linearly optimal roll structures are determined that maximize the transient temporal growth of streaks (Schmid & Henningson 2001) or the growth of the normal modes leads to secondary instabilities, eventually activating nonlinear effects (Floryan 1991). Because of their role in creating instabilities as well as in the modification of turbulence, streaks can be used to improve fluid mixing and heat transfer.

There is a lack of mixing intensification techniques suitable for low Reynolds number flows which do not incur high-pressure loss penalty (Bergless 1998, 2001; Ligrani, Oliveria & Blaskovich 2003; Siddiqui et al. 2010) – streaks could provide the means for the efficient stirring of such flows. The naturally occurring streaks require high Reynolds numbers, and their structure cannot be directly controlled. Streaks can be created intentionally at low Reynolds numbers and in a controlled manner using an external forcing whose pattern determines the streaks' distribution. It has been recently shown that streaks created by longitudinal grooves lead to a new instability mode (Moradi & Floryan 2014; Mohammadi, Moradi & Floryan 2015) whose growth leads to chaotic stirring at small Reynolds numbers while, at the same time, pressure losses decrease beneath those for flows without grooves.
Using uniform heating leads to natural convection in the form of Rayleigh-Bénard (RB) rolls (Bénard 1990; Rayleigh 1916), whose critical wavenumber determines the form of the rolls in the fully developed shear layers (Kelly 1994). Similar rolls can be created in developing shear layers (Akiyama, Hwang & Cheng 1971; Wu & Cheng 1976; Moutsoglou, Chen & Cheng 1981; Chen & Chen, 1984). Rolls may be created by other natural instabilities (Floryan 2006; Moradi, Budiman & Floryan 2017), but their spatial structure is dictated by the critical wavenumber at the onset. Patterned heating provides good control of roll formation as it creates a horizontal field of buoyancy forces with the desired spatial distribution. The rolls are driven by horizontal temperature gradients and occur regardless of the heating intensity – they represent a forced response rather than a bifurcation from a conductive state, as is the case for RB convection.

The spanwise heating patterns can produce large spanwise changes in the velocity profile despite weak convective flows, leading to the formation of rolls and streaks. Therefore, such one-dimensional patterns are of interest for instability. A detailed description of the primary and secondary convection driven by heating patterns characterized by a single Fourier mode applied at the lower wall is given in Hossain & Floryan (2013).

Limited information is available on the direct effects of streaks on heat transfer and fluid stirring. Experimental assessments of the effects of hot streaks on high-pressure turbine vanes can be found in Povey et al. (2007) and Barrigozi et al. (2017). Demonstrations that rolls can locally increase heat transfer in laminar boundary layers above that found in turbulent flows are presented in Méndez, Shadloo & Hadjadi (2020) and Marques, Rogenski & De Souza (2021). An analysis of the effects of streaks in flames is described in Miller et al. (2018). The significant role of rolls in atmospheric boundary layers is discussed in Foster (2005) and Jayaraman & Brasseur (2021).
1.3.4 Modelling of Heating & Roughness Patterns

The use of heating and roughness patterns for the purpose of flow control or mixing intensification essentially requires the analysis of thousands of patterns and their combinations. It is known that friction and pressure effects are responsible for the resistance experienced by the fluid. Accurate prediction of pressure drag requires a very accurate evaluation of pressure at the complex wall surface, while the prediction of friction requires a very accurate determination of velocity gradients in the vicinity of these walls. At this, one needs to use an algorithm capable of resolving flow fields in a multitude of complex geometric configurations with high accuracy and minimal user involvement.

The literature discusses two distinct methodologies for handling small amplitude grooves. The first one is the effective boundary condition concept (or the equivalent surface concept) that replaces the complex surface geometry with a smooth surface and augments the no slip and no penetration boundary conditions to account for the complex geometry. While it is argued that this concept is effective when the scale of flow features is large compared to surface features, it is unclear how to make meaningful comparisons. This concept has a long history with many forms of implementation (Nye 1969; Richardson 1973; Miksis & Davis 1994; Tuck & Kouzoubov 1995; Sarkar & Prosperetti 1996; Ponomarev & Meyerovich 2003; Bazant & Vinogradova 2008). It is used in numerical simulations with the no slip conditions replaced by a partial slip (Rothstein, 2009). The validity of this concept cannot be assessed until one is able to determine the actual flow over a corrugated wall. The second concept is domain perturbation. It takes advantage of the small groove amplitude and relies on the boundary conditions transfer procedure, which leads to linearization about the mean wall position. Such linearization may not be acceptable as it truncates flow physics governed by nonlinear effects. Attempts to firm up the basis of this approach by including higher-order terms in the transfer procedure (Kamrin, Bazant & Stone 2010) were met with marginal success (Cabal, Szumbarski & Floryan 2001).

The equivalent surface concept truncates the flow physics, and it is not known if the resulting error is acceptable. The correct approach is to determine the flow for the actual wall geometry. The first possible approach is to use grid-based methods—their use to
quantify mixing intensification in longitudinally grooved channel can be found in Gepner et al. (2020) and Yadav et al. (2017). Grids are computed numerically to map the geometry, but this process is labor intensive and prohibitively expensive if many geometries need to be investigated. It also becomes computationally expensive if one needs to use very fine grids to reduce error. Spectral elements can improve the accuracy but solution matching between different elements leads to difficulties when very deformed geometries are of interest (Karniadakis & Sherwin 2013; Cantwell et al. 2015). An analytical mapping provides an alternative suitable for certain classes of geometries but an increased complexity of the field equations leading to a full coefficient matrix makes it computationally expensive (Cabal, Szumbarski & Floryan 2001; Hamed & Floryan 1998; Cabal, Szumbarski & Floryan 2002). There are specialized methods able to provide very accurate geometry models, for example, the Schwarz-Christoffel transformation, where one does not map the geometry directly but rather evaluates mapping parameters (Floryan 1985; Floryan & Zemach 1987, 1988, 1993). The final concept developed in the context of moving boundary problems involves the use of fictitious or immersed boundaries (Floryan & Rasmussen, 1989) and provides the means for an efficient analysis of multiple geometries. More recent implementations can be found in Peskin (1981, 2001), Mittal & Iaccarino (2005), and Taira & Colonius (2007). These methods use low-order spatial discretization as well local fictitious forces to enforce the no-penetration and no-slip conditions which may result in inaccurate prediction of wall shear stresses.

For the sake of present work, spectral method has been selected due to their high spatial accuracy. Their applicability is limited to regular geometries, but this limitation can be overcome using the immersed boundary conditions (IBC) concept (Szumbarski & Floryan 1999). The IBC method uses formal construction of boundary constraints to generate the required closing relations. The spatial discretization uses global basis functions (Fourier and Chebyshev expansions) and provides the means to reduce the discretization error to machine level. The method uses global basis functions spanning the complete solution domain and does not rely on any grids. The method involves two types of Fourier expansions, one for the field equations and one for the boundary relations and, thus, the rate of convergence of both expansions determines the limits of its applicability. The programming effort associated with accounting for changes in geometry and thermal
boundary conditions is reduced to the specification of a set of Fourier coefficients which need to be provided as an input.

1.3.5 Algorithm for Linear Stability Analysis

Flow transitions are generally studied either using linear stability analysis (tracing if the infinitesimal perturbation amplifies exponentially at an asymptotically long time; Drazin & Reid 2004) or transient growth analysis of an initial perturbation (Schmid & Henningson 2001). The transition of modulated flows to secondary states restricts the effectiveness of modulations when drag reduction is of interest. This transition may be desired when mixing intensification is of interest (Rousta, Shirani & Habibi 2022). The current literature shows that flow transitions in complex geometries can be captured using direct numerical simulations (DNS) (Reed 2009), but such a strategy's limitations are not acknowledged.

The computational cost of simulations is known to be significant, but the real limitation lies elsewhere. Natural transition starts with the growth of disturbances, but answering the stability question requires analysis of all possible disturbances. Suppose that the spatial distribution of flow modulations is characterized by wavenumber \( \alpha \) and the spatial distribution of flow disturbances is characterized by wavenumber \( \delta \). The properties of the complete flow depend on the ratio of these wavenumbers. Integer-valued ratios describe simple commensurate states with either subharmonic or super harmonic properties and can be handled using DNS. Non-integer-valued but rational ratios may lead to more complex commensurate states where a slight change of this ratio results in an order-of-magnitude change in the overall system wavelength, as illustrated for a simple two-wavenumber system in Fig 1.3. Commensurate states form a countable set. Irrational ratios lead to aperiodic systems, forming an uncountable set (the product of an irrational and rational number is irrational).

DNS can handle only a restrictive class of commensurate systems (Blancher, Le Guer & El Omari 2015; Gepner & Floryan, 2016) as the size of the computational box is limited by the available memory. Extremely powerful computers shall help to overcome this issue. Continuous change of the disturbance wavenumber is not possible as it results in huge
variations of the computational domain, with the stability results being a function of the size of the computational box (Blancher, Le Guer & El Omari 2015; Gepner & Floryan, 2016).

![Figure 1.3](image.png)

**Figure 1.3** Wavelength $\lambda$ of a disturbed flow system as a function of disturbance wavenumber $\delta$. Red circles and blue triangles provide wavelengths of the complete system modulated with wavenumbers $\alpha = 1$, and $\alpha = 1.1$, respectively. Vertical dotted lines identify two possible incommensurate states where the system is aperiodic.

Re-gridding required by changes in the box length represents another significant difficulty due to labor cost. Irrational wavenumber ratios lead to aperiodic incommensurate states, which require an infinite computational box; such states are not accessible to numerical computations due to the inherent truncation error. One may conclude that DNS is suitable for a detailed exploration of particular cases, as the computational cost can be justified; however, it cannot address the stability question in general and is unsuitable for exploring the full parameter space required for identifying the most effective modulations either from the drag reduction or mixing intensification points of view.
Identifying the most effective modulations requires their systematic analysis, which can be accomplished using a process that starts with identifying stationary states, then determining their stability properties, and culminates with establishing saturation states and their domains of attraction. The most promising configurations can then be studied in detail using DNS. Such a strategy requires the development of a stability algorithm. Additional complications may arise due to the ability of spatial modulations to activate the spatial parametric resonance (Hossain & Floryan 2013; Hossain & Floryan 2022), which is poorly understood. Providing information about the flow topology past the bifurcation point is another stability analysis’ benefit.

Stability analysis requires a formulation capable of dealing with various commensurate systems and an accurate numerical method to solve the disturbance equations. The stability of unmodulated shear layers led to the Orr-Sommerfeld equation, whose spectrally accurate solution was given by Orszag (1971). The formulation of the stability problem when modulations do not change geometry was given by Floryan (1997), who represented disturbance quantities as waves with amplitudes modulated by base flow variations. This formulation avoids the need to use large computational boxes and frees stability results from their dependence on the size of the computational box. Cabal et al. (2002) implemented this concept in an algorithm that handles modulations associated with surface topographies. This algorithm relied on domain transformation, mapping a complex channel geometry into a smooth slot, and using spectral discretization. It is cumbersome to use as it involves very complex field equations. Therefore, development of a linear stability algorithm would be useful that is flexible enough to accurately and efficiently handle a large class of geometric and physical modulations in a relatively simple manner.

1.3.6 Flow Instabilities & Mixing

Efficient mixing is essential in many technological processes. It is a two-stage process involving diffusion overlaid on top of mechanical stirring (Eckart 1948; Ottino 1989; Villermaux 2019). The former is generally slow as it relies on molecular-level mechanisms. The latter, i.e., stirring, promote mixing as it leads to the stretching and folding of fluid
interfaces leading to increased concentration gradients and, thus, enabling the otherwise slow diffusion to act more rapidly and across shorter distances. Stirring can be controlled either through the creation of proper fluid movements or modification of the existing movements. The literature contains many contributions focused on increasing mixing efficiency through the intensification of stirring (Bergles 1998).

Mixing in laminar flows is inefficient due to well-organized fluid movement (Bergles 2001; Ligrani, Oliveira & Blaskovich 2003; Siddique et al. 2010), so it requires stirring. Stirring implies energy cost expressed as additional pressure losses in the case of passive devices or direct energy expenditure in the case of active devices. These costs could be high due to a high level of dissipation in low Reynolds number flows (Bergles 1998, 2001), even if stirring is induced through activation of chaos (Aref 1984; Stroock et al. 2002). The energy cost could be reduced by taking advantage of natural flow instabilities where the flow transition is triggered using spatially distributed forcing with the magnitude of the response controlled by the spatial pattern rather than the amplitude of modulations. As infinite possible actuation patterns exist, one must identify those producing secondary states advantageous to stirring promotion. The use of longitudinal grooves provides an example where natural instability leads to the onset of chaotic stirring, with the overall energy cost being reduced below that required by maintenance of the unmodified flow or marginally increased (Gepner & Floryan 2020). The onset of this stirring relies on an inviscid instability mechanism activated by introducing surface modifications discovered by (Moradi & Floryan 2014; Mohammadi, Moradi & Floryan 2015).

Using spatial modulations for flow control attracted much attention in recent years and led to the discovery of new effects. However, there is a dearth of data dealing with using spatial modulation patterns for stirring intensification. Since it is known that streamwise grooves create spanwise flow structures, which create instabilities that lead to chaos in a saturation state (Gepner & Floryan 2020), it is justifiable to explore the effects of spanwise periodic heating on the stability of channel flow as this may provide an alternative route to chaotic stirring. Heating patterns produce vortices in the cross-plane, which transport low-speed fluid away from the wall (upwash) and high-speed fluid towards the wall (downwash),
creating longitudinal streaks (Floryan 1991). The primary role of streaks in mixing processes is to activate instabilities leading to saturation states with the desired properties.

There is a significant body of literature describing the naturally occurring rolls in large-Re flows as they play significant roles in shear layer instabilities and the transition to turbulence (Butler & Farrell 1992; Waleffe 1997, 2003; Chernyshenko & Baig 2005; Jiménez 2013; Park, Hwang & Cossu 2011). But rolls which lead to the instabilities in small and medium Re flows are yet to be identified. An external forcing may be required to generate rolls in such flows. This constraint is legitimate as naturally occurring rolls do not provide the means for controlling their structure and their instabilities. Patterned heating provides this convenience through the changes in the heating intensity and heating pattern.

Uniform heating creates a vertical temperature gradient and leads to natural convection in the form of Rayleigh-Bénard (RB) rolls (Bénard 1900; Rayleigh 1916), with the RB instability dictating the form of these rolls through the critical disturbance wavenumber (Kelly 1994; Akiyama, Hwang & Cheng 1971; Wu & Cheng, 1976; Moutsoglou, Chen & Cheng 1981; Chen & Chen 1984; Wang 1982). On the contrary, patterned heating creates horizontal temperature gradients effective in creating well-controlled rolls, which occur regardless of the heating intensity. A good amount of literature can be found on the flow stability conditions due to uniform heating as well as other types of modulations (Beaumont 1981, Floryan, Szumbarski & Wu 2002, Rivera-Alvarez & Ordonez 2013); however, no information is available on the stability characteristics of flows modulated by spanwise patterned heating or the combination of spanwise heating and roughness patterns.

### 1.4 Overview of the Present Work

The effect of spanwise heating and roughness patterns in the laminar channel flow has been analyzed from a fundamental perspective. The present analysis paves the way for flow control strategies and mixing intensification in low Reynolds number flows. Heating patterns are observed to create rolls, leading to streaks formation in the laminar channel.
flows. Streaks are known to play a vital role in driving instability and can improve mixing and heat transfer rates.

A spectrally accurate and very efficient algorithm suitable for the prediction of pressure losses in heated grooved channels has been developed. Heating and topography patterns are used to create spatial flow modulations resulting in a pattern interaction problem. Search for combinations of patterns resulting in the reduction of pressure losses requires the development of a very accurate and efficient algorithm. The proposed algorithm uses a combination of the Fourier expansions in the horizontal directions and the Chebyshev expansions in the vertical direction to provide an excellent resolution of the near wall regions. The immersed boundary conditions (IBC) method is used for the flow boundary conditions at the complex boundaries. The resulting gridless discretization can be easily adapted to handle a wide range of topography patterns. Various tests demonstrate that the algorithm delivers spectral accuracy. Comparisons with the standard open-source codes based either on the finite volume or the spectral element discretization demonstrate several orders of magnitude better efficiency of the proposed algorithm. This algorithm can only solve stationary states and uses a fixed-point iteration scheme to progress to convergence.

In addition, a fully implicit time-dependent algorithm has been developed that uses spectral spatial discretization and up to sixth-order temporal discretization. The algorithm allows the analysis of time-dependent temperature profiles in grooved channels as well as transition to the secondary states. Different tests confirm that the algorithm provides the expected accuracy. The use of extrapolation to determine the initial approximation for nonlinear terms essentially cuts down computational costs by up to 85%. This algorithm can also be employed as an alternative iterative scheme to the first algorithm for steady-state flows.

Streaks and rolls are of interest in mixing intensification. Streaks naturally occur in large Reynolds numbers. It is shown that they can be created in laminar flows using spatially distributed heating. The structure and strength of streaks can be controlled by adjusting the heating pattern. Change in flow losses and heat transfer across the shear layer due to streaks formation has been determined. The creation of streaks using the Rayleigh-Bénard (RB)
instability was also studied — a heating intensity exceeding the critical Rayleigh number was required, with the spatial structure of the streaks dictated by the critical wavenumber.

The use of streamwise grooves for the intensification of streaks created by heating in shear layers has been investigated. Three different ranges of groove wavenumbers were analyzed. It is shown that uniform heating of a grooved surface produces intense streaks only when the groove wavenumber is near the critical RB wavenumber and the heating intensity exceeds the critical RB intensity. Long wavelength grooves reduce flow losses, but the resulting streaks are less intense. The use of heating patterns tuned with groove patterns can produce very intense streaks whose spatial distribution is easily controlled through the selection of the patterns' wavenumber. An increase in flow losses due to patterned heating can be compensated using spatial groove distributions with drag-reducing capabilities. It has been demonstrated that the most effective wavenumber producing high-intensity streaks at low flow losses is between the RB wavenumber and the drag-reducing wavenumbers; this optimal wavenumber has been identified.

An algorithm for analyzing the linear stability of flows modulated by grooves and heating patterns has been developed. This work is motivated by the interest in using streaks and rolls to intensify flow mixing at low Reynolds numbers. Identifying unstable modes through stability analysis is the primary step to forecast chaotic stirring. The algorithm avoids the limitations of classical DNS-based approaches and can deal with the pattern interaction effects activated by patterns of different physical quantities. Various tests demonstrate that the algorithm delivers spectral accuracy for eigenvalues and eigenfunctions. The algorithm can analyze a large number of geometric patterns with a minimal commitment to the user's time.

A stability analysis of thermally modulated channel flow has been carried out. A smooth channel subjected to periodic heating from the bottom was considered. Two different types of unstable modes were identified. One is a modified Tollmien-Schlichting mode, and the other is a new instability mode that exists at low Reynolds number. It is shown that the new instability mode connects to the Squire spectrum in the isothermal limit. The distribution of disturbance velocities suggests that movements are strong in the vicinity of the walls for
the Tollmien-Schlichting instability modified by the heating, while intense motions are found in the middle of the channel for the new instability mode. This new instability mode is of interest for mixing at a low Reynolds number. Detailed analysis of the new instability mode has been performed, and critical conditions have been determined. It is further demonstrated that the new instability is driven by the inviscid mechanism related to the inflection point.

1.5 Outline of the dissertation

This dissertation is organized into eight chapters. Chapter 1 provides the main objectives and motivations of the current work, as well as the relevant literature review. Chapter 2 describes the development of a spectrally accurate algorithm for the systematic steady-state analysis of spatially modulated flow, by surface roughness and heating patterns, in a three-dimensional channel. A highly accurate algorithm for the transient analysis of the modulated flow is presented in chapter 3. Chapter 4 discusses the effect of periodic and uniform heating on the formation of streaks and heat transfer characteristics. Effects of surface roughness along with the uniform and non-uniform heating are analyzed in Chapter 5. Chapter 6 describes and verifies a linear stability algorithm capable of tracing the onset of instability in spatially modulated shear layers. The transition to the secondary state due to periodic heating is described in Chapter 7. Chapter 8 summarizes the main conclusions and provides suggestions for future work.
Chapter 2

2 Algorithm for Analysis of Pressure Losses in Heated Channels\textsuperscript{1}

2.1 Introduction

This chapter is focused on the development of a spectrally accurate algorithm suitable for a systematic analysis of flows within conduits modified by surface corrugations and spatial heating patterns. The algorithm uses Fourier expansions in the horizontal directions and Chebyshev expansions in the wall-normal direction for discretization of the field equations and boundary conditions. The resulting gridless discretization can be easily adapted to handle a wide range of topography patterns with a very good resolution of near wall region. The Immersed Boundary Conditions (IBC) method is used to enforce flow boundary conditions at the geometrically irregular boundaries. Section 2.2 provides description of a model problem used to present the algorithm, Section 2.3 presents the analysis of the Flow Problem, Section 2.3.1 describes the spanwise convection Problem, Section 2.3.2 discusses the streamwise flow problem, Section 2.4 describes extraction of surface forces and heat fluxes and discusses other postprocessing issues. Section 2.5 discusses algorithm performance. Section 2.6 summarizes the main conclusions.

2.2 Problem Formulation

Consider a channel formed by two horizontal corrugated walls (see Figure 2.1) with longitudinal grooves periodic in the spanwise direction.

\textsuperscript{1} A version of this chapter has been published as –

These groove geometries are described by Fourier expansions of the form

\[ y_L^*(x^*) = -h^* + \sum_{n=-N_G}^{n=N_G} \hat{H}_L^{*(n)} e^{in\alpha^* x^*}, \]  

(2.2.1a)

\[ y_U^*(x^*) = h^* + \sum_{n=-N_G}^{n=N_G} \hat{H}_U^{*(n)} e^{in(\alpha^* x^*+\Omega^*_U)} \]  

(2.2.1b)

where the subscripts \( L \) and \( U \) refer to the lower and upper walls, respectively, \( \hat{H}_L^{*(n)} \) and \( \hat{H}_U^{*(n)} \) are the coefficients of the Fourier expansions describing the shape of corrugations at the lower and upper walls, respectively, \( N_G \) is the number of Fourier modes required to describe the geometry, \( \Omega^*_U \) stands for the phase shift between both corrugation systems, \( \alpha^* \) and \( \lambda^* \) denote their wave number and wavelength, respectively, and stars denote dimensional quantities. The expansion coefficients satisfy the reality conditions, i.e., \( \hat{H}_L^{*(-n)} \) and \( \hat{H}_U^{*(-n)} \) are the complex conjugates of \( \hat{H}_L^{*(n)} \) and \( \hat{H}_U^{*(n)} \), respectively. The channel is periodic in the spanwise \( x \)-direction with the wavelength \( \lambda^* = 2\pi/\alpha^* \) and extends to \( \pm\infty \) in both the \( x \)- and \( z \)-directions with the mean distance between the walls being \( 2h^* \). Scaling of the geometry using \( h^* \) as the length scale and extracting the difference between the crests and troughs of the upper and lower grooves \( B_L \) and \( B_U \) lead to geometry description in the form of

\[ y_L(x) = -1 + B_L \sum_{n=-N_G}^{n=N_G} H_L^{(n)} e^{in\alpha x} , \]  

(2.2.2a)

\[ y_U(x) = 1 + B_U \sum_{n=-N_G}^{n=N_G} H_U^{(n)} e^{in(\alpha x+\Omega_U)} \]  

(2.2.2b)

where

\[ -\frac{1}{2} \leq \sum_{n=-N_G}^{n=N_G} H_L^{(n)} e^{in\alpha x} = \sum_{n=-N_G}^{n=N_G} \frac{\hat{H}_L^{*(n)}}{B_L} e^{in\alpha x} \leq \frac{1}{2}, \]

\[ -\frac{1}{2} \leq \sum_{n=-N_G}^{n=N_G} H_U^{(n)} e^{in(\alpha x+\Omega_U)} = \sum_{n=-N_G}^{n=N_G} \frac{\hat{H}_U^{*(n)}}{B_U} e^{in(\alpha x+\Omega_U)} \leq \frac{1}{2}. \]

The gravitational acceleration \( g^* \) is acting in the negative \( y \)-direction. The steady, incompressible flow of a Newtonian fluid is driven in the positive \( z \)-direction by an externally imposed pressure gradient. The fluid has thermal conductivity \( k^* \), specific heat
$c^*$, thermal diffusivity $\kappa^* = k^*/\rho^*c^*$, kinematic viscosity $\nu^*$, dynamic viscosity $\mu^*$, thermal expansion coefficient $\beta^*$ and variations of the density $\rho^*$ follow the Boussinesq approximation. The reference temperature to be used for evaluation of material properties is defined later in this presentation.

![Figure 2.2 Schematic diagram of the flow system.](image)

The lower and upper walls are subject to the $x$-periodic heating resulting in the temperatures of the lower ($T_L^*$) and the upper ($T_U^*$) walls of the form

$$T_L^*(x^*) = T_{L, mean}^* + \sum_{n=\pm N_T}^{n=\pm N_T} \tilde{T}_{P,L}^*(n)e^{i\Omega_T^* x^*} e^{i\omega_T^* x^*},$$

(2.2.3a)

$$T_U^*(x^*) = T_{U, mean}^* + \sum_{n=\pm N_T}^{n=\pm N_T} \tilde{T}_{P,U}^*(n)e^{i\Omega_T^* x^*} e^{i\omega_T^* x^*},$$

(2.2.3b)

where $\tilde{T}_{P,L}^*(n)$ and $\tilde{T}_{P,U}^*(n)$ are the coefficients of Fourier expansions describing periodic components of temperatures at the lower and upper walls, respectively, $N_T$ is the number of Fourier modes required to describe temperature distribution, $\Omega_T^*$ stands for the phase shift between the lower corrugation and the lower temperature patterns, $\Omega_T^*$ stands for the phase shift between the lower corrugation and the upper temperature pattern, and $T_{L, mean}^*$
and $T_{U,\text{mean}}$ are the mean temperatures of the lower and upper walls, respectively. We select $T_{U,\text{mean}}^*$ as the reference temperature and define the relative temperature $\theta^* = T^* - T_{U,\text{mean}}^*$ resulting in the walls’ temperatures of the form

$$\theta_L^*(x^*) = \theta_{uni}^* + \sum_{n=-N_T,n \neq 0}^{n=N_T} \tilde{\theta}_L^{(n)} e^{i n (\alpha x^* + \Omega_T)}$$  \hspace{1cm} (2.2.4a)$$

$$\theta_U^*(x^*) = \sum_{n=-N_T,n \neq 0}^{n=N_T} \tilde{\theta}_U^{(n)} e^{i n (\alpha x^* + \Omega_T)}$$  \hspace{1cm} (2.2.4b)$$

where $\theta_{uni}^* = T_{U,\text{mean}}^* - T_{U,\text{mean}}^*$, $\tilde{\theta}_L^{(n)} = \tilde{T}_{L,n}^{(n)}$, $\tilde{\theta}_U^{(n)} = \tilde{T}_{U,n}^{(n)}$. We extract the difference between the maximum and minimum $\theta_{P,L}^*$ and $\theta_{P,U}^*$ of the lower and upper periodic temperature components, respectively, resulting in the walls’ temperatures of the form

$$\theta_L^*(x^*) = \theta_{uni}^* + \theta_{P,L}^* \sum_{n=-N_T,n \neq 0}^{n=N_T} \theta_L^{(n)} e^{i n (\alpha x^* + \Omega_T)}$$  \hspace{1cm} (2.2.5a)$$

$$\theta_U^*(x^*) = \theta_{P,U}^* \sum_{n=-N_T,n \neq 0}^{n=N_T} \theta_U^{(n)} e^{i n (\alpha x^* + \Omega_T)}$$  \hspace{1cm} (2.2.5b)$$

where

$$\frac{1}{2} \leq \sum_{n=-N_T,n \neq 0}^{n=N_T} \theta_L^{(n)} e^{i n (\alpha x^* + \Omega_T)} = \sum_{n=-N_T,n \neq 0}^{n=N_T} \frac{\tilde{\theta}_L^{(n)}}{\theta_{P,L}^*} e^{i n (\alpha x^* + \Omega_T)} \leq \frac{1}{2}$$

$$\frac{1}{2} \leq \sum_{n=-N_T,n \neq 0}^{n=N_T} \theta_U^{(n)} e^{i n (\alpha x^* + \Omega_T)} = \sum_{n=-N_T,n \neq 0}^{n=N_T} \frac{\tilde{\theta}_U^{(n)}}{\theta_{P,U}^*} e^{i n (\alpha x^* + \Omega_T)} \leq \frac{1}{2}$$

Use of $h^*$ as the length scale and $\kappa^* \nu^* / (g^* \beta^* h^* 3)$ as the temperature scale results in the dimensionless expressions for temperatures of the form

$$\theta_L(x) = Ra_{uni} + Ra_{P,L} \sum_{n=-N_T,n \neq 0}^{n=N_T} \theta_L^{(n)} e^{i n (\alpha x + \Omega_T)}$$  \hspace{1cm} (2.2.6a)$$

$$\theta_U(x) = Ra_{P,U} \sum_{n=-N_T,n \neq 0}^{n=N_T} \theta_U^{(n)} e^{i n (\alpha x + \Omega_T)}$$  \hspace{1cm} (2.2.6b)$$

where $Ra_{uni} = g^* \beta^* h^* 3 \theta_{uni}^* / (\kappa^* \nu^*)$ is the uniform Rayleigh number measuring the intensity of the uniform component of heating, $Ra_{P,L} = g^* \beta^* h^* 3 \theta_{P,L}^* / (\kappa^* \nu^*)$ is the lower periodic Rayleigh number measuring the intensity of the lower periodic heating and
\[ Ra_{p,U} = g^* \beta^* h^* \theta_{p,U} / (\kappa^* \nu^*) \] is the upper periodic Rayleigh number measuring the intensity of the upper periodic heating.

The dimensionless field equations can be written as

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \quad (2.2.7a) \]

\[ u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = -\frac{\partial p}{\partial x} + \nabla^2 u, \quad (2.2.7b) \]

\[ u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} = -\frac{\partial p}{\partial y} + \nabla^2 v + Pr^{-1} \theta, \quad (2.2.7c) \]

\[ u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} = -\frac{\partial p}{\partial z} + \nabla^2 w, \quad (2.2.7d) \]

\[ u \frac{\partial \theta}{\partial x} + v \frac{\partial \theta}{\partial y} + w \frac{\partial \theta}{\partial z} = Pr^{-1} \nabla^2 \theta \quad (2.2.7e) \]

where \( \bar{u} = (u, v, w) \) is the velocity vector with components in the \((x, y, z)\)-directions scaled with \( U_v^* = \nu^* / h^* \) as the velocity scale, \( p \) stands for the pressure scaled with \( \rho^* U_v^* \) as the pressure scale, \( \theta \) is the relative temperature scale with \( \kappa^* \nu^*/(g^* \beta^* h^* \nu^*) \) as the temperature scale and \( Pr = \nu^*/\kappa^* \) is the Prandtl number. The relevant boundary conditions at the walls have the form

\[ u[y_L(x)] = u[y_U(x)] = 0, \quad v[y_L(x)] = v[y_U(x)] = 0, \quad (2.2.8a,b) \]

\[ w[y_L(x)] = w[y_U(x)] = 0, \quad (2.2.8c) \]

\[ \theta[y_L(x)] = \theta_L(x), \quad \theta[y_U(x)] = \theta_U(x). \quad (2.2.8d-e) \]

One needs to specify constraints in order to close the system. These constraints may have either the form of specification of the mean pressure gradients in the \(x\)- and \(z\)-directions, i.e.

\[ \frac{\partial p}{\partial x}|_{mean} = \rho_x, \quad \frac{\partial p}{\partial z}|_{mean} = \rho_z, \quad (2.2.9a,b) \]

or the form of the mean flow rates in the \(x\)- and \(z\)-directions, i.e.
\begin{align}
Q_x(x) |_{\text{mean}} &= Q_x, \\
Q_z(x) |_{\text{mean}} &= Q_z. 
\end{align}

(2.2.10a,b)

The question if the heating and grooves can reduce streamwise pressure losses is posed as the question of determination of changes in the pressure gradient required to drive the same flow rate through the modified channel as through the unmodified reference channel. The plane Poiseuille flow of the form

\begin{align}
\bar{u}_0(x, y, z) &= [1 - y^2, 0, 1 - y^2], \\
p_{0z}(x, y, z) &= -\frac{2z}{Re_z}, \\
p_{0x}(x, y, z) &= -\frac{2x}{Re_x}, \\
Q_{z0} &= \frac{4}{3}, \\
Q_{x0} &= \frac{4}{3} 
\end{align}

is taken as the reference flow. In the above, the subscript 0 identifies the reference quantities, the x-velocity component has been scaled with its maximum \( U_{max}^* \) as the velocity scale, the z-velocity component has been scaled with its maximum \( W_{max}^* \) as the velocity scale, \( Q_{z0} \) and \( Q_{x0} \) stand for the flow rates in the z- and x-directions scaled with the z- and x-velocity scales, respectively, the z- and x-pressure components \( p_{0z} \) and \( p_{0x} \) have been scaled using \( \rho^*W_{max}^2 \) and \( \rho^*U_{max}^2 \), respectively, \( Re_z = W_{max}^*h^*/\nu^* \) and \( Re_x = U_{max}^*h^*/\nu^* \) are the z- and x-Reynolds numbers. The fixed flow rate constraint (2.2.10a, b) can now be specified as

\begin{align}
Q_x &= \left[ \int_{y_L(x)}^{y_U(x)} u(x, y) dy \right]_{\text{mean}} = \frac{4}{3} Re_x, \\
Q_z &= \left[ \int_0^{\lambda_x} \int_{y_L(x)}^{y_U(x)} w(x, y) dy dx \right]_{\text{mean}} = \frac{4}{3} Re_z
\end{align}

(2.2.12a)

(2.2.12b)

where \( Q_x \) and \( Q_z \) have been scaled with the viscous velocity scale used in (2.2.7). Alternatively, one may inquire what is the mean streamwise flow rate generated by the same pressure gradient in the original and modified channels, which leads to (2.2.9a, b) in the form

\begin{align}
\varphi_x &= -2Re_x, \\
\varphi_z &= -2Re_z. 
\end{align}

(2.2.13)
2.3 Analysis of the Flow Problem

Since the surface topography as well as the heating patterns do not depend on the streamwise direction, the unknowns can be represented in the following form

\[ u = u(x, y), v = v(x, y), w = w(x, y), p = Ax + Bz + P(x, y), \theta = \theta(x, y) \] (2.3.1)

which results in the decoupling of the continuity, the \( x \)- and the \( y \)-momentum and the energy equations from the \( z \)-momentum equation. The solution process starts with the former one, i.e. the convection problem in the \((x, y)\)-plane and is followed by the latter, i.e. solution for the streamwise flow.

2.3.1 The Spanwise Convection Problem

The field equations for the convection problem have the following form:

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \] (2.3.2a)

\[ u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \] (2.3.2b)

\[ u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + Pr^{-1} \theta, \] (2.3.2c)

\[ u \frac{\partial \theta}{\partial x} + v \frac{\partial \theta}{\partial y} = Pr^{-1} \left( \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} \right) \] (2.3.2d)

and are supplemented with (2.2.8a-b) and (2.2.8d-e). Introduction of stream function defined in the usual manner as \( \frac{\partial \psi}{\partial x} = -v, \frac{\partial \psi}{\partial y} = u \) and elimination of pressure result in

\[ \nabla^4 \psi - Pr^{-1} \frac{\partial \theta}{\partial x} = N_{\psi v}, \quad \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = Pr N_{\psi \theta} \] (2.3.3a,b)

where

\[ N_{\psi v} = \frac{\partial}{\partial y} \left( \frac{\partial}{\partial x} \tilde{u} \tilde{u} + \frac{\partial}{\partial y} \tilde{v} \right) - \frac{\partial}{\partial x} \left( \frac{\partial}{\partial x} \tilde{u} \tilde{v} + \frac{\partial}{\partial y} \tilde{v} \tilde{v} \right), \] (2.3.3c)
\[ N_{v\theta} = \frac{\partial}{\partial x} \widehat{u\theta} + \frac{\partial}{\partial y} \widehat{v\theta} \]  

and hats denote products of quantities under the hat. The relevant boundary conditions are written as

\[ \frac{\partial \psi}{\partial y} \bigg|_{y_L(x)} = 0, \quad \frac{\partial \psi}{\partial y} \bigg|_{y_U(x)} = 0, \quad \frac{\partial \psi}{\partial x} \bigg|_{y_L(x)} = 0, \quad \frac{\partial \psi}{\partial x} \bigg|_{y_U(x)} = 0, \]  

\[ \theta \big|_{y_L(x)} = \theta_L, \quad \theta \big|_{y_U(x)} = \theta_U \]  

and their numerical implementation will be discussed later in this presentation.

Since the grooves and the heating are periodic in the \( x \)-direction, all unknowns can be expressed as Fourier expansions of the form

\[ q(x, y) = \sum_{m=-N_M}^{N_M} q^{(m)}(y)e^{imax} \]  

where \( q \) stands for any of the following quantities: \( \psi, v, u, P, \theta, \widehat{uv}, \widehat{uu}, \widehat{uv}, \widehat{u\theta}, \widehat{v\theta} \), and the modal functions \( q^{(m)}(y) \) satisfy the reality conditions, i.e. \( q^{(m)} \) is the complex conjugate of \( q^{(-m)} \), and \( N_M > \max(N_G, N_F) \) with its acceptable value to be determined through numerical convergence studies. We shall use Chebyshev expansions to represent modal functions as they provide very good resolution in the near boundary regions. Since our preference is to use the standard definitions of these polynomials, we introduce transformation of the form

\[ \widehat{y} = 2 \left[ \frac{y-(1+y_L)}{y_t+y_b} \right] + 1 \]  

which maps the strip \( y \in (-1-y_b, 1+y_t) \) in the \( y \)-direction into the strip \( \widehat{y} \in (-1,1) \) in the \( \widehat{y} \)-direction. Here \( y_t \) and \( y_b \) denote locations of extremities of the upper and lower walls, respectively, and wall positions are given as

\[ \widehat{y}_L(x) = 1 - \Gamma(2+y_t) + \Gamma B_L \sum_{n=-N_G}^{n=N_G} H_L^{(n)} e^{inax}, \]  

\[ \widehat{y}_U(x) = 1 - \Gamma y_t + \Gamma B_U \sum_{n=-N_G}^{n=N_G} H_U^{(n)} e^{in(ax+\Omega a)}, \]
\[ \Gamma = \frac{d\psi}{dy} = 2/(y_t + y_b + 2), \quad (2.3.6d) \]

\[ \hat{y}_L(x) = \sum_{n=-N_G}^{n=N_G} A_L^{(n)} e^{in\alpha x}, A_L^{(0)} = 1 + \Gamma(-2 - y_t + B_L H_L^{(0)}), \quad (2.3.6e,f) \]

\[ A_L^{(n)} = \Gamma B_L H_L^{(n)} \quad \text{for } n \neq 0, \quad (2.3.6g) \]

\[ \hat{y}_U(x) = \sum_{n=-N_G}^{n=N_G} A_U^{(n)} e^{in\alpha x}, A_U^{(0)} = 1 + \Gamma(-y_t + B_U H_U^{(0)}), \quad (2.3.6h,i) \]

\[ A_U^{(n)} = \Gamma B_U H_U^{(n)} \quad \text{for } n \neq 0. \quad (2.3.6j) \]

As a result, the flow domain is completely immersed in the computational domain. Substitution of (2.3.5) into (2.3.3), introduction of transformation (2.3.6) and separation of Fourier components lead to the following modal equations

\[ [\Gamma^4 D^4 - 2m^2\alpha^2 \Gamma^2 D^2 + m^4\alpha^4] \psi^{(m)}(\hat{y}) - im\alpha Pr^{-1} \theta^{(m)}(\hat{y}) = im\alpha \Gamma D \tilde{u}(m)(\hat{y}) + \]

\[ [\Gamma^2 D^2 + m^2\alpha^2] \tilde{u}(m)(\hat{y}) - im\alpha \Gamma D \tilde{v}(m)(\hat{y}), \quad (2.3.7a) \]

\[ \Gamma^2 D^2 \theta^{(m)}(\hat{y}) - m^2\alpha^2 \theta^{(m)}(\hat{y}) = Pr[im\alpha \tilde{u}(m)(\hat{y}) + \Gamma D \tilde{v}(m)(\hat{y})] \quad (2.3.7b) \]

where \( D = \frac{d}{d\hat{y}}, \quad -N_M \leq m \leq N_M \) and terms on the right-hand side originate from the nonlinearities. The above system is solved iteratively with the right-hand side taken from previous iteration. To convert (2.3.7) into algebraic equations, the modal functions are represented as Chebyshev expansions of the form

\[ [\psi^{(m)}, \theta^{(m)}, u^{(m)}, v^{(m)}, p^{(m)}, \tilde{u}(m), \tilde{u}(m), \tilde{v}(m), \tilde{v}(m), \tilde{v}(m)](\hat{y}) \approx \sum_{k=0}^{N_K-1} [G \psi^{(m)}_k, \]

\[ G \theta^{(m)}_k, G u^{(m)}_k, G v^{(m)}_k, G p^{(m)}_k, G \tilde{u}(m)_k, G \tilde{u}(m)_k, G \tilde{v}(m)_k, G \tilde{v}(m)_k], \]

\[ G \tilde{u}(m)_k, G \tilde{v}(m)_k] T_k(\hat{y}) \quad (2.3.8) \]

(2.3.8) is substituted into (2.3.7) and the Galerkin projection method is used to construct the linear algebraic equations for the Chebyshev expansion coefficients. The algebraic equations have the following form
\[
\sum_{k=0}^{N_K-1} [\frac{d^4}{d T_x^4} (T_j, D^4 T_k) - 2m^2 \alpha^2 \frac{d^2}{d T_x^2} (T_j, D^2 T_k) + m^4 \alpha^4 (T_j, T_k)] G \psi_k^{(m)} - \\
i m \alpha Pr^{-1} (T_j, T_k) G \theta_k^{(m)} = \sum_{k=0}^{N_K-1} [i m \alpha \Gamma (T_j, D T_k) G \text{u}_k^{(m)} + (\Gamma^2 (T_j, D^2 T_k) + \\
m^2 \alpha^2 (T_j, T_k)) G \text{u}_k^{(m)} - i m \alpha \Gamma (T_j, D T_k) G \text{v}_k^{(m)}],
\]

(2.3.9a)

\[
\sum_{k=0}^{N_K-1} [\frac{d^2}{d T_x^2} (T_j, D^2 T_k) - m^2 \alpha^2 (T_j, T_k)] G \theta_k^{(m)} = Pr \sum_{k=0}^{N_K-1} [i m \alpha (T_j, T_k) G \text{u}_k^{(m)} + \\
\Gamma (T_j, D T_k) G \text{v}_k^{(m)}],
\]

(2.3.9b)

where \(0 \leq k, j \leq N_K - 1\), \(-N_M \leq m \leq N_M\) and all inner products \(<f(\tilde{y}), g(\tilde{y})>\) are defined in Appendix A.

We shall now discuss discretization of boundary conditions and constraints. We select an arbitrary constant in the definition of the stream function so that \(\psi\) becomes zero at the lower wall. The fixed flow rate constraint (2.2.10a) implies that \(\psi = Q_x\) at the upper wall leading to boundary conditions of the form

\[
\sum_{m=-N_M}^{m=N_M} \psi^{(m)} e^{i m \alpha} |_{y_L(x)} = 0, \quad \sum_{m=-N_M}^{m=N_M} \psi^{(m)} e^{i m \alpha} |_{y_U(x)} = Q_x,
\]

(2.3.10a,b)

\[
\sum_{m=-N_M}^{m=N_M} D \psi^{(m)} e^{i m \alpha} |_{y_L(x)} = 0, \quad \sum_{m=-N_M}^{m=N_M} D \psi^{(m)} e^{i m \alpha} |_{y_U(x)} = 0,
\]

(2.3.10c,d)

\[
\sum_{m=-N_M}^{m=N_M} \theta^{(m)} e^{i m \alpha} |_{y_L(x)} = Ra_{uni} + Ra_{P,L} \sum_{m=-N_T}^{m=N_T} \theta^{(m)} e^{i m \Omega_L t} e^{i m \alpha},
\]

(2.3.10e)

\[
\sum_{m=-N_M}^{m=N_M} \theta^{(m)} e^{i m \alpha} |_{y_U(x)} = Ra_{P,U} \sum_{m=-N_T}^{m=N_T} \theta^{(m)} e^{i m \Omega_T t} e^{i m \alpha}.
\]

(2.3.10f)

In the case of the fixed pressure gradient constraint \(\varphi_x = A\), condition (2.3.10b) needs to be replaced. We start with (2.3.2b), substitute (2.3.5) and extract mode zero to arrive at the relation \(\Gamma^3 D^3 \varphi^{(0)} - A = \Gamma D \text{u}^{(0)}\) which, after one integration, gives the form of this constraint suitable for numerical implementation, i.e.

\[
D^2 \psi^{(0)} (1) - D^2 \psi^{(0)} (-1) = 2 \Gamma^{-3} \varphi_x + \Gamma^{-2} [\text{u}^{(0)} (1) - \text{u}^{(0)} (-1)].
\]

(2.3.10g)
This constraint involves both ends of the solution domain. Enforcement of conditions (2.3.10a-f) requires use of the Immersed Boundary Conditions (IBC) concept (Cabral, Szumbarski & Floryan 2001; Abtahi, Hossain & Floryan 2016; Husain & Floryan 2008, 2010, 2013). We shall explain this concept for the lower wall as implementation at the upper wall is analogous.

Substitute (2.3.8) into (2.3.10 a, c, e) to get

\[ \sum_{m=-N_M}^{m=+N_M} \sum_{k=0}^{k=N_K-1} G \psi_k^{(m)} T_k^{(e)} \hat{y}_L(x) e^{im\alpha x} = 0, \]  
\[ \sum_{m=-N_M}^{m=+N_M} \sum_{k=0}^{k=N_K-1} G \psi_k^{(m)} DT_k^{(e)} \hat{y}_L(x) e^{im\alpha x} = 0, \]  
\[ \sum_{m=-N_M}^{m=+N_M} \sum_{k=0}^{k=N_K-1} G \theta_k^{(m)} T_k^{(e)} \hat{y}_L(x) e^{im\alpha x} = Ra_{unL} + Ra_{PL} \sum_{m=-N_M}^{m=+N_M} \theta_k^{(m)} e^{im\Omega_{TL} e^{im\alpha x}}. \]  

Values of Chebyshev polynomials and their derivatives evaluated at \( \hat{y}_L(x) \) represent functions periodic in \( x \) which can be expressed as Fourier expansions of the form

\[ T_k^{(e)} \hat{y}_L(x) = \sum_{s=-\infty}^{s=\infty} (E_L^{(s)})_k e^{is\alpha x}, \quad DT_k^{(e)} \hat{y}_L(x) = \sum_{s=-\infty}^{s=\infty} (d_L^{(s)})_k e^{is\alpha x}. \]  

Determination of coefficients \( (E_L^{(s)})_k \) and \( (d_L^{(s)})_k \) in these expansions is explained in Appendix B. Substitution of (2.3.12) into (2.3.11), extraction of Fourier modes and re-arrangement of indices result in the form of boundary conditions suitable for numerical implementation at the lower wall, i.e.

\[ \sum_{j=-N_M}^{j=+N_M} \sum_{k=0}^{k=N_K-1} G \psi_k^{(j)} (E_L^{(m-j)})_k = 0, \quad -N_M \leq m \leq N_M, \]  
\[ \sum_{j=-N_M}^{j=+N_M} \sum_{k=0}^{k=N_K-1} G \psi_k^{(j)} (d_L^{(m-j)})_k = 0, \quad -N_M \leq m \leq N_M, \]  
\[ \sum_{j=-N_M}^{j=+N_M} \sum_{k=0}^{k=N_K-1} G \theta_k^{(j)} (E_L^{(m-j)})_k = Ra_{PL} \theta_k^{(m)} e^{im\Omega_{TL}}, \quad 1 \leq |m| \leq N_M, \]
\[ \sum_{j=-N_M}^{j=N_M} \sum_{k=0}^{k=N_K-1} G \theta_k^{(j)} (E_{L})_k^{(-j)} = R_{auni}, \quad m = 0, \]  

(2.3.13d)

and a similar set of boundary conditions for the upper wall, i.e.

\[ \sum_{j=-N_M}^{j=N_M} \sum_{k=0}^{k=N_K-1} G \psi_k^{(j)} (E_{U})_k^{(-j)} = 0, \quad 1 \leq |m| \leq N_M, \]  

(2.3.14a)

\[ \sum_{j=-N_M}^{j=N_M} \sum_{k=0}^{k=N_K-1} G \psi_k^{(j)} (E_{U})_k^{(-j)} = Q_x, \quad m = 0, \]  

(2.3.14b)

\[ \sum_{j=-N_M}^{j=N_M} \sum_{k=0}^{k=N_K-1} G \psi_k^{(j)} (d_{U})_k^{(m-j)} = 0, \quad N_M \leq m \leq N_M, \]  

(2.3.14c)

\[ \sum_{j=-N_M}^{j=N_M} \sum_{k=0}^{k=N_K-1} G \theta_k^{(j)} (E_{U})_k^{(m-j)} = \]  

\[ Ra_{p,U} \theta_U^{(m)} e^{im\Omega_T U}, \quad 1 \leq |m| \leq N_M, \]  

(2.3.14d)

\[ \sum_{j=-N_M}^{j=N_M} \sum_{k=0}^{k=N_K-1} G \theta_k^{(j)} (E_{U})_k^{(-j)} = 0, \quad m = 0. \]  

(2.3.14e)

In the case of the fixed pressure gradient constraint, (2.3.14b) needs to be replaced by (2.3.10g). The boundary relations (2.3.13) - (2.3.14) are implemented using Tau procedure, i.e. equations for the four highest Chebyshev polynomials in each of (2.3.9a) and equations for the two highest polynomials in each of (2.3.9b) are replaced with the boundary relations resulting in a well-posed system of linear algebraic equations which can be efficiently solved (Husain & Floryan 2013). Potential accuracy gains associated with the use of a larger number of boundary relations leading to an over constraint system (Husain, Floryan & Szumbarski 2009; Husain & Floryan 2014) have not been explored in this work.

Pressure is evaluated from the known stream function. Use of the x-momentum equation written in the \((x, \tilde{y})\) coordinates leads to

\[ \frac{\partial p}{\partial x} = -\frac{\partial (uu)}{\partial x} - \Gamma \frac{\partial (uv)}{\partial \tilde{y}} + \frac{\partial^2 u}{\partial x^2} + \Gamma^2 \frac{\partial^2 u}{\partial \tilde{y}^2}. \]  

(2.3.15)

Substitution of (2.3.1) and (2.3.5) into (2.3.15) and separation of Fourier modes results in

\[ A + im\alpha P^{(m)}(\tilde{y}) = -im\alpha \langle uu \rangle^{(m)}(\tilde{y}) - \Gamma D(\langle uv \rangle^{(m)}(\tilde{y}) - m^2 \alpha^2 \Gamma D\psi^{(m)}(\tilde{y}) + \]
\[ \Gamma^3 D^3 \psi^{(m)}(\hat{y}) \]. \hspace{1cm} (2.3.16)

The above relation written for \( m = 0 \) gives expression for the pressure gradient \( A \) for the fixed flow rate constraint, i.e.

\[ A = \Gamma^3 D^3 \psi^{(0)}(\hat{y}) - \Gamma D(\nu \nu)^{(0)}(\hat{y}). \hspace{1cm} (2.3.17) \]

Pressure modal functions \( P^{(m)} \) for \( m \neq 0 \) can be determined from (2.3.16). The \( y \)-momentum equation written in the \( (x, \hat{y}) \) coordinates, i.e.

\[ \Gamma \frac{\partial p}{\partial \hat{y}} = -\frac{\partial(\nu \nu)}{\partial x} + \Gamma \frac{\partial^2 \nu}{\partial x^2} + \Gamma^2 \frac{\partial^2 v}{\partial \hat{y}^2} + Pr^{-1} \theta, \hspace{1cm} (2.3.18) \]

needs to be used for determination of \( P^{(0)} \). Substitution of (2.3.1) and (2.3.5) into (2.3.18) and separation of Fourier modes result in

\[ \Gamma D P^{(m)}(\hat{y}) = -ima \langle \nu \nu \rangle^{(m)}(\hat{y}) - \Gamma D(\nu \nu)^{(m)}(\hat{y}) + im^3 \alpha^3 \psi^{(m)}(\hat{y}) - \]

\[ im \alpha \Gamma^2 D^2 \psi^{(m)}(\hat{y}) + Pr^{-1} \theta^{(m)}. \hspace{1cm} (2.3.19) \]

Equation (2.3.19) written for the zeroth mode takes the form

\[ DP^{(0)}(\hat{y}) = \Gamma^{-1} Pr^{-1} \theta^{(0)}(\hat{y}) - D(\nu \nu)^{(0)}(\hat{y}) \hspace{1cm} (2.3.20a) \]

which, after integration, becomes

\[ P^{(0)}(\hat{y}) = \Gamma^{-1} Pr^{-1} \int_{-1}^{\hat{y}} \theta^{(0)}(\hat{y}) d\hat{y} - \langle \nu \nu \rangle^{(0)}(\hat{y}) + C_1 \hspace{1cm} (2.3.20b) \]

where \( C_1 \) stands for the constant required to bring the mean value \( P_{mean} \) of the periodic pressure component \( P(x, \hat{y}) \) to zero. Pressure normalization is explained in Appendix C. Substitution of the proper Chebyshev expansions brings this expression to the following form

\[ P^{(0)}(\hat{y}) = \Gamma^{-1} Pr^{-1} \sum_{k=0}^{N \nu} \theta_k^{(0)} J_k - \sum_{k=0}^{N \nu} \nu \nu_k^{(0)} T_k(\hat{y}) \hspace{1cm} (2.3.20c) \]

where \( J_k = \int_{-1}^{\hat{y}} T_k(\hat{y}) d\hat{y} \). The individual integrals can be evaluated as follows.
\[ J_0 = T_1(\varphi) + 1, \quad J_1 = \frac{T_2(\varphi)-1}{4}, \]
\[ J_k = \frac{1}{2} \left[ \frac{T_{k+1}(\varphi)-T_{k+1}(-1)}{k+1} - \frac{T_{k-1}(\varphi)-T_{k-1}(-1)}{k-1} \right] \text{ for } k > 1. \] (2.3.20d)

Discretization of the field equations, boundary conditions and the relevant constraints lead to a nonlinear system of algebraic equations for the Chebyshev expansion coefficients. This system is solved in an iterative manner by assuming that terms on the RHS of (2.3.9) are known. The iteration process begins with an initial approximation for the unknowns, using this approximation for determination of the nonlinear terms on the RHS of (2.3.9), solving the linear system to get a better approximation for the unknowns, using this information to update the nonlinear terms and repeating this process until convergence is achieved resulting in a fix-point method. A very large system of linear equations must be solved at each iteration step. Significant efficiencies are gained by taking advantage of the decoupling between the energy equation (2.3.9b) and the momentum equation (2.3.9a). Accordingly, two separate matrices of coefficients are constructed, and the energy equation is solved first followed by the momentum equation.

Structure of the coefficient matrix \( N \) of size \( q \times q \) for the energy equation, where \( q = (2N_M + 1)N_K \), is illustrated in Fig. 2.2A with the diagonal blocks corresponding to the field equation and the horizontal stripes arising from the boundary relations providing coupling between different modes. Before carrying out solution, this matrix is re-arranged to gain efficiency by taking advantage of its structure (Husain & Floryan 2013). All rows associated with the boundary relations are moved to the bottom forming a block diagonal matrix \( N_1 \) of size \( r \times q \) with \( r = (2N_M + 1)(N_K - 2) \) at the top and a full matrix \( N_2 \) of size \( s \times q \) with \( s = 2(2N_M + 1) \) at the bottom. Then the largest possible square matrix \( N_A \) is extracted from \( N_1 \) by relocating two unknown Chebyshev coefficients of the two lowest polynomials for each Fourier mode to the end of the vector of unknowns. The resulting matrix structure is shown in Fig. 2.2B. Matrices \( N_A \) and \( N_B \) retain the block diagonal structure with each block being of the size \((N_K - 2) \times (N_K - 2)\) and \((N_K - 2) \times 2\), respectively. The rectangular matrix \( N_C \) has size \( s \times r \) and the square matrix \( N_D \) has size \( s \times s \).
Figure 2.2 Structure of the coefficient matrix for the energy equation for $N_M = 2$ and $N_K = 15$. Fig 2.2A displays matrix before its re-arrangement with the data entered starting with mode -2 and ending with mode 2. Black symbols mark the non-zero elements. Fig 2.2B displays matrix after re-arrangement.

The resulting system can be written as

$$N_A \theta_1 + N_B \theta_2 = R_{\theta 1} \quad (2.3.21a)$$
$$N_C \theta_1 + N_D \theta_2 = R_{\theta 2} \quad (2.3.21b)$$

where vector $\theta_1$ contains unknowns $G\theta_k^{(n)}$ for $n \in [-N_M, N_M]$, $k \in [2, N_K - 1]$, vector $\theta_2$ contains unknowns $G\theta_k^{(n)}$ for $n \in [-N_M, N_M]$, $k \in [0,1]$, $R_{\theta 1}$ contains part of the right side vector $R_\theta$ for $n \in [-N_M, N_M]$, $k \in [2, N_K - 1]$ and $R_{\theta 2}$ contains part of the right side vector $R_\theta$ for $n \in [-N_M, N_M]$, $k \in [0,1]$. The solution of the above system can be expressed as

$$\theta_2 = [N_D - N_C N_A^{-1} N_B]^{-1} (R_{\theta 2} - N_C N_A^{-1} R_{\theta 1}), \quad (2.3.22a)$$
$$\theta_1 = N_A^{-1} [R_{\theta 1} - N_B \theta_2]. \quad (2.3.22b)$$
Memory requirement can be significantly reduced by storing only the diagonal blocks of \( N_A \) and \( N_B \). Additional computational efficiency is achieved by replacing inversion of one big matrix with that of several small block matrices located on the diagonal of the big matrix, and only diagonal elements need to be stored.

A similar procedure is applied to the momentum equation. The structure of the coefficient matrix \( L \) of size \( q \times q \) for the momentum equation is illustrated in Fig 2.3A with the diagonal blocks corresponding to the field equation and the horizontal stripes arising from the boundary relations providing coupling between different modes. The re-arrangement begins with moving all the boundary relations to the bottom forming a block diagonal matrix \( L_1 \) of size \( r \times q \) with \( r = (2N_M + 1)(N_K - 4) \) and a full matrix \( L_2 \) of size \( s \times q \) with \( s = 4(2N_M + 1) \). Then the largest possible square matrix \( L_A \) is extracted from \( L_1 \) by relocating four unknown Chebyshev coefficients of the four lowest polynomials for each Fourier mode to the end of the vector of unknowns. The resulting matrix structure is shown in Fig 2.3B. Matrices \( L_A \) and \( L_B \) retain the block diagonal structure. Each block of the matrix \( L_A \) and \( L_B \) is of the size \((N_K - 4) \times (N_K - 4)\) and \((N_K - 4) \times 4\), respectively. The rectangular matrix \( L_C \) has size \( s \times r \) and the square matrix \( L_D \) has size \( s \times s \). The resulting system can be written as

\[
L_A \psi_1 + L_B \psi_2 = R \psi_1, \tag{2.3.23a}
\]

\[
L_C \psi_1 + L_D \psi_2 = R \psi_2 \tag{2.3.23b}
\]

where vector \( \psi_1 \) contains unknowns \( G \psi_k^{(n)} \) for \( n \in [-N_M, N_M], k \in [4, N_K - 1] \), vector \( \psi_2 \) contains unknowns \( G \psi_k^{(n)} \) for \( n \in [-N_M, N_M], k \in [0,3] \), \( R \psi_1 \) contains part of the right side vector and the buoyancy term \( -M \theta + R \psi \) for \( n \in [-N_M, N_M], k \in [4, N_K - 1] \) and \( R \psi_2 \) contains part of the right side vector and the buoyancy term \( -M \theta + R \psi \) for \( n \in [-N_M, N_M], k \in [0,3] \). The solution of the above system can be expressed as

\[
\psi_2 = [L_D - L_C L_A^{-1} L_B]^{-1} (R \psi_2 - L_C L_A^{-1} R \psi_1), \tag{2.3.24a}
\]

\[
\psi_1 = L_A^{-1} [R \psi_1 - L_B \psi_2] \tag{2.3.24b}
\]
where the inverse operations can be carried out block by block and only diagonal blocks need to be stored.

Under-relaxation was used when necessary to achieve convergence. The updates of the nonlinear terms at each iteration involved transferring data into the physical space, evaluating all products on a grid system based on a set of collocation points in the $\hat{y}$-direction and a uniformly distributed set of points in the $x$-direction, constructing Fourier expansions for all required products using Fast Fourier Transforms at each $\hat{y}$-collocation point, using this information for construction of discrete forms of all modal functions appearing in the products, representing these functions in terms of Chebyshev expansions and constructing the final form of the RHS appearing in (2.3.9).

![Figure 2.3](image-url)

**Figure 2.3** Structure of the coefficient matrix for the momentum equation for $N_M = 2$ and $N_K = 15$. Fig 2.3A displays matrix before its re-arrangement with the data entered starting with mode -2 and ending with mode 2. Black symbols mark the non-zero elements. Fig 2.3B displays matrix after re-arrangement.

The aliasing error was controlled using "padding" (Canuto et al. 1991), i.e., using a discrete FFT transform with $N_p$, rather than $N_M$ points, where $N_p \geq 3N_M / 2$. Zeros were added for the additional Fourier modes as required.
2.3.2 The Streamwise Flow Problem

The field equation for the streamwise flow problem is linear, has the form

\[
\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} - u \frac{\partial w}{\partial x} - v \frac{\partial w}{\partial y} = \frac{\partial p}{\partial z} \tag{2.3.25}
\]

and is supplemented with (2.2.8c) and (2.2.12b). The streamwise velocity is represented as a Fourier expansion of the form

\[
w(x, y) \approx \sum_{m=-N_M}^{m=N_M} w^{(m)}(y) e^{im\alpha x} \tag{2.3.26}
\]

with its components satisfying the usual reality conditions. This expansion together with expansions (2.3.5) for \( u \) and \( v \), (2.3.1) for pressure and transformation (2.3.6) are substituted into (2.3.25), and the Fourier modes are extracted resulting on the following set of modal equations for \( m \neq 0 \)

\[
\left[ \Gamma^2 D^2 - (m\alpha)^2 \right] w^{(m)}(\hat{y}) - \Gamma \alpha \sum_{n=-N_M}^{n=N_M} n D \psi^{(m-n)}(\hat{y}) - (m - n) \\
\psi^{(m-n)}(\hat{y}) D] w^{(n)}(\hat{y}) = 0 \tag{2.3.27a}
\]

and a special equation for \( m = 0 \)

\[
\Gamma^2 D^2 w^{(0)}(\hat{y}) - \Gamma \alpha \sum_{n=-N_M}^{n=N_M} n [D \psi^{(-n)}(\hat{y}) + \psi^{(-n)}(\hat{y}) D] w^{(n)}(\hat{y}) = B. \tag{2.3.27b}
\]

Equations (2.3.27) represent a system of linear differential equations with known variable coefficients \( \psi^{(m)}(\hat{y}) = \sum_{k=0}^{N_K-1} G \psi^{(m)}_k T_k(\hat{y}) \) and \( D \psi^{(m)}(\hat{y}) = \sum_{k=0}^{N_K-1} G \psi^{(m)}_k D T_k(\hat{y}) \). The unknown modal functions are represented as Chebyshev expansions of the form

\[
w^{(m)}(\hat{y}) \approx \sum_{k=0}^{N_K-1} G w^{(m)}_k T_k(\hat{y}), \tag{2.3.28}
\]

(2.3.28) is inserted into (2.3.27) and products of Chebyshev expansions are expressed as Chebyshev expansions. Equations for the unknown expansion coefficients \( G w^{(m)}_k \) are extracted using the Galerkin projection method, i.e. (2.3.27) is multiplied by \( T_j \) with \( j =
\((0, N_K - 1)\) resulting in a system of linear algebraic equations for the expansion coefficients of the following form

\[
\sum_{k=0}^{N_K-1} \left[ [r^2 < T_j, D^2T_k > -m^2 a^2 < T_j, T_k > ] Gw_k^{(m)} - \right.
\]

\[
i \Gamma a \sum_{m=-N_m}^{n=N_M} \sum_{l=0}^{l=N_K-1} G \psi_i^{(m-n)} [n < T_j, DT_l T_k > - (m-n) < T_j, T_l DT_k > ] Gw_k^{(n)} \right] = 0, \quad m \neq 0,
\]

\[
\sum_{k=0}^{N_K-1} \left[ \Gamma^2 < T_j, D^2T_k > Gw_k^{(0)} - i \Gamma a \sum_{m=-N_m}^{n=N_M} \sum_{l=0}^{l=N_K-1} [nG \psi_i^{(-n)} < T_j, DT_l T_k > + < T_j, T_l DT_k > ] Gw_k^{(n)} \right] = B < T_j, T_0 > , \quad m = 0,
\]

where \(0 \leq k, j, l \leq N_K - 1\) and evaluation of inner products is explained in Appendix A. Use of (2.3.26) leads to boundary conditions in the form of

\[
\sum_{m=-N_M}^{m=N_M} w^{(m)} e^{imax} \bigg|_{\hat{y}_L(x)} = 0, \quad \sum_{m=-N_M}^{m=N_M} w^{(m)} e^{imax} \bigg|_{\hat{y}_U(x)} = 0 \quad (2.3.30a,b)
\]

which, when combined with (2.3.28), result in

\[
\sum_{m=-N_M}^{m=N_M} \sum_{k=0}^{k=N_K-1} \sum_{k=0}^{k=N_K-1} Gw_k^{(m)} T_k [\hat{y}_L(x)] e^{imax} = 0, \quad (2.3.31a)
\]

\[
\sum_{m=-N_M}^{m=N_M} \sum_{k=0}^{k=N_K-1} \sum_{k=0}^{k=N_K-1} Gw_k^{(m)} T_k [\hat{y}_U(x)] e^{imax} = 0. \quad (2.3.31b)
\]

Values of Chebyshev polynomials evaluated at the walls can be represented as Fourier expansions, similarly as discussed previously (see Appendix B), leading to boundary relations suitable for numerical implementation

\[
\sum_{j=-N_M}^{j=N_M} \sum_{k=0}^{k=N_K-1} Gw_k^{(j)} (E_L)^{m-j}_k = 0, \quad -N_M \leq m \leq N_M, \quad (2.3.32a)
\]

\[
\sum_{j=-N_M}^{j=N_M} \sum_{k=0}^{k=N_K-1} Gw_k^{(j)} (E_U)^{m-j}_k = 0, \quad -N_M \leq m \leq N_M. \quad (2.3.32b)
\]

The above boundary relations are implemented using tau procedure, i.e., equations for the two highest Chebyshev polynomials in each of Eqs (2.3.29) are replaced with the boundary
relations resulting in a well posed system of linear algebraic equations. Implementation of
the fixed pressure gradient constraint is simple as \( B = \varphi_z \) on the RHS of (2.3.29b) is
known. Implementation of the fixed flow rate constraint is more complex as \( B \) is not
known. The required additional algebraic equation is constructed from (2.2.10b) with
details explained in Appendix D. The resulting algebraic system has the form

\[
A_w w = B_w
\]  

(2.3.33)

where the matrix of coefficients \( A_w \) illustrated in Fig 2.4 has size \( t \times t \) with \( t = (2N_M + 1)N_K, w \) is the \( t \) dimensional vector of unknowns \( Gw_k^{(m)} \) and \( B_w \) is the \( t \) dimensional RHS vector which contains information either about the flowrate constraint or
the pressure gradient constraint. The matrix has a banded structure with a large width which
does not offer much potential for efficiency gains. Since this system needs to be solved
only once, it is solved using a standard solver.

**Figure 2.4** Structure of the coefficient matrix for the streamwise flow problem for
\( N_M = 5 \) and \( N_K = 30 \). The data is entered starting with mode -5 and ending with mode
5. Black symbols mark the non-zero elements. Horizontal strips correspond to
boundary relation while blocks correspond to the field equation.
2.4 Extraction of the Physically Relevant Data and Post-Processing

Several physical quantities need to be extracted from the known velocity, pressure and temperature fields. They include various forces acting at the walls as well as heat fluxes which need to be known in the physical \((x, y, z)\)-plane while computations are carried out in the computational plane \((x, \hat{y}, z)\)-plane. We start discussion with the lower wall.

Evaluation of forces acting on the fluid at the lower wall begins with determination of the stress vector \(\vec{\sigma}_L\), i.e.

\[
\vec{\sigma}_L = [\sigma_{x,L} \sigma_{y,L} \sigma_{z,L}] = [n_{x,L} n_{y,L} n_{z,L}] \begin{bmatrix}
2 \frac{\partial u}{\partial x} - p & \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} & \frac{\partial u}{\partial z} + \frac{\partial v}{\partial z} \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} & 2 \frac{\partial v}{\partial y} - p & \frac{\partial v}{\partial z} + \frac{\partial w}{\partial z} \\
\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} & \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} & 2 \frac{\partial w}{\partial z} - p
\end{bmatrix}_{y_L}
\] (2.4.1)

and the normal unit vector \(\vec{n}_L\) pointing outwards, i.e.

\[
\vec{n}_L = (n_{x,L}, n_{y,L}, n_{z,L}) = N_L \left( \frac{dy_L}{dx}, -1, 0 \right), \quad N_L = \left[ 1 + \left( \frac{dy_L}{dx} \right)^2 \right]^{-\frac{1}{2}}. \] (2.4.2)

The components of the stress vector at the lower wall can be expressed as

\[
\sigma_{x,L} = \sigma_{xv,L} + \sigma_{xp,L} = N_L \left[ 2 \frac{dy_L}{dx} \frac{\partial u}{\partial x} \right]_{y_L} - \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)_{y_L} - N_L \frac{dy_L}{dx} p \right|_{y_L}, \] (2.4.3a)

\[
\sigma_{y,L} = \sigma_{yv,L} + \sigma_{yp,L} = N_L \left[ \frac{dy_L}{dx} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right]_{y_L} - 2 \frac{\partial v}{\partial y} \right|_{y_L} + N_L p \right|_{y_L}, \] (2.4.3b)

\[
\sigma_{z,L} = \sigma_{zv,L} = N_L \left[ \frac{dy_L}{dx} \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) \right]_{y_L} - \left( \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right)_{y_L} \] (2.4.3c)

where \((\sigma_{xv,L}, \sigma_{yv,L}, \sigma_{zv,L})\) and \((\sigma_{xp,L}, \sigma_{yp,L})\) denote the viscous and pressure contributions, respectively. Similarly, the normal \((\sigma_{n,L})\) and tangential in the \(x-\) \((\sigma_{tx,L})\) and \(z-\) \((\sigma_{tz,L})\) directions components of the stress vector at the lower wall can be expressed as
\[
\sigma_{n,L} = \sigma_{nv,L} + \sigma_{np,L} = 2N_L^2 \left[ \frac{d^2y_L}{dx^2} \frac{\partial u}{\partial x} \bigg|_{y_L} - \frac{d^2y_L}{dx^2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \bigg|_{y_L} + \frac{\partial v}{\partial x} \right] - p|_{y_L}, \quad (2.4.4a)
\]

\[
\sigma_{tx,L} = N_L^2 \left\{ 2 \frac{d^2y_L}{dx^2} \left( \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) \bigg|_{y_L} - \left[ 1 - \left( \frac{d^2y_L}{dx^2} \right)^2 \right] \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \bigg|_{y_L} \right\}, \quad (2.4.4b)
\]

\[
\sigma_{tz,L} = N_L \left[ \frac{dy_L}{dx} \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial z} \right) \bigg|_{y_L} - \left( \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \bigg|_{y_L} \right] \quad (2.4.4c)
\]

where \( \sigma_{nv,L} \) and \( \sigma_{np,L} \) denote the viscous and pressure contributions, respectively. The component of the total force acting on the fluid at the lower wall per its unit length in the \( x \)-direction \( (F_{x,L}) \) is defined as

\[
F_{x,L} = F_{xv,L} + F_{xp,L} = \lambda^{-1} \int_{x_0}^{x_0+\lambda} \left[ 2 \frac{dy_L}{dx} \frac{\partial u}{\partial x} \bigg|_{y_L} - \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \bigg|_{y_L} \right] dx - \lambda^{-1} \int_{x_0}^{x_0+\lambda} \frac{dy_L}{dx} \bigg|_{y_L} (Ax + Bz + P) \bigg|_{y_L} dx \quad (2.4.5a)
\]

where the pressure has been explicitly divided into the linear and periodic components (see 2.3.1), and a component of this force in the \( z \)-direction \( (F_{z,L}) \) is defined as

\[
F_{z,L} = F_{zv,L} = \lambda^{-1} \int_{x_0}^{x_0+\lambda} \left[ \frac{dy_L}{dx} \left( \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) \bigg|_{y_L} - \left( \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \bigg|_{y_L} \right] dx \quad (2.4.5b)
\]

where \( x_0 \) is a convenient reference point and integration extends over one wavelength, \( F_{xp,L} \) denotes the pressure contribution while \( F_{xv,L} \) and \( F_{zv,L} \) stand for the viscous contributions in the \( x \)- and \( z \)-directions, respectively. The above relations reduce for the smooth isothermal walls to \( \sigma_{x,L} = -2Re_x, \sigma_{y,L} = -2Re_x, \sigma_{z,L} = -2Re_z, F_{x,L} = -2Re_x, F_{z,L} = -2Re_z. \)

The heat transfer characteristics are expressed in terms of the local Nusselt number \( Nu_{loc,L} \) defined as

\[
Nu_{loc,L} = \bar{n}_L \cdot \bar{q}_L = n_{x,L} q_{x,L} + n_{y,L} q_{y,L} \quad (2.4.6)
\]
where \( \tilde{q}_L = (q_{x,L}, q_{y,L}, 0) = \left( \frac{\partial \theta}{\partial x}, \frac{\partial \theta}{\partial y}, 0 \right) \bigg|_{y_L} \) stands for the temperature gradient at the wall.

The net heat flux leaving the wall per unit length in the \( x \)- and \( z \)-directions is expressed in terms of the mean Nusselt number \( Nu_{av,L} \) defined as

\[
Nu_{av,L} = \lambda^{-1} \int_{x_0}^{x_0 + \lambda} Nu_{loc,L} N_L^{-1} dx. \tag{2.4.7}
\]

Evaluation of the local surface forces at this wall requires determination of several constituting quantities in the computational plane \( (x, \hat{y}) \):

\[
\bar{N}_L = \left[ 1 - \alpha^2 \left( \sum_{n=N_G}^{n=N_G} n \tilde{\psi}_L^{(n)} e^{i\max} \right) \right]^{\frac{1}{2}}, \tag{2.4.8}
\]

\[
\frac{\partial u(x, \hat{y}_L)}{\partial x} = \Gamma \sum_{m=-N_M}^{m=N_M} \sum_{k=0}^{N_F-1} G \psi_k^{(m)} D_i \tilde{T}_k (\hat{y}_L) e^{i\max}, \tag{2.4.9}
\]

\[
\frac{\partial u(x, \hat{y}_L)}{\partial y} = \Gamma^2 \sum_{m=-N_M}^{m=N_M} \sum_{k=0}^{N_F-1} G \psi_k^{(m)} D_i^2 \tilde{T}_k (\hat{y}_L) e^{i\max}, \tag{2.4.10}
\]

\[
\frac{\partial v(x, \hat{y}_L)}{\partial x} = \alpha^2 \sum_{m=-N_M}^{m=N_M} \sum_{k=0}^{N_F-1} G \psi_k^{(m)} D_i T_k (\hat{y}_L) e^{i\max}, \tag{2.4.11}
\]

\[
\frac{\partial v(x, \hat{y}_L)}{\partial y} = -i \alpha \Gamma \sum_{m=-N_M}^{m=N_M} \sum_{k=0}^{N_F-1} G \psi_k^{(m)} D_i T_k (\hat{y}_L) e^{i\max}, \tag{2.4.12}
\]

\[
\frac{\partial w(x, \hat{y}_L)}{\partial x} = i \alpha \sum_{m=-N_M}^{m=N_M} \sum_{k=0}^{N_F-1} G \psi_k^{(m)} D_i W_k (\hat{y}_L) e^{i\max}, \tag{2.4.13}
\]

\[
\frac{\partial w(x, \hat{y}_L)}{\partial y} = \Gamma \sum_{m=-N_M}^{m=N_M} \sum_{k=0}^{N_F-1} G \psi_k^{(m)} D_i T_k (\hat{y}_L) e^{i\max}, \tag{2.4.14}
\]

\[
p(x, \hat{y}_L, z) = Ax + Bz + P(x, \hat{y}_L)
= Ax + Bz + \sum_{m=-N_M}^{m=N_M} \sum_{k=0}^{N_F-1} G P_k^{(m)} T_k (\hat{y}_L) e^{i\max}. \tag{2.4.15}
\]

Substitution of the above expressions into (2.4.3) results in relations suitable for numerical implementation:
\[ \sigma_{x,L}(x, \hat{y}_L) \hat{N}_L^{-1} = \]
\[ -2\Gamma \alpha^2 \left( \sum_{n=-N_G}^{n=N_G} n \hat{y}_L^{(n)} e^{imax} \right) \sum_{m=-N_M}^{m=N_M} \sum_{k=0}^{k=N_K-1} mG \psi_k^{(m)} DT_k(\hat{y}_L) e^{imax} - \]
\[ h \sum_{m=-N_M}^{m=N_M} \sum_{k=0}^{k=N_K-1} \left( \Gamma^2 G \psi_k^{(m)} D^2 T_k(\hat{y}_L) + \alpha^2 m^2 G \psi_k^{(m)} T_k(\hat{y}_L) \right) e^{imax} - \]
\[ i\alpha \left( \sum_{n=-N_G}^{n=N_G} n \hat{y}_L^{(n)} e^{imax} \right) \left( Ax + Bz + \sum_{m=-N_M}^{m=N_M} \sum_{k=0}^{k=N_K-1} (\Gamma^2 G \psi_k^{(m)} D^2 T_k(\hat{y}_L) + \right. \]
\[ m^2 \alpha^2 G \psi_k^{(m)} T_k(\hat{y}_L)) e^{imax} + 2i\alpha \Gamma \sum_{m=-N_M}^{m=N_M} m \sum_{k=0}^{k=N_K-1} G \psi_k^{(m)} DT_k(\hat{y}_L) e^{imax} + \]
\[ Ax + Bz + \sum_{m=-N_M}^{m=N_M} \sum_{k=0}^{k=N_K-1} G P_k^{(m)} T_k(\hat{y}_L) e^{imax}, \]  
(2.4.16)

\[ \sigma_{y,L}(x, \hat{y}_L) \hat{N}_L^{-1} = \]
\[ i\alpha \left( \sum_{n=-N_G}^{n=N_G} n \hat{y}_L^{(n)} e^{imax} \right) \sum_{m=-N_M}^{m=N_M} \sum_{k=0}^{k=N_K-1} (\Gamma^2 G \psi_k^{(m)} D^2 T_k(\hat{y}_L) + \right. \]
\[ \right. \]  
\[ m^2 \alpha^2 G \psi_k^{(m)} T_k(\hat{y}_L)) e^{imax} + \]
\[ \]  
\[ \]  
\[ Ax + Bz + \sum_{m=-N_M}^{m=N_M} \sum_{k=0}^{k=N_K-1} G \psi_k^{(m)} T_k(\hat{y}_L) e^{imax} - \]
\[ \Gamma \sum_{m=-N_M}^{m=N_M} \sum_{k=0}^{k=N_K-1} G \psi_k^{(m)} DT_k(\hat{y}_L) e^{imax}. \]  
(2.4.17)

\[ \sigma_{z,L}(x, \hat{y}_L) \hat{N}_L^{-1} = \]
\[ -\alpha^2 \left( \sum_{n=-N_G}^{n=N_G} n \hat{y}_L^{(n)} e^{imax} \right) \sum_{m=-N_M}^{m=N_M} \sum_{k=0}^{k=N_K-1} G \psi_k^{(m)} T_k(\hat{y}_L) e^{imax} - \]
\[ \Gamma \sum_{m=-N_M}^{m=N_M} \sum_{k=0}^{k=N_K-1} G \psi_k^{(m)} DT_k(\hat{y}_L) e^{imax}. \]  
(2.4.18)

Evaluation of the x- and z-components of the total force acting on the fluid per unit length involves integration described by (2.4.5). To get the x-component of the total force acting on the fluid at the lower wall per unit length, insert (2.4.9 – 2.4.11) & (2.4.15) into (2.4.5a) and express values of Chebyshev polynomials and their derivatives evaluated at the walls using Fourier expansions described in Appendix B. Express multiplications of Fourier expansions in terms of a single Fourier expansion and carry out the integration. Only mode zero of these expansions result in non-zero integrals leading to the following spectrally accurate expression for the forces

\[ F_{xv,L} = 2\Gamma \sum_{s=-N_M}^{s=N_M} \sum_{n=-N_M}^{n=N_M} \sum_{k=N_K-1}^{k=N_K} n(n-s) \alpha^2 \gamma_L^{(s-n)} G \psi_k^{(m)} a_L^{(-s)} - \]
\[ \Gamma^2 \sum_{s=-N_M}^{s=N_M} \sum_{k=0}^{k=N_K-1} G \psi_k^{(s)} R_{L,k}^{(-s)} - \sum_{s=-N_M}^{s=N_M} \sum_{k=0}^{k=N_K-1} s^2 \alpha^2 G \psi_k^{(s)} E_{L,k}^{(-s)}, \]  
(2.4.19)
where \( F_{xp,L,\text{linear}} \) arises due to the mean pressure gradient in the \( x \)-direction and \( F_{xp,L,\text{periodic}} \) arises due to the periodic pressure component. Similarly, the \( z \)-component of the total force can be expressed as

\[
F_{z,L} = \sum_{s=-N_M}^{s=N_M} \sum_{n=-N_M}^{n=N_M} \sum_{k=0}^{k=N_K-1} n(n-s)\alpha^2 x_L^{(n)} G\theta_k^{(m)} E_{L,k}^{(-s)} - \Gamma \sum_{s=-N_M}^{s=N_M} \sum_{k=0}^{k=N_K-1} Gw_k^{(s)} d_{L,k}^{(-s)} .
\]

(2.4.21)

Evaluation of the local \( Nu_{\text{loc,L}} \) and average \( Nu_{\text{av,L}} \) Nusselt numbers require the determination of the following constituting quantities:

\[
\frac{\partial \theta(x,\theta_L)}{\partial x} = i\alpha \sum_{m=-N_G}^{m=N_G} \sum_{k=0}^{k=N_K-1} G\theta_k^{(m)} T_k(\theta_L) e^{imax},
\]

(2.4.22)

\[
\frac{\partial \theta(x,\theta_L)}{\partial y} = \Gamma \sum_{m=-N_M}^{m=N_M} \sum_{k=0}^{k=N_K-1} G\theta_k^{(m)} DT_k(\theta_L) e^{imax}. \]

(2.4.23)

Substitution into (2.4.6) and (2.4.7) results in

\[
Nu_{\text{loc,L}}(x,\theta_L) \left[1 - \alpha^2 \left( \sum_{n=-N_G}^{n=N_G} n \theta_L^{(n)} e^{imax} \right)^2 \right] \frac{1}{z} =
\]

\[
-\alpha^2 \left( \sum_{n=-N_G}^{n=N_G} n \theta_L^{(n)} e^{imax} \right) \sum_{m=-N_M}^{m=N_M} \sum_{k=0}^{k=N_K-1} mG\theta_k^{(m)} T_k(\theta_L) e^{imax}
\]

\[
-\Gamma \sum_{m=-N_M}^{m=N_M} \sum_{k=0}^{k=N_K-1} G\theta_k^{(m)} DT_k(\theta_L) e^{imax} .
\]

(2.4.24a)

\[
Nu_{\text{av,L}} = \alpha^2 \sum_{s=-N_M}^{s=N_M} \sum_{n=-N_M}^{n=N_M} \sum_{k=0}^{k=N_K-1} n(n-s)\theta_L^{(s-n)} G\theta_k^{(n)} E_{L,k}^{(-s)} -
\]

\[
\Gamma \sum_{s=-N_M}^{s=N_M} \sum_{k=0}^{k=N_K-1} G\theta_k^{(s)} d_{L,k}^{(-s)} .
\]

(2.4.24b)

A similar analysis can be carried out for the upper wall.
2.5 Discussion and Algorithm Testing

The main advantage of the proposed algorithm is its spectral accuracy. Below we demonstrate through a set of numerical tests that the algorithm does deliver such accuracy despite the irregular flow domain. All tests selected for reporting have been carried out with the fixed pressure gradient constraint in the \( x \)-direction \( \phi_x = -2Re_x \) with \( Re_x = 1 \) and with the fixed flow rate constraint in the \( z \)-direction \( Q_z = \frac{4}{3} Re_z \) with \( Re_z = 15 \) for \( Pr = 0.71 \), \( y_u(x) = 1 + \frac{B_u}{2} \cos (\alpha x + \Omega_c) \), \( y_L(x) = 1 + \frac{B_L}{2} \cos (\alpha x) \), \( \theta_L(x) = Ra_{uni} + \frac{Ra_{pl}}{2} \cos (\alpha x + \Omega_{TL}) \), \( \theta_U(x) = \frac{Ra_{pl}}{2} \cos (\alpha x + \Omega_{TU}) \), \( \alpha = 2 \), \( \Omega_c = 0 \), \( \Omega_{TL} = \pi/2 \), \( \Omega_{TU} = \pi/2 \), unless otherwise noted.

The first set of tests deals with the accuracy of discretization of the field equations. Figure 2.5 displays variations of the error norm defined as

\[
\| Er_q \| = \frac{\| q_{ref} - q \|^2}{\| q_{ref} \|^2} \quad (2.5.1)
\]

where \( q \) stands for the quantity of interest, \( \| q_{ref} - q \|^2 \) is the \( L^2 \) norm of the difference between the reference solution (calculated with \( N_M = 30 \) and \( N_K = 80 \)) and solution obtained with a particular set of \( N_M \)’s and \( N_K \)’s while \( \| q_{ref} \|^2 \) is the \( L^2 \) norm of the reference solution which is evaluated on a grid of 201 equidistant points in the \( x \)-direction and 101 collocations points in the \( y \)-direction. The reader may note that the \( L^2 \) norm of a matrix \( M \) is equal to its largest singular value, i.e. \( \| M \|^2 = max [SVD(M)] \). Results displayed in Fig 2.5A-C demonstrate exponential decrease of the error when the number of Fourier modes used in the computations increases. A very large number of Chebyshev polynomials was used in these tests (\( N_K = 80 \)) in order to reduce error associated with the Chebyshev expansions below the machine error. Results displayed in Fig 2.6A-C demonstrate exponential decrease of the error when the number of Chebyshev polynomials used in the computations increases. \( N_M = 30 \) of Fourier modes was used in these tests in order to reduce error associated with the Fourier expansions below the machine error.
Figure 2.5 Variations of the error norms $\|E_{r_u}\|$ (Fig 2.5A), $\|E_{r_w}\|$ (Fig 2.5B) and $\|E_{r_\theta}\|$ (Fig 2.5C) as functions of the number $N_M$ of Fourier modes used in the computations for $N_K = 80$. Here, $Ra_{p,L} = Ra_{p,U} = Ra_{p}$. 

10^{-5} \quad 10^{-7} \quad 10^{-9} \quad 10^{-11} 

$N_M$ 

$10 \quad 15$ 

$(B_L = B_U, Ra_p) = (0.1, 1000)$ 

$(0.05, 500)$ 

$(0.03, 300)$ 

$(B_L = B_U, Ra_p) = (0.1, 1000)$ 

$(0.05, 500)$ 

$(0.03, 300)$ 

$(B_L = B_U, Ra_p) = (0.1, 1000)$ 

$(0.05, 500)$ 

$(0.03, 300)$
Figure 2.6 Variations of the error norms $\|E_{r_u}\|$ (Fig 2.6A), $\|E_{r_w}\|$ (Fig 2.6B) and $\|E_{r_\theta}\|$ (Fig 2.6C) as functions of the number $N_K$ of Chebyshev polynomials used in the computations for $N_M = 30$. Here, $Ra_{P,L} = Ra_{P,U} = Ra_p$. 

47
Another way to demonstrate spectral convergence is to evaluate Chebyshev norms defined as
\[ \| q^{(m)} \|_\omega = \left\{ \int_{-1}^{+1} [q^{(m)}(\tilde{\omega}) q^{(-m)}(\tilde{\omega}) \omega(\tilde{\omega})] d\tilde{\omega} \right\}^{1/2} \] (2.5.2)

where \( q^{(m)} \) stands for the modal function of choice and to determine its variations as a function of the mode number. A sample of such tests displayed in Fig 2.7 confirms the spectral convergence of the computed results. \( N_M = 30 \) Fourier modes were used in these tests and norms for 5 - 25 modes were plotted. \( N_K = 80 \) Chebyshev polynomials were used to reduce error associated with the Chebyshev expansions below the machine error.

Investigation of a numerical error in the evaluation of the force balance on a control volume extending over one wavelength in the \( x \)-direction and one unit in the \( z \)-direction provides useful measure of the physically significant error. The error norms of the \( x \)- (\( \| F \|_x \)) and \( z \)- (\( \| F \|_z \)) components of the force balance are defined as
\[ \| F \|_x = \left| F_{x,L} + F_{x,U} - \left( \frac{dp}{dx} \cdot Area_{xz} \right) \right|, \quad \| F \|_z = \left| F_{z,L} + F_{z,U} - \left( \frac{dp}{dz} \cdot Area_{xy} \right) \right| \] (2.5.3)

The expected values of both \( \| F \|_x \) and \( \| F \|_z \) are zero so their numerical evaluation provides a direct measure of the error. Results displayed in Fig 2.8 demonstrate exponential error reduction as the number of Fourier modes and Chebyshev polynomials used in the computations increase.

The final set of tests presented here deals with the evaluation of changes in the \( z \)-pressure gradient and the \( x \)-flow rate. The relevant error norms are defined as
\[ \| \varrho \|_z = \left| \varrho - \varrho_{z,\text{actual}} \right|, \quad \| Q \|_x = \left| Q - Q_{x,\text{actual}} \right| \] (2.5.4)

where \( \varrho_{z,\text{actual}} \) and \( Q_{x,\text{actual}} \) stand for the actual values which were determined using \( N_M = 30 \) Fourier modes and \( N_K = 80 \) Chebyshev polynomials; such \( N_M \) and \( N_K \) reduce the discretization error below the machine error as shown in the tests discussed previously. Variations of \( \| \varrho \|_z \) and \( \| Q \|_x \) displayed in Figs 2.9 and 2.10, respectively, do confirm the exponential convergence of numerical results.
Figure 2.7 Variation of the Chebyshev norms $\|u\|_\omega$ (Fig 2.7A), $\|w\|_\omega$ (Fig 2.7B) and $\|\theta\|_\omega$ (Fig 2.7C) as functions of the mode number $m$ for $N_K = 80, N_M = 30$. Here, $Ra_{P,L} = Ra_{P,U} = Ra_P$. 
Figure 2.8 Variation of the force error norms $\|F\|_x$ and $\|F\|_z$ (see text for details) as functions of the number $N_M$ of Fourier modes (Figs 2.8A and 2.8B, respectively) and as functions of the number of Chebyshev polynomials $N_K$ (Figs 2.8C and 2.8D, respectively) used in the computations. $N_K = 80$ Chebyshev polynomials were used in the former case and $N_M = 30$ Fourier modes in the latter case. Here, $Ra_{P,L} = Ra_{P,U} = Ra_p$. 

50
Problems considered in this analysis were solved by standard open-source code to assess the relative efficiency of different approaches. Two methods were used for comparative studies, i.e. the second-order accurate finite-volume discretization implemented within the OpenFoam framework and the spectral element method implemented within Nektar++ framework (Cantwell et al. 2015). Both these frameworks rely on the numerical grid generation for geometry modelling. The required grids were constructed using GMSH open source framework (Geuzaine & Remacle 2009).

![Graph A](image1.png)  ![Graph B](image2.png)

**Figure 2.9** Variation of the pressure error norm \( \| \mathbf{\phi} \|_z \) as a function of the number of Fourier modes \( N_M \) (Fig 2.9A) and as a function of the number of Chebyshev polynomials \( N_K \) (Fig 2.9B) used in the computations. See text for details. Here, \( Ra_{P,L} = Ra_{P,U} = Ra_P \).

The primary interest is in the use of small amplitude grooves to reduce the unwanted costs associated with pressure effects and reduction of the effective channel opening. Accurate modeling of such grooves requires use of a very fine mesh in their vicinity which leads to very large algebraic problems. A typical case that can be successfully handled using less
than one hour of computing time on the current top of the line desktop machine required 24+ hours of computing time using OpenFoam on the same machine. The execution time would differ, depending on details of an implementation, but it would be at least two orders of magnitude longer than that achieved using the algorithm proposed here. The memory costs are also two orders of magnitude larger than the current method primarily due to the size of matrices involved. Various gridding transitions were tested in order to reduce the size of numerical problem, but no significant reduction of computational costs was achieved.

Figure 2.10 Variation of the flow rate error norm $\|\mathbb{Q}\|$ as a function of the number of Fourier modes $N_M$ (Fig 2.10A) and as a function of the number of Chebyshev polynomials $N_K$ (Fig 2.10B) used in the computations. See text for details. Here, $Ra_{p,L} = Ra_{p,U} = Ra_p$.

Implementation using spectral elements is more efficient as such fine grids are not required but suffers from similar constraints associated with the size of the matrices. It is approximately an order of magnitude more expensive than the IBC methods both in terms...
of execution time and in terms of memory size. The final argument in favor of the proposed algorithm is its geometric flexibility. Change of the groove geometry requires only input of a different set of Fourier coefficients while it requires grid regeneration and testing when using either OpenFoam or Nektar++. Analysis of a large class of geometries could be automatized using the present method but requires a major labor investment when using grid-based approaches.

2.6 Summary

A very accurate and efficient algorithm to handle flow problems in grooved heated channels has been proposed and tested. The algorithm is designed to accurately simulate effects created by small amplitude grooves and to provide a very good resolution in the near wall regions. It uses Fourier expansions in the horizontal directions and Chebyshev expansions in the wall-normal direction for discretization of the field equations. The algebraic equations for the Chebyshev expansions coefficients are constructed using a Galerkin projection method. The imposition of the boundary conditions in irregular flow domains associated with the groove presence relies on the immersed boundary concept. These conditions are transformed into constraints which are implemented using the Tau concept. The problem formulation is completed using either the fixed flow rate constraint or the fixed pressure gradient constraint. Implementation of both types of constraints within the immersed boundary conditions concept is presented. The discretized equations are solved using iterative procedure with the nonlinear terms approximated using information available from the previous iteration resulting in a fixed-point iterative method. Numerous tests were carried out to demonstrate exponential convergence of results with an increase of the number of Fourier modes and Chebyshev polynomials used in the discretization. Comparisons with the open-source software frameworks based either on the finite volume or on the spectral element discretization demonstrate at least an order of magnitude better performance of the proposed algorithm. This algorithm avoids the need for numerical grid generation and thus provides potential for automatization for studies of effects of various groove systems.
Chapter 3

3 Time-Dependent Flows in Grooved Non-Isothermal Channels

3.1 Introduction

A highly accurate and fully implicit algorithm for analyses of transient effects in heated grooved channels, including chaotic responses and transition to secondary states, has been presented in this chapter. This algorithm can also be used as an alternative iteration scheme for the stationary heating problem as well as a method of verification for the algorithm presented in Chapter 2. The algorithm uses spectral spatial discretization and up to sixth-order temporal discretization, providing the means to deliver machine accuracy. Extensive testing demonstrates that the algorithm achieves the theoretically predicted accuracy. Section 3.2 provides the formulation of the model problem used for algorithm presentation. Section 3.3 discusses the initiation of simulations. Section 3.4 presents the method for advancing time in the simulation by one step. Section 3.4.1 shows how to handle the convection component including spatial discretization (Section 3.4.1.1), the strategy used in solving the resulting nonlinear system (Section 3.4.1.2) and extraction of physically relevant information (Section 3.4.1.3). Section 3.4.2 shows how to handle the streamwise component with Section 3.4.2.1 explaining how to extract the physically relevant information. Section 3.5 discusses the performance of the algorithm while Section 3.6 provides a short summary of the main conclusions.

____________________

2 A version of this chapter has been published as –

3.2 Problem Formulation

Consider channel formed by two horizontal walls extending to $\pm\infty$ in the $x$- and $z$-directions as shown in Fig 3.1. Our primary interest in the pressure-gradient driven flow in the $z$-direction but, for generality, we also admit pressure-gradient driven flow in the $x$-direction. The gravitational acceleration $g$ is acting in the negative $y$-direction. The fluid has thermal conductivity $k$, specific heat $c$, thermal diffusivity $\kappa = k / \rho c$, kinematic viscosity $\nu$, dynamic viscosity $\mu$, thermal expansion coefficient $\beta$ and variations of its density $\rho$ follow the Boussinesq approximation. The velocity and pressure fields, and the flow rates can be expressed in the absence of any heating and corrugation as

$$
\bar{u}(x, y, z) = [1 - y^2, 0, 1 - y^2], \quad p_{0z}(x, y, z) = -\frac{2x}{Re_x}, \quad p_{0x}(x, y, z) = -\frac{2x}{Re_x}, \quad Q_{z0} = \frac{4}{3}, \quad Q_{x0} = \frac{4}{3}
$$

where the maxima of the $x$- and $z$-velocity components ($U_{max}, W_{max}$) have been used as velocity scales for the $x$- and $z$-directions, $Q_{z0}$ and $Q_{x0}$ stand for the $z$- and $x$-direction flow rates scaled with the respective velocity scales, $p_{0z}$ and $p_{0x}$ are the $z$- and $x$-pressure components scaled using $\rho W_{max}^2$ and $\rho U_{max}^2$, the $z$- and $x$-Reynolds numbers are defined as $Re_z = W_{max} h / \nu$ and $Re_x = U_{max} h / \nu$, and the channel half height $h$ has been used as the length scale.

Modify the above channel by introducing longitudinal grooves (grooves parallel to the $z$-direction; see Fig 3.1) periodic in $x$-direction whose geometries are expressed as

$$
y_L(x) = -1 + B_L \sum_{n=-N_G}^{n=N_G} H_L^{(n)} e^{inx}, \quad y_U(x) = 1 + B_U \sum_{n=-N_G}^{n=N_G} H_U^{(n)} e^{inx}
$$

where

$$
-\frac{1}{2} \leq \sum_{n=-N_G}^{n=N_G} H_L^{(n)} e^{inx} \leq \frac{1}{2}, \quad -\frac{1}{2} \leq \sum_{n=-N_G}^{n=N_G} H_U^{(n)} e^{inx} \leq \frac{1}{2},
$$

the subscripts $U$ and $L$ refer to the upper and lower walls, respectively, $B_L$ and $B_U$ are the groove amplitudes, $H_L^{(n)}$ and $H_U^{(n)}$ are Fourier expansions coefficients satisfying the reality condition, i.e. $H_L^{(-n)}$ and $H_U^{(-n)}$ are the complex conjugates of $H_L^{(n)}$ and $H_U^{(n)}$, $N_G$ denotes
the number of Fourier modes, \( \alpha \) and \( \lambda \) stand for their wave number and wavelength, respectively.

![Schematic diagram of the flow system.](image)

**Figure 3.1** Schematic diagram of the flow system.

The temperatures of the lower (\( \theta_L \)) and the upper (\( \theta_U \)) walls are specified as

\[
\theta_L(x, t) = Ra_{uni} + Ra_{P,L} \sum_{n=-N_T,n \neq 0}^{n=N_T} \theta_L^{(n)}(t)e^{inax}, \quad (3.2.3a)
\]

\[
\theta_U(x, t) = Ra_{P,U} \sum_{n=-N_T,n \neq 0}^{n=N_T} \theta_U^{(n)}(t)e^{inax}, \quad (3.2.3b)
\]

where

\[
-\frac{1}{2} \leq \sum_{n=-N_T,n \neq 0}^{n=N_T} \theta_L^{(n)}(t)e^{inax} \leq \frac{1}{2}, \quad -\frac{1}{2} \leq \sum_{n=-N_T,n \neq 0}^{n=N_T} \theta_U^{(n)}(t)e^{inax} \leq \frac{1}{2}, \quad (3.2.3c)
\]

\( \theta = T - T_{U,mean} \) is the relative temperature, \( T \) is the absolute temperature, \( T_{U,mean} \) is the mean of the upper wall temperature which serves as the reference temperature, \( \theta_L^{(n)} \) and \( \theta_U^{(n)} \) are Fourier coefficients satisfying the reality conditions, i.e. \( \theta_L^{(-n)} \) and \( \theta_U^{(-n)} \) are the complex conjugates of \( \theta_L^{(n)} \) and \( \theta_U^{(n)} \), respectively, \( N_T \) is the number of Fourier modes, \( T_{L,mean} \) is the mean temperature of the lower wall, \( Ra_{uni} = \frac{g \beta h^3 \theta_{uni}}{(\kappa \nu)} \) where \( \theta_{uni} = \ldots \)
\( T_{L,mean} - T_{U,mean} \) is the uniform Rayleigh number, \( Ra_{P,L} = g \beta h^3 \theta_{P,L}/(\kappa \nu) \) and \( Ra_{P,U} = g \beta \theta_{P,U}/(\kappa \nu) \) are the lower and upper periodic Rayleigh numbers, \( \theta_{P,L} \) and \( \theta_{P,U} \) are the temperature amplitudes and \( \kappa \nu/(g \beta h^3) \) is the temperature scale.

The fluid movement in the channel modified by grooves and heating is described by the field equations of the form

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{3.2.4a}
\]

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = -\frac{\partial p}{\partial x} + \nabla^2 u, \tag{3.2.4b}
\]

\[
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} = -\frac{\partial p}{\partial y} + \nabla^2 v + Pr^{-1} \theta, \tag{3.2.4c}
\]

\[
\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} = -\frac{\partial p}{\partial z} + \nabla^2 w, \tag{3.2.4d}
\]

\[
\frac{\partial \theta}{\partial t} + u \frac{\partial \theta}{\partial x} + v \frac{\partial \theta}{\partial y} + w \frac{\partial \theta}{\partial z} = Pr^{-1} \nabla^2 \theta, \tag{3.2.4e}
\]

where \( \vec{u} = (u, v, w) \) is the velocity vector with components in the \((x, y, z)\)-directions made dimensionless using the viscous velocity scale \( U_v = \nu/h \), \( p \) stands for the pressure scaled with \( \rho U_v^2 \) and \( Pr = \nu/\kappa \) is the Prandtl number. The relevant boundary conditions at the walls have the form

\[
u[y_L(x), t] = u[y_U(x), t] = 0, \quad \nu[y_L(x), t] = \nu[y_U(x), t] = 0, \quad \tag{3.2.5a-b}
\]

\[
w[y_L(x), t] = w[y_U(x), t] = 0, \tag{3.2.5c}
\]

\[
\theta[y_L(x), t] = \theta_L(x, t), \quad \theta[y_U(x), t] = \theta_U(x, t). \tag{3.2.5d-e}
\]

Problem formulation is closed by specifying either the mean pressure gradients constraints in the \(x\)- and \(z\)-directions, i.e.

\[
\left. \frac{\partial p(x,t)}{\partial x} \right|_{mean} = \varphi_x, \quad \left. \frac{\partial p(x,t)}{\partial z} \right|_{mean} = \varphi_z, \tag{3.2.6a,b}
\]

or the mean flow rate constraints in the \(x\)- and \(z\)-directions, i.e.
\[ Q_x(x,t)_{\text{mean}} = Q_x, \quad Q_z(t)_{\text{mean}} = Q_z. \] (3.2.7a,b)

Determination if the heating and groove induced flow modulation can result in a reduction of pressure losses is made through determination of pressure gradients required to drive the same flow rate through the reference and modified channels. The flow rate constraint can now be expressed as

\[ Q_x = \left[ \int_{y_L(x)}^{y_U(x)} u(x,y,t) \, dy \right]_{\text{mean}} = \frac{4}{3} Re_x, \] (3.2.8a)

\[ Q_z = \left[ \int_0^{\lambda_x} \int_{y_L(x)}^{y_U(x)} w(x,y,t) \, dy \, dx \right]_{\text{mean}} = \frac{4}{3} Re_z. \] (3.2.8b)

The same question can be addressed by evaluating the flow rate driven by the same pressure gradient in the reference and modified channels, and this leads to the pressure gradients constraints in the form

\[ \varphi_x = -2Re_x, \quad \varphi_z = -2Re_z. \] (3.2.9)

### 3.3 Initiation of Computations

The above problem requires specification of initial conditions. For the sake of this discussion, we shall assume that simulations begin at time \( t = 0 \) which means that velocity and temperature fields needs to be specified at this time, i.e.

\[ u(x,y,0) = u_i(x,y), \quad v(x,y,0) = v_i(x,y), \quad w(x,y,0) = w_i(x,y), \quad \theta(x,y,0) = \theta_i(x,y) \] (3.3.1)

where \( u_i(x,y), v_i(x,y), w_i(x,y), \theta_i(x,y) \) are known. Such initial conditions may be difficult to secure in the actual simulations. There are two choices, i.e. (i) start with the zero initial conditions with assumption that the system response should become independent of the form of these conditions after the initial transient dies out, or (ii) start with the results of previous simulations. In the former case, the form of the initial conditions is irrelevant while in the latter case it has to be consistent with the output from
the previous simulations. We shall return to this question after completing discussion of discretization as this will dictate the form of the initial conditions.

Progression to the next time step involves discretization of time derivatives. We shall focus our attention on the implicit methods due to their superior stability characteristics. As our interests are in high accuracy simulations, we shall use high-order backward finite-difference formulae. These formulae require information about the unknowns at several previous time steps which makes the algorithm not self-starting. We shall therefore use the first-order backward discretization, which requires knowledge of the unknowns at only one previous time step, to make a few initial time steps before switching to more accurate methods. If the new simulations start from the previous simulations, the data required for restarting can be stored permitting the immediate use of the high-order formulae.

3.4 Time Progression

It is assumed that the complete information about the velocity and temperature fields up to time $t$ is available and these fields at time $t + \Delta t$ are sought. Use of the 3rd order backward finite-difference discretization for the time derivatives results in the following form of the governing equations:

$$
\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} + \frac{\partial w_1}{\partial z} = 0, \quad (3.4.1a)
$$

$$
\frac{11}{6} \Delta t^{-1} u_1 + u_1 \frac{\partial u_1}{\partial x} + v_1 \frac{\partial u_1}{\partial y} + w_1 \frac{\partial u_1}{\partial z} + \frac{\partial p_1}{\partial x} - \nabla^2 u_1 = 3\Delta t^{-1} u_0 - \frac{3}{2} \Delta t^{-1} u_{-1} + \frac{1}{3} \Delta t^{-1} u_{-2}, \quad (3.4.1b)
$$

$$
\frac{11}{6} \Delta t^{-1} v_1 + u_1 \frac{\partial v_1}{\partial x} + v_1 \frac{\partial v_1}{\partial y} + w_1 \frac{\partial v_1}{\partial z} + \frac{\partial p_1}{\partial y} - \nabla^2 v_1 - Pr^{-1} \theta_1 = 3\Delta t^{-1} v_0 - \frac{3}{2} \Delta t^{-1} v_{-1} + \frac{1}{3} \Delta t^{-1} v_{-2}, \quad (3.4.1c)
$$

$$
\frac{11}{6} \Delta t^{-1} w_1 + u_1 \frac{\partial w_1}{\partial x} + v_1 \frac{\partial w_1}{\partial y} + w_1 \frac{\partial w_1}{\partial z} + \frac{\partial p_1}{\partial z} - \nabla^2 w_1 = 3\Delta t^{-1} w_0 - \frac{3}{2} \Delta t^{-1} w_{-1} + \frac{1}{3} \Delta t^{-1} w_{-2},
$$
\[
\frac{1}{3} \Delta t^{-1} w_{-2}, \quad (3.4.1d)
\]

\[
\frac{11}{6} \Delta t^{-1} \theta_1 + u_1 \frac{\partial \theta_1}{\partial x} + v_1 \frac{\partial \theta_1}{\partial y} + w_1 \frac{\partial \theta_1}{\partial z} - Pr^{-1} \nabla^2 \theta_1 = 3 \Delta t^{-1} \theta_0 - \frac{3}{2} \Delta t^{-1} \theta_{-1} + \frac{1}{3} \Delta t^{-1} \theta_{-2}, \quad (3.4.1e)
\]

\[
u_1(y_L(x)) = u_1(y_U(x)) = 0, \quad v_1(y_L(x)) = v_1(y_U(x)) = 0, \quad (3.4.2a-b)
\]

\[
w_1(y_L(x)) = w_1(y_U(x)) = 0, \quad (3.4.2c)
\]

\[
\theta_1(y_L(x)) = \theta_L(x, t_1), \quad \theta_1(y_U(x)) = \theta_U(x, t_1). \quad (3.4.2d,e)
\]

\[
\left( \frac{\partial p_1}{\partial x} \right)_{\text{mean}} = \rho_x, \quad \left( \frac{\partial p_1}{\partial z} \right)_{\text{mean}} = \rho_z. \quad (3.4.3a,b)
\]

\[
\left[ \int_{y_L(x)}^{y_U(x)} u_1(x, y) dy \right]_{\text{mean}} = Q_x, \quad \left[ \int_{y_L(x)}^{y_U(x)} w_1(x, y) dy \right]_{\text{mean}} = Q_z \quad (3.4.4a,b)
\]

where subscripts 1, 0, -1, -2 refer to time instances \( t + \Delta t, t, t - \Delta t, t - 2\Delta t \), respectively, left-hand sides of (3.4.1a-e) are unknown and the right-hand sides are known. Formulae for the 1st, 2nd, 4th, 5th and 6th order discretization schemes are presented in Appendix E.

Determination of the form of the flow at time \( t_1 = t + \Delta t \) requires solution of system (3.4.1) – (3.4.4). The unknowns can be represented in view of the character of surface topography and the form of the heating as

\[
u_1 = u_1(x, y), \quad v_1 = v_1(x, y), \quad w_1 = w_1(x, y), \quad p_1 = Ax + Bz + P_1(x, y), \quad \theta_1 = \theta_1(x, y), \quad (3.4.5)
\]

which results in the decoupling of (3.4.1a-c, e) from (3.4.1d). The solution process can be therefore divided into two steps, i.e. the convection component consisting of (3.4.1a-c, e) followed by the streamwise component consisting of (3.4.1d).
### 3.4.1 Convection Component

Equations (3.4.1a-c, e) are simplified by introducing stream function defined in the usual manner as $\frac{\partial \psi}{\partial x} = -v$, $\frac{\partial \psi}{\partial y} = u$ and removing pressure from (3.4.1b,c) resulting in the following problem

$$\frac{11}{6} \Delta t^{-1} (\nabla^2 \psi_1) - \nabla^4 \psi_1 + Pr^{-1} \frac{\partial \theta_1}{\partial x} = \frac{\partial}{\partial x} \left( \frac{\partial (u_1 v_1)}{\partial x} + \frac{\partial (v_1 v_1)}{\partial y} \right)$$

$$- \frac{\partial}{\partial y} \left( \frac{\partial (u_1 u_1)}{\partial x} + \frac{\partial (u_1 v_1)}{\partial y} \right) + 3 \Delta t^{-1} \left( \frac{\partial^2 \psi_0}{\partial y^2} + \frac{\partial^2 \psi_0}{\partial x^2} \right) - \frac{3}{2} \Delta t^{-1} \left( \frac{\partial^2 \psi_{-1}}{\partial y^2} + \frac{\partial^2 \psi_{-1}}{\partial x^2} \right)$$

$$+ \frac{1}{3} \Delta t^{-1} \left( \frac{\partial^2 \psi_{-2}}{\partial y^2} + \frac{\partial^2 \psi_{-2}}{\partial x^2} \right), \hspace{1cm} (3.4.6a)$$

$$\frac{11}{6} \Delta t^{-1} \theta_1 - Pr^{-1} \nabla^2 \theta_1 = - \frac{\partial}{\partial x} (\overline{u_1 \theta_1}) - \frac{\partial}{\partial y} (\overline{v_1 \theta_1}) + 3 \Delta t^{-1} \theta_0 -$$

$$\frac{3}{2} \Delta t^{-1} \theta_{-1} + \frac{1}{3} \Delta t^{-1} \theta_{-2}. \hspace{1cm} (3.4.6b)$$

Here, hat denotes the products of functions under the hat. The constraints and boundary conditions are expressed as

$$y = y_L(x): \quad \frac{\partial \psi_1}{\partial y} = 0, \quad \frac{\partial \psi_1}{\partial x} = 0, \quad \theta_1 = \theta_L(x, t_1), \hspace{1cm} (3.4.7a-c)$$

$$y = y_U(x): \quad \frac{\partial \psi_1}{\partial y} = 0, \quad \frac{\partial \psi_1}{\partial x} = 0, \quad \theta_1 = \theta_U(x, t_1), \hspace{1cm} (3.4.7d-f)$$

$$\left[ \int_{y_L(x)}^{y_U(x)} u_1(x, y) dy \right]_{mean} = Q_x, \quad \left( \frac{\partial \psi_1}{\partial x} \right)_{mean} = \phi_x. \hspace{1cm} (3.4.8a-b)$$

#### 3.4.1.1 Spatial Discretization

All unknowns in (3.4.6) are periodic in the $x$-direction and can therefore be expressed in terms of Fourier expansions as
\[
[\psi_1, \theta_1, P_1, u_1 \overline{u}_1, v_1 \overline{v}_1, v_1 \overline{v}_1, u_1 \theta_1, v_1 \theta_1](x, y) \approx \sum_{n=-N_M}^{N_M} [\psi_1^{(n)}, \theta_1^{(n)}, P_1^{(n)}, u_1 \overline{u}_1^{(n)}, u_1 \overline{v}_1^{(n)}, v_1 \overline{v}_1^{(n)}, u_1 \theta_1^{(n)}, v_1 \theta_1^{(n)}](y)e^{in\beta x}
\] (3.4.9)

where the modal functions satisfy the reality condition, i.e. \(\psi_1^{(n)}, \theta_1^{(n)}, P_1^{(n)}, u_1 \overline{u}_1^{(n)}, v_1 \overline{v}_1^{(n)}, u_1 \theta_1^{(n)}, v_1 \theta_1^{(n)}\) are the complex conjugate of \(\psi_1^{(-n)}, \theta_1^{(-n)}, P_1^{(-n)}, u_1 \overline{u}_1^{(-n)}, v_1 \overline{v}_1^{(-n)}, u_1 \theta_1^{(-n)}, v_1 \theta_1^{(-n)}\), respectively. These functions are to be represented in terms of Chebyshev expansions which provide a very good resolution near boundaries. To use the standard definitions of Chebyshev polynomials, we introduce transformation of the form

\[
\hat{y} = 2 \frac{y-(1+\gamma_L)}{\gamma_T+\gamma_B} + 1
\] (3.4.10a)

which maps the strip \(y \in (-1 - y_b, 1 + y_t)\) in the \(y\)-direction into the strip \(\hat{y} \in (-1, 1)\) in the \(\hat{y}\)-direction where \(y_t\) and \(y_b\) stand for locations of extremities of the upper and lower walls, respectively. The wall positions are given in terms of \(\hat{y}\) as

\[
\hat{y}_L(x) = \sum_{n=-N_G}^{N_G} A_L^{(n)} e^{inax}, \quad A_L^{(0)} = 1 + \Gamma(-2 - y_t + B_L H_L^{(0)}),
\]

\[
A_L^{(n)} = \Gamma B_L H_L^{(n)} \quad \text{for} \quad n \neq 0,
\] (3.4.10b)

\[
\hat{y}_U(x) = \sum_{n=-N_G}^{N_G} A_U^{(n)} e^{inax}, \quad A_U^{(0)} = 1 + \Gamma(-y_t + B_U H_U^{(0)}),
\]

\[
A_U^{(n)} = \Gamma B_U H_U^{(n)} \quad \text{for} \quad n \neq 0
\] (3.4.10c)

where \(\Gamma = \frac{d\gamma}{dy} = 2/(y_t + y_b + 2)\). (3.4.10d)

Substitution of (3.4.9) into (3.4.6), the introduction of transformation (3.4.10a) and separation of Fourier components result in the following modal equations:

\[
\left[\frac{11}{6} \Delta t^{-1} \Gamma^2 D^2 - \frac{11}{6} \Delta t^{-1} n^2 \alpha^2 - n^4 \alpha^4 + 2n^2 \alpha^2 \Gamma^2 D^2 - \Gamma^4 D^4\right] \psi_1^{(n)} + in\alpha Pr^{-1} \theta_1^{(n)} = in\alpha \Gamma D \overline{v}_1 v_1^{(n)} - [\Gamma^2 D^2 + n^2 \alpha^2] \overline{u}_1 \overline{v}_1^{(n)} - in\alpha \Gamma D \overline{u}_1 \overline{u}_1^{(n)} + 3\Delta t^{-1} \left[\Gamma^2 D^2 - n^2 \alpha^2\right] \psi_0^{(n)}
\]
\[-\frac{3}{2} \Delta t^{-1} [\Gamma^2 D^2 - n^2 \alpha^2] \psi_{-1}^{(n)} + \frac{1}{3} \Delta t^{-1} [\Gamma^2 D^2 - n^2 \alpha^2] \psi_{-2}^{(n)}, \quad (3.4.11a)\]

\[
\left[ \frac{11}{6} \Delta t^{-1} + n^2 \alpha^2 P r^{-1} - \Gamma^2 P r^{-1} D^2 \right] \theta_1^{(n)} = -i n \alpha \bar{\theta}_1^{(n)} - \Gamma D \bar{\theta}_1^{(n)} + \frac{3}{2} \Delta t^{-1} \bar{\theta}_0^{(n)} - \frac{3}{2} \Delta t^{-1} \bar{\theta}_1^{(n)} + \frac{1}{3} \Delta t^{-1} \bar{\theta}_2^{(n)} \quad (3.4.11b)\]

where \( D^n = \frac{d^n}{dy^n} \) and \(-N_M \leq n \leq N_M \). We express modal functions in terms of Chebyshev expansions of the form

\[
\left[ \psi_1^{(n)}, \theta_1^{(n)}, P_1^{(n)}, \bar{u}_1 \bar{u}_1^{(n)}, \bar{v}_1 \bar{v}_1^{(n)}, \bar{v}_1 \bar{v}_1^{(n)}, \bar{u}_1 \bar{\theta}_1^{(n)}, \bar{v}_1 \bar{\theta}_1^{(n)} \right] (\bar{y}) \approx \sum_{k=0}^{N_K-1} \left[ G \psi_{1,k}^{(n)}, G \theta_{1,k}^{(n)}, G P_{1,k}^{(n)}, G \bar{u}_1 \bar{u}_1^{(n)}, G \bar{u}_1 \bar{v}_1^{(n)}, G \bar{v}_1 \bar{v}_1^{(n)}, G \bar{u}_1 \bar{\theta}_1^{(n)}, G \bar{v}_1 \bar{\theta}_1^{(n)} \right] T_k(\bar{y}) \quad (3.4.12)\]

where \( N_K \) denotes the number of Chebyshev polynomials. The linear algebraic equations for the Chebyshev expansion coefficients are constructed by substituting (3.4.12) into (3.4.11) and using Galerkin projection method resulting in the following relations

\[
\sum_{k=0}^{N_K-1} \left\{ \left[ \frac{11}{6} \Delta t^{-1} \Gamma^2 - T_j, D^2 T_k > - \frac{11}{6} \Delta t^{-1} n^2 \alpha^2 - T_j, T_k > -n^4 \alpha^4 > T_j, T_k > + 2n^2 \alpha^2 \Gamma^2 > T_j, D^2 T_k > - \Gamma^4 > T_j, D^4 T_k > \right] G \psi_{1,k}^{(n)} + i n \alpha P r^{-1} - T_j, T_k > G \theta_{1,k}^{(n)} \right\} =
\]

\[
N_{uu} + \sum_{k=0}^{N_K-1} \left\{ 3 \Delta t^{-1} \left[ \Gamma^2 < T_j, D^2 T_k > -n^2 \alpha^2 < T_j, T_k > \right] G \psi_{0,k}^{(n)} + \frac{3}{2} \Delta t^{-1} \right\}
\]

\[
\left[ \Gamma^2 < T_j, D^2 T_k > -n^2 \alpha^2 < T_j, T_k > \right] G \psi_{-1,k}^{(n)} + \frac{1}{3} \Delta t^{-1} \left[ \Gamma^2 < T_j, D^2 T_k > -n^2 \alpha^2 < T_j, T_k > \right] G \psi_{-2,k}^{(n)}
\]

\[
(3.4.13a)
\]

\[
\sum_{k=0}^{N_K-1} \left[ \frac{11}{6} \Delta t^{-1} < T_j, T_k > + n^2 \alpha^2 P r^{-1} < T_j, T_k > - \Gamma^2 P r^{-1} < T_j, D^2 T_k > \right] G \theta_{1,k}^{(n)} =
\]

\[
N_{u\theta} + \sum_{k=0}^{N_K-1} \left[ 3 \Delta t^{-1} < T_j, T_k > G \theta_{0,k}^{(n)} - \frac{3}{2} \Delta t^{-1} < T_j, T_k > G \theta_{-1,k}^{(n)} + \frac{1}{3} \Delta t^{-1} < T_j, T_k > G \theta_{-2,k}^{(n)} \right]
\]

\[
(3.4.13b)
\]
Where

\[
N_{uu} = \sum_{k=0}^{N_K-1} \left\{ \text{in} \alpha \Gamma < T_j, DT_k > G \nu_1 \nu_1^{(n)} - \left[ r^2 < T_j, D^2 T_k > + n^2 \alpha^2 < T_j, T_k > \right] \right. \\
\left. G \nu_1 \nu_1^{(n)} - \text{in} \alpha \Gamma < T_j, DT_k > G \nu_1 \nu_1^{(n)} \right\}.
\]

(3.4.14a)

\[
N_{u\theta} = \sum_{k=0}^{N_K-1} \left\{ -\text{in} \alpha < T_j, T_k > G \nu_1 \theta_1 \nu_1^{(n)} - \Gamma < T_j, DT_k > G \nu_1 \theta_1 \nu_1^{(n)} \right\}.
\]

(3.4.14b)

In the above, \(0 \leq k, j \leq N_K - 1\), \(-N_M \leq n, m \leq N_M\) and explicit definitions of inner products \(< f(\tilde{y}), g(\tilde{y}) >\) can be found in Appendix A.

The concept of Immersed Boundary Condition (IBC) method is used for the enforcement of boundary conditions. The following presentation is limited to a brief outline as details can be found in (Szumbarzski & Floryan 1999). The discretization process is discussed in the context of the upper wall as process for the lower wall is identical. We start by expressing dependent variables at the wall using spatial discretization described above, i.e.

\[
\frac{\partial \psi_1}{\partial y} \bigg|_{\hat{y}_U(x)} = \sum_{n=-N_M}^{n=N_M} \psi_1^{(n)}(\hat{y}_U)e^{inax} = \sum_{n=-N_M}^{n=N_M} \sum_{k=0}^{N_K-1} G \psi_1^{(n)}DT_k(\hat{y}_U)e^{inax}, \quad (3.4.15a)
\]

\[
\psi_1|_{\hat{y}_U(x)} = \sum_{n=-N_M}^{n=N_M} \psi_1^{(n)}(\hat{y}_U)e^{inax} = \sum_{n=-N_M}^{n=N_M} \sum_{k=0}^{N_K-1} G \psi_1^{(n)}T_k(\hat{y}_U)e^{inax}, \quad (3.4.15b)
\]

\[
\theta_1|_{\hat{y}_U(x)} = \sum_{n=-N_M}^{n=N_M} \theta_1^{(n)}(\hat{y}_U)e^{inax} = \sum_{n=-N_M}^{n=N_M} \sum_{k=0}^{N_K-1} G \theta_1^{(n)}T_k(\hat{y}_U)e^{inax}.
\]

(3.4.15c)

Values of Chebyshev polynomials and their derivatives at the wall are periodic functions of \(x\) and, thus, can be expressed in terms of Fourier expansion as follows

\[
T_k[\hat{y}_U(x)] = \sum_{s=-\infty}^{\infty} (E_U)_k^{(s)} e^{isax}, \quad DT_k[\hat{y}_U(x)] = \sum_{s=-\infty}^{\infty} (d_U)_k^{(s)} e^{isax}.
\]

(3.4.16)

Evaluation of coefficients \((E_U)_k^{(s)}\) and \((d_U)_k^{(s)}\) in these expansions is explained in Appendix B. Substitution of (3.4.16) into (3.4.15) and separation of Fourier components lead to boundary relations suitable for numerical implementation, i.e.

\[
\sum_{j=-N_M}^{j=N_M} \sum_{k=0}^{N_K-1} G \psi_1^{(j)}(E_U)_k^{(n-j)} = 0, \quad 1 \leq |n| \leq N_M,
\]

(3.4.17a)
\[ \sum_{j=-N_{M}}^{j=N_{M}} \sum_{k=0}^{N_{K}-1} G_{1,k}^{(j)} (E_{u})_{k}^{(n-j)} = Q_{x}, \quad n = 0, \]  
\[ \sum_{j=-N_{M}}^{j=N_{M}} \sum_{k=0}^{N_{K}-1} G_{1,k}^{(j)} (d_{u})_{k}^{(n-j)} = 0, \quad -N_{M} \leq n \leq N_{M}, \]  
\[ \sum_{j=-N_{M}}^{j=N_{M}} \sum_{k=0}^{N_{K}-1} G_{1,k}^{(j)} (E_{u})_{k}^{(n-j)} = Ra_{p,u} \theta_{u}^{(n)}, \quad 1 \leq |n| \leq N_{M}, \]  
\[ \sum_{j=-N_{M}}^{j=N_{M}} \sum_{k=0}^{N_{K}-1} G_{1,k}^{(j)} (E_{L})_{k}^{(n-j)} = 0, \quad n = 0. \]  

A similar process applied to boundary conditions at the upper wall results in boundary relations of the form:

\[ \sum_{j=-N_{M}}^{j=N_{M}} \sum_{k=0}^{N_{K}-1} G_{1,k}^{(j)} (E_{L})_{k}^{(n-j)} = 0, \quad -N_{M} \leq n \leq N_{M}, \]  
\[ \sum_{j=-N_{M}}^{j=N_{M}} \sum_{k=0}^{N_{K}-1} G_{1,k}^{(j)} (d_{L})_{k}^{(n-j)} = 0, \quad -N_{M} \leq n \leq N_{M}, \]  
\[ \sum_{j=-N_{M}}^{j=N_{M}} \sum_{k=0}^{N_{K}-1} G_{1,k}^{(j)} (E_{L})_{k}^{(n-j)} = Ra_{p,L} \theta_{L}^{(n)}, \quad 1 \leq |n| \leq N_{M}, \]  
\[ \sum_{j=-N_{M}}^{j=N_{M}} \sum_{k=0}^{N_{K}-1} G_{1,k}^{(j)} (E_{L})_{k}^{(n-j)} = Ra_{wul}, \quad n = 0. \]

In the above, the arbitrary constant in the definition of stream function was selected by assuming that \( \psi_{1} = 0 \) at the lower wall. Imposition of the fixed flow rate constraint results in setting \( \psi_{1} = Q_{x} \) at the upper wall. Imposition of the fixed pressure gradient constraint is more complex as one needs to replace (3.4.17b) with a relation imposing (3.4.3a). This process involves insertion of (3.4.9) and (3.4.12) into (3.4.1b) and extraction of mode zero resulting in a relation of the form

\[ \sum_{k=0}^{N_{K}-1} \left[ \frac{11}{6} \Gamma t^{-1} DT_{k}(-1) - \Gamma^{3} D^{3} T_{k}(-1) \right] G_{\psi_{1,k}}^{(0)} = \sum_{k=0}^{N_{K}-1} \left[ 3 \Gamma t^{-1} DT_{k}(-1) G_{\psi_{0,k}}^{(0)} - \frac{3}{2} \Gamma t^{-1} DT_{k}(-1) G_{\psi_{-1,k}}^{(0)} + \frac{1}{3} \Gamma t^{-1} DT_{k}(-1) G_{\psi_{-2,k}}^{(0)} - \Gamma DT_{k}(-1) \bar{u}_{1} \bar{v}_{1,k}^{(0)} \right] - A. \]  

\[ (3.4.19) \]
This relation can be written for any \( \hat{y} \) as the value of the mean pressure gradient is constant across the channel.

The above boundary relations and constraints are included in the algebraic system (3.4.13) using the Tau method. Accuracy of this discretization can be improved by utilizing additional boundary relations which leads to an over determined system (Husain & Floryan 2014).

3.4.1.2 Solution Strategy

The solution strategy relies on the fixed-point iterative method where it is assumed that the \( RHS \) of (3.4.13) is known. The \( RHS \) consists of both the unknown nonlinear terms as well as the known terms resulting from discretization. We start with a zero-initial approximation for the nonlinear terms, solve the linear system and use this solution to update the nonlinear terms for the next iteration. This update is carried out by transferring data to the physical space using grid in the \((x, \hat{y})\) plane which is composed of \( N_P + 1 \) equidistant points in the \( x \)-direction and \( N_K \) collocation points in the \( \hat{y} \) direction, where \( N_P \geq 1.5N_M \) and the last point in the \( x \)-direction is discarded due to periodicity, evaluating all the products there and, finally, moving those results back to the Fourier space. During the transfer of data back to the Fourier space, Fast Fourier Transformation (FFT) is used to compute Fourier expansions expressing these products. Aliasing error which occurs due to the Fourier transformation is controlled by the implementation of padding which determines the number of points \( N_P \) needs to be used in the \( x \)-direction.

This iterative process is repeated until convergence is achieved. The relative change of the unknown expansion coefficients is defined as

\[
R_1 = \frac{\| G_{\psi_{1,k}}^{(n)} \|_{2} - \| G_{\psi_{1,k}}^{(n)} \|_{2}}{\| G_{\psi_{1,k}}^{(n)} \|_{2}}, \quad R_2 = \frac{\| G_{\theta_{1,k}}^{(n)} \|_{2} - \| G_{\theta_{1,k}}^{(n)} \|_{2}}{\| G_{\theta_{1,k}}^{(n)} \|_{2}}, \quad (3.4.20a, b)
\]

where \( \| V \|_2 \) is the \( L^2 \) norm of vector \( V \) with \( N_V = (N_M + 1) \times N_K \) elements defined as
\[ \sqrt{\left( \sum_{m=1}^{m=N_v}|V_m|^2 \right)} \] and \( j \) denotes the iteration number. The calculation is stopped when both \( R_1 \) and \( R_2 \) decrease below the specified tolerance. Relaxation factor is used to control the iterative process when necessary.

A very large system of linear equations (3.4.13) with suitable boundary relations needs to be solved at every iteration. Since (3.4.13b) is decoupled from (3.4.13a), the solution process consists of at first solving (3.4.13b) followed by solution of (3.4.13a). The coefficient matrices have special structures (see Fig. 3.2) which permits use of specialized solvers (Husain & Floryan 2013) reducing both the execution time and memory requirements by two orders of magnitude when compared with the available solvers for sparse matrices.

![Figure 3.2](image)

**Figure 3.2** Structure of the coefficient matrix for the energy (Fig 3.2A) and momentum (Fig 3.2B) equations \( (N_M = 2, N_K = 10) \). Black color identifies the non-zero elements.

Computational time can be reduced using extrapolation to provide good initial approximation for the nonlinear terms, i.e.
\[(N_{uu})_1 = 3(N_{uu})_0 - 3(N_{uu})_{-1} + (N_{uu})_{-2} + O(h^3), \quad (3.4.21a)\]

\[(N_{u\theta})_1 = 3(N_{u\theta})_0 - 3(N_{u\theta})_{-1} + (N_{u\theta})_{-2} + O(h^3). \quad (3.4.21b)\]

Higher-order extrapolation formulae are given in Appendix F.

### 3.4.1.3 Spectrally Accurate Extraction of Physically Relevant Information

Physically relevant information is sought at each time step. This includes determination of forces acting at the wall as well as heat fluxes. Pressure evaluation begins with the x-momentum equation which is arranged in the form of

\[
\frac{\partial p_1}{\partial x} = -\frac{11}{6} \Delta t^{-1} u_1 + 3 \Delta t^{-1} u_0 - \frac{3}{2} \Delta t^{-1} u_{-1} + \frac{1}{3} \Delta t^{-1} u_{-2} - \frac{\partial u_1 u_1}{\partial x} - \Gamma \frac{\partial u_1}{\partial \tilde{y}} + \frac{\partial^2 u_1}{\partial x^2} + \Gamma^2 \frac{\partial^2 u_1}{\partial \tilde{y}^2}. \quad (3.4.22)\]

Substitution of (3.4.5) and (3.4.9) into (3.4.22) and separation of Fourier modes lead to

\[
A + ina P_1^{(n)} (\tilde{y}) = -\frac{11}{6} \Delta t^{-1} \Gamma D\psi_1^{(n)} (\tilde{y}) + 3 \Delta t^{-1} \Gamma D\psi_0^{(n)} (\tilde{y}) - \frac{3}{2} \Delta t^{-1} \Gamma D\psi_{-1}^{(n)} (\tilde{y}) +
\]

\[
\frac{1}{3} \Delta t^{-1} \Gamma D\psi_{-2}^{(n)} (\tilde{y}) - ina \tilde{u}_1 \tilde{u}_1^{(n)} (\tilde{y}) - \Gamma D\tilde{u}_1 \tilde{v}_1^{(n)} (\tilde{y}) - n^2 \alpha^2 \Gamma D\psi_1^{(n)} (\tilde{y})
\]

\[
+ \Gamma^3 D^3 \psi_1^{(n)} (\tilde{y}) \quad (3.4.23)\]

which can be used to determine all pressure modal functions except \(P^{(0)}\). \(A\) in the above is known if fixed pressure gradient constraint is used. When the fixed flow rate constraint is used, \(A\) can be evaluated by extracting mode zero from (3.4.23) leading to the following relation

\[
A = -\frac{11}{6} \Delta t^{-1} \Gamma D\psi_1^{(0)} (\tilde{y}) + 3 \Delta t^{-1} \Gamma D\psi_0^{(0)} (\tilde{y}) - \frac{3}{2} \Delta t^{-1} \Gamma D\psi_{-1}^{(0)} (\tilde{y}) + \frac{1}{3} \Delta t^{-1} \Gamma D\psi_{-2}^{(0)} (\tilde{y})
\]

\[
- \Gamma D\tilde{u}_1 \tilde{v}_1^{(0)} (\tilde{y}) + \Gamma^3 D^3 \psi_1^{(0)} (\tilde{y}). \quad (3.4.24)\]
Determination of \( P^{(0)} \) starts with the \( y \)-momentum equation arranged to the following form

\[
\Gamma \frac{\partial p_1}{\partial \hat{y}} = -\frac{11}{6} \Delta t^{-1} v_1 + 3 \Delta t^{-1} v_0 - \frac{3}{2} \Delta t^{-1} v_{-1} + \frac{1}{3} \Delta t^{-1} v_{-2} - \frac{\partial \Delta \hat{\nu}}{\partial x} - \Gamma \frac{\partial \hat{\nu}}{\partial \hat{y}} + \frac{\partial^2 \nu_1}{\partial \hat{y}^2} + \Gamma^2 \frac{\partial^2 \nu_1}{\partial \hat{y}^2} + Pr^{-1} \theta_1.
\]  

(3.4.25)

The substitution of (3.4.5) and (3.4.9) into (3.4.25) and separation of Fourier modes lead to

\[
\Gamma D P_1^{(n)}(\hat{y}) = \frac{11}{6} \Delta t^{-1} i n \beta \psi_1^{(n)}(\hat{y}) - 3 \Delta t^{-1} i n \alpha \psi_0^{(n)}(\hat{y}) + \frac{3}{2} \Delta t^{-1} i n \alpha \psi_{-1}^{(n)}(\hat{y}) - \frac{1}{3} \Delta t^{-1} i n \alpha \psi_{-2}^{(n)}(\hat{y}) - i n \alpha \bar{u}_1 \psi_1^{(n)}(\hat{y}) - \Gamma D \bar{v}_1 \psi_1^{(n)}(\hat{y}) + i n^3 \alpha^2 \psi_1^{(n)}(\hat{y})
\]

\[-i n \alpha \Gamma^2 D^2 \psi_1^{(n)}(\hat{y}) + Pr^{-1} \theta_1^{(n)}(\hat{y}).
\]  

(3.4.26)

Extraction of mode zero results in

\[
D P_1^{(0)}(\hat{y}) = -D \bar{v}_1 \psi_1^{(0)}(\hat{y}) + \Gamma^{-1} Pr^{-1} \theta_1^{(0)}(\hat{y})
\]  

(3.4.27)

which, after integration, leads to

\[
P_1^{(0)}(\hat{y}) = -\bar{v}_1 \psi_1^{(0)}(\hat{y}) + \Gamma^{-1} Pr^{-1} \int_{-1}^{\hat{y}} \theta_1^{(0)}(\hat{\nu}) d\hat{\nu} + C
\]  

(3.4.28)

where constant \( C \) is selected to bring the mean of the periodic pressure component to zero. Explicit evaluation of the modal functions requires introduction of Chebyshev expansions, i.e. substitution of (3.4.12) into the (3.4.28), which lead to

\[
P_1^{(0)}(\hat{y}) = -\sum_{k=0}^{N_{K-1}} G \bar{v}_1 \psi_1^{(0)} T_k(\hat{y}) + \Gamma^{-1} Pr^{-1} \sum_{k=0}^{N_{K-1}} G \theta_1^{(0)} J_k
\]  

(3.4.29)

where \( J_k = \int_{-1}^{\hat{y}} T_k(\hat{\nu}) d\hat{\nu} \). The individual integrals can be evaluated analytically as follows:

\[
J_0 = T_1(\hat{y}) + 1, \quad J_1 = \frac{T_2(\hat{y})-1}{4},
\]

\[
J_k = \frac{1}{2} \left[ \frac{T_{k+1}(\hat{y})-T_{k+1}(-1)}{k+1} - \frac{T_{k-1}(\hat{y})-T_{k-1}(-1)}{k-1} \right] \text{ for } k > 1,
\]  

(3.4.30)
which completes evaluation of pressure \( P_1 \).

We shall now discuss evaluation of wall forces. The \( x \)-components of the forces acting on the fluid at the lower \((F_{x,L})\) and upper \((F_{x,U})\) walls per unit length in the \( x \)-direction can be evaluated as

\[
F_{x,L} = F_{xv,L} + F_{xp,L} = \lambda^{-1} \int_{x_0}^{x_0 + \lambda} \left[ 2 \frac{\partial u_1}{\partial x} \right] n_{x,L} + \left( \frac{\partial u_1}{\partial y} + \frac{\partial v_1}{\partial x} \right) n_{y,L} \text{d}x,
\]

\[
-(Ax + Bz + P_1)|_{y_L} n_{x,L} N_L^{-1} \text{d}x,
\]

\[
F_{x,U} = F_{xv,U} + F_{xp,U} = \lambda^{-1} \int_{x_0}^{x_0 + \lambda} \left[ 2 \frac{\partial u_1}{\partial x} \right] n_{x,U} + \left( \frac{\partial u_1}{\partial y} + \frac{\partial v_1}{\partial x} \right) n_{y,U} \text{d}x,
\]

\[
-(Ax + Bz + P_1)|_{y_U} n_{x,U} N_U^{-1} \text{d}x
\]

where the unit normal vectors pointing outwards at the lower \((\vec{n}_L)\) and upper \((\vec{n}_U)\) walls are defined as

\[
\vec{n}_L = (n_{x,L}, n_{y,L}, n_{z,L}) = N_L \left( \frac{dy_L}{dx}, -1, 0 \right), \quad N_L = \left[ 1 + \left( \frac{dy_L}{dx} \right)^2 \right]^{-\frac{1}{2}}
\]

\[
\vec{n}_U = (n_{x,U}, n_{y,U}, n_{z,U}) = N_U \left( \frac{dy_U}{dx}, 1, 0 \right), \quad N_U = \left[ 1 + \left( \frac{dy_U}{dx} \right)^2 \right]^{-\frac{1}{2}}.
\]

Use of (3.4.9) and (3.4.12), expressing Chebyshev polynomials at the walls by the relevant Fourier expansions and execution of the relevant integrations result in the following expressions for the forces:

\[
F_{x,L} = 2\pi \sum_{s=-N_M}^{s=N_M} \sum_{m=-N_m}^{m=N_m} \sum_{k=0}^{k=N_k-1} m(m-s)\alpha^2 y_L^{(s-m)} G\psi_{1,k}^{(m)} d_{L,k} - \pi^2 \sum_{s=-N_M}^{s=N_M} \sum_{k=0}^{k=N_k-1} s^2 \alpha^2 G\psi_{1,k}^{(s)} E_{L,k} - \sum_{n=-N_0}^{n=N_0} A y_L^{(n)} e^{i\alpha x_0}
\]

\[
\sum_{s=-N_M}^{s=N_M} \sum_{m=-N_m}^{m=N_m} \sum_{k=0}^{k=N_k-1} i(s-m)\alpha y_L^{(s-m)} G\psi_{1,k}^{(m)} E_{L,k}^{(-s)},
\]
$$F_{x,U} = -2\Gamma \sum_{s=-N_M}^{s=N_M} \sum_{m=-N_m}^{m=N_m} \sum_{k=0}^{k=N_K-1} m(m - s) \alpha^2 y_u^{(s-m)} G\psi_{1,k}^{(m)} d_{U,k}^{(-s)} + \Gamma^2 \sum_{s=-N_M}^{s=N_M} \sum_{m=-N_m}^{m=N_m} \sum_{k=0}^{k=N_K-1} G\psi_{1,k}^{(s)} n^{(-s)} + \sum_{s=-N_M}^{s=N_M} \sum_{m=-N_m}^{m=N_m} \sum_{k=0}^{k=N_K-1} s^2 2 \alpha^2 G\psi_{1,k}^{(s)} E_{U,k}^{(-s)} + \sum_{n=-N_G}^{n=N_G} A y_u^{(n)} e^{i n \alpha x_o} +$$

$$\sum_{s=-N_M}^{s=N_M} \sum_{m=-N_m}^{m=N_m} \sum_{k=0}^{k=N_K-1} i(s - m) \alpha y_u^{(s-m)} G\psi_{1,k}^{(m)} G\psi_{1,k}^{(m)} E_{U,k}^{(-s)}, \quad (3.433b)$$

which are characterized by the spectral accuracy consistent with other parts of the algorithm.

The heat transfer characteristic is expressed in terms of Nusselt numbers. The average Nusselt numbers at the lower ($Nu_{av,L}$) and upper ($Nu_{av,U}$) walls are defined as

$$Nu_{av,L} = \lambda^{-1} \int_{x_0}^{x_0+\lambda} \left[ n_{x,L} \frac{\partial \theta}{\partial x} + n_{y,L} \frac{\partial \theta}{\partial y} \right] N_L^{-1} dx, \quad (3.434a)$$

$$Nu_{av,U} = \lambda^{-1} \int_{x_0}^{x_0+\lambda} \left[ n_{x,U} \frac{\partial \theta}{\partial x} + n_{y,U} \frac{\partial \theta}{\partial y} \right] N_U^{-1} dx. \quad (3.434b)$$

Substitution of (3.4.9) and (3.4.12) into (3.4.34), expressing values of Chebyshev polynomials at the walls by the relevant Fourier expansions and execution of integrations result in

$$Nu_{av,L} = \alpha^2 \sum_{s=-N_M}^{s=N_M} \sum_{m=-N_m}^{m=N_m} \sum_{k=0}^{k=N_K-1} m(m - s) y_u^{(s-m)} G\theta_{1,k}^{(m)} E_{L,k}^{(-s)} -$$

$$\Gamma \sum_{s=-N_M}^{s=N_M} \sum_{k=0}^{k=N_K-1} G\theta_{1,k}^{(s)} d_{L,k}^{(-s)}, \quad (3.435a)$$

$$\Gamma \sum_{s=-N_M}^{s=N_M} \sum_{k=0}^{k=N_K-1} G\theta_{1,k}^{(s)} d_{U,k}^{(-s)} \quad (3.435b)$$

which are characterized by the spectral accuracy consistent with other parts of the algorithm.
3.4.2 Streamwise Flow Component

The streamwise flow problem consists of (3.4.1d) and the relevant boundary conditions and constraints, and its solution leads to the determination of \( w_1 \). The unknown is represented as

\[
 w_1(x, y) \approx \sum_{n=-N_M}^{n=N_M} w_{1}^{(n)}(y) e^{in\alpha x} \tag{3.4.36}
\]

where the modal functions satisfy the reality condition. Substitution of (3.4.36) into (3.4.1d), use of transformation (3.4.10a) and separation of Fourier modes result in the following equations

\[
 \left[ \frac{11}{6} \Delta t^{-1} + (n\alpha)^2 - \Gamma^2 D^2 \right] w_1^{(n)} + i\Gamma \alpha \sum_{m=-N_M}^{m=N_M} m D \psi_1^{(n-m)} - (n-m) \psi_1^{(n-m)} D ]
\]

\[
 = 3\Delta t^{-1} w_0^{(n)} - \frac{3}{2} \Delta t^{-1} w_{-1}^{(n)} + \frac{1}{3} \Delta t^{-1} w_{-2}^{(n)}, \quad n \neq 0, \tag{3.4.37a}
\]

\[
 \left[ \frac{11}{6} \Delta t^{-1} - \Gamma^2 D^2 \right] w_1^{(0)} + i\Gamma \alpha \sum_{m=-N_M}^{m=N_M} m [ D \psi_1^{(-m)} + \psi_1^{(-m)} D ] w_1^{(m)}(\tilde{y}) = 3\Delta t^{-1} w_0^{(0)}
\]

\[
 - \frac{3}{2} \Delta t^{-1} w_{-1}^{(0)} + \frac{1}{3} \Delta t^{-1} w_{-2}^{(0)} - B, \quad n = 0 \tag{3.4.37b}
\]

where \(-N_M \leq n \leq N_M\) and the known \( u_1 \) and \( v_1 \) are expressed as

\[
 u_1(x, \tilde{y}) \approx \Gamma \sum_{n=-N_M}^{n=N_M} D \psi_1^{(n)}(\tilde{y}) e^{in\alpha x}, \quad v_1(x, \tilde{y}) \approx in\alpha \sum_{n=-N_M}^{n=N_M} \psi_1^{(n)}(\tilde{y}) e^{in\alpha x}. \tag{3.4.38}
\]

The modal function \( w_1^{(n)} \) are expressed in terms of Chebyshev expansions of the form

\[
 w_1^{(n)}(\tilde{y}) \approx \sum_{k=0}^{N_K-1} Gw_{1,k}^{(n)} T_k(\tilde{y}). \tag{3.4.39}
\]

Substitution of (3.4.39) into (3.4.37) and the implementation of the Galerkin projection method result in the linear algebraic equations for the Chebyshev expansion coefficients of the form

\[
 \sum_{k=0}^{N_K-1} \left[ \frac{11}{6} \Delta t^{-1} < T_j, T_k > + n^2\alpha^2 < T_j, T_k > - \Gamma^2 < T_j, D^2 T_k > \right] Gw_{1,k}^{(n)} +
\]

72
\[ i\Gamma \alpha \sum_{m=-N_M}^{m=N_M} \sum_{l=0}^{N_l-1} G\psi_{1,l}^{(n-m)} \left[ m < T_j, DT_i T_k > -(n - m) < T_j, T_i DT_k > \right] GW_{1,k}^{(m)} \]

\[
\sum_{k=0}^{N_K-1} \left[ 3\Delta t^{-1} < T_j, T_k > GW_{0,k}^{(n)} - \frac{3}{2} \Delta t^{-1} < T_j, T_k > GW_{0,k}^{(n)} + \frac{1}{3} \Delta t^{-1} < T_j, T_k > GW_{-2,k}^{(n)} \right], \quad n \neq 0, \quad (3.4.40a)
\]

\[
\sum_{k=0}^{N_K-1} \left[ \left( \frac{11}{6} \Delta t^{-1} < T_j, T_k > -\Gamma^2 < T_j, DT^2 T_k > \right) GW_{1,k}^{(0)} + i\Gamma \alpha \sum_{m=-N_M}^{m=N_M} \sum_{l=0}^{N_l-1} mG\psi_{1,l}^{(-m)} \right]
\]

\[
\left[ < T_j, DT_i T_k > + < T_j, T_i DT_k > \right] GW_{1,k}^{(m)} = \sum_{k=0}^{N_K-1} \left[ 3\Delta t^{-1} < T_j, T_k > GW_{0,k}^{(0)} - \frac{3}{2} \Delta t^{-1} < T_j, T_k > GW_{-2,k}^{(0)} \right] - B \left( T_j, T_0 \right), \quad n = 0, \quad (3.4.40b)
\]

where \( 0 \leq j, k, l \leq N_K - 1, \quad -N_M \leq n, m \leq N_M \) and all inner products \( < f(\hat{y}), g(\hat{y}) > \) can be found in Appendix A. The above algebraic system is supplemented with the boundary conditions (3.4.2c) which are discretized following method described in Section 3.4.1.1. The final form of the resulting boundary relations suitable for numerical implementation is as follows:

\[
\sum_{j=-N_M}^{j=N_M} \sum_{k=0}^{N_K-1} GW_{j,k}^{(n-j)} (E_L)_k^{(n-j)} = 0, \quad -N_M \leq n \leq N_M, \quad (3.4.41a)
\]

\[
\sum_{j=-N_M}^{j=N_M} \sum_{k=0}^{N_K-1} GW_{j,k}^{(n-j)} (E_U)_k^{(n-j)} = 0, \quad -N_M \leq n \leq N_M \quad (3.4.41b)
\]

where determination of coefficients \((E_L)_k^{(n-j)}\) and \((E_U)_k^{(n-j)}\) is described in Appendix B.

These conditions are imposed using the Tau procedure. The fixed pressure gradient constraint can be imposed directly as \( B = \varphi_x \) is known. Imposition of the fixed flow rate constraint is more complex as explained in Appendix F. The resulting system is solved using a general solver as the coefficient matrix illustrated in Fig 3.3 does not offer potential for the development of specialized solvers.
3.4.2.1 Spectrally Accurate Extraction of Physically Relevant Information

The $z$-components of the forces acting on the fluid at the lower ($F_{z,L}$) and upper ($F_{z,U}$) walls per unit length in the $x$-direction can be evaluated as

$$F_{z,L} = F_{zv,L} = \lambda^{-1} \int_{x_0}^{x_0 + \lambda} \left[ \left( \frac{\partial w_1}{\partial x} + \frac{\partial u_1}{\partial z} \right) y_L n_{x,L} + \left( \frac{\partial v_1}{\partial z} + \frac{\partial w_1}{\partial y} \right) y_L n_{y,L} \right] N_L^{-1} dx, \quad (3.4.42a)$$

$$F_{z,U} = F_{zv,U} = \lambda^{-1} \int_{x_0}^{x_0 + \lambda} \left[ \left( \frac{\partial w_1}{\partial x} + \frac{\partial u_1}{\partial z} \right) y_U n_{x,U} + \left( \frac{\partial v_1}{\partial z} + \frac{\partial w_1}{\partial y} \right) y_U n_{y,U} \right] N_U^{-1} dx. \quad (3.4.42b)$$

Substitution of (3.4.36), (3.4.38) and (3.4.39) into the above equation, expressing values of Chebyshev polynomials at the walls in terms of the relevant Fourier expansions and execution of the relevant integrations result in the following expressions for the forces

$$F_{z,L} = \sum_{s=-N_M}^{s=N_M} \sum_{m=-N_M}^{m=N_M} \sum_{k=0}^{k=N_K-1} m(m-s) \alpha^2 y_L^{(m)} G_{w_{1,k}}^{(s-m)} E_{L,k}^{(-s)} -$$

$$\Gamma \sum_{s=-N_M}^{s=N_M} \sum_{k=0}^{k=N_K-1} G_{w_{1,k}}^{(s)} d_{L,k}^{(-s)}, \quad (3.4.43a)$$
\[ F_{z,U} = -\sum_{s=-N_M}^{s=+N_M} \sum_{m=-N_M}^{m=+N_M} \sum_{k=0}^{k=N_K-1} m(m-s)\alpha^2 y_U^{(m)} G_{1,k}^{(s-m)} E_{U,k}^{(-s)} + \]
\[ \Gamma \sum_{s=-N_M}^{s=+N_M} \sum_{k=0}^{k=N_K-1} G_{1,k}^{(s)} d_{U,k}^{(-s)}. \]  

(3.4.43b)

3.5 Discussion and Algorithm Performance

The main advantage of the proposed algorithm is its high spatial accuracy and a wide choice of temporal accuracies. It is based on a fully implicit formulation which provides superior numerical stability properties. Instabilities may be created at the boundaries, and it is not known how IBC concept may affect them. Such instabilities were encountered but only when insufficient spatial accuracy was used. There was no need to investigate them in detail as it was found that they could be easily controlled by increasing the number of Fourier modes and Chebyshev polynomials.

We illustrate the temporal accuracy through tests carried out with the fixed flow rate constraints in both the x- and z-directions for \( Re_x = 5, Re_z = 3, Pr = 0.71, y_U(x) = 1, y_L(x) = -1 + 0.05 \cos(\alpha x), \theta_L(x) = Ra_{uni} + \frac{Ra_{p,L}}{2} \cos(\alpha x) \sin(\omega t), \theta_U(x) = 0, \alpha = 2, B_U = 0, Ra_{uni} = 50, Ra_{p,L} = 200, Ra_{p,U} = 0 \) and \( \omega = 2\pi \), i.e. the temporal effect were driven by the time periodic heating applied at the lower wall. The reference solution was produced using fifth-order method with \( \Delta t = 0.001 \). Computations started with trivial initial conditions using the first-order method for the first five steps and then were continued over six heating periods using the fifth-order method – this time interval was enough for elimination of transient effects associated with the step-like introduction of movement and heating. An arbitrary time point, \( t_{init} \), was selected during the 7th period to serve as an initial point for the tests, and computations continued from \( t_{init} \) for one period to produce the reference solution. In the actual tests, computations were started at \( t_{init} \) using different methods and were continued for one full period. Availability of accurate solution for \( t < t_{init} \) provided means for avoiding transient effects associated with the initiations of the computations using the not-self-starting methods (2nd, ..., 6th order methods). The number of Fourier modes and Chebyshev polynomials, and the convergence
criteria, were selected to reduce the spatial discretization and iteration errors to machine level, i.e. $N_M = 30$ Fourier modes and $N_K = 80$ Chebyshev polynomials were used and convergence criterion (relative change) was set at $10^{-14}$. We used error in the evaluation of the $x$-pressure gradient defined as

$$
\| E_{r_t} \| = \left| \frac{dp}{dx} - \left( \frac{dp}{dx} \right)_{ref} \right|
$$

as a measure of the accuracy of each method. Results displayed in Fig 3.4 demonstrate that the algorithm delivers the theoretically predicted accuracy and that a significant reduction of the absolute error with an increase of the order of temporal discretization can be achieved. The absolute accuracy advantage of the higher-order methods increases with reduction of the time step with the $4^{th}$ and higher order methods being able to deliver machine level accuracy with time step of 0.005.

Figure 3.4 Variations of the error $\| E_{r_t} \|$ (see Eq. 3.5.1) in the evaluation of the pressure gradient as a function of the time step $\Delta t$. 

76
Tests for the spatial accuracy were carried out by starting with zero initial conditions and continuing computations until transient effects died out, i.e. computations were carried out until differences between Chebyshev expansion coefficients in two sequential time steps were below a preset limit as this would guarantee eliminations of any errors associated with time discretization. Such tests were carried out for the fixed flow rate constraint in both the $x$- and $z$-directions with $Re_x = 3$, $Re_z = 8$, $Pr = 0.71$, $y_u(x) = 1 + \frac{B_u}{2} \cos(\alpha x)$, $y_L(x) = -1 + \frac{B_L}{2} \cos(\alpha x)$, $\theta_L(x) = Ra_{uni} + \frac{Ra_p}{2} \cos(\alpha x)$, $\theta_U(x) = \frac{Ra_p}{2} \cos(\alpha x)$, $\alpha = 2$ and $Ra_{uni} = 0$. The reference solution was obtained by carrying out computations with $N_M = 30$ Fourier modes and $N_K = 80$ Chebyshev polynomials as this reduced the spatial discretization error below machine level for flow conditions used in the tests.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3.5.png}
\caption{Figure 3.5 Variations of errors $\|E_{r_u}\|$ (Fig 3.5A) and $\|E_{r_w}\|$ (Fig 3.5B) as functions of the number of Fourier modes $N_M$ used in the computations. $N_K = 80$ Chebyshev polynomials which reduced the Chebyshev truncation error below machine accuracy.}
\end{figure}
The error was defined as

$$\|Er_q\| = \frac{\|q_{ref} - q\|^2}{\|q_{ref}\|^2}$$

where $q$ stands for any quantity of interest, $q_{ref}$ denotes the reference solution and $\|\ldots\|^2$ is the $L^2$ norm of a matrix $M$ resulting from evaluation of this quantity on a grid of 201 collocations points in the $γ$-direction and 201 equidistant points in the $x$-direction, and which is equal to the largest singular value of this matrix, i.e. $\|M\|^2 = max [SVD(M)]$.

Results displayed in Fig 3.5 demonstrate exponential reduction of error in the evaluation of the $u$- and $w$-velocity components with an increase of the number of Fourier modes used in the computations. Results displayed in Fig 3.6 demonstrate similar reduction of error with an increase of the number of Chebyshev polynomials.

Figure 3.6 Variations of errors $\|Er_u\|$ (Fig 3.6A) and $\|Er_w\|$ (Fig 3.6B) as functions of the number of Chebyshev polynomials $N_K$ used in the computations. $N_M = 30$ Fourier modes were used which reduced the Fourier truncation error below machine accuracy.
The relative efficiency of different time discretization methods, as compared to the first-order method, were determined by looking at a time required for advancing solution by one time step with $\Delta t = 0.005$. The initial conditions used in these tests were taken from data used for demonstration of temporal accuracy. Results displayed in Table 3.1 demonstrate a nearly identical performance of all methods when no extrapolation was used to determine the initial approximation for nonlinear terms, which should not be surprising as the equations to be solved at each time step are nearly identical. The higher order methods become much more efficient when advantage is taken of the extrapolation technique which provides a better initial approximation for the nonlinear terms – the 6th order method requires only 16% of time used by the first order method.

Table 3.1 Comparison of the computational cost of advancing solution by one time step. Cost of the 1st order method is used as the reference.

<table>
<thead>
<tr>
<th>Order of Method and extrapolation</th>
<th>No Extrapolation</th>
<th>With Extrapolation</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$\sim 1$</td>
<td>0.77</td>
</tr>
<tr>
<td>3</td>
<td>$\sim 1$</td>
<td>0.61</td>
</tr>
<tr>
<td>4</td>
<td>$\sim 1$</td>
<td>0.42</td>
</tr>
<tr>
<td>5</td>
<td>$\sim 1$</td>
<td>0.31</td>
</tr>
<tr>
<td>6</td>
<td>$\sim 1$</td>
<td>0.16</td>
</tr>
</tbody>
</table>

Development of algorithm presented in this paper was stimulated by interest in the analysis of effects of small grooves and identification of conditions when such grooves lead to a reduction of pressure losses. Use of grid-based solvers requires construction of very fine meshes to resolve flow in the vicinity of grooved wall resulting in very large algebraic systems, use of distributed memory systems and long execution times. Geometric flexibility represents the best advantage of the proposed algorithm as the geometry modelling requires provision of a small number of Fourier coefficients, and it completely avoids grid generation and eliminates the need for grid convergence studies. Meeting absolute accuracy criteria involves a simple increase either of the number of Chebyshev
Figure 3.7 Instantaneous vectors lines for a flow in a conduit with geometry of the form 
\[ y_U(x) = 1 + 0.025 \cos(\alpha x), \quad y_L(x) = -1 + 0.05 \cos(\alpha x), \quad \alpha = 2, \]  
for a flow rate in the \( z \)-direction corresponding to \( Re_z = 2 \) and the zero pressure gradient in the \( x \)-direction \( (Re_x = 0) \) 
for a fluid with \( Pr = 0.71 \). The conduit is exposed to heating resulting in the lower wall 
temperature of the form 
\[ \theta_L(x) = Ra_{uni} + \frac{Ra_{pl}}{2} \cos(\alpha x) \sin(\omega t) \]  
with \( Ra_{uni} = 50, \quad Ra_{pl} = 800, \quad \omega = 2\pi \) and the upper wall temperature of the form 
\[ \theta_U(x) = \frac{Ra_{pl}}{2} \cos(\alpha x) \]  
with \( Ra_{pl} = 800. \quad T = 1 \) stands for the time-period and \( t_0 = 6T \). Color coding refers to the \( w \)-velocity 
component.
polynomial or the number of Fourier modes. Analysis of large number of geometries avoids labor associated with gridding and grid convergence studies and can be automatized providing means for optimization of groove geometry.

Fig 3.7 illustrates complex structure of velocity field created by a combination of groove and time dependent heating patterns. This field was created by starting with the trivial initial conditions, carrying out simulations until transient effect associated with the start conditions died out so that the flow had an opportunity to settle down to its time periodic form (about six periods), and then continuing for one more period and recording results. The structure of velocity field underlines the need for accuracy and proper resolution of the near-wall regions.

3.6 Summary

A highly accurate algorithm for analyses of transient effects in grooved conduits, including transition to secondary states as well as the potential onset of chaotic responses, has been developed. Combinations of geometric and heating patterns lead to pattern interaction problems which could be beneficial in the development of energy efficient stirring systems. The algorithm is designed to accurately model physical effects associated with small amplitude grooves. It uses spectral spatial discretization and up to sixth-order temporal discretization and can deliver a near machine accuracy if desired. The temporal discretization is fully implicit to take advantage of its superior stability properties. The numerical instabilities associated with the special immersed-boundary-conditions (IBC) treatment of boundary conditions can be easily controlled by increasing the spatial accuracy. The spatial discretization uses Chebyshev expansions in the vertical direction to provide very good resolution and accuracy in the near wall regions. Discretization in the horizontal directions is based on Fourier expansions which automatically satisfy the periodicity conditions. The algebraic equations for the Chebyshev expansion coefficients are constructed using the Galerkin projection method. The imposition of boundary conditions in irregular flow domains relies on the IBC concept; the flow boundary
conditions are transformed into constraints and are implemented using the Tau concept. The IBC concept is also used to implement either the fixed flow rate or the fixed pressure gradient constraint. The discretized equations are solved at each time step using a fixed-point iterative method. Information about the surface topography is provided in the form of Fourier coefficients and this eliminates the need for numerical grid generation. The algorithm provides the geometric flexibility required for efficient analyses of a wide range of topography patterns. It delivers the theoretically predicted accuracy for both the spatial and temporal discretizations as demonstrated by numerical tests. The computational costs of various time discretization schemes are nearly the same but can be reduced by up to 80% using extrapolation to provide a better initial approximation of nonlinear terms. The algorithm is very efficient as demonstrated by comparisons with finite-volume open-source codes as well as spectral element open-source codes – the computing time for a similar case study is at least an order of magnitude shorter. It delivers superior spatial and temporal accuracy which is not available in the open-source codes.
Chapter 4

4 Creation of Streaks in a Smooth Channel using Heating Patterns

4.1 Introduction

The primary role of streaks in mixing process is either to induce known instabilities or to create new instabilities leading to saturation states with desired properties. In this chapter, we have shown that streaks can be created in a controlled manner in a smooth channel using spatially distributed heating with their spatial distribution dictated by the heating pattern. The energy costs of streak formation were determined both in terms of additional pressure losses required to drive the same flow rate in the heated and isothermal channels and in terms of the reduction of the flow rate if the pressure gradient remained unaltered. The creation of streaks using the Rayleigh-Bénard effect was studied for completeness. Algorithms discussed in Chapter 2 & 3 were used for this analysis. Section 4.2 provides the formulation of the model problem. Section 4.3 discusses streak creation using the RB convection. Section 4.4 explains streaks driven by sinusoidal heating applied at the lower wall. Section 4.5 considers streaks induced by a combination of sinusoidal and uniform heating at the lower wall. Section 4.6 describes streaks generated by sinusoidal heating applied at the upper wall. Section 4.7 analyses streaks formed by sinusoidal heating at both walls. Section 4.8 gives an assessment of how variations of Prandtl number affect the system response. Section 4.9 provides a short summary of the main conclusions.

\[ \text{A version of this chapter has been published as –} \]

4.2 Problem Formulation

Consider fluid flow in a channel formed by two horizontal plates placed at a distance $2h^*$ apart and extending to $\pm \infty$ in both the $x$- and $z$-directions (see Fig 4.1) with gravity being acted in the negative $y$-direction. The fluid is assumed to be incompressible, Newtonian and variation of density follows the Boussinesq approximation. It has thermal conductivity $k^*$, specific heat $c^*$, thermal diffusivity $\kappa^* = k^*/\rho^*c^*$, kinematic viscosity $\nu^*$, dynamic viscosity $\mu^*$ and thermal expansion coefficient $\beta^*$ and it is driven in the positive $z$-direction by a fixed pressure gradient. Both walls are exposed to the $x$-periodic heating leading the walls’ relative temperatures $(\theta^* = T^* - T_{U,\text{mean}}^*)$ of the form

$$\theta^*_L(x^*) = \theta^*_\text{uni} + \frac{\theta^*_{P,L}}{2} \cos(\alpha^*x^*), \quad (4.2.1a)$$

$$\theta^*_U(x^*) = \frac{\theta^*_{P,U}}{2} \cos(\alpha^*x^* + \Omega^*) \quad (4.2.1b)$$

where the heating wavelength is defined as $\lambda^* = 2\pi/\alpha^*$ and $\Omega^*$ stands for the phase shift between the upper and lower wall heating. Use of half of the channel height $h^*$ as the length scale and $\kappa^*\nu^*/(g^*\beta^*h^3)$ as the temperature scale result in

$$\theta_L(x) = Ra_{\text{uni}} + \frac{Ra_{P,L}}{2} \cos(\alpha x), \quad (4.2.2a)$$

$$\theta_U(x) = \frac{Ra_{P,U}}{2} \cos(\alpha x + \Omega) \quad (4.2.2b)$$

where the uniform Rayleigh number $Ra_{\text{uni}} = g^*\beta^*h^3\theta_{\text{uni}}^*/(\kappa^*\nu^*)$ determines the intensity of the uniform component of heating, the lower periodic Rayleigh number $Ra_{P,L} = g^*\beta^*h^3\theta_{P,L}^*/(\kappa^*\nu^*)$ determines the intensity of the lower wall periodic heating and the upper periodic Rayleigh number $Ra_{P,U} = g^*\beta^*h^3\theta_{P,U}^*/(\kappa^*\nu^*)$ determines the intensity of the upper wall periodic heating.
The governing equations in the dimensionless form are

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 ,
\]

\[(4.2.3a)\]

\[
 u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = -\frac{\partial p}{\partial x} + \nabla^2 u ,
\]

\[(4.2.3b)\]

\[
 u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} = -\frac{\partial p}{\partial y} + \nabla^2 v + Pr^{-1} \theta ,
\]

\[(4.2.3c)\]

\[
 u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} = -\frac{\partial p}{\partial z} + \nabla^2 w ,
\]

\[(4.2.3d)\]

\[
 u \frac{\partial \theta}{\partial x} + v \frac{\partial \theta}{\partial y} + w \frac{\partial \theta}{\partial z} = Pr^{-1} \nabla^2 \theta
\]

\[(4.2.3e)\]

where \(U_v^* = v^*/h^*\) is the velocity scale, \(\rho^*U_v^{*2}\) is the pressure scale and \(Pr = \nu^*/\kappa^*\) is the Prandtl number. The corresponding boundary conditions have the form

\[
u = v = w = 0 \quad \text{at} \quad y = \pm 1,
\]

\[(4.2.4a-c)\]

\[
\theta(x,-1,z) = \theta_L(x), \quad \theta(x,1,z) = \theta_U(x).
\]

\[(4.2.4d-e)\]

Effect of heating is quantified by comparing properties of the modified flow with the isothermal flow which has the form
\( \vec{u}_0(x, y, z) = [0, 0, w_0] = [0, 0, Re_z(1 - y^2)], \quad p_0(x, y, z) = -2 z Re_z, \)

\[ Q_0 = \frac{4}{3} Re_z \quad (4.2.5) \]

where subscript 0 denotes the isothermal quantities and the Reynolds number is defined as
\( Re_z = W_{max}^* h^* / \nu^* = W_{max}^* / U_v^* \) where \( W_{max}^* \) denotes the maximum of the \( z \) velocity component. The problem formulation is closed by imposing the zero mean pressure gradient constraint in the \( x \)-direction, i.e.

\[ \frac{\partial p}{\partial x}_{\text{mean}} = 0 \quad (4.2.6) \]

and either by keeping the mean pressure gradient in the \( z \)-direction the same for the isothermal and non-isothermal cases, i.e.

\[ \frac{\partial p}{\partial z}_{\text{mean}} = -2 Re_z \quad (4.2.7a) \]

or by keeping the same flowrate in the \( z \)-direction for the isothermal and non-isothermal cases, i.e.

\[ Q = \frac{4}{3} Re_z. \quad (4.2.7b) \]

The above system was solved numerically with spectral accuracy using the algorithm discussed in Chapter 2 & 3. The heating pattern does not depend on the streamwise direction which leads to decoupling of Eqs (4.2.3a-c, e) and (4.2.3d). The system was solved in two steps where the solution started with the spanwise convective problem in the \((x, y)\)-plane and followed by solution of the streamwise flow. Discretization relied on the Fourier expansions in the horizontal directions and the Chebyshev expansions in the \( y \)-direction. The pressure gradient in the case of the fixed flow rate constraint was evaluated directly as a part of the solution process while the flow rate in the case of the fixed pressure gradient constraint was evaluated during postprocessing as

\[ Q = \lambda^4 \int_0^2 \int_{-1}^{+1} w \, dy \, dx. \quad (4.2.8) \]
In the following sections, all results are presented for air \((Pr = 0.71)\) unless otherwise noted.

### 4.3 Rayleigh-Bénard Effect

Formation of streaks requires creation of streamwise rolls which produce spanwise variations of velocity field. The most obvious method is the use of the Rayleigh-Bénard (RB) instability which is activated when intensity of the uniform heating exceeds the critical uniform Rayleigh number \(Ra_{uni,c} = 213.5\). These rolls occur as a bifurcation from a purely conductive state, their size is characterized by the critical wavenumber \(\alpha_c = 1.5585\), which limits the type of rolls which can be formed, and their spatial locations are not controlled. Even a very weak rotational motion in the \((x, y)\)-plane, such as that coming from the RB instability near the critical conditions, can create significant spanwise distortion of the velocity field as it convects low velocity particles away from the walls (upwash) and high velocity particles towards the walls (downwash). Figure 4.2A illustrates these rolls and the resulting spiral particle trajectories. Increase of the heating intensity beyond the critical conditions results in an increase of streaks’ strength as illustrated in Fig 4.2B. The type of fluid has a major effect as a decrease of the Prandtl number increase streaks’ strength as demonstrated in Fig 4.2C. This should not be surprising as conduction dominates convection for low \(Pr\) fluids.

Creation of a complex flow field increases the energy cost of maintaining such flow due to an increase of dissipation. This energy can be measured either by an increase of the mean pressure gradient required to maintain the same flow rate in the isothermal and heated channels or by a reduction of the flow rate if the mean pressure gradient is kept unchanged. Increase of \(Ra_{uni}\) by about 10% above its critical value results in an increase of pressure losses by \(\sim 2.6\%\) (see Fig 4.3A) or reduction of the flow rate by \(\sim 2.6\%\) (see Fig 4.3B) for air. Rolls results in an increase of transverse mixing and in an increase of the heat flow between the walls which can be quantified using the average Nusselt number defined as

\[
Nu_{av} = \frac{1}{\lambda} \int_0^L \left(-\frac{d\theta}{dy}\right)_{y=-1} \, dx. \tag{4.3.1}
\]
Figure 4.2 (A) Flow topology created by the Rayleigh-Bénard instability ($\alpha = 1.5585$, $Ra_{uni} = 250$, $Re_z = 15$). Golden and turquoise colors identify stream tubes corresponding to $\psi = 0.45, -0.45$, respectively, and dashed lines show particle trajectories. Colors in the $(x, y)$-plane illustrate the temperature field while black solid lines illustrate vector lines. (B) Variations of the spanwise gradient of the streamwise velocity component $dw/dx$ at $y = 0$ for $\alpha = 1.5585$, $Ra_{uni} = 200, 220, 250, Pr = 0.71$. (C) Variations of the spanwise gradient of the streamwise velocity component $dw/dx$ at $y = 0$ for $\alpha = 1.5585$, $Ra_{uni} = 250$, $Pr = 0.025, 0.71, 7.56$. 
Variations of $B = \frac{dp}{dz}/Re_z$ (Fig 4.3A), (ii) the flowrate $Q/Re_z$ (Fig 4.3B) and (iii) the average Nusselt number $Nu_{av}$ (Fig 4.3C) as functions of the uniform Rayleigh number $Ra_{uni}$ for $\alpha = 1.5585$.

Variations of $Nu_{av}$ as a function of $Ra_{uni}$ illustrated in Fig 4.3C exhibit its linear increases for $Ra_{uni} < Ra_{uni,c}$ where heat is carried by conduction, and a nearly linear but much
more rapid increase for $Ra_{uni} > Ra_{uni,c}$ where heat is carried mostly by convection. Computations show that transition occurs at $Ra_c = 213.5$ which matches data available in the literature (Reid & Harris 1958; Koschmieder 1993). The stronger convection for small $Pr$’s has minor effect on $Nu_{av}$ as it marginally effects the temperature field.

4.4 Sinusoidal Heating at the Lower Wall

To gain ability to control streak size, we explore use of sinusoidal heating where the roll size is set through selection of the heating wavenumber. Introduction of a spanwise temperature gradient leads to a streak formation through a forced system response regardless of the heating intensity.

The flow topology (Fig 4.4A) is qualitatively like that shown in Fig 4.2A but the streak locations are determined by positions on the hot and cold spots. The fluid is advected away from the lower wall at the hot spots (upwash) and towards this wall at the cold spots (downwash). The streamwise flow has parabolic velocity distribution which means that the roll motion brings slow fluid above the hot spot towards the channel center creating a local velocity deficit, and fast fluid from the center towards the upper wall creating a local velocity excess. The situation in the zone above the cold spot is opposite with the slow fluid from vicinity of the upper wall being brought down to the channel center and the fast fluid from the center being brought to the lower wall. The upward motion is concentrated above the hot spot while the downward motion is distributed in the spanwise direction around the cold spot as illustrated in Fig 4.4B displaying the streamwise velocity modifications $w_m = (w - w_0)$. As a result, the streaks have a complex structure with alternating velocity excesses and velocity deficits in the upper part of the channel and a similar configuration but moved by half wavelength in the spanwise direction in the lower part of the channel. The overall flow topology is characterized by formation of two high velocity stream tubes per one heating wavelength as illustrated in Fig 4.4C.
Figure 4.4 (A) Flow topology - golden and turquoise colors identify stream tubes corresponding to $\psi = 1.2, -1.2$; colors in the $(x, y)$-plane illustrate the temperature field; dashed lines illustrate particle trajectories. (B) Distributions of the $w$-velocity modifications $w_m = (w - w_0)$ in the $(x, y)$ – plane; colors identify magnitudes of velocity modifications. (C) Distributions of the $w$-velocity in the $(x, y)$ – plane; colors identify magnitudes of velocity. In all subfigures, $\alpha = 1, Ra_{P,L} = 1000, Ra_{P,U} = Ra_{uni} = 0, Re_z = 15$, black lines illustrate vector lines.

The strength of the streaks can be deduced from plots of the spanwise gradient of streamwise velocity $dw/dx$ displayed in Fig 4.5. Increase of the periodic Rayleigh number increases the streak intensity as illustrated in Fig 4.5A with much higher intensity concentrated around the hot spots – the increase of intensity is more tempered in the return zone where fluid flows downwards. Effect of the heating wave number is more complex
with the most intense streaks created by $\alpha \approx 1$ as shown in Fig 4.5B. Excessively short heating wavelength results in confining convective motion to within a boundary layer forming in the vicinity of heated wall (Hossain & Floryan 2013) and reduction of the streak intensity. Use of an excessively long heating wavelength also reduces this intensity but may lead to a transition to a secondary state through a local RB instability in the vicinity of the hot spot if $Ra_{p,L}$ is too large (Asgarian, Hossain & Floryan 2016; Floryan, Hossain & Bassom 2019). The roll structure near the hot spot is illustrated in Fig 4.6A for conditions preventing formation of secondary rolls and in Fig 4.6B for the same conditions but with $Ra_{p,L}$ increased to the level required for the onset of the local RB instability. The intensity of the local secondary rolls is nearly equal to the intensity of the primary rolls created with $\alpha \approx 1$ (see Fig 4.5B).

Figure 4.5 Variations of the spanwise gradient of the streamwise velocity component $dw/dx$ at $y = 0$ for $\alpha = 1$ as a function of $Ra_{p,L}$ (Fig 4.5A) and for $Ra_{p,L} = 1000$ as a function of $\alpha$ (Fig 4.5B).
Figure 4.6 Flow topologies for $\alpha = 0.2, Re_z = 15, Ra_{P,U} = Ra_{uni} = 0$ for $Ra_{P,L} = 100$ (Fig 4.6A) and for $Ra_{P,L} = 1000$ (Fig 4.6B). Yellow (ocher) and turquoise colors identify stream tubes corresponding to $\psi = 0.2, -0.2$ in Fig 4.6A, and to $\psi = 1, -1$ in Fig 4.6B. Colors in the $(x,y)$-plane illustrate the temperature field while the black solid lines illustrate vector lines.

Formation of streaks increases energy loses associated with maintaining the modified flow. When the flow rate is fixed, the pressure losses increase with an increase of the streak intensity (increase of $Ra_{P,L}$) and $\alpha$ which leads to the maximum losses shifts at the same time from $\alpha \approx 1.5$ for $Ra_{P,L} = 100$, which is very similar to $\alpha_c$ for the RB convection, to $\alpha \approx 2$ for $Ra_{P,L} = 1000$, which is the largest $Ra_{P,L}$ used in this analysis. Losses rapidly decrease when $\alpha$ moves away from its most effective value - they decrease proportionally to $\alpha^4$ when $\alpha \to 0$ and proportionally to $\alpha^{-6}$ when $\alpha \to \infty$ (see Fig 4.7A). These limits can be determined analytically (Floryan, Shadman & Hossain 2018) – details not given due to their length. There is a noticeable increase of losses for small $\alpha$’s for conditions leading to formation of local secondary rolls around the hot spots as discussed previously.

Losses can also be measured using reduction of the flow rate in the heated channel as compared to the flow rate driven by the same pressure gradient in the isothermal channel. The character of variations of these losses is very similar to pressure losses as shown in Fig 4.7B. Presence of rolls creates transverse fluid movement which leads to an increase of the
transverse heat flux as demonstrated in Fig 4.7C. The largest heat flux occurs for \( \alpha \approx 1 \) and it rapidly decreases when \( \alpha \) increases as well as when \( \alpha \) decreases - \( Nu_{av} = O(\alpha^2) \) for \( \alpha \to 0 \) and \( Nu_{av} = O(\alpha^{-3}) \) for \( \alpha \to \infty \) in agreement with (Hossain & Floryan 2013a).

\[
Nu_{av} \approx O(\alpha^2) \quad \text{for} \quad \alpha \to 0 \\
Nu_{av} \approx O(\alpha^{-3}) \quad \text{for} \quad \alpha \to \infty
\]

**Figure 4.7** Variations of the mean pressure gradient \( B = (dp/dz)/Re_z \) (Fig 4.7A), the flowrate \( Q/Re_z \) (Fig 4.7B) and the average Nusselt number \( Nu_{av} \) (Fig 4.7C) as functions of the wavenumber \( \alpha \) for \( Ra_{P,L} = 100, 500, 1000, Ra_{P,U} = Ra_{uni} = 0 \).
Analysis of streaks was limited to $Ra_{p,L} \leq 1000$ to avoid conditions leading to formation of secondary rolls for long wavelength heating (Hossain & Floryan 2013) as they would unnecessarily increase flow losses without producing meaningful benefits in terms of streak formation.

4.5 Combined Sinusoidal and Uniform Heating at the Lower Wall

A better range of streak strengths may be achieved by combining the uniform and sinusoidal heating. Results displayed in Fig 4.8A demonstrate that streak’s strength increases as one moves from cooling of the lower wall (negative $Ra_{uni}$) towards its heating. Modifications of the streamwise velocity component are marginal in the case of cooling (see Fig 4.8B) while formation of high velocity stream tubes is significantly enhanced by heating (see Fig 4.8C). The spanwise shear can be easily doubled by combining uniform and periodic heating (see Fig 4.8A). Enhancement of streak strength can be measured by increase of losses associated with maintain such flows. Figure 4.9A demonstrates a monotonic increase of these losses as uniform cooling is replaced by progressively stronger heating. The most effective wavenumber of the periodic heating component changes in the same manner as in the case of a pure periodic heating, i.e., it changes from $\alpha \approx 1$ for cooling with $Ra_{uni} = -200$ to $\alpha \approx 2$ for heating with $Ra_{uni} = 250$.

Formation of secondary rolls for small $\alpha$’s is enhanced by uniform heating as documented in Fig 4.9A. Losses expressed in terms of reduction of flow rate driven by the same pressure gradient in the isothermal and heated channels change in a similar manner as illustrated in Fig 4.9B. There is a significant enhancement of the heat flow between channel walls by heating while cooling leads to a reverse in the direction of heat flow with the magnitude of this flow significantly increasing with the rate of cooling as documented in Fig 4.9C. It is remarkable that increase of $Ra_{uni}$ above $Ra_c$ required to initiate the RB convection does not produce any exceptional change in the fluid movement and in the resulting heat transfer.
Figure 4.8 (A) Variations of the spanwise gradient of the streamwise velocity component $dw/dx$ at $y = 0$. Distributions of the $w$-velocity in the $(x, y)$ – plane for (B) $Ra_{uni} = -200$ and (C) $Ra_{uni} = 250$. The same color scale identifying the magnitude of velocity is used in Figs 4.8 B, C. Black lines illustrate vector lines. In all subfigures, $\alpha = 1$ and $Ra_{P,L} = 500$. 

(a)

(b)

(c)
Figure 4.9 Variations of the mean pressure gradient $B = (dp/dz)/Re_z$ (Fig 4.9A), the flowrate $Q/Re_z$ (Fig 4.9B) and the modulus of the average Nusselt number $|Nu_{av}|$ (Fig 4.9C) as functions of the wavenumber $\alpha$ for $Ra_{p,\perp} = 500$ and $Ra_{uni} = -200, -100, 0, 100, 200, 250$. Thin dotted lines in Figs 4.9 B, C illustrate the mean pressure gradient and the flowrate for the isothermal flow. Solid (dashed) lines in Fig 4.9C represent positive (negative) values of $Nu_{av}$. 
4.6 Sinusoidal Heating at the Upper Wall

Sinusoidal heating applied at the upper wall produces identical response as when this heating is applied at the lower wall (Section 4.4) with an up-down symmetry of the flow field for both types of heating. This similarity can also be demonstrated by reversing the direction of gravity (Hossain & Floryan 2014). It is interesting that the characteristics of the flow, including flow losses, are not affected by the location of heating.

4.7 Sinusoidal Heating at Both Walls

The final concept to be explored for potential increase of streak’s intensity is the use of sinusoidal heating at both walls. We start discussion with the perfectly tuned heating patterns, i.e., heating distributions at both walls are characterized by the same wavenumber, and we shall explore the effect of change in their relative positions as quantified by the phase difference $\Omega$ (see Fig 4.1). It is sufficient to investigate $\Omega \in (0, \pi)$ due to periodicity of the system with $\Omega = 0$ corresponding to hot spots on the upper wall aligned with hot spots on the lower wall and $\Omega = \pi$ corresponding to hot spots on the upper wall aligned with hot cold spots on the lower wall.

Results displayed in Fig 4.10A demonstrate formation of a regular pattern of well-developed streaks for $\Omega = 0$. This pattern becomes distorted when $\Omega$ increases resulting in a pattern of alternating weak and strong streaks as illustrated in Fig 4.10B. Streaks for $\Omega = 0.999\pi$ (Fig 4.10C) can be viewed as totally disorganized with their form becoming re-organized in a different manner for $\Omega = \pi$ (see Fig 4.11D). The process of weakening of streaks is also illustrated in Fig 4.11 displaying variations of the spanwise gradient of the streamwise velocity component $dw/dx$. These variations are very regular with a large amplitude for $\Omega = 0$, formation of pairs of strong and weak streaks is visible for $\Omega = \pi/2$, and the overall weakening of streak is very visible for $\Omega = 0.999\pi$. The reader may note that change of the streak irregularity cannot be captured by such one-dimensional cut.
through velocity field while it is well visible in Fig 4.10C which provides a snapshot of this field.

![Diagram](image)

**Figure 4.10** Flow topologies for $\alpha = 1, Ra_p,L = Ra_p,U = 500, Re_{\infty} = 15, Ra_{uni} = 0$. Results displayed in Figs 4.10A-D correspond to phase differences $\Omega = 0, 0.5\pi, 0.999\pi, \pi$, respectively. Yellow (ocher) and green (forest) colors correspond to $\psi = 0.7, -0.7$ in Fig. 4.10A, to $\psi = 0.6, -0.6$ in Fig 4.10B, to $\psi = 0.3, -0.3, -0.8$ in Fig. 4.10C, and to $\psi = 0.2, -0.2$ in Fig 4.10D. Colors in the $(x, y)$-plane illustrate the temperature field while black solid lines illustrate vector lines.

The same temperature color scale is used in all subfigures.

Transition in flow properties with $\Omega$ is also illustrated in Fig 4.12 using changes in the flow rate when the flow is driven by the same pressure gradient through the isothermal and heated channels. The minimum of $Q$ corresponds to $\Omega = 0$ which produces the strongest
streaks. The flow rate increases as $\Omega$ increases suggesting weakening of streaks as their pattern becomes more disorganized (see Fig 4.10). Point $\Omega = \pi$ is special and it represents a removable singularity for function $Q = Q(\Omega)$ as $Q(\pi) = 1.317$ exists while $Q \to 1.283$ as $\Omega \to \pi$ from above and from below, i.e., this limit also exists. This point corresponds to a special organization of streaks (see Fig 4.10D) but it is only of theoretical interest as a small deviation from $\Omega = \pi$ results in a loss of coherence of the streaks. Flow topology for this special point has strong left and right symmetries as well as up and down symmetries and is characterized by the lowest energy loss required for maintaining flow with such streaks.

![Figure 4.11](image)

**Figure 4.11** Variations of the spanwise gradient of the streamwise velocity component $dw/dx$ at $y = 0$ for $\alpha = 2$. 
Figure 4.12 Variations of the flow rate $Q/Re_x$ as a function of the phase difference $\Omega$ for $\alpha = 1$ and $Ra_{p,L} = Ra_{p,U} = 500$. The insert illustrates variations of the difference between the flow rates at $\Omega = \pi$ and at any other $\Omega$ as a function of distance $(\pi - \Omega)$ expressed in the logarithmic scale. Points A, B, C, D correspond to flow conditions used in Fig 4.10 A, B, C, D.

The energy requirements for flows with streaks are illustrated in Fig 4.13A displaying variations of the pressure gradient required to maintain the same flow rate in the isothermal and heated channels. The pressure losses always increase with heating, and strongly depend on the heating wavenumber and the phase shift. The maximum increase can be reduced by almost 75% by moving the upper heating from position where its hot spots overlap with the hot spots of the lower heating ($\Omega = 0$) to a position where its hot spots overlap with the cold spots of the lower wall ($\Omega = \pi$). Another important feature is the occurrence of the local RB instability for the long wavelength heating. Increase of $\Omega$ from 0 to $\pi$ increases $\alpha$ required to produce secondary rolls from $\sim 0.2$ to $\sim 1.4$ for conditions used in this test. Variations of the flow rate obtained in the heated channel while maintaining the same pressure gradient as in the isothermal channel as a function of $\alpha$ exhibit similar trends as documented in Fig 4.13B. The largest heat flow between both walls is achieved for $\Omega = 0$. 
and it decreases significantly as $\Omega$ increases to $\Omega = \pi$ as documented in Fig 4.13C. The most effective heating wavenumber as far as the heat transfer is concerned is $\alpha = \sim 1.2$ and changes marginally with $\Omega$.

**Figure 4.13** Variations of the mean pressure gradient $B = (dp/dz)/Re_z$ (Fig 4.13A), the flowrate $Q/Re_z$ (Fig 4.13B) and average Nusselt number $Nu_{av}$ (Fig 4.13C) as functions of the wavenumber $\alpha$ for $Ra_{P,L} = Ra_{P,U} = 500, Ra_{uni} = 0$ and the phase shifts $\Omega = 0, 0.25\pi, 0.5\pi, 0.75\pi, 0.99\pi, \pi$. Dotted lines correspond to the one-wall heating with $Ra_{P,L} = 500$. Solid and dashed lines are used alternatively to increase the figure readability. Thin dotted lines in Figs 4.13 A-B illustrate the mean pressure gradient and the flowrate for the isothermal flow.
The final question to be considered is the effect of detuning between the upper and lower heating patterns, i.e., the system response when the heating patterns are described by different wavenumbers. We shall use simple temperature profiles to illustrate the wealth of possible states, i.e.

\[ \theta_L(x) = Ra_{uni} + \frac{Ra_{p.L}}{2} \cos(gx) , \quad \theta_U(x) = \frac{Ra_{p.U}}{2} \cos(\mathcal{H}x + \Omega). \] (4.7.1a, b)

It is clear from the preceding discussion that it is the distribution of the hot and cold spots along each wall which determines the local streak pattern with a transition between these patterns occurring somewhere in the interior of the channel. Existence of different patterns near each wall results in weaker and less organized streaks, as already illustrated in Fig 4.10, and is undesired. To illustrate these patterns (Eq 4.7.1), we consider wall temperatures for different combination of \( g \) and \( \mathcal{H} \) taking \( Ra_{p,L} = Ra_{p,U} = 1 \) for simplicity. Properties of the resulting system can be categorized in terms of the commensurability index \( CI \) defined as

\[ CI = \frac{\mathcal{H}}{g}. \] (4.7.2)

Values of \( CI \) expressed either as \( n \) or as \( 1/n \), where \( n \) is an integer, correspond to periodic systems formed by combinations of subharmonic/superharmonic components and represent special forms of commensurable systems. More complex rational values result in complicated commensurable periodic systems with wavelengths which can vary by several orders of magnitude. The irrational values describe the non-commensurable (aperiodic) systems. Table 4.1 and Fig 4.14 illustrate how periodicity of the channel subject to a detuned heating of both walls vary depending on the selection of \( g \) and \( \mathcal{H} \), and how the number of hot spots forming at each wall per system wavelength changes. Different numbers of hot spots at each wall suggests formation of complex streak patterns and this is undesired. An example of streak complexity is given in Fig 4.14C for a rather simple case of \( CI = 1/3 \) where \( \mathcal{H} \) represents a subharmonic of \( g \). The streak structure in the vicinity of each wall is dictated by heating pattern applied at this wall with a transition between both patterns taking place in the channel interior. While details for more complex cases are
not given, it is obvious that formation of strong regular streaks requires use of well-tuned heating with a proper alignment of the hot and cold spots at both walls.

**Table 4.1** System wavelength $\lambda$ and number of the hot and cold spots at each wall for selected values of the commensurability index CI.

<table>
<thead>
<tr>
<th>CI</th>
<th>System Wavelength($\lambda$)</th>
<th>No of hotspots per system wavelength (lower wall)</th>
<th>No of hotspots per system wavelength (upper wall)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.15</td>
<td>$\lambda = 20\lambda_g = 3\lambda_{\text{H}}$</td>
<td>21</td>
<td>4</td>
</tr>
<tr>
<td>1/3</td>
<td>$\lambda = 3\lambda_g = \lambda_{\text{H}}$</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>$1/\sqrt{2}$</td>
<td>$\lambda = \infty$</td>
<td>$\infty$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>0.8</td>
<td>$\lambda = 5\lambda_g = 4\lambda_{\text{H}}$</td>
<td>6</td>
<td>5</td>
</tr>
<tr>
<td>1</td>
<td>$\lambda = \lambda_g = \lambda_{\text{H}}$</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>1.3</td>
<td>$\lambda = 10\lambda_g = 13\lambda_{\text{H}}$</td>
<td>11</td>
<td>14</td>
</tr>
<tr>
<td>$\sqrt{2}$</td>
<td>$\lambda = \infty$</td>
<td>$\infty$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>1.4</td>
<td>$\lambda = 5\lambda_g = 7\lambda_{\text{H}}$</td>
<td>6</td>
<td>8</td>
</tr>
<tr>
<td>1.8</td>
<td>$\lambda = 5\lambda_g = 9\lambda_{\text{H}}$</td>
<td>6</td>
<td>10</td>
</tr>
<tr>
<td>2</td>
<td>$\lambda = 2\lambda_g = \lambda_{\text{H}}$</td>
<td>3</td>
<td>5</td>
</tr>
</tbody>
</table>
Figure 4.14 Distributions of the upper (top) and lower (bottom) wall temperatures for the commensurability index $CI = 1.3$ (Fig 4.14A) and $CI = 0.15$ (Fig 4.14B) for $\Omega = 0$. Figure 4.14C illustrates the temperature (color) and flow (solid lines) fields for $CI = 1/3$ and $Ra_{PL} = 500, Ra_{PU} = 500, Ra_{uni} = 0$.

4.8 Effects of Prandtl Number

The effectiveness of using heating for creation of streaks depends on the fluid thermal properties. Results presented in Fig 4.15 illustrate this dependence for the most effective heating wavenumber $\alpha = 1$ and three types of heating strategies, i.e., periodic heating of one wall, a combination of periodic and uniform heating of the lower wall and periodic heating of both walls with $\Omega = 0$. In all cases increase of $Pr$ reduces streak intensity with heating being unable to produce meaningful streaks for $Pr > 10$. 
Figure 4.15 Variations of the mean pressure gradient $B = (dp/dz)/Re_z$ (Fig 4.15A), the flowrate $Q/Re_z$ (Fig 4.15B) and the average Nusselt number $Nu_{av}$ (Fig 4.15C) as functions of the Prandtl number $Pr$ for $\alpha = 1$ for different heating conditions, i.e. periodic heating at the lower wall with $Ra_{p,L} = 500, Ra_{p,U} = 0, Ra_{uni} = 0$ (solid lines); combination of periodic and uniform heating at the lower wall with $Ra_{p,L} = 500, Ra_{p,U} = 0, Ra_{uni} = 250$ (dotted lines); periodic heating of both walls with $Ra_{p,L} = 500, Ra_{p,U} = 500, Ra_{uni} = 0, \Omega = 0$ (dashed lines).
This loss of effectiveness is underscored by the reduction of $dw/dx$ with an increase of $Pr$ illustrated in Fig 4.16(A). Figure 4.16(B) illustrates the streamwise velocity profile at different spanwise location for $Pr = 0.71$. Substantial changes in velocity profile can be noticed in comparison with the reference case due to the different heating strategies. The streak structure and the reduction of its intensity with an increase of $Pr$ are also well illustrated using plots of $w_m = (w - w_0)$ displayed in Fig 4.17 - the streaks are regular and very pronounced for fluids with small $Pr$’s but become marginal for fluids with large $Pr$’s.

![Figure 4.16](image.png)

**Figure 4.16** (A) Variations of the spanwise gradient of the streamwise velocity component $dw/dx$ at $y = 0$ for $\alpha = 1$. (B) Variations of the streamwise velocity component $w$ at different $x$-locations for $\alpha = 1$. Green lines are multiplied by 0.9 for better readability. Black dash-dotted line corresponds to the reference velocity profile (Poiseuille). For both the subfigures, solid, dotted, and dashed lines correspond to the same heating strategies as used in Fig 4.15.
Figure 4.17 Distributions of velocity modifications $w_m = (w - w_0)$ in the $(x, y)$-plane; colors identify magnitudes of $w_m$. First row corresponds to $Pr = 0.1$, second row corresponds to $Pr = 0.71$, and third row corresponds to $Pr = 10$.

4.9 Summary

The analysis of the use of heating for the creation of rolls and streaks in fully developed shear layers has been carried out, including the determination of the energy cost of such streaks as measured either by using the increase of pressure gradient required to drive the same flow rate through the heated and isothermal channels or by using the reduction of the flow rate if the available pressure gradient is fixed. The formation of such streaks is of interest in intensification of mixing, especially in low $Re$ flows which are known to be very difficult to mix. Several heating strategies were explored with the goal of creating strong streaks. The basic strategy involved the use of spanwise periodic heating which produces streaks for any heating intensity – these streaks represent a forced system response. The
range of heating wavenumbers $\alpha$ resulting in well-developed streaks has been determined. It was shown that heating with $\alpha = O(1)$ is of the most interest, as short and long wavelength heating lead to very weak streaks. Periodic heating results in the same response whether the heating is applied at the upper or lower walls. Uniform heating applied at the lower wall creates streaks through the Rayleigh-Bénard instability but without the means for controlling their size and distribution, and only if the heating intensity exceeds critical conditions. A combination of uniform and periodic heating provides the means for increasing the strength of streaks formed by periodic heating. The use of periodic heating at both walls provides the means for creation of the strongest streaks but only if the upper and lower heating are well tuned, i.e., they have the same wavenumber, and their relative position is properly selected, i.e., the upper and lower hot spots are aligned with each other. The use of very long wavelength heating leads to the formation of local streaks near wall hot spots, but only if the heating intensity is large enough. It was shown that heating is an effective tool for streak formation for fluids with small Prandtl numbers $Pr$ and becomes ineffective when $Pr > 10$. 
5.1 Introduction

It is shown in Chapter 4 that streaks can be created in a controlled manner in the smooth channel. In this chapter, it has been investigated if the addition of grooves can result in more intense streaks as well as determined the effect of heated grooves on flow losses. Three ranges of groove wavenumbers were of interest: wavenumbers near the critical wavenumber of the Rayleigh-Bénard (RB) instability, wavenumbers characterizing drag-reducing grooves, and the optimal wave numbers. Section 5.2 describes model problem – flow in a channel equipped with longitudinal grooves and exposed to a combination of uniform and spanwise-periodic heating. Streaks created by uniform heating of grooved wall are characterized in Section 5.3. Section 5.4 describes streaks created by spanwise-periodic heating of grooves, while Section 5.5 represents streaks created by a combination of uniform and periodic heating of grooved surfaces. In Section 5.6, a short summary of the main conclusions is provided.

5.2 Problem Formulation

Consider flow in a channel formed by two horizontal plates extending to ±∞ in the x- and z-directions (see Fig 5.1), with the lower wall being equipped with longitudinal grooves and the upper wall being smooth, and with gravity acting in the negative y-direction.

---

4 A version of this chapter has been published as –

The mean distance between the walls is $2h^*$. The geometry of the channel is given as:

$$y_U = 1, \quad y_L(x) = -1 + \frac{b_L}{2} \cos(ax). \quad (5.2.1)$$

where subscripts $L, U$ refer to the lower and upper walls, respectively, $B_L$ stands for the groove amplitude, $\alpha$ denotes the wavenumber, symbol $\lambda = 2\pi/\alpha$ is used to denote wavelength, $h^*$ is used as the length scale and stars denote dimensional quantities.

![Schematic diagram of the flow configuration.](image)

**Figure 5.1** Schematic diagram of the flow configuration.

The upper wall is isothermal while the lower wall is heated with temperature variations being at most of $O(10)$ which leads to acceptability of the Boussinesq fluid model (Paolucci 1982). The fluid has thermal conductivity $k^*$, specific heat $c^*$, thermal diffusivity $\kappa^* = k^*/\rho^*c^*$, kinematic viscosity $\nu^*$, dynamic viscosity $\mu^*$ and thermal expansion coefficient $\beta^*$ and it is driven in the positive $z$-direction by a fixed pressure gradient.

The relative wall temperatures are

$$\theta_U = 0, \quad \theta_L(x) = Ra_{uni} + \frac{Ra_{PL}}{2} \cos(\alpha x + \Omega_{TL}) \quad (5.2.2)$$
where \( \theta = T - T_u \) is the relative temperature with respect to temperature of the upper wall, \( T \) stands for the absolute temperature, \( \kappa^* \nu^*/(g^* \beta^* h^* \dot{\theta}_u) \) is the temperature scale, the uniform Rayleigh number \( Ra_{uni} = g^* \beta^* h^* \dot{\theta}_u/(\kappa^* \nu^*) \) determines the intensity of the uniform component of heating, the periodic Rayleigh number \( Ra_{p,L} = g^* \beta^* h^* \dot{\theta}_p/(\kappa^* \nu^*) \) determines the intensity of the periodic heating component, and \( \Omega_{T_L} \) is the phase shift between the topography and temperature patterns. The groove and temperature patterns are perfectly tuned, i.e., they are described by the same wavenumber, with the pattern interaction effect being driven by spatial positioning of these patterns.

Formation of rolls and streaks is described by the continuity, Navier-Stokes and energy equations of the form

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 , \quad (5.2.3a)
\]

\[
u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = -\frac{\partial p}{\partial x} + \nabla^2 u , \quad (5.2.3b)
\]

\[
u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} = -\frac{\partial p}{\partial y} + \nabla^2 v + Pr \theta, \quad (5.2.3c)
\]

\[
u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} = -\frac{\partial p}{\partial z} + \nabla^2 w , \quad (5.2.3d)
\]

\[
u \frac{\partial \theta}{\partial x} + v \frac{\partial \theta}{\partial y} + w \frac{\partial \theta}{\partial z} = Pr^{-1} \nabla^2 \theta \quad (5.2.3e)
\]

where \( U_{\nu^*} = \nu^*/h^* \) is the velocity scale, \( \rho^* U_{\nu^*}^2 \) is the pressure scale and \( Pr = \nu^*/\kappa^* \) is the Prandtl number. The relevant boundary conditions are

\[
u = v = w = 0 \quad \text{at} \quad y = y_L \quad \text{and} \quad y = 1 \quad (5.2.4a)
\]

\[	heta(x, y_L, z) = \theta_L(x), \quad \theta(x, 1, z) = 0. \quad (5.2.4b)
\]

The geometry and heating conditions result in \( \frac{\partial \theta}{\partial z} = 0 \) thus decoupling (5.2.3a-c, e) from (5.2.3d) – this leads to a two-step solution process starting with solution of the nonlinear problem (5.2.3a-c, e) followed by solution of a linear problem (5.2.3d). The linearity of (5.2.3d) permits eliminate of \( Re_z \) as a parameter through a simple scaling. “Cost” of streak
formation is quantified by comparing properties of flow in the z-direction with the isothermal form of this flow in a smooth channel. This reference flow has the following form

\[ \tilde{u}_0(x, y, z) = [0, 0, w_0] = [0, 0, Re_z(1 - y^2)], \quad p_0(x, y, z) = -2z \text{Re}_z. \]

\[ Q_0 = \frac{4}{3} \text{Re}_z. \quad (5.2.5) \]

In the above, subscript 0 denotes the isothermal quantities and the Reynolds number is defined as \( Re_z = W_{max}^* h^* / \nu^* = W_{max}^* / U^*_v \) where \( W_{max}^* \) denotes the maximum of the z-velocity component.

We use two methods of assessing the “cost” of streaks’ creation. In the first one, we assume that pressure gradient in the z-direction remains the same with and without heating and grooves, i.e., we impose the fixed pressure gradient constraint of the form

\[ \frac{\partial p}{\partial z}_{\text{mean}} = -2Re_z, \quad (5.2.6) \]

and determine change of the flow rate \( Q_c \), i.e.,

\[ Q_c = Q - Q_0, \quad Q = \lambda^{-1} \int_0^1 \int_{y_L}^{y_L+1} w \, dy \, dx \quad (5.2.7) \]

where positive \( Q_c \) corresponds to an increase of the flow rate. Alternatively, we impose the fixed flow rate constraint of the form

\[ Q = \frac{4}{3} \text{Re}_z \quad (5.2.8) \]

and determine the reduction of the required pressure gradient \( P_c \), i.e.,

\[ P_c = \frac{\partial p}{\partial z}_{\text{mean}} - \frac{dp_0}{dz}, \quad (5.2.9) \]

with the positive \( P_c \) corresponding to the reduction of pressure losses. We eliminate any external forces which might drive the flow in the x-direction by imposing the zero mean pressure gradient constraint of the form
\[ \frac{\partial p}{\partial x} \bigg|_{\text{mean}} = 0. \]  

System (5.2.3) - (5.2.10) is solved numerically using spectrally accurate algorithm discussed in Chapter 2 & 3. It is of interest to monitor the change in heat fluxes owing to the use of grooves. These fluxes are presented in terms of the Nusselt number correction $Nu_c$. This correction has been determined by evaluating the Nusselt number $Nu$ for the grooved channel and then subtracting the Nusselt number associated with conduction in a smooth channel $Nu_{cond}$ from it, i.e.,

\[ Nu_c = Nu - Nu_{cond}, \quad Nu_{cond} = \frac{1}{2} Ra_{uni}. \]  

Positive $Nu_c$ corresponds to an increase of the heat flow. Our interest is in creation of strong streaks while minimizing “cost” expressed either in terms of pressure losses or in terms of flow loses. We shall explore three strategies: (i) use of the uniform heating of grooved surfaces, (ii) spanwise periodic heating of grooved surfaces, and (iii) use of a combination of uniform and periodic heating of grooved surfaces. We note that the spatial distribution of streaks is dictated by the groove and heating patterns.

### 5.3 Uniform Heating - Isothermal Grooves

Both walls are isothermal with temperature of the lower wall being higher than temperature of the upper wall. Uniform Rayleigh number $Ra_{uni}$ measures the temperature difference between these walls. Two ranges of groove wavenumbers are of interest for uniform heating: (i) the long wavelength grooves which are known to reduce flow losses (Mohammadi & Floryan 2013; Chen et al. 2016), and (ii) grooves with $\alpha \approx 1.57$ which is the critical wavenumber for the Rayleigh-Bénard ($RB$) instability (Bénard 1900; Rayleigh 1916). The former one is of interest as it implies lower flow losses associated with formation of streaks. The latter one is of interest as it takes advantage of the $RB$ instability to increase streak intensity - the heating intensity must however reach the critical value ($Ra_{uni} = 213.5$) required for the instability onset. We shall refer to the later range of wavenumbers as the $RB$ range. Flow topology which can be created using isothermal
grooves in the latter case is illustrated in Fig. 5.2. Temperature contours indicate the presence of the $x$-temperature gradients, which generate transverse movement (Abtahi & Floryan 2017) resulting in the formation of rolls/streaks. Particle trajectories illustrate rolling up the fluid layers in spirals. The spiral movement allows stretching of the fluid layers which is feature characteristic for the chaotic mixing (Gepner & Floryan 2020). Topologies for the small-$\alpha$ streaks are similar and thus are not shown.

**Figure 5.2** Flow topology in an isothermal grooved channel for $\alpha = 1.57$, $B_L = 0.4$, $Ra_{uni} = 150$, $Ra_{p,L} = 0$, $Pr = 0.71$ and $Re_z = 5$. Colors in the front $(x, y)$-plane illustrate the temperature field while black solid line illustrates vector lines, colors in the rear $(x, y)$-plane represent the $w$-velocity field, and dash-dotted lines show particle trajectories.

The strength of the streaks is measured in two ways: (i) using change of the fluid kinetic energy

$$\Delta E_k = E_k - E_{k,0} = \lambda^{-1} \int_0^\lambda \int_{y_L}^{y_{L+1}} (u^2 + v^2 + w^2) \, dy \, dx - \int_{-1}^{+1} w_0^2 \, dy$$

(5.3.1)

and (ii) using the maximum of the spanwise gradient of the longitudinal velocity component

$$\xi = \max \left( \frac{dw}{dx} / Re_z \right)$$

(5.3.2)
in the midsection of the channel. The latter one is of interest as such shear is responsible for flow instabilities leading to chaos (Gepner & Floryan 2020). In the above, $E_k$ stands for the kinetic energy of the actual flow while $E_{k,0}$ stands for the kinetic energy of the reference flow.

Figure 5.3 (A) Variations of the change in kinetic energy $\Delta E_k$ (Eq. 5.3.1) as a function of $Ra_{uni}$ for $\alpha = 1.57$, $Ra_{p,L} = 0, Re_z = 5$ and selected $B_L$’s. (B) Variation of the maximum of the spanwise velocity gradient $\xi$ (Eq. 5.3.2) as a function of $\alpha$ for $B_L = 0.06$, $Ra_{p,L} = 0$, and selected $Ra_{uni}$’s. Dashed lines provide results for isothermal grooved channel. Red dotted line in Fig 5.3B shows results for a smooth channel with $Ra_{p,L} = 220$. Grey color identifies $\alpha$’s leading to a reduction of pressure losses in a grooved isothermal channel.

Variations of $\Delta E_k$ displayed in Fig 5.3A for $\alpha = 1.57$ demonstrate fairly weak streaks for small $Ra_{uni}$’s. Their intensity rapidly increases when $Ra_{uni}$ approaches the critical value of 213.5 with grooves of higher amplitude producing stronger streaks.
Figure 5.4 Variation of the flow rate correction $Q_c$ (Eq. 5.2.7; Fig 5.4A), the Nusselt number correction $Nu_c$ (Eq. 5.2.11; Fig 5.4B) and the pressure gradient correction $P_c$ (Eq. 5.2.9; Fig 5.4C) as functions of $\alpha$ for $B_I = 0.06$ and $Ra_{P,L} = 0$. The dashed lines in Figs 5.4A and 5.4C correspond to the isothermal grooved channel; grey color identifies $\alpha$’s leading to a reduction of pressure losses (increase of the flow rate) in such channel.
Variations of $\xi$ displayed in Fig 5.3B illustrates the effect of the $RB$ mechanism on intensification of streaks for $\alpha$ near $\alpha = 1.57$ for a sufficiently large $Ra_{uni}$. Reduction of $\xi$ at large $\alpha$’s is associated with formation of convection boundary layer near the grooved wall while $\xi$ is evaluated in the channel midsection. Reduction of $\xi$ at small $\alpha$’s is due to the reduction of horizontal $x$-temperature gradients. The reference results for isothermal grooved channel and uniformly heated smooth channel demonstrate a large advantage offered by uniformly heated grooves.

Formation of streaks increases flow losses as illustrated in Fig 5.4 – reduction of the flow rate is illustrated in Fig 5.4A while increase of the pressure is gradient is illustrated in Fig 5.4C – this data measures the energy “cost” associated with streak creation. Heating increases losses over the whole range of $\alpha$’s with a local peak forming for $\alpha \approx 1.57$.

Figure 5.5 Variations of the maximum of the spanwise velocity gradient $\xi$ (Eq. 5.3.2; dashed lines) and the flow rate correction $Q_c$ (Eq. 5.2.7; solid lines) as functions of the groove amplitude $B_L$ for $\alpha = 0.6$ (green lines, the drag reduction zone) and $\alpha = 1.53$ (red lines, the $RB$ zone) for $Ra_{uni} = 205$, $Ra_{P,L} = 0$. Grey color identifies conditions leading to an increase of the flow rate.
Figure 5.4B illustrates the heat transfer consequences of streak formation. Region marked using grey color identifies conditions where introduction of unheated grooves reduces flow losses. Addition of heating marginally increases losses for such $\alpha'$s. It is possible to use such heated grooves for streak creation without paying a penalty in the form of increased losses - the overall energy “cost” of flows with streaks is still below the energy “cost” of flow without heating and grooves. Data displayed in Fig 5.5 demonstrates increase of the effectiveness of streak formation for larger groove amplitudes for both the drag reducing $\alpha'$s as well as for $\alpha'$s in the RB zone. The energy “cost” of using $\alpha'$s in the RB zone monotonically increases with $B_L$ while use of $\alpha'$s in the drag reducing zone initially reduces losses but its excessive increase reverses this trend, and the losses begin to increase rapidly.

![Graph A](image1.png)  ![Graph B](image2.png)

**Figure 5.6** Spanwise distributions of the $w$-velocity component at different $y$-locations (A) and the transverse distributions of the $w$-velocity component at different $x$-locations (B) for $\alpha = 1.57, Ra_{uni} = 200, Ra_{P,L} = 0, B_L = 0.1$.

Streaks are used to create spanwise shear layers which are expected to trigger a new kind of instability (Moradi & Floryan 2014; Mohammadi, Moradi & Floryan 2015). This is step one towards chaotic stirring (Gepner & Floryan 2020). The spanwise distributions of the
streamwise velocity component are displayed in Fig 5.6A for $\alpha$ in the RB zone. They demonstrate velocity increase near the groove trough and reduction around the groove crest resulting in the formation of transverse shear layers. The spanwise gradients of the $w$-velocity component decreases with distance away from the grooved wall – these gradients are responsible for activation of the inviscid instability mechanism described in Mohammadi, Moradi & Floryan (2015). The wall-normal distributions of the $w$-velocity component displayed in Fig 5.6B demonstrate small changes in vertical shear concentrated closed to the grooved wall. They produce a minor modification of the classical shear-driven instability (Moradi & Floryan 2014). Plots of the $w$-velocity component for $\alpha$’s in the loss reduction zone have a qualitatively similar form and thus are not shown. The spanwise velocity gradient for such $\alpha$’s is much smaller, but this can be compensated by using larger groove amplitudes (see Fig 5.5).

5.4 Periodic Heating and Pattern Interaction Effect

Presence of groves creates spanwise modulations in the flow and its uniform heating produces streaks with pattern dictated by the groove wavenumber. In this Section we investigate use of periodic heating of grooved surface which provides means for creation of potentially stronger streaks (Chapter 4). We focus attention on heating patterns perfectly tuned with the groove patterns where both effects can potentially reinforce each other. Presence of patterns of distinct physical quantities activates the pattern interaction effect (Floryan & Inasawa 2021) which may either weaken or amplify the streak formation process. In this case, interaction of groove and heating patterns creates net spanwise flow which may be directed either to the left or to the right depending on the relative positions of both patterns.

The relative position of both patterns is measured using phase difference $\Omega_{TL}$ with $\Omega_{TL} = 0$ corresponding to hot spots overlapping with the groove crests and $\Omega_{TL} = \pi$ corresponding to hot spots overlapping with groove troughs – the pattern interaction effect is not active for these two special configurations.
Figure 5.7 Variations of the change in kinetic energy $\Delta E_k$ (Eq. 5.3.1; Fig 5.7A), the maximum of the spanwise velocity gradient $\xi$ (Eq. 5.3.2; Fig 5.7B) and the flow rate correction $Q_c$ (Eq. 5.2.7; Fig 5.7C) as functions of $\Omega_{TL}$ for $\alpha = 1.57$ (the RB zone; solid blue lines) and $\alpha = 0.6$ (the drag reduction zone; solid red lines), $Ra_{uni} = 0$, $B_L = 0.1$, and $Ra_{P,L}$’s are specified in the figures. Reference quantities for a smooth channel exposed to the same periodic heating are illustrated using dotted lines. The reference quantities for a grooved isothermal channel are: $\alpha = 1.57$: $\Delta E_k = -0.0515$, $\xi = 0.0326$, $Q_c / Re_z = -0.0014$; $\alpha = 1$: $\Delta E_k = 0.0095$, $\xi = 0.0334$, $Q_c / Re_z = -0.0001$; $\alpha = 0.6$: $\Delta E_k = 0.0507$, $\xi = 0.0259$, $Q_c / Re_z = 0.0007$. 
Results displayed in Fig 5.7A demonstrate that the strongest streaks are obtained for the hot spots placed half-way between the groove crests and troughs for low heating, i.e., $Ra_{p,L} = 200$. Increase of heating to $Ra_{p,L} = 400$ shows preference for placing hot spots closer to the groove trough for $\alpha = 0.6$ (drag reducing zone) and $\alpha = 1$ but retains the previous preference for $\alpha = 1.57$ (RB zone). Use of $\alpha \approx 1$ leads to the strongest streaks so $\alpha = 1$ is viewed as optimal and is added to the further discussion of potential gains due to periodic heating of grooved walls. Distributions of $\xi$ displayed in Fig 5.7B lead to somewhat different conclusions as all $\alpha$’s show preference for placing hot spots approximately halfway between the groove crests and troughs. Addition of grooves to a periodically heated wall increases flow losses for $\alpha$’s in the RB-zone and can decrease losses for $\alpha = 0.6$ (drag reducing zone) and $\alpha = 1$ depending on $\Omega_{TL}$ as illustrated in Fig 5.7C.

**Figure 5.8** Topology of flow in an isothermal grooved channel for $\alpha = 1$, $B_L = 0.1$, $Ra_{uni} = 0$, $Ra_{p,L} = 200$, $\Omega_{TL} = \pi/2$ and $Re_z = 1$. Colors in the front $(x, y)$-plane illustrates the temperature field while the black solid line illustrates velocity vector lines, colors in the rear $(x, y)$-plane represents $w$-velocity field. Black dotted and purple solid lines inside the plotted box show particle trajectories.
The above discussion shows that the configuration of interest is $\alpha = 1$ as it generates very strong streaks without paying excessive penalty in terms of flow losses. The reader may note that positioning of hot spots halfway between the groove crests and troughs, which is the most effective configuration at small heating rates, results in the strongest thermal drift (Abtahi & Floryan 2017, 2018). The overall flow topology for such conditions is presented in Fig 5.8 showing net flow to the left.

Results displayed in Fig 5.9 permit assessment of effects of the pattern wavenumbers and identification of the most effective $\alpha$. Variations of $\Delta E_k$ displayed in Fig 5.9A demonstrate that streak intensity changes significantly as a function of $\alpha$ with $\alpha \approx 1$ being the most effective, regardless of whether the smooth or grooved heated surfaces are used. Use of grooved isothermal channel produces an order of magnitude weaker streaks while use of heated smooth channel produces competitive streaks but weaker than those obtained with heated grooved channel with a proper relative position of both patterns. The same conclusions can be reached on the bases of variations of $\xi$ illustrated in Fig 5.9B.

Variations of $Q_c$ illustrated in Fig 5.9C demonstrate small increase of flow losses associated with addition of grooves for $\alpha$’s in the $RB$ zone for the best relative position of both patterns. Use of $\alpha$’s in the drag reducing zone shows reduction of flow losses when grooves are combined with periodic heating with relative position of both patterns playing a minor role. It becomes obvious that use of $\alpha \approx 1$ produces the most intense streaks according to Figs 5.9A and 5.9B but the energy “cost” is smaller than the maximum “cost” according to data in Fig 5.9C.

The reader may recall that weak streaks produced by isothermal grooves are sufficient to produce chaotic stirring (Gepner & Floryan 2020) which suggests that use of heating in combination with grooves should be a powerful technique for stirring intensification.
Figure 5.9 Variations of the change in kinetic energy $\Delta E_k$ (Eq. 5.3.1; Fig 5.9A), the maximum of the spanwise velocity gradient $\xi$ (Eq. 5.3.2; Fig 5.9B) and the flow rate correction $Q_c$ (Eq. 5.2.7; Fig 5.9C) as functions of $\alpha$ for $Ra_{uni} = 0$, $Ra_{P,L} = 250$, $B_L = 0.1$ and selected $\Omega_{TL}$’s. Dashed and dotted lines give reference results for the grooved isothermal and smooth periodically heated channels, respectively. Grey color identifies $\alpha$’s leading to a reduction of pressure losses in a grooved isothermal channel. Green lines mark $\alpha = 1$. 

124
5.5 Combined Periodic and Uniform Heating

The final question to be addressed is assessment of potential gains associated with combining the uniform and periodic heating of grooved surfaces. Use of $\alpha$’s in the $RB$-zone shows a rapid increase of $\Delta E_k$ as $Ra_{uni}$ increases but the difference between heating of smooth and grooved surfaces is minor with a small preference for heating smooth surfaces at higher $Ra_{uni}$’s (see Fig 5.10A). Use of $\alpha = 1$, which represents the optimal wavenumbers, shows advantage at small $Ra_{uni}$’s but use of $\alpha = 1.57$ (the $RB$ zone) is more effective at larger $Ra_{uni}$’s. Use of $\alpha = 0.6$ (drag reducing zone) produces much weaker streaks with a strong advantage of grooved surfaces. Use of $\xi$ shows a similar pattern of variations (see Fig 5.10B) with optimal $\alpha$’s and $\alpha$’s in the $RB$ zone being equally competitive. The flow losses for the grooved and smooth surfaces are nearly similar as shown in Fig 5.10C and increase with $Ra_{uni}$. The advantage of using the optimal $\alpha$ is due to its ability to produce strong streaks at a much lower energy “cost” that increases with an increase of $Ra_{uni}$.

The reader should note that reduction of $\alpha$ leads to a reduction of flow losses (Mohammadi & Floryan 2013) but its excessive reduction combined with an excessive heating lead to formation of secondary convection near the hot spots which generates localized streaks (Asgarian, Floryan & Hossain 2016; Floryan, Hossain & Bassom 2019). Formation of such streaks was not of interest in this study.
Figure 5.10 Variations of the change in kinetic energy $\Delta E_k$ (Eq. 5.3.1; Fig 5.10A), the maximum of the spanwise velocity gradient $\xi$ (Eq. 5.3.2; Fig 5.10B) and the flow rate correction $Q_c$ (Eq. 5.2.7; Fig 5.10C) as functions of $Ra_{uni}$ for $B_L = 0.1$, $Ra_{P,L} = 250$, $\Theta_{TL} = \pi/2$. Blue color identifies $\alpha = 1.57$ (the RB zone), red color identifies $\alpha = 0.6$ (the drag reducing zone) and green color identifies $\alpha = 1$. Dotted and dashed lines illustrate results for smooth periodically heated channels and for isothermal grooved channels, respectively. Grey color in Fig 5.10C identifies conditions leading to a reduction of flow losses.
5.6 Summary

The creation of streaks in low Reynolds number laminar shear flows is of interest for the intensification of stirring. This analysis is focused on the use of grooves for the intensification of streaks created by heating. Grooves play two roles: (i) they produce spanwise flow modulations which contribute to streak creation and (ii) they can reduce flow losses if their wavelength is long enough. Heating patterns applied to smooth surfaces are known to produce streaks. The combination of such heating and groove patterns leads to the formation of more intense streaks with smaller losses compared to heated smooth surfaces. The interest in this analysis was focused on low heating intensity and small groove amplitudes to control the energy “cost” associated with streak formation, and on heating patterns perfectly tuned with groove patterns.

The model problem involves pressure-gradient-driven channel flow with the lower wall fitted with streamwise grooves and exposed to a combination of uniform and spanwise periodic heating with the spatial distribution matching the groove distribution. The model allows for different relative positions of the groove and heating patterns. The flow equations were solved numerically with spectral accuracy with geometry modelling carried out using the immersed boundary conditions concept which uses constraints equivalent to the imposition of boundary conditions at the borders of the flow domain. The efficiency of the streak creation was measured either by the pressure gradient correction when the fixed flow rate constraint was used or by the flow rate correction when the fixed pressure gradient constraint was used.

It is shown that isothermal grooves create relatively weak streaks. Uniform heating of these grooves results in the formation of intense streaks but only when the groove wavenumber is near the critical RB wavenumber and when the heating intensity either meets or exceeds the critical uniform Rayleigh number. The flow losses caused by such heating increase above the losses associated with isothermal grooves. The need to match the critical RB wavenumber limits the ability to create intense streaks with different spatial distributions. The need to match the critical heating intensity prevents the use of this method for streak creation using weak heating. The use of long-wavelength grooves reduces losses below
those found in an isothermal smooth channel, but the resulting streaks have a much lower intensity. The streak intensity can be increased by increasing the groove amplitude.

Patterned heating of grooved surfaces demonstrates the potential for a significant increase of streak intensity as compared to patterned heating of smooth surfaces and uniform heating of grooved surfaces. It also provides the means for control of the spatial distributions of streaks and for the creation of such streaks using low-intensity heating. The strongest streaks are formed using $\alpha \approx 1$ with these streaks being stronger than streaks formed by the RB instability mechanism and having a much lower “cost” in terms of flow losses. Streaks formed by long-wavelength patterns remain of interest as they have a respectable intensity but much lower energy “cost”. The relative position of both patterns plays an important role with the best effect obtained when hot spots are located approximately halfway between the groove crest and trough.
Chapter 6

6 Linear Stability of Spatially Modulated Flows

6.1 Introduction

In Chapters 4 and 5, characteristics of the primary state for modulated flows have been discussed. In this chapter, an algorithm is presented for analyzing the linear stability of flows modulated by grooves and heating patterns with a view to tracing the transition to the secondary states. This transition is desired when mixing intensification is of interest. The algorithm can handle commensurate and incommensurate states, which are not accessible to classical DNS-based approaches. It uses spectral discretization of the field equations combined with the spectrally accurate Immersed Boundary Conditions (IBC) method to handle the irregularity of the solution domain. The algorithm can deal with the pattern interaction effects activated by patterns of different physical quantities. The presentation of this algorithm is organized into six parts. Section 6.2 describes the model problem used to present the algorithm. Section 6.3 discusses the determination of stationary states, which is the first step in the overall analysis. The linear stability analysis is presented in Section 6.4, with Section 6.4.1 describing the modal equations and Section 6.4.2 discussing the discretization of the boundary conditions., and Section 6.4.3 describing the complete discretized linear system. Section 6.5 discusses the numerical solution to the eigenvalue problem and eigenvalue tracing procedures. The accuracy testing and demonstration of the spectral convergence are described in Section 6.6. Section 6.7 gives a summary of the main conclusions.

5 A version of this chapter has been submitted for publication as –

6.2 Problem Formulation

A model problem is selected consisting of flow between two parallel plates extending to \( \pm \infty \) in the \( x \)- and \( z \)-directions. The flow is spatially modulated by distributed heating and a pattern of surface grooves. This presentation is limited to modulations that are the spanwise \( x \)-coordinate functions, however, modulation in \( z \)-direction does not pose any conceptual difficulties. The conduit is oriented arbitrarily to the gravity vector \( \vec{g} \) (Fig 6.1).

The fluid is assumed to be incompressible, while the density (\( \rho \)) difference owing to the temperature variation is accounted for using the Boussinesq approximation. It has thermal conductivity \( k \), specific heat \( c \), thermal diffusivity \( \alpha = k/\rho c \), kinematic viscosity \( \nu \), dynamic viscosity \( \mu \), and thermal expansion coefficient \( \beta \). The non-dimensional equations of motion are

\[
\nabla \vec{V} = 0, \\
\frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla) \vec{V} = -\nabla p + \nabla^2 \vec{V} + Pr^{-1} \theta \vec{g}, \\
\frac{\partial \theta}{\partial t} + (\vec{V} \cdot \nabla) \theta = Pr^{-1} \nabla^2 \theta
\]

where half of the channel height \( h \) is used as length scale, velocity vector \( \vec{V} \) is normalized by the viscous velocity scale \( U_v = \nu/h \), kinematic pressure \( p \) is scaled with \( \rho U_v^2 \), relative temperature \( \theta \) is scaled with \( \kappa \nu/|\vec{g}| \beta h^3 \), \( Pr = \nu/k \) is the Prandtl number, and \( \vec{g} \) is the gravity vector with \( g_x = g \sin \Lambda \cos \gamma, g_y = g \cos \Lambda, g_z = g \sin \Lambda \sin \gamma \) being its components in the \( (x, y, z) \) directions with angles \( \Lambda \) and \( \gamma \) being defined in Fig 6.1 (see Appendix H for details). The relevant boundary conditions are

\[
\vec{V}[y_L(x), t] = \vec{V}[y_U(x), t] = 0, \quad (6.2.2a, b) \\
\theta[y_L(x), t] = \theta_L(x, t), \quad \theta[y_U(x), t] = \theta_U(x, t) \quad (6.2.2c, d)
\]

where the subscripts \( L \) and \( U \) refer to the lower and upper plates, respectively.
The above system is closed by specifying constraints either in the form of mean pressure gradients in the $x$- and $z$-directions, that is,

$$
\frac{\partial p}{\partial x}\bigg|_{\text{mean}} = \varphi_x, \quad \frac{\partial p}{\partial z}\bigg|_{\text{mean}} = \varphi_z, 
$$

or in the form of the mean flow rates in the $x$- and $z$-directions, that is

$$
Q_x\big|_{\text{mean}} = Q_x, \quad Q_z\big|_{\text{mean}} = Q_z. 
$$

Fourier expansions of the form

$$
y_L(x) = -1 + B_L Y_L(x) = -1 + B_L \sum_{n=-N_L}^{n=N_L} H_L^{(n)} e^{inax}, \quad (6.2.4a)
$$

$$
y_U(x) = 1 + B_U Y_U(x) = 1 + B_U \sum_{n=-N_U}^{n=N_U} H_U^{(n)} e^{in(ax+\Omega)}, \quad (6.2.4b)
$$
describe the surface topographies. In the above, $Y_L(x)$ and $Y_U(x)$ are the shape functions describing plates topographies satisfying conditions.
\[ \max[Y_L(x)] - \min[Y_L(x)] = 1, \quad \max[Y_U(x)] - \min[Y_U(x)] = 1, \quad (6.2.4c) \]

\( B_L \) and \( B_U \) are the peak-to-bottom groove amplitudes at the lower and upper plates, respectively, \( H_L^{(n)} \) and \( H_U^{(n)} \) are the coefficients of the Fourier expansions describing the shape of grooves at the lower and upper plates, respectively, \( N_G \) is the number of Fourier modes required to describe the geometry, \( \alpha \) is the modulation wave number, and \( \Omega_G \) stands for the phase shift between the upper and lower groove systems.

The plates’ temperatures are expressed as Fourier expansions of the form

\[
\begin{align*}
\theta_L(x) &= Ra_{uni} + Ra_{p,L} \Theta_L(x) \\
&= Ra_{uni} + Ra_{p,L} \sum_{n=-N_T,n \neq 0}^{n=N_T} \theta_L^{(n)} e^{i\alpha x + \Omega_{TL}} \quad (6.2.5a) \\
\theta_U(x) &= Ra_{p,U} \Theta_U(x) \\
&= Ra_{p,U} \sum_{n=-N_T,n \neq 0}^{n=N_T} \theta_U^{(n)} e^{i\alpha x + \Omega_{TU}} \quad (6.2.5b)
\end{align*}
\]

where \( \Theta_L(x) \) and \( \Theta_U(x) \) are the shape functions describing temperature distributions at the lower and upper plates, respectively, satisfying conditions

\[ \max[\Theta_L(x)] - \min[\Theta_L(x)] = 1, \quad \max[\Theta_U(x)] - \min[\Theta_U(x)] = 1, \quad (6.2.5c) \]

\( \theta_L^{(n)} \) and \( \theta_U^{(n)} \) are the coefficients of the Fourier expansions describing the temperature profile at the lower and upper plates, respectively, \( N_G \) is the number of Fourier modes required to describe the form of heating, \( Ra_{uni} = |\bar{g}| \beta h^3 \theta_{uni} / (\kappa \nu) \) is the uniform Rayleigh number measuring the intensity of the uniform component of heating, \( Ra_{p,L} = |\bar{g}| \beta h^3 \theta_{p,L} / (\kappa \nu) \) is the lower periodic Rayleigh number measuring the intensity of the lower periodic heating and \( Ra_{p,U} = |\bar{g}| \beta h^3 \theta_{p,U} / (\kappa \nu) \) is the upper periodic Rayleigh number measuring the intensity of the upper periodic heating, \( \theta_{p,L} \) and \( \theta_{p,U} \) are the differences between the maximum and minimum of the lower and upper periodic temperature components, \( \Omega_{TL} \) stands for the phase shift between the lower groove and the lower temperature distribution, and \( \Omega_{TU} \) stands for the phase shift between the lower
groove and the upper-temperature distribution. Heating and groove modulations are perfectly tuned; these modulations are described by the same wavenumber $\alpha$.

The stability problem represents an initial value problem that can be solved using direct numerical simulation (DNS) for any set of the initial disturbance velocity and temperature fields. The Fourier-Chebyshev expansions combined with Immersed Boundary Conditions (IBC) method provide the required high-accuracy discretization capable of handling complex geometry (Chapter 3). The spectral element method described in (Cantwell et al. 2015) represents another alternative. DNS requires the specification of a computational box containing an integer number of modulation wavelengths and an integer number of disturbance wavelengths. Such a box may have to be very long. Its size limits possible investigations to a few combinations of groove and heating wavelengths (Blancher, Le Guer & El Omari 2015; Gepner & Floryan 2016) and prevent analysis of arbitrary disturbances. Here we pursue an alternative strategy that involves the identification of stationary solutions and the determination of their stability properties leading to the determination of the critical conditions resulting in the initiation of bifurcation.

The following section briefly discusses the algorithm required for determining stationary states. This algorithm had been described in Chapter 2 & 3, so the presentation is limited to basic concepts needed in describing the stability algorithm.

## 6.3 Determination of Stationary States

Stationary states are described by the stationary version of (6.2.1) – (6.2.2). It is helpful to restate these equations for clarity of the presentation:

\[ \nabla \cdot \vec{V}_B = 0, \quad (\vec{V}_B \cdot \nabla) \vec{V}_B = -\nabla p_B + \nabla^2 \vec{V}_B + Pr^{-1} \theta_B \vec{g}, \]

\[ (\vec{V}_B \cdot \nabla) \theta_B = Pr^{-1} \nabla^2 \theta_B, \quad (6.3.1a-c) \]

\[ \vec{V}_B[y_L(x)] = \vec{V}_B[y_U(x)] = 0, \quad \theta_B[y_L(x)] = \theta_L(x), \quad \theta_B[y_U(x)] = \theta_U(x) \quad (6.3.2a-d) \]
where subscript $B$ denotes stationary quantities. One needs to add either the pressure gradient constraint $\varphi_x$ or the flow rate constraint $Q_x$ for the $x$-direction, and either the pressure constraint $\varphi_z$ or the flow rate constraint $Q_z$ for the $z$-direction. Most of the computations have been carried out with $\varphi_x = 0$ and $\varphi_z = -2Re_z$ where the Reynolds number is defined as $Re_z = (w_B)_{max} h/\nu$.

Spectrally accurate spatial discretization capable of reducing the discretization error to machine accuracy is used. This discretization relies on Fourier expansions in the $z$-streamwise and $x$-spanwise directions and Chebyshev expansions in the $y$-transverse direction. The algebraic equations are constructed using the Galerkin projection method. The irregularity of the flow domain is handled using the spectrally accurate Immersed Boundary Conditions (IBC) method, with flow boundary conditions replaced by constraints which are implemented using the tau method.

For simplicity of presentation of the stability algorithm, we express velocity components, pressure, and temperature of stationary states in the following form

$$
[u_B, v_B, w_B, p_B, \theta_B](x, y) = \sum_{n=-N_B}^{N_B} [f_u^{<n>}, f_v^{<n>}, f_w^{<n>}, f_p^{<n>}, f_\theta^{<n>}(y)] e^{in\alpha x}.
$$

(6.3.3)

We have also defined the vorticity $\bar{\omega}_B = (\xi_B, \eta_B, \phi_B) = \left( \frac{\partial w_B}{\partial y} - \frac{\partial v_B}{\partial z}, \frac{\partial u_B}{\partial z} - \frac{\partial w_B}{\partial x}, \frac{\partial u_B}{\partial x} \right)$ and write its components as Fourier expansions of the form

$$
[\xi_B, \eta_B, \phi_B](x, y) = \sum_{n=-N_B}^{N_B} [f_\xi^{<n>}, f_\eta^{<n>}, f_\phi^{<n>}(y)] e^{in\alpha x}.
$$

(6.3.4)

### 6.4 Linear Stability Analysis

The stability analysis begins with the governing equations expressed in terms of the vorticity transport, energy, and continuity equations in the following form
\[
\frac{\partial \bar{\omega}}{\partial t} - (\bar{\omega} \cdot \nabla) \bar{\nabla} + (\bar{\nabla} \cdot \bar{\omega}) = \nabla^2 \bar{\omega} + \nabla \times (Pr^{-1} \bar{\nabla} \bar{\theta}), \quad (6.4.1a)
\]
\[
\frac{\partial \theta}{\partial t} + (\bar{\nabla} \cdot \bar{\theta}) \nabla = Pr^{-1} \nabla^2 \theta, \quad (6.4.1b)
\]
\[
\nabla \cdot \bar{\nabla} = 0, \quad \bar{\omega} = \nabla \times \bar{\nabla}. \quad (6.4.1c,d)
\]

Unsteady three-dimensional disturbances are superposed on the stationary flow in the form

\[
\bar{\omega} = \bar{\omega}_B(x, y) + \bar{\omega}_D(x, y, z, t), \quad (6.4.2a)
\]
\[
\bar{\nabla} = \bar{\nabla}_B(x, y) + \bar{\nabla}_D(x, y, z, t), \quad (6.4.2b)
\]
\[
\theta = \theta_B(x, y) + \theta_D(x, y, z, t) \quad (6.4.2c)
\]

where components of vorticity vectors are \( \bar{\omega}_B = (\xi_B, \eta_B, \phi_B) \), \( \bar{\omega}_D = (\xi_D, \eta_D, \phi_D) \) and subscript \( D \) denotes disturbance quantities. The flow quantities (6.4.2) are substituted into (6.4.1), the mean parts are subtracted, and the equations are linearized, yielding the linear disturbance equations of the form

\[
\frac{\partial \bar{\omega}_D}{\partial t} - (\bar{\omega}_B \cdot \bar{\nabla}) \bar{\nabla}_D - (\bar{\omega}_D \cdot \bar{\nabla}) \bar{\nabla}_B + (\bar{\nabla}_B \cdot \bar{\omega}) \bar{\omega}_D + (\bar{\nabla}_D \cdot \bar{\omega}) \bar{\omega}_B
\]
\[
= \nabla^2 \bar{\omega}_D + \nabla \times (Pr^{-1} \bar{\nabla} \bar{\theta}_D), \quad (6.4.3a)
\]
\[
\frac{\partial \theta_D}{\partial t} + (\bar{\nabla}_B \cdot \bar{\theta}) \nabla + (\bar{\nabla}_D \cdot \bar{\theta}) \bar{\nabla} = Pr^{-1} \nabla^2 \theta_D, \quad (6.4.3b)
\]
\[
\nabla \cdot \bar{\nabla}_D = 0, \quad \bar{\omega}_D = \nabla \times \bar{\nabla}_D. \quad (6.4.3c,d)
\]

The homogeneous boundary conditions of the form

\[
\bar{\nabla}_D(x, y, z, t) = 0, \quad \theta_D(x, y, z, t) = 0 \text{ at } y = y_L(x) \text{ and } y = y_U(x) \quad (6.4.4a,b)
\]

complete the formulation.

The solution of (6.4.3) can be written in the form

\[
[\bar{\nabla}_D, \bar{\omega}_D, \theta_D](x, y, z, t) = [\bar{\nabla}_D, \bar{\omega}_D, \theta_D](x, y)e^{i(\delta x + \mu z - \sigma t)} + c.c. \quad (6.4.5a-c)
\]
where $\delta$ and $\mu$ are the real wavenumbers in the $x$- and $z$-directions, respectively, representing components of the disturbance wave vector, $\sigma = \sigma_r + i\sigma_i$ is the complex frequency with $\sigma_i$ describing the rate of growth of disturbances and $\sigma_r$ describing their frequency and $c.c.$ stands for the complex conjugate. $\tilde{G}_D(x,y)$, $\tilde{\Omega}_D(x,y)$ and $\kappa_D(x,y)$ are the $x$-periodic amplitude functions accounting for the groove and heating-induced modulations. Substitution of (6.4.5) into (6.4.3) leads to partial differential equations for the amplitude functions. Since these functions are $x$-periodic, they can be expressed in terms of the Fourier series of the form

$$\tilde{G}_D(x,y) = \sum_{m=-N_D}^{+N_D} [g_u^{<m>}(y), g_v^{<m>}(y), g_w^{<m>}(y)]e^{im\alpha x} + c.c., \quad (6.4.6a)$$

$$\tilde{\Omega}_D(x,y) = \sum_{m=-N_D}^{+N_D} [g_\xi^{<m>}(y), g_\eta^{<m>}(y), g_\phi^{<m>}(y)]e^{im\alpha x} + c.c., \quad (6.4.6b)$$

$$\kappa_D(x,y) = \sum_{m=-N_D}^{+N_D} [g_\theta^{<m>}(y)]e^{im\alpha x} + c.c. \quad (6.4.6c)$$

where $g_u^{<m>}(y)$, $g_v^{<m>}(y)$, $g_w^{<m>}(y)$, $g_\xi^{<m>}(y)$, $g_\eta^{<m>}(y)$, $g_\phi^{<m>}(y)$, $g_\theta^{<m>}(y)$ stand for the modal functions.

A combination of (6.4.5) and (6.4.6) provides the final expressions for $\tilde{V}_D$, $\tilde{\omega}_D$, $\tilde{\theta}_D$ of the form

$$\tilde{V}_D(x,y,z,t) = \sum_{m=-N_D}^{+N_D} [g_u^{<m>}(y), g_v^{<m>}(y)]e^{i[(\delta+ma)x+\mu z-\sigma t]} + c.c., \quad (6.4.7a)$$

$$\tilde{\omega}_D(x,y,z,t) = \sum_{m=-N_D}^{+N_D} [g_\xi^{<m>}(y), g_\eta^{<m>}(y)]e^{i[(\delta+ma)x+\mu z-\sigma t]} + c.c. \quad (6.4.7b)$$

$$\tilde{\theta}_D(x,y,z,t) = \sum_{m=-N_D}^{+N_D} [g_\phi^{<m>}(y)]e^{i[(\delta+ma)x+\mu z-\sigma t]} + c.c. \quad (6.4.7c)$$

The above formulation provides means for continuous variations of the disturbance wave numbers $\delta$ and $\mu$ for a specific modulation wave number $\alpha$, i.e., it bypasses problems associated with proper resolution of arbitrary commensurable states.
Substitution of (6.4.7) into (6.4.3) leads to \((2 \ast N_D + 1)\) ordinary differential equations (ODE) for the unknown modal functions for each partial differential equation (PDE) in the above system. Separation of Fourier modes and extensive rearrangements provide an explicit form of these ODEs expressed in terms of \(g_v^{<m>}, g_\eta^{<m>}, g_\theta^{<m>}\) suitable for computations, i.e.,

\[
T^{<m>}{g_v^{<m>}} - F^{<m>}{g_\theta^{<m>}} = - \sum_{n=-N_M}^{N_M} [(T_1^{<m-n>} + T_2^{<m-n>} + T_3^{<m-n>})g_v^{<m-n>} \\
+ (T_4^{<m-n>} - T_5^{<m-n>}) g_\eta^{<m-n>}]
\]  

(6.4.8a)

\[
S^{<m>}{g_\eta^{<m>}} + M^{<m>}{g_\theta^{<m>}} = \sum_{n=-N_M}^{N_M} [(S_1^{<m-n>} + S_2^{<m-n>} + S_3^{<m-n>})g_v^{<m-n>} + \\
(S_4^{<m-n>} - S_5^{<m-n>}) g_\eta^{<m-n>}],
\]  

(6.4.8b)

\[
Q^{<m>}{g_\theta^{<m>}} = \sum_{n=-N_M}^{N_M} [Q_1^{<m-n>} g_\eta^{<m-n>} + Q_2^{<m-n>} g_v^{<m-n>} + \\
+ \frac{\mu n a f_\theta^{<n>}}{k_m^{<m-n>}} g_\eta^{<m-n>}]
\]  

(6.4.8c)

where,

\[
D q = \frac{dq}{dyq},
\]

\[
t_m = \delta + m\alpha,
\]

\[
k_m^2 = t_m^2 + \mu^2,
\]

\[
T^{<m>} = (D^2 - k_m^2)^2 + i\sigma(D^2 - k_m^2),
\]

\[
S^{<m>} = (D^2 - k_m^2) + i\sigma,
\]

\[
Q^{<m>} = Pr^{-1}(D^2 - k_m^2) + i\sigma,
\]

\[
F^{<m>} = Pr^{-1}(i\mu \sin\Lambda \sin\gamma D + k_m^2 \cos\Lambda + it_m \sin\Lambda \cos\gamma D),
\]

\[
M^{<m>} = Pr^{-1}(\mu \sin\Lambda \cos\gamma - t_m \sin\Lambda \sin\gamma),
\]
The coupling terms in these ODEs contain indices \( <m-n> \) with the relevant terms truncated for \( |m-n| \geq N_D \).
The boundary conditions can be written as

\[ \sum_{m=-N_D}^{N_D} [g_u^{<m>}, g_v^{<m>}, g_w^{<m>}, g_{\theta}^{<m>}] e^{i m \alpha} = 0 \text{ at } y = y_L(x) \text{ and } y = y_U(x). \quad (6.4.9) \]

One needs to extract boundary conditions for different Fourier modes as such conditions are required for the modal equations. This process is complex due to the irregularity of the flow domain, which provides coupling between different modes. We shall provide a detailed explanation in Section 6.4.2.

In the limit of no modulations, the stationary state is reduced to mode zero of \( u_B \) and \( w_B \). The disturbance wavenumber \( \delta \) in the \( x \)-direction is taken as \( \alpha \), and the system is reduced after re-arranging indices \( t_m = (m+1)\alpha = J\alpha \), to the following form

\[ T^{<J>} g_v^{<J>} - F^{<J>} g_{\theta}^{<J>} + \]

\[ \left\{ [i \mu D^2 - i \mu (D^2 - k_j^2)] f_w^{<0>} + [i J \alpha D^2 - i J \alpha (D^2 - k_j^2)] f_u^{<0>} \right\} g_v^{<J>} = 0, \quad (6.4.10a) \]

\[ S^{<J>} g_{\eta}^{<J>} + M^{<J>} g_{\theta}^{<J>} - i (\mu f_{\eta}^{<0>} + J \alpha f_u^{<0>}) g_v^{<J>} \]

\[ + i (J \alpha D f_w^{<0>} - \mu D f_u^{<0>}) g_v^{<J>} = 0, \quad (6.4.10b) \]

\[ Q^{<J>} g_{\theta}^{<J>} - (i J \alpha f_{\theta}^{<0>} + i \mu f_w^{<0>}) g_v^{<J>} - D f_{\theta}^{<0>} g_v^{<J>} = 0 \quad (6.4.10c) \]

where \( k_j^2 = (J \alpha)^2 + \mu^2 \) and \( |J| = 0, 1, 2, 3, ..., (2N_D + 1) \). \( (6.4.10a) \) is the Orr-Sommerfeld operator with thermal coupling term \( F^{<J>} g_{\theta}^{<J>} \), \( (6.4.10b) \) is the Squire operator where \( i (J \alpha D f_w^{<0>} - \mu D f_u^{<0>}) g_v^{<J>} \) is the coupling term with the Orr-Sommerfeld operator and \( M^{<J>} g_{\theta}^{<J>} \) is the thermal coupling term, and \( (6.4.10c) \) is the energy equation with \( D f_{\theta}^{<0>} g_v^{<J>} \) providing coupling with the Orr-Sommerfeld operator.
6.4.1 Discretization of the Modal Equations

Determination of the amplitude functions requires a numerical solution of a system of ODE’s described in the previous section. The modal functions are to be represented in terms of Chebyshev expansions. Our preference for the standard definition of the ODE’s described in the previous section. The modal functions are to be represented in terms of Chebyshev expansions. Our preference for the standard definition of the Chebyshev polynomials requires a preliminary mapping of the solution domain $y \in (-1, 1)$ onto $\hat{y} \in (-1, 1)$ using transformation $\hat{y} = \Gamma[y - (1 + y_t)] + 1$ with $\Gamma = \frac{2}{2+y_t+y_b}$ where $y_t$ and $y_b$ identify locations of extremities of the upper and lower walls, respectively. After transformation, the locations of the walls are specified as

\[
\hat{y}_L(x) = 1 - \Gamma(2 + y_t) + \Gamma B_L \sum_{n=-N_G}^{n=N_G} H_L^{(n)} e^{in\alpha x}, \quad (6.4.11a)
\]

\[
\hat{y}_U(x) = 1 - \Gamma y_t + \Gamma B_U \sum_{n=-N_G}^{n=N_G} H_U^{(n)} e^{in\alpha x + \Omega_G}. \quad (6.4.11b)
\]

It is convenient to cast them in a shorter form for simplicity of the presentation, i.e.,

\[
\hat{y}_L(x) = \sum_{n=-N_G}^{n=N_G} A_L^{(n)} e^{in\alpha x}, \quad A_L^{(0)} = 1 + \Gamma \left(-2 - y_t + B_L H_L^{(0)}\right),
\]

\[
A_L^{(n)} = \Gamma B_L H_L^{(n)} \quad \text{for } n \neq 0, \quad (6.4.11c)
\]

\[
\hat{y}_U(x) = \sum_{n=-N_G}^{n=N_G} A_U^{(n)} e^{in\alpha x}, \quad A_U^{(0)} = 1 + \Gamma \left(-y_t + B_U H_U^{(0)}\right),
\]

\[
A_U^{(n)} = \Gamma B_U H_U^{(n)} \quad \text{for } n \neq 0. \quad (6.4.11d)
\]

Equations (6.4.8) expressed in terms of $\hat{y}$ assume the following form

\[
\hat{\tau}^{<m>} g_v^{<m>} - \hat{\tau}^{<m>} g_\theta^{<m>} = -\sum_{n=-N_M}^{n=N_M} \left[(\hat{\tau}_1^{<m-n>} + \hat{\tau}_2^{<m-n>} + g_v^{<m-n>}) + (\hat{\tau}_4^{<m-n>} - \hat{\tau}_5^{<m-n>}) g_\eta^{<m-n>}\right], \quad (6.4.12a)
\]

\[
\hat{\xi}^{<m>} g_\eta^{<m>} + \hat{M}^{<m>} g_\theta^{<m>} = \sum_{n=-N_M}^{n=N_M} \left[(\hat{\xi}_1^{<m-n>} + \hat{\xi}_2^{<m-n>} + \hat{\xi}_3^{<m-n>})g_v^{<m-n>} + (\hat{\xi}_4^{<m-n>} - \hat{\xi}_5^{<m-n>})g_\eta^{<m-n>}\right], \quad (6.4.12b)
\]
\[ \tilde{Q}^{<m>} g^{<m>}_\theta = \sum_{n=-N_M}^{N_M} \left[ \tilde{Q}_1^{<m-n>} g^{<m-n>}_\theta + \tilde{Q}_2^{<m-n>} g^{<m-n>}_v \right] + \frac{\mu \alpha f^{<n>}_m}{k_{m-n}^2} g^{<m-n>}_\eta \]  

(6.4.12c)

where,

\[ \tilde{D} q = \frac{d q}{d y q^*} \]

\[ \tilde{\tau}^{<m>} = (R^2 \tilde{D}^2 - k_m^2))^2 + i\sigma(R^2 \tilde{D}^2 - k_m^2), \]

\[ \tilde{\sigma}^{<m>} = (R^2 \tilde{D}^2 - k_m^2) + i\sigma, \]

\[ \tilde{Q}^{<m>} = Pr^{-1}(R^2 \tilde{D}^2 - k_m^2) + i\sigma, \]

\[ \tilde{\nu}^{<m>} = Pr^{-1}(\mu \sin \Delta \sin \gamma \Gamma \tilde{D} + k_m^2 \cos \Delta + it_m \sin \Delta \cos \gamma \Gamma \tilde{D}), \]

\[ \tilde{M}^{<m>} = Pr^{-1}(\mu \sin \Delta \cos \gamma - t_m \sin \Delta \sin \gamma), \]

\[ \tilde{\epsilon}_1^{<m-n>} = \frac{it_m-n}{k_{m-n}^2} \left( -t_m^2 \Gamma^2 \tilde{D} f^{<n>}_u + i n a k_m^2 \Gamma^2 \tilde{D} f^{<n>}_v - t_m \Gamma f^{<n>}_w \tilde{D}^2 \right) + i t_m \Gamma^3 \tilde{D} f^{<n>}_v + i t_m \Gamma^2 \tilde{D} f^{<n>}_u + \]

\[ \tilde{\epsilon}_2^{<m-n>} = i k_m^2 t_m n f^{<n>}_u + k_m^2 \Gamma^2 \tilde{D} f^{<n>}_v + i t_m \Gamma^2 \tilde{D} f^{<n>}_u + \]

\[ \tilde{\epsilon}_3^{<m-n>} = \frac{i \mu}{k_{m-n}^2} \left( -t_m \Gamma_n f^{<n>}_u \Gamma^2 \tilde{D}^2 + i \mu f^{<n>}_v \Gamma^3 \tilde{D}^3 - t_m \Gamma_n \Gamma^2 \tilde{D} f^{<n>}_u \right), \]

\[ \tilde{\epsilon}_4^{<m-n>} = \frac{it_m-n}{k_{m-n}^2} \left( -t_m \Gamma_n f^{<n>}_u \Gamma \tilde{D} + i t_m \Gamma_n \gamma^2 \tilde{D} f^{<n>}_w \right) - \mu^2 f^{<n>}_w \Gamma^2 \tilde{D}^2, \]

\[ \tilde{\epsilon}_5^{<m-n>} = \frac{i \mu}{k_{m-n}^2} \left( -t_m \Gamma_n f^{<n>}_u \Gamma \tilde{D} + i t_m \Gamma f^{<n>}_v \Gamma^2 \tilde{D}^2 - t_m \Gamma_n \Gamma^2 \tilde{D} f^{<n>}_u \right) - \mu^2 f^{<n>}_w \Gamma \tilde{D}. \]
\[ + it_m f_v^{<n>} \Gamma^2 \hat{D}^2 - 2\mu n \alpha \Gamma \hat{D} f_w^{<n>} - \mu t_{m+n} f_w^{<n>} \Gamma \hat{D}, \]

\[ \tilde{\mathbf{s}}_{1<\mathbf{m-n>}} = \frac{i \mu}{k_{m-n}^2} (t_{m-n} t_m f_u^{<n>} \Gamma \hat{D} - it_m f_v^{<n>} \Gamma^2 \hat{D}^2 + \mu t_{m-n} f_w^{<n>} \Gamma \hat{D}), \]

\[ \tilde{\mathbf{s}}_{2<\mathbf{m-n>}} = i \mu \Gamma \hat{D} f_u^{<n>} - it_{m-n} \Gamma \hat{D} f_w^{<n>} + in \alpha f_w^{<n>} \Gamma \hat{D} - in \alpha \Gamma \hat{D} f_w^{<n>}, \]

\[ \tilde{\mathbf{s}}_{3<\mathbf{m-n>}} = \frac{it_{m-n}}{k_{m-n}^2} (i \mu f_v^{<n>} \Gamma^2 \hat{D}^2 - \mu t_m f_u^{<n>} \Gamma \hat{D} + (n^2 \alpha^2 - \mu^2) f_w^{<n>} \Gamma \hat{D}), \]

\[ \tilde{\mathbf{s}}_{4<\mathbf{m-n>}} = \frac{it_{m-n}}{k_{m-n}^2} (t_{m-n} t_m f_u^{<n>} - it_m f_v^{<n>} \Gamma \hat{D} + \mu t_{m-n} f_w^{<n>}), \]

\[ \tilde{\mathbf{s}}_{5<\mathbf{m-n>}} = \frac{i \mu}{k_{m-n}^2} (i \mu f_v^{<n>} \Gamma \hat{D} - \mu t_m f_u^{<n>} + (n^2 \alpha^2 - \mu^2) f_w^{<n>}), \]

\[ \tilde{q}_{1<\mathbf{m-n>}} = it_m f_u^{<n>} + f_v^{<n>} \Gamma \hat{D} + i \mu f_w^{<n>}, \]

\[ \tilde{q}_{2<\mathbf{m-n>}} = \Gamma \hat{D} f_\theta^{<n>} - \frac{n \alpha f_w^{<n>} t_{m-n}}{k_{m-n}^2} \Gamma \hat{D}. \]

System (6.4.12) has variable coefficients involving base flow – in the first step of the numerical solution, these coefficients are represented as Chebyshev expansions of the form

\[ [u_B, v_B, w_B](x, \hat{y}) = \sum_{n=-N_B}^{N_B} \sum_{r=0}^{N_C-1} [G_{u,r}^{<n>}, G_{v,r}^{<n>}, G_{w,r}^{<n>}] T_r(\hat{y}) e^{imx} \quad (6.13.4) \]

where \(G_{u,r}^{<n>}, G_{v,r}^{<n>}, G_{w,r}^{<n>}\) are the known. In the second step, the unknown modal functions are expressed in terms of the Chebyshev expansions of the form

\[ [g_v^{<m>}, g_\eta^{<m>}, g_\theta^{<m>}] (\hat{y}) = \sum_{k=0}^{N_k-1} [G_{k,v}^{<m>}, G_{k,\eta}^{<m>}, G_{k,\theta}^{<m>}] T_k(\hat{y}) \quad (6.14.4) \]

where \(G_{k,v}^{<m>}, G_{k,\eta}^{<m>}\) and \(G_{k,\theta}^{<m>}\) are unknown and are to be determined. Substitution of (6.13.4) – (6.14.4) into (6.12.4) and application of the Galerkin projection method result, after a rather lengthy algebra, in a system of algebraic equations for these coefficients of the following form

\[ \sum_{k=0}^{N_{T-1}} [A^{<m>} G_{k,v}^{<m>} - H^{<m>} G_{k,\theta}^{<m>}] + \sum_{n=-N_m}^{N_m} \sum_{k=0}^{N_{k-1}} \sum_{r=0}^{N_T-1} [P^{<m-n>} G_{k,\eta}^{<m-n>}] = 0, \quad (6.15.4a) \]
\[\sum_{k=0}^{N_T-1} \{D^{<m>}G_{k,\eta}^{<m>} + L^{<m>}G_{k,\theta}^{<m>} \} + \sum_{n=-N_M}^{N_M} \sum_{k=0}^{N_T-1} \{K^{<m-n>}G_{k,v}^{<m-n>} + Q^{<m-n>}G_{k,\eta}^{<m-n>} \} = 0, \quad (6.4.15b)\]

\[\sum_{k=0}^{N_T-1} I^{<m>}G_{k,\theta}^{<m>} + \sum_{n=-N_M}^{N_M} \sum_{k=0}^{N_T-1} \sum_{r=0}^{N_T-1} \{U^{<m-n>}G_{k,\theta}^{<m-n>} + V^{<m-n>}G_{k,v}^{<m-n>} + \]

\[W^{<m-n>}G_{k,\eta}^{<m-n>} \} = 0 \quad (6.4.15c)\]

where,

\[A^{<m>} = \Gamma^4 < T_j, D^4 T_k > - 2\Gamma^2 k_m^2 < T_j, D^2 T_k > + k_m^4 < T_j, T_k > + i\sigma(\Gamma^2 \]

\[< T_j, D^2 T_k > - k_m^2 < T_j, T_k >,\]

\[D^{<m>} = \Gamma^2 < T_j, D^2 T_k > - k_m^2 < T_j, T_k > + i\sigma < T_j, T_k >,\]

\[I^{<m>} = Pr^{-1}\Gamma^2 < T_j, D^2 T_k > - Pr^{-1}k_m^2 < T_j, T_k > + i\sigma < T_j, T_k >,\]

\[H^{<m>} = Pr^{-1}[i\mu \sin\Lambda \sin\gamma \Gamma < T_j, DT_k > + k_m^3 \cos\Lambda < T_j, T_k > + \]

\[+it_m \sin\Lambda \cos\gamma \Gamma < T_j, DT_k >,\]

\[L^{<m>} = Pr^{-1}[\mu \sin\Lambda \cos\gamma < T_j, T_k > - t_m \sin\Lambda \sin\gamma < T_j, T_k >],\]

\[p^{<m-n>} = \frac{-it_m^2 t_{m-n} \Gamma^2}{k_m^2} G_{u,r}^{<n>} < T_j, DT_r DT_k > - \frac{n\alpha k_m^2 t_m n \Gamma^2}{k_m^2} G_{v,r}^{<n>} < T_j, DT_r DT_k > \]

\[-\frac{it_m^2 t_{m-n} \Gamma^2}{k_m^2} G_{u,r}^{<n>} < T_j, T_r D^2 T_k > - \frac{t_m n t_m \Gamma^3}{k_m^2} G_{v,r}^{<n>} < T_j, DT_r D^2 T_k > - \frac{t_m n t_m \Gamma^3}{k_m^2} G_{u,r}^{<n>} < T_j, DT_r DT_k > + it_m \Gamma^2 G_{u,r}^{<n>} < T_j, DT_r DT_k > + \]

\[< T_j, D^2 T_k > + ik_m t_{m-n} G_{u,r}^{<n>} < T_j, T_r T_k > + k_m^2 \Gamma G_{v,r}^{<n>} < T_j, T_r DT_k > + it_m \Gamma^2 G_{u,r}^{<n>} < T_j, DT_r DT_k > + \]
\[ i\mu k_m^2 G_{w,r}^{<n>} < T_j, T_r T_k > - \frac{i\mu^2 t_{m-2n} \Gamma^2}{k_m^2} G_{u,r}^{<n>} < T_j, T_r D^2 T_k > - \frac{\mu^2 \Gamma^3}{k_m^2} G_{v,r}^{<n>} < T_j, T_r D^2 T_k > , \]

\[ < T_j, T_r D^3 T_k > + \frac{i\mu^2 t_{m-n} \Gamma^2}{k_m^2} G_{u,r}^{<n>} < T_j, DT_r DT_k > - \frac{i\mu^3 \Gamma^2}{k_m^2} G_{w,r}^{<n>} < T_j, T_r D^2 T_k > , \]

\[ R^{<m-n>} = - \frac{it_m n \mu t_{m-2n} \Gamma}{k_m^2} G_{u,r}^{<n>} < T_j, T_r DT_k > - \frac{t_m \mu t_{m-n} \Gamma}{k_m^2} G_{w,r}^{<n>} < T_j, T_r D^2 T_k > + \]

\[ \frac{it_m n t_{m-n} \Gamma}{k_m^2} G_{u,r}^{<n>} < T_j, DT_r DT_k > - \frac{it_m \mu t_{m-n} \Gamma}{k_m^2} G_{w,r}^{<n>} < T_j, T_r DT_k > + \frac{i \mu^2 t_m \Gamma}{k_m^2} G_{u,r}^{<n>} < T_j, T_r DT_k > + \frac{\mu t_m \Gamma^2}{k_m^2} \]

\[ G_{v,r}^{<n>} < T_j, DT_r DT_k > + \frac{\mu t_m \Gamma}{k_m^2} G_{v,r}^{<n>} < T_j, T_r D^2 T_k > + \frac{2 i \mu^2 n \alpha \Gamma}{k_m^2} G_{w,r}^{<n>} < T_j, DT_r T_k > \]

\[ + \frac{i \mu^2 t_{m+n} \Gamma}{k_m^2} G_{w,r}^{<n>} < T_j, T_r DT_k > , \]

\[ K^{<m-n>} = - \frac{it_m n t_m \mu t_{m-n} \Gamma}{k_m^2} G_{u,r}^{<n>} < T_j, T_r DT_k > - \frac{\mu t_m \Gamma^2}{k_m^2} G_{v,r}^{<n>} < T_j, T_r D^2 T_k > \]

\[ - \frac{i \mu^2 t_{m-n} \Gamma}{k_m^2} G_{w,r}^{<n>} < T_j, T_r DT_k > - i \mu \Gamma G_{u,r}^{<n>} < T_j, DT_r T_k > + it_{m-n} \Gamma G_{w,r}^{<n>} \]

\[ < T_j, DT_r T_k > - in \alpha \Gamma G_{w,r}^{<n>} < T_j, T_r DT_k > + in \alpha \Gamma G_{w,r}^{<n>} < T_j, DT_r T_k > \]

\[ + \frac{t_{m-n} \mu \Gamma^2}{k_m^2} G_{v,r}^{<n>} < T_j, T_r D^2 T_k > + \frac{it_{m-n} \mu t_m \Gamma}{k_m^2} G_{u,r}^{<n>} < T_j, T_r DT_k > - \frac{it_{m-n} n^2 \alpha^2}{k_m^2} G_{w,r}^{<n>} \]

\[ < T_j, T_r DT_k > + \frac{it_{m-n} \mu \Gamma^2}{k_m^2} G_{w,r}^{<n>} < T_j, T_r DT_k > , \]

\[ Q^{<m-n>} = - \frac{it_{m-n} t_m}{k_m^2} G_{u,r}^{<n>} < T_j, T_r T_k > - \frac{t_{m-n} t_m \Gamma}{k_m^2} G_{v,r}^{<n>} < T_j, T_r DT_k > \]
\begin{align*}
- \frac{i t_{m-n}^2 \mu}{k_{m-n}^2} G_{\omega,r}^{<n>} < T_j, T_r T_k > & - \frac{\mu t}{k_{m-n}^2} G_{\omega,r}^{<n>} < T_j, T_r DT_k > - \frac{i \mu t_m}{k_{m-n}^2} G_{u,r}^{<p>} < T_j, T_r T_k > \\
+ \frac{i \mu a^2}{k_{m-n}^2} G_{\omega,r}^{<n>} < T_j, T_r T_k > & - \frac{i \mu}{k_{m-n}^2} G_{u,r}^{<n>} < T_j, T_r T_k >,
\end{align*}

\[ U^{<m-n>} = - i t_{m-n} G_{u,r}^{<n>} < T_j, T_r T_k > - \Gamma G_{\omega,r}^{<n>} < T_j, T_r DT_k > - i \mu G_{\omega,r}^{<n>} < T_j, T_r T_k >, \]

\[ V^{<m-n>} = \Gamma G_{u,r}^{<n>} < T_j, T_r DT_k > - \frac{n a t_{m-n} \Gamma}{k_{m-n}} G_{\theta,r}^{<n>} < T_j, T_r DT_k >, \]

\[ W^{<m-n>} = - \frac{\mu a}{k_{m-n}} G_{\theta,r}^{<l>} < T_j, T_r T_k >. \]

The inner products appearing in these equations are defined as

\[ < T_j, D^p T_r D^q T_k > = \int_{-1}^{1} T_j(\tilde{y}) D^p T_r(\tilde{y}) D^q T_k(\tilde{y}) \omega(\tilde{y}) d\tilde{y} \quad (6.4.16) \]

where \( \omega(\tilde{y}) = (1 - \tilde{y}^2)^{-1/2} \) denotes the weight function.

### 6.4.2 Discretization of Boundary Conditions

Thermal, no-slip, and no-penetration boundary conditions (6.4.4) at the lower wall are restated as

\[ \bar{V}_D(x, \hat{y}_L(x), z, t) = \sum_{m=-N_D}^{+N_D} \left[ g_u^{<m>} (\hat{y}_L(x)) , g_v^{<m>} (\hat{y}_L(x)) \right], \]

\[ g_w^{<m>} (\hat{y}_L(x)) e^{i[(\delta + ma)x + \mu z - \sigma t]} = 0, \quad (6.4.17a) \]

\[ \theta_D(x, \hat{y}_L(x), z, t) = \sum_{m=-N_D}^{+N_D} \left[ g_{\theta}^{<m>} (\hat{y}_L(x)) \right] e^{i[(\delta + ma)x + \mu z - \sigma t]} = 0. \quad (6.4.17b) \]

To be consistent with the modal equations, they must be expressed in terms of the vertical velocity and vorticity components. This is done using expressions presented below:

\[ u_D(x, \hat{y}_L(x), z, t) = \sum_{m=-N_D}^{N_D} \left[ \frac{i t m}{k_m^2} g_u^{<m>} (\hat{y}_L(x)) - \frac{i \mu}{k_m^2} g_{\theta}^{<m>} (\hat{y}_L(x)) \right] \]
\[ e^{i[(\delta + ma)x + \mu z - \sigma t]} = 0, \]  
\(^{(6.4.18a)}\)

\[ v_D[x, \hat{y}_L(x), z, t] = \sum_{m=-N_D}^{N_D} g_v^{<m>}(\hat{y}_L(x)) e^{i[(\delta + ma)x + \mu z - \sigma t]} = 0, \]  
\(^{(6.4.18b)}\)

\[ w_D[x, \hat{y}_L(x), z, t] = \sum_{m=-N_D}^{N_D} \left[ \frac{i\mu r}{k_m^2} g_v^{<m>}(\hat{y}_L(x)) + \frac{it_m}{k_m^2} g_\theta^{<m>}(\hat{y}_L(x)) \right] e^{i[(\delta + ma)x + \mu z - \sigma t]} = 0, \]  
\(^{(6.4.18c)}\)

\[ \theta_D[x, \hat{y}_L(x), z, t] = \sum_{m=-N_D}^{N_D} g_\theta^{<m>}(\hat{y}_L(x)) e^{i[(\delta + ma)x + \mu z - \sigma t]} = 0. \]  
\(^{(6.4.18d)}\)

Substitution of the Chebyshev expansion (6.4.14) into (6.4.18) results in

\[ u_D[x, \hat{y}_L(x), z, t] = \sum_{m=-N_D}^{N_D} \sum_{k=0}^{N_K-1} \left[ \frac{i\mu r}{k_m^2} D T_k[\hat{y}_L(x)] G_{k,v}^{<m>} - \frac{i\mu}{k_m^2} T_k[\hat{y}_L(x)] G_{k,\theta}^{<m>} \right] e^{i[(\delta + ma)x + \mu z - \sigma t]} = 0, \]  
\(^{(6.4.19a)}\)

\[ v_D[x, \hat{y}_L(x), z, t] = \sum_{m=-N_D}^{N_D} \sum_{k=0}^{N_K-1} T_k[\hat{y}_L(x)] G_{k,v}^{<m>} e^{i[(\delta + ma)x + \mu z - \sigma t]} = 0, \]  
\(^{(6.4.19b)}\)

\[ w_D[x, \hat{y}_L(x), z, t] = \sum_{m=-N_D}^{N_D} \sum_{k=0}^{N_K-1} \left[ \frac{i\mu r}{k_m^2} D T_k[\hat{y}_L(x)] G_{k,v}^{<m>} + \frac{it_m}{k_m^2} T_k[\hat{y}_L(x)] G_{k,\theta}^{<m>} \right] e^{i[(\delta + ma)x + \mu z - \sigma t]} = 0, \]  
\(^{(6.4.19c)}\)

\[ \theta_D[x, \hat{y}_L(x), z, t] = \sum_{m=-N_D}^{N_D} \sum_{k=0}^{N_K-1} T_k[\hat{y}_L(x)] G_{k,\theta}^{<m>} e^{i[(\delta + ma)x + \mu z - \sigma t]} = 0. \]  
\(^{(6.4.19d)}\)

The above boundary relations involve values of Chebyshev polynomials and their derivatives at the grooved wall. These values represent periodic functions of the x-coordinate and, thus, can be expressed in terms of Fourier expansions as follows

\[ T_k[\hat{y}_L(x)] = \sum_{p=-N_j}^{N_j} (E_L)_k^{<p>} e^{ipax}, \]  
\(^{(6.4.20a)}\)

\[ D T_k[\hat{y}_L(x)] = \sum_{p=-N_j}^{N_j} (d_L)_k^{<p>} e^{ipax}. \]  
\(^{(6.4.20b)}\)
Appendix B explains the determination of coefficients $E_L$ and $d_L$. Substitution of (6.4.20) into (6.4.19), extraction of Fourier modes, and retention of the first $N_D$ modes result in the boundary relations of the form

$$\sum_{p=-N_D}^{N_D} \sum_{k=0}^{N_K-1} \left[ \frac{[\tau_p]}{k_p^2} (d_L)_k <m-p> G_{k,v}^{p} - \frac{[\mu]}{k_p^2} (E_L)_k <m-p> G_{k,\eta}^{p} \right] = 0,$$  \hspace{1cm} (6.4.21a)

$$\sum_{p=-N_D}^{N_D} \sum_{k=0}^{N_K-1} (E_L)_k <m-p> G_{k,v}^{p} = 0,$$  \hspace{1cm} (6.4.21b)

$$\sum_{p=-N_D}^{N_D} \sum_{k=0}^{N_K-1} \left[ \frac{[\tau_p]}{k_p^2} (d_L)_k <m-p> G_{k,v}^{p} + \frac{[\mu]}{k_p^2} (E_L)_k <m-p> G_{k,\eta}^{p} \right] = 0,$$  \hspace{1cm} (6.4.21c)

$$\sum_{p=-N_D}^{N_D} \sum_{k=0}^{N_K-1} (E_L)_k <m-p> G_{k,\theta}^{p} = 0.$$  \hspace{1cm} (6.4.21d)

where truncation $|m - p| \leq N_D$ is needed to maintain consistency with the modal equations. Increasing the number of retained boundary relations is possible as it improves spatial accuracy but leads to over-constrained formulation (Husain, Floryan & Szumbarski 2009). A similar process leads to boundary relations on the upper wall of the form

$$\sum_{p=-N_D}^{N_D} \sum_{k=0}^{N_K-1} \left[ \frac{[\tau_p]}{k_p^2} (d_U)_k <m-p> G_{k,v}^{p} - \frac{[\mu]}{k_p^2} (E_U)_k <m-p> G_{k,\eta}^{p} \right] = 0,$$  \hspace{1cm} (6.4.22a)

$$\sum_{p=-N_D}^{N_D} \sum_{k=0}^{N_K-1} (E_U)_k <m-p> G_{k,v}^{p} = 0,$$  \hspace{1cm} (6.4.22b)

$$\sum_{p=-N_D}^{N_D} \sum_{k=0}^{N_K-1} \left[ \frac{[\tau_p]}{k_p^2} (d_U)_k <m-p> G_{k,v}^{p} + \frac{[\mu]}{k_p^2} (E_U)_k <m-p> G_{k,\eta}^{p} \right] = 0,$$  \hspace{1cm} (6.4.22c)

$$\sum_{p=-N_D}^{N_D} \sum_{k=0}^{N_K-1} (E_U)_k <m-p> G_{k,\theta}^{p} = 0.$$  \hspace{1cm} (6.4.22d)

Relations (6.4.21) and (6.4.22) provide intermodal coupling associated with the grooves.

\section*{6.4.3 The Linear Algebraic System}

We include boundary relations in the system (6.4.15) using the Tau method (Canuto \textit{et al.} 1991), i.e., equations corresponding to the highest order Chebyshev polynomials in each
of Eqs (6.4.15) are replaced with the boundary relations (6.4.21) and (6.4.22). The linear system used for computations has a simple form

$$ \mathbf{Ax} = 0 $$

(6.4.23)

where $\mathbf{A}$ represents the coefficient matrix, and $\mathbf{x}$ identifies the vector of unknowns.

![Figure 6.2 Structure of the coefficient matrix $\mathbf{A}$ for $N_D = 2$ and $N_k = 10$ (Fig 6.2A).](image)

Green lines identify off-diagonal blocks providing coupling between different unknowns. Figure 6.2B displays the structure of a single block with green shading identifying entries corresponding to boundary relations. Black symbols mark the non-zero elements.

The structure of the coefficient matrix is illustrated in Fig 6.2A where the large blocks identified by green lines correspond to different sets of unknowns (i.e., $v, \eta, \theta$) – the off-diagonal blocks provide coupling between these unknowns. The structure of a single block for $v$, displayed in Fig 6.2B illustrates couplings between Fourier modes with the diagonal sub-blocks corresponding to a specific modal function and the off-diagonal sub-blocks corresponding to other modes.
providing couplings between the modes. Rows with green shading identify positions of
boundary relations that provide additional intermodal coupling.

6.5 Method of Solution

A homogeneous system (6.4.23) has a nontrivial solution only for certain combinations of
parameters which are inter-related through a dispersion relation of the form

$$\Theta(\alpha, \mu, \delta, \sigma, Re_z, Ra_{uni}, Ra_{p, L}, Ra_{p, U}, Pr, B_L, B_U, H_L^{(n)}, H_U^{(n)}, \theta_L^{(n)}, \theta_U^{(n)}) = 0.$$ (6.5.1)

All but two real quantities can be selected arbitrarily. Determining these two quantities
requires finding zeroes of (6.5.1). The explicit form of (6.5.1) is not given. The dispersion
relation can be posed in various ways for computational purposes. The first form involves
posing the system as an algebraic eigenvalue problem for \(\sigma\) in the form

$$CE = \sigma BE \tag{6.5.2}$$

where \(E\) denotes the eigenvector and \(A\) and \(B\) are the coefficient matrices. The \(\sigma\)-spectrum
can be computed using a general eigenvalue solver (Moler 2004). This process can be
computationally expensive, and the computed spectrum may suffer from accuracy
problems as our stability problem leads to large algebraic systems. The cost can be reduced
by evaluating only a spectrum segment using the Arnoldi method (Saad 2003). All these
methods are suitable for locating eigenvalues of interest but unsuitable for their tracing
through parameter space.

Local solutions are more computationally efficient and more accurate but typically produce
just one eigenvalue (Moler 2004). The first method used in this study, the inverse iterations
method (Demmel 1997), is suitable for tracing complex frequencies. One starts with an
initial approximation for the eigenpair \((E_p, \sigma_p) = (E^{(0)}, \sigma_0)\) and improve it iteratively.
The iteration process uses the following relation

$$(A - \sigma_0 B)E^{(b+1)} = BE^{(b)} \tag{6.5.3}$$
which starts with $b = 0$ and determines the next approximation for the eigenpair as

$$
\sigma_p^{(b+1)} = E_p^{(b)T} A E_p^{(b)} / E_p^{(b)*} B E_p^{(b)}
$$

(6.5.4)

where the asterisk denotes the complex conjugate transpose. If $\sigma_p$ is the eigenvalue closest to $\sigma_0$, then $E^{(b)}$ converges to $E_p$. The initial approximation $E^{(0)}$ must be consistent with boundary conditions.

The Newton-Raphson method, posed either in terms of one complex variable or in terms of two real variables, offers various alternatives. Search for zeros of the determinant of $A$ is one of them. This method is ineffective as the determinant varies by several orders of magnitude as a function of the unknown eigenvalue, which leads to numerical difficulties. The alternative approach involves converting the homogeneous problem (6.4.23) into an inhomogeneous one. To do so, one replaces a homogeneous boundary condition with an inhomogeneous condition, making the system inhomogeneous and easy to solve. The omitted homogeneous boundary condition provides a test verifying if the correct eigenvalue has been selected. This process involves a search for zero of the omitted boundary condition. A simple example of this process in the case of a smooth surface consists of replacing the homogeneous boundary condition for $Dv^{(1)} = 0$ at one of the walls with an inhomogeneous condition $D^2v^{(1)} = 1$, which, at the same time, imposes the normalization condition. In general, $Dv^{(1)}$ is not zero, so one must implement the Newton-Raphson search process to find eigenvalues that will bring this condition to zero. Any boundary relation can be used as the test condition. A good initial guess is required to achieve convergence.

### 6.6 Testing of the Algorithm

The main feature of the algorithm is its high (spectral) accuracy and ability to handle arbitrary temperature and groove patterns in a very efficient manner. The solution process consists of two steps, i.e., the determination of the stationary state and the determination of the stability characteristics of this state with spectral accuracy. Chapter 2 discusses the
stationary state solver. The present discussion is focused on the testing and description of the performance of the stability solver. In the tests, the stationary solutions were determined with machine accuracy to eliminate the error of these solutions, potentially affecting the accuracy of stability results.

**Figure 6.3** Spectrum for the traveling wave instability in a plane Poiseuille flow for $B_L = 0, Ra_{uni} = Ra_{PL} = Ra_{PU} = 0, \delta = 0, \mu = 1, Re_z = 10000$ (plane isothermal Poiseuille flow). The results were obtained with $N_D = 1$ Fourier modes and $N_K = 120$ Chebyshev polynomials.

All the tests reported here have been carried out with zero pressure gradient constraints in the $x$-direction. The surface and thermal patterns were of the form $y_L = -1 + \frac{B_L}{2} \cos(\alpha x)$, $y_U = +1$, $\theta_L = Ra_{uni} + \frac{Ra_{PL}}{2} \cos(\alpha x + \Omega_L)$, and $\theta_U = \frac{Ra_{PU}}{2} \cos(\alpha x + \Omega_{TU})$. We considered (i) two-dimensional disturbances ($\delta = 0$) and (ii) three-dimensional disturbances ($\delta \neq 0, \mu \neq 0$) to demonstrate the algorithm performance.

The testing begins by demonstrating that the proposed algorithm reproduces results for special cases available in the literature. The first case involves the stability of plane
Poiseuille flow in a smooth isothermal channel which corresponds to $B_L = 0$, $Ra_{unt} = Ra_{P,L} = Ra_{P,U} = 0$, $\delta = 0$. The computed complex amplification rate for $\mu = 1$ and $Re_z = 10000$ was $2375.2649 + 37.3967i$ which, after re-scaling, matches the classical result (Orszag 1971). The spectrum computed for the same conditions (Fig 6.3) matched a similar spectrum available in the literature (Schmid & Henningson 2001). Spurious eigenvalues have been discarded from the spectrum by retaining only eigenvalues independent of the number of Chebyshev polynomials ($N_K$) used in the computations.

![Graphs](image)

**Figure 6.4** Spectrum for flow in an isothermal channel with streamwise grooves for (A) $\alpha = 1$, $\mu = 0.4$, $\delta = 0$, $B_L = 0.1$, $B_U = 0$, $Re_z = 3000$ and (B) $\alpha = 1$, $\mu = 1$, $\delta = 0$, $B_L = 0.1$, $B_U = 0$, $Re_z = 7000$. The results were obtained with $N_D = 10$ Fourier modes and $N_K = 150$ Chebyshev polynomials.

The second case entails traveling wave instability in a channel with isothermal longitudinal grooves. Two types of waves are possible, i.e., waves driven by viscous shear (Moradi & Floryan 2014) and waves driven by an inviscid mechanism (Mohammadi, Moradi & Floryan 2015). There was only one unstable eigenvalue in the former case with the complex
amplification $\sigma = 1763.6239 + 10.8425i$ and one unstable eigenvalue in the latter case with the complex amplification $\sigma = 1179.0851 + 1.3345i$. The third case deals with the Rayleigh-Bénard (RB) instability which occurs as a bifurcation from a conductive state when a fluid layer is uniformly heated from below. This case corresponds to $B_L = B_U = 0, Ra_{p,L} = Ra_{p,U} = 0, Re_z = 0, \mu = 0$. This algorithm successfully reproduced the critical conditions, which are $Ra_{uni,cr} = 213.5$ and $\delta_{cr} = 1.558$ (Rayleigh 1916; Chandrasekhar 1961).

The fourth case deals with an infinite horizontal layer subjected to periodic heating and corresponds to $B_L = B_U = 0, Ra_{p,U} = Ra_{uni} = 0, Re_z = 0$. It is known that the secondary convection may have the form of either longitudinal, transverse, or oblique rolls depending on the heating conditions. This algorithm successfully reproduced the critical conditions for each type of roll given by (Hossain & Floryan 2013).

The fifth case deals with an infinite stationary horizontal layer bounded from below by a corrugated wall and exposed to periodic heating. The test was carried out for $Ra_{p,L} = Ra_{p,U} = 1600, Ra_{uni} = 100, \alpha = 3, \Omega_{TU} = \pi, B_L = 0.07, B_U = 0, Re_z = 0, \delta = 1.1, \mu = 1.1$. Two unstable eigenvalues $(0, 4.6247i), (0, 0.72128i)$ existing under such conditions were matched with the data from the literature (Moradi & Floryan 2019). The zero real parts of these eigenvalues indicate that the secondary convection has the form of stationary rolls. The corresponding spectrum displayed in Fig 6.5 also matches the literature.

A general case of flow through a slot formed by grooved walls exposed to periodic heating with $Ra_{p,L} = 500, Ra_{p,U} = Ra_{uni} = 0, \alpha = 1, B_L = 0.2, B_U = 0, \Omega_{TL} = \pi/2, Re_z = 1000, \delta = 0.8, \mu = 0.5$ is discussed. The spectrum presented in Fig 6.6 has a significantly different structure compared to spectra of special cases, with its width depending on the number of Fourier modes ($N_D$) used in the computation. There is one unstable eigenvalue for these conditions $(\sigma_r + i \sigma_i) = (458.6385, 7.6468)$. 

153
Figure 6.5 Spectrum for convection in a horizontal slot with corrugated lower wall and exposed to periodic heating with $Ra_{P,L} = Ra_{P,U} = 1600$, $Ra_{uni} = 100$, $\alpha = 3$, $\Omega_{TU} = \pi$, $B_L = 0.07$, $B_U = 0$, $Re_z = 0$, $\delta = 1.1$, $\mu = 1.1$. $N_D = 20$ Fourier modes and $N_K = 60$ Chebyshev polynomials were used in these computations.

We shall now discuss the accuracy of the determination of the unstable eigenvalue. This accuracy depends on the number of Fourier modes and Chebyshev polynomials used in the stability computations. The reader may recall that the stationary state was determined with machine accuracy, so its accuracy does not affect the stability results.
Figure 6.6 Spectrum for flow in a channel with longitudinal grooves exposed to spanwise periodic heating with $\alpha = 1$, $\delta = 0.8$, $\mu = 0.5$, $Ra_{P,L} = 500$, $\Omega_{TL} = \pi/2$, $B_L = 0.1$, $Re_z = 1000$. $N_D = 10$ Fourier modes and $N_K = 120$ Chebyshev polynomials were used in these computations.

We shall demonstrate the accuracy of results using error $\Delta \sigma_i$ defined as

$$\Delta \sigma_i = |\sigma_i - \sigma_{i,ref}|$$

(6.6.1) where $\sigma_i$ stands for the computed imaginary part of the eigenvalue and $\sigma_{i,ref}$ is its exact value. Since the exact eigenvalue is not known, $\sigma_{i,ref}$ has been determined by its evaluation with machine accuracy, which, for conditions of this test, required the use of $N_D = 25$ Fourier modes and $N_K = 80$ Chebyshev polynomials.
Figure 6.7 Variations of the error $\Delta \sigma_i$ as a function of the number of Fourier modes $N_D$ used in the computations for (A) a two-dimensional traveling wave disturbance ($\delta = 0, \mu = 0.3$) for $\alpha = 1, B_L = 0.07, B_U = 0$, $Re_z = 1000$ and (B) a three-dimensional travelling wave disturbance ($\delta = 0.8, \mu = 0.5$) for $\alpha = 1, B_L = 0.1, B_U = 0, \Omega_{TL} = \frac{\pi}{2}, \delta = 0.8, \mu = 0.5$. The heating conditions used in the tests were $Ra_{uni} = 0$ and $Ra_{P,L} = Ra_{P,U} = 50$ and $Ra_{P,L} = Ra_{P,U} = 75$. All results were obtained with $N_k = 80$ Chebyshev polynomials.

Figure 6.7 illustrates variations of $\Delta \sigma_i$ as a function of the number of Fourier modes $N_D$ used in the computations. These computations used $N_k = 80$ Chebyshev polynomials which reduced the error of this part of discretization to the machine level. Results for two-dimensional traveling waves show an exponential decrease of the error as the number of Fourier modes increases demonstrating the spectral accuracy of the algorithm. The absolute value of the error increases with an increase in the heating intensity but the character of its exponential decrease is not affected. A similar exponential reduction of error can be observed for the three-dimensional disturbances in Fig 6.7B. These results indicate that a near machine accuracy can be achieved using $N_D = 15$ Fourier modes.
Results displayed in Fig 6.8 illustrate variations of $\Delta \sigma_i$ as a function of the number $N_K$ of Chebyshev polynomials used in the computations. These computations used $N_D = 20$ Fourier modes to reduce the error associated with this part of discretization to the machine level. The results demonstrate an exponential decrease in the error as the number of Chebyshev polynomials increases. They also illustrate the need to use a larger number of Chebyshev polynomials to achieve the same absolute accuracy as in the case of a two-dimensional wave. In most cases, forty Chebyshev polynomials are sufficient to achieve machine accuracy.

**Figure 6.8** Variation of $\Delta \sigma_i$ as a function of the number of Chebyshev polynomials $N_K$ for $\alpha = 1, B_L = 0.07, B_U = 0, Re_x = 500$ for (A) two-dimensional disturbance with $\delta = 0, \mu = 0.4$ and (B) three-dimensional disturbance with $\delta = 0.4, \mu = 0.4$. The heating conditions used in the tests were $Ra_{uni} = 0$ and $(Ra_{P,L}, Ra_{P,U}) = (100,0)$ and $(Ra_{P,L}, Ra_{P,U}) = (200,200)$. All results were obtained with $N_D = 20$ Fourier modes.

The Immersed Boundary Conditions (IBC) method raises the question of how well the homogeneous boundary conditions are enforced at the grooved walls. Since the expected
values of the disturbance velocity components and the disturbance temperature are zero, all these quantities evaluated at the boundaries represent the absolute errors. They vary in an oscillatory manner along the wall as illustrated in Fig 6.9. The relative position of the groove and heating pattern determines the location of the maximum error, but the absolute value of the error is never higher than $10^{-10}$.

Figure 6.9 Distributions of $u_D$ (dashed-dotted line), $v_D$ (solid line), $w_D$ (dashed line) and $\theta_D$ (dotted line) at the lower wall for $\alpha = 1$, $B_L = 0.07$, $B_U = 0$, $Re_z = 1000$, $Ra_{p,L} = 100$, $Ra_{p,U} = 0$, $Ra_{uni} = 0$, $\mu = 0.4$ and (A) $\delta = 0$ and (B) $\delta = 0.4$. All results were obtained with $N_D = 15$ Fourier modes and $N_K = 40$ Chebyshev polynomials.

To quantify the boundary error and its variations as a function of the number of Fourier modes used in the computations, we measured this error using the norm defined as

$$||B.E.|| = \max(q) |\gamma_L$$

(6.6.2)

where $q$ is the flow quantities of interest. Eigenfunctions used in (6.6.2) were normalized by bringing their Euclidean norm defined as
\[
\|E\|_2 = \sqrt{E_1^2 + E_2^2 + \cdots + E_{2N_D+1}^2} = 1
\]  
(6.6.3)

to unity. Here \( E \) is an eigenvector that includes each disturbance quantity appearing in Eq (6.5.2) with length \( 3N_K(2N_D + 1) \). Results displayed in Fig 6.10 demonstrate that \( \|B \cdot E.\| \) decreases exponentially with \( N_D \). The convergence rate varies between different disturbance quantities, but all follow an exponential trend.

![Graphs showing exponential decrease of boundary errors with increasing \( N_D \)]

**Figure 6.10** Variations of the boundary errors as functions of the number \( N_D \) of Fourier modes used in the computations for \( a = 1, B_L = 0.07, Re_x = 1000, Ra_{p,L} = 100, \mu = 0.4, \delta = 0 \) (Fig. 6.10A) and \( \delta = 0.4 \) (Fig. 6.10B). All results were obtained with \( N_K = 80 \) Chebyshev polynomials.

Another way to demonstrate the spectral convergence of the algorithm is to determine variations of the Chebyshev norm defined as

\[
\left\| \Phi_q^{(m)} (\tilde{y}) \right\| = \left\{ \int_{-1}^{1} \left| c_{k,q}^{(m)} (\tilde{y}) \right|^2 \omega(\tilde{y}) d\tilde{y} \right\}^{\frac{1}{2}}
\]  
(6.6.4)
where $q$ stands for the modal function of choice. Figure 6.11 illustrates variations of these norms with an increase of the mode number $m$ for two-dimensional and three-dimensional disturbances. These results confirm the exponential reduction of the Chebyshev norm as the mode number increases for all the quantities.

**Figure 6.11** Variations of the Chebyshev norms as functions of the mode number for $\alpha = 1, B_L = 0.07, B_U = 0, Re_z = 1000, Ra_{P,L} = 100, Ra_{P,U} = Ra_{unt} = 0, \mu = 0.4$ and (A) $\delta = 0$ and (B) $\delta = 0.4$. All results were obtained with $N_D = 15$ Fourier modes and $N_K = 80$ Chebyshev polynomials.

All the tests reported in this section validate the spectral accuracy of the algorithm. The algorithm has the major advantage in efficiently handling geometry and heating patterns’ variations compared to the grid-based approaches. Its gridless nature makes it suitable for a parametric study involving the analysis of any parameter that defines geometric and heating patterns. Mesh construction and grid independence study make such analysis labor and time extensive for the conventional methods.
Table 6.1 Variations of the leading eigenvalue $\sigma$ as a function of the number of Fourier modes $N_D$ and the number of Chebyshev polynomials $N_K$ used in the computations for an isothermal system with $Ra_{p,L} = 0$, $B_L = 0.1$, $B_U = 0$, $\alpha = 1$, $\delta = 0$ and $Re_z = 3000$, $\mu = 0.4$, and $Re_z = 7000$, $\mu = 1$.

<table>
<thead>
<tr>
<th>$N_D$ ($N_K = 60$)</th>
<th>$Re_z = 3000$</th>
<th>$N_D$ ($N_K = 60$)</th>
<th>$Re_z = 7000$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>1179.08682020717 + 1.33381083909094i</td>
<td>6</td>
<td>1763.64640299571 + 10.8681148089647i</td>
</tr>
<tr>
<td>6</td>
<td>1179.08498197117 + 1.33456960594078i</td>
<td>8</td>
<td>1763.64610939885 + 10.8678105329962i</td>
</tr>
<tr>
<td>7</td>
<td>1179.08510116443 + 1.33455994127634i</td>
<td>10</td>
<td>1763.64606374314 + 10.8679040517608i</td>
</tr>
<tr>
<td>8</td>
<td>1179.08509619457 + 1.33455925533565i</td>
<td>12</td>
<td>1763.64605135703 + 10.8678624874547i</td>
</tr>
<tr>
<td>9</td>
<td>1179.08509624864 + 1.33455931431154i</td>
<td>14</td>
<td>1763.64605461566 + 10.8678514117845i</td>
</tr>
<tr>
<td>10</td>
<td>1179.08509628793 + 1.33455927875359i</td>
<td>16</td>
<td>1763.64605374314 + 10.8679040517608i</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$N_K$ ($N_D = 10$)</th>
<th>$N_K$ ($N_D = 14$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>25</td>
<td>1179.08295432967 + 1.34398417980985i</td>
</tr>
<tr>
<td>30</td>
<td>1179.08506375439 + 1.33458741175872i</td>
</tr>
<tr>
<td>35</td>
<td>1179.08509638874 + 1.334559141318i</td>
</tr>
<tr>
<td>40</td>
<td>1179.08509629011 + 1.33455928309432i</td>
</tr>
<tr>
<td>50</td>
<td>1179.08506374314 + 1.334559141318i</td>
</tr>
<tr>
<td>55</td>
<td>1179.08509629011 + 1.33455928309432i</td>
</tr>
<tr>
<td>60</td>
<td>1179.08506374314 + 1.334559141318i</td>
</tr>
</tbody>
</table>
Table 6.2 Variations of the leading eigenvalue $\sigma$ as a function of the number of Fourier modes $N_D$ and the number of Chebyshev polynomials $N_K$ used in the computations for a nonisothermal system with $Re_z = 1000$, $Ra_{p,L} = 500$, $B_L = 0.1$, $B_U = 0$, $\alpha = 1$, $\delta = 0$, $\mu = 1$.

<table>
<thead>
<tr>
<th>$N_D$</th>
<th>$N_K = 65$</th>
<th>$N_K$</th>
<th>$N_D = 14$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>456.597615922004 + 1.71429057304040i</td>
<td>25</td>
<td>456.187820164092 + 1.968264060666i</td>
</tr>
<tr>
<td>7</td>
<td>456.522771220942 + 1.89523539873341i</td>
<td>30</td>
<td>457.487644813961 + 3.36758615065911i</td>
</tr>
<tr>
<td>9</td>
<td>456.336670596876 + 1.78932405422752i</td>
<td>35</td>
<td>456.337724757423 + 1.809323020794i</td>
</tr>
<tr>
<td>11</td>
<td>456.3399875058724 + 1.81097532610426i</td>
<td>40</td>
<td>456.337764893252 + 1.8096687579122i</td>
</tr>
<tr>
<td>13</td>
<td>456.337647502655 + 1.80980390186439i</td>
<td>45</td>
<td>456.337764845658 + 1.80966865294930i</td>
</tr>
<tr>
<td>15</td>
<td>456.337814751728 + 1.80967202603477i</td>
<td>50</td>
<td>456.337764550277 + 1.8096683961425i</td>
</tr>
<tr>
<td>17</td>
<td>456.337818273923 + 1.80968513952038i</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Tables 1 and 2 provide numerical values illustrating the convergence of the test eigenvalue determined using different numbers of Chebyshev polynomials and Fourier modes. The comparison data set uses a form similar to Orszag (1971), which others can use for verification of the accuracy of their results.

6.7 Summary

An algorithm suitable for analyzing the stability of spatially modulated flows, including modulations induced by surface topography and associated with heating patterns, has been proposed. The algorithm relies on spatial discretization based on Fourier expansions in the streamwise and spanwise directions and Chebyshev expansions in the wall-normal...
direction. The linear system of algebraic equations for the expansion coefficients has been determined using the Galerkin projection method. The homogeneous boundary conditions to be imposed along the corrugated walls have been replaced by constraints (the Immersed Boundary Conditions (IBC) method) which were included in the complete system of linear equations using the Tau concept. The resulting eigenvalue problem was solved using standard methods, determining full and partial (Arnoldi method) spectra. The eigenvalue tracing procedures based on inverse iterations, the Newton-Raphson procedure for one complex unknown, and the Newton-Raphson procedure for two real unknowns have been implemented. Numerous tests demonstrated that the algorithm provides spectral accuracy. The proposed algorithm is gridless and requires minimal user time to adapt to new geometries of the bounding walls and new heating patterns. It bypasses the need for cumbersome grid convergence studies to verify grid-based methods' accuracy. The number of Fourier modes and Chebyshev polynomials used in the computations sets the absolute accuracy. The stability algorithm works with a spectrally accurate algorithm based on the IBC concept used to determine stationary states.
Chapter 7

7 Instabilities of the Thermally Modulated Shear Layers

7.1 Introduction

It is shown in Chapters 4 and 5 that rolls can be created in small and medium Reynolds number flows using heating and roughness patterns. Destabilization of these rolls is necessary from the perspective of mixing intensification. In this chapter, the stability analysis of these rolls is presented. Channel flow subject to spanwise thermal modulations is considered. Modulations create streamwise streaks and rolls, producing three-dimensional flow structures. It is shown that these structures induce a new type of instability which persists at low Reynolds numbers. Detailed characterization and quantification of this instability are given, including an explanation of its mechanism. The heating intensity and spatial distribution can control this instability. All presented results in this chapter are for the Prandtl number $Pr = 0.71$ unless explicitly stated otherwise. The model problem used in the analysis is explained in Section 7.2 - it involves a horizontal channel exposed to periodic heating at the lower wall. The primary state involving rolls and streaks is described in Section 7.3. The linear stability problem is formulated in Section 7.4. Different instability modes are discussed in Section 7.5, and the properties of the new instability mode are described in Section 7.6. Instability mechanisms are described in Section 7.7. Flow instabilities resulting from the heating applied at the upper wall are discussed in Section 7.8. Section 7.9 provides a summary of the main conclusions.

---

6 A version of this chapter has been submitted for publication as –

7.2 Problem Formulation

Consider fully developed laminar flow in a channel formed by two horizontal plates placed at a distance \(2 h^*\) apart and extending to \(\pm \infty\) in both the \(x\)- and \(z\)-directions (see Fig 7.1) with gravitational acceleration acting in the negative \(y\)-direction. The fluid is assumed to be incompressible, Newtonian, and density follows the Boussinesq approximation, which is appropriate for low heating rates being of interest in this analysis (Paolucci 1982). The fluid has the thermal conductivity \(k^*\), the specific heat \(c^*\), the thermal diffusivity \(\kappa^* = k^*/\rho^*c^*\), the kinematic viscosity \(\nu^*\), the dynamic viscosity \(\mu^*\) and the thermal expansion coefficient \(\beta^*\) where stars denote dimensional quantities, and a constant pressure gradient drives the fluid in the positive \(z\)-direction.

The lower wall is exposed to the \(x\)-periodic heating resulting in the relative walls' temperatures \((\theta^* = T^* - T_{\text{w,mean}})\) of the form

\[
\theta_L(x^*) = \frac{\theta^*_{\text{in}}}{2} \cos(\alpha^* x^*), \quad \theta_U(x^*) = 0
\]

Figure 7.1 Schematic diagram of the flow configuration.
where $\alpha^*$ stands for the heating wavenumber, the heating wavelength is defined as $\lambda^* = 2\pi/\alpha^*$, and $\theta_{p,L}$ stands for the (peak to bottom) amplitude of the heating. Use of half of the channel height $h^*$ as the length scale and $\kappa^*\nu^*/(g^*\beta^*h^*)^2$ as the temperature scale result in

$$\theta_L(x) = \frac{Ra_{p,L}}{z} \cos (\alpha x), \quad \theta_U(x) = 0 \quad (7.2.2)$$

where $Ra_{p,L} = g^*\beta^*h^*\theta_{p,L}^*/(\kappa^*\nu^*)$ is the periodic Rayleigh number.

The heating-induced flow modifications can be determined by solving the following dimensionless field equations

$$\nabla \cdot \vec{V} = 0, \quad (7.2.3a)$$

$$\frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla)\vec{V} = -\nabla p + \nabla^2 \vec{V} + Pr^{-1} \theta \vec{g}, \quad (7.2.3b)$$

$$\frac{\partial \theta}{\partial t} + (\vec{V} \cdot \nabla)\theta = Pr^{-1} \nabla^2 \theta \quad (7.2.3c)$$

where $\vec{V} = [u, v, w]$ is the velocity vector with components in the $[x, y, z]$-directions, $p$ stands for the pressure, $U_v^* = \nu^*/h^*$ is the velocity scale, $\rho^*U_v^{*2}$ is the pressure scale and $Pr = \nu^*/\kappa^*$ is the Prandtl number. The corresponding boundary conditions have the form

$$y = 1: \quad \vec{V} = 0, \quad \theta = 0; \quad y = -1: \quad \vec{V} = 0, \quad \theta_L = \frac{Ra_{p,L}}{z} \cos (\alpha x). \quad (7.2.4)$$

In the absence of heating, the velocity and pressure field have the form

$$\vec{u}_0(x,y,z) = [0,0,w_0] = [0,0,Re_z(1-y^2)],$$

$$p_0(x,y,z) = -2zRe_z, \quad Q_0 = \frac{4}{3} Re_z. \quad (7.2.5)$$

In the above, subscript 0 denotes the isothermal quantities, and the Reynolds number is defined as $Re_z = W_{z,\text{max}}h^*/\nu^* = W_{z,\text{max}}^*/U_v^*$ where $W_{z,\text{max}}$ denotes the maximum of the $z$-velocity component. The heating does not introduce any pressure gradient, which leads to the following constraints
It is known that patterned heating creates streaks that may either break into complex secondary forms, potentially involving chaotic mixing, or could initiate transition to turbulence. Streaks breakdown can be predicted by solving (7.2.3) - (7.2.6) as an initial value problem – this process is cumbersome and computationally expensive as it requires establishing dependence of breakdown on initial conditions. This analysis aims to determine heating conditions leading to the most effective transition, which compounds difficulties as it implies analysis of many heating patterns. The linear stability theory is used to determine bifurcation points to reduce the analysis cost. A review of these points provides bases for identifying the most promising heating strategies, which can then be explored using DNS to determine saturation states. The process bypasses the issue of initial conditions, and its first step involves determining stationary states.

7.3 Stationary State

The introduction of heating modifies the Poiseuille flow with modifications having the forms of longitudinal streaks whose spanwise structure is dictated by the heating wave number. These streaks are determined by solving the stationary version of (7.2.3) – (7.2.4), i.e.,

\[ \nabla \bar{V}_B = 0, \quad (7.3.1a) \]

\[ (\bar{V}_B \cdot \nabla)\bar{V}_B = -\nabla p_B + \nabla^2 \bar{V}_B + Pr^{-1} \theta_B \bar{g}, \quad (7.3.1b) \]

\[ (\bar{V}_B \cdot \nabla)\theta_B = Pr^{-1} \nabla^2 \theta_B, \quad (7.3.1c) \]

\[ y = 1: \quad \bar{V}_B = 0, \quad \theta_B = 0; \quad y = -1: \quad \bar{V}_B = 0, \quad \theta_B = \frac{RapL}{2} \cos (\alpha x), \quad (7.3.2a-d) \]

\[ \frac{\partial p_B}{\partial z} \bigg|_{\text{mean}} = -2Re_z, \quad \frac{\partial p_B}{\partial x} \bigg|_{\text{mean}} = 0 \quad (7.3.3) \]
where subscript $B$ denotes the stationary quantities. The numerical solution uses spatial discretization in the form of Fourier expansions in the spanwise and streamwise directions, i.e.,

$$
[u_B, v_B, w_B, p_B, \theta_B](x, y) = \sum_{n=-N_B}^{N_B} [f_u^{<n>}, f_v^{<n>}, f_w^{<n>}, f_p^{<n>}, f_\theta^{<n>}] (y) e^{i n x},
$$

(7.3.4a)

with modal functions represented as Chebyshev expansions in the transverse direction of the form

$$
q_B^{(n)}(y) = \sum_{k=0}^{N_K-1} G q_{k,B}^{(n)} T_k(y)
$$

(7.3.4b)

where $q_B^{(n)}$ stands for any of the modal functions $f_u^{<n>}, f_v^{<n>}, f_w^{<n>}, f_p^{<n>}, f_\theta^{<n>}$ and $G q_{k,B}^{(n)}$ are the unknown expansion coefficients. This discretization provides spectral accuracy. A detailed description of the algorithm and its testing can be found in Chapter 2. It suffices to state that the results' absolute accuracy is set by selecting the proper number of Chebyshev polynomials $N_K$ and Fourier modes $N_B$ to be used in the computations. These numbers were determined by requiring at least four-digit accuracy for the eigenvalues determined as a part of the stability analysis.

The vorticity field and its components required in the stability analysis are represented as

$$
\tilde{\omega}_B = (\xi_B, \eta_B, \phi_B) = \left(\frac{\partial w_B}{\partial y}, \frac{\partial v_B}{\partial z}, \frac{\partial u_B}{\partial x}, -\frac{\partial w_B}{\partial x}, -\frac{\partial v_B}{\partial x}, \frac{\partial u_B}{\partial y}\right),
$$

(7.3.5a)

$$
[\xi_B, \eta_B, \phi_B](x, y) = \sum_{n=-N_B}^{N_B} [f_\xi^{<n>}, f_\eta^{<n>}, f_\phi^{<n>}] (y) e^{i n x}.
$$

(7.3.5b)

The modal functions in (7.3.5b) are represented by Chebyshev expansions similar to (7.3.4b).

The heating introduces an $x$-horizontal temperature gradient into the flow field. Fluid tends to move upward above the hot spots and down above the cold spots, resulting in the formation of counter-rotating streamwise rolls (Fig 7.2). These rolls represent a forced system response, and they exist for any $Ra_{p,L}$; their intensity increases with $Ra_{p,L}$. The rolls create flow modifications characterized by zones of increased and decreased
streamwise velocity creating inflection points in the spanwise distribution of the $w_B$-velocity component (Fig 7.3A) and changing shear distribution across the channel (Fig 7.3B). The zones of high $w_B$-gradients are concentrated near the hot points. The inflection points may activate an inviscid instability mechanism similar to that documented by (Mohammadi et al. 2015). Changes in vertical shear are expected to affect the classical traveling wave instability. The conditions leading to the potential onset and changes of these instabilities can be controlled by the heating wavenumber $\alpha$ and the heating intensity, as shown in Chapter 4.

**Figure 7.2** Flow topology for $\alpha = 1, Ra_{P,L} = 1000, Re_x = 350$. Colors in the $(x, y)$-cross-sections illustrate sequentially from left to right the temperature field, the $u$-velocity field, the $v$-velocity field, and the $w$-velocity field. Black solid lines added to the temperature contour plot illustrate the velocity vector lines. The grey color illustrates stream tubes.
Figure 7.3 Variations of the spanwise (A) and transverse (B) gradients of the streamwise velocity component at different $y$-locations for $\alpha = 1$, $Ra_{p,L} = 1000$, and $Re_x = 100$. Dashed lines give reference values for the isothermal channel.

The following section describes the formulation of the linear stability problem for the streaks-modified flow.

### 7.4 Linear Stability

The heating spatially modulates the stationary state. Formulation of its stability problem begins with governing equations expressed in terms of the vorticity transport, energy, and continuity equations, i.e.,

$$
\frac{\partial \vec{\omega}}{\partial t} - (\vec{\omega} \cdot \nabla) \vec{V} + (\vec{V} \cdot \nabla) \vec{\omega} = \nabla^2 \vec{\omega} + \nabla \times (Pr^{-1} \theta \vec{g}),
$$

(7.4.1a)

$$
\frac{\partial \theta}{\partial t} + (\vec{V} \cdot \nabla) \theta = Pr^{-1} \nabla^2 \theta,
$$

(7.4.1b)
\[ \nabla. \vec{V} = 0, \quad \vec{\omega} = \nabla \times \vec{V}. \]  

(7.4.1c,d)

Unsteady three-dimensional disturbances are added to the stationary state in the form

\[ \vec{\omega} = \vec{\omega}_B(x, y) + \vec{\omega}_D(x, y, z, t), \]  

(7.4.2a)

\[ \vec{V} = \vec{V}_B(x, y) + \vec{V}_D(x, y, z, t), \]  

(7.4.2b)

\[ \theta = \theta_B(x, y) + \theta_D(x, y, z, t) \]  

(7.4.2c)

where subscript \( D \) denotes disturbance quantities, the flow quantities (7.4.2) are substituted into (7.4.1), the base part is subtracted, and the equations are linearized. The resulting linear disturbance equations have the form

\[ \frac{\partial \vec{\omega}_D}{\partial t} - (\vec{\omega}_B . \nabla) \vec{V}_D - (\vec{\omega}_D . \nabla) \vec{V}_B + (\vec{V}_B . \nabla) \vec{\omega}_D + (\vec{V}_D . \nabla) \vec{\omega}_B = \nabla^2 \vec{\omega}_D + \nabla \times (Pr^{-1} \theta_D \vec{g}), \]  

(7.4.3a)

\[ \frac{\partial \theta_D}{\partial t} + (\vec{V}_B . \nabla) \theta_D + (\vec{V}_D . \nabla) \theta_B = Pr^{-1} \nabla^2 \theta_D, \]  

(7.4.3b)

\[ \nabla. \vec{V}_D = 0, \quad \vec{\omega}_D = \nabla \times \vec{V}_D. \]  

(7.4.3c,d)

\[ y = \pm 1 \quad : \quad \vec{V}_D(x, y, z, t) = 0, \quad \theta_D(x, y, z, t) = 0 \]  

(7.4.3e,f)

Since heating modulates the stationary state, the disturbance quantities can be expressed as waves with amplitudes modulated in the \( x \)-direction, i.e.,

\[ [\vec{V}_D, \vec{\omega}_D, \theta_D](x, y, z, t) = [\vec{G}_D, \vec{\Omega}_D, \kappa_D](x, y) e^{i(\delta x + \mu z - \sigma t)} + c. c. \]  

(7.4.4a-c)

where \( \delta \) and \( \mu \) are the real disturbance wavenumbers in the \( x \)- and \( z \)-directions, respectively, \( \sigma = \sigma_r + i\sigma_i \) is the complex frequency with \( \sigma_i \) describing the rate of growth of disturbances and \( \sigma_r \) describing their frequency, and \( c. c. \) stands for the complex conjugate. The amplitude functions \( \vec{G}_D(x, y), \vec{\Omega}_D(x, y) \) and \( \kappa_D(x, y) \) are \( x \)-periodic as dictated by the type of modulation. Accordingly, these functions can be expressed in terms of the Fourier expansions of the form
\[ \bar{G}_D(x,y) = \sum_{m=-N_D}^{+N_D} [g_u^{<m>(y)}, g_v^{<m>(y)}, g_w^{<m>(y)}] e^{i\max} + \text{c.c.,} \quad (7.4.5a) \]

\[ \bar{\Omega}_D((x,y) = \sum_{m=-N_D}^{+N_D} [g_\xi^{<m>(y)}, g_\eta^{<m>(y)}, g_\phi^{<m>(y)}] e^{i\max} + \text{c.c.,} \quad (7.4.5b) \]

\[ \kappa_D(x,y) = \sum_{m=-N_D}^{+N_D} [g_\theta^{<m>(y)}] e^{i\max} + \text{c.c.} \quad (7.4.5c) \]

The modal functions are quantities in the square brackets on the right-hand side of (7.4.5).

The corresponding boundary conditions have the form

\[ y = \pm 1: \quad g_u^{<m>} = g_v^{<m>} = g_w^{<m>} = g_\theta^{<m>} = 0 \quad . \quad (7.4.6) \]

Combination of (7.4.4) and (7.4.5) results in

\[ \bar{V}_D(x,y,z,t) = \sum_{m=-N_D}^{+N_D} [g_u^{<m>(y)}, g_v^{<m>(y)}, g_w^{<m>(y)}] e^{i[(\delta + m\alpha)x + \mu z - \sigma t]} + \text{c.c.,} \quad (7.4.7a) \]

\[ \bar{\omega}_D(x,y,z,t) = \sum_{m=-N_D}^{+N_D} [g_\xi^{<m>(y)}, g_\eta^{<m>(y)}, g_\phi^{<m>(y)}] e^{i[(\delta + m\alpha)x + \mu z - \sigma t]} + \text{c.c.,} \quad (7.4.7b) \]

\[ \theta_D(x,y,z,t) = \sum_{m=-N_D}^{+N_D} [g_\theta^{<m>(y)}] e^{i[(\delta + m\alpha)x + \mu z - \sigma t]} + \text{c.c.} \quad (7.4.7c) \]

Substitution of (7.4.7) into (7.4.3) and separating Fourier modes lead to a system of coupled ordinary differential equations for the modal functions. These equations can be combined into a system involving \( g_v^{<m>}, g_\eta^{<m>}, g_\theta^{<m>} \) in the form

\[ T^{<m>} g_v^{<m>} - Pr^{-1}k^2 g_\theta^{<m>} = \sum_{n=-N_D}^{N_D} [(T_1^{<m-n>} + T_2^{<m-n>} + T_3^{<m-n>}) g_v^{<m-n>} + \\
( T_4^{<m-n>} - T_5^{<m-n>}) g_\eta^{<m-n>}] , \quad (7.4.8a) \]

\[ S^{<m>} g_\eta^{<m>} = \sum_{n=-N_D}^{N_D} [(S_1^{<m-n>} + S_2^{<m-n>} + S_3^{<m-n>}) g_v^{<m-n>} + \notag \\
(S_4^{<m-n>} - S_5^{<m-n>}) g_\theta^{<m-n>}] , \quad (7.4.8b) \]
\[ Q^{<m> g_{\theta}^{<m>}} = \sum_{n=-N_D}^{+N_D} \left[ Q_1^{<m-n> g_{\theta}^{<m-n>}} + Q_2^{<m-n> g_v^{<m-n>}} + \frac{\mu n a f_{\theta}^{<n>}}{k_{m-n}^2} g_\eta^{<m-n>} \right] \]

(7.4.8c)

\[ y = \pm 1: \quad \frac{it_m}{k_m^2} g_v^{<m>} - \frac{i\mu}{k_m^2} g_\eta^{<m>} = g_v^{<m>} = \frac{i\mu}{k_m^2} D g_\eta^{<m>} + \frac{it_m}{k_m^2} g_\eta^{<m>} = g_\theta^{<m>} = 0 \]

(7.4.8d)

for \( N_D \leq m \leq -N_D \) with all operators defined in the Chapter 6 (Section 6.4). The above system was discretized by representing modal functions in the form of Chebyshev expansions, i.e.,

\[ q_{D}^{(m)}(y) = \sum_{k=0}^{N_k-1} G q_{k,D}^{(m)} T_k(y) \]

(7.4.9)

where \( q_{D}^{(m)}(y) \) stands for any of the modal functions and \( G q_{k,D}^{(m)} \) are the expansion coefficients. The complete discretization procedure involving a combination of (7.4.7) and (7.4.9) provides spectral accuracy. The use of the Galerkin projection method combined with the tau procedure for incorporation of the boundary conditions led to an eigenvalue problem for a system of linear algebraic equations for the unknowns \( G q_{k,D}^{(m)} \). The dispersion relation has to be determined numerically. The solution procedure and its accuracy testing can be found in Chapter 6. All eigenvalues presented in the discussion below were determined with an accuracy of no less than four digits. The same numbers \( N_k \) of Chebyshev polynomials were used to solve the stationary flow problem as well as the stability problem.

### 7.5 Instability Mode

Two types of instability were identified, i.e., the classical Tollmien-Schlichting (TS) instability modified by the heating and a new instability activated by the heating. Figure 7.4A displays a typical spectrum for the modified "two-dimensional" TS wave for \( Re_x = 7000 \) and low heating with \( Ra_{pL} = 50 \). The term 'two-dimensional' is given in quote marks, as disturbances are always three-dimensional due to heating-imposed modulations.
The spectrum consists of the $A$ (wall modes), $P$ (center modes), and $S$ branches (Mack 1976) with the unstable eigenvalue $\sigma = 1768.648 + 11.134i$ located at the tip of the $A$ branch and the disturbance wave moves in the flow directions with the phase speed $\frac{\sigma_r}{\sigma_z} = 0.25266$, which is about 25% of the maximum flow velocity. The top eigenvalue at branch $P$ is stable with $\sigma = 6929.139 - 44.387i$. Increase the heating to $Ra_{P,L} = 200$ results in the spectrum displayed in Fig 7.4B, similar to the spectrum from Fig 7.4A but with all branches becoming "thicker". There are two critical changes – heating stabilized the top eigenvalue in branch $A$ ($\sigma = 1741.033 - 1.498i$) and destabilized the top eigenvalue in branch $P$ ($\sigma = 6809.698 + 20.0668i$).

**Figure 7.4** Disturbance spectrum for (A) $Ra_{P,L} = 50$ and (B) $Ra_{P,L} = 200$ for $\alpha = 1, \delta = 0, \mu = 1$ and $Re_z = 7000$. Tracing of the unstable eigenvalues labeled as (I), (II), (III) is illustrated in Fig 7.5. The displayed results were obtained with $N_D = N_B = 10$ Fourier modes and $N_K = 120$ Chebyshev polynomials.
The unstable wave moves in the streamwise direction with the phase speed of \( \frac{\sigma_p}{\sqrt{7000}} = 0.9728 \), which is very close to the maximum flow velocity.

Figure 7.5 Variations of the unstable eigenvalues labeled as (I), (II), (III) in Fig 7.4 as functions of \( Ra_{p,L} \). Figs 7.5 A, B, C show the variation of eigenvalues (I), (II), (III), respectively. The isothermal limit corresponds to \( Ra_{p,L} = 0 \).
The unstable eigenvalue from the top of branch $A$ connects in the isothermal limit to the Orr-Sommerfeld (OS) spectrum, while the unstable eigenvalue from the top of branch $P$ connects to the Squire spectrum, as demonstrated in Fig 7.5.

The disturbance velocity eigenfunctions for the $TS$ wave at $Ra_{pL} = 50$, $Re = 7000$ displayed in Fig 7.6 show the formation of wall layers with high-intensity movements in the $x$- and $z$-directions near the heated wall and similar but less intense layers near the unheated wall. The $w$- and $v$-components are significantly larger than the $u$-component suggesting that the dominant motion takes place in the $(y,z)$-plane. Distributions of temperature disturbances show a similar high-intensity layer near the heated wall and a less intense layer near the unheated wall. These eigenfunctions are qualitatively similar to the eigenfunctions for the isothermal $TS$ waves.

Eigenfunctions for the new mode at the same Reynolds number are displayed in Fig 7.7. The high-intensity movement is concentrated near the channel center with the $u$- and $w$-components being significantly larger than the $v$-component suggesting a qualitatively different disturbance flow topology, with the dominant motion taking place in the $(x,z)$-plane. Temperature disturbances are also concentrated near the midsection of the channel. The leading mode has a form similar to the isothermal Squire mode.
Figure 7.6 Distributions of the real and imaginary parts of the disturbance velocity and temperature eigenfunctions (A) $g_u^{(m)}$, (B) $g_v^{(m)}$, (C) $g_w^{(m)}$, and (D) $g_\theta^{(m)}$ ($m = 0,1,2$), corresponding to the unstable eigenvalue $\sigma = 1768.648 + 11.134i$ for $Ra_{p,\perp} = 50$, $\alpha = 1$, $\delta = 0$, $\mu = 1$ and $Re_z = 7000$. Eigenfunctions are normalized with condition $\max |g_w^{(0)}| = 1$. The black dashed lines stand for the isothermal case.
Figure 7.7 Distributions of the real and imaginary parts of the disturbance velocity and temperature eigenfunctions (A) $g_u^{(m)}$, (B) $g_v^{(m)}$, (C) $g_w^{(m)}$, and (D) $g_\theta^{(m)}$ ($m = 0,1,2$), corresponding to the unstable eigenvalue $\sigma = 6809.698 + 19.481i$ for $Ra_{p,l} = 200$, $\alpha = 1$, $\delta = 0$, $\mu = 1$ and $Re_z = 7000$. Eigenfunctions are normalized with condition $\max |g_u^{(0)}| = 1$. The black dashed line stands for the isothermal case.
7.6 Properties of the New Instability

The new instability, driven by the heating, is of the main interest as it allows mixing intensification at low Re’s. Figures 7.8 A-B illustrate a typical spectrum for such a case. While its structure barely resembles spectra at high Re, the unstable eigenvalue is located at its right side, connecting to the Squire spectrum in the isothermal limit, as illustrated in Fig 7.9.

Figure 7.8 (A) Disturbance spectrum for the two-dimensional traveling wave for $Ra_{p,L} = 1000, \alpha = 1, \delta = 0, \mu = 1$ and $Re_z = 350$. (B) Enlargement of the red box from (A) with green and blue dots identifying isothermal Squire and OS spectra, respectively. All presented results were obtained with $N_D = N_B = 10$ Fourier modes and $N_K = 120$ Chebyshev polynomials.
Figure 7.9 Variations of the unstable eigenvalue from Fig 7.8 as $Ra_p$ is reduced.

Topologies of the disturbance velocity and temperature fields illustrated in Fig. 7.10 demonstrate that the instability's activity is concentrated in the zone of the upward fluid movement, i.e., above the hot spots where heated plumes are formed.
Figure 7.10 Topology of the disturbance velocity and temperature fields corresponding to the unstable eigenvalue $\sigma = 297.18 + 1.163i$ for $Ra_{P,L} = 1000, \alpha = 1, \delta = 0, \mu = 1$ and $Re_z = 350$. (A) Spanwise disturbance velocity (iso-surfaces for $u_D = 0.25, -0.25$). (B) normal disturbance velocity (iso-surfaces for $v_D = 0.04, -0.04$). (C) streamwise disturbance velocity (iso-surfaces for $w_D = 0.3, -0.3$). (D) disturbance temperature (iso-surfaces for $\theta_D = 0.8, -0.8$). The red arrows show the locations of hot spots.
The disturbance activity is minimal above the cold spots. The disturbance flow field consists of z-periodic blobs moving downstream with a phase speed slightly lower than the maximum flow velocity.

The discussion so far considered two-dimensional disturbances. These disturbances correspond to either $\delta = 0$ in Eq 7.4.7 or $\delta = \pm n\alpha$ as the latter one leads to the renumbering of terms in Fourier expansions. Analysis of three-dimensional waves requires specification of $\mu$, and care must be exercised in tracing eigenvalues as functions of $\delta$ to follow the proper Brillouin zone (Bloch 1929). Figure 7.11 illustrates variations of the amplification rate as the disturbance obliqueness increases. The two-dimensional wave corresponds in this figure to $\delta = \alpha$. The results demonstrate that the two-dimensional waves play a critical role.

![Figure 7.11 Variations of the amplification rate as a function of $\delta$ for $\alpha = 1, Ra_{p,L} = 1000$, and $Re_z = 350$.]
The streamwise variations of disturbances are characterized by the wave number \( \mu \), which raises the question of its effect on the flow stability. Data displayed in Fig. 7.12 for two-dimensional waves demonstrate that the most unstable \( \mu \) changes marginally as a function of \( Re_z \) and \( Ra_{p,\perp} \) in the range of parameters used in this analysis. The most unstable streamwise wave number is in the range \( \mu \in (1\cdots1.2) \).

The next question is the determination of the most effective heating wave number as far as flow destabilization is concerned.

![Figure 7.12](image)

**Figure 7.12** Neutral curves (A) in the \((Re_z, \mu)\) - plane and (B) in the \((Re_z, \sigma_r)\) - plane for two-dimensional traveling waves for \( \alpha = 1 \) and \( \delta = 0 \).

Data displayed in Fig 7.13 demonstrates that effective heating wave number varies in the range \( \alpha \in (0.8, 0.9) \) and slightly increases with an increase of \( Re_z \). The corresponding critical streamwise wave number varies in the range \( \mu \in (0.6, 0.7) \) with the critical \( \sigma_r/Re_z \) being nearly unchanged at \( \sim 0.6 \).
Figure 7.13 The global critical curves for $Re_z = 350, 500$. 
7.7 Mechanism of Instability

We first demonstrate that an inviscid mechanism drives instability. Figure 7.14 displays variations of the amplification rate as a function of $Re_z$ for the new mode and the TS wave. If the flow conditions are kept constant, an increase of $Re_z$ can be viewed as reducing fluid viscosity. An increase of $Re_z$ eventually stabilizes TS waves, as it is known that these waves are driven by viscous shear. These waves can be unstable in the limit of $Re_z \to \infty$ only if the transverse vorticity component has local extrema with the vorticity dynamics driving the instability through the so-called inflection point instability (inflection in the vertical distribution of the $w$-velocity component; Fjørtoft 1950). The flow studied here does not have such properties, and thus, these waves become stabilized at large enough $Re_z$.

![Figure 7.14 Variations of the amplification rate $\sigma_i$ as a function of $Re_z$ for $\alpha = 1, \delta = 0, \mu = 1$.]
Reduced viscosity has a qualitatively different effect on the new waves. The growth rates of these waves increase with $Re_z$ as documented in Fig 7.14, which suggests that they are driven by an inviscid mechanism associated with the extrema of the vertical vorticity component (or inflection points in the spanwise distribution of the $w$-velocity component). Reduction of viscosity reduces dissipation, thus allowing faster growth of this instability. The growth rate should become independent of $Re$ for larger enough $Re_z$ when dissipation become negligible.

![Figure 7.15 Variations of the amplification rate as a function of $Ra_{p,L}$ for $\alpha = 1, \delta = 0, \mu = 1$ and $Re_z = 350$. The solid line shows results from the complete stability analysis. The dashed line shows results from the isothermal stability analysis of the flow field established by the heating (energy equation eliminated from the stability system).](image)

It is evident from the preceding discussion that the heating drives the new instability. We shall demonstrate that the primary role of the heating is to set up a velocity field subjected
to instabilities not directly related to the heating. Figure 7.15 illustrates variations of the amplification rate as a function of \( Ra_{p,L} \) for the complete stability problem and for a simplified problem with thermal effects omitted – thermal effects were used to establish the primary state but were omitted from the stability analysis. These results demonstrate that the elimination of temperature effects destabilizes the flow and increases the disturbance growth rate while retaining the qualitative character of their variations with \( Ra_{p,L} \). The system response is similar to that observed in the case of TS waves exposed to heating discussed in Section 7.5. In that case, the addition of heating stabilized these waves while retaining the same qualitative character of variations of amplification rate with \( Ra_{p,L} \).

Since the primary role of heating is to set up a velocity field, analysis of mechanical energy transfers associated with this field identifies flow features responsible for the energy transfer from the stationary state to disturbances. This analysis starts with the isothermal field equations, i.e.

\[
\frac{\partial \vec{V}}{\partial t} + \vec{V} \cdot \nabla \vec{V} + \nabla p = \nabla^2 \vec{V}, \quad (7.7.1a)
\]

\[
\nabla \cdot \vec{V} = 0. \quad (7.7.1b)
\]

The variables are separated into saturation state and disturbances, i.e.,

\[
\vec{V} = \vec{V}_B + \vec{v}_D, \quad p = p_B + p_D. \quad (7.7.2)
\]

Substitution of (7.7.2) into (7.7.1) and linearization results in

\[
\frac{\partial \vec{v}_D}{\partial t} + \vec{V}_B \cdot \nabla \vec{v}_D + \nu \nabla \vec{V}_B + \nabla p_D = \nabla^2 \vec{v}_D, \quad (7.7.3a)
\]

\[
\nabla \cdot \vec{v}_D = 0. \quad (7.7.3b)
\]

Mechanical energy functional is constructed by taking a scalar product of the momentum equation with the disturbance velocity vector and integrating it over a control volume extending across the channel and over one wavelength in the \( x \)- and \( z \)-directions. The resulting relation expressed in index notation has the form
\[
\int_{-1}^{1} \int_{0}^{\lambda_x} \int_{0}^{\lambda_y} \left( \frac{1}{2} \frac{\partial}{\partial t} (v_i v_i) + v_i v_j \frac{\partial v_i}{\partial x_j} + v_i v_j \frac{\partial v_j}{\partial x_i} + v_i \frac{\partial p_d}{\partial x_i} \right) dx \, dz \, dy = \\
\int_{-1}^{1} \int_{0}^{\lambda_x} \int_{0}^{\lambda_y} (\nabla^2 v_i) \cdot v_i \, dx \, dz \, dy
\]

where \( \bar{v}_D = (v_1, v_2, v_3) \). The use of the periodicity conditions and Green's theorem leads to

\[
\int_{-1}^{1} \int_{0}^{\lambda_x} \int_{0}^{\lambda_y} \frac{1}{2} \frac{\partial}{\partial t} (v_i v_i) \, dx \, dz \, dy = -\int_{-1}^{1} \int_{0}^{\lambda_x} \int_{0}^{\lambda_y} \frac{\partial v_i}{\partial x_j} \frac{\partial v_j}{\partial x_i} \, dx \, dz \, dy - \\
\int_{-1}^{1} \int_{0}^{\lambda_x} \int_{0}^{\lambda_y} v_i v_j \frac{\partial v_i}{\partial x_j} \, dx \, dz \, dy.
\]

Growth of disturbances requires the right-hand side to be positive. Since the first term of the right-hand side describes dissipation, which is always positive, the second term determines the stability conditions. The explicit form of this term expressed using the notation used in the stability analysis described previously has the form

\[
\int_{-1}^{1} \int_{0}^{\lambda_x} \int_{0}^{\lambda_y} (u_D u_D \frac{\partial u_B}{\partial x} + u_D v_D \frac{\partial u_B}{\partial y} + v_D u_D \frac{\partial v_B}{\partial x} + v_D v_D \frac{\partial v_B}{\partial y} + \\
w_D u_D \frac{\partial w_B}{\partial x} + w_D v_D \frac{\partial w_B}{\partial y}) dx \, dz \, dy.
\]

Equation (7.7.6) was simplified by taking advantage of the independence of the stationary state on \( z \). Negative terms in (7.7.6) are responsible for potential instability. Detailed analysis was carried out with eigenfunctions normalized by setting dissipation integral to unity. Table 7.1 gives values of these terms for the TS instability at high \( Re_z \) (row 1), the new instability at large \( Re_z \) (row 2), and the new instability at low \( Re_z \) (row 3). Data in the first row demonstrate that the energy transfer from the stationary state to disturbances is, in the case of TS waves, driven by the vertical gradient of the streamwise velocity component (term 6: \( \frac{\partial w_B}{\partial y} \), streamwise shear) while spanwise shear provides a slight stabilization. The vertical shear is sufficiently strong to drive the instability, with other components providing marginal contributions. The onset of the new instability requires more vigorous heating at the same \( Re_z \), as illustrated by data in row 2. The energy transfer
is dominated by the spanwise gradient of the streamwise velocity component (term 5: $\frac{\partial w_B}{\partial x}$, spanwise shear). Again, the horizontal shear is sufficient to drive the instability by itself, with other components providing marginal contributions. Reduction of Re eliminates TS waves but does not eliminate the new instability.

Energy transfers in this case (row 3) are driven equally by the vertical and horizontal gradients of the streamwise velocity component (terms 5 and 6) and require the "cooperation" of both effects to overcome dissipation; none of these shears can drive the instability by itself. The remaining components provide a marginal contribution.

<table>
<thead>
<tr>
<th>Table 7.1 Energy transfer from the stationary state to disturbances.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Term</td>
</tr>
<tr>
<td>TS mode: $Re = 7000, Ra_p = 50$</td>
</tr>
<tr>
<td>New mode: $Re = 7000, Ra_p = 200$</td>
</tr>
<tr>
<td>New mode: $Re = 350, Ra_p = 1000$</td>
</tr>
</tbody>
</table>

### 7.8 Heating at the Upper Wall

Consider switching heating from the lower to the upper wall. Re-write (7.2.2) as

$$\theta_L(x) = \frac{1}{2} Ra_{p,L} \cos(\alpha x), \quad \theta_U(x) = 0 \quad (7.7.1a,b)$$

189
and then formulate the new thermal boundary condition corresponding to the heated upper wall, i.e.,

\[
\theta_L(x) = 0, \quad \theta_U(x) = \frac{1}{2} Ra_{p,U} \cos(ax).
\] (7.7.2a,b)

The governing systems for the two problems are closely related. If we take the problem of a heated lower wall with \( Ra_{p,L} = B \) and \( Ra_{p,U} = 0 \) and then make the transformation \( Ra_{p,L} \rightarrow 0, \ Ra_{p,U} \rightarrow B, \ u \rightarrow -U, \ v \rightarrow -V, \ p \rightarrow P, \ \theta \rightarrow -\theta, \ x \rightarrow -X + \pi, \ y \rightarrow -Y, \) it can be shown that the underlying equations remain unchanged, while the thermal boundary conditions are reversed in sign. Given this relationship between the two cases, all the properties of the flow system created by heating the upper wall can be inferred directly from the results when the lower wall is heated.

### 7.9 Conclusions

The stability of horizontal channel flow was considered. The channel lower wall was exposed to a spanwise periodic heating characterized by the wave number \( \alpha \) and the amplitude expressed in terms of the periodic Rayleigh number \( Ra_{p,L} \). Heating created longitudinal rolls and streaks, which resulted in the flow three-dimensionalization. Analysis of the stability of this flow led to the identification of two instability modes. The first mode is the classical Tollmien-Schlichting (TS) waves which are stabilized by the heating and require a high Reynolds number for their activation. The new mode is a wave traveling streamwise with a velocity nearly three times larger than the TS wave, with the most intense motion occurring in the middle channel section. This wave is driven by an inviscid instability mechanism associated with inflection points in the spanwise distribution of the streamwise velocity component and can persist to Reynolds numbers \( Re_z = 0(100) \) if sufficiently intense heating is used. The most effective heating wavenumber for the creation of this instability is \( \alpha \in (0.8, 0.9) \). The critical wave has the streamwise wavenumber \( \mu \in (0.6, 0.7) \). It is shown that the primary role of the heating was to set up the three-dimensional flow structures whose instability is driven by the streamwise
and spanwise gradients of the streamwise velocity component. The flow response is the same regardless of whether the heating is applied to the lower or upper plates. This instability has many potential applications, as its intensity and onset can be controlled by controlling the heating intensity and spatial distribution. Detailed results were presented for the fluid’s Prandtl number $Pr = 0.71$. 
Chapter 8

8 Conclusions and Recommendations

8.1 Conclusions

In this dissertation, the effects of spatially modulated flow have been numerically analyzed. Modulations in the form of spanwise differential heating and surface roughness are considered. As the first step, a spectrally accurate algorithm has been developed to analyze flows in heated grooved channels periodic in the spanwise direction. The algorithm can model any form of heating and roughness patterns. It uses the combination of Fourier expansions in the horizontal direction and Chebyshev polynomials in the vertical direction to discretize the field equations and boundary conditions. Fourier expansion satisfies the spanwise periodicity, while the Chebyshev polynomial provides an excellent resolution of near-wall regions. The solution is carried out in a fixed computational domain, and irregularities at the boundaries are submerged within the domain. The Immersed Boundary Condition (IBC) method is implemented to enforce the boundary conditions, which is conceptually different and provides closing conditions for the field equations in terms of constraints. The Galerkin projection method is imposed to construct the linear algebraic equations for the Chebyshev expansion coefficients.

Simultaneously, the boundary conditions are incorporated into the system of equations following the Tau concept. The resulting nonlinear system of algebraic equations is solved iteratively, assuming the nonlinear terms are known. It requires an initial approximation for the nonlinear terms to initiate the iteration process. The system is solved with an initial guess, and the solution is used to approximate the unknowns better. The process continues until the desired convergence is achieved. The rearrangement of the coefficient matrix structure gains significant efficiencies. Various tests ensure the spectral rate of convergence of the algorithm. However, the absolute error increases with an increase in roughness amplitude. Test results show that errors associated with the larger groove
amplitude can be partially controlled using more Fourier modes and Chebyshev polynomials. In particular, the algorithm eliminates the requirement of mesh generation. This gridless identity presents the opportunity to efficiently analyze numerous possible heating and roughness profiles which is imperative to reach general conclusions about modulated flows. Comparison with the standard open-source software clearly demonstrates the advantage over execution time and memory size.

A highly accurate and fully implicit time-dependent algorithm has been developed for the transient analysis in a non-isothermal corrugated channel. The algorithm can be utilized to study the effect of temporal thermal modulations, together with chaos and transition to the secondary states. Concurrently, it can be employed as an alternative iteration scheme for stationary heating problems while verifying the numerical results obtained from the first algorithm. The algorithm implements spectral spatial discretization in the form of Fourier expansions and Chebyshev polynomials and uses up to sixth-order temporal discretization. Information about irregular boundaries is provided in terms of Fourier coefficients, thereby eliminating the need for grid generation. Time derivatives are discretized using the backward finite-difference formulae, which requires information about unknowns at several prior time steps for higher-order methods. As a result, computations for higher-order methods primarily start with the first-order method, which only requires knowledge about one previous time step. The availability of information about multiple previous steps can readily implement higher-order schemes. Numerical procedures to generate the algebraic equations are similar to that used in the first algorithm. The test illustrates that the algorithm delivers the theoretically predicted accuracy in the event of time-periodic heating. The absolute error reduces significantly with an increase in the order of temporal discretization. Sufficient accuracy can be achieved in the sixth-order method even with the time step of 0.01. Variation of boundary errors as functions of Fourier modes and Chebyshev polynomials confirms the spectral convergence. It is also clear from the test that the use of higher groove amplitude as well as heating intensity demands high numbers of Fourier modes and Chebyshev polynomials. Computational time has been significantly improved using extrapolation for a better approximation of nonlinear terms. To advance the solution by one step, the use of the second-order extrapolation method cuts the computational cost by approximately 33%, the third-order extrapolation method cuts the
cost by 39%, the fourth-order extrapolation method cuts the cost by 58%, the fifth-order extrapolation method cuts the cost by 69%, and sixth-order extrapolation method cuts the cost by 84%. The algorithm performs better over the finite volume and spectral element-based open-source codes in terms of user involvement.

The use of periodic heating for the creation of rolls and streaks has been analyzed. Pressure-driven reference flow is considered in a smooth channel. At first, uniform heating is applied at the lower wall to mimic the classical RB convection, where critical conditions dictate the roll’s structure. An increase in heating intensity beyond the critical Rayleigh number leads to a hike in streaks’ strength as well as flow losses. Heat transfer between the walls improves since the rolls create transverse motion in the system. Streaks can be created in a more controlled manner by employing sinusoidal heating at the lower wall. Spanwise temperature gradient leads to the formation of rolls regardless of the heating intensity, while the heating wave number of $O(1)$ results in the most intense streaks. Excessively small and large heating wavelengths lessen the usefulness. The streak intensity increases with the periodic Rayleigh number, and the maximum strength is observed on the top of the hot spots. A local RB instability can be spotted for very small heating wavenumber if the periodic Rayleigh number is too large. Flow losses increase for all the combinations of heating wave numbers and heating intensities, reaching a peak at $\alpha \approx 1.5$. Maximum heat flow occurs at $\alpha \approx 1$ for sinusoidal heating, rapidly decreasing when $\alpha$ increases and decreases. The response remains similar should the sinusoidal heating is applied at the upper wall.

The addition of a uniform heating component and sinusoidal heating can simply double the spanwise shear. Adding uniform heating components monotonically increases the flow losses. The potential of the sinusoidal heating at both walls has been assessed. The relative position between two heating patterns is vital in determining the rolls’ structure. The streak strength decreases when the hot spots at the upper wall are precisely located on top of the lower wall cold spots. Maximum strength reaches when the upper wall hot spots align with the lower wall hot spots. Detuning between upper and lower wall heating patterns has also been examined. It is concluded that the formation of strong regular streaks needs a well-tuned heating pattern. Thermal properties of the fluids affect the creation of streaks. An
increase in the $Pr$ number reduces streaks’ strength, with $Pr > 10$ completely eradicating the effect of all heating arrangements.

The use of spanwise grooves at the lower wall to intensify streaks created by heating in shear layers has been investigated. At first, the isothermal groove is considered where the lower wall temperature is higher than the temperature of the upper wall. When the groove wave number is identical to the critical wavenumber of the RB instability, a rapid increment occurs in kinetic energy as $Ra_{uni}$ approaches the critical value for RB configuration. The maximum spanwise gradient of the streamwise velocity component also demonstrates a similar response. The streaks’ strength decreases for small $\alpha$ owing to small spanwise temperature gradients when the flow losses and heat transfer are maximum for the most substantial streaks. Long wavelength grooves are of interest due to their drag-reducing characteristic. It is shown that heating of such grooves produces weak streaks with marginal change in the flow losses. The effectiveness of streak formation increases with groove amplitude for all ranges of $\alpha$’s. The heated groove evidently exhibits an advantage compared to the uniformly heated smooth channel.

A combination of streamwise groove and heating patterns activates the pattern interaction effect and creates stronger streaks. Net spanwise flow is achieved in this case either to the left or right based on the relative position of both patterns. The pattern interaction effect is not activated in cases where the hot spots are located on the peak or bottom of the groove. The hot spot must be placed halfway between the groove trough and crest for relatively small heating to achieve the strongest streaks. For higher heating intensity, the preferable position of hot spots depends on the wavenumbers. Variations of kinetic energy and spanwise velocity gradient as a function of wavenumber indicate that the optimal wavenumber lies around $\alpha \approx 1$ for all $\Omega_{TL}$. The intense steak is produced at $\alpha \approx 1$ with smaller flow losses than the maximum losses.

Finally, the effect of combined periodic and uniform heating has been addressed. The use of $\alpha$ in RB zones depicts a jump in kinetic energy as the uniform Rayleigh number increases. However, the difference in streaks’ strength is marginal when a comparison is made with the periodically heated smooth channels. Variation of spanwise velocity
gradient shows a similar trend. It is expected that a combination of heating and groove should be a powerful tool for mixing intensification.

A linear stability algorithm suitable for analyzing the stability characteristic of modulated shear layers has been developed. The algorithm can handle modulations created by surface topography and heating patterns. It uses Fourier expansions and Chebyshev polynomials for discretization in horizontal and wall-normal directions, respectively. The linear system of algebraic equations for unknown coefficients is formed by the Galerkin projection method, and irregularities at the boundaries are treated with Immersed Boundary Condition method. Including homogeneous boundary conditions into the system of algebraic equations results in an eigenvalue problem, and it is solved using standard methods. Various tests demonstrate the exponential convergence of the computed eigenvalues and eigenfunctions with an increase in Fourier Modes and Chebyshev polynomials. The algorithm bypasses the difficulties associated with DNS in investigating the transition to the secondary state.

The stability of channel flow periodically heated from the below has been analyzed. It is found that heating stabilizes the classical Tollmien-Schlichting (TS) wave; however, it also introduces a new instability mode associated with the inflection point. This new mode exists at a very low Reynolds number compared to the critical Reynolds number for Tollmien-Schlichting instability. A value of $Ra_{p,\mu} = 1800$ can remarkably reduce the critical Reynolds number to $Re_{crit} \approx 260$. The distribution of disturbance quantities illustrates intense activity in the middle of the channel for the new mode. At the same time, movements are mainly concentrated in the near wall region for heating modified TS wave. Dominant disturbance motion occurs in the $x$-$z$ plane and $y$-$z$ plane for the new instability mode and modified TS mode, respectively. A comparison of spectrums exhibits thicker branches for the new mode. In the zero heating (isothermal) limit, the new instability mode connects to the square spectrum, whereas the modified TS mode connects to the Orr-Sommerfeld (OS) spectrum. Further analysis shows that two-dimensional disturbance plays a critical role with $\mu \in (1\sim1.2)$ being the most unstable. The critical heating wavenumber for the onset of instability has also been determined.
Additional analysis confirms that an inviscid mechanism drives the new instability mode. Reduction of viscosity eventually stabilizes viscous shear-driven instability. At the same time, the growth rate of the new mode increases with the reduction of viscosity before it becomes independent when the dissipation is negligible. It has been shown that the primary role of heating is to set a velocity field prone to instabilities. Discarding the energy equation from the stability analysis does not necessarily alter the qualitative responses. A detailed energy analysis has been carried out to find the gradients responsible for the potential instability. It is concluded that streamwise shear (vertical gradient of the streamwise velocity component) plays a vital role in the case of TS instability. On the other hand, the spanwise gradient of the streamwise velocity component is the dominant mean of energy transfer for the new mode at a high Reynolds number, while both vertical and spanwise gradients almost equally contribute to overcoming dissipation at a low Reynolds number. Finally, it is presented that placing heating at the upper wall results in exactly similar responses.

8.2 Recommendations for Future Work

Spatially modulated flows and their transition to the secondary states have been analyzed in this dissertation. To advance the understanding of this particular class of flows, the following suggestions can be considered for future work:

- The present work investigates the effect of transverse roughness and heating patterns (modulations do not change in the flow direction). The analysis can further be extended to the modulation that simultaneously varies in the spanwise (perpendicular to the flow) and streamwise (parallel to the flow) directions. The current algorithm can easily be adjusted to incorporate streamwise variation without conceptual complexity. However, the computation cost may rise due to the larger size of the coefficient matrix.
• Time-dependent temperature profiles are often encountered in nature. The transient algorithm proposed in this dissertation can be implemented to investigate flows subjected to time-dependent boundary conditions.

• The use of heated grooves creates intense streaks. Analysis has been carried out only for the cases where the lower wall is equipped with heating and roughness profiles. The placement of heated grooves at both walls can produce stronger streaks with smaller flow losses.

• Stability analysis of the thermally modulated flow in a smooth channel has been analyzed. It is found that a new instability mode exists when the sinusoidal heating profile is applied at the lower wall. Literature shows that the isothermal groove can also achieve this new mode in a much weaker form, which is sufficient to produce chaotic stirring. As a result, it is tempting to infer that pattern heating can lead to chaotic mixing, but the simulation of saturation states must support this statement. It is imperative to analyze the saturation state to confirm the usefulness of patterned heating as a noble mixing technique.

• The stability algorithm, developed as a part of this dissertation, can be implemented to study the transition to the secondary states for a system facilitated with pattern heating applied at both walls, the combination of uniform and periodic heating, and heated grooves. Based on the analysis of the stationary state, one can assume that these systems will remarkably reduce the critical heating intensity to achieve instability at much lower Reynolds numbers.

• The stability algorithm can also be implemented to study the Prandtl number dependence of the secondary states, which has the potential to unveil interesting flow physics.
References


Gepner, S. W., & Floryan, J. M. (2020). Use of Surface Corrugations for Energy-Efficient Chaotic Stirring in Low Reynolds Number Flows. *Scientific Reports, 10*(1). [https://doi.org/10.1038/s41598-020-66800-5](https://doi.org/10.1038/s41598-020-66800-5)


Appendices

Appendix A

Appendix A: Evaluation of the Inner Products

Chebyshev polynomials of the first kind are defined as

\[ T_0(\hat{y}) = 1, \quad T_1(\hat{y}) = \hat{y}, \quad T_{k+1}(\hat{y}) = 2\hat{y}T_k(\hat{y}) - T_{k-1}(\hat{y}) \quad \text{for} \quad k \geq 1. \quad (A.1) \]

Derivatives of Chebyshev polynomials can be expressed in terms of Chebyshev polynomials:

\[ DT_k = 2k \sum_{m=0}^{N_k-1} \frac{1}{c_m} T_m, \quad k - m = odd, \quad k \geq m + 1, \quad (A.2) \]

\[ D^2T_k = k \sum_{m=0}^{N_k-2} \frac{1}{c_m} (k^2 - m^2) T_m, \quad k - m = even, \quad k \geq m + 2, \quad (A.3) \]

\[ D^3T_k = D(D^2T_k) = \]

\[ 2k \sum_{m=0}^{N_k-2} \frac{1}{c_m} m (k^2 - m^2) \sum_{n=0}^{N_k-1} \frac{1}{c_n} T_n, \]

\[ k - m = even, \quad k \geq m + 2, \quad m - n = odd, \quad m \geq n + 1, \quad (A.4) \]

\[ D^4T_k = D^2(D^2T_k) = \]

\[ k \sum_{m=0}^{N_k-2} \frac{1}{c_m} m (k^2 - m^2) \sum_{n=0}^{N_k-2} \frac{1}{c_n} (m^2 - n^2) T_n, \]

\[ k - m = even, \quad k \geq m + 2, \quad m - n = even, \quad m \geq n + 2. \quad (A.5) \]

In the above, \( C_k = 2 \) when \( k = 0 \) and \( C_k = 1 \) when \( k \geq 1 \). Inner product of two Chebyshev polynomials is defined as

\[ < T_j(\hat{y}), T_k(\hat{y}) > = \int_{-1}^{1} T_j(\hat{y}) T_k(\hat{y}) \omega(\hat{y}) d\hat{y} \quad (A.6) \]
where \( \omega(\hat{y}) = \frac{1}{\sqrt{1-\hat{y}^2}} \) is the weight function. The orthogonality properties of Chebyshev polynomials result in the following properties of the inner product:

\[
<T_j, T_k> = \frac{\pi}{2} C_k \delta_{j,k} \begin{cases} 
\pi, & j = k = 0, \\
\frac{\pi}{2}, & j = k \geq 1, \\
0, & j \neq k
\end{cases}
\]

(A.7)

where \( \delta_{j,k} \) is the Kronecker delta. Inner product of the first derivative of Chebyshev polynomial with Chebyshev polynomial \( <T_j, DT_k> \) can be evaluated by inserting (A.2) into its definition and taking advantage of (A.7) resulting in

\[
<T_j, DT_k> = k \pi, \quad k - j = odd, \quad k \geq j + 1.
\]

(A.8)

Inner product of the second derivative of Chebyshev polynomial with Chebyshev polynomial \( <T_j, D^2T_k> \) can be evaluated by inserting (A.3) and taking advantage of (A.7) resulting in

\[
<T_j, D^2T_k> = k(k^2 - j^2) \frac{\pi}{2}, \quad k - j = even, \quad k \geq j + 2.
\]

(A.9)

Inner product of the third derivative of Chebyshev polynomial with Chebyshev polynomial \( <T_j, D^3T_k> \) can be evaluated as by inserting (A.4) into its definition and taking advantage of (A.7) resulting in

\[
<T_j, D^3T_k> = k \pi \sum_{n=0}^{N_k-2} \frac{n}{c_n} (k^2 - n^2),
\]

\[
k - n = even, \quad k \geq n + 2; \quad n - j = odd, \quad n \geq j + 1.
\]

(A.10)

Inner products of the fourth derivative of Chebyshev polynomial with Chebyshev polynomial \( <T_j, D^4T_k> \) can be evaluated by inserting (A.5) into its definitions and taking advantage of (A.7) resulting in

\[
<T_j, D^4T_k> = \frac{k \pi}{2} \sum_{n=0}^{N_k-2} \frac{1}{c_n} n(k^2 - n^2)(n^2 - j^2),
\]

\[
k - n = even, \quad k \geq n + 2 \geq j + 4; \quad n - j = even, \quad n \geq j + 2.
\]

(A.11)
Inner product of Chebyshev polynomial with a product of Chebyshev polynomial and derivative of Chebyshev polynomial is defined as

\[ < T_j, DT_l * T_k > = \int_{-1}^{+1} T_j DT_l T_k \omega \, dy. \] (A.12)

Use of the Chebyshev multiplication property \( T_j T_k = \frac{1}{2} (T_{j+k} + T_{j-k}) \) leads to

\[ < T_j, DT_l * T_k > = \frac{1}{2} \left[ \int_{-1}^{+1} T_{j+k} DT_l \omega \, dy + \int_{-1}^{+1} T_{j-k} DT_l \omega \, dy \right] \] (A.13)

which can be expressed using (A.6) as

\[ < T_j, DT_l * T_k > = \frac{1}{2} \left[ < T_{j+k}, DT_l > + < T_{j-k}, DT_l > \right]. \] (A.14)

Similarly

\[ < T_j, T_l * DT_k > = \frac{1}{2} \left[ < T_{j+l}, DT_k > + < T_{j-l}, DT_k > \right], \] (A.15)

\[ < T_j, D^2 T_l * T_k > = \frac{1}{2} \left[ < T_{j+k}, D^2 T_l > + < T_{j-k}, D^2 T_l > \right], \] (A.16)

\[ < T_j, T_l * D^2 T_k > = \frac{1}{2} \left[ < T_{j+l}, D^2 T_k > + < T_{j-l}, D^2 T_k > \right], \] (A.17)

\[ < T_j, DT_l * DT_k > = \frac{1}{4} \left[ < T_j, D^2 T_{k+l} > + < T_j, D^2 T_{k-l} > - < T_{j+k}, D^2 T_l > 
- < T_{j-k}, D^2 T_l > \right] - < T_{j+l}, D^2 T_k > - < T_{j-l}, D^2 T_k > \] (A.18)
Appendix B: Evaluation of Coefficients of Fourier Expansions Describing Variations of Values of Chebyshev Polynomials Evaluated along the Walls

Values of Chebyshev polynomials at the lower wall can be represented as a Fourier expansion of the form

\[ T_k[\hat{y}_L(x)] = \sum_{s=-\infty}^{+\infty} E_{L,k}^{(s)} e^{i s \beta x}. \]  

(B.1)

Evaluation of coefficients \((w_L)_k^{(s)}\) begins with the lowest order Chebyshev polynomial, i.e.

\[
T_0 = 1 \Rightarrow \sum_{s=-\infty}^{+\infty} E_{L,0}^{(s)} e^{i s \beta x} = 1 \Rightarrow \begin{cases} E_{L,0}^{(0)} = 1, \\ E_{L,0}^{(s)} = 0, s \neq 0, \end{cases} \]  

(B.2)

\[ T_1[\hat{y}_L(x)] = \hat{y}_L(x) \Rightarrow \sum_{s=-\infty}^{+\infty} E_{L,1}^{(s)} e^{i s \beta x} = \sum_{s=-\infty}^{+\infty} A_L^{(s)} e^{i s \beta x} \Rightarrow E_{L,1}^{(s)} = \hat{y}_L^{(s)}. \]  

(B.3)

The remaining coefficients \(E_{L,k}^{(s)} (k \geq 2)\) can be computed using the Chebyshev recursion relation which results in

\[
E_{L,k+1}^{(s)} = 2 \sum_{n=-\infty}^{+\infty} A_L^{(n)} E_{L,k}^{(s-n)} - E_{L,k-1}^{(s)}, \quad k \geq 2. \]  

(B.4)

The first derivative of Chebyshev polynomials is represented as a Fourier expansion of the form

\[ DT_k[\hat{y}_L(x)] = \sum_{s=-\infty}^{+\infty} d_{L,k}^{(s)} e^{i s \beta x}. \]  

Evaluation of coefficients \((d_L)_k^{(s)}\) begins with the lowest order polynomial, i.e.

\[
DT_0 = 0 \Rightarrow \sum_{s=-\infty}^{+\infty} d_{L,0}^{(s)} e^{i s \beta x} = 0 \Rightarrow d_{L,0}^{(s)} = 0, \]  

(B.5)

\[
DT_1 = 1 \Rightarrow \sum_{s=-\infty}^{+\infty} d_{L,1}^{(s)} e^{i s \beta x} = 1 \Rightarrow \begin{cases} d_{L,1}^{(0)} = 1, \\ d_{L,1}^{(s)} = 0, s \neq 0, \end{cases} \]  

(B.6)
\[ DT_2 [\tilde{\gamma}_L(x)] = 4 \tilde{\gamma}_L(x) \Rightarrow \sum_{\beta=-\infty}^{+\infty} d_{L,2}^{(s)} e^{is\beta x} = \sum_{\beta=-\infty}^{+\infty} 4A_{L}^{(s)} e^{is\beta x} \Rightarrow d_{L,2}^{(s)} = 4A_{L}^{(s)}. \] (B.7)

The remaining coefficients \( d_{L,k}^{(s)} (k \geq 3) \) can be computed using the Chebyshev recursive formula and have the following form
\[ d_{L,k+1}^{(s)} = 2 \sum_{n=-\infty}^{+\infty} A_{L}^{(n)} d_{L,k}^{(s-n)} - d_{L,k-1}^{(s)} + 2E_{L,k}, \quad k \geq 3. \] (B.8)

The second derivative of Chebyshev polynomials is represented as a Fourier expansion of the form
\[ D^2 T_k [\tilde{\gamma}_L(x)] = \sum_{\beta=-\infty}^{+\infty} R_{L,k}^{(s)} e^{is\beta x}. \] (B.9)

Evaluation of coefficients \( R_{L,k}^{(s)} \) begins with the lowest order polynomial, i.e.
\[ D^2 T_0 = 0 \Rightarrow \sum_{\beta=-\infty}^{+\infty} R_{L,0}^{(s)} e^{is\beta x} = 0 \Rightarrow R_{L,0}^{(s)} = 0, \] (B.10)
\[ D^2 T_1 = 0 \Rightarrow \sum_{\beta=-\infty}^{+\infty} R_{L,1}^{(s)} e^{is\beta x} = 0 \Rightarrow R_{L,1}^{(s)} = 0, \] (B.11)
\[ D^2 T_2 = 4 \Rightarrow \sum_{\beta=-\infty}^{+\infty} R_{L,2}^{(s)} e^{is\beta x} = 4 \Rightarrow \begin{cases} R_{L,2}^{(0)} = 4, \\ R_{L,2}^{(s)} = 0, s \neq 0. \end{cases} \] (B.12)

The remaining coefficients \( R_{L,k}^{(s)} (k \geq 2) \) can be computed using the Chebyshev recursive formula and have the following form
\[ R_{L,k+1}^{(s)} = 2 \sum_{n=-\infty}^{+\infty} A_{L}^{(s-n)} R_{L,k}^{(n)} + 4d_{L,k}^{(s)} - R_{L,k-1}^{(s)}, \quad k \geq 2. \] (B.13)

Replacement of labels \( L \) with labels \( U \) provides Fourier expansions describing values of Chebyshev polynomials at the upper wall.
Appendix C

Appendix C: Pressure Normalization

Pressure in (2.3.1) is normalized by bringing the mean value of its periodic component $P(x, \hat{y})$ to zero. The mean value is defined as

$$P_{\text{mean}} = \frac{1}{\text{Area}} \int_{0}^{\lambda x} \int_{y_l(x)}^{y_u(x)} P(x, y) \, dy \, dx = \frac{1}{\Gamma \cdot \text{Area}} \int_{0}^{\lambda x} \frac{\hat{y}}{y_l(x)} \hat{y} \, d\hat{y} \, dx \quad (C.1)$$

where $\text{Area} = \frac{1}{\Gamma} \int_{0}^{\lambda x} \int_{y_l(x)}^{y_u(x)} d\hat{y} \, dx$.

Substitution of (2.3.5) and (2.3.8) into (C.1) results in

$$P_{\text{mean}} = \frac{1}{\text{Area}} \int_{0}^{\lambda x} \sum_{m=-N_M}^{m=+N_M} \sum_{k=0}^{k=N_K-1} G P_k^{(n)} I_k(x) e^{imbx} \, dx \quad (C.2)$$

where $I_k(x) = \int_{y_l(x)}^{y_u(x)} T_k(\hat{y}) \, d\hat{y}$.

Integrals $I_k(x)$ can be evaluated analytically, i.e.

$$I_0(x) = T_1[\hat{y}_u(x)] - T_1[\hat{y}_l(x)] \quad , \quad I_1(x) = \frac{T_2[\hat{y}_u(x)] - T_2[\hat{y}_l(x)]}{4} \quad , \quad (C.3a,b)$$

$$I_k(x) = \frac{1}{2} \left\{ \frac{T_{k+1}[\hat{y}_u(x)] - T_{k+1}[\hat{y}_l(x)]}{k+1} - \frac{T_{k+1}[\hat{y}_u(x)] - T_{k+1}[\hat{y}_l(x)]}{k-1} \right\} \quad , \quad k > 1 \quad . \quad (C.3c)$$

Values of Chebyshev polynomials along the boundaries can be expressed using Fourier expansions as explained in Appendix B. Accordingly, (C.3) can be written as

$$I_k(x) = \sum_{s=-N_M}^{s=+N_M} \hat{I}_k^{(s)} e^{isbx} \quad (C.4a)$$

Where

$$\hat{I}_0^{(s)} = E_{U,1}^{(s)} - E_{L,1}^{(s)} \quad , \quad \hat{I}_1^{(s)} = \frac{1}{4} \left[ E_{U,2}^{(s)} - E_{L,2}^{(s)} \right]$$

$$\hat{I}_k^{(s)} = \frac{1}{2} \left[ \frac{E_{U,k+1}^{(s)} - E_{L,k+1}^{(s)}}{k+1} - \frac{E_{U,k-1}^{(s)} - E_{L,k-1}^{(s)}}{k-1} \right] \quad , \quad k > 1 \quad . \quad (C.4b)$$

with all $E_k^{(s)}$ given by (B.2) - (B.4). Substitution of (C.3) into (C.2) results in
\[ P_{\text{mean}} = \frac{1}{\text{Area}} \frac{1}{\lambda} \sum_{m=-N_M}^{m=N_M} \sum_{s=-N_M}^{s=N_M} \sum_{k=0}^{k=N_K-1} GP_k^{(s)} \hat{I}_k^{(m-s)} \int_0^{\lambda x} e^{im\beta x} dx \]  (C.5)

which, after evaluation of all integrals, reduces to

\[ P_{\text{mean}} = \frac{1}{\text{Area}} \frac{\lambda x}{\lambda} \sum_{s=-N_M}^{s=N_M} \sum_{k=0}^{k=N_K-1} GP_k^{(s)} \hat{I}_k^{(s)*}. \]  (C.6)

where star denotes complex conjugate. It can be shown that

\[ \overline{\text{Area}} = \frac{\lambda x}{\lambda} \left( A_U^{(0)} - A_L^{(0)} \right), \]  (C.7)

which leads to the final expression for the mean value of the form

\[ P_{\text{mean}} = \frac{1}{\overline{\text{Area}}} \sum_{s=-N_M}^{s=N_M} \sum_{k=0}^{k=N_K-1} GP_k^{(s)} \hat{I}_k^{(s)*}. \]  (C.8)

The expression of the periodic pressure component normalized by setting its mean value to zero is

\[ P(x, \hat{y}) = \sum_{m=-N_M}^{m=N_M} P^{(m)}(\hat{y}) e^{im\beta x} - \frac{1}{\overline{\text{Area}}} \sum_{s=-N_M}^{s=N_M} \sum_{k=0}^{k=N_K-1} GP_k^{(s)} \hat{I}_k^{(s)*}. \]  (C.9)
Appendix D: Discretization of the Fixed Flow Rate Constraint in the Streamwise Flow

Implementation of the fixed flow rate constraint for the streamwise flow begins with (2.3.29b) with the unknown mean pressure gradient \( B \) placed on the left-hand side resulting in

\[
\sum_{k=0}^{N_K-1} \left\{ n \Gamma^2 \left< T_j, D^2 T_k \right> \bar{G}_k^{(0)} - i \Gamma \beta \sum_{n=-N_m}^{N_m} \sum_{l=0}^{N_K-1} n G \psi_l^{(n)} \left< T_j, DT_l T_k \right> \right.
\]

\[
\left. + \left< T_j, T_l DT_k \right> \right\} \bar{G}_k^{(n)} - B < T_j, T_0 >= 0, \tag{D.1}
\]

The additional equation required to close the system comes from the flow rate constraint (2.2.12) which has the following form

\[
\bar{Q}_z = \left( \int_{\bar{y}_L(x)}^{\bar{y}_U(x)} w \ dy \ dx \right)_{\text{mean}} = \frac{1}{r} \left( \int_{\bar{y}_L(x)}^{\bar{y}_U(x)} w \ dy \ dx \right)_{\text{mean}}. \tag{D.2}
\]

Insertion of (2.3.26) and (2.3.28) into (D.2) results in

\[
\bar{Q}_z = \left[ \frac{1}{r} \sum_{m=-N_M}^{N_M} \sum_{k=0}^{N_K-1} Gw_k^{(m)} \int_0^{\lambda x} l_k(x) e^{im\beta x} \ dx \right]_{\text{mean}}. \tag{D.3}
\]

where \( l_k(x) = \int_{\bar{y}_L(x)}^{\bar{y}_U(x)} T_k(\bar{y})d\bar{y} \).

The integrals are evaluated analytically in the same manner as in Appendix C.

Insertion of (C.4) into (D.3) leads to

\[
\bar{Q}_z = \left[ \frac{1}{r} \sum_{m=-N_M}^{N_M} \sum_{s=-N_M}^{N_M} \sum_{k=0}^{N_K-1} Gw_k^{(s)} \int_0^{\lambda x} e^{im\beta x} \ dx \right]_{\text{mean}}. \tag{D.4}
\]

Evaluation of integrals in (D.4) and extraction of mode zero give the final form of this constraint suitable for the numerical implementation, i.e.

\[
\bar{Q}_z = \frac{1}{r} \sum_{s=-N_M}^{N_M} \sum_{k=0}^{N_K-1} Gw_k^{(s)} \hat{l}_k^{(s)*}. \tag{D.5}
\]
The above equation is added to the algebraic system (2.3.33) and the velocity $w$ and the mean pressure gradient $B$ are determined simultaneously.
Appendix E

Appendix E: Temporal Discretization

First-order (one-step) method:

\[
\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} + \frac{\partial w_1}{\partial z} = 0, \tag{E.1}
\]

\[
\Delta t^{-1} u_1 + u_1 \frac{\partial u_1}{\partial x} + v_1 \frac{\partial u_1}{\partial y} + w_1 \frac{\partial u_1}{\partial z} + \frac{\partial p_1}{\partial x} - \nabla^2 u_1 = \Delta t^{-1} u_0, \tag{E.2}
\]

\[
\Delta t^{-1} v_1 + u_1 \frac{\partial v_1}{\partial x} + v_1 \frac{\partial v_1}{\partial y} + w_1 \frac{\partial v_1}{\partial z} + \frac{\partial p_1}{\partial y} - \nabla^2 v_1 - Pr^{-1} \theta_1 = \Delta t^{-1} v_0, \tag{E.3}
\]

\[
\Delta t^{-1} w_1 + u_1 \frac{\partial w_1}{\partial x} + v_1 \frac{\partial w_1}{\partial y} + w_1 \frac{\partial w_1}{\partial z} + \frac{\partial p_1}{\partial z} - \nabla^2 w_1 = \Delta t^{-1} w_0, \tag{E.4}
\]

\[
\Delta t^{-1} \theta_1 + u_1 \frac{\partial \theta_1}{\partial x} + v_1 \frac{\partial \theta_1}{\partial y} + w_1 \frac{\partial \theta_1}{\partial z} - Pr^{-1} \nabla^2 \theta_1 = 3 \Delta t^{-1} \theta_0, \tag{E.5}
\]

\[
u_1(y_L(x)) = u_1(y_U(x)) = 0, \quad v_1(y_L(x)) = v_1(y_U(x)) = 0, \tag{E.6,7}
\]

\[
w_1(y_L(x)) = w_1(y_U(x)) = 0, \tag{E.8}
\]

\[
\theta_1(y_L(x)) = \theta_L(x, t_1), \quad \theta_1(y_U(x)) = \theta_U(x, t_1), \tag{E.9,10}
\]

\[
\left( \frac{\partial p_1}{\partial x} \right)_{\text{mean}} = \delta \phi, \quad \left( \frac{\partial p_1}{\partial z} \right)_{\text{mean}} = \delta \phi, \tag{E.11,12}
\]

\[
\left[ \int_{y_L(x)}^{y_U(x)} u_1(x,y)\,dy \right]_{\text{mean}} = Q_x, \quad \left[ \int_{y_L(x)}^{y_U(x)} w_1(x,y)\,dy \right]_{\text{mean}} = Q_z. \tag{E.13,14}
\]

Second-order (two-step) method:

\[
\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} + \frac{\partial w_1}{\partial z} = 0, \tag{E.15}
\]

\[
\frac{3}{2} \Delta t^{-1} u_1 + u_1 \frac{\partial u_1}{\partial x} + v_1 \frac{\partial u_1}{\partial y} + w_1 \frac{\partial u_1}{\partial z} + \frac{\partial p_1}{\partial x} - \nabla^2 u_1 = 2 \Delta t^{-1} u_0 - \frac{1}{2} \Delta t^{-1} u_{-1}, \tag{E.16}
\]
\[
\frac{3}{2} \Delta t^{-1} v_1 + u_1 \frac{\partial v_1}{\partial x} + v_1 \frac{\partial v_1}{\partial y} + w_1 \frac{\partial v_1}{\partial z} + \frac{\partial p_1}{\partial y} - \nabla^2 v_1 - P r^{-1} \theta_1 =
\]
\[
2 \Delta t^{-1} v_0 - \frac{1}{2} \Delta t^{-1} v_{-1}, \quad (E.17)
\]
\[
\frac{3}{2} \Delta t^{-1} w_1 + u_1 \frac{\partial w_1}{\partial x} + v_1 \frac{\partial w_1}{\partial y} + w_1 \frac{\partial w_1}{\partial z} + \frac{\partial p_1}{\partial y} - \nabla^2 w_1 = 2 \Delta t^{-1} w_0 - \frac{1}{2} \Delta t^{-1} w_{-1}, \quad (E.18)
\]
\[
\frac{3}{2} \Delta t^{-1} \theta_1 + u_1 \frac{\partial \theta_1}{\partial x} + v_1 \frac{\partial \theta_1}{\partial y} + w_1 \frac{\partial \theta_1}{\partial z} - P r^{-1} \nabla^2 \theta_1 = 2 \Delta t^{-1} \theta_0 - \frac{1}{2} \Delta t^{-1} \theta_{-1}, \quad (E.19)
\]
\[
u_1(y_L(x)) = u_1(y_U(x)) = 0, \quad v_1(y_L(x)) = v_1(y_U(x)) = 0, \quad (E.20,21)
\]
\[
w_1(y_L(x)) = w_1(y_U(x)) = 0, \quad (E.22)
\]
\[
\theta_1(y_L(x)) = \theta_i(x,t_1), \quad \theta_1(y_U(x)) = \theta_U(x,t_1). \quad (E.23,24)
\]
\[
\left(\frac{\partial p_1}{\partial x}\right)_{\text{mean}} = \varphi_x, \quad \left(\frac{\partial p_1}{\partial z}\right)_{\text{mean}} = \varphi_z, \quad (E.25,26)
\]
\[
\left[\int_{y_L(x)}^{y_U(x)} u_1(x,y)dy\right]_{\text{mean}} = \varphi_x, \quad \left[\int_{y_L(x)}^{y_U(x)} w_1(x,y)dy\right]_{\text{mean}} = \varphi_z. \quad (E.27,28)
\]

Forth-order (four-step) method:
\[
\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} + \frac{\partial w_1}{\partial z} = 0, \quad (E.29)
\]
\[
\frac{25}{12} \Delta t^{-1} u_1 + u_1 \frac{\partial u_1}{\partial x} + v_1 \frac{\partial u_1}{\partial y} + w_1 \frac{\partial u_1}{\partial z} + \frac{\partial p_1}{\partial x} - \nabla^2 u_1 =
\]
\[
4 \Delta t^{-1} u_0 - 3 \Delta t^{-1} u_{-1} + \frac{4}{3} \Delta t^{-1} u_{-2} - \frac{1}{4} \Delta t^{-1} u_{-3}, \quad (E.30)
\]
\[
\frac{25}{12} \Delta t^{-1} v_1 + u_1 \frac{\partial v_1}{\partial x} + v_1 \frac{\partial v_1}{\partial y} + w_1 \frac{\partial v_1}{\partial z} + \frac{\partial p_1}{\partial y} - \nabla^2 v_1 - P r^{-1} \theta_1 =
\]
\[
4 \Delta t^{-1} v_0 - 3 \Delta t^{-1} v_{-1} + \frac{4}{3} \Delta t^{-1} v_{-2} - \frac{1}{4} \Delta t^{-1} v_{-3}, \quad (E.31)
\]
\[
\frac{25}{12} \Delta t^{-1} w_1 + u_1 \frac{\partial w_1}{\partial x} + v_1 \frac{\partial w_1}{\partial y} + w_1 \frac{\partial w_1}{\partial z} + \frac{\partial p_1}{\partial z} - \nabla^2 w_1 =
\]

\[ 4\Delta t^{-1}w_0 - 3\Delta t^{-1}w_1 + \frac{4}{3}\Delta t^{-1}w_2 - \frac{1}{4}\Delta t^{-1}w_3, \quad (E.32) \]

\[ \frac{25}{12}\Delta t^{-1}\theta_1 + u_1 \frac{\partial \theta_1}{\partial x} + v_1 \frac{\partial \theta_1}{\partial y} + w_1 \frac{\partial \theta_1}{\partial z} - P_r^{-1} \nabla^2 \theta_1 = 4\Delta t^{-1}\theta_0 - 3\Delta t^{-1}\theta_1 + \frac{4}{3}\Delta t^{-1}\theta_2 - \frac{1}{4}\Delta t^{-1}\theta_3, \quad (E.33) \]

\[ u_1(y_L(x)) = u_1(y_U(x)) = 0, \quad v_1(y_L(x)) = v_1(y_U(x)) = 0, \quad (E.34, 35) \]

\[ w_1(y_L(x)) = w_1(y_U(x)) = 0, \quad (E.36) \]

\[ \theta_1(y_L(x)) = \theta_L(x, t_1), \quad \theta_1(y_U(x)) = \theta_U(x, t_1), \quad (E.37, 38) \]

\[ \left( \frac{\partial p_1}{\partial x} \right)_{\text{mean}} = \varrho_x, \quad \left( \frac{\partial p_1}{\partial z} \right)_{\text{mean}} = \varrho_z, \quad (E.39, 40) \]

\[ \left[ f_{y_L(x)} u_1(x, y) dy \right]_{\text{mean}} = Q_x, \quad \left[ f_{y_L(x)} w_1(x, y) dy \right]_{\text{mean}} = Q_z. \quad (E.41, 42) \]

Fifth-order (five-step) method:

\[ \frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} + \frac{\partial w_1}{\partial z} = 0, \quad (E.43) \]

\[ \frac{137}{60} \Delta t^{-1} u_1 + u_1 \frac{\partial u_1}{\partial x} + v_1 \frac{\partial u_1}{\partial y} + w_1 \frac{\partial u_1}{\partial z} + \nabla^2 u_1 = 5\Delta t^{-1} u_0 - 5\Delta t^{-1} u_{-1} + \frac{10}{3} \Delta t^{-1} u_{-2} - \frac{5}{4} \Delta t^{-1} u_{-3} + \frac{1}{5} \Delta t^{-1} u_{-4}, \quad (E.44) \]

\[ \frac{137}{60} \Delta t^{-1} v_1 + u_1 \frac{\partial v_1}{\partial x} + v_1 \frac{\partial v_1}{\partial y} + w_1 \frac{\partial v_1}{\partial z} - \nabla^2 v_1 - P_r^{-1} \theta_1 = 5\Delta t^{-1} v_0 - 5\Delta t^{-1} v_{-1} + \frac{10}{3} \Delta t^{-1} v_{-2} - \frac{5}{4} \Delta t^{-1} v_{-3} + \frac{1}{5} \Delta t^{-1} v_{-4}, \quad (E.45) \]

\[ \frac{137}{60} \Delta t^{-1} w_1 + u_1 \frac{\partial w_1}{\partial x} + v_1 \frac{\partial w_1}{\partial y} + w_1 \frac{\partial w_1}{\partial z} - \nabla^2 w_1 = 5\Delta t^{-1} w_0 - 5\Delta t^{-1} w_{-1} + \]

223
\[ \frac{10}{3} \Delta t^{-1}w_{-2} - \frac{5}{4} \Delta t^{-1}w_{-3} + \frac{1}{5} \Delta t^{-1}w_{-4}, \]
(E.46)

\[ \frac{137}{60} \Delta t^{-1}\theta_1 + u_1 \frac{\partial \theta_1}{\partial x} + v_1 \frac{\partial \theta_1}{\partial y} + w_1 \frac{\partial \theta_1}{\partial z} - Pr^{-1} \nabla^2 \theta_1 = 5 \Delta t^{-1}\theta_0 - 5 \Delta t^{-1}\theta_{-1} + \]

\[ \frac{10}{3} \Delta t^{-1}\theta_2 - \frac{5}{4} \Delta t^{-1}\theta_3 + \frac{1}{5} \Delta t^{-1}\theta_{-4}, \]
(E.47)

\[ u_1(y_L(x)) = u_1(y_U(x)) = 0, \quad v_1(y_L(x)) = v_1(y_U(x)) = 0, \quad \text{E.48,49} \]

\[ w_1(y_L(x)) = w_1(y_U(x)) = 0, \quad \text{E.50} \]

\[ \theta_1(y_L(x)) = \theta_L(x,t_1), \quad \theta_1(y_U(x)) = \theta_U(x,t_1), \quad \text{E.51,52} \]

\[ \left( \frac{\partial p_1}{\partial x} \right)_{mean} = \varphi_x, \quad \left( \frac{\partial p_1}{\partial z} \right)_{mean} = \varphi_z, \quad \text{E.53,54} \]

\[ \left[ \int_{y_L(x)}^{y_U(x)} u_1(x,y)dy \right]_{mean} = \varphi_x, \quad \left[ \int_{y_L(x)}^{y_U(x)} w_1(x,y)dy \right]_{mean} = \varphi_z. \quad \text{E.55,56} \]

Sixth-order (six-step) method:

\[ \frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} + \frac{\partial w_1}{\partial z} = 0, \quad \text{E.57} \]

\[ \frac{49}{20} \Delta t^{-1}u_1 + u_1 \frac{\partial u_1}{\partial x} + v_1 \frac{\partial u_1}{\partial y} + w_1 \frac{\partial u_1}{\partial z} + \nabla^2 u_1 = 6 \Delta t^{-1}u_0 - \frac{15}{2} \Delta t^{-1}u_{-1} + \]

\[ \frac{20}{3} \Delta t^{-1}u_{-2} - \frac{15}{4} \Delta t^{-1}u_{-3} + \frac{6}{5} \Delta t^{-1}u_{-4} - \frac{1}{6} \Delta t^{-1}u_{-5}, \quad \text{E.58} \]

\[ \frac{49}{20} \Delta t^{-1}v_1 + u_1 \frac{\partial v_1}{\partial x} + v_1 \frac{\partial v_1}{\partial y} + w_1 \frac{\partial v_1}{\partial z} - \nabla^2 v_1 - Pr^{-1} \theta_1 = 6 \Delta t^{-1}v_0 - \]

\[ \frac{15}{2} \Delta t^{-1}v_{-1} + \frac{20}{3} \Delta t^{-1}v_{-2} - \frac{15}{4} \Delta t^{-1}v_{-3} + \frac{6}{5} \Delta t^{-1}v_{-4} - \frac{1}{6} \Delta t^{-1}v_{-5}, \quad \text{E.59} \]

\[ \frac{49}{20} \Delta t^{-1}w_1 + u_1 \frac{\partial w_1}{\partial x} + v_1 \frac{\partial w_1}{\partial y} + w_1 \frac{\partial w_1}{\partial z} - \nabla^2 w_1 = 6 \Delta t^{-1}w_0 - \frac{15}{2} \Delta t^{-1}w_{-1} + \]
\[
\frac{20}{3} \Delta t^{-1} w_{-2} - \frac{15}{4} \Delta t^{-1} w_{-3} + \frac{6}{5} \Delta t^{-1} w_{-4} - \frac{1}{6} \Delta t^{-1} w_{-5}, \quad (E.60)
\]

\[
\frac{49}{20} \Delta t^{-1} \theta_1 + u_1 \frac{\partial \theta_1}{\partial x} + v_1 \frac{\partial \theta_1}{\partial y} + w_1 \frac{\partial \theta_1}{\partial z} - Pr^{-1} \nabla^2 \theta_1 = 6 \Delta t^{-1} \theta_0 - \frac{15}{2} \Delta t^{-1} \theta_{-1} +
\]

\[
\frac{20}{3} \Delta t^{-1} \theta_{-2} - \frac{15}{4} \Delta t^{-1} \theta_{-3} + \frac{6}{5} \Delta t^{-1} \theta_{-4} - \frac{1}{6} \Delta t^{-1} \theta_{-5}, \quad (E.61)
\]

\[
u_1(y_L(x)) = u_1(y_U(x)) = 0, \quad v_1(y_L(x)) = v_1(y_U(x)) = 0, \quad (E.62, 63)
\]

\[
w_1(y_L(x)) = w_1(y_U(x)) = 0, \quad (E.64)
\]

\[
\theta_1(y_L(x)) = \theta_L(x, t_1), \quad \theta_1(y_U(x)) = \theta_U(x, t_1). \quad (E.65, 66)
\]

\[
\left( \frac{\partial p_1}{\partial x} \right)_{mean} = \varrho_x, \quad \left( \frac{\partial p_1}{\partial z} \right)_{mean} = \varrho_z, \quad (E.67, 68)
\]

\[
\left[ \int_{y_L(x)}^{y_U(x)} u_1(x, y) dy \right]_{mean} = Q_x, \quad \left[ \int_{y_L(x)}^{y_U(x)} w_1(x, y) dy \right]_{mean} = Q_z. \quad (E.69, 70)
\]
Appendix F

Appendix F: Extrapolation Formulae

Second-order extrapolation:

\[(N_{uu})_1 = 2(N_{uu})_0 - (N_{uu})_{-1} + O(h^2), \quad (F.1a)\]
\[(N_{u\theta})_1 = 2(N_{u\theta})_0 - (N_{u\theta})_{-1} + O(h^2). \quad (F.1b)\]

Fourth-order extrapolation:

\[(N_{uu})_1 = 4(N_{uu})_0 - 6(N_{uu})_{-1} + 4(N_{uu})_{-2} - (N_{uu})_{-3} + O(h^4), \quad (F.2a)\]
\[(N_{u\theta})_1 = 4(N_{u\theta})_0 - 6(N_{u\theta})_{-1} + 4(N_{u\theta})_{-2} - (N_{u\theta})_{-3} + O(h^4). \quad (F.2b)\]

Fifth-order extrapolation:

\[(N_{uu})_1 = 5(N_{uu})_0 - 10(N_{uu})_{-1} + 10(N_{uu})_{-2} - 5(N_{uu})_{-3} + (N_{uu})_{-4} + O(h^5), \quad (F.3a)\]
\[(N_{u\theta})_1 = 5(N_{u\theta})_0 - 10(N_{u\theta})_{-1} + 10(N_{u\theta})_{-2} - 5(N_{u\theta})_{-3} + (N_{u\theta})_{-4} + O(h^5). \quad (F.3b)\]

Sixth-order extrapolation:

\[(N_{uu})_1 = 6(N_{uu})_0 - 15(N_{uu})_{-1} + 20(N_{uu})_{-2} - 15(N_{uu})_{-3} + 6(N_{uu})_{-4} - (N_{uu})_{-5} + O(h^6), \quad (F.4a)\]
\[(N_{u\theta})_1 = 6(N_{u\theta})_0 - 15(N_{u\theta})_{-1} + 20(N_{u\theta})_{-2} - 15(N_{u\theta})_{-3} + 6(N_{u\theta})_{-4} - (N_{u\theta})_{-5} + O(h^6). \quad (F.4b)\]
Appendix G

Appendix G: Fixed Flow Rate Constraint in the Streamwise Flow (Time – Dependent Algorithm)

The unknown mean pressure gradient $B$ in (3.4.40b) is placed on the left-hand side resulting in the following relation

$$\sum_{k=0}^{N_K-1}\{\Gamma^2 < T_j, D^2 T_k > Gw_{1,k}^{(0)} - i\Gamma \beta \sum_{n=-N_m}^{n=N_m} \sum_{l=0}^{N_K-1} n G\psi_{l,n}^{(-n)} \} < T_j, DT_t, T_k >$$

$$+ < T_j, DT_k > ]Gw_{1,k}^{(n)} + B < T_j, T_0 > = \sum_{k=0}^{N_K-1}\left[3\Delta t^{-1} < T_j, T_k > Gw_{0,k}^{(0)} - \frac{3}{2}\Delta t^{-1} < T_j, T_k > Gw_{-1,k}^{(0)} + \frac{1}{3}\Delta t^{-1} < T_j, T_k > Gw_{-2,k}^{(0)} \right] . \quad (G.1)$$

The flow rate constraint (3.2.8) provides condition required to determine $B$, i.e.

$$Q_x = \left( \int_0^\lambda \int_{y_L(x)}^{y_U(x)} w_1 \, dy \, dx \right)_{\text{mean}} = \frac{1}{r} \left( \int_0^\lambda \int_{y_L(x)}^{y_U(x)} w_1 \, d\hat{y} \, dx \right)_{\text{mean}} . \quad (G.2)$$

Insertion of (3.4.36) and (3.4.39) into (G.2) results in

$$Q_x = \left[ \frac{1}{r} \sum_{m=-N_M}^{N_M} \sum_{k=0}^{N_K-1} Gw_{1,k}^{(m)} \int_0^\lambda l_k(x) \, e^{im\beta x} \, dx \right]_{\text{mean}} \quad (G.3)$$

where $l_k(x) = \int_{y_L(x)}^{y_U(x)} T_k(\hat{y}) \, d\hat{y}$.

Integrals $l_k(x)$ can be evaluated analytically, i.e.

$$I_0(x) = T_1[\hat{y}_U(x)] - T_1[\hat{y}_L(x)] , \quad I_1(x) = \frac{T_2[\hat{y}_U(x)] - T_2[\hat{y}_L(x)]}{4} , \quad (G.4a,b)$$

$$I_k(x) = \frac{1}{2} \left( \frac{T_{k+1}[\hat{y}_U(x)] - T_{k+1}[\hat{y}_L(x)]}{k+1} - \frac{T_{k-1}[\hat{y}_U(x)] - T_{k-1}[\hat{y}_L(x)]}{k-1} \right) , \quad k > 1 , \quad (G.4c)$$

and the above relations require evaluation of Chebyshev polynomials along the walls. The relevant process is explained in Appendix B. Equations (G.4) can be re-written as
\[ I_k(x) = \sum_{s=N+1}^{N-1} I_k^{(s)} e^{i \beta x} \quad (G.5a) \]

where \( I_0^{(s)} = E_{U,1}^{(s)} - E_{L,1}^{(s)} \), \( I_1^{(s)} = \frac{1}{4} [ E_{U,2}^{(s)} - E_{L,2}^{(s)} ] \).

\[ \hat{I}_k^{(s)} = \frac{1}{2} \left[ \frac{E_{U,k+1}^{(s)} - E_{L,k+1}^{(s)}}{k+1} - \frac{E_{U,k-1}^{(s)} - E_{L,k-1}^{(s)}}{k-1} \right] k > 1 \quad (G.5b) \]

Substitution of (G.4) into (G.3) results in

\[ Q_z = \left[ \frac{1}{r} \sum_{m=-N+1}^{N-1} \sum_{s=-N+1}^{N-1} \sum_{k=0}^{\infty} G_{w,1k}^{(s)} \hat{I}_k^{(m-s)} \int_0^\Lambda e^{im \beta x} dx \right]_{\text{mean}}. \quad (G.6) \]

The last step involves evaluation of integrals and extraction of mode zero to arrive at the final form of the volume flow rate constraint

\[ Q_z = \frac{1}{r} \sum_{s=-N+1}^{N-1} \sum_{k=0}^{\infty} G_{w,1k}^{(s)} \hat{I}_k^{(s)}. \quad (G.7) \]

The above equation is added to the algebraic system formed by (3.4.40a) and (G.1) with \( B \) being determined directly as a part of the solution process.
Appendix H

Appendix H: Orientation of Gravity Vector

Range of angles (see Fig. 6.1): $0 \leq \Lambda \leq \pi, \quad -\pi \leq \gamma \leq \pi$

Special cases:

$\Lambda = 0,$
Gravity vector acts along the negative $y$-axis for any value of $\gamma$.

$\Lambda = \pi,$
Gravity vector acts along the positive $y$-axis for any value of $\gamma$.

$\Lambda = \pi/2, \quad \gamma = 0$
Gravity vector is directed along the negative $x$-axis.

$\Lambda = \pi/2, \quad \gamma = \pi$
Gravity vector is directed along the positive $x$-axis.

$\Lambda = \pi/2, \quad \gamma = \pi/2$
Gravity vector acts along the negative the $z$-axis.

$\Lambda = \pi/2, \quad \gamma = -\pi/2$
Gravity vector acts along the positive $z$-axis.
Appendix I

Appendix I: Copyright Releases

I.1 John Wiley and Sons
This Agreement between Mr. Shoyon Panday ("You") and John Wiley and Sons ("John Wiley and Sons") consists of your license details and the terms and conditions provided by John Wiley and Sons and Copyright Clearance Center.

License Number: 5494571064137
License date: Feb 23, 2023
Licensed Content Publisher: John Wiley and Sons
Licensed Content Publication: International Journal for Numerical Methods in Fluids
Licensed Content Title: An algorithm for analysis of pressure losses in heated channels
Licensed Content Author: J. M. Floryan, S. Panday
Licensed Content Date: Nov 8, 2020
Licensed Content Volume: 93
Licensed Content Issue: 5
Licensed Content Pages: 28
Type of use: Dissertation/Thesis
Requestor type: Author of this Wiley article
Format: Print and electronic
Portion: Full article
Will you be translating? No
Title: Control of Shear Layers Using Heating Patterns
Institution name: Western University
Expected presentation date: Apr 2023
Total: 0.00 USD
I.2 John Wiley and Sons

This Agreement between Mr. Shoyon Panday ("You") and John Wiley and Sons ("John Wiley and Sons") consists of your license details and the terms and conditions provided by John Wiley and Sons and Copyright Clearance Center.

License Number: 5494580153785
License date: Feb 23, 2023
Licensed Content Publisher: John Wiley and Sons
Licensed Content Publication: International Journal for Numerical Methods in Fluids
Licensed Content Title: Time-dependent flows in grooved non-isothermal channels
Licensed Content Author: Jerzy Maciej Floryan, Shoyon Panday
Licensed Content Date: Aug 16, 2021
Licensed Content Volume: 93
Licensed Content Issue: 12
Licensed Content Pages: 25
Type of use: Dissertation/Thesis
Requestor type: Author of this Wiley article
Format: Print and electronic
Portion: Full article
Will you be translating? No
Title: Control of Shear Layers Using Heating Patterns
Institution name: Western University
Expected presentation date: Apr 2023
Total: 0.00 USD
I.3 American Institute of Physics

This Agreement between Mr. Shoyon Panday ("You") and AIP Publishing ("AIP Publishing") consists of your license details and the terms and conditions provided by AIP Publishing and Copyright Clearance Center.

License Number: 5494580667414
License date: Feb 23, 2023
Licensed Content Publisher: AIP Publishing
Licensed Content Publication: Physics of Fluids
Licensed Content Title: Creation of streaks using heating patterns
Licensed Content Author: S. Panday, J. M. Floryan
Licensed Content Date: Aug 1, 2021
Licensed Content Volume: 33
Licensed Content Issue: 8
Type of use: Thesis/Dissertation
Requestor type: Author (original article)
Format: Print and electronic
Portion: Excerpt (> 800 words)
Will you be translating? No
Title: Control of Shear Layers Using Heating Patterns

Institution name: Western University
Expected presentation date: Apr 2023
Portions: Full Article
Total: 0.00 USD
I.4 American Physical Society
This license agreement between the American Physical Society ("APS") and Shoyon Panday ("You") consists of your license details and the terms and conditions provided by the American Physical Society and SciPris.

License Content Information
License Number: RNP/23/FEB/063440
License date: 23-Feb-2023
DOI: 10.1103/PhysRevFluids.7.083502
Title: Streak creation using groove and heating patterns
Author: S. Panday and J. M. Floryan
Publication: Physical Review Fluids
Publisher: American Physical Society
Cost: USD $ 0.00

Request Details
Does your reuse require significant modifications: No
Specify intended distribution locations: Worldwide
Reuse Category: Reuse in a thesis/dissertation
Requestor Type: Author of requested content
Items for Reuse: Whole Article
Format for Reuse: Print and Electronic
Total number of print copies: Up to 10000

Information about New Publication
University/Publisher: Western University
Title of dissertation/thesis: Control of Shear Layers Using Heating Patterns

Author(s): Shoyon Panday

Expected completion date: Apr. 2023
Curriculum Vitae

Name: Shoyon Panday

Post-secondary Education and Degrees: 
- Military Institute of Science & Technology (MIST) Dhaka, Bangladesh. 2013-2016 BSc
- Western University London, Ontario, Canada. 2018-2023 Ph.D.

Honours and Awards: 
- Chancellor Gold Medal Bangladesh University of Professionals, 2017.

Related Work Experience:
- Teaching Assistant Western University 2019-2023
- Research Assistant Western University 2018-2023

Journal Publications:


Conference Articles and Abstracts:


